MASTERS IN MANAGEMENT FINANCE AND INVESTMENTS

Valuation and hedging of Loan Prepayment risk: An options pricing approach

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Abstract

This thesis discusses loan prepayment risk and describes a binomial model that can be used to value and hedge this risk. Borrowers have an option to repay lenders before the respective loans reach maturity. In the particular case of fixed rate loans and mortgages, this poses risks to lenders. If prepayment occurs when market interest rates have declined, lenders are forced to replace the high interest loans with low interest ones, thereby reducing their net interest margin. The option to repay loans early is an American call option on a bond. Bonds with embedded options can be valued using a binomial interest rate tree. In this thesis, the Black Derman Toy binomial model is calibrated to the South African continuous term structure of interest rates and interest rate cap volatilities. Prepayment risk is then valued as the embedded call option for fixed rate loans of various maturities. Option greeks delta, vega and gamma are described and calculated. Various instruments that exist in the South African fixed income market and can be used to hedge prepayment risk are presented.

Declaration

I Mduduzi Dumisani Dube declare that this Thesis is my own, unaided work. It is being submitted in partial fulfilment for the Degree of Master of Management in Finance and Investments at the Wits Business School, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signature of Candidate

07 Oct 2022

Date

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I would like to thank my family and friends for all the encouragement and support.

Contents

1 Introduction **6**
1.1 Overview of Loan markets . . . 6
1.2 South African Loan Market . . . 6
1.3 Structure of the thesis . . . 7

2 Prepayment risk **8**
2.1 Empirical studies . . . 8
2.2 Contingent claims approach . . . 8

3 The Short Rate Model **10**
3.1 Outline . . . 10
3.2 Implementation . . . 11
3.3 Market Inputs . . . 13
3.3.1 Term structure of interest rates . . . 13
3.3.2 Volatility . . . 14

4 Option valuation **16**
4.1 Approach . . . 16
4.2 Results . . . 17

5 Sensitivity Analysis **19**
5.1 Interest rate delta . . . 19
5.2 Vega . . . 20
5.3 Gamma . . . 21

6 Conclusion **22**
6.1 Hedging . . . 22
6.2 Further work . . . 22

Appendices **25**

List of Tables

3.3.1 Interest rate tree out to 1.5yrs 15
3.3.2 Calibration results 15

4.1.1 Interest rate tree out to 1yr 16
4.1.2 Cashflows of a bond with no embedded option 16
4.1.3 Cashflows of a bond with an embedded call option 17
4.2.1 Option Values 18

.0.1 Term structure of continuous rates 26
.0.2 ATM caplet forward volatility 26

List of Figures

1.2.1 South African Banks Balance Sheets Feb 2022 . . . 7

3.1.1 forward rates . . . 10
3.2.1 Binomial tree of short rates out to $3\Delta t$ . . . 11
3.3.1 Bloomberg curve bootstrapping . . . 14
3.3.2 Bloomberg forward caplet volatilities . . . 14

4.2.1 Option values for increasing maturity . . . 18

5.1.1 IR Delta by maturity . . . 20
5.2.1 Vega by maturity . . . 21
5.3.1 Gamma . . . 21

Chapter 1

Introduction

1.1 Overview of Loan markets

Banks offer a variety of loan financing options for their clients. These loans commonly come in the form of amortizing loans where principal and interest are serviced at regular intervals or bullet style loans consisting of regular interest payments and principal at maturity of the loan. Implicit in these loans is the ability for the borrower to repay the loans earlier before the full term.

Corporate borrowers tend to refinance debt when their credit quality improves or when market interest rates drop and borrowing becomes cheaper even after taking transaction costs associated with refinancing into account. They also tend to pay off debt by issuing equity in a bid to optimize their capital structure to reduce weighted average cost of capital. The repayment behaviour of personal borrowers is usually driven by factors unrelated to interest rate market conditions for example insurance pay outs in case of death.

The option for a borrower to repay a loan early is a risk to the lender. The lender forgoes future interest income and must replace that loan in its portfolio of assets. Depending on prevailing interest rate market conditions, if rates have since decreased from the time the loan was extended, the lender might be forced to replace with a lower margin loan leading to a reduction in net interest income. Loans are of either fixed or variable interest rates type. Fixed rate loans pose the most prepayment risk to lenders as borrowers are most likely to refinance when interest rates decline. Variable interest rate loans reference a benchmark e.g. the outgoing LIBOR[1] published by the Intercontinental Exchange (ICE).

1.2 South African Loan Market

Traditional commercial banking institutions dominate the loan market in South Africa. The majority of these loans are of floating interest rate[2] e.g. prime linked loans for consumers and JIBAR[3] linked loans for corporations. The balance sheets (in ZAR millions) for ABSA (ABG), Firstrand (FSR) Nedbank (NED), Standard Bank (SBK) and Capitec (CPI) respectively is given in Fig 1.2.1 courtesy of the South African Reserve Bank BA900 monthly returns [4]. By lending at a floating rate, the future costs or benefits of the level of interest rates are passed directly onto the borrower which leads to no incentive for refinancing due to decreases in market rates and thereby reducing prepayment risk for the lender.

The biggest lenders in the South African economy are listed and their end of financial year results are made publicly available. In the respective risk management sections of the results, FirstRand Bank Limited,[5] ABSA[6] and Standard

[1] London Interbank Rate, see https://www.theice.com/iba/libor for more information on this benchmark

[2] See https://www.resbank.co.za/en/home/what-we-do/statistics/key-statistics/current-market-rates for the definition of various South African market interest rates and historical data

[3] Johannesburg Interbank Average Rate

[4] https://www.resbank.co.za/en/home/what-we-do/statistics/releases/banking-sector-information/banks-ba900-economic-returns

[5] https://www.firstrand.co.za/media/investors/financial-results/firstrand-bank-annual-report-2021.pdf

[6] https://www.absa.africa/content/dam/africa/absaafrica/pdf/2021/absa-group-annual-consolidated-and-separate-financial-statements.pdf

Bank Group[7] disclose concerns around prepayment risks on the fixed rate loans in their books. They mention the use of interest rate swaps to hedge this risk and will periodically rebalance these hedges. The notional size and maturity profiles of the fixed rate loans in their balance sheets are considered proprietary information and banks will not divulge this information as competitors will gain insights into their interest rate risk profiles and therefore profit and loss drivers.

Rbn - February 2022	System	ABG	FSR	NED	SBK	CPI
Core loans	**4,055.5**	**910.0**	**840.3**	**754.1**	**991.1**	**83.9**
Total retail advances	**1,882.2**	**423.7**	**404.2**	**327.8**	**529.6**	**75.3**
Mortgages	1,105.5	262.2	219.4	156.6	385.7	3.6
Instalment finance	332.3	77.5	81.0	121.5	46.2	0.2
Unsecured retail	444.4	84.0	103.9	49.7	97.7	71.5
Credit Card	134.4	34.9	33.2	15.8	34.2	6.7
Overdrafts	44.6	13.0	11.7	3.7	15.8	0.3
Other unsecured	265.4	36.1	59.0	30.2	47.8	64.5
Non-retail advances	**2,169.5**	**486.3**	**436.1**	**426.3**	**461.5**	**8.7**
Commercial property	558.7	94.9	43.8	192.4	104.8	3.1
Non-retail instalment finance	158.5	34.0	37.1	21.0	52.2	0.7
Overdrafts	184.3	52.5	56.5	17.9	35.5	1.7
General Commercial	1,268.0	305.0	298.8	195.0	269.0	3.2
Provisions	**-166.6**	**-32.4**	**-35.6**	**-23.8**	**-41.3**	**-17.5**
Private sector debt instruments	167.8	7.4	59.5	20.8	52.2	0.0
Repos and intebanks assets	723.7	140.3	119.0	68.6	179.2	29.0
Liquid assets	1,164.3	173.2	282.4	181.5	225.6	63.0
Total Assets	**6,703.1**	**1,352.9**	**1,432.1**	**1,095.2**	**1,619.9**	**175.5**
Total deposits	**4,602.4**	**955.3**	**998.0**	**829.1**	**1,050.5**	**133.5**
Retail deposits	1,523.2	333.4	349.8	245.6	302.3	126.2
Commercial deposits	1,509.5	282.9	421.1	253.3	357.3	5.8
Wholesale funding	1,398.8	251.2	217.6	308.7	356.5	1.4
Interbank funding	170.9	87.8	9.5	21.5	34.4	0.1
Foreign Currency Funding	282.2	55.8	61.3	17.7	68.7	1.3
Repo & Collat borrowing	168.6	87.1	21.6	25.1	13.4	0.0
Total Equity	**549.5**	**100.9**	**104.8**	**78.7**	**107.3**	**32.3**

Figure 1.2.1: South African Banks Balance Sheets Feb 2022

1.3 Structure of the thesis

In the following chapter 2, we outline prepayment risk and review previous literature on the subject. In chapter 3, the Black Derman Toy model of the evolution of short term interest rates developed by Black et al. (1990) is presented. Its implementation using binomial trees as done by Hull (2020) is also outlined. In chapter 4 we then show how call options on fixed rate bonds of varying maturities and coupons are valued using a binomial tree that has been calibrated to market prices. Thereafter in chapter 5 results are discussed and a sensitivity analysis is conducted to illustrate how different variables influence the magnitude of prepayment risk. A discussion on the current available market instruments in the South African market that can be used to hedge this risk and the conclusion follow in chapter 6. After the hedging discussion, the *python* code developed to carry out the numerical computation is presented in the Appendix for readers who wish to delve into the technical details.

[7] https://thevault.exchange/?get_group_doc=18/1615894015-SBG2020AnnualFinancialStatements.pdf

Chapter 2

Prepayment risk

2.1 Empirical studies

Various methods of assessing prepayment risk exist. Schwartz and Torous (1989) have proposed a model based on empirical study. They fit past prepayment rates to a set of explanatory variables and use this resultant function with coefficients calculated from historical data to explain future prepayment behaviour. David Ho and Su (2006) study the structural and behavioural experience of the risk of prepayment in the Chinese residential mortgage life insurance market. They use 4 years' worth of data on 1000 Shanghai residential mortgagors between January 1999 and December 2003 and propose a hazard rate is model dependant on factors that include monthly incomes of borrowers, growth in GDP etc. By using macroeconomic variables and financial ratios for banks over the 1976 to 2006 period, Fayman and He (2011) conduct a study on the effect of prepayment risk in the performance of USA commercial Banks and find that prepayment risk significantly impacts return on loans, return on equity and real estate loans to total loans ratios.

2.2 Contingent claims approach

Emprical studies do not address the question of what value the lender must be compensated for taking on the risk of replacing the fixed rate assets with lower margin loans as a result of the decrease in market rates. Sherris (2002) shows that taking a contingent claims approach, the early repayment option a borrower has is equivalent to a call option on the bond. This is because as interest rates decrease, bond prices increase and the borrower has the option to repay their loan at the inception price, making this option in the money, the opposite is true for when rates rise. The value of this option is the difference between the value of the outstanding loan payments at the new market rate and the value of the same outstanding loan payments at the original interest rate i.e. at inception of the loan. The call option is of American type as it can be exercised at any time. The lender has to be compensated for the risks they take on due to the prepayment option. Cossin and Lu (2004) also propose that prepayment penalties must be viewed as fees in arrears reflecting the value of a call option at origination. This contingent claims approach enables us to place a monetary value on prepayment risk.

To value an option, we must consider the potential future states of the risk factor underlying the option. In the case of bond options, we are concerned with future states of the term structure of interest rates. The volatility of future interest rates is of vital importance as it will determine the probability distribution of the underlying risk factor. Increasing volatility implies an increase in the uncertainty of future levels of interest rates. This means odds are higher that rates will either decrease or increase by a large magnitude. Increased odds of future rates declining means the risk of prepayment increases as well. As the COVID-19 pandemic set in, Global Central banks became active in the primary and secondary markets for bonds by cutting rates in order to avert total economic collapse by improving liquidity in credit markets to As the world better understands the COVID-19 pandemic, levels of economic activity have increased. This has led to interest rate hikes being undertaken and also predicted for the future. This increase in interest rate volatility increases the magnitude of prepayment risk present on lenders' balance sheets.

Methods have been put forward for valuing American options on Bonds. For example, Wilmott (2013) begins from stochastic processes for the short term rate and by using replication and delta hedging derives the partial differential equations for the value of the options, which are adaptations of the Black Scholes equation used to value options on non-dividend paying stocks. The short coming of this type of method is that it is not easily amenable to varying cashflow structures of the underlying bond. The binomial lattice method easily adapts to either bullet style or amortising loans. Following the approach by Kalotay et al. (1993), the value of such bond options is calculated as the difference between the value of a bond with this option embedded and the value of the same bond without the option.

Chapter 3

The Short Rate Model

3.1 Outline

The valuation of contingent claims requires information about the volatility of interest rates. We have to model the evolution of term structure of interest rates. This is done using the concept of a future short term rate or the spot interest rate. Theoretically this is the rate that applies in the future for an instantaneous maturity. Practically, the duration is taken to be the shortest maturity with sufficient liquidity. The assumption made is that the future short term interest rate is a stochastic process that follows Brownian motion.

An illustration of forward rates is shown in figure 3.1.1.

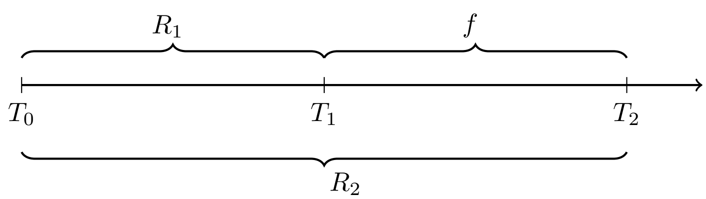


Figure 3.1.1: forward rates

If R_1 and R_2 are continuously compounded zero rates that apply between time T_0 and T_1 and T_2 respectively, and the forward rate f applies between T_1 and T_2, then for no arbitrage the following relationship must hold,

$$e^{R_1 T_1} e^{f(T_2 - T_1)} = e^{R_2 T_2}$$

which implies,

$$f = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1} \tag{3.1.1}$$

For a cashflow that occurs at T_2, its value at T_1 is found by discounting at rate f for the period $T_2 - T_1$.

There are two main categories of short rate models, namely **Equilibrium models** such as the earliest model by Vasicek (1977) and Cox et al. (1985) and **No Arbitrage models** for example Ho and Lee (1986) and Black et al. (1990). Equilibrium models make assumptions about the economic variables and then derive a process for the short rate r. Bond and option prices are an output of the model. No arbitrage models on the other hand have parameters which are calibrated to the market price of traded securities to ensure that the term structure model produces values for zero coupon bonds which are equal to the observed market values. This makes them preferable over Equilibrium models.

We make use of the Black-Derman-Toy (BDT) model outlined in the paper by Black et al. (1990). This is based on a binomial tree implementation where the future short term rate follows a lognormal process. The advantage with this assumption is it precludes negative interest rates which is consistent with South Africa and emerging markets in general. Svoboda (2002) also finds that the BDT model is more applicable in the South African market due to the nature of the instruments that trade in the local market and the ease of calibration. The BDT model was also developed by practitioners and this makes it a popular choice among practitioners.

The analytical BDT model is given by

$$d \ln r = [\theta(t) - a(t) \ln r] \, dt + \sigma(t) dZ \tag{3.1.2}$$

where

- r is the short rate
- $\theta(t)$ and $a(t)$ are functions chosen to match the current term structure of interest rates and volatility
- $\sigma(t)$ is the volatility, taken to be a deterministic function of time, i.e. no randomness
- dZ is a Weiner process

3.2 Implementation

We follow the discrete time implementation of the BDT model outlined by Hull (2020) as follows:

- we set step length of Δt, taken to be 3 months as this is a tenor with liquid instruments in the South African market
- zero coupon rates for all periods are known, the rate for maturity $n\Delta t$ is R_n
- in each Δt, the probability of an increase or a decrease in the short rate is 50% as we are valuing in a risk-neutral world

The binomial model is given below.

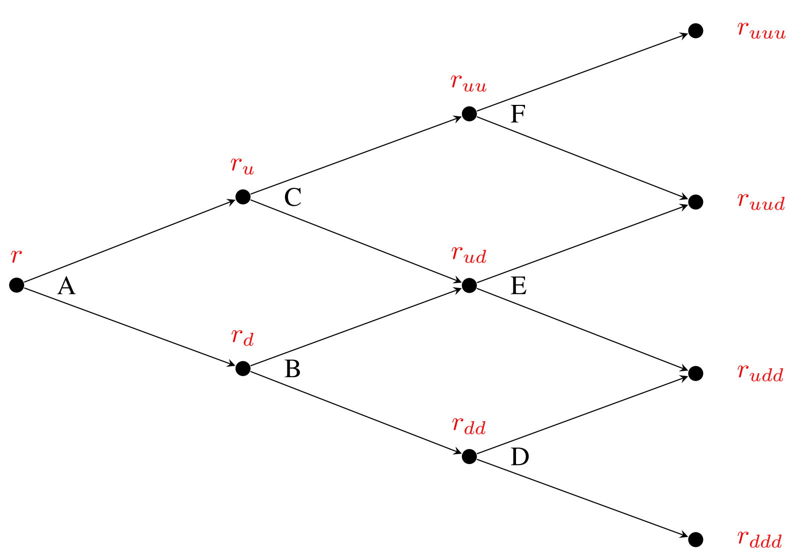


Figure 3.2.1: Binomial tree of short rates out to $3\Delta t$

The value of a zero coupon bond that matures at $2\Delta t$ with face value ZAR 1.00 is given by

$$e^{-R_2 2\Delta t}$$

After 1 time step Δt, at node B, the value is given by

$$e^{-r_d \Delta t}$$

and at node C by

$$e^{-r_u \Delta t}$$

$\therefore$ the value of this bond today, i.e. node A is given below

$$e^{-r\Delta t}\left[0.5e^{-r_d\Delta t}+0.5e^{-r_u\Delta t}\right]=e^{-R_2 2\Delta t} \tag{3.2.1}$$

We need to calibrate volatility to the market. The standard deviation of the log of the rate at time t must be $\sigma_1\sqrt{\Delta t}$. The rate can take on two possible values, r_u and r_d, i.e. this is a random variable with 2 possible outcomes. The standard deviation of this variable will be given by

$$\begin{aligned}
\text{Var (X)} &= \mathbb{E}[X^2]-\mathbb{E}[X]^2\\
&= \frac{1}{2}\left[(\ln r_u)^2+(\ln r_d)^2\right]-\left[\frac{\ln r_u+\ln r_d}{2}\right]^2\\
&= \frac{1}{2}\left(\ln r_u-\ln r_d\right)\\
\therefore \text{s.d} &= \frac{\ln r_u-\ln r_d}{2}
\end{aligned}$$

which leads to.

$$\sigma_1\sqrt{\Delta t}=\frac{1}{2}\ln\left(\frac{r_u}{r_d}\right) \tag{3.2.2}$$

Equations 3.2.2 and 3.2.1 can then be solved simultaneously to calculate r_u and r_d.

To calculate r_{uu}, r_{ud} and r_{dd} first we observe that the standard deviation is calculated as below

$$\frac{1}{2}\ln\left(\frac{r_{uu}}{r_{ud}}\right)=\sigma_2\sqrt{\Delta t} \tag{3.2.3}$$

$$\frac{1}{2}\ln\left(\frac{r_{ud}}{r_{dd}}\right)=\sigma_2\sqrt{\Delta t} \tag{3.2.4}$$

From equations 3.2.3 and 3.2.4, the below results hold

$$\begin{aligned}
r_{ud} &= r_{dd}e^{2\sigma_2\sqrt{\Delta t}}\\
r_{uu} &= r_{ud}e^{2\sigma_2\sqrt{\Delta t}}\\
&= r_{dd}e^{4\sigma_2\sqrt{\Delta t}}
\end{aligned} \tag{3.2.5}$$

For a bond that matures at $3\Delta t$ with face value ZAR 1.00, its value at node B is given by

$$e^{-r_d\Delta t}\left(0.5e^{-r_{dd}\Delta t}+0.5e^{-r_{ud}\Delta t}\right) \tag{3.2.6}$$

and at node C the value is given by

$$e^{-r_u\Delta t}\left(0.5e^{-r_{ud}\Delta t}+0.5e^{-r_{uu}\Delta t}\right) \tag{3.2.7}$$

$\therefore$ at T_0 the value is given by the below

$$\begin{aligned} & e^{-r\Delta t}\left[0.5e^{-r_d\Delta t}\left(0.5e^{-r_{dd}\Delta t}+0.5e^{-r_{ud}\Delta t}\right)\right] \\ & \qquad + e^{-r\Delta t}\left[0.5e^{-r_u\Delta t}\left(0.5e^{-r_{ud}\Delta t}+0.5e^{-r_{uu}\Delta t}\right)\right] = e^{-R_3 3\Delta t} \end{aligned} \tag{3.2.8}$$

Since we already calculated r_u and r_d, we can use equations (3.2.3) (3.2.4) and (3.2.8) to calculate r_{dd}, r_{ud} and r_{uu}. Solving this system of equations analytically would be cumbersome therefore the Newton-Raphson numerical root finding algorithm is used. All interest rates for a particular node are linked by the volatility relationship (see example in equation 3.2.5). We can then choose a suitable first guess for the rate r_{dd} and then goal-seek to find r_{ud} and r_{uu}. This procedure is followed to extend the interest rate tree further including more nodes. Hull (2020) uses annual compounding for rates, in this implementation, continuous compounding is used.

3.3 Market Inputs

3.3.1 Term structure of interest rates

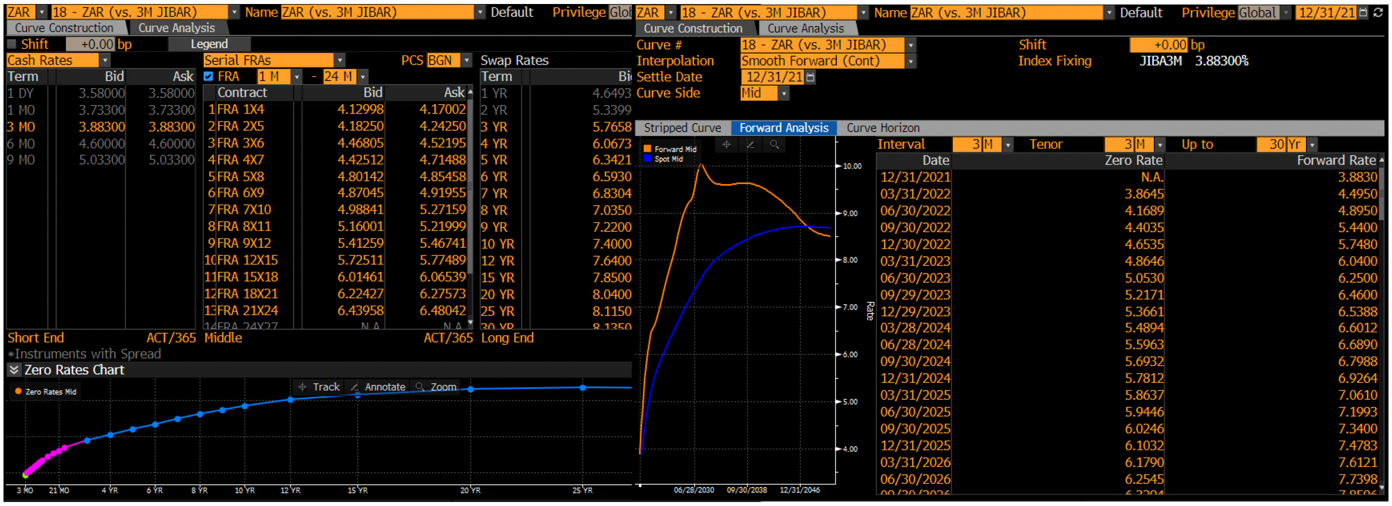
The term structure of zero rates, or zero curve is calculated from ZAR interest rate derivatives in a process called *bootstrapping*. For the short end of the curve, i.e.1 day out to 3 months, money market loans and deposits are used. The mid-section of the curve is populated with zero rates calculated from FRAs[1]. FRAs are over the counter agreements to pay or receive a future interest rate based on the JIBAR rate fixing. These contracts are used to hedge interest rate risk that is anticipated in the future and/or speculative purposes.

From 3 years out to 30 years, interest rate swaps are used. These are contracts where the counterparties agree to exchange fixed rates for JIBAR every quarter. Market rates are sourced from Bloomberg. The ZAR Swap curve is used as the term structure of interest rates.

Banks calculate their firm specific funding costs taking into account different sources of funds. In calculating the JIBAR rate, the major South African banks are polled for deposit rates for tenors of 1,3,6,9, and 12 months. The highest and the lowest are discarded and the rest averaged. This makes the Swap curve, which is stripped off instruments paying 3M JIBAR, a good indication of the average funding costs of banks in the economy.

The bootstrapping procedure is not outlined in this project as the focus will be on modelling future states of the term structure and using these variations to calculated the expected option value. Mahomed (2015) is a good source the outline of the South African fixed income market and the mechanics of the derivatives that trade in that market. For curve construction literature, Hagan and West (2006) explains how to recover zero rates from market instruments and the different interpolation techniques. The Bloomberg **ICVS** service provides this functionality and is used to obtain the South African swap zero curve as at **31 December 2021.** The term structure is given in table .0.1 of the appendix. Continuously compounded rates are used for the binomial tree.

[1] Forward Rate Agreements



(a) curve inputs (b) curve outputs

Figure 3.3.1: Bloomberg curve bootstrapping

3.3.2 Volatility

The approach taken by Boyle et al. (2001) is used. To make calibration arbitrage free and simpler, JIBAR at the money (ATM) forward cap volatilities are used. Caps trade in a liquid market in South Africa. A caplet is a financial contract that gives the holder the right but not obligation to pay a known interest rate over a set period in the future. A cap is a series of adjacent caplets. The ability to choose whether or not to exercise differentiates a caplet from the FRA. Bloomberg's **VCUB** service allows us to obtain this market data. The market data page is shown in figure 3.3.2. We receive annualised volatility from Bloomberg and have to convert these to match the interval being used in the model by scaling with the square root of the chosen time interval. For the set of vols obtained in Bloomberg $\{\sigma_i, i = 1, 2, 3....\}$, the volatilities used in the model are given by $\{\sigma_i \times \sqrt{\Delta t}, i = 1, 2, 3...\}$. The forward caplet volatilities are given in Table .0.2 of the appendix.

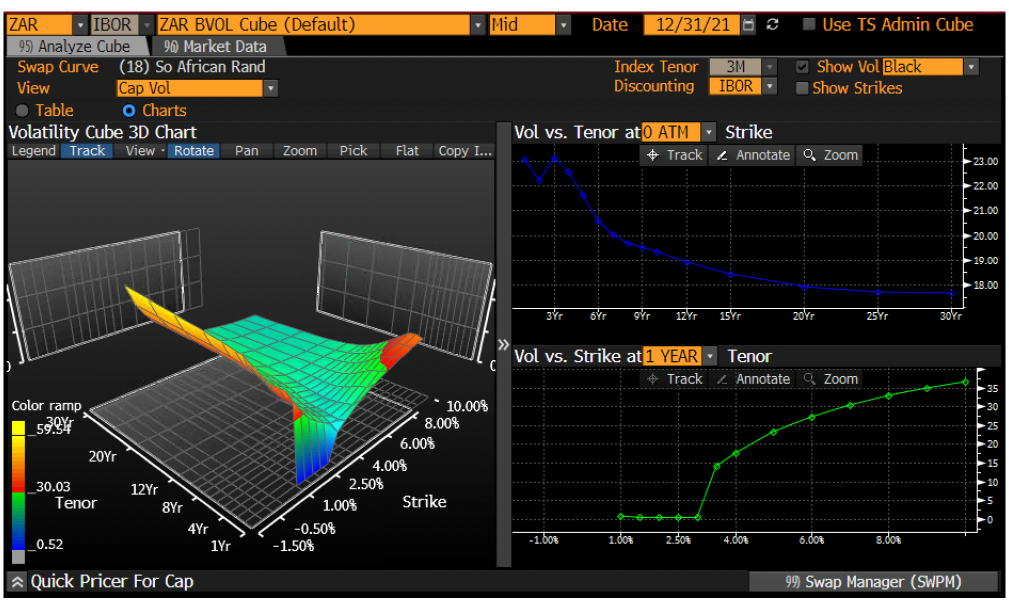


Figure 3.3.2: Bloomberg forward caplet volatilities

Calibration results

As an illustration, the interest rate tree out to 1.5 years is shown below.

Node	1	2	3	4	5	6
						8.8704%
					7.8459%	7.5370%
				6.8436%	6.6495%	6.4040%
			5.7362%	5.8084%	5.6355%	5.4413%
		4.6689%	4.8382%	4.9298%	4.7762%	4.6234%
	3.8645%	4.2779%	4.0808%	4.1842%	4.0479%	3.9284%

Table 3.3.1: Interest rate tree out to 1.5yrs

Zero coupon bonds with face value ZAR$10,000,000$ are used. A comparison of the bond prices generated by the calibrated model and those from discounting off the term structure is given below. Minimal differences can be seen therefore our model is consistent with market prices.

	Node														
Maturity	1	2	3	4	5	6	7	...	73	74	75	76	77	78	79	80
Model PV	9 903 851.98	9 793 710.66	9 675 129.90	9 545 311.27	9 410 039.30	9 270 058.53	9 127 439.25	...	2 111 155.26	2 061 810.06	2 013 680.04	1 966 600.17	1 920 968.12	1 876 338.02	1 832 708.32	1 790 262.40
Market PV	9 903 851.98	9 793 710.66	9 675 129.90	9 545 311.27	9 410 039.30	9 270 058.53	9 127 439.26	...	2 111 155.22	2 061 810.06	2 013 680.04	1 966 600.17	1 920 968.12	1 876 338.02	1 832 708.32	1 790 262.40
Difference	0.00	0.00	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	...	0.04	0.00	0.00	(0.00)	0.00	(0.00)	(0.00)	0.00

Table 3.3.2: Calibration results

Chapter 4

Option valuation

4.1 Approach

The approach taken to value the embedded options is given by

Value of option = Value of option free bond - Value of bond with embedded option

We illustrate this by means of an example. A 1 year bond with ZAR $1,000,000$ face value and a coupon rate of 5% per annum compounded quarterly is used. Although we use a bullet style loan with fixed interest payments, the binomial tree can be used to value loans with any cashflow structure as it uses periodic discounting. We value the case of a pure bond with no embedded option and then the same bond but with the call option where the holder can sell it for face value, which represents the prepayment risk inherent.

The interest rate tree out to 1yr is as given below:

Node	1	2	3	4
				6.8436%
			5.7362%	5.8084%
		4.6689%	4.8382%	4.9298%
	3.8645%	4.2779%	4.0808%	4.1842%

Table 4.1.1: Interest rate tree out to 1yr

The bond cashflows are illustrated below.

Nodes			
1	2	3	4
			995 324.48
		994 746.31	
	997 828.06		997 903.69
1 003 178.62		999 339.92	
	1 003 007.20		1 000 097.91
		1 003 243.53	
			1 001 964.03

Table 4.1.2: Cashflows of a bond with no embedded option

At maturity i.e. node 4, the bond pays the face value ZAR$1,000,000$ plus the coupon ZAR$1,000,000 \times \frac{5\%}{4} =$ ZAR$12,500$. Using backward induction, at the start of node 4 the bond can take on 4 possible values since our rates tree has 4 possible observations of the one period forward rate.

Using the upper most observation as an example, the value of the bond is given by

$$1\,012\,500\,\mathrm{e}^{-6.8436\%\times 0.25} = \text{ZAR}\,995\,324.48$$

Prior to maturity, at each node the value of the bond is its discounted expected value. Using the upper most value in node 3 as an example, the value of the bond is given by

$$\left(\frac{995\,324.48 + 997\,903.69}{2} + 12\,500\right)\,\mathrm{e}^{\,-5.7362\%\times 0.25} = \text{ZAR}\,994\,746.31$$

Continuing with backward induction, the bond value is calculated to be ZAR1 003 178.63.

A similar approach is taken in valuing a bond with an embedded option, with the important difference that at any node on the valuation tree, if the value of the bond is greater than its face value, which is what the borrower owes to the lender, the borrower can refinance and repay the initial obligation i.e. the face value of the bond. The borrower can refinance at any point in the life of the bond, making the option of American type. This valuation approach is illustrated below.

Nodes			
1	2	3	4
			995 324.59
		994 746.31	
	997 828.06		997 903.69
1 001 677.64		999 339.92	
	~~1 001 378.76~~ 1 000 000.00		~~1 000 097.91~~ 1 000 000.00
		~~1 002 223.02~~ 1 000 000.00	
			~~1 001 964.03~~ 1 000 000.00

Table 4.1.3: Cashflows of a bond with an embedded call option

The bond with the embedded call option is valued at ZAR1 001 677.64. This leads to a value of the option of ZAR1 003 178.63 − ZAR1 001 677.64 = **ZAR 1 500.98**

4.2 Results

The values per maturity are given in table 4.2.1

Maturity	Value (ZAR)
1	1 500.98
2	650.70
3	885.67
4	758.76
5	434.69
6	170.09
7	48.52
8	16.31
9	11.60
10	10.20
11	16.20
12	28.70
13	47.25
14	68.88
15	93.87
16	120.84
17	150.04
18	182.25
19	217.68
20	257.79

Table 4.2.1: Option Values

Figure 4.2.1 shows option values for various maturities with the forward volatility profile super imposed. For high volatility we observe high option values. This is to be expected, as rates become more uncertain, the probability of the bond option ending in the money also increases. For constant volatility, we observe option prices increasing with an increase in maturity. The more longer dated an option is, the higher the probability of it maturing in the money, keeping volatility constant.

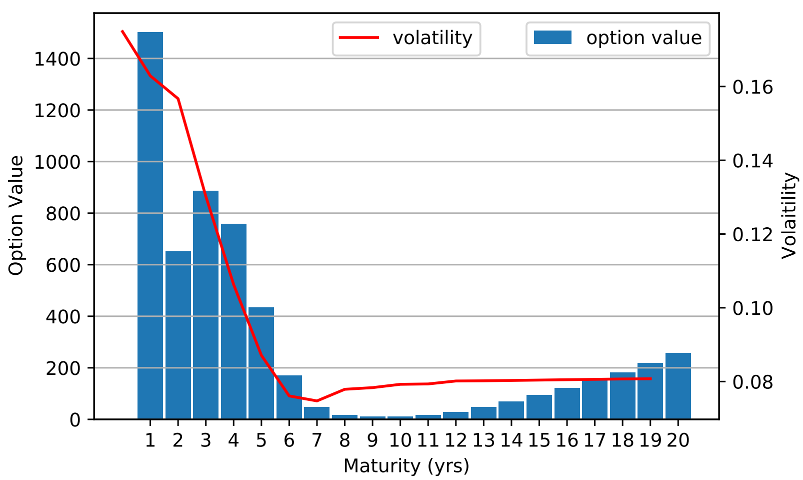


Figure 4.2.1: Option values for increasing maturity

Chapter 5

Sensitivity Analysis

The option value V can be represnted by a function and its variables as below

$$V = V\left(K, r(t), \sigma(t), \tau, fv, c\right)$$

where

- K is the option strike, in case of prepayment risk this is the face capital owed on a loan at any point in its life
- $r(t)$ is the market term structure of zero rates
- $\sigma(t)$ is the volatility, taken to be a deterministic function of time, i.e. no randomness
- τ is the maturity of the underlying bond
- fv is the face value of the underlying bond
- c is the coupon rate of the underlying bond

The sensitivities of the value of the option to the different variables are known in option literature as "greeks". Greeks are important for the hedging of option exposures i.e. risk management of the portfolio. These are calculated as partial derivatives of the multi-variable function. The valuation function is non-linear and therefore will have first and higher order partial derivatives.

For a function f, with dependents (x, y), the partial derivative with respect to variable x is defined as

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\delta \to 0} \frac{f(x+\delta, y) - f(x,y)}{\delta}$$

By using a suitably small δ, we can find the derivatives of a non-analytic function such as the binomial option valuation model numerically. Burden and Faires (2010) is a good reference for numerical differentiation.

The sensitivities calculated in the following illustrations are based on a bullet style loan with a 5% p.a. compounded quarterly coupons and a face value of ZAR $1\,000\,000$.

5.1 Interest rate delta

The delta is defined to be the change in the option value for a given shift in the term structure of interest rates. The industry standard in measuring this is called the delta (Δ) or PV01, i.e. *Present Value of 1 basis point*. This involves shifting the term structure of interest up uniformly by 1 basis point[1]. The binomial tree is then recalibrated to the

[1] one hundredth of a percentage point

shifted term structure and the option is revalued. The difference between the revaluation and the base value is equal to delta. The numerical computation is given by

$$\Delta = V\left(K, r(t) + 0.01\%, \sigma(t), \tau, fv, c\right) - V\left(K, r(t), \sigma(t), \tau, fv, c\right) \tag{5.1.1}$$

The results are illustrated in figure 5.1.1.

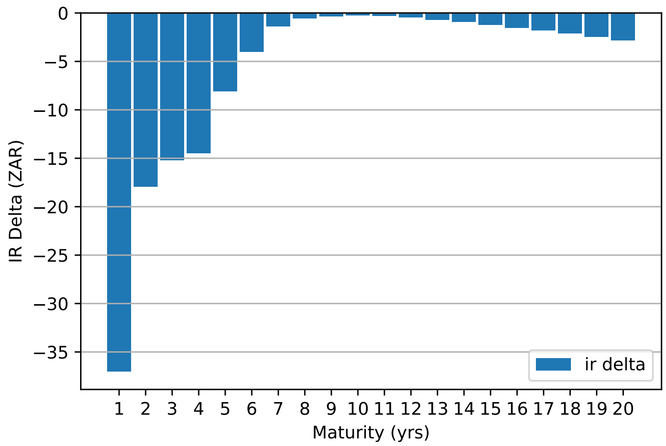


Figure 5.1.1: IR Delta by maturity

As interest rates rise, the bond option goes out of the money. This is because exercising i.e. refinancing a bond is suboptimal in a high interest rate environment. We observe the negative ir delta i.e. change in values as rates increase. The opposite is true for when interest rates decrease.

5.2 Vega

Vega is the change in the option value for a change in the volatility, keeping all other parameters constant. In this numerical procedure, it is calculated by shifting the forward volatilities by a shift factor δ and recalibrating the binomial tree then revaluing the option, then taking the difference between the revaluation and the original value. Industry practice is to choose δ = 1% and to not scale up the resultant sensitivity by δ. The sensitivity number tells us how much the value of the option would change by for a 1% parallel move in the volatility term structure. The numerical computation would be given by

$$\text{vega} = V\left(K, r(t), \sigma(t) + 1\%, \tau, fv, c\right) - V\left(K, r(t), \sigma(t), \tau, fv, c\right)$$

The results are shown in figure 5.2.1

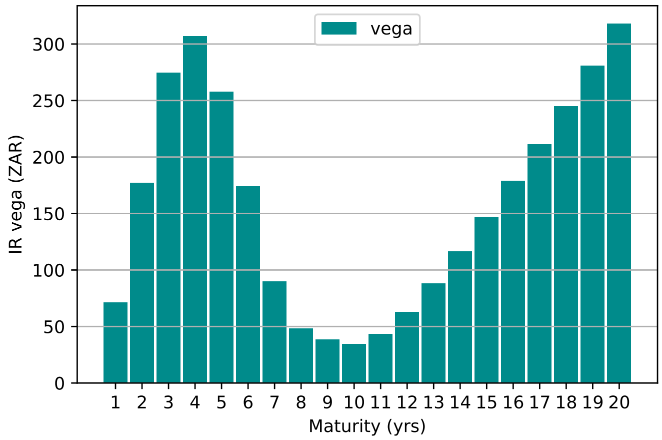


Figure 5.2.1: Vega by maturity

The value of the option will increase with an increase in volatility as the probability of ending in the money increases.

5.3 Gamma

Gamma is the second order change in the value of the option for a move in the term structure of interest rates. It tells us how the option delta will behave as rates move. Taking equation 5.1.1 as the delta function, gamma will be calculated as below

$$\text{gamma} = \Delta\left(K, r(t) + 0.01\%, \sigma(t), \tau, fv, c\right) - \Delta\left(K, r(t), \sigma(t), \tau, fv, c\right)$$

The results are given in figure 5.3.1. As rates increase, the delta of the option also increases. Delta is negative, and approaches zero as rates rise. This is because the option is moving out of the money and is becoming worthless as the probability of it being exercised is decreasing.

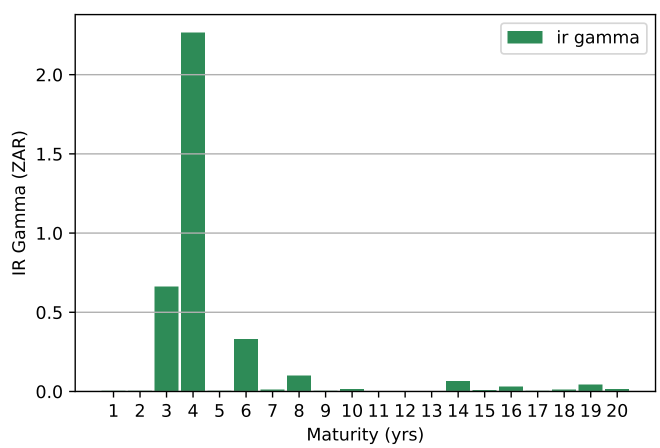


Figure 5.3.1: Gamma

Chapter 6

Conclusion

6.1 Hedging

The most accurate instrument to hedge prepayment risk would be for the lender to buy bond options where the underlying bond issuer has identical credit risk to the borrower. This is not always possible. In South Africa, only bond options with the underlying being the sovereign have a liquid market. Lenders also pool in a large number of fixed rate loans into a single portfolio and manage that risk on a portfolio basis in order to have a fuller picture of bankwide risks and to reduce the cost of hedging.

The most practical approach would be to use the same instruments that are used in the valuation of prepayment risk to hedge. FirstRand Bank Limited,[1] ABSA[2] and Standard Bank Group[3] disclose in their financial statements the use of interest rate swaps. This is an example of a delta hedging strategy. On the interest rate swap, the lender will agree to receive a fixed rate of interest and pay JIBAR. As interest rates decrease, the lender experiences mark to market losses on the prepayment options but makes profit from the interest rate swaps. Interest rate swaps are matched to the option exposures by PV01 and maturity of underlying loans. This immunises the portfolio against parallel moves of the zero curve.

This strategy will be periodically rebalanced as the delta of the prepayment option will change as rates move. The frequency of rebalancing will depend on the magnitude of the gamma. Option gamma peaks for At The Money options. The further in or out of the money an option is, the less its gamma.

Delta hedging will not protect lenders against volatility risk however. They will experience losses as volatility increases. To hedge this volatility risk, the lender would have to buy interest rate floors. The resultant residual delta can then be hedged using swaps. Buying interest rate options however could be expensive compared to the charge the lender must levy the borrower for the prepayment risk.

6.2 Further work

This thesis has outlined the nature of prepayment risk and its equivalence to an American call option on a bond. Prepayment risk is significant in fixed rate loans and its value depends on prevailing interest rates and implied volatilities. The Black-Derman-Toy one factor model for short term interest rates and its computational implementation have been outlined. The binomial tree has been used to value bullet style bonds and American call options on the bonds. It is not limited to this style of cashflows and can be used to value any structure of cashflows and optionality embedded in them. The paper has also discussed the current South African market for fixed income instruments which are used for price discovery. These instruments are used in the valuation and hedging of prepayment risk. The valuation model is simple to implement as the paper hopefully illustrated. With further refinements such as the factoring in of

[1] https://www.firstrand.co.za/media/investors/financial-results/firstrand-bank-annual-report-2021.pdf
[2] https://www.absa.africa/content/dam/africa/absaafrica/pdf/2021/absa-group-annual-consolidated-and-separate-financial-statements.pdf
[3] https://thevault.exchange/?get_group_doc=18/1615894015-SBG2020AnnualFinancialStatements.pdf

transaction costs and using a finer spacing of the nodes on the binomial tree, this method should be used to calculate and help manage the risk from prepayment options.

Bibliography

Black, F., Derman, E., and Toy, W. (1990). A one-factor model of interest rates and its application to treasury bond options. *Financial analysts journal*, 46(1):33–39.

Boyle, P., Tan, K., and Tian, W. (2001). Calibrating the black-derman-toy model: Some theoretical results. *Applied Mathematical Finance*, 8:27–48.

Burden, R. and Faires, J. (2010). *Numerical Analysis*. Cengage Learning.

Cossin, D. and Lu, H. (2004). Pricing prepayment option in c and i loans at origination. *SSRN Electronic Journal*.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

David Ho, K. H. and Su, H. (2006). Structural prepayment risk behavior of the underlying mortgages for residential mortgage life insurance in a developing market. *Journal of Housing Economics*, 15:257–278.

Fayman, A. and He, L. T. (2011). Prepayment risk and bank performance. *The Journal of Risk Finance*, 12:26–40.

Hagan, P. and West, G. (2006). Interpolation methods for curve construction. *Applied Mathematical Finance*, 13(2):89–129.

Ho, T. S. Y. and Lee, S.-B. (1986). Term structure movements and pricing interest rate contingent claims. *The Journal of Finance*, 41(5):1011–1029.

Hull, J. C. (2020). The Black, Derman and Toy Model. http://www-2.rotman.utoronto.ca/~hull/TechnicalNotes/TechnicalNote23.pdf.

Kalotay, A. J., Williams, G. O., and Fabozzi, F. J. (1993). A Model for Valuing Bonds and Embedded Options. *Financial Analysts Journal*, 49:35–46.

Mahomed, O. (2015). *South African Financial Markets*. University of CapeTown, MPhil Financial Mathematics course notes, Cape Town.

Schwartz, E. S. and Torous, W. N. (1989). Prepayment and the valuation of mortgage-backed securities. *The Journal of Finance*, 44(2):375–392.

Sherris, M. (2002). Pricing And Hedging Loan Prepayment Risk. *actuaries.org*.

Svoboda, S. (2002). *An investigation of various interest rate models and their calibration in the South African market*. PhD thesis, University of the Witwatersrand.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188.

Wilmott, P. (2013). *Paul Wilmott on Quantitative Finance*. Wiley, England.

Appendices

Maturity Date	Zero Rate	Maturity Date	Zero Rate	Maturity Date	Zero Rate	Maturity Date	Zero Rate	Maturity Date	Zero Rate	Maturity Date	Zero Rate
31-Dec-21	0.0000%	31-Dec-26	6.4026%	31-Dec-31	7.6951%	31-Dec-36	8.3108%	31-Dec-41	8.6011%	31-Dec-46	8.6999%
31-Mar-22	3.8645%	31-Mar-27	6.4727%	31-Mar-32	7.7469%	31-Mar-37	8.3300%	31-Mar-42	8.6102%	29-Mar-47	8.7006%
30-Jun-22	4.1689%	30-Jun-27	6.5426%	30-Jun-32	7.7944%	30-Jun-37	8.3488%	30-Jun-42	8.6189%	28-Jun-47	8.7008%
30-Sep-22	4.4035%	30-Sep-27	6.6131%	30-Sep-32	7.8388%	30-Sep-37	8.3674%	30-Sep-42	8.6272%	30-Sep-47	8.7008%
30-Dec-22	4.6535%	31-Dec-27	6.6839%	31-Dec-32	7.8798%	31-Dec-37	8.3853%	31-Dec-42	8.6350%	31-Dec-47	8.7004%
31-Mar-23	4.8646%	31-Mar-28	6.7545%	31-Mar-33	7.9172%	31-Mar-38	8.4024%	31-Mar-43	8.6422%	31-Mar-48	8.6998%
30-Jun-23	5.0530%	30-Jun-28	6.8252%	30-Jun-33	7.9526%	30-Jun-38	8.4192%	30-Jun-43	8.6490%	30-Jun-48	8.6990%
29-Sep-23	5.2171%	29-Sep-28	6.8955%	30-Sep-33	7.9863%	30-Sep-38	8.4356%	30-Sep-43	8.6554%	30-Sep-48	8.6979%
29-Dec-23	5.3661%	29-Dec-28	6.9647%	30-Dec-33	8.0178%	31-Dec-38	8.4515%	31-Dec-43	8.6614%	31-Dec-48	8.6966%
28-Mar-24	5.4894%	30-Mar-29	7.0325%	31-Mar-34	8.0479%	31-Mar-39	8.4666%	31-Mar-44	8.6669%	31-Mar-49	8.6951%
28-Jun-24	5.5963%	29-Jun-29	7.0984%	30-Jun-34	8.0766%	30-Jun-39	8.4813%	30-Jun-44	8.6720%	30-Jun-49	8.6935%
30-Sep-24	5.6932%	28-Sep-29	7.1620%	29-Sep-34	8.1041%	30-Sep-39	8.4956%	30-Sep-44	8.6766%	30-Sep-49	8.6917%
31-Dec-24	5.7812%	31-Dec-29	7.2251%	29-Dec-34	8.1305%	30-Dec-39	8.5093%	30-Dec-44	8.6808%	31-Dec-49	8.6898%
31-Mar-25	5.8637%	29-Mar-30	7.2817%	30-Mar-35	8.1558%	30-Mar-40	8.5225%	31-Mar-45	8.6846%	31-Mar-50	8.6879%
30-Jun-25	5.9446%	28-Jun-30	7.3389%	29-Jun-35	8.1801%	29-Jun-40	8.5351%	30-Jun-45	8.6880%	30-Jun-50	8.6858%
30-Sep-25	6.0246%	30-Sep-30	7.3981%	28-Sep-35	8.2036%	28-Sep-40	8.5473%	29-Sep-45	8.6910%	30-Sep-50	8.6837%
31-Dec-25	6.1032%	31-Dec-30	7.4573%	31-Dec-35	8.2269%	31-Dec-40	8.5594%	29-Dec-45	8.6935%	30-Dec-50	8.6816%
31-Mar-26	6.1790%	31-Mar-31	7.5172%	31-Mar-36	8.2488%	29-Mar-41	8.5702%	30-Mar-46	8.6957%	31-Mar-51	8.6794%
30-Jun-26	6.2545%	30-Jun-31	7.5782%	30-Jun-36	8.2699%	28-Jun-41	8.5808%	29-Jun-46	8.6975%	30-Jun-51	8.6772%
30-Sep-26	6.3294%	30-Sep-31	7.6383%	30-Sep-36	8.2907%	30-Sep-41	8.5913%	28-Sep-46	8.6989%	29-Sep-51	8.6750%

Table .0.1: Term structure of continuous rates

Start Date	End Date	Vol	Start Date	End Date	Vol	Start Date	End Date	Vol	Start Date	End Date	Vol
31-Dec-22	31-Mar-22	36.90%	31-Dec-26	31-Mar-27	17.42%	31-Dec-31	31-Mar-32	15.85%	31-Dec-36	31-Mar-37	16.08%
31-Mar-22	30-Jun-22	34.98%	31-Mar-27	30-Jun-27	17.42%	31-Mar-32	30-Jun-32	15.85%	31-Mar-37	30-Jun-37	16.08%
30-Jun-22	30-Sep-22	34.05%	30-Jun-27	30-Sep-27	17.43%	30-Jun-32	30-Sep-32	15.86%	30-Jun-37	30-Sep-37	16.09%
30-Sep-22	30-Dec-22	32.80%	30-Sep-27	31-Dec-27	17.43%	30-Sep-32	31-Dec-32	15.86%	30-Sep-37	31-Dec-37	16.09%
30-Dec-22	31-Mar-23	33.09%	31-Dec-27	31-Mar-28	15.21%	31-Dec-32	31-Mar-33	15.87%	31-Dec-37	31-Mar-38	16.10%
31-Mar-23	30-Jun-23	32.58%	31-Mar-28	30-Jun-28	15.22%	31-Mar-33	30-Jun-33	15.87%	31-Mar-38	30-Jun-38	16.10%
30-Jun-23	29-Sep-23	32.30%	30-Jun-28	29-Sep-28	15.22%	30-Jun-33	30-Sep-33	15.87%	30-Jun-38	30-Sep-38	16.10%
29-Sep-23	29-Dec-23	32.04%	29-Sep-28	29-Dec-28	15.23%	30-Sep-33	30-Dec-33	15.88%	30-Sep-38	31-Dec-38	16.11%
29-Dec-23	28-Mar-24	31.32%	29-Dec-28	30-Mar-29	14.95%	30-Dec-33	31-Mar-34	16.02%	31-Dec-38	31-Mar-39	16.12%
28-Mar-24	28-Jun-24	31.34%	30-Mar-29	29-Jun-29	14.95%	31-Mar-34	30-Jun-34	16.03%	31-Mar-39	30-Jun-39	16.12%
28-Jun-24	30-Sep-24	31.37%	29-Jun-29	28-Sep-29	14.96%	30-Jun-34	29-Sep-34	16.03%	30-Jun-39	30-Sep-39	16.12%
30-Sep-24	31-Dec-24	31.39%	28-Sep-29	31-Dec-29	14.96%	29-Sep-34	29-Dec-34	16.04%	30-Sep-39	30-Dec-39	16.13%
31-Dec-24	31-Mar-25	26.02%	31-Dec-29	29-Mar-30	15.57%	29-Dec-34	30-Mar-35	16.04%	30-Dec-39	30-Mar-40	16.13%
31-Mar-25	30-Jun-25	26.04%	29-Mar-30	28-Jun-30	15.58%	30-Mar-35	29-Jun-35	16.04%	30-Mar-40	29-Jun-40	16.14%
30-Jun-25	30-Sep-25	26.05%	28-Jun-30	30-Sep-30	15.58%	29-Jun-35	28-Sep-35	16.05%	29-Jun-40	28-Sep-40	16.14%
30-Sep-25	31-Dec-25	26.07%	30-Sep-30	31-Dec-30	15.58%	28-Sep-35	31-Dec-35	16.05%	28-Sep-40	31-Dec-40	16.14%
31-Dec-25	31-Mar-26	21.25%	31-Dec-30	31-Mar-31	15.67%	31-Dec-35	31-Mar-36	16.06%	31-Dec-40	29-Mar-41	16.15%
31-Mar-26	30-Jun-26	21.26%	31-Mar-31	30-Jun-31	15.67%	31-Mar-36	30-Jun-36	16.06%	29-Mar-41	28-Jun-41	16.15%
30-Jun-26	30-Sep-26	21.26%	30-Jun-31	30-Sep-31	15.67%	30-Jun-36	30-Sep-36	16.07%	28-Jun-41	30-Sep-41	16.16%
30-Sep-26	31-Dec-26	21.27%	30-Sep-31	31-Dec-31	15.67%	30-Sep-36	31-Dec-36	16.07%	30-Sep-41	31-Dec-41	16.16%

Table .0.2: ATM caplet forward volatility

Code for project

[1]:
```
import pandas as pd
import numpy as np
import math
from scipy.optimize import newton #newton raphson
from matplotlib import pyplot as plt

pd.options.display.float_format = '{:.4f}'.format
import statistics
```

[2]:
```
fv = 10000000 #face value for bond in calibrating tree
maturity_yrs = 20
dt = 0.25
nodes = int(maturity_yrs/dt) #number of nodes in the binomial tree
tolerance = 0.001 #tolerance for newton raphson solver
```

[3]:
```
#Import Bloomberg data saved in spreadsheet
market_data = pd.read_excel("BBG Data.xlsx",sheet_name = 'Caplet␣
↪cashflows',parse_dates=True)
vol = list(market_data['Cap Vols'][1:]*math.sqrt(dt)/100)
r = list(market_data['Zero Rate'][1:]/100)
fwds = list(market_data['3M FwdRate'][1:]/100)
maturity_dates = list(market_data['ExpiryDate'][1:])
```

[4]:
```
#ZCB NPVs
market_pv = []
for i in range(0,len(r)):
    market_pv.append(fv*(math.exp(-r[i]*(i+1)*dt)))
```

[5]:
```
#Generates factors that multiply the fwd vol per period
def vol_multi(time):
    vol_multi =[[]]
    temp = []
    for x in range(time):
        if x == 0:
            vol_multi[0].append(0)
        else:
            for y in range(0,x+1):
                if y == 0:
                    temp.append(1)
                else:
                    temp.append(y * 2)
                    x = x - 2
            vol_multi.append(temp)
            temp = []
    return vol_multi
```

[6]:
```
state = vol_multi(nodes)
```

[34]:
```
fv = 10000000#Bond face value used for calibrating the model
def rate_calc(guess):
    rd = guess
    ru = rd*math.exp(2*vol[0]*dt)

    bond_value = market_pv[0]*(0.5*fv*math.exp(-rd*dt) + 0.5*fv*math.exp(-ru*dt))
    diff = bond_value - fv*market_pv[1]

    return diff
```

[35]:
```
two_period_rates = [[]]
two_period_rates[0] = r[0]
rd = newton(rate_calc,0.01,rtol=tolerance)
ru = rd*math.exp(2*vol[0]*dt)

two_period_rates.append([rd,ru])
```

[36]:
```
#Create binomial interest rate tree and calibrate to market prices and␣
 ↪volatilities
rates_tree = []
npv_diffs = []
pvs = [] #value of fv at each node

splits = 2
rates_tree.append(r[0])
rates_tree.append(two_period_rates[1])
check = []
for i in range(2,len(state)):

    def rate_solver(guess):

        temp_rates = []
        temp_rates.append(guess)

        for j in range(1,len(state[i])):
                temp_rates.append(temp_rates[0]*math.exp(state[i][j]*math.
 ↪sqrt(dt)*vol[i-1]))

        temp_pvs = [fv*math.exp(-j*dt) for j in temp_rates]

        while len(temp_pvs)>2:

            temp_pvs2 = [sum(temp_pvs[e:e + splits])/splits for e in␣
 ↪range(len(temp_pvs) - splits + 1)] #averaging
```

```
            temp_pvs2 = [a*math.exp(-b*dt) for a,b in␣
↪zip(temp_pvs2,rates_tree[(len(temp_pvs2)-1)])] #discounting
            temp_pvs = temp_pvs2

        #NPV at node 1
        npv = [sum(temp_pvs[e:e + splits])/splits for e in range(len(temp_pvs) -␣
↪splits + 1)]
        npv = npv[0]*math.exp(-rates_tree[0]*dt)

        pvs.append(npv)
        npv_diff = (npv - market_pv[i])

        return npv_diff

    r_test = newton(rate_solver,0.1,rtol=tolerance)

    temp_rates = []
    temp_rates.append(r_test)

    for j in range(1,len(state[i])):
        temp_rates.append(temp_rates[0]*math.exp(state[i][j]*math.
↪sqrt(dt)*vol[i-1]))
    rates_tree.append(temp_rates)
```

[10]:
```
#Calibration check by finding difference of Bond npvs via tree vs via market␣
↪rates

npv_diffs = []
pvs = [] #value of fv at each node

for i in range(1,len(rates_tree)):

    temp_pvs = [fv*math.exp(-j*dt) for j in rates_tree[i]]

    while len(temp_pvs)>2:

        k = len(temp_pvs)
        temp_pvs2 = [sum(temp_pvs[e:e + splits])/splits for e in␣
↪range(len(temp_pvs) - splits + 1)] #averaging
        #temp_pvs2 = [a/(1+b) for a,b in␣
↪zip(temp_pvs2,rates_test[(len(temp_pvs2)-1)])] #discounting
        temp_pvs2 = [a*math.exp(-b*dt) for a,b in␣
↪zip(temp_pvs2,rates_tree[(len(temp_pvs2)-1)])] #discounting
        temp_pvs = temp_pvs2

    #NPV at node 1
```

```
    npv = [sum(temp_pvs[e:e + splits])/splits for e in range(len(temp_pvs) -
 ↪splits + 1)]

    npv = npv[0]*math.exp(-rates_tree[0]*dt)

    pvs.append(npv)
    npv_diffs.append(npv - market_pv[i])
```

[37]:
```
rates_tree[0] = [rates_tree[0]]
```

Option Valuation

[38]:
```
#write up loan cashflow tree

coupon = 0.05/4
#frequency = 1 #how many pmts per annum
face_value = 1000000
maturity = 20
nodes2 = int(maturity/dt)
```

[43]:
```
#value a simple bond with interest pmts and bullet principal

pv_simple =  [a*math.exp(-b*dt) for a,b in zip([face_value +
 ↪(face_value*coupon)]*len(rates_tree[nodes2-1]),rates_tree[nodes2-1])]
pv_tracker = []
pv_tracker.append(pv_simple)

for i in range(nodes2-2,-1,-1):
    pv_simple = [x+coupon*face_value for x in pv_simple] #add coupon
    pv_simple = [sum(pv_simple[e:e + splits])/splits for e in
 ↪range(len(pv_simple) - splits + 1)] #averaging
    pv_simple = [a*math.exp(-b*dt) for a,b in zip(pv_simple,rates_tree[i])]
 ↪#discounting

    pv_tracker.append(pv_simple)

#value a call option embedded bond with interest pmts and bullet principal
k = face_value #strike price
pv_embedded =  [a*math.exp(-b*dt) for a,b in zip([face_value +
 ↪(face_value*coupon)]*len(rates_tree[nodes2-1]),rates_tree[nodes2-1])]
pv_embedded = [min(a,k) for a in pv_embedded]
pv_tracker_embedded = []
pv_tracker_embedded.append(pv_embedded)

#Discount till first node
for i in range(nodes2-2,0,-1):
```

```
    pv_embedded = [x+coupon*face_value for x in pv_embedded] #add coupon
    pv_embedded = [sum(pv_embedded[e:e + splits])/splits for e in␣
↪range(len(pv_embedded) - splits + 1)] #averaging
    pv_embedded = [a*math.exp(-b*dt) for a,b in zip(pv_embedded,rates_tree[i])]␣
↪#discounting
    pv_embedded = [min(a,k) for a in pv_embedded] #compare bond value to the␣
↪strike price of the underlying option

    pv_tracker_embedded.append(pv_embedded)

pv_embedded = [x+coupon*face_value for x in pv_embedded]
pv_embedded = [a*math.exp(-b*dt) for a,b in zip([sum(pv_embedded[e:e + splits])/
↪splits for e in range(len(pv_embedded) - splits + 1)],rates_tree[0])]

pv_tracker_embedded.append(pv_embedded)
option_value  = [b-a for a,b in zip(pv_tracker_embedded[-1], pv_tracker[-1])]
option_value[0]
```

[43]: 257.79011557542253

Varying maturity

[47]:
```
option_values = []

for j in range(1,maturity+1):

    #value a simple bond with interest pmts and bullet principal
    pv_simple =  [a*math.exp(-b*dt) for a,b in zip([face_value +␣
↪(face_value*coupon)]*len(rates_tree[int((j/dt)-1)]),rates_tree[int((j/dt)-1)])]
    pv_tracker = []
    pv_tracker.append(pv_simple)

    for i in range(int(j/dt)-2,-1,-1):
        #print('i equals',i)
        pv_simple = [x+coupon*face_value for x in pv_simple] #add coupon
        pv_simple = [sum(pv_simple[e:e + splits])/splits for e in␣
↪range(len(pv_simple) - splits + 1)] #averaging
        pv_simple = [a*math.exp(-b*dt) for a,b in zip(pv_simple,rates_tree[i])]␣
↪#discounting

        pv_tracker.append(pv_simple)

    k = face_value #strike price
    pv_embedded =  [a*math.exp(-b*dt) for a,b in zip([face_value +␣
↪(face_value*coupon)]*len(rates_tree[int((j/dt)-1)]),rates_tree[int((j/dt)-1)])]
    pv_embedded = [min(a,k) for a in pv_embedded]
```

```
    pv_tracker_embedded = []
    pv_tracker_embedded.append(pv_embedded)

    #Discount till first node
    for i in range(int(j/dt)-2,0,-1):
        pv_embedded = [x+coupon*face_value for x in pv_embedded] #add coupon
        pv_embedded = [sum(pv_embedded[e:e + splits])/splits for e in␣
↪range(len(pv_embedded) - splits + 1)] #averaging
        pv_embedded = [a*math.exp(-b*dt) for a,b in␣
↪zip(pv_embedded,rates_tree[i])] #discounting
        pv_embedded = [min(a,k) for a in pv_embedded] #compare bond value to the␣
↪strike price of the underlying option

        pv_tracker_embedded.append(pv_embedded)

    pv_embedded = [x+coupon*face_value for x in pv_embedded]
    pv_embedded = [a*math.exp(-b*dt) for a,b in zip([sum(pv_embedded[e:e +␣
↪splits])/splits for e in range(len(pv_embedded) - splits + 1)],rates_tree[0])]

    pv_tracker_embedded.append(pv_embedded)
    pv_tracker_embedded
    option_value  = [b-a for a,b in zip(pv_tracker_embedded[-1], pv_tracker[-1])]
    option_values.append(option_value[0])
option_values
```

[50]:
```
x = list(np.arange(1,maturity+1))
plt.bar(x,option_values,width = 0.9,label='Line 1')
#plt.plot(market_data['Cap Vols'])
plt.grid('True',axis = 'y')
plt.xticks(ticks=x,labels = x)
plt.xlabel('Maturity (yrs)')
plt.ylabel('Option Value')
plt.legend(['option value'],loc='best')
#plt.title('Option Values per Maturity')

axes2 = plt.twinx()
axes2.plot(vol[::4],color='r')
axes2.set_ylabel('Volaitility')
axes2.legend(["volatility"],loc='upper center')
#plt.savefig('graph.pdf')
```

Vega

We bump up the caplet volatilities by 1%, and re calculate the interest rate tree

[14]:

```
#Create binomial interest rate tree and calibrate to market prices and␣
↪volatilities
rates_tree2 = []
npv_diffs = []
pvs = [] #value of fv at each node
bump= 1/100

splits = 2
rates_tree2.append(r[0])
rates_tree2.append(two_period_rates[1])
check = []
for i in range(2,len(state)):

    def rate_solver(guess):

        temp_rates = []
        temp_rates.append(guess)

        for j in range(1,len(state[i])):
                temp_rates.append(temp_rates[0]*math.exp(state[i][j]*math.
↪sqrt(dt)*(vol[i-1]+bump)))

        temp_pvs = [fv*math.exp(-j*dt) for j in temp_rates]

        while len(temp_pvs)>2:

            temp_pvs2 = [sum(temp_pvs[e:e + splits])/splits for e in␣
↪range(len(temp_pvs) - splits + 1)] #averaging

            temp_pvs2 = [a*math.exp(-b*dt) for a,b in␣
↪zip(temp_pvs2,rates_tree[(len(temp_pvs2)-1)])] #discounting
            temp_pvs = temp_pvs2

        #NPV at node 1
        npv = [sum(temp_pvs[e:e + splits])/splits for e in range(len(temp_pvs) -␣
↪splits + 1)]
        npv = npv[0]*math.exp(-rates_tree2[0]*dt)

        pvs.append(npv)
        npv_diff = (npv - market_pv[i])

        return npv_diff

    r_test = newton(rate_solver,0.1,rtol=tolerance)

    temp_rates = []
    temp_rates.append(r_test)
```

```
    for j in range(1,len(state[i])):
        temp_rates.append(temp_rates[0]*math.exp(state[i][j]*math.
↪sqrt(dt)*(vol[i-1]+bump)))
    rates_tree2.append(temp_rates)
```

[19]:
```
option_values_vol_bump = []

for j in range(1,maturity+1):

    #value a simple bond with interest pmts and bullet principal

    pv_simple =  [a*math.exp(-b*dt) for a,b in zip([face_value +␣
↪(face_value*coupon)]*len(rates_tree[int((j/dt)-1)]),rates_tree2[int((j/
↪dt)-1)])]
    pv_tracker = []
    pv_tracker.append(pv_simple)

    for i in range(int(j/dt)-2,-1,-1):
        #print('i equals',i)
        pv_simple = [x+coupon*face_value for x in pv_simple] #add coupon
        pv_simple = [sum(pv_simple[e:e + splits])/splits for e in␣
↪range(len(pv_simple) - splits + 1)] #averaging
        pv_simple = [a*math.exp(-b*dt) for a,b in zip(pv_simple,rates_tree2[i])]␣
↪#discounting

        pv_tracker.append(pv_simple)

    pv_tracker
    #print(rates_test[i])

#value a call option embedded bond with interest pmts and bullet principal

    k = face_value #strike price
    pv_embedded =  [a*math.exp(-b*dt) for a,b in zip([face_value +␣
↪(face_value*coupon)]*len(rates_tree2[int((j/dt)-1)]),rates_tree2[int((j/
↪dt)-1)])]
    pv_embedded = [min(a,k) for a in pv_embedded]
    pv_tracker_embedded = []
    pv_tracker_embedded.append(pv_embedded)

    #Discount till first node
    for i in range(int(j/dt)-2,0,-1):
        #print('i equals',i)
        pv_embedded = [x+coupon*face_value for x in pv_embedded] #add coupon
```

```
        pv_embedded = [sum(pv_embedded[e:e + splits])/splits for e in␣
↪range(len(pv_embedded) - splits + 1)] #averaging
        pv_embedded = [a*math.exp(-b*dt) for a,b in␣
↪zip(pv_embedded,rates_tree2[i])] #discounting
        pv_embedded = [min(a,k) for a in pv_embedded] #compare bond value to the␣
↪strike price of the underlying option

        pv_tracker_embedded.append(pv_embedded)

    pv_embedded = [x+coupon*face_value for x in pv_embedded]
    pv_embedded = [a*math.exp(-b*dt) for a,b in zip([sum(pv_embedded[e:e +␣
↪splits])/splits for e in range(len(pv_embedded) - splits + 1)],rates_tree2[0])]

    pv_tracker_embedded.append(pv_embedded)

    pv_tracker_embedded

    option_value  = [(b-a) for a,b in zip(pv_tracker_embedded[-1],␣
↪pv_tracker[-1])]

    option_values_vol_bump.append(option_value[0])
option_values_vol_bump
```

[51]:
```
vega =[(a - b) for a,b in zip(option_values_vol_bump,option_values)]
#vega
```

[52]:
```
x = list(np.arange(1,maturity+1))
plt.bar(x,vega,width = 0.9,color='orangered')
plt.grid('True',axis = 'y')
plt.xticks(ticks=x,labels = x)
plt.xlabel('Maturity (yrs)')
plt.ylabel('vega')
plt.legend(['vega'],loc='best')
#plt.savefig('vega.pdf')
```

IR Delta

[23]:
```
#ZCB NPVs
market_pv_bumped = []
ir_bump = 1/10000 #term structure shift factor
r2 = [r+ir_bump for r in r]
for i in range(0,len(r2)):
    market_pv_bumped.append(fv*(math.exp(-r2[i]*(i+1)*dt)))
```

[54]:
```python
#recalibrating the IR tree to shifted term strcture

fv = 10000000#Bond face value used for calibrating the model
def rate_calc2(guess):
    rd = guess
    ru = rd*math.exp(2*vol[0]*dt)

    bond_value = market_pv_bumped[0]*(0.5*fv*math.exp(-rd*dt) + 0.5*fv*math.
 ↪exp(-ru*dt))
    diff = bond_value - fv*market_pv_bumped[1]

    return diff

two_period_rates = [[]]
two_period_rates[0] = r2[0]
rd = newton(rate_calc2,0.01,rtol=tolerance)
ru = rd*math.exp(2*vol[0]*dt)

two_period_rates.append([rd,ru])
#two_period_rates

#Create binomial interest rate tree and calibrate to market prices and␣
 ↪volatilities
rates_tree = []
npv_diffs = []
pvs = [] #value of fv at each node

splits = 2
rates_tree.append(r2[0])
rates_tree.append(two_period_rates[1])
check = []
for i in range(2,len(state)):

    def rate_solver(guess):

        temp_rates = []
        temp_rates.append(guess)

        for j in range(1,len(state[i])):
                temp_rates.append(temp_rates[0]*math.exp(state[i][j]*math.
 ↪sqrt(dt)*vol[i-1]))

        temp_pvs = [fv*math.exp(-j*dt) for j in temp_rates]

        while len(temp_pvs)>2:
```

```
            temp_pvs2 = [sum(temp_pvs[e:e + splits])/splits for e in␣
↪range(len(temp_pvs) - splits + 1)] #averaging

            temp_pvs2 = [a*math.exp(-b*dt) for a,b in␣
↪zip(temp_pvs2,rates_tree[(len(temp_pvs2)-1)])] #discounting
            temp_pvs = temp_pvs2

        #NPV at node 1
        npv = [sum(temp_pvs[e:e + splits])/splits for e in range(len(temp_pvs) -␣
↪splits + 1)]
        npv = npv[0]*math.exp(-rates_tree[0]*dt)

        pvs.append(npv)
        npv_diff = (npv - market_pv_bumped[i])

        return npv_diff

    r_test = newton(rate_solver,0.1,rtol=tolerance)

    temp_rates = []
    temp_rates.append(r_test)

    for j in range(1,len(state[i])):
        temp_rates.append(temp_rates[0]*math.exp(state[i][j]*math.
↪sqrt(dt)*vol[i-1]))
    rates_tree.append(temp_rates)
```

[55]:
```
#revalue options with new tree
rates_tree[0] = [rates_tree[0]]
option_value_ir_bumped = []

for j in range(1,maturity+1):

    #value a simple bond with interest pmts and bullet principal

    pv_simple =  [a*math.exp(-b*dt) for a,b in zip([face_value +␣
↪(face_value*coupon)]*len(rates_tree[int((j/dt)-1)]),rates_tree[int((j/dt)-1)])]
    pv_tracker = []
    pv_tracker.append(pv_simple)

    for i in range(int(j/dt)-2,-1,-1):
        #print('i equals',i)
        pv_simple = [x+coupon*face_value for x in pv_simple] #add coupon
        pv_simple = [sum(pv_simple[e:e + splits])/splits for e in␣
↪range(len(pv_simple) - splits + 1)] #averaging
```

```
        pv_simple = [a*math.exp(-b*dt) for a,b in zip(pv_simple,rates_tree[i])]␣
↪#discounting

        pv_tracker.append(pv_simple)

    pv_tracker

#value a call option embedded bond with interest pmts and bullet principal
    k = face_value #strike price
    pv_embedded =  [a*math.exp(-b*dt) for a,b in zip([face_value +␣
↪(face_value*coupon)]*len(rates_tree[int((j/dt)-1)]),rates_tree[int((j/dt)-1)])]
    pv_embedded = [min(a,k) for a in pv_embedded]
    pv_tracker_embedded = []
    pv_tracker_embedded.append(pv_embedded)

    #Discount till first node
    for i in range(int(j/dt)-2,0,-1):
        #print('i equals',i)
        pv_embedded = [x+coupon*face_value for x in pv_embedded] #add coupon
        pv_embedded = [sum(pv_embedded[e:e + splits])/splits for e in␣
↪range(len(pv_embedded) - splits + 1)] #averaging
        pv_embedded = [a*math.exp(-b*dt) for a,b in␣
↪zip(pv_embedded,rates_tree[i])] #discounting
        pv_embedded = [min(a,k) for a in pv_embedded] #compare bond value to the␣
↪strike price of the underlying option

        pv_tracker_embedded.append(pv_embedded)

    pv_embedded = [x+coupon*face_value for x in pv_embedded]
    pv_embedded = [a*math.exp(-b*dt) for a,b in zip([sum(pv_embedded[e:e +␣
↪splits])/splits for e in range(len(pv_embedded) - splits + 1)],rates_tree[0])]

    pv_tracker_embedded.append(pv_embedded)

    pv_tracker_embedded

    option_value  = [b-a for a,b in zip(pv_tracker_embedded[-1], pv_tracker[-1])]

    option_value_ir_bumped.append(option_value[0])
#option_value_ir_bumped
```

[56]:

```
ir_delta =[(a - b) for a,b in zip(option_value_ir_bumped,option_values)]
```

[57]:

```
x = list(np.arange(1,maturity+1))
plt.bar(x,ir_delta,width = 0.9,label='Line 1')
plt.grid('True',axis = 'y')
plt.xticks(ticks=x,labels = x)
```

```
plt.xlabel('Maturity (yrs)')
plt.ylabel('IR Delta')
plt.legend(['ir delta'],loc='best')
#plt.savefig('ir_delta.pdf')
```