A symmetry perspective of third-order polynomial evolution equations

Mr Bongumusa Gwaxa

Supervisor: Prof. Sameerah Jamal

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

A thesis submitted to the Faculty of Science, University of the Witwatersrand, in requirement for the degree Doctor of Philosophy, May, 2024.

Declaration

I declare that the contents of this thesis are original, except where due references have been made. It is submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution.

B. Gwaxa

Signed at __Johannesburg__ on the __6^{th}__ day of __May__ 2024.

Publications

Details publications that form part of the research presented in this thesis are:

On the conservation laws, Lie symmetry analysis and power series solutions of a class of third-order polynomial evolution equations, Arabian Journal of Mathematics, (2023) 12:553–564 [10].

On the optimal system and series solution of fifth-order Fujimoto-Watanabe equations, Malaysian Journal of Mathematical Sciences, (2023) 17(4): 557–573 [11].

Conserved vectors and symmetry invariant solutions to polynomial wave models, accepted to appear in Journal of Applied Nonlinear Dynamics, (2024) [12].

Abstract

In this thesis, we analyse the full class of ten Fujimoto-Watanabe equations. In particular, these are highly nonlinear third-order and two fifth-order equations. With the aid of computer algebra software such as Mathematica, we calculate symmetries for these equations and we construct their commutator tables. The one dimensional system of optimal subalgebras is obtained via adjoint operators. Finally, we reduce these higher-order partial differential equations into ordinary differential equations, derive their solutions via a power series solution method and show how convergence may be tested. Lastly, we determine some conservation laws.

Acknowledgements

I am struggling to find adequate words to convey my appreciation for Professor Sameerah Jamal, my supervisor, who has provided unwavering support throughout my five-year research journey in mathematics - for my honours, masters and PhD. Her extensive knowledge and guidance have been invaluable to my success in achieving this significant milestone.

Furthermore, I am grateful to my mother, Nomapa Valencia Gwaxa and my aunt, Nomzamo Ngxale, for their continuous assistance throughout my academic pursuits.

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ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

Chapter 1

Introduction

In many branches of mathematics, physics, engineering and other disciplines, differential equations are essential [35, 21, 14, 24]. Differential equations are significant because they allow for the mathematical description and prediction about the behaviour of complex phenomena [14]. It can be a difficult task to solve differential equations and you often need advanced mathematical tools and techniques [16]. Symmetries are one of the most fundamental ideas in differential equations, and are one of the most important concepts in mathematics as a whole [46]. Symmetries give a powerful tool for finding exact solutions to differential equations, reducing their complexity and understanding their behaviour [4, 5, 15].

Conservation laws and the concept of symmetries are closely related. In physics, conservation laws explain the fundamental principle that governs the behaviour of physical systems. According to conservation laws, certain quantities within a physical system are conserved, which means they can only be changed from one form to another and cannot be created or be destroyed [30]. Conservation laws include quantities such as momentum, angular momentum and energy that remains the same over time. The conservation of energy, for instance is one of the most well known conservation laws. According to this law, a closed system's overall energy remains constant over time. The conservation of momentum also results from the spatial translation symmetry of physical laws. According to this law, a closed system's overall momentum remains constant over time unless it is affected by the external force. Different symmetries laid a foundation on to other conservation laws. For instance, the rotational symmetry of physical laws leads to the conservation of angular momentum.

Figure 1.1: Sophus Lie

Sophus Lie developed the theory of Lie groups and Lie algebras in the 19th century. The modern theory of symmetries in differential equations was made possible thanks to Lie's work. Finding a unifying framework for the study of differential equations and developing a technique to produce solutions to differential equations were the driving forces behind Lie's research.

Mathematicians like Elie Cartan, Hermann Weyl and Vladimir Arnold made significant advances to the study of symmetries in differential equations in the 20th century [21, 24, 14]. A wide range of problems have been tackled using the theory of symmetries, such as the study of integrable systems, the theory of solitons and the study of fluid dynamics [17, 18, 31].

A transformation that leaves the differential equation invariant is a symmetry of the differential equation [6]. In geometry, for instance, a sphere retains its appearance even after being rotated by a specified angle. Hence, the sphere has rotational symmetry as a result of this feature.

We can easily list more applications of symmetries; namely: the determination of conservation laws and its admitted solutions [13, 28, 32, 7], linearization techniques [27], finding integrating factors [29] and canonical coordinates. Therefore, the study of conservation laws and symmetries are fundamental in comprehending the universe and its underlying physical laws. It is impossible to exaggerate the significance of symmetries and conservation laws in the study of differential equations. The theory of symmetries and conservation laws in differential equations will be examined in this thesis, for a family of ten PDEs called the Fujimoto-Watanabe equations.

1.1 Outline of Chapters

There are ten polynomial evolutionary equations under the Fujimoto-Watanabe class of equations. However some of these equations share common symmetry properties.

Hence, based on this fact, we divide the study into three parts.

In Chapter 3, we consider the special class of three third-order polynomial evolutionary equations. The symmetry and group properties are given and the adjoints are presented in a table. The optimal system of subalgebras are derived and the subsequent reduced ODEs. The reduced equations are solved by applying powers series and a sample convergence test is discussed. Some graphical illustrations of truncated solutions are found. Conservation laws are determined via multipliers.

Chapter 3 has been accepted for publication [12].

In Chapter 4, we consider the special class of four third-order polynomial Fujimoto-Watanabe equations. Similar to the previous chapter, the symmetries, invariants, group properties, adjoints, reduced equations, power series solutions and conservation laws are all determined.

Chapter 4 has been published [10].

Finally, in Chapter 5, the two fifth-order polynomial evolutionary equations from Fujimoto-Watanabe are studied in a similar manner.

Chapter 5 has been published [11].

Chapter 2

Notation and Theory

In this chapter, we discuss the important theory underpinning this thesis.

2.1 Lie Symmetry Analysis

In general, a symmetry is a transformation, mapping all solutions of a given equation to another solution of the very same equation. Consider q unknown functions u^a which depend on p independent variables x^i, i.e. $u = (u^1, \ldots, u^q)$, $x = (x^1, \ldots, x^p)$, with indices $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$. Let

$$G_\alpha(x, u^{(k)}) = 0, \tag{2.1}$$

be a system of nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x. A one-parameter Lie group of transformations with ε as the group parameter, invariant under (2.1), is given by

$$\bar{x} = \Xi(x, u; \varepsilon) \quad \bar{u} = \Phi(x, u; \varepsilon). \tag{2.2}$$

Expanding (2.2) around the identity $\varepsilon = 0$, generates the infinitesimal transformations:

$$\bar{x}^i = x^i + \varepsilon\xi^i(x, u) + O(\varepsilon^2), \tag{2.3}$$
$$\bar{u}^\alpha = u^\alpha + \varepsilon\eta^\alpha(x, u) + O(\varepsilon^2). \tag{2.4}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following vector field

$$X = \xi^i\partial_{x^i} + \eta^\alpha\partial_{u^\alpha}. \tag{2.5}$$

The infinitesimal criterion for invariance is given by

$$X[G_\alpha(x, u^{(k)})] = 0, \quad \text{when} \quad G_\alpha(x, u^{(k)}) = 0, \tag{2.6}$$

where X is extended to all derivatives appearing in the equation through an appropriate prolongation. The operator in Eq. (2.5) defines a Lagrange system or characteristic equation

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha},$$

whose solution provides zero-order invariants

$$W^{[0]}(x^i, u^\alpha). \tag{2.7}$$

The calculation of Lie symmetries above has been automated by many software packages. We omit any detailed calculations of symmetries below, but we refer the reader to [43] for a simple discussion or [36, 33, 45] for further examples.

2.2 One-Dimensional Optimal System

The optimal system of subalgebras is a special set of algbegras that provides a systematic procedure for constructing nonequivalent classes of invariant solutions for equations.

To this end, we recall the Lie brackets or commutators which are given by

$$[X_i, X_j] = X_i X_j - X_j X_i.$$

The approach for the one-dimensional optimal system is provided in [35]. Its procedure is where one takes a general element from the Lie algebra and then reduce it to its simplest like form by applying carefully chosen adjoint transformations

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{1}{2}\,\varepsilon^2[X_i, [X_i, X_j]] - \cdots .$$

2.3 Conservation Laws

Multipliers [44], Λ_α, are integrating factors, satisfying the relation

$$D_i T^i = \Lambda_\alpha G_\alpha, \tag{2.8}$$

where Λ_α is derived by equations satisfying

$$E(\Lambda_\alpha G_\alpha) = 0, \tag{2.9}$$

where E is the Euler operator. Now having the multipliers, they give rise to a conserved vector $T = (T^1, ..., T^n)$ that satisfies the conservation relation

$$D_i T^i = 0, \tag{2.10}$$

along the solutions of a given equation.

2.4 Power Series Solutions

An interesting solving tool is that of power series [2], especially for nonlinear ODEs that are difficult to solve otherwise. Power series are a useful tool to find approximate solutions, and the solutions may be made exact by finding a recurrence relation of the series.

We list the steps in employing this technique below.

- Set up an infinite power series with unknown coefficients, say a_r, that are to be determined, say $p(s)$.
- Substitute the power series representation $p(s)$ into the ODE. A set of equations involving the coefficient of the power series terms will result from this.
- Compare coefficients of the powers of s to calculate some of the first few unknown coefficients a_r. This will lead to a general formula for higher-order coefficients a_r.
- Examine the behaviour of the coefficients to determine whether the power series is converging. Most convergence tests involve analyzing the behaviour of the coefficient as the index of the terms increase. We use a particular

test, which is algorithmic. The convergence test involves the implicit function theorem. This requires showing that if a particular majorant series is analytic and convergent in the neighbourhood of a point, then so too is the constructed power series.

- A sufficient number of terms must be used to evaluate the power series in order to obtain the desired accuracy if the series is determined to converge.

Chapter 3

Fujimoto-Watanabe's polynomial wave models: Conserved vectors and symmetry invariant solutions

3.1 Introduction

Fujimoto-Watanabe, in a remarkable paper, showed that a special class of third-order polynomial equations exist such that they possess nontrivial Lie-Bäcklund symmetries [9]. These evolution equations appear in the study of waves and oceans, as well as several engineering applications. It is known that via differential substitutions, these equations are connected to the KdV, Sawada-Kotera and Kaup equations [39]. Due to these properties, these equations have applications in fluid dynamics, or specifically ion-acoustic waves in plasmas, stratified internal waves and other nonlinear wave phenomena. Such models, among others, have applications in wave propagation studies [23, 19, 3]. This family of equations possess Lie algebras that are commutative. They are:

$$u_t = u_x^3 u_{xxx} + \alpha u_x^3, \tag{3.1}$$

$$u_t = u^3 u_{xxx} + \frac{3}{2} u^2 u_x u_{xx} + \alpha u^2 u_x, \tag{3.2}$$

$$u_t = u^3 u_{xxx} + \alpha u^3 u_x, \tag{3.3}$$

where $u(x, t)$ generally represents a displacement from rest (for example, the height of a fluid above or below rest), and $\alpha \neq 0$ is an arbitrary constant. In the literature, the above types of equations and ones related to them, have been analysed from the point of view of bifurcations, phase portraits, periodic solutions etc. [26, 42, 22, 41]. Symmetry methods and conservation laws are fundamental components of the analysis of PDEs [29, 33, 18]. The theoretical aspects relating to symmetries is well known and is best studied from the book [43]. Here, symmetry methods are combined with power series to arrive at solutions. The PDEs are reduced to ODEs, but are difficult to solve with common techniques. However, power series are a brilliant tool to find approximate solutions, and the solutions may be exact if a recurrence relation can be found.

The plan of the chapter follows. In Section 3.2, we present the symmetry invariant properties possessed by the three equations, with the commutator, adjoint and group details. Section 3.3, contains the conservation laws and multipliers calculated for the equations. In Section 3.4, we present reductions and power series solutions and a demonstration of the convergence of a solution. The discussion and conclusion follows in Section 3.5 and 3.6, respectively.

3.2 Invariant Properties

Suppose we induce a Lie group of infinitesimal ($\varepsilon \ll 1$) transformations:

$$x \to x + \varepsilon\xi(x,t,u), \quad t \to t + \varepsilon\tau(x,t,u), \quad u \to u + \varepsilon\phi(x,t,u), \tag{3.4}$$

with symmetry vector field

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u.$$

The coefficients ξ, τ and ϕ satisfy the well known Lie symmetry condition, so that the Lie point symmetries of Eq. (3.1) and Eq. (3.2) are

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_t, \\ X_3 &= x\partial_x - 5t\partial_t + 3u\partial_u. \end{aligned} \tag{3.5}$$

The Lie brackets of (3.5) are given in Table 3.1, and the corresponding one-parameter groups K_m ($m = 1, 2, 3$) generated by the symmetries X_m, as

$$\begin{aligned} K_1 &: (x,t,u) \mapsto (x+\varepsilon, t, u), \\ K_2 &: (x,t,u) \mapsto (x, t+\varepsilon, u), \\ K_3 &: (x,t,u) \mapsto (e^{\varepsilon}x, e^{5\varepsilon}t, e^{3\varepsilon}u). \end{aligned} \tag{3.6}$$

Moreover, we require the adjoints to calculate the one-dimensional subalgebras from Table 3.2. We consider

$$X = a_1X_1 + a_2X_2 + a_3X_3, \tag{3.7}$$

where the a_i's are arbitrary constants, and simplify it by adjoint maps. Suppose first that $a_3 \neq 0$, then we can assume that $a_3 = 1$. If we act on X by $\mathrm{Ad}(\exp(a_1X_1))$ we get

$$X' = X_3 + a_2X_2.$$

Now acting on X' by $\mathrm{Ad}(\exp(-a_2/5)X_2)$ results in X_3. If $a_3 = 0$, $a_2 = 1$, then we act on $X = a_1X_1 + X_2$ by $\mathrm{Ad}(\exp(c_1X_3)$ resulting in

$$X_2,\ X_2 \pm X_1.$$

Next suppose that $a_3 = a_2 = 0$, $a_1 = 1$ which gives X_1.

Table 3.1: Lie brackets of (3.5)

$[,]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	$-5X_2$
X_3	$-X_1$	$5X_2$	0

Table 3.2: Adjoint Representation of Subalgebras of (3.5)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \varepsilon X_1$
X_2	X_1	X_2	$5\varepsilon X_2 + X_3$
X_3	$e^{\varepsilon}X_1$	$e^{-5\varepsilon}X_2$	X_3

Hence, we obtain in this case, the optimal system of one-dimensional subalgebras spanned by

$$X_1,\ X_2 + \epsilon X_1,\ X_3,$$

where $\epsilon = 0, \pm 1$. On the other hand, the Lie point symmetries of (3.3) are

$$X_1,\ X_2,\ \bar{X}_3 = -3t\partial_t + u\partial_u,$$

$$X_4 = u\cos\left(\sqrt{\alpha}x\right)\partial_u + \frac{\sin\left(\sqrt{\alpha}x\right)\partial_x}{\sqrt{\alpha}},$$

$$X_5 = u\sin\left(\sqrt{\alpha}x\right)\partial_u - \frac{\cos\left(\sqrt{\alpha}x\right)\partial_x}{\sqrt{\alpha}}. \tag{3.8}$$

The Lie brackets of (3.8) are given in Table 3.3. The one-parameter groups for this

Table 3.3: Lie brackets of (3.8)

[,]	X_1	X_2	$\bar{X}_3$	X_4	X_5
X_1	0	0	0	$-\sqrt{\alpha}X_5$	$\sqrt{\alpha}X_4$
X_2	0	0	$-3X_2$	0	0
$\bar{X}_3$	0	$3X_2$	0	0	0
X_4	$\sqrt{\alpha}X_5$	0	0	0	$\frac{X_1}{\sqrt{\alpha}}$
X_5	$-\sqrt{\alpha}X_4$	0	0	$-\frac{X_1}{\sqrt{\alpha}}$	0

case are

$$\bar{K}_3 : (x,t,u) \mapsto (x, e^{-3\varepsilon}t, e^{\varepsilon}u), \tag{3.9}$$

$$K_4 : (x,t,u) \mapsto \left(\frac{2\cot^{-1}\left(e^{-\varepsilon}\cot\left(\frac{\sqrt{\alpha}x}{2}\right)\right)}{\sqrt{\alpha}}, t, \frac{e^{\varepsilon}u\left(\cot^2\left(\frac{\sqrt{\alpha}x}{2}\right)+1\right)}{e^{2\varepsilon}+\cot^2\left(\frac{\sqrt{\alpha}x}{2}\right)}\right), \tag{3.10}$$

$$K_5 : (x,t,u) \mapsto \Big(\frac{\sin^{-1}\left(\tanh\left(\varepsilon - \tanh^{-1}\left(\sin\left(\sqrt{\alpha}x\right)\right)\right)\right)}{-\sqrt{\alpha}}, t, \frac{u\operatorname{sech}\left(\varepsilon - \tanh^{-1}\left(\sin\left(\sqrt{\alpha}x\right)\right)\right)}{\sqrt{1-\sin^2\left(\sqrt{\alpha}x\right)}}\Big). \tag{3.11}$$

The adjoints are given in Table 3.4, and for this equation, the optimal system of

Table 3.4: Adjoint Representation of Subalgebras of (3.8)

Ad	X_1	X_2	$\bar{X}_3$	X_4	X_5
X_1	X_1	X_2	$\bar{X}_3$	$X_5\sin(\sqrt{\alpha}\varepsilon)$ $+X_4\cos(\sqrt{\alpha}\varepsilon)$	$X_5\cos(\sqrt{\alpha}\varepsilon)$ $-X_4\sin(\sqrt{\alpha}\varepsilon)$
X_2	X_1	X_2	$3\varepsilon X_2+\bar{X}_3$	X_4	X_5
$\bar{X}_3$	X_1	$e^{-3\varepsilon}X_2$	$\bar{X}_3$	X_4	X_5
X_4	$X_5\left(\frac{1}{2}\sqrt{\alpha}e^{-\varepsilon}-\frac{\sqrt{\alpha}e^{\varepsilon}}{2}\right)$ $+\left(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2}\right)X_1$	X_2	$\bar{X}_3$	X_4	$X_1\left(\frac{e^{-\varepsilon}}{2\sqrt{\alpha}}-\frac{e^{\varepsilon}}{2\sqrt{\alpha}}\right)$ $+\left(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2}\right)X_5$
X_5	$X_4\left(\frac{\sqrt{\alpha}e^{\varepsilon}}{2}-\frac{1}{2}\sqrt{\alpha}e^{-\varepsilon}\right)$ $+\left(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2}\right)X_1$	X_2	$\bar{X}_3$	$X_1\left(\frac{e^{\varepsilon}}{2\sqrt{\alpha}}-\frac{e^{-\varepsilon}}{2\sqrt{\alpha}}\right)$ $+\left(\frac{e^{-\varepsilon}}{2}+\frac{e^{\varepsilon}}{2}\right)X_4$	X_5

one-dimensional subalgebras is found as follows. Here we consider

$$X = a_1X_1 + a_2X_2 + a_3\bar{X}_3 + a_4X_4 + a_5X_5, \tag{3.12}$$

where the a_i's are arbitrary constants, and we simplify this by the applications of adjoint maps. Suppose firstly that $a_3 \neq 0$, then we can assume that $a_3 = 1$. If we act on X by adjoint maps of X_1, X_2, X_4 and X_5 respectively, with appropriate values of a_i's, we get

$$\bar{X}_3 + aX_1.$$

If $a_4 = 1$, $a_3 = 0$, then acting on X by adjoint maps of X_4, X_5 respectively, with suitable values of a_i's results in

$$X_4, aX_4 + X_2.$$

If $a_5 = 1$, $a_4 = a_3 = 0$, then further acting on X by an adjoint map of $\bar{X}_3$ with appropriate values of a_i gives rise to

$$X_5 + bX_1,\ X_5 + bX_1 + \delta X_2.$$

Now if $a_1 = 1$, $a_5 = a_4 = a_3 = 0$ and acting on X by adjoint maps of X_3 with appropriate values of a_i gives rise to

$$X_2 + \epsilon X_1.$$

If $a_2 = 1$, $a_1 = a_5 = a_4 = a_3 = 0$, we get X_2. Therefore, for this equation, the optimal system of one-dimensional subalgebras is given by ($a(\neq 0)$, b are constants, $\delta = \pm 1$ and $\epsilon = 0, \pm 1$),

$$X_2,\ X_4,\ \bar{X}_3 + aX_1,\ X_2 + aX_4,\ X_5 + bX_1,\ X_5 + bX_1 + \delta X_2,\ X_2 + \epsilon X_1.$$

3.3 Conservation Laws and Components

In this section, we derive the conservation laws associated with Eqs. (3.1)-(3.3).

Eq. (3.1) admits two conservation laws, corresponding to the multipliers:

$$\Lambda^1(x, t, u, u_x, u_{xx}) = -\frac{1}{2}\frac{-2u_{xx}x + u_x}{u_x^3},$$

with

$$\begin{aligned} T_t^1 &= -\frac{1}{2}\frac{x}{u_x}, \\ T_x^1 &= \frac{1}{2}\frac{-2\alpha u_x^3 x - u_x^2 u_{xx}^2 x + 3\alpha u u_x^2 + u_x^3 u_{xx} - u_t x}{u_x^2}, \end{aligned} \tag{3.13}$$

and

$$\Lambda^2(x, t, u, u_x, u_{xx}) = \frac{u_{xx}}{u_x^3},$$

$$T_t^2 = -\frac{1}{2u_x},$$

$$T_x^2 = -\frac{1}{2}\frac{2\alpha u_x^3 + u_x^2 u_{xx}^2 + u_t}{u_x^2}. \tag{3.14}$$

Eq. (3.2) admits four conservation laws, viz.

$$\Lambda^3(x,t,u,u_x,u_{xx}) = \frac{1}{u^{\frac{3}{2}}},$$

admits

$$T_t^3 = -\frac{2}{\sqrt{u}},$$

$$T_x^3 = -u^{3/2}u_{xx} - \frac{2}{3}\alpha u^{3/2}. \tag{3.15}$$

$$\Lambda^4(x,t,u,u_x,u_{xx}) = \frac{-2}{u^2},$$

admits

$$T_t^4 = \frac{2}{u},$$

$$T_x^4 = 2uu_{xx} + \frac{1}{2}u_x^2 - +2\alpha u. \tag{3.16}$$

$$\Lambda^5(x,t,u,u_x,u_{xx}) = \frac{2}{3}\alpha + u_{xx},$$

admits

$$T_t^5 = \frac{2}{3}\,\alpha\,u - \frac{1}{2}\,{u_x}^2,$$

$$T_x^5 = u_t\,u_x - \frac{2}{3}\,\alpha\,u^3 u_{xx} - \frac{2}{9}\,\alpha^2 u^3 - \frac{1}{2}\,u^3{u_{xx}}^2. \tag{3.17}$$

$$\Lambda^6(x,t,u,u_x,u_{xx}) = \frac{2\,\alpha\,tu^2 + 3\,tu^2 u_{xx} + 2\,x}{u^2},$$

admits

$$T_t^6 = \tfrac{1}{2}\,\tfrac{4\,\alpha\,tu^2 - 3\,tuu_x{}^2 - 4\,x}{u},$$

$$T_x^6 = -\tfrac{2}{3}\,\alpha^2 tu^3 - \tfrac{3}{2}\,tu^3 u_{xx}{}^2 - 2\,\alpha\,ux - 2\,uu_{xx}\,x - \tfrac{1}{2}\,xu_x{}^2 + 2\,uu_x + 3\,tu_t\,u_x - 2\,\alpha\,tu^3 u_{xx}. \tag{3.18}$$

Eq. (3.3) admits five conservation laws, viz.

$$\Lambda^7(x,t,u,u_x,u_{xx}) = \frac{1}{u^2},$$

provides the conserved quantities

$$\begin{aligned} T_t^7 &= -\frac{1}{u}, \\ T_x^7 &= \frac{1}{2}u_x^2 + \frac{1}{2}\alpha u^2 - uu_{xx}. \end{aligned} \tag{3.19}$$

$$\Lambda^8(x,t,u,u_x,u_{xx}) = \frac{1}{u^3},$$

provides the conserved quantities

$$\begin{aligned} T_t^8 &= -\frac{1}{2u^2}, \\ T_x^8 &= -\alpha u - u_{xx}. \end{aligned} \tag{3.20}$$

$$\Lambda^9(x,t,u,u_x,u_{xx}) = \frac{\sin(\sqrt{\alpha}x)}{u^3},$$

gives

$$\begin{aligned} T_t^9 &= -\frac{1}{2}\frac{\sin(\sqrt{\alpha}x)}{u^2}, \\ T_x^9 &= \cos(\sqrt{\alpha}x)u_x\sqrt{\alpha} - \sin(\sqrt{\alpha}x)u_{xx}, \end{aligned} \tag{3.21}$$

$$\Lambda^{10}(x,t,u,u_x,u_{xx}) = \frac{\cos(\sqrt{\alpha}x)}{u^3},$$

admits the components

$$\begin{aligned} T_t^{10} &= -\frac{1}{2}\frac{\cos(\sqrt{\alpha}x)}{u^2}, \\ T_x^{10} &= -\sin(\sqrt{\alpha}x)u_x\sqrt{\alpha} - \cos(\sqrt{\alpha}x)u_{xx}. \end{aligned} \tag{3.22}$$

$$\Lambda^{11}(x,t,u,u_x,u_{xx}) = \frac{1}{2}\,\frac{\alpha\,u^2 + 2\,uu_{xx} - {u_x}^2}{u^2},$$

admits the components

$$\begin{aligned} T_t^{11} &= \frac{1}{2}\,\frac{\alpha\,u^2 - {u_x}^2}{u}, \\ T_x^{11} &= -\frac{1}{8}\,\frac{\alpha^2u^5 + 4\,\alpha\,u^4u_{xx} - 2\,\alpha\,u^3{u_x}^2 + 4\,u^3{u_{xx}}^2 - 4\,u^2{u_x}^2u_{xx} + uu_x{}^4 - 8\,u_t\,u_x}{u}. \end{aligned} \tag{3.23}$$

The above conserved quantities may be expressed using the notation in [1], as follows. Let $\bar{\epsilon}$ denote the space of solutions of a given model, and consider the spatial domain $\Omega(-\infty,\infty)$, then the integration of a non-trivial conservation law over Ω produces a conserved integral

$$C = \int_\Omega T_t\ dx\ |_{\bar{\epsilon}},$$

satisfying

$$\frac{dC}{dt} = -T_x|_{\partial\Omega}\ \Big|_{\bar{\epsilon}}\ .$$

That is, we may denote the conserved integrals as

$$C^k = \int_{-\infty}^{\infty} T_t^k\ dx,$$

where $k = 1, \ldots, 11$. We remark that the physical interpretation of these conservation laws would be more elegant under a Hamiltonian formulation, and while these equations may not all possess an apparent Hamiltonian function, we shall present an approach to solve this problem elsewhere in the near future.

3.4 Power Law Solutions

In this section, we draw on the optimal system of one-dimensional subalgebras to reduce Eqs. (3.1)-(3.3) to ordinary differential equations. Some of the subsequent ODEs are highly nonlinear and may be solved using the power series method [2, 25]. Reducing Eqs. (3.1)-(3.3) by the generator X_1 gives

$$y'(t) = 0,$$

where $u(x,t) = y(t)$. Solving gives the obvious solution

$$y(t) = c_0,$$

where c_0 is an arbitrary constant. Hence, each of the three equations possess the constant solution

$$u(x,t) = c_0.$$

3.4.1 Series solutions for Eq. (3.1)

Reducing the equation by $X_2 + \epsilon X_1$ gives

$$\epsilon^4 - 3p(s)^2(\alpha\epsilon^3 + p'''(s)) = 0, \tag{3.24}$$

where $s = \frac{-x+t\epsilon}{\epsilon}$ and $u(x,t) = p(s)$. In search of series solutions, a substitution of the power series

$$p(s) = \sum_{r_1=0}^{\infty} a_{r_1} s^{r_1}, \tag{3.25}$$

is done for the reduced equation. Hence, we find the series (we set $\epsilon = 1$ for simplicity)

$$\begin{aligned} p(s) \;=\; & a_0 + a_1 s + a_2 s^2 + \frac{s^3\left(3\alpha a_0^2+1\right)}{18a_0^2} + \frac{s^4\left(\alpha a_1 - \frac{a_1\left(3\alpha a_0^2+1\right)}{3a_0^2}\right)}{12a_0} + \\ & \frac{s^5\left(\alpha a_1^2 - \frac{a_1^2\left(3\alpha a_0^2+1\right)}{3a_0^2} - 4a_1\left(\alpha a_1 - \frac{a_1\left(3\alpha a_0^2+1\right)}{3a_0^2}\right) + 2\alpha a_0 a_2 - \frac{2a_2\left(3\alpha a_0^2+1\right)}{3a_0}\right)}{60a_0^2} \\ & + \ldots . \end{aligned} \tag{3.26}$$

Reducing Eq. (3.1) by X_3 gives the ODE

$$\begin{aligned} & 9y(z)^3\left(\alpha + 525z^2y''(z) + 125z^3y'''(z) + 340zy'(z)\right) \\ & +15zy(z)^2y'(z)\left(\alpha + 525z^2y''(z) + 125z^3y'''(z) + 330zy'(z)\right) \\ & -y'(z) + 54y(z)^4 = 0, \end{aligned} \tag{3.27}$$

where $z = tx^5$ and $u(x,t) = x^3y(z)$.

A substitution of the power series

$$y(z) = \sum_{r_1=0}^{\infty} a_{r_1} z^{r_1}, \tag{3.28}$$

into (3.27), provides the power series solution

$$\begin{aligned}
y(z) &= a_0 + 9z\left(\alpha a_0^3 + 6a_0^4\right) + 21z^2\left(702a_0^3\left(\alpha a_0^3 + 6a_0^4\right) + 9\alpha a_0^2\left(\alpha a_0^3 + 6a_0^4\right)\right) \\
&\quad + z^3\Bigg(110502a_0^3\left(702a_0^3\left(\alpha a_0^3 + 6a_0^4\right) + 9\alpha a_0^2\left(\alpha a_0^3 + 6a_0^4\right)\right) \\
&\quad + 390258a_0^2\left(\alpha a_0^3 + 6a_0^4\right)^2 \\
&\quad + 399\alpha a_0^2\left(702a_0^3\left(\alpha a_0^3 + 6a_0^4\right) + 9\alpha a_0^2\left(\alpha a_0^3 + 6a_0^4\right)\right) \\
&\quad + 1539\alpha a_0\left(\alpha a_0^3 + 6a_0^4\right)^2\Bigg) + \ldots. \qquad (3.29)
\end{aligned}$$

3.4.2 Series solutions for Eq. (3.2)

Reducing the equation by $X_2 + \epsilon X_1$ gives

$$p'(s)(2\epsilon^3 + p(s)^2(2\alpha\epsilon^2 + 3p''(s))) + 2p(s)^3p'''(s) = 0, \qquad (3.30)$$

where $s = \frac{-x+t\epsilon}{\epsilon}$ and $u(x,t) = p(s)$. Similar to above, the series solution for (4.13) is

$$\begin{aligned}
p(s) &= a_0 + a_1 s + a_2 s^2 + \frac{s^3\left(-\alpha a_1 a_0^2 - 3a_1 a_2 a_0^2 - a_1\right)}{6a_0^3} \\
&\quad \frac{s^4\left(-2\alpha a_2 a_0^2 - 2\alpha a_1^2 a_0 - \frac{9a_1\left(-\alpha a_1 a_0^2 - 3a_1 a_2 a_0^2 - a_1\right)}{2a_0} - 6a_2^2 a_0^2 - 6a_1^2 a_2 a_0 - 2a_2\right)}{24a_0^3} + \ldots. \qquad (3.31)
\end{aligned}$$

Reducing the equation by X_3 gives

$$\begin{aligned}
&\Bigg(12y(z)^4 - 2y'(z) + 3zy(z)^2y'(z)\Big(2\alpha + 54zy'(z) + 27z^2y''(z)\Big) + y(z)^3 \\
&\qquad \Big(4\alpha + 246zy'(z) + 270z^2y''(z) + 54z^3y'''(z)\Big)\Bigg) = 0, \qquad (3.32)
\end{aligned}$$

where $z = tx^3$ and $u(x,t) = x^2 y(z)$. In this case, we have the series solution

$$\begin{aligned} y(z) &= a_0 + 2z\left(\alpha a_0^3 + 3a_0^4\right) + \frac{3}{2}z^2\left(98a_0^3\left(\alpha a_0^3 + 3a_0^4\right) + 6\alpha a_0^2\left(\alpha a_0^3 + 3a_0^4\right)\right) \\ &\quad + 2z^3\Bigg(135a_0^3\left(98a_0^3\left(\alpha a_0^3 + 3a_0^4\right) + 6\alpha a_0^2\left(\alpha a_0^3 + 3a_0^4\right)\right) + 324a_0^2\left(\alpha a_0^3 + 3a_0^4\right)^2 \\ &\quad + 3\alpha a_0^2\left(98a_0^3\left(\alpha a_0^3 + 3a_0^4\right) + 6\alpha a_0^2\left(\alpha a_0^3 + 3a_0^4\right)\right) + 8\alpha a_0\left(\alpha a_0^3 + 3a_0^4\right)^2\Bigg) + \ldots. \end{aligned} \tag{3.33}$$

3.4.3 Series solutions for Eq. (3.3)

A reduction of the equation by the symmetry combination $X_2 + \epsilon X_1$ gives

$$\epsilon^2\Big(\epsilon + \alpha p(s)^3\Big)p'(s) + p(s)^3 p'''(s) = 0, \tag{3.34}$$

where $s = \frac{-x+t\epsilon}{\epsilon}$ and $u(x,t) = p(s)$. Thus, the series solution is

$$\begin{aligned} p(s) = a_0 + a_1 s + a_2 s^2 + \frac{s^3\left(-\alpha a_1 a_0^3 - a_1\right)}{6a_0^3} \\ + \frac{s^4\left(-2\alpha a_2 a_0^3 - 3\alpha a_1^2 a_0^2 - \frac{3a_1\left(-\alpha a_1 a_0^3 - a_1\right)}{a_0} - 2a_2\right)}{24a_0^3} + \ldots \end{aligned} \tag{3.35}$$

A reduction by X_4 yields $y'(t) = 0$, where

$$u(x,t) = \sin\left(\sqrt{\alpha}x\right) y(t).$$

Hence a solution is

$$u(x,t) = \sin\left(\sqrt{\alpha}x\right) c_1.$$

A reduction by $\bar{X}_3 + aX_1$ yields

$$-3zy(z)^3\left(\left(\alpha a^2 + 21\right)y'(z) + 9z\left(4y''(z) + zy^{(3)}(z)\right)\right) + a^3 y'(z) - \left(\alpha a^2 + 1\right)y(z)^4 = 0, \tag{3.36}$$

where $u(x,t) = e^{x/a} y\left(z\right)$ and $z = te^{\frac{3x}{a}}$, having power series

$$y(z) = a_0 + \frac{a_0^4 z\left(\alpha a^2+1\right)}{a^3} + \frac{z^2\left(\frac{7\alpha a_0^7\left(\alpha a^2+1\right)}{a} + \frac{67a_0^7\left(\alpha a^2+1\right)}{a^3}\right)}{2a^3} +$$
$$\frac{z^3}{3a^3}\left(\frac{15\alpha a_0^{10}\left(\alpha a^2+1\right)^2}{a^4} + \frac{195a_0^{10}\left(\alpha a^2+1\right)^2}{a^6} + \frac{5\alpha a_0^3\left(\frac{7\alpha a_0^7\left(\alpha a^2+1\right)}{a} + \frac{67a_0^7\left(\alpha a^2+1\right)}{a^3}\right)}{a}\right.$$
$$\left. + \frac{173a_0^3\left(\frac{7\alpha a_0^7\left(\alpha a^2+1\right)}{a} + \frac{67a_0^7\left(\alpha a^2+1\right)}{a^3}\right)}{a^3}\right) + \ldots \quad (3.37)$$

A reduction by $X_2 + aX_4$ yields

$$a^2 y'(z)\left(a - \alpha^{3/2} y(z)^3\right) + \alpha^{3/2} y(z)^3 y^{(3)}(z) = 0, \quad (3.38)$$

where

$$z = \frac{at - \log\left(\sin\left(\frac{\sqrt{\alpha}x}{2}\right)\right) + \log\left(\cos\left(\frac{\sqrt{\alpha}x}{2}\right)\right)}{a}$$

and $u(x,t) = \sin\left(\sqrt{\alpha}x\right) y(z)$. The corresponding power series is

$$\begin{aligned} y(z) \;&=\; a_0 + a_1 z + a_2 z^2 + \frac{z^3\left(a^2\alpha^{3/2}a_0^3 a_1 - a^3 a_1\right)}{6\alpha^{3/2}a_0^3} + \\ &\quad \frac{z^4\left(3a^2a_0^2a_1^2\alpha^{3/2} + 2a^2a_0^3a_2\alpha^{3/2} - \frac{3a_1\left(a^2\alpha^{3/2}a_0^3a_1 - a^3a_1\right)}{a_0} - 2a^3a_2\right)}{24\alpha^{3/2}a_0^3} + \ldots. \end{aligned} \quad (3.39)$$

For this case, X_2 produces a reduced equation $y^{(3)}(x) + \alpha y'(x) = 0$, where $u(x,t) = y(x)$ and we solve to find

$$y(x) = -\frac{C_2\cos\left(\sqrt{\alpha}x\right)}{\sqrt{\alpha}} + \frac{C_1\sin\left(\sqrt{\alpha}x\right)}{\sqrt{\alpha}} + C_3,$$

with C_1, C_2, C_3 as arbitrary. A reduction by $X_5 + bX_1$ gives us

$$u(x,t) = \left(\sqrt{\alpha}b - \cos\left(\sqrt{\alpha}x\right)\right) y(t),$$

where $y'(t) = 0$. Hence the invariant solution is

$$u(x,t) = \left(\sqrt{\alpha}b - \cos\left(\sqrt{\alpha}x\right)\right) c_0,$$

where c_0 is an arbitrary constant as before. Finally, a reduction by $X_5 + bX_1 + \delta X_2$ yields the equation

$$\alpha^{3/2}\delta\left(\alpha b^2 - 1\right) y(z)^3 + 1\big) y'(z) + \alpha^{3/2}\delta^3 y(z)^3 y^{(3)}(z) = 0,$$

where $u(x,t) = (\sqrt{\alpha}b - \cos(\sqrt{\alpha}x))\, y(z)$ and

$$z = \frac{2\delta \tanh^{-1}\left(\frac{\left(\sqrt{\alpha}b+1\right)\tan\left(\frac{\sqrt{\alpha}x}{2}\right)}{\sqrt{1-\alpha b^2}}\right)}{\sqrt{1-\alpha b^2}} + t.$$

A solution to the reduced equation via power series is

$$\begin{aligned} y(z) = a_0 + a_1 z + a_2 z^2 + \frac{z^3\left(a_0^3 a_1 \alpha^{3/2}\delta - a_0^3 a_1 \alpha^{5/2} b^2 \delta - a_1\right)}{6\alpha^{3/2} a_0^3 \delta^3} \\ + \frac{z^4}{24\alpha^{3/2} a_0^3 \delta^3}\Big(3a_0^2 a_1^2 \alpha^{3/2}\delta + 2a_0^3 a_2 \alpha^{3/2}\delta - 3a_0^2 a_1^2 \alpha^{5/2} b^2 \delta \\ -2a_0^3 a_2 \alpha^{5/2} b^2 \delta - \frac{3a_1(a_0^3 a_1 \alpha^{3/2}\delta - a_0^3 a_1 \alpha^{5/2} b^2 \delta - a_1)}{a_0} - 2a_2\Big) + \ldots \end{aligned} \tag{3.40}$$

3.4.4 Testing for Convergence

Each of the series solutions given above may be expressed as a recurrence relation. We illustrate this below for Eq. (3.24) with the series (3.25). Eq. (3.24) can be expressed as

$$1+3\alpha\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}a_{r_1-r_2}a_{r_2}s^{r_1}$$

$$-3\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)\,a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}s^{r_1}=0. \tag{3.41}$$

From (3.41) comparing coefficients of the power of s, we find that a_0, a_1, a_2 are arbitrary,

$$a_3=\frac{(3\alpha a_0^2+1)}{18a_0^2}, \tag{3.42}$$

and in general for $r_1 \geq 1$, we have

$$\begin{aligned} a_{r_1+3} &= \frac{-\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}}{(r_1+3)(r_1+2)(r_1+1)a_0^2} \\ &+ \frac{\alpha\sum_{r_2=0}^{r_1}a_{r_1-r_2}a_{r_2}}{(r_1+3)(r_1+2)(r_1+1)a_0^2}. \end{aligned} \tag{3.43}$$

Therefore, from (3.43), we can obtain the constants in the power series solution. Next we prove the convergence of the power series solution (3.25). From (3.43) we obtain

$$|a_{r_1+3}|\leq M\left(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}|a_{r_1-r_2+3}||a_{r_2-r_3}||a_{r_3}|+\sum_{r_2=0}^{r_1}|a_{r_1-r_2}||a_{r_2}|\right), \tag{3.44}$$

where $r_1 = 1, 2, 3...$, and $M = max\{\frac{1}{a_0^2}, \frac{|\alpha|}{a_0^2}\}$. Suppose we have the power series $\mu = R(s) = \Sigma_{r_1=0}^{\infty} p_{r_1} s^{r_1}$ where

$$p_k=|a_k| \;\; k=0,1,\ldots,3 \tag{3.45}$$

and

$$p_{r_1+3} = M\left(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3} + \sum_{r_2=0}^{r_1} p_{r_1-r_2}p_{r_2}\right). \tag{3.46}$$

Hence $|a_{r_1}| \leq p_{r_1},\ r_1 = 0, 1, 2...$ Next we prove that μ is convergent in a neighbourhood of a point, and μ is a majorant series of equation (3.25), hence:

$$\begin{aligned}
R(s) &= p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \sum_{r_1=1}^{\infty} p_{r_1+3}s^{r_1+3} \\
&= p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \\
&\quad M\sum_{r_1=1}^{\infty}\left(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3} + \sum_{r_2=0}^{r_1} p_{r_1-r_2}p_{r_2}\right)s^{r_1+3} \\
&= p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \\
&\quad M\left(\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3}s^{r_1+3} + \sum_{r_1=1}^{\infty}\sum_{r_2=0}^{r_1} p_{r_1-r_2}p_{r_2}s^{r_1+3}\right) \\
&= p_0 + p_1 s + p_2 s^2 + p_3 s^3 \\
&\quad + M[R^3(s) + R^2(s^3 - p_1 s - p_2 s^2) + \nu(s)],
\end{aligned} \tag{3.47}$$

where $\nu(s)$ is a polynomial with respect to s, that is

$$\nu(s) = -2p_0p_1p_2s^3 - 3p_0^2p_3s^3 - p_0p_1^2s^2 - 2p_0^2p_2s^2 - p_0^2s^3 - 2p_0^2p_1s - p_0^3.$$

Let

$$\begin{aligned}
F(s,\mu) &= \mu - p_0 - p_1 s - p_2 s^2 - p_3 s^3 \\
&\quad - M[R^3(s) + R^2(s^3 - p_1 s - p_2 s^2) + \nu(s)],
\end{aligned} \tag{3.48}$$

be the implicit function equation, where we obtain that $F(0, p_0) = 0$ and $F_\mu(0, r_0) = 1 - 3Mp_0^2 \neq 0$. By the implicit function theorem [37], $\mu = R(s)$ is analytic and convergent in a neighbourhood of the point $(0, p_0)$ in the plane and with a positive radius, then the power series solution (3.25) is convergent in the neighbourhood of a point $(0, p_0)$. Therefore (3.25) can be written as

$$p(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \Sigma_{r_1=1}^{\infty} a_{r_1+3} s^{r_1+3}$$

$$= a_0 + a_1 s + a_2 s^2 + a_3 s^3 +$$

$$\Sigma_{r_1=1}^{\infty}\left(\frac{-\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}}{(r_1+3)(r_1+2)(r_1+1)a_0^2}\right.$$

$$\left.+\frac{\alpha\sum_{r_2=0}^{r_1} a_{r_1-r_2}a_{r_2}}{(r_1+3)}(r_1+2)(r_1+1)a_0^2\right)s^{r_1+3}. \quad (3.49)$$

Thus the exact power series solution of equation (3.1) can be written as

$$u(x,t) = a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^3 + \Sigma_{r_1=1}^{\infty} a_{r_1+3}(t-x)^{r_1+3}$$

$$= a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^4 +$$

$$\Sigma_{r_1=1}^{\infty}\left(\frac{-\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}}{(r_1+3)(r_1+2)(r_1+1)a_0^2}\right.$$

$$\left.+\frac{\alpha\sum_{r_2=0}^{r_1} a_{r_1-r_2}a_{r_2}}{(r_1+3)(r_1+2)(r_1+1)a_0^2}\right)(t-x)^{r_1+3}, \quad (3.50)$$

where $a_0 \neq 0, a_1, a_2,$ are arbitrary constants, a_3 is given in (3.42) and the rest of the constants may be determined by (3.43). This produces the solution (3.26).

3.5 Graphical Results

In this section, we plot truncated solutions for some of the results found above. In Figure 3.1, we have graphical solutions for the ODEs from reductions of Eq.

(3.1) and the resulting graphical solutions for Eq. (3.1) itself. The solution (3.26) corresponds to a travelling wave reduction and we observe that the solution (3.29) grows much larger than (3.26). The graphical solutions for Eq. (3.2) are quite similar to those of Eq. (3.1). In Figures 3.2 and 3.3 , we have graphical solutions for the ODEs from reductions of Eq. (3.3) and the resulting graphical solutions for Eq. (3.3) itself. The truncated solution of (3.35) is positive at small values of the independent variable but becomes negative at higher values while the solution for (3.37) grows the fastest. We have shown that (3.26) converges but, similarly, and very tediously, we can test for convergence to all reduced equations in this chapter. The solutions can then be transformed back into original variables, given the invertible transformations stated for each reduction.

3.6 Discussion and Conclusion

In this study, we have compiled the reversible transformations that render a class of polynomial equations invariant. The reduced equations are difficult to solve exactly using most techniques, but with power series, solutions are obtainable. In fact, we observe that a power series is effective in solving ODEs admitted by the Lie symmetry reduction approach. The convergence of such series may be tested algorithmically in a manner shown above. The optimal system of subalgebras and conserved quantities were also derived to gain further insights into the family of equations.

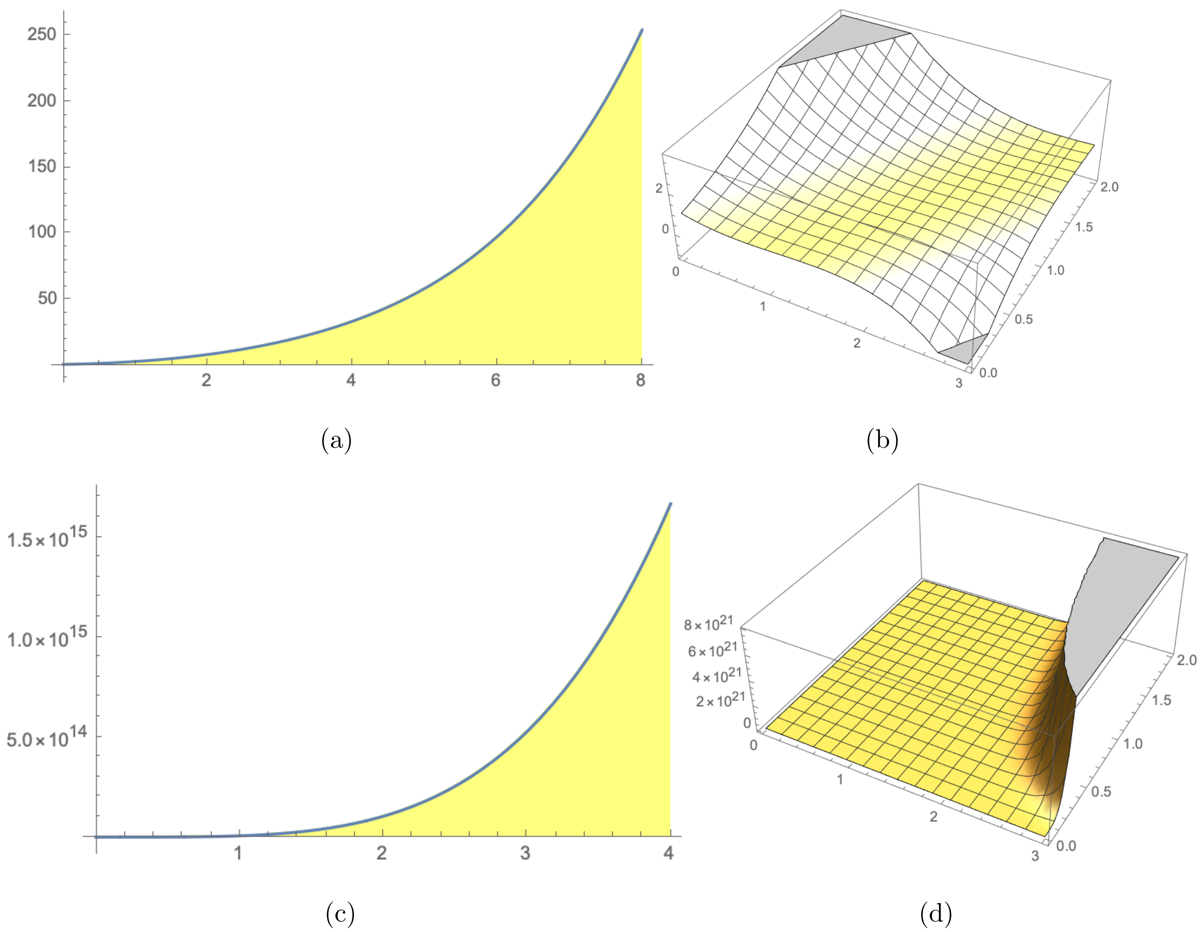


Figure 3.1: Graphical illustration of the analytical solutions are depicted, we select the parameter values $\alpha = a_k = 1$: (a) Plot of Eq. (3.26); (b) 3D plot of Eq. (3.26); (a) Plot of Eq. (3.29); and (b) 3D plot of Eq. (3.29).

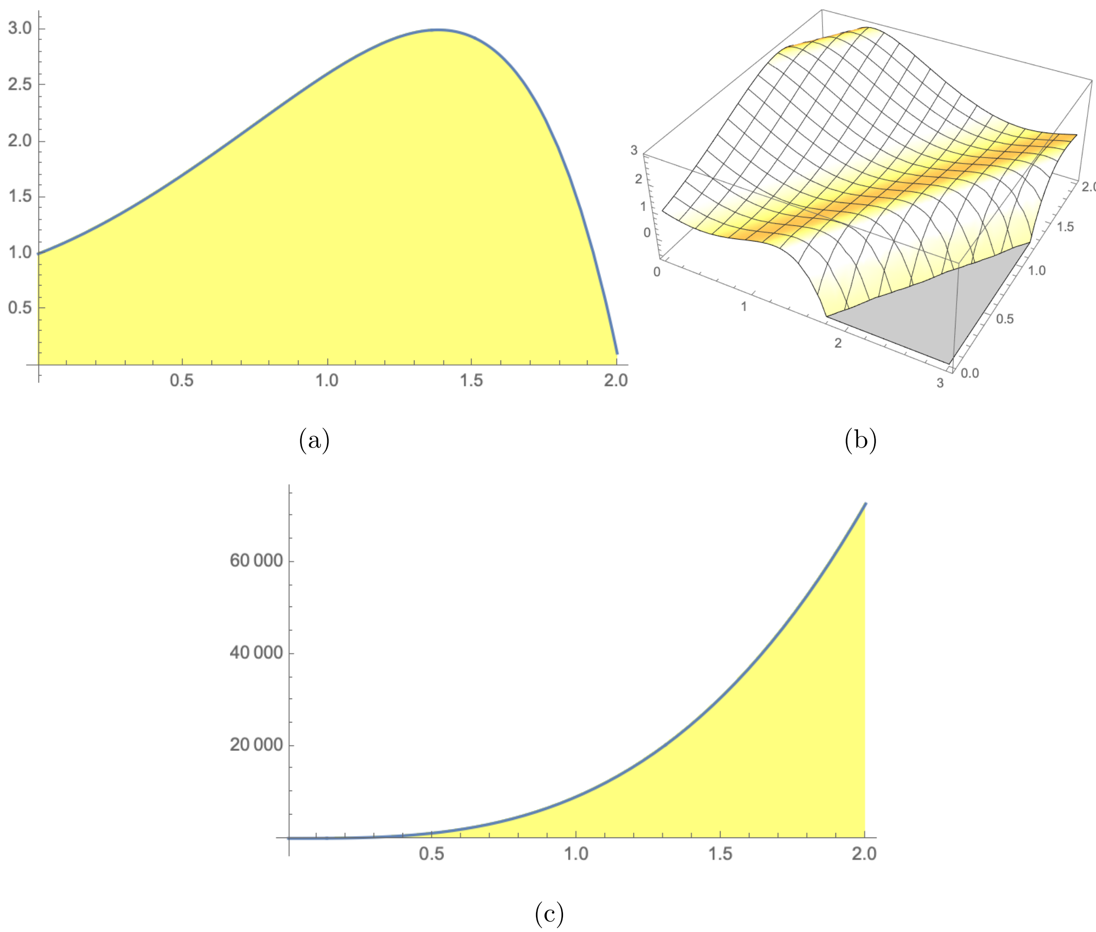
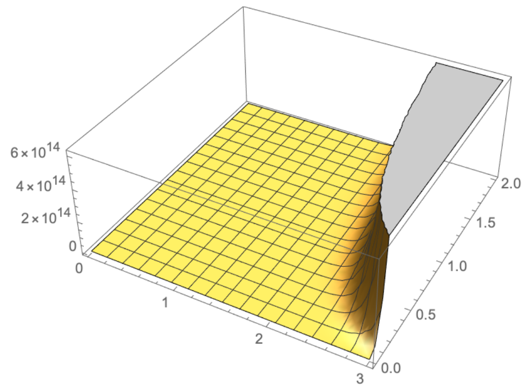
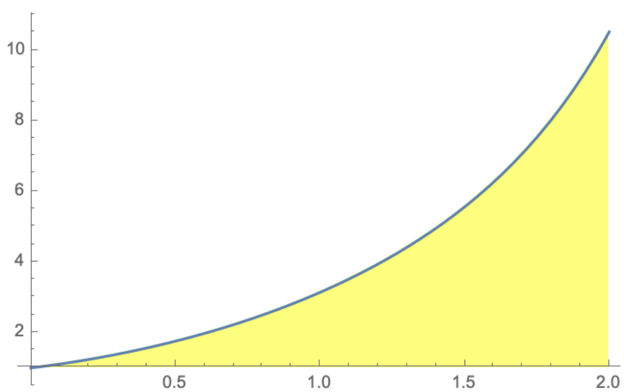


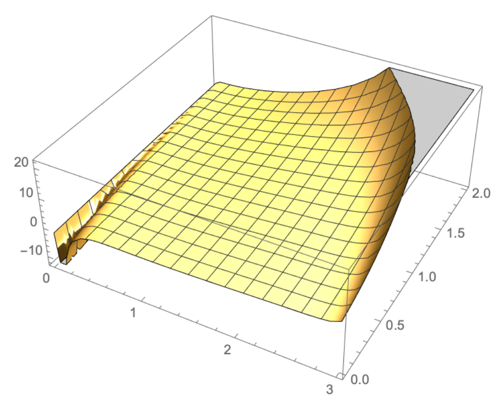
Figure 3.2: Graphical illustration of the analytical solutions, we select the parameter values $\alpha = a_k = 1$: (a) Plot of Eq. (3.35); (b) 3D plot of Eq. (3.35); (a) Plot of Eq. (3.37).



(a)



(b)



(c)

Figure 3.3: Graphical illustration of the analytical solutions, we select the parameter values $\alpha = a_k = 1$: (b) 3D plot of Eq. (3.37); (b) Plot of Eq. (3.39); and (c) 3D plot of Eq. (3.39).

Chapter 4

Fujimoto-Watanabe's four third-order polynomial evolution equations

4.1 Introduction

Many of the Fujimoto-Watanabe equations [9], possess a recursion operator. The four equations of interest in this chapter are:

$$u_t = u_x^3 u_{xxx} + \alpha u_x^4, \tag{4.1}$$

$$u_t = u^3 u_{xxx} + 3u^2 u_x u_{xx} + \alpha(u^3 u_{xx} + u^2 u_x^2) + \frac{2}{9}\alpha^2 u^3 u_x, \tag{4.2}$$

$$u_t = u^3 u_{xxx} + 3u^2 u_x u_{xx} + 4\alpha u^3 u_x, \tag{4.3}$$

$$u_t = u^3 u_{xxx} + \frac{3}{2}u^2 u_x u_{xx} + \alpha(u^3 u_{xx} + u^2 u_x^2) + \frac{2}{9}\alpha^2 u^3 u_x, \tag{4.4}$$

where $\alpha \neq 0$ is an arbitrary constant.

In the literature, some Fujimoto-Watanabe equations have been studied , but not necessarily the ones listed above. Liu [26] used a complete discrimination system to obtain the classifications of traveling wave solutions of a Fujimoto-Watanabe equation. By the method of dynamical systems, the bifurcations and dynamics of traveling wave solutions of a Fujimoto-Watanabe equation were analysed [42, 41], and also phase portraits and periodic solutions were described. In [8], a Fujimoto-Watanabe equation (not from above) was studied from the perspective of Lie symmetries.

As previously mentioned, symmetry methods and conservation laws are deeply intertwined in the analysis of PDEs [29, 33, 18]. In this paper we explore how symmetry methods combine with power series to arrive at solutions. The resulting equations obtained through Lie reductions are difficult to solve, and most other techniques will not work.

The plan of the chapter follows. In Section 4.2, we present the symmetry details possessed by the four equations, together with the commutator, adjoint and group properties. Section 4.3, contains the reductions and power series solutions, while the next section demonstrates the convergence of a solution. The conserved vector components are given in Section 4.5, and a conclusion follows in Section 4.6.

4.2 Symmetries and Groups

First of all, let us consider a one-parameter Lie group of infinitesimal transformations:

$$x \to x + \varepsilon\xi(x, t, u), \tag{4.5}$$

$$t \to t + \varepsilon\tau(x, t, u), \tag{4.6}$$

$$u \to u + \varepsilon\phi(x, t, u), \tag{4.7}$$

with the infinitesimal $\varepsilon \ll 1$. The vector field associated with the above transformations can be written as

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{4.8}$$

The symmetry group of the equation will be generated by the vector field of the above form. Applying the prolongation to the given equations, we find the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ that must satisfy the Lie symmetry condition.

The Lie point symmetries of Eqs (4.1)-(4.4) are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= -3t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \end{aligned} \tag{4.9}$$

with Lie brackets $[X_i, X_j] = X_iX_j - X_jX_i$, given in Table 4.1. Thus, we have the

Table 4.1: Lie brackets of (4.9)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	$-3X_2$
X_3	0	$3X_2$	0

corresponding one-parameter groups G_i $(i = 1, 2, 3)$ generated by the symmetries X_i of the four equations, as

$$G_1 : (x,t,u) \mapsto (x+\varepsilon, t, u), \tag{4.10}$$

$$G_2 : (x,t,u) \mapsto (x, t+\varepsilon, u), \tag{4.11}$$

$$G_3 : (x, t, u) \mapsto (x, e^{-3\varepsilon}t, e^{\varepsilon}u). \tag{4.12}$$

We can see that G_1 is a space translation, G_2 is a time translation, and G_3 is a scaling transformation.

The adjoints corresponding to (4.9) are given in Table 4.2, where we require the ad-

Table 4.2: Adjoint Representation of Subalgebras (4.9)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$3\varepsilon X_2 + X_3$
X_3	X_1	$e^{-3\varepsilon}X_2$	X_3

joints to calculate the one-dimensional subalgebras. Thus, for the class of equations (4.1)-(4.4), we obtain the optimal system of one-dimensional subalgebras

$$X_1,\ X_3 + aX_1,\ X_2 + \epsilon X_1,$$

where a is a constant and $\epsilon = 0, \pm 1$.

4.3 Solutions via Power Series

In this section, we draw on the optimal system of one-dimensional subalgebras to reduce Eqs. (4.1)-(4.4) to ODEs. Some of the subsequent ODEs are highly nonlinear and may be solved using the power series method.

Reducing Eqs. (4.1)-(4.4) by the generator X_1 gives

$$y'(t) = 0, \tag{4.13}$$

where $u(x,t) = y(t)$. Solving (4.13) gives

$$y(t) = c_1, \tag{4.14}$$

where c_1 is an arbitrary constant. Hence, each of the four equations possess the constant solution $u(x,t) = c_1$.

4.3.1 The Case of Eq. (4.1)

Reducing equation (4.1) by $X_2 + \epsilon X_1$ gives the travelling wave reduction

$$-3p(s)^2 p'''(s) + 4\alpha\epsilon^3 p(s)^3 + \epsilon^4 = 0, \tag{4.15}$$

where $u(x,t) = p(s)$ and $s = \frac{t\epsilon - x}{\epsilon}$.

A reduction of (4.1) by $X_3 + aX_1$ gives

$$\left((3+4a^3\alpha)y(z)^4 - a^4 y'(z) + 81z^2 y(z)^2 y^{(1)}(z)(7y'(z) + 3z(4y''(z) + zy'''(z))\right) + \\ 3zy(z)^3\left((66+4a^3\alpha)y'(z) + 27z(4y''(z) + zy'''(z))\right) = 0, \tag{4.16}$$

where $u(x,t) = e^{\frac{x}{a}} y(z)$ and $z = e^{\frac{3x}{a}} t$.

Let us consider the reduced equation (4.15) with $\epsilon = 1$. A substitution of the power series

$$p(s) = \sum_{r_1=0}^{\infty} a_{r_1} s^{r_1}, \tag{4.17}$$

with a_{r_1} as unknown constants to be determined, into (4.15) provides

$$1+4\alpha\left(a_0^3+3sa_0^2a_1+3s^2(a_0a_1^2+a_0^2a_2)+\sum_{r_1=3}^{\infty}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}a_{r_1-r_2}a_{r_2-r_3}a_{r_3}\right)$$

$$-3\sum_{r_1=3}^{\infty}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)\,a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}=0. \tag{4.18}$$

From (4.18) comparing coefficients of the power of s, we find that a_0, a_1, a_2 are arbitrary,

$$a_3=\frac{4\alpha a_0^3+1}{18a_0^2},\quad a_4=\frac{\alpha a_0a_1-\frac{a_1\left(4\alpha a_0^3+1\right)}{6a_0^2}}{6a_0}, \tag{4.19}$$

and

$$a_5=\frac{2\alpha a_2a_0^2+2\alpha a_1^2a_0-4a_1\left(\alpha a_0a_1-\frac{a_1\left(4\alpha a_0^3+1\right)}{6a_0^2}\right)-\frac{a_2\left(4\alpha a_0^3+1\right)}{3a_0}-\frac{a_1^2\left(4\alpha a_0^3+1\right)}{6a_0^2}}{30a_0^2}. \tag{4.20}$$

In general for $r_1 \geq 3$, we have

$$\begin{aligned} a_{r_1+3} &= \frac{-3\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}}{3(r_1+3)(r_1+2)(r_1+1)a_0^2} \\ &+ \frac{4\alpha\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}a_{r_1-r_2}a_{r_2-r_3}a_{r_3}}{3(r_1+3)(r_1+2)(r_1+1)a_0^2}. \end{aligned} \tag{4.21}$$

Therefore from (4.19) - (4.21), for arbitrary constants $a_0 \neq 0, a_1, a_2$ we obtain the power series solution for equation (4.15), viz.

$$
\begin{aligned}
p(s) \quad = \quad & a_0 + a_1 s + a_2 s^2 + \frac{s^3\left(4\alpha a_0^3+1\right)}{18a_0^2} + \frac{s^4\left(\alpha a_0 a_1 - \frac{a_1\left(4\alpha a_0^3+1\right)}{6a_0^2}\right)}{6a_0} \\
& + \frac{s^5}{30a_0^2}\left(2\alpha a_2 a_0^2 + 2\alpha a_1^2 a_0 - 4a_1\left(\alpha a_0 a_1 - \frac{a_1\left(4\alpha a_0^3+1\right)}{6a_0^2}\right) - \frac{a_2\left(4\alpha a_0^3+1\right)}{3a_0}\right. \\
& \left. - \frac{a_1^2\left(4\alpha a_0^3+1\right)}{6a_0^2}\right) \\
& + \ldots \qquad (4.22)
\end{aligned}
$$

An inversion of the transformation gives the solution to the PDE (4.1), namely

$$
\begin{aligned}
u(x,t) \quad = \quad & a_0 + a_1(t-x) + a_2(t-x)^2 + \frac{(t-x)^3\left(4\alpha a_0^3+1\right)}{18a_0^2} + \\
& \frac{(t-x)^4\left(\alpha a_0 a_1 - \frac{a_1\left(4\alpha a_0^3+1\right)}{6a_0^2}\right)}{6a_0} + \frac{(t-x)^5}{30a_0^2}\left(2\alpha a_2 a_0^2 + 2\alpha a_1^2 a_0\right. \\
& -4a_1\left(\alpha a_0 a_1 - \frac{a_1\left(4\alpha a_0^3+1\right)}{6a_0^2}\right) - \frac{a_2\left(4\alpha a_0^3+1\right)}{3a_0} \\
& \left. - \frac{a_1^2\left(4\alpha a_0^3+1\right)}{6a_0^2}\right) \\
& + \ldots \qquad (4.23)
\end{aligned}
$$

Similarly, and very tediously, we can construct power series solutions to all reduced equations. We omit the details for all other cases. The proof of convergence of the solution is discussed later.

A solution to (4.16) is found from substituting

$$y(z) = \sum_{r_1=0}^{\infty} a_{r_1} z^{r_1},$$

into (4.16) and similar to the above discussion, we get

$$
\begin{aligned}
y(z) \;=\;& a_0 + a_1 z + 7z^2\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) + \\
& \frac{1}{3}z^3\Big(280\alpha a_0^3\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) + 7392a_0^3\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) \\
& +60\alpha a_1^2 a_0^2 + 1179a_1^2 a_0^2\Big)\frac{1}{2}z^4\Big(\frac{26}{3}\alpha a_0^3\big(280\alpha a_0^3\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) \\
& +7392a_0^3\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) + 60\alpha a_1^2 a_0^2 + 1179a_1^2 a_0^2\big) \\
& +506a_0^3\big(280\alpha a_0^3\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) \\
& +7392a_0^3\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) + 60\alpha a_1^2 a_0^2 + 1179a_1^2 a_0^2\big) \\
& +546\alpha a_1 a_0^2\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) \\
& +27909a_1 a_0^2\left(2\alpha a_1 a_0^3 + 15a_1 a_0^3\right) + 26\alpha a_1^3 a_0 + 870a_1^3 a_0\Big) \\
& + \ldots
\end{aligned} \tag{4.24}
$$

4.3.2 The Case of Eq. (4.2)

In the same vein, reducing equation (4.2) by $X_2 + \epsilon X_1$ gives

$$
p'(s)(9\epsilon^3 + 2\alpha(9+\alpha)\epsilon^2 p(s)^3 + 27p(s)^2 p''(s)) + 9p(s)^3(-\alpha\epsilon p''(s) + p'''(s)) = 0, \tag{4.25}
$$

while $X_3 + aX_1$ leads to

$$
\begin{aligned}
-e^{\frac{4x}{a}}\big((36 + 9a\alpha + 2a^2\alpha(9+\alpha)\big)y(z)^4 - a^3 y'(z) \\
+243z^2 y(z)^2 y'(z)(5y'(z) + 3zy''(z)) \\
+3zy(z)^3\Bigg((351 + 45a\alpha + 2a^2\alpha(9+\alpha)y'(z) \\
+27z((15 + a\alpha)y''(z) + 3zy'''(z))\Bigg) = 0.
\end{aligned} \tag{4.26}
$$

For this case, we find the solutions

$$
\begin{aligned}
p(s) \;=\;& a_0 + a_1 s + a_2 s^2 \\
&+\frac{s^3\left(-2\alpha^2 a_1 a_0^3 - 18\alpha a_1 a_0^3 + 18\alpha a_2 a_0^3 - 54 a_1 a_2 a_0^2 - 9a_1\right)}{54 a_0^3} \\
&-\frac{s^4}{108 a_0^3}\Bigg(2\alpha^2 a_2 a_0^3 + 3\alpha^2 a_1^2 a_0^2 - \frac{1}{2}\alpha\Big(- 2\alpha^2 a_1 a_0^3 - 18\alpha a_1 a_0^3 \\
&+18\alpha a_2 a_0^3 - 54 a_1 a_2 a_0^2 - 9a_1\Big) \\
&+\frac{3a_1\left(-2\alpha^2 a_1 a_0^3 - 18\alpha a_1 a_0^3 + 18\alpha a_2 a_0^3 - 54 a_1 a_2 a_0^2 - 9a_1\right)}{a_0} \\
&+18\alpha a_2 a_0^3 + 27\alpha a_1^2 a_0^2 - 27\alpha a_1 a_2 a_0^2 + 54 a_2^2 a_0^2 + 54 a_1^2 a_2 a_0 + 9a_2\Bigg) \\
&+\ldots, \qquad (4.27)
\end{aligned}
$$

and

$$
\begin{aligned}
y(z) \;=\;& a_0 + \frac{1}{9}\left(2\alpha^2 + 27\alpha + 36\right) a_0^4 z + \\
&\frac{1}{18} z^2\Big(\frac{14}{9}\alpha^2\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + 33\alpha\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + \\
&133\Big(2\alpha^2 + 27\alpha + 36\Big) a_0^7\Big) + \frac{1}{27} z^3\Bigg(\frac{10}{27}\alpha^2\left(2\alpha^2 + 27\alpha + 36\right)^2 a_0^{10} + \\
&\frac{170}{3}\left(2\alpha^2 + 27\alpha + 36\right)^2 a_0^{10} + 9\alpha\left(2\alpha^2 + 27\alpha + 36\right)^2 a_0^{10} + \\
&\frac{10}{9}\alpha^2 a_0^3\Big(\frac{14}{9}\alpha^2\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + 33\alpha\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + \\
&133\left(2\alpha^2 + 27\alpha + 36\right) a_0^7\Big) + 36\alpha a_0^3\Big(\frac{14}{9}\alpha^2\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + \\
&33\alpha\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + 133\left(2\alpha^2 + 27\alpha + 36\right) a_0^7\Big) \\
&+260 a_0^3\Big(\frac{14}{9}\alpha^2\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + 33\alpha\left(2\alpha^2 + 27\alpha + 36\right) a_0^7 + \\
&133\left(2\alpha^2 + 27\alpha + 36\right) a_0^7\Big)\Bigg) + \ldots, \qquad (4.28)
\end{aligned}
$$

respectively.

4.3.3 The Case of Eq. (4.3)

Reducing the equation by $X_2 + \epsilon X_1$ yields

$$p'(s)\left(\epsilon^3 + 4\alpha\epsilon^2 p(s)^3 + 3p(s)^2 p''(s)\right) + p(s)^3 p'''(s) = 0, \tag{4.29}$$

and $X_3 + aX_1$ gives

$$\begin{aligned}(-4 - 4a^2\alpha)y(z)^4 + a^3 y'(z) - 27z^2 y(z)^2 y'(z)(5y'(z) + 3zy''(z)\\ -3zy(z)^3((39 + 4a^2\alpha)y'(z) + 9z(5y''(z) + zy'''(z)) = 0.\end{aligned} \tag{4.30}$$

In this case, we obtain

$$\begin{aligned}p(s) \quad = \quad & a_0 + a_1 s + a_2 s^2\\ & + \frac{s^3\left(-4\alpha a_1 a_0^3 - 6a_1 a_2 a_0^2 - a_1\right)}{6a_0^3}\\ & \frac{s^4}{12a_0^3}\Bigg(- 4\alpha a_2 a_0^3 - 6\alpha a_1^2 a_0^2 - \frac{3a_1\left(-4\alpha a_1 a_0^3 - 6a_1 a_2 a_0^2 - a_1\right)}{a_0}\\ & -6a_2^2 a_0^2 - 6a_1^2 a_2 a_0 - a_2\Bigg)\\ & + \ldots,\end{aligned} \tag{4.31}$$

and

$$
\begin{aligned}
y(z) \;\; = \;\; & \frac{1}{2}z^2\left(16\alpha(\alpha+1)a_0^7+532(\alpha+1)a_0^7\right)+4(\alpha+1)a_0^4 z+a_0\\
&-\frac{2}{3}z^3\Bigg(96\alpha(\alpha+1)^2a_0^{10}-4080(\alpha+1)^2a_0^{10}+\\
&+2\alpha a_0^3\big(16\alpha(\alpha+1)a_0^7+532(\alpha+1)a_0^7\big)\\
&-130a_0^3\left(16\alpha(\alpha+1)a_0^7+532(\alpha+1)a_0^7\right)\Bigg)+\\
&+\frac{1}{4}z^4\Bigg(-1280\alpha(\alpha+1)^3a_0^{13}+40768(\alpha+1)^3a_0^{13}\\
&-120\alpha(\alpha+1)a_0^6\big(16\alpha(\alpha+1)a_0^7\\
&+532(\alpha+1)a_0^7\big)+5226(\alpha+1)a_0^6\left(16\alpha(\alpha+1)a_0^7+532(\alpha+1)a_0^7\right)\\
&-\frac{2678}{3}a_0^3\Bigg(96\alpha(\alpha+1)^2a_0^{10}-4080(\alpha+1)^2a_0^{10}+2\alpha a_0^3\big(16\alpha(\alpha+1)a_0^7+\\
&532(\alpha+1)a_0^7\big)-130a_0^3\left(16\alpha(\alpha+1)a_0^7+532(\alpha+1)a_0^7\right)\Bigg)\\
&+\frac{40}{3}\alpha a_0^3\Bigg(96\alpha(\alpha+1)^2a_0^{10}-4080(\alpha+1)^2a_0^{10}+2\alpha a_0^3\big(16\alpha(\alpha+1)a_0^7\\
&+532(\alpha+1)a_0^7\big)+-130a_0^3\big(16\alpha(\alpha+1)a_0^7+532(\alpha+1)a_0^7\big)\Bigg)\Bigg)\\
&+\ldots
\end{aligned}
\tag{4.32}
$$

for the above ODEs.

4.3.4 The Case of Eq. (4.4)

Similarly, for Eq. (4.4), $X_2+\epsilon X_1$, the travelling wave reduction produces

$$
18\epsilon^3-4\alpha(9+\alpha)\epsilon^2p(s)^3-27p(s)^2p''(s))+18p(s)^3(\alpha\epsilon p''(s)+p'''(s)=0, \tag{4.33}
$$

and $X_3 + aX_1$ gives

$$\begin{aligned}\Big((45+18a\alpha+4a^2\alpha(9+\alpha)\Big)y(z)^4 - 18a^3y'(z)\\ +243z^2y(z)^2y'(z)(5y'(z)+3zy''(z))\\ +3zy(z)^3\left(540+90a\alpha+4a^2\alpha(9+\alpha)\right)y'(z)\\ +27z\left((27+2a\alpha)y''(z)+6zy'''(z))\right) = 0,\end{aligned} \tag{4.34}$$

with corresponding solutions

$$\begin{aligned} p(s) \;=\; & a_0 + a_1 s + a_2 s^2 \\ & + \frac{s^3\left(-2\alpha^2 a_1 a_0^3 - 18\alpha a_1 a_0^3 + 18\alpha a_2 a_0^3 - 27 a_1 a_2 a_0^2 - 9a_1\right)}{54a_0^3} \\ & - \frac{s^4}{216a_0^3}\Big(4\alpha^2 a_2 a_0^3 + 6\alpha^2 a_1^2 a_0^2 - \alpha\big(-2\alpha^2 a_1 a_0^3 - 18\alpha a_1 a_0^3 + 18\alpha a_2 a_0^3 - 9a_1 \\ & - 27a_1 a_2 a_0^2\big) + \frac{9a_1\left(-2\alpha^2 a_1 a_0^3 - 18\alpha a_1 a_0^3 + 18\alpha a_2 a_0^3 - 27a_1 a_2 a_0^2 - 9a_1\right)}{2a_0} \\ & + 36\alpha a_2 a_0^3 + 54\alpha a_1^2 a_0^2 - 54\alpha a_1 a_2 a_0^2 + 54a_2^2 a_0^2 + 54a_1^2 a_2 a_0 + 18a_2\Big) \\ & + \ldots, \end{aligned} \tag{4.35}$$

and finally,

$$
\begin{aligned}
y(z) \quad = \quad & a_0 + \frac{1}{18}\left(4\alpha^2 + 54\alpha + 45\right) a_0^4 z + \frac{1}{18} z^2 \biggl(\frac{7}{9}\alpha^2 \left(4\alpha^2 + 54\alpha + 45\right) a_0^7 \\
& + \frac{33}{2}\alpha \left(4\alpha^2 + 54\alpha + 45\right) a_0^7 + 50 \left(4\alpha^2 + 54\alpha + 45\right) a_0^7\biggr) \\
& + \frac{1}{54} z^3 \Biggl(\frac{5}{27}\alpha^2 \bigl(4\alpha^2 + 54\alpha + 45\bigr)^2 a_0^{10} + \frac{235}{12}\left(4\alpha^2 + 54\alpha + 45\right)^2 a_0^{10} \\
& + \frac{9}{2}\alpha \left(4\alpha^2 + 54\alpha + 45\right)^2 a_0^{10} + \frac{20}{9}\alpha^2 a_0^3 \biggl(\frac{7}{9}\alpha^2 \left(4\alpha^2 + 54\alpha + 45\right) a_0^7 \\
& + \frac{33}{2}\alpha \left(4\alpha^2 + 54\alpha + 45\right) a_0^7 + 50\bigl(4\alpha^2 + 54\alpha + 45\bigr) a_0^7\biggr) \\
& + 72\alpha a_0^3 \Bigl(\frac{7}{9}\alpha^2 \left(4\alpha^2 + 54\alpha + 45\right) a_0^7 + \frac{33}{2}\alpha \left(4\alpha^2 + 54\alpha + 45\right) a_0^7 \\
& + 50 \left(4\alpha^2 + 54\alpha + 45\right) a_0^7\Bigr) + 433 a_0^3 \Bigl(\frac{7}{9}\alpha^2 \bigl(4\alpha^2 + 54\alpha \\
& + 45\bigr) a_0^7 + \frac{33}{2}\alpha \left(4\alpha^2 + 54\alpha + 45\right) a_0^7 + 50 \left(4\alpha^2 + 54\alpha + 45\right) a_0^7\Bigr)\Biggr) \\
& + \ldots.
\end{aligned} \tag{4.36}
$$

4.4 Convergence - Exact Power Series

Next we prove the convergence of the power series solution (4.17). The rest of the series solutions above may be tested for convergence in the same manner. Consider, the case of Eq. (4.1), the first reduction. From (4.21) we obtain

$$
|a_{r_1+3}| \leq M \left(\sum_{r_2=1}^{r_1} \sum_{r_3=0}^{r_2} |a_{r_1-r_2+3}||a_{r_2-r_3}||a_{r_3}| + \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} |a_{r_1-r_2}||a_{r_2-r_3}||a_{r_3}| \right), \tag{4.37}
$$

where $r_1 = 3, 4, 5...$, and $M = max\{\frac{1}{a_0^2}, \frac{4|\alpha|}{3a_0^2}\}$.

Suppose we have the power series $\mu = R(s) = \Sigma_{r_1=0}^{\infty} p_{r_1} s^{r_1}$ where

$$p_k = |a_k|, \ k = 0, 1, \ldots, 5 \tag{4.38}$$

and

$$p_{r_1+3} = M\left(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3} + \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2}p_{r_2-r_3}p_{r_3}\right), \tag{4.39}$$

for $r_1 = 3, 4, 5....$ Hence

$$|a_{r_1}| \leq p_{r_1}, \ r_1 = 0, 1, 2... \tag{4.40}$$

Next we prove that μ is convergent in a neighborhood of a point. In fact, μ is a majorant series of equation (4.17) and it can be written as follows:

$$\begin{aligned}
R(s) &= p_0 + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + p_5 s^5 + \sum_{r_1=3}^{\infty} p_{r_1+3} s^{r_1+3} \\
&= p_0 + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + p_5 s^5 + \\
&\quad M\sum_{r_1=3}^{\infty}\left(\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3} + \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2}p_{r_2-r_3}p_{r_3}\right) s^{r_1+3} \\
&= p_0 + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + p_5 s^5 + \\
&\quad M\left(\sum_{r_1=3}^{\infty}\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2+3}p_{r_2-r_3}p_{r_3} s^{r_1+3} + \sum_{r_1=3}^{\infty}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2} p_{r_1-r_2}p_{r_2-r_3}p_{r_3} s^{r_1+3}\right) \\
&= p_0 + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + p_5 s^5 \\
&\quad + M[R^3(s)(s^3+1) - p_0^2 R - R^2(p_1 s + p_1 s^2) + \nu(s)],
\end{aligned} \tag{4.41}$$

where $\nu(s)$ is a polynomial with respect to s, that is

$\nu(s) = -3p_0p_1^2s^5 - 3p_0^2p_2s^5 - p_1^2p_3s^5 - 4p_0p_2p_3s^5 - 4p_0p_1p_4s^5 - 2p_0^2p_5s^5 - p_0p_2^2s^4 - 3p_0^2p_1s^4 - 4p_0p_1p_3s^4 - 2p_0^2p_4s^4 - 2p_0p_1p_2s^3 - 2p_0^2p_3s^3 - p_0p_1^2s^2 - p_0^2p_2s^2 - p_0^3s^3 - p_0^2p_1s$

Let the implicit function equation be:

$$\begin{aligned}
F(s,\mu) &= \mu - p_0 - p_1 s - p_2 s^2 - p_3 s^3 - p_4 s^4 - p_5 s^5 \\
&\quad - M[R^3(s)(s^3+1) - p_0^2 R(s) - R^2(s)(p_1 s + p_1 s^2) + \nu(s)],
\end{aligned} \tag{4.42}$$

where we obtain that $F(0,p_0) = 0$ and $F_\mu(0,r_0) = 1 - 2Mp_0^2 \neq 0$. By virtue of the implicit function theorem [37], $\mu = R(s)$ is analytic and convergent in a neighbourhood of the point $(0,p_0)$ in the plane and with a positive radius. Then the power series solution (4.17) is convergent in the neighbourhood of a point $(0,p_0)$.

Therefore (4.17) can be written as

$$\begin{aligned} p(s) &= a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + \Sigma_{r_1=3}^{\infty} a_{r_1+3} s^{r_1+3} \\ &= a_0 + a_1 s + a_2 s^2 + a_3 s^4 + a_4 s^4 + a_5 s^5 + \\ &\Sigma_{r_1=3}^{\infty}\left(\frac{-3\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}}{3(r_1+3)(r_1+2)(r_1+1)a_0^2}\right. \\ &\left.+\frac{4\alpha\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}a_{r_1-r_2}a_{r_2-r_3}a_{r_3}}{3(r_1+3)(r_1+2)(r_1+1)a_0^2}\right)s^{r_1+3}. \end{aligned} \quad (4.43)$$

Thus the exact power series solution of equation (4.1) can be written as

$$\begin{aligned} u(x,t) &= a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^3 + a_4(t-x)^4 + a_5(t-x)^5 \\ &\quad + \Sigma_{r_1=3}^{\infty} a_{r_1+3}(t-x)^{r_1+3} \\ &= a_0 + a_1(t-x) + a_2(t-x)^2 + a_3(t-x)^4 + a_4(t-x)^4 + a_5(t-x)^5 + \\ &\Sigma_{r_1=3}^{\infty}\left(\frac{-3\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)a_{r_1-r_2+3}a_{r_2-r_3}a_{r_3}}{3(r_1+3)(r_1+2)(r_1+1)a_0^2}\right. \\ &\left.+\frac{4\alpha\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}a_{r_1-r_2}a_{r_2-r_3}a_{r_3}}{3(r_1+3)(r_1+2)(r_1+1)a_0^2}\right)(t-x)^{r_1+3} \end{aligned} \quad (4.44)$$

where $a_0 \neq 0, a_1, a_2$, are arbitrary constants, a_3, a_4, a_5 are given in (4.19)-(4.20) and the rest of the constants are to be determined by (4.21).

4.5 Conservation Laws

In this section, we derive the conservation laws associated with Eqs. (4.1)-(4.4). In the Tables 4.3 and 4.4, we list the conservation laws for each of our four cases.

4.6 Discussion and Conclusion

A special class of third-order evolutionary equations were investigated. Several distinct ODEs were obtained by reduction with the optimal system. In particular, these equations are highly nonlinear with solutions best found by power series. Through the results obtained, it can be found that the power series method is effective in solving ODEs admitted by the Lie group reduction method. A standard technique for testing the convergence of such solutions is demonstrated. Finally, the conservation laws are found for the class of equations.

Table 4.3: Conserved Components Eq. (4.1)- (4.2)

Eq. #	Multiplier	Density & Flux
Eq.(4.1)	$\Lambda^1(x,t,u,u_x,u_{xx}) = \frac{u_{xx}}{u_x^3}$	$T_t^1 = \frac{-1}{2u_x}$ $T_x^1 = -\frac{1}{2}\frac{\alpha u_x^4+u_x^2+u_{xx}^2+u_t}{u_x^2}$
	$\Lambda^2(x,t,u,u_x,u_{xx}) =$ $\frac{-\sqrt{\alpha}\cos(2\sqrt{\alpha}x)u_x+\sin(2\sqrt{\alpha}x)u_{xx}}{u_x^3}$	$T_t^2 = \frac{-\sin(\sqrt{\alpha}x)+\cos(\sqrt{\alpha}x)}{u_x}$ $T_x^2 = \frac{1}{u_x^2}\Bigg(\sin(\sqrt{\alpha}x)\cos(\sqrt{\alpha}x)\alpha u_x^4$ $+2\cos(\sqrt{\alpha}x)^2\sqrt{\alpha}u_x^3u_{xx}$ $-u_{xx}\sin(\sqrt{\alpha}x)u_x^2 - \sqrt{\alpha}u_x^3u_{xx}$ $-\sin(\sqrt{\alpha}x)\cos(\sqrt{\alpha}x)u_t\Bigg)$
	$\Lambda^3(x,t,u,u_x,u_{xx}) =$ $\frac{\sqrt{\alpha}\sin(2\sqrt{\alpha}x)+\cos(2\sqrt{\alpha}x)u_{xx}}{u_x^3}$	$T_t^3 = -\frac{1}{2}\frac{2\cos(\sqrt{\alpha}x)^2-1}{u_x}$ $T_x^3 = -\frac{1}{2}\frac{1}{u_x^2}\Bigg(4\sin(\sqrt{\alpha}x)\cos(\sqrt{\alpha}x)\sqrt{\alpha}u_x^3u_{xx}$ $-2\cos(\sqrt{\alpha}x)^2\alpha u_x^4 - u_{xx}^2u_x^2 + 2\cos(\sqrt{\alpha}x)^2u_t$ $-u_t + 2u_{xx}\cos(\sqrt{\alpha}x)^2u_x^2 + \alpha u_x^4\Bigg)$
Eq.(4.2)	$\Lambda^1(x,t,u) = e^{\frac{2}{3}\alpha x}$	$T_t^1 = e^{\frac{2}{3}\alpha x}u$ $T_x^1 = -\frac{1}{3}e^{\frac{2}{3}\alpha x}u^3(\alpha u_x + 3u_{xx})$
	$\Lambda^2(x,t,u) = \frac{1}{u^2}$	$T_t^2 = -\frac{1}{u}$ $T_x^2 = -\frac{1}{9}\alpha^2u^2 - uu_{xx} - u_x^2 - \alpha uu_x$
	$\Lambda^3(x,t,u) = \frac{e^{\frac{1}{3}\alpha x}}{u^2}$	$T_t^3 = -\frac{e^{\frac{1}{3}\alpha x}}{u}$ $T_x^2 = -2\alpha u^2 - uu_{xx} - u_x^2$ $T_x^3 = -\frac{1}{3}e^{\frac{1}{3}\alpha x}\left(2\alpha uu_x + 3uu_{xx} + 3u_x^2\right)$
	$\Lambda^4(x,t,u) = \frac{e^{\frac{2}{3}\alpha x}}{u^2}$	$T_t^4 = -\frac{e^{\frac{2}{3}\alpha x}}{u}$ $T_x^4 = -\frac{1}{3}e^{\frac{2}{3}\alpha x}(\alpha uu_x + 3uu_{xx} + 3u_x^2)$

Table 4.4: Conserved Components Eq. (4.3)- (4.4)

Eq. #	Multiplier	Density & Flux
Eq.(4.3)	$\Lambda^1(x,t,u) = 1$	$T_t^1 = u$ $T_x^1 = -\alpha u^4 - u^3 u_{xx}$
	$\Lambda^2(x,t,u) = \frac{1}{u^2}$	$T_t^2 = -\frac{1}{u}$ $T_x^2 = -2\alpha u^2 - uu_{xx} - u_x^2$
	$\Lambda^3(x,t,u) = \frac{\sin(2\sqrt{\alpha}x)}{u^2}$	$T_t^3 = -\frac{1}{u}\Bigg(2(2u_x^2\sqrt{\alpha}t\cos(\sqrt{\alpha}x)^2 u - \sqrt{\alpha}\cos(2\sqrt{\alpha}x)tuu_x^2$ $-u_x^2\sqrt{\alpha}tu + \cos(\sqrt{\alpha}x)\sin(\sqrt{\alpha}x))\Bigg)$ $T_x^2 = -2\alpha u^2 - uu_{xx} - u_x^2$ $T_x^3 = -2\sin\Bigg(2\sqrt{\alpha}x)u_x^2 + 4\sqrt{\alpha}uu_x\cos(\sqrt{\alpha}x)^2$ $-2uu_{xx}\cos(\sqrt{\alpha}x)\sin(\sqrt{\alpha}x) - 2\sqrt{\alpha}uu_x\cos(\sqrt{\alpha}x\Bigg)^2$
	$\Lambda^4(x,t,u) = \frac{\cos(2\sqrt{\alpha}x)}{u^2}$	$T_t^4 = -\frac{1}{u}\Bigg(-4u_x^2\sqrt{\alpha}t\cos(\sqrt{\alpha}x)u + 2\sin(2\sqrt{\alpha}x)\sqrt{\alpha}tuu_x^2$ $-1 + 2\cos(\sqrt{\alpha}x)^2\Bigg)$ $T_x^4 = -\cos(2\sqrt{\alpha}x)u_x^2 + uu_{xx} - 2uu_{xx}\cos(\sqrt{\alpha}x)^2$ $-4\sqrt{\alpha}uu_x\cos(\sqrt{\alpha}x)\sin(\sqrt{\alpha}x)$
Eq.(4.4)	$\Lambda^1(x,t,u) = \frac{1}{u^2}$	$T_t^1 = -\frac{1}{u}$ $T_x^1 = -\frac{1}{9}\alpha^2 u^2 - \frac{1}{4}u_x^2 - u_x\alpha u - uu_{xx}$
	$\Lambda^2(x,t,u) = \frac{e^{\frac{1}{3}\alpha x}}{u^{3/2}}$	$T_t^2 = -\frac{2e^{\frac{1}{3}\alpha x}}{\sqrt{u}}$ $T_x^2 = -\frac{1}{3}u^{\frac{3}{2}}e^{\frac{1}{3}\alpha x}(2\alpha u_x + 3u_{xx})$

Chapter 5

On the optimal system and series solutions of fifth-order Fujimoto-Watanabe equations

5.1 Introduction

Fujimoto-Watanabe [9], explored two fifth-order equations

$$A : u_t = u^5 u_{xxxxx} + 5u^4(u_x u_{xxxx} + 2u_{xx}u_{xxx}), \tag{5.1}$$

$$B : u_t = u^5 u_{xxxxx} + 5u^4(u_x u_{xxxx} + \frac{1}{2}u_{xx}u_{xxx}) + \frac{15}{4}u^3 u_x^2 u_{xxx}. \tag{5.2}$$

The full classification of fifth-order evolution equations [9] is too long and tedious, that it was not found, unlike the case of third-order evolution equations. As a special consideration, (5.1) and (5.2) may be mapped onto the Sawada-Kotera [40] and Kaup [20] equations via a Schwarzian derivative [38], and via the inverse scattering transform technique at zero spectral parameter [39]. Due to these properties, the above equations have applications in fluid dynamics, ion-acoustic waves in plasmas and nonlinear shallow water wave phenomena, to name a few. These equations are also important to study from other perspectives, as interesting notions may apply, such as the inverse scattering technique.

As an instrument in the study of all possible types of differential equations, transformations are the heart and soul of analytical solution techniques. Furthermore, transformative methods may sometimes be used in conjunction with numerical approaches. Explicitly, transformations simplify the equation, either by creating an easier form of the equation, or through combining variables, and thereby decreasing the number of variables. The only challenge, of course, is the calculation of such transformations. However, these days, symmetry methods are the most reliable method to generate transformations of *most* differential equations.

In terms of PDEs, the invariance under a one-parameter Lie group of transformations enacts a reduction of the number of independent variables, while with ODEs, the order may be reduced. The infinitesimal generator of the transformation is applied so that solutions arise constructively.

In this chapter, we explore Lie symmetry reductions of this family of equations, which are naturally of high order, and therefore must be solved using special techniques. As we have seen, such a technique that works well, is the power series approach.

The organisation of the chapter is provided next. In the next section, we display the symmetry invariant properties admitted by the first equation, with its commutator, adjoint and group specifications. Section 5.3 has the associated symmetry details for the second equation. In Section 5.4, we have a demonstration of the convergence of solutions for our fifth-order reduced equations. The conclusion is given in Section 5.5.

5.2 Case A

Let us invoke a Lie group of (one-parameter) infinitesimal ($\varepsilon \ll 1$) mappings, for x, t, u to

$$x + \varepsilon\, \hat{\xi},$$

$$t + \varepsilon\, \hat{\tau},$$

and

$$u \to u + \varepsilon\, \hat{\phi},$$

respectively and with $\hat{\phi}$, $\hat{\xi}$, and $\hat{\tau}$ as functions of (x, t, u). Now, a symmetry vector field is

$$X = \hat{\phi}\partial_u + \hat{\xi}\partial_x + \hat{\tau}\partial_t,$$

where the unknown coefficient functions in (x, t, u) are given by the standard Lie symmetry determining condition. Omitting the detailed calculations, the Lie point symmetries of (5.1) are four, viz.

$$\begin{aligned}
X_1 &= \partial_x, \\
X_2 &= \partial_t, \\
X_3 &= u\partial_u + x\partial_x, \\
X_4 &= 5t\partial_t + x\partial_x, \\
X_5 &= 2ux\partial_u + x^2\partial_x.
\end{aligned} \tag{5.3}$$

The Lie brackets of (5.3) are given in Table 5.1 where $[X_j, X_k]$ is the commutator given by

$$[X_j, X_k] = X_j X_k - X_k X_j.$$

The one-parameter groups M_j $(j = 1, \ldots, 4)$ of the symmetries X_j, are

Table 5.1: Lie brackets of (5.3)

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_1	X_1	$2X_3$
X_2	0	0	0	$5X_2$	0
X_3	$-X_1$	0	0	0	X_5
X_4	$-X_1$	$-5X_2$	0	0	X_5
X_5	$-2X_3$	0	$-X_5$	$-X_5$	0

$$M_1 : (x, t, u) \mapsto (x + \varepsilon, t, u), \tag{5.4}$$

$$M_2 : (x, t, u) \mapsto (x, t + \varepsilon, u), \tag{5.5}$$

$$M_3 : (x, t, u) \mapsto (e^{\varepsilon} x, t, e^{\varepsilon} u), \tag{5.6}$$

$$M_4 : (x, t, u) \mapsto (e^{\varepsilon} x, e^{5\varepsilon} t, u), \tag{5.7}$$

$$M_5 : (x, t, u) \mapsto \left(\frac{x}{-1 + x\varepsilon}, t, \frac{u}{(-1 + x\varepsilon)^2} \right). \tag{5.8}$$

The adjoints are given in Table 5.2 and we define the optimal system of one-dimensional subalgebras for Cases A.

Here we consider

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5, \tag{5.9}$$

where the a_is are arbitrary constants, and simplify this by the applications of adjoint maps.

Table 5.2: Adjoint Representation of Subalgebras of (5.3)

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	$X_3-\varepsilon X_1$	$X_4-\varepsilon X_1$	$\varepsilon^2X_1-2\varepsilon X_3+X_5$
X_2	X_1	X_2	X_3	$X_4-5\varepsilon X_2$	X_5
X_3	$e^{\varepsilon}X_1$	X_2	X_3	X_4	$e^{-\varepsilon}X_5$
X_4	$e^{\varepsilon}X_1$	$e^{5\varepsilon}X_2$	X_3	X_4	$e^{-\varepsilon}X_5$
X_5	$\varepsilon^2X_5+2\varepsilon X_3+X_1$	X_2	εX_5+X_3	εX_5+X_4	X_5

Suppose firstly that $a_3 \neq 0$, then we can assume that $a_3 = 1$. If we act on X by $\mathrm{Ad}(\exp(1/(2a_5))X_1)$ we get

$$X^{'} = a_1^1X_1 + a_2X_2 + a_4X_4 + a_5X_5,$$

$a_5 \neq 0$, and the scalar a_1^1 depends on a_1, a_4, a_5. Now acting on $X^{'}$ by $\mathrm{Ad}(\exp(a_2/(5a_4))X_2)$ results in $X^{''} = a_1^1X_1 + a_4X_4 + a_5X_5$, $a_4 \neq 0$. Next $\mathrm{Ad}(\exp(c_3X_3))$ on $X^{''}$ yields

$$X_4,\ aX_4 \pm X_1,\ aX_4 \pm X_5,\ aX_4 \pm X_5 \pm X_1. \tag{5.10}$$

If $a_5 = 0$, then $X = a_1X_1+a_2X_2+X_3+a_4X_4$. We act on X by $\mathrm{Ad}(\exp(a_2/(5a_4))X_2)$ and obtain

$$X^{'} = a_1X_1 + X_3 + a_4X_4,$$

$a_4 \neq 0$. Further acting on $X^{'}$ by $\mathrm{Ad}(\exp(c_2X_3)$ gives

$$X_3 + aX_4,\ X_3 + aX_4 \pm X_1. \tag{5.11}$$

If $a_4 = 0$, then we act on $X = a_1X_1 + a_2X_2 + X_3$ by $\mathrm{Ad}(\exp(a_1X_1))$ and obtain $X^{'} = X_3 + a_2X_2$, and further acting on $X^{'}$ by $\mathrm{Ad}(\exp(c_2X_4))$ gives rise to

$$X_3,\ X_3 \pm X_2. \tag{5.12}$$

If $a_5 \neq 0$, $a_4 = 0$, then $X^{'} = a_1^1 X_1 + a_2 X_2 + a_5 X_5$. Acting on $X^{'}$ by $\mathrm{Ad}(\exp(c_2 X_3))$ one obtains

$$bX_2 \pm X_5,\ bX_2 \pm X_5 \pm X_1. \tag{5.13}$$

If $a_3 = 0$, $a_4 = 1$, then we act on $X = a_1X_1 + a_2X_2 + X_4 + a_5X_5$ by $\mathrm{Ad}(\exp((a_2/5)X_2))$ and get $X^{'} = a_1X_1 + X_4 + a_5X_5$. Further acting on $X^{'}$ by $\mathrm{Ad}(\exp(c_3X_3))$ gives

$$X_4,\ X_4 \pm X_1,\ X_4 \pm X_5,\ X_4 \pm X_5 \pm X_1. \tag{5.14}$$

Assume now $a_4 = a_3 = 0$, $a_5 = 1$, then we act on $X = a_1X_1 + a_2X_2 + X_5$ by $\mathrm{Ad}(\exp(c_1X_3))$ which yields X_5, $X_5 \pm X_1$, $X_5 \pm X_2$, $X_5 \pm X_1 \pm X_2$. Now assume that $a_5 = a_4 = a_3 = 0$, $a_1 = 1$, we act on $X = X_1 + a_2X_2$ by $\mathrm{Ad}(\exp(c_1X_3))$ which gives rise to X_1, $X_1 \pm X_2$. Next suppose that $a_1 = a_5 = a_4 = a_3 = 0$, $a_2 = 1$ that yields X_2. Now the above discussion gives rise to the optimal system of one-dimensional subalgebras for this case,

$$\begin{aligned} &aX_4 + \delta X_5 + \epsilon X_1, X_2,\ aX_4 + X_3 + \epsilon X_1,\ bX_2 + \delta X_5 + \epsilon X_1, \\ &X_3 + \epsilon X_2,\ X_4 + \epsilon X_1,\ X_1 + \epsilon X_2, \end{aligned}$$

where $a(\neq 0)$, b are constants, $\delta = \pm 1$ and $\epsilon = 0, \pm 1$.

5.2.1 Symmetry Invariant Reductions

In this subsection, we apply a Lie reduction technique, whereby zero-order invariants transform the order of the given PDE. This procedure results in reduced equations that are difficult to solve by most analytical techniques. We instead apply series to determine explicit solutions for (5.1).

A reduction of the equation by the symmetry vectors $X_1 + X_2$ ($\epsilon = 1$) produces the

ODE

$$
\begin{aligned}
y'(z)\Big(+ 5y(z)^4 y''''(z)\Big) \\
+y(z)^4\Big(5y''(z)y'''(z) + y(z)y^{(5)}(z)\Big) = 0,
\end{aligned} \tag{5.15}
$$

where $z = -x + t$ and $u(x,t) = y(z)$. Next, the power series

$$
y(z) = \sum_{r_1=0}^{\infty} a_{r_1} z^{r_1}, \tag{5.16}
$$

is entered into (5.15). We see that a_0, a_1, a_2, a_3, a_4 are arbitrary constants and that a solution reads as

$$
\begin{aligned}
y(z) \;=\; & a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \\
& + \frac{\left(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right) z^5}{120a_0^5} \\
& \frac{z^6}{360a_0^5}\Bigg(- 90a_3^2a_0^4 - 240a_2a_4a_0^4 - 120a_1a_2a_3a_0^3 - 240a_1^2a_4a_0^3 - a_2 - \\
& \frac{5a_1\left(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right)}{a_0}\Bigg) + \ldots .
\end{aligned} \tag{5.17}
$$

Hence a solution for (5.1) may be obtained by reversing the transformation, namely,

$$
\begin{aligned}
u(x,t) \;=\; & a_0 + a_1\left(-x+t\right) + a_2\left(-x+t\right)^2 + a_3\left(-x+t\right)^3 + \\
& + a_4\left(-x+t\right)^4 + \frac{\left(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right)\left(-x+t\right)^5}{120a_0^5} \\
& \frac{\left(-x+t\right)^6}{360a_0^5}\Bigg(- 90a_3^2a_0^4 - 240a_2a_4a_0^4 - 120a_1a_2a_3a_0^3 - 240a_1^2a_4a_0^3 - a_2 - \\
& \frac{5a_1\left(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right)}{a_0}\Bigg) + \ldots
\end{aligned} \tag{5.18}
$$

The question arises of whether such a series solution converges or not. We answer this question in Section 5.4.

Suppose we consider the next reduction from the list of optimal one-dimensional subalgebras, that is, $aX_4 + \delta X_5 + \epsilon X_1$. This symmetry produces the invariant $z = t \exp\left(-\frac{10a\tan^{-1}\left(\frac{a}{\sqrt{4\delta\epsilon - a^2}} + \frac{2\delta x}{\sqrt{4\delta\epsilon - a^2}}\right)}{\sqrt{4\delta\epsilon - a^2}}\right)$, where

$$u(x,t) = \left(ax + \delta x^2 + \epsilon\right) \exp\left(-\frac{2a\tan^{-1}\left(\frac{a+2\delta x}{\sqrt{4\delta\epsilon - a^2}}\right)}{\sqrt{4\delta\epsilon - a^2}}\right) y\left(z\right). \tag{5.19}$$

In this instance, (5.1) reduces to the high-order nonlinear ODE

$$\begin{aligned}
&16a\delta\epsilon\left(3a^2 + 4\delta\epsilon\right)y(z)^6 + y'(z) + 125a^3\delta^2 y(z)^4\left(84\left(10a^2 + \delta\epsilon\right)y'(z)^2\right.\\
&50a^2\delta^2 y''(z)\left(18y''(z) + 5zy'''(z)\right) + 5y'(z)\left(12\left(48a^2 + 8\epsilon\right)y''(z) + 5a^2\delta\right.\\
&\left.\left.+5a^2 z\left(48y'''(z) + 5zy''''(z)\right)\right)\right) + 5azy(z)^5\left(4\left(756a^4 + 327a^2\delta\epsilon + 16\delta^2\epsilon^2\right)y'(z)\right.\\
&25a^2\delta\left(84\left(10a^2 + 8\epsilon\right)y''(z) + 5z\left(4\left(48a^2 + 8\epsilon\right)y'''(z)\right.\right.\\
&\left.\left.\left.5a^2 z\left(12y'''''(z) + zy^{(z)}(z)\right)\right)\right)\right) = 0.
\end{aligned} \tag{5.20}$$

By the same procedure, that is the power series (5.16), we find the series solution

$(\epsilon = 1)$

$$
\begin{aligned}
y(z) \;=\;& a_0 - 16a\left(3a^2+4\right)a_0^6 z - 2\left(3a^2+4\right)\Big(- 60480a^6a_0^{11} - 27312a^4a_0^{11} \\
& -2816a^2a_0^{11}\Big)z^2 - \frac{8}{3}\Big(5779200a^7\left(3a^2+4\right)^2 a_0^{16} + 1405440a^5 \times \\
& \left(3a^2+4\right)^2 a_0^{16} + 81920a^3\left(3a^2+4\right)^2 a_0^{16} - 60060a^5\left(3a^2+4\right) \times \\
& \Big(- 60480a^6a_0^{11} - 27312a^4a_0^{11} - 2816a^2a_0^{11}\Big)a_0^5 - 8592a^3 \times \\
& \left(3a^2+4\right)\left(-60480a^6a_0^{11} - 27312a^4a_0^{11} - 2816a^2a_0^{11}\right)a_0^5 \\
& -256a\left(3a^2+4\right)\left(-60480a^6a_0^{11} - 27312a^4a_0^{11} - 2816a^2a_0^{11}\right) \times \\
& a_0^5\Big)z^3 + \ldots
\end{aligned}
\tag{5.21}
$$

which may be transformed into a solution in $u(x,t)$, using (5.19), for equation (5.1).

Similarly, a reduction of equation (5.1) by the symmetry vector X_2 gives

$$y(z)y^{(5)}(z) + 5y^{(4)}(z)y'(z) + 5y^{(3)}(z)y''(z) = 0,$$

where $u = y(z),\ z = x$, and the series solution is

$$
\begin{aligned}
y(z) \;=\;& a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + \\
& + \frac{\left(-a_2a_3 - 2a_1a_4\right)z^5}{2a_0} + \frac{\left(-3a_3^2 - 8a_2a_4 - \frac{6a_1(-a_2a_3-2a_1a_4)}{a_0}\right)z^6}{12a_0} \\
& \frac{\left(-\frac{7\left(-3a_3^2-8a_2a_4-\frac{6a_1(-a_2a_3-2a_1a_4)}{a_0}\right)a_1}{4a_0} - 12a_3a_4 - \frac{8a_2(-a_2a_3-2a_1a_4)}{a_0}\right)z^7}{21a_0} + \ldots
\end{aligned}
\tag{5.22}
$$

Repeating this idea, with the symmetry $X_4 + \epsilon X_1$, we have the transformation

$z = \frac{t}{(x+\epsilon)^5}$ where $u = y(z)$. Hence the reduced equation is $(\epsilon = 1)$

$$\begin{aligned} & y'(z)\left(625z^3y(z)^4\left(576y''(z)+5z\left(48y^{(3)}(z)+5zy^{(4)}(z)\right)\right)+15120zy(z)^5+1\right) \\ & \quad +105000z^2y(z)^4y'(z)^2+625z^2y(z)^4\Big(10z^2y''(z)\left(18y''(z)+5zy^{(3)}(z)\right) \\ & \quad +y(z)\left(5z^3y^{(5)}(z)+60z^2y^{(4)}(z)+168y''(z)+192zy^{(3)}(z)\right)\Big)=0, \end{aligned} \tag{5.23}$$

with series solution

$$\begin{aligned} y(z) &= a_0 + a_1 z - 7560a_1a_0^5z^2 - 280\left(215a_0^4a_1^2 - 2162160a_0^{10}a_1\right)z^3 \\ & \quad -120\Big(-38064600a_1^2a_0^9 - 813960\left(215a_0^4a_1^2 - 2162160a_0^{10}a_1\right)a_0^5 \\ & \quad +1190a_1^3a_0^3\Big)z^4 + \ldots \end{aligned} \tag{5.24}$$

Next, the symmetry combination $X_3 + \epsilon X_2$ results in the invariant $z = t - \epsilon\log(x)$, $u(x,t) = xy(z)$, so that the reduced equation is

$$\begin{aligned} & y'(z)\left(1+4\epsilon y(z)^5+5y(z)^4\Big(-3\epsilon y''(z)+\epsilon^5 y''''(z)\Big)\right) \\ & +\epsilon^3y(z)^4\left(10\epsilon^2y''(z)y'''(z)+y(z)\Big(-5y'''(z)+\epsilon^2y^{(5)}(z)\Big)\right)=0. \end{aligned} \tag{5.25}$$

Here, we find the solution $(\epsilon = 1)$

$$y(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + \frac{\left(-4a_1a_0^5 + 30a_3a_0^5 + 30a_1a_2a_0^4 - 120a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right)z^5}{120a_0^5} + \ldots \tag{5.26}$$

In the penultimate reduction, we use the vector $bX_2 + \delta X_5 + \epsilon X_1$, so that $z = \frac{\sqrt{\delta}t\sqrt{\epsilon} - b\tan^{-1}\left(\frac{\sqrt{\delta}x}{\sqrt{\epsilon}}\right)}{\sqrt{\delta}\sqrt{\epsilon}}$, and $u(x,t) = \left(\delta x^2+\epsilon\right)y(z)$. Eq. (5.1) becomes

$$\begin{aligned} & \Bigg(y'(z)\Big(1+64b\delta^2\epsilon^2y(z)^5+5y(z)^4\big(12b^3\delta\epsilon y''(z)+b^5y'''(z)\big)\Big) \\ & +b^3y(z)^4\Big(10b^2y''(z)y'''(z)+y(z)\big(20\delta\epsilon y'''(z)+b^2y^{(5)}(z)\big)\Big)\Bigg)=0, \end{aligned} \tag{5.27}$$

with solution ($\epsilon = 1$)

$$
\begin{aligned}
y(z) \quad = \quad & a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \\
& \frac{z^5\left(-120a_0^4 a_1 a_2 b^3 \delta - 120a_0^5 a_3 b^3 \delta - 120a_0^4 a_2 a_3 b^5 - 120a_0^4 a_1 a_4 b^5 - 64a_0^5 a_1 b \delta^2 - a_1\right)}{120a_0^5 b^5} \\
& + \ldots
\end{aligned}
\tag{5.28}
$$

Finally, the symmetry $aX_4 + X_3 + \epsilon X_1$, with $z = t(ax + x + \epsilon)^{-\frac{5a}{a+1}}$, and $u(x,t) = (ax + x + \epsilon)^{\frac{1}{a+1}} y\,(z)$, produces the reduction

$$
\begin{aligned}
& 4a\big(1 + 4a + a^2 - 6a^3\big)y(z)^6 + y'(z) + 125a^3 z^2 y(z)^4 \times \\
& \Bigg(21\big(-1 - 21 + 40a^2\big)y'(z)^2 + 50a^2 z^2 y''(z)\Big(18y''(z) + 5zy'''(z)\Big) \\
& +5zy'(z)\Bigg(\Big(-3 - 6a + 576a^2\Big)y''(z) + 5a^2 z\Big(48y'''(z) + 5zy''''(z)\Big)\Bigg)\Bigg) \\
& 5azy(z)^5\Bigg(\Big(4 + 16a - 311a^2 - 654a^3 + 3024a^4\Big)y'(z) \\
& 25a^2 z\Big(21\big(-1 - 2a + 40a^2\big)y'''(z) + 5z\Big(\big(-1 - 2a + 192a^2\big)y'''(z) \\
& +5a^2 z\big(12y''''(z) + zy^{(5)}\big)\Big)\Big)\Bigg) = 0,
\end{aligned}
\tag{5.29}
$$

with solution

$$
\begin{aligned}
y(z) \;=\;& a_0 + a_1 z + \\
& \frac{1}{2}\left(-15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\right)z^2 \\
& -\frac{4}{3}\Bigg(45150a_0^4a_1^2a^5 + 30030a_0^5\big(-15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 \\
& -176a^2a_1a_0^5 - 44aa_1a_0^5\big)a^5 - 5490a_0^4a_1^2a^4 - 2148a_0^5 \times \\
& \left(-15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\right)a^4 \\
& -2585a_0^4a_1^2a^3 - 1042a_0^5\big(-15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 \\
& -176a^2a_1a_0^5 - 44aa_1a_0^5\big)a^3 + 160a_0^4a_1^2a^2 + 32a_0^5\big(-15120a^5a_1a_0^5 \\
& +3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\big)a^2 + \\
& 8a_0^5\left(-15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\right)a \\
& +40a_0^4a_1^2a\Bigg)z^3 + \ldots
\end{aligned}
\tag{5.30}
$$

5.3 Case B

We now turn to Case B with a similar analysis. The Lie point symmetries of Eq. (5.2) are

$$
\begin{aligned}
X_1 &= \partial_x, \\
X_2 &= \partial_t, \\
X_3 &= 5t\partial_t - u\partial_u,
\end{aligned}
\tag{5.31}
$$

and the Lie commutator brackets of (5.31) are displayed in Table 5.3.

Thus, we have the relevant M_j of this case are

$$M_1 : (x, t, u) \mapsto (x + \varepsilon, t, u), \tag{5.32}$$

Table 5.3: Lie brackets of (5.31)

$[,]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	$5X_2$
X_3	0	$-5X_2$	0

$$M_2 : (x, t, u) \mapsto (x, t + \varepsilon, u), \tag{5.33}$$

$$M_3 : (x, t, u) \mapsto (x, e^{5\varepsilon} t, e^{\varepsilon} u). \tag{5.34}$$

The adjoints are given in Table 5.4. We consider

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3, \tag{5.35}$$

where the a_is are arbitrary constants, and as before simplify it by adjoint maps. Suppose first that $a_3 \neq 0$, then we can assume that $a_3 = 1$. If we act on X by $\mathrm{Ad}(\exp(a_2/5)X_2)$ we get

$$X_3 + aX_1. \tag{5.36}$$

If $a_3 = 0$, $a_2 = 1$, then we act on $X = a_1X_1 + X_2$ by $\mathrm{Ad}(\exp(c_1X_3)$ results in X_2, $X_2 \pm X_1$. Next suppose that $a_3 = a_2 = 0$, $a_1 = 1$ gives X_1. Now the above procedure gives for this case, the one dimensional subalgebras

$$X_1,\ X_3 + aX_1,\ X_2 + \epsilon X_1,$$

where a is a constant and $\epsilon = 0, \pm 1$.

5.3.1 Symmetry Invariant Reductions

A reduction by X_1 gives

$$y'(t) = 0, \tag{5.37}$$

Table 5.4: Adjoint Representation of Subalgebras of (5.31)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$X_3 - 5\varepsilon X_2$
X_3	X_1	$e^{5\varepsilon}X_2$	X_3

with $u(x,t) = y(t)$. If one solves this, we get

$$y(t) = c_1, \tag{5.38}$$

which is a trivial constant solution.

Reducing the equation (5.2) by $X_2 + \epsilon X_1$ gives

$$\begin{aligned} y'(z)\Big(2\epsilon^5 - 15\epsilon y(z)^3 y'''(z) + 10y(z)^4 y''''(z)\Big) + y(z)^4 \\ \Big(5y''(z)y'''(z) + 2y(z)y'''''(z)\Big) = 0, \end{aligned} \tag{5.39}$$

where $z = \frac{-x+t\epsilon}{\epsilon}$ and $u(x,t) = y(z)$. The power series solution we find is ($\epsilon = 1$)

$$\begin{aligned} y(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 \\ &\quad \frac{\left(-30a_2a_3a_0^4 + 45a_1a_3a_0^3 - a_1\right) z^4}{120a_0^4a_1} + \ldots \end{aligned} \tag{5.40}$$

Reducing the equation by $X_3 + aX_1$ gives the ordinary differential equation

$$\begin{aligned}
17y(z)^6 + 2a^5y'(z) - 375z^2y(z)^3y'(z)\big(43y'(z) + 5z(18y''(z) + 5zy'''(z))\big) \\
+25zy(z)^4\Bigg(4095zy'(z)^2 + 5z\Big(-3ae^{\frac{x}{a}} + 25z^2y''(z)\Big)\Big(18y''(z) + 5zy'''(z)\Big) \\
+y'(z)\Big(-132a + 16275z^2y''(z) + 9375z^3y'''(z) + 1250z^4y''''(z)\Big)\Bigg) \\
+5y(z)^5\Bigg(-3a + 5960zy'(z) + 41625z^2y''(z) + 47875z^3y'''(z) \\
+15000z^4y''''(z) + 1250z^5y'''''(z)\Bigg) = 0,
\end{aligned} \tag{5.41}$$

where $z = e^{\frac{5x}{a}}t$ and $u = e^{\frac{x}{a}}y(z)$. Hence a solution is

$$\begin{aligned}
y(z) \;=\; & a_0 + \frac{1}{2}\left(15a_0^5 - 17a_0^6\right)z \\
& + \frac{1}{4}\left(\frac{3375}{2}a_0^4\left(15a_0^5 - 17a_0^6\right) - 14951a_0^5\left(15a_0^5 - 17a_0^6\right)\right)z^2 \\
& \frac{1}{6}\Bigg(-118988\left(\frac{3375}{2}a_0^4\left(15a_0^5 - 17a_0^6\right) - 14951a_0^5\left(15a_0^5 - 17a_0^6\right)\right)a_0^5 + \\
& \frac{20175}{4}\left(\frac{3375}{2}a_0^4\left(15a_0^5 - 17a_0^6\right) - 14951a_0^5\left(15a_0^5 - 17a_0^6\right)\right)a_0^4 \\
& -\frac{125815}{2}\left(15a_0^5 - 17a_0^6\right)^2a_0^4 + \frac{29475}{4}\left(15a_0^5 - 17a_0^6\right)^2a_0^3\Bigg)z^3 + \ldots
\end{aligned} \tag{5.42}$$

5.4 Convergence of Series

In this section, we demonstrate the convergence properties [2] associated with the above solution types. Suppose we observe Eq. (5.15) that is substituted with (5.16), viz.

$$
\begin{aligned}
&\sum_{r_1=0}^{\infty}\Bigg((r_1+1)a_{r_1+1}+\\
&\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Big(\sum_{r_6=0}^{r_5}a_{r_6}a_{r_5-r_6}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}\\
&\times a_{r_1-r_2+5}(r_1-r_2+4)(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)\Big)+\\
&5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}(r_2+2)(r_2+1)a_{r_2+2}\\
&\times(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)a_{r_1-2r_2+3}\\
&+5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}a_{r_2+1}a_{r_1-2r_2+4}\\
&\times(r_1-2r_2+4)(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)\Bigg)z^{r_1}=0.
\end{aligned}
\tag{5.43}
$$

Therefore, by extensive manipulation, the coefficients are generated by

$$
\begin{aligned}
a_{r_1+5}=&-\frac{(r_1+1)a_{r_1+1}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\\
&-\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Bigg(5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}(r_2+2)(r_2+1)a_{r_2+2}\\
&\times\frac{(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)a_{r_1-2r_2+3}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\\
&+5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}a_{r_2+1}a_{r_1-2r_2+4}\\
&\times\frac{(r_1-2r_2+4)(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\Bigg)\\
&-\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5}a_{r_6}a_{r_5-r_6}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}\\
&\times\frac{a_{r_1-r_2+5}(r_1-r_2+4)(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}.
\end{aligned}
\tag{5.44}
$$

for $r_1 \geq 0$, such that (5.17) reads as

$$y(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \sum_{r_1=0}^{\infty} a_{r_1+5} z^{r_1+5}. \tag{5.45}$$

Next, from Eq. (5.44) we obtain

$$\begin{aligned} |a_{r_1+5}| \leq\; & M\Bigg(|a_{r_1+1}| \\ & + \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Big(|a_{r_5}||a_{r_4-r_5}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_2+2}||a_{r_1-2r_2+3}| \\ & + |a_{r_5}||a_{r_4-r_5}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_2+1}||a_{r_1-2r_2+4}|\Big) \\ & + \sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5}|a_{r_6}||a_{r_5-r_6}||a_{r_4-r_5}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_1-r_2+5}|\Bigg), \end{aligned} \tag{5.46}$$

where $r_1 = 0, 1, 2...$, and $M = \frac{1}{a_0^5}$.

Consider the series $\hat{\mu} = R(z) = \sum_{r_1=0}^{\infty} p_{r_1} z^{r_1}$ where

$$p_k = |a_k|, \;\; k = 0, \ldots, 4 \tag{5.47}$$

and

$$\begin{aligned} p_{r_1+5} =\; & M\Bigg(p_{r_1+1} \\ & + \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Big(p_{r_5}p_{r_4-r_5}p_{r_3-r_4}p_{r_2-r_3}p_{r_2+2}p_{r_1-2r_2+3} \\ & + p_{r_5}p_{r_4-r_5}p_{r_3-r_4}p_{r_2-r_3}p_{r_2+1}p_{r_1-2r_2+4}\Big) \\ & + \sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5}p_{r_6}p_{r_5-r_6}p_{r_4-r_5}p_{r_3-r_4}p_{r_2-r_3}p_{r_1-r_2+5}\Bigg). \end{aligned} \tag{5.48}$$

Hence

$$|a_{r_1}| \leq p_{r_1},\ r_1 = 0, 1, 2... \tag{5.49}$$

We are now required to show that $\hat{\mu}$ converges. The series $\hat{\mu}$ is a majorant series of equation (5.45) so that:

$$\begin{aligned} R(z) &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \sum_{r_1=0}^{\infty} p_{r_1+5} z^{r_1+5} \\ &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 \\ &\quad + M[R^5 + R\gamma(z) + \nu(z)], \end{aligned} \tag{5.50}$$

where $\nu(z) = \theta(z) - p_0^5$, with $\theta(z)$ and $\gamma(z)$ are polynomial expressions such that every term contains a power of at least one in z. Then let

$$\begin{aligned} H(z, \hat{\mu}) &= \hat{\mu} - p_0 - p_1 z - p_2 z^2 \\ &\quad - M[R^5 + R\gamma(z) + \nu(z)], \end{aligned} \tag{5.51}$$

be an implicit function such that $H(0, p_0) = 0$ and $H_{\hat{\mu}}(0, p_0) = -5Mp_0^4 + 1 \neq 0$. We now invoke the implicit function theorem [37], seeing that $\hat{\mu}$ is convergent and analytic in the neighbourhood of $(0, p_0)$ in the plane, and with a positive radius. We conclude that (5.45) is convergent in the same neighbourhood.

Hence, solution (5.45) is expressed as

$$
\begin{aligned}
y(z) \;=\;& a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \\
& \sum_{r_1=0}^{\infty}\Bigg(-\frac{(r_1+1)a_{r_1+1}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\
& -\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Bigg(5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}(r_2+2)(r_2+1)a_{r_2+2} \\
& \times\frac{(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)a_{r_1-2r_2+3}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\
& +5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}a_{r_2+1}a_{r_1-2r_2+4} \\
& \times\frac{(r_1-2r_2+4)(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\Bigg) \\
& -\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5} a_{r_6}a_{r_5-r_6}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3} \\
& \times\frac{a_{r_1-r_2+5}(r_1-r_2+4)(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\Bigg) z^{r_1+5}
\end{aligned}
\tag{5.52}
$$

Consequently, the exact series of (5.1) is constructed as follows

$$
\begin{aligned}
u(x,t) \;=\;& a_0 + a_1\left(-x+t\right) + a_2\left(-x+t\right)^2 \\
& + a_3\left(-x+t\right)^3 + a_4\left(-x+t\right)^4 + \\
& \sum_{r_1=0}^{\infty}\Bigg(- \frac{(r_1+1)a_{r_1+1}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\
& - \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Bigg(5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}(r_2+2)(r_2+1)a_{r_2+2} \\
& \times \frac{(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)a_{r_1-2r_2+3}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\
& + 5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}a_{r_2+1}a_{r_1-2r_2+4} \\
& \times \frac{(r_1-2r_2+4)(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\Bigg) \\
& - \sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5} a_{r_6}a_{r_5-r_6}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3} \\
& \times \frac{a_{r_1-r_2+5}(r_1-r_2+4)(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\Bigg) \times \\
& \left(-x+t\right)^{r_1+5},
\end{aligned}
\tag{5.53}
$$

where $a_0 \neq 0, a_1, a_2, a_3, a_4$ are arbitrary constants. Eq. (5.44) may then be applied to obtain the rest of the constants. This systematic approach generates the solution (5.17).

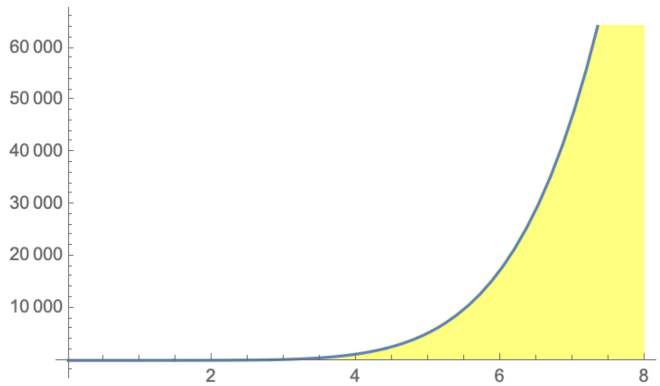
Analogously, albeit tediously, it is easy to test for convergence of the power series solutions that correspond to the reduced equations derived above. Furthermore, these series solutions are also easily transformed into the original variables of the given equations. Due to the voluminous convergence procedure above, we omit the detailed testing for the other cases. In Figures 5.1-5.2 we display plots of selected solutions.

The reductions involving (5.17) and (5.40) include a travelling wave variable, based on the invariant obtained from the particular combination of symmetries X_1 and X_2, but these plots are of the truncated series solutions, and hence are, at best, approximate in nature.

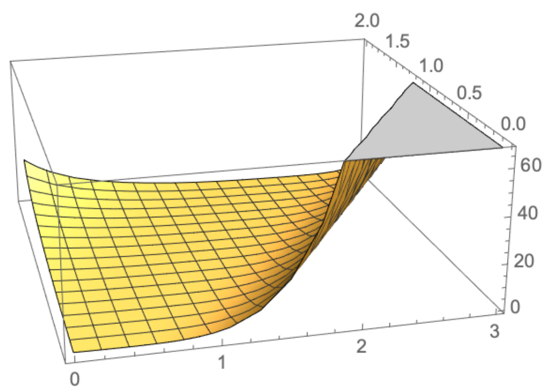
We observe that Case A's truncated solutions grows faster and larger than Case B's truncated solution. It is difficult to make any further physical interpretations for the truncated solutions.

5.5 Discussion and Conclusion

This is the first time that a symmetry analysis of the fifth-order Fujimoto-Watanabe equation have been performed. Nonlinear equations are always a challenging topic of interest. Secondly, the power series approach has not been applied to these highly nonlinear models before, hence such solutions have been reported here only. There are many studies of the third-order models but not the fifth-order ones above. Therefore this study offers new knowledge of the invariant properties of the fifth-order models.

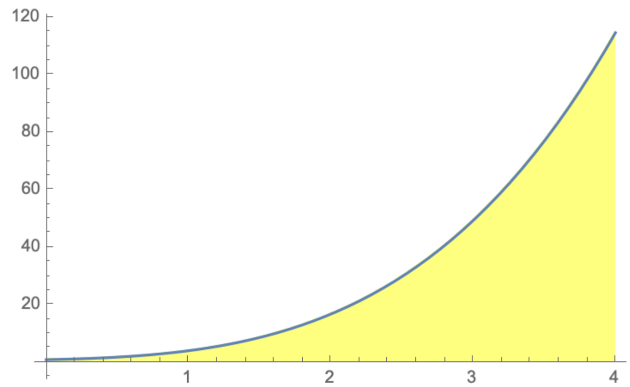


(a)

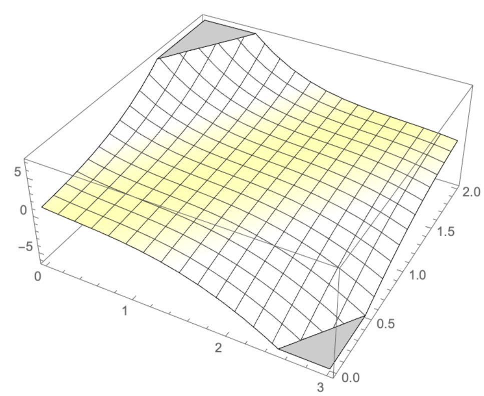


(b)

Figure 5.1: Graphical description of the analytical solutions with parameter values $\alpha = a_k = 1$: (a) Plot of Eq. (5.17); (b) 3D plot of Eq. (5.17)



(a)



(b)

Figure 5.2: Graphical description of the analytical solutions with parameter values $\alpha = a_k = 1$: (a) Plot of Eq. (5.40); and (b) 3D plot of Eq. (5.40).

Conclusion

Nonlinear PDEs have remained a topic of great interest, but the difficulty in their solution process often hinders the analysis of such equations. In this regard, vast work has been done on the basis of various structures known as point symmetries. In particular, we have studied the Fujimoto-Watanabe equations, the third-order polynomial equations and two fifth-order equations that are known to admit generalized symmetries. The Lie point symmetries and their commutator tables plus adjoints were calculated for the set of ten equations.

We also derived the conservation laws for each equation via the multiplier method given that these equations may not admit a Lagrangian. Equations that possess a Lagrangian will admit conservation laws via Noether's theorem.

The optimal one dimensional system of subalgebras was obtained for each group of Lie symmetries, and therefore invariants were determined to reduce the nonlinear PDEs to ODEs. However, the reduction process yielded extremely nonlinear equations as well, and the problem of finding solutions became more complicated. In such a situation, power series act as an active medium. The obtained series solutions have been verified by Mathematica and truncated solutions were plotted in some cases.

Exploring solutions via power series is a simple technique but is hugely time consuming. Nonetheless, we have successfully demonstrated that symmetries combined with power series are a significant duo in unraveling solutions to difficult equations, especially those that are highly nonlinear, and an added bonus is that there exist ways to address the associated convergence test of the resulting solutions.

In future research, we plan to explore the solutions provided by the above conservation laws by applying double reductions. Secondly, we plan to create interactive software worksheets that aid the derivation of power series solutions and their convergence testing procedure. While the procedure can be very tedious, software automation will enhance the efficiency of the test and promote further analysis of high-order nonlinear equations.

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