

THEORY AND MEASUREMENT OF
SAMPLING AND COARSE QUANTIZATION
EFFECTS IN A SIGNAL MEMORY

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University of the Witwatersrand, Johannesburg, in fulfillment of
the requirements for the degree of Master of Science in
Engineering

Somerset West, 1990

DECLARATION

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in Engineering in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.



(Ryszard Zbigniew Bil)

25th day of JUNE 1990

ABSTRACT

The effects of sampling and coarse quantisation on signal fidelity at the output of a Digital Radio Frequency Memory (DRFM) memory module have been considered. Fourier analysis on deterministic signals after quantisation and sampling has been contrasted and compared to the generalised statistical approach of randomly distributed quantisation errors within the quantising intervals. It has been shown theoretically and by measurement, that the spectra of coarsely quantised signals contain quantisation noise that is coloured, as opposed to white, especially if the sample correlation coefficients are high. Coarse quantisation produces spurs (harmonics) that appear in the message band either as aliased or unaliased spurs. Harmonic power summation yielded quantisation noise powers that correlated well with the generalised approach from three bits of quantisation upwards, and near identical results for six bits. Spectral measurements were made on a one bit quantiser and the results obtained compared favourably with the theoretical predictions, showing the harmonic spurs and the effect of spectral fold-over.

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The requirement for coarse quantisation in Digital Radio Frequency Memories (DRFMs) originates from the technological complexities and limitations involved in implementing multi-bit, extremely high speed (500MHz +) analogue-to-digital (A/D) and digital-to-analogue (D/A) conversion. The DRFM is hardware that is able to store "Radio Frequency" (RF) signals digitally for any length of time that is desired. The accuracy, or fidelity with which such a signal may be reproduced at the output of a DRFM depends upon numerous factors. These include the number of quantisation levels in the A/D converter, which affect the nature and spectrum of the quantisation error, and the combined effects of sampling, which results in aliasing of the quantised spectrum.

Other effects have not been addressed here in any detail, but will be considered briefly later. The spectra that result from coarse quantisation, combined with sampling, are of great importance in DRFM applications. These spectra were investigated and have been shown to be contrary to the standard quantisation theory of random, white quantisation noise.

The DRFM is used to store and then reproduce RF signals. It is also used to generate RF signals, or to provide an interface between the fast RF world and the relatively slow, off-line digital signal processing and instrumentation worlds. A more detailed explanation of the DRFM, its applications and its technological complexities is given below, to provide the reader

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with an understanding for the source of the problem of coarse quantisation coupled to high speed sampling.

This dissertation is concerned with the principles of operation of the DRFM and not any specific technology. However, it is because of technological considerations that the theory of operation is important. It will be shown that due to these technological problems, very coarse quantisation must be used.

1.1 Background to the Problem - The DRFM

The exact definition of RF in the abbreviation DRFM, is somewhat vague. RF may be anything from a few Megahertz to a few Gigahertz. The upper end of this range is where the technological problem lies today, and as a result is the source of the problem for this dissertation.

1.1.1 Functional Description of the DRFM

A basic block diagram of a typical DRFM is shown in Figure 1.1. It consists of a receiver, a down-conversion stage, a baseband stage and an up-conversion stage. The purpose of the receiver is to split the incoming signal(s) into distinct bands, with a bandwidth corresponding to that of the baseband DRFM, or memory module. With the absolute state of the art this bandwidth borders on a Gigahertz. More typically though, the bandwidth is currently of the order of a few hundred Megahertz.

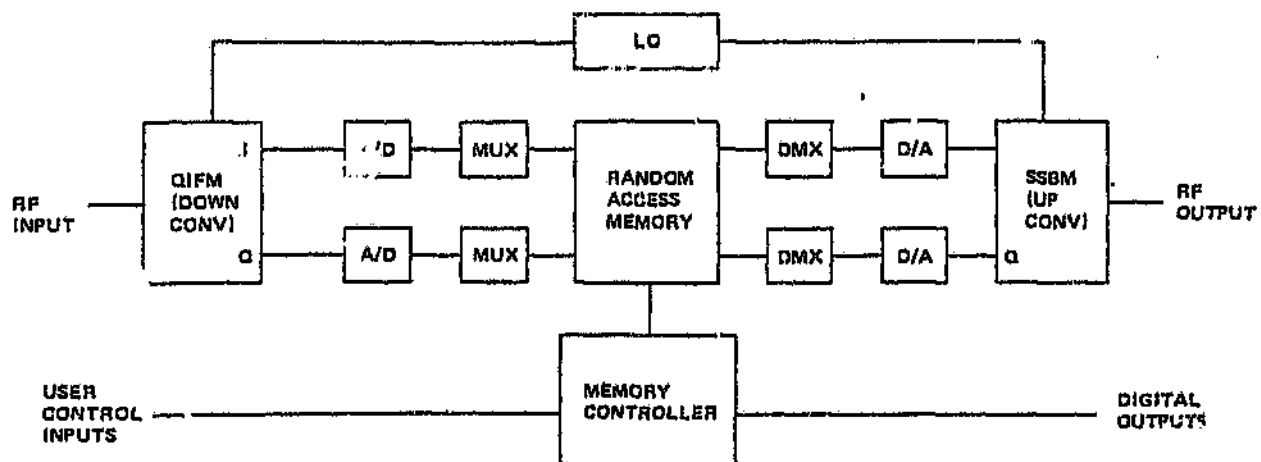


Figure 1.1 Block Diagram of a Basic DRFM - Schneider (1986)

Once split up into contiguous bands, each band is demodulated by means of a mixer and a phase-locked local oscillator. The result is a baseband of approximately $0\text{Hz} \sim F_B\text{Hz}$ (the memory module bandwidth or message band). At this point the signal enters the memory module and is passed through an A/D converter with one bit, i.e. two quantisation levels. In some of the later versions, and depending upon the application, a larger number of bits have been used. The memory module, but specifically the operation of the coarse quantisation A/D converter in it, are the subject of this dissertation.

Assuming that one bit is used, the signal is now demultiplexed by means of an N bit shift register arrangement, resulting in a parallel word of N bits. The signal is effectively "slowed down", making it more compatible with the operating speed of the memory module's random access memory (RAM). Clearly, the larger N is, the slower the clocking rate of the memory. The disadvantage is, however, that the delay between the input signal and the output signal increases, during regeneration. To overcome this problem, peripheral circuitry may be required, in order to maintain a specific minimum delay between the input and output signals, if needed for the application. Every N sampling cycles, the word is latched and then stored in RAM.

Once the bit stream has been read into memory it remains there to be read out at any time, altered or unaltered in form, as desired. During regeneration, the output bit stream from the memory is converted from parallel to serial form by a shift register clocked at the same rate as the input shift register. The digitized waveform is then passed through a digital-analog (D/A) converter, ideally resulting in a baseband signal that is identical or suitably modified compared to the input baseband

signal. The distortion in this signal as a result of coarse quantisation has been addressed in the dissertation that follows. This signal is then up-converted to its original radio-frequency (RF) by means of a mixer and the same phase-locked local oscillator as used for the down-conversion process. Finally, the various outputs from the contiguous bands are summed, and emitted or sent to further processors, if necessary.

1.1.2 Technology Involved in the DRFM

With bandwidths approaching a Gigahertz, very fast digital circuits are needed in order to achieve the sampling rates required in a DRFM. These speeds are beyond the capabilities of TTL or even ECL logic. To solve the problem of Gigahertz sampling rates, new technology GaAs logic must be used.

GaAs is capable of achieving much higher electron mobilities at significantly lower voltage levels. The propagation delays are also much lower, typically less than 300ps. GaAs has been shown to be able to operate at speeds of up to 750MHz without any problems. Laboratory developments are producing hardware with speeds of up to 2GHz. This makes it possible to transition from the analogue to the digital world of signal processing at much higher frequencies, making fully digital receivers possible.

Up till now, only broadband DRFMs with one bit quantisation have been dealt with. This is directly linked to the availability of high speed digital technology. As will be discussed later, the problem of signal fidelity lies with the coarse quantisers that are available and the resultant generation of spurious signals. To reduce the spurious signal levels, it is necessary to quantise the input signal more accurately. GaAs technology makes it

possible to achieve this goal to a reasonable degree. Hughes has demonstrated a 6-bit 750MHz A/D conversion process and by 1989, was expected to have a 10-bit DRFM operating at 1,2GHz by quadruplexing four 300MHz channels. The typical cost of a "high-performance" DRFM was quoted at about \$45 000 (Waller, 1986). It is probable that this has dropped slightly since 1986.

1.1.3 Considerations in the Design of a DRFM

Considering a sine wave input that is converted to a square wave by means of a D-flip flop or a similar device, it may be seen that the zero crossings are constrained to occur at integral sampling clock intervals. Thus a digital stream of ones and zeros that is intended to represent the input signal, is a square wave with an average period (over a few cycles) that differs from the original period. The peak error of this new period has been shown to be $\hat{\epsilon} = \pm F_u / (F_c \cdot T)$ by Horowitz & Hill (1982), where T =measurement gate width, $F_c \gg F_u$, F_c =clock frequency and F_u =input frequency.

If the input signal is a radar pulse, then the essential information, namely frequency, can be recovered using this technique. Amplitude and pulsewidth are excluded. For more exacting applications, such as television or digital oscilloscopes, better amplitude quantisation would be a necessity.

As stated earlier, there will be a large amount of spectral distortion due to the generation of square wave harmonics in a one bit quantiser. By adding parallel channels and determining when the input signal crosses through zero-points relative to a 90° phase-shifted clock, more information is obtained.

In the DRFM described in Figure 1.1, it is possible to obtain increased operating bandwidths by means of multiplexing. A variation of this is to use in-phase (I) and quadrature (Q) mixers. This effectively generates two independent samples for each sampling period, as will be shown later. Hence, the operating bandwidth may be doubled, or alternatively, the sampling rate requirements halved. Similarly, it is possible to extend this approach to N parallel channels.

1.1.4 Spurious Signal Generation

In some currently available DRFMs the spurious signal level is as high as -10dBc, which in most applications is far from ideal. The basic generators of spurious signals in a DRFM are briefly covered below for the sake of completeness:

a.) Local Oscillator Leakage. This is a practical limitation as a result of the available technology. The state of the art at present is about -30dBc for the up-conversion process.

b.) Image Frequency. Image frequency generation is a direct function of the I and Q mixing process. In practice it has been found to be difficult to achieve better than -30dBc. An example of this is the relative timing precision that must be achieved between the I and Q digital memory outputs. In order to maintain image suppression of better than 35dB, the relative timing precision must be better than two degrees. At a frequency of 200MHz this corresponds to 0.03ns!

c.) Harmonics. Harmonics will be generated due to the imperfect quantisation. In the limiting case of single bit quantisation, square wave harmonics will be obtained.

d.) Aliasing. The problem of aliasing is related to the sampling rate and the bandwidth of the input signal. This is also a function of the number of quantisation levels used in the A/D converter. Considering a one bit A/D converter, if the desired spurious signal level due to aliasing is to be better than -30dBc in the message band, the sampling frequency must exceed 62 times the true input RF sinewave frequency. This is because the 31st harmonic of a square wave has an amplitude of -29,8dBc, and all higher harmonics will fold back into the message band due to sampling.

e.) Intermodulation Products. When simultaneous input signals occur, signal suppression or capture takes place if the relative signal levels are substantially different. In the other case when the relative levels are approximately equal, intermodulation products may occur.

Harmonics and the combined effect of aliasing are the two major effects studied and reported in this dissertation.

1.1.5 Typical Parameters of DRFMs Circa 1985

Typical parameters for some DRFMs are given in Table 1.1 overleaf.

Table 1.1 Typical DRFM Parameters, Koos (1985)

Bandwidth (MHz)	20	100	460
Memory Length (us)	10000	200	150
Spurious (dBc)	-30(6bit)	-10(1bit)	-10(1bit)
D.R. In (dB)	3	10	10
D.R. Out (dB)	3	3	3
Nominal Input (dBm)	0	0	0
Independent Memory Blocks	16	16	16

1.1.6 Applications of the DRFM

One of the great advantages of the DRFM is that complex, phase coded waveforms may be handled with ease. Digital data may be manipulated according to any program as desired. The stored signal may be read out at any time or as often as desired without any degradation as a function of time. Continuous wave (CW) signals may be generated by continuous read-out of a single signal cycle.

It is also possible for stored data to be read out from memory at speeds compatible with more typical digital signal processing speeds, allowing the use of off-line Fast Fourier Transform (FFT) techniques in digital spectrum analysers.

The versatility offered by the DRFM makes it extremely useful for numerous applications, some of which have been mentioned earlier. For the reasons mentioned above, the particular aspect of coarse quantisation coupled to high speed sampling has been investigated.

Some applications of the DRFM include the storing of short RF pulses, television pictures and also the digital oscilloscope. In addition, the DRFM may be used to synthesize RF signals. It has been found in this case, that the frequency spectra produced from synthesized signals are indistinguishable from those produced after storage of a signal emanating from a signal generator, Lowenschuss (1980).

1.2 Statement of the Problem

As stated above, a DRFM has the ability to store or generate RF signals digitally. Inherent in this statement is the fact that the ANALOGUE RF signals must be converted to DIGITAL form and/or vice versa. This implies two processes, namely sampling of the analogue signal and secondly, digitisation of the signal into discrete levels. Although the principles of these processes are well known for the general case, it is not technologically possible, except under laboratory conditions, to sample and digitise RF signals fast enough and with the required degree of accuracy to ensure that the commonly assumed theories of NYQUIST SAMPLING and UNIFORMLY DISTRIBUTED RANDOM ERROR QUANTISATION may be used.

Although there are a number of other important system aspects in the design of a DRFM, only the above two mentioned characteristics will be considered any further. These are the two major characteristics of the memory module portion of a typical DRFM.

This dissertation analyses the spectra that are generated when signals are sampled and then coarsely quantised, as in a typical DRFM.

In most standard texts on sampling theory, it is firstly assumed that the input signal to a sampler is ideally bandlimited. The second assumption is that the sampling frequency is higher than the Nyquist rate of twice the signal bandwidth, thus avoiding the well-known phenomenon of aliasing. Similarly, standard quantisation theory firstly assumes that there are many quantisation levels. Secondly, this approach assumes that the error, between the actual signal level at any particular sampling point in time and its appropriate quantised level, is random and uniformly distributed within that quantising interval. Thus it is shown that the quantising noise, as the error is usually termed, is of a random Gaussian nature.

On its own, it may appear that the Nyquist criterion may be satisfied by a suitably high sampling rate and/or limiting of the signal bandwidth. It will, however, be shown that with "coarse" quantisation, neither is the quantisation noise random or white, nor is the Nyquist criterion necessarily satisfied. The term "coarse" will also be defined in more detail. Specifically coarse quantisation, or any quantisation for that matter, being a non-linear process, will be shown to spread the spectrum of the input waveform, or generate harmonics. Thus the coarser the quantisation, the more non-linear the process, and the greater the spectral spreading and generation of spurious signals.

1.3 Scope of the Investigation

The chapters that follow cover the essentials of sampling (Chapter Two) and general quantisation theory (Chapter Three), including lowpass sampling, bandpass sampling and multi-channel sampling to reduce the Nyquist sampling rate requirements. Chapter Three covers the nature of quantisation error, the

standard random noise approach, under-sampling or aliasing of the quantisation error and prefiltering.

Following on from this theoretical background, the use of Fourier Transform analysis to determine the spectra of quantisation noise is investigated in Chapter Four. Initially one bit quantisation effects are covered, but these are later extended to the N bit case. The effect of input signal amplitude relative to the dynamic range of the quantiser has been covered in terms of maximum spurious signal levels and total quantising noise power. Finally, to verify the theory, measurements have been made on a one bit quantiser and these are compared with the theoretical results in Chapter Five.

This chapter addresses numerous aspects of sampling, both theory and practice. It investigates what has been learnt in this field since the days of Shannon and Nyquist and considers some of the more advanced theories that deal with multi-channel sampling and bandpass sampling. All the topics covered are particularly relevant to the DRFM. Sampling is very important in conjunction with coarse quantisation effects, as is the technological importance of bandwidth reduction by means of multi-channel sampling.

2.1 Sampling Theory for Lowpass Signals

2.1.1 Ideal Impulse Sampling of Lowpass Signals

Consider a bandlimited lowpass signal that has the following relation. The signal itself is given by $x(t)$ and its Fourier transform by $X(f)$, such that $X(f)=0$ for $|f|>f_x$, where f_x is the signal bandwidth.

Ideal sampling of $x(t)$ is by definition instantaneous, and may in theory be accomplished by using an impulse train as the sampling function, namely $x_s(t)=x(t) \cdot s_s(t)$, where $s_s(t)$ is the sampling pulse train given by $s_s(t)=\sum_{-\infty}^{\infty} \delta(t-kT_s)$.

It then follows that:

$$\begin{aligned} x_{\delta}(t) &= x(t) \cdot \sum_{-\infty}^{\infty} \delta(t - kT_s) \\ x_{\delta}(t) &= \sum_{-\infty}^{\infty} x(kT_s) \cdot \delta(t - kT_s) \end{aligned} \quad (2.1)$$

Thus:

$$X_{\delta}(f) = X(f) * S_{\delta}(f)$$

and since:

$$S_{\delta}(f) = f_s \cdot \sum_{-\infty}^{\infty} \delta(f - nf_s)$$

then:

$$X_{\delta}(f) = f_s \cdot \sum_{-\infty}^{\infty} X(f - nf_s) \quad (2.2)$$

Figure 2.1, overleaf, illustrates the process. In order to recover the signal so that it is unaliased due to the overlap of the spectral replicas, as indicated in equation 2.2 above, it is well known that the original signal $x(t)$ must be sampled at a rate at least twice the maximum significant frequency f_x , in the spectrum of $x(t)$ (i.e. $X(f)$), according to the Nyquist criterion. Secondly, the baseband signal is recoverable by means of an ideal lowpass filter, as illustrated overleaf, if the filter has a bandwidth defined by B , such that $f_x \leq B \leq f_s - f_x$.

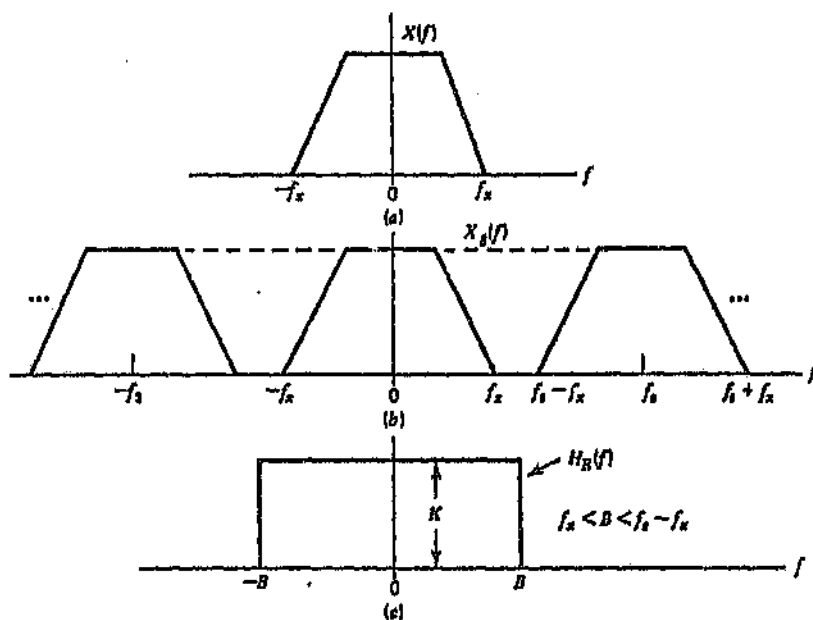


Figure 2.1 Impulse Sampling Process - Shanmugam (1979)

If the filter gain is allowed to be $K=1/f_s$, then from Figure 2.1 and the above analysis, it follows that $X(f)=X_\delta(f) \cdot H_R(f)$. Thus:

$$x(t)=F^{-1}\{X_\delta(f) \cdot H_R(f)\}$$

and:

$$x(t) = [h_R(t) * x_\delta(t)]$$

where $h_R(t)$ is the impulse response of the reconstruction filter.

Since $H_R(f)$ is given by:

$$H_R(f) = T_s \cdot \text{rect}(f/2B)$$

then:

$$h_R(t) = 2BT_s \cdot \text{sinc}(2Bt)$$

Therefore:

$$\begin{aligned} x(t) &= [2BT_s \cdot \text{sinc}(2Bt)] * \left[\sum_{k=-\infty}^{\infty} x(kT_s) \cdot \delta(t - kT_s) \right] \\ &= 2BT_s \cdot \sum_{k=-\infty}^{\infty} x(kT_s) \text{sinc}[2B(t - kT_s)] \end{aligned} \quad (2.3)$$

This results in the classic form of the uniform sampling theorem, Shanmugam(1979).

2.1.2 Finite Aperture Width Sampling of Lowpass Signals

The analysis above described the well known case of ideal impulse sampling. However, in practice it is found that the sampling train function consists of repetitive flat topped pulses with a finite aperture or pulse width. In the ideal case the spectral replicas have equal amplitudes for all corresponding frequencies. This is due to the fact that the Fourier coefficients of an impulse train are the same for all harmonic frequencies. However, in the case of a finite aperture width pulse train, this is no longer the case. The harmonics follow the form of the classic sampling function, $\sin(x)/x$ (c.f. Appendix A). This implies that the spectral replicas have decreasing amplitudes with increasing frequency, as shown overleaf. This decaying spectrum helps to reduce aliasing effects when the signal is possibly undersampled, as a result of it not being strictly bandlimited. With narrow aperture flat topped sampling pulses (i.e natural sampling, not

resulting in flat topped samples) the resulting spectrum is thus a good approximation to ideal impulse sampling, Shanmugam (1979).

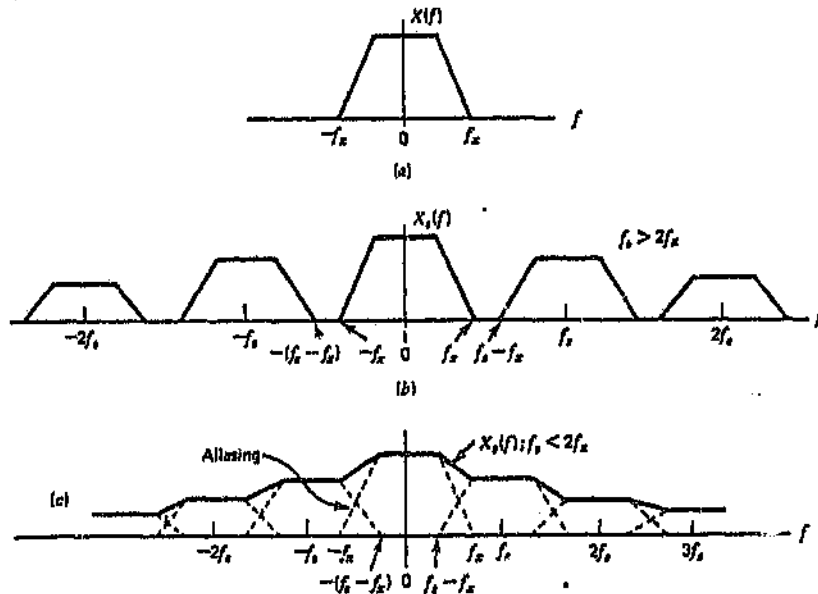


Figure 2.2 Sampled Spectrum with Narrow Finite Aperture Width
Sampling - Shanmugam (1979)

With a sample-and-hold which results in flat-topped samples, the finite aperture samples result in the attenuation of the high frequency components of the baseband signal, i.e. baseband distortion. A reasonable tradeoff should be made of this attenuation versus aliasing improvement. The attenuation is known as the aperture effect and may be compensated by equalization of the digital channel after reconversion to analogue form has been completed. In practice, if the sampling pulse train pulsewidth is small enough ($\tau \ll 1/f_s$), then it does not introduce any significant amplitude distortion in the spectrum of the sampled signal over the message band and so no equalisation may be required.

2.2 General Aspects of the Sampling Theorem

Jerri (1977) has provided a tutorial review on the various forms and applications of the sampling theorem. For the purposes of any

future analysis, the theoretical developments in 2.1.1 above will form the basis of all arguments. Jerri's (1977) review paper has many interesting results and numerous formulations of the sampling theorem. However, these cloud the issue if introduced in too much depth and so only results that are relevant to the DRFM problem will be shown.

An extension of the basic sampling theorem has been presented by Fogel and illustrated by Jerri (1977). He states that if a function contains no frequency higher than $f_x(\text{Hz})$, then it is determined by giving M function derivative values at each of a series of points extending throughout the time domain and with a sampling interval given by $T_s = (M/2f_x)$. Thus if M facts about a function are known at any sampling instant, it is possible to reduce the sampling rate requirements by the factor M .

This approach is used in aircraft navigation systems in order to determine course position from velocity and acceleration at half the nominal Nyquist sampling rate, by the knowledge of two derivatives of the course position. Obtaining M derivatives of a function to reduce sampling rate requirements is not directly applicable to the DRFM, primarily due to the extremely fast processing speeds required, leaving little time to evaluate derivatives of the input function at each sampling point. However, an alternative is to use multiplexed (parallel channel) sampling. In this case, the sampling rate is inversely proportional to the number of channels used and is directly proportional to the Nyquist rate, Jerri (1977).

All of the above are termed explicit sampling, i.e. the bandlimited function is represented by its sample values $x(t_n)$ at preselected instants of time (t_n) which are independent of $x(t)$.

Another approach is implicit sampling. In this case, the function is represented in terms of the instants of time (t_n) at which the function assumes a predetermined value, e.g. zero-crossings or its crossings with a cosine or other function.

Other aspects of the sampling problem are errors introduced by various inaccuracies. Truncation errors are caused by using only a finite number of samples. In general, in the case of the DRFM, most of the signals that are analysed are pulsed waveforms of one type or another. Thus as long as the memory available is sufficiently long to store the complete pulse, this is not a serious problem. Truncation error is analysed more fully in Appendix B for the sake of completeness.

A further related aspect is that of quantization or amplitude error. This is a result of the uncertainty in measuring the amplitude of the sampled function values. It may intuitively be seen that the coarser the degree of quantization, the larger and hence the more significant will be the harmonics that are, as a result, generated. Hence the concept of bandlimitedness will be violated further, resulting in more aliasing error. This aspect is considered in greater depth in Chapter Three. Another type of error is jitter error. This is caused by sampling at instants that differ slightly from the ones actually required.

2.3 Sampling Theorem for Random Lowpass Processes

Let $\{X(t)\}$ be a continuous-time wide-sense stationary random process with a bandlimited power spectral density (PSD). It is well known that the PSD is the Fourier transform of the autocorrelation function $R_{XX}(\tau)$.

Hence:

$$S_{XX}(w)=0, \quad |w| > w_B, \quad w_B = 2\pi f_X$$

where:

$$S_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-jw\tau) d\tau$$

and:

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

Since the power of the random process is finite, for non-zero τ , there is an upper bound on the autocorrelation function:

$$E[X^2(t)] = R_{XX}(0); \quad |R_{XX}(\tau)| \leq R_{XX}(0)$$

Individual sample functions of the process do not possess Fourier transforms because their energies are not finite Jayant & Noll (1984). However, from samples $X(kT_s)$ the ideal $X'(kT_s)$ can be reconstructed using:

$$X'(t) = \sum_K X(kT_s) \text{Sa}\{(\pi/T_s) \cdot (t - kT_s)\}; \quad (2.4)$$

$X'(t)$ may be considered to be a sample function of a random process $\{X'(t)\}$. The error function is $e(t) = X(t) - X'(t)$. This has a mean-square value given by:

$$\sigma_e^2 = E[e^2(t)] = E[(X(t) - X'(t))^2] = 0 \quad \text{if } T_s \leq 1/2f_X \quad (2.5)$$

The proof of this statement is given in Appendix C and follows from Jayant & Noll (1984). The result is that both stationary random lowpass processes and deterministic lowpass signals may be handled identically.

2.4 Multi-Channel Sampling of Lowpass Signals

Brown (1981) considered multi-channel sampling of lowpass signals. This topic is relevant to the DRFM because of the very large bandwidths under consideration, and hence the very fast sampling rates. The outcome of Brown's analysis results in

important conclusions if $x(t)$ is bandlimited to $|w| \leq w_B$. It is assumed that $x(t)$ is passed through m linear time-invariant filters (channels) to obtain m outputs $z_1(t), z_2(t), \dots, z_m(t)$. If the filters are independent in a sense that has been defined by Brown (1981), then $x(t)$ may be reconstructed from samples of the outputs $\{z_k\}$, each output being sampled at $w_B/(m\pi)$ samples per second or $1/m$ times the rate associated with the input signal, as determined by Nyquist's theorem.

In general, the sampling rate may be reduced by a factor m , if m independent pieces of information are available about a band-limited signal at each of the sampling points. This differs slightly from the concept of multiplexed sampling which has yet to be considered. Brown's approach merely states that $x(t)$, as defined, and its $m-1$ derivatives, each at a rate as defined above, suffices for complete signal reconstruction. This concept was also discussed by Jerri (1977) in his review paper without proof. Brown (1981) proved this concept, which he also mentions as being stated by Shannon without proof.

As stated earlier, this technique, although it conceptually reduces the sampling rate requirements, does not really help to alleviate the speed problem in the DRFM, since it is still necessary to determine the $m-1$ derivatives of $x(t)$. This unfortunately adds to the processing time. The more ideal concept of multiplexed sampling, will be considered in the following section on more general bandpass signals.

2.5 Sampling Theory for Bandpass Signals

Since in the case of the DRFM, bandpass signals are normally considered at some fairly high nominal radio frequency, e.g. a

band of bandwidth 100MHz, centered at a frequency of 500MHz, bandpass sampling is of particular relevance. It is possible to effectively "undersample" the signal at a rate determined by the signal bandwidth and its centre frequency. This rate is a fraction of what would normally be required in terms of standard lowpass sampling theory and hence holds obvious advantages for the DRFM. The requirements for using this approach will now be investigated.

2.5.1 Intuitive Approach to Bandpass Sampling Requirements

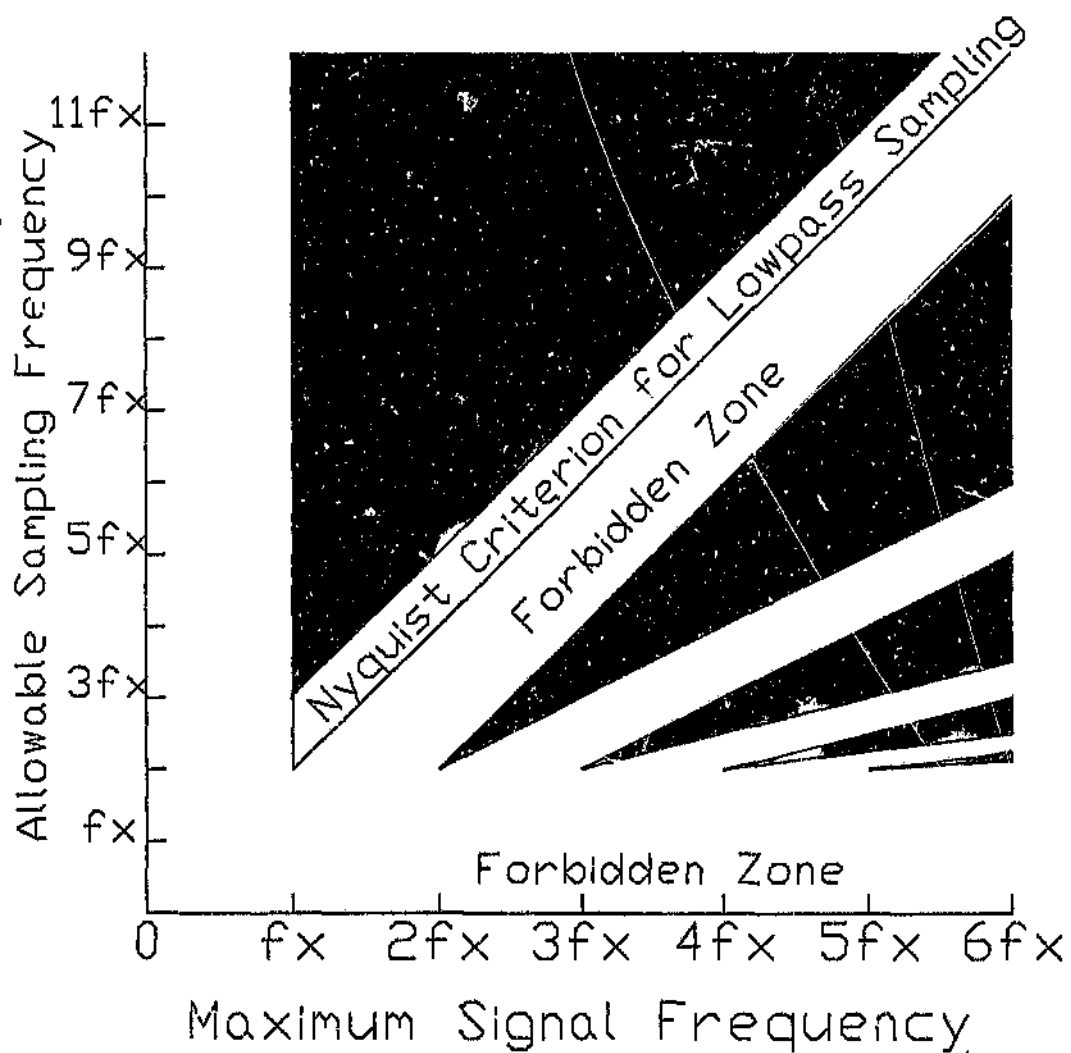
Shanmugam (1979) and Stremmer (1979) both show diagrams of the minimum sampling frequency required for a signal described by a bandwidth and the highest or lowest signal frequency, respectively. If the bandwidth is f_x , then it may be shown that the sampling rate of $2f_{xu}/m$, where the upper signal frequency limit is f_{xu} and m is the largest integer not exceeding f_{xu}/f_x , will ensure that $x(t)$ may be represented by its sample values of $x(kT_s)$.

Shanmugam (1979) further states that it may not always be possible to use higher sampling rates, unless these exceed the fundamental lowpass requirement that the minimum sampling rate is $2f_{xu}$. In order to quickly establish the implications of this statement, some basic theory was considered together with the use of illustrations. The details of this intuitive approach are given in Appendix D and the results of this approach are shown overleaf in Figure 2.3, in what is an extension of the basic illustrations shown in Shanmugam (1979) and Stremmer (1979).

From Figure 2.3 it is clear that the minimum sampling frequency for bandpass signals lies somewhere between two and four times

the signal bandwidth. However, there are definite areas of forbidden operation, as shown. The higher the centre frequency, the narrower the area of valid operation for "sub-Nyquist" sampling. Clearly, in the case of the DRFM, the centre frequency is high compared to the signal bandwidth and so the danger of slipping into the forbidden area is significant.

Hence, because the exact input signal frequency and characteristics are not known, the bandpass sampling approach is not particularly valuable in DRFMs.



f_x =signal bandwidth

Figure 2.3 Sampling Frequency Requirements for a Bandpass Signal

2.5.2 Quadrature Sampling of Bandpass Signals

Brown (1979) considered the quadrature sampling of bandpass signals, which is basically an extension of the intuitive analysis shown in 2.5.1 above. The signal under consideration has a non-zero signal spectrum for $0 \leq \omega_0 - \omega_B/2 \leq \omega \leq \omega_0 + \omega_B/2$, $\omega_B > 0$, as shown below in Figure 2.4.

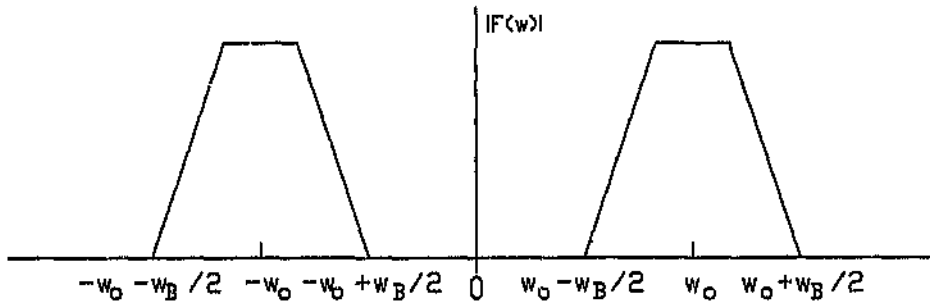


Figure 2.4 Bandpass Signal

Brown (1979) also stated that Grace & Pitt require uniform sampling of both the bandpass signal and its quarter wavelength translation (based on nominal ω_0) at a common sampling rate, depending upon the exact relationship between ω_0 and ω_B . If the intersample spacing is chosen "properly", then the bandpass signal may be entirely reconstructed from the known sample values. In the lowpass case it was observed that $x(t) = \sum_n x(n\pi/\omega_B) \text{Sa}(\omega_B(t - n\pi/\omega_B))$; if $-\omega_B \leq \omega \leq \omega_B$, there are ω_B/π samples per second, the sampling interval is $T_s = \pi/\omega_B$ and ω_B = angular bandwidth for positive frequencies only.

In the bandpass case, the angular bandwidth is still ω_B , but $T_s = \pi/\omega_B$ no longer characterises the signal uniquely. If and only if $\omega_0 + \omega_B/2 = n\omega_B$ with $n > 0$ then $T_s = \pi/\omega_B$ suffices, Brown (1979).

In the absence of the above-mentioned positioning assumption, reconstruction can only take place if $\pi/\omega_B < T_s < \pi/2\omega_B$, depending on

the exact relationship between w_0 and w_B . Another approach is to utilise the Hilbert transform, in which case $x(t)$ and its transform $\hat{x}(t)$ can each be sampled at a uniform rate of $w_B/2\pi$, i.e. at half the Nyquist rate. Unfortunately it is still necessary to produce the Hilbert transform and as a result this method is not of much use for the DRFM. It is worth noting that this is termed quadrature sampling by some, although this is not strictly speaking true.

Of greater importance is quadrature sampling, basically being a two channel multiplexed system. Both $x(t)$ and $x(t-f_0/4)$ are each sampled at a rate depending on the exact positioning of the signal band as mentioned earlier. The sampling rate may be reduced to half the nominal Nyquist rate, i.e. w_B/π samples per second, if and only if $w_0 = mw_B/2$ and m is an even positive integer.

2.5.3 Multi-Channel Bandpass Sampling

Figueiras-Vidal et al. (1982) extended this concept to more than two channels. If there are N channels, then if w_0 is not equal to $kw_B/2N$ and w_0 must be maintained because it is a carrier frequency or because there is some symmetry associated with w_0 , then one may use the smallest $w_B' > w_B$ such that $w_0 = kw_B'/2N$. This value of w_B' is given by $w_B' = 2Nw_0/[2Nw_0/w_B]$, where the operation $[.]$ implies "the largest integer contained in", e.g. $[\pi] = 3$.

The corresponding new sampling rate is thus given by:

$$1/T_s' = w_B'/2\pi N = w_0/\pi[2Nw_0/w_B]. \quad (2.6)$$

Figueiras-Vidal et al. (1982) also state that in general, band limitation is not a physical possibility, as is widely known, and therefore $X(w)$ does not really have a centre frequency associated

with it. It is however possible to specify a minimum sampling rate, corresponding to a new angular bandwidth w_B'' , without maintaining w_0 . Hence $w_B'' = (2Nw_0 + w_B) / [(2Nw_0 + w_B) / w_B]$, giving $T_s'' = 2\pi N / w_B''$ and $w_0'' = w_0 + w_B / 2 - w_B'' / 2$.

Therefore, it is possible to reconstruct the signal from the samples of the outputs of $2N$ linear, time-invariant systems driven by it at a sampling rate $1/2N$ times the nominal Nyquist rate. There are some conditions associated with this statement, namely that the transfer functions of the system have a non-singular matrix, the centre angular frequency and the signal bandwidth are related by $w_0 = kw_B / 2N$ (k is an integer) and finally that the impulse responses of N systems have zero quadrature components, while the other N systems have zero in-phase components.

It has also been shown that this concept may be extended when the special relationship between w_0 and w_B no longer holds. An illustrative example is the case of a four channel system when all the linear systems are pure delays. In this case the channels are all 90° out of phase with respect to each other in terms of the centre frequency w_0 . Thus the sampling rate is a quarter of the nominal Nyquist rate and this is of extreme relevance to the DRFM application.

2.6 Summary

An investigation of sampling theory and practice, with special relevance to the DRFM has been carried out. Impulse sampling has been used as the underlying theory of the investigation. Differences between actual finite aperture width sampling and impulse sampling have been considered and it was shown that for

most practical purposes, the ideal case is a good enough approximation of reality.

Special theories relating to different forms of sampling theory that aim to reduce the sampling rate were investigated. Some of these theories consider the use of signal derivatives and Hilbert transforms of the input. This approach is not practical for the DRFM because time is a serious constraint.

A variation on this theme is the use of multi-channel sampling such as four channels, where the channels sample signals that are time-shifted by 90° at the nominal signal centre frequency. This effectively generates four independent samples within one nominal signal cycle, but slows down the sampling rate requirements in each channel by an equivalent amount (four in this case).

Another variation is to time phase the sampling clock relative to the input signal, thus again generating four independent samples per clock cycle, while still maintaining the slower sampling rate in each channel. It must be noted though, that the relative timing precision must be well controlled so as not to generate serious errors through channel cross-talk.

Bandpass sampling was considered, since it is possible to reduce the sampling rate requirements, based on lowpass theory, in special cases. Thus the sampling frequency is a function of the bandwidth rather than the maximum signal frequency. Unfortunately, in order to exploit this method it is necessary to maintain special relationships between the signal maximum or minimum frequencies and the signal bandwidth. If this is not achieved, then aliasing will occur. It was also shown that the special requirement becomes difficult to meet at higher frequencies and hence not very useful. Mixing the signal down to

a baseband will help to alleviate this problem when dealing with DRFMs.

Quantization theory as applied to deterministic and random signals will be dealt with. The specific case of one bit quantization has been considered in depth, as it is vital to the analysis of DRFMs. Both the generation and sampling of signals by means of the one bit technique have been investigated.

Having considered sampling theory for various theoretical situations in Chapter Two, it may be said that all of the above cases assume an exact analogue or extremely accurately quantized replica of the sampled signal amplitude. In the case of the DRFM, it is at present not easy, or at all technologically possible, to obtain a fast analogue-to-digital (A/D) converter with more than a "few" bits. "Few" can vary from one to as much as six, depending strictly upon cost and/or bandwidth.

The question of how the number of bits affects the generation of spurious signals and to what degree this affects the desired signal fidelity remains to be answered. Both theory and practice will be covered and it is assumed that the most basic concepts of quantisation noise are understood. For completeness, this topic has been covered in Appendix E.

3.1 General Aspects Of Quantization

An important consideration when analyzing quantization, is to consider the real life situation. In practice one tends to sample

and then quantize the signal. On the other hand, when considering what the output signal waveform will be like following an A/D plus D/A conversion block plus sampling process, it is often easier to quantize an analogue signal mathematically first, into a series of rectangular pulses that have an amplitude dependent upon the quantizer characteristic and are valid while the input signal remains in the quantising interval. This gives a discrete amplitude, but continuous time waveform.

After sampling, the waveform becomes discrete amplitude-discrete time and is identical to the sampled and then quantised waveform. It then follows that Fourier analysis will allow one to obtain the "baseband" spectrum of the quantized signal, without introducing the complications of sampling. Once the baseband spectrum has been obtained, the concepts associated with sampling, namely aliasing or intermodulation due to spectral fold-over will be determined, as a function of sampling frequency.

This concept allows the portrayal of the unsampled, discrete amplitude-continuous time signal spectrum, as opposed to the added complications associated with the Fast Fourier Transform (FFT), such as aliasing, picket fence effects and leakage, when the FFT is used to evaluate the spectrum of a quantised and sampled signal. Thus, in a nutshell, the "quantization before sampling" analysis allows the determination of errors introduced by quantization first (harmonic distortion), and then the effects of aliasing separately. The FFT is, however, useful for determining the overall sampled and quantised signal spectrum.

The statistics of the input signal, the degree of quantization, and the sampling rate, together influence the aliasing

distortion, since the smaller the degree of quantization, the greater the harmonic content of the quantized signal and hence the greater the influence of the higher harmonics on the desired message band once spectral foldover has occurred as a result of sampling.

The fundamental concept here is that of the mathematical commutativity of sampling and quantization. The problem is that the two functions are generally handled totally independently. Jayant and Noll (1984) state, without proof, that the operations of sampling and quantization could be performed in either order, but in practice sampling precedes quantization. A non-rigorous approach to proving this concept is presented in Appendix F. Jayant and Noll (1984) also state that amplitude quantization determines the overall distortion and bit rate.

If an input signal to a quantizer is $x(t)$ and the analogue reconstruction is denoted by $y(t)$, then the quantization error (for ideal quantization) may be given by:

$$q = x - y = x - Q(x)$$

where $y \in \{y_1, y_2, y_3, \dots, y_L\}$ for L levels of quantization. Hence $y = y_k$ if $x \in I_k$ and $y = Q(x)$ is the mapping function. Thus the quantization error is totally deterministic for a given x , because $x \in I_k$ leads to a single output y_k . However, if x is an outcome of a random variable X , q may be considered to be the outcome of another random variable Q .

This leads to the concept of quantization noise to describe quantization errors. Also, quantization may be modeled analytically as additive noise if the quantization is very fine. With coarse quantization, the errors are input-dependent. To make the errors appear like input-independent additive noise,

dithering is used. In this case, quantization noise is deliberately introduced and in general is signal-correlated, Jayant and Noll (1984).

3.2 Quantization Error Variance

Quantisation error variance is considered for the specific case of the uniform quantizer in Appendix E. However, it is useful at this stage to consider some of the statistical aspects of quantization. Up to this point in time, the random variable X has been considered to have a zero mean. If X has a non-zero mean, then one can subtract this value before quantization and add it back after quantization, or shift the centre of the quantizer to the mean value.

As stated above, the quantizer maps X into the discrete random variable $Y=Q(x)=y_k$ if X is an element of the interval I_k . Therefore the quantizer error $Q=X-Y$ is also a random variable with probability density function (pdf) $p_Q(\cdot)$ and the variance is given by:

$$\sigma_Q^2 = E[q^2] = \int_{-\infty}^{\infty} q^2 p_Q(q) dq$$

or because $q=x-Q(x)$ is a function of x , then:

$$\sigma_Q^2 = \int_{-\infty}^{\infty} (x-Q(x))^2 p_X(x) dx \quad (3.1)$$

By breaking up the region of integration into L intervals I_k and considering a uniform quantizer as before, the same results as in Appendix E are obtained. The assumptions required to obtain these results are more rigorously specified by Jayant and Noll (1984). The quantization intervals or step sizes are given by:

$$\Delta = 2x_{\max}/2^R \quad (3.2)$$

where $x_{\max}$ is the amplitude of $x(t)$ if $x \in [-x_{\max}, x_{\max}]$ and R is the number of bits. Hence the quantization error variance is

given by:

$$\sigma_q^2 = x_{\max}^2 \cdot (2^{-2R})/3 \quad (3.3)$$

Thus it is clear that the rms value of the quantization noise decreases exponentially with increasing bit rate.

3.3 Granular Noise and Overload Distortion

If x_{ol} is the maximum quantizing value or overload amplitude, then for all $|x| > x_{ol}$, the signal is part of the overload region. Errors in the finite quantizable range $(-\Delta/2, \Delta/2)$ are called granular errors, Jayant and Noll (1984).

Overload errors have certain characteristics associated with them. These are bursts of $q(n)$ samples of identical polarity that occur if the overloading input signal $x(n)$ is slowly varying. This is generally the case with most Nyquist sampled and over-sampled waveforms.

Overload error is, in general serious, because the error samples are unbounded for an unbounded input. By ensuring a proper quantizer design, the overload probability may be reduced, or even made zero if the input pdf is bounded to $\pm x_{\max}$ and the overload amplitude is designed to be $x_{ol} \geq x_{\max}$.

Granularity errors refer to the rapid oscillatory or "fine-grained" structure of $q(x)$ in the non-overload region. The difference in the two error spectra is reflected by significant differences in the perceptual annoyance of the two error types, in the case of speech waveforms.

In the case of a uniform midrise quantizer, $x_{ol} = 2^{R-1} \cdot \Delta$; and for a uniform midtread quantizer, $x_{ol} = (2^{R-1} - 0.5) \cdot \Delta$. The total quantization error variance is:

$$\sigma_q^2 = \sigma_q^2(\text{granular}) + \sigma_q^2(\text{overload}) \quad (3.4)$$

$$\sigma_q^2(\text{granular}) = \sum_{k=2}^{L-1} \int_{x_k}^{x_{k+1}} (x - y_k)^2 p_X(x) dx + 2 \int_{x_L}^{x_{01}} (x - y_L)^2 p_X(x) dx \quad (3.5)$$

and:

$$\sigma_q^2(\text{overload}) = 2 \int_{x_{01}}^{\infty} (x - y_L)^2 p_X(x) dx \quad (3.6)$$

Jayant and Noll (1984) also state that the probability of overload for a random variable X with a Gaussian pdf is $P(|x| > x_{01}) = 1 - \text{erf}(x_{01}/\sqrt{2} \cdot \sigma_X)$. For example, if $x_{01} = 4\sigma_X$, then $P_{01} = 0,000\ 064$ and if $x_{01} = 2\sigma_X$, then $P_{01} = 0,046$.

It is possible to define a loading factor $f_1 = x_{01}/\sigma_X = LA/2\sigma_X$ which defines the half-amplitude range of the quantizer in multiples of the standard deviation of the signal to be quantized. In terms of signal-to-quantizing noise ratio then, it may be shown that for a random Gaussian variable and uniform granular error:

$$\text{SNR} = \sigma_X^2 / \sigma_q^2 = 3L^2 / f_1^2$$

and if the number of quantisation levels is 2^R then:

$$\text{SNR}(\text{dB}) = 6,02R - 10 \log_{10}(f_1^2/3); R \gg 1 \quad (3.7)$$

in the case of uniform quantization with zero or negligible overload.

For a uniform granular error pdf with $x_{\max} = x_{01}$ and a random variable with an arcsine pdf, it may be shown that if:

$$f_1 = x_{\max}/\sigma_X = \sqrt{2}$$

then:

$$\text{SNR}(\text{dB}) = 6,02R + 1,76 \quad (3.8)$$

Jayant and Noll (1984) also state that equation 3.7 may be used if there is an unbounded input with negligible overload probability. In this case the overload errors can be neglected.

Hence, if $f_1 = 4$, then:

$$\text{SNR}(\text{dB}) = 6,02R - 7,27 \quad (3.9)$$

From the above examples of different pdfs, it follows that the choice of $f_1=4$ does not necessarily give the minimum overall (granular + overload) noise. Therefore the quantizer characteristic may be optimized in order to minimize this noise as a function of the signal pdf and the number of quantization bits.

3.4 Statistical Properties of Quantization Errors

It was stated earlier that for a specific input, x , the output of a quantization process, y , may be given deterministically by $y=x-q$, where q is the quantization error. But, if x is a realization of a random variable X , then q may be considered to be a realization of a random variable Q . Jayant and Noll (1984) then show that the assumption of a uniform pdf for q is valid if $R \geq 2$ and that the assumption of a white quantization error is valid if $R \geq 2$ and if the adjacent sample correlation of the input, ρ_{xx} is not too close to 1. An intuitive approach to this problem is illustrated in Figure 3.1 below.

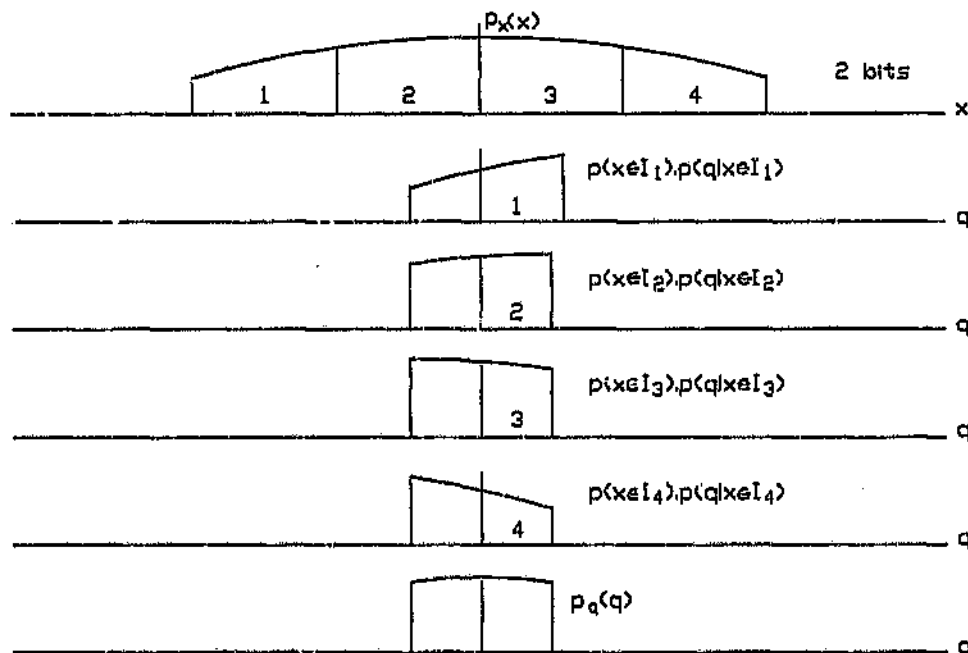


Figure 3.1 Intuitive Approach to Quantization Error Statistics

Sripad and Snyder (1977) show that for a random Gaussian variable, if the ratio of the standard deviation to the quantisation interval is greater than 0,7, then the pdf of the quantisation error may be assumed to be uniform across the quantisation interval. With one bit quantisation, this requirement will not be satisfied. Assume a Gaussian variable, such that the $\pm 3\sigma$ limits are within the quantiser limits, i.e. no significant overload distortion. With a one bit quantiser then, $\sigma/\Delta = 0,33$. Similarly with a two bit quantiser, $\sigma/\Delta = 0,67$. For quantisers with more than two bits, $\sigma/\Delta > 0,7$, satisfying Sripad and Snyder's (1977) requirements for a uniformly distributed quantisation error within the quantising interval, as well as the assumptions of Jayant and Noll (1984) above.

As far as the correlation between the quantisation noise and the input signal is concerned, Sripad and Snyder (1977) state that the correlation becomes negligible when $\sigma/\Delta \gg 1$. This is only satisfied when the number of quantisation levels is reasonably large. It will be shown later, that even though for three bit quantisation the total quantisation noise power is adequately predicted by the uniform pdf model, the actual spectrum of the quantisation noise is still well correlated with the input signal.

Up until now, all analysis has assumed that if a uniform quantizer is used, then the probability density of the error within each quantization interval is of a uniform nature. At this stage it will be useful to consider the problem in more detail. Reference may be made to Figure 3.1. Considering a quantizer that is not necessarily uniform, with 2^R levels, then the errors produced in each decision or quantization interval I_k contribute to the total quantization error and are a function of the

probability $P_k = P(X \in I_k)$, which is the probability that one finds the amplitude of the input signal in this interval. Therefore the total pdf $p_q(\cdot)$ is the sum of the L conditional PDFs weighted by the corresponding probabilities, since the events $(X \in I_j)$ and $(X \in I_k)$ are mutually exclusive.

Hence:

$$p_q(q) = \sum_{k=1}^L P_k \cdot p_q(q | X \in I_k) \quad (3.10)$$

$$p_q(q | X \in I_k) = p_n(q + y_k | X \in I_k) \quad (3.11)$$

but:

$$p_x(x | X \in I_k) = (1/P_k) p_x(x); \quad X \in I_k \quad (3.12a)$$

$$= 0; \text{ otherwise.} \quad (3.12b)$$

Thus $p_q(q)$ is a superposition of sections of the input pdf $p_x(x)$. These sections are given by the decision intervals I_k and are recentered by shifting I_k by a distance y_k (c.f. Figure 3.1). The PDF $p_q(q)$ is guaranteed to be flat for all symmetric input PDFs if the PDF within a segment is a straight line, if the quantization is uniform, if the reconstruction levels are the midpoints of the corresponding decision intervals and if the input is bounded in amplitude, Jayant and Noll (1984).

Further, even if the the input random variable is unbounded, a flat-topped PDF is a useful approximation if the number of output levels is large. Thus:

$$p_q(q) = (1/\Delta) \text{rect}(q/\Delta) \text{ for large } L. \quad (3.13)$$

If $Q(m)$ and $Q(n)$ are random variables defining quantization errors at times m and n , then $R_{qq}(m-n) = E[Q(m)Q(n)]$, and taking Fourier Transforms yields $S_{qq}(w)$, the power spectral density. This may be determined if the joint PDF $p_{q(m)q(n)}[q(m)q(n)]$ is known. If L is large, there are many changes in sign of the error

waveform. If error samples are taken at the waveform Nyquist rate, then they are statistically independent, unless the waveform does not change much in amplitude in a Nyquist interval (i.e. $p_{xx}(1) \rightarrow 1$), or if the waveform has periodicities that correlate distant error samples, or if the quantizer only has a few intervals, so that small error oscillations do not exist.

This is of particular relevance to the DRFM because first of all the number of quantization intervals is in general quite small, and secondly because the input signals are quite often of a pseudo-deterministic nature (pulse modulated sinusoidal waveforms) with significant sample correlation coefficients. A useful interpretation of the above may still be made, even with reference to the DRFM. In the case of statistical independence there is the implication that the joint PDF is a product of the individual PDFs, thus:

$$P_{q(m)q(n)}[q(m), q(n)] = 1/\Delta^2; \quad -\Delta/2 \leq q(m), q(n) \leq \Delta/2 \quad (3.14a)$$

$$= 0; \text{ otherwise.} \quad (3.14b)$$

The spectral interpretation of equation 3.14, Jayant and Noll (1984), is that the unsampled error waveform has a spectrum that is much broader than the signal spectrum. Sampling of the quantized signal implies an undersampling of the error waveform and thus aliasing must occur. Therefore the total noise spectrum has many components that will fold back into the message band. As a result it may be said that the noise spectrum flattens and also that the baseband power equals the variance of the unsampled error waveform.

An analysis of the quantization noise must take into account the fact that the noise is not purely additive and not uncorrelated with the input. In particular, at low bit rates, say $R=1$

bit/sample, the errors in quantization must necessarily be considered as belonging to two classes, one caused by a change in level and the other caused by uncorrelated noise, Jayant and Noll (1984).

3.5 Spurious Signals and Errors

A number of sources of spurious signals are considered, such as the error due to bandlimiting a signal with prefilters or anti-alias filters, the error due to aliasing and so forth.

3.5.1 Effective Bandwidth

A signal $x(t)$ has a Fourier Transform $X(w)$. Now if $x(t)$ is bandlimited to a bandwidth $f_x (=B)$, then one obtains a new signal $\hat{x}(t)$. Hence the error signal is:

$$e(t) = x(t) - \hat{x}(t)$$

and the corresponding error signal energy is:

$$E_e(w) = 1/(2\pi) \int_{-\infty}^{\infty} |X(w) - \hat{X}(w)|^2 dw \quad (3.15)$$

This is illustrated in Figure 3.2 below, Lathi (1983).

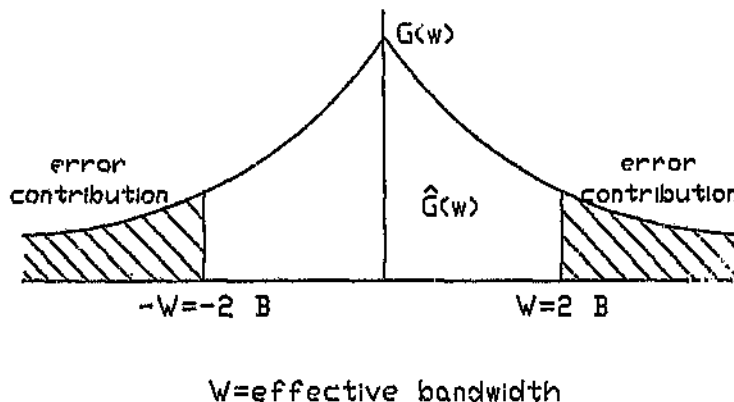


Figure 3.2 Effective Bandwidth After Prefiltering

A specific example is warranted here for illustrative purposes.

Assume for the moment that the signal is:

$$x(t) = 2a/(t^2 + a^2) \text{ with a Fourier transform } X(w) = 2\pi \cdot \exp(-a|w|)$$

The corresponding error signal energy is thus:

$$\begin{aligned} E_e &= 1/\pi \cdot \int_{2\pi f_x}^{\infty} |2\pi \cdot \exp(-aw)|^2 dw \quad (\text{due to symmetry}) \\ &= (2\pi/a) \exp(-4\pi a f_x) \end{aligned}$$

The energy contained in the original signal $x(t)$ is:

$$E_g = \lim_{f_x \rightarrow 0} E_e = 2\pi/a \quad (3.16)$$

If it is required that the error signal is less than 1% of E_x , then $E_e = 0,01 E_x$ or $(2\pi/a) \cdot \exp(-4\pi a f_x) = 0,01 \cdot 2\pi/a$. Thus $f_x = 0,36/a$ and in order to satisfy the requirement specified above, it is necessary that $f_x \geq 0,36/a$ and $f_s \geq 0,72/a$.

3.5.2 Aliasing Effects

Aliasing occurs due to non-ideal prefiltering or bandlimiting. Another source of aliasing type effects is that due to time jitter and non-linearities after sampling, such as quantization. In most practical cases, it is possible for these errors to be made small enough and so they are less serious than others due to insufficient prefiltering. A word of caution is needed here in connection with the DRFM, since the number of quantization levels is small. Sideband (higher order components of the fundamental signal) overlap due to aliasing causes high-frequency distortion if the message band is not too large. However, if the far out sidebands or harmonics have appreciable levels, then the aliasing distortion is far from limited to the high-frequency components.

Once reconstruction of the signal has taken place, then the signal $x'(t)$ is strictly bandlimited to $w_s/2$, but $x'(t) \neq x(t)$.

Sample values $x'(kT_s)$ are however equivalent to respective samples $x(kT_s)$, Jayant and Noll (1984).

3.5.3 Ideal Pre-Filtering of a Non-Bandlimited Signal

Let σ_f^2 be the error variance due to prefiltering of a non-bandlimited stochastic signal $\{X(t)\}$ with a power spectral density (PSD) $S_{xx}(w)$. Because of sampling, the PSD becomes:

$$S_{xx}^*(w) = \sum_K S_{xx}(w - kw_s) \quad (3.17)$$

The input process may be thought of as two processes $\{X_l(t)\}$ which is passed through an ideal lowpass filter, with a cut-off frequency $w_s/2$ and $\{X_u(t)\}$ which is suppressed. The lower frequency band component with a PSD $S_{xx}(w)\text{rect}(w/w_s)$ can be sampled and reconstructed with zero mean-squared error. The upper band with PSD $S_{xx}(w)[1-\text{rect}(w/w_s)]$ is missing. The mean square error is then:

$$\sigma_f^2 = (1/\pi) \cdot \int_{w_s/2}^{\infty} S_{xx}(w) dw \quad (3.18)$$

The signal-to-prefiltering error ratio is σ_x^2/σ_f^2 . As the roll-off of the input spectrum increases, it will be found that the prefiltering error decreases for a given bandwidth (w_B) of the baseband, where $w_s = 2w_B \gg w_c$, the cut-off frequency of the input spectrum. This is because the sampling frequency is double the nominal bandwidth (as determined by the prefilter cut-off frequency), which is significantly larger than the upper frequency cut-off of the input spectrum.

Therefore as the input spectrum decays more sharply with frequency (i.e. the more bandlimited the input spectrum is), so the less the amount of spectral power that remains beyond the prefilter cut-off frequency. Hence the prefiltering error will be less.

3.5.4 Non-Ideal Prefiltering of a Non-Bandlimited Signal

In the ideal case above, of a cut-off frequency w_B and Nyquist sampling, $f_s = w_B/\pi$, the only source of error is that due to σ_f^2 . With a non-ideal (non-brickwall type) prefilter, there will always be a finite amount of power that exists above the cut-off frequency w_B . The result is that aliasing distortion or fold-over distortion will occur, causing an error σ_a^2 . The total contribution of all the aliasing components is described by:

$$S_{aa}^*(w) = \sum_{n \neq 0} S_{xx}(w - nw_s) \quad (3.19)$$

The actual contribution to the message band is known as the aliasing error and has a power spectral density given by:

$$S_{aa}(w) = S_{aa}^*(w) \text{rect}(w/w_s) \quad (3.20)$$

For $n = \pm 1, \pm 2, \text{etc.}$, the repetitions contribute successively to the aliasing spectrum. By folding these contributions back to higher frequencies, it may be seen that the overall contribution is equivalent to the signal baseband spectrum above $w_s/2$, as illustrated in Figure 3.3 below, Jayant and Noll (1984).

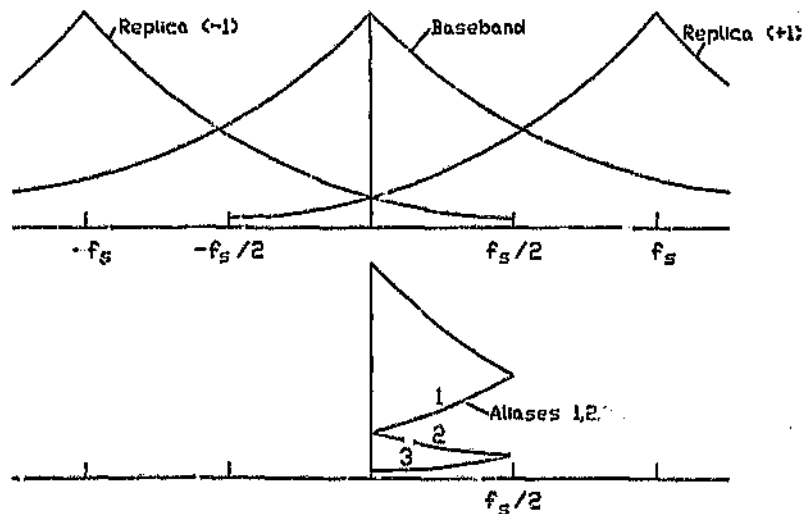


Figure 3.3 Aliasing Error Contributions Due to Spectral Replication and Foldback

Thus the prefiltering error due to an ideal prefilter and that due to aliasing (without prefiltering) involve the same areas in the illustration above, and are equivalent quantitatively, but not qualitatively. These two types of distortion are perceptually quite different in the case of speech, the aliasing error being far more objectionable, Jayant and Noll (1984).

Without prefiltering, the total error variance includes both the prefiltering error (due to the reconstruction filter) and the aliasing error. Hence this is twice the error with prefiltering, in which case the aliasing error is zero. In the case of speech and video signals which tend to have spectra that cut-off reasonably sharply, it is possible to utilise imperfect reconstruction filters which reduce the contributions due to the higher frequencies where the most significant aliasing components reside, Jayant and Noll (1984). In the case of the DRFM this will have to be analysed as a function of the specific application.

3.6 Summary

The theory and statistics of generalised quantization error have been investigated. The phenomenon of aliasing related to quantisation was investigated from the theoretical point of view initially. The generation of aliasing signals with and without prefiltering was considered and the foldover process was illustrated. This effect is paramount to the understanding of coarse quantisation.

Following on from the investigation of sampling and quantization theory, the effects of quantization on the output spectrum of deterministic signals as well as the effects of sampling will be analysed by means of Fourier Transform theory.

It will be shown that quantisation and aliasing errors are linked, and that in order to achieve a specified level of distortion it is necessary to clearly specify the number of levels of quantization and the sampling frequencies. These are related by what has already been termed the significant or effective bandwidth of the input signal.

Two methods will be compared, namely the Fast Fourier Transform (FFT) and the analogue Fourier Transform approaches. Also, it is useful to analyse two standard signals, namely the fixed frequency sinusoid and the linearly frequency-modulated sinewave (LFMSW), since the characteristics of these two signals are well known and they are also easily generated for test purposes.

4.1 The One Bit Case

4.1.1 Spectrum of a One Bit Quantized Waveform

Johnston (1984) presented a technique for the generation of analogue FM pulses using one level of quantization for use in radar transmitters. Although this does not address all the problems of the DRFM, the generation of such signals certainly relates to the DRFM. Johnston (1984) also analysed a one bit quantised linear frequency modulated sinewave that was pulse modulated, i.e. modulated by a pulse of a certain width, by means of the analogue Fourier Transform. Johnston's (1984) approach has been adapted for analysing sampled and quantised waveforms by means of the analogue Fourier Transform. The reason for using this analogue technique is that it has definite advantages over FFT techniques for spectral analysis.

In most cases, explicit sampling techniques are used, where the signal is quantized into a number of discrete levels at specific sampling points (impulse sampling) and is then recovered by a D/A converter and an ideal low-pass filter. This process requires very high speed D/A converters, especially so when trying to digitise RF signals. In order to dispense with the D/A converter, it is necessary to resort to implicit sampling.

In the case of a radar transmitter or a communications transmitter using FM techniques, it is of no consequence that the signal is of a constant amplitude, since the information is contained in the signal frequency. In fact this improves the transmitter efficiency. The information content of such a signal may be extracted from its zero-crossings, which leads to the concept of a frequency modulated square wave (FMSW).

If, for example, the input signal is $\cos(\omega_0 t)$, then the associated one bit equivalent is $\text{sgn}(\cos(\omega_0 t))$, as shown in Figure 4.1 below. The spectrum effectively consists of the Fourier Series coefficients of a square wave with its odd harmonics decreasing proportionally to the increasing frequency. If the periodic function is $f_T(t)$, then its Fourier Transform is $2\pi \cdot \sum_n F_n \cdot \delta(\omega - n\omega_0)$, where F_n are the Fourier Series coefficients.

If the signal is now truncated (or pulse modulated by a pulse that is much longer than the carrier period), since this will be shown to be convenient for analysis purposes and also approximates a radar pulse, the result is that the spectrum which previously consisted of a number of discrete lines, now consists of these lines which have been spread by the convolution of the original spectrum with the spectrum of the sampling function as defined by the Fourier Transform of the modulating pulse, as shown in Figure 4.1 below.

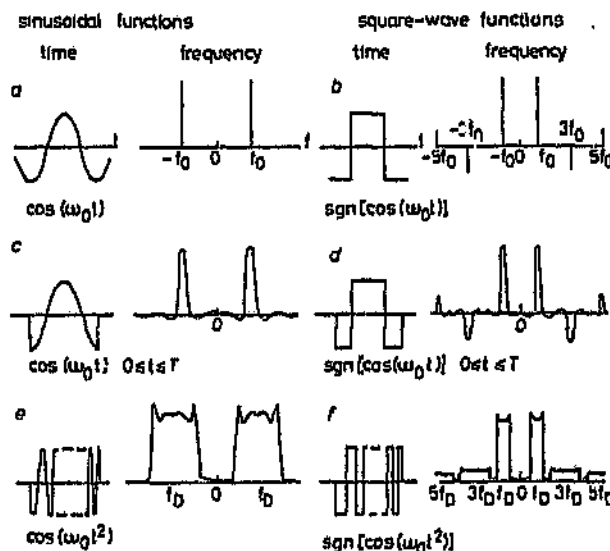


Figure 4.1 Intuitive Development of the FMSW Spectrum - Johnston (1984)

If the original sinusoid is frequency modulated, and if the time-bandwidth product is large, it may be shown that the resulting spectrum is reasonably rectangular, Johnston (1984). There are applications where a rectangular or uniform spectrum is advantageous in terms of efficient use of the available bandwidth. If this signal is quantized to two levels, then it will be shown and may also be seen intuitively that the resulting spectrum will also contain harmonics that are related to the input signal, as stated earlier when considering the requirements for decorrelation of the quantisation noise spectrum from the input signal.

If the fundamental of the FMSW occupies the band from f_1 to f_2 , then the n^{th} harmonic must occupy the frequency range nf_1 to nf_2 . Also, it then follows that the n^{th} harmonic must occupy a bandwidth that is n times the bandwidth of the fundamental. Since the fundamental has a fixed time-bandwidth product, the n^{th} harmonic also has a time-bandwidth product that is n times that of the fundamental. For a harmonic n , it may be seen that the harmonic spectra will start to overlap, as a function of f_1 and f_2 . This occurs when $(n+2)f_1 < nf_2$. A detailed analysis of the FMSW spectrum is given in Appendix G.

Software has been written to implement Equation G3 for the FMSW spectrum as shown in Appendix G. The software listing is given in Appendix H. In order to validate the software, the specific example quoted in Johnston (1984) was considered. The results are graphically compared overleaf in Figure 4.2.

Figures 4.3 and 4.4 illustrate the effects of decreasing time-bandwidth product. In these illustrations, the input signal was a cosinusoid swept linearly between 55Hz and 75Hz within a variable

truncating or pulse modulating function with a pulsewidth of one second and 0.1s respectively. Further, the signal was hardlimited. It is clear that for small time-bandwidth products (substantially less than about 20), the uniformity of the spectrum across any particular harmonic's bandwidth begins to be severely degraded. The spectrum no longer maintains its rectangular shape, but indicates that there is a tendency for the greater part of the signal energy to "dwell" near the centre frequency of the FMSW signal, with increasing ripple in the harmonic passbands.

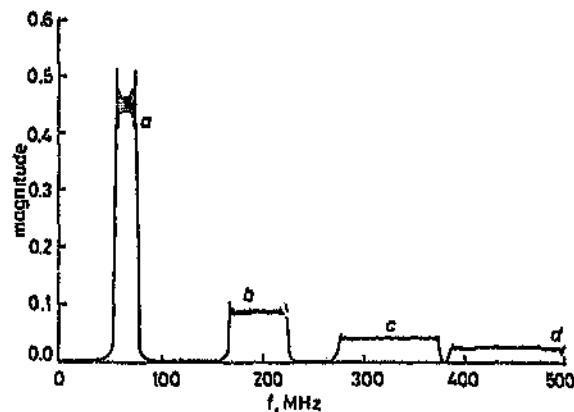
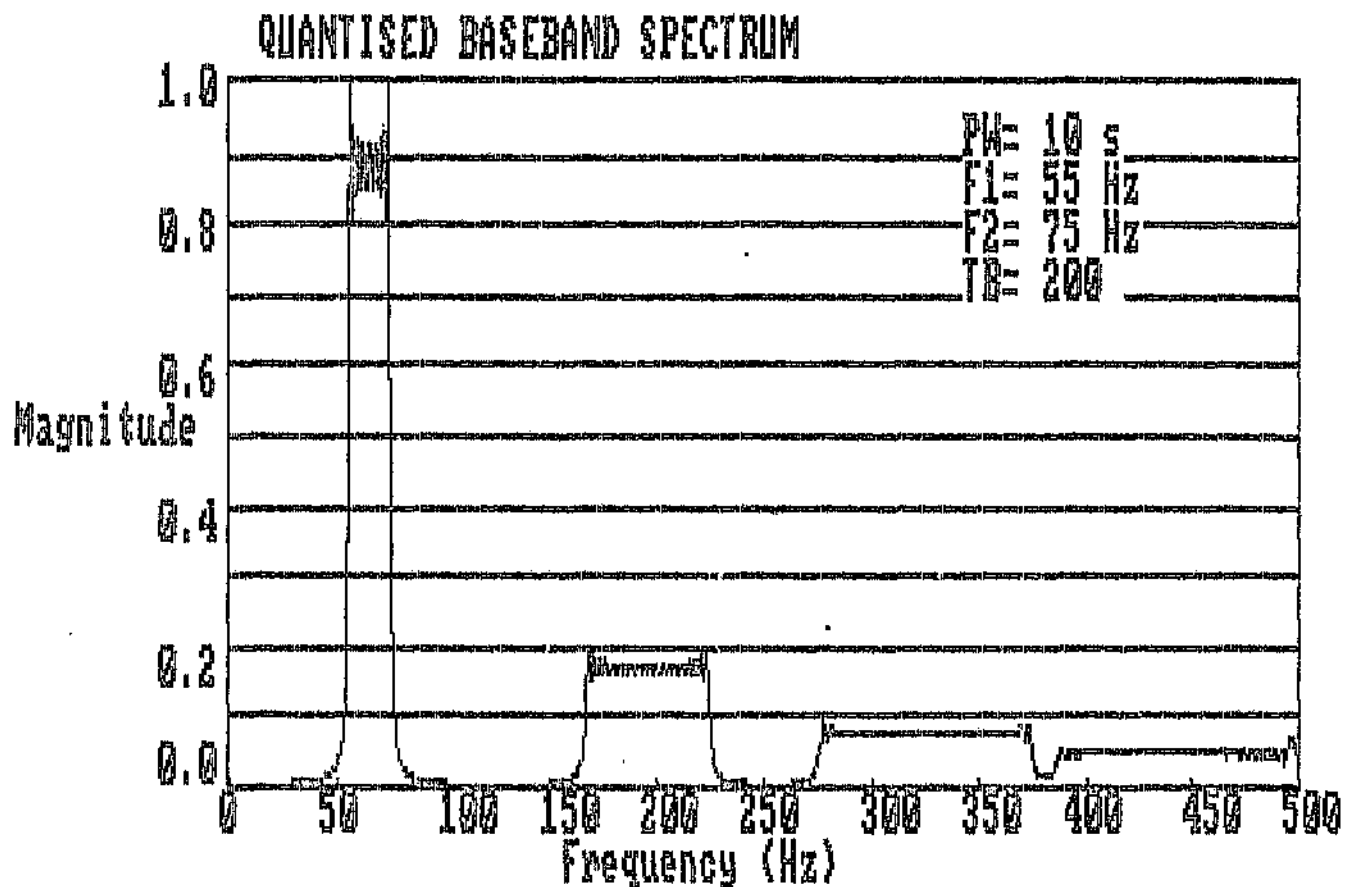
In general the fundamental of the signal would be used to form the output, however there may be some cases where it is desirable to make use of the increased bandwidths and thus time-bandwidth products of the higher harmonics. These may be filtered out and amplified as necessary. In the case of a constant frequency square wave, it is known that the harmonics decrease in amplitude with $1/n$. This is not the case for the FMSW and a proof of this fact is given by Johnston (1984). If one considers Parseval's theorem, which is basically one of energy conservation, then it may be intuitively seen that the energy of each harmonic of the FMSW must follow the relationship of that of the unmodulated signal.

Hence, as the fundamental bandwidth increases, so do the harmonic bandwidths. In order to maintain energy balance in the harmonics, the amplitude must decrease accordingly. Johnston (1984) thus shows that the harmonics of the FMSW decrease with $n^{-3/2}$ rather than $1/n$.

The most significant problem with calculating the FMSW spectrum by means of the above technique is that it takes a vast amount of

computer time, which is a function of the time-bandwidth product, and thus the number of zero-crossings. It was found that the computer time required for a time-bandwidth product of the order of a hundred was in excess of 7 hours on an IBM PC AT without co-processor or a tenth of this figure with a co-processor.

If it is only necessary to determine the spectrum of the fundamental signal, then it is possible to reduce the bandwidth of the spectrum to be displayed. Another problem which is also directly related to the time-bandwidth product, is the requirement for computer memory. The greater the time-bandwidth product, the greater the number of zero-crossings and hence data points.



Johnston (1984) - F1=55MHz, F2=75MHz, PW=10us, TB=200

Figure 4.2 Validation of the FMSW Spectrum Software

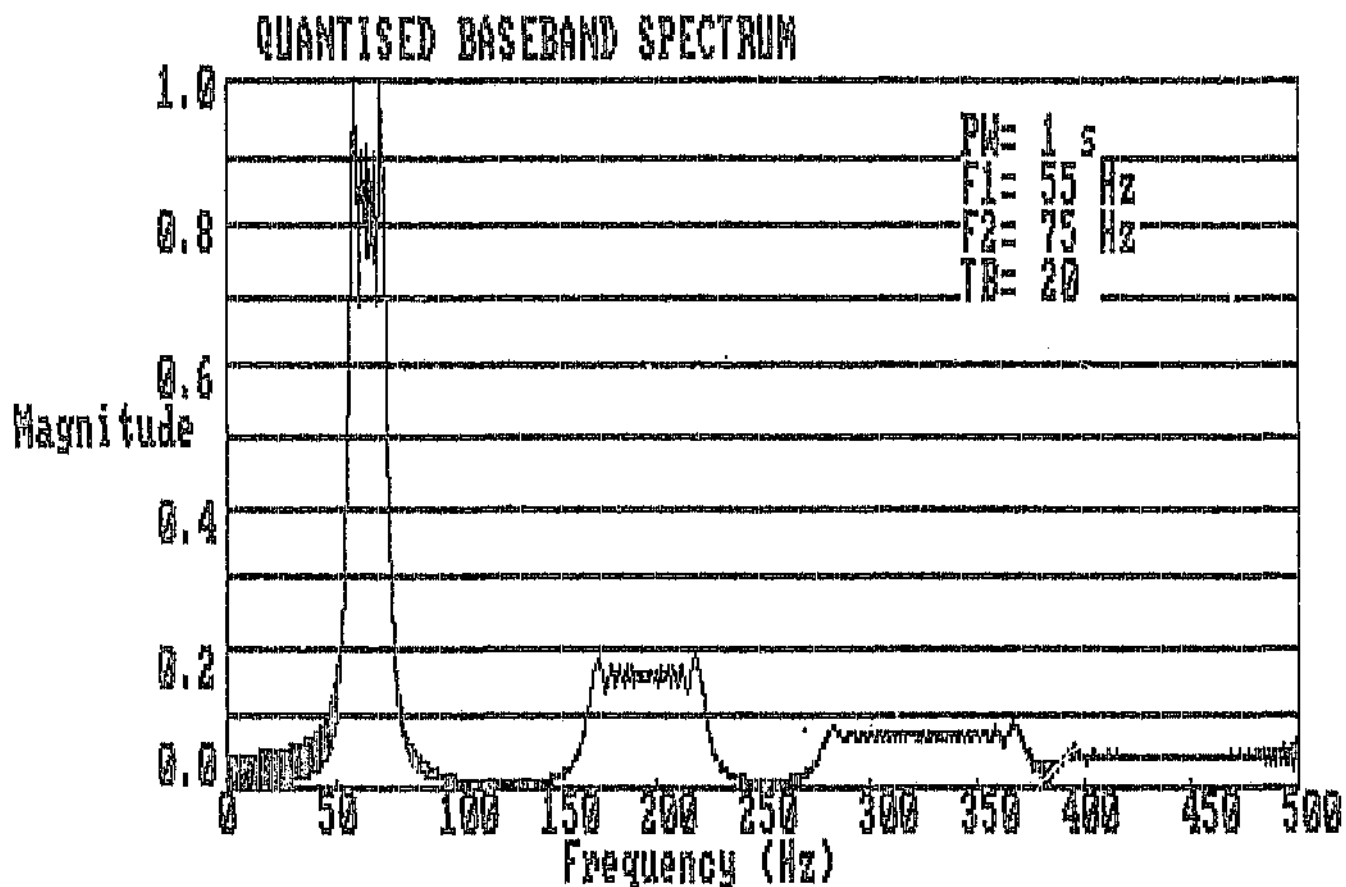


Figure 4.3 FMSW Spectrum for TB=20

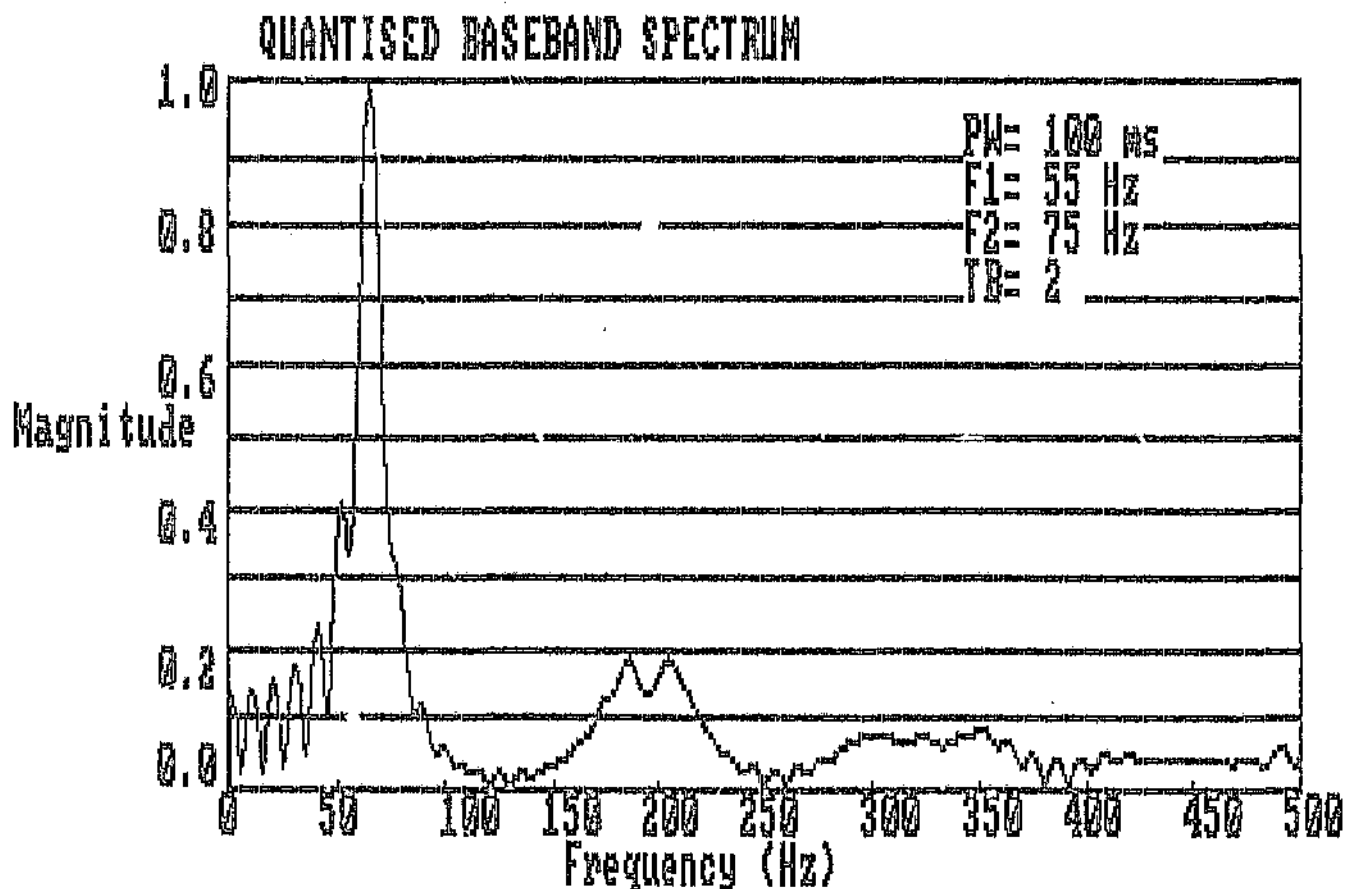


Figure 4.4 FMSW Spectrum for TB=2

It is worth noting, however, that the spectrum as calculated is free from the effects of aliasing, and provided that a sufficient number of plot points is chosen, the picket fence type effect may be overcome. The overall conclusion concerning this method, is that it trades speed for spectral accuracy compared to the FFT.

So far the FMSW has mainly been considered, however a single frequency sinusoid has also been mentioned. This type of signal is simply a degraded FMSW signal with the rate of change of frequency set to zero. As expected the spectrum is the equivalent of the harmonic Fourier Series spectrum of a square wave, as shown in Figure 4.5.

This approach of generating spectra forms the basis of all further spectral analysis for the DRFM. As a matter of interest, it has been found that the largest harmonic ($n=3$) with a value of -10dBc (c.f. Figure 4.5) is the quoted level of spurious signals for typical one bit DRFMs. Hence, even at this early stage in the argument, it may be seen that the analysis ties up with what has been found in practice. The implication is that the additional sources of error, such as aliasing and so forth, are negligible compared to that due to quantization. These effects have been considered in more detail later.

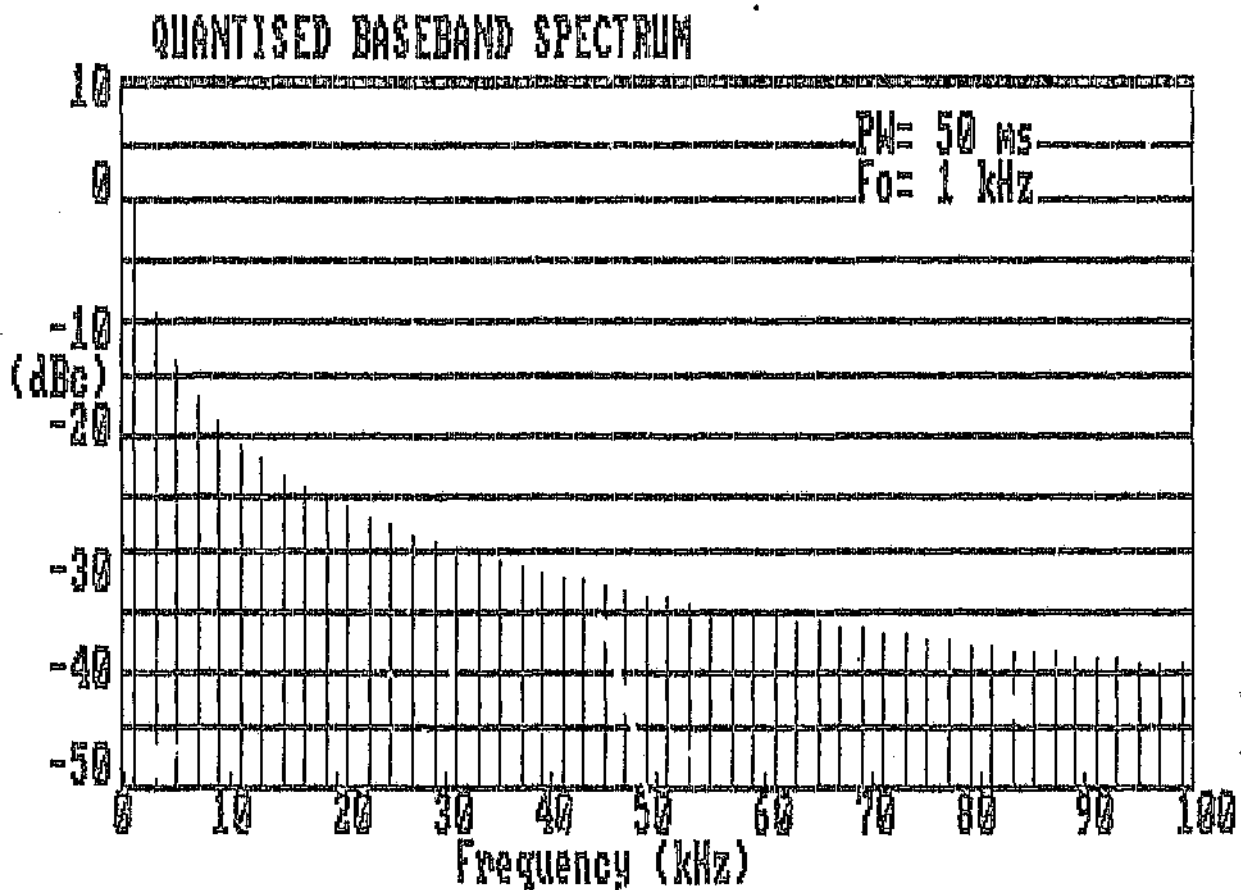


Figure 4.5 One Bit Quantized Spectrum of a Single Tone Signal

4.1.2 The FFT Approach

As mentioned above, the analogue Fourier Transform approach to obtaining the signal spectrum has definite accuracy advantages over the DFT or FFT approach and also in terms of the ability to separate quantization and aliasing effects, but at a cost of computation speed. This is stated on the assumption of a fixed memory limit. As the memory limit falls away, it can be expected that the sampling frequency for the FFT approach can be increased, which will improve the accuracy, but once again at the cost of speed. Given the available computing power, it was found that the maximum degree of accuracy available from FFT computations was insufficient to establish what the sources of various errors were with sufficient clarity. This is demonstrated for the case of a single tone input in Figure 4.6. The problem here is that the quantization and aliasing errors are combined, even when fairly high sampling frequencies are used. Leakage effects are evident as a result of the inherent windowing. In addition, the FFT implies time quantisation, which is not present in the case of the implicit sampling approach described earlier.

As far as DRFM design goes then, it becomes difficult to trade the effects of quantization for aliasing and sampling frequency so as to achieve some specified spurious signal level. The "analogue" technique, if it may be so called, as described in 4.1.1, makes it possible to establish the errors due to harmonic or quantization distortion and then to specify a sampling frequency which will determine the nature of the aliasing error, or aliased quantisation error.

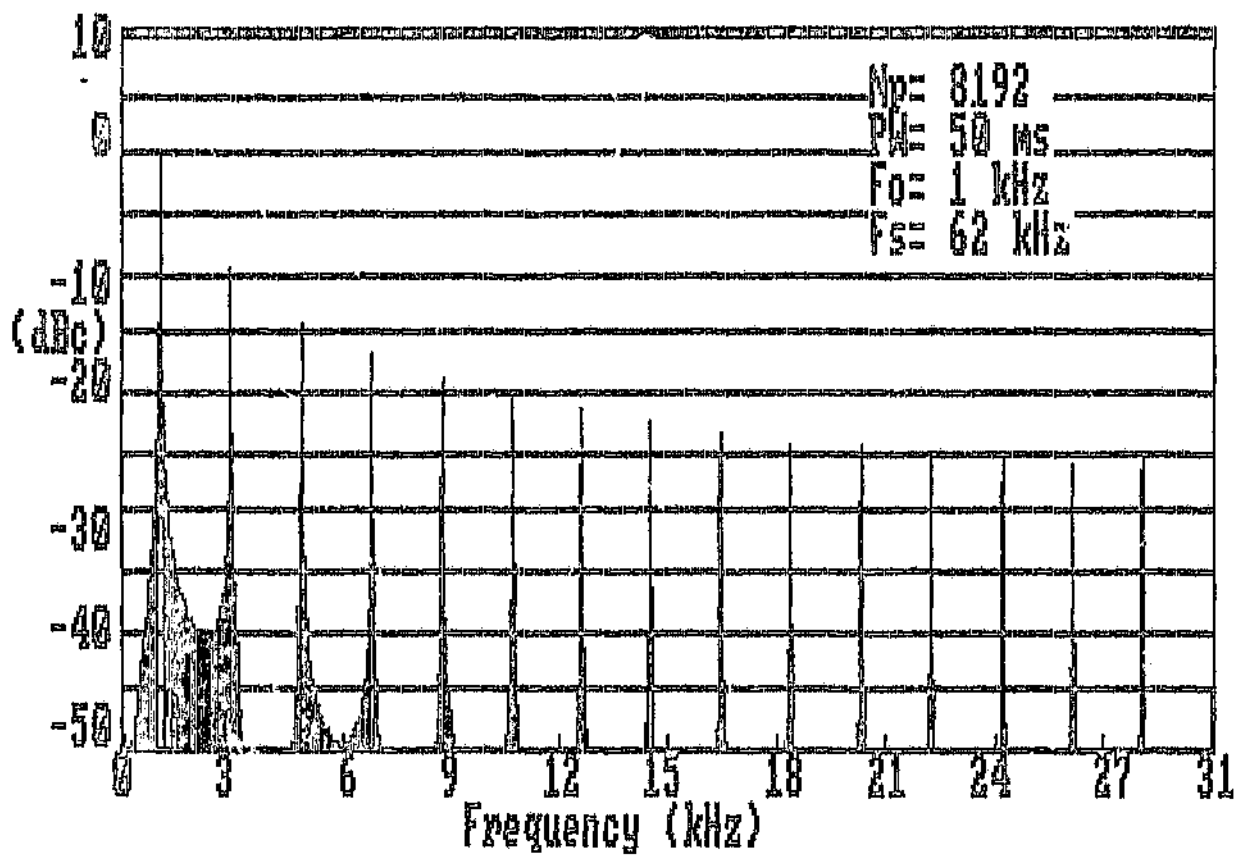


Figure 4.6 FFT of Single Tone One Bit Quantized Signal

4.1.3 The DRFM as a Signal Synthesizer

One of the applications previously mentioned for the DRFM, is that of a frequency or signal synthesizer. Johnston (1984) describes the specific theory behind this method. In order to utilize a one bit technique, hardware is produced that will generate a series of ones and zeros (+1s and -1s) at specific intervals of time $n \cdot \delta t$. Now it is most unlikely that the multiples of the sampling interval will coincide exactly with the zero-crossings, unless of course the signal is a sub-harmonic of the clock frequency exactly. Thus each zero-crossing must be quantized in time so as to coincide with the closest integral multiple of the clock period.

The result is that the error in time quantization must be less than $\pm \delta t/2$, and this is equivalent to changing the instantaneous frequency around each zero-crossing, causing an error in the output spectrum, Johnston (1984). If the timing or sampling interval is kept sufficiently small, then the error may be sufficiently minimised. The nature of this error is analysed by Johnston (1984) and is described in Appendix I.

It should be noted that the signal being generated is totally deterministic with a priori knowledge of the zero-crossings. Thus for a specific required zero-crossing, it is time quantised to the nearest $n \cdot \delta t$. The four possible errors are described in Appendix I. Equation I2, describing the error spectrum is similar to that for the signal spectrum of 4.1.1. The difference is that the nature of the two signals is different. The error signal consists of a series of narrow pulses with amplitudes ± 2 , depending upon the exact timing between the actual zero-crossings and the quantized zero-crossings.

The result is that for small δt , the error signal is approximated by a train of impulses. Now it is known that the spectrum of an impulse is broad, i.e. white in nature over the band of interest. Hence the error spectrum may be considered to approximate white noise. If the error signal amplitude is small, then it will not significantly degrade the actual signal spectrum.

Although this error is only due to time quantisation of the zero-crossings, the resulting error waveform is impulsive in the time domain, and hence the error spectrum is broad in nature (white). The size of this error is small because a priori information about the signal is known. In the case of signal generation, there is not real sampling in the true sense of the word. Rather, if the signal at the output of a one bit digital signal synthesiser were monitored, it would be found that it is of a continuous nature as opposed to a series of samples that still have to be passed through a reconstruction filter. Thus the former signal will produce an unaliased baseband spectrum with errors due to inaccurate time quantisation, whereas the latter will contain aliasing components due to violation of bandlimitedness and due to sampling.

To clarify this issue, assume that a single tone input is hardlimited, i.e. quantized to one bit. The result is a square wave that is to be produced using the technique of digital signal generation. The baseband spectrum of this quantized signal may be determined according to the procedures of 4.1.1. This signal is modified slightly so that the actual zero-crossings coincide with the closest integral multiple of the timing interval, which is small compared to the time between zero-crossings.

The result is that the signal has an error spectrum (also

baseband) associated with it. The output of the digital signal synthesiser hardware is this same signal, continuous in time and of a square wave nature. In contrast, in the sampling case, the quantized baseband signal is first sampled, causing spectral replication and aliasing, and then reconstructed. Hence in the latter case, the message band spectrum will contain those parts of the spectral replicas that fall into the message band (up to half the sampling frequency) region.

These aliasing components may add up to considerably more than the error due to the instantaneous frequency changes associated with digital signal generation. Note, however that the basic principle of computing an analogue baseband signal spectrum applies in both cases. The differences lie in the way timing and sampling errors are generated.

When generating the signal, it is possible to determine the "clean" baseband spectrum. Time quantisation error may be introduced into the digital signal generation process afterwards. In the sampling case, the spectral replicas of the amplitude quantised baseband spectrum must be introduced at the sampling rate.

In order to understand the process of digital signal synthesis and the associated error, the software used for the process of Equation G3 was extended to incorporate the error function of Equation I2, due to time quantisation, and is listed in Appendix J. The practical example of Johnston (1984) was used to verify the software. Figure 4.7 illustrates the perfect baseband signal on both linear and logarithmic (dB) scales. Figure 4.8 shows the computed error spectrum and the combined (phase and magnitude considered) signal plus error spectrum.

It may be seen that the spurious signals due to time quantization are of the order of -40dBc or less. The most serious form of error as far as spurious signals are concerned is that due to amplitude quantization or harmonic distortion. The effect of reduction of the time-bandwidth product is once again illustrated in Figure 4.9.

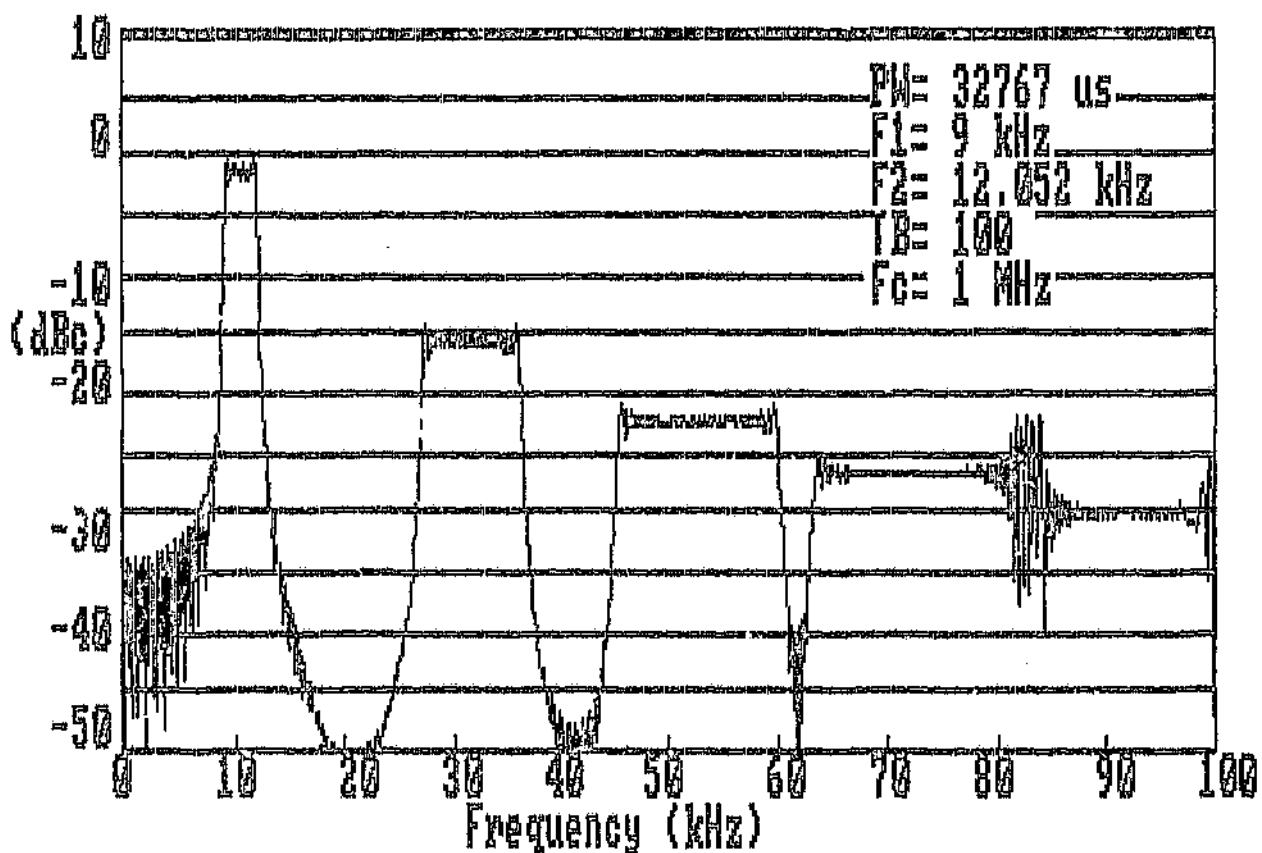
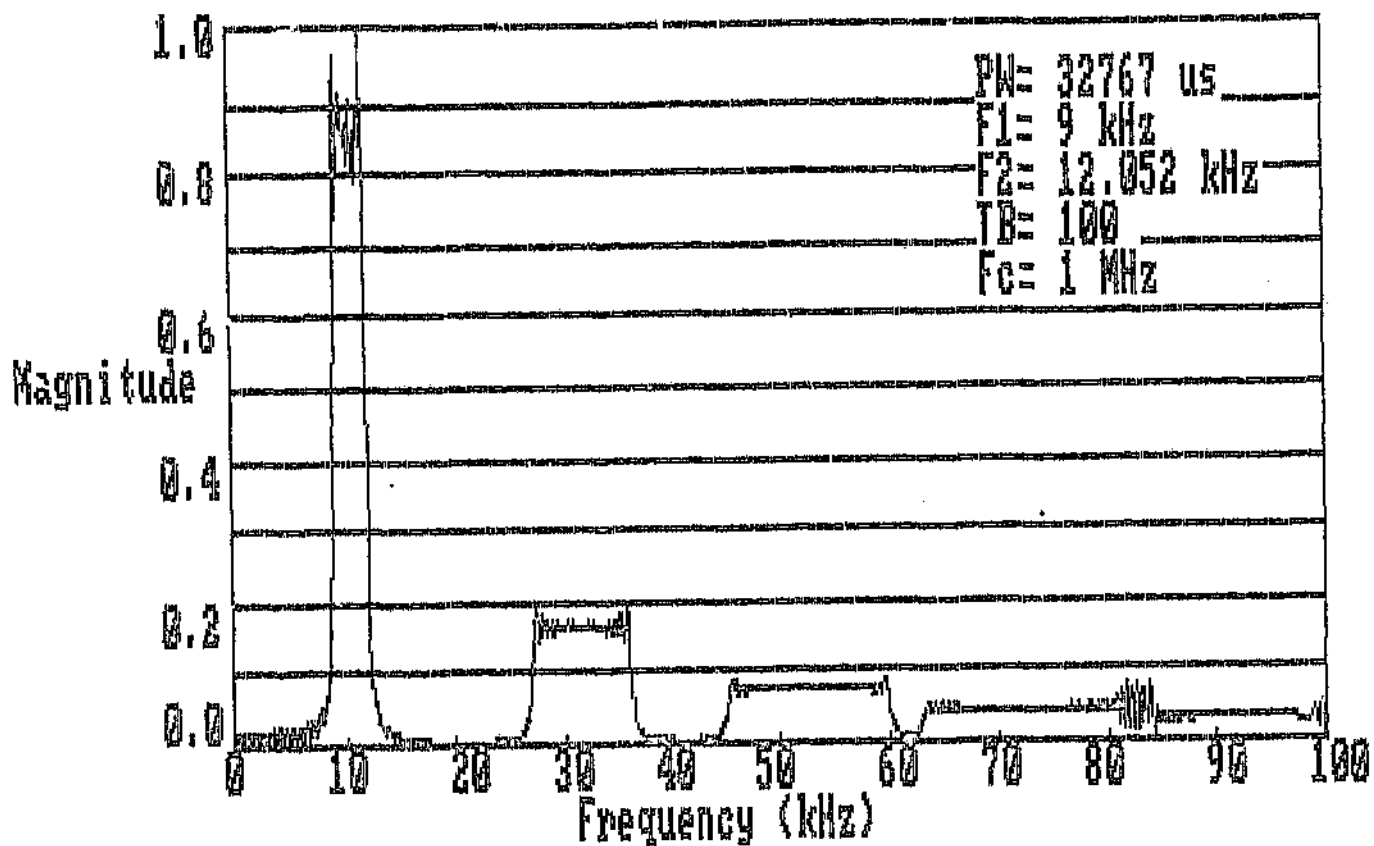


Figure 4.7 FMSW Spectrum on Linear and Logarithmic Scales

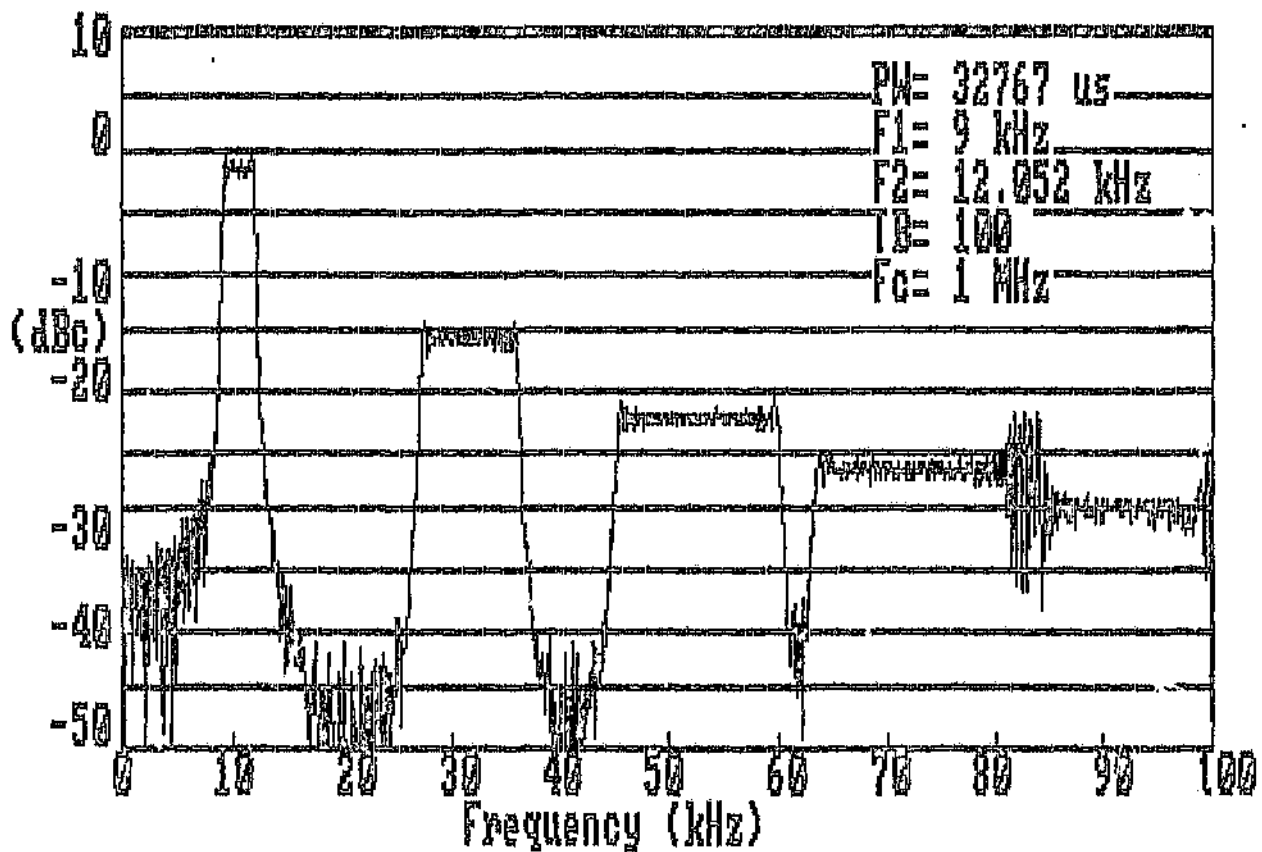
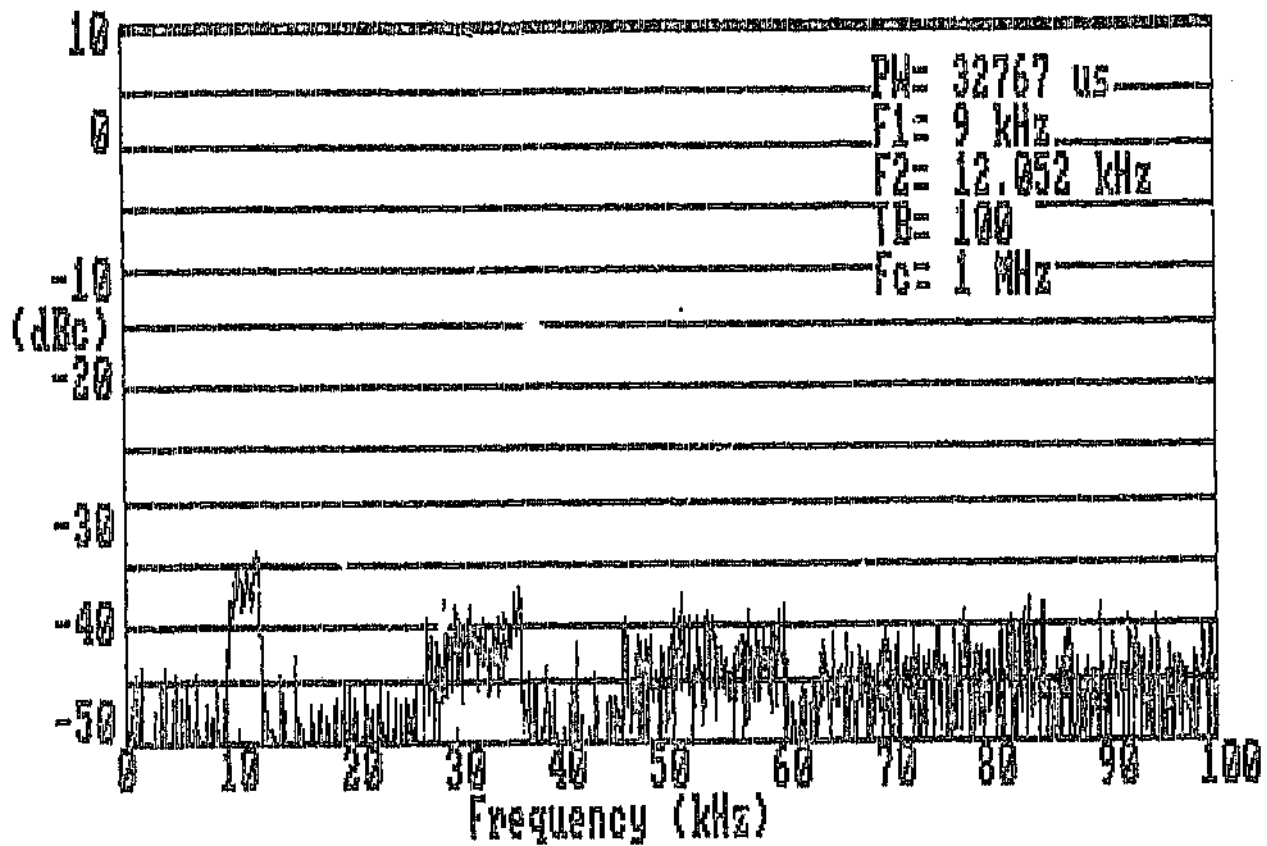
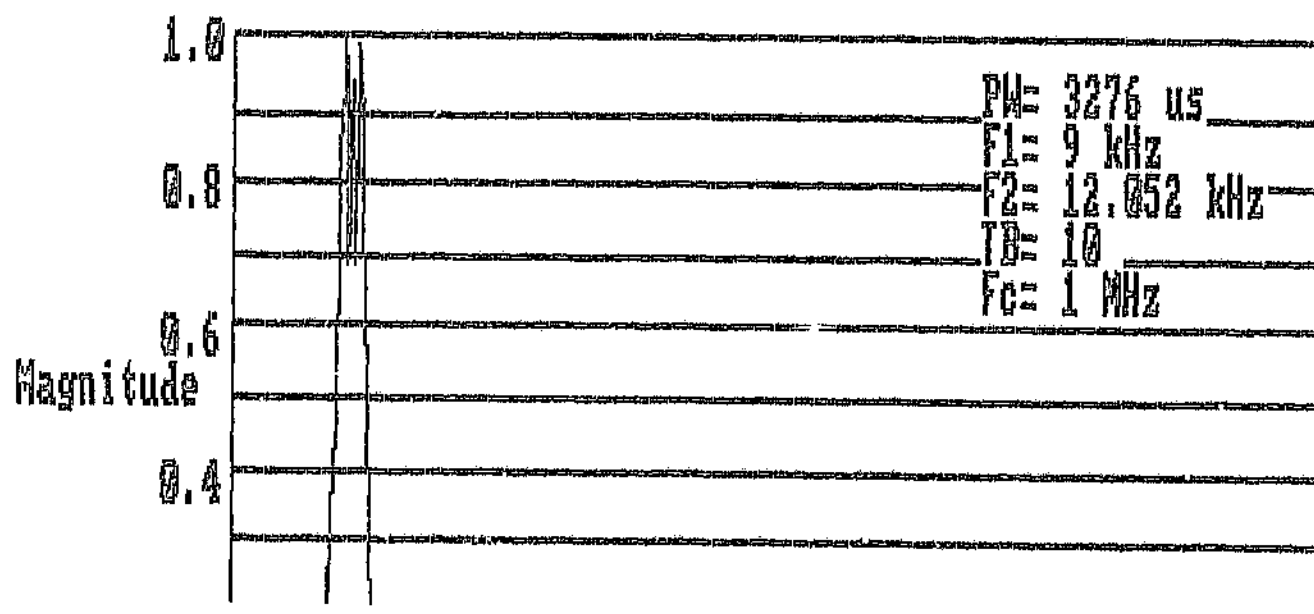
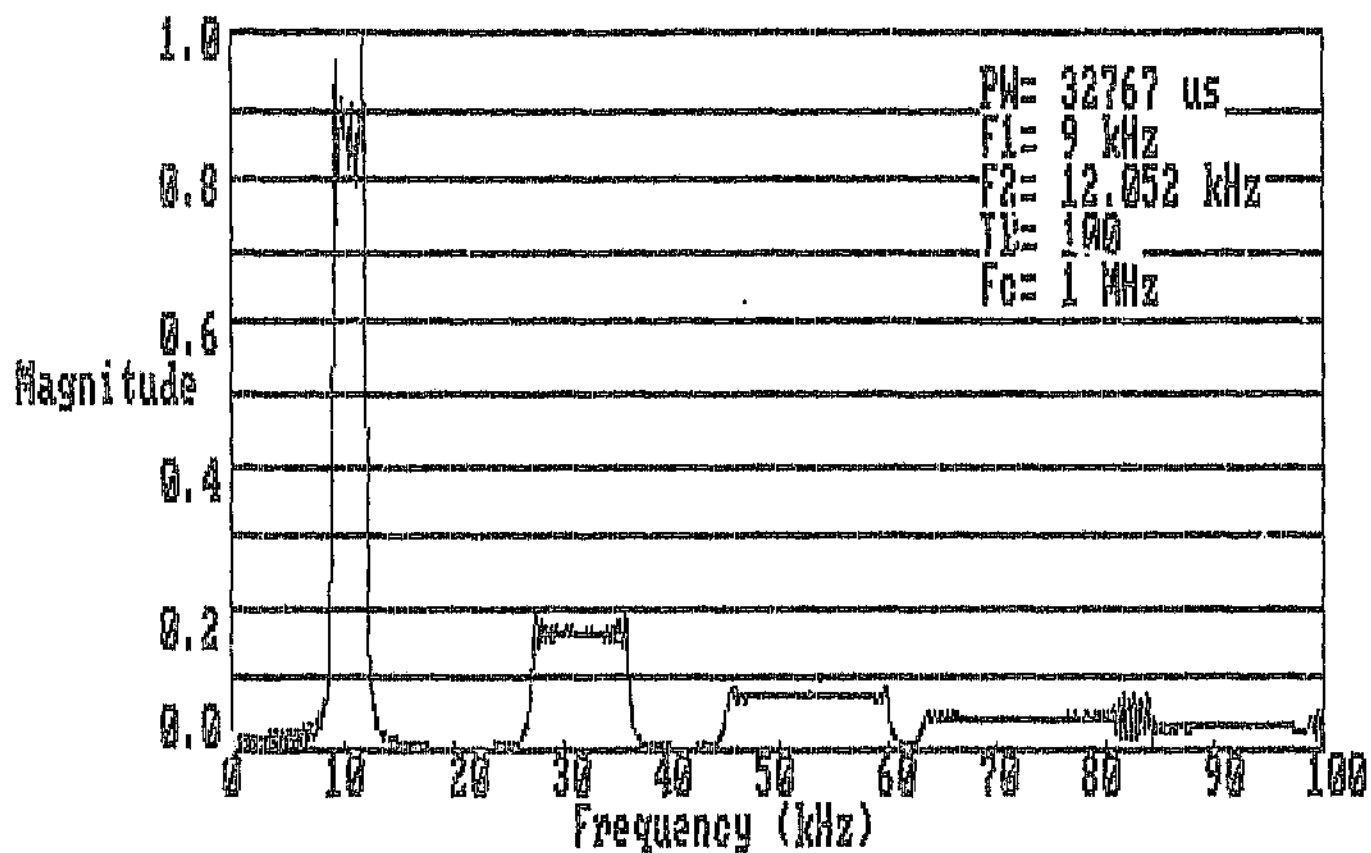


Figure 4.3 Error and Combined Spectra for Signal of Figure 4.7



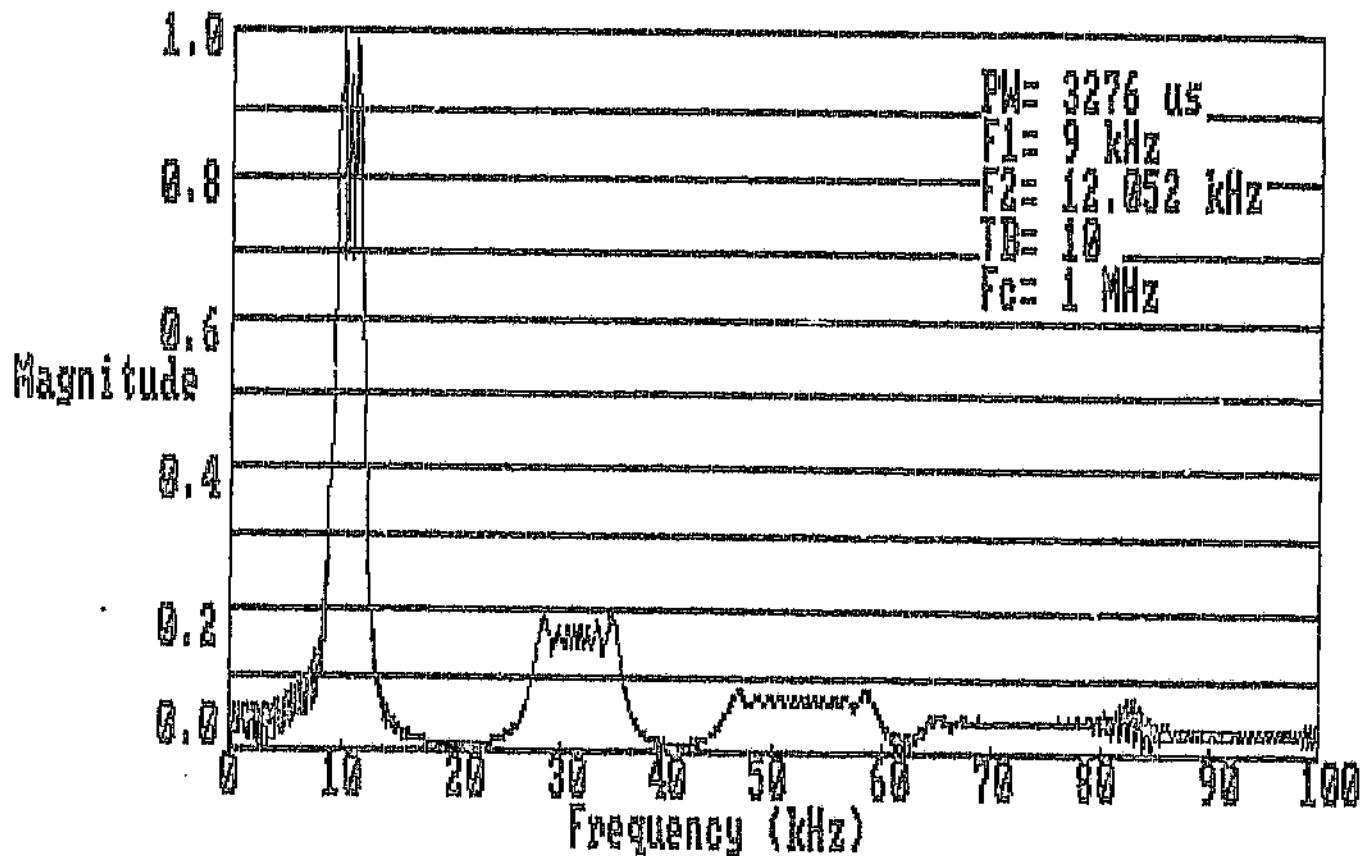
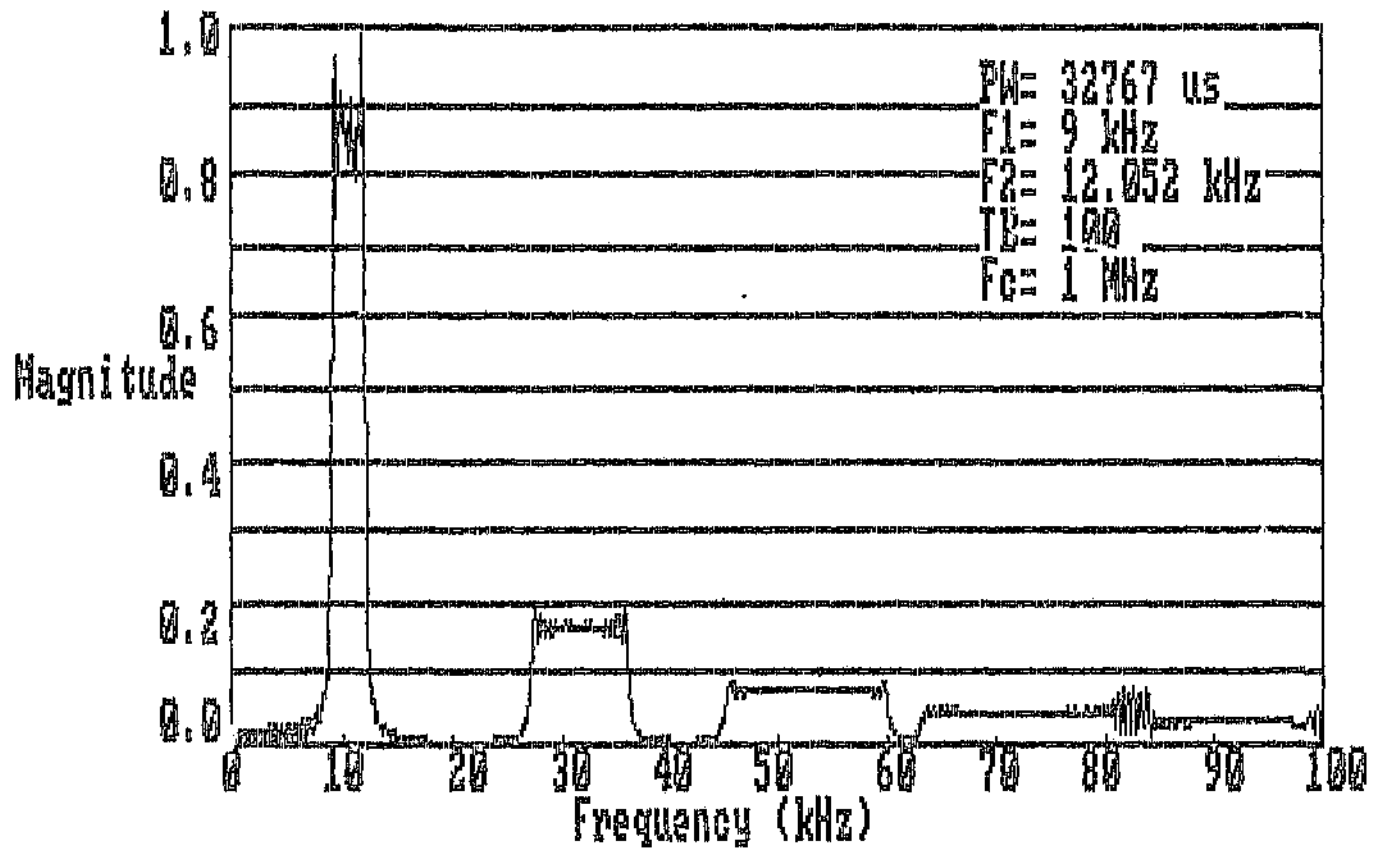


Figure 4.9 Effect of Time-Bandwidth Product on Signal Synthesis

The procedure of 4.1.1 allows one to determine the baseband spectrum of any signal up to any desired frequency, given sufficient computing time and memory. The only errors here are due to quantization alone and are easily seen on a plot of the quantized signal spectrum. By specifying any desired sampling frequency, the effects of aliasing may be determined.

As an example consider a single tone input at unit frequency. After quantization to one bit has taken place, it has been shown that there will be odd harmonics decreasing as function of $1/n$. Thus the signal spectrum is defined as:

$$|X(f)| = (2/n\pi) \cdot \delta(n-f) \text{ for } n=1,3,5,\dots \quad (4.1)$$

since the Fourier coefficients of a square wave are given by:

$$F_n = \text{Sa}(n\pi/2) ; \quad n \neq 0 \quad (4.2)$$

If the sampling frequency is chosen to be 10 units, then the signal spectrum will be replicated at centre frequencies of integral multiples of 10 units. Considering the envelope of the harmonic spectrum, as shown in Figure 4.10 overleaf, it may be seen that the quantisation error effectively results from the generation of harmonics, causing the quantised signal spectrum to spread beyond the bandlimitedness originally considered, i.e. beyond half the sampling frequency.

Beyond this frequency, the quantised baseband spectrum is successively folded back into the message band region (defined by half the sampling frequency). Thus considering the example above, it may be seen that the overall quantisation error is formed by considering all the spectral replicas introduced by sampling. An equivalent of this is to successively fold the baseband spectrum that extends beyond the message band (5 frequency units) back into the message band, as shown in Figure 4.11 overleaf for the

case of a spectrum consisting of discrete lines. Because the quantisation error is a broad spectrum, it is effectively undersampled, that is to say aliased.

By utilising this foldback technique, a computer program that evaluates the quantisation error for any given quantised baseband spectrum and a given sampling frequency, has been written.

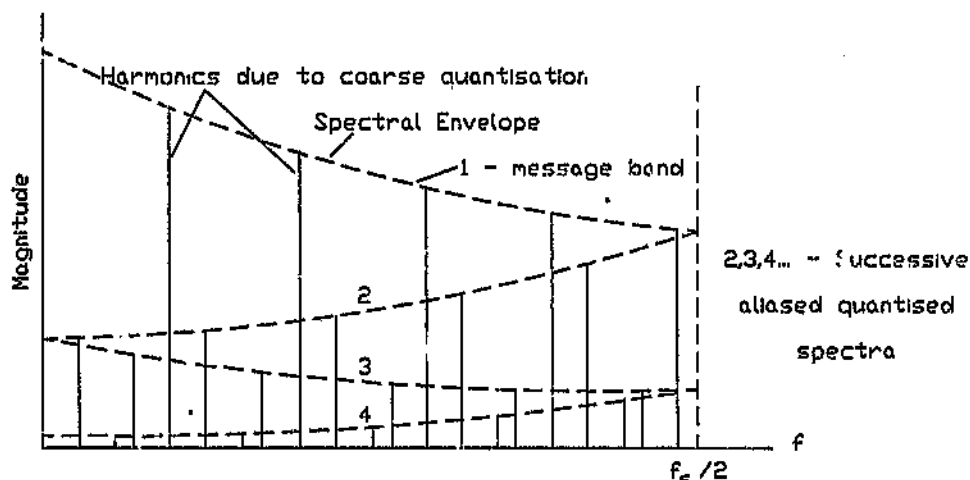


Figure 4.10 Illustration of the Generation of Aliasing

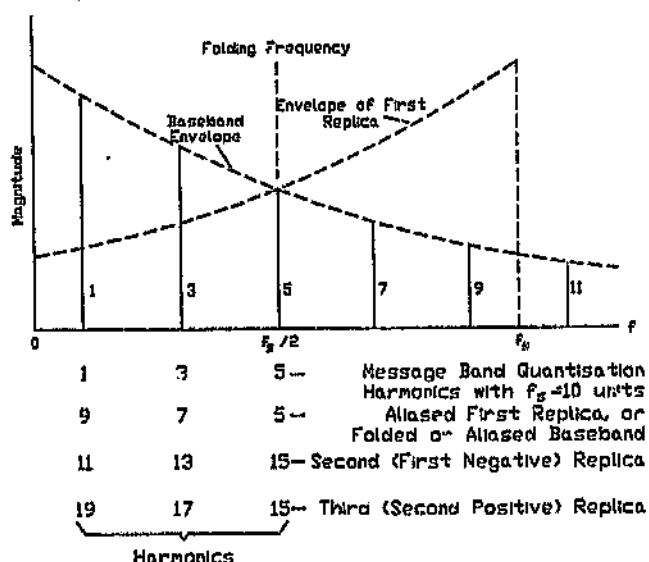


Figure 4.11 Illustration of Aliasing Error Formation for Example in Text

The practical aspects of this procedure must be considered with care, since the frequency spectrum of the baseband signal is discrete. Take the example mentioned above. The original signal had a frequency component at 1 unit and after quantization at 1,3,5,7,9,etc. units. The sampling frequency was specified to be 10 units. Thus it may be seen that the first spectral replica will overlap with the desired signal at 5 units, the first negative frequency replica at 1 unit, the second positive frequency replica at 5 units, and so forth.

Hence, at a frequency of 1 unit, the aliasing contributions equivalent to 9,11,19,etc. units will be obtained. At a frequency of 3 units the aliasing contributions will be due to the components at 7,13,17,etc. units and so-on, as illustrated in Figure 4.11. Therefore, as long as the sampling frequency is an even multiple of the fundamental frequency, it will be found that the odd harmonics will overlap with odd harmonics.

If the sampling frequency is an odd multiple of the fundamental frequency, then baseband odd harmonics will overlap with both even and odd harmonics. Taking this a step further, it may be seen that if the above two special cases of sampling frequency are not adhered to, then the harmonics outside the desired band will not necessarily overlap with any other harmonic at all. This effect occurs in practice and might result in peak spurious signals due to quantisation that are very much reduced, even though the total quantisation error power is still the same, as will be shown later. The difference lies in the nature of the spectrum.

The above argument was pursued for the case of a single tone input since it explicitly and very simply illustrates the type of

problem that could arise when using the foldback approach. When dealing with more realistic signals, such as the frequency modulated sinusoid, it is known that the spectrum is essentially continuous, rather than a series of discrete lines, as in the single tone case. The harmonics still exist, but consist of continuous bands of activity.

Since a discrete spectrum has been calculated because of the use of computer techniques, it is imperative to ensure that the sampling frequency is an integral multiple of the frequency interval used to determine the baseband spectrum. If one does not do this, then clearly the foldback will cause some of the aliased quantisation error contributions to fall between actual frequency points used in the baseband spectrum. This becomes problematic with computer simulation, and in this case it is better to use the FFT.

In the extreme, it is possible that none of the aliasing contributions appear on the plot, resulting in a spectrum that is difficult to interpret. The limitation imposed above is not severe in practice, since the frequency interval used is generally quite small. On this basis, software was written to determine the spectrum of the aliased quantisation error and is an extension of the basic software implementing Equation G3. The listing is given in Appendix K.

The quantisation error spectrum is expected to be reasonably flat, since it essentially consists of a series at any specified frequency and this series is of the form of decreasing values according to the Fourier series coefficients. Thus the sum of n terms of the relevant coefficients at any point is approximately equal. This was shown to be valid for specific cases when

evaluated by hand. Using standard mathematics concerning convergence of series, it is difficult to show that some convergence value will be reached by means of the comparison, integral or ratio tests for convergence or divergence of series. In order to illustrate this effect a number of test cases are shown.

Figure 4.12(a) illustrates the single tone input of 1kHz with the baseband spectrum calculated up to 200kHz with 1000 data points. The pulsewidth of the signal was set at 100ms to ensure that the spectral spreading effect of pulse modulation was insignificant. The next illustration (b) shows the desired quantised message band spectrum with a sampling frequency of 62kHz, i.e. a. expanded up to 31kHz. The validity of an earlier statement, that in order to ensure spurious levels due to aliasing of less than -30dBc, it is necessary to sample at 62 times the signal frequency, may be decided. Because the 31st harmonic is -29,8dBc with one bit quantization, the 33rd harmonic which will alias, will be below -30dBc.

However, this only considers the effect of the first replica. Thus it may be expected that the actual level of the spurious signals due to aliasing will be somewhat greater than -30dBc. In the above example (as shown in Figure 4.12(c)) the first three positive and the first two negative replicas were considered. The aliased quantisation error spectrum has a flat nature as believed and the level is of the order of -24 dBc. The magnitude and the phase of the contributing components were considered, although in the case of a single tone input the components are purely additive. The total signal within the desired frequency region is shown in 4.12(d).

The second example considered had a similar baseband spectrum to the example of Figure 4.12. The basic differences were that the number of data points were increased to 1500 and the modulating pulsewidth was reduced to 10ms. Thus the spectral lines will have a finite width associated with them. This spectrum is shown in Figure 4.13(a), but this effect is not clear due to the expanse of spectrum illustrated. In this case the sampling frequency was set to 20kHz, much lower than before. The desired message band thus extends to 10kHz and is illustrated in Figure 4.13(b). Here the effect of the narrow modulation pulsewidth is more evident, although the resolution on the frequency axis is poor.

The peak level of the aliased quantisation error, now taking into account a more significant number of replicas is greatly increased, being of the order of -17dBc. If only the first replica was considered, this level should be of the order of -20 dBc. From Figures 4.12 and 4.13 it is clear that there is a fundamental and important link between quantization error, aliasing and sampling frequency.

A final example shown in Figure 4.14 concerns Johnston's (1984) example of the 9-12kHz FMSW signal. The baseband spectrum is shown up to 2MHz for 1500 data points. A sampling frequency of 1MHz was stipulated and the quantised message band spectrum is shown in Figure 4.14(b). The corresponding aliased quantisation error is shown in Figure 4.14(c). The maximum level of error is of the order of -35dBc and based on the single tone harmonic values, the predicted level would be about -38dBc. Finally, reducing the sampling frequency to 200kHz (c.f. Figure 4.15), the typical level of aliasing error is about -20 to -25dBc. Again based on single tone harmonic values, the predicted level would be about -25dBc.

hence it may be seen that reasonable correlation is obtained between single tone simple predictions and frequency modulated signals. Further, it may be seen that with one bit of quantization, the most significant source of error is that due to quantization as a result of the rather severe levels of harmonic distortion that are generated. This influences the level of aliased quantisation error in the message band which is dependent upon the sampling frequency.

Thus the prime source of spurious signals in the DRFM is due to coarse quantization. Specifically, in the case of the one bit DRFM then, it may be seen that the third harmonic generates the largest spurious signal. In the case of a single tone input, which is an industry reference, the level is -9,5dBc. All sources of quoted data indicate levels of -10dBc for one bit DRFMs. Thus confirmation of the theory, although limited, is positive. The link between this foldback and Fourier transform approach compared to the standard theory of uniformly distributed quantisation error within each quantisation level, will still be shown, after the extension of this approach to the N bit quantisation case.

Figure 4.12 Aliasing Example #1

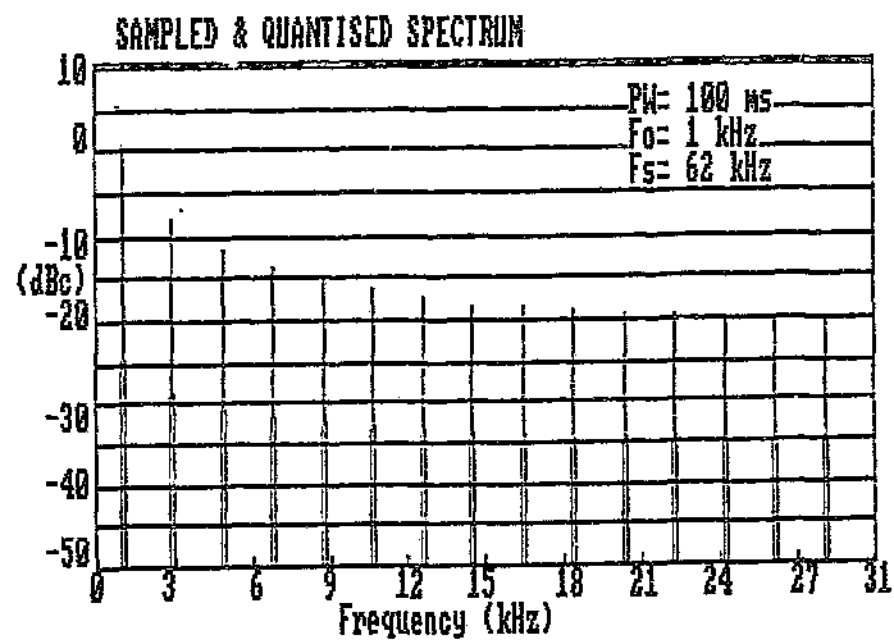
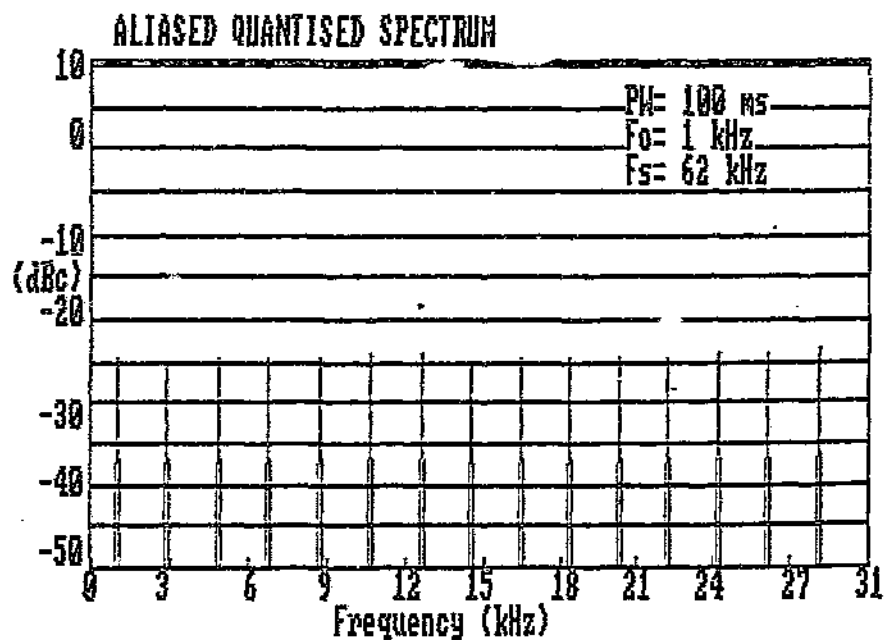
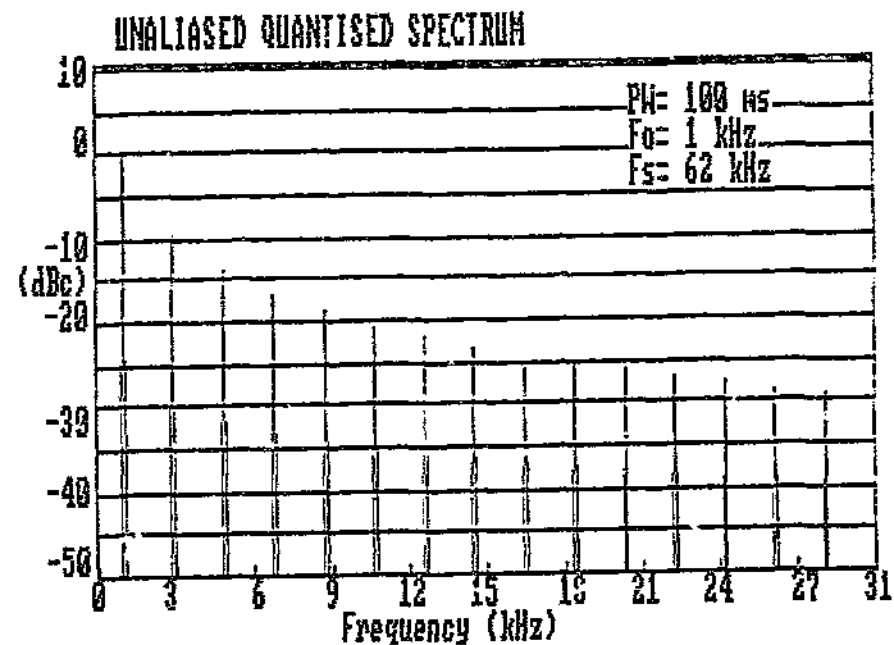
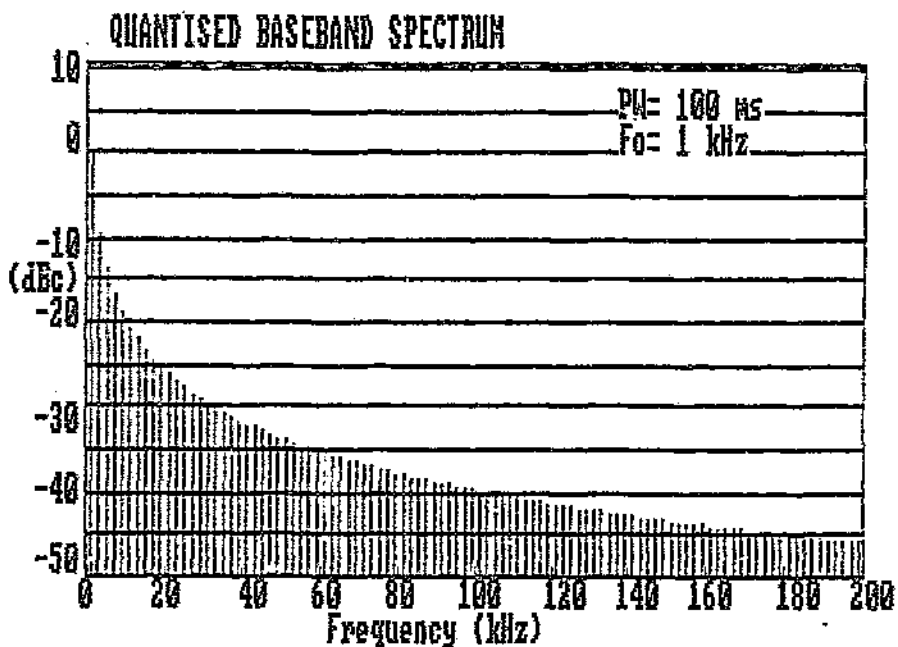


Figure 4.13 Aliasing Example #2

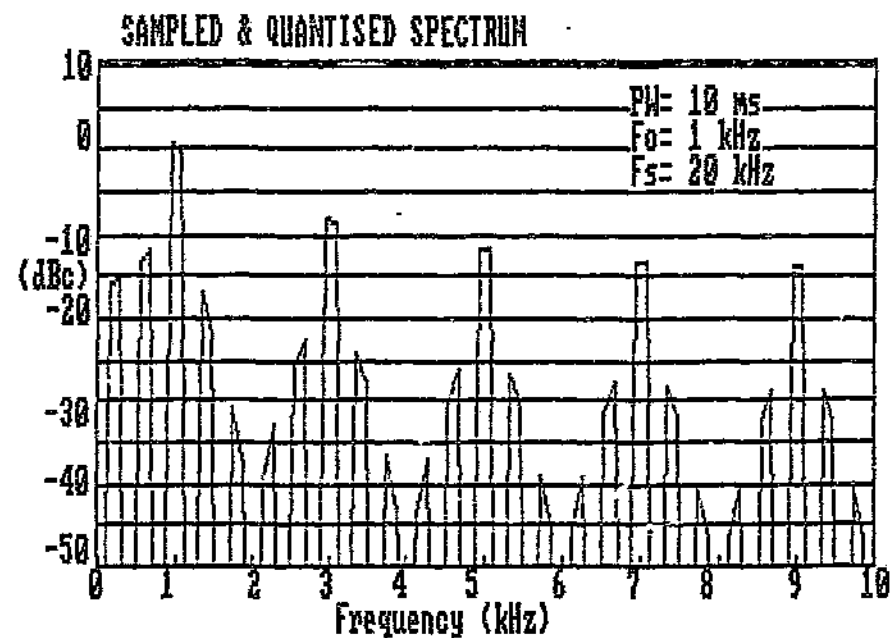
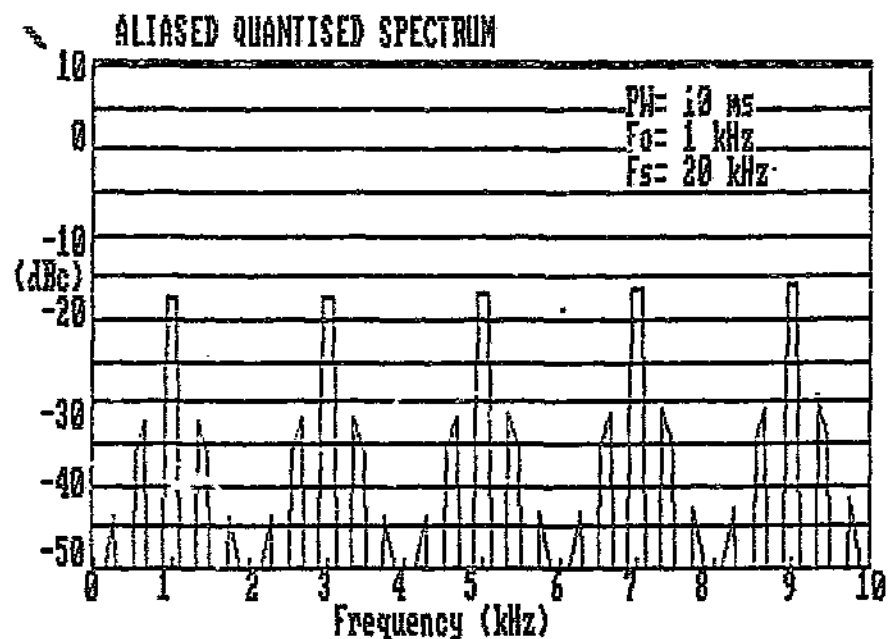
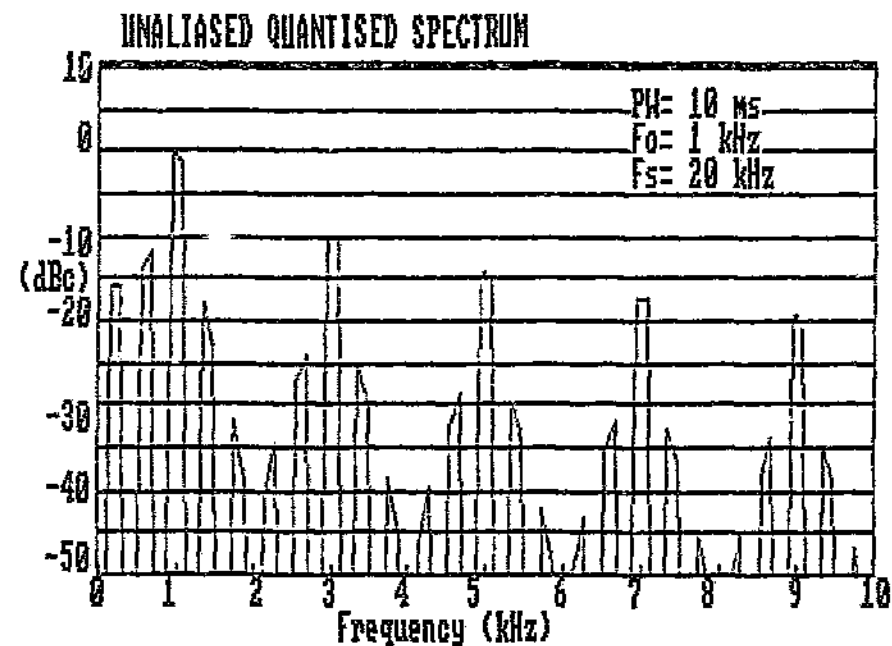
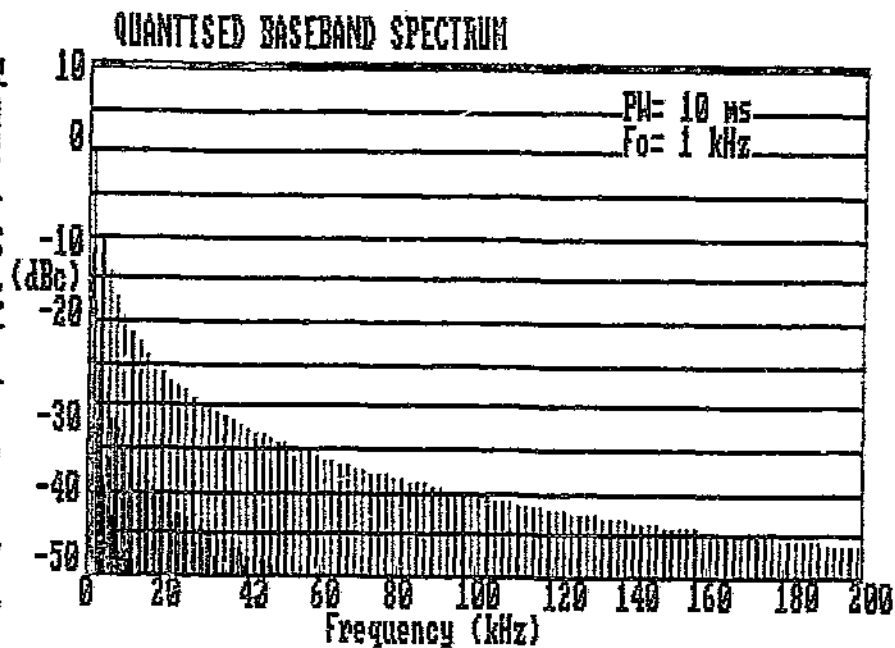


Figure 4.14 Aliasing Example #3

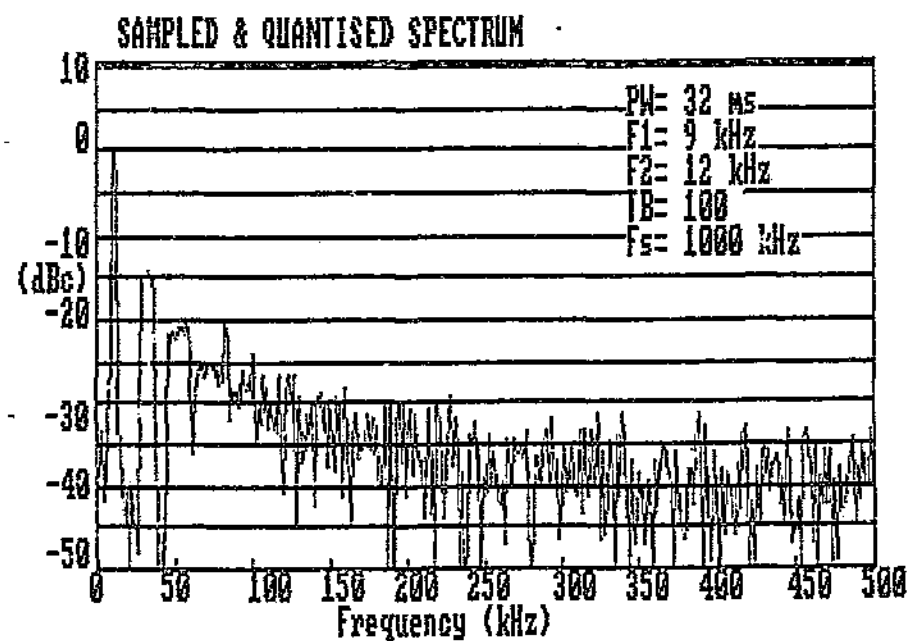
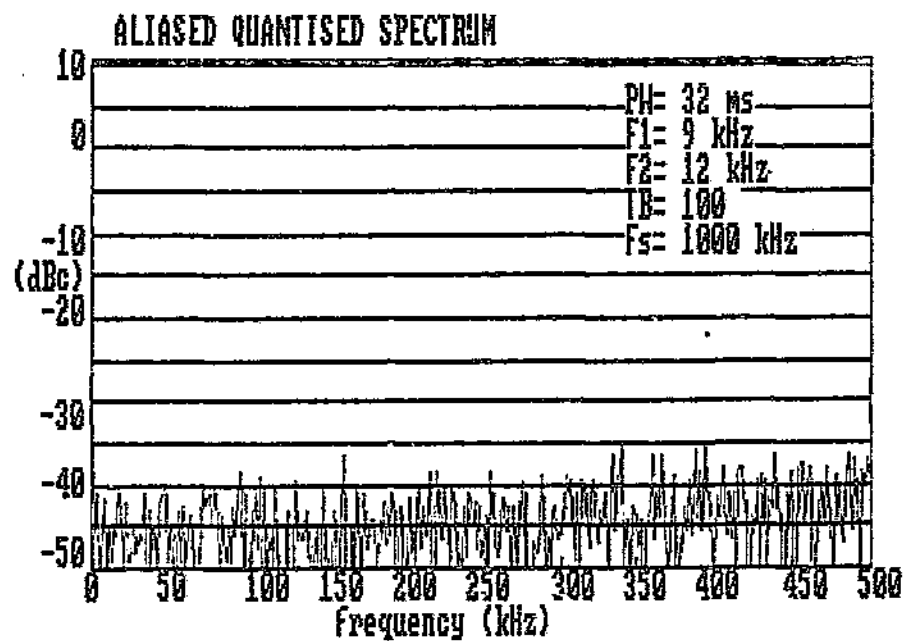
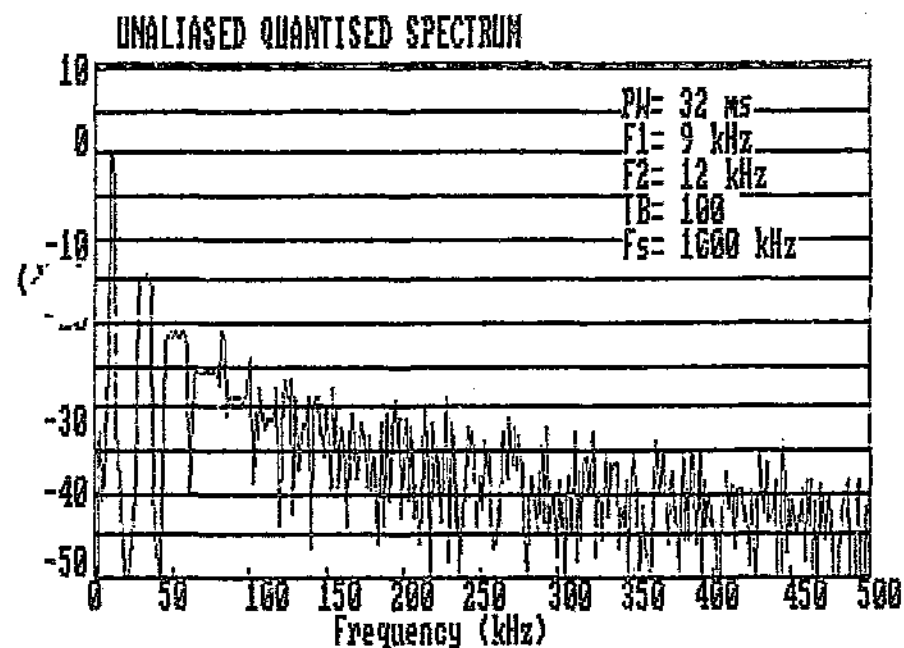
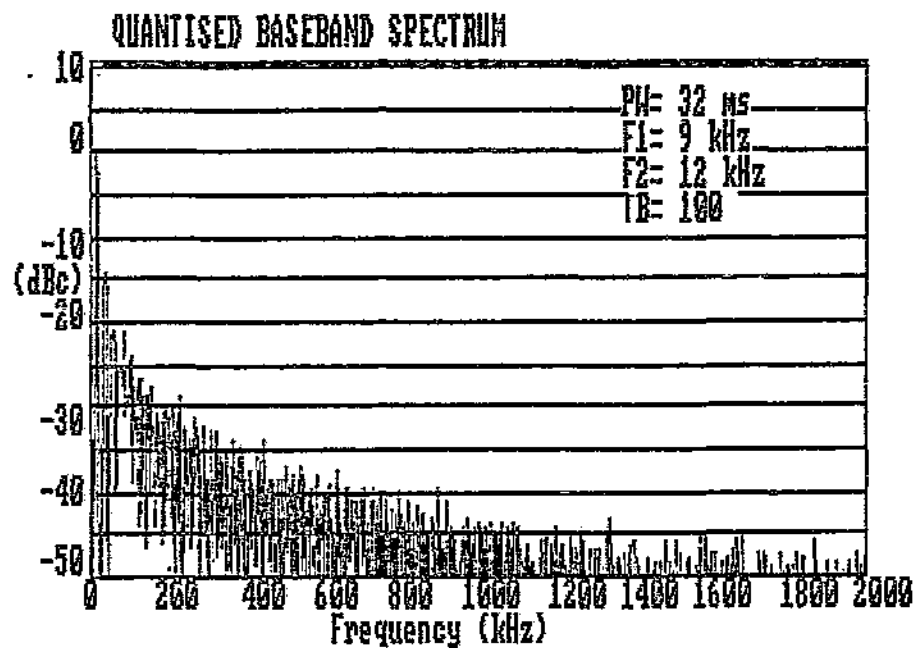
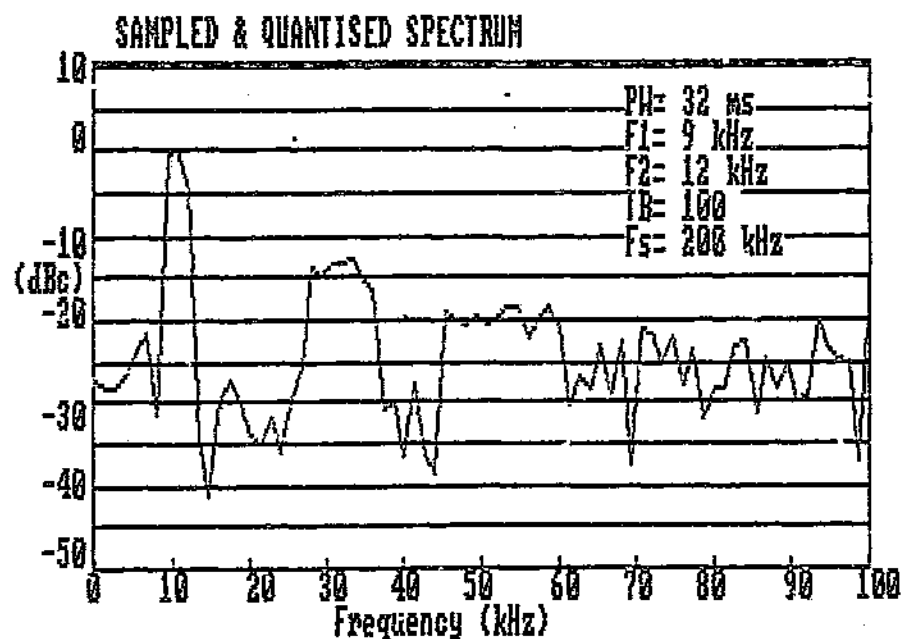
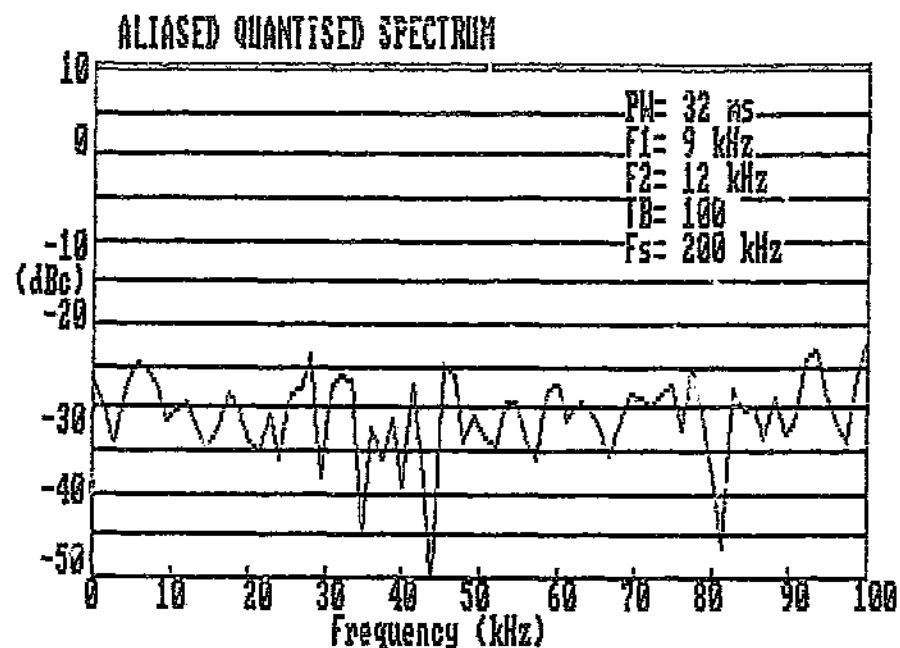
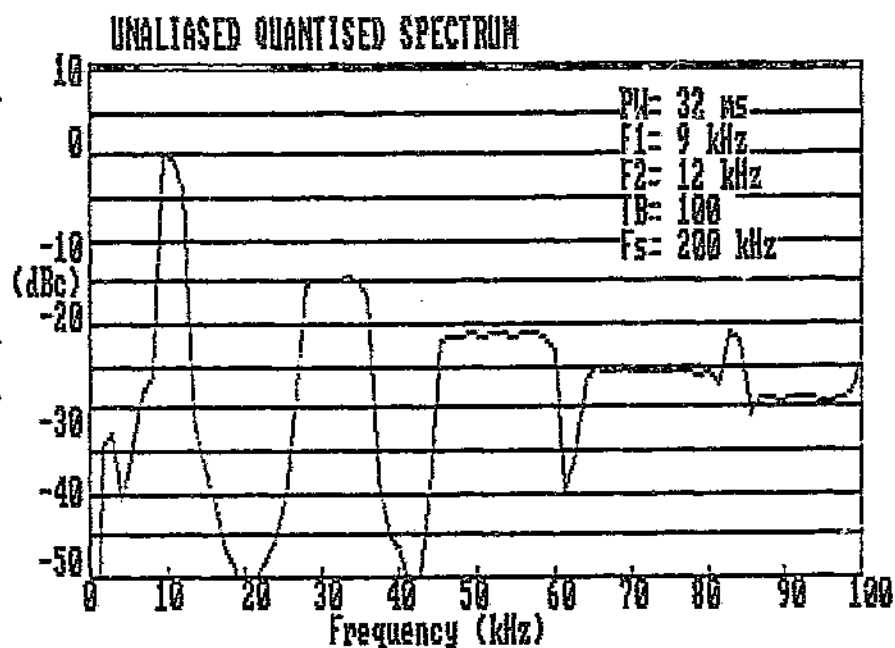


Figure 4.15 Aliasing Example #4



4.2 Extension to N Bit Quantisation

The theory of one bit quantization has been covered in 4.1 above, but the link to general quantisation theory remains to be established. Certain conclusions have been made, and although not yet explicitly stated, it is clear that if the number of levels of quantization increases and the level of the associated harmonics decreases, so the quantisation error and the aliased quantisation error levels also decrease for any given sampling frequency. Hence the principles involved in the extension to N bits have been considered within the limitation of the available computing power.

Software was written to evaluate the frequency spectrum of a signal processed by a multi-bit A/D converter. A link between harmonic generation and the general theory of quantization will be illustrated. Software was also written to evaluate the effect of quantization on the largest spurious signal produced when the input signal is not matched exactly to the dynamic range of the quantizer.

4.2.1 Background Information for the N Bit Case

Up till now, one bit quantization has required that the zero crossings be calculated so that the Fourier Transform of the quantized signal may be determined. This implies that the times at which the signal crosses from one quantization interval to the next, i.e. level crossings, must be evaluated. This concept may be extended to the N bit case and is illustrated in Figure 4.16 overleaf.

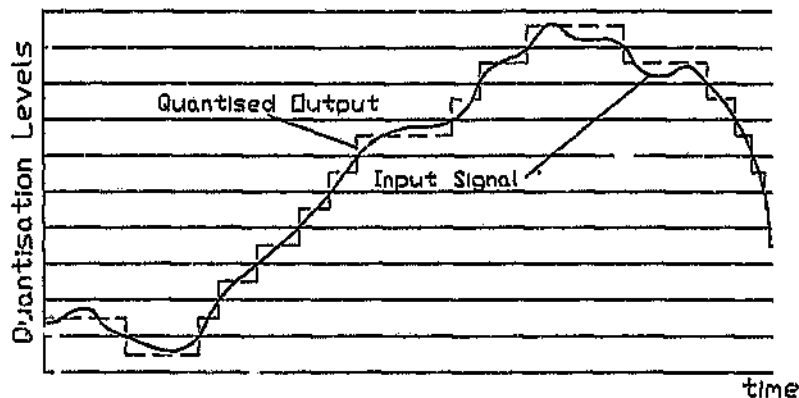


Figure 4.16 Extension of Spectral Theory to N Bits

The theory is easy to prove and is shown in Appendix M. The differences between the one bit and general cases are outlined below. With reference to Johnston (1984) and 4.1 above, the signal $s(t)$ will take on values x_n that have values determined by the number of levels of quantization, instead of just ± 1 . In addition, the values t_n are not the zero-crossings anymore, but the level crossings. Hence it may be seen that Equation M2 for the Fourier Transform is identical in all respects, with the exception that x_n is multi-valued and the number of time data points will have increased in proportion to the number of levels of quantization.

Thus it may be seen that the Fourier Transform of a signal may be determined, by means of Equation M3, to any degree of quantization accuracy desired (limited only by computational resources), in the limit of an infinite number of quantization levels, tending towards the exact Fourier Transform. As stated earlier, this method has certain advantages over the FFT in terms of accuracy and the ability to separate the effects of quantisation and aliasing, but at the cost of computation speed.

With particular reference to the analysis of quantization effects, it is believed that this method is far superior to the FFT because of the ability to separate the effects of quantization from sampling. Also, to achieve similar results with the FFT would require very fast data sampling so as not to influence the signal with aliasing effects. In turn, this requires a vast number of data points and so this latter approach takes just as long if not perhaps longer than the Fourier Transform approach. With the FFT method, it is difficult to isolate the aliasing error once realistic sampling is introduced. The FFT approach is, however, more useful when determining the combined effects of sampling and quantisation, because of the advantage of computation speed.

Since the extension to N bits required far more computing power, an interim solution was sought. To obtain an idea of the general levels of quantization error for a single tone input, as an indicator to the spurious signal levels that could be achieved for a given number of bits, a general expression for the Fourier Series coefficients of a single-tone input as a function of the number of levels of quantization was obtained (c.f. Equation L7), as described in Appendix L.

Tables 4.1 and 4.2 show the levels of the first 99 harmonics for quantization between one and ten bits. It may be seen that for one bit quantization the third harmonic is at -9,54dBc and for two bits it is at -18,3dBc. An advertisement referring to a two bit 500MHz bandwidth DRFM indicates that the spurious signal levels are typically -17dBc.

It has already been stated that for one bit quantization, the quantization error is in general more significant than any

aliasing error. In the case of two bits, it is clear that because the level of the quantization error is considerably lower, the influence of the aliased quantisation error terms becomes more significant. This is demonstrated by the fact that the spurious signal levels exceed the third harmonic levels by about 1,3dB. Hence the aliasing terms must be around the -23dBc level. The third harmonic levels then underestimate the spurious signal levels as the number of bits increases, but this is still a useful indication of the degree of quantization error. Software has also been written to determine the various spectra as for the one bit case, based on the results of Appendix M.

Table 4.1 Fourier Series Coefficients - 1 to 5 Bit Quantization

HARMONIC	SPURIOUS LEVELS VERSUS NO. OF BITS (dBc)				
	1 BITS	2 BITS	3 BITS	4 BITS	5 BITS
1	0	0	0	0	0
3	-9.542425	-18.27221	-27.04456	-35.89192	-44.80696
5	-13.9794	-25.41834	-29.21966	-36.77035	-45.20194
7	-16.90196	-28.34092	-34.88713	-38.46618	-45.87577
9	-19.08485	-27.81461	-51.36521	-41.71711	-46.94489
11	-20.82785	-20.82785	-35.32438	-49.10348	-48.61635
13	-22.27887	-22.27887	-39.64908	-57.00993	-51.29912
15	-23.52183	-32.25164	-36.26034	-45.89701	-56.05908
17	-24.60898	-36.04791	-33.73892	-44.95197	-69.12496
19	-25.57507	-37.01404	-33.54594	-53.23287	-61.946
21	-26.44438	-35.17412	-32.08073	-48.42366	-55.56643
23	-27.23455	-27.23455	-27.08024	-42.43506	-53.86748
25	-27.9588	-27.95881	-33.07975	-47.60281	-55.32904
27	-28.62728	-37.35711	-40.75333	-45.60131	-62.85056
29	-29.24796	-40.68688	-38.82564	-39.5875	-63.13923
31	-29.82724	-41.26623	-64.27265	-53.34694	-53.93942
33	-30.37028	-39.09993	-40.58244	-39.10855	-51.87956
35	-30.88136	-30.88135	-49.09756	-42.27639	-54.82488
37	-31.36403	-31.36405	-38.17121	-40.96829	-79.30233
39	-31.82129	-40.55114	-47.42995	-44.30002	-52.62145
41	-32.25568	-43.69458	-36.04242	-40.48615	-49.23323
43	-32.66937	-44.10841	-57.90069	-46.87111	-52.52099
45	-33.06425	-41.79389	-37.68041	-41.24782	-64.12597
47	-33.44196	-33.44193	-38.99071	-35.03027	-49.136
49	-33.80393	-33.80394	-39.28725	-39.72071	-48.49174
51	-34.1514	-42.88127	-36.85843	-42.63716	-61.82603
53	-34.48552	-45.92441	-45.11164	-42.26911	-50.98422
55	-34.80726	-46.24631	-44.20502	-57.11347	-49.38535
57	-35.1175	-43.84713	-63.80327	-62.87539	-73.6421
59	-35.41704	-35.41702	-40.63477	-46.22716	-51.39721
61	-35.7066	-35.70662	-51.02824	-52.53675	-58.77294
63	-35.98681	-44.71675	-42.12545	-50.23709	-51.40046
65	-36.25827	-47.69713	-46.01136	-47.40759	-52.71831
67	-36.5215	-47.96056	-57.91316	-65.40486	-51.09648
69	-36.77698	-45.50664	-57.71027	-49.49082	-47.24867
71	-37.02517	-37.02514	-45.62041	-84.58635	-59.57989
73	-37.26646	-37.2665	-37.58228	-55.21326	-48.13011
75	-37.50122	-46.23115	-40.14023	-52.55241	-62.42237
77	-37.72981	-49.16864	-56.97576	-55.96798	-54.55017
79	-37.95254	-49.39159	-54.54028	-45.38908	-48.47113
81	-38.1697	-46.89931	-44.99354	-52.42025	-56.7569
83	-38.38156	-38.38151	-46.82022	-44.03821	-45.1574
85	-38.58838	-38.58841	-54.3064	-57.4456	-71.05437
87	-38.79038	-47.52042	-53.57685	-50.30895	-52.21114
89	-38.9878	-50.42665	-60.62869	-53.86535	-47.96928
91	-39.18083	-50.6199	-47.82614	-47.56663	-49.42535
93	-39.36966	-48.09916	-54.7709	-56.51848	-48.96941
95	-39.55447	-39.55444	-42.10229	-42.42649	-46.44624
97	-39.73543	-39.73548	-39.68139	-45.96649	-46.09968
99	-39.91271	-48.64265	-50.91228	-48.3007	-44.22345

Table 4.2 Fourier Series Coefficients - 6 to 10 Bit Quantization

HARMONIC	SPURIOUS LEVELS VERSUS NO. OF BITS (dBc)				
	6 BITS	7 BITS	8 BITS	9 BITS	10 BITS
1	0	0	0	0	0
3	-53.77019	-62.7639	-71.77654	-80.79998	-89.85079
5	-53.95705	-62.8538	-71.81754	-80.81281	-89.82548
7	-54.25722	-62.9957	-71.88797	-80.85208	-89.85176
9	-54.69424	-63.19254	-71.98023	-80.89426	-89.86481
11	-55.30591	-63.45417	-72.1016	-80.95453	-89.90138
13	-56.14564	-63.78996	-72.25063	-81.02281	-89.92927
15	-57.29575	-64.21518	-72.43388	-81.10945	-89.97704
17	-58.88729	-64.74697	-72.65091	-81.20489	-90.01475
19	-61.15776	-65.41185	-72.91087	-81.32151	-90.07534
21	-64.62075	-66.24133	-73.21555	-81.45018	-90.13068
23	-70.9145	-67.28402	-73.57433	-81.60102	-90.20317
25	-95.80091	-68.60818	-73.99272	-81.76712	-90.27339
27	-70.67121	-70.3268	-74.48248	-81.95676	-90.2934
29	-65.70937	-72.64081	-75.05523	-82.16834	-90.44774
31	-63.54733	-75.98134	-75.72858	-82.40785	-90.55114
33	-63.01324	-81.6227	-76.51967	-82.6724	-90.65689
35	-64.03467	-101.9276	-77.45833	-82.96935	-90.77711
37	-67.35538	-83.97924	-78.585	-83.30131	-90.90646
39	-77.79891	-77.88438	-79.95575	-83.67308	-91.04846
41	-72.93796	-74.82238	-81.6613	-84.08544	-91.19784
43	-64.9545	-73.0895	-83.86591	-84.55025	-91.36614
45	-61.84664	-72.23394	-86.89274	-85.06913	-91.54319
47	-61.13959	-72.12221	-91.61355	-85.65517	-91.73886
49	-62.83977	-72.78715	-102.4652	-86.31531	-91.94776
51	-69.28603	-74.4446	-99.45718	-87.06389	-92.17354
53	-74.65946	-77.73956	-91.09714	-87.91841	-92.41676
55	-62.78306	-85.40808	-87.18321	-88.90409	-92.68164
57	-59.14276	-89.70536	-84.75926	-90.05469	-92.96961
59	-58.71508	-78.31516	-83.14066	-91.4182	-93.28202
61	-61.74222	-73.85852	-82.0672	-93.06136	-93.61178
63	-76.76391	-71.44	-81.41678	-95.13465	-93.97877
65	-63.81412	-70.37693	-81.14287	-97.86941	-94.37238
67	-58.11524	-70.35637	-81.23558	-101.8839	-94.80364
69	-57.27561	-71.56296	-81.72229	-109.4515	-95.27246
71	-60.74791	-74.651	-82.67945	-118.2829	-95.78679
73	-85.55563	-82.7684	-84.2641	-104.9334	-96.34785
75	-61.66854	-84.25678	-86.8229	-100.0666	-96.96791
77	-57.99986	-74.123	-91.33664	-97.11972	-97.6463
79	-60.0227	-70.05025	-103.7752	-95.05851	-98.40909
81	-74.39003	-68.1382	-96.23186	-93.54534	-99.25822
83	-63.84983	-67.72483	-88.50794	-92.39865	-100.221
85	-60.73935	-68.83746	-84.62868	-91.53643	-101.3076
87	-67.59919	-72.22065	-82.2091	-90.90204	-102.5876
89	-66.23933	-82.42191	-80.64136	-90.47544	-104.0834
91	-59.74824	-78.51548	-79.70593	-90.23353	-105.9085
93	-63.80051	-70.41271	-79.31778	-90.17484	-108.2145
95	-66.46736	-67.24606	-79.47246	-90.30021	-111.3827
97	-56.95875	-66.30051	-80.23908	-90.62756	-116.2785
99	-57.80951	-67.24665	-81.8105	-91.17216	-128.5682

4.2.2 Harmonics and Quantisation Noise for the N Bit Case Using the Fourier Transform

DRFMs, because of the use of quantization, in the most primitive sense, one bit quantization, cause intermodulation effects when sampling is introduced. Thus every harmonic of the quantised signal is present in the message band spectrum, either as the original, or as a folded down or aliased version.

In terms of general quantization and sampling theory, which has been dealt with in Chapters Two and Three, the signal to be sampled must satisfy the Nyquist requirement and for the quantization noise to be white, it is assumed that there are a substantial number of quantizing levels and a reasonable input signal.

In a DRFM however, even if one limits the signal bandwidth by utilizing an anti-aliasing filter prior to sampling and quantization, the coarse quantization process still increases the signal bandwidth by introducing harmonics of the fundamental. The level of the harmonics is a function of the number of bits in the quantizer. Thus the more bits used, the lower the level of any particular harmonic relative to the one bit case, in general.

The figures overleaf show the quantized, but unsampled, spectrum when a 1kHz sinewave is modulated by a 50ms pulse for the one, two and three bit cases.

QUANTISED BASEBAND SPECTRUM

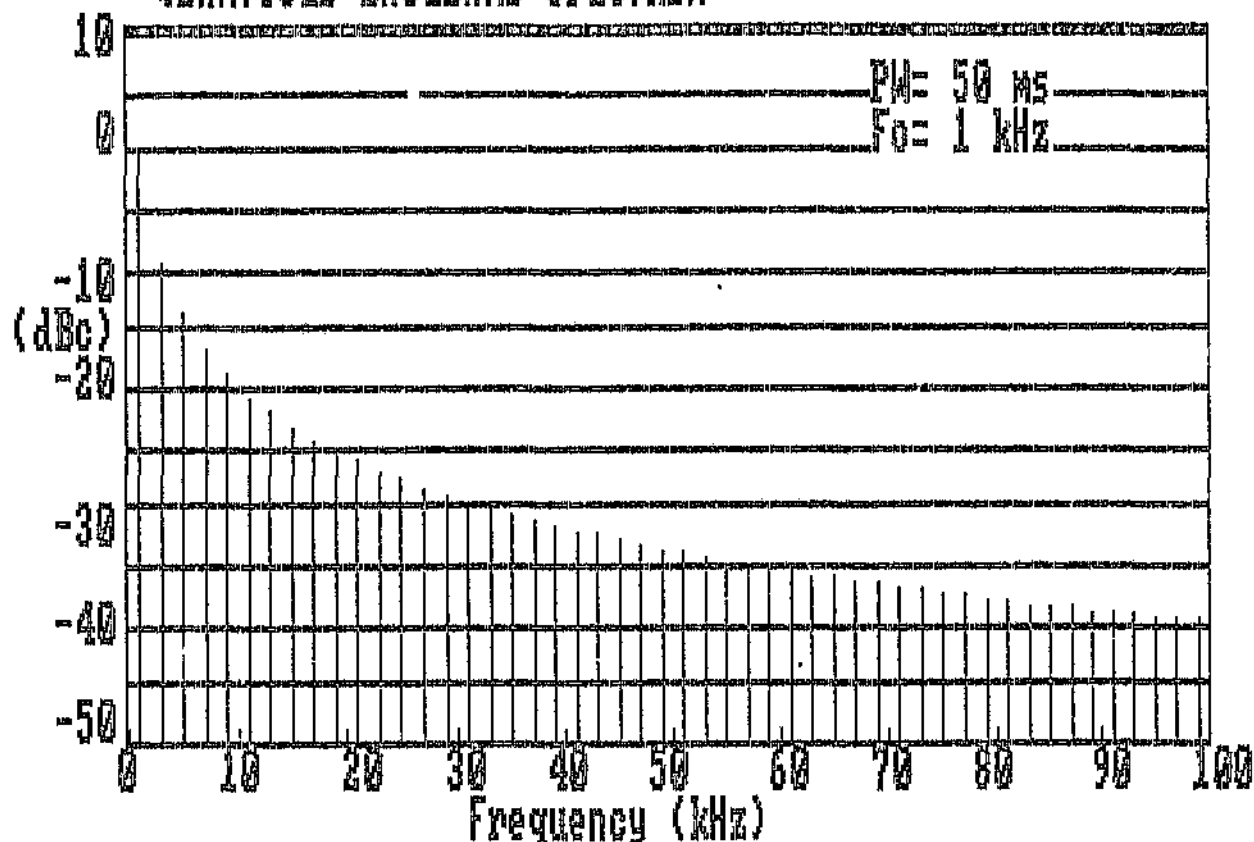


Figure 4.17 One Bit Quantised Unsampling Sinewave Spectrum

QUANTISED BASEBAND SPECTRUM

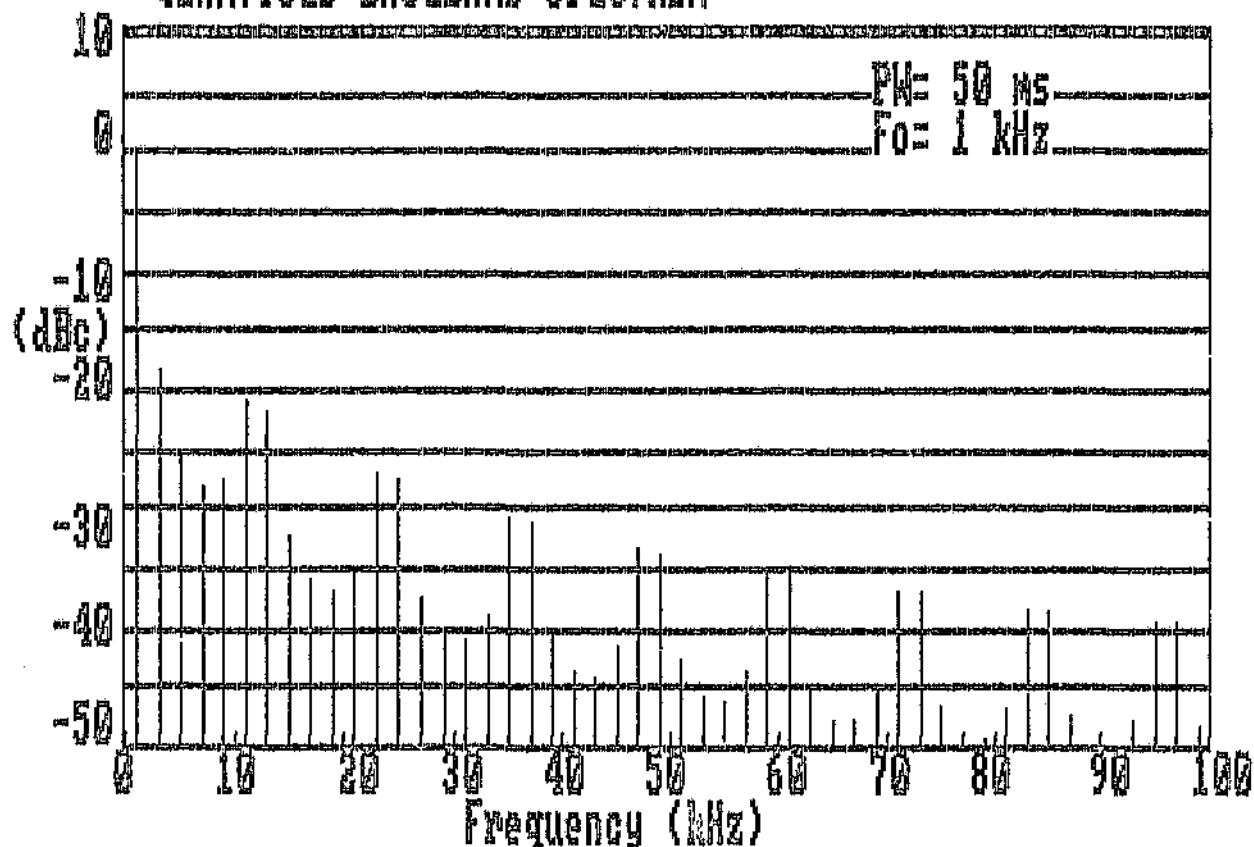


Figure 4.18 Two Bit Quantised Unsampling Sinewave Spectrum

QUANTISED BASEBAND SPECTRUM

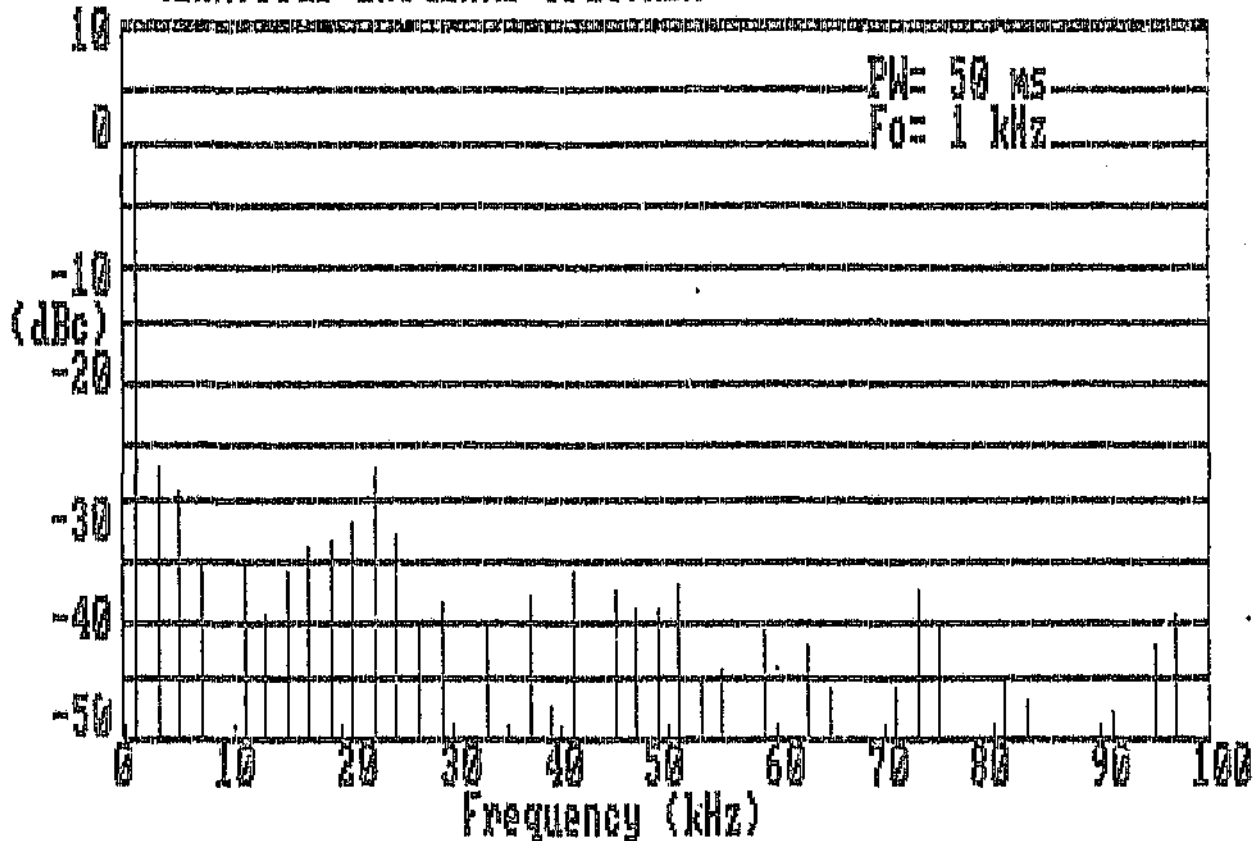


Figure 4.19 Three Bit Quantised Unsampled Sinewave Spectrum

As may be seen, the most significant harmonic's power decreases considerably when the number of quantization bits increases. The results are tabulated below in Table 4.3.

Table 4.3 Largest Harmonic Versus Number of Bits

NUMBER OF BITS	LARGEST HARMONIC	RELATIVE POWER (dBc)
1	3	-9,5
2	3	-18
3	3	-27

Quantization, by the nature of the process, causes a spreading of the spectrum, which is normally ignored by the statistical analysis approaches of generalised quantization theory. The relation of quantizing noise with coarse quantisers to that

predicted by the generalised approach remains to be defined. Standard statistical quantizing noise theory indicates that the quantizing noise is "white" in nature. Care must, however, be exercised when noting under what assumptions these deductions are made. There are several references to indicate otherwise for the coarse quantization case, including Sripad and Snyder (1977) quoted earlier!

Hanrahan (1984) shows that in a typical PCM system, as illustrated overleaf in Figure 4.20, the spectrum of the quantizing noise follows a sinc^2 function, and that after the reconstruction filter, the noise spectrum is flat. The reconstruction filter response is shown in Figure 4.21, deviating from the ideal brickwall response to equalise the sinc^2 shape, introduced by the sample and hold process. Further, it is stated that the major assumptions used are that the quantizing error arising from each encoder interval is uniformly distributed across the interval, and also that successive samples of the quantizing noise are uncorrelated. These assumptions do not always, unfortunately, apply to the coarse quantization case. Reference is also made to a paper by Tsividis and Zee here, which makes use of a technique that will be illustrated soon and that is supported by other references, namely harmonic power summation.

As a start, in Hanrahan's analysis of the PCM system, the assumption is made that the input signal is band-limited. This is a very reasonable assumption that is more than satisfied when using a simple single-tone signal or band-limited speech for analysis and demonstration purposes. The second assumption is that the PDF of the signal is Gaussian, but this is not necessarily true for the coarse quantisation case, although it should not influence the argument too much.

Each replica introduced by the sampling process, will undergo spectral spreading caused by coarse quantisation. As may be visualized, and as is attempted to be illustrated below in Figure 4.21, this leads to an aliasing problem (undersampling of the quantisation error), that is a direct result of the coarse quantization and sampling process.

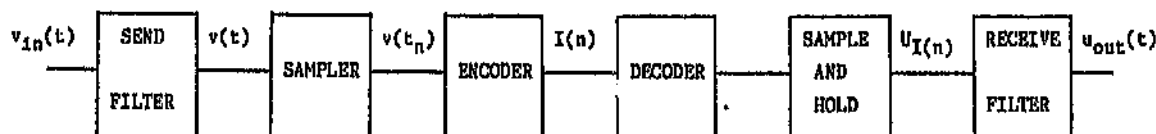


Figure 4.20 Typical PCM System Elements - Hanrahan (1984)

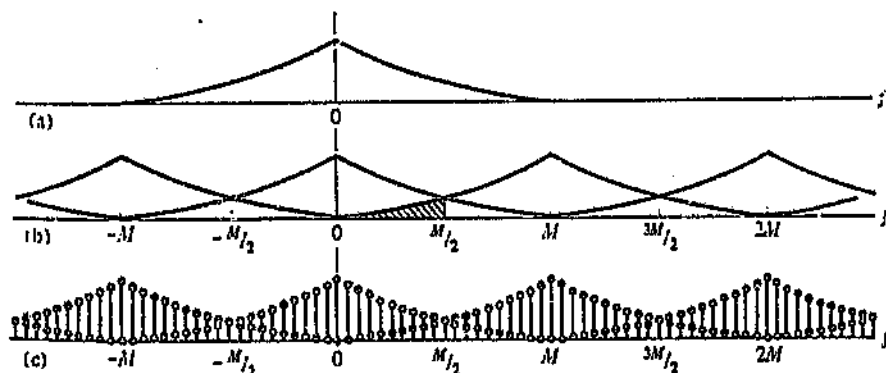


Figure 4.21 Aliasing of Quantised Signal - Stremier (1979)

Another underlying assumption made in standard quantization theory is that the quantizing error is assumed to be uniformly distributed across each quantizing interval. With coarse quantization and certain signals, this cannot be said to be true.

The criterion for the quantisation error to be white, given by Hanrahan (1984), is that the ratio of the standard deviation of the signal to the quantizing interval should be $\gg 1$. For a sine-wave and a one bit quantizer, this assumption is not valid. It is agreed, however, that this assumption is valid for normal A and u-law quantisers with a large number of bits.

By summing the power in the harmonics (excluding the fundamental), it will be found for the one bit case that the power in the harmonics (spur power) is -6,3dBc. This implies that the "noise" power is only 6,3dB below the desired signal.

By adjusting the duty ratio of the square wave that results from the one bit quantization of a sine wave to $1/N$, where N is the harmonic that must be eliminated, it will be found that the N th harmonic will be eliminated, Schneider (1986). By removing the third harmonic, this technique yields a highest harmonic level of -14dBc, namely 4dB lower. This also reduces the total spur power to -9,1dB, (2,8dB lower than before). With this process however, even harmonics are now generated. By alternating the duty cycle around the 50% mark, the phase of the even harmonics is changed alternately, and thus by superposition over a period of time, the even harmonics are eliminated, while reducing the level of the highest (3rd) harmonic. The so-called reduction in spur power shown above is only apparent, because the power in the removed 3rd harmonic is channeled into the generation of even harmonics. Thus total spur power in fact remains the same at any instant. Only temporal integration removes this unwanted spur power.

Oppenheim and Schaffer (1975) note that the assumptions made with standard quantizing theory do not always hold. They state that the general assumptions made, lead to a rather simple analysis of

quantization effects. Such assumptions are not always valid for all signals. However, they go on to say that when the signal is a complicated one, such as speech or music, where the signal fluctuates rapidly in a somewhat unpredictable manner, the assumptions are more realistic. Further, experiments have shown that the more complicated the signal, the less the correlation between the signal and the quantization error, and the error also becomes uncorrelated. Finally, they state that the assumptions of the statistical model appear to be valid if the signal is sufficiently complex and the quantization steps sufficiently small so that the amplitude of the signal is likely to traverse many quantization steps in going from sample to sample. This heuristically states what has essentially been shown to be true in simulations previously.

Bennett (1948) (as quoted by Claassen and Jongepier (1981)), gave an indication of when the spectrum of quantization noise is white. The basic result obtained by him, was that when a quantizer is sufficiently loaded, i.e. when the signal attains an appreciable number of quantizer levels, then the errors can be regarded as noise with a white spectrum over the signal band.

Claassen and Jongepier (1981) state that this result has been used to model quantization errors for both analog and digital signals, and that in general, the model is quite accurate. A number of others have given conditions under which the spectrum of quantization noise of stochastic and deterministic signals is white. Claassen and Jongepier (1981) state that even for a sinusoidal signal, if it has sufficient variation, the quantizing error (noise) spectrum is white.

They give a formula $|f| \leq \pi \cdot L \cdot f_0$, which must be obeyed if the quantizing noise spectrum is to be white, where f is the band of interest and the signal of interest has a frequency f_0 . L is the number of quantizing levels. Thus if the input frequency is 100Hz and the band of interest is 20kHz, as for audio, then the number of quantizing levels must exceed 63,66. Hence the quantizer must have at least 6 bits for the quantizing noise to be white in this 20kHz interval.

By taking into account sampling, as well as the quantization effects described previously, Claasen and Jongepier (1981) state that as long as the sampling frequency is well below $f = f_0 \cdot 2\pi L$, then the error (noise) spectrum that results is sufficiently aliased to introduce enough foldover to create a white spectrum. Thus if the input frequency is 100Hz and $L=2$ levels (one bit), then the sampling frequency must be well below 1256Hz, to ensure a white spectrum. If the sampling frequency is a tenth of this value, namely 125Hz, then the Nyquist criterion for even the fundamental will not be satisfied.

Considering a sampling frequency of 200Hz, say, then the number of quantisation levels would have to exceed three, i.e. at least two bits for the noise to appear white. This is in agreement with statements made by Jayant & Noll (1984).

Widrow (1956), describes rough amplitude quantization as an extension of Nyquist sampling theory. He states that if the samples of a continuous variable are random and statistically independent of each other, then a first order PDF, $W(X)$, completely describes this process. Similarly, the Fourier Transform of the PDF is known as the characteristic function of $W(X)$.

Thus:

$$F_X(\beta) = \int_{-\infty}^{\infty} W(X) \cdot \exp(-jx\beta) dx \quad (4.3)$$

The signal at the input to the quantizer may take on an infinite number of magnitudes, while the output of the quantizer may only assume a fixed number of discrete states. The PDF of the output, $W'(X')$, therefore consists of a series of impulses that are uniformly spaced along the amplitude axis. Each of these impulses is centered in a quantization box. Uniform quantization is assumed. Figure 4.22 below shows how the output PDF is defined.

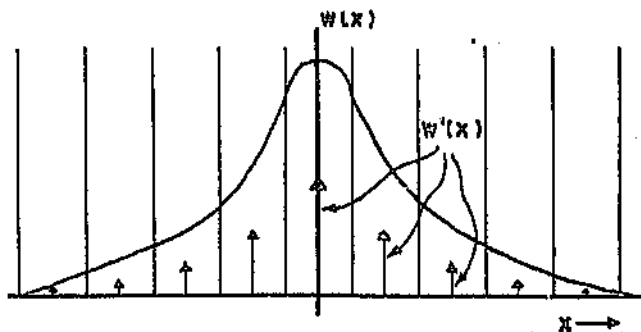


Figure 4.22 Output PDF by Means of Area Sampling - Widrow (1956)

Because any event occurring in any particular quantization box is always defined to produce an output at the centre of that box (quantizing interval), then the amplitude of the output impulse PDF for any box must be equal to the area under the input PDF within the quantizing interval in question. The characteristic function of $W'(X')$, i.e. its Fourier Transform, is periodic, because it is the Fourier Transform of a series of impulses with spacing q , the quantizing interval.

With normal time sampling, the process of periodically sampling a function $f(t)$ is identical to multiplying it by series of unit area impulses, that are uniformly spaced. The impulse train may be represented by a Fourier Series as:

$f^*(t) = [f(t)][\text{impulse carrier}] = [f(t)][1/T] \sum_{-\infty}^{\infty} \exp(jn\Omega t) \quad (4.4)$
as shown before.

Thus it is the sum of an infinite number of sinusoidal carriers with uniform frequency spacing $2\pi/T_s$, where T_s is the sampling period. The signal $f(t)$ then modulates these carriers to create identical sidebands about each frequency $2n\pi/T_s$. This has been shown previously in a discussion of sampling theory. The results are illustrated below in Figure 4.23.

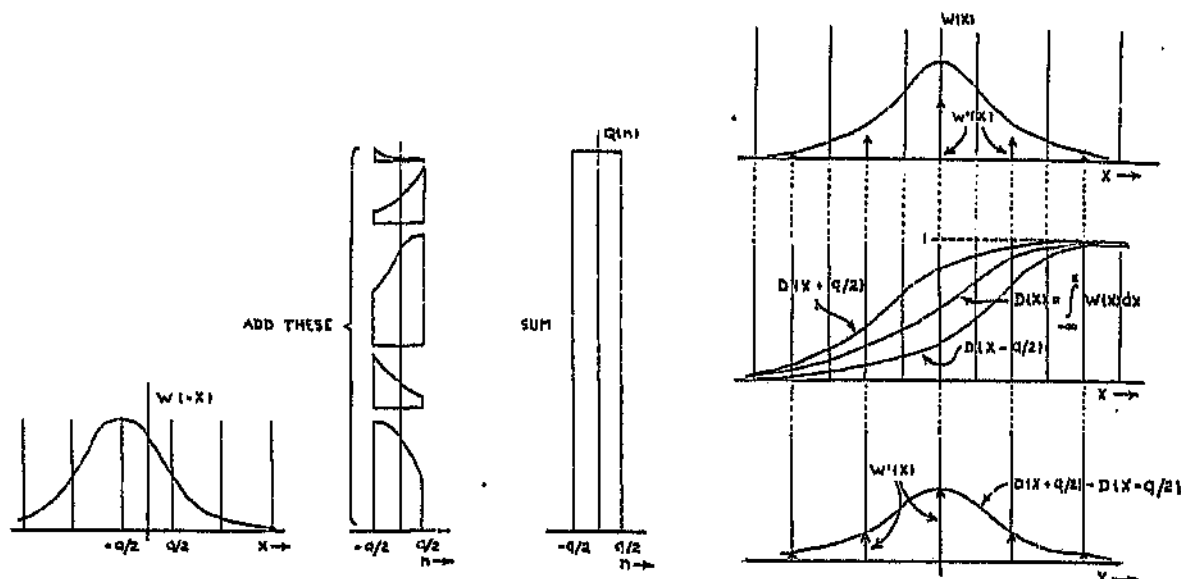


Figure 4.23 Ideal Impulse Sampling and Construction of Area Samples - Widrow (1956)

By separating the zeroth array in the array of sections illustrated, say by using an ideal low-pass filter, it is possible to recover the envelope of the input signal, $f(t)$, from its samples. The Nyquist criterion for sampling may thus be deduced, that the input signal must have no significant harmonic content above half the sampling frequency for signal recovery to take place.

If we have, as stated above, an input signal with a PDF $W(X)$, then the quantizer output PDF, $W'(X')$, may be thought of as an area sampler acting on the input. The cumulative distribution of $W(X)$, $D(x)$, may be obtained by integrating the input PDF. Therefore, considering any quantizing interval within the dynamic range of the quantizer, it is possible to calculate $D(x+q/2)$ and $D(x-q/2)$, as shown in Figure 4.23 above. $W'(X')$ is then the difference between these two cumulative distributions. Widrow (1956) defined a quantity known as the radian fineness, which is $2\pi/q$. When this radian fineness is twice as high as the radian "frequency" of the highest "frequency" component contained in the shape of $W(X)$, then it is possible to recover $W(X)$ from the quantized distribution $W'(X')$.

The understanding of the quantizer would be complete if it were true that the quantization noise, namely the difference between the input and the output, were independent of the quantizer input. This is unfortunately not true. Not only is the quantization noise statistically related to the input, but it is also causally related. Because the output of a quantizer is a single-valued function of the input, any input PDF gives a definite output signal and a definite noise.

The distribution density of the noise itself is independent of the input PDF if and only if the Nyquist criterion for quantization, as defined above, is satisfied. This criterion is satisfied when the quantization grain is fine "enough". Qualitatively, Widrow (1956) states that if the dynamic range of an input variable extends over several quantizing intervals, then the quantization noise will be uniformly distributed and it will be a first order process. He also states that the quantization noise is practically uncorrelated until the quantizing interval

size is one standard deviation and the input correlation is 95%, or until the box size is two standard deviations and the correlation is 80%. The latter box size is considered to be very rough quantization, Widrow (1956).

Summarizing then, if the Nyquist criterion for SAMPLING is satisfied, then the function is recoverable. If the Nyquist criterion for QUANTIZATION is satisfied, then the statistics of the function are recoverable, and the quantization noise is well understood, i.e. white. In the case of well correlated input signals, such as pulse modulated sinusoids, it has been shown that the quantisation noise is highly correlated with the input, especially for very coarse quantisation, such as a one or two bit quantiser, where the sample amplitude does not vary much from one sample to the next.

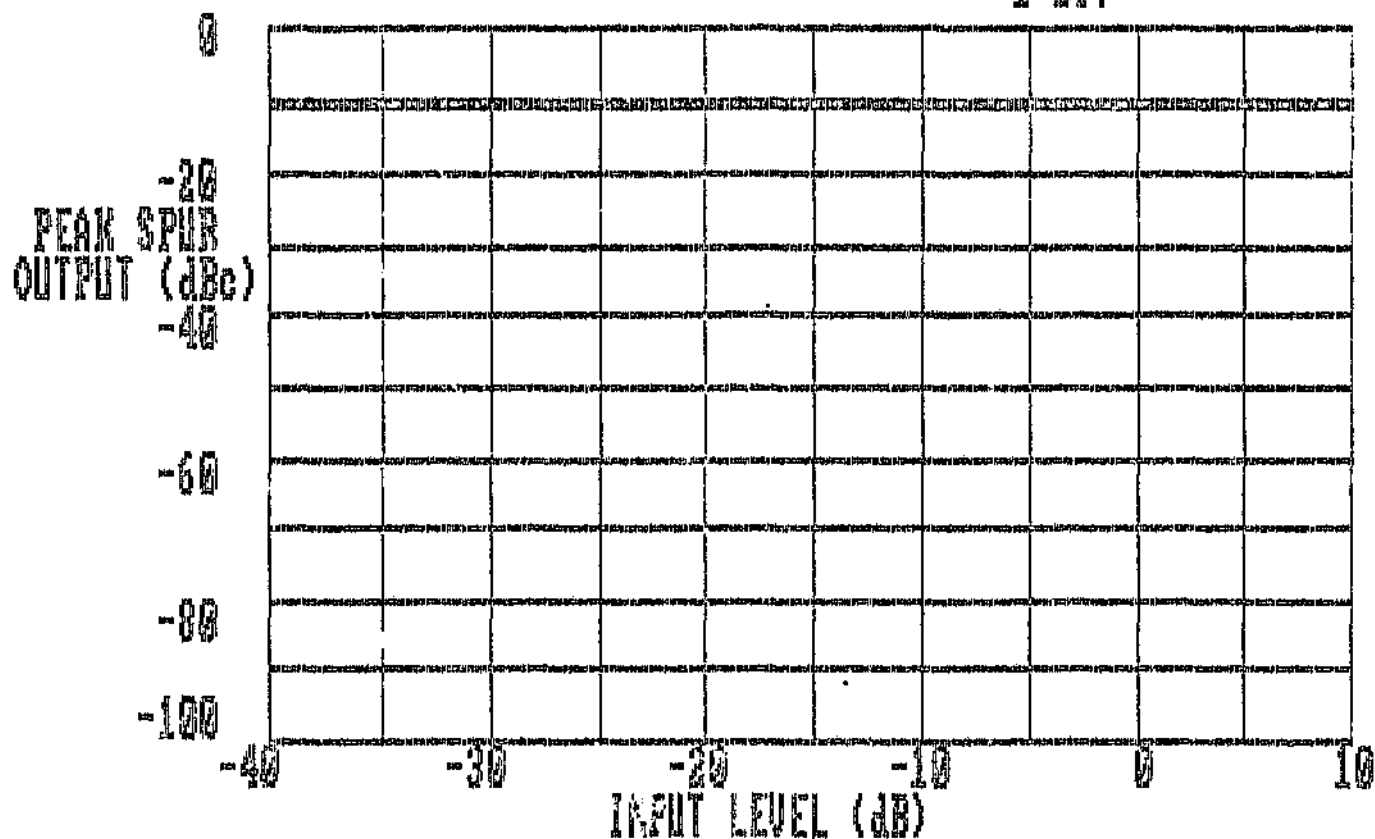
4.2.3 Quantiser Dynamic Range Effects

Software was written to predict the level of the largest spurious signal versus the level of the input signal relative to the ideal 0dB full quantizer dynamic range. The largest spurious signal is effectively the third harmonic for a sinewave input. One to six bits have been used.

In the one bit case, it is clear that the spurious level remains at -9,5dB, regardless of the input level. For the two bit case, the level of the third harmonic varies, depending on the input level. For a 0dB input level, the spurious level corresponds to the levels predicted by the earlier FFT and Fourier Transform analysis.

As the input level drops below 0dB, so the spurious level changes, until only two levels of the quantizer are being used, reverting to the -9,5dB spurious level of the one bit quantizer. Above 0dB, the quantizer saturates, until the input signal becomes so large that it once again hard limits. Thus the maximum spurious level once again becomes -9,5dB. The results are shown for one to six bits. In each case the results are similar, only the quantizer having much more dynamic range before the -9,5dB spurious signal level is reached with saturation at one end of the quantiser's dynamic range and using only two quantisation levels at the other end.

1 BIT



2 BITS

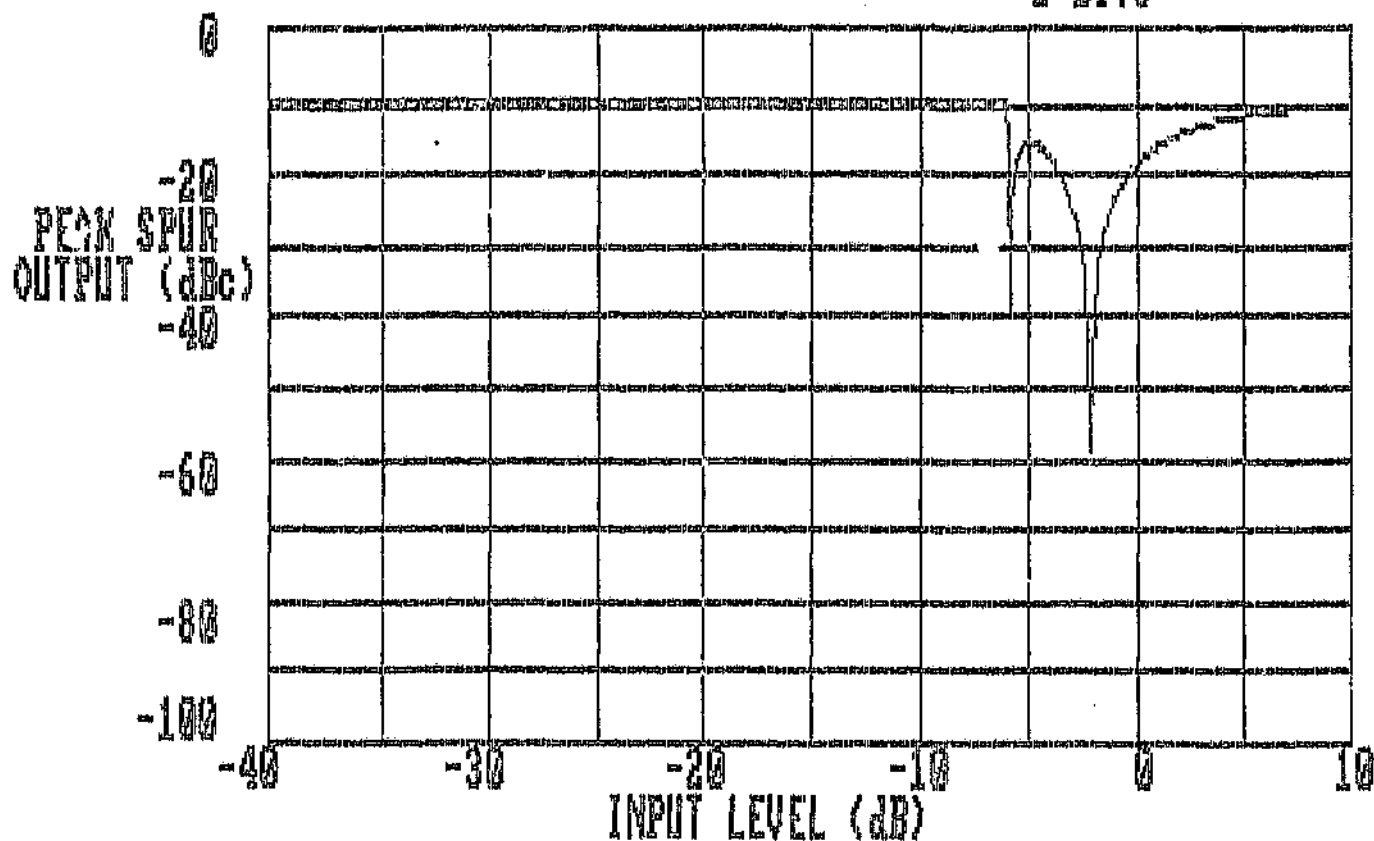


Figure 4.24 Peak Spurious Signal Versus Input Level : a.) 1-Bit
b.) 2-Bits

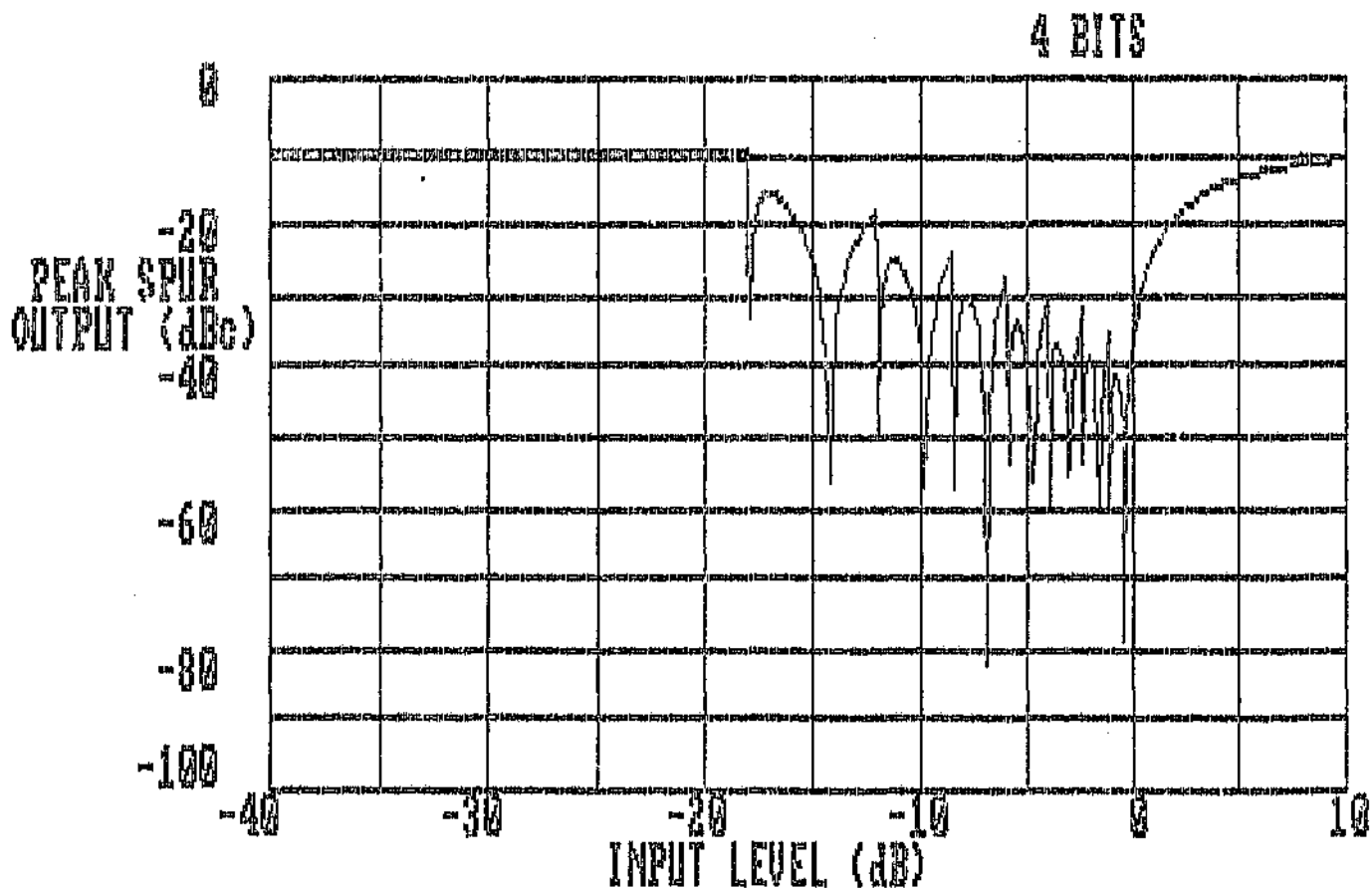
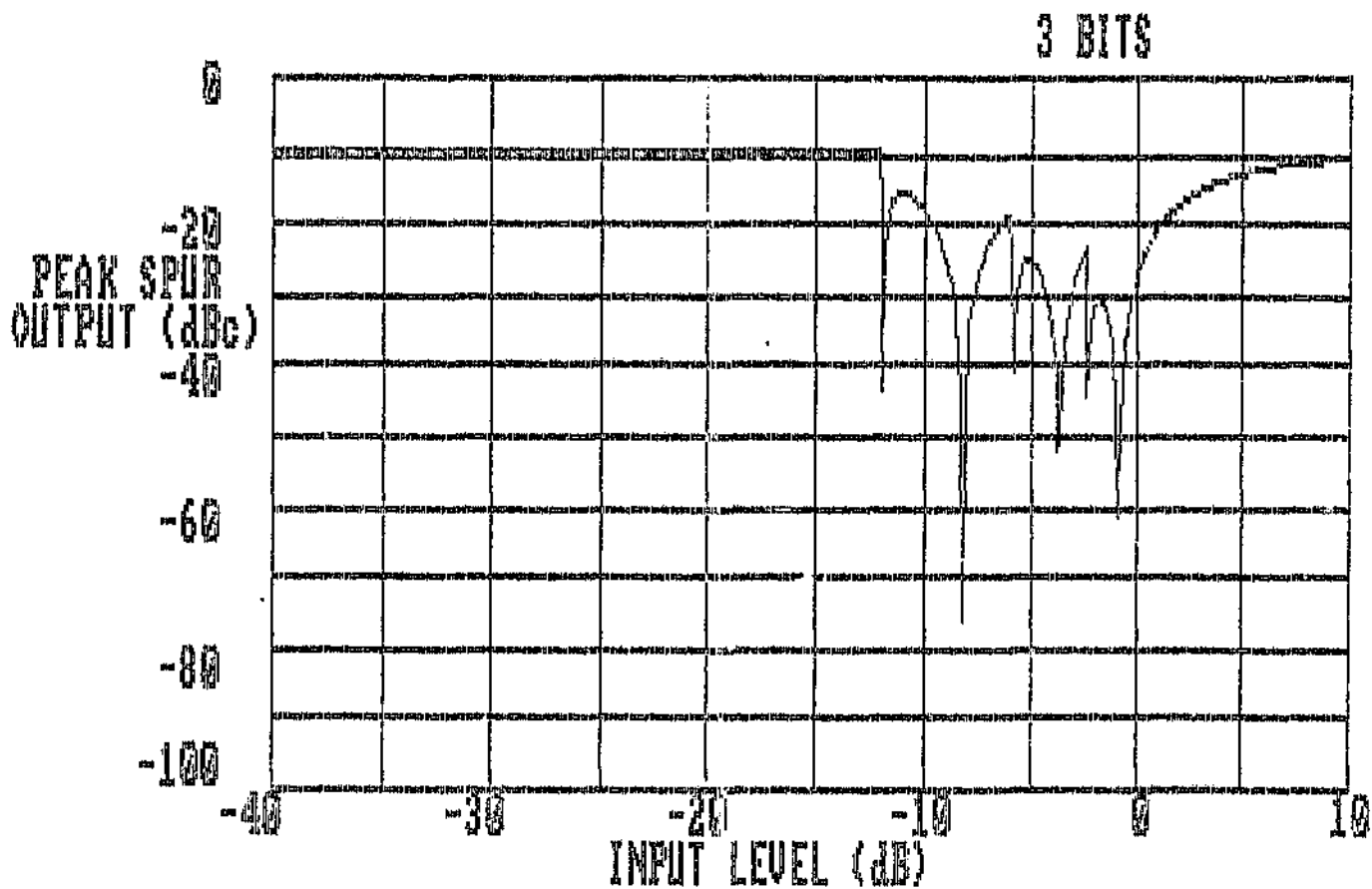


Figure 4.25 Peak Spurious Signal Versus Input Level : a.) 3-Bits
b.) 4-Bits

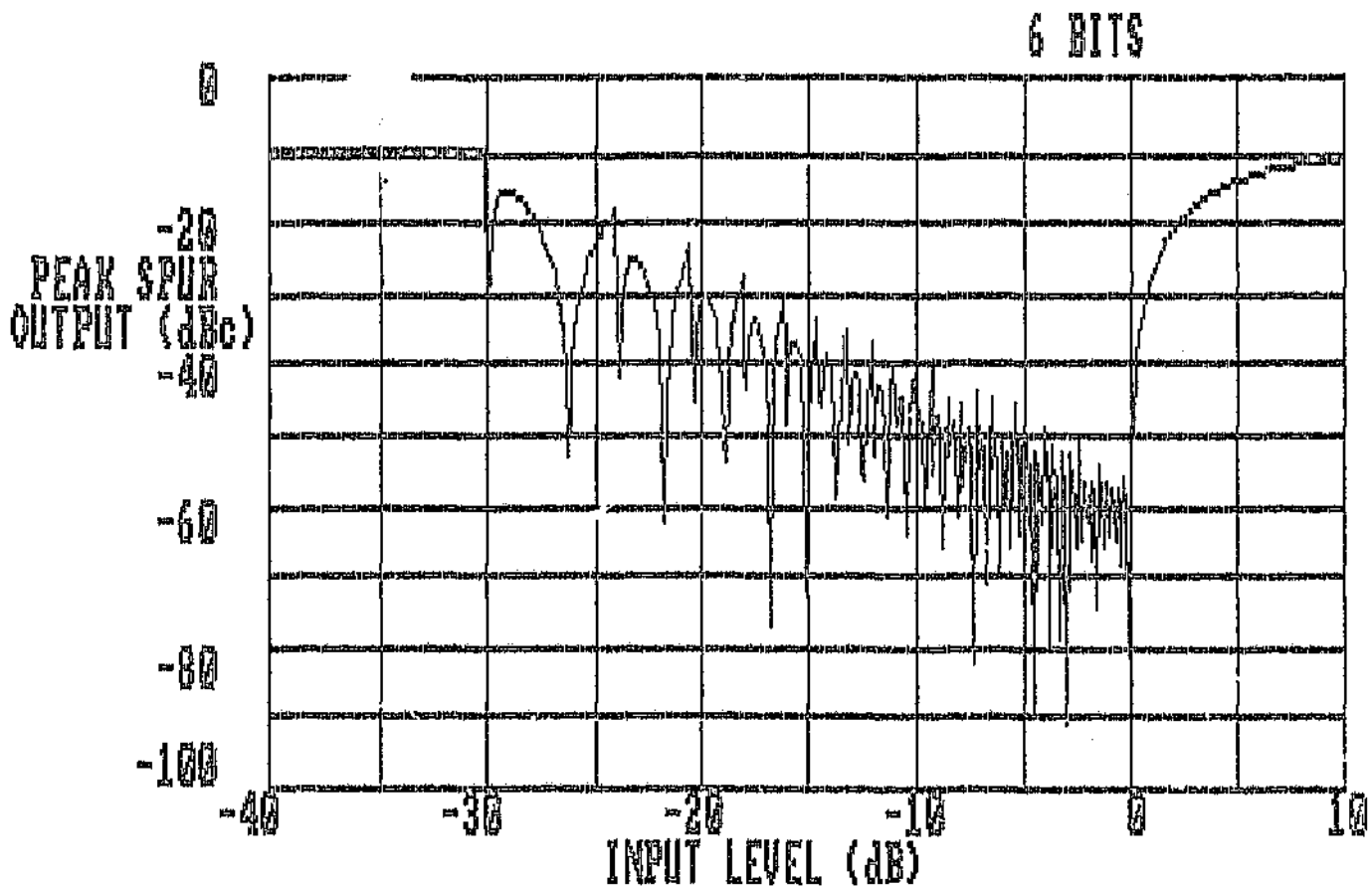
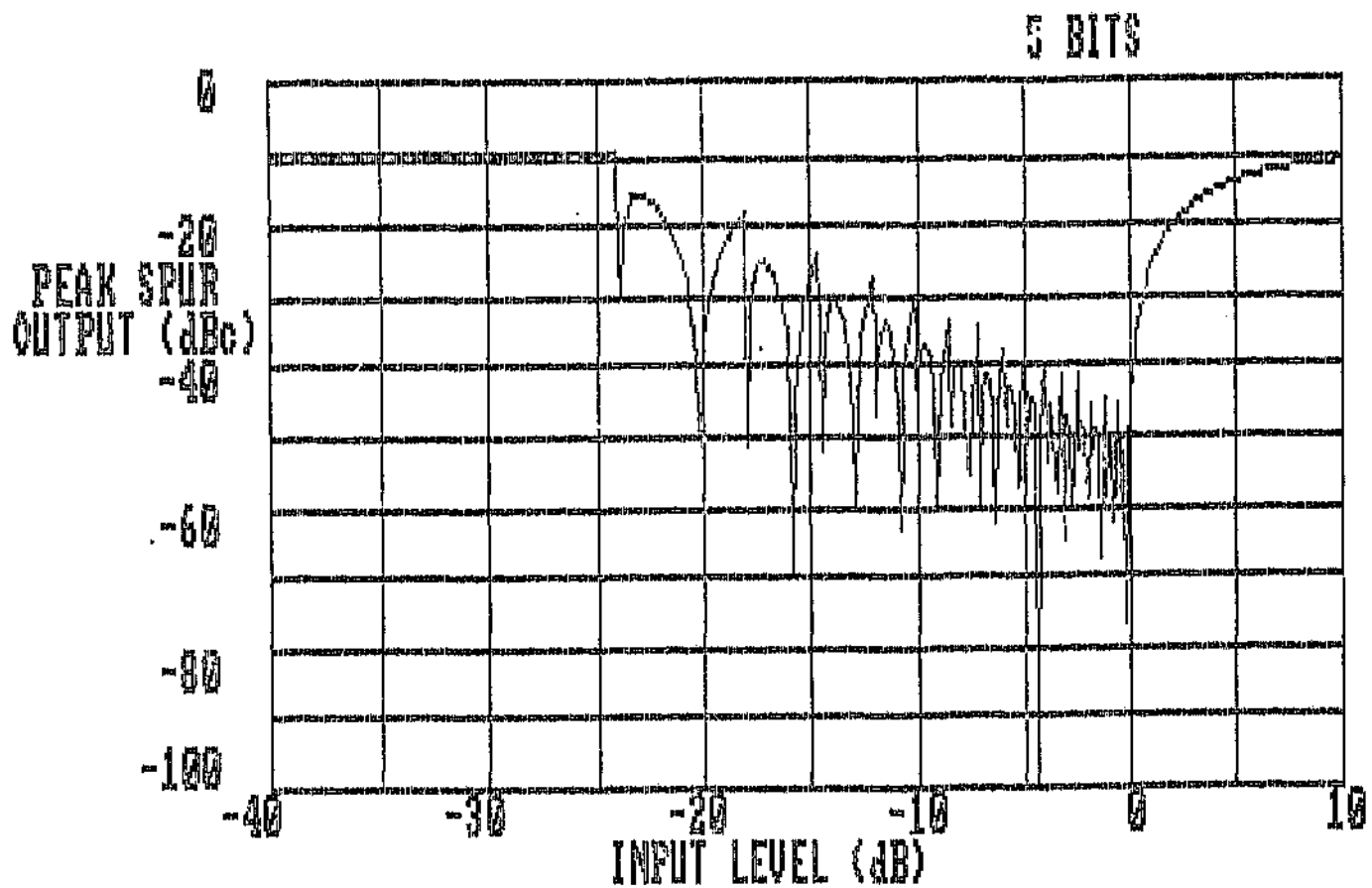


Figure 4.26 Peak Spurious Signal Versus Input Level : a.) 5-Bits
b.) 6-Bits

The link between harmonic generation and quantising noise power, is the total spur or harmonic power versus the number of quantization bits and signal input level. The pattern is similar to the maximum spurious signal level shown above. Of most interest, however, is the level of the total spur power, or quantizing noise power, for a 0dB input signal, as predicted by the Fourier Transform harmonic generation approach, compared to the level normally predicted by general quantizing noise theory ($-20\log[1,22(2^n-1)]$, Schneider (1986)). The results are summarized below in Table 4.4.

Table 4.4 Quantising Noise Power Versus Number Of Bits

Number Of Bits	General Theory(dB)	Fourier Approach(dB)
1	-1,7	-6,3
2	-11,3	-13,0
3	-18,6	-19,5
4	-25,2	-26,0
5	-31,6	-32,0
6	-37,7	-38,0

Thus it may be seen that from two bits upwards, the general theory and the more exact Fourier Transform approach, converge rapidly. From six bits upwards, it may be said that both approaches are identical in predicting the level of the quantizing noise power. However, only the Fourier Transform approach or FFT approach allows prediction of the sampled and quantised signal spectrum. The plots of total spur power or quantisation noise power are illustrated below as a function of the input signal level relative to the quantiser input dynamic range for one to six bit quantisers.

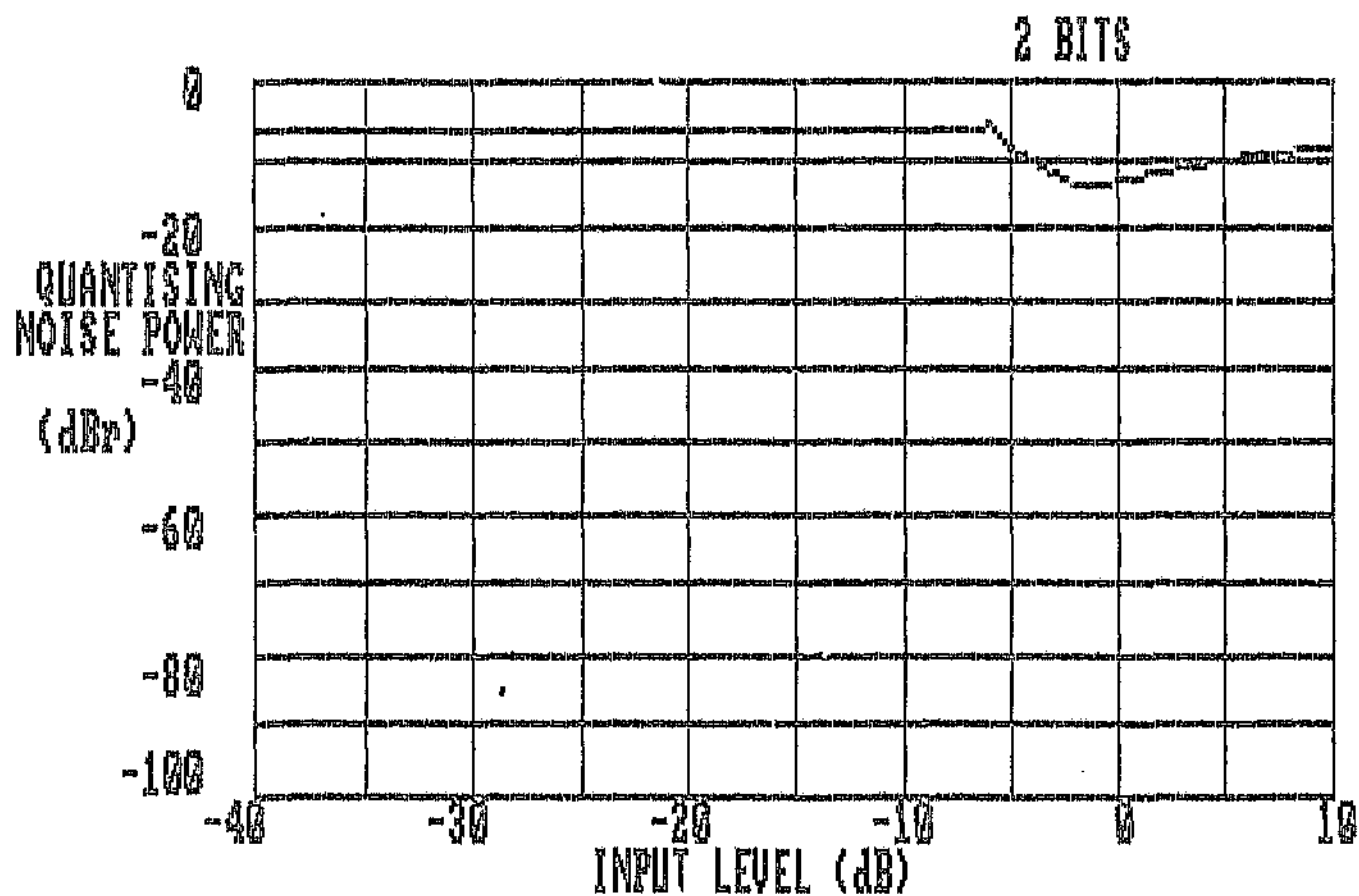
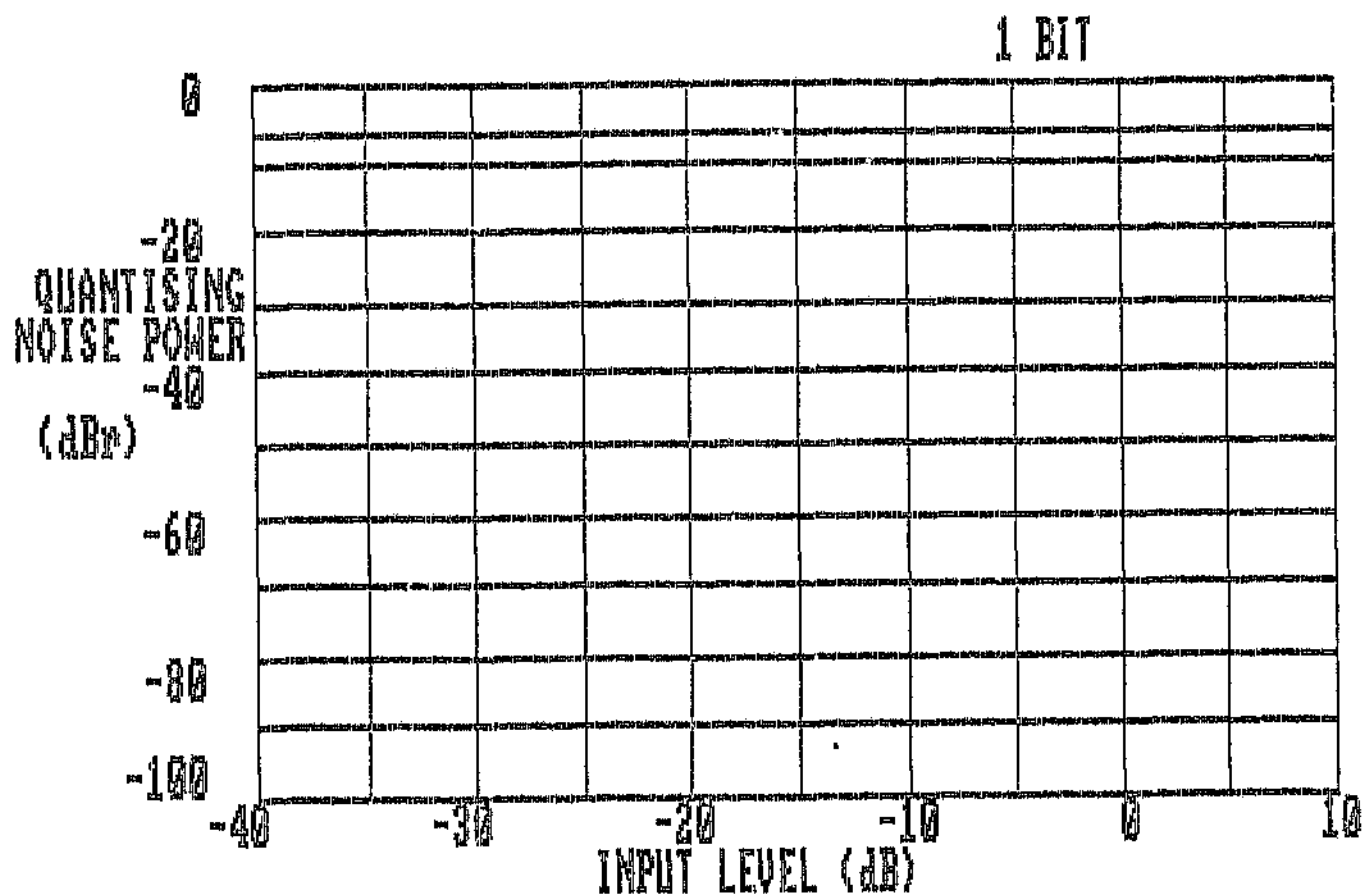


Figure 4.27 Quantising Noise Power Versus Input Level : a.) 1-Bit
b.) 2-Bits

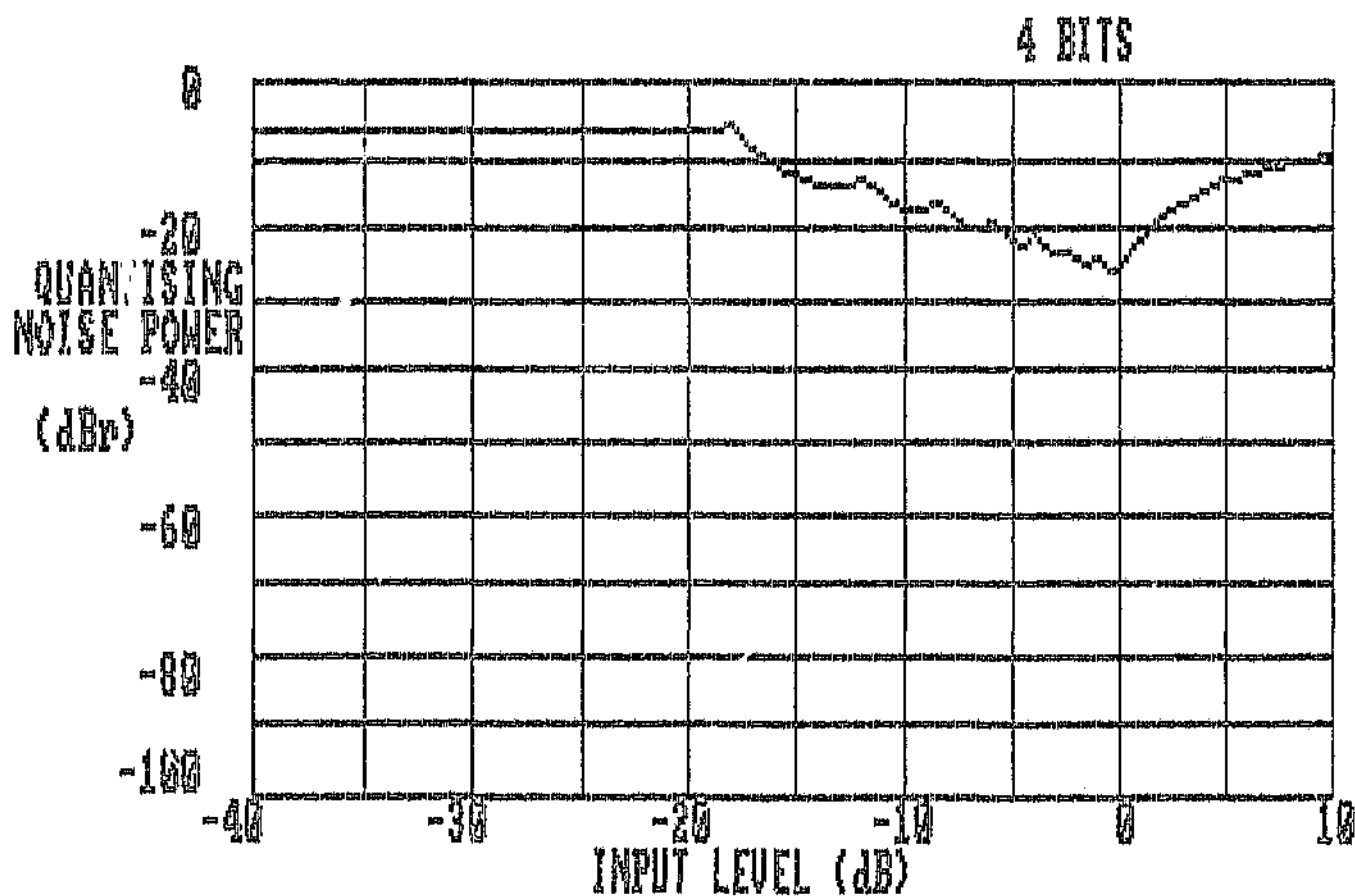
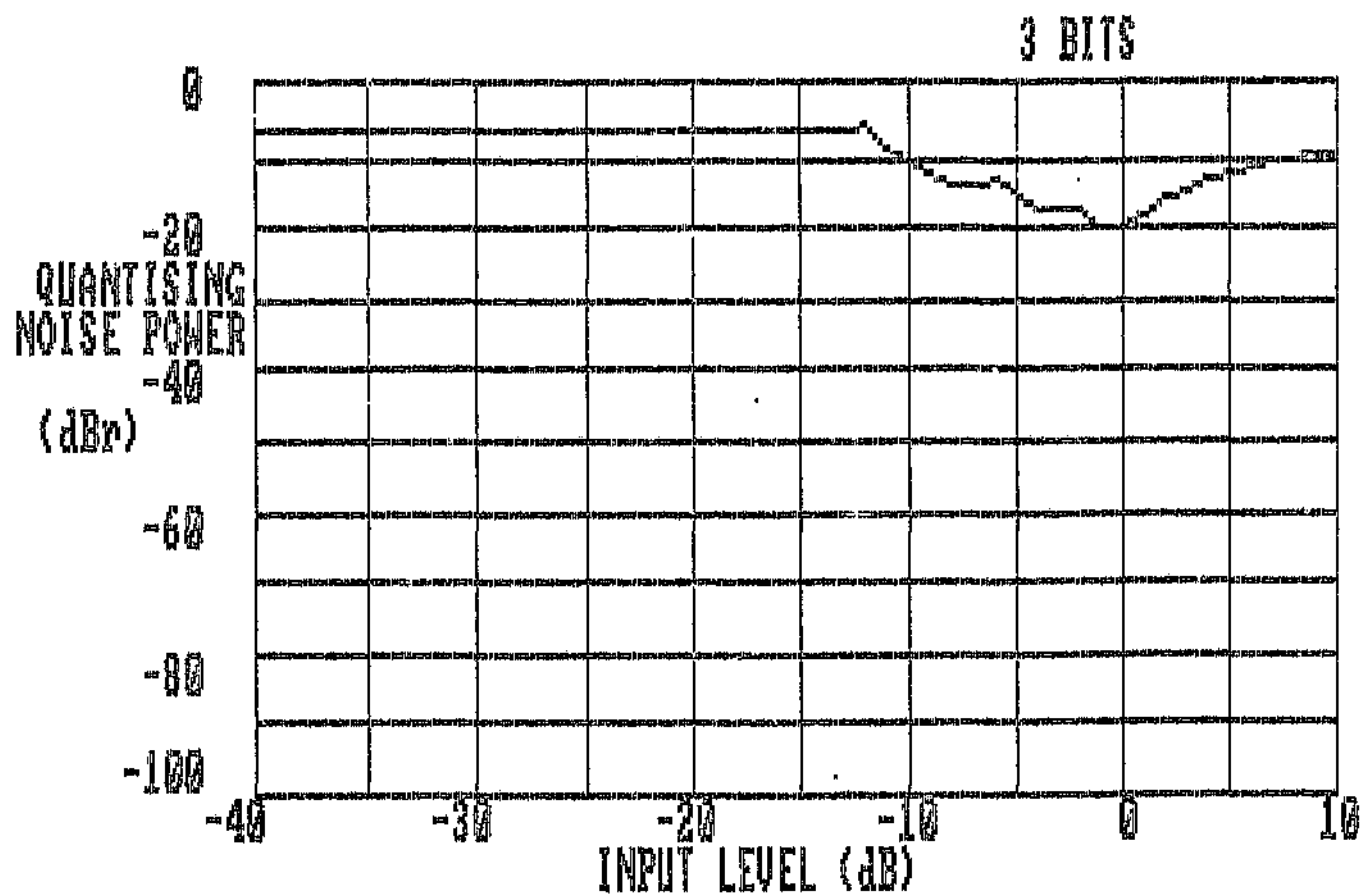


Figure 4.28 Quantising Noise Power Versus Input Level : a.) 3-Bits b.) 4-Bits

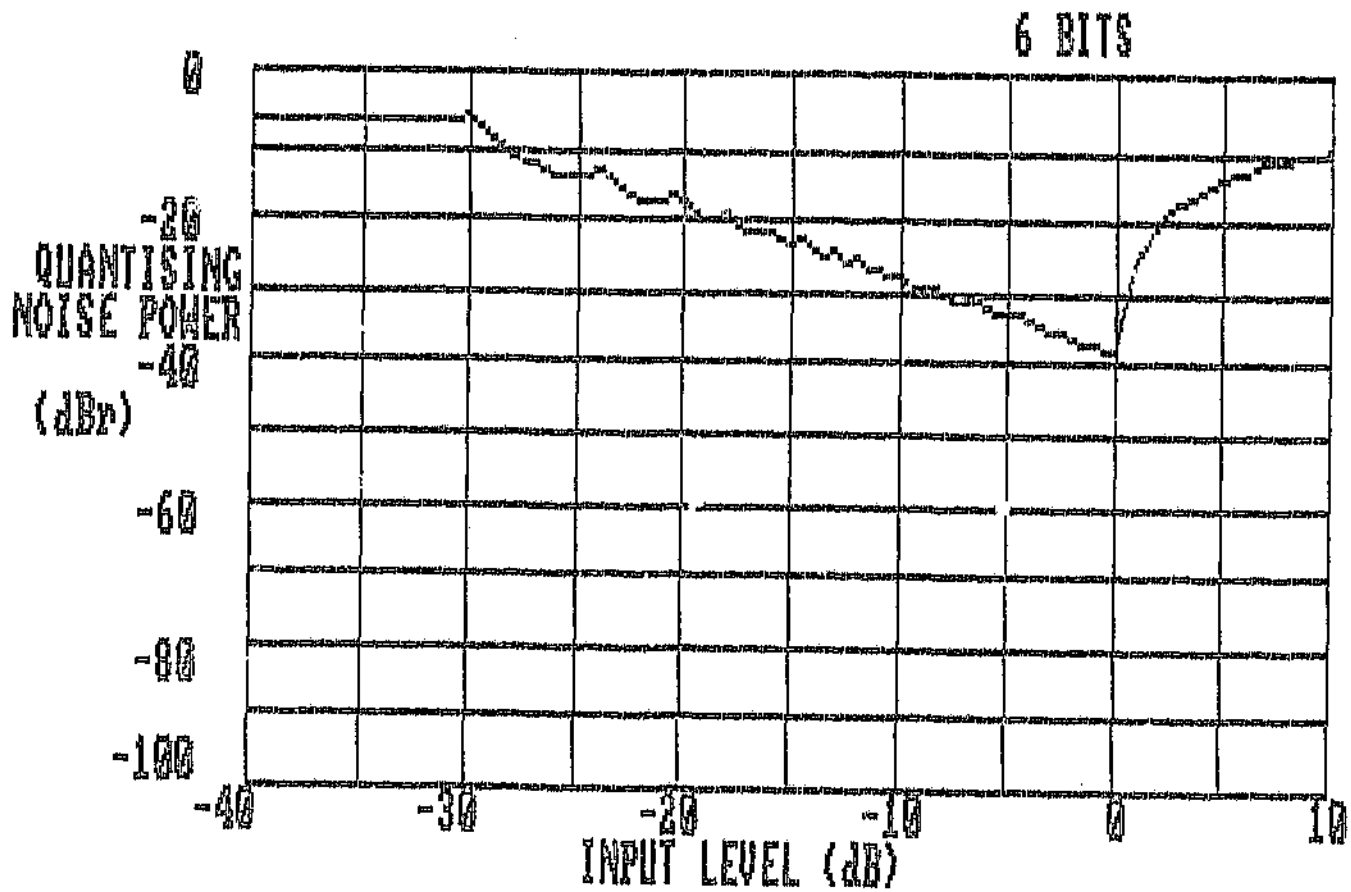
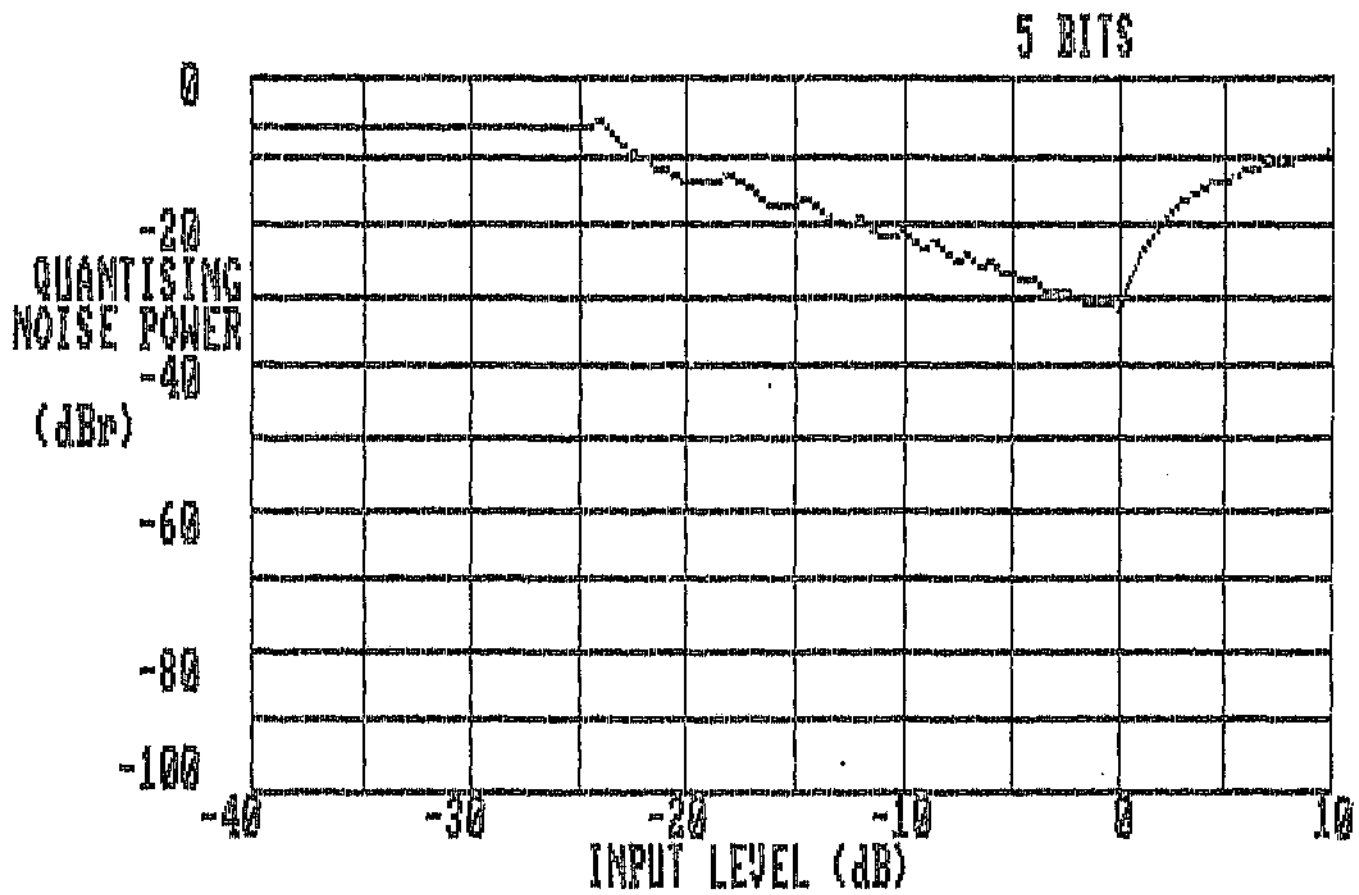


Figure 4.29 Quantising Noise Power Versus Input Level : a.) 5-Bits b.) 6-Bits

4.3 Summary

The spectrum of a one bit quantized signal was studied and a computer program was written to evaluate the spectrum of a quantised single tone and a FMSW signal. This method of evaluating the signal spectrum was compared with the FFT technique and the tradeoff of computation speed versus accuracy was stated. It was found that the "analogue" technique is much better for illustrating the effects of aliasing and quantisation separately, but slower. In order to obtain as accurate a spectrum using the FFT requires an excessive amount of memory and also increases processing time significantly. If greater degrees of quantization are required, then the "analogue" method also requires excessive computer memory.

The errors associated with the DRFM as a signal synthesizer were considered as well as when used as a memory. It was shown that in the former case it is possible to maintain lower spurious signal levels (excluding quantization distortion). This is because one has a priori knowledge of the signal, thus reducing the error of positioning zero-crossings to a minimum and also because there is no aliasing distortion as such, in the true sense of the word.

A number of test cases were illustrated, including comparison with Johnston's (1984) results, in order to validate the software. Some other specific signals, namely the single tone input and the FMSW were used as test cases with various combinations of sampling frequency. It was deduced that for one bit quantization, the quantization error is of greater significance than the aliasing error. The level of spurious signals was found to compare with quoted data.

Consideration was given to extending the process to N bits. The Fourier series coefficients of a single tone input quantized to N bits were evaluated. It was found that the results for two bits correlated with one quoted example and it was also deduced that the aliasing error took on more significance in this case. The above techniques allow one to independently analyse quantization and aliasing error.

Reference to others work on the subject of coarse quantization has also shown that the coloured quantization noise effect obtained here is normal. These references also indicate under which circumstances the quantizing noise may be assumed to be white. This typically occurs when there are at least three bits in the quantizer characteristic, and the input signal is reasonably uncorrelated from sample to sample.

Spectra for pulse modulated sinewaves were generated for up to three bits of quantisation, illustrating that the spectra definitely contain spurs and are not white. The level of the peak spur power was plotted for one to six bits of quantisation as a function of the signal input level relative to the quantiser dynamic range. Similarly, the total spur power, or quantisation noise power was plotted for one to six bits of quantisation as a function of the input signal level relative to the quantiser input dynamic range.

It was found that the quantising noise power started to converge rapidly with the results predicted by the statistical approach of generalised quantisation theory from three bits upwards, and that the results could be considered identical from six bits upwards. Only the Fourier Transform or FFT approaches have the ability to predict the quantisation noise spectra though.

5 ANALYTICAL AND MEASURED RESULTS COMPARED

As stated in Chapter Four, it is possible, by generating a Fourier Transform of any quantised signal first, prior to introducing sampling, to obtain the spectrum of that sampled and quantised signal. This approach to the problem requires a significant amount of computer time, but it also allows the separation of sampling and quantization effects. Specific analytical cases will be considered and these will be compared with practical measured results for verification.

5.1 Example #1 - Pulse Modulated Sinewave

Taking a sinewave as an example, by quantizing it, a monotonically decaying series of lines at odd multiples of the input frequency is obtained, namely a series of odd harmonics decaying with $1/n$ in terms of voltage. The first 99 harmonics are illustrated overleaf in Figure 5.1. By sampling the quantised signal at 62 times the input frequency, the spectrum will fold about a frequency equal to 31 times the input frequency. The first 31 harmonics will thus form what is termed the unaliased message band quantised signal. By summing the remaining harmonics taking account of the spectral folding, the aliased quantisation error is obtained, forming a part of the total quantizing error. The remainder of the quantizing error consists of the remainder of the harmonics, excluding the fundamental, in the unaliased message band spectrum. By adding all the components of the

quantisation error (all components including the fundamental), it is possible to obtain the overall quantized and sampled spectrum of any input signal.

As stated earlier, this process requires a large amount of computation time on a PC-AT compatible. Further, not enough harmonics are used in general, to obtain an accurate picture of what is happening. This procedure does, however, illustrate the mechanisms that are occurring with sampling and quantization clearly. The spectra mentioned above, are illustrated below.

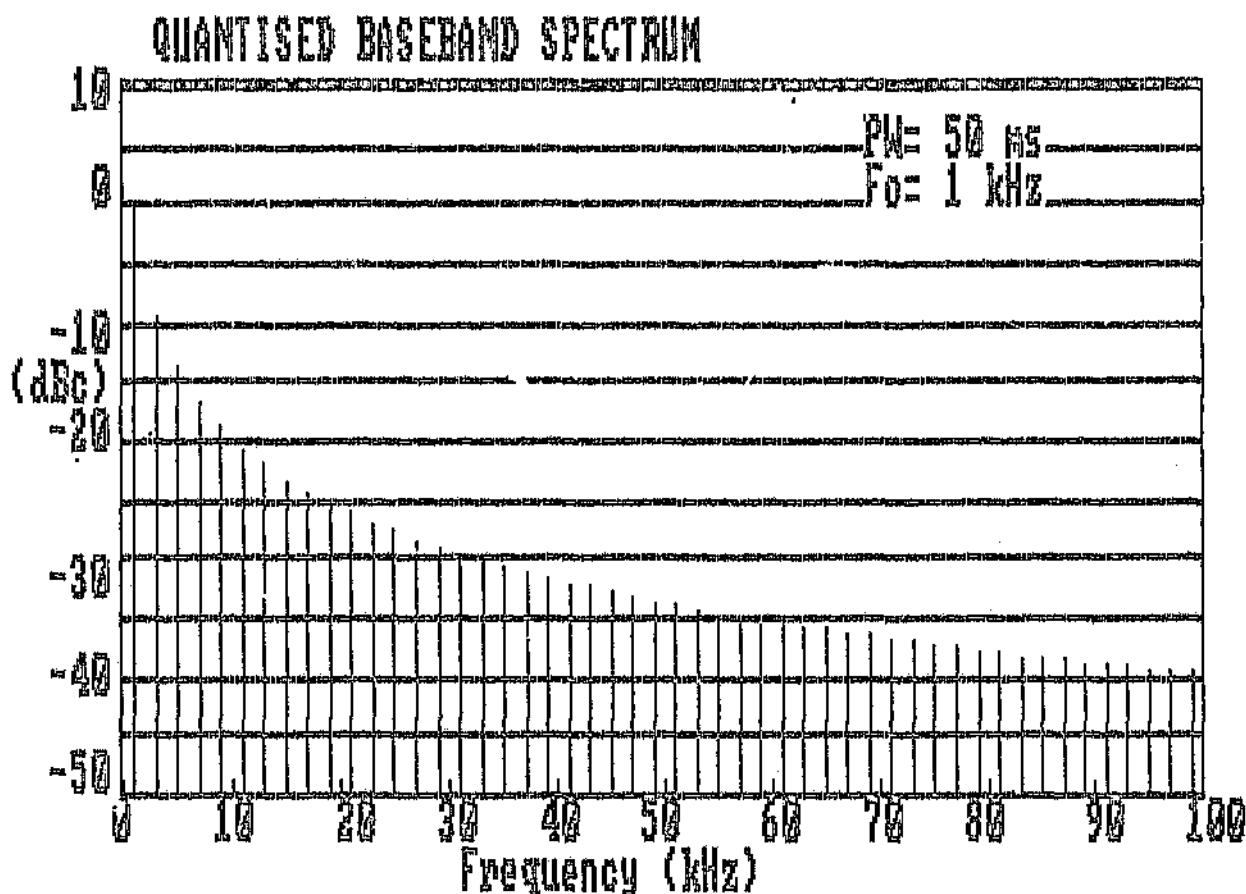


Figure 5.1 Unsampld Quantised Spectrum

UNALIASED QUANTISED SPECTRUM

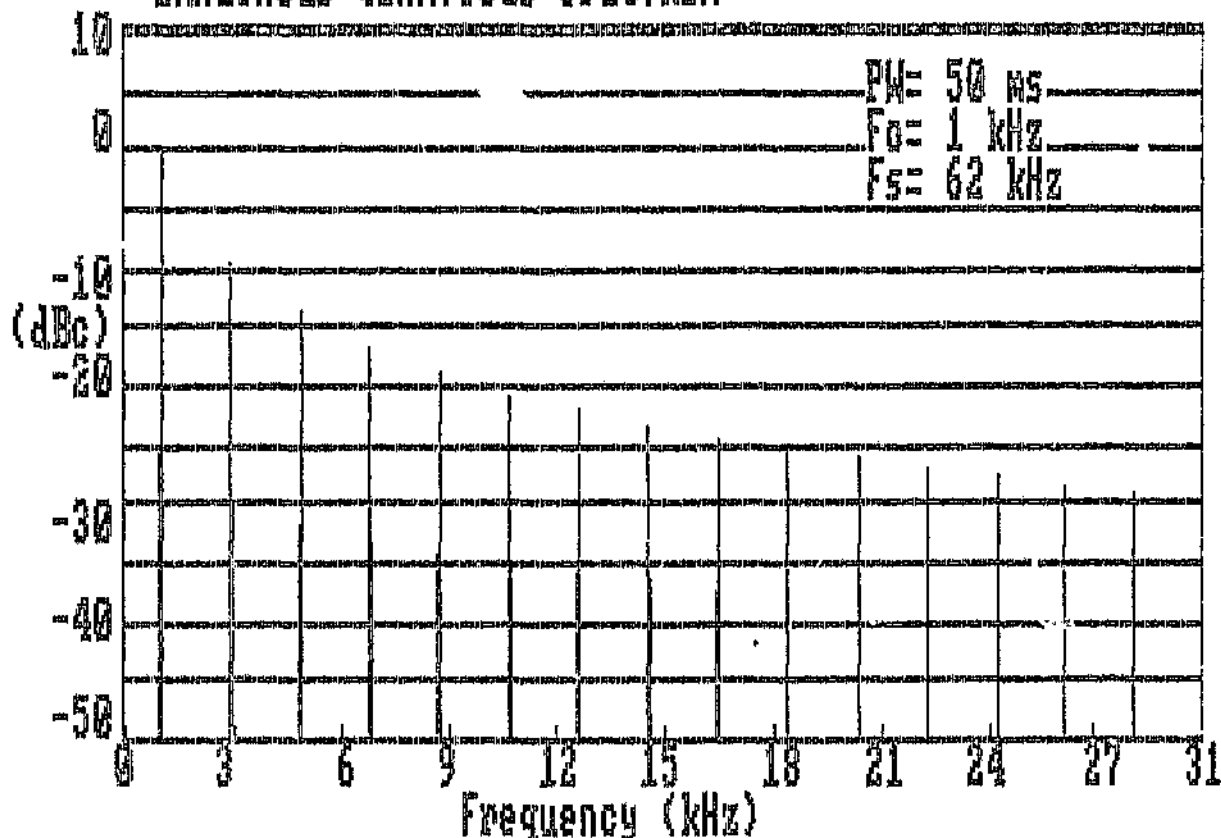


Figure 5.2 Unaliased Quantised Spectrum

ALIASED QUANTISED SPECTRUM

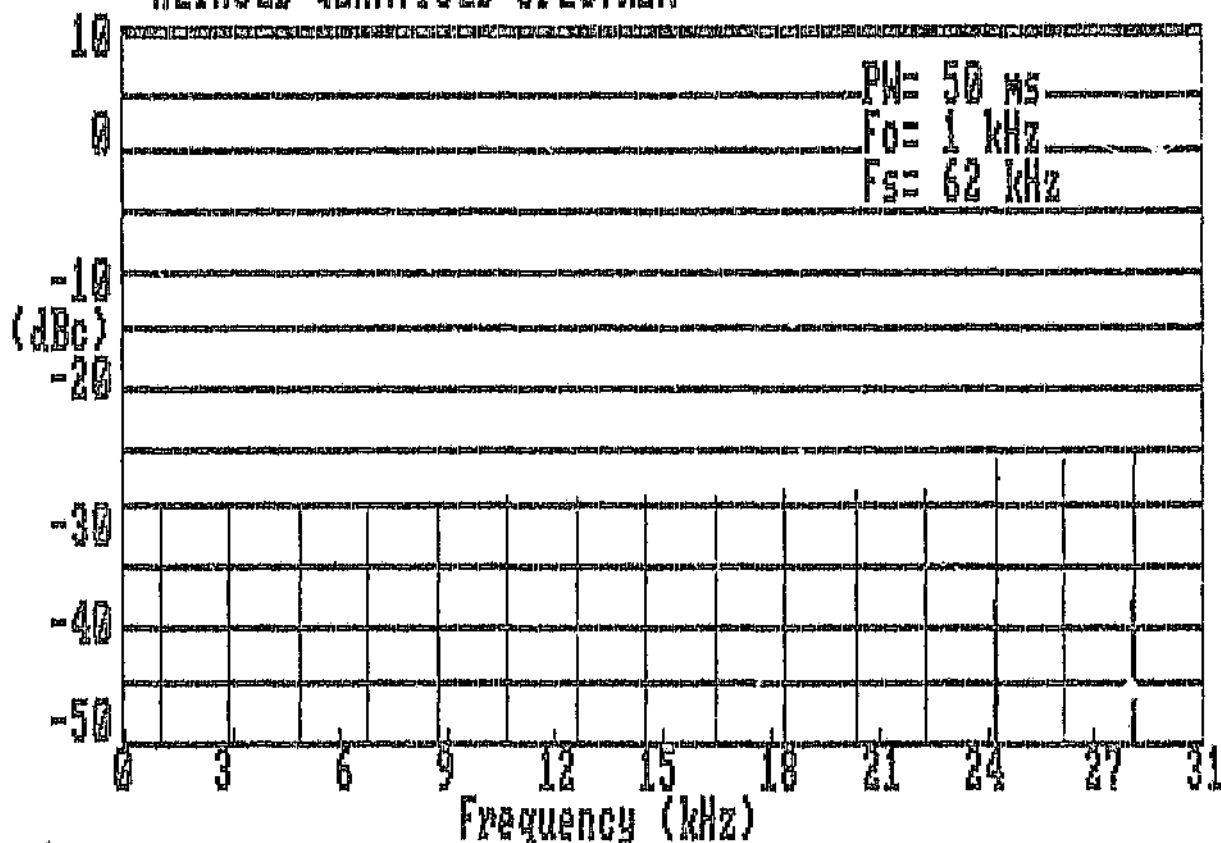


Figure 5.3 Aliased Quantised Message Band Spectrum

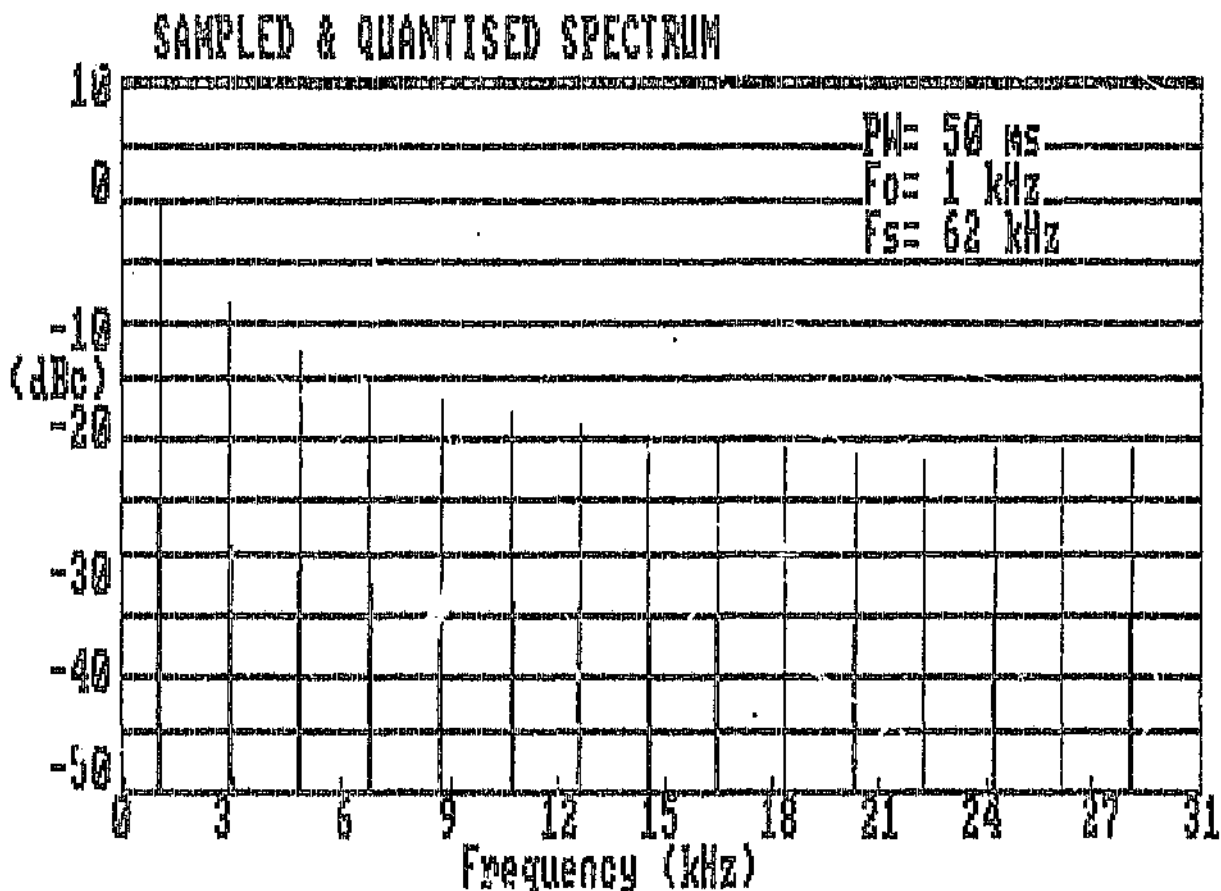


Figure 5.4 Overall Message Band Quantised Spectrum

A much easier approach to obtaining the spectrum of a quantized and sampled sinewave, is to make use of the FFT. The identical input signal, namely a 50ms pulse modulated 1kHz sinewave, was used as an example. 8192 points were used for the FFT to avoid FFT peculiarities, such as the picket fence effect. Thus the predicted spectra are very close to the real spectra. The number of bits in the quantizer must be specified. Thus the input signal is first quantized. Then, the sampling frequency in the DRFM must be specified. This corresponds to the FFT sampling frequency. The FFT is then performed. The resulting spectrum is the overall sampled and quantised spectrum, as shown overleaf in Figure 5.5, and corresponds closely to the spectra predicted above.

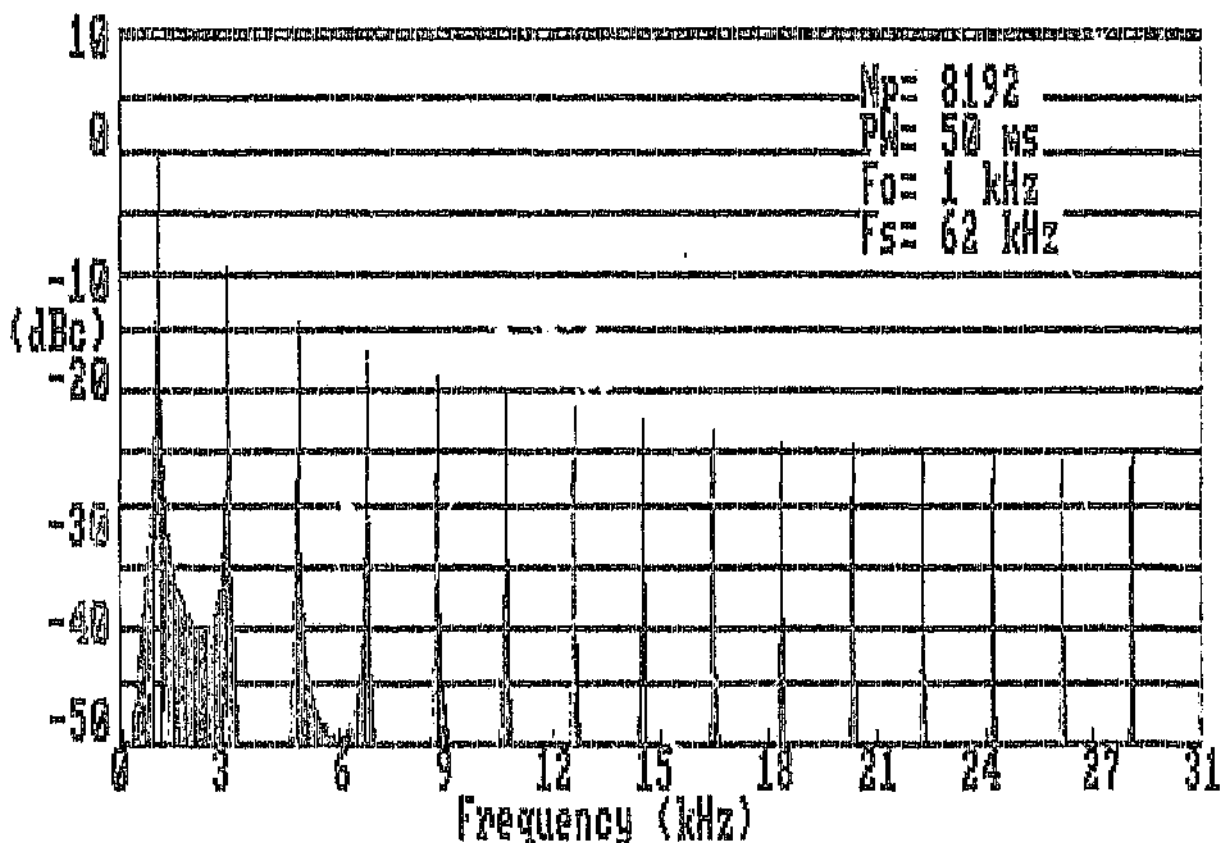


Figure 8.5 Overall Quantised and Sampled Spectrum Using FFT

In order to verify this theoretical approach, a one bit DREM emulator, consisting of a one bit A/D-D/A block (a D flip-flop circuit) was built. Although no storage was performed, this simulates the sampling and quantization process adequately. The D flip-flop has a certain amount of hysteresis on the threshold when comparing the 0-->1 threshold to the 1-->0 threshold, but represents an acceptable one bit digitiser. One of the problems, though, was to obtain sufficient stability from the sampling clock, which was a WAVETEK 188 pulse generator. This generator had a tendency to drift very slowly in frequency with time, making it difficult to set up an exact sampling frequency. The principles of coarse quantization and sampling are well illustrated in Figures 5.6 to 5.8, though.

The output of the one bit DRFM emulator was passed into a TEKTRONIX 7603 mainframe with 715 plug-in spectrum analyzer. The filter bandwidth used was set to 300Hz, and the frequency span per division was set to 100kHz, 5kHz and 10kHz per division, respectively. The sampling frequency was set very high initially, about 10MHz, in order to illustrate just the quantizing effect, with insignificant spectral fold-over. The input signal was a 50ms pulse modulated 5kHz sinewave, in all cases. As expected, the levels of the harmonics correspond to the theoretical levels and are tabulated below. As may be seen, the theoretical and measured results correspond very closely.

It should be observed that the D flip-flop may be modeled as an ideal impulse sampling plus a one bit quantisation process, followed by a sample-and-hold circuit. Thus for a sinewave input, a square wave output, as opposed to impulse samples of a square wave are obtained. The sample-and-hold circuit has a frequency response characterised by $H(f) = \text{Sa}(\pi f T_s)$, where T_s is the sampling period. The signal spectrum at the output of the sample-and-hold circuit is thus modified by its frequency response. At a frequency equal to half the sampling frequency, the sample-and-hold circuit will introduce a loss of 3,92dB compared to the ideal impulse sampling case. At a frequency of 0,2 times the sampling frequency, the loss introduced by the sample-and-hold would be about 0,58dB.

Thus instead of having a monotonically decaying line spectrum, as shown in Figure 5.1, a droop effect will be experienced at higher frequencies in the message band spectrum, as half the sampling frequency is approached. With the very high sampling frequencies used to generate Figures 5.6, 5.7 and 5.8, this effect will be negligible. For all practical purposes, with sampling frequencies

of the order of 10MHz in Figure 5.6, not more than about 0,2dB of droop should be expected at the higher frequencies.

The conclusion is that not accounting for the sample-and-hold in certain cases, such as the above, makes no significant impact on the sampled and quantised spectrum when assuming idealised impulse sampling. It will be seen from Figure 5.6, that a very slight droop effect may be noticed at the highest frequencies portrayed. However, the basic form of the quantised spectrum remains as predicted. The coarse quantisation process introduces extreme non-linearities, producing harmonics of the input spectrum, as predicted by the theory of Chapter Four, and not a white uniform spectrum for the quantising noise as predicted by general quantisation theory.

Table 5.1 Theoretical and Measured Spurious Levels

Harmonic	Theoretical(dBc)	Measured(dBc)
1	0	0
3	-9,5	-9,5
5	-14	-14
7	-16,5	-16,5
9	-19	-18,2

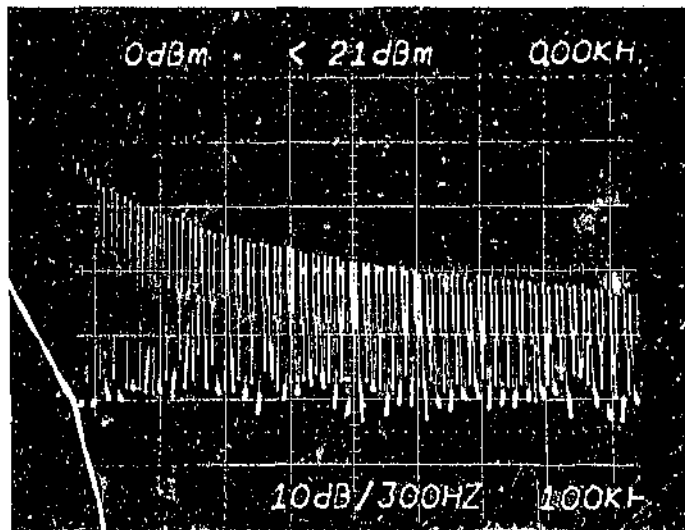


Figure 5.6 Measured "Unsampled" Quantised Spectrum

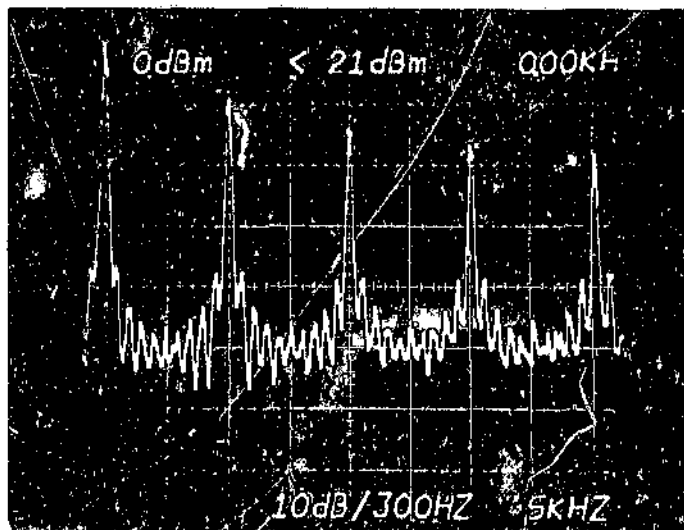


Figure 5.7 First Nine Harmonics of Figure 5.6

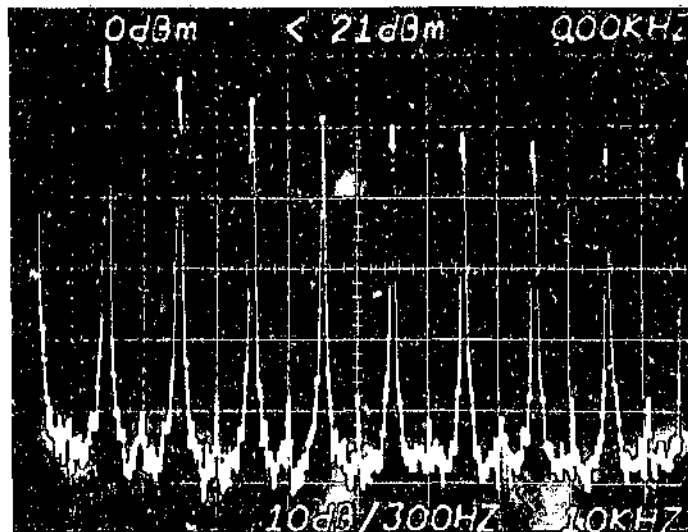


Figure 5.8 First 19 Harmonics of Figure 5.6

By adjusting the sampling frequency slightly away from a multiple of the input frequency, it is possible to obtain the foldover effect a little more clearly. The theoretical results for a 62,5kHz, 62,1kHz and a 62,3kHz sampling frequency are shown overleaf in Figures 5.9, 5.10 and 5.11.

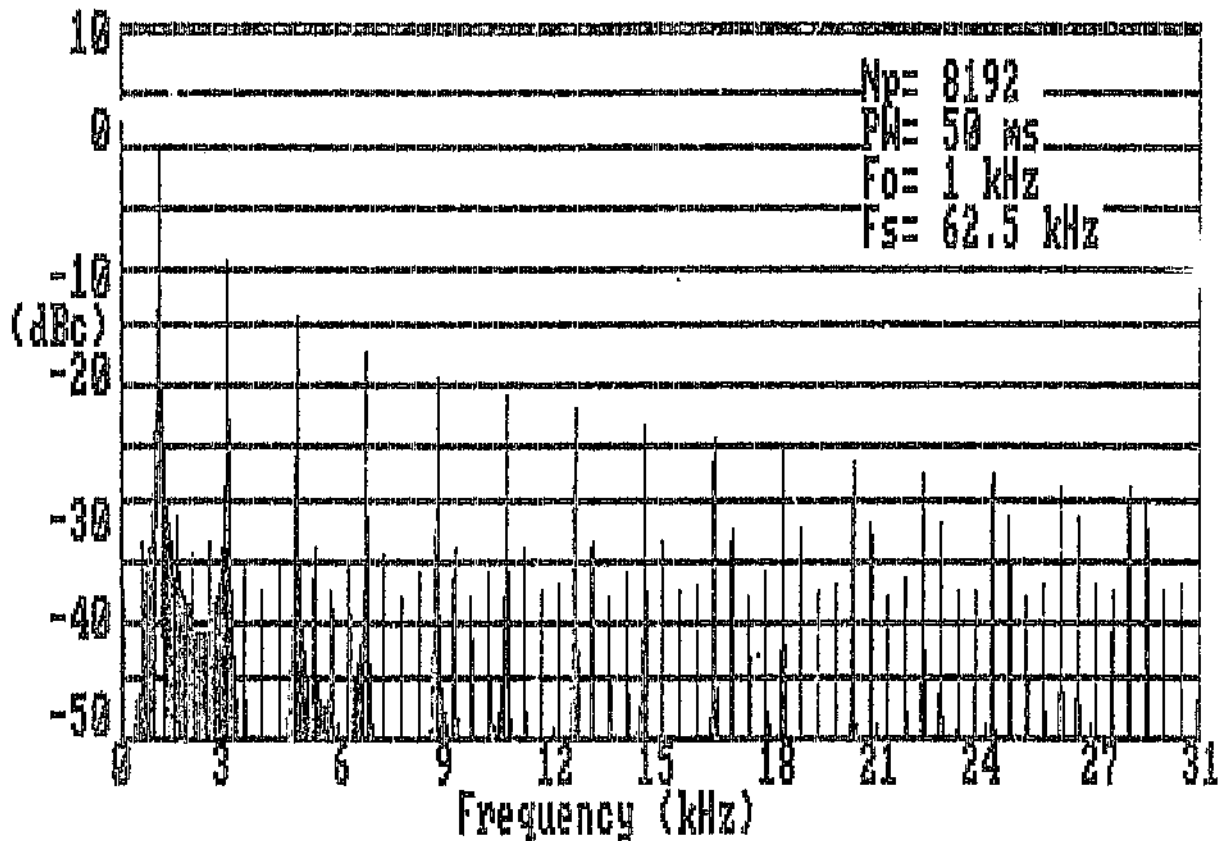


Figure 5.9 Sampled and Quantised Spectrum with $f_s = 62.5 \text{ kHz}$

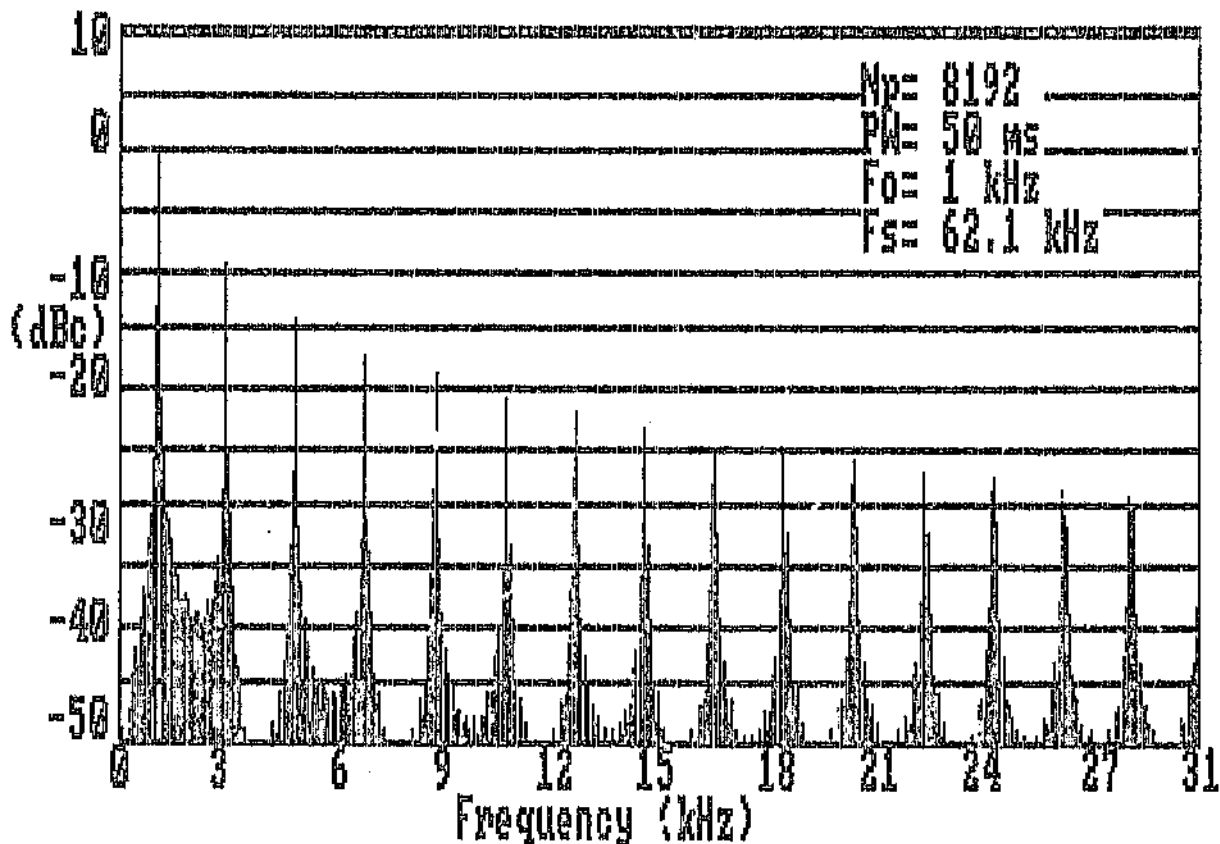


Figure 5.10 Sampled and Quantised Spectrum with $f_s = 62.1 \text{ kHz}$

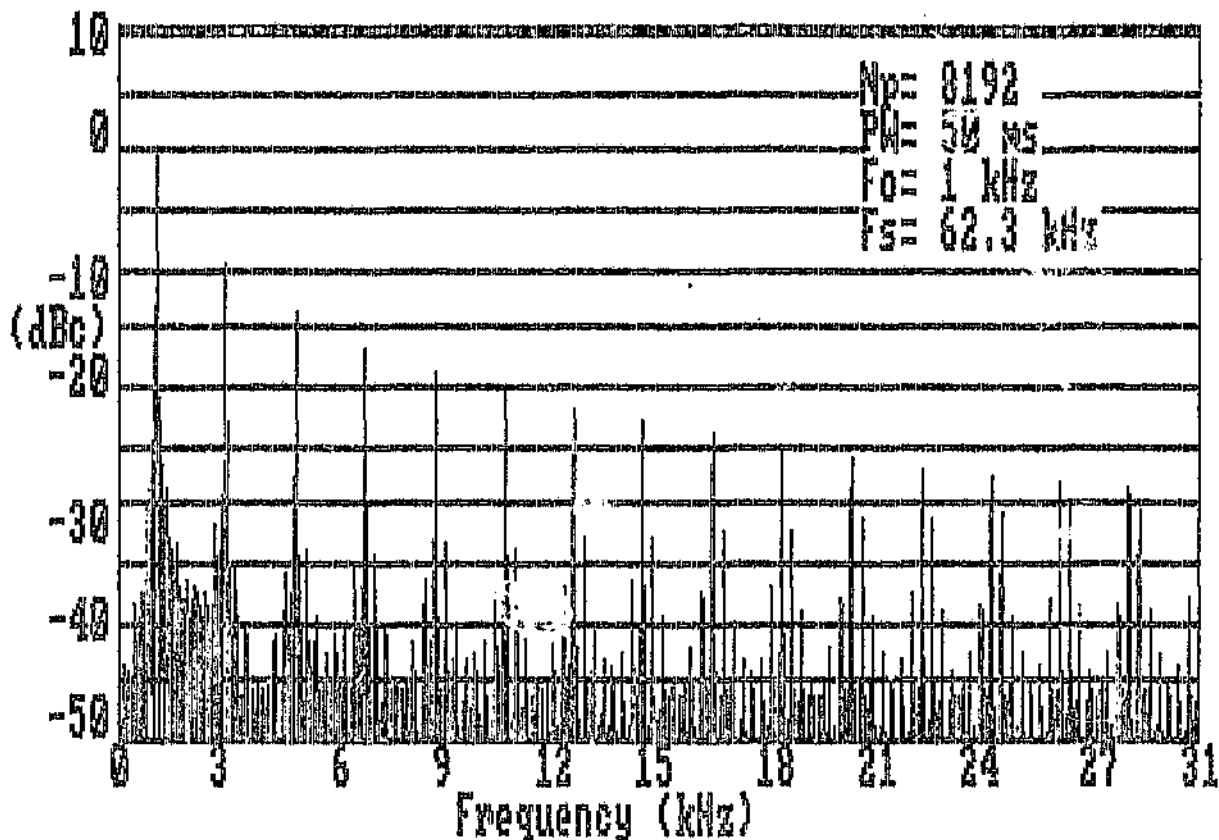


Figure 5.11 Sampled and Quantised Spectrum with $f_s = 62.3 \text{ kHz}$

Measured results for similar situations are shown overleaf, and may be seen to correspond fairly well to the predicted results. The exact same sampling frequencies have not been used, as these are difficult to set up with the available equipment. However, the foldover effect of sampling and the coloured quantizing noise spectrum is clearly illustrated. The edges of the photographs do not unfortunately show the full spectrum analyzer screen in all cases, due to photographic adjustment problems. The effect of the sample-and-hold introduced by the D flip-flop is slightly visible in Figures 5.13 and 5.14.

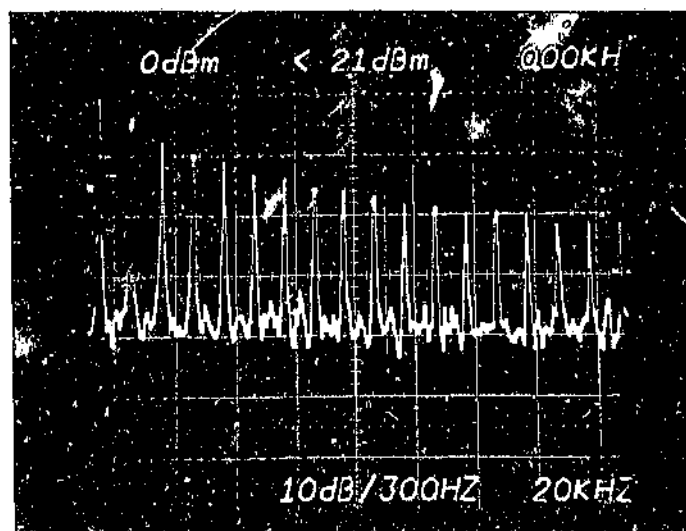


Figure 5.12 Measured Sampled and Quantised Spectrum with Sampling Frequency Approximately a Multiple of the Input Frequency

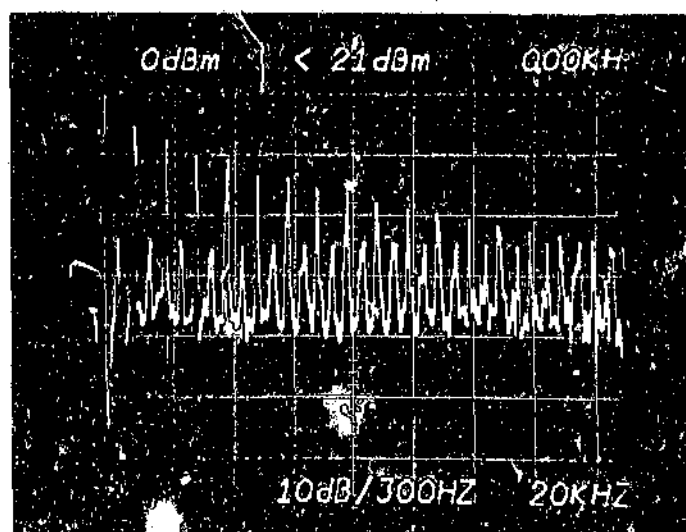


Figure 5.13 Measured Sampled and Quantised Spectrum with Sampling Frequency Not Equal to a Multiple of the Input Frequency

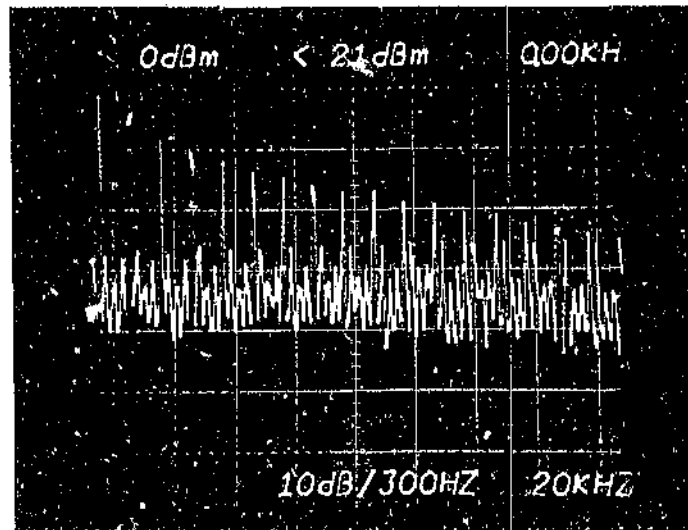


Figure 5.14 Measured Sampled and Quantised Spectrum with Sampling Frequency Not Equal to a Multiple of the Input Frequency and Different to Figure 5.13

5.2 Theoretical Results for Two and Three Bit Quantisers

The theoretical results for a two bit and three bit quantized sinewave are shown overleaf, and again it may be seen that the quantizing error spectrum is not white. There are definite "spurs" or harmonics, only their level is significantly lower.

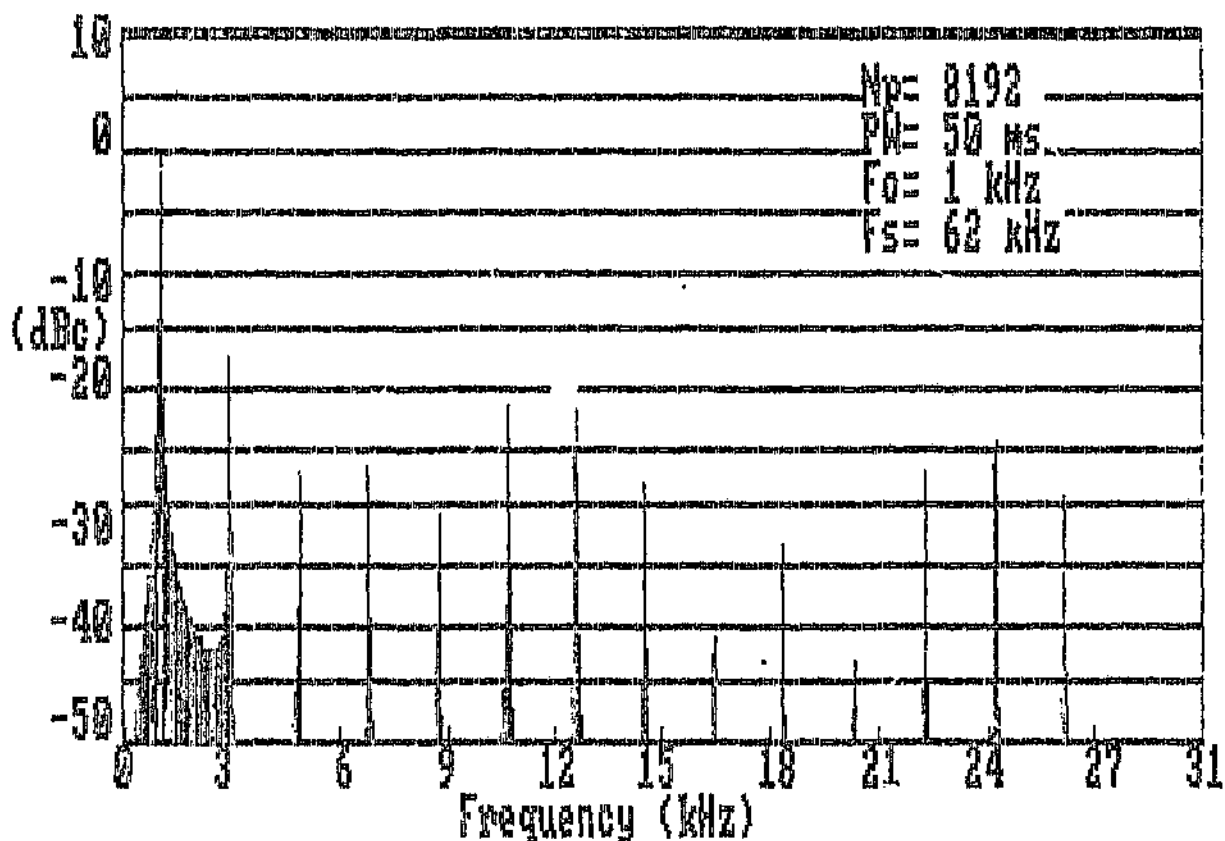


Figure 5.15 Sampled and Two Bit Quantised Spectrum of Sinewave

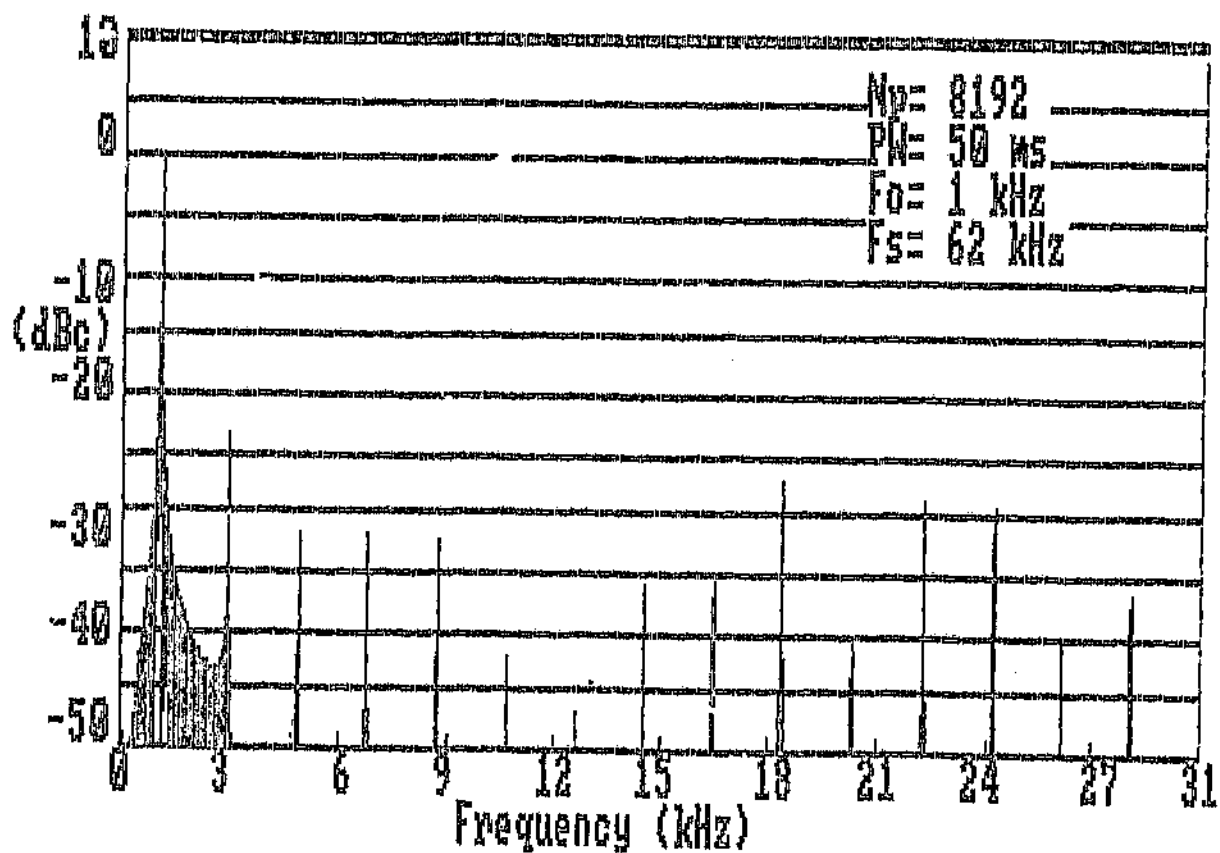


Figure 5.16 Sampled and Three Bit Quantised Spectrum of Sinewave

In order to take a much more complicated input signal, a 50ms pulse modulated, linear frequency modulated sinewave (LFMSW) was used. The start frequency was 1kHz and the stop frequency was 5kHz. Appropriate FFT analysis was done on the signal. Three bit quantization yielded an approximately white quantizing error spectrum, as expected, and as predicted by the references to coarse quantization effects. Even with two bit quantization, the error spectrum is approximately white, although it is possible to distinguish some ripple in the spectrum. Looking at the one bit quantization case in Figure 5.19, the harmonic effects start to become clearer again, resulting in steps in the output error spectrum.

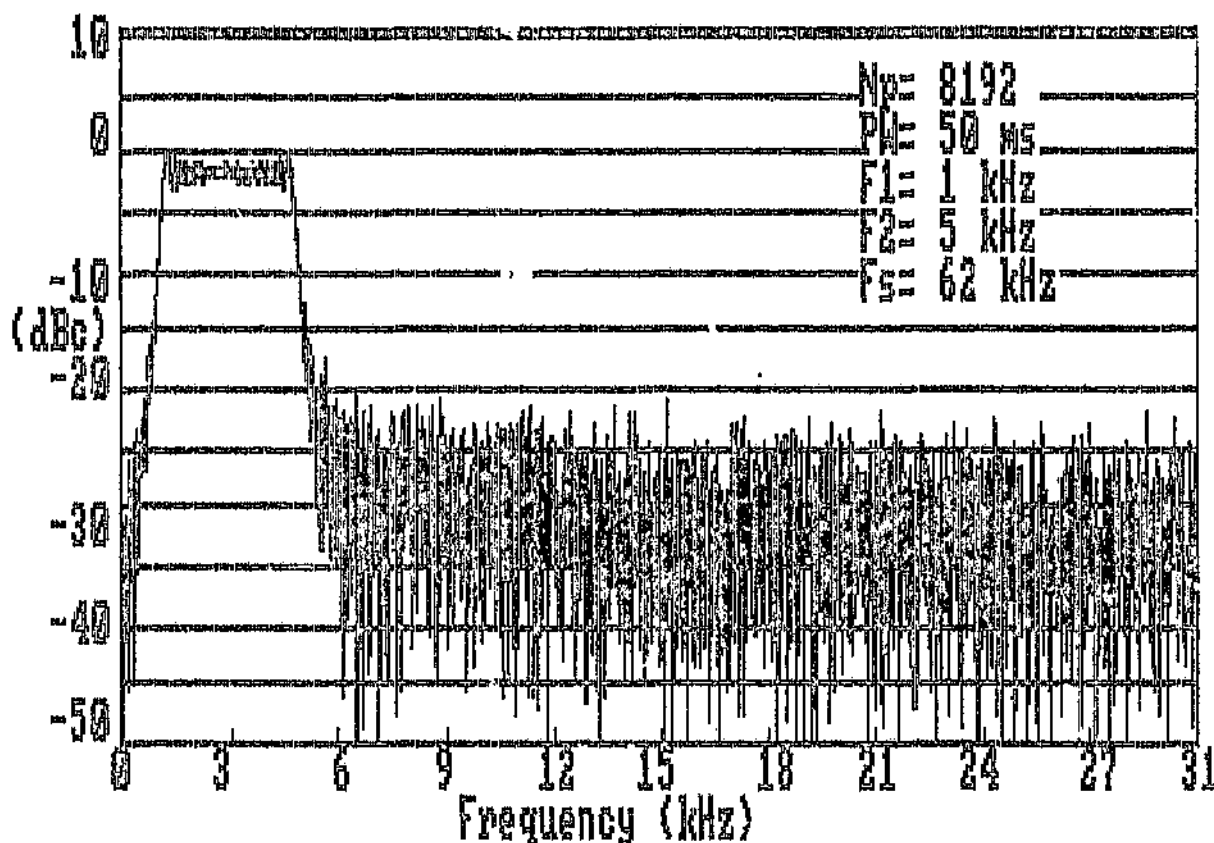


Figure 5.17 Sampled and Three Bit Quantised LFMSW

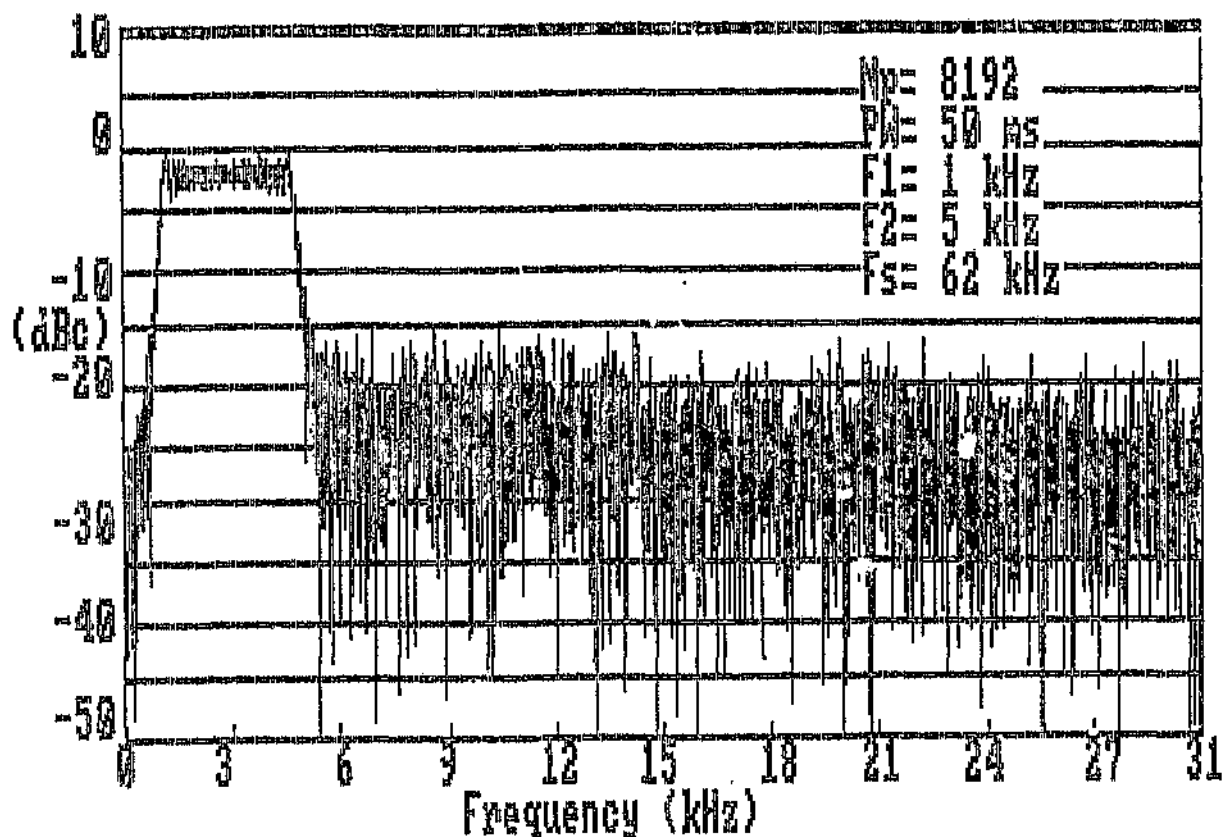


Figure 5.18 Sampled and Two Bit Quantised Spectrum of LFMSW

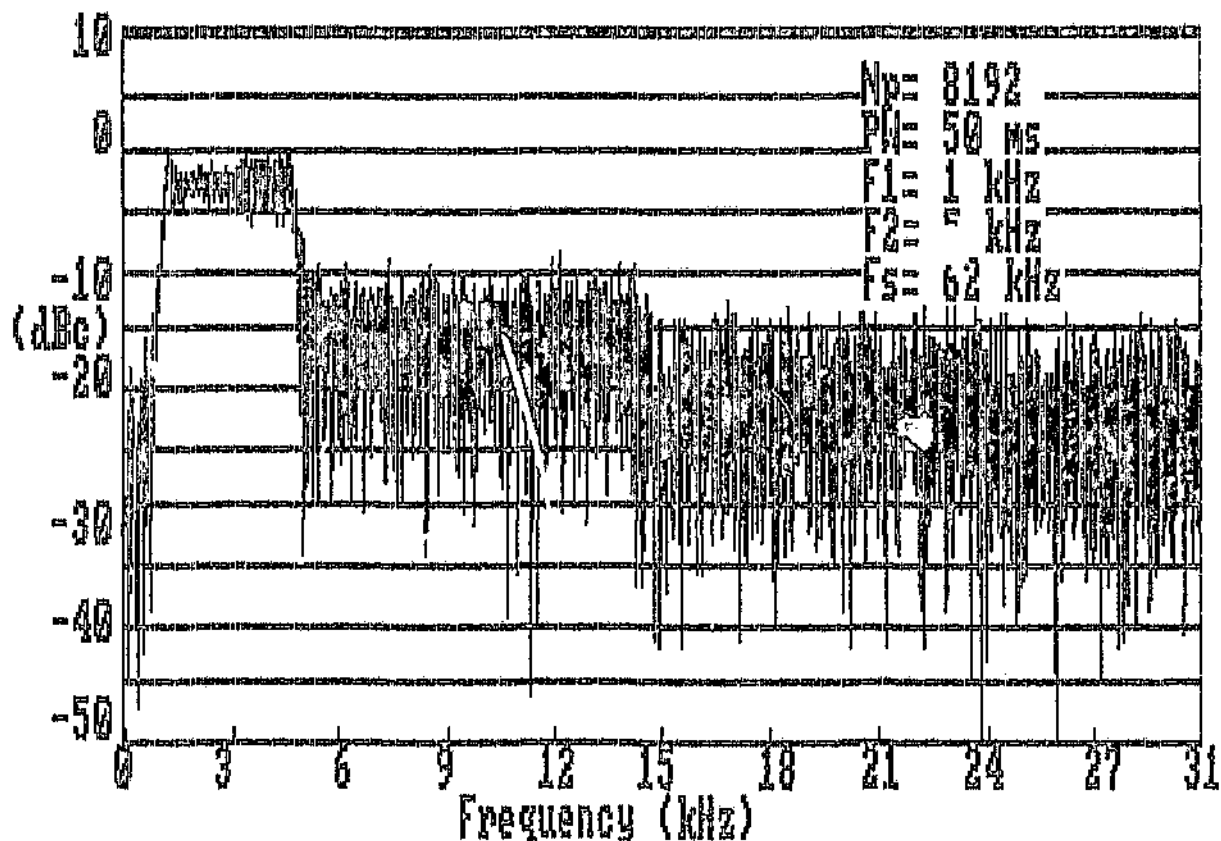


Figure 5.19 Sampled and One Bit Quantised Spectrum of LFMSW

5.3 Summary

From the theoretical predictions and the measured results illustrated above, it may be categorically stated that the quantizing error spectrum is definitely not white under all circumstances. Further, depending on the coarseness of the quantization, and the complexity of the input signal, it is possible for the quantizing noise to be highly correlated with the input signal. This observation has also been made by Widrow (1956), Sripad and Snyder (1977) and others, and has been clearly illustrated above, both theoretically and by measurement. The foldback effect of the sampling process on coarsely quantised spectra was also clearly illustrated by measurement.

6 CONCLUSIONS AND RECOMMENDATIONS

6.1 Discussion

A brief description of the principles of operation of a DRFM have been covered, together with the specific technological problems involved in the digitisation of radio frequency and microwave signals with sufficient accuracy. The problems of coarse quantisation and aliasing have been considered, since these two effects are amongst the major issues that affect the performance and implementation of a DRFM.

To this end, a detailed investigation of the literature was carried out, in order to look at the specific problem of sampling theory and practice. The aim of this was to see if there are any special techniques available that might lend themselves to implementation in a DRFM. Lowpass, bandpass and multi-channel sampling were investigated. Multi-channel sampling, by means of quadrature or more channels allows significant bandwidth reductions to be implemented. This reduces the sampling rate requirements.

The theory of quantisation was studied, although little was found to have been said on the subject of coarse quantisation, especially in quantitative terms. Johnston's (1984) approach was exploited to further the understanding of quantisation and sampling effects and to separate these two effects. Quantisation noise spectra may be generated by means of this technique

(Equation G3), with sampling effects introduced later.

The FFT was also exploited to generate spectra for signals that are quantised and sampled, without the need for separating the two effects. Quantisation effects were modeled and software was written to generate spectra, initially only for one bit quantisation, but later this was extended to N bits.

The link between the Fourier analysis approach of harmonic generation and the general quantisation theory was obtained, as also suggested by other researchers from the literature survey.

The effect of signal input level, relative to the quantiser dynamic range was also investigated in terms of the maximum spurious signal level generated, and the total quantising noise power, for varying degrees of quantisation accuracy. Theoretical results, using the coarse quantisation approach of harmonic summation correlated with the results of generalised quantisation theory. Correlation was reasonable for the three bit case and improved to nearly 100% correlation for the six bit case.

Measurements were made on a one bit DRFM emulator, and the spectra were photographed on a spectrum analyser display. These were compared to the theoretical results and were found to compare well.

6.2 Conclusions

Following from the studies on sampling and quantisation in Chapters Two and Three, which specifically looked at techniques for bandwidth reduction, or sampling rate reduction, by means of multi-channel processing and bandpass sampling, it was concluded that the most critical part of a DRFM that affects the signal

fidelity at its output, is the A/D and D/A conversion process within the memory module.

The quantisation process in the A/D and D/A conversion process is coarse, and this is due to the current technological limitations of producing multi-bit Gigahertz sampling rate A/D and D/A converters.

There is no doubt that in the future, this problem will be resolved. However, looking back historically at the evolution of these converters, it will be seen that they developed from a "few" bits with each new technology, as it came into being. Whereas then, we were striving for kilohertz sampling rates, now we are striving for Gigahertz sampling rates. In the future, it is likely that this will be pushed to tens and hundreds of Gigahertz, or even higher, with technology of the future. Thus the approaches outlined in this dissertation should be applicable for the years ahead.

The commutative nature of the sampling and quantisation processes, as stated by Jayant and Noll (1984), was proved. Together with Johnston's (1984) approach for generating the spectra of one bit quantised signals, this technique was exploited to produce the spectra of sampled and quantised signals. Sinusoids and pulse modulated sinusoids were used for this analysis, due to their ease of reproduction for test purposes. This technique (Equation G3) allows the separation of the effects of quantisation and sampling. It was shown that because quantisation is a non-linear process, it generates harmonics of the input signal. This effectively results in spectral spreading. The FFT was also used to generate combined sampled and quantised spectra.

The above technique was extended to N bit quantisers (Equation M3) and it was shown that the spectra still contain harmonics. By adding the power in these harmonics, it is possible to obtain the total quantisation noise power, which correlates with the generalised statistical approach from three bits upwards. The results are nearly identical for six bit quantisation.

It was concluded that the generalised approach is incapable of predicting the quantisation noise power accurately for less than six bits, and even less so, the quantisation noise spectra. As the number of bits and quantisation levels decreases, so the variation in the quantised signal decreases from sample to sample. The implication of this sample to sample signal correlation is that the error from sample to sample is no longer random, resulting in coloured quantisation noise spectra. This effect is accentuated as the sampling frequency is increased, as also found by Claasen and Jongepier (1981).

Spectra were calculated for up to three bits for sinusoids and linearly frequency modulated sinusoids. Measurements were made on a one bit quantiser and these verify the theoretical predictions for sinusoids.

Summarising, the Fourier Transform or FFT techniques offer a different approach to the generation of quantisation noise. These methods allow for a better understanding of the processes by which quantisation noise is generated. This is especially so because it is possible to separate the effects of sampling and quantisation and to generate spectra by one or both techniques, even for very coarse quantisation, without assumptions about the quantisation error statistics.

6.3 Recommendations for Future Work

Future work is required to verify the spectra predicted by the multi-bit theory. This entails building a multi-bit back to back A/D and D/A block, so that the input signal level can be varied from the bottom to the top ends of the quantiser's dynamic range, while observing the spectrum on a spectrum analyser. More complicated signals should also be evaluated with the one bit quantiser.

FINITE APERTURE WIDTH SAMPLING

With sample and hold type sampling, the sampled waveform is given by:

$$\begin{aligned} x_s(t) &= \sum_{-\infty}^{\infty} x(kT_s) p(t - kT_s) \\ &= [p(t)] * \left[\sum_{-\infty}^{\infty} x(kT_s) \delta(t - kT_s) \right] \end{aligned} \quad (A1)$$

where $p(t)$ is a flat-topped pulse of duration τ , Shanmugam (1979).

By taking Fourier Transforms, it follows that:

$$X_s(f) = P(f) X_\delta(f) = P(f) \left[f_s \cdot \sum_n X(f - nf_s) \right] \quad (A2)$$

where $P(f)$ is the Fourier transform of $p(t)$ and $X_\delta(f)$ is the Fourier transform of the ideal sampled waveform as given in 2.1, Shanmugam(1979). $P(f)$ is a sinc function, defined by the Fourier transform of a single pulse of width τ . This type of sampling leads to distortion in the message band.

An alternative derivation for use with natural sampling, which does not result in flat-topped samples as in the above case, makes use of the Fourier series equivalent of a periodic pulse train. If $k(t)$ is the sampling pulse train given by:

$$k(t) = \sum_n K_n \exp(jn\omega_s t), \quad \omega_s = 2\pi/T_s \quad (A3)$$

then:

$$\begin{aligned} K_n &= 1/(\pi n) \cdot \sin(n\pi\tau/T_s), \quad \tau = \text{pulsewidth} \\ &= \tau/T_s \cdot \text{Sa}(n\pi\tau/T_s) \end{aligned} \quad (A4)$$

The sampled waveform is then:

$$\begin{aligned} g_s(t) &= g(t) k(t) \\ &= g(t) \cdot \sum_n K_n \exp(jn\omega_s t) \\ &= \sum_n K_n g(t) \exp(jn\omega_s t) \end{aligned} \quad (A5)$$

Hence:

$$G_s(\omega) = \sum_n K_n G(\omega - n\omega_s) \quad (A6)$$

TRUNCATION ERROR

There are three approaches to this problem reviewed by Jerri (1977), namely:

a.) If $f(t)$ is a bandlimited function given by:

$$f(t) = \int_{-a}^a \exp(j\omega t) F(\omega) d\omega$$

$$= \sum_n f(n\pi/a) \cdot \sin(at - n\pi) / (at - n\pi) \quad (B1)$$

$$\epsilon_T = \text{truncation error} = f(t) - f_N(t) \quad (B2)$$

where $f_N(t)$ is the partial sum considering only $2N+1$ terms of the infinite sampling series.

$$\text{Therefore } \epsilon_T = \left| \sum_{n > N} f(n\pi/a) \cdot \sin(at - n\pi) / (at - n\pi) \right|$$

This is bounded as:

$$|\epsilon_T(t)| \leq \sqrt{2/\pi} \cdot E |\sin(\pi t/\delta t)| \cdot \sqrt{(T\delta t/(T^2 - t^2))} \quad (B3)$$

for $-T \leq t \leq T$

$\delta t < 1/W$

W = highest frequency of $f(t)$

& E = total finite energy carried by $f(t) = \int_{-W}^W |F(\omega)|^2 d\omega$

b.) If $f_N(t) = \sum_{K-N}^{K+N} f(n/2W) \cdot \sin(2W\pi t - n\pi) / (2W\pi t - n\pi)$ and $0 < N < \infty$, where a above is given by $a = 2\pi W$, K is an integer such that $2Wt - 1/2 \leq K(t) \leq 2Wt + 1/2$ and N is a fixed integer, then $\epsilon_T(t) \rightarrow 0$ as $N \rightarrow \infty$ if $f(t)$ is bandlimited to $a = 2\pi W$.

Considering a bandlimit $rW < W$ with $0 < r < 1$, an improvement on the bound for the truncation error is obtained:

$$|\epsilon_T(t)| \leq 4M / (\pi^2 N(1-r)) = 4M / (\pi^2 Nq) \quad \text{for } -\infty < t < \infty \quad (B4)$$

and $M = \max |f(t)|$ for all t , $q = 1 - r$. Similarly $\epsilon_T \leq 4M |\sin 2\pi Wt| / (\pi Nq)$

c.) If $f(t)$ is approximated by an asymmetrical partial sum:

$$f_{N(1), N(2)}(t) = \sum_{K-N_1}^{K-N_2} f(n/2W) \cdot \sin(2W\pi t - n\pi) / (2W\pi t - n\pi) \quad (B5)$$

then the bound on the truncation error is:

$$|\epsilon_T(t)| \leq 2M/(\pi^2(1-r)) \cdot (1/N_1 + 1/N_2) \quad \text{for } -\infty < t < \infty \quad (B6)$$

SAMPLING OF RANDOM LOWPASS PROCESSES

The mean-square value of the error function was shown to have the form:

$$E[e^2(t)] = E[(X(t) - X'(t))^2] \quad (\text{c.f. Equation 2.5}) \quad (C1)$$

and hence:

$$\begin{aligned} E[e^2(t)] &= E[(X(t) - \sum_k X(kT_s) \text{Sa}((\pi/T_s)(t - kT_s)))^2] \\ &= R_{XX}(0) - 2E[X(t) \sum_k X(kT_s) \text{Sa}((\pi/T_s)(t - kT_s))] \\ &\quad + E[\sum_k \sum_j X(jT_s) X(kT_s) \text{Sa}((\pi/T_s)(t - jT_s)) \text{Sa}((\pi/T_s)(t - kT_s))] \end{aligned} \quad (C2)$$

By interchanging the order of expectation and summation, one obtains:

$$\begin{aligned} E[e^2(t)] &= R_{XX}(0) - 2 \sum_k R_{XX}(t - kT_s) \text{Sa}((\pi/T_s)(t - kT_s)) \\ &\quad + \sum_j \text{Sa}((\pi/T_s)(t - jT_s)) \sum_k R_{XX}(jT_s - kT_s) \text{Sa}((\pi/T_s)(t - kT_s)) \end{aligned} \quad (C3)$$

However $R_{XX}(\cdot)$ is a deterministic bandlimited function with a non-zero PSD for $|w| \leq w_B$ only. The second term in the above equation is $2R_{XX}(0)$, because $R_{XX}(t - kT_s) = R_{XX}(kT_s - t)$ and also because a time-shifted function $x(t - \alpha)$ can be reconstructed from its time-shifted samples $x(kT_s - \alpha)$. Now, by applying the sampling theorem twice to the third term in the above equation, it will be found that this is equivalent to $R_{XX}(t - t) = R_{XX}(0)$. Therefore

$$E[e^2(t)] = \sigma_e^2 = 0. \quad (\text{c.f. Equation 2.5}) \quad (C4)$$

The above proof is not strictly rigorous because the error function is zero at specific instants, namely the sampling instants kT_s . Thus $(e(t))$ is not in fact wide-sense stationary, but these points are a set of zero probability and will therefore not affect the expectation, Jayant & Noll (1984).

THEORY AND ILLUSTRATION OF BANDPASS SAMPLING

Consider a lowpass signal $f_m(t)$ with a bandwidth W_1 . Also assume that this signal modulates a sinusoid $\cos(w_1 t)$. Hence we have a bandpass signal centered on $w=w_1$. This bandpass signal has a bandwidth $W=2W_1$. The sampling waveform is given by:

$$p_T(t) = \sum_n P_n \exp(jnw_0 t) \quad (D1)$$

The sampled waveform is given by:

$$\begin{aligned} f_s(t) &= f(t) p_T(t) \\ &= f_m(t) \cos(w_1 t) \sum_n P_n \exp(jnw_0 t) \end{aligned} \quad (D2)$$

Taking Fourier transforms:

$$F_s(w) = \sum_n P_n \cdot \{F.T.\} (f_m(t) \cos(w_1 t) \exp(jnw_0 t)) \quad (D3)$$

where the operation (F.T.) denotes taking Fourier transforms.

Now:

$$F(w) = \{F.T.\} (f_m(t) \cos(w_1 t)) = 1/2 \cdot [F_m(w+w_1) + F_m(w-w_1)] \quad (D4)$$

and:

$$\begin{aligned} F_s(w) &= \sum_n P_n \cdot \{F.T.\} (f(t) \exp(jnw_0 t)) \\ &= \sum_n P_n F(w - nw_0) \end{aligned} \quad (D5)$$

where:

$$f(t) = f_m(t) \cos(w_1 t) \quad (D6)$$

Therefore:

$$F_s(w) = \sum_n P_n / 2 [F_m(w-w_1-nw_0) + F_m(w+w_1-nw_0)] \quad (D7)$$

This process is illustrated in Figure D1 overleaf. The conditions for avoiding aliasing must now be established. By drawing a series of pictures for various conditions, it is possible to establish bounds on the sampling rate requirements. Figures D2 to D5 illustrate some specific situations. From these it is possible to establish the results of Figure 2.3, thus clarifying Shanmugam's (1979) statement on the restrictions associated with

bandpass sampling. Note that as w_g increases, so the replicas spread out and pass through the fundamental baseband signal, as illustrated.

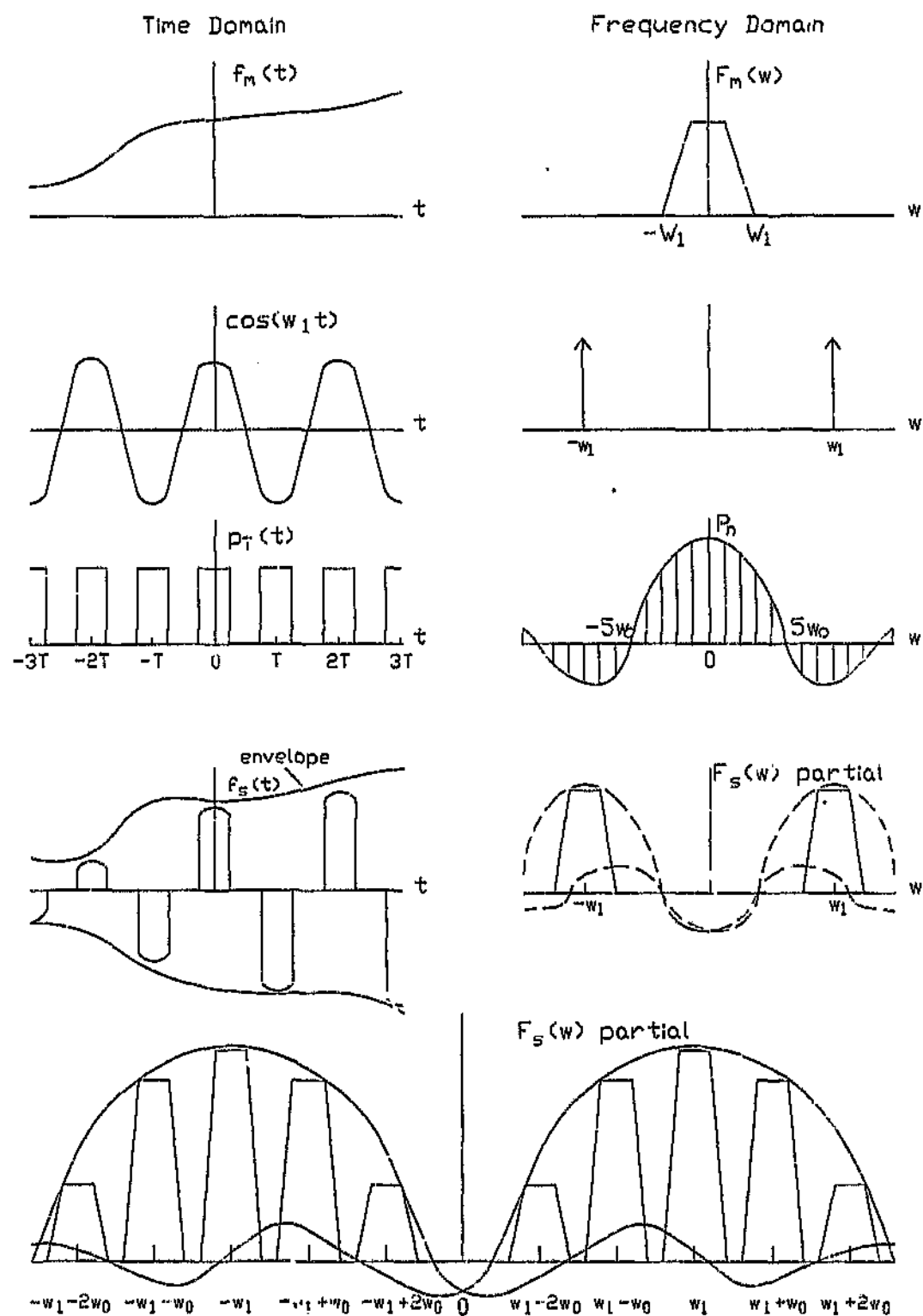


Figure D1 Illustration of the Bandpass Sampling Process

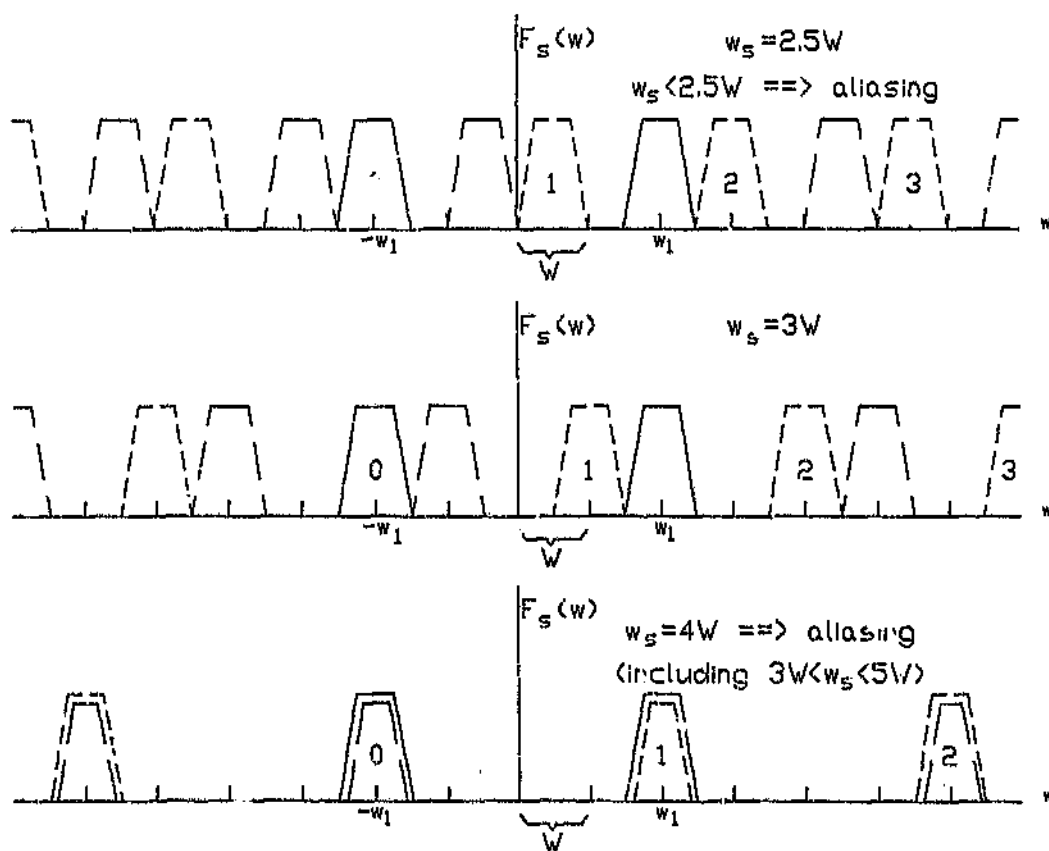


Figure D2 Bandpass Sampling $w_1 = 2W$

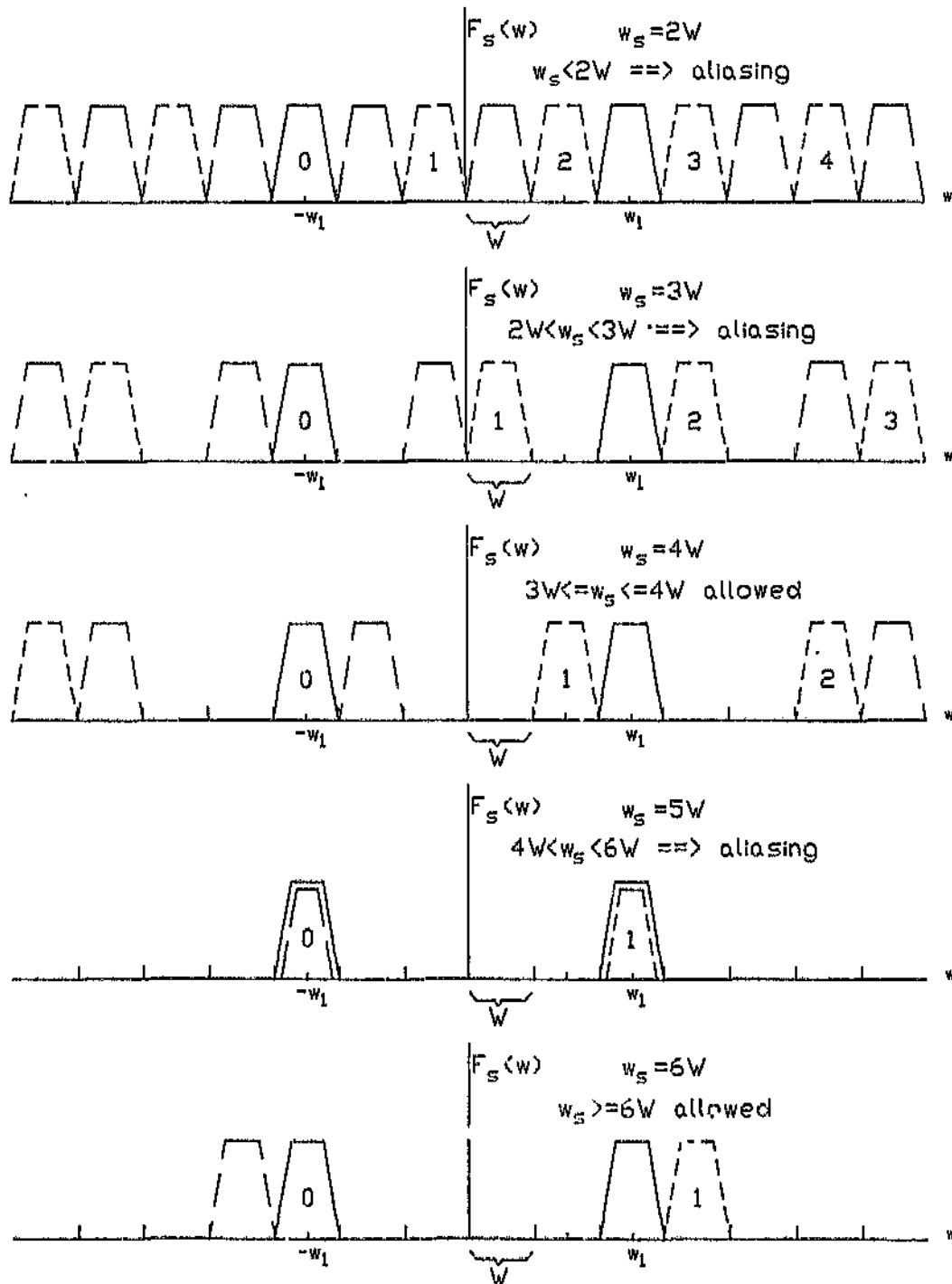
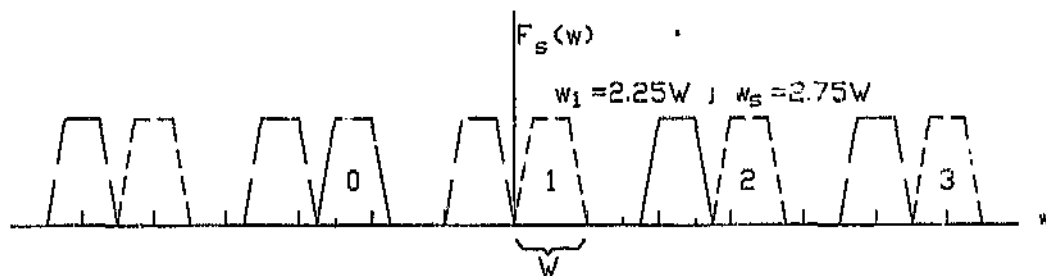
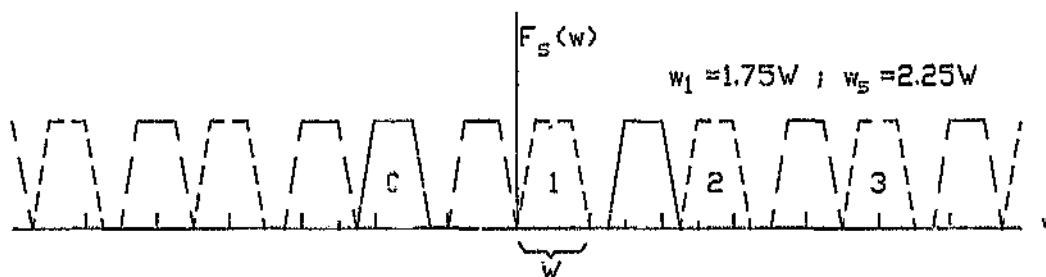


Figure D3 Bandpass Sampling $w_1 = 2, 5W$



$2.75W \leq \omega_s \leq 3.5W$
 $\omega_s \geq 5.5W$

no aliasing



$2.25W \leq \omega_s \leq 2.5W$
 $\omega_s \geq 4.5W$

no aliasing

Figure D4 Illustrating the Conditions for Aliasing of Bandpass Sampled Signals for $1.5W \leq \omega_1 \leq 2.5W$

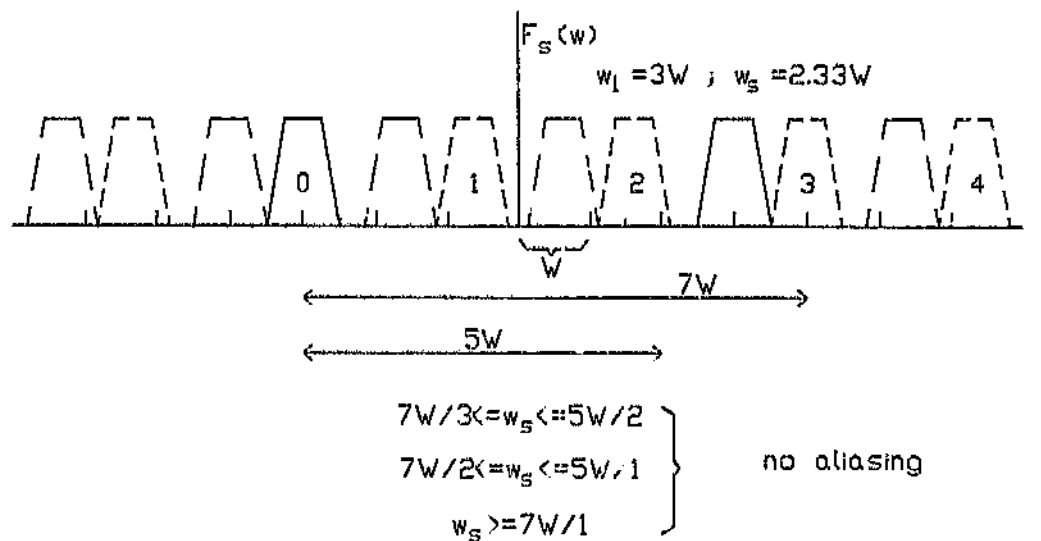
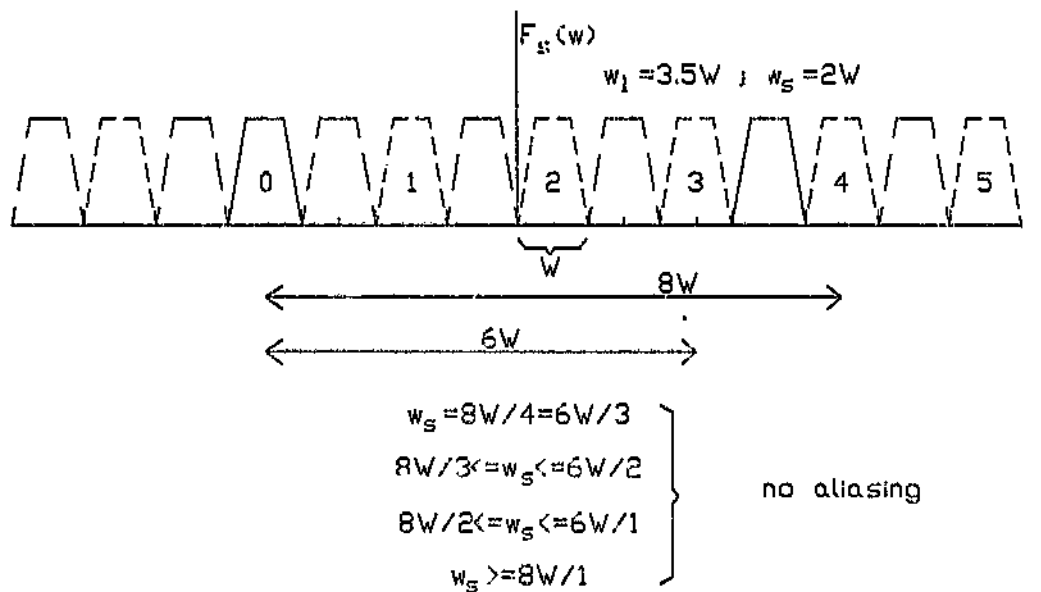


Figure D5 Illustrating the Conditions for Aliasing of Bandpass Sampled Signals for $2.5W \leq w_1 \leq 3.5W$

QUANTISATION NOISE

Consider a random process $X(t)$ at the input to a quantizer as shown in Figure E1 below. The signal is sampled at intervals T_s satisfying the conditions of Nyquist and results in $X(kT_s)$. The quantizer quantizes the signal to 1 of Q allowable levels according to a rule, as in Figure E1, namely $m_1, m_2, m_3, \dots, m_Q$. Hence $X_q(kT_s) = m_i$ if $x_{i-1} \leq X(kT_s) < x_i$ and $x_0 = -\infty$; $x_Q = +\infty$. Thus $X_q(t) = X_q(kT_s)$ for $kT_s \leq t < (k+1)T_s$. Now, S_q/N_q = output signal to quantizing noise power ratio and is given by:

$$E[X_q^2(kT_s)] / E[(X(kT_s) - X_q(kT_s))^2] \quad (E1)$$

$$= E[X_q^2] / E[(X - X_q)^2] \quad (E2)$$

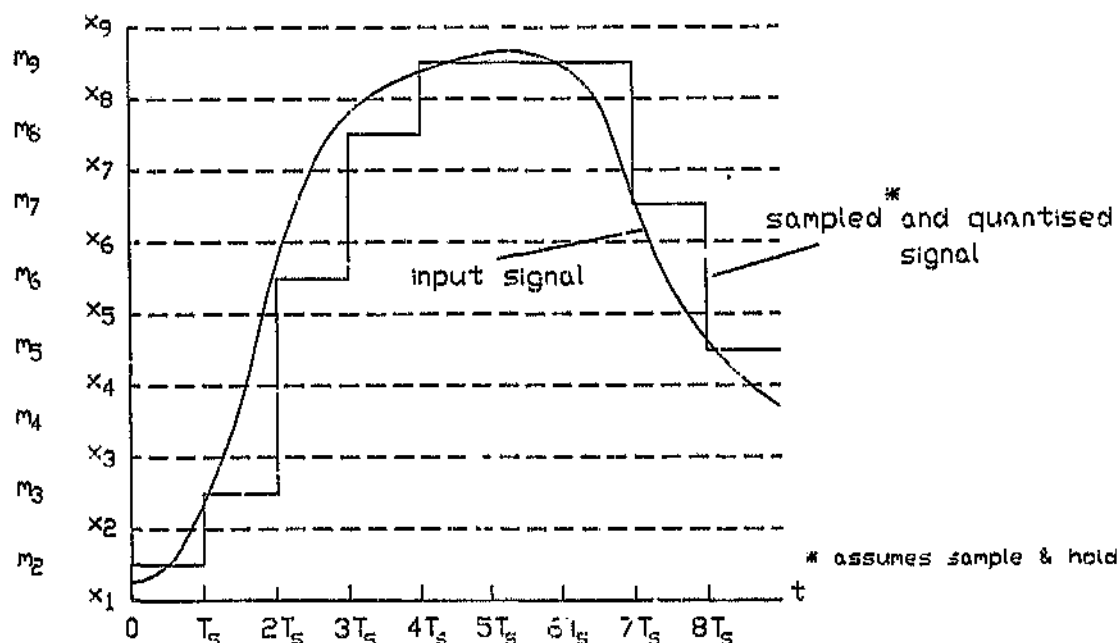


Figure E1 Illustration of the Quantization Process

Taking a uniform quantizer now with step sizes $=\Delta$, Q levels of quantization and a signal X with dynamic range $b-a$, then:

$$\Delta = (b-a)/Q \quad (E3)$$

$$x_i = a + i \Delta \quad (E4)$$

$$\text{and } m_i = (x_{i-1} + x_i) / 2, \quad i = 1, 2, \dots, Q \quad (E5)$$

Therefore:

$$\begin{aligned} N_q &= E[(X - X_q)^2] \\ &= \int_a^b (x - x_q)^2 p_X(x) dx \\ &= \sum_{i=1}^Q \int_{x_{i-1}}^{x_i} (x - m_i)^2 \cdot 2 p_X(x) dx \end{aligned} \quad (E6)$$

where $x_0 = a$, $x_Q = b = x_0 + Q$, $x_i = a + i\Delta$ and $m_i = a + i\Delta - \Delta/2$

Now:

$$\begin{aligned} S_q &= E[X_q^2] \\ &= \int_a^b x_q^2 p_X(x) dx \\ &= \sum_{i=1}^Q \int_{x_{i-1}}^{x_i} m_i^2 p_X(x) dx \end{aligned} \quad (E7)$$

Also, it follows that $p_X(x) = k$ over the interval $[-a; a]$, where $k = 1/2a$, since $\int_{-a}^a p_X(x) dx = 1$.

By evaluating the above integrals, it is a simple procedure to show that $N_q = \Delta^2/12$ and $S_q = (Q^2 - 1)\Delta^2/12$. Hence, the signal-to-quantizing noise power ratio is $S_q/N_q = Q^2 - 1$, Shanmugam (1979).

THE COMMUTATIVE NATURE OF SAMPLING AND QUANTISATION

In general, ideal impulse sampling theory has $s_\delta(t) = \sum_k \delta(t - kT_s)$ as the impulse sampling function.

If a function $x(t)$ is sampled, then the resulting function is $x_\delta(t)$ given by:

$$\begin{aligned} x_\delta(t) &= x(t) \sum_k \delta(t - kT_s) \\ &= \sum_k x(kT_s) \delta(t - kT_s) \end{aligned} \quad (F1)$$

and the spectrum will be:

$$X_\delta(f) = f_s \sum_n X(f - nf_s) \quad (F2)$$

Considering the case of one bit quantization, the quantized signal $x_q(t)$ (if quantization occurs before sampling), is given by:

$$x_q(t) = \text{sgn}(x(t)). \quad (F3)$$

If this signal is then sampled by an impulse train, the result is:

$$\begin{aligned} x_{q\delta}(t) &= x_q(t) \cdot s_\delta(t) \\ &= x_q(t) \cdot \sum_k \delta(t - kT_s) \\ &= \text{sgn}(x(t)) \cdot \sum_k \delta(t - kT_s) \\ &= \sum_k \text{sgn}(x(kT_s)) \cdot \delta(t - kT_s) \end{aligned} \quad (F4)$$

because of the properties of the Dirac-delta function. Similarly $x_\delta(t)$ was shown to be:

$$x_\delta(t) = \sum_k x(kT_s) \cdot \delta(t - kT_s) \quad (F5)$$

Now quantising the sampled signal at each of the sampling points, results in:

$$x_{\delta q}(t) = \sum_k \text{sgn}(x(kT_s)) \cdot \delta(t - kT_s) \quad (F6)$$

Again, because of the properties of the Dirac-delta function:

$$x_{\delta q}(t) = \sum_k \text{sgn}(x(kT_s)) \cdot \delta(t - kT_s) \quad (F7)$$

which as seen from Equation F4, is equivalent to $x_{q\delta}(t)$. Thus for the one bit quantisation case, it may be seen that sampling and quantisation are commutative, as stated for the general case by Jayant and Noll (1984).

Likewise, the spectra of the two functions may be determined as follows:

$X_{q\delta}(f) = X_q(f) * S_\delta(f)$ [$= X_{\delta q}(f)$ because the impulse train is not quantisable in amplitude], where:

$$S_\delta(f) = f_s \sum_k \delta(f - nf_s) \quad (F8)$$

Therefore:

$$\begin{aligned} X_{q\delta}(f) &= X_q(f) * f_s \sum_k \delta(f - nf_s) \\ &= f_s \sum_k X_q(f - nf_s) \text{ by the properties of the Dirac-delta function.} \end{aligned}$$

This equivalence, is illustrated graphically in Figure F1 overleaf. In the case of the most basic form of the DRFM, a one bit A/D conversion process is used, however the above analysis may easily be extended to the N bit case, as illustrated in Figure F1. The above analysis is clearly valid for the case of ideal impulse sampling and the properties associated with the Dirac-delta function. No attempt has been made to consider this analysis with a sampling pulse train.

The above one bit analysis may be extended to the general case as shown below. With quantisation followed by sampling:

$$\begin{aligned} x_{q\delta}(t) &= x_q(t) \cdot s_\delta(t) \\ &= x_q(t) \cdot \sum_k \delta(t - kT_s) \\ &= \sum_k x_q(kT_s) \cdot \delta(t - kT_s) \end{aligned} \quad (F9)$$

With quantisation preceded by sampling:

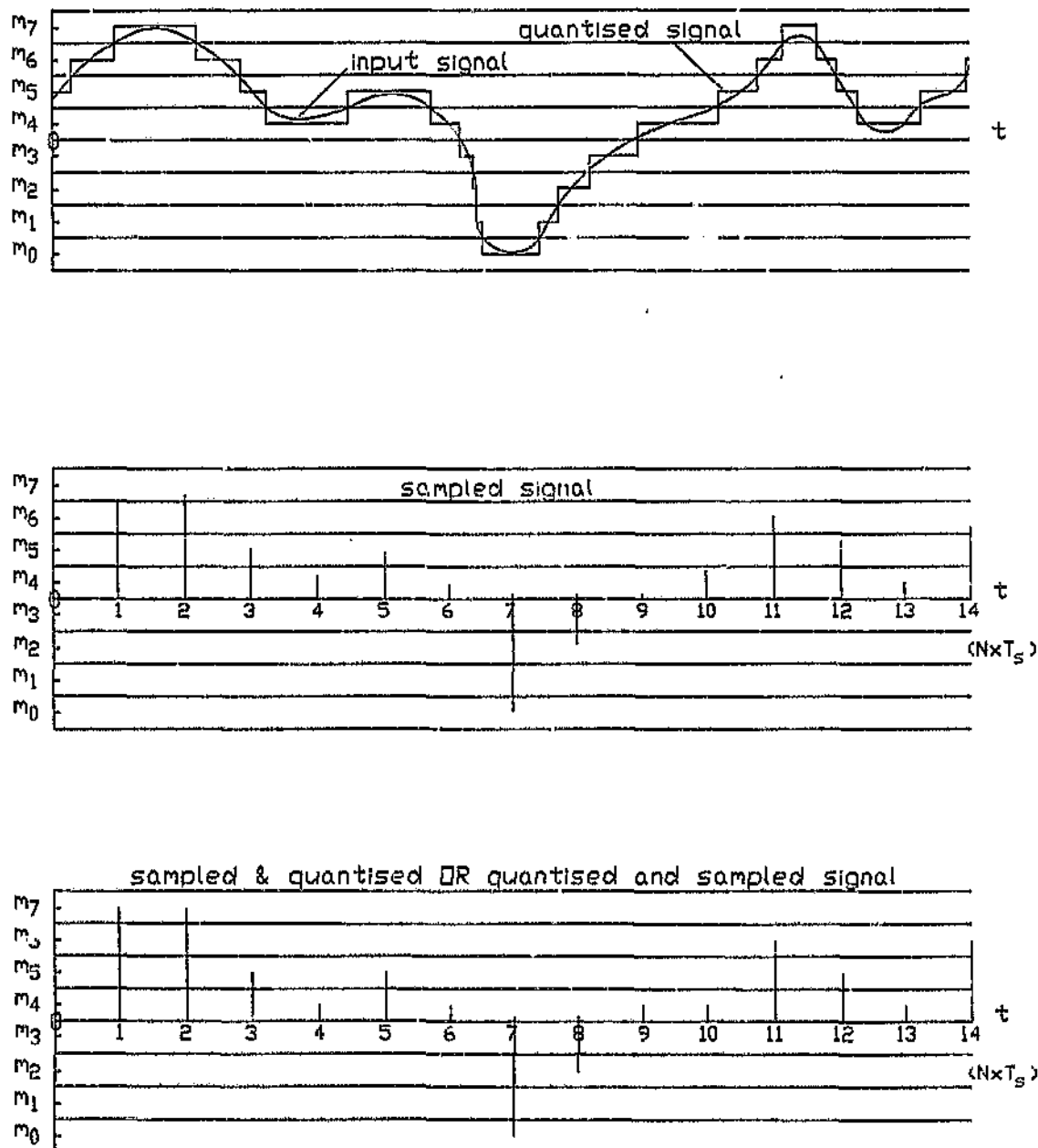
$$x_{\delta q}(t) = \sum_k (x(kT_s) \cdot \delta(t - kT_s))_q \quad (F10)$$

where "q" denotes the quantisation process. Since the Dirac-delta

function cannot be quantised:

$$x_{\delta q}(t) = \sum_k x_q(kT_s) \cdot \delta(t - kT_s) \quad (F11)$$

which as before is equivalent to $x_{q\delta}(t)$ (Equation F9) above.



* impulse sampling is assumed

Figure F1 Commutative Properties of Sampling and Quantisation

ANALYSIS OF THE FMSW SPECTRUM

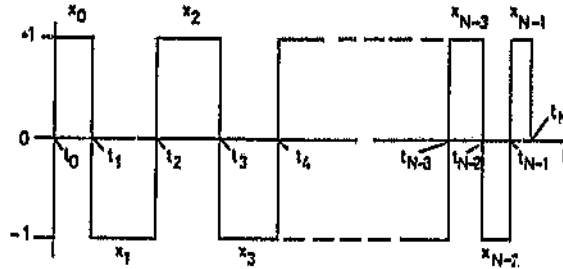


Figure G1 Time Representation of the FMSW Spectrum - Johnston (1984)

Consider the FMSW signal illustrated in Figure G1 above. The signal is represented by a series of rectangular pulses of amplitude ± 1 . Each of these pulses may be described as follows:

$$s(t) = x_n \quad t_n \leq t \leq t_{n+1} \quad x_n \in [-1; +1] \quad (G1)$$

$$n = 0, 1, 2, \dots, N-1$$

Hence the total signal may be represented as:

$$s(t) = \sum_{n=0}^{N-1} x_n \Big|_{t_n}^{t_{n+1}} \quad (G2)$$

The spectrum of this signal is now obtained by taking Fourier transforms, thus:

$$S(w) = \sum_{n=0}^{N-1} x_n \cdot \int_{t_n}^{t_{n+1}} \exp(-j\omega t) dt$$

$$= \sum_{n=0}^{N-1} (x_n/w) ((\sin(\omega t_{n+1}) - \sin(\omega t_n)) + j(\cos(\omega t_{n+1}) - \cos(\omega t_n))) \quad (G3)$$

Equation G3 yields itself readily to computer implementation.

SOFTWARE LISTINGS FOR COMPUTATION OF THE FMSW SPECTRUM

```

1 'FMSW
10 REM this program considers 1-bit digitally generated signals
20 REM ref. j.a. johnston;proc. ieee,pt f,no. 4,july 1984
21 PI=3.141593
30 DIM T(1500),F(1000),X(1500),SM(1000),TERMR(1500),TERMI(1500)
40 INPUT "ENTER f1,f2,pw,no. of points,fp1,fp2?",F1,F2,PW,NP,FP1,FP2
41 IF NP>1000 THEN PRINT "np>1000-try again!!!"
42 IF NP>1000 THEN GOTO 40
50 GOSUB 1000
60 GOSUB 2000
70 GOSUB 3000
80 STOP
90 END
100 '
110 '
1000 'this subroutine evaluates t(n),x(n)
1010 N=2*PW*((F2+F1)/2)
1011 N=CINT(N)
1020 IF N<=1500 THEN 1040
1030 PRINT "n>1500":STOP:END
1031 PRINT N
1040 K=(F2-F1)/PW
1041 'lprint n
1050 FOR I=0 TO N
1060 IF K<=0 THEN T(I)=(-F1+SQR(F1^2+K*I))/K ELSE T(I)=I/2/F1
1070 NEXT I
1080 FOR I=0 TO N-1
1090 X(I)=SGN(SIN((2*PI*F1+PI*K/2*(T(I)+T(I+1))))*(T(I)+T(I+1))/2))
1091 'lprint i,t(i),x(i)
1100 NEXT I
1110 RETURN
1120 '
1130 '
2000 'this subroutine evaluates s(f)
2010 FOR J=0 TO NP
2011 PRINT J
2020 F(J)=J*(FP2-FP1)/NP+FP1
2021 IF F(J)<=0 THEN 2030
2022 SM(J)=0:F(J)=0
2023 GOTO 2110
2030 SUMREAL=0:SUMIMAG=0
2040 FOR I=0 TO N-1
2050 TERMR(I)=X(I)/(2*PI*F(J))*(SIN(2*PI*F(J)*T(I+1))-SIN(2*PI*F(J)*T(I)))
2060 SUMREAL=SUMREAL+TERMR(I)
2070 TERMI(I)=X(I)/(2*PI*F(J))*(COS(2*PI*F(J)*T(I+1))-COS(2*PI*F(J)*T(I)))
2080 SUMIMAG=SUMIMAG+TERMI(I)
2081 'lprint i,termr(i),termi(i)
2090 NEXT I
2100 SM(J)=SQR(SUMREAL^2+SUMIMAG^2)
2101 'lprint sumreal,sumimag,sm(j)
2110 NEXT J
2120 RETURN
2130 '
2140 '

```

```

3000 'plot routine
3010 CLS
3020 SCREEN 3:SCREEN 2:'high-res. graphics-640X200 pixels
3030 COLOR 7:'black on white
3040 VIEW (20,10)-(600,180),,1:'defines graphics area with border
3050 CLS
3060 WINDOW (FP1,0)-(FP2,1)
3070 FOR J=0 TO NP-1
3080 LINE (F(J),SM(J))-(F(J+1),SM(J+1))
3090 NEXT J
3091 DF=FP2-FP1
3092 FOR J1=0 TO 1 STEP .1
3093 LINE (J1*DF+FP1,0)-(J1*DF+FP1,.01)
3094 LINE (FP1,J1)-(.01*DF+FP1,J1)
3095 NEXT J1
3100 RETURN
3110 '
3120 '
4000 'this subroutine normalizes the output data to lie between [-1,1]
4010 SMAX=0
4020 FOR J=0 TO NP
4030 IF SM(J)>SMAX THEN SMAX=SM(J)
4040 NEXT J
4050 'lprint smax
4060 FOR J=0 TO NP
4070 SM(J)=SM(J)/SMAX
4080 NEXT J
4090 RETURN

```

ANALYSIS OF THE ERROR SPECTRUM OF A ONE BIT DIGITALLY SYNTHESISED SIGNAL

Figure I1 below shows the four possible types of error associated with time quantization of a signal that is digitally synthesized by means of a one bit process.

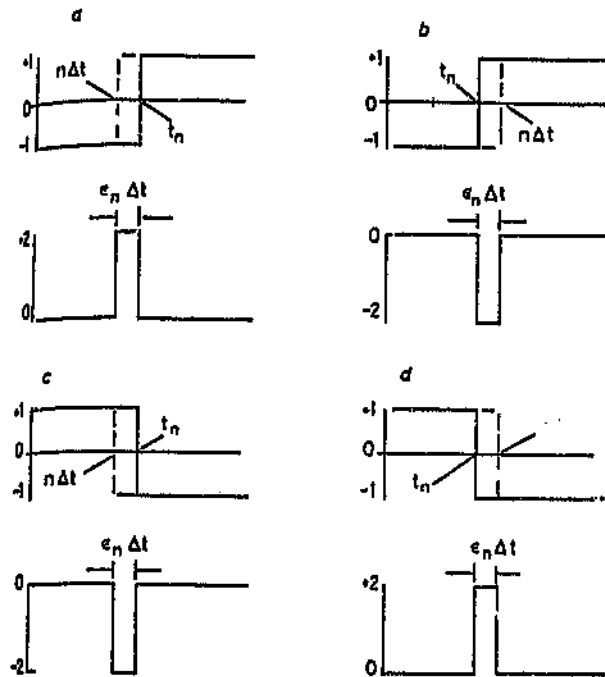


Figure I1 Time Quantization Errors in Digital Signal Synthesis - Johnston (1984)

The error signal may be represented by a series of pulses of amplitude ± 2 , depending on the exact relative timing of the desired zero-crossings and the closest integral multiple of the timing quantization interval. The width of these pulses varies from 0 to $\delta t/2$. The typical error signal may thus be represented as follows (c.f. Figure I2):

$$e(t) = \sum_{n=1}^{N-1} e_n \left| \frac{t_n + \epsilon_n \delta t}{t_n} \right| ; e_n \in [-2; +2] ; \epsilon_n \in [-\frac{1}{2}; +\frac{1}{2}] \quad (I1)$$

Similar to the baseband signal, taking Fourier Transforms gives:

$$E(w) = \sum_{n=1}^{N-1} e_n \cdot \int_{t_n}^{t_n + \epsilon_n \delta t} \exp(-j\omega t) dt$$

which after manipulation gives:

$$E(w) = \sum_{n=1}^{N-1} (e_n / w) \{ (\sin(w(t_n + \epsilon_n \delta t)) - \sin(wt_n)) + j(\cos(w(t_n + \epsilon_n \delta t)) - \cos(wt_n)) \} \quad (12)$$

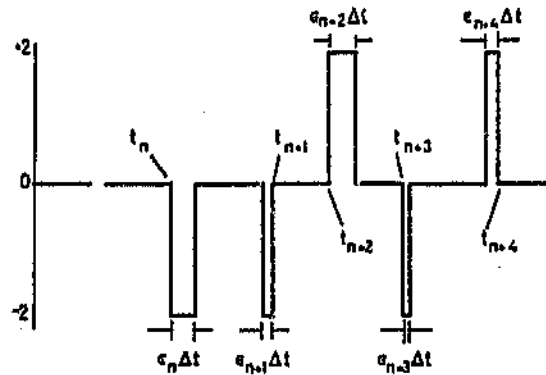


Figure 12 Typical Error Signal - Johnston (1984)

SOFTWARE LISTING FOR SPECTRUM AND ERROR OF A DIGITALLY SYNTHESISED SIGNAL

```

1 'FMSW2
10 REM this program considers 1-bit digitally generated signals
20 REM ref. j.a. johnston;proc. lee,pt.f,no. 4,july 1984 "FMSW2"
21 PI=3.141593
30 DIM T(1500),F(1000),X(1500),SM(1000),TE(1500),E(1500),EM(1000),STM(1000)
40 INPUT "ENTER f1,f2,pw,no. of points,fp1,fp2?",F1,F2,PW,NP,FP1,FP2
41 IF NP>1000 THEN PRINT "np>1000-try again!!!"
42 IF NP>1000 THEN GOTO 40
43 INPUT "DO YOU WANT AN ERROR SPECTRUM AS WELL-Y/N?",A1$
44 IF A1$="N" THEN 50
45 IF A1$="Y" THEN INPUT "ENTER CLOCK FREQUENCY,FC?",FC
46 IF A1$<>"Y" THEN 43
50 GOSUB 1000
51 IF A1$="Y" THEN GOSUB 5000
60 GOSUB 2000
61 GOSUB 4000
62 INPUT "WHICH PLOT DO YOU WANT,SIGNAL(1),ERROR(2),TOTAL(3),END(4)?",A1
63 IF A1>4 THEN 62
64 IF A1=4 THEN 80
65 INPUT "DO YOU WANT A LINEAR(1) OR LOG(2) PLOT?",A2
66 IF A2>2 THEN 65
67 IF A2=1 THEN GOSUB 3000
68 IF A2=2 THEN GOSUB 7000
71 GOTO 62
80 STOP
90 END
100 '
110 '
1000 'this subroutine evaluates t(n),x(n)
1010 N=2*PW*((F2+F1)/2)
1011 N=CINT(N)
1020 IF N<=1500 THEN 1040
1030 PRINT "n>1500" STOP:END
1031 PRINT N
1040 K=(F2-F1)/PW
1041 'lprint n
1050 FOR I=0 TO N
1060 IF K<>0 THEN T(I)=(-F1+SQR(F1^2+K*I))/K ELSE T(I)=I/2/F1
1070 NEXT I
1080 FOR I=0 TO N-1
1090 X(I)=SGN(SIN((2*PI*F1+PI*K/2*(T(I)+T(I+1))))*(T(I)+T(I+1))/2))
1091 'lprint i,t(i),x(i)
1100 NEXT I
1110 RETURN
1120 '
1130 '
2000 'this subroutine evaluates S(f) & accesses subroutine that evals. E(f)
2010 FOR J=0 TO NP
2011 PRINT J
2020 F(J)=J*(FP2-FP1)/NP+FP1
2021 IF F(J)<>0 THEN 2030
2022 SM(J)=0:F(J)=0:EM(J)=0:STM(J)=0
2023 GOTO 2110
2030 SUMREAL=0:SUMIMAG=0:SUMEREAL=0:SUMEIMAG=0

```

```

2040 FOR I=0 TO N-1
2050 TERMR=X(I)/((2*PI*F(J))*(SIN(2*PI*F(J)*T(I+1))-SIN(2*PI*F(J)*T(I)))
2060 SUMREAL=SUMREAL+TERMR
2070 TERMI=X(I)/((2*PI*F(J))*(COS(2*PI*F(J)*T(I+1))-COS(2*PI*F(J)*T(I)))
2080 SUMIMAG=SUMIMAG+TERMI
2081 IF A1$="Y" THEN GOSUB 6000
2082 'lprint i,termr,termi
2090 NEXT I
2100 SM(J)=SQR(SUMREAL^2+SUMIMAG^2)
2101 IF A1$="Y" THEN EM(J)=SQR(SUMEREAL^2+SUMEIMAG^2)
2102 IF A1$="Y" THEN STM(J)=SQR((SUMEREAL+SUMREAL)^2+(SUMEIMAG+SUMIMAG)^2)
2103 'lprint sumreal,sumimag,sm(j)
2110 NEXT J
2120 RETURN
2130 '
2140 '
3000 'linear plot routine
3010 CLS
3020 SCREEN 3:SCREEN 2:'high-res. graphics-640X200 pixels
3030 COLOR 7:'black on white
3040 VIEW (20,10)-(600,180),,1:'defines graphics area with border
3050 CLS
3060 WINDOW (FP1,0)-(FP2,1)
3070 FOR J=0 TO NP-1
3071 ON A1 GOTO 3072,3074,3076
3072 Y0=SM(J):Y1=SM(J+1)
3073 GOTO 3080
3074 Y0=EM(J):Y1=EM(J+1)
3075 GOTO 3080
3076 Y0=STM(J):Y1=STM(J+1)
3080 LINE (F(J),Y0)-(F(J+1),Y1)
3090 NEXT J
3091 DF=FP2-FP1
3092 FOR J1=0 TO 1 STEP .1
3093 LINE (J1*DF+FP1,0)-(J1*DF+FP1,.01)
3094 LINE (FP1,J1)-(.01*DF+FP1,J1)
3095 NEXT J1
3100 RETURN
3110 '
3120 '
4000 'this subroutine normalizes the output data to lie between [-1,1]
4010 SMAX=0
4020 FOR J=0 TO NP
4030 IF SM(J)>SMAX THEN SMAX=SM(J)
4040 NEXT J
4050 'lprint smax
4060 FOR J=0 TO NP
4070 SM(J)=SM(J)/SMAX
4071 IF A1$="Y" THEN STM(J)=STM(J)/SMAX
4072 IF A1$="Y" THEN EM(J)=EM(J)/SMAX
4080 NEXT J
4090 RETURN
5000 'this subroutine evaluates e(n) and te(n)
5001 DT=1/FC
5010 FOR I=1 TO NP-1
5020 A=T(I)/DT
5030 M=GINT(A)
5040 TE(I)=M*DT-T(I)
5050 IF TE(I)>=0 THEN 5080
5060 IF X(I)=1 THEN E(I)=2
5061 IF X(I)=-1 THEN E(I)=-2
5070 GOTO 5090

```

```

5080 IF X(I)=1 THEN E(I)=-2
5081 IF X(I)=-1 THEN E(I)=2
5090 NEXT I
5100 RETURN
6000 'this subroutine evaluates E(f)
6010 IF I=0 THEN RETURN
6020 TERMER=E(I)/(2*PI*F(J))*(SIN(2*PI*F(J)*(T(I)+TE(I)))-SIN(2*PI*F(J)*T(I)))
6030 SUMEREAL=SUMEREAL+TERMER
6040 TERMEI=E(I)/(2*PI*F(J))*(COS(2*PI*F(J)*(T(I)+TE(I)))-COS(2*PI*F(J)*T(I)))
6050 SUMEIMAG=SUMEIMAG+TERMEI
6060 RETURN
6070 '
6080 '
7000 'dB plot routine
7010 CLS
7020 SCREEN 3:SCREEN 2
7030 COLOR 7
7040 VIEW (20,10)-(600,180),,1
7050 CLS
7060 WINDOW (FP1,-50)-(FP2,10):'60dB DYNAMIC RANGE
7061 L10=LOG(10)
7070 FOR J=0 TO NP-1
7071 ON A1 GOTO 7072,7074,7076
7072 IF SM(J)=0 THEN Y0=-100 ELSE Y0=20*LOG(SM(J))/L10
7073 IF SM(J+1)=0 THEN Y1=-100:GOTO 7080 ELSE Y1=20*LOG(SM(J+1))/L10:GOTO 7080
7074 IF EM(J)=0 THEN Y0=-100 ELSE Y0=20*LOG(EM(J))/L10
7075 IF EM(J+1)=0 THEN Y1=-100:GOTO 7080 ELSE Y1=20*LOG(EM(J+1))/L10:GOTO 7080
7076 IF STM(J)=0 THEN Y0=-100 ELSE Y0=20*LOG(STM(J))/L10
7077 IF STM(J+1)=0 THEN Y1=-100 ELSE Y1=20*LOG(STM(J+1))/L10
7080 LINE (F(J),Y0)-(F(J+1),Y1)
7090 NEXT J
7091 DF=FP2-FP1
7092 FOR J1=0 TO 1 STEP .1
7093 LINE(J1*DF+FP1,-50)-(J1*DF+FP1,-49)
7094 NEXT J1
7095 FOR J1=-50 TO 10 STEP 5
7096 LINE (FP1,J1)-(.01*DF+FP1,J1)
7097 NEXT J1
7100 RETURN
7110 '
7120 '

```

SOFTWARE LISTING FOR GENERATION OF ALIASED QUANTISATION ERROR FOR N BIT QUANTISATION

```

'DRFM2
' this program considers N bit digitally SAMPLED signals
' ref. j.a. johnston;proc. ieee,pt f,no. 4,july 1984 "DRFM2"
'SOFTWARE AUTHOR-R.Z.BIL
'VERSION 2.1 (1988)
'*****
'MAIN PROGRAM

      PI=3.141593:A3=0
      DIM T(13000),F(1000),X(13000),SM(1000),EM(1000),STM(1000)
      DIM STMAG(1000),SREAL(1000),Q(1024),M(1024)
10    INPUT "ENTER NO. OF PLOT POINTS FOR BASEBAND SPECTRUM?",NP
      IF NP>2000 THEN
        PRINT "NP>2000-try again!!!":GOTO 10
      ELSE
        GOTO 20
      END IF
20    PRINT "ENTER BASEBAND SPECTRUM TYPE TO BE CONSIDERED-"
      INPUT "LOWER FREQUENCY IS 0 Hz OR NOT? (Y/N)",FF1=0
      INPUT "ENTER TYPE OF SIGNAL- LINEAR(1) OR LOG(2)?",T1
      PRINT "GENERATING QUANTIZER- QUANT"
      PRINT "GENERATING ZERO-CROSSING TIMES & QUANTIZED VALUES OF X(T)"
      ON T1 GOSUB LSVXTIM,LEV.TIM
      PRINT "EVALUATING BASEBAND SPECTRUM":GOSUB BA "EAND
40    INPUT "DO YOU WANT AN ALIASING ERROR SPECTRUM? (Y/N)",A1$
50    IF A1$="N" THEN
      A1=1:PRINT "NORMALIZING DATA":GOSUB NORMAL:GOTO 30
    ELSEIF A1$<>"Y" THEN
      GOTO 40
    ELSE
      DFP=FF1/NP
      PRINT "ENTER AN INTEGER VALUE NC SUCH THAT THE SAMPLING"
      PRINT "FREQUENCY=NS*";DFP:INPUT "NS?",NS
      FC=NS*DFP:FFOLD=FC/2
    'END IF
    END IF
    PRINT "EVALUATING ALIASING COMPONENTS":GOSUB ALIASING
    PRINT "NORMALIZING DATA":GOSUB NORMAL
50    PRINT "WHICH PLOT? BASEBAND SIGNAL(1),UNALIASED SIGNAL(2),ALIASING"
    INPUT "ERROR(3),TOTAL(4),END(5)?",A1
    IF A1>5 THEN 50
    IF A1=5 THEN 60
30    INPUT "DO YOU WANT A LINEAR(1) OR LOG(2) PLOT?",A2
    IF A2>2 THEN 30
    IF A2=1 THEN GOSUB LINPLOT
    IF A2=2 THEN GOSUB LOGPLOT
    GOTO 50
60    'SCREEN 0,0,0
80    INPUT "DO YOU WANT A DIFFERENT SAMPLING FREQUENCY,Y/N?",A2$
    IF A2$="N" THEN
      GOTO 70
    ELSEIF A2$<>"Y" THEN
      GOTO 80
    ELSE
      A3=1:GOTO 90
    'END IF
    END IF
70    STOP
    END

'*****
'SUBROUTINES
'*****
'this subroutine evaluates zero-crossing times for a sinusoid or LFM signal

```

'N-bit quantization for values of signal x(t)

LEVXTIM:

```

110 IF T1=1 THEN INPUT "ENTER FREQUENCY OF SINUSOID?",F1:F2=F1
    IF T1=2 THEN INPUT "ENTER START & STOP FREQUENCIES OF LFM SIGNAL?",F1,F2
    INPUT "ENTER PULSEWIDTH OF PULSE MODULATED SIGNAL?",PW
    N=2*PW*((F2+F1)/2)
    N=CINT(N):NX=(NBIT*(NBIT-1)+1)*N
    IF NX<-13000 THEN
        GOTO 100
    ELSE
        PRINT "NO. OF LEVEL-CROSSINGS EXCEEDS 13000-"
        PRINT "REDUCE PULSEWIDTH OR FREQUENCY!!!":GOTO 110
    END IF
100 PRINT N,NX,NBIT
    K=(F2-F1)/PW
    FOR J=NLEV/2 TO NLEV-1
        XTEMP=SQR(1-Q(J)^2)
        QTEMP=Q(J)/XTEMP:'TAN(THETA)
        THETA=ATN(QTEMP)
        NTEMP=J-NLEV/2
        FOR I=0 TO N STEP 2
            ILX=I*(NLX+1)+NTEMP:'I even, +VE CYCLE
            BETA=I*PI+THETA
            IF ILX<=NX THEN
                IF K<0 THEN
                    T(ILX)=(-F1+SQR(F1^2+K*BETA/PI))/K:X(ILX)=M(J)
                ELSE
                    T(ILX)=BETA/(2*PI*F1):X(ILX)=M(J)
                END IF
            END IF
            IF NTEMP=0 THEN 999
            ILX=I*(NLX+1)-NTEMP:'I even, -VE CYCLE
            BETA=I*PI-THETA
            IF ILX<=NX THEN
                IF K<0 THEN
                    T(ILX)=(-F1+SQR(F1^2+K*BETA/PI))/K:X(ILX)=M(NLEV/2-NTEMP)
                ELSE
                    T(ILX)=BETA/(2*PI*F1):X(ILX)=M(NLEV/2-NTEMP)
                END IF
            END IF
999 NEXT I
        FOR I=1 TO N STEP 2
            IF NTEMP=0 THEN 1000
            ILX=I*(NLX+1)-NTEMP:'I odd, +VE CYCLE
            BETA=I*PI-THETA
            IF ILX<=NX THEN
                IF K<0 THEN
                    T(ILX)=(-F1+SQR(F1^2+K*BETA/PI))/K:X(ILX)=M(J-1)
                ELSE
                    T(ILX)=BETA/(2*PI*F1):X(ILX)=M(J-1)
                END IF
            END IF
1000 ILX=I*(NLX+1)+NTEMP:'I odd, -VE CYCLE
            BETA=I*PI+THETA
            IF ILX<=NX THEN
                IF K<0 THEN
                    T(ILX)=(-F1+SQR(F1^2+K*BETA/PI))/K:X(ILX)=M(NLEV/2-NTEMP-1)
                ELSE
                    T(ILX)=BETA/(2*PI*F1):X(ILX)=M(NLEV/2-NTEMP-1)
                END IF
            END IF
        NEXT I
    NEXT J

```

```

        FOR I=0 TO NX
        PRINT I,T(I),X(I)
        NEXT I
RETURN

'*****
' this subroutine evaluates S(f), THE BASEBAND SPECTRUM OF X(T)
BASEBAND:
        FOR J=0 TO NP
        PRINT J
        F(J)=J*(FP2-FP1)/NP+FP1
        IF F(J)<0 THEN 120
        SM(J)=0:F(J)=0
        GOTO 130
120      SUMREAL=0:SUMIMAG=0
        FOR I=0 TO NX-1
        TERMR=X(I)/(2*PI*F(J))*(SIN(2*PI*F(J)*T(I+1))-SIN(2*PI*F(J)*T(I)))
        SUMREAL=SUMREAL+TERMR
        TERMI=X(I)/(2*PI*F(J))*(COS(2*PI*F(J)*T(I+1))-COS(2*PI*F(J)*T(I)))
        SUMIMAG=SUMIMAG+TERMI
        NEXT I
        SREAL(J)=SUMREAL:SIMAG(J)=SUMIMAG
        SM(J)=SQR(SUMREAL^2+SUMIMAG^2)
130      NEXT J
RETURN

'*****
' linear plot routine
LINEPLOT:
        CLS
        SCREEN 2:'high-res. graphics-640X200 pixels
        VIEW (100,10)-(600,150),,1:'defines graphics area with border
        CLS
        IF A1=1 THEN WINDOW (FP1,0)-(FP2,1)
        IF A1<>1 THEN WINDOW (FP1,0)-(FFOLD,1)
        IF A1=1 THEN
        K1=NP-1
        ELSE
        K1=IFOLD
        END IF
        FOR J=0 TO K1
        ON A1 GOTO 140,140,150,160
140      YO=SM(J):Y1=SM(J+1):FO=F(J):F1=F(J+1)
        GOTO 170
150      YO=EM(J):Y1=EM(J+1):FO=F(J):F1=F(J+1)
        GOTO 170
160      YO=STM(J):Y1=STM(J+1):FO=F(J):F1=F(J+1)
170      LINE (FO,YO)-(F1,Y1)
        NEXT J
        IF A1=1 THEN
        DF=FP2-FP1
        ELSE
        DF=FFOLD-FP1
        END IF
        FOR J1=0 TO 1 STEP .1
        LINE (J1*DF+FP1,0)-(J1*DF+FP1,.01)
        LINE (FP1,J1)-(DF+FP1,J1)
        NEXT J1
RETURN

'*****
' this subroutine normalizes the output data to lie between [0,1]

```

```

NORMAL:
  IF A3=1 THEN 180
  SMAX=0
  FOR J=0 TO NP
    IF SM(J)>SMAX THEN SMAX=SM(J)
  NEXT J
180  FOR J=0 TO NP
    IF A3=0 THEN SM(J)=SM(J)/SMAX
    IF A1$="Y" THEN STM(J)=STM(J)/SMAX
    IF A1$="Y" THEN EM(J)=EM(J)/SMAX
  NEXT J
RETURN

'*****
'FOLD-OVER SUBROUTINE : ALIASING

ALIASING:
  IFOLD=NS/2
  IMAX=INT(NP/IFOLD)+1
  FOR J=0 TO IFOLD
    EREAL=0:EIMAG=0
    FOR I=1 TO IMAX STEP 2
      J1=IFOLD*(I+1)-J
      IF J1>NP THEN 190
      EREAL=EREAL+SREAL(J1):EIMAG=EIMAG+SIMAG(J1)
190    NEXT I
    FOR I=2 TO IMAX STEP 2
      J1=I*IFOLD+J
      IF J1>NP THEN 200
      EREAL=EREAL+SREAL(J1):EIMAG=EIMAG+SIMAG(J1)
200    NEXT I
    EM(J)=SQR(EREAL^2+EIMAG^2)
    STM(J)=SQR((EREAL+SREAL(J))^2+(EIMAG+SIMAG(J))^2)
  NEXT J
RETURN

'*****
'dB plot routine

LOGPLOT:
  CLS
  SCREEN 2
  VIEW (100,10)-(600,150),,1
  CLS
  IF A1=1 THEN WINDOW (FP1,-50)-(FP2,10):'60dB DYNAMIC RANGE
  IF A1<>1 THEN WINDOW (FP1,-50)-(FFOLD,10)
  IF A1=1 THEN
    K1=NP-1
  ELSE
    K1=IFOLD
  END IF
  L10=LOG(10)
  FOR J=0 TO K1
    ON A1 GOTO 210,210,220,230
210  IF SM(J)=0 THEN
    Y0=-100:F0=F(J)
  ELSE
    Y0=20*LOG(SM(J))/L10:F0=F(J)
  END IF
  IF SM(J+1)=0 THEN
    Y1=-100:F1=F(J+1):GOTO 240
  ELSE
    Y1=20*LOG(SM(J+1))/L10:F1=F(J+1):GOTO 240

```

```

220      END IF
      IF EM(J)=0 THEN
        YO=-100:FO=F(J)
      ELSE
        YO=20*LOG(EM(J))/L10:FO=F(J)
      END IF
      IF EM(J+1)=0 THEN
        Y1=-100:F1=F(J+1):GOTO 240
      ELSE
        Y1=20*LOG(EM(J+1))/L10:F1=F(J+1):GOTO 240
      END IF
230      IF STM(J)=0 THEN
        YO=-100:FO=F(J)
      ELSE
        YO=20*LOG(STM(J))/L10:FO=F(J)
      END IF
      IF STM(J+1)=0 THEN
        Y1=-100:F1=F(J+1)
      ELSE
        Y1=20*LOG(STM(J+1))/L10:F1=F(J+1)
      END IF
240      LINE (FO,YO)-(F1,Y1)
      NEXT J
      IF A1=1 THEN
        DF=FP2-FP1
      ELSE
        DF=PFOLD-FP1
      END IF
      FOR J1=0 TO 1 STEP .1
        LINE(J1*DF+FP1,-50)-(J1*DF+FP1,-49)
      NEXT J1
      FOR J1=-50 TO 10 STEP 5
        LINE (FP1,J1)-(DF+FP1,J1)
      NEXT J1
RETURN

'*****
'QUANTIZATION ROUTINE

QUANT:
  INPUT "ENTER NUMBER OF BITS (UP TO 10) USED IN QUANTIZER?";NBIT
  NLEV=2^NBIT:'NLEV-NO. OF LEVELS,NBIT-NO. OF BITS
  FOR J=0 TO NLEV
    Q(J)=-1+J*2/NLEV:PRINT Q(J)
  NEXT J
  FOR J=0 TO NLEV-1
    M(J)=(Q(J+1)+Q(J))/2:PRINT M(J)
  NEXT J
  'N-NO. OF ZERO-CROSSINGS
  'NX-NO. OF LEVEL CROSSINGS,NLX-NO. OF LEVEL-X BETWEEN (ZERO-X)-1
  NLX=NLEV-2
RETURN

'*****

```

FOURIER COEFFICIENTS OF A SINGLE TONE INPUT AS A FUNCTION OF THE DEGREE OF QUANTISATION

The development of the expression for the Fourier coefficients is based on Figure L1 below.

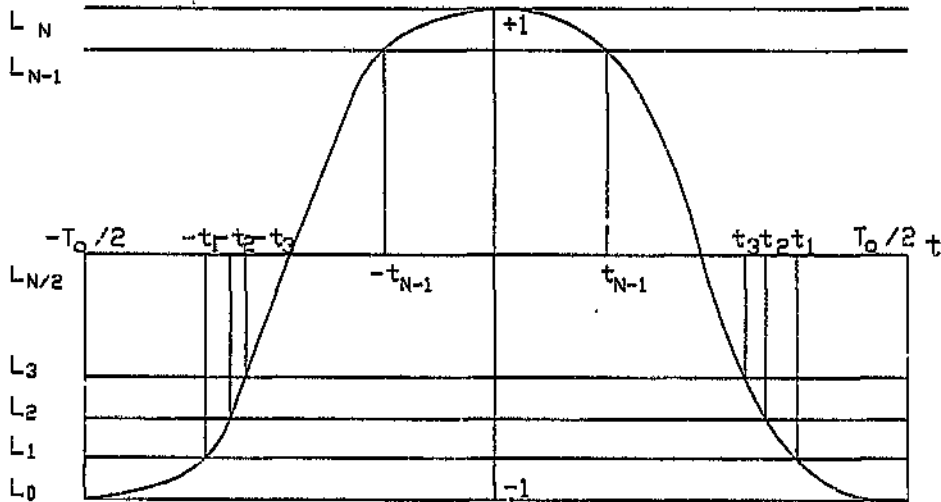


Figure L1 Illustration of the Quantization Process

The number of bits for the quantization characteristic is given by R . Hence the number of quantization levels is $N=2^R$. The lowest level is -1 and the highest is $+1$, assuming a cosinusoidal input as shown. Hence:

$$L_0 = -1, L_N = 1, L_{N/2} = 0 \text{ and } L_i = L_0 + 2i/N \quad (L1)$$

The quantized values take on the midpoint values of Q_i , if the signal lies in the range L_{i-1} to L_i . Thus:

$$Q_i = (L_i + L_{i-1})/2 \quad (L2)$$

The Fourier series coefficients are given by:

$$c_n = 1/T_0 \int_{-T_0/2}^{T_0/2} f(t) \exp(-jn\omega_0 t) dt \quad (L3)$$

If the input signal is a cosinusoid, $x(t) = \cos(\omega_0 t)$ as shown above, then $\omega_0 = 2\pi/T_0$. The times at which the signal crosses from one quantization interval to the next, may be determined from the

values a_k , where:

$$a_k = w_0 t_k = \cos^{-1}(L_i)$$

and:

$$0 \leq \theta_k \leq \pi, \quad t_k = a_k / w_0$$

Thus:

$$\begin{aligned} c_n = 1/T_0 [& \int_{-T_0/2}^{t_1} Q_1 \exp(-jnw_0 t) dt + \int_{-t_1}^{-t_2} Q_2 \exp(-jnw_0 t) dt \\ & + \int_{-t_2}^{-t_3} Q_3 \exp(-jnw_0 t) dt + \dots + \int_{-t_{N-1}}^{-t_{N-2}} Q_N \exp(-jnw_0 t) dt \\ & + \int_{t_{N-1}}^{T_0/2} Q_{N-1} \exp(-jnw_0 t) dt + \dots + \int_{t_1}^{T_0/2} Q_1 \exp(-jnw_0 t) dt] \quad (L4) \end{aligned}$$

Now $t_0 = T_0/2$, since $a_0 = \pi$. Therefore:

$$\begin{aligned} c_n = -1/(jn.2\pi) . [& Q_1 (\exp(jna_1) - \exp(jn\pi)) + \\ & Q_2 (\exp(jna_2) - \exp(jna_1)) + \\ & Q_3 (\exp(jna_3) - \exp(jna_2)) + \dots \\ & + Q_N (\exp(jna_{N-1}) - \exp(-jna_{N-1})) + \\ & Q_{N-1} (\exp(-jna_{N-2}) - \exp(-jna_{N-1})) + \dots \\ & + Q_1 (\exp(-jn\pi) - \exp(-jna_1))] \quad (L5) \end{aligned}$$

Associating terms in $\exp(jna_k)$ and $\exp(-jna_k)$, gives:

$$\begin{aligned} c_n = -1/(jn.2\pi) . [& -1/2^{R-1}] . [(\exp(jna_1) - \exp(-jna_1)) + \quad (L6) \\ & (\exp(jna_2) - \exp(-jna_2)) + \dots + (\exp(jna_{N-1}) - \exp(-jna_{N-1}))] \end{aligned}$$

and finally:

$$c_n = 1/(n\pi.2^{R-1}) . [\sin(na_1) + \sin(na_2) + \dots + \sin(na_{N-1})] \quad (L7)$$

A computer program was written to evaluate c_n for $n=1,2,3,\dots,99$

and $R=1$ to 10.

MULTI-BIT QUANTISATION

Consider a signal $s(t)=x_n$ for $t_n \leq t < t_{n+1}$ and $x_n \in m_i$. m_i are the midpoints of the quantisation intervals q_i , such that $m_i = (q_i + q_{i+1})/2$, if $q_i \leq x_n < q_{i+1}$, $n=0,1,2,\dots,N-1$. Thus t_n are the level crossing instants. Hence:

$$s(t) = \sum_{n=0}^{N-1} x_n \Big|_{t_n}^{t_{n+1}} \quad (M1)$$

To find the spectrum of $s(t)$, take the Fourier transform as for the one bit case, giving:

$$\begin{aligned} S(w) &= \sum_{n=0}^{N-1} x_n \int_{t_n}^{t_{n+1}} \exp(-j\omega t) dt \quad (M2) \\ &= \sum_{n=0}^{N-1} (j x_n / \omega) \cdot [\exp(-j\omega t_{n+1}) - \exp(-j\omega t_n)] \\ &= \sum_{n=0}^{N-1} (j x_n / \omega) \cdot [\cos(\omega t_{n+1}) - j \sin(\omega t_{n+1}) - \cos(\omega t_n) + j \sin(\omega t_n)] \\ &= \sum_{n=0}^{N-1} (x_n / \omega) \cdot [(\sin(\omega t_{n+1}) - \sin(\omega t_n)) + j(\cos(\omega t_{n+1}) - \cos(\omega t_n))] \quad (M3) \end{aligned}$$

This is identical to Equation G3 for the one bit case, apart from the fact that now, the x_n are elements of the m_i and the t_n are now the quantisation level crossing instants, instead of the zero-crossing instants.

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