# Integrability in Giant Graviton Dynamics

**Augustine Larweh Mahu**
[aglarweh@gmail.com]

April 2018

*A thesis submitted to the Faculty of Science, University of The Witwatersrand, in fulfillment of the requirements for the degree of Doctor of Philosophy*

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

# Declaration

I, the undersigned, hereby declare that the work contained in this PhD thesis is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly. This thesis has not been submitted before for any degree or examination in any other university.

(**Augustine Larweh Mahu**)

# Abstract


In this thesis we will study a system of giant gravitons with strings stretched between them. These strings can assume an excited state and these excitations can be described as particles known as magnons. We discuss this system in the AdS/CFT context, where we focus on the $SU(2)$ and $SU(3)$ subsectors of the $\mathcal{N} = 4$ super Yang-Mills theory.

We consider the anomalous dimensions of restricted Schur polynomials constructed using $n \sim \mathcal{O}(N)$ complex adjoint scalars $Z$ and $m$ complex adjoint scalars $Y$. These operators belong to the $SU(2)$ sector of the theory. We fix $m \ll n$ so that our operators are almost half BPS. At leading order in $\frac{m}{n}$ this system corresponds to a dilute gas of $m$ free magnons. Adding the first correction of order $\frac{m}{n}$ to the anomalous dimension, which arises at two loops, we find nonzero magnon interactions. The form of this new operator mixing is studied in detail for a system of two giant gravitons with four strings attached.

We also consider computing the anomalous dimensions for operators belonging to the $SU(3)$ sector of the theory, and with bare dimension of order $N$. These operators are built out of $n$ $Z$, $m$ $Y$ and $p$ $X$ complex adjoint scalars. $n$, $m$ and $p$ all scale as $N$ in the large $N$ limit, and we fix $n \gg p+m$ and $\frac{m}{p} \sim 1$ so that our operators are again almost half BPS. For these operators the large $N$ limit and the planar limit are distinct and summing only the planar diagrams will not capture the large $N$ dynamics. Previous studies have computed the spectrum of anomalous dimensions for this class of operators, but neglected terms which were argued to be small. Dropping these terms and diagonalizing the dilatation operator reduces to diagonalizing a set of decoupled oscillators. We compute explicitely the terms which were neglected and show that diagonalizing the dilatation operator still reduces to diagonalizing a set of decoupled oscillators. Our key insight is that the problem is equivalent to bosons hopping on a lattice.

# Acknowledgements

I dedicate this work to my wife, Mary Warmann, for the many sacrifices she made during my PhD programme. I appreciate her patience and immense support through challenging times in my studies.

I am also grateful to my supervisor, Robert de Mello Koch, for giving me good guidance in my research work. I truly appreciate the effort he put into my education especially the advanced courses he gave to me and my colleagues. I also thank my office mates Phumudzo Rabambi, Lwazi Nkumani, and Hasina Tahiridimbisoa Nirina Maurice for the many insightful discussions we had on Physics.

I also thank the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation as well for supporting my research for the full three years.

I am grateful to God for seeing me through this research.

# Contents

# List of Figures

# Chapter 1

# Introduction

## 1.1 Overview of AdS/CFT correspondence

The Anti-de Sitter Space/Conformal Field Theory (AdS/CFT) Correspondence proposed by Maldacena [1] is a promising approach towards a definition of quantum gravity. The correspondence provides new ways of performing calculations where more conventional methods fail. It states that the 10-dimensional type IIB superstring theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with gauge group $SU(N)$, living in the flat 4-dimensional boundary of $AdS_5$. This equivalence implies one-to-one correspondence between all aspects of the theories including the observables, correlation functions and global symmetries. Apart from the perturbative spectrum of closed strings, there are $D$-branes and their open string excitations, as well as other spacetime geometries, living in the gauge theory. An interesting class of $D$-branes are the giant graviton branes [2, 3, 4]. We now have a good idea of how to describe the operators that correspond to certain examples of these branes. The examples we have in mind are almost $\frac{1}{2}$-Bogomol'nyi-Prasad-Sommerfield (BPS) giant gravitons. Importantly, the regimes in which it is possible to do calculations easily do not coincide on both sides of the correspondence. To state it differently, weakly coupled calculations on the String Theory side correspond to strongly coupled calculations on the CFT side and vice versa.

Following from Maldacena's conjecture, Witten in his paper [5] and the paper by Gubser, Klebanov and Polyakov (GKP) in [6] elaborated on the idea and proposed a precise correspondence between Conformal Field Theory observables and those of Supergravity. In particular, scaling dimensions of $\mathcal{N} = 4$ super Yang-Mills conformal field theory have a dual interpretation as energies in type IIB string theory on an asymptotically $AdS_5 \times S^5$ background. For this reason, we are interested in the dilatation operator which allows the computation of anomalous dimensions in a CFT. We will investigate the anomalous dimensions of a system of operators with bare dimension of $\mathcal{O}(N)$. Schrödinger's solution of the hydrogen atom problem found an energy spectrum, which played an important role in the development of quantum mechanics while in quantum electrodynamics the value of the Lamb shift played an important role in its development. Similarly, the eigenvalues of the dilatation operator gives the spectrum of anomalous dimensions of $\mathcal{N} = 4$ super Yang-Mills theory, which corresponds to the energy spectrum in the dual type IIB String Theory and we are of the view that this will also play an important role in quantum gravity. In the planar limit, the spectrum is completely determined [7]. One of the goals of this work is to take the first steps towards developing a similar understanding of large $N$ but non-planar limits of the theory.

Giant gravitons are a particular class of D-branes [2, 3, 4] and the corresponding $\frac{1}{2}$-BPS operators are of central interest in this thesis. The CFT operators are built from complex scalar fields $X$, $Y$ and $Z$ of the $\mathcal{N} = 4$ super Yang-Mills theory. Order $N$ fields are used in the construction so that these operators have a large dimension in the large $N$ limit. For operators with such a large classical dimension the usual large $N$ techniques, i.e. an expansion organised by genus of contributing ribbon graphs, is not possible [8]. New techniques to study the large $N$ limit have been developed. For the free theory, bases of operators that diagonalize the two point function to all orders in $\frac{1}{N}$ have been identified[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. This technology is applied first for descriptions in which the giant graviton plus open string system is treated using words in the gauge theory to represent the open string [20, 21, 22, 23, 24, 25, 26] and secondly descriptions which treat all fields in the operator democratically [27, 28, 29, 30].

Motivated largely by the AdS/CFT correspondence [1, 5, 6] dramatic progress in the computation of the spectrum of anomalous dimensions of $\mathcal{N} = 4$ super Yang-Mills theory has been made. This is due primarily, to the discovery of integrability in the planar limit of the theory [7, 31, 32]. In this thesis we seek to further explore integrability beyond the planar limit and this is part of our motivation for this study.

We will focus on operators built using $n \sim \mathcal{O}(N)$ $Z$s and $m$ $Y$s with $n \gg m$. This implies that we are close to the $\frac{1}{2}$-BPS giant graviton and that we have a new small parameter $\frac{m}{n}$. The condition that $n \gg m$ is crucial since a systematically small deformation of a BPS operator is simpler than a generic operator. Thus, we can consider, for example, the first order correction in a systematic $\frac{m}{n}$ expansion. This will lead to the study of the first subleading corrections in $\frac{m}{n}$ to the anomalous dimensions which we published in[33]. This paper [33] forms the entire chapter 5. We will also consider the $SU(3)$ sector of the $\mathcal{N} = 4$ super Yang-Mills theory where operators are built out of $n \sim \mathcal{O}(N)$ $Z$, $m \sim Y$ and $p \sim X$ complex adjoint scalars fields with $n \gg m + p$ and $\frac{m}{p} \sim 1$. For these operators, the leading large $N$ limit receives contributions from nonplanar diagrams [8]. For the set of problems we will study, the large $N$ limit and the planar limit are distinct and summing only the planar diagrams simply does not capture the large $N$ dynamics. This is the second problem we have considered in chapter 6. Group representation theory techniques have been effective in solving this problem [9, 11, 12, 13, 14, 15, 16, 18, 25, 26, 34, 35, 36].

The one loop dilatation operator in a large $N$ but non-planar limit has been diagonalized exactly [28, 29, 30]. A key insight of that work was the demonstration that the dilatation operator reduces to a collection of decoupled oscillators, thereby proving that it defines an integrable system. Since our system has a bare dimension of order $N$, the usual $\frac{1}{N}$ expansion is not applicable. Even the problem of computing the leading large $N$ limit is a formidable task. In [28, 37] it was understood that this problem can effectively be solved using group representation theory.

In chapter 2 we present relevant background in type IIB String theory, emphasizing those aspects of string theory relevant for holography. We also discusss one of the many possible ways by which we can obtain $\mathcal{N} = 4$ super Yang-Mills Theory by dimensional reduction of $d = 10$-dimensional $\mathcal{N} = 1$ super Yang-Mills Theory. A section is dedicated each to $AdS$ spacetime with negative curvature and a brief introduction to the Conformal group. The last section discusses Matrix models.

Chapter 3 discusses groups and their representations, with emphasis on the symmetric group and the unitary group. The study of these two groups are fundamental to technology we will use in computing the anomalous dimension, motivated by the Schur-Weyl duality. We will also discuss Schur's Lemmas and the fundamental orthogonality relation from which we are able to perform a Fourier transform on the Schur and restricted Schur polynomial bases.

In chapter 4 we introduce group theory technology which we will apply to the $\mathcal{N} = 4$ super Yang-Mills theory. Detailed discussion of Schur polynomials and restricted Schur polynomials are given. An appropriate and important approximation scheme called, the displaced corners approximation, is introduced together with an intertwinning map. The last section briefly discusses the dilatation operator in the $SU(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory.

The entire chapter 5 is a paper published in a peer review journal titled "Interacting double coset magnons" [33]. In chapter 6 we have solved the problem of determining the action of the one loop dilatation operator on restricted Schur polynomial basis in the $SU(3)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory. This action is diagonalized by mapping the problem into a system of particles hopping on a lattice [38]. In chapter 7 we give our conclusions and some discussion of our results. Background and technical details are collected in three Appendices to the thesis.

# Chapter 2

# Background

## 2.1 Type IIB String Theory

In this section we consider type IIB String Theory and its field content since it plays an essential role in the example AdS/CFT correspondence that we consider. Here we will study the Rammond, Neveu and Schwarz (RNS) formalism [39, 40] to quantize the Polyakov action, which employs worldsheet fermions.

The basic building blocks of String Theory include strings and membranes. We can have an open string or a closed string. An open string has two end points, while the closed string is a loop with no end. As the development of the theory matured, people realized that the theory required objects other than just strings. These objects came in the form of sheets or branes. Strings can attach at one or both ends to these branes. In general we can have a $D_p$-brane where a $D_p$-brane is a $p$-dimensional brane.

### 2.1.1 Open string

The worldsheet of a free open string is a strip. There are two types of boundary conditions that are possible for open strings. These boundary conditions appear when solving the partial differential equations as a result of varying the action $S$, and demanding that the variation $\delta S = 0$. The first is called Neumann or free endpoint boundary conditions. Consider the Nambu-Goto action

$$S_{NG} = \int d^2\gamma \mathcal{L} = -T \int d\tau d\sigma \sqrt{-\gamma} = -T \int d\tau d\sigma \sqrt{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2},$$

where $\dot{X} = \frac{\partial X^\mu}{\partial \tau}$, $X' = \frac{\partial X^\mu}{\partial \sigma}$ and

$$\gamma_{\alpha\beta} = \begin{bmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ X' \cdot \dot{X} & X'^2 \end{bmatrix}.$$

The boundary condition can be written as

$$\frac{\delta \mathcal{L}}{\delta X'^{\mu}}\Big|_{\sigma=0,\pi} = 0 \quad \text{or} \quad P^{\sigma}_{\mu}\Big|_{\sigma=0,\pi} = 0,$$
$$\Rightarrow \frac{\partial X^{\mu}}{\partial \sigma} = 0 \quad \text{where} \quad \sigma = 0, \pi.$$

Suppose the ends of the open string are fixed instead. In that case we have Dirichlet or fixed point boundary conditions. They are given by

$$\frac{\delta \mathcal{L}}{\delta \dot{X}^{\mu}}\Big|_{\sigma=0,\pi} = 0 \quad \text{or} \quad \frac{\partial X^{\mu}}{\partial \tau}\Big|_{\sigma=0,\pi} = 0.$$

Note that these boundary conditions are for the bosonic string. This boundary condition leads naturally to the concept of $D$-branes which are objects upon which open strings end.

### 2.1.2 Closed string

A closed string is a little loop moving through space-time. In this case, the worldsheet is a cylinder or a tube. The boundary conditions are periodic, described by

$$X^{\mu}(\tau, \sigma) = X^{\mu}(\tau, \sigma + \pi).$$

The bosonic string is short of describing nature since the spectrum does not contain any fermions and so there is no hope of recovering the standard model of particle Physics. Both the open and closed string spectrum contain a problematic tachyon state, and the critical dimension $d = 26$ is somewhat far from the 4-dimensional world we live in.

### 2.1.3 Ramond-Neveu-Schwarz formalism (RNS)

Introduction of worldsheet supersymmetry into the theory remedies the problem of the tachyon. Worldsheet supersymmetry forces us to include fermions in our theory. This leads to an enlarged Hilbert space, that can be reduced using the Gliozzi, Scherk and Olive (GSO) projection [41]. The GSO projection removes the tachyon from the Hilbert space. Thus the unwanted tachyon state is removed and our critical dimension of $d = 26$ reduces to $d = 10$. Modifying the Polyakov action to include fermions is a procedure called Ramond-Neveu-Schwarz (RNS) formalism [39, 40]. This approach is supersymmetric on the worldsheet. From the Polyakov action in conformal gauge

$$S_P = \frac{-T}{2} \int d\tau d\sigma \partial_{\alpha} X^{\mu} \partial^{\alpha} X_{\mu},$$

we include free fermions in the theory using the RNS formalism by adding a kinetic energy term for a Dirac field to the Lagrangian. This adds $d$ free fermionic fields $\psi^\mu$ to the action so that it assumes the form

$$S_{P'} = \frac{-T}{2} \int d\tau d\sigma \Big( \partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \Big),$$

where $\rho^\alpha$ are Dirac matrices on the worldsheet. These are given by

$$\rho^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \rho^1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

using an appropriate choice of basis. The fields $\psi^\mu = \psi^\mu(\sigma, \tau)$ are two-component Majorana spinors on the worldsheet. We use the notation $\psi^\mu_A$ where $\mu = 0, 1, \cdots, d-1$ and $A = \pm$ is the spinor index. The fermionic field $\psi$ can also be written as a column vector

$$\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}.$$

Note that $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ and $\bar{\psi}^\mu = (\psi^\dagger)^\mu \rho^0$. Using light cone coordinates $\sigma^\pm \equiv \tau \pm \sigma$, the fermionic part of the Polyakov action is given by

$$S_F = iT \int d\tau d\sigma \left( \psi^\mu_- \partial_+ \psi_{-\mu} + \psi^\mu_+ \partial_- \psi_{+\mu} \right).$$

The corresponding equation of motion is

$$\partial_+ \psi^\mu_- = \partial_- \psi^\mu_+ = 0.$$

This implies we have separate right and left moving waves, $\psi^\mu_-$ and $\psi^\mu_+$ respectively, as in the bosonic case. By applying boundary conditions to the fermionic part of the action we learn for the open string, the Ramond(R)-sector constitute integer spins and the Neveu-Schwarz(NS)-sector constitute half integer spins. The NS-NS and R-R sectors both gives spacetime bosons while NS-R and R-NS gives spacetime fermions. The same is achieved when we consider the closed string. After the GSO projection, we obtain a theory with spacetime supersymmetry.

In summary our type IIB string theory includes supersymmetry, and open and closed strings. These open strings have their ends attached to D-branes and fermions in this theory are chiral. This theory will contain massless fermions that do not have partners of the opposite handedness, as is observed in our world today.

One of the string theory backgrounds allowed by the equations of motion is a product space of $AdS_5$ with the sphere $S^5$. This compactification corresponds to the maximally supersymmetric case which is of interest to us. $AdS_5$ is a 5-dimensional Anti-de Sitter space with a negative curvature.

Anti-de Sitter space can be visualised as being saddle-shaped, and has a boundary at spatial infinity. It is on this 4-dimensional boundary that the quantum gravity dynamics are holographically described by the superconformal Yang-Mills theory: a field theory with supersymmetric conformal symmetry, or superconformal symmetry. The isometry group of $AdS_5$ can be identified with the conformal symmetry of the boundary theory. A closer look reveals that the symmetries of the two theories match exactly, as they must for this identification to hold.

## 2.2 $\mathcal{N} = 4$ super Yang-Mills theory

This section summarizes one of the many possible ways by which we can obtain $\mathcal{N} = 4$ super Yang-Mills Theory in Minkowski space. It is obtained as a dimensional reduction of a $d = 10$-dimensional $\mathcal{N} = 1$ super Yang-Mills Theory. We also discuss the field content of the theory.

Four dimensional $\mathcal{N} = 4$ super Yang-Mills theory is a very special quantum field theory. It is a four dimensional theory that has maximum supersymmetry. The possible gauge groups are subsets of the general linear group. This theory can be obtained by applying the method of dimensional reduction to the $9 + 1$-dimensional supersymmetric Yang-Mills theory. The action of the latter is given by

$$S = \int d^{10}x \,\mathrm{Tr}\Big( -\frac{1}{4} F_{MN}F^{MN} + i\bar{\Psi}\Gamma_M D^M \Psi\Big),$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N],$$

with gauge field $A_M$ and $g$ the coupling strength. The field strength $F_{MN}$ is antisymmetric and it satisfies the Bianchi identity

$$D_M F_{NP} + D_N F_{PM} + D_P F_{MN} = 0.$$

$D_M$ is the covariant derivative defined as follows

$$D_M \mathcal{O} = \partial_M \mathcal{O} - ig[A_M, \mathcal{O}],$$

where $\mathcal{O}$ is an element of the Lie algebra of the gauge group. $\Psi$ is a Majorana-Weyl spinor in $(9+1)$-dimensions which has 16 real components. Note that $M, N = 0, \cdots, 9$. The dimensional reduction takes $d^{10}x \to d^4x$ but it also renames 6 reduced spatial directions of $A_M$ as six scalars $\Phi_a$ in $d = 4$. The spatial derivatives of the corresponding 6 directions are set to zero.

The Dirac matrices can be written as

$$\gamma^\mu = \begin{bmatrix} 0_2 & \bar{\sigma}^\mu_{\alpha\dot{\beta}} \\ \sigma^{\mu\alpha\dot{\beta}} & 0_2 \end{bmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{bmatrix},$$

where $\sigma^{\mu\alpha\dot{\beta}} = (1, \vec{\sigma})$ and $\bar{\sigma}^{\mu}_{\alpha\dot{\beta}} = (1, -\vec{\sigma})$ with the vector $\vec{\sigma}$ of Pauli matrices. In six-dimensional Euclidean space with signature of the metric $\eta = \mathrm{diag}(-,-,-,-,-,-)$, the Clifford algebra is $\{\hat{\gamma}^a, \hat{\gamma}^b\} = -2\delta^{ab}$, and we can write the corresponding Dirac matrices as

$$\hat{\gamma}^a = \begin{bmatrix} 0_4 & \Sigma^{aAB} \\ \bar{\Sigma}^a_{AB} & 0_4 \end{bmatrix}, \quad \hat{\gamma}^7 = i\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\hat{\gamma}^4\hat{\gamma}^5\hat{\gamma}^6 = \begin{bmatrix} 1_4 & 0_4 \\ 0_4 & -1_4 \end{bmatrix},$$

where $\Sigma$ and $\bar{\Sigma}$ denote sigma matrices defined by

$$\begin{aligned} (\Sigma^{1AB}, \cdots, \Sigma^{6AB}) &= (\eta_{1AB}, \eta_{2AB}, \bar{\eta}_{3AB}, i\bar{\eta}_{1AB}, i\bar{\eta}_{2AB}, i\eta_{3AB}), \\ (\bar{\Sigma}^1_{AB}, \cdots, \bar{\Sigma}^6_{AB}) &= (\eta_{1AB}, \eta_{2AB}, \bar{\eta}_{3AB}, -i\bar{\eta}_{1AB}, -i\bar{\eta}_{2AB}, -i\eta_{3AB}), \end{aligned} \quad (2.1)$$

and these are defined in terms of the 't Hooft symbols

$$\begin{aligned} \eta_{iAB} &:= \epsilon_{iAB} + \delta_{iA}\delta_{4B} - \delta_{iB}\delta_{4A}, \\ \bar{\eta}_{iAB} &:= \epsilon_{iAB} - \delta_{iA}\delta_{4B} + \delta_{iB}\delta_{4A}. \end{aligned}$$

We will make use of the anti-symmetric four tensor $\epsilon_{ABCD}$ which is normalized to $\epsilon_{1234} = 1$. This is then used to raise and lower indices according to the definition in equation (2.1). The anti-symmetric four tensor satisfies

$$\epsilon_{DABC}\epsilon_{DKLM} = \delta^{KLM}_{ABC} + \delta^{KLM}_{BCA} + \delta^{KLM}_{CAB} - \delta^{KLM}_{CBA} - \delta^{KLM}_{BAC} - \delta^{KLM}_{ACB},$$

where the delta symbols are defined to be 1 if the lower indices match with the upper indices, and 0 otherwise. These sigma matrices defined above satisfy the following identities

$$\begin{aligned} \bar{\Sigma}^a_{AB} = (\Sigma^{aAB})^* = \frac{1}{2}\varepsilon_{ABCD}\Sigma^{aCD}, \quad \Sigma^{aAB} = \frac{1}{2}\varepsilon_{ABCD}\bar{\Sigma}^a_{CD}, \\ \bar{\Sigma}^a_{AB}\bar{\Sigma}^a_{CD} = 2\varepsilon_{ABCD}, \quad \Sigma^{aAB}\bar{\Sigma}^b_{CD} = 4\delta^{ab}. \end{aligned}$$

A close look at the first equation tells us that $\bar{\Sigma}^a_{AB} = -\bar{\Sigma}^a_{BA}$. Let us define

$$\phi^{AB} := \frac{1}{\sqrt{2}}\Sigma^{aAB}\phi^a, \quad \bar{\phi}_{AB} := \frac{1}{\sqrt{2}}\Sigma^a_{AB}\phi^a.$$

These fields are related as follows

$$\bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2}\varepsilon_{ABCD}\phi^{CD}, \quad \phi^{AB} = (\bar{\phi}_{AB})^* = \frac{1}{2}\varepsilon_{ABCD}\bar{\phi}_{CD}.$$

Now we let $\xi$ and $\Psi$ be the Majorana-Weyl spinors. Then

$$\begin{aligned} \tilde{\xi}\Gamma^\mu\Psi &= \tilde{\xi}_{\dot{\alpha}A}\sigma^{\mu\dot{\alpha}\beta}\psi^A_\beta + \xi^{\alpha A}\bar{\sigma}^\mu_{\alpha\dot{\beta}}\tilde{\psi}^{\dot{\beta}}_A, \\ \tilde{\xi}\Gamma^{a+3}\Psi &= -\Sigma^{aAB}\tilde{\xi}_{\dot{\alpha}A}\tilde{\psi}^{\dot{\alpha}}_B + \Sigma^a_{AB}\xi^{\alpha A}\psi^B_\alpha, \end{aligned}$$

where

$$\tilde{\xi} = \xi^\dagger \Gamma^0 = (1 \quad 0) \otimes (0 \quad \tilde{\xi}_{\dot{\alpha}A}) + (0 \quad 1) \otimes (\xi^{\alpha A} \quad 0).$$

Note that $(\psi^A_\alpha)^* = \tilde{\psi}_{\dot{\alpha}A}$, and $A = 1, 2, 3, 4$. Thus, we have,

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big( &-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi^{AB})(D^\mu\bar{\phi}_{AB}) + \frac{1}{8}g^2[\phi^{AB}, \phi^{CD}][\bar{\phi}_{AB}, \bar{\phi}_{CD}] \\ &+ 2i\tilde{\psi}_{\dot{\alpha}A}\sigma_\mu^{\dot{\alpha}\beta}D^\mu\psi^A_\beta - \sqrt{2}g\psi^{\alpha A}[\bar{\phi}_{AB}, \psi^B_\alpha] - \sqrt{2}g\tilde{\psi}_{\dot{\alpha}A}[\phi^{AB}, \tilde{\psi}^{\dot{\alpha}}_B]\Big).\end{aligned}$$

## 2.3 Anti-de Sitter spacetime

In this section we will study the spacetime which has a constant negative curvature, $R < 0$, called anti-de Sitter($AdS$) spacetime. According to the AdS/CFT correspondence, quantum gravity in AdS spacetime is completely equivalent to the dynamics of a conformal field theory which lives on the boundary of the spacetime. Conformally compactifying $AdS$, the boundary at $\theta = \frac{\pi}{2}$, is identical to conformally compactified Minkowski space. This is one of the key points of the AdS/CFT correspondence – the boundary of $AdS_5$ is the conformal compactification of Minkowski spacetime in $(3+1)$-dimensions, where the conformal field theory, $\mathcal{N} = 4$ super Yang-Mills theory lives. For this reason we will study the conformal structure of both AdS spacetime and Minkowski spacetime.

The $(p+2)$-dimensional anti-de Sitter spacetime ($AdS_{p+2}$) is the surface

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = R^2, \tag{2.2}$$

embedded in $(p+3)$-dimensional spacetime, with $R$ the constant radius of curvature. Depending on the choice of co-ordinate system one can realize different forms of the $AdS$ metric. The metric of the $(p+3)$-dimensions of the embedding space is

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2. \tag{2.3}$$

To solve equation (2.2) consider the co-ordinate system defined by

$$\begin{aligned} X_0 &= R\cosh\rho\cos\tau, \\ X_{p+2} &= R\cosh\rho\sin\tau, \\ X_i &= R\sinh\rho\,\Omega_i, \end{aligned}$$

where $i = 1, \cdots, p+1$ and $\sum_{i=1}\Omega_i^2 = 1$. The induced metric on $AdS_{p+2}$ is

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_p^2). \tag{2.4}$$

The entire hyperboloid is covered when we consider $0 \leq \rho$ and $0 \leq \tau < 2\pi$ and so we call $(\tau, \rho, \Omega_i)$ the global coordinates of $AdS$. Near $\rho = 0$ the metric is given by $ds^2 \simeq R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega_p^2)$ and the hyperboloid has the topology $S^1 \times \mathbb{R}^{p+1}$ where $S^1$ represents closed timelike curves in the $\tau$ direction.

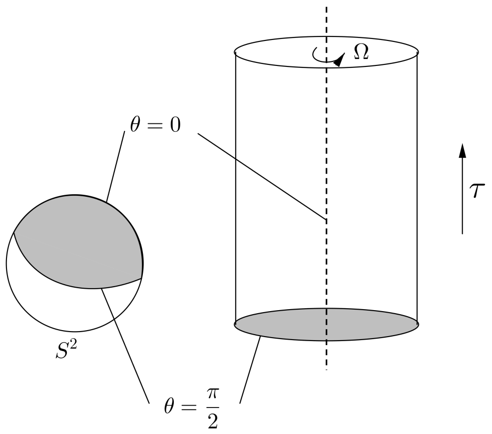



Figure 2.1: $AdS_3$ can be conformally mapped into one half of the Einstein static universe $\mathbb{R} \times S^2$.

### 2.3.1 Conformal structure of (p+2)-dimensional Minkowski spacetime

Consider a $p+2$-dimensional Minkowski spacetime, $\mathbb{R}^{1,p+1}$ with the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_p^2, \tag{2.5}$$

where $d\Omega_p^2$ is the spatial metric defined on the $p$-sphere. The radial coordinate is only defined for $r \geq 0$. Consider the transformation

$$\begin{aligned} v &= t + r, \\ u &= t - r. \end{aligned}$$

This implies $v \geq u$ and $-\infty < u, v < +\infty$. Substituting into equation (2.5) we have

$$ds^2 = -dudv + \frac{(v-u)^2}{4} d\Omega_p^2.$$

Next if we apply the transformation $u = \tan \tilde{u}$ and $v = \tan \tilde{v}$ so that $-\frac{\pi}{2} \leq \tilde{u}, \tilde{v} < \frac{\pi}{2}$ our metric above takes the form

$$ds^2 = \frac{1}{\cos^2 \tilde{u} \cos^2 \tilde{v}} \Big( - d\tilde{u} d\tilde{v} + \frac{1}{4} \sin^2(\tilde{v} - \tilde{u}) d\Omega_p^2 \Big). \tag{2.6}$$

Finally, if we make the transformation $\tau = \tilde{v} + \tilde{u}$ and $\chi = \tilde{v} - \tilde{u}$ to obtain

$$ds^2 = \frac{1}{4\cos^2(\frac{\tau+\chi}{2})\cos^2(\frac{\tau-\chi}{2})}\Big( - d\tau^2 + d\chi^2 + \sin^2\chi d\Omega_p^2\Big). \qquad (2.7)$$

This can be written as

$$ds'^2 = \Big( - d\tau^2 + d\chi^2 + \sin^2\chi d\Omega_p^2\Big),$$

where

$$ds^2 = \frac{1}{4\cos^2(\frac{\tau+\chi}{2})\cos^2(\frac{\tau-\chi}{2})}ds'^2.$$

This observation will be important when we look at the conformal compactification of $AdS_{p+2}$ in the next subsection

### 2.3.2 Conformal structure of $\mathbf{AdS}_{p+2}$

Let us focus on studying the causal structure of $AdS_{p+2}$. Towards this end, make the following co-ordinate transformations, $\tan\theta = \sinh\rho$ and $0 \leq \theta < \frac{\pi}{2}$. We learn that

$$\tan^2\theta + 1 = \sinh^2\rho + 1,$$
$$\Rightarrow \frac{1}{\cos^2\theta} = \cosh^2\rho,$$

and so our metric takes the form

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_p^2).$$

The metric of the Einstein static universe is obtained by multiplying the above metric with $R^{-2}\cos^2\theta$. Thus, the prefactor is removed by conformal transformation, and what we have left is the conformally transformed metric given by

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_p^2.$$

The original global coordinate $\rho$ was defined for $\rho \geq 0$, and therefore $0 \leq \theta \leq \frac{\pi}{2}$. However, as previously noted in the Minkowski case, the metric is now regular for $\theta = \frac{\pi}{2}$ and the space can be conformally compactified to include this point - we can now see that $AdS_{p+2}$ is conformal to one-half of the Einstein Static Universe. Just like flat space, if we can conformally compactify a spacetime and obtain the same conformal structure as $AdS$ then we call it asymptotically $AdS$. Conformally compactifying $AdS$ by including $\theta = \frac{\pi}{2}$, we find that the boundary is identical to conformally compactified Minkowski space. This is one of the key points of the AdS/CFT correspondence – the boundary of $AdS_5$ is 3 + 1-dimensional Minkowski spacetime, where the conformal field theory, $\mathcal{N} = 4$ super Yang-Mills lives.

### 2.3.3 Poincaré patch

It is useful to introduce another set of coordinates $(u, t, \vec{x})$ for $u > 0, \vec{x} \in \mathbb{R}^p$ defined by

$$\begin{aligned} X_0 &= \frac{1}{2u}\Big(1 + u^2(R^2 + \vec{x}^2 - t^2)\Big), \\ X_{p+2} &= Rut, \\ X^i &= Rux^i \quad (i = 1, \cdots, p), \\ X^{p+1} &= \frac{1}{2u}\Big(1 - u^2(R^2 - \vec{x}^2 + t^2)\Big). \end{aligned}$$

We show below that the surface $u = 0$ is the Poincaré Killing horizon and $u \to \infty$ approaches the boundary of $AdS_{p+2}$ spacetime. Thus, unlike the global coordinates associated with the induced metric in equation (2.4), the Poincaré coordintates $(u, t, \vec{x})$ only cover half of the hyperboloid in equation (2.2). The half of spacetime which is not covered corresponds to $u \leq 0$.

$$\begin{aligned} dX_0 &= \frac{1}{2}\Big(-\frac{1}{u^2} + R^2 + \vec{x}^2 - t^2\Big)du + u(\vec{x}d\vec{x} - tdt), \\ dX_0^2 &= \frac{1}{4}\Big(-\frac{1}{u^2} + R^2 + \vec{x}^2 - t^2\Big)^2 du^2 + u^2(\vec{x}d\vec{x} - tdt)^2 \\ &\quad + u\Big(-\frac{1}{u^2} + R^2 + \vec{x}^2 - t^2\Big)(\vec{x}d\vec{x} - tdt)du, \end{aligned} \tag{2.8}$$

$$\begin{aligned} dX_{p+1} &= -\frac{1}{2}\Big(\frac{1}{u^2} + R^2 - \vec{x}^2 + t^2\Big)du - u(-\vec{x}d\vec{x} + tdt), \\ dX_{p+1}^2 &= \frac{1}{4}\Big(\frac{1}{u^2} + R^2 - \vec{x}^2 + t^2\Big)^2 du^2 + u^2(-\vec{x}d\vec{x} + tdt)^2 \\ &\quad + u\Big(\frac{1}{u^2} + R^2 - \vec{x}^2 + t^2\Big)(-\vec{x}d\vec{x} + tdt)du, \end{aligned} \tag{2.9}$$

$$\begin{aligned} dX_i &= R(x_i du + u dx_i), \\ dX_i^2 &= R^2(x_i^2 du^2 + u^2 dx_i^2 + 2x_i u dx_i du), \end{aligned} \tag{2.10}$$

$$\begin{aligned} dX_{p+2} &= R(tdu + udt), \\ dX_{p+2}^2 &= R^2(t^2 du^2 + u^2 dt^2 + 2tudtdu), \end{aligned} \tag{2.11}$$

In terms of the new co-ordinates

$$
\begin{aligned}
ds^2 &= -dX_0^2 + dX_{p+1}^2 - dX_{p+2}^2 + \sum_{i=1}^{p} dX_i^2 \\
&= R^2\Big(\frac{1}{u^2} - \vec{x}^2 + t^2\Big)du^2 + 2R^2u(-\vec{x}d\vec{x} + tdt)du - R^2(t^2du^2 + u^2dt^2 + 2tudtdu) \\
&\quad + \sum_{i=1}^{p} R^2(x_i^2du^2 + u^2dx_i^2 + 2x_iudx_idu) \\
&= \frac{R^2}{u^2}du^2 - R^2u^2dt^2 + R^2u^2d\vec{x}^2 + 2R^2u(-\vec{x}d\vec{x} + tdt)du - 2R^2u(-\vec{x}d\vec{x} + tdt)du \\
&= R^2\Big(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2)\Big), \qquad (2.12)
\end{aligned}
$$

where in the last two lines we have used the fact that

$$\sum_{i=1}^{p} R^2(x_i^2du^2 + u^2dx_i^2 + 2x_iudx_idu) = R^2(\vec{x}^2du^2 + u^2d\vec{x}^2 + 2\vec{x}\cdot d\vec{x}\ udu).$$

We can write equation (2.12) in a useful way by introducing the transformation $u = \frac{1}{z}$ which leads to

$$ds^2 = \frac{R^2}{z^2}\Big(dz^2 + (-dt^2 + d\vec{x}^2)\Big).$$

This metric makes manifest the isometries given by the subgroups $ISO(1,p)$ and $SO(1,1)$ of the $SO(2,p+1)$ isometry, where the $ISO(1,p)$ is the Poincaré transformation on $(t,\vec{x})$ while $SO(1,1)$ transforms as follows

$$(t,\vec{x},u) \to (ct, c\vec{x}, c^{-1}u), \quad c > 0.$$

This last transformation is identified in the AdS/CFT correspondence as transformations generated by the dilatation operator $D$ in the conformal symmetry group of $\mathbb{R}^{1,p}$.

## 2.4 Conformal group and algebra

Conformal Field Theory (CFT) is Quantum Field Theory (QFT) which is invariant under the *conformal group*. In QFT the renormalization group describes how to go from the ultra violet (UV) to the infra red (IR) degrees of freedom [42]. The endpoints of this renormalization group flow are described by CFTs. The conformal group is characterized as the set of transformations which preserves angles which allows a local rescaling of the metric. A conformal transformation of the coordinates is an invertible mapping $x \to x'$, which leaves the metric tensor invariant up to a scale [43]

$$g'_{\mu\nu}(x') = \Omega^2(x)g_{\mu\nu}(x). \qquad (2.13)$$

The Poincaré transformations which are the Lorentz transformations, together with translations, is a subgroup of conformal group. This corresponds to a scale factor $\Omega^2(x) = 1$. Consider real transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{2.14}$$

In (2.14) the matrices $\Lambda^\mu_\nu$ defines three independent boosts and three independent rotations, which gives the Lorentz transformations. The term $a^\mu$ describes translations in spacetime with four independent components. Thus, Poincaré transformations have ten generators. We have assumed the Minkowski metric $\eta_{\mu\nu}$ with signature $(+,-,-,-)$.

We define the invariant distance between two spacetime points separated by $dx^\nu$ as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{2.15}$$

Another important transformation called a *scale transformation*, acts by rescaling or zooming in or out. Under this transformation co-ordinates transform as follows

$$x^\mu \to \lambda x^\mu. \tag{2.16}$$

Using (2.15) and infinitesimal form of (2.16) we have,

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}(\lambda dx^\mu)(\lambda dx^\nu), \\ &= \lambda^2 \eta_{\mu\nu} dx^\mu dx^\nu, \end{aligned} \tag{2.17}$$

and thus, comparing (2.15) and (2.17) it is obvious that the metric $\eta_{\mu\nu}$ transforms according to the transformation rule

$$\eta_{\mu\nu} \to \lambda^2 \eta_{\mu\nu}. \tag{2.18}$$

According to (2.18) scale transformations locally rescale the metric. In order to ensure the Heisenberg uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$ is satisfied after the scaling (2.16), we require transformation law for conjugate momentum $p^\mu$ to be of the form

$$p'^\mu = -i\frac{\partial}{\partial x^\mu} = -i\frac{\partial x^\mu}{\partial x'^\mu}\frac{\partial}{\partial x^\mu} = -i\frac{1}{\lambda}\frac{\partial}{\partial x^\mu} = \frac{1}{\lambda}p^\mu.$$

The Heisenberg uncertainty principle in the transformed co-ordinate is

$$\Delta(\lambda x)\Delta(\frac{1}{\lambda}p) \geq \frac{\hbar}{2}$$

and the commutation relation for the conjugate variables $x^\mu$ and $p^\mu$ is

$$[\lambda x^\mu, \frac{1}{\lambda}p^\mu] = -i\hbar.$$

In addition to the above set of transformations, we consider special conformal transformations parametrized by a constant vector $b^\mu$ given by

$$x^\mu \to x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2 x^2}. \tag{2.19}$$

A simple way of arriving at the above transformation involves introducing the inversion map

$$I : x^\mu \to x'^\mu = \frac{x^\mu}{x^\rho x_\rho} = \frac{x^\mu}{x^2}.$$

A special conformal transformation can be realized as the composition of the transformations

$$I \circ T(b) \circ I : x^\mu \to x'^\mu,$$

with $T(b)$ the transformation for translation. Indeed, a simple computation shows

$$\begin{aligned}(I \circ T(b) \circ I)(x^\mu) &= (I \circ T(b)) \left(\frac{x^\mu}{x^2}\right)\\ &= I\left(\frac{x^\mu}{x^2} + b^\mu\right)\\ &= \frac{\frac{x^\mu}{x^2} + b^\mu}{\frac{1}{x^2} + 2\frac{b\cdot x}{x\cdot x} + b^2}.\end{aligned}$$

The infinitesimal form of the special conformal transformation is obtained by neglecting quadratic terms in $b$, which gives

$$\begin{aligned}(I \circ T(b) \circ I)(x^\mu) \approx \frac{x^\mu + x^2 b^\mu}{1 + 2(b\cdot x)} &\approx \left(x^\mu + x^2 b^\mu\right)(1 - 2(b\cdot x)),\\ &= x^\mu - 2(b\cdot x)x^\mu + x^2 b^\mu.\end{aligned} \tag{2.20}$$

Given the transformations we have discussed so far we anticipate the solution to the conformal Killing equation to be of the form

$$\epsilon^\mu = a^\mu + \Lambda^\mu{}_\nu x^\mu + \lambda x^\mu + x^2 b^\mu - 2x^\mu x_\alpha b^\alpha. \tag{2.21}$$

### 2.4.1 Condition for conformal invariance

We will study infinitesimal coordinate transformations. Towards this end, consider

$$x'^\rho = x^\rho + \epsilon^\rho(x) + \mathcal{O}(\epsilon^2), \tag{2.22}$$

and using the metric tensor $\eta_{\mu\nu}$, we lower the index of $\epsilon^\mu$ as $\epsilon_\mu = \eta_{\mu\nu}\epsilon^\nu$. From the right hand side of (2.18) we write the transformation for a metric tensor of rank (2), $\eta_{\mu\nu}$ as

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} \quad \text{and} \quad \eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Omega^2(x)\eta_{\mu\nu}. \tag{2.23}$$

Note that $\Omega^2(x)$ is playing the role of $\lambda^2$ in (2.18) for the *scale transformation*. Substituting (2.22) into (2.23) we get,

$$\begin{aligned}\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} &= \eta_{\rho\sigma}\left(\frac{\partial x^{\rho}}{\partial x^{\mu}}+\frac{\partial \epsilon^{\rho}}{\partial x^{\mu}}+\mathcal{O}(\epsilon^2)\right)\left(\frac{\partial x^{\sigma}}{\partial x^{\nu}}+\frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}}+\mathcal{O}(\epsilon^2)\right)\\ &= \eta_{\mu\nu}+\eta_{\mu\sigma}\frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}}+\eta_{\rho\nu}\frac{\partial \epsilon^{\rho}}{\partial x^{\mu}}+\mathcal{O}(\epsilon^2)\\ &= \eta_{\mu\nu}+\left(\frac{\partial \epsilon_{\mu}}{\partial x^{\nu}}+\frac{\partial \epsilon_{\nu}}{\partial x \mu}\right)+\mathcal{O}(\epsilon^2).\end{aligned} \tag{2.24}$$

We want to know the conditions on $\epsilon_\mu$ in order that we obtain a conformal transformation. From (2.24) we write,

$$\partial_\nu\epsilon_\mu+\partial_\mu\epsilon_\nu=\bar{\Omega}^2(x)\eta_{\mu\nu}. \tag{2.25}$$

We determine the function $\bar{\Omega}^2(x)=\Omega^2(x)-1$ by contracting with the metric $\eta^{\mu\nu}$, over $\mu$ and $\nu$ indices as

$$\begin{aligned}\eta^{\mu\nu}\left(\partial_\nu\epsilon_\mu+\partial_\mu\epsilon_\nu\right) &= \bar{\Omega}^2(x)\eta^{\mu\nu}\eta_{\mu\nu},\\ 2\partial^\mu\epsilon_\mu &= d\bar{\Omega}^2(x).\end{aligned} \tag{2.26}$$

Thus, we obtain the conformal Killing equation [43]

$$\partial_\nu\epsilon_\mu+\partial_\mu\epsilon_\nu=\frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu}. \tag{2.27}$$

Next take the derivative $\partial_\rho$ of (2.27) and apply cyclic permutation to find

$$\partial_\rho\partial_\nu\epsilon_\mu+\partial_\rho\partial_\mu\epsilon_\nu=\frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial\cdot\epsilon), \tag{2.28}$$

$$\partial_\nu\partial_\mu\epsilon_\rho+\partial_\nu\partial_\rho\epsilon_\mu=\frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial\cdot\epsilon), \tag{2.29}$$

$$\partial_\mu\partial_\rho\epsilon_\nu+\partial_\mu\partial_\nu\epsilon_\rho=\frac{2}{d}\eta_{\nu\rho}\partial_\mu(\partial\cdot\epsilon). \tag{2.30}$$

Adding the last two equations and subtracting the first equation from it gives

$$2\partial_\mu\partial_\nu\epsilon_\rho=\frac{2}{d}(\eta_{\rho\mu}\partial_\nu+\eta_{\nu\rho}\partial_\mu-\eta_{\mu\nu}\partial_\rho)(\partial\cdot\epsilon). \tag{2.31}$$

Act with $\partial_\nu$ on both sides of equation (2.31)

$$2\partial_\mu\partial_\nu(\partial\cdot\epsilon)=\frac{4}{d}\partial_\mu\partial_\nu(\partial\cdot\epsilon)-\frac{2}{d}\eta_{\mu\nu}\Box(\partial\cdot\epsilon). \tag{2.32}$$

This implies that

$$\partial_\mu\partial_\nu(\partial\cdot\epsilon)=0 \qquad (d\neq 0), \tag{2.33}$$

since $\eta_{\mu\nu}\Box(\partial \cdot \epsilon) = 0$. Differentiate equation (2.28) with respect to $\partial_\alpha$, subtract the last equation from the first two equations and use equation (2.33) to find

$$2\partial_\alpha\partial_\rho\partial_\nu\epsilon_\mu = 0. \tag{2.34}$$

This implies that $\epsilon_\mu$ is a quadratic polynomial. Thus, we can write the most general solutions to the conformal Killing equation (2.27) as

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho. \tag{2.35}$$

Fixing the coefficients in (2.35) by requiring that (2.26) holds we can recover (2.21).

### 2.4.2 Conformal group in $D \geq 3$ dimensions

The table below summarizes finite conformal transformations and their corresponding generators.

| Transformations | | Notation | Generators |
|---|---|---|---|
| Translation | $x'^\mu = x^\mu + a^\mu$ | $P_\mu$ | $-i\partial_\mu$ |
| Dilation | $x'^\mu = \lambda x^\mu$ | $D$ | $-ix^\mu\partial_\mu$ |
| Rotation | $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ | $M_{\mu\nu}$ | $i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ |
| SCT | $x'^\mu = \frac{x^\mu - (x\cdot x)b^2}{1-2(b\cdot x)+(x.x)b^2}$ | $K_\mu$ | $-i\Big(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\Big)$. |

## 2.5 Matrix models

In this section we will use the path integral quantization. Our fields are matrices hence the name matrix model. We will try to motivate the claim that string theory, defined by a perturbative expansion, can be considered a dual description of non-Abelian gauge theories when the rank of the gauge group is large [5, 44, 45, 46]. Consider a 0-dimensional theory of the matrix valued fields $M$, which are $N \times N$ hermitian matrices transforming in the adjoint representation of the Unitary group $U(N)$

$$M \to M' = U^\dagger M U.$$

A 0-dimensional universe has only a single point, so there is no notion of a local symmetry. However, we declare that physical observables are invariant under the above transformation. In this sense, we say that our observables are gauge invariant. Consider the action

$$S = \frac{\omega}{2}\,\mathrm{Tr}(M^2),$$

and the corresponding partition function defined as

$$Z = \int [dM] e^{-S}.$$

For any QFT we are interested in the correlation functions. We define correlation functions as

$$\langle M_{ij} M_{kl} \cdots M_{pq} \rangle = \mathcal{N} \int [dM] e^{-S} M_{ij} M_{kl} \cdots M_{pq}.$$

Introducing a source term $\mathrm{Tr}(JM)$ into the partition function gives the generating function

$$Z[J] = \int [dM] e^{-S + \mathrm{Tr}(JM)}. \tag{2.36}$$

From equation (2.36) we compute the correlators as follows

$$\langle M_{ij} M_{kl} \cdots M_{pq} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \cdots \frac{d}{dJ_{qp}} Z[J]|_{J=0}.$$

The integral in (2.36) is over $N^2$ independent parameters since $M$ is hermitian. Let $\mathrm{Re}[M_{ij}]$ be the real part and $\mathrm{Im}[M_{ij}]$ the imaginary part of matrix element $M_{ij}$. We can write the integration measure as

$$[dM] = \mathcal{N} \prod_{i=1}^{N} dM_{ii} \prod_{i<j}^{N} d(\mathrm{Re}[M_{ij}]) d(\mathrm{Im}[M_{ij}]),$$

and

$$\mathrm{Tr}(M^2) = \sum_i M_{ii}^2 + \sum_{\substack{i,j \\ i \neq j}} \Big( (\mathrm{Re}[M_{ij}])^2 + (\mathrm{Im}[M_{ij}])^2 \Big).$$

We fix the normalization $\mathcal{N}$ of the measure by requiring that

$$\begin{aligned} Z[J=0] &= 1 \\ &= \mathcal{N} \int \prod_{i=1}^{N} dM_{ii} \prod_{i<j}^{N} d(\mathrm{Re}[M_{ij}]) d(\mathrm{Im}[M_{ij}]) e^{-\frac{\omega}{2}\left(\sum_i M_{ii}^2 + \sum_{\substack{i,j \\ i\neq j}} \left(\mathrm{Re}[M_{ij}])^2 + (\mathrm{Im}[M_{ij}])^2\right)\right)} \\ &= \mathcal{N} \left( \sqrt{\frac{2\pi}{\omega}} \right)^N \left( \sqrt{\frac{\pi}{\omega}} \right)^{\frac{1}{2}N(N-1)}. \end{aligned} \tag{2.37}$$

Hence, $\mathcal{N} = \left(\frac{1}{\sqrt{2}}\right)^N \left(\sqrt{\frac{\omega}{\pi}}\right)^{N^2}$. The generating function we obtain after performing the Gaussian integral in equation (2.36) is

$$Z[J] = e^{\frac{1}{2\omega} \mathrm{Tr}(J^2)}. \tag{2.38}$$

The two point correlation function is given by

$$\begin{aligned}\langle M_{ij} M_{kl} \rangle &= \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} Z[J]|_{J=0} \\ &= \Big( \frac{1}{\omega} \delta_{jk} \delta_{il} + \frac{J_{kl}}{\omega^2} J_{ij} \Big) e^{\frac{1}{2\omega} \operatorname{Tr}(J^2)} |_{J=0} \\ &= \frac{1}{\omega} \delta_{jk} \delta_{il}.\end{aligned}$$

Using this rule, the 2-point function of the traced operator is

$$\begin{aligned}\langle \mathcal{O}_2 \rangle &= \langle \operatorname{Tr}(M^2) \rangle \\ &= \langle M_{ij} M_{ji} \rangle \\ &= \frac{N^2}{\omega},\end{aligned} \tag{2.39}$$

where repeated indices are summed so that $\delta_{ii} = N$. Similarly

$$\begin{aligned}\langle \mathcal{O}_4 \rangle &= \langle \operatorname{Tr}(M^4) \rangle \\ &= \langle M_{ij} M_{jk} M_{kl} M_{li} \rangle \\ &= \frac{1}{\omega^2} (\delta_{jj} \delta_{ik} \delta_{ll} \delta_{ki} + \delta_{jk} \delta_{il} \delta_{kl} \delta_{ji} + \delta_{jl} \delta_{ii} \delta_{kk} \delta_{jl}) \\ &= \frac{1}{\omega^2} (2N^3 + N).\end{aligned} \tag{2.40}$$

From the examples above, we can read off the Feynman rules.

1. For each matrix element $M_{ij}$ draw a pair of dots labelled by the matrix indices.

2. Put all of the dots on a dashed line.

3. Join pairs of dots by ribbons such that

   - ribbons stay above the dashed line,
   - ribbon ends join ribbon ends,
   - ribbons are not twisted.

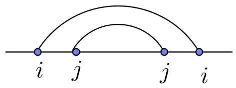
4. Each of the two lines composing the ribbon join pairs of dots. Each line translates into a Kronecker delta which equates the labels of the endpoints of the line.

5. For each trace, join matching indices by a line.

6. For each ribbon (propagator) there is a factor of $\frac{1}{\omega}$.

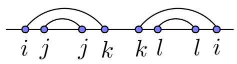
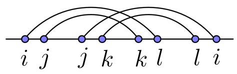
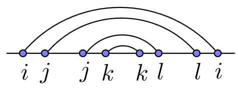
The rules are illustrated in the following three examples

$$\langle Tr\big(M^2\big)\rangle = \langle M_{ij}M_{ji}\rangle$$

$=$ 

$$= \frac{1}{\omega^2}\left(\delta_{ii}\delta_{jj}\right) = \frac{N^2}{\omega^2}\,,$$

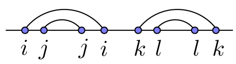
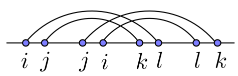
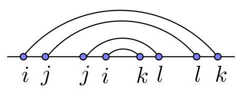
$$\langle Tr\big(M^4\big)\rangle = \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle$$

$=$  $+$  $+$ 

$$= \frac{1}{\omega^2}\left(\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll} + \delta_{il}\delta_{jk}\delta_{ji}\delta_{kl} + \delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk}\right)$$

$$= \frac{1}{\omega^2}\left(2N^3 + N\right),$$

$$\langle Tr\big(M^2\big)Tr\big(M^2\big)\rangle = \langle M_{ij}M_{ji}M_{kl}M_{lk}\rangle$$

$=$  $+$  $+$ 

$$= \frac{1}{\omega^2}\left(\delta_{ii}\delta_{jj}\delta_{kk}\delta_{ll} + \delta_{il}\delta_{jk}\delta_{jl}\delta_{ik} + \delta_{ik}\delta_{jl}\delta_{jl}\delta_{ik}\right)$$

$$= \frac{1}{\omega^2}\left(N^4 + 2N^2\right)$$

### 2.5.1 The large $N$ limit and factorization

There are important simplifications that take place in the $N \to \infty$ limit. To motivate these simplifications consider

$$\langle \mathcal{O}_2\rangle = \frac{N^2}{\omega}, \qquad \langle \mathcal{O}_2\mathcal{O}_2\rangle = \frac{1}{\omega^2}\Big(N^4 + 2N^2\Big).$$

Thus, at large $N$ we have

$$\begin{aligned}\langle \mathcal{O}_2\mathcal{O}_2\rangle &= \frac{1}{\omega^2}(N^4+2N^2)\\ &= \frac{N^4}{\omega^2}\Big(1+O(\frac{1}{N^2})\Big)\\ &= \langle \mathcal{O}_2\rangle\langle \mathcal{O}_2\rangle.\end{aligned}$$

This is an example of a general property of the large $N$ limit: the expectation value of the product of gauge invariant operators is equal to the product of the expectation values. In general, factorization implies

$$\langle \prod_i \mathcal{O}_i\rangle = \prod_i \langle \mathcal{O}_i\rangle\Big[1+O(\frac{1}{N^2})\Big].$$

We will now argue that factorization implies that the theory is described by classical dynamics i.e there is a classical theory we can use to study the large $N$ limit of the quantum field theory. To make this argument, consider the problem of computing expectation values given some probability density describing our system. Suppose that $\mu_i$ is the probability to be in the $i^{\text{th}}$ state. Since the system must be in some state $\sum_i \mu_i = 1$. The expectation value of observable $\mathcal{O}_n$ is

$$\langle \mathcal{O}_n\rangle = \sum_i \mu_i \mathcal{O}_n(i),$$

where $\mathcal{O}_n(i)$ is the value of $\mathcal{O}_n$ in the $i^{\text{th}}$ state. Phrased in this language, factorization says

$$\sum_i \mu_i \mathcal{O}_{n_1}(i)\mathcal{O}_{n_2}(i)\cdots \mathcal{O}_{n_k}(i) = \sum_{i_1}\mu_{i_1}\mathcal{O}_{n_1}(i_1)\sum_{i_2}\mu_{i_2}\mathcal{O}_{n_2}(i_2)\cdots\sum_{i_k}\mu_{i_k}\mathcal{O}_{n_k}(i_k),$$

which implies [47] that $\mu_i = 1$ for $i = i^*$ and $\mu_i = 0$ for $i \neq i^*$ . It is precisely because only one configuration contributes that we claim we are in a classical limit. Indeed, in the usual path integral approach, the correlation functions of the theory

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \int [D\phi] e^{\frac{iS}{\hbar}}\phi(x_1)\cdots\phi(x_n),$$

can be evaluated using a stationary phase approximation. In this case, the only configuration which contributes obeys

$$\frac{\delta S}{\delta \phi} = 0.$$

This is the classical configuration.

At this point we should point out one subtle point: we will see that when we allow the number of fields in our operator to grow with $N$ as we take $N \to \infty$, factorization is spoiled. Wick contractions between fields in different observables are still suppressed by $\frac{1}{N}$, but the huge number of possible contractions give enormous combinatoric factors that overpower the $\frac{1}{N}$ suppression. If the number of fields in the observable grows as $\sqrt{N}$ (or more rapidly), factorization is spoiled.

### 2.5.2 Interacting matrix model

We have demonstrated that factorization is a property of the large $N$ limit for a class of correlators of the free theory. One might then wonder if this conclusion holds after interactions are included. This question is answered in this subsection. Introducing a quartic interaction with coupling strength $g$, the generating function of correlators is

$$Z[J] = \int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2) - g\,\mathrm{Tr}(M^4) + \mathrm{Tr}(JM)}.$$

Correlators computed using this generating function receive contributions from vacuum graphs. To eliminate the vacuum graphs we need to choose a different normalization condition. The generating function

$$\tilde{Z}[J] = \frac{\int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2) - g\,\mathrm{Tr}(M^4) + \mathrm{Tr}(JM)}}{\int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2) - g\,\mathrm{Tr}(M^4)}},$$

will generate correlators without vacuum graph contributions. The 2-point correlation function to order $g$ is given by

$$\langle \mathrm{Tr}(M^2) \rangle = \frac{N^2}{\omega} - \frac{g}{\omega^3}[8N^3 + 4N] + O(g^2).$$

The large $N$ limit of the above correlation function is now dominated by the term of order $g$. If we compute the $O(g^2)$ term, it behaves as $N^4$ so that it would dominate. This pattern continues: higher powers of $g$ come multiplied by ever increasing powers of $N$. The large $N$ limit of this correlator (or any other correlator) simply does not make sense. 't Hooft solved this problem [48] by introducing a new parameter

$$\lambda = gN,$$

where $\lambda$ is held fixed as we take $N \to \infty$. This is called a double scaling limit because we are scaling both $g \to 0$ and $N \to \infty$ while holding $\lambda = gN$ fixed. In the 't Hooft limit the two point correlator becomes

$$\langle \mathrm{Tr}(M^2) \rangle = \frac{N^2}{\omega} - \frac{8\lambda N^2}{\omega^3}[1 + \frac{1}{2N^2}] + O(\lambda^2). \tag{2.41}$$

This expansion in the two small parameters, $\frac{1}{N^2}$ and $\lambda$, has an interesting interpretation which strongly suggests a connection to string theory. Developing this interpretation is the goal of the next subsection.

### 2.5.3 Powers of $N$, topologicaly invariant quantities and the loop expansion

To develop some intuition for the 't Hooft expansion of correlation functions we need to explore the $N$ dependence of each term in the expansion. Each term in the expansion comes from a sum of ribbon graphs. Our first task will be to find a simple way to determine the $N$ dependence coming from a given graph. It will prove to be useful to explore the $N$ dependence of ribbon graphs after we rescale the field $M = \sqrt{N}M'$. After this rescaling

$$-\frac{\omega}{2}\,\mathrm{Tr}(M^2) - g\,\mathrm{Tr}(M^4) = -\frac{\omega N}{2}\,\mathrm{Tr}(M'^2) - \lambda N\,\mathrm{Tr}(M'^4),$$

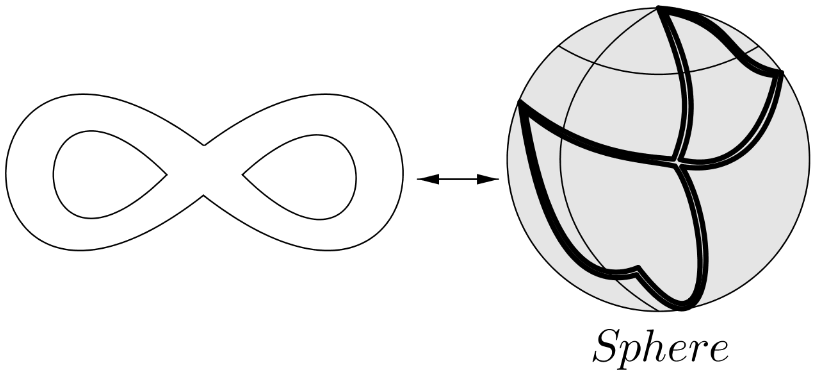
and the corresponding generating function is

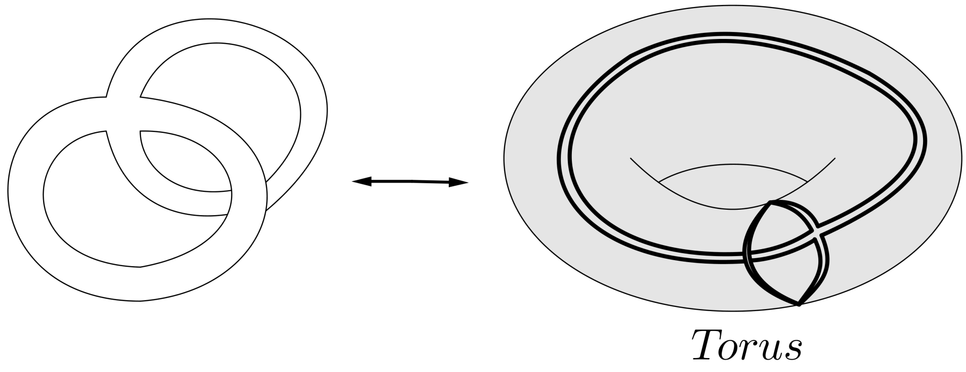
$$Z[J] = \int [dM'] e^{-\frac{\omega N}{2}\,\mathrm{Tr}(M'^2) - \lambda N\,\mathrm{Tr}(M'^4) + \mathrm{Tr}(JM')}. \tag{2.42}$$

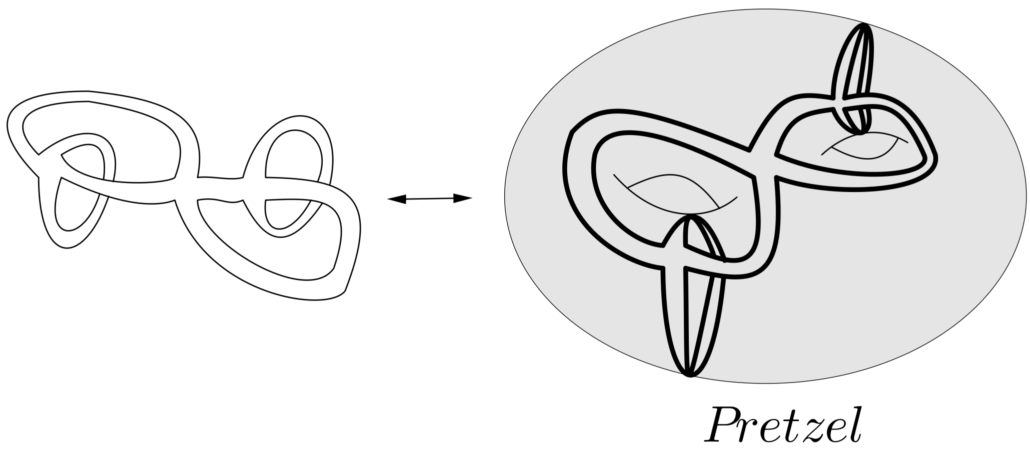
Our ribbon rules change as follows

1. Each closed loop comes with a factor of $N$.
2. Each ribbon comes with a factor of $\frac{1}{\omega N}$.
3. Each vertex comes with a factor of $-\lambda N$.

The ribbon graph can be understood as the triangulation of a two dimensional surface. Here are a few examples of the relation between ribbon graphs and surfaces:






From the above relation it is clear that each ribbon becomes an edge of the triangulation, each closed loop in the ribbon graph becomes a face of the triangulation and each vertex of the ribbon graph becomes a vertex of the triangulation. Denoting the number of ribbons by $E$, the number of closed loops by $F$ and the number of vertices by $V$ the ribbon graph evaluates to

$$(\frac{1}{\omega N})^E (-\lambda N)^V (N)^F = (-\lambda)^V \omega^{-E} N^{V-E+F},$$

so that the $N$ dependence is given by

$$N^{V-E+F}.$$

The integer $\chi = V - E + F$ is a topological invariant of the surface triangulated by the ribbon graph called the Euler characteristic. One can prove that

$$\chi = 2 - 2H - B,$$

where $H$ is the number of handles, and $B$ is the number of boundaries of the surface.

In QFT we sum over all possible worldlines that a particle can follow in spacetime, when one considers path integral quantization of a particle. For path integral quantization of a string, we sum over all possible worldsheets that a string can follow in spacetime. We have learned that summing ribbon graphs amounts to summing over surfaces, which suggests that a matrix model is related to a string theory.

To summarize the discussion, ribbon graphs can be drawn on surfaces with different topologies in such a way that the ribbons do not intersect. The graph triangulates a particular topological surface. The topology of the surface it triangulates gives the $N$ dependence of each ribbon diagram in the 't Hooft limit [48]. Planar diagrams can be drawn in the plane without any of the ribbons crossing. They triangulate a sphere. It is the planar diagrams that determine the large $N$ correlation functions when we hold the number of matrices in each observable fixed as we take $N \to \infty$.

In this thesis we will focus on observables built from order $N$ matrices as we take $N \to \infty$. To study the correlation functions of these observables, we need to sum the complete set of ribbon graphs. This can only be accomplished with new methods that exploit powerful results available in group theory. In the next chapter we will review the relevant background.

# Chapter 3

# Group theory

We will discuss groups and their representations, with an emphasis on the symmetric group and the unitary group. The symmetric group provides a powerful way to sum over all possible Feynman diagrams. We will also study $U(p)$ groups, motivated by Schur-Weyl duality. This duality between the unitary group $U(p)$ and the symmetric group $S_n$ implies deep and fundamental connections between the irreducible representations(irreps) of $U(p)$ and $S_n$. Using the duality, we will be able to trade a complicated problem in $S_n$ representation theory for a solved problem in $U(p)$ representation theory. The action of $U(p)$ and $S_n$ on the vector space $V_p^{\otimes m}$ commute, hence they must be simultaneously diagonalizable on $V_p^{\otimes m}$. This implies they share the same eigenvectors. This is the key accomplishment of [28]. Also we shall look at Schur's Lemmas and their proofs and subsequently obtain the fundamental orthogonality relation. The fundamental orthogonality relation is important as it will be one of the neccessary identities needed to implement the Fourier transform on the Schur polynomial basis, the restricted Schur polynomial basis and the Gauss graph basis which we learn about in later chapters. As a result the two point function in the bases enumerated is simple to compute. We shall also look at the delta function on the group.

## 3.1 Basic definitions and notation

This section begins with the definition of a group. Permutations are then discussed and the cycle notation for elements of the symmetric group is introduced.

**Definition 3.1.1** *A group is a non-empty set $G$, together with a binary operation "$\cdot$", called the group composition law, such that the following axioms are satisfied;*

*(a) For every $g_1, g_2 \in G$, $g_1 \cdot g_2 \in G$.*

*(b) The group composition law is associative, that is for any $g_1, g_2, g_3 \in G$, $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.*

*(c) There is an element $e \in G$, called the identity, such that $g \cdot e = g = e \cdot g$ for every $g \in G$.*

*(d) Every element $g \in G$ has an inverse $g^{-1}$ which belongs to the set $G$ such that it satisfies $g^{-1}g = e = gg^{-1}$.*

We will write $\sigma\tau$ instead of $\sigma \cdot \tau$ for the composition law of permutations. Consider the set $S = \{1, \cdots, n\}$. A permutation rearranges the elements of $S$. For $n = 5$ a permutation might rearrange the 5 objects as follows

$$\begin{aligned} 1 &\longrightarrow 2 \\ 2 &\longrightarrow 3 \\ 3 &\longrightarrow 1 \\ 4 &\longrightarrow 5 \\ 5 &\longrightarrow 4 \end{aligned}$$

A convenient notation for this permuation called cycle notation, is to list the elements in the order they appear under repeated action of the permutation. For the example above, we would write

$$(1, 2, 3)(4, 5).$$

The permutation we are considering permutes $1, 2$ and $3$ and $4, 5$ separately. This is reflected in the notation by the appearance of a product of two lists of integers. Each such list is called a cycle. Thus the permutation we are considering is composed of two cycles, one 2-cycle $(4, 5)$ and one 3-cycle $(1, 2, 3)$. Cycle notation is redundant because distinct cycles can be used to label a single permutation. Indeed as permutations, $\tau = (1, 2, 3) = (2, 3, 1) = (3, 1, 2)$. Consider the permutations $\sigma = (1, 2)$ and $\tau = (2, 3)$. The composition of these cycles is

$$\sigma\tau = (1, 2)(2, 3) = (1, 2, 3).$$

This composition of the permutations follows by noting

1: $\tau$ maps 1 to 1, and $\sigma$ maps 1 to 2. So for the product we have $(1, 2, \cdots)$

2: $\tau$ maps 2 to 3, and $\sigma$ maps 3 to 3. So for the product we have $(1, 2, 3, \cdots)$

3: $\tau$ maps 3 to 2, and $\sigma$ maps 2 to 1. So for the product we have $(1, 2, 3)$

The inverse of $\rho(x) = y$ satisfies the equation $x = \rho^{-1}(y)$ for $x, y \in S$. In cycle notation the inverse cycle is given by reversing the order of the elements in each cycle. The inverse of a permutation is given by inverting each cycle in the permutation. For the permutation we considered above,

$$\Big((1,2,3)(4,5)\Big)^{-1} = (3,2,1)(5,4).$$

A permutation composed of 1-cycles and a single 2-cycle is called a transposition. Every permutation can be written as a product of transpositions.

**Definition 3.1.2** *When two cycles have no elements in common, they are said to be disjoint.*

The permutations $\alpha = (1,3,2)$ and $\beta = (4,5)$ are disjoint. Disjoint cycles commute. This implies that when multiplying disjoint cycles, it doesn't matter in which order the product is taken: $\alpha\beta = (1,3,2)(4,5) = (4,5)(1,3,2) = \beta\alpha$.

The **cycle structure** of a permutation is defined after writing it as a disjoint product of cycles, and is unique up to the permutation of these disjoint cycles. Thus, $\sigma = (1,3,2)(4,5) = (4,5)(1,3,2) \in S_5$ has two disjoint cycles and its cycle structure is one 3-cycle and one 2-cycle.

## 3.2 Symmetric group $S_n$

The symmetric group $S_n$ is the set of all permutations of $n$ objects. The order of the group is $|S_n| = n!$. Consider the symmetric group $S_3$. Using cycle notation, the elements of the group are

$$S_3 = \{\mathbb{1}, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}.$$

In what follows, we will make extensive use of the fact that conjugacy classes of $S_n$ are given by the subsets of permutations that share the same cycle structure. We will also make extensive use of matrix representations of both the symmetric group and of the unitary group. A matrix representation associates to each element of the group $\mathcal{G}$ (denoted $g$) a matrix (denoted $\Gamma(g)$) such that

$$\Gamma(g_1 \cdot g_2) = \Gamma(g_1)\Gamma(g_2). \tag{3.1}$$

Notice that on the right hand side of the above equation we have matrix multiplication, while on the left hand side $g_1$ and $g_2$ are composed using the group composition law. For a matrix representation of the symmetric group, $g_1$ and $g_2$ would be some permutations written in cycle notation and their composition would be exactly as we described it above.

The identity of the group is given by the identity matrix $\Gamma(\mathbb{1}) = \mathbb{1}_{d\times d}$ and $\Gamma(g_1^{-1}) = [\Gamma(g_1)]^{-1}$. If $\Gamma(g)$ are $d \times d$ matrices, we say we have a $d$-dimensional matrix representation of the group. We will usually simply say we have a $d$-dimensional representation. Notice that (3.1) can be understood as a homomorphism between the group $\mathcal{G}$ and the group given by the set of matrices $\Gamma(g)$ with group composition given by matrix multiplication.

The matrices $\Gamma(g)$ of the representation act on a vector space $V$ called the carrier space of the representation. It may happen that a vector $\vec{v}$ which belongs to some subspace $V_1 \subset V$ remains in the subspace under the action of the group, that is,

$$\Gamma(g)\vec{v} \in V_1 \quad \forall g \in \mathcal{G}.$$

We say that the subspace $V_1$ is invariant under the action of $\mathcal{G}$. There are two trivial invariant subspace $V_1 = 0$ and $V_1 = V$. If $V_1$ is a proper subspace, we say that vector space $V$ has a non-trivial invariant subspace under the action of the representation $\Gamma$.

**Definition 3.2.1** *A representation $\Gamma$ of a group $\mathcal{G}$ on a vector space $V$ is irreducible if the vector space $V$ does not contain a non-trivial $\mathcal{G}$ invariant subspace.*

Given a matrix representation $\Gamma(g)$ and an invertible matrix $S$ we can construct a new representation $\tilde{\Gamma}(g)$, defined by

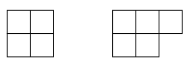
$$\tilde{\Gamma}(g) = S^{-1}\Gamma(g)S.$$

It is simple to check this

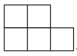
$$\begin{aligned}
\tilde{\Gamma}(g_1)\tilde{\Gamma}(g_2) &= S^{-1}\Gamma(g_1)SS^{-1}\Gamma(g_2)S \\
&= S^{-1}\Gamma(g_1)\Gamma(g_2)S \\
&= S^{-1}\Gamma(g_1 \cdot g_2)S \\
&= \tilde{\Gamma}(g_1 \cdot g_2).
\end{aligned}$$

We say that $\Gamma$ and $\tilde{\Gamma}$ are equivalent representations. The building blocks for the complete set of representations of a group are the inequivalent irreducible representations. Any finite group has a finite number of inequivalent irreducible representations. The irreducible inequivalent representations of the symmetric group $S_n$ can be labelled by Young diagrams. A Young diagram denoted by $R \vdash n$, is a diagram with $n$ boxes arranged in left-adjusted rows and stacked so that the row lengths are in weakly decreasing order.

Here are examples of valid Young diagrams



Here is an example of a Young diagram that is not valid

.

Since the length of row 2 is greater than the length of row 1, the row lengths do not weakly decrease.

The set of all possible Young diagrams with $n$ boxes labels the set of all inequivalent irreducible representations of the symmetric group $S_n$.

Elements of the group $S_n$ that have the same cycle structure, for example $(1, 2, 3)$ and $(1, 3, 2)$, belong to the same conjugacy class. We can associate each cycle structure with a partition of $n$. These different partitions of $n$ are also represented by Young diagrams. Thus, Young diagrams can be used to label both conjugacy classes and irreducible representations. Further, there is a one-to-one correspondence between conjugacy classes and irreducible representations. This correspondence holds for any finite group.

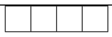
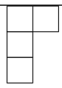
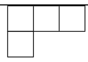
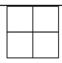
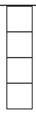
| Irreps of $S_4$ | Conjugacy Class Representative | Size of Conjugacy Class |
|---|---|---|
|  | $(1)(2)(3)(4)$ | 1 |
|  | $(1, 2)(3)(4)$ | 6 |
|  | $(1, 2, 3)(4)$ | 8 |
|  | $(1, 2)(3, 4)$ | 3 |
|  | $(1, 2, 3, 4)$ | 6 |

Table 3.1

In the above table we determine the number of elements in a conjugacy classes $C$ of $S_n$, corresponding to a particular cycle structure in the following theorem.

**Theorem 3.2.1** *Suppose $\sigma \in S_n$, and let $m_1, m_2, \cdots, m_r$ be the distinct integers in the cycle type of $\sigma$, and let there be $k_i$ cycles of order $m_i$ in $\sigma$. Thus $\sum k_i m_i = n$. Then the number of conjugates of $\sigma$ is exactly*

$$|C| = \frac{n!}{\prod_i k_i! m_i^{k_i}}.$$

To realize this formula choose a cycle type in some fixed order. By fixed order we mean the index $i$ equals the length of the cycle type (i.e. length is the number of distinct elements in the cycle). Consider the $n!$ possible assignments of the integers from 1 to $n$ into the slots in the cycles. If two such arrangement define the same permutation then they are equal. Consider a particular arrangement (i.e permutation $\sigma \in S_n$), and consider the $k_i$ cycles of order $m_i$ in $\sigma$. Each of the cycles can be written in $m_i$ different ways (by starting with a different element). Also, the cycles themselves can appear in any of $k!$ possible orders while defining the same permuation, $\sigma$. For example, if $k_i = 2$ and the first cycle contains $a_1, \cdots, a_l$ while the second contains $b_1, \cdots, b_l$, that is the same permutation as if the first cycle contained $b_1, \cdots, b_l$ and the second contained $a_1, \cdots, a_l$. So there are $k_i! m_i^{k_i}$ equivalent permutations considering the cycles of order $m_i$. The total number of permutations, considering each of the possible cycle orders, equivalent to the given permuation $\sigma$ is

$$(k_1! m_1^{k_1})(k_2! m_2^{k_2}) \cdots (k_r! m_r^{k_r}).$$

**Example 3.2.1** *Let $n = 4$ and let us compute the size of the conjugacy class containing the permuation $\sigma = (1,2)(3,4)$. In this case, $r = 1$,*

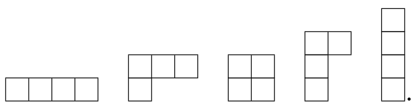
$$m_2 = 2 \qquad k_2 = 2,$$

*and so we have*

$$|C| = \frac{4!}{2!2^2} = 3.$$

*Note that this agrees with the table* (3.1).

**Example 3.2.2** *For $n = 4$, all possible Young diagrams are,*



From the example above $S_4$ has 5 inequivalent irreducible representations and 5 conjugacy classes. Each $S_4$ representation will be written as $\Gamma_R(\sigma)$ where $R$ is any of the five Young diagrams and $\sigma \in S_4$. If $R$ is an irreducible representation of $\mathcal{G}$ we will use $\Gamma_R(g)$ to denote the matrix representing $g \in \mathcal{G}$ in irreducible representation $R$. The dimension of the representation is easily summarized using the Young diagram itself. To do this we introduce the concept of a **hook length**. The hook length of a box $x$ in a Young diagram is the number of boxes in the same row to the right of box $x$, plus the number of boxes in the same column below box $x$, plus the box itself.

**Example 3.2.3** *Hook lengths for the Young diagram, $R \vdash 4$: We have filled in the hook lengths of each box in the Young diagram given below.*

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \; .$$

Now we can write the dimension of any inequivalent irreducible representation of $S_n$ as follows [49, 50].

$$d_R = \frac{n!}{\text{hooks}_R} \qquad \text{where} \quad \text{hooks}_R = \prod_{x \in R} \text{hooks}(x). \tag{3.2}$$

Apart from providing a formula for the dimension of an irreducible representation, the Young diagram can be decorated in such a way that the decorated Young diagrams label a complete basis for the carrier space of the representation. The decorated Young diagram is called a Young-Yamanouchi symbol and is denoted by a ket vector $|R\rangle$. The rules for constructing Young-Yamanouchi symbols is as follows:

1. Each box in $R \vdash n$ is assigned a unique integer $k$, with $1 \leq k \leq n$.
2. Starting from the top to bottom and left to right, each entry must be decreasing.

**Example 3.2.4** *Let* $R = \yng(2,1)$ *be an irreducible representation of the* $S_3$ *group. Then the Young-Yamanouchi states are*

$$\young(32,1)\,, \qquad \young(31,2)\,.$$

It is clear that the number of possible ways of decorating a Young diagram must give the dimension $d_R$ of the irreducible representation labelled by the Young diagram $R$. This is indeed the case.

**Example 3.2.5** *We will compute the dimension of the irreducible representation labelled by the Young diagram* $\yng(2,1)$*. This must match the number of Young-Yamanouchi states given in example 3.2.4. Applying* (3.2) *we find*

$$d_{\yng(2,1)} = \frac{3!}{3.1.1} = 2.$$

For the applications we will consider, the most important use of the Young-Yamanouchi states is in constructing the matrices for a given irreducible representation. There is an algorithm that can be followed to carry out this construction. To state the algorithm, we need an additional concept called the **content of a box** of a Young diagram. The box in row $i$ and column $j$, has the content $j - i$.

**Example 3.2.6** *The content for each box in* $R \vdash 10$ *is shown below*

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} -2 & -1 & \multicolumn{2}{c}{} \\ \cline{1-2} -3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\,.$$

Recall that a specific Young-Yamanouchi state associates to each box in a Young diagram $R \vdash n$ an integer $i$ with $1 \leq i \leq n$. To state the algorithm which determines the matrix elements of a given irreducible representation, it is useful to introduce the notation $|R,(i,i+1)\rangle$ for the Young-Yamanouchi state obtained by swapping the boxes associated to $i$ and $i+1$ in the state $|R\rangle$. The action of $\Gamma_R((i,i+1))$ on the state $|R\rangle$ is given by

$$\Gamma_R((i,i+1))\,|R\rangle = \frac{1}{c_i - c_{i+1}}\,|R\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}}\,|R,(i,i+1)\rangle\,,$$

where $c_i$ is the content of the box associated to $i$ and $c_{i+1}$ is the content of the box associated to $(i+1)$. Notice that the algorithm for constructing the matrices gives the matrix elements of adjacent transpositions, that is, transpositions of the form $(i,i+1)$. Since we can generate all of the elements of the symmetric group by multiplying adjacent transpositions, we do indeed have an algorithm which determines all of the matrix elements of irreducible representation $R$.

**Example 3.2.7** *Using the states given in example 3.2.4, we find*

$$\Gamma_{\ydiagram{2,1}}((1,2))\left|\ytableaushort{31,2}\right\rangle = \frac{1}{2}\left|\ytableaushort{31,2}\right\rangle + \sqrt{1-\frac{1}{(2)^2}}\left|\ytableaushort{32,1}\right\rangle = \frac{1}{2}\left|\ytableaushort{31,2}\right\rangle + \frac{\sqrt{3}}{2}\left|\ytableaushort{32,1}\right\rangle$$
$$\Gamma_{\ydiagram{2,1}}((1,2))\left|\ytableaushort{32,1}\right\rangle = -\frac{1}{2}\left|\ytableaushort{32,1}\right\rangle + \sqrt{1-\frac{1}{(2)^2}}\left|\ytableaushort{31,2}\right\rangle = -\frac{1}{2}\left|\ytableaushort{32,1}\right\rangle + \frac{\sqrt{3}}{2}\left|\ytableaushort{31,2}\right\rangle$$

*so that*

$$\Gamma_{\ydiagram{2,1}}((1,2)) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

*In a similar way we find*

$$\Gamma_{\ydiagram{2,1}}((2,3)) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

*Using these results we can now generate the remaining elements of the group*

$$\Gamma_{\ydiagram{2,1}}((1,3)) = \Gamma_{\ydiagram{2,1}}((2,3)(1,2)(2,3)) = \Gamma_{\ydiagram{2,1}}((2,3))\Gamma_{\ydiagram{2,1}}((1,2))\Gamma_{\ydiagram{2,1}}((2,3)) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$\Gamma_{\ydiagram{2,1}}((1,2,3)) = \Gamma_{\ydiagram{2,1}}((1,2)(2,3)) = \Gamma_{\ydiagram{2,1}}((1,2))\Gamma_{\ydiagram{2,1}}((2,3)) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$\Gamma_{\ydiagram{2,1}}((1,3,2)) = \Gamma_{\ydiagram{2,1}}((2,3)(1,2)) = \Gamma_{\ydiagram{2,1}}((2,3))\Gamma_{\ydiagram{2,1}}((1,2)) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

*In any representation, the identity of the group maps to the identity matrix so that we have*

$$\Gamma_{\tiny\yng(2,1)}(\mathbb{1}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have given a rather complete description of the construction of the matrix representations of the symmetric group. As will become clear later, the representation theory problem we will need to solve entails finding matrix representations of the subgroup $S_n \times S_m \subset S_{n+m}$ within the carrier space of an irreducible representation of $S_{n+m}$. Since we need to scale $n + m = \mathcal{O}(N) \to \infty$, the algorithm we have given above is not powerful enough to solve this problem. Ultimately we will use Schur-Weyl duality to map the above problem into a much easier $U(p)$ representation theory problem. With this goal in mind, we now review the relevant $U(p)$ representation theory.

## 3.3 $U(p)$ group

The $U(p)$ group for $p \geq 1$ is isomorphic to the set of all $p \times p$ unitary matrices $U$ with matrix multiplication as the group multiplication law. Recall that unitary matrices obey $UU^\dagger = \mathbb{1}$. For the representations of the symmetric group $S_n$, we have seen that states in the carrier space of a representation $R$ can be labelled by Young-Yamanouchi symbols, which are constructed using the Young diagram $R$ labelling the representation. States in the carrier space of a $U(p)$ representation can be labelled by Gelfand-Tsetlin patterns which are also constructed out of the Young diagram $R$. We will discuss Gelfand-Tsetlin patterns in detail in section 3.5 below.

An important tool in our work is the deep relationship between the representation theory of the symmetric group and the general linear group (of which $U(p)$ is subgroup) called **Schur-Weyl duality** [49, 50]. This duality arises when we consider the tensor product of $m$ copies of the $p$-dimensional vector $V$ denoted by $V_p^{\otimes m}$. Here $V$ is the fundamental representation of $U(p)$. It comes about because the action of the unitary group, $U(p)$ on $V^{\otimes m}$ commutes with the action of $S_m$ on $V^{\otimes m}$. We will now demonstrate that the action of $U(p)$ on $V^{\otimes m}$ commutes with the action of $S_m$ on $V^{\otimes m}$.

The action of $\sigma \in S_m$ on $V^{\otimes m}$ is defined by

$$\sigma : V^{\otimes m} \to V^{\otimes m}$$
$$\sigma\left(V_p^{\otimes m}\right) = \vec{v}(\sigma(1)) \otimes \vec{v}(\sigma(2)) \otimes \cdots \otimes \vec{v}(\sigma(m)).$$

Thus, $\sigma \in S_m$ will move the vector in the $i^{\text{th}}$ slot to the $\sigma(i)^{\text{th}}$ slot, but does not change its value.

Let $D(U)$ be a unitary $p \times p$ matrix representing $U \in U(p)$ in the fundamental representation. The action of $U(p)$ on $V_p^{\otimes m}$ is given as

$$U\Big(V_p^{\otimes m}\Big) = D(U)\vec{v}(1) \otimes D(U)\vec{v}(2) \otimes \cdots \otimes D(U)\vec{v}(m).$$

From these two definitions we will show that the action of the two groups $S_m$ and $U(p)$ on $V_p^{\otimes m}$ commute. Composing the two groups $S_m$ and $U(p)$ we have,

$$\begin{aligned}
\sigma\Big(U(\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m))\Big) &= \sigma\Big(D(U)\vec{v}(1) \otimes D(U)\vec{v}(2) \otimes \cdots \otimes D(U)\vec{v}(m)\Big) \\
&= D(U)\vec{v}(\sigma(1)) \otimes D(U)\vec{v}(\sigma(2)) \otimes \cdots \otimes D(U)\vec{v}(\sigma(m)) \\
&= D(U)\Big(\vec{v}(\sigma(1)) \otimes \vec{v}(\sigma(2)) \otimes \cdots \otimes \vec{v}(\sigma(m))\Big) \\
&= \Big(D(U)\sigma(\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)\Big).
\end{aligned}$$

Since the action of the $S_m$ and $U(p)$ groups on $V_p^{\otimes m}$ commute, they must be simultaneously diagonalizable on $V_p^{\otimes m}$. This implies that $S_m$ and $U(p)$ share the same eigenvectors. This explains why the Young diagram also labels irreducible representation of $U(p)$. This is Schur-Weyl duality. From the Schur-Weyl duality we can write $V_p^{\otimes m}$ as

$$V_p^{\otimes m} = \oplus_s V_s^{U(p)} \otimes V_s^{S_n},$$

and $s$ runs over all Young diagrams with $m$ boxes. The formula above implies that

$$p^m = \sum_s \mathrm{Dim}_s d_s, \tag{3.3}$$

where $\mathrm{Dim}_s = \frac{f_R}{\mathrm{hooks}_R}$ is the dimension of the irreducible representation of $U(p)$ and $d_s$ is the dimension of $s$, an irreducible representation of $S_m$ [49, 50]. The factor $f_R$ for $U(p)$ is given by $p$ plus the content of the box and $\mathrm{hooks}_R$ defined in equation (3.2) is the hooks of $R$.

**Example 3.3.1** *Consider $R \vdash 3$*

| irreps of $S_3$ | $\mathrm{hooks}_R$ | $d_R$ | $\mathrm{Dim}_R$ |
|---|---|---|---|
| $\ydiagram{3}$ | 6 | 1 | $\frac{p(p+1)(p+2)}{6}$ |
| $\ydiagram{2,1}$ | 3 | 2 | $\frac{(p-1)(p)(p+1)}{3}$ |
| $\ydiagram{1,1,1}$ | 6 | 1 | $\frac{(p-2)(p-1)(p)}{6}$ |

*Applying the formula in equation* (3.3) *above we have,*

$$\begin{aligned}\sum_s \mathrm{Dim}_s d_s &= \frac{p(p+1)(p+2)}{6} + 2\frac{(p-1)(p)(p+1)}{3} + \frac{(p-2)(p-1)(p)}{6} \\ &= \frac{1}{3}p(p^2+2+2p^2-2) \\ &= p^3 = \dim(V_p^{\otimes 3}).\end{aligned}$$

Thus, the vector space $V_p^{\otimes 3}$ under the action of $S_3$ and $U(p)$ reduces to three irreducible $S_3 \times U(p)$ subspaces labelled by three Young diagrams in the above table.

## 3.4 Partially labelled Young diagram

The notion of partially labelled Young diagram is particularly useful when working with representations of $S_n \times S_m$ that have been deduced from representations of $S_{n+m}$. The irreducible representations of $S_n \times S_m$ are labelled by two Young diagrams $(r, s)$, one for $S_n$ and one for $S_m$ respectively.

Consider a Young diagram $R \vdash (n+m)$ with $p$ rows and $m$-impurities. The number of boxes $n+m$ in $R$ are of order $N$, with $n$ boxes order $\mathcal{O}(N)$ and $m$ order $\mathcal{O}(1)$ such that $m \ll n$. For this reason we choose to call all boxes labelled with $m$ as impurities among the many boxes labelled with $n$. These impurities are labelled starting from the right hand corners of the Young diagram $R \vdash (n+m)$ to the left hand side, such that the shape of the unlabelled Young diagram $r \vdash n$ remaining is a valid Young diagram. Representations $r$, that are subduced by $R$ are those with Young diagrams that can be obtained by removing $m$ boxes from $R$. Note that if we remove the same set of $m$ boxes in different orders, it leads to different subspaces that all have the same irreducible representation $r$. Since these subspaces carry the same irreducible representation of $S_n$, we have a multiplicity problem and we need to distinguish them. They are distinguished by the order in which boxes are labelled. To deal with this let us specify an order in which the boxes are removed from $R$ with label ranging from 1 to $m$ [28]. We must ensure that valid Young diagrams are obtained by removing boxes at each step. Another type of multiplicity problem occurs when we want to organize these boxes into irreducible representations of $S_m$. We find that a given irreducible representation can appear more than once. This is resolved by the unitary group $U(p)$, for a diagram with $p$ rows. Note that this second type of multiplicity label is identified with Gelfand-Tsetlin patterns [28].

Let us consider an example where $n = 3$ and $m = 2$. The partially labelled Young tableaux for this particular representation would be

$$\begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 2 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 1 \\ \cline{1-2}\end{array} \;. \tag{3.4}$$

Each of these corresponds to a collection of states spanning a $r$ subspace and you can view the partly labelled Young diagram as a multiplicity label that distinguishes these subspaces. It should also be clear that in this example, $r$ is an irreducible representation of the $S_3$ group that permutes $4, 5$ and 6. Completing the first partially labelled Young diagram gives the Young tableau

$$\begin{array}{|c|c|c|}\hline 5&4&1\\ \hline 3&2\\ \cline{1-2}\end{array}\qquad\qquad \begin{array}{|c|c|c|}\hline 5&3&1\\ \hline 4&2\\ \cline{1-2}\end{array}\quad .$$

Observe that each of the above Young tableau represents a vector in the carrier space of the $S_5$ irreducible representation labelled by the Young diagram $\yng(3,2)$. By dropping all labelled boxes from equation (3.4) we obtain the Young diagram $\yng(2,1)$. A general rule which may be extracted from this example is:

*A partially labelled Young diagram that has $n + m$ boxes, $m$ of which are labelled, stands for a collection of states which furnish the basis for an irreducible representation of $S_n \times (S_1)^m$. The Young diagram that labels the representation of the $S_n$ subgroup is given by dropping all labelled boxes [28].*

## 3.5 Constructing Gelfand-Tsetlin patterns

An inequivalent irreducible representation for $U(p)$ is uniquely given by specifying the sequence of $p$ integers

$$\vec{m} = (m_{1p}, m_{2p}, \cdots, m_{pp}),$$

satisfying $m_{k,p} \geq m_{k+1,p}$, for $1 \leq k \leq p - 1$. This sequence is called the weight of the irreducible representation. It can be identified with the row lengths of a Young diagram $R$, that has no more than $p$ rows. This Young diagram labels a $U(p)$ irreducible representation. Recall that we can use the same labelling for irreducible representations of the symmetric groups and the unitary groups. This is again an indication of Schur-Weyl duality. Note that irreducible representations of $S_n$ are labelled by Young diagrams with $n$ boxes, while irreducible representations of $U(p)$ are labelled by Young diagrams with no more than $p$ rows. Restricting this irreducible representation to the subgroup $U(p-1)$ the resulting representation is reducible. It decomposes into a direct sum of $U(p - 1)$ irreducible representations with highest weights [49, 50]

$$\vec{m}' = (m_{1,p-1}, m_{2,p-1}, \cdots, m_{p,p-1}),$$

for which the "betweenness" conditions

$$m_{k,p} \geq m_{k,p-1} \geq m_{k+1,p} \quad \text{for} \quad 1 \leq k \leq p - 1,$$

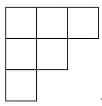
are satisfied. The Gelfand-Tsetlin patterns, denoted by $M$, are

$$M = \begin{bmatrix} m_{1,p} & & m_{2,p} & \cdots & m_{p-1,p} & & m_{p,p} \\ & m_{1,p-1} & & m_{2,p-1} & \cdots & m_{p-1,p-1} & \\ & & \cdots & \cdots & \cdots & & \\ & & m_{1,2} & & m_{2,2} & & \\ & & & m_{1,1} & & & \end{bmatrix}.$$

In $M$ the top row contains the weights that specify the irreducible representations of the state and the entries of the lower rows are subjected to the betweenness condition. The very first line of the Gelfand-Tsetlin pattern tells us the shape of the Young diagram. For example

$$M = \begin{bmatrix} 3 & & 2 & & 1 \\ & 2 & & 1 & \\ & & 2 & & \end{bmatrix},$$

the Young diagram has 3 boxes in the first row, 2 boxes in the second row and 1 box in the third row. Hence the shape is



.

In our construction we will make use of two important weights, the $\Sigma$-weights and $\Delta$ weights. We define the row sum of $M$ as

$$\sigma_l(M) = \sum_{k=1}^{l} m_{k,l}.$$

The $\Sigma$-weight is defined as the sequence of row sums

$$\Sigma(M) = \Big(\sigma_p(M), \sigma_{p-1}(M), \cdots, \sigma_1(M)\Big).$$

These $\Sigma$-weights do not provide a unique label for the states in the carrier space. It is possible that $\Sigma(M) = \Sigma(M')$, but $M \neq M'$. The number of states $\vec{v}(M)$ in the carrier space that have the same $\Sigma$-weight, $\Sigma = \Sigma(M)$ is called the inner multiplicity $I(\Sigma)$ of the state. $I(\Sigma)$ will play an important role below.

The $\Delta$ weights are defined in terms of differences between row sums

$$\begin{aligned}\Delta(M) &= \Big(\sigma_p(M) - \sigma_{p-1}(M), \sigma_{p-1}(M) - \sigma_{p-2}(M), \cdots, \sigma_1(M) - \sigma_0(M)\Big) \\ &\equiv \Big(\delta_p(M), \delta_{p-1}(M), \cdots, \delta_1(M)\Big).\end{aligned}$$

where $\sigma_0(\cdot) = 0$. These weights will also play an important role below.

## 3.6 Relation between Gelfand-Tsetlin patterns and Young diagrams

Given a Gelfand-Tsetlin pattern $M$ we can construct a corresponding semi-standard Young tableaux. For a $U(p)$ irreducible representation there are at most $p$ rows in the associated Young diagram. A semi-standard Young tableau is a Young diagram with labelled boxes. These boxes are labelled such that each box contains an integer $i$ with $1 \leq i \leq p$. Starting from left to right, the numbers in each row weakly increase. The numbers in each column strictly increase from top to bottom. The relation between Gelfand-Tsetlin patterns and semi-standard Young tableau is most easily developed with a concrete example

$$M = \begin{bmatrix} 3 & & 2 & & 1 \\ & 2 & & 1 & \\ & & 2 & & \end{bmatrix}.$$

We fill the empty Young diagram by first considering the last row of the Gelfand-Tsetlin patterns. In this example we have a 2 in the last row which translates into a Young diagram with one row of 2 boxes. Superpose the shape of two boxes in the topmost leftmost corner of the full Young diagram. Label these two boxes with 1 since the last row of the Gelfand-Tsetlin pattern tells us which boxes are labelled with a 1.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & \phantom{1} \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} .$$

The first two rows of the pattern tells us which boxes are labelled with a two. The second last row of $M$ corresponds to a Young diagram with 2 boxes in the first row and 1 box in the second row. Again superpose this new diagram at the topmost leftmost corner of the above partially labelled diagram and label each empty boxes with a 2.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & \phantom{1} \\ \hline 2 & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} .$$

Repeat this process until all rows of the Gelfand-Tsetlin pattern are used. The result for our example is the semi-standard Young tableau shown below

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} .$$

Consider row $k$ in the tableau above. The number of boxes containing the number $l$ is given by $m_{k,l} - m_{k,l-1}$ and we set $m_{kl} \equiv 0$ if $k > l$.

The components $\delta_l(M)$ of the $\Delta$ weight of a Gelfand-Tsetlin pattern $M$, are the number of boxes containing $l$ in the tableau corresponding to $M$. Thus, the tableau corresponding to two patterns with the same $\Delta$ weight contain the same set of entries (that is, the same number of $l$-boxes) but arranged in different ways. Counting the number of ways to arrange the relevant fixed set of entries in the tableau is one interpretation for the inner multiplicity.

## 3.7 Clebsch-Gordan coefficients

This section introduces two bases and the transformation matrix that takes us between these bases. This transformation matrix is what we refer to as the Clebsch-Gordan coefficients. The explicit form of this transformation matrix is discussed in detail. The Clebsch-Gordan coefficients are particularly important when we compute the projectors appearing in the action of dilatation operator on restricted Schur polynomials in chapter 4, in the displaced corners approximation.

Consider two irreducible unitary representations $R$ and $S$ of the $U(p)$ group. The tensor product of these two representations decomposes into a direct sum of irreducible components

$$R \otimes S = \bigoplus_T \nu(T)T. \tag{3.5}$$

A specific irreducible representation $T$ may appear more than once in the product $R \otimes S$. The multiplicity of this decomposition is given by $\nu(T)$. We will only need the tensor product of an arbitrary representation with weight $\vec{m}_m$, with the defining representation (also called the fundamental representation) which has weight $(1, \vec{0})$. This product has all multiplicities set to 1 so we will not need any multiplicity labels. Denote the weight of the irreducible representation with $\vec{m}_R$ and use $M_R$ to denote the Gelfand-Tsetlin pattern for a particular state in the carrier space of this irreducible representation. The product $R \otimes S$ has two natural bases. The first of them is obtained by taking the direct product of the states spanning the carrier spaces of $R$ and $S$. We will label states in this basis with bra/ket notation as

$$|\vec{m}_R, M_R; \vec{m}_S, M_S\rangle .$$

The second natural basis, is given as a direct sum over the bases of the carrier spaces for the irreducible representations $T$ appearing in the sum on the right hand side of (3.5). The states in this basis are labelled as

$$|\vec{m}_T, M_T\rangle ,$$

where $T$ runs over all irreducible representations appearing in the sum on the right hand side of (3.5). The transformation matrix that takes us between the two bases is given by the Clebsch-Gordan coefficients [28]. They are written as

$$\langle \vec{m}_R, M_R; \vec{m}_S, M_S \rangle \, \vec{m}_T, M_T.$$

Because the specific irreducible representations we consider are uniquely labelled by the weight which is recorded in the first row of the corresponding Gelfand-Tsetlin pattern, the labels $R, S, T$ will be dropped. We can write the Clebsch-Gordan coefficients of $U(p)$ as

$$\left\langle \vec{m}_p, M; \vec{m}'_p, M' \right\rangle \vec{m}''_p, M'' = \begin{pmatrix} \vec{m}_p & \vec{m}'_p|\vec{m}''_p \\ \vec{m}_{p-1} & \vec{m}'_{p-1}|\vec{m}''_{p-1} \end{pmatrix} \left\langle \vec{m}_{p-1}, M_1; \vec{m}'_{p-1}, M'_1 \right\rangle \vec{m}''_{p-1}, M''_1.$$

The above equation shows a relation between Clebsch-Gordan coefficients of $U(p)$ and $U(p-1)$. On the left hand side of the equation we have Clebsch-Gordan coefficients of the group $U(p)$ and on the right hand side of the equation we have Clebsch-Gordan coefficients of the group $U(p-1)$. The weights $\vec{m}_p, \vec{m}'_p, \vec{m}''_p$ label irreducible representations of $U(p)$, while weights $\vec{m}_{p-1}, \vec{m}'_{p-1}, \vec{m}''_{p-1}$ label irreducible representations of $U(p-1)$. The Gelfand-Tsetlin patterns $M_1$, $M'_1$ and $M''_1$ are obtained from $M$, $M'$ and $M''$ respectively by removing the first row. Thus, the weights $\vec{m}_{p-1}, \vec{m}'_{p-1}, \vec{m}''_{p-1}$ correspond with the second rows in $M$, $M'$ and $M''$. The coefficients

$$\begin{pmatrix} \vec{m}_p & \vec{m}'_p|\vec{m}''_p \\ \vec{m}_{p-1} & \vec{m}'_{p-1}|\vec{m}''_{p-1} \end{pmatrix},$$

are called the scalar factors of the Clebsch-Gordan coefficients

$$\left\langle \vec{m}_{p-1}, M_1; \vec{m}'_{p-1}, M'_1 \right\rangle \vec{m}''_{p-1}, M''_1.$$

We can write the Clebsch-Gordan coefficients as a product of scalar factors by employing the chain of subgroups referenced by the Gelfand-Tsetlin patterns. By doing this we obtain

$$\begin{aligned} \left\langle \vec{m}_p, M; \vec{m}'_p, M' \right\rangle \vec{m}''_p, M'' = & \begin{pmatrix} \vec{m}_p & \vec{m}'_p|\vec{m}''_p \\ \vec{m}_{p-1} & \vec{m}'_{p-1}|\vec{m}''_{p-1} \end{pmatrix} \begin{pmatrix} \vec{m}_{p-1} & \vec{m}'_{p-1}|\vec{m}''_{p-1} \\ \vec{m}_{p-2} & \vec{m}'_{p-2}|\vec{m}''_{p-2} \end{pmatrix} \\ & \times \begin{pmatrix} \vec{m}_{p-2} & \vec{m}'_{p-2}|\vec{m}''_{p-2} \\ \vec{m}_{p-3} & \vec{m}'_{p-3}|\vec{m}''_{p-3} \end{pmatrix} \cdots . \end{aligned}$$

Clebsch-Gordan coefficients obey a selection rule. They vanish unless

$$\sum_{i=1}^{j} m_{ij} + \sum_{i=1}^{j} m'_{ij} = \sum_{i=1}^{j} m''_{ij}, \quad \text{where} \quad j = 1, 2, \cdots, p.$$

Note that the Clebsch-Gordan coefficients we will need come from taking products off some general representation $\vec{m}_p$ with the fundamental representation. The weight of the fundamental representation is $(1, 0, \cdots, 0)$ with $(p-1)$ 0s appearing. The product we will consider is

$$\vec{m}_p \otimes (1, \vec{0}) = \sum_{i=1}^{m} \vec{m}_p^{+i}.$$

Knowing $\vec{m}_p$ we can obtain $\vec{m}_p^{+i}$ by replacing $m_{ip}$ by $m_{ip}+1$. One must check that this replacement leads to a valid Gelfand-Tsetlin pattern. These Clebsch-Gordan coefficients factor into products of scalar factors of the form

$$\begin{pmatrix} \vec{m}_p & (1,\vec{0})_p | \vec{m}_p^{+i} \\ \vec{m}_{p-1} & (1,\vec{0})_{p-1} | \vec{m}_{p-1}^{+j} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \vec{m}_p & (1,\vec{0})_p | \vec{m}_p^{+i} \\ \vec{m}_{p-1} & (0,\vec{0})_{p-1} | \vec{m}_{p-1} \end{pmatrix}.$$

The explicit formulas for these factors are known [28]

$$\begin{pmatrix} \vec{m}_p & (1,\vec{0})_p | \vec{m}_p^{+i} \\ \vec{m}_{p-1} & (1,\vec{0})_{p-1} | \vec{m}_{p-1}^{+j} \end{pmatrix} = S(i,j) \left| \frac{\prod_{k\neq j}^{p-1}(l_{k,p-1} - l_{i,p} - 1) \prod_{k\neq i}^{p}(l_{k,p} - l_{j,p-1})}{\prod_{k\neq i}^{p}(l_{k,p} - l_{i,p}) \prod_{k\neq j}^{p-1}(l_{k,p-1} - l_{j,p-1} - 1)} \right|^{\frac{1}{2}}$$

$$\begin{pmatrix} \vec{m}_p & (1,\vec{0})_p | \vec{m}_p^{+i} \\ \vec{m}_{p-1} & (0,\vec{0})_{p-1} | \vec{m}_{p-1} \end{pmatrix} = \left| \frac{\prod_{j=1}^{p-1}(l_{j,p-1} - l_{i,p} - 1)}{\prod_{j\neq i}^{p}(l_{j,p} - l_{i,p})} \right|^{\frac{1}{2}},$$

where $l_{s,k} = m_{s,k} - s$, $S(i,j) = 1$ if $i \leq j$ and $S(i,j) = -1$ if $i > j$.

To conclude this section, we will explain the common foundation of Young-Yamanouchi labelling and Gelfand-Tsetlin labelling. Both methods of labelling describe the basis state in terms of its representation content with respect to a nested chain of subgroups. For example, the state

$$\begin{ytableau} 4 & 2 \\ 3 & 1 \end{ytableau}$$

is in the following irreducible representations

$$\ydiagram{2,2} \qquad \ydiagram{2,1} \qquad \ydiagram{1,1} \qquad \ydiagram{1}$$

with respect to the subgroups $S_4 \supset S_3 \supset S_2 \supset S_1$. The state

$$\begin{bmatrix} 3 & & 2 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix}$$

is in the following irreducible representations

$$\ydiagram{3,2,1} \qquad \ydiagram{2,1} \qquad \ydiagram{1}$$

with respect to the subgroups $U(3) \supset U(2) \supset U(1)$.

## 3.8 Schur's Lemmas

In this section we discuss Schur's two Lemmas and prove them. From these Lemmas we will establish the fundamental orthogonality relation which is an essential identity that would be used throughout this thesis. This identity enables us to compute the two point function of the Schur polynomials, restricted Schur polynomials and the Gauss graph operators which we shall come to in later chapters.

**Lemma 3.8.1** *Let $R$ be a $d_R$-dimensional vector space. If $\Gamma_R$ is an irreducible representation of a finite group $\mathcal{G}$ and if there exist a $d_R \times d_R$ matrix $M$ such that it commutes with all matrices of $\Gamma_R$, i.e*

$$M\Gamma_R(g) = \Gamma_R(g)M \quad \forall g \in \mathcal{G}, \tag{3.6}$$

*then $M$ must be proportional to the matrix identity, i.e $M = \alpha I_{d_R \times d_R}$.*

**Proof 3.8.1** *If all eigenvalues of $M$ are zero, then from equation (3.6) $\alpha = 0$. If there exist at least one non-zero eigenvalue of $M$, then*

$$M |v\rangle = \alpha |v\rangle \quad \textit{where} \quad |v\rangle \in R. \tag{3.7}$$

*From (3.6) and (3.7) we find*

$$M\Gamma_R(g) |v\rangle = \Gamma_R(g)M |v\rangle = \alpha\Gamma_R(g) |v\rangle \quad \forall g \in \mathcal{G}.$$

*Since $\Gamma_R$ is an irreducible representation of $\mathcal{G}$, the subspace of vectors spanned by $\Gamma_R(g) |v\rangle$ $\forall g \in \mathcal{G}$, is the whole space. Thus, every vector is an eigenvector of $M$ associated to the eigenvalue $\alpha$. The only matrix that satisfies this is $M = \alpha I_{d_R \times d_R}$.*

**Lemma 3.8.2** *The only solution to the equation*

$$M\Gamma_R(g) = \Gamma_R(g)M \quad \forall g \in \mathcal{G}, \tag{3.8}$$

*with $R$ and $S$ two inequivalent irreducible representations is $M = 0$.*

**Proof 3.8.2** *Let $d_R$ and $d_S$ be the dimensions of the irreducible representations $R$ and $S$ respectively.*

**Case 1**: $d_R = d_S$
*$M$ cannot be invertible. If we assume that $M$ is invertible then*

$$M\Gamma_R(g) = \Gamma_S(g)M \Leftrightarrow \Gamma_R(g) = M^{-1}\Gamma_S(g)M \quad \forall g \in \mathcal{G}.$$

*The last equation establishes a contradiction since it implies that $\Gamma_R$ and $\Gamma_S$ are equivalent. Hence $M$ has at least one zero eigenvalue, i.e there is a non-zero vector $|n\rangle$ such that*

$$M |n\rangle = 0.$$

*From* (3.8) *we find*

$$M\Gamma_R(g)\,|n\rangle = \Gamma_S(g)M\,|n\rangle = 0.$$

*$M$ annihilates the subspace of vectors spanned by $\Gamma_R(g)\,|n\rangle$ $\forall g \in \mathcal{G}$. Since $\Gamma_R$ is an irreducible representation of $\mathcal{G}$, the subspace by $\Gamma_R(g)\,|n\rangle$ $\forall g \in \mathcal{G}$ is the whole space. Hence $M = 0$.*

**Case 2**: *$d_R > d_S$*
*$M$ is a map from $R$ to $S$. Let $|i\rangle$ and $|j\rangle$ be an orthonormal basis of the irreducible representations of $R$ and $S$ respectively. We write $M$ in this basis as*

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1d_R} \\ M_{11} & M_{12} & \cdots & M_{1d_R} \\ \vdots & \vdots & \vdots & \vdots \\ M_{11} & M_{12} & \cdots & M_{1d_R} \end{pmatrix},$$

*where $M_{ji} = \langle j|\,M\,|i\rangle$. $M$ has more columns than rows. The rows of $M$ are a collection of $d_S < d_R$ vectors in a $d_R$ dimensional space. One can find a null vector of $M$ which is orthogonal to all the rows of $M$, i.e $M\,|i\rangle > 0$. From* (3.8)

$$M\Gamma_R(g)\,|i\rangle = \Gamma_S(g)M\,|i\rangle = 0. \tag{3.9}$$

*The subspace of vectors spanned by $\Gamma_R(g)\,|i\rangle$ $\forall g \in G$ are annihilated by $M$. Since $\Gamma_R$ is an irreducible representation of $\mathcal{G}$, the only solution to* (3.9) *is $M = 0$.*

**Case 3**: *$d_R < d_S$*
*For any arbitrary vector $|v\rangle$, using* (3.8) *we find that*

$$M\Gamma_R(g)\,|v\rangle = \Gamma_S(g)M\,|v\rangle = \Gamma_S(g)\,|\tilde{v}\rangle\,,$$

*where $|\tilde{v}\rangle = M\,|v\rangle$. Since $S$ is an irreducible representation, the right hand side of the above equation can only be written, as $g$ varies over $\mathcal{G}$, using a basis of $d_S$ linearly independent vectors. $\Gamma_R(g)\,|v\rangle$ on the left hand side can be expressed in terms of a basis of $d_R$ linearly independent vectors. Since $d_R < d_S$ this is a contradiction and it implies that $M = 0$.*

In the section that follows we will use Schur's Lemmas to establish the orthogonality relation between two irreducible representations. This is important because the orthogonality relation provides a means for computing the two point function on Schur polynomials, restricted Schur polynomials and the Gauss graph which we discuss in later chapters.

## 3.9 Fundamental orthogonality relation

Let $\Gamma_R$ and $\Gamma_S$ be two irreducible representations defined over a group $\mathcal{G}$. Let $M(R,S,b,c)$ be $d_R \times d_S$ dimensional matrices defined as

$$M(R,S,b,c)_{ad} = \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{cd}, \tag{3.10}$$

where $a,b = 1,\cdots,d_R$ are indices of the representation $R$ and $c,d = 1,\cdots,d_S$ are indices of the representation $S$. The matrices satisfy the identity

$$M(R,S,b,c)\Gamma_R(g) = \Gamma_S(g)M(R,S,b,c) \quad \forall g \in \mathcal{G}. \tag{3.11}$$

**Proof 3.9.1** *Using equation* (3.10) *we prove the statement in equation* (3.11) *as follows*

$$\begin{aligned} M(R,S,b,c)_{ad}\Gamma_S(g_1)_{de} &= \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{cd}\Gamma_S(g_1)_{de} \\ &= \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(gg_1)_{ce}. \end{aligned} \tag{3.12}$$

*Using the fact that $\Gamma_S$ is an irreducible representation of $\mathcal{G}$ we easily obtain the last line. Next change the summation variable from $g$ to $h = gg_1$ so that equation* (3.12) *takes the form*

$$\begin{aligned} M(R,S,b,c)_{ad}\Gamma_S(g_1)_{de} &= \sum_{h\in\mathcal{G}} \Gamma_R(g_1h^{-1})_{ab}\Gamma_S(h)_{ce} \\ &= \sum_{h\in\mathcal{G}} \Gamma_R(g_1)_{ad}\Gamma_R(h^{-1})_{db}\Gamma_S(h)_{ce} \\ &= \Gamma_R(g_1)_{ad}\sum_{h\in\mathcal{G}} \Gamma_R(h^{-1})_{db}\Gamma_S(h)_{ce} \\ &= \Gamma_R(g_1)_{ad}M(R,S,b,c)_{de}. \end{aligned}$$

We have shown that the matrices satisfy (3.11) and so by Schur's second Lemma we have

$$M(R,S,b,c)_{ad} = \delta_{RS}\delta_{ad}A(R,b,c). \tag{3.13}$$

We compute $A(R,b,c)$ by setting $R = S$, $a = d$ and sum over $a$ and from (3.10) to obtain

$$\begin{aligned} \sum_{a=1}^{d_R} M(R,R,b,c)_{aa} &= \sum_{g\in\mathcal{G}}\sum_{a=1}^{d_R} \Gamma_R(g^{-1})_{ab}\Gamma_R(g)_{ca} \\ &= \sum_{g\in\mathcal{G}} \Gamma_R(gg^{-1})_{cb} \\ &= \sum_{g\in\mathcal{G}} \Gamma_R(I_{\mathcal{G}})_{cb} \\ &= |\mathcal{G}|\delta_{bc}. \end{aligned} \tag{3.14}$$

From equation (3.13) we also have

$$\sum_{a=1}^{d_R} M(R,S,b,c)_{aa} = \delta_{RR} \sum_{a=1}^{d_R} \delta_{aa} A(R,b,c)$$
$$= d_R A(R,b,c). \tag{3.15}$$

Comparing equations (3.14) and (3.15) we can write

$$A(R,b,c) = \frac{|G|}{d_R}\delta_{bc}.$$

The fundamental orthogonality condition is

$$M(R,S,b,c)_{ad} = \sum_{g\in\mathcal{G}} \Gamma_R(g^{-1})_{ab}\Gamma_S(g)_{cd} = \frac{|G|}{d_R}\delta_{RS}\delta_{ad}\delta_{bc}. \tag{3.16}$$

This is a very useful result which will be used many times in the sections ahead.

## 3.10 Delta function on the group

In quantum mechanics, the delta function is important whenever we perform a Fourier transform between position space and momentum space. The delta function we introduce in this section is used in a similar way. For example we will use the delta function to Fourier transform Schur polynomials and restricted Schur polynomials into multi trace structures which are gauge invariant.

To introduce the idea of a delta function on the group, let us first construct the **regular representation**. This is a reducible representation that is obtained directly from the multiplication table of the group. The construction of the regular representation is based on arranging the multiplication table of a group so that the unit element appears along the main diagonal of the table. Within such an arrangement the columns (or rows) of the table are labelled by the group elements, arranged in any order, and the corresponding inverses label the rows (or columns). As an example consider the $S_3$ multiplication table

| | **(123)** | **(132)** | $\mathbb{1}$ | **(12)** | **(13)** | **(23)** |
|---|---|---|---|---|---|---|
| **(132)** | $\mathbb{1}$ | (123) | (132) | (23) | (12) | (13) |
| **(123)** | (132) | $\mathbb{1}$ | (123) | (13) | (23) | (12) |
| $\mathbb{1}$ | (123) | (132) | $\mathbb{1}$ | (12) | (13) | (23) |
| **(12)** | (23) | (13) | (12) | $\mathbb{1}$ | (132) | (123) |
| **(13)** | (12) | (23) | (13) | (123) | $\mathbb{1}$ | (132) |
| **(23)** | (13) | (12) | (23) | (132) | (123) | $\mathbb{1}$ |

One can read off the regular representation matrix $\Gamma_{rr}((1,2))$, for example, by regarding the multiplication table as a $|\mathcal{G}| \times |\mathcal{G}|$ array from which the representation of the group element $(1,2)$ is assembled by putting a '1' where the element appears in the multiplication table and '0' elsewhere. Thus

$$\Gamma_{rr}(\mathbb{1}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \Gamma_{rr}((1,2)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To each group element we associate a basis vector. For the example above we have

$$|\mathbb{1}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |(1,2)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \cdots \quad |(1,3,2)\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

**Definition 3.10.1** *The regular representation $\Gamma_{rr}(g)\ \forall g \in \mathcal{G}$ is defined by*

$$\Gamma_{rr}(g_2)\,|g_3\rangle = |g_2 \cdot g_3\rangle\,.$$

This representation contains every irreducible representation of a group at least once. Using the associativity of the group product, we verify that this defines a group representation

$$\begin{aligned} \Gamma_{rr}(g_1)\Gamma_{rr}(g_2)\,|g_3\rangle &= \Gamma_{rr}(g_1)\,|g_2 \cdot g_3\rangle \\ &= |g_1 \cdot (g_2 \cdot g_3)\rangle \\ &= |(g_1 \cdot g_2) \cdot g_3\rangle \\ &= \Gamma_{rr}(g_1 \cdot g_2)\,|g_3\rangle\,. \end{aligned}$$

Since the above equality holds for any vector $|g_3\rangle$, we have

$$\Gamma_{rr}(g_1)\Gamma_{rr}(g_2) = \Gamma_{rr}(g_1 \cdot g_2),$$

which does indeed confirm that we have a representation. From the matrix representation just described table we learn that the character $\chi_{rr}(g)\ \forall g \in \mathcal{G}$ is

$$\chi_{rr}(g) = \mathrm{Tr}(\Gamma_{rr}(g)) = |\mathcal{G}|\delta(g), \tag{3.17}$$

where the delta function

$$\delta(g) = \begin{cases} 1 & g \in \mathcal{G} \quad \text{is the identity,} \\ 0 & \text{otherwise.} \end{cases}$$

Since the regular representation is reducible, let us denote the irreducible representations of $\mathcal{G}$ by $R$. We find that $R$ appears $n_R = d_R$ times as demonstrated below.

$$\begin{aligned} n_R &= \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi_{rr}(g) \chi_R(g^{-1}) \\ &= \sum_{g \in \mathcal{G}} \delta(g) \chi_R(g^{-1}) \\ &= \chi_R(\mathbb{1}) \\ &= d_R. \end{aligned}$$

The character $\chi_{rr}(g)$ of the regular representation is related to the $\chi_R(g)$ as follows

$$\begin{aligned} \chi_{rr}(g) &= \sum_R n_R \chi_R(g) \\ &= \sum_R d_R \chi_R(g). \end{aligned}$$

Using the above equation and (3.17) we obtain an equation for the delta function on the group $\mathcal{G}$

$$\delta(g) = \sum_R \frac{d_R}{|\mathcal{G}|} \chi_R(g). \tag{3.18}$$

In conclusion we have discussed the $S_n$ group and $U(p)$ group, and demonstrated the duality between them when they act on the vector space $V_p^{\otimes m}$. The actions of the two groups on $V_p^{\otimes m}$ commute and hence can be simultaneously diagonalized. We also discussed the fundamental orthogonality relation and the delta function. In the next chapter we will see how the $S_n$ group enters into the construction of the Schur polynomials and restricted Schur polynomials. We will also see the role played by the fundamental orthogonality relation and the delta function when we compute the two point function of the Schur polynomials and the restricted Schur polynomials.

# Chapter 4

# Group theory techniques applied to $\mathcal{N} = 4$ SYM theory

## 4.1 Schur polynomials

In this chapter we want to explain how group representation theory can be used to construct a basis of local operators of $\mathcal{N} = 4$ super Yang-Mills theory. For operators constructed from a single matrix, these are known as Schur polynomials [9, 8]. For operators constructed using more than one matrix, these are known as the restricted Schur polynomials [25, 22]. The simplest case to consider is the $\frac{1}{2}$ BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory corresponding to operators constructed from a single complex adjoint scalar.

Our observables are constructed out of the complex matrix $Z$. The most general observable is a product of traces of powers of $Z$. The complete list of observables we can build using 3 $Z$ fields is given by

$$\mathrm{Tr}(Z^3), \quad \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Z), \quad \mathrm{Tr}(Z)^3.$$

We can introduce a convenient language to describe these observables. Towards this end, consider the action of the operator $Z$ on the $N$-dimensional vector space $V_N$

$$Z : V_N \to V_N.$$

By tensoring $n$ copies of $Z$ we obtain an operator $Z^{\otimes n} \equiv Z \otimes Z \otimes \cdots \otimes Z$ which acts on the space $V_N^{\otimes n}$ obtained by tensoring $n$ copies of $V_N$. It will prove to be useful to collect indices $i_1 \cdots i_n$ and $j_1 \cdots j_n$ indexing states in each copy of $V_N$, into indices $I$ and $J$ defined as follows

$$(Z^{\otimes n})^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n}.$$

Recall that we have defined an action of $\sigma \in S_n$ of the symmetric groups on $V_N^{\otimes n}$ as follows

$$\sigma : V_N^{\otimes n} \to V_N^{\otimes n}$$
$$\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(n) \mapsto \vec{v}(\sigma(1)) \otimes \vec{v}(\sigma(2)) \otimes \cdots \otimes \vec{v}(\sigma(n)).$$

It is a simple task to verify that this defines $\sigma$ as a matrix acting on $V_N^{\otimes n}$ with matrix elements given by

$$(\sigma)^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \cdots \delta^{i_n}_{j_{\sigma(n)}}.$$

Using this matrix representation we can easily verify that

$$(\sigma_1)^I_J (\sigma_2)^J_K = (\sigma_2 \cdot \sigma_1)^I_K,$$

where $\sigma_2 \cdot \sigma_1$ is the usual composition of permutations, as described in section 3.1. Any observable built from the single matrix $Z$ can be written as

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = (\sigma)^I_J (Z^{\otimes n})^J_I = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}. \tag{4.1}$$

Two permutations which belong to the same congucacy class define the same observable.

The action of the symmetric group on $V_N^{\otimes n}$ is reducible. We can introduce projection operators that project onto the irreducible representations contained in $V_N^{\otimes n}$. It is well known that the number of conjugacy classes of a group matches the number of irreducible representations of the group. There is a duality between conjugacy classes and irreducible representations. Using this duality we can associate observables to irreducible representations. A rather direct way to construct this association uses projection operators. To define the projection operators we need the definition of a character.

**Definition 4.1.1** $\chi_R(\sigma) = \mathrm{Tr}(\Gamma_R(\sigma))$ *is the character of group element* $\sigma \in S_n$ *in irreducible representation* $R$.

The projection operator is defined as

$$P_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \sigma \quad \Leftrightarrow \quad P_R{}^I_J = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \sigma^I_J.$$

A projection operator $\hat{P}$ must satisfy the condition $\hat{P}^2 = \hat{P}$. In this spirit we show that indeed $P_R$ above is a projection operator.

$$\begin{aligned} P_R P_T &= \frac{1}{n!m!} \sum_{\sigma \in S_n} \sum_{\tau \in S_m} \chi_T(\tau) \chi_R(\sigma) \sigma \tau \\ &= \frac{1}{n!m!} \sum_{\gamma \in S_n} \sum_{\tau \in S_m} \chi_T(\tau) \chi_R(\gamma \tau^{-1}) \gamma \\ &= \frac{1}{n!m!} \sum_{\gamma \in S_n} \sum_{\tau \in S_m} \Gamma_T(\tau)_{aa} \Gamma_R(\gamma)_{bc} \Gamma_R(\tau^{-1})_{cb} \gamma. \end{aligned} \tag{4.2}$$

As commented above, permuatations that belong to the same conjugacy class define the same observable. In other words these observables are unique up to conjugacy classes of $\sigma$. We prove this statement as follows.

**Proof 4.1.1** *Let* $\sigma = \gamma\tau\gamma^{-1}$ $\forall \sigma, \tau, \gamma \in S_n$ *where* $\sigma$ *and* $\tau$ *are conjugate pairs. Equation* (4.1) *takes the form*

$$\begin{aligned}\mathrm{Tr}(\gamma\tau\gamma^{-1}Z^{\otimes n}) &= \sum_R \chi_R(\gamma\tau\gamma^{-1})\chi_R(Z^{\otimes n}) \\ &= \sum_R \mathrm{Tr}\left(\Gamma_R(\gamma)\Gamma_R(\tau)\Gamma_R(\gamma^{-1})\right)\chi_R(Z^{\otimes n}).\end{aligned}$$

*By the cyclicity of the trace the above equation gives*

$$\mathrm{Tr}(\gamma\tau\gamma^{-1}Z^{\otimes n}) = \sum_R \mathrm{Tr}\left(\Gamma_R(\tau)\right)\chi_R(Z^{\otimes n}). \tag{4.3}$$

*Since* $\sigma, \tau \in S_n$ *we learn that indeed permutations that belong to the same conjugacy class define the same observable.*

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \mathrm{Tr}(\gamma\tau\gamma^{-1}Z^{\otimes n}),$$

*where* $\sigma$ *and* $\tau$ *are conjugate pairs.*

Equation (4.2) above is summed over $\tau$ using the fundamental orthogonality relation (3.16), to obtain

$$\begin{aligned}P_R P_T &= \frac{\delta_{RT}}{d_R}\delta_{ab}\delta_{ac}\frac{1}{n!}\sum_{\tau\in S_m}\Gamma_R(\gamma)_{bc}\gamma \\ &= \frac{\delta_{RT}}{d_R}P_R.\end{aligned}$$

This verifies that, up to an easily determined normalization, $P_R$ defined a projection operator.

The observables associated to irreducible representations are the Schur polynomials. The Schur polynomials are defined by

$$\begin{aligned}\chi_R(Z^{\otimes n}) &= \mathrm{Tr}(P_R Z^{\otimes n}) = {P_R}^I_J {Z^{\otimes n}}^J_I \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma^I_J {Z^{\otimes n}}^J_I \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}} \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\,\mathrm{Tr}(\sigma Z^{\otimes n}).\end{aligned} \tag{4.4}$$

In the above definition $R$ is a Young diagram with $n$ boxes, which labels an irreducible representation of $S_n$. In the free field theory, the Schur polynomials diagonalize the two point function [9] exactly, as we demonstrate below. In free field theory Wick's theorem says that

$$\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \rangle = \sum_{\sigma \in S_n} (\sigma)^I_L (\sigma^{-1})^K_J.$$

$$\begin{aligned}
\langle \chi_R(Z^{\otimes n}) \chi_T(Z^{\dagger \otimes n}) \rangle &= \sum_{\sigma \in S_n} (P_R)^I_J (P_T)^K_L (\sigma)^J_K (\sigma^{-1})^L_I \\
&= \sum_{\sigma \in S_n} \mathrm{Tr}(P_R \sigma P_T \sigma^{-1}) \\
&= n! \, \mathrm{Tr}(P_R P_T) \\
&= n! \frac{\delta_{RT}}{d_R} \, \mathrm{Tr}(P_R).
\end{aligned}$$

To compute the trace of the projection operator we need to use the fact that the Schur polynomial is a $U(N)$ character

$$\begin{aligned}
\chi_R(U) &= \frac{1}{n} \sum_{\sigma \in S_n} \chi_R(\sigma) U^{i_1}_{i_{\sigma(1)}} \cdots U^{i_n}_{i_{\sigma(n)}} \\
&= \mathrm{Tr}(P_R U^{\otimes n}),
\end{aligned}$$

which gives a concrete relation between the characters of the symmetric and the unitary groups. This relation is a consequence of Schur-Weyl duality. Thus, the trace is

$$\mathrm{Tr}(P_R) = \chi_R(U = \mathbb{1}) = \mathrm{Dim}_R.$$

Finally, we have the two point function given by

$$\langle \chi_R(Z^{\otimes n}) \chi_T(Z^{\dagger \otimes n}) \rangle = n! \delta_{RT} \frac{\mathrm{Dim}_R}{d_R}.$$

where $\mathrm{Dim}_R$ is the dimension of an irreducible representation $R$ of $U(N)$. Further, Schur polynomials are complete in the sense that any observable can be written as a linear combination of Schur polynomials, a fact we illustrate below.

**Proof 4.1.2** *From equation* (4.4)

$$\chi_R(Z^{\otimes n}) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \, \mathrm{Tr}(\sigma Z^{\otimes n}),$$

*we claim that our observable can be written as*

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \sum_R N^\sigma_R \chi_R(Z^{\otimes n}), \tag{4.5}$$

*where we will determine the coefficient $N_R^\sigma$. Note that $\chi_R(\sigma) = \chi_R(\sigma^{-1})$ since $\sigma$ and $\sigma^{-1}$ belong to the same conjugacy class.*

$$\begin{aligned}
\chi_R(Z^{\otimes n}) &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma^{-1})\,\mathrm{Tr}(\sigma Z^{\otimes n}) \\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma^{-1})\sum_T N_T^\sigma \chi_T(Z^{\otimes n}) \\
&= \sum_T \Big(\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma^{-1})N_T^\sigma\Big)\chi_T(Z^{\otimes n}).
\end{aligned}$$

*We now require that*

$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma^{-1})N_T^\sigma = \delta_{RT},$$

*which when compared to the fundamental orthogonality relation of characters*

$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma^{-1})\chi_T(\sigma) = \delta_{RT},$$

*implies that $N_T^\sigma = \chi_T(\sigma)$. Hence equation (4.5) takes the form*

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \sum_R \chi_R(\sigma)\chi_R(Z).$$

## 4.2 Restricted Schur polynomials

We now consider observables built using two complex matrices, $Z$ and $Y$. This is known as the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory. Permutations again provide a convenient way to describe the observables of the theory. Indeed the most general gauge invariant observables built out of $Z$ and $Y$ can be written as

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}, \tag{4.6}$$

where $\sigma \in S_{n+m}$.

In this multimatrix setting we need to generalize the Schur polynomials to the restricted Schur polynomials. This section will focus on restricted Schur polynomials built using $n \sim \mathcal{O}(N) Z$ and $m \sim \mathcal{O}(N) Y$ fields where $m \ll n$. We will refer to the $Y$ fields as "impurities". These fields belong to the $SU(2)$ sector of the theory. A restricted Schur polynomial is a generalization of the Schur polynomial which is defined [25] as

$$\begin{aligned}
\chi_R,(r,s)_{\alpha\beta}(Z,Y) &= \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)_{\alpha\beta}}(\sigma) Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_n}_{i_{\sigma(n)}} \\
&= \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)_{\alpha\beta}}(\sigma)\,\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}).
\end{aligned} \tag{4.7}$$

These operators provide a basis for the local gauge invariant operators of a matrix model. In the above equation $R$ labels an irreducible representation of $S_{n+m}$ with $n+m$ boxes. $r$ is a Young diagram with $n$ boxes which labels an irreducible representation of $S_n$ and $s$ is a Young diagram with $m$ boxes which labels an irreducible representation of $S_m$. $(r,s)$ labels an irreducible representation of the subgroup $S_n \times S_m$ of the group $S_{n+m}$. The object $\chi_{R,(r,s)_{\alpha\beta}}(\sigma)$ is called the restricted character. A particular irreducible representation of the $S_n \times S_m$ subgroup may be subduced more than once. Hence we introduce multiplicity labels $\alpha$ and $\beta$ to keep track of the different copies subduced. Similar to computing the character $\chi_R(\sigma) = \mathrm{Tr}(\Gamma_R(\sigma))$, $\chi_{R,(r,s)_{\alpha\beta}}(\sigma)$ is computed by taking the trace of row index of $\Gamma_R(\sigma)$ over the subspace associated to the $\alpha$-th copy of $(r,s)$ and the column index over the subspace associated to the $\beta$-th copy of $(r,s)$.

In what follows we demonstrate the fundamental orthogonality relation for the restricted character. We will define the projection operator and prove that it is a projector before deriving a generalization of the orthogonality relation for characters. We define

$$\begin{aligned} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) &= \mathrm{Tr}_{R,(r,s)_{\alpha\beta}}(\sigma) \\ &= \sum_I \langle R,(r,s)_\alpha ; I| \, \Gamma_R(\sigma) \, |R,(r,s)_\beta ; I\rangle \, . \end{aligned} \tag{4.8}$$

The projection operator is defined as

$$P_{R,(r,s)_{\alpha\beta}} = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma)\sigma,$$

and acts as an orthogonal projection operates in $V^{\otimes n+m}$. Next we show that indeed this is a projector. Towards this end consider

$$P_{R,(r,s)_{\alpha\beta}} P_{T,(t,u)_{\gamma\delta}} = \frac{1}{(n!m!)^2} \sum_{\sigma, rho \in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) \chi_{T,(t,u)_{\gamma\delta}}(\rho)\sigma\rho.$$

Let $\sigma = \tau\rho^{-1}$, then we have

$$\begin{aligned} P_{R,(r,s)_{\alpha\beta}} P_{T,(t,u)_{\gamma\delta}} &= \frac{1}{(n!m!)^2} \sum_{\sigma, rho \in S_{n+m}} \sum_{I,J} \langle R,(r,s)_\alpha ; I| \, \Gamma_R(\tau)\Gamma_R(\rho^{-1}) \, |R,(r,s)_\beta ; I\rangle \\ &\quad \times \langle T,(t,u)_\gamma ; J| \, \Gamma_T(\rho) \, |T,(t,u)_\delta ; J\rangle \\ &= \frac{(n+m)!}{d_R} \delta_{RS} \sum_{I,J} \langle R,(r,s)_\alpha ; I| \, \Gamma_R(\tau) \, |T,(t,u)_\delta ; J\rangle \, \langle T,(t,u)_\gamma ; J\rangle \, R,(r,s)_\beta ; I \\ &= \frac{(n+m)!}{d_R} \delta_{RS}\delta_{rt}\delta_{su}\delta_{\gamma\beta} \sum_I \langle R,(r,s)_\alpha ; I| \, \Gamma_R(\tau) \, |R,(r,s)_\beta ; I\rangle \\ &= \frac{(n+m)!}{d_R} \delta_{RS}\delta_{rt}\delta_{su}\delta_{\gamma\beta} \chi_{R,(r,s)_{\alpha\delta}}(\tau). \end{aligned}$$

Thus, we find [15]

$$P_{R,(r,s)_{\alpha\beta}}P_{T,(t,u)_{\gamma\delta}} = \frac{(n+m)!}{d_R n!m!}\delta_{RS}\delta_{rt}\delta_{su}\delta_{\gamma\beta}\sum_{\tau\in S_{n+m}}P_{R,(r,s)_{\alpha\delta}}.$$

With the above property of the projector established, we argue that the restricted characters are orthogonal on the restricted conjugacy classes.

$$\begin{aligned}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)_{\alpha\beta}}(\sigma)\chi_{T,(t,u)_{\gamma\delta}}(\sigma^{-1}) &= \sum_{\sigma\in S_{n+m}}\sum_{I,J}\langle R,(r,s)_\alpha;I|\,\Gamma_R(\sigma)\,|R,(r,s)_\beta;I\rangle \\ &\quad\times\langle T,(t,u)_\gamma;J|\,\Gamma_T(\sigma^{-1})\,|T,(t,u)_\delta;J\rangle \\ &= \sum_{\sigma\in S_{n+m}}\sum_{I,J}\langle R,(r,s)_\alpha;I|\,\Gamma_R(\sigma)\Gamma_T(\sigma^{-1})\,|T,(t,u)_\delta;J\rangle \\ &\quad\times\langle T,(t,u)_\gamma;J\rangle\, R,(r,s)_\beta;I.\end{aligned}$$

From the orthogonality condition in equation (3.16) the above equation becomes [51]

$$\begin{aligned}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)_{\alpha\beta}}(\sigma)\chi_{T,(t,u)_{\gamma\delta}}(\sigma^{-1}) &= \frac{(n+m)!}{d_R}\sum_{I,J}\langle R,(r,s)_\alpha;I\rangle\, T,(t,u)_\delta;J \\ &\quad\times\langle T,(t,u)_\gamma;J\rangle\, R,(r,s)_\beta;I \\ &= \frac{(n+m)!}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}\sum_{I,J}\delta_{IJ} \\ &= \frac{(n+m)!}{d_R}d_r d_s\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}. \qquad (4.9)\end{aligned}$$

This is the orthogonality relation for restricted characters. In the free field theory, the restricted Schur polynomials diagonalize the two point function exactly [25]. The Wick contraction formula for a single matrix generalizes to

$$\langle (Y^{\otimes m}Z^{\otimes n})^I_J (Y^{\otimes m}Z^{\otimes n})^K_L\rangle = \sum_{\sigma\in S_n\times S_m}\sigma^I_L(\sigma^{-1})^K_J.$$

We contract $Zs$ and $Z^\dagger s$ and $Ys$ and $Y^\dagger s$ which is reflected in the sum over permutations over $S_n\times S_m$. The two point function of restricted Schur polynomials is

$$\begin{aligned}\langle\chi_{R,(r,s)_{\alpha\beta}}(Z,Y)\chi_{T,(t,u)_{\delta\gamma}}(Z^\dagger,Y^\dagger)\rangle &= \langle \mathrm{Tr}(P_{R,(r,s)_{\alpha\beta}}Y^{\otimes m}Z^{\otimes n})\,\mathrm{Tr}(P_{T,(t,u)_{\delta\gamma}}Y^{\dagger\otimes m}Z^{\dagger\otimes n})\rangle \\ &= (P_{R,(r,s)_{\alpha\beta}})^I_J(P_{T,(t,u)_{\delta\gamma}})^K_L\langle (Y^{\otimes m}Z^{\otimes n})^J_I(Y^{\dagger\otimes m}Z^{\dagger\otimes n})^L_K\rangle \\ &= \sum_{\sigma\in S_n\times S_m}\mathrm{Tr}(P_{R,(r,s)_{\alpha\beta}}\sigma P_{T,(t,u)_{\delta\gamma}}\sigma^{-1}) \\ &= n!m!\,\mathrm{Tr}(P_{R,(r,s)_{\alpha\beta}}P_{T,(t,u)_{\delta\gamma}}) \\ &= \frac{(n+m)!}{d_R}\delta_{RS}\delta_{rt}\delta_{su}\delta_{\gamma\beta}\,\mathrm{Tr}(P_{R,(r,s)_{\alpha\delta}}). \qquad (4.10)\end{aligned}$$

Now we compute the trace $\mathrm{Tr}(P_{R,(r,s)_{\alpha\beta}})$. Denote partially labelled Young diagrams corresponding to subspace $r$ by $|r, YY, a\rangle$ with the index $a$ a label for the states in $r$, i.e $a = 1, 2, \cdots, d_r$, where $d_r$ is the dimension of the irreducible representation labelled by Young diagram $r$. In general, we write the states as

$$|r, a; s, b; \alpha\rangle = \sum_{YY} \alpha_{YY,b,\alpha} |r, YY, a\rangle , \tag{4.11}$$

where $YY$ denotes the sum over all partially labelled diagrams.

$$\begin{aligned}
\mathrm{Tr}(P_{R,(r,s)_{\alpha\delta}}) &= \frac{1}{(n!m!)^2} \sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)_{\alpha\delta}}(\sigma)\,\mathrm{Tr}(\sigma) \\
&= \frac{1}{(n!m!)^2} \sum_{\sigma\in S_{n+m}} \sum_{a,b} \langle r, a; s, b; \alpha|\, \Gamma_R(\sigma)\, |r, a; s, b; \delta\rangle\,\mathrm{Tr}(\sigma) \\
&= \frac{1}{(n!m!)^2} \sum_{\sigma\in S_{n+m}} \sum_{YY,YY',a,b} \alpha_{YY,b,\alpha}\alpha_{YY',b,\delta} \langle r, YY, a|\, \Gamma_R(\sigma)\, |r, YY', a\rangle\,\mathrm{Tr}(\sigma) \\
&= \frac{1}{n!m!} \sum_{YY,YY',b} \alpha_{YY,b,\alpha}\alpha_{YY',b,\delta}\,\mathrm{Tr}(P_{r,r,YY,YY'}) \\
&= \frac{1}{n!m!} \sum_{YY,YY',b} \alpha_{YY,b,\alpha}\alpha_{YY',b,\delta} d_r f_R \\
&= \frac{1}{n!m!} d_r d_s f_R \delta_{\alpha\delta}.
\end{aligned} \tag{4.12}$$

As a result, equation (4.10) becomes

$$\langle \chi_{R,(r,s)_{\alpha\beta}}(Z, Y)\chi_{T,(t,u)_{\delta\gamma}}(Z^\dagger, Y^\dagger)\rangle = \frac{(n+m)!}{d_R n!m!} d_r d_s f_R \delta_{RS}\delta_{rt}\delta_{su}\delta_{\gamma\beta}\delta_{\alpha\delta}. \tag{4.13}$$

If we Fourier transform equation (4.7) we expect

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \sum_{R,(r,s)_{\alpha\beta}} Q^{R,(r,s)_{\alpha\beta}}(\sigma^{-1})\chi_{R,(r,s)_{\alpha\beta}}(Z, Y). \tag{4.14}$$

Our task now is to determine the exact form of $Q^{R,(r,s)_{\alpha\beta}}(\sigma^{-1})$. To do this we substitute the above equation into (4.7) to obtain

$$\begin{aligned}
\chi_R, (r, s)_{\alpha\beta}(Z, Y) &= \frac{1}{n!m!} \sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) \sum_{R,(r,s)_{\alpha\beta}} Q^{R,(r,s)_{\alpha\beta}}(\sigma^{-1})\chi_{R,(r,s)_{\alpha\beta}}(Z, Y) \\
&= \frac{1}{n!m!} \sum_{R,(r,s)_{\alpha\beta}} \Big( \sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) Q^{R,(r,s)_{\alpha\beta}}(\sigma^{-1})\Big)\chi_{R,(r,s)_{\alpha\beta}}(Z, Y).
\end{aligned}$$

If we now require

$$\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)_{\alpha\beta}}(\sigma) Q^{R,(r,s)_{\alpha\beta}}(\sigma^{-1}) = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma},$$

and compare with the orthogonality relation in (4.9), we learn that

$$Q^{R,(r,s)_{\alpha\beta}}(\sigma^{-1}) = \frac{d_R n!m!}{(n+m)!d_r d_s}\chi_{T,(t,u)_{\gamma\delta}}(\sigma^{-1}).$$

Therefore we have

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \sum_{R,(r,s)_{\alpha\beta}} \frac{d_R n!m!}{(n+m)!d_r d_s}\chi_{R,(r,s)_{\alpha\beta}}(\sigma^{-1})\chi_{R,(r,s)_{\alpha\beta}}(Z,Y). \tag{4.15}$$

Note that permutations which can be related as

$$\gamma\tau\gamma^{-1} = \sigma \qquad \gamma \in S_n \times S_m,$$

define the same physical observable. A proof is provided below.

**Proof 4.2.1** *We show that*

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \mathrm{Tr}(\gamma\tau\gamma^{-1} Z^{\otimes n} Y^{\otimes m}) \quad \forall \sigma,\tau \in S_{n+m} \quad \forall\gamma \in S_n \times S_m.$$

*By the cyclicity of the trace we can write the above equation as*

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \mathrm{Tr}(\tau\gamma^{-1} Z^{\otimes n} Y^{\otimes m}\gamma).$$

*Observe that*

$$\gamma^{-1} Z^{\otimes n} Y^{\otimes m}\gamma = Z^{i_{\gamma(1)}}_{i_{\gamma(1)}} \cdots Z^{i_{\gamma(n)}}_{i_{\gamma(n)}} Y^{i_{\gamma(n+1)}}_{i_{\gamma(n+1)}} \cdots Y^{i_{\gamma(n+m)}}_{i_{\gamma(n+m)}}.$$

*The matrices $Z$ and $Y$ have only been shifted to a new slot by the left and right action of $\gamma \in S_n \times S_m$. Since they are bosons, the fields can be shifted back to their original positions so that*

$$\begin{aligned}
\mathrm{Tr}(\tau\gamma^{-1} Z^{\otimes n} Y^{\otimes m}\gamma) &= \mathrm{Tr}(Z^{i_{\gamma(1)}}_{i_{\tau\gamma(1)}} \cdots Z^{i_{\gamma(n)}}_{i_{\tau\gamma(n)}} Y^{i_{\gamma(n+1)}}_{i_{\tau\gamma(n+1)}} \cdots Y^{i_{\gamma(n+m)}}_{i_{\tau\gamma(n+m)}}) \\
&= \mathrm{Tr}(Z^{i_{(1)}}_{i_{\tau(1)}} \cdots Z^{i_{(n)}}_{i_{\tau(n)}} Y^{i_{(n+1)}}_{i_{\tau(n+1)}} \cdots Y^{i_{(n+m)}}_{i_{\tau(n+m)}}) \\
&= \mathrm{Tr}(\tau Z^{\otimes n} Y^{\otimes m}).
\end{aligned}$$

## 4.3 Young's orthogonal representation simplified

Let $R, T \vdash (n+m)$ and $r, t \vdash n$ be Young diagrams with $p$ rows. We denote by $l_{R_i}$ , where $i = 1, \cdots, p$, the number of boxes in the $i$-th row of $R$. The simplification we consider arises when

$$\begin{cases} (n+m) \approx \mathcal{O}(N) & \text{for } m = \xi n,\ \xi \ll 1 \\ p \approx \mathcal{O}(1) \\ l_{R_i} - l_{R_{i+1}} \approx \mathcal{O}(N), \end{cases}$$

and $l_{R_1}$ and $l_{R_p}$ are respectively the longest and shortest rows in $R$.

It is a highly non-trivial problem to explicitly evaluate the expression for the dilatation operator. To carry out this step, we need to develop new approximation techniques. Towards this end, consider the Young diagram $R \vdash 16$ with two rows. The first row has the last box to the right labelled 1 and the last box to the second row labelled 2 as shown below.

$$R: \quad \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & 1 \\ \hline \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} & 2 \\ \cline{1-6} \end{array}$$

Let the difference between row one and row two (i.e $l_{R_i} - l_{R_{i+1}}$) be of order $N$ and consider

$$\Gamma_R((1,2))\,|R\rangle = \frac{1}{c_1 - c_2}\,|R\rangle + \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}\,\left|R_{(2,1)}\right\rangle ,$$

where $c_1$ is the content of a box in the 1[st] location and $c_2$ is the content of a box in the 2[nd] location. This is discussed in [28]. Also $\left|R_{(j,i)}\right\rangle$ is the state labelled by Young-Yamanouchi symbol obtained from $|R\rangle$ by swapping the labels of boxes $i$ and $j$. Acting with $\Gamma_R((1,2))$ on the ket vector $|R\rangle$ for the case when the boxes 1 and 2 are in the same row gives

$$\Gamma_R((1,2))\,|R\rangle = |R\rangle\,.$$

This is true since $c_2$ sits to the left of $c_1$ and so $c_1 - c_2 = 1$. For $c_1 - c_2 = \mathcal{O}(N)$, taking $N$ to infinity implies that the factor $\frac{1}{c_1 - c_2} \to 0$. Therefore, in such a limit, we have

$$\Gamma_R((1,2))\,|R\rangle = \left|R_{(2,1)}\right\rangle .$$

In general, if $i$ and $j$ are in the same row

$$\Gamma_R((i,j))\,|\text{state}R\rangle = |\text{state}R\rangle\,.$$

On the other hand if $i$ and $j$ are in different rows

$$\Gamma_R((i,j))\,|\text{state}R\rangle = |\text{different state}R\rangle = \left|R_{(j,i)}\right\rangle .$$

The above result found in [28] follows since $c_i - c_j = \mathcal{O}(N)$ and sending $N$ to infinity implies that the factor $\frac{1}{c_i - c_j} \to 0$.

## 4.4 Representing intertwiners

When $S_n$ acts on $V^{\otimes n}$ for $n > 1$ it furnishes a reducible representation. Imagine that this includes the irreducible representations $R$ and $S$. Representing the action of $\sigma$ as a matrix $\Gamma(\sigma)$, in a suitable basis we can write

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_R(\sigma) & 0 & \cdots \\ 0 & \Gamma_S(\sigma) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} . \tag{4.16}$$

If we restrict ourselves to an $S_{n-1}$ subgroup of $S_n$, then in general both $R$ and $S$ will subduce a number of representations. For the sake of this discussion let us assume that $R$ subduces $R'_1$ and $R'_2$ and that $S$ subduces $S'_1$ and $S'_2$. Then, for $\sigma \in S_{n-1}$ we have

$$\Gamma_R(\sigma) = \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{r'_3}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & \Gamma_{R'_4}(\sigma) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{4.17}$$

Imagine that as Young diagrams $S'_1 = R'_1$, that is, one of the irreducible representations subduced by $R$ is isomorphic to one of the representations subduced by $S$. Then, a simple application of the fundamental orthogonality relation gives

$$\begin{aligned} &\sum_{\sigma \in S_{n-1}} \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ij} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ab} \\ &= \frac{(n-1)!}{dR'_1} \delta_{R'_1 S'_1} \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ib} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{aj} \\ &= \frac{(n-1)!}{dR'_1} \delta_{R'_1 S'_1} (I_{R'_1 S'_1})_{ib} (I_{S'_1 R'_1})_{aj}. \end{aligned} \tag{4.18}$$

The intetwiners are the maps between isomorphic spaces as defined. For case where $R \neq S$ we write

$$\begin{aligned} I_{R'S'} &= E_{ij} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\ I_{S'R'} &= E_{ji} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}. \end{aligned}$$

and for the case when $R = S$ we have

$$I_{R'S'} = E_{kk} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} = I_{S'R'} \qquad k = 1, 2, \cdots, p.$$

In what follows we show how to realize permutations in terms of the basis $E_{ij}$ of $U(N)$. We discuss the operator $E_{ij}$ together with its properties. We will then show that these operators can be used to represent intertwiners in our calculation. Let $|i\rangle$, $|j\rangle$, $|k\rangle$ and $|l\rangle$ be basis vectors associated with each row of a Young diagram. Choose $(1,3) \in S_3$ and consider the (by now) natural action of the symmetric group such that

$$(1,3)\, |i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle = |k\rangle \otimes |j\rangle \otimes |i\rangle \otimes |l\rangle\, .$$

Acting with permutation group element $(1,3)$, simply swaps slots for vectors $|i\rangle$ and $|k\rangle$. Introduce the matrix $E^{(a)}_{ij}$ which acts on the $a^{\text{th}}$ vector in the tensor product. All entries of $E^{(a)}_{ij}$ are zero except for the entry in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column which is 1. Now consider the operator $\mathrm{Tr}\Big(E^{(1)}E^{(3)}\Big)$ on the state we have constructed above. Thus, we have

$$\begin{aligned}
\mathrm{Tr}(E^{(1)}E^{(3)})\,|i\rangle\otimes|j\rangle\otimes|k\rangle\otimes|l\rangle &= \sum_{a,b} E^{(1)}_{ab}E^{(3)}_{ba}\,|i\rangle\otimes|j\rangle\otimes|k\rangle\otimes|l\rangle \\
&= \sum_{a,b} E^{(1)}_{ab}\,|i\rangle\otimes|j\rangle\otimes E^{(3)}_{ba}\,|k\rangle\otimes|l\rangle \\
&= \sum_{a} |a\rangle\otimes|j\rangle\otimes E^{(3)}_{ia}\,|k\rangle\otimes|l\rangle \\
&= |k\rangle\otimes|j\rangle\otimes|i\rangle\otimes|l\rangle\,.
\end{aligned}$$

In the last two steps summing over $a$ and $b$ to obtain a non-zero results requires that $b=i$ and $a=k$. Here we have learned that the action of a permutation group element $(i,j)\in S_3$ for $i<j$ is the same as acting with $\mathrm{Tr}\Big(E^{(i)}E^{(j)}\Big)$ on the state $|i\rangle\otimes|j\rangle\otimes|k\rangle\otimes|l\rangle$. A detailed discussion can be found in [28].

Consider two Young diagrams, $R$ and $T$. Drop a box from $R$ to obtain $R'$ and drop a box from $T$ to obtain $T'$. If the shape of $R'$ matches the shape of $T'$, we know the corresponding carrier spaces are isomorphic so that we can define a map between them. This map is the intertwiner $I_{R'T'}$

$$I_{R'T'} : T' \to R'.$$

To see a concrete example, consider

$$R = \begin{ytableau} ~ & ~ & ~ & ~ & ~ & ~ & ~ & ~ & ~ \\ ~ & ~ & ~ & ~ & ~ \end{ytableau} \qquad T = \begin{ytableau} ~ & ~ & ~ & ~ & ~ & ~ & ~ & ~ \\ ~ & ~ & ~ & ~ & ~ & ~ \end{ytableau}\,.$$

We remove a box each from row one of $R$ and row two of $T$ to achieve $R'$ and $T'$ respectively.

$$R' = \begin{ytableau} ~ & ~ & ~ & ~ & ~ & ~ & 2 & 1 \\ ~ & ~ & ~ & 4 & 3 \end{ytableau} \quad : \qquad \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes|r,I\rangle$$

$$T' = \begin{ytableau} ~ & ~ & ~ & ~ & ~ & ~ & ~ & 2 \\ ~ & ~ & 4 & 3 & 1 \end{ytableau} \quad : \qquad \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes|r,I\rangle\,.$$

The $R$-state and $T$-state differ by the first slot in the tensor product. Making a transition from the $R$-state to $T$-state requires the operator

$$\begin{aligned} E_{12}^{(1)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes |r, I\rangle &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes |r, I\rangle \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes |r, I\rangle \,. \end{aligned}$$

Thus, $E_{12}^{(1)} : V_{T'} = V_{R'}$. In general removing a labelled box from $i^{\text{th}}$ location by our convention will take $R$ to $R'$ and removing a box from the $j^{\text{th}}$ location will take $T$ to $T'$. The map which takes $T'$ to $R'$ or $R'$ to $T'$ is called **intertwiner**. We will explore the properties of the $E$ operator with some simple examples.

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that

$$E_{12}E_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{12}E_{23} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_{13}.$$

Now consider the action of the operator $E_{ij}$ on the basis vectors,

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives us,

$$E_{12}V_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = E_{12}V_3 \qquad E_{13}V_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

In general we have the useful algebra

$$E_{ij}E_{kl} = \delta_{jk}E_{il}, \quad E_{ij}^{(a)}E_{ji}^{(b)} = (a, b), \quad \text{and} \quad E_{ij}\,|k\rangle = \delta_{jk}\,|i\rangle \,.$$

## 4.5 Dilatation operator in the $SU(2)$ sector

The eigenvalues of the dilatation operator are the scaling dimensions of the local operators in the Conformal Field Theory. The dilatation operator is one of the generators of the conformal algebra. Expanding the dilatation operator in powers of $g_{YM}^2$ we have [31]

$$D = \sum_{j=0}^{\infty} \Big(\frac{g_{YM}^2}{16\pi^2}\Big)^j D_{2j} = D_0 + \frac{g_{YM}^2}{16\pi^2} D_2 + \Big(\frac{g_{YM}^2}{16\pi^2}\Big)^2 D_4 + \mathcal{O}(g_{YM}^2). \tag{4.19}$$

In this section we are interested in the one loop dilatation operator denoted by $D_2$ in the $SU(2)$ sector. In the $SU(2)$ sector the $D_0$, $D_2$ and $D_4$ are given by

$$\begin{aligned}
D_0 &= \mathrm{Tr}\Big(Y\frac{\partial}{\partial Y}\Big) + \mathrm{Tr}\Big(Z\frac{\partial}{\partial Z}\Big)\\
D_2 &= \mathrm{Tr}([Z,Y][\tfrac{\partial}{\partial Z},\tfrac{\partial}{\partial Y}])\\
D_4 &=: \mathrm{Tr}\bigg(\Big[[Y,Z],\frac{\partial}{\partial Z}\Big]\Big[[\tfrac{\partial}{\partial Y},\tfrac{\partial}{\partial Z}],Z\Big]\bigg) : + : \mathrm{Tr}\bigg(\Big[[Y,Z],\frac{\partial}{\partial Y}\Big]\Big[[\tfrac{\partial}{\partial Z},\tfrac{\partial}{\partial Y}],Y\Big]\bigg) : +\\
&\quad : \mathrm{Tr}\bigg(\Big[[Y,Z],T^a\Big]\Big[[\tfrac{\partial}{\partial Z},\tfrac{\partial}{\partial Y}],T^a\Big]\bigg) :,
\end{aligned} \tag{4.20}$$

where $Y$ and $Z$ are scalar fields in $\mathcal{N} = 4$ SYM. In the above expression $T^a$ are the generators of the $U(N)$ Lie algebra. The normal ordering notation :: indicates that the derivatives do not act on the fields enclosed by ::. We call $D_0$ the tree level dilatation generator, $D_2$ the one loop dilatation generator, $D_4$ the two loop dilatation generator and so on. A detailed description of the dilatation operator in $SU(2)$ is in [31].

So far in chapters 1, 2, 3 and 4, we have given the background from String theory and $\mathcal{N} = 4$ super Yang-Mills theory needed to appreciate Maldacena's AdS/CFT correspondence, from which eigenvalues of the dilatation operator in conformal field theory have an interpretation as an energy in string theory. Powerful group theory technology was discussed in this chapter. This background introduced places importance on the study of the action of the dilatation operator on the restricted Schur polynomial, which we will discuss in later chapters.

# Chapter 5

# Interacting double coset magnons

## 5.1 Introduction

In this chapter we will study the first subleading corrections in $\frac{m}{n}$ to the anomalous dimension. This arises at two loops, and we find non-zero magnon interactions.

Consider operators built using $n \sim \mathcal{O}(N)$ $Z$s and $m$ $Y$s with $n \gg m$, which implies that we are close to the $\frac{1}{2}$-BPS giant graviton and that we have a new small parameter $\frac{m}{n}$ in the game. The condition that $n \gg m$ is crucial for our approximations which is not too surprising: a systematically small deformation of a BPS operator will be simpler than the generic operator. Part of the motivation for this chapter is to consider the first order correction in a systematic $\frac{m}{n}$ expansion.

In both the open string description where the open string and giant graviton are distinguished and in the more democratic description, there is a close connection [52] between the dynamics of the $Y$ fields and the LLM plane [53] description of giant magnon dynamics in the dual string theory [54, 55, 56]. In this study we are interested in the anomalous dimensions of gauge theory operators corresponding to restricted Schur polynomials that treat the fields democratically. This corresponds to a dilute gas of magnons. The spectrum of anomalous dimensions has been computed to all loops at large $N$ and to leading order in the small parameter $\frac{m}{n}$ [52, 57]. This all loop answer is possible thanks to supersymmetry [58]. The result agrees with explicit one loop computations in [28, 30], two loop computations in [59] and even three loop computations performed in [60] using a collective coordinate [61, 62, 63, 64] approach. The result has an interesting structure which is worth understanding to appreciate the results we report here. The operators we study are labelled by irreducible representations of the symmetric group $S_n$, that is, by Young diagrams with $n$ boxes. To specify the Young diagram we can write a list of row lengths. The row lengths must sum to $n$, so that the Young diagram is determined by some partition of $n$. Denoting the Young diagram by $r$ the notation "$r \vdash n$" says "$r$" is a partition of "$n$". The operator we are interested in is defined by

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \mathrm{Tr}\,(\sigma Z^{\otimes n} \otimes Y^{\otimes m}). \tag{5.1}$$

The labels $R \vdash n+m$, $r \vdash n$, $s \vdash m$ label irreducible representations of $S_{n+m}$, $S_n$ and $S_m$ respectively. $(r,s)$ labels a representation of the subgroup $S_n \times S_m$. The above polynomial is only nonzero if $(r,s)$ arises from $R$ after restricting to the $S_n \times S_m$ subgroup. Since it may arise more than once, we need multiplicity labels (denoted $\alpha$ and $\beta$ above) to keep track of the copy we are considering. One way to ensure that $(r,s)$ arises after restricting to the subgroup is to realize $r$ by removing $m$ boxes from Young diagram $R$. These removed boxes are then reassembled to give $s$. Use $m_i$ to count the number of boxes that must be removed from row $i$ of $R$ to get $r$. Assemble the $m_i$ to produce the vector $\vec{m}$. The vector $\vec{m}$ is conserved to leading order in $\frac{m}{n}$.

In this article we will study the first subleading corrections in $\frac{m}{n}$ to the anomalous dimension. This explores the first contributions which induce magnon interactions for the dilute magnon gas. If we are ever to understand the nonperturbative sectors of string theory using the gauge theory/gravity correspondence, it seems that we must move beyond small deformations of $\frac{1}{2}$ -BPS operators. One way to do this is to construct a good understanding of this system, beyond the leading order in $\frac{m}{n}$ . This is a key motivation for this chapter.

There are already corrections of order $\frac{m}{n}$ to the one loop anomalous dimension. These corrections do not lead to new operator mixing–they only correct the anomalous dimension. The first nontrivial corrections appear at two loops. Consequently, in Section 5.2 we review the structure of the two loop dilatation operator. It is useful to rewrite the action of the two loop dilatation operator in a basis, the Gauss graph basis, that diagonalizes the leading order. In this way it will become apparent that the subleading correction induces new operator mixing. In the process of transforming to the Gauss graph basis we encounter new types of traces that have not been computed before. In Appendix B we develop techniques powerful enough to evaluate the most general trace we could encounter, which is much more general than the traces that appear at two loops. These results will be useful in further studies of the dynamics of Gauss graph operators. To illustrate our results, in Section 5.3 we consider a state of two giant gravitons with four strings attached. Our results show a number of interesting features. Operators labelled by different Gauss graphs start to mix implying that we do indeed have an interacting system of magnons. As a consequence of these interactions, the vector $\vec{m}$ is no longer conserved and operators with different $\vec{m}$ labels mix. Finally, at the leading order in $\frac{m}{n}$ and at large $N$, the all loop dilatation operator factorizes into an action on the $Y$s times an action on the $Z$s. Although this factorization was only exhibited at one loop in [28] and at two loops in [59], the arguments of [57] as well as the form of the all loop anomalous dimension [58] implies that this factorization holds to all loops; see [52] for further discussion. The subleading term that we have evaluated does not factorize into an action on the $Y$s times an action on the $Z$s. This proves that the action of the dilatation operator only factorizes into an action on the $Y$s times an action on the $Z$s at the leading order in a systematic $\frac{m}{n}$ expansion. In Section 5.4 we discuss these results and suggest a number of interesting directions in which the present study can be extended.

## 5.2 The two loop dilatation operator

The complete two loop dilatation operator in the $SU(2)$ is given by [31]

$$D_4 = D_4^{(1)} + D_4^{(2)} + D_4^{(3)}, \tag{5.2}$$

where

$$\begin{aligned} D_4^{(1)} &= -2g^2 : \mathrm{Tr}\left(\Big[[Y,Z], \frac{\partial}{\partial Z}\Big]\Big[\Big[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\Big], Z\Big]\right) : \\ D_4^{(2)} &= -2g^2 : \mathrm{Tr}\left(\Big[[Y,Z], \frac{\partial}{\partial Y}\Big]\Big[\Big[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\Big], Y\Big]\right) : \\ D_4^{(3)} &= -2g^2 : \mathrm{Tr}\left([[Y,Z], T^a]\Big[\Big[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\Big], T^a\Big]\right) :, \end{aligned} \tag{5.3}$$

and

$$g = \frac{g_{YM}^2}{16\pi^2}. \tag{5.4}$$

The sum over $a$ in $D_4^{(3)}$ is easily performed with the help of the identity

$$\mathrm{Tr}(T^a A T^a B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B), \tag{5.5}$$

which follows from the completeness of the $T^a \in U(N)$. The size of these three terms is easily estimated as follows:$D_4^{(1)}$ has two derivatives with respect to $Z$ which act on $n$ $Z$ fields and one with respect to $Y$ which acts on $m$ $Y$ fields. The size of this term is thus $\sim n^2 m$. Similarly, we estimate that $D_4^{(2)} \sim nm^2$ and $D_4^{(3)} \sim Nnm \sim n^2 m$. Thus, the leading order comes from $D_4^{(1)}$ and $D_4^{(3)}$, while the first $\mathcal{O}(\frac{m}{n})$ correction to the leading term comes from $D_4^{(2)}$. In subsection 5.2.1 we will review the results for the action of $D_4^{(1)}$ and $D_4^{(3)}$. In the process we will introduce the basis of operators, the

Gauss graph operators, that diagonalizes the action of the leading dilatation operator. Following this, we study $D_4^{(2)}$ in subsection 5.2.2. To transform this action to the Gauss graph basis requires that we develop new techniques to evaluate certain traces that appear. These techniques are developed in Appendix B and the action of $D_4^{(2)}$ in the Gauss graph basis is discussed in subsection 5.2.2.

### 5.2.1 Leading contribution

Acting on a restricted Schur polynomial, the action of $D_4^{(1)}$ is

$$D_4^{(1)}\chi_{R,(r,s)\alpha\beta} = g^2 \sum_{T,(t,u)\mu\nu} (L^a_{T,(t,u)\mu\nu;R,(r,s)\gamma\delta} + L^b_{T,(t,u)\mu\nu;R,(r,s)\gamma\delta})\chi_{T,(t,u)\gamma\delta}, \tag{5.6}$$

where

$$\begin{aligned} L^a_{T,(t,u)\mu\nu;R,(r,s)\gamma\delta} = &- \sum_{R''T''} \frac{d_T n(n-1)m}{d_t d_u d_{R''}(n+m)(n+m-1)} c_{R,R'}c_{R',R''} \\ &\times \Big[ \mathrm{Tr}\Big( I_{T''R''}(2,m+2,m+1)P_{R,(r,s)\alpha\beta}[(1,m+2),(1,m+1)] \\ &\times (2,m+2)I_{R''T''}(2,m+2)P_{T,(t,u)\delta\gamma}(2,m+2)[(m+1,2,1) \\ &- (2,m+1,1)]\Big) + \mathrm{Tr}\Big( I_{T''R''}(2,m+2)[(1,m+1)-(m+2,m+1)] \\ &\times P_{R,(r,s)\alpha\beta}(1,m+2,2)I_{R''T''}\Big((1,m+1)(2,m+2)P_{T,(t,u)\delta\gamma} \\ &\times (2,m+2)(2,m+1) - (m+1,2,m+2)P_{t,u}\delta\gamma(2,m+2)(2,1)\Big)\Big)\Big], \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} L^b_{T,(t,u)\mu\nu;R,(r,s)\gamma\delta} = &\sum_{R'T'} \frac{d_T n(n-1)m}{d_t d_u d_{R'}(n+m)} c_{R,R'} \Big[ \mathrm{Tr}\Big( I_{T'R'}[(1,m+2,m+1)P_{R,(r,s)\alpha\beta} \\ &- (m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+1)]I_{R'T'}[(1,m+1)P_{T,(t,u)\delta\gamma} \\ &- P_{T,(t,u)\delta\gamma}(1,m+1)]\Big) + \mathrm{Tr}\Big( I_{T'R'}[(1,m+1,m+2)P_{R,(r,s)\alpha\beta} \\ &- (m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+2)]I_{R'T'}[(1,m+1)P_{T,(t,u)\delta\gamma} \\ &- P_{T,(t,u)\delta\gamma}(1,m+1)]\Big)\Big]. \end{aligned} \tag{5.8}$$

In the above expression the traces run over the direct sum of the carrier spaces $R \oplus T$. The Young diagrams $R$ and $T$ both label irreducible representations of $S_{n+m}$. Primes denote Young diagrams obtained by dropping boxes, with one box dropped for each prime. Thus, for example, $T''$ is an irreducible representation of $S_{n+m-2}$, obtained by dropping two boxes from

$T$. The factors $I_{T'R'}$ and $I_{T''R''}$ are intertwining maps mapping from the carrier space $T'$ to $R'$ and from $T''$ to $R''$ respectively. $c_{RR'}$ is the factor of the box that must be dropped from $R$ to get $R'$. We use a little letter to denote dimensions of irreducible representations of the symmetric group so that, for example, $d_R$ is the dimension of the symmetric group representation labelled by Young diagram $R$. Finally, $P_{R,(r,s)\alpha\beta}$ denotes the intertwining maps which correctly restrict the trace in $R$ to the subspace relevant for the restricted character, that is $\chi_{R,(r,s)\alpha\beta}(\sigma) = \mathrm{Tr}(P_{R,(r,s)\alpha\beta}\Gamma^R(\sigma))$.

The above result is exact in the sense that all orders in $\frac{1}{N}$ are included. The traces appearing in the above expression run over the direct sum of carrier spaces $R \oplus T$. To exploit the simplifications of the large $N$ limit, we now employ the distant corners approximation. In this approximation, the traces over $R \oplus T$ are reduced to a trace over the tensor product of the direct sum of the carrier spaces $r \oplus t$ and $V_p^{\otimes m}$ where $R$ has a total of $p$ rows and $V_p$ is a $p$ dimensional vector space. The trace over $r \oplus t$ is rather straightforward. The bulk of the work then entails tracing over $V_p^{\otimes m}$. From now on we will work with normalized restricted Schur polynomials $\mathcal{O}_{R,(r,s)\alpha\beta}$ which are a scaled version of the $\chi_{R,(r,s)\alpha\beta}$

$$\mathcal{O}_{R,(r,s)\alpha\beta}(Z,Y) = \sqrt{\frac{\mathrm{hooks}_r \mathrm{hooks}_s}{f_R \mathrm{hooks}_R}}\chi_{R,(r,s)\alpha\beta}(Z,Y). \tag{5.9}$$

To denote the length of a given row of a Young diagram, we will indicate the Young diagram label with a subscript which identifies the row. Thus $r_1$ is the length of the first row of $r$ and $T_2$ is the length of the second row in $T$. After tracing over $r \oplus t$, we have

$$\begin{aligned} D^{(1)_4}\mathcal{O}_{R,(r,s)\alpha\beta}(Z,Y) = -2g^2 \sum_{i<j} \frac{m}{\sqrt{d_u d_s}} \Big( & \mathrm{Tr}_{V_p^{\otimes m}}(E_{ii}^{(1)} P_{R,(r,s)\alpha\beta} E_{jj}^{(1)} P_{T,(t,u)\delta\gamma}) \\ & + \mathrm{Tr}_{V_p^{\otimes m}}(E_{jj}^{(1)} P_{R,(r,s)\alpha\beta} E_{ii}^{(1)} P_{T,(t,u)\delta\gamma})\Big)\Delta_{ij}\mathcal{O}_{T,(t,u)\gamma\delta}(Z,Y), \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} &\Delta_{ij}\mathcal{O}_{T,(t,u)\gamma\delta}(Z,Y) \\ &\equiv \Big(\sqrt{(N+R_2)(N+R_2-1)(N+R_1-1)(N+R_1-2)}\delta_{T''_{jj},R''_{ii}}\delta_{t''_{jj},r''_{ii}} \\ &\quad + \sqrt{(N+R_2-2)(N+R_2-3)(N+R_1)(N+R_1+1)}\delta_{T''_{jj},R''_{ii}}\delta_{t''_{jj},r''_{ii}} \\ &\quad - (2N-3)\Big(\sqrt{(N+R_1-1)(N+R_2-1)}\delta_{T'_j,R'_i}\delta_{t'_j,r'_i} \\ &\quad + \sqrt{(N+R_2-2)(N+R_1)}\delta_{T'_i,R'_j}\delta_{t'_i,r'_j}\Big) \\ &\quad + [2(N+R_1-1)(N+R_2-2)-(n-1)(2N+n-3)]\delta_{T,R}\delta_{t,r}\Big)\mathcal{O}_{T,(t,u)\gamma\delta}(Z,Y). \end{aligned} \tag{5.11}$$

and of course $n = r_1 + r_2$. The delta functions which appear are 1 if the Young diagram labels have the same same shape and 0 otherwise. In the above formula, the matrices $E_{ij}^{(A)}$ which appear are a basis for the representation of $U(p)$ on $V_p^{\otimes m}$. Concretely, $E_{ij}$ is a matrix with every entry equal to zero except for the entry in the $i^{\text{th}}$ row and $j^{\text{th}}$ column, which is equal to 1. In terms of $E_{ij}$ we can write

$$E_{ij}^{(A)} = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes E_{ij} \otimes \cdots \otimes \mathbb{1}, \tag{5.12}$$

where $E_{ij}$ appears in the $A^{\text{th}}$ factor of the tensor product and $\mathbb{1}$ is the $p$ dimensional unit matrix. To obtain the result (5.10) we have used

$$\sqrt{\frac{\text{hooks}_T \text{hooks}_r}{\text{hooks}_R \text{hooks}_t}} = 1\Big(1 + \mathcal{O}\Big(\frac{m}{n}\Big)\Big), \tag{5.13}$$

and we have only kept the leading order. The action of $D_4^{(1)}$ is a product of two factors: an action on Young diagrams $R$, $r$ and an independent action on Young diagram $s$. To evaluate this second action explicitly, we need to trace over $V_p^{\otimes m}$. This is most easily achieved by moving to the basis of Gauss graph operators. Each Gauss graph operator is labelled by an element of the double coset $H \setminus S_m / H$ where $H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p}$. The relation between the Gauss graph operator ($\mathcal{O}_{R,r}(\sigma)$) and the normalized restricted Schur polynomial is

$$\mathcal{O}_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s \vdash m} \sum_{\mu_1, \mu_2} \sqrt{d_s} \Gamma_{jk}^{(s)}(\sigma) B_{j\mu_1}^{s \to \mathbb{1}_H} B_{k\mu_2}^{s \to \mathbb{1}_H} \mathcal{O}_{R,(r,s)\mu_1\mu_2}. \tag{5.14}$$

This transformation can be understood as a Fourier transform applied to the double coset. The branching coefficients $B_{j\mu_1}^{s \to \mathbb{1}_H}$ give a resolution of the projector from the irreducible representation $s$ of $S_m$ to the trivial representation of $H$

$$\frac{1}{|H|} \sum_{\sigma \in H} \Gamma_{ik}^{(s)}(\sigma) = \sum_{\mu} B_{i\mu}^{s \to \mathbb{1}_H} B_{k\mu}^{s \to \mathbb{1}_H}. \tag{5.15}$$

In terms of the Gauss graph operator, we find

$$D^{(1)_4} \mathcal{O}_{R,r}(\sigma) = -2g^2 \sum_{i<j} n_{ij}(\sigma) \Delta_{ij} \mathcal{O}_{R,r}(\sigma). \tag{5.16}$$

The numbers $n_{ij}(\sigma)$ can be read off of the element of the double coset element $\sigma$. For more details see Appendix A.

After using (5.5), the action of $D_4^{(3)}$ reduces to the action of the one loop dilatation operator. Consequently, we will not discuss this term further.

### 5.2.2 Subleading contribution

There are a number of different sources for the subleading contribution. Firstly, the leading two loop terms computed above receive corrections–see equation (5.3). These corrections do not lead to additional mixing. They only imply a correction to the anomalous dimension. Similarly, even the one loop anomalous dimension receives an $\frac{m}{n}$ correction (also from using the approximation given in (5.3)), without any additional operator mixing. The first correction that implies new operator mixing comes from the leading contribution to $D_4^{(2)}$ and it will be the focus of this subsection. This term has not been considered before.

Acting on a normalized restricted Schur polynomial, we find

$$\begin{aligned} D_4^{(2)} \mathcal{O}_{R,(r,s)\alpha\beta}(Z,Y) = \sum_{T,(t,u)\mu\nu} & \Big( M^{(a)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} - M^{(b)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} \\ & - M^{(c)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} + M^{(d)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} \Big) \mathcal{O}_{T,(t,u)\mu\nu}(Z,Y), \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} M^{(a)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} = \sum_{R',R''} & \frac{d_T nm(m-1) c_{RR'} c_{R'R''}}{d_t d_u (n+m)(n+m-1) d_{R''}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}} \\ & \times \operatorname{Tr} \Big( I_{T''R''} \Gamma_R((2,m+1)) [\Gamma_R((1,m+1)), P_{R,(r,s)\alpha\beta}] \\ & \times I_{R''T''} \Big( \Gamma_T((1,m+1)) [\Gamma_T((2,m+1)), P_{T,(t,u)\mu\nu}] \Big) \Big) \end{aligned} \tag{5.18}$$

$$\begin{aligned} M^{(b)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} = \sum_{R'} & \frac{d_T nm(m-1) c_{RR'}}{d_t d_u (n+m) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}} \\ & \times \operatorname{Tr} \Big( \Gamma_R((2,m+1)) [\Gamma_R((1,m+1)), P_{R,(r,s)\alpha\beta}] I_{R'T'} \\ & \times [\Gamma_T((1,m+1)), P_{T,(t,u)\mu\nu}] I_{T'R'} \Big) \end{aligned} \tag{5.19}$$

$$\begin{aligned} M^{(c)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} = \sum_{R'} & \frac{d_T nm(m-1) c_{RR'}}{d_t d_u (n+m) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}} \\ & \times \operatorname{Tr} \Big( [\Gamma_R((1,m+1)), P_{R,(r,s)\alpha\beta}] \Gamma_R((2,m+1)) I_{R'T'} \\ & \times [\Gamma_T((1,m+1)), P_{T,(t,u)\mu\nu}] I_{T'R'} \Big) \end{aligned} \tag{5.20}$$

and

$$M^{(d)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} = \sum_{R',R''} \frac{d_T nm(m-1)c_{RR'}c_{R'R''}}{d_t d_u (n+m)(n+m-1)d_{R''}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}}$$
$$\times \operatorname{Tr}\Big(I_{T''R''}[\Gamma_R((1,m+1)), P_{R,(r,s)\alpha\beta}]\Gamma_R((2,m+1))$$
$$\times I_{R''T''}\Big([\Gamma_T((1,m+1)), P_{T,(t,u)\mu\nu}]\Gamma_T((2,m+1))\Big)\Big). \tag{5.21}$$

The traces appearing above again run over the direct sum of carrier spaces $R \oplus T$ and the action given above is again correct to all orders in $\frac{1}{N}$. To take advantage of the simplifications of the large $N$ limit, we again employ the distant corners approximation, which again leads to an expression that has a single trace over $V_p^{\otimes m}$ remaining

$$M^{(a)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta}$$
$$= \sum_{R',R''} \delta_{R''T''} \frac{d_T nm(m-1)c_{RR'}c_{R'R''}}{d_t d_u (n+m)(n+m-1)d_{R''}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}} d_{r'_i} \delta_{r'_i t'_l}$$
$$\times [\operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{kj} E^{(2)}_{ll} p_{s\alpha\beta} E^{(1)}_{ii} E^{(2)}_{jk} p_{u\mu\nu}) - \operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{kj} p_{s\alpha\beta} E^{(1)}_{ii} E^{(2)}_{jk} p_{u\mu\nu})\delta_{kl}$$
$$- \operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ki} E^{(2)}_{ll} p_{s\alpha\beta} E^{(2)}_{ik} p_{u\mu\nu})\delta_{ij} + \operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ki} p_{s\alpha\beta} E^{(2)}_{ik} p_{u\mu\nu})\delta_{ij}\delta_{kl}] \tag{5.22}$$

$$M^{(b)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta}$$
$$= \sum_{R'} \delta_{R'T'} \frac{d_T nm(m-1)c_{RR'}}{d_t d_u (n+m)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}} d_{r'_i} \delta_{r'_i t'_k}$$
$$\times [\operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ka} E^{(2)}_{ak} p_{s\alpha\beta} E^{(1)}_{ii} p_{u\mu\nu}) - \operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ba} E^{(2)}_{ab} p_{s\alpha\beta} E^{(1)}_{ii} p_{u\mu\nu})\delta_{ik}$$
$$- \operatorname{Tr}_{V_p^{\otimes m}}(p_{s\alpha\beta} E^{(1)}_{ki} E^{(2)}_{ik} p_{u\mu\nu}) + \operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{bi} E^{(2)}_{ib} p_{s\alpha\beta} E^{(1)}_{kk} p_{u\mu\nu})] \tag{5.23}$$

$$M^{(c)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta}$$
$$= \sum_{R'} \delta_{R'T'} \frac{d_T nm(m-1)c_{RR'}}{d_t d_u (n+m)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}} d_{r'_i} \delta_{r'_i t'_k}$$
$$\times [\operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{kk} p_{s\alpha\beta} E^{(1)}_{ic} E^{(2)}_{ci} p_{u\mu\nu}) - \operatorname{Tr}_{V_p^{\otimes m}}(p_{s\alpha\beta} E^{(1)}_{ik} E^{(2)}_{ki} p_{u\mu\nu})$$
$$- \operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ii} p_{s\alpha\beta} E^{(1)}_{ab} E^{(2)}_{ba} p_{u\mu\nu})\delta_{ik} + \operatorname{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ii} p_{s\alpha\beta} E^{(1)}_{ck} E^{(2)}_{kc} p_{u\mu\nu})] \tag{5.24}$$

and

$$
\begin{aligned}
&M^{(d)}_{T,(t,u)\mu\nu;R,(r,s)\alpha\beta} \\
&= \sum_{R',R''} \delta_{R''T''} \frac{d_T nm(m-1)c_{RR'}c_{R'R''}}{d_t d_u (n+m)(n+m-1)d_{R''}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_T \text{hooks}_R \text{hooks}_t \text{hooks}_u}} d_{r'_i} \delta_{r'_i t'_l} \\
&\times [\mathrm{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ka} E^{(2)}_{ai} p_{s\alpha\beta} E^{(1)}_{ib} E^{(2)}_{bk} p_{u\mu\nu}) \delta_{ij}\delta_{kl} - \mathrm{Tr}_{V_p^{\otimes m}}(E^{(1)}_{la} E^{(2)}_{ai} p_{s\alpha\beta} E^{(1)}_{ik} E^{(2)}_{kl} p_{u\mu\nu}) \delta_{ij} \\
&- \mathrm{Tr}_{V_p^{\otimes m}}(E^{(1)}_{ki} E^{(2)}_{ij} p_{s\alpha\beta} E^{(1)}_{jb} E^{(2)}_{bk} p_{u\mu\nu}) \delta_{lk} + \mathrm{Tr}_{V_p^{\otimes m}}(E^{(1)}_{li} E^{(2)}_{ij} p_{s\alpha\beta} E^{(1)}_{jk} E^{(2)}_{kl} p_{u\mu\nu})].
\end{aligned} \tag{5.25}
$$

To compute the remaining trace over $V_p^{\otimes m}$ we will again move to the Gauss graph basis. This requires computing traces that have not been considered in previous works. Schematically, these traces are of the form

$$
\mathrm{Tr}(A p_{s\mu\nu} B p_{u\gamma\delta}), \tag{5.26}
$$

where $A$ and $B$ can be any product of $E^{(A)}_{ab}$s. The details of how to compute these traces in general are given in Appendix B. To summarize the key ideas, consider an interwining map $p_{s\mu\nu}$ built on the state $|\bar{v}_1, \vec{m}_1\rangle$ with symmetry group $H_1$ and the map $p_{t\gamma\delta}$ built on the state $|\bar{v}_2, \vec{m}_2\rangle$ with symmetry group $H_2$. The transformation to Gauss graph basis is given by

$$
\begin{aligned}
T &= \sum_{s\mu\nu lm} \sum_{u\gamma\delta np} \mathrm{Tr}(A p_{s\mu\nu} B p_{u\gamma\delta}) B^{s\to 1\!\!1_H}_{m\mu} B^{s\to 1\!\!1_H}_{l\nu} \Gamma^{(s)}_{ml}(\sigma_1) B^{u\to 1\!\!1_H}_{p\gamma} B^{u\to 1\!\!1_H}_{n\delta} \Gamma^{(u)}_{pn}(\sigma_2) \\
&= \frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \langle v_2| \sigma_2^{-1} (\psi_2^{-1} A \psi_2) \psi_1 |v_1\rangle \langle v_2| (\psi_2^{-1} B^T \psi_2) \psi_1 \sigma_1 |v_1\rangle .
\end{aligned} \tag{5.27}
$$

The final sum over $\psi_1$ and $\psi_2$ then gives

$$
T = \frac{(m-2)!}{|H_1||H_2|} |\mathcal{O}(\sigma_1)|^2 n_{ab}(\sigma_2) n_{cd}(\sigma_2) \delta_{AB}([\sigma_1, \sigma_2]). \tag{5.28}
$$

The indices $a$, $b$, $c$, $d$ are read from $A$ and $B$ as explained in Appendix B and

$$
\begin{aligned}
|\mathcal{O}(\sigma_1)|^2 &= \prod_{i=1}^{p} n_{ii}(\sigma_1)! \prod_{k,l=1,l\neq k}^{p} n_{kl}(\sigma_1)! \\
&= \left\langle \mathcal{O}_{R,r}(\sigma_1)^\dagger \right\rangle \mathcal{O}_{R,r}(\sigma_1).
\end{aligned} \tag{5.29}
$$

The delta function $\delta_{AB}([\sigma_1, \sigma_2])$ is defined using $A$ and $B$. This is also explained in Appendix B. Using these results, we find

$$
D_4^2 \mathcal{O}_{R,r}(\sigma_1) = \sum_{T,t,\sigma_2} (M^{1,\sigma_1,\sigma_2}_{T,t;R,r} - M^{2,\sigma_1,\sigma_2}_{T,t;R,r} - M^{3,\sigma_1,\sigma_2}_{T,t;R,r} + M^{4,\sigma_1,\sigma_2}_{T,t;R,r}) \mathcal{O}_{T,t}(\sigma_2), \tag{5.30}
$$

where

$$
\begin{aligned}
M^{1,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R',R''} \delta_{R''T''}\delta_{r'_i t'_l}\sqrt{\frac{C_{RR'}C_{R'R''}C_{TT'}C_{T'T''}}{R_j T_k}}|\mathcal{O}_{R,r}(\sigma_1)|^2\delta_{AB}([\sigma_1],[\sigma_2]) \\
&\times \Big[n_{ik}(\sigma_2)n_{kl}(\sigma_2) - \delta_{kl}\sum_{a=1}^{p} n_{ik}(\sigma_2)n_{ka}(\sigma_2) - \delta_{ij}\sum_{b=1}^{p} n_{bk}(\sigma_2)n_{kl}(\sigma_2) \\
&+ \delta_{ij}\delta_{kl}\sum_{a,b=1}^{p} n_{bk}(\sigma_2)n_{ka}(\sigma_2)\Big]. \qquad (5.31)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
M^{2,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R'} \delta_{R'T'}\delta_{r'_i t'_k}\sqrt{C_{RR'}C_{TT'}}|\mathcal{O}_{R,r}(\sigma_1)|^2\delta_{AB}([\sigma_1],[\sigma_2]) \\
&\times \Big[\sum_{a,b=1}^{p} n_{ik}(\sigma_2)n_{ba}(\sigma_2) - \delta_{ik}\sum_{a,b,c=1}^{p} n_{ib}(\sigma_2)n_{ca}(\sigma_2) - \sum_{a,b=1}^{p} n_{ak}(\sigma_2)n_{bi}(\sigma_2) \\
&+ \sum_{a,b=1}^{p} n_{kb}(\sigma_2)n_{ai}(\sigma_2)\Big] \qquad (5.32)
\end{aligned}
$$

$$
\begin{aligned}
M^{3,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R'} \delta_{R'T'}\delta_{r'_i t'_k}\sqrt{C_{RR'}C_{TT'}}|\mathcal{O}_{R,r}(\sigma_1)|^2\delta_{AB}([\sigma_1],[\sigma_2]) \\
&\times \Big[\sum_{a,b=1}^{p} n_{bk}(\sigma_2)n_{ia}(\sigma_2) - \sum_{a,b=1}^{p} n_{ka}(\sigma_2)n_{ib}(\sigma_2) - \delta_{ik}\sum_{a,b,c=1}^{p} n_{bi}(\sigma_2)n_{ac}(\sigma_2) \\
&+ \sum_{a,b=1}^{p} n_{ki}(\sigma_2)n_{ba}(\sigma_2)\Big] \qquad (5.33)
\end{aligned}
$$

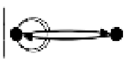
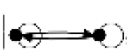
$$
\begin{aligned}
M^{4,\sigma_1,\sigma_2}_{T,t;R,r} &= \sum_{R',R''} \delta_{R''T''}\delta_{r'_i t'_l}\sqrt{\frac{C_{RR'}C_{R'R''}C_{TT'}C_{T'T''}}{R_j T_l}}|\mathcal{O}_{R,r}(\sigma_1)|^2\delta_{AB}([\sigma_1],[\sigma_2]) \\
&\times \Big[\delta_{ij}\delta_{kl}\sum_{a,b=1}^{p} n_{bl}(\sigma_2)n_{la}(\sigma_2) - \delta_{ij}\sum_{a=1}^{p} n_{kl}(\sigma_2)n_{la}(\sigma_2) - \delta_{kl}\sum_{b=1}^{p} n_{bl}(\sigma_2)n_{li}(\sigma_2) \\
&+ n_{kl}(\sigma_2)n_{li}(\sigma_2)\Big]. \qquad (5.34)
\end{aligned}
$$

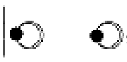
Notice that both $M^{1,\sigma_1,\sigma_2}_{T,t;R,r}$ and $M^{4,\sigma_1,\sigma_2}_{T,t;R,r}$ depend on the length of the rows of the Young diagrams $R$ and $T$ that participate. Since these lengths determine the angular momentum of the giant gravitons, they determine the radius to which the giant gravitons will expand. This is the first dependence of the anomalous dimensions on the geometry of the giant graviton. To illustrate the form of operator mixing captured in the above formulas, it is helpful to consider a specific example. This is the content of the next section.

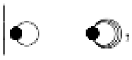
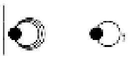
## 5.3 Example: A 2 giant graviton bound state with 4 strings attached

In this section we will consider the simplest nontrivial system that exhibits the general structure of the subleading operator mixing problem. This problem is a system of two giant gravitons with four strings stretched between them. There are $\frac{m}{n}$ corrections to the anomalous dimension from the one loop contribution as well as from $D_4^{(1)}$ and $D_4^{(3)}$ at two loops. These corrections do not induce extra operator mixing, so that the Gauss graph operators continue to have a good anomalous dimension. The only extra operator mixing comes from the subleading contribution $D_4^{(2)}$, and this is what we aim to explore.

Each element of the Gauss graph basis is labelled by two Young diagrams $R, r$ as well as a Gauss graph. We will number the states according to their Gauss graph labelling as shown below

$|1,R,r\rangle = \Big|$  $, R, r\Big\rangle,$ $|2,R,r\rangle = \Big|$  $, R, r\Big\rangle,$ $|3,R,r\rangle = \Big|$  $, R, r\Big\rangle,$

$|4,R,r\rangle = \Big|$  $, R, r\Big\rangle,$ $|5,R,r\rangle = \Big|$  $, R, r\Big\rangle,$ $|6,R,r\rangle = \Big|$  $, R, r\Big\rangle,$

$|7,R,r\rangle = \Big|$  $, R, r\Big\rangle,$ $|8,R,r\rangle = \Big|$  $, R, r\Big\rangle,$ $|9,R,r\rangle = \Big|$  $, R, r\Big\rangle.$

The states $|i\rangle$ for $i = 5, 6, 7, 8, 9$ are BPS at the leading order in $\frac{m}{n}$. The subleading corrections to the anomalous dimension coming from one loop, as well as from $D_4^{(1)}$ and $D_4^{(3)}$ at two loops is a multiplicative order $\frac{m}{n}$ correction and vanishes because the leading order anomalous dimension vanishes. Evaluating the subleading order contribution coming from $D_4^{(2)}$, we find

$$D_4^{(2)} |i, R, r\rangle = 0 \qquad i = 5, 6, 7, 8, 9. \tag{5.35}$$

so that the states that are BPS at the leading order do not receive a subleading correction. This is not peculiar to the example we consider and is to be expected generally, since for the BPS states we have $n_{ab}(\sigma_2) = 0$ for $a \neq b$.

The state $|4\rangle$ also does not mix with other states. However, for this state we have a nontrivial correction to the eigenvalue since

$$
\begin{aligned}
D_4^{(2)} |4, R, r\rangle = \sum_{T,t} 64 \Bigg[ & \delta_{RT} \Big( \frac{\delta_{r'_1 t'_1}}{R_2} + \frac{\delta_{r'_2 t'_2}}{R_1} \Big) \delta_{R''_{12} T''_{12}} (N + R_1 - 1)(N + R_2 - 2) \\
& + \delta_{RT} \frac{\delta_{r'_1 t'_1}}{R_1} \delta_{R''_{11} T''_{11}} (N + R_1 - 1)(N + R_1 - 2) + \delta_{RT} \frac{\delta_{r'_2 t'_2}}{R_2} \delta_{R''_{22} T''_{22}} (N + R_2 - 2)(N + R_2 - 3) \\
& - \delta_{RT} \delta_{r'_1 t'_1} \delta_{R'_1 T'_1} (N + R_1 - 1) - \delta_{RT} \delta_{r'_2 t'_2} \delta_{R'_2 T'_2} (N + R_2 - 2) \\
& - \delta_{R'_1 T'_2} \frac{\delta_{r'_1 t'_2}}{\sqrt{R_1 (R_1 - 1)}} \delta_{R''_{11} T''_{12}} (N + R_1 - 2) \sqrt{(N + R_1 - 1)(N + R_2 - 1)} \\
& - \delta_{R'_2 T'_1} \frac{\delta_{r'_2 t'_1}}{\sqrt{R_1 (R_1 + 1)}} \delta_{R''_{12} T''_{11}} (N + R_1 - 1) \sqrt{(N + R_1)(N + R_2 - 2)} \\
& - \delta_{R'_2 T'_1} \frac{\delta_{r'_2 t'_1}}{\sqrt{R_2 (R_2 - 1)}} \delta_{R''_{22} T''_{12}} (N + R_2 - 3) \sqrt{(N + R_1)(N + R_2 - 2)} \\
& - \delta_{R'_1 T'_2} \frac{\delta_{r'_1 t'_2}}{\sqrt{R_2 (R_2 + 1)}} \delta_{R''_{12} T''_{22}} (N + R_2 - 2) \sqrt{(N + R_1 - 1)(N + R_2 - 1)} \\
& + \delta_{R'_1 T'_2} \delta_{r'_1 t'_2} \sqrt{(N + R_1 - 1)(N + R_2 - 1)} + \delta_{R'_2 T'_1} \delta_{r'_2 t'_1} \sqrt{(N + R_1)(N + R_2 - 2)} \Bigg] |4, T, t\rangle \, .
\end{aligned}
\tag{5.36}
$$

The remaining states, $|i\rangle$ with $i = 1, 2, 3$ mix under the action of $D_4^{(2)}$. Using the matrix notation

$$
D_4^{(2)} |i, R, r\rangle = \sum_{T,t} (D_4^{(2)})_{ij} |j, T, t\rangle \qquad i, j = 1, 2, 3. \tag{5.37}
$$

the action of $D_4^{(2)}$ in the subspace is given by

$$
D_4^{(2)} = A \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} + B \begin{pmatrix} 0 & 1 & 0 \\ 1 & & 1 \\ 0 & 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + C^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \tag{5.38}
$$

where the coefficients $A$, $B$ and $C$ are quoted in Appendix B.6. The coefficients $A$, $B$ and $C$ are operators that have a nontrivial action of the $R$, $r$ labels of the Gauss graph operators. It is straightforward to check that the matrix coefficients of these operators do not commute and hence they are not simultaneously diagonalizable. This implies that the action of the dilatation operator no longer factorizes into commuting actions on the $Z$ and $Y$ fields. It is this failure of factorization that we were referring to when we talked about the general structure of the mixing problem.

## 5.4 Conclusions

Our results have a number of interesting features that deserve comment. In the $\frac{m}{n} = 0$ limit, the action of the dilatation operator factorizes into an action on the $Z$ fields and an action on the $Y$ fields. The subleading correction has spoiled this factorization of the dilatation operator. This is rather natural: in the limit $\frac{m}{n} = 0$ we consider a giant graviton built with an infinite number ($n = \infty$) of $Z$ fields, so that the backreaction of the magnons on the giant graviton can be neglected. Without backreaction, we expect the dynamics of the giant graviton is completely decoupled from the dynamics of the magnons and this is the root of the factorized action of the dilatation operator. By adding the first correction, we are saying that $n$ is large but not infinite. In this situation, although backreaction is small, it is not zero. The magnons will now provide a small perturbation to the dynamics of the giant gravitons; the action of the dilatation operator will no longer factorize into an action on the giant graviton (i.e. on the $Z$s) times an action on the magnons (i.e. on the $Y$s).

The subleading correction spoiled the factorization of the dilatation operator by introducing further operator mixing. Another interesting result of our analysis is that the subleading corrections did not induce extra mixing for the BPS operators. Indeed, after accounting for the complete $\frac{m}{n}$ correction to two loops, we found our BPS operators remain uncorrected and continue to have a vanishing anomalous dimension. Although our computation is performed in a specific example, we argued that we expect this conclusion to be general since for the BPS operators we have $n_{ab}(\sigma) = 0$ for $a \neq b$. Looking at the result (5.30), it is clear that vanishing $n_{ab}(\sigma)$ implies a vanishing action $D_4^{(2)}$.

The form of the action of the dilatation operator implies that when the correction to the anomalous dimension is nonzero it will depend on the length of the rows of the Young diagrams labelling the operator. Since these lengths determine the angular momentum of the giant gravitons, they determine the radius to which the giant gravitons will expand. This implies that the anomalous dimensions start to depend on the geometry of the giant graviton.

The dynamics of open strings on the worldvolume of a giant graviton is expected to give rise to a Yang-Mills theory at low energy. The lightest mode of the open strings attached to the giant graviton becomes the gauge boson of the theory. This suggests that within $\mathcal{N} = 4$ super Yang-Mills theory, we should see classes of operators whose dynamics is captured by a new emergent gauge theory. The word emergent is particularly apt in this case because this new Yang-Mills theory will be local on a space that is distinct from the space of the original spacetime of the $\mathcal{N} = 4$ super Yang-Mills theory. The gauge symmetry which determines the interactions of the theory is a local symmetry with respect to this new space and the time of the original spacetime. The space of the emergent Yang-Mills theory is the worldvolume of the giant graviton, which itself is built from the $Z$s. So this space and an associated local gauge invariance is to emerge from the dynamics of the $Z$ matrices in the large $N$ limit. For the operators dual to giant gravitons studied in this article, it is natural to think that the magnons themselves will become the gauge bosons. Indeed, the allowed state space of the magnons is parametrized by a double coset. The structure of this double coset is determined by the expected Gauss law of the emergent gauge theory. To really understand the mechanism behind this emergence it is important that we get a good handle on how the magnons interact. It is by studying these interactions that we may hope to recognize the Yang-Mills theory that must emerge. In this chapter we have computed the first of these interactions and we have developed tools that allow us to study these interactions in general.

# Chapter 6

# Anomalous dimensions from boson lattice models

## 6.1 Outline of chapter

Operators dual to strings attached to giant graviton branes in $\mathrm{AdS}_5\times\mathrm{S}^5$ can be described rather explicitly in the dual $\mathcal{N}=4$ super Yang-Mills theory. They have a bare dimension of order $N$ so that for these operators the large $N$ limit and the planar limit are distinct: summing only the planar diagrams will not capture the large $N$ dynamics. Focusing on the one-loop $SU(3)$ sector of the theory, we consider operators that are a small deformation of a $\frac{1}{2}$-BPS multi-giant graviton state. The diagonalization of the dilatation operator at one loop has been carried out, but explicit formulas for the operators of a good scaling dimension are only known when certain terms which were argued to be small, are neglected. In this article we include the terms which were neglected. The diagonalization is achieved by a novel mapping which replaces the problem of diagonalizing the dilatation operator with a system of bosons hopping on a lattice. The giant gravitons define the sites of this lattice and the open strings stretching between distinct giant gravitons define the hopping terms of the Hamiltonian. Using the lattice boson model, we argue that the lowest energy giant graviton states are obtained by distributing the momenta carried by the $X$ and $Y$ fields evenly between the giant gravitons with the condition that any particular giant graviton carries only $X$ or $Y$ momenta, but not both.

## 6.2 Introduction

Motivated by the AdS/CFT correspondence[1, 5, 6], there has been dramatic progress in computing the planar spectrum of anomalous dimensions in $\mathcal{N}=4$ super Yang-Mills theory. The planar spectrum is now known, in principle, to all orders in the 't Hooft coupling [65]. This has been possible thanks to the discovery of integrability[7, 32] in the planar limit of the theory. This spectrum of anomalous dimensions reproduces classical string energies on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime, in the dual string theory[66].

Much less is known about $\mathcal{N} = 4$ super Yang-Mills theory outside the planar limit. There are many distinct large N but non-planar limits of the theory that could be considered and these correspond to a variety of fascinating physical problems. For example, the problem of considering new spacetime geometries (including black hole solutions) corresponds to considering operators with a bare dimension of order $N^2$[53], while giant graviton branes[2, 3, 4] are dual to operators with a bare dimension of order $N$. The planar limit does not correctly capture the dynamics of these operators[8, 21].

Although much less is known about these large $N$ but non-planar limits, some progress has been made. Approaches based on group representation theory provide a powerful tool, essentially because they allow us to map the problem of the dynamics of the non-planar limit - summing the ribbon graphs contributing to correlation functions - into a purely algebraic problem in group theory. Typically, it can be phrased as the construction of a collection of projection operators and their properties. Once the algebraic problem is properly formulated, systematic approaches to it can be developed. As an example of this approach, bases of local gauge invariant operators have been given[9, 11, 12, 13, 15, 14, 16, 18]. These bases provide a good starting point from which the anomalous dimensions can be studied. This is basically because they diagonalize the free field two point function and, at weak coupling, operator mixing is highly constrained[25, 26, 67, 68, 37]. The resulting operators have a complicated multi-trace structure, quite different to the single trace structure relevant for the planar limit and its mapping to an integrable spin chain. The spectrum of anomalous dimensions has been computed for operators that are small deformations of $\frac{1}{2}$-BPS operators. Problems with 2 distinct characters have been solved: It is possible to simply treat all fields in the operator on the same footing, construct the basis and then diagonalize [27, 28, 29, 30] or alternatively, one can build operators that realize a spacetime geometry or a giant graviton brane and use words constructed from the fields of the CFT to describe string excitations[25, 52, 69]. In the approach that treats all fields on the same footing, one simply defines the operators of the basis and considers the diagonalization of the dilatation operator with no physical input from the dual gravity description. When considering states dual to systems of giant gravitons, the Gauss Law of the dual giant graviton world volume gauge theory emerges, so that in this approach we see open string and membranes are present in the CFT Hilbert space. When using words to describe string excitations, computations in the CFT reproduce the classical values of energies computed in string theory[52, 69], the worldsheet S-matrix[58] and has lead to the discovery of integrable subsectors for closed string excitations of certain LLM backgrounds[69]. Clearly, this is a rich problem with hidden simplicity, so that further study of these limits are bound to be fruitful. The existence of this hidden simplicity is not unexpected: conventional lore of the large $N$ limit identifies $\frac{1}{N}$ as the gravitational interaction, so that the $N \to \infty$ limit, in which this interaction is turned off, should be a simple limit.

One next step that can be contemplated, is to go beyond small perturbations of the $\frac{1}{2}$-BPS sector. This problem is our main motivation in this study, and we will take a small step in this direction. We will study operators constructed from three complex adjoint scalars $X$, $Y$, $Z$ of $\mathcal{N}=4$ super Yang-Mills theory. Operators that are a small perturbation of a $\frac{1}{2}$-BPS operator are constructed using mainly $Z$ fields. For these operators, interactions between the $X, Y$ fields are subdominant to interactions between $X, Z$ and between $Y, Z$ fields and can hence be neglected. As we move further from the original $\frac{1}{2}$-BPS operator, more and more $X, Y$ fields are added. At some point the interactions between the $X, Y$ fields can no longer be neglected. Dealing with these interactions is the focus of our study. We will argue that this is a well defined problem, that can be solved, often explicitly. This is accomplished by phrasing the new $X, Y$ interactions as a lattice model, for essentially free bosons. Thus, we finally land up with a simple problem that is familiar and can be solved. This is the basic achievement of this chapter.

Our results show a fascinating structure that deserves to be discussed. The mapping to the lattice model associates a harmonic oscillator to both the $X$ field and to the $Y$ field. Earlier results [29] treating the leading term, performed the diagonalization by associating a harmonic oscillator to the $Z$ field, so that in the end we seem to be seeing an equality in the description of the three scalar fields. An even-handed treatment of all three fields is a big step towards being able to treat operators constructed with equal numbers of $X$, $Y$ and $Z$ fields. This would most certainly go beyond the $\frac{1}{2}$-BPS sector, the main motivation for our study.

In the next section we review the action of the one loop dilatation operator $D_2$. The action of $D_2$ in the $SU(3)$ sector, in the Schur polynomial basis, has been evaluated previously[70] and we simply quote and use the result. We then move to the Gauss graph basis of [30], in which the terms in $D_2$ arising from $Z, Y$ or $Z, X$ interactions are diagonal. Again, this is a known result and we simply use it. The Gauss graph basis has a natural interpretation in terms of giant graviton branes and their open string excitations. We will often use this language of branes and strings. We then come to the central term of interest: the term in $D_2$ arising from $X, Y$ interactions. Denote this term by $D_2^{XY}$. We will carefully evaluate this term, arriving at a rather simple formula, which is the starting point for section 6.4. The explicit expression for $D_2^{XY}$ can easily be identified with a lattice model for a collection of bosons. The giant gravitons define the sites of this lattice, and the open string excitations determine the lattice Hamiltonian. Section 6.5 diagonalizes the dilatation operator for a number of giant gravitons plus open string configurations, arriving at detailed and explicit expressions both for the anomalous dimensions and for the operators of a definite scaling dimension. Our conclusions and some discussion are given in section 6.6.

## 6.3 Action of the One Loop Dilatation Operator

We combine the 6 hermitian adjoint scalars of $\mathcal{N} = 4$ super Yang-Mills theory into three complex combinations, denoted $X, Y, Z$. The operators we consider are constructed using $n$ $Z$s, $m$ $Y$s and $p$ $X$s. Operators that are dual to giant graviton branes are constructed using $n + m + p \sim N$ fields. We will focus on operators that are small deformations of $\frac{1}{2}$-BPS operators, achieved by choosing $n \gg m+p$. We will fix $\frac{m}{p} \sim 1$ as $N \to \infty$ and treat $\frac{m}{n}$ as a small parameter. The collection of operators constructed using $X, Y, Z$ fields are often referred to as the $SU(3)$ sector. This is not strictly speaking correct since these operators do mix with operators containing fermions. At one loop however, this is a closed sector.

Our starting point is the action of the one loop dilatation operator of the $SU(3)$ sector

$$D_2 = D_2^{YZ} + D_2^{XZ} + D_2^{XY}, \tag{6.1}$$

where

$$D_2^{AB} \equiv g_{YM}^2 \mathrm{Tr}\left([A, B][\partial_A, \partial_B]\right), \tag{6.2}$$

on the restricted Schur polynomial basis. This has been evaluated in [70]. Further, the terms $D_2^{YZ}$ and $D_2^{XZ}$ have been diagonalized. The operators of a definite scaling dimension $O_{R,r}(\sigma)$, called Gauss graph operators[28, 30], are labeled by a pair of Young diagrams $R \vdash n+m+p$ and $r \vdash n$ as well as a permutation $\sigma \in S_m \times S_p$. Although these labels arise when diagonalizing $D_2^{YZ}$ and $D_2^{XZ}$ in the CFT, they have a natural interpretation in the dual gravitational description in terms of giant graviton branes plus open string excitations. A Young diagram $R$ that has $q$ rows corresponds to a system of $q$ giant gravitons. The $Y$ and $X$ fields describe the open string excitations of these giant gravitons, so that there are $m + p$ open strings in total. We can describe the state of the system using a graph, with nodes of the graph representing the branes (and hence rows of $R$) and directed edges of the graph describing the open string excitations (represented by $X$ and $Y$ fields in the CFT). Each directed edge ends on any two (not necessarily distinct) of the $q$ branes. The only configurations that appear when $D_2^{YZ}$ and $D_2^{XZ}$ are diagonalized have the same number of strings starting or terminating on any given giant graviton, for the $X$ and $Y$ strings separately[30, 70]. Thus the Gauss Law of the brane world volume theory implied by the fact that the giant graviton has a compact world volume[22] emerges rather naturally in the CFT description. Since every terminating edge endpoint can be associated to a unique emanating endpoint, we can give a nice description of how the open strings are connected to the giant gravitons by specifying how the terminating and emanating endpoints are associated. The permutation $\sigma \in S_m \times S_p$ describes how the $m$ $Y$'s and the $p$ $X$'s are draped between the $q$ giant gravitons by describing this association[30, 70]. The explicit form of the Gauss graph operators is[30, 70]

$$\begin{aligned} O^{\vec{m},\vec{p}}_{R,r}(\sigma) = \frac{|H_X \times H_Y|}{\sqrt{p!m!}} \sum_{j,k} \sum_{s\vdash m} \sum_{t\vdash p} \sum_{\vec{\mu}_1,\vec{\mu}_2} \sqrt{d_s d_t} \Gamma^{(s,t)}_{jk}(\sigma) \\ \times B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}_1} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\mu}_2} O_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}. \end{aligned} \tag{6.3}$$

Each box in $R$ is associated with one of the complex fields. $r$ is a label for the $Z$ fields. The graph $\sigma$ encodes important information. The number of $Y$ (or $X$) strings terminating on the $i$th node which equals the number of $Y$ (or $X$) strings emanating from the $i$th node is denoted by $m_i$ (or $p_i$). $m_i$ (or $p_i$) also counts the number of boxes in the $i$th row of $R$ that correspond to $Y$ (or $X$) fields. We will often assemble $m_i$ and $p_i$ into the vectors $\vec{m}$ and $\vec{p}$. The number of $Y$ (or $X$) strings stretching between nodes $i$ and $k$ is denoted $m_{ik}$ (or $p_{ik}$), while the number of strings stretching from node $i$ to node $k$ are denoted $m_{i\to k}$ (or $p_{i\to k}$). A Young diagram with $k$ boxes $a \vdash k$ labels an irreducible representation of $S_k$ with dimension $d_a$. The branching coefficients $B^{(s,t)\to 1_{H_X\times H_Y}}_{j\vec{\mu}_1}$ resolve the operator that projects from $(s,t)$, with $s \vdash m$, $t \vdash p$, an irreducible representation of $S_m \times S_p$, to the trivial (identity) representation of the product group $H_Y \times H_X$ with $H_Y = S_{m_1} \times S_{m_2} \times \cdots S_{m_q}$ and $H_X = S_{p_1} \times S_{p_2} \times \cdots S_{p_q}$ , i.e.

$$\frac{1}{H_X \times H_Y} \sum_{\gamma \in H_X \times H_Y} \Gamma^{(s,t)}_{ik}(\gamma) = \sum_{\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{i\vec{\mu}} B^{(s,t)\to 1_{H_X\times H_Y}}_{k\vec{\mu}}. \tag{6.4}$$

$\Gamma^{(s,t)}_{jk}(\sigma)$ is a matrix (with row and column indices $jk$) representing $\sigma \in S_m \times S_p$ in irreducible representation $(s,t)$. The operators $O_{R,(r,s,t)\vec{\mu}_1\vec{\mu}_2}$ are normalized versions of the restricted Schur polynomials [15]

$$\chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}(Z,Y,X) = \frac{1}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}(\sigma) \mathrm{Tr}\,(\sigma Z^{\otimes n} Y^{\otimes m} X^{\otimes p}), \tag{6.5}$$

which themselves provide a basis for the gauge invariant operators of the theory. The restricted characters $\chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}(\sigma)$ are defined by tracing the matrix representing group element $\sigma$ in representation $R$ over the subspace giving an irreducible representation $(r,s,t)$ of the $S_n \times S_m \times S_p$ subgroup.

There is more than one choice for this subspace and the multiplicity labels $\vec{\mu}_1\vec{\mu}_2$ resolve this ambiguity, for the row and column index of the trace. The operators $O_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}$ given by

$$O_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2} = \sqrt{\frac{\text{hooks}_r\text{hooks}_s\text{hooks}_t}{\text{hooks}_R f_R}}\chi_{R,(t,s,r)\vec{\mu}_1\vec{\mu}_2}, \tag{6.6}$$

have unit two point function. $\text{hooks}_r$ stands for the product of hook lengths of Young diagram $r$ and $f_R$ stands for the product of the factors of Young diagram $R$. The action of the dilatation operator on the Gauss graph operators is [28, 30, 70]

$$\begin{aligned} D_2^{YZ}O_{R,r}^{\vec{m},\vec{p}}(\sigma) &= -g_{YM}^2\sum_{i<j} m_{ij}(\sigma)\Delta_{ij}O_{R,r}^{\vec{m},\vec{p}}(\sigma) \\ D_2^{XZ}O_{R,r}^{\vec{m},\vec{p}}(\sigma) &= -g_{YM}^2\sum_{i<j} p_{ij}(\sigma)\Delta_{ij}O_{R,r}^{\vec{m},\vec{p}}(\sigma), \end{aligned} \tag{6.7}$$

where $\Delta_{ij} = \Delta_{ij}^- + \Delta_{ij}^0 + \Delta_{ij}^+$[29]. We will now spell out the action of the operators $\Delta_{ij}^+$, $\Delta_{ij}^0$ and $\Delta_{ij}^-$. Denote the row lengths of $r$ by $l_{r_i}$. The Young diagram $r_{ij}^+$ is obtained by deleting a box from row $j$ and adding it to row $i$. The Young diagram $r_{ij}^-$ is obtained by deleting a box from row $i$ and adding it to row $j$. In terms of these Young diagrams we have

$$\Delta_{ij}^0 O_{R,r}^{\vec{m},\vec{p}}(\sigma) = -(2N + l_{r_i} + l_{r_j})O_{R,r}^{\vec{m},\vec{p}}(\sigma) \tag{6.8}$$

$$\Delta_{ij}^+ O_{R,r}^{\vec{m},\vec{p}}(\sigma) = \sqrt{(N + l_{r_i})(N + l_{r_j})}O_{R_{ij}^+,r_{ij}^+}^{\vec{m},\vec{p}}(\sigma) \tag{6.9}$$

$$\Delta_{ij}^- O_{R,r}^{\vec{m},\vec{p}}(\sigma) = \sqrt{(N + l_{r_i})(N + l_{r_j})}O_{R_{ij}^-,r_{ij}^-}^{\vec{m},\vec{p}}(\sigma). \tag{6.10}$$

Notice that $D_2^{YZ}$ and $D_2^{XZ}$ in (6.7) are not yet diagonal: they still mix operators with different $R, r$ labels. This last diagonalization however, is rather simple: it maps into diagonalizing a collection of decoupled oscillators as demonstrated in [29]. We will call these $Z$ oscillators, since they are associated to the $r$ label which organizes the $Z$ fields. It is clear that $D_2^{XY}$ does not act on the $r$ label so that in the end, the contribution from $D_2^{XY}$ simply shifts the ground state eigenvalue of the $Z$ oscillators.

We will now focus on the term $D_2^{XY}$. Recall that our operators are built with many more $Z$ fields, than $X$ or $Y$ fields ($n \gg p + m$). Since this term contains no derivatives with respect to $Z$ it is subleading (of order $\frac{m}{n}$) when compared to $D_2^{YZ}$ and $D_2^{XZ}$. Diagonalizing this operator is the main goal of this article, so it is useful to sketch the derivation of the matrix elements of $D_2^{XY}$ in the Gauss graph basis. We will simply quote existing results that we need, giving complete details only for the final stages of the evaluation, which are novel. The reader will find useful background material in [70]. The action of this term on the restricted Schur polynomial basis was computed in [70]. The result is

$$D_2^{XY} O_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \sum_{R'} \sum_{T,(y,x,w)\vec{\alpha}\vec{\beta}} \mathcal{C}$$
$$\mathrm{Tr}_{\,R\oplus T}\left(\left[P_1, \Gamma^R(1,p+1)\right] I_{R'T'} \left[P_2, \Gamma^T(1,p+1)\right] I_{T',R'}\right) O_{T,(y,x,w)\vec{\beta}\vec{\alpha}},$$

where

$$\begin{aligned} \mathcal{C} &= -g_{YM}^2 c_{RR'} \frac{d_T m p}{d_x d_y d_w (n+m+p) d_{R'}} \sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_w \mathrm{hooks}_x \mathrm{hooks}_y}} \\ P_1 &= P_{R,(t,s,r)\vec{\mu}\vec{\nu}} \qquad\qquad P_2 = P_{T,(y,x,w)\vec{\alpha}\vec{\beta}}. \end{aligned} \tag{6.11}$$

$\Gamma^S(\sigma)$ is the matrix representing $\sigma \in S_{n+m+p}$ in irreducible representation $S \vdash n+m+p$. Young diagram $R'$ is obtained from Young diagram $R$ by dropping a single box, with $c_{RR'}$ denoting the factor of this box. $I_{T'R'}$, $I_{R'T'}$, $P_1$ and $P_2$ are intertwining maps. $I_{T'R'}$ maps from the carrier space of $R'$ to the carrier space of $T'$. It is only non-vanishing if $T'$ and $R'$ are equal as Young diagrams implying that operators labelled by $R$ and $T$ can only mix if they differ by the placement of a single box. The operators $P_1$ and $P_2$ are the intertwining maps used in the construction of the restricted Schur polynomials. It is challenging to evaluate the above expression explicitly, basically because it is difficult to construct $P_1$ and $P_2$. However, the above expression has not yet employed the simplifications of large $N$. To do this, following [28] we will use the displaced corners approximation. After applying the approximation we obtain[70]

$$D_2^{XY} O_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \sum_{T,(w,v,u)\vec{\alpha}\vec{\beta}} \tilde{M}_{R,(t,s,r)\vec{\mu}\vec{\nu}\ T,(w,v,u)\vec{\alpha}\vec{\beta}} O_{T,(w,v,u)\vec{\alpha}\vec{\beta}}, \tag{6.12}$$

where

$$\begin{aligned} \tilde{M}_{R,(t,s,r)\vec{\mu}\vec{\nu}\ T,(w,v,u)\vec{\alpha}\vec{\beta}} = -g_{YM}^2 \sum_{R'} \delta_{R_i' T_k'} \delta_{ru} \frac{pm}{\sqrt{d_s d_t d_w d_v}} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}} \times \\ \mathrm{Tr}\Big[E_{ki}^{(1)} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E_{ik}^{(p+1)} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2} - E_{ci}^{(1)} E_{kc}^{(p+1)} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E_{ak}^{(1)} E_{ia}^{(p+1)} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2} \\ -E_{kc}^{(1)} E_{ci}^{(p+1)} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E_{ia}^{(1)} E_{ak}^{(p+1)} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2} + E_{ki}^{(p+1)} P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2} E_{ik}^{(1)} P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}\Big]. \end{aligned} \tag{6.13}$$

The trace in this expression is over the tensor product $V_p^{\otimes n+m}$ where $V_p$ is the fundamental representation of $U(p)$. The intertwining maps used to define the restricted Schur polynomials ($P_1$ and $P_2$ above) factor into an action on the boxes associated to the $Z$ fields, an action on the boxes associated to the $Y$ fields and an action on the boxes associated to the X fields. The intertwining maps[1] $P^{(\vec{p},\vec{m})}_{t\alpha_1\beta_1;s\alpha_2\beta_2}$ and $P^{(\vec{p}',\vec{m}')}_{w\mu_1\nu_1;v\mu_2\nu_2}$ are the actions

[1] A very explicit algorithm for the construction of these maps has been given in [28].

of the intertwining maps on the $X$ and $Y$ fields only. This happens because the trace over the Z field indices, which is simple as the dilatation operator $D_2^{XY}$ does not act on the Z fields, has been performed. Young diagram $R_i'$ is obtained from $R$ by dropping a single box from row $i$ and $T_k'$ from $T$ by dropping a single box from row $k$.

The result (6.13) gives the $D_2^{XY}$ term in the dilatation operator, as a matrix that must be diagonalized. As we will see, all three terms in $D_2$ are simultaneously diagonalizable at large $N$ so that it is convenient to employ the Gauss graph basis which already diagonalizes both $D_2^{ZY}$ and $D_2^{ZX}$. The problem of diagonalizing $D_2^{XY}$ then amounts to a diagonalization on degenerate subspaces of $D_2^{ZY}$ and $D_2^{ZX}$. Thus, the original diagonalization of an enormous matrix is replaced by diagonalizing a number of smaller matrices - a significant simplification. Applying the results of [70], we find that, after the change in basis

$$D_2^{XY}\hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma_1)=M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2}\hat{O}^{\vec{m},\vec{p}}_{T,t}(\sigma_2), \tag{6.14}$$

where

$$\begin{aligned}
&M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2}=-g_{YM}^2\frac{1}{\sqrt{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}}\times\\
&\sum_{R'}\frac{\delta_{R_i'T_k'}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}}\ \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}}\\
&\Big[\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ki}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle\\
&-\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2|\vec{p}',\vec{m}'\rangle\\
&-\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(1)}_{kc}E^{(p+1)}_{ci}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ia}E^{(p+1)}_{ak}\psi_2|\vec{p}',\vec{m}'\rangle\\
&+\langle\vec{p}',\vec{m}'|\sigma_2\psi_2^{-1}E^{(p+1)}_{ki}\psi_1|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ik}\psi_2|\vec{p}',\vec{m}'\rangle\Big].
\end{aligned}\tag{6.15}$$

Detailed calculation to arrive at this result can be found in Appendix C. Here the Gauss graph operators $\hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma_1)$ are normalized to have a unit two point function. They are related to the operators introduced in (6.3) as follows

$$O^{\vec{m},\vec{p}}_{R,r}(\sigma)=\sqrt{\prod_{i=1}^{q}m_{ii}(\sigma)!m_{ii}(\sigma)!\prod_{k,l,k\neq l}m_{k\to l}(\sigma)!p_{k\to l}(\sigma)!}\ \hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma). \tag{6.16}$$

Introduce the vectors $(v^{(i)})_a=\delta_{ia}$ which form a basis for $V_p$. The vector $|\vec{p},\vec{m}\rangle$ is defined as follows

$$|\vec{p},\vec{m}\rangle=|\vec{p}\rangle\otimes|\vec{m}\rangle, \tag{6.17}$$

where

$$\begin{aligned} |\vec{p}\rangle &= (v^{(1)})^{\otimes p_1} \otimes \cdots \otimes (v^{(q)})^{\otimes p_q} \\ |\vec{m}\rangle &= (v^{(1)})^{\otimes m_1} \otimes \cdots \otimes (v^{(q)})^{\otimes m_q}. \end{aligned} \tag{6.18}$$

We will now explain how the sums over $\psi_1$ and $\psi_2$ in (6.15) can be evaluated. This discussion is novel and is one of the new contributions of this chapter. Consider the term

$$T_1 = \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}', \vec{m}' | \sigma_2 \psi_2^{-1} E_{ki}^{(1)} \psi_1 | \vec{p}, \vec{m} \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ik}^{(p+1)} \psi_2 | \vec{p}', \vec{m}' \rangle.$$

The dependence on the permutations $\sigma_1, \sigma_2$ can be simplified with the following change of variables: replace $\psi_2$ with $\tilde{\psi}_2$ where

$$\tilde{\psi}_2 = \psi_2 \sigma_2^{-1} \qquad \Rightarrow \qquad \tilde{\psi}_2^{-1} = \sigma_2 \psi_2^{-1}. \tag{6.19}$$

After relabeling $\tilde{\psi}_2 \to \psi_2$ and taking the transpose of the first factor which is a real number, we find

$$T_1 = \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}, \vec{m} | \psi_1^{-1} E_{ik}^{(1)} \psi_2 | \vec{p}', \vec{m}' \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ik}^{(p+1)} \psi_2 \sigma_2 | \vec{p}', \vec{m}' \rangle.$$

If $i \neq k$, the matrix element $\langle \vec{p}, \vec{m} | \psi_1^{-1} E_{ik}^{(1)} \psi_2 | \vec{p}', \vec{m}' \rangle$ is only non-vanishing if $\vec{p} \neq \vec{p}'$ and $\vec{m} = \vec{m}'$, while the matrix element $\langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ik}^{(p+1)} \psi_2 \sigma_2 | \vec{p}', \vec{m}' \rangle$ is only non-vanishing if $\vec{p} = \vec{p}'$ and $\vec{m} \neq \vec{m}'$. Thus, $T_1$ vanishes for $i \neq k$. Indicate this explicitly as follows

$$T_1 = \delta_{ik} \sum_{\psi_1, \psi_2 \in S_{\vec{p}} \times S_{\vec{m}}} \langle \vec{p}, \vec{m} | \psi_1^{-1} E_{ii}^{(1)} \psi_2 | \vec{p}, \vec{m} \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ii}^{(p+1)} \psi_2 \sigma_2 | \vec{p}, \vec{m} \rangle.$$

To simplify this expression further, note that $E_{ii}^{(1)} |\vec{p}, \vec{m}\rangle$ is only non-zero if vector $v^{(i)}$ occupies slot one in the vector $|\vec{p}\rangle$. In this case $E_{ii}^{(1)} |\vec{p}, \vec{m}\rangle = |\vec{p}, \vec{m}\rangle$. Since $\psi_1$ and $\psi_2$ shuffle the vectors in $|\vec{p}, \vec{m}\rangle$ into all possible locations, $E_{ii}^{(1)}$ will in the end count how many times the vector $v^{(i)}$ appears in $|\vec{p}, \vec{m}\rangle$. This is given by $p_i$ introduced above. A similar argument applies to $E^{(p+1)} |\vec{p}, \vec{m}\rangle$, Thus, we obtain

$$\begin{aligned} T_1 &= \delta_{ik} \frac{p_i}{p} \frac{m_i}{m} \sum_{\psi_1, \psi_2 \in S_{\vec{p}} \times S_{\vec{m}}} \langle \vec{p}, \vec{m} | \psi_1^{-1} \psi_2 | \vec{p}, \vec{m} \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} \psi_2 \sigma_2 | \vec{p}, \vec{m} \rangle \\ &= \delta_{ik} \frac{p_i}{p} \frac{m_i}{m} \sum_{\psi_1, \psi_2 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{h_1, h_2 \in H_X \times H_Y} \delta(\psi_1^{-1} \psi_2 h_1) \delta(\sigma_1^{-1} \psi_1^{-1} \psi_2 \sigma_2 h_2). \end{aligned}$$

Now, perform the following change of summation variables $\psi_1 \to \tilde{\psi}_1$ with

$$\psi_1 = \psi_2 \tilde{\psi}_1. \tag{6.20}$$

The summand is now independent of $\psi_2$ so that after summing over $\psi_2$ and relabeling $\tilde{\psi}_1 \to \psi_1$ we find

$$T_1 = \delta_{ik}(p-1)!(m-1)!p_i m_i \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{h_1, h_2 \in H_X \times H_Y} \delta(\psi_1 h_1)\delta(\sigma_1^{-1}\psi_1 \sigma_2 h_2).$$

Summing over $\psi_1$ now gives

$$T_1 = \delta_{ik}(p-1)!(m-1)!p_i m_i \sum_{h_1, h_2 \in H_X \times H_Y} \delta(\sigma_1^{-1} h_1^{-1} \sigma_2 h_2). \tag{6.21}$$

We also need to consider the term

$$\begin{aligned} T_4 &= \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}', \vec{m}' | \sigma_2 \psi_2^{-1} E_{ki}^{(p+1)} \psi_1 | \vec{p}, \vec{m} \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ik}^{(1)} \psi_2 | \vec{p}', \vec{m}' \rangle \\ &= \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ik}^{(1)} \psi_2 | \vec{p}', \vec{m}' \rangle \langle \vec{p}, \vec{m} | \psi_1^{-1} E_{ik}^{(p+1)} \psi_2 \sigma_2^{-1} | \vec{p}', \vec{m}' \rangle. \end{aligned}$$

Changing variables $\psi_1^{-1} \to \sigma_1^{-1}\psi_1^{-1}$ shows that $T_4 = T_1$ and hence

$$T_1 + T_4 = 2\delta_{ik}(p-1)!(m-1)!p_i m_i \sum_{h_1, h_2 \in H_X \times H_Y} \delta(\sigma_1^{-1} h_1^{-1} \sigma_2 h_2). \tag{6.22}$$

The next sum we consider is

$$T_2 = \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}', \vec{m}' | \sigma_2 \psi_2^{-1} E_{ci}^{(1)} E_{kc}^{(p+1)} \psi_1 | \vec{p}, \vec{m} \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ak}^{(1)} E_{ia}^{(p+1)} \psi_2 | \vec{p}', \vec{m}' \rangle.$$

Changing variables $\psi_2^{-1} \to \tilde{\psi}_2^{-1}$ with

$$\tilde{\psi}_2^{-1} = \sigma_2 \psi_2^{-1} \qquad \Rightarrow \qquad \tilde{\psi}_2 = \psi_2 \sigma_2^{-1}, \tag{6.23}$$

the sum becomes

$$\begin{aligned} T_2 = & \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}', \vec{m}' | \psi_2^{-1} E_{ci}^{(1)} E_{kc}^{(p+1)} \psi_1 | \vec{p}, \vec{m} \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} \psi_1^{-1} E_{ak}^{(1)} E_{ia}^{(p+1)} \psi_2 \sigma_2 | \vec{p}', \vec{m}' \rangle \\ = & \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}', \vec{m}' | \psi_2^{-1} \psi_1 E_{ci}^{\psi_1^{-1}(1)} E_{kc}^{\psi_1^{-1}(p+1)} | \vec{p}, \vec{m} \rangle \\ & \times \langle \vec{p}, \vec{m} | \sigma_1^{-1} E_{ak}^{\psi_1^{-1}(1)} E_{ia}^{\psi_1^{-1}(p+1)} \psi_1^{-1} \psi_2 \sigma_2 | \vec{p}', \vec{m}' \rangle. \end{aligned}$$

Change variables $\psi_2 \to \rho$ with $\rho = \psi_1^{-1}\psi_2$ and relabel $\rho \to \psi_2$ to find

$$T_2 = \sum_{\psi_1 \in S_{\vec{p}} \times S_{\vec{m}}} \sum_{\psi_2 \in S_{\vec{p}'} \times S_{\vec{m}'}} \langle \vec{p}', \vec{m}' | \psi_2^{-1} E_{ci}^{\psi_1^{-1}(1)} E_{kc}^{\psi_1^{-1}(p+1)} | \vec{p}, \vec{m} \rangle \langle \vec{p}, \vec{m} | \sigma_1^{-1} E_{ak}^{\psi_1^{-1}(1)} E_{ia}^{\psi_1^{-1}(p+1)} \psi_2 \sigma_2 | \vec{p}', \vec{m}' \rangle.$$

We will use $\hat{b}$ to denote the $q$ dimensional vector that has all entries zero except the $b$th entry which is 1. For a non-zero contribution the first factor requires

$$\begin{aligned} \vec{p}-\hat{i}+\hat{c} &= \vec{p}' \\ \vec{m}-\vec{c}+\vec{k} &= \vec{m}', \end{aligned} \tag{6.24}$$

and the second factor requires

$$\begin{aligned} \vec{m}-\hat{i}+\hat{a} &= \vec{m}' \\ \vec{p}-\vec{a}+\vec{k} &= \vec{p}', \end{aligned} \tag{6.25}$$

There are two solutions:

Case 1: $\hat{c}=\hat{i}$ and $\hat{a}=\hat{k}$. In this case $\vec{p}=\vec{p}'$ and $\vec{m}-\hat{i}+\hat{k}=\vec{m}'$.
Case 2: $\hat{c}=\hat{k}$ and $\hat{a}=\hat{i}$. In this case $\vec{m}=\vec{m}'$ and $\vec{p}-\hat{i}+\hat{k}=\vec{p}'$.

For case 1

$$T_2 = \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}} \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}'}} \langle \vec{p}',\vec{m}'|\psi_2^{-1}E_{ii}^{\psi_1^{-1}(1)}E_{ki}^{\psi_1^{-1}(p+1)}|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}E_{kk}^{\psi_1^{-1}(1)}E_{ik}^{\psi_1^{-1}(p+1)}\psi_2\sigma_2|\vec{p}',\vec{m}'\rangle.$$

Consider the sum over $\psi_1$. Due to the factor $E_{ki}^{\psi_1^{-1}(p+1)}$ we get a non-zero contribution from the slots $p+1, p+2,\cdots,p+m$ (a $Y$ string) if a string starts from node $k$ and ends at node $i$. Thus, the sum over $\psi_1$ gives

$$\begin{aligned} T_2 &= (p-1)!(m-1)!p_{i\to k}m_{ii} \sum_{\psi_2\in S_{\vec{p}}\times S_{\vec{m}'}} \langle\vec{p},\vec{m}'|\psi_2^{-1}|\vec{p},\vec{m}'\rangle\langle\vec{p},\vec{m}'|\sigma_1^{-1}\psi_2\sigma_2|\vec{p},\vec{m}'\rangle \\ &= (p-1)!(m-1)!p_{i\to k}m_{ii} \sum_{\psi_2\in S_{\vec{p}}\times S_{\vec{m}'}} \sum_{h_1,h_2\in H_X\times H_Y} \delta(\psi_2^{-1}h_1)\delta(\sigma_1^{-1}\psi_2\sigma_2h_2) \\ &= (p-1)!(m-1)!p_{i\to k}m_{ii} \sum_{h_1,h_2\in H_X\times H_Y} \delta(\sigma_1^{-1}h_1\sigma_2h_2). \end{aligned} \tag{6.26}$$

For case 2

$$T_2 = \sum_{\psi_1\in S_{\vec{p}}\times S_{\vec{m}}} \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}}} \langle \vec{p}',\vec{m}|\psi_2^{-1}E_{ki}^{\psi_1^{-1}(1)}E_{kk}^{\psi_1^{-1}(p+1)}|\vec{p},\vec{m}\rangle\langle\vec{p},\vec{m}|\sigma_1^{-1}E_{ik}^{\psi_1^{-1}(1)}E_{ii}^{\psi_1^{-1}(p+1)}\psi_2\sigma_2|\vec{p}',\vec{m}\rangle.$$

Consider the sum over $\psi_1$. We get a non-zero contribution for each $Y$ string starting from node $k$ which ends at node $i$. After summing over $\psi_1$ we have

$$\begin{aligned} T_2 &= (p-1)!(m-1)!p_{ii}m_{k\to i} \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}}} \langle\vec{p}',\vec{m}|\psi_2^{-1}|\vec{p},\vec{m}'\rangle\langle\vec{p}',\vec{m}|\sigma_1^{-1}\psi_2\sigma_2|\vec{p}',\vec{m}\rangle \\ &= (p-1)!(m-1)!p_{ii}m_{k\to i} \sum_{\psi_2\in S_{\vec{p}'}\times S_{\vec{m}}} \sum_{h_1,h_2\in H_X\times H_Y} \delta(\psi_2^{-1}h_1)\delta(\sigma_1^{-1}\psi_2\sigma_2h_2) \\ &= (p-1)!(m-1)!p_{ii}m_{k\to i} \sum_{h_1,h_2\in H_X\times H_Y} \delta(\sigma_1^{-1}h_1\sigma_2h_2). \end{aligned} \tag{6.27}$$

Armed with these sums, we now obtain a rather explicit expression for the matrix elements of $D_2^{XY}$ in the Gauss graph basis

$$\begin{aligned} M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} &= -g_{YM}^2 \frac{\delta_{ru}}{\sqrt{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}} \sum_{R'} \delta_{R'_i T'_k} \sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}} \\ &\times \ [2\delta_{ik}p_i m_i - p_{ki}m_{ii} - p_{ii}m_{ik}] \sum_{h_1,h_2\in H_X\times H_Y} \delta(\sigma_1^{-1}h_1\sigma_2 h_2). \end{aligned} \tag{6.28}$$

This is the key result of this section and one of the key results of this chapter. We will now describe how the above matrix can be diagonalized.

## 6.4 Boson Lattice

Our goal in this section is to diagonalize (6.28). This is achieved by interpreting (6.28) as the matrix elements of a Hamiltonian for bosons on a lattice. Towards this end, first note that the matrix elements $M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2}$ are only non-zero if we can choose coset representatives such that $\sigma_1$ and $\sigma_2$ describe the same element of $S_m \times S_p$. This implies that the brane-string systems described by $\sigma_1$ and $\sigma_2$ differ only in the number of strings with both ends attached to the same brane, but not in the number of string stretching between distinct branes. This already implies that the contribution $D_2^{XY}$ only mixes eigenstates of $D_2^{XZ}$ and $D_2^{YZ}$ that are degenerate and hence that all three are simultaneously diagonalizable. In this case the matrix element in (6.28) simplifies to

$$\begin{aligned} M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} &= -g_{YM}^2 \sqrt{\frac{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2}{|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}} \delta_{ru}\delta_{R'_i T'_k} \sqrt{\frac{(N+l_{R_i})(N+l_{T_k})}{l_{R_i}l_{T_k}}} \\ &\times \ \Big[2\delta_{ik}p_i(\sigma_2)m_i(\sigma_2) - p_{ki}m_{ii}(\sigma_2) - p_{ii}(\sigma_2)m_{ik}\Big]. \end{aligned} \tag{6.29}$$

The number of strings stretching between the branes $m_{ik}$ (for $Y$ strings) and $p_{ki}$ (for $X$ strings) are the same for both systems so that

$$m_{ik}(\sigma_1) = m_{ik}(\sigma_2) \equiv m_{ik} \qquad p_{ik}(\sigma_1) = p_{ik}(\sigma_2) \equiv p_{ik}. \tag{6.30}$$

It is the number of closed loops ($m_{ii}$ for $Y$ loops and $p_{ii}$ for $X$ loops) that can differ between the operators that mix. Finally, we have introduced the notation

$$p_i(\sigma) = \sum_{k\neq i} p_{ik} + p_{ii}(\sigma) \qquad m_i(\sigma) = \sum_{k\neq i} m_{ik} + m_{ii}(\sigma). \tag{6.31}$$

From the structure of the operator mixing problem, we would expect that $M^{\vec{m},\vec{p}}_{R,r,\sigma_1\ T,t,\sigma_2} = M^{\vec{m},\vec{p}}_{T,t,\sigma_2\ R,r,\sigma_1}$. This is indeed the case, as a consequence of the easily checked identity

$$\sqrt{\frac{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2}{|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}}\Big[2\delta_{ik}p_i(\sigma_2)m_i(\sigma_2)-p_{ki}m_{ii}(\sigma_2)-p_{ii}(\sigma_2)m_{ik}\Big]$$
$$=\sqrt{\frac{|O^{\vec{m},\vec{p}}_{T,t}(\sigma_2)|^2}{|O^{\vec{m},\vec{p}}_{R,r}(\sigma_1)|^2}}\Big[2\delta_{ik}p_i(\sigma_1)m_i(\sigma_1)-p_{ki}m_{ii}(\sigma_1)-p_{ii}(\sigma_1)m_{ik}\Big]. \tag{6.32}$$

which holds for any $i,k$.

The lattice model consists of two distinct species of bosons, one for $X$ and one for $Y$, hopping on a lattice, with a site for every brane, or equivalently, a site for every row in the Young diagram $R$ labeling the Gauss graph operator $\hat{O}^{\vec{m},\vec{p}}_{R,r}(\sigma)$. The bosons are described by the following commuting sets of operators

$$[a_i,a_j^\dagger]=\delta_{ij}\qquad [a_i^\dagger,a_j^\dagger]=0=[a_i,a_j]$$
$$[b_i,b_j^\dagger]=\delta_{ij}\qquad [b_i^\dagger,b_j^\dagger]=0=[b_i,b_j]. \tag{6.33}$$

Using these boson oscillators, we have

$$m_{ii}=a_i^\dagger a_i\qquad p_{ii}=b_i^\dagger b_i \tag{6.34}$$

$$m_i=\sum_k m_{ik}+a_i^\dagger a_i\qquad p_i=\sum_k p_{ik}+b_i^\dagger b_i. \tag{6.35}$$

The vacuum of the Fock space $|0\rangle$ obeys

$$a_i|0\rangle=0=b_i|0\rangle\qquad i=1,2,\cdots,q. \tag{6.36}$$

The Hamiltonian of the lattice model is given by

$$H=\sum_{i,j=1}^{q}\sqrt{\frac{(N+l_{R_i})(N+l_{R_j})}{l_{R_i}l_{R_j}}}\bigg(2\delta_{ij}\Big(\sum_{l\neq i}p_{il}+b_i^\dagger b_i\Big)\Big(\sum_{l\neq i}m_{il}+a_i^\dagger a_i\Big)$$
$$-p_{ji}a_j^\dagger a_i-m_{ji}b_j^\dagger b_i\bigg). \tag{6.37}$$

Notice that this Hamiltonian is quadratic in each type of oscillator. It has a nontrivial repulsive interaction given by the $\sum_i a_i^\dagger a_i b_i^\dagger b_i$ term, which makes it energetically unfavorable for $a$ and $b$ type particles to sit on the same site. Also, the full Fock space is a tensor product between the Fock space for the $a$ oscillator and the Fock space for the $b$ oscillator. We will use the occupation number representation to describe the boson states. To complete the mapping to the lattice model, we need to explain the correspondence between Gauss graph operators and states of the boson lattice. This map is given by reading the boson occupation numbers for each site from the number of closed strings with both ends attached to the node corresponding to that site. In the next subsection we consider an example which nicely illustrates this map.

Finally, lets make an important observation regarding (6.37). Although the eigenvalues of this Hamiltonian are subleading contributions to the anomalous dimension, there is an important situation in which this correction is highly significant: for BPS states the leading contribution to the anomalous dimension vanishes and this subleading correction is important. The BPS operators are labelled by Gauss graphs that have $p_{ik} = m_{ik} = 0$ whenever $i \neq k$, i.e. there are no strings stretching between branes. In this case, it is clear that (6.37) vanishes so that the BPS operators remain BPS.

### 6.4.1 Example

In this subsection we will consider an example for which $R$ has $q = 3$ rows and $p = m = 3$. In this problem, 10 operators mix. The Gauss graph labels for the operators that mix are displayed in Figure 6.1.

For the Gauss graph operators shown, we have the following correspondence with boson lattice states

$$\begin{aligned}
|1\rangle &= a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle \qquad |2\rangle = a_1^\dagger \frac{(a_2^\dagger)^2}{\sqrt{2!}}|0\rangle \\
|3\rangle &= a_3^\dagger \frac{(a_2^\dagger)^2}{\sqrt{2!}}|0\rangle \qquad |4\rangle = a_2^\dagger \frac{(a_1^\dagger)^2}{\sqrt{2!}}|0\rangle \\
|5\rangle &= a_3^\dagger \frac{(a_1^\dagger)^2}{\sqrt{2!}}|0\rangle \qquad |6\rangle = a_1^\dagger \frac{(a_3^\dagger)^2}{\sqrt{2!}}|0\rangle \\
|7\rangle &= a_2^\dagger \frac{(a_3^\dagger)^2}{\sqrt{2!}}|0\rangle \qquad |8\rangle = \frac{(a_3^\dagger)^3}{\sqrt{3!}}|0\rangle \\
|9\rangle &= \frac{(a_2^\dagger)^3}{\sqrt{3!}}|0\rangle \qquad |10\rangle = \frac{(a_1^\dagger)^3}{\sqrt{3!}}|0\rangle.
\end{aligned} \tag{6.38}$$

It is now rather straight forwards to compute matrix elements of the lattice Hamiltonian. For example

$$\langle 1|H|2\rangle = -\sqrt{\frac{(N + l_{R_3})(N + l_{R_2})}{l_{R_2} l_{R_3}}}\sqrt{2}. \tag{6.39}$$

It is instructive to compare this to the answer coming from (6.28). To move from state 2 to state 1, a string must detach from node 2 and reattach to node 3. Thus, we should plug $i = 2$ and $k = 1$ into (6.28). The Gauss graph $\sigma_1$ corresponds to $|1\rangle$ while $\sigma_2$ corresponds to $|2\rangle$. In addition $R_2' = T_1'$ and from the Gauss graphs we read off $p_{32} = 1$ and $m_{22}(\sigma_1) = 1$. It is now

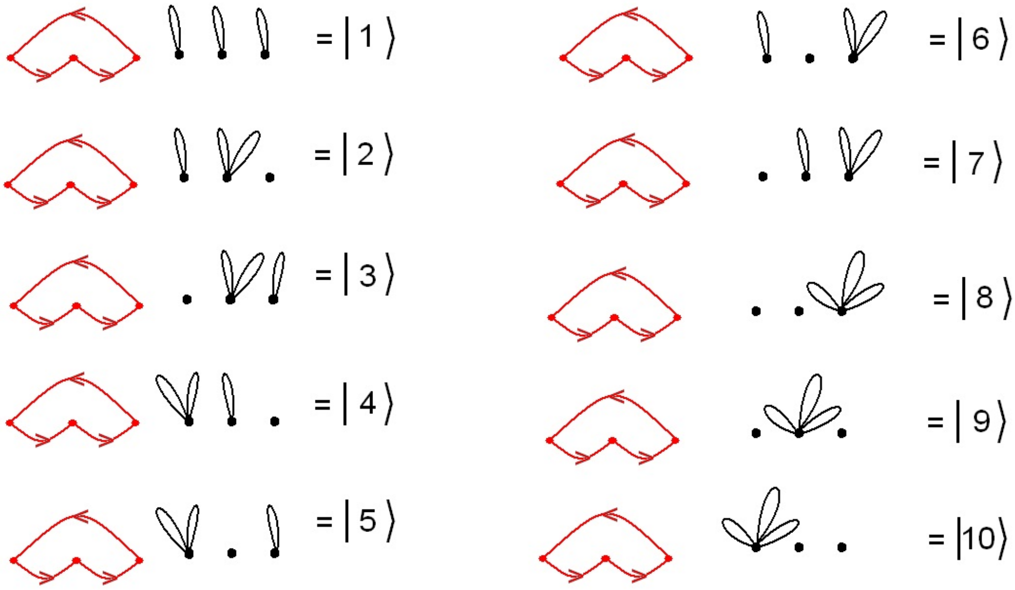



Figure 6.1: Each Gauss graph label is composed of two graphs, the first for the $X$ strings and the second for the $Y$ strings. Each graph has 3 nodes (because $q = 3$). There are no $b$ type particles because there are no closed $X$ strings. There are 3 $a$ type particles because there are three closed $Y$ strings. All operators share the same $r$ label.

simple to see that (6.28) is in complete agreement with the above matrix element.

Finally, the state corresponding to the Gauss graph in Figure 6.2 is

$$\frac{(a_1^\dagger)^3}{\sqrt{3!}}\frac{(a_3^\dagger)^3}{\sqrt{3!}}a_5^\dagger a_6^\dagger b_1^\dagger \frac{(b_2^\dagger)^2}{\sqrt{2!}}\frac{(b_3^\dagger)^2}{\sqrt{2!}}b_4^\dagger b_5^\dagger b_6^\dagger|0\rangle. \tag{6.40}$$

## 6.5 Diagonalization

In this section we will consider a class of examples that can be diagonalized explicitly. Our main motivation is to show that working with the lattice is simple, so the mapping we have found is useful.

### 6.5.1 Exact Eigenstates

For these examples take

$$p_{ki} = p_{ik} = \delta_{k,i+1}B \qquad m_{ki} = m_{ik} = \delta_{k,i+1}A. \tag{6.41}$$

with $A$ and $B$ two positive integers. For examples of Gauss graphs that obey this condition, see Figure 6.3. There are two cases we will consider: we

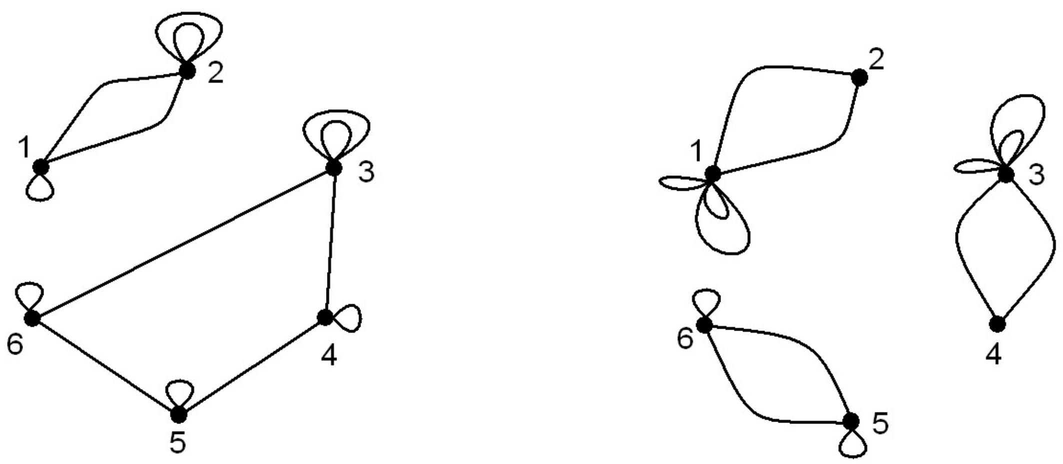



Figure 6.2: An example of a Gauss graph with non-zero a and b occupation numbers.

will fix the number of $a$ particles to zero and leave the number of $b$ particles arbitrary, or, fix the number of $b$ particles to zero and leave the number of $a$ particles arbitrary. We will also specialize to labels $R$ that have the difference between any two row lengths $l_{R_i} - l_{R_j} \sim N$, but $\frac{l_{R_i}-l_{R_j}}{l_{R_i}} \approx 0$. In this case our lattice Hamiltonian simplifies to

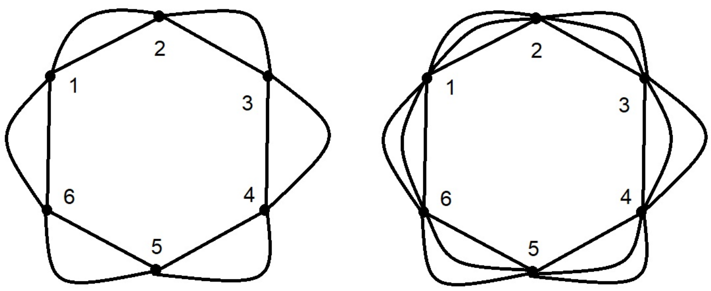



Figure 6.3: An example of a Gauss graph that is easily solvable. The example shown has $A = 2$ and $B = 3$.

$$H = \frac{(N + l_{R_1})}{l_{R_1}} \sum_{i=1}^{q} \Big( 2 \left( B + b_i^\dagger b_i \right) \left( A + a_i^\dagger a_i \right) \\ -B(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) - A(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) \Big). \tag{6.42}$$

This Hamiltonian is easily diagonalized by going to Fourier space. Indeed, in terms of the new oscillators

$$\tilde{a}_n = \frac{1}{\sqrt{q}} \sum_{k=1}^{q} e^{i\frac{2\pi kn}{q}} a_k \qquad \tilde{b}_n = \frac{1}{\sqrt{q}} \sum_{k=1}^{q} e^{i\frac{2\pi kn}{q}} b_k \qquad n = 0, 1, \cdots, q-1 \tag{6.43}$$

the Hamiltonian becomes (we have set the number of $a$ particles to zero)

$$H = A\frac{(N + l_{R_1})}{l_{R_1}} \sum_{n=0}^{q-1} \left(2 - 2\cos\left(\frac{2\pi n}{q}\right)\right) \tilde{b}_n^\dagger \tilde{b}_n + 2ABq\frac{(N + l_{R_1})}{l_{R_1}}. \tag{6.44}$$

Eigenstates of the lattice Hamiltonian are given by arbitrary momentum space excitations

$$\prod_{n=0}^{q-1} \frac{(\tilde{a}_n^\dagger)^{\alpha_n}}{\sqrt{\alpha_n!}} |0\rangle \qquad \text{or} \qquad \prod_{n=0}^{q-1} \frac{(\tilde{b}_n^\dagger)^{\beta_n}}{\sqrt{\beta_n!}} |0\rangle. \tag{6.45}$$

where the occupation numbers $\alpha_n, \beta_n$ are arbitrary. This state can be translated back into the Gauss graph language to give operators of a definite scaling dimension.

### 6.5.2 General Properties of Low Energy Eigenstates

In this subsection we will sketch the features of generic low energy states of the lattice Hamiltonian. We begin by relaxing the constraint that only one species is hopping. In the end we will also make comments valid for the general Gauss graph configuration. The Hamiltonian becomes

$$H = H_a + H_b + H_{ab} + E_0 \tag{6.46}$$

$$H_a = \frac{(N + l_{R_1})}{l_{R_1}} B \sum_{i=1}^{q} \left(2a_i^\dagger a_i - a_i^\dagger a_{i+1} - a_{i+1}^\dagger a_i\right) \tag{6.47}$$

$$H_b = \frac{(N + l_{R_1})}{l_{R_1}} A \sum_{i=1}^{q} \left(2b_i^\dagger b_i - b_i^\dagger b_{i+1} - b_{i+1}^\dagger b_i\right) \tag{6.48}$$

$$H_{ab} = \frac{(N + l_{R_1})}{l_{R_1}} \sum_{i=1}^{q} 2b_i^\dagger b_i a_i^\dagger a_i. \tag{6.49}$$

The constant $E_0 = 2ABq\frac{(N+l_{R_1})}{l_{R_1}}$ is not important for the dynamics but must be included to obtain the correct anomalous dimensions. To start, consider $H_a$ which is a kinetic term for the $a$ particles. The first term in the Hamiltonian implies that it costs energy to have an $a$ particle occupying a site, while the second and third terms tell us this energy can be lowered by hopping between sites $i$ and $i+1$. Consequently, to minimize $H_a$, the $a$ particles will spread out as much as is possible. This is in perfect accord with the results of the last section. The lowest energy single particle state is the zero momentum state, which occupies each site with the same probability: the particle spreads out as much as is possible. Very similar reasoning for $H_b$ implies that the $b$ particles will also spread out as much as is possible. Finally, the term $H_{ab}$ is a repulsive interaction, telling us that it costs energy to have $a$s and $b$s occupying the same site. So there is a competition going on: The terms $H_a$ and $H_b$ want to spread the $a$s and $b$s uniformly on the lattice which would certainly distribute $a$s and $b$s to the same site. The term $H_{ab}$ wants to ensure that any particular site will have only $a$s or $b$s but not both. Who wins?

Consider a thermodynamic like limit where we consider a very large number of both species of particles, $n_a$ and $n_b$. In the end, the low energy state will be a "demixed" state with no sites holding both $a$s and $b$s. To see this, note that $H_a$ grows like $n_a$ and $H_b$ like $n_b$. This is much smaller than the growth of the term $H_{ab}$ which grows like $n_a n_b$, so the repulsive interaction wins. This conclusion is nicely borne out by numerical results for the two component Bose-Hubbard model [71, 72]. The ground state phase diagram of the Hamiltonian of [71], shows four distinct phases: double super fluid phase, supercounterflow phase, demixed Mott insulator phase and a demixed superfluid phase. Comparing our Hamiltonian to that of [71], we are always in the demixed superfluid phase: the $a$ and $b$ particles do not mix, but are free to move in their respective domains.

For the generic Gauss graph, with any choices for the values of $m_{ik}$ and $p_{ik}$, it is clear that $H_a$ and $H_b$ will still cause the $a$ and $b$ particles to spread out as much as possible. The term $H_{ab}$ will again dominate when we have large numbers of $a$s and $b$s so we again expect a demixed gas. We can translate this structure of the generic state back into the language of the giant graviton description. Up to now we have considered dual giant gravitons which correspond to operators labelled by Young diagrams with long rows. Recall that dual giant gravitons wrap an $\mathrm{S}^3 \subset \mathrm{AdS}_5$. In this context, $l_{R_1}$ is the momentum of each giant graviton and $N+l_{R_1}$ is the radius on the LLM plane at which the giant graviton orbits. The Hamiltonian for giant gravitons, which wrap $\mathrm{S}^3 \subset \mathrm{S}^5$ is given by

$$\begin{aligned} H = \frac{(N-l_{R_1})}{l_{R_1}} \sum_{i=1}^{q} \Big( 2 \left(B + b_i^\dagger b_i\right) \left(A + a_i^\dagger a_i\right) \\ -B(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) - A(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) \Big). \end{aligned} \tag{6.50}$$

These operators are labelled by Young diagrams with long columns. The giant gravitons orbit on the LLM plane with a radius of $N - l_{R_1}$. The $X$

and $Y$ fields are each charged under different $U(1)$s of the $\mathcal{R}$-symmetry group. The $\mathcal{R}$-symmetry of the CFT translates into angular momentum of the dual string theory, so that attaching the particles to a given giant graviton corresponds to giving the giant graviton angular momentum. The lowest energy giant graviton states are obtained by distributing the momenta carried by the $X$ and $Y$ fields evenly between the giant gravitons with the condition that any particular giant graviton carries only $X$ or $Y$ momenta, but not both. These conclusions hold for the generic state where there are enough $p_{ik}$ and $m_{ik}$ non-zero, allowing the $X$s and $Y$s to hop between any two giant gravitons, possibly by a complicated path. Thus in the end we see that the mapping to the boson lattice model has allowed a rather detailed understanding of the operator mixing problem.

## 6.6 Conclusions

In this article we have studied the action of the one loop dilatation operator $D_2$ on Gauss graph operators $O^{\vec{m},\vec{p}}_{R,r}(\sigma)$ which belong to the $SU(3)$ sector. The term we have studied, $D_2^{XY}$ , is diagonal in the $r$ label, mixing operators labelled by distinct graphs. It makes a subleading contribution as compared to $D_2^{XZ}$ and $D_2^{YZ}$ when $n \gg m+p$. The two leading terms mix operators labelled by distinct $r$s. Diagonalizing the action of $D_2^{XZ}$ and $D_2^{YZ}$ on $r$ leads to a collection of decoupled harmonic oscillators, which we refer to as the $Z$ oscillators, since the $r$ label is associated with $Z$. The spectrum of the $Z$ oscillators gives the leading contribution to the anomalous dimensions. The new contribution that we have studied in this chapter can also be mapped to a collection of oscillators, describing a lattice boson model. This is done by introducing two sets of oscillators, the $X$ and $Y$ oscillators associated to the $X$ and $Y$ fields. Diagonalizing the $X$ and $Y$ oscillators breaks degeneracies among different copies of $Z$ oscillators and leads to a constant addition to their ground state energy. This is then a constant shift of the anomalous dimension. Although this shift is subleading (it is of order $\frac{m}{n}$), it could potentially show that certain states are not in fact BPS. This was investigated in detail and it turns out that states that are BPS (their leading order anomalous dimension vanishes) at leading order, remain BPS when the subleading correction is computed (it too vanishes).

The mapping that we have found to a lattice boson model has achieved an enormous simplification of the operator mixing problem and we have managed to understand it in some detail. Indeed, using the lattice boson model, we have argued that the lowest energy giant graviton states are obtained by distributing the momenta carried by the $X$ and $Y$ fields evenly between the giant gravitons with the condition that any particular giant graviton carries only $X$ or $Y$ momenta, but not both. Since states with two charges are typically $\frac{1}{4}$-BPS while states with 3 charges are typically $\frac{1}{8}$-BPS, it maybe that the solution is locally trying to maximize susy. It would be interesting to arrive at the same picture, employing the dual string theory description.

Perhaps the most interesting consequence of our results is that they suggest ways in which one can go beyond the $\frac{1}{2}$-BPS sector. Indeed, all three types of fields considered have been mapped to oscillators, so perhaps there is a more general description of this sector that treats all three types of oscillators on the same footing. This would relax the constraint $n \gg p + m$ which allows for operators that are far from the $\frac{1}{2}$-BPS limit. Deriving this picture is a fascinating open problem, since it will require that we go beyond the displaced corners approximation, or alternatively, that we generalize it.

As a final comment, recall that Mikhailov [73] has constructed an infinite family of $\frac{1}{8}$ BPS giant graviton branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Quantizing the space of Mikhailov's solutions leads to $N$ non-interacting bosons in a harmonic oscillator[74, 75, 76]. It is tempting to speculate that it is precisely these oscillators that we are uncovering in our study; for evidence in harmony with this suggestion see [77]. It would be interesting to make this speculation precise.

# Chapter 7

# Conclusions

In this thesis we focused on computing anomalous dimensions using the AdS/CFT correspondence. We considered type IIB superstring theory and $U(N)$, $\mathcal{N} = 4$ super Yang-Mills theory in a large $N$ but non-planar limit. A powerful group theory approach to the problem has been developed. Using this technology, we computed anomalous dimesions by acting the two loop dilatation operator in the $SU(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory and the one loop dilatation operator in the $SU(3)$ sector of the $\mathcal{N} = 4$ super Yang-Mills theory, on the restricted Schur polynomial basis. The restricted Schur polynomial is dual to a system of excited strings stretched between giant gravitons. String excitations are identified as magnon particles.

Our results in chapter 5 indicates that in the $\frac{m}{n} = 0$ limit, the action of the dilatation operator factorizes into an action on the $Z$ fields and an action on the $Y$ fields. The subleading correction has spoiled this factorization of the dilatation operator by introducing further operator mixing. Another insteresting result of our analysis is that the subleading corrections did not induce extra mixing for the BPS operators. After accounting for the complete $\frac{m}{n}$ correction to two loops, we found our BPS operators remain uncorrected and continue to have a vanishing anomalous dimension.

In chapter 6 we studied the action of the one loop dilatation operator $D_2$ on Gauss graph operators $O^{\vec{m},\vec{p}}_{R,r}(\sigma)$ which belong to the $SU(3)$ sector. The term we have studied, $D_2^{XY}$, is diagonal in the $r$ label, mixing operators labelled by distinct graphs. This term gave a subleading contribution as compared to $D_2^{XZ}$ and $D_2^{YZ}$ when $n \gg m + p$. Diagonalizing the action of $D_2^{XZ}$ and $D_2^{YZ}$ on $r$ led to a collection of decoupled harmonic oscillators, which we refer to as the $Z$ oscillators, since the $r$ label is associated with $Z$. The new contribution $D_2^{XY}$ was mapped to a collection of oscillators, describing a lattice boson model. This was done by introducing two sets of oscillators, the $X$ and $Y$ oscillators associated to the $X$ and $Y$ fields. Diagonalizing the $X$ and $Y$ oscillators breaks degeneracies among different copies of $Z$ oscillators and leads to a constant addition to their ground state energy.

Although this shift is subleading (it is of order $\frac{m}{n}$), it could potentially show that certain states are not in fact BPS. This was investigated in detail and it turns out that states that are BPS (their leading order anomalous dimension vanishes) at leading order, remain BPS when the subleading correction is computed (it too vanishes). The mapping that we have found to a lattice boson model has achieved an enormous simplification of the operator mixing problem and we have managed to understand it in some detail. There is a demixing effect whereby the lowest energy state allows a given giant to carry $X$ excitations or $Y$ excitations, but not both. Since states with two charges are typically $\frac{1}{4}$-BPS while states with 3 charges are typically $\frac{1}{8}$-BPS, it may be that the solution is locally trying to maximize susy. It would be interesting to arrive at the same picture, employing the dual string theory description.

Perhaps the most interesting consequence of our results is that they suggest ways in which one can go beyond the $\frac{1}{2}$-BPS sector. Indeed, all three types of fields considered have been mapped to oscillators, so perhaps there is a more general description of this sector that treats all three types of oscillators on the same footing. This would relax the constraint $n \gg p + m$ which allows for operators that are far from the $\frac{1}{2}$-BPS limit. Deriving this picture is a fascinating open problem, since it will require that we go beyond the displaced corners approximation, or alternatively, that we generalize it.

As a final comment, recall that Mikhailov [73] has constructed an infinite family of $\frac{1}{8}$-BPS giant graviton branes in $\mathrm{AdS}_5\times\mathrm{S}^5$. Quantizing the space of Mikhailov's solutions leads to $N$ non-interacting bosons in a harmonic oscillator[74, 75, 76]. It is tempting to speculate that it is precisely these oscillators that we are uncovering in our study; for evidence in harmony with this suggestion see [77]. It would be interesting to make this speculation precise.

# Appendix A

# Double coset background

The double coset ansatz was formulated in [30] by diagonalizing the one loop dilatation operator. In this appendix we will review those aspects of [30] that are relevant for our study.

At the most basic level, the double coset ansatz follows from the fact that there are two ways to decompose $V_p^{\otimes m}$. To start, refine $V_p^{\otimes m}$ by the $U(1)^p$ charges measured by $E_{ii}$, as follows:

$$V_p = \oplus_{i=1}^{p} V_i. \tag{A.1}$$

The vector space $V_i$ is a one-dimensional space. It is spanned by the eigenstate of $E_{ii}$ with eigenvalue one. Consequently if $v_i \in V_i$ we have

$$\begin{aligned} E_{ii} v_j &= \delta_{ij} v_i \\ E_{ij} v_k &= \delta_{jk} v_i, \end{aligned} \tag{A.2}$$

In the restricted Schur polynomial construction of [28] for long rows, a state in $V_i$ corresponds to a $Y$-box in the $i^{\text{th}}$ row. The $U(1)$ charges of a restricted Schur polynomial can be collected into the vector $\vec{m}$, which corresponds to a vector with $m_1$ copies of $v_1$, $m_2$ copies of $v_2$ etc.

$$|\bar{v}, \vec{m}\rangle \equiv \left| v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots \otimes v_p^{\otimes m_p} \right\rangle. \tag{A.3}$$

A general state with these charges is given by acting with a permuation

$$|v_\sigma\rangle \equiv \sigma \left| v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots \otimes v_p^{\otimes m_p} \right\rangle, \tag{A.4}$$

where

$$\sigma \left| v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_p} \right\rangle = \left| v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots \otimes v_{i_{\sigma(p)}} \right\rangle. \tag{A.5}$$

This description enjoys a symmetry under

$$H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p}, \tag{A.6}$$

and as a consequence, not all $\sigma$ give independent vectors

$$|v_\sigma\rangle = |v_{\sigma\gamma}\rangle\,, \tag{A.7}$$

if $\gamma \in H$.

The restricted Schur polynomials are organized by reduction multiplicities of $U(p)$ to $U(1)^p$, which are counted by the Kotska numbers and resolved by the Gelfand-Tsetlin patterns. It is possible to prove the equality of Kostka numbers and the branching multiplicity of $S_m \to H$. This is a very direct indication that there are two possible ways to organize the local operators of the theory.

We can develop the steps above at the level of a basis for $V_p^{\otimes m}$. In terms of the branching coefficients, defined by

$$\frac{1}{|H|}\sum_{\gamma\in H}\Gamma^{(s)}_{ik}(\gamma) = \sum_\mu B^{s\to 1_H}_{i\mu} B^{s\to 1_H}_{k\mu}\,, \tag{A.8}$$

we have

$$|\vec{m}, s, \mu; i\rangle \equiv \sum_j B^{s\to 1_H}_{j\mu}\, |v_{s,i,j}\rangle = B^{s\to 1_H}_{j\mu} \sum_{\sigma\in S_m} \Gamma^{(s)}_{ij}(\sigma)\, |v_\sigma\rangle\,. \tag{A.9}$$

The $\mu$ index is a multiplicity for the reduction of $S_m$ into $H$. We also have

$$\langle \vec{m}, u, \nu; j| = \sum_{\tau\in S_m} \langle \bar{v}, \vec{m}|\, \tau^{-1}\Gamma^{(u)}_{jk}(\tau) B^{u\to 1_H}_{k\nu}\,, \tag{A.10}$$

which ensures the correct normalization

$$\langle \vec{m}, u, \nu; j\rangle\, \vec{m}, s, \mu; i = \delta_{\vec{m}\vec{n}}\delta_{us}\delta_{ji}\delta_{\mu\nu}. \tag{A.11}$$

Lastly, the group-theoretic coefficients

$$C^s_{\mu_1\mu_2}(\tau) = |H|\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl} B^{s\to 1_H}_{k\mu_1} B^{s\to 1_H}_{l\mu_2}\,, \tag{A.12}$$

provide an orthogonal transformation between double coset elements labelled by $\sigma$ and the restricted Schur polynomials labelled by an irreducible representation $s \vdash m$ and a pair of multiplicities $\mu_1$, $\mu_2$.

# Appendix B

# How to compute traces

In this appendix we will compute the traces needed to evaluate the action of $D_4^{(2)}$ in the Gauss graph basis. The generic form of the trace we need to evaluate is

$$\mathrm{Tr}(A p_{s\mu\nu} B p_{u\gamma\delta}), \tag{B.1}$$

with $A$ and $B$ any arbitrary product of the $E_{ij}^{(A)}$s. If we are able to compute this trace, we are able to evaluate the action of any differential operator that does not change the number of $Z$ and $Y$ fields, on the Gauss graph operator in the displaced corners approximation. This therefore provides a general method to exploit the simplifications of the large $N$ limit, for this class of operators.

The Fourier transform we want to consider maps between functions labelled by an irreducible representation and a pair of multiplicity labels and functions that take values on the double coset $H \setminus S_m / H$. We can choose a permutation $\sigma$ to represent each class of the coset $[\sigma]$. The transform is then

$$\tilde{f}([\sigma]) = \sum_{s,\alpha\beta} \Gamma^s_{ab}(\sigma) B^{s\to 1_H}_{a\alpha} B^{s\to 1_H}_{b\beta} f(s,\alpha,\beta). \tag{B.2}$$

For further details the reader is referred to [30].

## B.1 Projector transformed

In this section we will Fourier transform the intertwining map used to define the restricted Schur polynomial. The projector that participates in the trace (B.1) can be expressed as

$$p_{s\mu\nu} = \sum_a |\vec{m}, s, \mu; a\rangle \langle \vec{m}, s, \nu; a| \,. \tag{B.3}$$

We will make use of the relations

$$\langle \vec{m}, t, \nu; j| = \frac{d_t}{m!|H|} \sum_{\tau \in S_m} \langle \bar{v}, \vec{m}| \, \tau^{-1} \Gamma^{(t)}_{jk}(\tau) B^{t\to 1_H}_{k\nu}, \tag{B.4}$$

and

$$|\vec{m}, s, \mu; i\rangle = \sum_{\sigma \in S_m} \Gamma^{(s)}_{il}(\gamma) B^{s\to 1_H}_{l\mu} \sigma \, |\bar{v}, \vec{m}\rangle \,, \tag{B.5}$$

as well as

$$\langle \bar{v}, \vec{m}| \, \sigma \, |\bar{v}, \vec{m}\rangle = \sum_{\gamma \in H} \delta(\sigma\gamma), \tag{B.6}$$

and

$$\sum_{\nu} B^{s\to 1_H}_{c\nu} B^{s\to 1_H}_{d\nu} = \frac{1}{|H|} \sum_{\gamma \in H} \Gamma^{(s)}_{cd}(\gamma). \tag{B.7}$$

We will use the notation $|v_\sigma\rangle = \sigma \, |\bar{v}\rangle$. It is then simple to show that

$$\begin{aligned} \sum_{s\mu\nu lm} p_{s\mu\nu} B^{s\to 1_H}_{m\mu} B^{s\to 1_H}_{l\nu} \Gamma^{(s)}_{ml}(\sigma) &= \sum_{s\mu\nu lma} |\vec{m}, s, \mu; a\rangle \langle \vec{m}, t, \nu; a| \, B^{s\to 1_H}_{m\mu} B^{s\to 1_H}_{l\nu} \Gamma^{(s)}_{ml}(\sigma) \\ &= \frac{1}{|H|^3} \sum_{\psi, \tau \in S_m} \sum_{\gamma_1, \gamma_2 \in H} |v_\psi\rangle \langle v_\tau| \, \delta(\tau^{-1}\psi\gamma_2\sigma\gamma_1). \end{aligned} \tag{B.8}$$

This last equation implies that the permutation applied to the ket and the permutation applied to the bra are related by multiplication by a permutation representing the double coset element.

## B.2 Summing over $H$

We consider an interwining map $p_{s\mu\nu}$ built on the state $|\bar{v}_1, \vec{m}_1\rangle$ with symmetry group $H_1$ and interwining map $p_{t\gamma\delta}$ built on the state $|\bar{v}_2, \vec{m}_2\rangle$ with symmetry group $H_2$. We make no assumptions about $H_1$ and $H_2$. In general they will be different groups and hence we Fourier transform $p_{s\mu\nu}$ and $p_{t\gamma\delta}$ to different double cosets. Using the result obtained above, the Fourier transform of

$$\mathrm{Tr}(A p_{s\mu\nu} B p_{u\gamma\delta}), \tag{B.9}$$

is

$$\begin{aligned} T &= \frac{1}{|H_1|^3 |H_2|^3} \sum_{\gamma_i \in H_i} \sum_{\tau_i \in H_i} \sum_{\psi_i \in S_m} \langle v_2| \, \gamma_2 \sigma_2^{-1} \tau_2 \psi_2^{-1} A \psi_1 \, |v_1\rangle \langle v_1| \, \gamma_1 \sigma_1^{-1} \tau_1 \psi_1^{-1} B \psi_2 \, |v_2\rangle \\ &= \frac{1}{|H_1|^3 |H_2|^3} \sum_{\gamma_i \in H_i} \sum_{\tau_i \in H_i} \sum_{\psi_i \in S_m} \langle v_2| \, \gamma_2 \sigma_2^{-1} \tau_2 \psi_2^{-1} A \psi_1 \, |v_1\rangle \langle v_2| \, \psi_2^{-1} B^T \psi_1 \tau_1 \sigma_1 \gamma_1 \, |v_1\rangle \,. \end{aligned} \tag{B.10}$$

To get to the last line, we used the fact that the matrix element $\langle v_1|\,\gamma_1\sigma_1^{-1}\tau_1\psi_1^{-1}B\psi_2\,|v_2\rangle$ is a real number and the permutations are represented by matrices with real elements. To make the discussion concrete, it is useful to make a specific choice for $A$ and $B$. This will allow us to illustrate the argument in a very concrete setting. In the end we will state the general result. Choose, for example,

$$A = E^{(1)}_{ki}E^{(2)}_{ij} \qquad B = E^{(1)}_{jl}E^{(2)}_{lk}. \tag{B.11}$$

Using the facts that

$$\begin{aligned} \gamma\,|v_1\rangle &= |v_1\rangle && \forall\gamma\in H_1 \\ \beta\,|v_2\rangle &= |v_2\rangle && \forall\beta\in H_2 \\ \psi^{-1}E^{(a)}_{qr}\psi &= E^{\psi(a)}_{qr} && \forall\psi\in S_m, \end{aligned} \tag{B.12}$$

we readily find

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\,\sigma_2^{-1}E^{\psi_2(1)}_{ki}E^{\psi_2(2)}_{ij}\psi_1\,|v_1\rangle\,\langle v_2|\,E^{\psi_2(1)}_{lj}E^{\psi_2(2)}_{kl}\psi_1\sigma_1\,|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\,\sigma_2^{-1}\psi_2^{-1}A\psi_2\psi_1\,|v_1\rangle\,\langle v_2|\,\psi_2^{-1}B^T\psi_2\psi_1\sigma_1\,|v_1\rangle\,. \end{aligned} \tag{B.13}$$

where on the last line we have written the general result. Our next task is to compute the sums over $\psi_1$ and $\psi_2$.

## B.3 Summing over $S_m$

In this section we consider two increasingly difficult examples before we state the general result. The first example is closely related to the trace needed to obtain the one loop dilatation operator. Since we know the result of this trace, this example is a nice test of our ideas. The second example is a simple but nontrivial example which will illustrate how the general case works. In the following subsection we will quote the result for the general case.

### B.3.1 First Evaluation

Choose $A = E^{(1)}_{ab}$ and $B = E^{(1)}_{ba}$ to get

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i}\langle v_2|\,\sigma_2^{-1}E^{\psi_2(1)}_{ab}\psi_1\,|v_1\rangle\,\langle v_2|\,E^{\psi_2(1)}_{ab}\psi_1\sigma_1\,|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i}\langle v_1|\,\psi_1^{-1}E^{\psi_2(1)}_{ba}\sigma_2\,|v_2\rangle\,\langle v_2|\,E^{\psi_2(1)}_{ab}\psi_1\sigma_1\,|v_1\rangle\,. \end{aligned} \tag{B.14}$$

We must turn a $b$ vector in $|v_1\rangle$ into an $a$ vector (and possibly permute) to get $|v_2\rangle$. Since the ordering of the slots in $|v_1\rangle$ and $|v_2\rangle$ is arbitrary, we can remove this possible permutation by declaring

$$|v_2\rangle = E^{(1)}_{ab}\,|v_1\rangle \qquad |v_1\rangle = E^{(1)}_{ba}\,|v_2\rangle\,. \tag{B.15}$$

Thus (in the computation below we denote by $S^1_a(S^2_a)$ the set of all slots in $|v_1\rangle$ ($|v_2\rangle$) that are filled with an $a$ vector)

$$\begin{aligned}
T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i}\langle v_1|\,\psi_1^{-1}E^{\psi_2(1)}_{ba}\sigma_2 E^{(1)}_{ab}\,|v_2\rangle\,\langle v_2|\,E^{(1)}_{ba}E^{\psi_2(1)}_{ab}\psi_1\sigma_1\,|v_1\rangle \\
&= \frac{1}{|H_1||H_2|}\sum_{\psi_i}\langle v_1|\,\psi_1^{-1}\sigma_2 E^{\sigma_2\psi_2(1)}_{ba}E^{(1)}_{ab}\,|v_2\rangle\,\langle v_2|\,E^{(1)}_{ba}E^{\psi_2(1)}_{ab}\psi_1\sigma_1\,|v_1\rangle \\
&= \frac{1}{|H_1||H_2|}\sum_{\psi_i}\sum_{\gamma_i\in H_i}\delta\Big(\gamma_1\psi_1^{-1}\sigma_2(1,\sigma_2\psi_2(1))\Big)\delta\Big(\gamma_2\sigma_1^{-1}\psi_1^{-1}(1,\psi_2(1))\Big) \\
&\quad\times\sum_{x\in S^2_a}\delta(x,\sigma_2\psi_2(1))\sum_{y\in S^2_a}\delta(y,\psi_2(1)) \\
&= \frac{1}{|H_1||H_2|}\sum_{\psi_2}\sum_{\gamma_i\in H_1}\delta\Big(\gamma_1\sigma_1\gamma_2(1,\psi_2(1)\sigma_2(1,\sigma_2\psi_2(1))\Big) \\
&\quad\times\sum_{x\in S^2_a}\delta(x,\sigma_2\psi_2(1))\sum_{y\in S^2_a}\delta(y,\psi_2(1)).
\end{aligned} \tag{B.16}$$

Now consider the final sum over $\psi_2$. $\psi_2(1)$ is the start point of an oriented edge in Gauss graph $\sigma_2$ and $\sigma_2\psi_2(1)$ is the end point of the same edge. The delta functions on the last line ensure that both endpoints of this string are attached to node $a$ in the Gauss graph. This is swapped with the edge labelled 1 (i.e. the edge in the first slot) and compared to $\sigma_1$. According to (B.15), the edge in the first slot of $|v_1\rangle$ is attached to node $b$. Thus the above sum is ensuring that when a closed loop on node $a$ of $\sigma_2$ is removed and reattached to node $b$ of $\sigma_2$ we get $\sigma_1$. The above sum is nonzero only when $\sigma_1$ and $\sigma_2$ are related in this way. The deltas only fix $\psi_2(1)$, so summing over $S_m$ the remaining "unfixed" piece of $\psi_2$ gives $(m-1)!$. The first delta will, as usual, give the norm of Gauss graph $\sigma_1$ and we will get a nonzero contribution whenever $\psi_2(1)$ is one of the values in $S^2_a$. There are $n_{aa}(\sigma_2)$ possible values. Thus, when $T$ is nonzero it takes the value

$$T = (m-1)!n_{aa}(\sigma_2)|\mathcal{O}(\sigma_1)|^2, \tag{B.17}$$

where we have assumed that both $\sigma_1$ and $\sigma_2$ have a total of $p$ nodes and we denote the number of oriented line segments stretching from node $k$ to node $l$ of $\sigma$ by $n_{kl}(\sigma)$. We have denoted the "norm" of the Gauss graph operator by $|\mathcal{O}(\sigma_1)|^2$ . This is the value of the two point function of the Gauss graph operator

$$|\mathcal{O}(\sigma_1)|^2 = \prod_{i=1}^{p} n_{ii}(\sigma_1)! \prod_{k,l=1,l\neq k}^{p} n_{kl}(\sigma_1)! = \langle \mathcal{O}^\dagger_{R,r}(\sigma_1)\mathcal{O}_{R,r}(\sigma_1)\rangle. \qquad \text{(B.18)}$$

The value of the trace (B.17) is in perfect agreement with the known result [78].

### B.3.2 Second evaluation

For the second example we consider, we choose

$$A = E^{(1)}_{ki}E^{(2)}_{ij} \qquad B = E^{(1)}_{jl}E^{(2)}_{lk}. \qquad \text{(B.19)}$$

There is some freedom in the placement of the indices on $A$ and $B$. To see why this is the case, recall that we are evaluating the Fourier transform of

$$\mathrm{Tr}(Ap_{s\mu\nu}Bp_{u\gamma\delta}). \qquad \text{(B.20)}$$

The intertwining maps $p_{s\mu\nu}$ and $p_{u\gamma\delta}$ commute with any element of $S_m$. Consequently we have

$$\mathrm{Tr}(Ap_{s\mu\nu}Bp_{u\gamma\delta}) = \mathrm{Tr}(A\sigma\sigma^{-1}p_{s\mu\nu}Bp_{u\gamma\delta}) = \mathrm{Tr}(A\sigma p_{s\mu\nu}\sigma^{-1}Bp_{u\gamma\delta}), \qquad \text{(B.21)}$$

where $\sigma$ is any element in $S_m$. Choosing $\sigma = (1,2)$ and using the representation $(1,2) = E^{(1)}_{ab}E^{(2)}_{ba}$ where $a$ and $b$ are summed from 1 to $p$, as well as the product rule

$$E^A_{ab}E^B_{cd} = E^B_{cd}E^A_{ab} \qquad A \neq B \qquad E^A_{ab}E^A_{cd} = \delta_{bc}E^A_{ad}, \qquad \text{(B.22)}$$

we find

$$A(12) = E^{(1)}_{ki}E^{(2)}_{ij}(1,2) = E^{(1)}_{ki}E^{(2)}_{ij}E^{(1)}_{ab}E^{(2)}_{ba} = E^{(1)}_{kj}E^{(2)}_{ii} \qquad \text{(B.23)}$$

$$(1,2)B = (1,2)E^{(1)}_{jl}E^{(2)}_{lk} = E^{(1)}_{ab}E^{(2)}_{ba}E^{(1)}_{jl}E^{(2)}_{lk} = E^{(1)}_{ll}E^{(2)}_{jk}. \qquad \text{(B.24)}$$

This implies that we can rather consider $A = E^{(1)}_{kj}E^{(2)}_{ii}$ and $B = E^{(1)}_{ll}E^{(2)}_{jk}$ without changing the value of the trace. In this case we have (argue as we did above and use $|v_2\rangle = E^{(1)}_{kj}|v_1\rangle$)

$$\begin{aligned} T &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m} \langle v_2|\,\sigma_2^{-1}E^{\psi_2(1)}_{kj}E^{\psi_2(2)}_{ii}\psi_1\,|v_1\rangle\,\langle v_2|\,E^{\psi_2(1)}_{ll}E^{\psi_2(2)}_{kj}\psi_1\sigma_1\,|v_1\rangle \\ &= \frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m} \langle v_1|\,\psi_1^{-1}E^{\psi_2(1)}_{jk}E^{\psi_2(2)}_{ii}\sigma_2\,|v_2\rangle\,\langle v_2|\,E^{\psi_2(1)}_{ll}E^{\psi_2(2)}_{kj}\psi_1\sigma_1\,|v_1\rangle\,. \end{aligned} \qquad \text{(B.25)}$$

Notice that when $E_{ii}^{\psi_2(2)}$ acts on $|v_2\rangle$, it does not change the identity of any of the vectors appearing in $|v_2\rangle$. On the other hand, $E_{jk}^{\psi_2(1)}$ will turn an $e_k$ into an $e_j$. Thus, again up to an arbitrary permutation which we can always remove, we must have

$$|v_2\rangle = E_{kj}^{(1)} |v_1\rangle . \tag{B.26}$$

The trace now takes the value

$$\begin{aligned}
T &= \frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \langle v_1| \psi_1^{-1} E_{jk}^{\psi_2(1)} E_{ii}^{\psi_2(2)} \sigma_2 E_{kj}^{(1)} |v_1\rangle \langle v_1| E_{jk}^{(1)} E_{ll}^{\psi_2(1)} E_{kj}^{\psi_2(2)} \psi_1 \sigma_1 |v_1\rangle \\
&= \frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \langle v_1| \psi_1^{-1} \sigma_2 E_{jk}^{\sigma_2\psi_2(1)} E_{ii}^{\sigma_2\psi_2(2)} E_{kj}^{(1)} |v_1\rangle \langle v_1| E_{jk}^{(1)} E_{ll}^{\psi_2(1)} E_{kj}^{\psi_2(2)} \psi_1 \sigma_1 |v_1\rangle \\
&= \frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \sum_{\gamma_i} \delta\Big(\gamma_1 \psi_1^{-1} \sigma_2 (\sigma_2 \psi_2(1), 1)\Big) \delta\Big(\gamma_2 \sigma_1^{-1} \psi_1^{-1} (\psi_2(2), 1)\Big) \\
&\quad \times \sum_{x \in S_l^2} \delta(x, \psi_2(1)) \sum_{y \in S_k^2} \delta(y, \psi_2(2)) \sum_{w \in S_k^2} \delta(w, \sigma_2\psi_2(1)) \sum_{v \in S_i^2} \delta(v, \sigma_2\psi_2(2)) \\
&= \frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \sum_{\gamma_i} \delta\Big(\gamma_1 \sigma_1 \gamma_2 (1, \psi_2(2)) \sigma_2 (\sigma_2 \psi_2(1), 1)\Big) \\
&\quad \times \sum_{x \in S_l^2} \delta(x, \psi_2(1)) \sum_{y \in S_k^2} \delta(y, \psi_2(2)) \sum_{w \in S_k^2} \delta(w, \sigma_2\psi_2(1)) \sum_{v \in S_i^2} \delta(v, \sigma_2\psi_2(2)) \\
&= \frac{(m-2)!}{|H_1||H_2|} n_{lk}(\sigma_2) n_{ki}(\sigma_2) \prod_{q=1}^{p} n_{qq}(\sigma_1)! \prod_{r,s=1, r\neq s}^{p} n_{rs}(\sigma_1)!,
\end{aligned} \tag{B.27}$$

whenever it is nonzero.

## B.4 General result

Recall that both $T$ and $R$ have $p$ long rows or columns. For the general result we consider

$$\begin{aligned}
&\sum_{s\mu\nu lm} \sum_{u\gamma\delta np} \mathrm{Tr}(A p_{s\mu\nu}) B p_{u\delta\gamma} B_{m\mu}^{s\to 1_H} B_{l\nu}^{s\to 1_H} \Gamma_{ml}^{(s)}(\sigma_1) B_{p\gamma}^{u\to 1_H} B_{n\delta}^{u\to 1_H} \Gamma_{pn}^{(u)}(\sigma_2) \\
&\qquad = \frac{1}{|H_1||H_2|} \sum_{\psi_i \in S_m} \langle v_2| \sigma_2^{-1} (\psi_2^{-1} A \psi_2) \psi_1 |v_1\rangle \langle v_2| (\psi_2^{-1} B^T \psi_2) \psi_1 \sigma_1 |v_1\rangle ,
\end{aligned} \tag{B.28}$$

where $A$ and $B$ are each products of a collection of the $E_{ab}^{(\alpha)}$s with $1 \leq a$, $b \leq p$ and $1 \leq \alpha \leq m$. We say that $E_{ab}^{(\alpha)}$ occupies slot $\alpha$. The sum over $\psi_2$ sums over the possible choices for the slots into which we place the factors of $E_{ab}^{(\alpha)}$ in $A$ and $B$. Thus, the specific slots chosen for the factors in $A$ and

$B$ are arbitrary–we must simply respect the relative ordering of factors in $A$ and $B$, i.e factors sharing the same slot in one labelling share the same slot in all labellings. The sum over $\psi_1$ ensures that the relative labelling of vectors appearing in $|v_1\rangle$ and $|v_2\rangle$ is arbitrary. Thus, the specific labelling of the directed edges in the Gauss graph is arbitrary which ensures that the above sum is indeed defined on the relevant double cosets to which $\sigma_1$ and $\sigma_2$ belong. There are two pieces of information that we need to read from $\sigma_1$, $\sigma_2$, $A$ and $B$:

(1) When is the sum nonzero?

(2) What is the value of the sum?

It is simplest to begin with the second question first. Towards this end, consider the expressions for $A$ and $B$. After using the algebra for the $E_{ab}^{(\alpha)}$ if needed, we know that at most a single $E_{ab}^{(\alpha)}$ acts per slot in both $A$ and $B$. By inserting factors of

$$\sum_{a=1}^{p} E_{aa}^{(\alpha)} = \mathbb{1}, \tag{B.29}$$

as necessary, we can ensure that the same set of occupied slots appears in $A$ and $B$. For concreteness, assume that $q$ slots are occupied in both. Use $i_\alpha^r(i_\alpha^c)$ to denote the row(column) indices of the $E_{ab}$ in the $\alpha$th slot in $B$ and use $j_\alpha^r(j_\alpha^c)$ to denote the row(column) indices of $E_{ab}$ in the $\alpha$th slot in $A$. Thanks to the lessons we have learned from the examples treated above, when the sum is nonzero it is given by

$$\frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\,\sigma_2^{-1}(\psi_2^{-1}A\psi_2)\psi_1\,|v_1\rangle\,\langle v_2|\,(\psi_2^{-1}B^T\psi_2)\psi_1\sigma_1\,|v_1\rangle$$
$$= \frac{(m-q)!|\mathcal{O}(\sigma_1)|^2}{|H_1||H_2|}\prod_{\alpha=1}^{q} n_{i_\alpha^c j_\alpha^r}(\sigma_2). \tag{B.30}$$

If any particular $n_{ij}(\sigma_2)$ appears more than once, each new factor in the product is to be reduced by 1. For more example, $n_{12}(\sigma_2)^3$ would be replaced by $n_{12}(\sigma_2)(n_{12}(\sigma_2)-1)(n_{12}(\sigma_2)-2)$. By taking the transpose of (B.30), the value of the sum is not changed because it is a real number. However, the roles of $\sigma_1$ and $\sigma_2$, as well as of $A$ and $B$ are reversed on the left-hand side of (B.30). Consequently, we must also have

$$\frac{1}{|H_1||H_2|}\sum_{\psi_i\in S_m}\langle v_2|\,\sigma_2^{-1}(\psi_2^{-1}A\psi_2)\psi_1\,|v_1\rangle\,\langle v_2|\,(\psi_2^{-1}B^T\psi_2)\psi_1\sigma_1\,|v_1\rangle$$
$$= \frac{(m-q)!|\mathcal{O}(\sigma_2)|^2}{|H_1||H_2|}\prod_{\alpha=1}^{q} n_{j_\alpha^c i_\alpha^r}(\sigma_1). \tag{B.31}$$

The equality of (B.30) and (B.31) defines our delta function. We find that $\delta_{AB}([\sigma_1][\sigma_2]) = 1$ if

$$\frac{(m-q)!|\mathcal{O}(\sigma_1)|^2}{|H_1||H_2|}\prod_{\alpha=1}^{q} n_{i^c_\alpha j^r_\alpha}(\sigma_2) = \frac{(m-q)!|\mathcal{O}(\sigma_2)|^2}{|H_1||H_2|}\prod_{\alpha=1}^{q} n_{j^c_\alpha i^r_\alpha}(\sigma_1), \quad \text{(B.32)}$$

and it is zero otherwise.

We will sketch how the general result is proved. First, even if $A$ and $B$ straddle $qlm$ slots, by using (B.29) we can always introduce further $E^{(\alpha)}$s so that all $m$ slots are straddled. Thus, without loss of generality we can now focus on the $q = m$ case. In this case, it is easy to prove that if $\prod_{\alpha=1}^{q} n_{i^c_\alpha j^r_\alpha}(\sigma_2)$ is nonzero, it is given by

$$\prod_{\alpha=1}^{q} n_{i^c_\alpha j^r_\alpha}(\sigma_2) = |\mathcal{O}(\sigma_2)|^2, \quad \text{(B.33)}$$

which proves the result. Similarly, if $\prod_{\alpha=1}^{q} n_{j^c_\alpha i^r_\alpha}(\sigma_1)$ is nonzero, it is equal to

$$\prod_{\alpha=1}^{q} n_{j^c_\alpha i^r_\alpha}(\sigma_1) = |\mathcal{O}(\sigma_1)|^2, \quad \text{(B.34)}$$

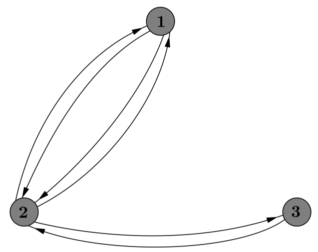
which again proves the general result.

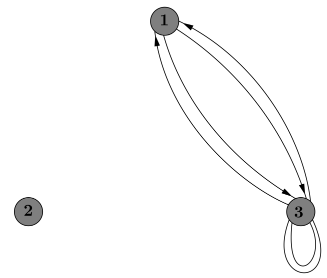
## B.5 Illustration of the general result

To illustrate the formula derived in the previous section consider computing the trace for the case that

$$\begin{aligned} A &= E^{(1)}_{31}E^{(2)}_{11}E^{(3)}_{32}E^{(4)}_{12}E^{(5)}_{32}E^{(6)}_{33} \\ B &= E^{(1)}_{23}E^{(2)}_{23}E^{(3)}_{11}E^{(4)}_{13}E^{(5)}_{31}E^{(6)}_{23}. \end{aligned} \quad \text{(B.35)}$$

For our example we have $m = 6$, $q = 6$ and $p = 3$ so that $(m-q)! = 1$. We choose $\sigma_1$ and $\sigma_2$ as illustrated below

$\sigma_1 =$ 


$\sigma_2 =$ 

From these Gauss graphs we easily read off $H_1 = S_2 \times S_3$ and $H_2 = S_4 \times S_2$. Consequently, $|H_1| = 12$ and $|H_2| = 48$. If we choose the permutation

$$\sigma_1 = (13)(24)(56) \in H_1 \setminus S_6/H_1, \tag{B.36}$$

to represent the first Gauss graph, then we can choose the first factor in $H_1$ to permute 1 and 2 and the second factor to permute 3, 4 and 5. If we choose the permutation

$$\sigma_2 = (12)(45), \tag{B.37}$$

to represent the second Gauss graph, then we can choose the first factor to permute 2 and 5 and the last factor to permute 1, 3, 4 and 6.

From the row indices of $A$ given by the ordered set $\{3, 1, 3, 1, 3, 3\}$, and the column indices of $B$ given by the ordered set $\{3, 3, 1, 3, 1, 3\}$, we read off

$$\begin{aligned}\prod_{\alpha=1}^{q} n_{i^c_\alpha j^r_\alpha} &= n_{33}(\sigma_2)n_{31}(\sigma_2)n_{13}(\sigma_2)n_{31}(\sigma_2)n_{13}(\sigma_2)n_{33}(\sigma_2)\\ &\to n_{33}(\sigma_2)(n_{33}(\sigma_2)-1)n_{31}(\sigma_2)(n_{31}(\sigma_2)-1)n_{13}(\sigma_2)(n_{13}(\sigma_2)-1)\\ &= |\mathcal{O}(\sigma_2)|^2 = 8.\end{aligned} \tag{B.38}$$

which indicates that the sum may indeed be nonzero. From the column indices of $A$ given by the ordered set $\{1, 1, 2, 2, 2, 3\}$, and the row indices of $B$ given by the ordered set $\{2, 2, 1, 1, 3, 2\}$, we read off

$$\begin{aligned}\prod_{\alpha=1}^{q} n_{j^c_\alpha i^r_\alpha} &= n_{12}(\sigma_1)n_{12}(\sigma_1)n_{21}(\sigma_1)n_{21}(\sigma_1)n_{23}(\sigma_1)n_{32}(\sigma_1)\\ &\to n_{12}(\sigma_1)(n_{12}(\sigma_1)-1)n_{21}(\sigma_1)(n_{21}(\sigma_1)-1)n_{23}(\sigma_1)n_{32}(\sigma_1)\\ &= |\mathcal{O}(\sigma_1)|^2 = 4,\end{aligned} \tag{B.39}$$

which indicates that the sum is indeed nonzero. We finally obtain

$$\frac{1}{|H_1||H_2|}\sum_{\psi_i \in S_m} \langle v_2|\,\sigma_2^{-1}(\psi_2^{-1}A\psi_2)\psi_1\,|v_1\rangle\,\langle v_2|\,(\psi_2^{-1}B^T\psi_2)\psi_1\sigma_1\,|v_1\rangle = \frac{32}{|H_1||H_2|} = \frac{1}{18}. \tag{B.40}$$

## B.6 Results for the 2-brane 4-string system

In this section we collect the operator valued coefficients that are relevant for the example described in Section 5.3. The coefficients are

$$
\begin{aligned}
A = {}& \delta_{RT}\Big(\frac{\delta_{r_1' t_1'}}{R_2} + \frac{\delta_{r_2' t_2'}}{R_1}\Big)\delta_{R_{12}'' T_{12}''}(N+R_1-1)(N+R_2-2) \\
& + \delta_{RT}\frac{\delta_{r_1' t_1'}}{R_1}\delta_{R_{11}'' T_{11}''}(N+R_1-1)(N+R_1-2) \\
& + \delta_{RT}\frac{\delta_{r_2' t_2'}}{R_2}\delta_{R_{22}'' T_{22}''}(N+R_2-2)(N+R_2-3) - 2\delta_{RT}\delta_{r_1' t_1'}\delta_{R_1' T_1'}(N+R_1-1) \\
& - 2\delta_{RT}\delta_{r_2' t_2'}\delta_{R_2' T_2'}(N+R_2-2) \\
& - \delta_{R_1' T_2'}\frac{\delta_{r_1' t_2'}}{\sqrt{R_1(R_1-1)}}\delta_{R_{11}'' T_{12}''}(N+R_2-2)\sqrt{(N+R_1-1)(N+R_2-1)} \\
& - \delta_{R_2' T_1'}\frac{\delta_{r_2' t_1'}}{\sqrt{R_1(R_1+1)}}\delta_{R_{12}'' T_{11}''}(N+R_1-1)\sqrt{(N+R_1)(N+R_2-2)} \\
& - \delta_{R_2' T_1'}\frac{\delta_{r_2' t_1'}}{\sqrt{R_2(R_2-1)}}\delta_{R_{22}'' T_{12}''}(N+R_2-3)\sqrt{(N+R_1)(N+R_2-2)} \\
& - \delta_{R_1' T_2'}\frac{\delta_{r_1' t_2'}}{\sqrt{R_2(R_2-1)}}\delta_{R_{12}'' T_{22}''}(N+R_2-2)\sqrt{(N+R_1-1)(N+R_2-1)} \\
& + 2\delta_{r_2' t_1'}\delta_{R_2' T_1'}\sqrt{(N+R_1)(N+R_2-2)} + 2\delta_{r_1' t_2'}\delta_{R_1' T_2'}\sqrt{(N+R_1-1)(N+R_2-1)}.
\end{aligned} \tag{B.41}
$$

$$
B = -8\delta_{r_2' t_1'}\delta_{RT}\delta_{R_{12}'' T_{12}''}\sqrt{\frac{(N+R_1-1)(N+R_2-1)}{R_1 R_2}}. \tag{B.42}
$$

$$
\begin{aligned}
C = {}& 8\frac{\delta_{r_2' t_1'}}{\sqrt{R_1(R_2+2)}}\delta_{R_{11}'' T_{22}''}\sqrt{(N+R_1-1)(N+R_1-2)(N+R_2-1)(N+R_2)} \\
& - 8\delta_{R_1' T_2'}\frac{\delta_{r_1' t_1'}}{\sqrt{R_1(R_2+1)}}\delta_{R_{11}'' T_{12}''}(N+R_1-2)\sqrt{(N+R_1-1)(N+R_2-1)} \\
& - 8\delta_{R_1' T_2'}\frac{\delta_{r_2' t_2'}}{\sqrt{R_1(R_2+1)}}\delta_{R_{12}'' T_{22}''}(N+R_2-2)\sqrt{(N+R_1-1)(N+R_2-1)}.
\end{aligned} \tag{B.43}
$$

$$
\begin{aligned}
C^{\dagger} = {} & 8\frac{\delta_{r'_2 t'_1}}{\sqrt{R_2(R_1+2)}}\delta_{R''_{22}T''_{11}}\sqrt{(N+R_1)(N+R_1+1)(N+R_2-2)(N+R_2-3)} \\
& - 8\delta_{R'_2 T'_1}\frac{\delta_{r'_1 t'_1}}{\sqrt{R_2(R_1+1)}}\delta_{R''_{12}T''_{11}}(N+R_1-1)\sqrt{(N+R_1)(N+R_2-2)} \\
& - 8\delta_{R'_2 T'_1}\frac{\delta_{r'_2 t'_2}}{\sqrt{R_2(R_1+1)}}\delta_{R''_{22}T''_{12}}(N+R_2-3)\sqrt{(N+R_1)(N+R_2-2)}.
\end{aligned}
\tag{B.44}
$$

# Appendix C

# One loop dilatation operator in the $SU(3)$ sector

## C.1 Evaluation of the action of the dilatation operator in the restricted Schur basis

The one loop dilatation operator in this sector is given by

$$\begin{aligned} D_2 = & -g_{YM}^2 \operatorname{Tr}\Big([Y,Z][Y^\dagger,Z^\dagger]\Big) - g_{YM}^2 \operatorname{Tr}\Big([X,Z][X^\dagger,Z^\dagger]\Big) \\ & - g_{YM}^2 \operatorname{Tr}\Big([Y,X][Y^\dagger,X^\dagger]\Big). \end{aligned} \tag{C.1}$$

Acting with the operator (C.1) on the restricted Schur polynomial yields

$$D_2\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) = -g_{YM}^2(D_2^{YZ} + D_2^{XZ} + D_2^{XY})\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X),$$

where

$$D_2^{YZ}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) \equiv \operatorname{Tr}\Big([Y,Z][Y^\dagger,Z^\dagger]\Big)\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) \tag{C.2}$$

$$D_2^{XZ}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) \equiv \operatorname{Tr}\Big([X,Z][X^\dagger,Z^\dagger]\Big)\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X, \tag{C.3}$$

$$D_2^{XY}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) \equiv \operatorname{Tr}\Big([Y,X][Y^\dagger,X^\dagger]\Big)\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X). \tag{C.4}$$

The leading order terms $D_2^{YZ}$ and $D_2^{XZ}$ mixing $X,Z$ and $Y,Z$ were computed in [30][28] are diagonalized in the Gauss graph basis. We consider a limit in which, there are more $Z$s than $X$s and there are more $Z$s than $Y$s. $D_2^{XY}$ is a subleading term which mixes $X$ and $Y$ fields, where both are of order $\mathcal{O}(1)$. This term was studied in [70]. We will evaluate the action of this term in detail. Towards this end, consider

$$
\begin{aligned}
&D_2^{XY}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X)\\
&=\mathrm{Tr}\Big((YX-XY)(Y^\dagger X^\dagger - X^\dagger Y^\dagger)\Big)\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X)\\
&=(YX-XY)^a_b\Big((Y^\dagger)^b_c(X^\dagger)^c_a-(X^\dagger)^b_c(Y^\dagger)^c_a\Big)\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X)\\
&=\frac{1}{p!m!n!}\sum_{\sigma\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)(YX-XY)^a_b\Bigg[(Y^\dagger)^b_c(X^\dagger)^c_a X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}-(X^\dagger)^b_c(Y^\dagger)^c_a X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\Bigg]\\
&\quad\times Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}\tag{C.5}
$$

Wick contracting the daggered matrices $X^\dagger$ and $Y^\dagger$ with $X$ and $Y$ respectively, on the right hand side of equation (C.5) gives

$$
\begin{aligned}
&D_2^{XY}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X)\\
&=\frac{1}{p!m!n!}\sum_{\sigma\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)(YX-XY)^a_b\Bigg[\Big(\delta^c_{i_{\sigma(1)}}\delta^{i_1}_a X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}\\
&\quad+X^{i_1}_{i_{\sigma(1)}}\delta^c_{i_{\sigma(2)}}\delta^{i_2}_a X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\delta^c_{i_{\sigma(p)}}\delta^{i_p}_a\Big)(Y^\dagger)^b_c\\
&\quad\times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}-\Big(\delta^b_{i_{\sigma(1)}}\delta^{i_1}_c X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}+X^{i_1}_{i_{\sigma(1)}}\delta^b_{i_{\sigma(2)}}\delta^{i_2}_c X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}\\
&\quad+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\delta^b_{i_{\sigma(p)}}\delta^{i_p}_c\Big)(Y^\dagger)^c_a Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\Bigg]Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&=\frac{1}{p!m!n!}\sum_{\sigma\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)(YX-XY)^a_b\Bigg[\Big(\delta^c_{i_{\sigma(1)}}\delta^{i_1}_a X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}\\
&\quad+X^{i_1}_{i_{\sigma(1)}}\delta^c_{i_{\sigma(2)}}\delta^{i_2}_a X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\delta^c_{i_{\sigma(p)}}\delta^{i_p}_a\Big)\\
&\quad\times\Big(\delta^b_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_c Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\delta^b_{i_{\sigma(p+2)}}\delta^{i_{p+2}}_c Y^{i_{p+3}}_{i_{\sigma(p+3)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\\
&\quad+\cdots+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^b_{i_{\sigma(p)}}\delta^{i_p}_c\Big)-\Big(\delta^b_{i_{\sigma(1)}}\delta^{i_1}_c X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}\\
&\quad+X^{i_1}_{i_{\sigma(1)}}\delta^b_{i_{\sigma(2)}}\delta^{i_2}_c X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}+\cdots+X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_{p-1}}_{i_{\sigma(p-1)}}\delta^b_{i_{\sigma(p)}}\delta^{i_p}_c\Big)\\
&\quad\times\Big(\delta^c_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_a Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\delta^c_{i_{\sigma(p+2)}}\delta^{i_{p+2}}_a Y^{i_{p+3}}_{i_{\sigma(p+3)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\\
&\quad+\cdots+Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m-1}}_{i_{\sigma(p+m-1)}}\delta^c_{i_{\sigma(p)}}\delta^{i_p}_a\Big)\Bigg]Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}\tag{C.6}
$$

Our next goal is to move the delta functions in equation (C.6) to slot 1. Recall that $p$ and $m$ are the number of slots associated with $X$ and $Y$ fields respectively in the restricted Schur polynomial. Consider the first term in (C.6) which we further simply as

$$
\begin{aligned}
&\frac{1}{p!m!n!}\sum_{\sigma\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)(YX-XY)^a_b\Big(\delta^c_{i_{\sigma(1)}}\delta^{i_1}_a\delta^b_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_c-\delta^b_{i_{\sigma(1)}}\delta^{i_1}_c\delta^c_{i_{\sigma(p+1)}}\delta^{i_{p+1}}_a\Big)\\
&\quad\times X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&\quad=\frac{1}{p!m!n!}\sum_{\sigma\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)\Big(\delta^{i_{p+1}}_{i_{\sigma(1)}}(YX-XY)^{i_1}_{i_{\sigma(p+1)}}-\delta^{i_1}_{i_{\sigma(p+1)}}(YX-XY)^{i_{p+1}}_{i_{\sigma(1)}}\Big)\\
&\quad\times X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+2}}_{i_{\sigma(p+2)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}\tag{C.7}
$$

Every other term in equation (C.6) can be simplified to obtain the result in (C.7). We demonstrate this fact for the second term in (C.6) as follows

$$
\begin{aligned}
&\frac{1}{p!m!n!}\sum_{\sigma\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)(YX-XY)^a_b\Big(X^{i_1}_{i_{\sigma(1)}}\delta^c_{i_{\sigma(2)}}\delta^{i_2}_a Y^{i_{p+1}}_{i_{\sigma(p+1)}}\delta^b_{i_{\sigma(p+2)}}\delta^{i_{p+2}}_c\\
&\quad-X^{i_1}_{i_{\sigma(1)}}\delta^b_{i_{\sigma(2)}}\delta^{i_2}_c Y^{i_{p+1}}_{i_{\sigma(p+1)}}\delta^c_{i_{\sigma(p+2)}}\delta^{i_{p+2}}_a\Big)X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+3}}_{i_{\sigma(p+3)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\\
&\quad\times Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}\\
&\quad=\frac{1}{p!m!n!}\sum_{\sigma\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)\Big(X^{i_1}_{i_{\sigma(1)}}\delta^{i_{p+2}}_{i_{\sigma(2)}}(YX-XY)^{i_2}_{i_{\sigma(p+2)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\\
&\quad-X^{i_1}_{i_{\sigma(1)}}\delta^{i_2}_{i_{\sigma(p+2)}}(YX-XY)^{i_{p+2}}_{i_{\sigma(2)}}Y^{i_{p+1}}_{i_{\sigma(p+1)}}\Big)X^{i_3}_{i_{\sigma(3)}}\cdots X^{i_p}_{i_{\sigma(p)}}Y^{i_{p+3}}_{i_{\sigma(p+3)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}\\
&\quad\times Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}.
\end{aligned}\tag{C.8}
$$

Let $\sigma=(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)$ for $\sigma\in S_{p+m+n}$. Summing over $\tau\in S_{p+m+n}$ is replaced by summing over $\sigma\in S_{p+m+n}$. In the transformation $\sigma\to\gamma\tau\gamma^{-1}$ $\sigma,\tau\in S_{p+m+n}$ and $\gamma\in S_p\times S_m$. Since

$$\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma)=\mathrm{Tr}_{(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\Gamma_R(\sigma)),$$

we then have

$$\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\gamma\sigma\gamma^{-1})=\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\sigma),$$

for any $\gamma \in S_p \times S_m$. Thus, we can write equation (C.8) as

$$\begin{aligned}
&\frac{1}{p!m!n!}\sum_{\tau\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}((p+1,p+2)(1,2)\tau(1,2)(p+1,p+2))\\
&\quad\times\Big(X^{i_1}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(1)}}\delta^{i_{p+2}}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(2)}}\\
&\quad\times(YX-XY)^{i_2}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(p+2)}}Y^{i_{p+1}}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(p+1)}}\\
&\quad-X^{i_1}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(1)}}\delta^{i_2}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(p+2)}}\\
&\quad\times(YX-XY)^{i_{p+2}}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(2)}}Y^{i_{p+1}}_{i_{(p+1,p+2)(1,2)\tau(1,2)(p+1,p+2)(p+1)}}\Big)\\
&\quad\times X^{i_3}_{i_{\tau(3)}}\cdots X^{i_p}_{i_{\tau(p)}}Y^{i_{p+3}}_{i_{\tau(p+3)}}\cdots Y^{i_{p+m}}_{i_{\tau(p+m)}}Z^{i_{p+m+1}}_{i_{\tau(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\tau(p+m+n)}}\\
&=\frac{1}{p!m!n!}\sum_{\tau\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\tau)\Big(X^{i_{(1,2)(p+1,p+2)(1)}}_{i_{\tau(1,2)(p+1,p+2)(1)}}\delta^{i_{(1,2)(p+1,p+2)(p+2)}}_{i_{\tau(1,2)(p+1,p+2)(2)}}\\
&\quad\times(YX-XY)^{i_{(1,2)(p+1,p+2)(2)}}_{i_{\tau(1,2)(p+1,p+2)(p+2)}}Y^{i_{(1,2)(p+1,p+2)(p+1)}}_{i_{\tau(1,2)(p+1,p+2)(p+1)}}-X^{i_{(1,2)(p+1,p+2)(1)}}_{i_{\tau(1,2)(p+1,p+2)(1)}}\\
&\quad\times\delta^{i_{(1,2)(p+1,p+2)(2)}}_{i_{\tau(1,2)(p+1,p+2)(p+2)}}(YX-XY)^{i_{(1,2)(p+1,p+2)(p+2)}}_{i_{\tau(1,2)(p+1,p+2)(2)}}Y^{i_{(1,2)(p+1,p+2)(p+1)}}_{i_{\tau(1,2)(p+1,p+2)(p+1)}}\Big)\\
&\quad\times X^{i_3}_{i_{\tau(3)}}\cdots X^{i_p}_{i_{\tau(p)}}Y^{i_{p+3}}_{i_{\tau(p+3)}}\cdots Y^{i_{p+m}}_{i_{\tau(p+m)}}Z^{i_{p+m+1}}_{i_{\tau(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\tau(p+m+n)}}\\
&=\frac{1}{p!m!n!}\sum_{\tau\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\tau)\Big(\delta^{i_{p+1}}_{i_{\tau(1)}}(YX-XY)^{i_1}_{i_{\tau(p+1)}}-\delta^{i_1}_{i_{\tau(p+1)}}(YX-XY)^{i_{p+1}}_{i_{\tau(1)}}\Big)\\
&\quad\times X^{i_2}_{i_{\tau(2)}}\cdots X^{i_p}_{i_{\tau(p)}}Y^{i_{p+2}}_{i_{\tau(p+2)}}\cdots Y^{i_{p+m}}_{i_{\tau(p+m)}}Z^{i_{p+m+1}}_{i_{\tau(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\tau(p+m+n)}}. \qquad \text{(C.9)}
\end{aligned}$$

Comparing equations (C.7) and (C.9), the result follows. Each of the $pm$ terms in equation (C.6) are thus identical and so (C.6) becomes

$$\begin{aligned}
&D_2^{XY}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X)\\
&=\frac{pm}{p!m!n!}\sum_{\tau\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\tau)\Big(\delta^{i_{p+1}}_{i_{\tau(1)}}(YX-XY)^{i_1}_{i_{\tau(p+1)}}-\delta^{i_1}_{i_{\tau(p+1)}}(YX-XY)^{i_{p+1}}_{i_{\tau(1)}}\Big)\\
&\quad\times X^{i_2}_{i_{\tau(2)}}\cdots X^{i_p}_{i_{\tau(p)}}Y^{i_{p+2}}_{i_{\tau(p+2)}}\cdots Y^{i_{p+m}}_{i_{\tau(p+m)}}Z^{i_{p+m+1}}_{i_{\tau(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\tau(p+m+n)}}\\
&=\frac{pm}{p!m!n!}\sum_{\psi\in S_{p+m+n}}\Big(\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}((1,p+1)\psi)-\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(\psi(1,p+1))\Big)\delta^{i_1}_{i_{\psi(1)}}\\
&\quad\times X^{i_2}_{i_{\psi(2)}}\cdots X^{i_p}_{i_{\psi(p)}}(YX-XY)^{i_{p+1}}_{i_{\psi(p+1)}}Y^{i_{p+2}}_{i_{\psi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\psi(p+m)}}Z^{i_{p+m+1}}_{i_{\psi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\psi(p+m+n)}}\\
&=\frac{pm}{p!m!n!}\sum_{\psi\in S_{p+m+n}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}([(1,p+1),\psi])\delta^{i_1}_{i_{\psi(1)}}X^{i_2}_{i_{\psi(2)}}\cdots X^{i_p}_{i_{\psi(p)}}\\
&\quad\times(YX-XY)^{i_{p+1}}_{i_{\psi(p+1)}}Y^{i_{p+2}}_{i_{\psi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\psi(p+m)}}Z^{i_{p+m+1}}_{i_{\psi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\psi(p+m+n)}}. \qquad \text{(C.10)}
\end{aligned}$$

To go from line 1 to line 2 consider the term $\delta^{i_{p+1}}_{i_{\tau(1)}}(YX-XY)^{i_1}_{i_{\tau(p+1)}}$ with $\delta^{i_{p+1}}_{i_1}$ kept in the $1^{st}$ slot and $(YX-XY)^{i_1}_{i_{\tau(p+1)}}$ is shifted to the $(p+1)^{th}$

slot by $\tau \to (1,p+1)\psi$. Similarly, for the term $\delta^{i_1}_{i_{\tau(p+1)}}(YX-XY)^{i_{p+1}}_{i_{\tau(1)}}$ we change variables $\tau \to \psi(1,p+1)$. The delta function $\delta^{i_1}_{i_{\psi(1)}}$ reduces the sum over $\psi \in S_{p+m+n}$ to a sum over $\psi \in S_{p+m+n-1}$ since it fixes $\psi(1)=1$. Now apply the reduction rule [11] by using the coset decomposition

$$\begin{aligned} S_{p+m+n} = \mathbb{1}S_{p+m+n-1} \oplus (1,2)S_{p+m+n-1} \oplus (1,3)S_{p+m+n-1} \oplus \cdots \\ \oplus (1,p+m+n-1)S_{p+m+n-1}. \end{aligned} \tag{C.11}$$

Plugging the above identity into equation (C.10) we have

$$\begin{aligned}
&D_2^{XY}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) \\
&= \frac{pm}{p!m!n!}\sum_{\phi\in S_{p+m+n-1}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}([(1,p+1),\phi])\delta^{i_1}_{i_{\phi(1)}}X^{i_2}_{i_{\phi(2)}}\cdots X^{i_p}_{i_{\phi(p)}} \\
&\quad\times(YX-XY)^{i_{p+1}}_{i_{\phi(p+1)}}Y^{i_{p+2}}_{i_{\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\phi(p+m)}}Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}} \\
&\quad+\frac{pm}{p!m!n!}\sum_{\phi\in S_{p+m+n-1}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}([(1,p+1),(1,2)\phi])\delta^{i_1}_{i_{(1,2)\phi(1)}}X^{i_2}_{i_{(1,2)\phi(2)}}\cdots X^{i_p}_{i_{(1,2)\phi(p)}} \\
&\quad\times(YX-XY)^{i_{p+1}}_{i_{(1,2)\phi(p+1)}}Y^{i_{p+2}}_{i_{(1,2)\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{(1,2)\phi(p+m)}}Z^{i_{p+m+1}}_{i_{(1,2)\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{(1,2)\phi(p+m+n)}} \\
&\quad+\cdots+\frac{pm}{p!m!n!}\sum_{\phi\in S_{p+m+n-1}}\chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}([(1,p+1),(1,p+m+n)\phi])\delta^{i_1}_{i_{(1,p+m+n)\phi(1)}} \\
&\quad\times X^{i_2}_{i_{(1,p+m+n)\phi(2)}}\cdots X^{i_p}_{i_{(1,p+m+n)\phi(p)}}(YX-XY)^{i_{p+1}}_{i_{(1,p+m+n)\phi(p+1)}} \\
&\quad\times Y^{i_{p+2}}_{i_{(1,p+m+n)\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{(1,p+m+n)\phi(p+m)}}Z^{i_{p+m+1}}_{i_{(1,p+m+n)\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{(1,p+m+n)\phi(p+m+n)}} \\
&= \frac{pm}{p!m!n!}\sum_{\phi\in S_{p+m+n-1}}\mathrm{Tr}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Big(\Gamma_R\Big(\Big[(1,p+1),\Big(N+\sum_{i=2}^{p+m+n}(1,i)\Big)\phi\Big]\Big)\Big) \\
&\quad\times X^{i_2}_{i_{\phi(2)}}\cdots X^{i_p}_{i_{\phi(p)}}(YX-XY)^{i_{p+1}}_{i_{\phi(p+1)}}Y^{i_{p+2}}_{i_{\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\phi(p+m)}}Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}}.
\end{aligned} \tag{C.12}$$

The Jucys-Murphy elements $\sum_{i=2}^{p+m+n}(1,i)$ is

$$\begin{aligned} C_2^{p+m+n} - C_2^{p+m+n-1} &= \sum_{i<j,i=1}^{j=n+m+p}(i,j) - \sum_{i<j,i=2}^{j=n+m+p}(i,j) \\ &= (1,2)+(1,3)+\cdots+(1,p+m+n), \end{aligned}$$

where $C_2^{p+m+n}$ and $C_2^{p+m+n-1}$ are Casimirs of $S_{p+m+n}$ and $S_{p+m+n-1}$ respectively. The Casimir acts on a state belonging to a representation labelled by a Young diagram $R$ and has a value given by

$$C_2^{p+m+n}|R,a\rangle = \left(\sum_i \frac{r_i(r_i-1)}{2} - \sum_j \frac{c_j(c_j-1)}{2}\right)|R,a\rangle\,.$$

The row lengths of $R$ are denoted by $r_i$ and the column lengths of $R$ denoted by $c_j$ where $i$ is the row index and $j$ is the column index. $a$ distinguishes the elements in this basis. $R'$ is the Young diagram obtained by removing a box from $R$. Using the branching rule $R = \bigoplus_{R'} R'$, it is clear that

$$\begin{aligned}
\Gamma_R(N + C_2^{p+m+n}) &= (N + \lambda_R)\mathbb{1}_{dR\times dR} = \bigoplus_{R'}(N + \lambda_R)\mathbb{1}_{dR'\times dR'} \\
\Gamma_R(C_2^{p+m+n-1}) &= \lambda_{R'}\mathbb{1}_{dR\times dR} = \bigoplus_{R'} \lambda_{R'}\mathbb{1}_{dR'\times dR'},
\end{aligned}$$

where

$$\lambda_R = \sum_i \frac{r_i(r_i-1)}{2} - \sum_j \frac{c_j(c_j-1)}{2}, \quad \text{and} \quad \lambda_{R'} = \sum_{i'} \frac{r'_{i'}(r'_{i'}-1)}{2} - \sum_{j'} \frac{c'_{j'}(c'_{j'}-1)}{2}.$$

Let

$$\begin{aligned}
c_{RR'} &= N + \lambda_R - \lambda_{R'} \\
&= N + \left(\sum_i \frac{r_i(r_i-1)}{2} - \sum_{i'} \frac{r'_{i'}(r'_{i'}-1)}{2}\right) - \left(\sum_j \frac{c_j(c_j-1)}{2} - \sum_{j'} \frac{c'_{j'}(c'_{j'}-1)}{2}\right).
\end{aligned}$$

One can verify that $\lambda_R - \lambda_{R'}$ is the content of the box that is removed from $R$ to produce $R'$. Hence $c_{RR'}$ is the factor of the box that must be removed to go from $R$ to $R'$. We can write equation (C.12) as

$$\begin{aligned}
&D_2^{XY}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) \\
&= \frac{pm}{p!m!n!}\sum_{R'}\sum_{\phi\in S_{p+m+n-1}} c_{RR'}\,\mathrm{Tr}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Big([\Gamma_R(1,p+1),\Gamma_{R'}(\phi)]\Big) \\
&\quad\times X^{i_2}_{i_{\phi(2)}}\cdots X^{i_p}_{i_{\phi(p)}}(YX-XY)^{i_{p+1}}_{i_{\phi(p+1)}}Y^{i_{p+2}}_{i_{\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\phi(p+m)}}Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}}.
\end{aligned} \tag{C.13}$$

From equation (C.13) we rewrite

$$X^{i_2}_{i_{\phi(2)}}\cdots X^{i_p}_{i_{\phi(p)}}(YX-XY)^{i_{p+1}}_{i_{\phi(p+1)}}Y^{i_{p+2}}_{i_{\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\phi(p+m)}}Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}},$$

as a trace so that it can be written in terms of restricted Schur polynomials. Using the fact that $\phi(1) = 1$,

$$\begin{aligned}
&X^{i_2}_{i_{\phi(2)}}\cdots X^{i_p}_{i_{\phi(p)}}(YX)^{i_{p+1}}_{i_{\phi(p+1)}}Y^{i_{p+2}}_{i_{\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\phi(p+m)}}Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}} \\
&= X^{i_2}_{i_{\phi(2)}}\cdots X^{i_p}_{i_{\phi(p)}}Y^{i_{p+1}}_{i_{\phi(1)}}X^{i_1}_{i_{\phi(p+1)}}Y^{i_{p+2}}_{i_{\phi(p+2)}}\cdots Y^{i_{p+m}}_{i_{\phi(p+m)}}Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}} \\
&= X^{i_2}_{i_{\phi(1,p+1)(2)}}\cdots X^{i_p}_{i_{\phi(1,p+1)(p)}}Y^{i_{p+1}}_{i_{\phi(1,p+1)(1)}}X^{i_1}_{i_{\phi(1,p+1)(p+1)}}Y^{i_{p+2}}_{i_{\phi(1,p+1)(p+2)}}\cdots Y^{i_{p+m}}_{i_{\phi(1,p+1)(p+m)}} \\
&\quad\times Z^{i_{p+m+1}}_{i_{\phi(1,p+1)(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\phi(1,p+1)(p+m+n)}} \\
&= \mathrm{Tr}\Big(\phi(1,p+1)X^{\otimes p}Y^{\otimes m}Z^{\otimes n}\Big).
\end{aligned} \tag{C.14}$$

Note that we have made use of the transformation $\phi \to \phi(1,p+1)$. Similarly, let $\phi \to (1,p+1)\phi$ so that we have

$$
\begin{aligned}
&X^{i_2}_{i_{\phi(2)}} \cdots X^{i_p}_{i_{\phi(p)}} (XY)^{i_{p+1}}_{i_{\phi(p+1)}} Y^{i_{p+2}}_{i_{\phi(p+2)}} \cdots Y^{i_{p+m}}_{i_{\phi(p+m)}} Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}} \cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}} \\
&= X^{i_2}_{i_{\phi(2)}} \cdots X^{i_p}_{i_{\phi(p)}} X^{i_{p+1}}_{i_{\phi(1)}} Y^{i_1}_{i_{\phi(p+1)}} Y^{i_{p+2}}_{i_{\phi(p+2)}} \cdots Y^{i_{p+m}}_{i_{\phi(p+m)}} Z^{i_{p+m+1}}_{i_{\phi(p+m+1)}} \cdots Z^{i_{p+m+n}}_{i_{\phi(p+m+n)}} \\
&= X^{i_2}_{i_{(1,p+1)\phi(2)}} \cdots X^{i_p}_{i_{(1,p+1)\phi(p)}} X^{i_{p+1}}_{i_{(1,p+1)\phi(1)}} Y^{i_1}_{i_{(1,p+1)\phi(p+1)}} Y^{i_{p+2}}_{i_{(1,p+1)\phi(p+2)}} \cdots Y^{i_{p+m}}_{i_{(1,p+1)\phi(p+m)}} \\
&\quad \times Z^{i_{p+m+1}}_{i_{(1,p+1)\phi(p+m+1)}} \cdots Z^{i_{p+m+n}}_{i_{(1,p+1)\phi(p+m+n)}} \\
&= \mathrm{Tr}\Big((1,p+1)\phi X^{\otimes p} Y^{\otimes m} Z^{\otimes n}\Big). \qquad \text{(C.15)}
\end{aligned}
$$

Using (C.14) and (C.15) we find

$$
\begin{aligned}
D_2^{XY} \chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) = \frac{pm}{p!m!n!} \sum_{R'} \sum_{\phi \in S_{p+m+n-1}} c_{RR'} \mathrm{Tr}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} \Big( [\Gamma_R(1,p+1), \Gamma_{R'}(\phi)] \Big) \\
\times \mathrm{Tr}\Big( [\phi,(1,p+1)] X^{\otimes p} Y^{\otimes m} Z^{\otimes n} \Big). \qquad \text{(C.16)}
\end{aligned}
$$

Next we will motivate the Fourier transform between $\mathrm{Tr}\Big(\alpha X^{\otimes p} Y^{\otimes m} Z^{\otimes n}\Big)$ and the restricted Schur polynomial basis $\chi_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X)$. We revisit the Fourier transform between a function in momentum space and a function in position space to motivate our analysis. One can use the equations

$$
f(x) = \mathcal{N} \int dk e^{-ik\cdot x} \tilde{f}(k), \quad \tilde{f}(k) = \int dx e^{ik\cdot x} f(x) \quad \text{and} \quad \int dk e^{ikx} = 2\pi\delta(x), \qquad \text{(C.17)}
$$

to fix the normalization $\mathcal{N}$. We obtain $\mathcal{N}$ as follows

$$
\begin{aligned}
f(x) &= \mathcal{N} \int dk e^{-ik\cdot x} \tilde{f}(k) \\
&= \mathcal{N} \int dk e^{-ik\cdot x} \Big( \int dy e^{ik\cdot y} f(y) \Big) \\
&= \mathcal{N} \int dk \int dy e^{ik\cdot(y-x)} f(y) \\
&= 2\pi\mathcal{N} \int dy \delta(y-x) f(y) \\
&= 2\pi\mathcal{N} f(x). \qquad \text{(C.18)}
\end{aligned}
$$

Thus $\mathcal{N} = \frac{1}{2\pi}$. To develop the Fourier transform of interest to us we will need the restricted Schur polynomial in (6.5) and the two point function

$$
\langle \chi_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X)^\dagger \chi_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X) \rangle = \frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t} \delta_{RT}\delta_{ru}\delta_{sv}\delta_{tw}\delta_{\vec{\alpha}\vec{\mu}}\delta_{\vec{\beta}\vec{\nu}}. \qquad \text{(C.19)}
$$

The Fourier transform of the restricted Schur polynomial is given by

$$\mathrm{Tr}\Big(\alpha X^{\otimes p}Y^{\otimes m}Z^{\otimes n}\Big) = \sum_{T,(w,v,u)} \frac{d_T n!m!p!}{d_u d_v d_w (p+m+n)!}\chi_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}(\alpha^{-1})\chi_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X). \tag{C.20}$$

One can make sense of the factor $\frac{d_T n!m!p!}{d_u d_v d_w (p+m+n)!}$ by following the approach for equation (C.18). This is an important idea, used throughout this thesis. Consider the defintion

$$\mathrm{Tr}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Big(\Gamma_R(\sigma)\Big) = \mathrm{Tr}\Big(P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R(\sigma)\Big),$$

and note that we have introduced the intertwining map $P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}$, explained in detail below. Plugging equation (C.20) into equation (C.16) we have

$$\begin{aligned}
&D_2^{XY}\chi_{R(t,s,r)_{\vec{\alpha}\vec{\beta}}}(Z,Y,X)\\
&\quad= \sum_{T,(w,v,u)}\sum_{R'}\sum_{\phi\in S_{n+m+p-1}} \frac{d_T pmc_{RR'}}{d_u d_v d_w (p+m+n)!}\,\mathrm{Tr}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Big([\Gamma_R(1,p+1),\Gamma_{R'}(\phi)]\Big)\Big)\\
&\quad\times \mathrm{Tr}_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Big([\Gamma_T(1,p+1),\Gamma_{T'}(\phi^{-1})]\Big)\chi_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X)\\
&\quad= \sum_{T,(w,v,u)}\sum_{R'}\sum_{\phi\in S_{n+m+p-1}} \frac{d_T pmc_{RR'}}{d_u d_v d_w (p+m+n)!}\\
&\quad\times\Bigg[\mathrm{Tr}\Big(P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R((1,p+1))\Gamma_{R'}(\phi)\Big)\,\mathrm{Tr}\Big(P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_T((1,p+1))\Gamma_{T'}(\phi^{-1})\Big)\\
&\quad-\mathrm{Tr}\Big(P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R((1,p+1))\Gamma_{R'}(\phi)\Big)\,\mathrm{Tr}\Big(P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_{T'}(\phi^{-1})\Gamma_T((1,p+1))\Big)\\
&\quad-\mathrm{Tr}\Big(P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_{R'}(\phi)\Gamma_R((1,p+1))\Big)\,\mathrm{Tr}\Big(P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_T((1,p+1))\Gamma_{T'}(\phi^{-1})\Big)\\
&\quad+\mathrm{Tr}\Big(P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_{R'}(\phi)\Gamma_R((1,p+1))\Big)\,\mathrm{Tr}\Big(P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_{T'}(\phi^{-1})\Gamma_T((1,p+1))\Big)\Bigg]\\
&\quad\times\chi_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X).
\end{aligned}\tag{C.21}$$

Next consider the fundamental orthogonality relation, which will enable us to compute the sum over the symmetric group $S_{p+m+n-1}$. This relation is

$$\sum_{\phi\in S_{p+m+n-1}}[\Gamma_{R'}(\phi)]_{ij}[\Gamma_{T'}(\phi^{-1})]_{ab} = \frac{(p+m+n-1)!}{d_{R'}}\delta_{R'T'}[I_{R'T'}]_{ib}[I_{T'R'}]_{aj},$$

where $I_{R'T'}$ ($I_{T'R'}$) is the intertwiner mapping the carrier space of $T'\vdash(p+m+n-1)(R'\vdash(p+m+n-1))$ into the carrier space of $R'\vdash(p+m+n-1)(T'\vdash(p+m+n-1))$. We evaluate the first term in equation

(C.21) as follows

$$
\begin{aligned}
&\sum_{T,(w,v,u)} \sum_{R'} \sum_{\phi\in S_{n+m+p-1}} \frac{d_T p m c_{RR'}}{d_u d_v d_w (p+m+n)!} \mathrm{Tr}\Big(P_{R,(t,s,r)_{\vec\alpha\vec\beta}} \Gamma_R((1,p+1))\Gamma_{R'}(\phi)\Big) \\
&\quad \times \mathrm{Tr}\Big(P_{T,(w,v,u)_{\vec\mu\vec\nu}} \Gamma_T((1,p+1))\Gamma_{T'}(\phi^{-1})\Big) \\
&= \sum_{T,(w,v,u)} \sum_{R'} \frac{d_T p m c_{RR'}}{d_u d_v d_w (p+m+n)!} \sum_{\phi\in S_{n+m+p-1}} (P_{R,(t,s,r)_{\vec\alpha\vec\beta}})_{ab} \Gamma_R((1,p+1))_{bc} \Gamma_{R'}(\phi)_{ca} \\
&\quad \times (P_{T,(w,v,u)_{\vec\mu\vec\nu}})_{ij} \Gamma_T((1,p+1))_{jk} \Gamma_{T'}(\phi^{-1})_{ki} \\
&= \sum_{T,(w,v,u)} \sum_{R'} \frac{d_T p m c_{RR'}}{d_{R'} d_u d_v d_w (p+m+n)} \delta_{R'T'} [I_{T'R'}]_{ka} (P_{R,(t,s,r)_{\vec\alpha\vec\beta}})_{ab} \Gamma_R((1,p+1))_{bc} [I_{R'T'}]_{ci} \\
&\quad \times (P_{T,(w,v,u)_{\vec\mu\vec\nu}})_{ij} \Gamma_T((1,p+1))_{jk} \\
&= \sum_{T,(w,v,u)} \sum_{R'} \frac{d_T p m c_{RR'}}{d_{R'} d_u d_v d_w (p+m+n)} \delta_{T'R'} \,\mathrm{Tr}\,\Big(I_{T'R'} P_{R,(t,s,r)_{\vec\alpha\vec\beta}} \Gamma_R((1,p+1)) \\
&\quad \times I_{R'T'} P_{T,(w,v,u)_{\vec\mu\vec\nu}} \Gamma_T((1,p+1))\Big). \qquad \text{(C.22)}
\end{aligned}
$$

It is easy to compute the remaining terms in the same way. In the end

$$
\begin{aligned}
D_2^{XY} \chi_{R(t,s,r)_{\vec\alpha\vec\beta}}(Z,Y,X) &= \sum_{T,(w,v,u)} \sum_{R'} \frac{d_T p m c_{RR'}}{d_{R'} d_u d_v d_w (p+m+n)} \delta_{R'T'} \\
&\times \Bigg( \mathrm{Tr}\,\Big(I_{T'R'} P_{R,(t,s,r)_{\vec\alpha\vec\beta}} \Gamma_R((1,p+1)) I_{R'T'} P_{T,(w,v,u)_{\vec\mu\vec\nu}} \Gamma_T((1,p+1))\Big) \\
&- \mathrm{Tr}\,\Big(I_{T'R'} P_{R,(t,s,r)_{\vec\alpha\vec\beta}} \Gamma_R((1,p+1)) I_{R'T'} \Gamma_T((1,p+1)) P_{T,(w,v,u)_{\vec\mu\vec\nu}}\Big) \\
&- \mathrm{Tr}\,\Big(I_{T'R'} \Gamma_R((1,p+1)) P_{R,(t,s,r)_{\vec\alpha\vec\beta}} I_{R'T'} P_{T,(w,v,u)_{\vec\mu\vec\nu}} \Gamma_T((1,p+1))\Big) \\
&+ \mathrm{Tr}\,\Big(I_{T'R'} \Gamma_R((1,p+1)) P_{R,(t,s,r)_{\vec\alpha\vec\beta}} I_{R'T'} \Gamma_T((1,p+1)) P_{T,(w,v,u)_{\vec\mu\vec\nu}}\Big)\Bigg) \\
&\times \chi_{T,(w,v,u)_{\vec\mu\vec\nu}}(Z,Y,X). \qquad \text{(C.23)}
\end{aligned}
$$

We will work with normalized restricted Schur polynomials $\mathcal{O}_{R,(t,s,r)_{\vec\alpha\vec\beta}}(Z,Y,X)$ given by

$$
\chi_{R,(t,s,r)_{\vec\alpha\vec\beta}}(Z,Y,X) = \sqrt{\frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_r}} \mathcal{O}_{R,(t,s,r)_{\vec\alpha\vec\beta}}(Z,Y,X).
$$

In terms of the normalized restricted Schur polynomials, equation (C.23) becomes

$$
D_2^{XY} \mathcal{O}_R,(t,s,r)_{\vec\alpha\vec\beta}(Z,Y,X) = -\sum_{T,(w,v,u)} M^{\vec p \vec m}_{R,(t,s,r)_{\vec\alpha\vec\beta}:T,(w,v,u)_{\vec\mu\vec\nu}} \mathcal{O}_{T(w,v,u)_{\vec\mu\vec\nu}}(Z,Y,X), \qquad \text{(C.24)}
$$

where

$$
\begin{aligned}
M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} = &\sum_{R'} \frac{d_T pmc_{RR'}}{d_{R'} d_u d_v d_w (n+m+p)} \delta_{R'T'} \sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_u \mathrm{hooks}_v \mathrm{hooks}_w}} \\
&\times \Bigg( \mathrm{Tr}\left( I_{T'R'} P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} \Gamma_R((1,p+1)) I_{R'T'} \Gamma_T((1,p+1)) P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \right) \\
&- \mathrm{Tr}\left( I_{T'R'} P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} \Gamma_R((1,p+1)) I_{R'T'} P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \Gamma_T((1,p+1)) \right) \\
&- \mathrm{Tr}\left( I_{T'R'} \Gamma_R((1,p+1)) P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} I_{R'T'} \Gamma_T((1,p+1)) P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \right) \\
&+ \mathrm{Tr}\left( I_{T'R'} \Gamma_R((1,p+1)) P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} I_{R'T'} P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \Gamma_T((1,p+1)) \right) \Bigg) \\
&\times \mathcal{O}_{T(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X). \qquad \text{(C.25)}
\end{aligned}
$$

Equation (C.24) is exact to one loop. A challenge in evaluating the above traces is to compute the projectors $P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}$. We overcome this problem by using the displaced corners approximation.

## C.2 Trace in the displaced corners approximation

We compute the traces over $R \oplus T$ that appear in equation (C.25) using the displaced corners approximation. The result is valid in the large $N$ limit. The displaced corners approximation can be applied valid when

$$
\begin{cases} (p+m+n) \approx \mathcal{O}(N) & \text{for } m=\eta N,\ p=\xi N,\ \eta,\xi \ll 1 \\ q \approx \mathcal{O}(1) \\ l_{R_i} - l_{R_{i+1}} \approx \mathcal{O}(N), \end{cases} \qquad \text{(C.26)}
$$

$l_{R_i}$ is the row length of a given Young diagram $R \vdash (p+m+n)$ in row $i$ and $q$ is the number of rows in $R$. Consider the case where $R'$ is obtained from $R$ by dropping a box in row $i$ and $T'$ is obtained from $T$ by dropping a box from row $k$, represented diagrammatically below

$$
R \xrightarrow{i^{th}\text{box}} R'
$$

$$
T \xrightarrow{k^{th}\text{box}} T'.
$$

The intertwiner is non-zero only if $R' = T'$, which is enforced by the delta function $\delta_{R'T'}$. The intertwiner map is

$$
I_{R'T'} = E^{(1)}_{ik} \otimes \mathbb{1}^{\otimes p-1} \otimes \mathbb{1}^{\otimes m} \qquad I_{T'R'} = E^{(1)}_{ki} \otimes \mathbb{1}^{\otimes p-1} \otimes \mathbb{1}^{\otimes m}. \qquad \text{(C.27)}
$$

For simplicity use the notation $I_{R'T'} = E^{(1)}_{ik}$ and $I_{T'R'} = E^{(1)}_{ki}$ throughout this appendix. We can simplify the expressions appearing in equation (C.24):

$$\begin{aligned}
\Gamma_T((1,p+1))I_{T'R'} &= E^{(1)}_{ci}E^{(p+1)}_{kc} &\qquad \Gamma_R((1,p+1))I_{R'T'} &= E^{(1)}_{ak}E^{(p+1)}_{ia}\\
I_{T'R'} &= E^{(1)}_{ki} &\qquad \Gamma_R((1,p+1))I_{R'T'}\Gamma_T((1,p+1)) &= E^{(p+1)}_{ik}\\
\Gamma_T((1,p+1))I_{T'R'}\Gamma_R((1,p+1)) &= E^{(p+1)}_{ki} &\qquad I_{R'T'} &= E^{(1)}_{ik}\\
I_{T'R'}\Gamma_R((1,p+1)) &= E^{(1)}_{kc}E^{(p+1)}_{ci} &\qquad I_{R'T'}\Gamma_T((1,p+1)) &= E^{(1)}_{ia}E^{(p+1)}_{ak}.
\end{aligned}$$

In the large $N$ limit we can to compute the restricted projector $P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}$. In terms of the state $|t,\alpha_1,a;s,\beta_1,b;r,c\rangle$, which gives a basis for the $(t,s,r)$ irreducible representation of $S_p \times S_m \times S_n$,

$$\begin{aligned}
P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} &= \sum_{a,b} |t,\alpha_1,a;s,\beta_1,b\rangle \langle t,\alpha_2,a;s,\beta_2,b| \otimes \sum_c |r,c\rangle\langle r,c|\\
&= P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} \otimes \mathbb{1}_r, \qquad\qquad \text{(C.28)}
\end{aligned}$$

where $P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} = \sum_{a,b}|t,\alpha_1,a;s,\beta_1,b\rangle\langle t,\alpha_2,a;s,\beta_2,b|$ and $\sum_c |r,c\rangle\langle r,c| = \mathbb{1}_r$. Once we think in terms of Young-Yamanouchi labels of the Young-diagram $R \vdash (p+m+n)$ for the impurities $p \sim X$ and $m \sim Y$ respectively, it is clear that $|r,c\rangle\langle r,c|$ is the identity $\mathbb{1}$. Similarly, we have

$$P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}} = P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}} \otimes \mathbb{1}_u.$$

$\vec{p}(\vec{p}')$ and $\vec{m}(\vec{m}')$ are vectors associated with boxes removed from each row of the Young diagram $R(T)$. From the above equations the traces in equation (C.24) are as follows

$$\begin{aligned}
&\mathrm{Tr}\Big(I_{T'R'}P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R((1,p+1))I_{R'T'}\Gamma_T((1,p+1))P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Big)\\
&= \mathrm{Tr}\Big(E^{(1)}_{ki}\cdot P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} \otimes \mathbb{1}_r \cdot E^{(p+1)}_{ik}\cdot P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}} \otimes \mathbb{1}_u\Big)\\
&= \mathrm{Tr}\Big(E^{(1)}_{ki}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\,\mathrm{Tr}\Big(\mathbb{1}_r\cdot\mathbb{1}_u\Big)\\
&= d_r\,\mathrm{Tr}\Big(E^{(1)}_{ki}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ru}. \qquad\qquad \text{(C.29)}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Tr}\Big(I_{T'R'}P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R((1,p+1))I_{R'T'}P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_T((1,p+1))\Big)\\
&= \mathrm{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}\cdot P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} \otimes \mathbb{1}_r \cdot E^{(1)}_{ak}E^{(p+1)}_{ia}\cdot P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}} \otimes \mathbb{1}_u\Big)\\
&= \mathrm{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ak}E^{(p+1)}_{ia}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\,\mathrm{Tr}\Big(\mathbb{1}_r\cdot\mathbb{1}_u\Big)\\
&= d_r\,\mathrm{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ak}E^{(p+1)}_{ia}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ru}. \qquad\qquad \text{(C.30)}
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{Tr}\Big(I_{T'R'}\Gamma_R((1,p+1))P_{R,(t,s,r)_{\vec\alpha\vec\beta}}I_{R'T'}\Gamma_T((1,p+1))P_{T,(w,v,u)_{\vec\mu\vec\nu}}\Big)\\
&=\mathrm{Tr}\Big(E^{(1)}_{kc}E^{(p+1)}_{ci}\cdot P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}\otimes \mathbb{1}_r\cdot E^{(1)}_{ia}E^{(p+1)}_{ak}\cdot P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\otimes\mathbb{1}_u\Big)\\
&=\mathrm{Tr}\Big(E^{(1)}_{kc}E^{(p+1)}_{ci}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ia}E^{(p+1)}_{ak}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\,\mathrm{Tr}\Big(\mathbb{1}_r\cdot\mathbb{1}_u\Big)\\
&=d_r\,\mathrm{Tr}\Big(E^{(1)}_{kc}E^{(p+1)}_{ci}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ia}E^{(p+1)}_{ak}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ru}.
\end{aligned}
\tag{C.31}
$$

$$
\begin{aligned}
&\mathrm{Tr}\Big(I_{T'R'}\Gamma_R((1,p+m+1))P_{R,(t,s,r)_{\vec\alpha\vec\beta}}I_{R'T'}P_{T,(w,v,u)_{\vec\mu\vec\nu}}\Gamma_T((1,p+m+1))\Big)\\
&=\mathrm{Tr}\Big(E^{(p+1)}_{ki}\cdot P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}\otimes\mathbb{1}_r\cdot E^{(1)}_{ik}\cdot P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\otimes\mathbb{1}_u\Big)\\
&=\mathrm{Tr}\Big(E^{(p+1)}_{ki}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ik}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\,\mathrm{Tr}\Big(\mathbb{1}_r\cdot\mathbb{1}_u\Big)\\
&=d_r\,\mathrm{Tr}\Big(E^{(p+1)}_{ki}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ik}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ru}.
\end{aligned}
\tag{C.32}
$$

Substituting equations (C.29),(C.30),(C.31) and (C.32) into (C.24) we obtain

$$
D_2^{XY}\mathcal{O}_R,(t,s,r)_{\vec\alpha\vec\beta}(Z,Y,X)=-\sum_{T,(w,v,u)}M^{\vec p\vec m}_{R,(t,s,r)_{\vec\alpha\vec\beta}:T,(w,v,u)_{\vec\mu\vec\nu}}\mathcal{O}_{T(w,v,u)_{\vec\mu\vec\nu}}(Z,Y,X),
\tag{C.33}
$$

where now

$$
\begin{aligned}
&M^{\vec p\vec m}_{R,(t,s,r)_{\vec\alpha\vec\beta}:T,(w,v,u)_{\vec\mu\vec\nu}}\\
&=\sum_{R'}\frac{c_{RR'}d_Tpm}{d_{R'}d_ud_vd_w(p+m+n)}\sqrt{\frac{f_T\mathrm{hooks}_T\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_t}{f_R\mathrm{hooks}_R\mathrm{hooks}_u\mathrm{hooks}_v\mathrm{hooks}_w}}\delta_{R'_iT'_k}\delta_{ru}d_r\\
&\times\Bigg[\mathrm{Tr}\Big(E^{(1)}_{ki}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ik}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)-\mathrm{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ak}E^{(p+1)}_{ia}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\\
&-\mathrm{Tr}\Big(E^{(1)}_{kc}E^{(p+1)}_{ci}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ia}E^{(p+1)}_{ak}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)+\mathrm{Tr}\Big(E^{(p+1)}_{ki}P^{(\vec p,\vec m)}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ik}P^{(\vec p',\vec m')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\Bigg].
\end{aligned}
$$

Next we consider simplifying the factor

$$
\delta_{R'_iT'_k}\delta_{ru}\frac{c_{RR'}d_Tpm}{d_{R'}d_ud_vd_w(p+m+n)}\sqrt{\frac{f_T\mathrm{hooks}_T\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_t}{f_R\mathrm{hooks}_R\mathrm{hooks}_u\mathrm{hooks}_v\mathrm{hooks}_w}}d_r.
\tag{C.34}
$$

Let $R,T$ be Young diagrams with $q$-rows. We denote by $l_{R_i}$, where $i,\cdots,q$, the number of boxes in the $i^{\text{th}}$ row of $R$. Suppose that $S\vdash a$ and $S'\vdash a-1$ where $S'$ is obtained from $S$ by removing the box at the corner of the $i^{\text{th}}$ row of $S$. If $S$ and $S'$ satisfy displaced corners approximation in (C.26), then

$$
\frac{\mathrm{hooks}_S}{\mathrm{hooks}_{S'}}=l_{S_i}.
\tag{C.35}
$$

Recall the dimension of $S$ is

$$d_S = \frac{a!}{\text{hooks}_S}.$$

Since the delta functions $\delta_{R'_i T'_k}$ and $\delta_{ru}$ set $R'_i = T'_k$ and $r = u$ respectively, we have

$$\begin{aligned} \frac{d_T}{d_{T'}(p+m+n)} &= \frac{\text{hooks}_{T'}}{\text{hooks}_T} \\ &= \frac{1}{l_{T_k}}, \end{aligned} \tag{C.36}$$

where $d_T = \frac{(p+m+n)!}{\text{hooks}_T}$ and $d_{T'} = \frac{(p+m+n-1)!}{\text{hooks}_{T'}}$. Similarly,

$$\begin{aligned} \frac{\text{hooks}_T}{\text{hooks}_R} &= \frac{\text{hooks}_T}{\text{hooks}_{T'}} \frac{\text{hooks}_{T'}}{\text{hooks}_{R'}} \frac{\text{hooks}_{R'}}{\text{hooks}_R} \\ &= l_{T_k} \cdot 1 \cdot \frac{1}{l_{R_i}} \\ &= \frac{l_{T_k}}{l_{R_i}}. \end{aligned} \tag{C.37}$$

It is a simple matter to show that

$$\frac{mp}{d_u d_v d_w} \sqrt{\frac{\text{hooks}_r \text{hooks}_s \text{hooks}_t}{\text{hooks}_u \text{hooks}_v \text{hooks}_w}} d_r = \frac{mp}{\sqrt{d_s d_v d_t d_w}}, \tag{C.38}$$

and

$$c_{RR'} \sqrt{\frac{f_T}{f_T}} = \sqrt{c_{RR'} c_{TT'}}.$$

Substituting equations (C.35),(C.36),(C.37) and (C.38) into (C.34) we have,

$$\begin{aligned} &\delta_{R'_i T'_k} \delta_{ru} \frac{c_{RR'} d_T mp}{d_{R'} d_u d_v d_w (n+m+p)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R \text{hooks}_u \text{hooks}_v \text{hooks}_w}} d_r \\ &\quad = \delta_{R'_i T'_k} \delta_{ru} \frac{mp}{\sqrt{d_s d_v d_t d_w}} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}}. \end{aligned} \tag{C.39}$$

Substituting this result into equation (C.33) we obtain

$$D_2^{XY} \mathcal{O}_R, (t,s,r)_{\vec{\alpha}\vec{\beta}}(Z,Y,X) = -\sum_{T,(w,v,u)} M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \mathcal{O}_{T(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X), \tag{C.40}$$

where

$$
\begin{aligned}
&M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\\
&=\sum_{R'}\delta_{R'_iT'_k}\delta_{ru}\frac{pm}{\sqrt{d_sd_vd_td_w}}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}}\\
&\times\Big[\mathrm{Tr}\Big(E^{(1)}_{ki}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)-\mathrm{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ak}E^{(p+1)}_{ia}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\\
&-\mathrm{Tr}\Big(E^{(1)}_{kc}E^{(p+1)}_{ci}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ia}E^{(p+1)}_{ak}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)+\mathrm{Tr}\Big(E^{(p+1)}_{ki}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\Big].
\end{aligned}
\tag{C.41}
$$

## C.3 Acting with $D_2^{YZ}$ and $D_2^{XZ}$ on restricted Schur polynomials

To evaluate $D_2^{YZ}$ and $D_2^{XZ}$ on restricted Schur polynomials in equations (C.2) and (C.3) we follow exactly the same steps as above for the action of $D_2^{YZ}$ above. The result is

$$
D_2^{YZ}\mathcal{O}_R,(t,s,r)_{\vec{\alpha}\vec{\beta}}(Z,Y,X)=-\sum_{T,(w,v,u)}M^{\vec{m}\vec{n}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\mathcal{O}_{T(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X),
\tag{C.42}
$$

where

$$
\begin{aligned}
&M^{\vec{m}\vec{n}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\\
&=-\sum_{R'}\frac{d_Tnmc_{RR'}}{d_{R'}d_ud_vd_w(n+m+p)}\delta_{R'T'}\sqrt{\frac{f_T\mathrm{hooks}_T\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_t}{f_R\mathrm{hooks}_R\mathrm{hooks}_u\mathrm{hooks}_v\mathrm{hooks}_w}}\\
&\times\mathrm{Tr}\Big(I_{T'R'}\Big[\Gamma_R((1,p+1))\Gamma_R((p+1,p+m+1))P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R((1,p+1))-\Gamma_R((1,p+1))\\
&\times P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R((p+1,p+m+1))\Gamma_R((1,p+1))\Big]I_{R'T'}\Big[\Gamma_T((1,p+m+1))\Gamma_T((1,p+1))\\
&\times P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_T((1,p+1))-\Gamma_T((1,p+1))P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_T((1,p+1))\Gamma_T((1,p+m+1))\Big]\Big).
\end{aligned}
$$

Again the result in equation (C.42) is exact. Next we simplify the factor

$$
\delta_{R'_iT'_k}\delta_{r'_iu'_k}\frac{c_{RR'}d_Tmn}{d_{R'}d_ud_vd_w(p+m+n)}\sqrt{\frac{f_T\mathrm{hooks}_T\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_t}{f_R\mathrm{hooks}_R\mathrm{hooks}_u\mathrm{hooks}_v\mathrm{hooks}_w}}d_{r'},
$$

given equations (C.35), (C.36), (C.37) and (C.38) and use the fact that

$$
\frac{mn}{d_ud_vd_w}\sqrt{\frac{f_T\mathrm{hooks}_T\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_t}{f_R\mathrm{hooks}_R\mathrm{hooks}_u\mathrm{hooks}_v\mathrm{hooks}_w}}d_{r'}=\frac{mn}{d_ud_vd_w}\sqrt{\frac{l_{T_k}}{l_{R_i}}l_{r_i}l_{u_k}},
$$

where $l_{r_i} = \frac{\text{hooks}_r}{\text{hooks}_{r'}}$ and $l_{u_k} = \frac{\text{hooks}_u}{\text{hooks}_{u'}}$. Thus

$$\delta_{R_i'T_k'}\delta_{r_i'u_k'}\frac{c_{RR'}d_T mn}{d_{R'}d_u d_v d_w(p+m+n)}\sqrt{\frac{f_T\text{hooks}_T\text{hooks}_r\text{hooks}_s\text{hooks}_t}{f_R\text{hooks}_R\text{hooks}_u\text{hooks}_v\text{hooks}_w}}d_{r'}$$
$$= \delta_{R_i'T_k'}\delta_{r_i'u_k'}\frac{m}{\sqrt{d_t d_s d_v d_w}}\sqrt{\frac{l_{r_i}l_{u_k}}{l_{R_i}l_{T_k}}c_{RR'}c_{TT'}}. \qquad (C.43)$$

Applying displaced corners approximation in (C.26) and the intertwiners in (C.27) we have

$$D_2^{YZ}\mathcal{O}_R,(t,s,r)_{\vec{\alpha}\vec{\beta}}(Z,Y,X) = -\sum_{T,(w,v,u)} M^{\vec{m}\vec{n}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\mathcal{O}_{T(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X), \qquad (C.44)$$

where

$$M^{\vec{m}\vec{n}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}$$
$$= \sum_{R',T'}\delta_{R_i'T_k'}\delta_{r_i'u_k'}\frac{m}{\sqrt{d_t d_s d_v d_w}}\sqrt{\frac{l_{r_i}l_{u_k}}{l_{R_i}l_{T_k}}c_{RR'}c_{TT'}}$$
$$\times\Bigg[\mathrm{Tr}\Big(P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik} - \mathrm{Tr}\Big(E^{(p+1)}_{kk}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)$$
$$-\mathrm{Tr}\Big(E^{(p+1)}_{ii}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{kk}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) + \mathrm{Tr}\Big(E^{(p+1)}_{ii}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik}\Bigg].$$

Similarly, equation (C.3) yields

$$D_2^{XZ}\mathcal{O}_R,(t,s,r)_{\vec{\alpha}\vec{\beta}}(Z,Y,X) = -\sum_{T,(w,v,u)} M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\mathcal{O}_{T(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X), \qquad (C.45)$$

where

$$M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}$$
$$= -\sum_{R'}\frac{d_T npc_{RR'}}{d_{R'}d_u d_v d_w(n+m+p)}\delta_{R'T'}\sqrt{\frac{f_T\text{hooks}_T\text{hooks}_r\text{hooks}_s\text{hooks}_t}{f_R\text{hooks}_R\text{hooks}_u\text{hooks}_v\text{hooks}_w}}$$
$$\times\mathrm{Tr}\Big(I_{T'R'}\Big[\Gamma_R((1,p+m+1))P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} - P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}}\Gamma_R((1,p+m+1))\Big]$$
$$\times I_{R'T'}\Big[\Gamma_T((1,p+m+1))P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}} - P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Gamma_T((1,p+m+1))\Big]\Big).$$

Again the result in equation (C.45) is exact. We use the displaced corners approximation scheme in (C.26) and intertwiners in (C.27) to obtain

$$D_2^{XZ}\mathcal{O}_R,(t,s,r)_{\vec{\alpha}\vec{\beta}}(Z,Y,X) = -\sum_{T,(w,v,u)} M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\mathcal{O}_{T(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X), \qquad (C.46)$$

where

$$
\begin{aligned}
&M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \\
&= \sum_{R',T'} \delta_{R'_i T'_k}\delta_{r'_i u'_k}\frac{p}{\sqrt{d_t d_s d_v d_w}}\sqrt{\frac{l_{r_i} l_{u_k}}{l_{R_i} l_{T_k}}c_{RR'}c_{TT'}} \\
&\quad\times\Bigg[\mathrm{Tr}\Big(P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik} - \mathrm{Tr}\Big(E^{(1)}_{kk}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) \\
&\quad - \mathrm{Tr}\Big(E^{(1)}_{ii}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{kk}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) + \mathrm{Tr}\Big(E^{(1)}_{kk}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik}\Bigg].
\end{aligned}
$$

Putting equations (C.44), (C.45) and (C.46) together, the one loop dilatation operator acting on the normalized restricted Schur polynomial basis, at large $N$, is

$$
\begin{aligned}
D\mathcal{O}_R,(t,s,r)_{\vec{\alpha}\vec{\beta}}(Z,Y,X) = g^2_{YM}\sum_{T,(w,v,u)}\Big(&M^{\vec{m}\vec{n}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} + M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \\
&+ M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}}\Big)\mathcal{O}_{T(w,v,u)_{\vec{\mu}\vec{\nu}}}(Z,Y,X),
\end{aligned}
\tag{C.47}
$$

where

$$
\begin{aligned}
&M^{\vec{m}\vec{n}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \\
&= \sum_{R',T'} \delta_{R'_i T'_k}\delta_{r'_i u'_k}\frac{m}{\sqrt{d_t d_s d_v d_w}}\sqrt{\frac{l_{r_i} l_{u_k}}{l_{R_i} l_{T_k}}c_{RR'}c_{TT'}} \\
&\quad\times\Bigg[\mathrm{Tr}\Big(P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik} - \mathrm{Tr}\Big(E^{(p+1)}_{kk}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) \\
&\quad - \mathrm{Tr}\Big(E^{(p+1)}_{ii}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{kk}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) + \mathrm{Tr}\Big(E^{(p+1)}_{ii}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \\
&= \sum_{R',T'} \delta_{R'_i T'_k}\delta_{r'_i u'_k}\frac{p}{\sqrt{d_t d_s d_v d_w}}\sqrt{\frac{l_{r_i} l_{u_k}}{l_{R_i} l_{T_k}}c_{RR'}c_{TT'}} \\
&\quad\times\Bigg[\mathrm{Tr}\Big(P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik} - \mathrm{Tr}\Big(E^{(1)}_{kk}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ii}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) \\
&\quad - \mathrm{Tr}\Big(E^{(1)}_{ii}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{kk}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) + \mathrm{Tr}\Big(E^{(1)}_{kk}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\delta_{ik}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&M^{\vec{p}\vec{m}}_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}:T,(w,v,u)_{\vec{\mu}\vec{\nu}}} \\
&= \sum_{R',T'} \delta_{R'_i T'_k} \delta_{ru} \frac{pm}{\sqrt{d_s d_v d_t d_w}} \sqrt{\frac{c_{RR'} c_{TT'}}{l_{R_i} l_{T_k}}} \\
&\times \Bigg[ \mathrm{Tr}\Big( E^{(1)}_{ki} P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} E^{(p+1)}_{ik} P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}} \Big) - \mathrm{Tr}\Big( E^{(1)}_{ci} E^{(p+1)}_{kc} P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} E^{(1)}_{ak} E^{(p+1)}_{ia} P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}} \Big) \\
&- \mathrm{Tr}\Big( E^{(1)}_{kc} E^{(p+1)}_{ci} P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} E^{(1)}_{ia} E^{(p+1)}_{ak} P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}} \Big) + \mathrm{Tr}\Big( E^{(p+1)}_{ki} P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}} E^{(1)}_{ik} P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}} \Big) \Bigg].
\end{aligned}
$$

## C.4 Change of basis from restricted Schur basis to the Gauss graph basis

In this section we consider the action of the term in the dilatation operator that mixes $X$ and $Y$ fields, on the Gauss operators. This entails a change of basis from the restricted Schur polynomial basis to the Gauss graph basis. A detailed account of the construction of Gauss operators built from two complex scalar fields can be found in [30]. An extension of this construction to three complex scalar fields is given in [70]. Consider the following subgroups

$$
H_X = S_{p_1} \times \cdots \times S_{p_q} \qquad H_Y = S_{m_1} \times \cdots \times S_{m_q}, \tag{C.48}
$$

where $H_X$ acts on boxes in the partly labelled Young diagrams that are labelled with an integer $i < p+1$. These boxes are associated with $X$ fields. $H_Y$ acts on boxes associated with $Y$ fields. These subgroups are symmetries that leave all partly labelled Young diagrams invariant. As a result, the partly labelled diagrams belong to $S_p \times S_m / H_X \times H_Y$. The Gauss graphs $\sigma = \sigma_x \otimes \sigma_y$, where $\sigma_x \in S_p$ and $\sigma_y \in S_m$, are in one-to-one correspondence with elements of the double coset

$$
H_X \times H_Y \setminus S_p \times S_m / H_X \times H_Y. \tag{C.49}
$$

The explicit form of our projector is given by

$$
\begin{aligned}
P_{R,(t,s,r)_{\vec{\alpha}\vec{\beta}}} &= \sum_a^{d_p} |t, \alpha_1, a\rangle \langle t, \beta_1, a| \otimes \sum_b^{d_m} |s, \alpha_2, b\rangle \langle s, \beta_2, b| \otimes \sum_c^{d_n} |r, c\rangle \langle r, c| \\
&= P_{t_{\alpha_1\beta_1}} \otimes P_{s_{\alpha_2\beta_2}} \otimes \mathbb{1}_r,
\end{aligned} \tag{C.50}
$$

where

$$
P^{(\vec{p})}_{t_{\alpha_1\beta_1}} = \sum_a^{d_p} |\vec{p}, t, \alpha_1; a\rangle \langle \vec{p}, t, \beta_1; a| \qquad P^{(\vec{m})}_{s_{\alpha_2\beta_2}} = \sum_b^{d_m} |\vec{m}, s, \alpha_2; b\rangle \langle \vec{m}, s, \beta_2; b| . \tag{C.51}
$$

Similarly, for a Young diagram $T$ and subgroups $S_{p'}$ and $S_{m'}$ we have

$$P_{T,(w,v,u)_{\vec{\mu}\vec{\nu}}} = P_{w_{\mu_1\nu_1}} \otimes P_{v_{\mu_2\nu_2}} \otimes \mathbb{1}_u,$$

where

$$P^{(\vec{p}')}_{w_{\mu_1\nu_1}} = \sum_e^{d_{p'}} |\vec{p}', w, \mu_1; e\rangle \langle \vec{p}', w, \nu_1; e| \qquad P^{(\vec{m}')}_{v_{\mu_2\nu_2}} = \sum_f^{d_{m'}} |\vec{m}', v, \mu_2; f\rangle \langle \vec{m}', v, \nu_2; f| . \tag{C.52}$$

The key identities that allow us to go from the restricted Schur polynomial basis to the Gauss graph basis [30] are

$$\begin{aligned}
|\vec{p}, t, \alpha_1; a\rangle &= \sum_{\psi_x \in S_p} \Gamma^{(t)}_{ax_1}(\psi_x) B^{t\to 1_{H_X}}_{x_1\alpha_1} \psi_x \left|\vec{V}, \vec{p}\right\rangle \\
\langle \vec{p}, t, \beta_1; a| &= \sum_{\phi_x \in S_p} \frac{d_t}{p!|H_X|} \left\langle \vec{V}, \vec{p}\right| \phi_x^{-1} \Gamma^{(t)}_{ay_1}(\phi_x) B^{t\to 1_{H_X}}_{y_1\beta_1} \\
|\vec{m}, s, \alpha_2; b\rangle &= \sum_{\psi_y \in S_m} \Gamma^{(s)}_{bx_2}(\psi_y) B^{s\to 1_{H_Y}}_{x_2\alpha_2} \psi_y \left|\vec{V}, \vec{m}\right\rangle \\
\langle \vec{m}, s, \beta_2; b| &= \sum_{\phi_y \in S_m} \frac{d_s}{m!|H_Y|} \left\langle \vec{V}, \vec{m}\right| \phi_y^{-1} \Gamma^{(s)}_{by_2}(\phi_y) B^{s\to 1_{H_Y}}_{y_2\beta_2},
\end{aligned} \tag{C.53}$$

where

$$\left|\vec{V}, \vec{p}\right\rangle \equiv \left|v_1^{\otimes p_1} \otimes v_2^{\otimes p_2} \otimes \cdots \otimes v_q^{\otimes p_q}\right\rangle \quad \text{and} \quad \left|\vec{V}, \vec{m}\right\rangle \equiv \left|v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots \otimes v_q^{\otimes m_q}\right\rangle .$$

We define

$$\left|\vec{V}, \vec{p}, \vec{m}\right\rangle \equiv \left|\vec{V}, \vec{p}\right\rangle \otimes \left|\vec{V}, \vec{m}\right\rangle = \left|v_1^{\otimes p_1} \otimes v_2^{\otimes p_2} \otimes \cdots \otimes v_q^{\otimes p_q} \otimes v_{q+1}^{\otimes m_1} \otimes v_{q+2}^{\otimes m_2} \otimes \cdots \otimes v_{2q}^{\otimes m_q}\right\rangle . \tag{C.54}$$

From equation (C.53) we write the ket vector as

$$\begin{aligned}
&|\vec{p}, t, \alpha_1, a\rangle \otimes |\vec{m}, s, \alpha_2, b\rangle \\
&= \sum_{\psi_x \in S_p} \Gamma^{(t)}_{ax_1}(\psi_x) B^{t\to 1_{H_X}}_{x_1\alpha_1} \psi_x \left|\vec{V}, \vec{p}\right\rangle \otimes \sum_{\psi_y \in S_m} \Gamma^{(s)}_{bx_2}(\psi_y) B^{s\to 1_{H_Y}}_{x_2\alpha_2} \psi_y \left|\vec{V}, \vec{m}\right\rangle \\
&= \sum_{\psi_x \in S_p} \sum_{\psi_y \in S_m} \Gamma^{(t)}_{ax_1}(\psi_x) \otimes \Gamma^{(s)}_{bx_2}(\psi_y) B^{t\to 1_{H_X}}_{x_1\alpha_1} \otimes B^{s\to 1_{H_Y}}_{x_2\alpha_2} \psi_x \left|\vec{V}, \vec{p}\right\rangle \otimes \psi_y \left|\vec{V}, \vec{m}\right\rangle \\
&= \sum_{\psi_x \otimes \psi_y \in S_p \otimes S_m} \Gamma^{(t,s)}_{ab;x_1x_2}(\psi_x \otimes \psi_y) B^{(t,s)\to 1_{H_X \otimes H_Y}}_{x_1x_2;\alpha_1\alpha_2} (\psi_x \otimes \psi_y) \left|\vec{V}, \vec{p}, \vec{m}\right\rangle \\
&= \sum_{\psi_1 \in S_p \times S_m} \Gamma^{(t,s)}_{ab;x_1x_2}(\psi_1) B^{(t,s)\to 1_{H_X \otimes H_Y}}_{x_1x_2;\alpha_1\alpha_2} (\psi_1) \left|\vec{V}, \vec{p}, \vec{m}\right\rangle ,
\end{aligned} \tag{C.55}$$

where $\psi_1 = \psi_x \otimes \psi_y$, $\psi_x \in S_p$, $\psi_y \in S_m$. Similarly, the bra vector is

$$
\begin{aligned}
&\langle \vec{p}, t, \beta_1; a| \times \langle \vec{m}, s, \beta_2; b| \\
&= \sum_{\phi_x \in S_p} \frac{d_t}{p!|H_X|} \left\langle \vec{V}, \vec{p} \right| \phi_x^{-1} \Gamma^{(t)}_{ay_1}(\phi_x) B^{t \to 1_{H_X}}_{y_1 \beta_1} \otimes \sum_{\phi_y \in S_m} \frac{d_s}{m!|H_Y|} \left\langle \vec{V}, \vec{m} \right| \phi_y^{-1} \Gamma^{(s)}_{by_2}(\phi_y) B^{s \to 1_{H_Y}}_{y_2 \beta_2} \\
&= \sum_{\phi_x \otimes \phi_y \in S_p \times S_m} \frac{d_t d_s}{p!m!|H_X||H_Y|} \left\langle \vec{V}, \vec{p}, \vec{m} \right| (\phi_x^{-1} \otimes \phi_y^{-1}) \Gamma^{(t,s)}_{ab;y_1 y_2}(\phi_x \otimes \phi_y) B^{(t,s) \to 1_{H_X \times H_Y}}_{y_1 y_2; \beta_1 \beta_2} \\
&= \sum_{\phi_1 \in S_p \times S_m} \frac{d_t d_s}{p!m!|H_X||H_Y|} \left\langle \vec{V}, \vec{p}, \vec{m} \right| (\phi_1^{-1}) \Gamma^{(t,s)}_{ab;y_1 y_2}(\phi_1) B^{(t,s) \to 1_{H_X \times H_Y}}_{y_1 y_2; \beta_1 \beta_2},
\end{aligned}
\tag{C.56}
$$

where $\phi_1 = \phi_x \otimes \phi_y$ and $\phi_1^{-1} = \phi_x^{-1} \otimes \phi_y^{-1}$ for $\phi_x \in S_p$, $\phi_y \in S_m$. Also, using equations (C.55) and (C.68) the projector can be written in the form

$$
\begin{aligned}
P^{(\vec{p},\vec{m})}_{t_{\alpha_1 \beta_1}; s_{\alpha_2 \beta_2}} &= P^{(\vec{p})}_{t_{\alpha_1 \beta_1}} \otimes P^{(\vec{m})}_{s_{\alpha_2 \beta_2}} \\
&= |\vec{p}, t, \alpha_1; a\rangle \langle \vec{p}, t, \beta_1; a| \otimes |\vec{m}, s, \alpha_2; b\rangle \langle \vec{m}, s, \beta_2; b| \\
&= |\vec{p}, t, \alpha_1, a\rangle \otimes |\vec{m}, s, \alpha_2, b\rangle \langle \vec{p}, t, \beta_1; a| \otimes \langle \vec{m}, s, \beta_2; b| \\
&= \sum_{\substack{\psi_1, \phi_1 \\ \in S_p \times S_m}} \frac{d_t d_s}{p!m!|H_X||H_Y|} \psi_1 \left| \vec{V}, \vec{p}, \vec{m} \right\rangle \left\langle \vec{V}, \vec{p}, \vec{m} \right| \phi_1^{-1} \Gamma^{(t,s)}_{ab;x_1 x_2}(\psi_1) \\
&\quad \times \Gamma^{(t,s)}_{ab;y_1 y_2}(\phi_1) B^{(t,s) \to 1_{H_X \times H_Y}}_{x_1 x_2; \alpha_1 \alpha_2} B^{(t,s) \to 1_{H_X \times H_Y}}_{y_1 y_2; \beta_1 \beta_2} \\
&= \sum_{\substack{\psi_1, \phi_1 \\ \in S_p \times S_m}} \frac{d_t d_s}{p!m!|H_X||H_Y|} \left| (\vec{V}, \vec{p}, \vec{m})_{\psi_1} \right\rangle \left\langle (\vec{V}, \vec{p}, \vec{m})_{\phi_1} \right| \Gamma^{(t,s)}_{ab;x_1 x_2}(\psi_1) \\
&\quad \times \Gamma^{(t,s)}_{ab;y_1 y_2}(\phi_1) B^{(t,s) \to 1_{H_X \times H_Y}}_{x_1 x_2; \alpha_1 \alpha_2} B^{(t,s) \to 1_{H_X \times H_Y}}_{y_1 y_2; \beta_1 \beta_2}.
\end{aligned}
\tag{C.57}
$$

For $\gamma_x \in H_X$ and $\gamma_y \in H_Y$ we have $\gamma_1 = \gamma_x \times \gamma_y \in H_X \times H_Y$. The branching coefficients are defined as follows

$$
\begin{aligned}
\frac{1}{|H_X \times H_Y|} \sum_{\gamma_1 \in H_X \times H_Y} \Gamma^{(s,t)}_{ai;bj}(\gamma_1) &= \frac{1}{|H_X|} \sum_{\gamma_x \in H_X} \Gamma^t_{ab}(\gamma_x) \otimes \frac{1}{|H_Y|} \sum_{\gamma_y \in H_Y} \Gamma^s_{ij}(\gamma_y) \\
&= \sum_{\mu\nu} B^{t \to 1_{H_X}}_{a\mu} B^{t \to 1_{H_X}}_{b\mu} \otimes B^{s \to 1_{H_Y}}_{i\nu} B^{s \to 1_{H_Y}}_{j\nu} \\
&= \sum_{\mu,\nu} B^{(t,s) \to 1_{H_X \times H_Y}}_{ai;\mu\nu} B^{(t,s) \to 1_{H_X \times H_Y}}_{bj;\mu\nu}.
\end{aligned}
\tag{C.58}
$$

From [30], for the case of two complex scalar fields,

$$
\sum_{s \vdash m} \frac{d_s}{m!} \Gamma^s_{ij}(\sigma) \Gamma^s_{ij}(\tau) = \delta(\sigma \tau^{-1}).
$$

The identity relevant for the study of three complex scalar fields is

$$\sum_{t\vdash p}\frac{d_t}{p!}\Gamma^t_{ij}(\sigma_x)\Gamma^t_{ij}(\tau_x)\otimes\sum_{s\vdash m}\frac{d_s}{m!}\Gamma^s_{ab}(\sigma_y)\Gamma^s_{ab}(\tau_y)=\sum_{\substack{t\vdash p\\ s\vdash m}}\frac{d_td_s}{p!m!}\Gamma^t_{ij}(\sigma_x)\otimes\Gamma^s_{ab}(\sigma_y)\cdot\Gamma^t_{ij}(\tau_x)\otimes\Gamma^s_{ab}(\tau_y)$$
$$=\sum_{\substack{t\vdash p\\ s\vdash m}}\frac{d_td_s}{p!m!}\Gamma^{(t,s)}_{ia;jb}(\sigma_x\otimes\sigma_y)\Gamma^{(t,s)}_{ia;jb}(\tau_x\otimes\tau_y)$$
$$=\sum_{\substack{t\vdash p\\ s\vdash m}}\frac{d_td_s}{p!m!}\Gamma^{(t,s)}_{ia;jb}(\sigma_1)\Gamma^{(t,s)}_{ia;jb}(\sigma_2)$$
$$=\delta(\sigma_1\sigma_2^{-1}), \qquad \text{(C.59)}$$

where the delta function is

$$\delta(\sigma_1\sigma_2^{-1})=\begin{cases}1 & \text{if } \sigma_1\sigma_2^{-1}=\mathbb{1},\\ 0 & \text{otherwise.}\end{cases}$$

Using the idea that branching coefficients provide a transformation between two bases, write the Gauss operators

$$|\mathcal{O}_{R,r}(\sigma_x)\rangle=\frac{|H_X|}{\sqrt{p!}}\sum_{i,j}\sum_{t\vdash p}\sum_{\alpha_1,\beta_1}\sqrt{d_t}\Gamma^{(t)}_{ij}(\sigma_x)B^{t\to 1_{H_X}}_{i\alpha_1}B^{t\to 1_{H_X}}_{j\beta_1}\left|\mathcal{O}_{R,(t,s,r)_{\alpha_1\beta_1}}\right\rangle$$

$$|\mathcal{O}_{R,r}(\sigma_y)\rangle=\frac{|H_Y|}{\sqrt{m!}}\sum_{k,l}\sum_{s\vdash m}\sum_{\alpha_2,\beta_2}\sqrt{d_s}\Gamma^{(s)}_{kl}(\sigma_y)B^{s\to 1_{H_Y}}_{k\alpha_2}B^{s\to 1_{H_Y}}_{l\beta_2}\left|\mathcal{O}_{R,(t,s,r)_{\alpha_2\beta_2}}\right\rangle.$$

Therefore

$$|\mathcal{O}_{R,r}(\sigma_1)\rangle=|\mathcal{O}_{R,r}(\sigma_x\otimes\sigma_y)\rangle=|\mathcal{O}_{R,r}(\sigma_x)\rangle\otimes|\mathcal{O}_{R,r}(\sigma_y)\rangle$$
$$=\frac{|H_X\times H_Y|}{\sqrt{p!m!}}\sum_{i,j,k,l}\sum_{\substack{t\vdash p\\ s\vdash m}}\sum_{\substack{\alpha_1,\beta_1\\ \alpha_2,\beta_2}}\sqrt{d_td_s}\Gamma^{(t)}_{ij}(\sigma_x)\otimes\Gamma^{(s)}_{kl}(\sigma_y)B^{t\to 1_{H_X}}_{i\alpha_1}B^{t\to 1_{H_X}}_{j\beta_1}$$
$$\times\otimes B^{s\to 1_{H_Y}}_{k\alpha_2}B^{s\to 1_{H_Y}}_{l\beta_2}\left|\mathcal{O}_{R,((t,s),r)_{\vec\alpha\vec\beta}}\right\rangle$$
$$=\frac{|H_X\times H_Y|}{\sqrt{p!m!}}\sum_{i,j,k,l}\sum_{\substack{t\vdash p\\ s\vdash m}}\sum_{\substack{\alpha_1,\beta_1,\\ \alpha_2,\beta_2}}\sqrt{d_td_s}\Gamma^{(t,s)}_{ik;jl}(\sigma_x\otimes\sigma_y)B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}$$
$$\times B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2}\left|\mathcal{O}_{R,((t,s),r)_{\vec\alpha\vec\beta}}\right\rangle$$
$$=\frac{|H_X\times H_Y|}{\sqrt{p!m!}}\sum_{i,j,k,l}\sum_{\substack{t\vdash p\\ s\vdash m}}\sum_{\substack{\alpha_1,\beta_1,\\ \alpha_2,\beta_2}}\sqrt{d_td_s}\Gamma^{(t,s)}_{ik;jl}(\sigma_1)B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2}$$
$$\times\left|\mathcal{O}_{R,((t,s),r)_{\vec\alpha\vec\beta}}\right\rangle, \qquad \text{(C.60)}$$

where the vector $\left|\mathcal{O}_{R,((t,s),r)_{\vec{\alpha}\vec{\beta}}}\right\rangle = \left|\mathcal{O}_{R,(t,s,r)_{\alpha_1\beta_1}}\right\rangle \otimes \left|\mathcal{O}_{R,(t,s,r)_{\alpha_2\beta_2}}\right\rangle$. The bra vector is given by

$$\begin{aligned}\left\langle\mathcal{O}^{\dagger}_{T,u}(\sigma_2^{-1})\right| &= \left\langle\mathcal{O}^{\dagger}_{T,u}((\tau_x^{-1}\otimes\tau_y^{-1})\right| = \left\langle\mathcal{O}^{\dagger}_{T,u}(\tau_x^{-1})\right| \otimes \left\langle\mathcal{O}^{\dagger}_{T,u}(\tau_y^{-1})\right| \\ &= \frac{|H_{p'}\times H_{m'}|}{\sqrt{p'!m'!}} \sum_{a,b,c,d}\sum_{\substack{w\vdash p'\\ v\vdash m'}}\sum_{\substack{\mu_1,\mu_2,\\ \nu_1,\nu_2}} \sqrt{d_w d_v}\,\Gamma^{(w,v)}_{ac;bd}((\tau_1\otimes\tau_2)^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2} \\ &\quad\times B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\left\langle\mathcal{O}^{\dagger}_{T,((w,v),u)_{\vec{\mu}\vec{\nu}}}\right|, \qquad \text{(C.61)}\end{aligned}$$

where $\sigma_2^{-1} = \tau_1^{-1}\otimes\tau_2^{-1}$. The matrix elements of $D_2^{XY}$ in Gauss graph basis are given by

$$\begin{aligned}M^{\vec{p}\vec{m}}_{T,u;R,r} &= \left\langle\mathcal{O}^{\dagger}_{T,u}(\sigma_2^{-1})\right| D_2^{XY}\left|\mathcal{O}_{R,r}(\sigma_1)\right\rangle \\ &= \frac{|H_X\times H_Y||H_{p'}\times H_{m'}|}{\sqrt{p!m!}\sqrt{p'!m'!}} \sum_{\substack{i,j,k,l,\\ a,b,c,d}}\sum_{\substack{t\vdash p\\ s\vdash m}}\sum_{\substack{w\vdash p'\\ v\vdash m'}}\sum_{\substack{\alpha_1,\alpha_2,\\ \beta_1,\beta_2}}\sum_{\substack{\mu_1,\mu_2\\ \nu_1,\nu_2}} \sqrt{d_t d_s}\sqrt{d_w d_v}\,\Gamma^{(t,s)}_{ik;jl}(\sigma_1) \\ &\quad\times B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2}\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2} \\ &\quad\times\left\langle\mathcal{O}^{\dagger}_{T,((w,v),u)_{\vec{\mu}\vec{\nu}}}\right| D_2^{XY}\left|\mathcal{O}_{R,((t,s),r)_{\vec{\alpha}\vec{\beta}}}\right\rangle, \qquad \text{(C.62)}\end{aligned}$$

where

$$\left\langle\mathcal{O}^{\dagger}_{T,((w,v),u)_{\vec{\mu}\vec{\nu}}}\right| D_2^{XY}\left|\mathcal{O}_{R,((t,s),r)_{\vec{\alpha}\vec{\beta}}}\right\rangle = \sum_{T,((w,v),u)} M^{XY}_{R,((t,s),r)_{\vec{\alpha}\vec{\beta}}:T,((w,v),u)_{\vec{\mu}\vec{\nu}}}\delta_{RT}\delta_{(t,s)(w,v)}\delta_{ru}\delta_{\vec{\alpha}\vec{\mu}}\delta_{\vec{\beta}\vec{\nu}}.$$

Hence equation (C.62) becomes

$$\begin{aligned}&M^{\vec{p}\vec{m}}_{T,u;R,r} \\ &= \frac{|H_X\times H_Y||H_{p'}\times H_{m'}|}{(p-1)!(m-1)!}\sum_{R'}\delta_{R'_iT'_k}\delta_{ru}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}} \sum_{\substack{i,j,k,l,\\ a,b,c,d}}\sum_{\substack{t\vdash p\\ s\vdash m}}\sum_{\substack{w\vdash p'\\ v\vdash m'}}\sum_{\substack{\alpha_1,\alpha_2,\\ \beta_1,\beta_2}}\sum_{\substack{\mu_1,\mu_2\\ \nu_1,\nu_2}} \\ &\quad\times\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\Gamma^{(t,s)}_{ik;jl}(\sigma_1)B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2} \\ &\quad\times\Bigg[\mathrm{Tr}\Big(E^{(1)}_{ki}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(p+1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) - \mathrm{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ak}E^{(p+1)}_{ia}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) \\ &- \mathrm{Tr}\Big(E^{(1)}_{kc}E^{(p+1)}_{ci}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ia}E^{(p+1)}_{ak}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big) + \mathrm{Tr}\Big(E^{(p+1)}_{ki}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ik}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\Bigg]. \\ &\qquad \text{(C.63)}\end{aligned}$$

Consider the second term in equation (C.63)

$$
\begin{aligned}
&\sum_{\substack{i,j,k,l,\\a,b,c,d}}\sum_{\substack{t\vdash p\\s\vdash m}}\sum_{\substack{w\vdash p'\\v\vdash m'}}\sum_{\substack{\alpha_1,\alpha_2,\\\beta_1,\beta_2}}\sum_{\substack{\mu_1,\mu_2\\\nu_1,\nu_2}}\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\Gamma^{(t,s)}_{ik;jl}(\sigma_1)\\
&\times B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2}\operatorname{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}P^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}E^{(1)}_{ak}E^{(p+1)}_{ia}P^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)\\
&=\frac{1}{|H_X\times H_Y||H_{p'}\times H_{m'}|}\sum_{\substack{i,j,k,l,\\a,b,c,d}}\sum_{\substack{t\vdash p\\s\vdash m}}\sum_{\substack{w\vdash p'\\v\vdash m'}}\sum_{\substack{\alpha_1,\alpha_2,\\\beta_1,\beta_2}}\sum_{\substack{\mu_1,\mu_2\\\nu_1,\nu_2}}\frac{d_td_sd_wd_v}{p!m!p'!m'!}\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}\\
&\times B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\Gamma^{(t,s)}_{ik;jl}(\sigma_1)B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2}\operatorname{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}\sum_{\substack{\psi_1,\phi_1\\\in S_p\times S_m}}\\
&\times|(V,\vec{p},\vec{m})_{\psi_1}\rangle\langle(V,\vec{p},\vec{m})_{\phi_1}|\,\Gamma^{(t,s)}_{rm;x_1x_2}(\psi_1)\Gamma^{(t,s)}_{rm;y_1y_2}(\phi_1)B^{(t,s)\to 1_{H_X\times H_Y}}_{x_1x_2;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_X\times H_Y}}_{y_1y_2;\beta_1\beta_2}\\
&\times E^{(1)}_{ak}E^{(p+1)}_{ia}\sum_{\substack{\psi_2,\phi_2\\\in S_{p'}\times S_{m'}}}|(V,\vec{p}',\vec{m}')_{\psi_2}\rangle\langle(V,\vec{p}',\vec{m}')_{\phi_2}|\,\Gamma^{(w,v)}_{np;x_3x_4}(\psi_2)\Gamma^{(w,v)}_{np;y_3y_4}(\phi_2)\\
&\times B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{x_3x_4;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{y_3y_4;\nu_1\nu_2}\Big).
\end{aligned}\tag{C.64}
$$

Using (C.58) the above equation becomes

$$
\begin{aligned}
&\frac{1}{|H_X\times H_Y|^3|H_{p'}\times H_{m'}|^3}\sum_{\substack{i,j,k,l,\\a,b,c,d}}\sum_{\substack{t\vdash p\\s\vdash m}}\sum_{\substack{w\vdash p'\\v\vdash m'}}\frac{d_td_sd_wd_v}{p!m!p'!m'!}\sum_{\substack{\gamma_1,\gamma_2\\\in H_X\times H_Y}}\sum_{\substack{\gamma_3,\gamma_4\\\in H_{p'}\times H_{m'}}}\sum_{\substack{\psi_1,\phi_1\\\in S_p\times S_m}}\sum_{\substack{\psi_2,\phi_2\\\in S_{p'}\times S_{m'}}}\\
&\times\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})\Gamma^{(t,s)}_{ik;jl}(\sigma_1)\operatorname{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}|(V,\vec{p},\vec{m})_{\psi_1}\rangle\langle(V,\vec{p},\vec{m})_{\phi_1}|\,\Gamma^{(t,s)}_{rm;x_1x_2}(\psi_1)\\
&\times\Gamma^{(t,s)}_{rm;y_1y_2}(\phi_1)\Gamma^{(t,s)}_{x_1x_2;ik}(\gamma_2)\Gamma^{(t,s)}_{y_1y_2;jl}(\gamma_1)E^{(1)}_{ak}E^{(p+1)}_{ia}|(V,\vec{p}',\vec{m}')_{\psi_2}\rangle\\
&\times\langle(V,\vec{p}',\vec{m}')_{\phi_2}|\,\Gamma^{(w,v)}_{np;x_3x_4}(\psi_2)\Gamma^{(w,v)}_{np;y_3y_4}(\phi_2)\Gamma^{(w,v)}_{x_3x_4;ac}(\gamma_3)\Gamma^{(w,v)}_{y_3y_4;bd}(\gamma_4)\Big)\\
&=\frac{1}{|H_X\times H_Y|^3|H_{p'}\times H_{m'}|^3}\sum_{\substack{j,l,\\b,d}}\sum_{\substack{t\vdash p\\s\vdash m}}\sum_{\substack{w\vdash p'\\v\vdash m'}}\frac{d_td_sd_wd_v}{p!m!p'!m'!}\sum_{\substack{\gamma_1,\gamma_2\\\in H_X\times H_Y}}\sum_{\substack{\gamma_3,\gamma_4\\\in H_{p'}\times H_{m'}}}\sum_{\substack{\psi_1,\phi_1\\\in S_p\times S_m}}\sum_{\substack{\psi_2,\phi_2\\\in S_{p'}\times S_{m'}}}\\
&\times\operatorname{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}|(V,\vec{p},\vec{m})_{\psi_1}\rangle\langle(V,\vec{p},\vec{m})_{\phi_1}|\,\Gamma^{(t,s)}_{rm;jl}(\psi_1\gamma_2\sigma_1)\Gamma^{(t,s)}_{rm;jl}(\phi_1\gamma_1)E^{(1)}_{ak}E^{(p+1)}_{ia}\\
&\times|(V,\vec{p}',\vec{m}')_{\psi_2}\rangle\langle(V,\vec{p}',\vec{m}')_{\phi_2}|\,\Gamma^{(w,v)}_{np;bd}(\psi_2\gamma_3\sigma_2^{-1})\Gamma^{(w,v)}_{np;bd}(\phi_2\gamma_4)\Big).
\end{aligned}\tag{C.65}
$$

Next apply equation (C.59) to obtain

$$\frac{1}{|H_X\times H_Y|^3|H_{p'}\times H_{m'}|^3}\sum_{\substack{\gamma_1,\gamma_2\\ \in H_X\times H_Y}}\sum_{\substack{\gamma_3,\gamma_4\\ \in H_{p'}\times H_{m'}}}\sum_{\substack{\psi_1,\phi_1\\ \in S_p\times S_m}}\sum_{\substack{\psi_2,\phi_2\\ \in S_{p'}\times S_{m'}}}\mathrm{Tr}\Big(E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1$$
$$\times |V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\phi_1^{-1}\delta(\psi_1\gamma_2\sigma_1\gamma_1^{-1}\phi_1^{-1})E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\langle V,\vec{p}',\vec{m}'|$$
$$\times\phi_2^{-1}\delta(\psi_2\gamma_3\sigma_2^{-1}\gamma_4^{-1}\phi_2^{-1})\Big)$$
$$=\frac{1}{|H_X\times H_Y|^3|H_{p'}\times H_{m'}|^3}\sum_{\substack{\gamma_1,\gamma_2\\ \in H_X\times H_Y}}\sum_{\substack{\gamma_3,\gamma_4\\ \in H_{p'}\times H_{m'}}}\sum_{\substack{\psi_1,\phi_1\\ \in S_p\times S_m}}\sum_{\substack{\psi_2,\phi_2\\ \in S_{p'}\times S_{m'}}}\langle V,\vec{p}',\vec{m}'|\,\phi_2^{-1}$$
$$\times\delta(\psi_2\gamma_3\sigma_2^{-1}\gamma_4^{-1}\phi_2^{-1})E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\phi_1^{-1}\delta(\psi_1\gamma_2\sigma_1\gamma_1^{-1}\phi_1^{-1})$$
$$\times E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2\,|V,\vec{p}',\vec{m}'\rangle$$
$$=\frac{1}{|H_X\times H_Y|^3|H_{p'}\times H_{m'}|^3}\sum_{\substack{\gamma_1,\gamma_2\\ \in H_X\times H_Y}}\sum_{\substack{\gamma_3,\gamma_4\\ \in H_{p'}\times H_{m'}}}\sum_{\substack{\psi_1\\ \in S_p\times S_m}}\sum_{\substack{\psi_2\\ \in S_{p'}\times S_{m'}}}\langle V,\vec{p}',\vec{m}'|\,\gamma_4\sigma_2\gamma_3^{-1}$$
$$\times\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\gamma_1\sigma_1^{-1}\gamma_2^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2\,|V,\vec{p}',\vec{m}'\rangle$$
$$=\frac{1}{|H_X\times H_Y||H_{p'}\times H_{m'}|}\sum_{\psi_1\in S_p\times S_m}\sum_{\psi_2\in S_{p'}\times S_{m'}}\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1\,|V,\vec{p},\vec{m}\rangle$$
$$\times\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\,. \tag{C.66}$$

In the last two lines above we used $\psi_2\gamma_3\sigma_2^{-1}\gamma_4^{-1}\phi_2^{-1}=\mathbb{1}$ and $\psi_1\gamma_2\sigma_1\gamma_1^{-1}\phi_1^{-1}=\mathbb{1}$. We also summed over $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$ producing the factor $|H_X\times H_Y|$ and $|H_{p'}\times H_{m'}|$ where the invariance of $|V,\vec{p},\vec{m}\rangle$ under $H_X\times H_Y$ was used. A more general equation is obtained by replacing the interwining maps with $A=E^{(1)}_{ci}E^{(p+1)}_{kc}$ and $B=E^{(1)}_{ak}E^{(p+1)}_{ia}$

$$\sum_{\substack{i,j,k,l,\\ a,b,c,d}}\sum_{\substack{t\vdash p\\ s\vdash m}}\sum_{\substack{w\vdash p'\\ v\vdash m'}}\sum_{\substack{\alpha_1,\alpha_2,\\ \beta_1,\beta_2}}\sum_{\substack{\mu_1,\mu_2\\ \nu_1,\nu_2}}\Gamma^{(w,v)}_{ac;bd}(\sigma_2^{-1})B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\Gamma^{(t,s)}_{ik;jl}(\sigma_1)$$
$$\times B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2}\mathrm{Tr}\Big(AP^{(\vec{p},\vec{m})}_{t_{\alpha_1\beta_1};s_{\alpha_2\beta_2}}BP^{(\vec{p}',\vec{m}')}_{w_{\mu_1\nu_1};v_{\mu_2\nu_2}}\Big)$$
$$=\frac{1}{|H_X\times H_Y||H_{p'}\times H_{m'}|}\sum_{\psi_1\in S_p\times S_m}\sum_{\psi_2\in S_{p'}\times S_{m'}}\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}A\psi_1\,|V,\vec{p},\vec{m}\rangle$$
$$\times\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}B\psi_2\,|V,\vec{p}',\vec{m}'\rangle\,. \tag{C.67}$$

Let $q$ be the number of rows in a Young diagrams $R$ and $T$. Let the vectors

$$\vec{p}=(p_1,p_2,\cdots,p_q)\qquad \vec{m}=(m_1,m_2,\cdots,m_q), \tag{C.68}$$

record how boxes are removed from Young diagram $R$ and let

$$\vec{p}'=(p'_1,p'_2,\cdots,p'_q)\qquad \vec{m}'=(m'_1,m'_2,\cdots,m'_q), \tag{C.69}$$

record how boxes are removed from Young diagram $T$. Thus, we have

$$p = \sum_{i=1}^{q} p_i = \sum_{i=1}^{q} p'_i = p' \quad \text{and} \quad m = \sum_{i=1}^{q} m_i = \sum_{i=1}^{q} m'_i = m'. \tag{C.70}$$

We can write equation (C.63) as

$$\begin{aligned}
&M^{\vec{p}\vec{m}}_{T,u;R,r} \\
&= \sum_{R'} \frac{\delta_{R'_i T'_k}\delta_{ru}}{(p-1)!(m-1)!}\sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i}l_{T_k}}} \sum_{\psi_1\in S_p\times S_m}\ \sum_{\psi_2\in S_p\times S_m} \\
&\quad \times \Big[ \langle V,\vec{p}',\vec{m}'|\, \sigma_2\psi_2^{-1}E^{(1)}_{ki}\psi_1\,|V,\vec{p},\vec{m}\rangle \,\langle V,\vec{p},\vec{m}|\, \sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ik}\psi_2\,|V,\vec{p}',\vec{m}'\rangle \\
&\quad - \langle V,\vec{p}',\vec{m}'|\, \sigma_2\psi_2^{-1}E^{(1)}_{ci}E^{(p+1)}_{kc}\psi_1\,|V,\vec{p},\vec{m}\rangle \,\langle V,\vec{p},\vec{m}|\, \sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ak}E^{(p+1)}_{ia}\psi_2\,|V,\vec{p}',\vec{m}'\rangle \\
&\quad - \langle V,\vec{p}',\vec{m}'|\, \sigma_2\psi_2^{-1}E^{(1)}_{kc}E^{(p+1)}_{ci}\psi_1\,|V,\vec{p},\vec{m}\rangle \,\langle V,\vec{p},\vec{m}|\, \sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ia}E^{(p+1)}_{ak}\psi_2\,|V,\vec{p}',\vec{m}'\rangle \\
&\quad + \langle V,\vec{p}',\vec{m}'|\, \sigma_2\psi_2^{-1}E^{(p+1)}_{ki}\psi_1\,|V,\vec{p},\vec{m}\rangle \,\langle V,\vec{p},\vec{m}|\, \sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ik}\psi_2\,|V,\vec{p}',\vec{m}'\rangle \Big].
\end{aligned} \tag{C.71}$$

We normalize the Gauss graph $\left|\hat{\mathcal{O}}_{R,r}(\sigma_1)\right\rangle$ as follows

$$\begin{aligned}
&\left\langle \mathcal{O}^\dagger_{T,u}(\sigma_2)\right\rangle \mathcal{O}_{R,r}(\sigma_1) \\
&= \frac{|H_X\times H_Y||H_{p'}\times H_{m'}|}{\sqrt{p!m!p'!m'!}} \sum_{\substack{a,b,c,d\\ i,j,k,l}}\ \sum_{\substack{t\vdash p\\ s\vdash m}}\ \sum_{\substack{w\vdash p'\\ v\vdash m'}}\ \sum_{\substack{\alpha_1,\alpha_2,\\ \beta_1,\beta_2}}\ \sum_{\substack{\mu_1,\mu_2\\ \nu_1,\nu_2}} \sqrt{d_td_s}\sqrt{d_wd_v}\,\Gamma^{(w,v)}_{ac;bd}(\sigma_2) \\
&\quad \times B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{ac;\mu_1\mu_2}\, B^{(w,v)\to 1_{H_{p'}\times H_{m'}}}_{bd;\nu_1\nu_2}\,\Gamma^{(t,s)}_{ik;jl}(\sigma_1) B^{(t,s)\to 1_{H_X\times H_Y}}_{ik;\alpha_1\alpha_2}\, B^{(t,s)\to 1_{H_X\times H_Y}}_{jl;\beta_1\beta_2} \\
&\quad \times \left\langle \mathcal{O}^\dagger_{T,((w,v),u)_{\vec{\mu}\vec{\nu}}}\right\rangle \mathcal{O}_{R,((t,s),r)_{\vec{\alpha}\vec{\beta}}} \\
&= \frac{1}{p!m!} \sum_{\substack{a,b,c,d\\ i,j,k,l}}\ \sum_{\substack{t\vdash p\\ s\vdash m}}\ \sum_{\substack{\gamma_1\\ \in H_X\times H_Y}}\ \sum_{\substack{\gamma_2\\ \in H_{p'}\times H_{m'}}} d_td_s\Gamma^{(t,s)}_{ac;bd}(\sigma_2)\Gamma^{(t,s)}_{ac;ik}(\gamma_1)\Gamma^{(t,s)}_{bd;jl}(\gamma_2)\Gamma^{(t,s)}_{ik;jl}(\sigma_1) \\
&\quad \times \delta_{RT}\delta_{ru}\delta_{wt}\delta_{vs}\delta_{\vec{\mu}\vec{\alpha}}\delta_{\vec{\nu}\vec{\beta}} \\
&= \sum_{a,c,j,l}\ \sum_{\substack{t\vdash p\\ s\vdash m}}\ \sum_{\substack{\gamma_1\\ \in H_X\times H_Y}}\ \sum_{\substack{\gamma_2\\ \in H_{p'}\times H_{m'}}} \frac{d_td_s}{p!m!}\Gamma^{(t,s)}_{ac;jl}\Big(\sigma_2\gamma_2\Big)\Gamma^{(t,s)}_{ac;jl}\Big(\gamma_1\sigma_1\Big) \\
&= \sum_{\substack{\gamma_1,\gamma_2\\ \in H_X\times H_Y}} \delta\Big(\gamma_1\sigma_1\gamma_2\sigma_2^{-1}\Big) \\
&= |\mathcal{O}(\sigma_1)|^2\delta_{[\sigma_1][\sigma_2]},
\end{aligned} \tag{C.72}$$

where the definition of $\delta_{[\sigma_1][\sigma_2]} = 1$ if $\sigma_1$ and $\sigma_2$ define the same Gauss graph and vanish otherwise, and we have used the fact that

$$\left\langle \mathcal{O}^\dagger_{T,((w,v),u)_{\vec{\mu}\vec{\nu}}}\right\rangle \mathcal{O}_{R,((t,s),r)_{\vec{\alpha}\vec{\beta}}} = \delta_{RT}\delta_{ru}\delta_{wt}\delta_{vs}\delta_{\vec{\mu}\vec{\alpha}}\delta_{\vec{\nu}\vec{\beta}}.$$

Our new normalized operator is given by

$$\left|\hat{\mathcal{O}}_{R,r}(\sigma_1)\right\rangle = \frac{1}{\sqrt{|\mathcal{O}(\sigma_1)|^2}} \left|\mathcal{O}_{R,r}(\sigma_1)\right\rangle.$$

The normalization factor is

$$|\mathcal{O}(\sigma_1)|^2 = \prod_{i=1}^{p} n_{ii}(\sigma_1)! \prod_{k,l=1,l\neq k}^{p} n_{kl}(\sigma_1)!. \tag{C.73}$$

Equation (C.73) is described in the paper [78]. The result for the one loop dilatation operator acting on the normalized Gauss graph basis is given by

$$D\left|\hat{\mathcal{O}}_{R,r}(\sigma_1)\right\rangle = g_{YM}^2 \sum_{T,u,(\sigma_2)} \left(\hat{M}^{\vec{m}\vec{n}}_{T,u;R,r} + \hat{M}^{\vec{p}\vec{n}}_{T,u;R,r} + \hat{M}^{\vec{p}\vec{m}}_{T,u;R,r}\right)\left|\hat{\mathcal{O}}_{T,u}(\sigma_2)\right\rangle, \tag{C.74}$$

where

$$\begin{aligned}\hat{M}^{\vec{m}\vec{n}}_{T,u;R,r} = &\frac{1}{\sqrt{|\mathcal{O}(\sigma_1)|^2|\mathcal{O}(\sigma_2)|^2}} \sum_{R'} \frac{\delta_{R_i'T_k'}\delta_{r_i'u_k'}}{p!(m-1)!}\sqrt{\frac{l_{r_i}l_{u_k}}{l_{R_i}l_{T_k}}}c_{RR'}c_{TT'} \sum_{\psi_1\in S_p\times S_m}\sum_{\psi_2\in S_p\times S_m}\\
&\Big[\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ii}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\\
&-\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}E^{(p+1)}_{kk}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{ii}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\\
&-\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}E^{(p+1)}_{ii}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}E^{(p+1)}_{kk}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\\
&+\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}E^{(p+1)}_{ii}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\Big]\end{aligned} \tag{C.75}$$

$$\begin{aligned}\hat{M}^{\vec{p}\vec{n}}_{T,u;R,r} = &\frac{1}{\sqrt{|\mathcal{O}(\sigma_1)|^2|\mathcal{O}(\sigma_2)|^2}} \sum_{R'} \frac{\delta_{R_i'T_k'}\delta_{r_i'u_k'}}{(p-1)!m!}\sqrt{\frac{l_{r_i}l_{u_k}}{l_{R_i}l_{T_k}}}c_{RR'}c_{TT'} \sum_{\psi_1\in S_p\times S_m}\sum_{\psi_2\in S_p\times S_m}\\
&\Big[\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ii}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\\
&-\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}E^{(1)}_{kk}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{ii}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\\
&-\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}E^{(1)}_{ii}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}E^{(1)}_{kk}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\\
&+\langle V,\vec{p}',\vec{m}'|\,\sigma_2\psi_2^{-1}E^{(1)}_{ii}\psi_1\,|V,\vec{p},\vec{m}\rangle\langle V,\vec{p},\vec{m}|\,\sigma_1^{-1}\psi_1^{-1}\psi_2\,|V,\vec{p}',\vec{m}'\rangle\Big]\end{aligned} \tag{C.76}$$

$$
\begin{aligned}
\hat{M}^{\vec{p}\vec{m}}_{T,u;R,r} = {}& \frac{1}{\sqrt{|\mathcal{O}(\sigma_1)|^2|\mathcal{O}(\sigma_2)|^2}} \sum_{R'} \frac{\delta_{R'_i T'_k}\delta_{ru}}{(p-1)!(m-1)!} \sqrt{\frac{c_{RR'}c_{TT'}}{l_{R_i} l_{T_k}}} \sum_{\psi_1 \in S_p \times S_m} \sum_{\psi_2 \in S_p \times S_m} \\
& \times \Big[ \langle V, \vec{p}', \vec{m}' | \, \sigma_2 \psi_2^{-1} E^{(1)}_{ki} \psi_1 \, | V, \vec{p}, \vec{m} \rangle \langle V, \vec{p}, \vec{m} | \, \sigma_1^{-1} \psi_1^{-1} E^{(p+1)}_{ik} \psi_2 \, | V, \vec{p}', \vec{m}' \rangle \\
& - \langle V, \vec{p}', \vec{m}' | \, \sigma_2 \psi_2^{-1} E^{(1)}_{ci} E^{(p+1)}_{kc} \psi_1 \, | V, \vec{p}, \vec{m} \rangle \langle V, \vec{p}, \vec{m} | \, \sigma_1^{-1} \psi_1^{-1} E^{(1)}_{ak} E^{(p+1)}_{ia} \psi_2 \, | V, \vec{p}', \vec{m}' \rangle \\
& - \langle V, \vec{p}', \vec{m}' | \, \sigma_2 \psi_2^{-1} E^{(1)}_{kc} E^{(p+1)}_{ci} \psi_1 \, | V, \vec{p}, \vec{m} \rangle \langle V, \vec{p}, \vec{m} | \, \sigma_1^{-1} \psi_1^{-1} E^{(1)}_{ia} E^{(p+1)}_{ak} \psi_2 \, | V, \vec{p}', \vec{m}' \rangle \\
& + \langle V, \vec{p}', \vec{m}' | \, \sigma_2 \psi_2^{-1} E^{(p+1)}_{ki} \psi_1 \, | V, \vec{p}, \vec{m} \rangle \langle V, \vec{p}, \vec{m} | \, \sigma_1^{-1} \psi_1^{-1} E^{(1)}_{ik} \psi_2 \, | V, \vec{p}', \vec{m}' \rangle \Big].
\end{aligned}
\tag{C.77}
$$

# References


[1] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999, hep-th/9711200. [Adv. Theor. Math. Phys.2,231(1998)].

[2] John McGreevy, Leonard Susskind, and Nicolaos Toumbas. Invasion of the giant gravitons from Anti-de Sitter space. *JHEP*, 06:008, 2000, hep-th/0003075.

[3] Marcus T. Grisaru, Robert C. Myers, and Oyvind Tafjord. SUSY and goliath. *JHEP*, 08:040, 2000, hep-th/0008015.

[4] Akikazu Hashimoto, Shinji Hirano, and N. Itzhaki. Large branes in AdS and their field theory dual. *JHEP*, 08:051, 2000, hep-th/0008016.

[5] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett.*, B428:105–114, 1998, hep-th/9802109.

[6] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998, hep-th/9802150.

[7] J. A. Minahan and K. Zarembo. The Bethe ansatz for N=4 superYang-Mills. *JHEP*, 03:013, 2003, hep-th/0212208.

[8] Vijay Balasubramanian, Micha Berkooz, Asad Naqvi, and Matthew J. Strassler. Giant gravitons in conformal field theory. *JHEP*, 04:034, 2002, hep-th/0107119.

[9] Steve Corley, Antal Jevicki, and Sanjaye Ramgoolam. Exact correlators of giant gravitons from dual N=4 SYM theory. *Adv. Theor. Math. Phys.*, 5:809–839, 2002, hep-th/0111222.

[10] Steven Corley and Sanjaye Ramgoolam. Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory. *Nucl. Phys.*, B641:131–187, 2002, hep-th/0205221.

[11] Robert de Mello Koch, Jelena Smolic, and Milena Smolic. Giant Gravitons - with Strings Attached (I). *JHEP*, 06:074, 2007, hep-th/0701066.

[12] Yusuke Kimura and Sanjaye Ramgoolam. Branes, anti-branes and brauer algebras in gauge-gravity duality. *JHEP*, 11:078, 2007, 0709.2158.

[13] Thomas William Brown, P. J. Heslop, and S. Ramgoolam. Diagonal multi-matrix correlators and BPS operators in N=4 SYM. *JHEP*, 02:030, 2008, 0711.0176.

[14] Thomas William Brown, P. J. Heslop, and S. Ramgoolam. Diagonal free field matrix correlators, global symmetries and giant gravitons. *JHEP*, 04:089, 2009, 0806.1911.

[15] Rajsekhar Bhattacharyya, Storm Collins, and Robert de Mello Koch. Exact Multi-Matrix Correlators. *JHEP*, 03:044, 2008, 0801.2061.

[16] Yusuke Kimura and Sanjaye Ramgoolam. Enhanced symmetries of gauge theory and resolving the spectrum of local operators. *Phys. Rev.*, D78:126003, 2008, 0807.3696.

[17] Yusuke Kimura. Non-holomorphic multi-matrix gauge invariant operators based on Brauer algebra. *JHEP*, 12:044, 2009, 0910.2170.

[18] Yusuke Kimura. Correlation functions and representation bases in free N=4 Super Yang-Mills. *Nucl. Phys.*, B865:568–594, 2012, 1206.4844.

[19] Jurgis Pasukonis and Sanjaye Ramgoolam. Quivers as Calculators: Counting, Correlators and Riemann Surfaces. *JHEP*, 04:094, 2013, 1301.1980.

[20] Vijay Balasubramanian, Min-xin Huang, Thomas S. Levi, and Asad Naqvi. Open strings from N=4 superYang-Mills. *JHEP*, 08:037, 2002, hep-th/0204196.

[21] David Berenstein. Shape and holography: Studies of dual operators to giant gravitons. *Nucl. Phys.*, B675:179–204, 2003, hep-th/0306090.

[22] Vijay Balasubramanian, David Berenstein, Bo Feng, and Min-xin Huang. D-branes in Yang-Mills theory and emergent gauge symmetry. *JHEP*, 03:006, 2005, hep-th/0411205.

[23] David Berenstein, Diego H. Correa, and Samuel E. Vazquez. Quantizing open spin chains with variable length: An Example from giant gravitons. *Phys. Rev. Lett.*, 95:191601, 2005, hep-th/0502172.

[24] David Berenstein, Diego H. Correa, and Samuel E. Vazquez. A Study of open strings ending on giant gravitons, spin chains and integrability. *JHEP*, 09:065, 2006, hep-th/0604123.

[25] Robert de Mello Koch, Jelena Smolic, and Milena Smolic. Giant Gravitons - with Strings Attached (II). *JHEP*, 09:049, 2007, hep-th/0701067.

[26] David Bekker, Robert de Mello Koch, and Michael Stephanou. Giant Gravitons - with Strings Attached. III. *JHEP*, 02:029, 2008, 0710.5372.

[27] Warren Carlson, Robert de Mello Koch, and Hai Lin. Nonplanar Integrability. *JHEP*, 03:105, 2011, 1101.5404.

[28] Robert de Mello Koch, Matthias Dessein, Dimitrios Giataganas, and Christopher Mathwin. Giant Graviton Oscillators. *JHEP*, 10:009, 2011, 1108.2761.

[29] Robert de Mello Koch, Garreth Kemp, and Stephanie Smith. From Large N Nonplanar Anomalous Dimensions to Open Spring Theory. *Phys. Lett.*, B711:398–403, 2012, 1111.1058.

[30] Robert de Mello Koch and Sanjaye Ramgoolam. A double coset ansatz for integrability in AdS/CFT. *JHEP*, 06:083, 2012, 1204.2153.

[31] N. Beisert, C. Kristjansen, and M. Staudacher. The Dilatation operator of conformal N=4 superYang-Mills theory. *Nucl. Phys.*, B664:131–184, 2003, hep-th/0303060.

[32] Niklas Beisert et al. Review of AdS/CFT Integrability: An Overview. *Lett. Math. Phys.*, 99:3–32, 2012, 1012.3982.

[33] Abdelhamid Mohamed Adam Ali, Robert de Mello Koch, Nirina Hasina Tahiridimbisoa, and Augustine Larweh Mahu. Interacting Double Coset Magnons. *Phys. Rev.*, D93(6):065057, 2016, 1512.05019.

[34] Jurgis Pasukonis and Sanjaye Ramgoolam. From counting to construction of BPS states in N=4 SYM. *JHEP*, 02:078, 2011, 1010.1683.

[35] Yusuke Kimura. Quarter BPS classified by Brauer algebra. *JHEP*, 05:103, 2010, 1002.2424.

[36] Jurgis Pasukonis and Sanjaye Ramgoolam. Quantum states to brane geometries via fuzzy moduli spaces of giant gravitons. *JHEP*, 04:077, 2012, 1201.5588.

[37] Vincent De Comarmond, Robert de Mello Koch, and Katherine Jefferies. Surprisingly Simple Spectra. *JHEP*, 1102:006, 2011, 1012.3884.

[38] Shaun de Carvalho, Robert de Mello Koch, and Augustine Larweh Mahu. Anomalous dimensions from boson lattice models. 2018, 1801.02822.

[39] Pierre Ramond. Dual Theory for Free Fermions. *Phys. Rev.*, D3:2415–2418, 1971.

[40] André Neveu and John H Schwarz. Tachyon-free dual model with a positive-intercept trajectory. *Physics Letters B*, 34(6):517–518, 1971.

[41] F. Gliozzi, Joel Scherk, and David I. Olive. Supersymmetry, Supergravity Theories and the Dual Spinor Model. *Nucl. Phys.*, B122:253–290, 1977.

[42] Slava Rychkov. EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions. 2016, 1601.05000.

[43] Philippe Di Francesco, Pierre Mathieu, and David Snchal. *Conformal field theory.* Graduate texts in contemporary physics. Springer, New York, NY, 1997.

[44] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000, hep-th/9905111.

[45] Edward Witten. Chern-Simons gauge theory as a string theory. *Prog. Math.*, 133:637–678, 1995, hep-th/9207094.

[46] Edward Witten. Perturbative gauge theory as a string theory in twistor space. *Commun. Math. Phys.*, 252:189–258, 2004, hep-th/0312171.

[47] Edward Witten. Baryons in the 1/n Expansion. *Nucl. Phys.*, B160:57–115, 1979.

[48] Gerard 't Hooft. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys.*, B72:461, 1974. [,337(1973)].

[49] W. Fulton and J. Harris. *Representation Theory: A First Course.* Graduate Texts in Mathematics. Springer New York, 1991.

[50] D.P. Zhelobenko. *Compact Lie Groups and Their Representations.* Translations of mathematical monographs. American Mathematical Society, 1973.

[51] Rajsekhar Bhattacharyya, Robert de Mello Koch, and Michael Stephanou. Exact Multi-Restricted Schur Polynomial Correlators. *JHEP*, 06:101, 2008, 0805.3025.

[52] Robert de Mello Koch, Nirina Hasina Tahiridimbisoa, and Christopher Mathwin. Anomalous Dimensions of Heavy Operators from Magnon Energies. *JHEP*, 03:156, 2016, 1506.05224.

[53] Hai Lin, Oleg Lunin, and Juan Martin Maldacena. Bubbling AdS space and 1/2 BPS geometries. *JHEP*, 10:025, 2004, hep-th/0409174.

[54] David Berenstein, Diego H. Correa, and Samuel E. Vazquez. All loop BMN state energies from matrices. *JHEP*, 02:048, 2006, hep-th/0509015.

[55] Diego M. Hofman and Juan Martin Maldacena. Reflecting magnons. *JHEP*, 11:063, 2007, 0708.2272.

[56] Diego M. Hofman and Juan Martin Maldacena. Giant Magnons. *J. Phys.*, A39:13095–13118, 2006, hep-th/0604135.

[57] Robert de Mello Koch, Stuart Graham, and Ilies Messamah. Higher Loop Nonplanar Anomalous Dimensions from Symmetry. *JHEP*, 02:125, 2014, 1312.6227.

[58] Niklas Beisert. The SU(2—2) dynamic S-matrix. *Adv. Theor. Math. Phys.*, 12:945–979, 2008, hep-th/0511082.

[59] Robert de Mello Koch, Garreth Kemp, Badr Awad Elseid Mohammed, and Stephanie Smith. Nonplanar integrability at two loops. *JHEP*, 10:144, 2012, 1206.0813.

[60] Eric Dzienkowski. Excited States of Open Strings From $\mathcal{N} = 4$ SYM. *JHEP*, 12:036, 2015, 1507.01595.

[61] David Berenstein. Giant gravitons: a collective coordinate approach. *Phys. Rev.*, D87(12):126009, 2013, 1301.3519.

[62] David Berenstein and Eric Dzienkowski. Open spin chains for giant gravitons and relativity. *JHEP*, 08:047, 2013, 1305.2394.

[63] David Berenstein and Eric Dzienkowski. Giant gravitons and the emergence of geometric limits in beta-deformations of $\mathcal{N} = 4$ SYM. *JHEP*, 01:126, 2015, 1408.3620.

[64] David Berenstein. On the central charge extension of the $\mathcal{N} = 4$ SYM spin chain. *JHEP*, 05:129, 2015, 1411.5921.

[65] Nikolay Gromov, Vladimir Kazakov, Sebastien Leurent, and Dmytro Volin. Quantum Spectral Curve for Planar $\mathcal{N} = 4$ Super-Yang-Mills Theory. *Phys. Rev. Lett.*, 112(1):011602, 2014, 1305.1939.

[66] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo. The Algebraic curve of classical superstrings on AdS(5) x S**5. *Commun. Math. Phys.*, 263:659–710, 2006, hep-th/0502226.

[67] Thomas William Brown. Permutations and the Loop. *JHEP*, 06:008, 2008, 0801.2094.

[68] Robert de Mello Koch, Grant Mashile, and Nicholas Park. Emergent Threebrane Lattices. *Phys. Rev.*, D81:106009, 2010, 1004.1108.

[69] Robert de Mello Koch, Christopher Mathwin, and Hendrik J. R. van Zyl. LLM Magnons. *JHEP*, 03:110, 2016, 1601.06914.

[70] Robert de Mello Koch, Stuart Graham, and Wandile Mabanga. Subleading corrections to the Double Coset Ansatz preserve integrability. *JHEP*, 02:079, 2014, 1312.6230.

[71] F. Lingua, M. Guglielmino, V. Penna, and B. Capogrosso Sansone. Demixing effects in mixtures of two bosonic species. *Phys. Rev. A*, 92:053610, Nov 2015.

[72] Tapan Mishra, Ramesh V. Pai, and B. P. Das. Phase separation in a two-species bose mixture. *Phys. Rev. A*, 76:013604, Jul 2007.

[73] Andrei Mikhailov. Giant gravitons from holomorphic surfaces. *JHEP*, 11:027, 2000, hep-th/0010206.

[74] Chris E. Beasley. BPS branes from baryons. *JHEP*, 11:015, 2002, hep-th/0207125.

[75] Indranil Biswas, Davide Gaiotto, Subhaneil Lahiri, and Shiraz Minwalla. Supersymmetric states of N=4 Yang-Mills from giant gravitons. *JHEP*, 12:006, 2007, hep-th/0606087.

[76] Gautam Mandal and Nemani V. Suryanarayana. Counting 1/8-BPS dual-giants. *JHEP*, 03:031, 2007, hep-th/0606088.

[77] Robert de Mello Koch and Lwazi Nkumane. From Gauss Graphs to Giants. 2017, 1710.09063.

[78] Robert de Mello Koch, Rocky Kreyfelt, and Stephanie Smith. Heavy Operators in Superconformal Chern-Simons Theory. *Phys. Rev.*, D90(12):126009, 2014, 1410.0874.

[79] Eric D'Hoker and Daniel Z. Freedman. Supersymmetric gauge theories and the AdS / CFT correspondence. In *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pages 3–158, 2002, hep-th/0201253.

[80] M. Luscher and G. Mack. Global Conformal Invariance in Quantum Field Theory. *Commun. Math. Phys.*, 41:203–234, 1975.

[81] Robert de Mello Koch. Geometries from Young Diagrams. *JHEP*, 11:061, 2008, 0806.0685.