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Research Article

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Inequalities of harmonic univalent functions with connections of hypergeometric functions

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Abstract: Let $\mathcal{SH}$ be the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f'(0) - 1 = 0$. In this paper, we introduce and study a subclass $\mathcal{H}(\alpha, \beta)$ of the class $\mathcal{SH}$ and the subclass $\mathcal{NH}(\alpha, \beta)$ with negative coefficients. We obtain basic results involving sufficient coefficient conditions for a function in the subclass $\mathcal{H}(\alpha, \beta)$ and we show that these conditions are also necessary for negative coefficients, distortion bounds, extreme points, convolution and convex combinations. In this paper an attempt has also been made to discuss some results that uncover some of the connections of hypergeometric functions with a subclass of harmonic univalent functions.

Keywords: Harmonic function, Analytic function, Univalent function, Unit disk

MSC: 30C45

1 Introduction

As a continuous complex-valued function, $f = u + iv$ in a simply connected domain $\mathcal{D}$ is said to be a harmonic if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain $\mathcal{D}$, we may write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$ such that h is called the analytic part and g is the co-analytic part of f. For f to be locally univalent and sense-preserving in $\mathcal{D}$, it is sufficiently agreeable that $|h'(z)| > |g'(z)|$, $z \in \mathcal{D}$. (see [1]).

Denote by $\mathcal{SH}$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f'(0) - 1 = 0$. So, for $f = h + \bar{g} \in \mathcal{SH}$, it can be expressed in the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1 \tag{1}$$

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Note that $\mathcal{SH}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of its member is zero. Consequently, the function $f(z)$ for this class can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Also, let $\mathcal{NH}$ be the subclass of $\mathcal{SH}$ consisting of functions $f = h + \bar{g}$ such that functions h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \;\; g(z) = -\sum_{n=1}^{\infty} |b_n| z^n, \;\; |b_1| < 1. \tag{2}$$

Clunie and Sheil-Small (1984) [1] studied the class $\mathcal{SH}$ with some its geometric subclasses and calculated coefficient bounds. For analytic functions $\phi(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their convolution is defined as $(\phi * \psi)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, $z \in \mathcal{U}$. In the harmonic function case, with $f = h + \bar{g}$ and $F = H + \bar{G}$, their harmonic convolution is defined as $f * F = h * H + \overline{g * G}$. If ϕ_1 and ϕ_2 are analytic and $f = h + \bar{g}$ is in $\mathcal{SH}$, Ahuja and Silverman (2004) [2] defined

$$f * \left(\phi_1 + \bar{\phi}_2\right) = h * \phi_1 + \overline{g * \phi_2}. \tag{3}$$

Let $F(a, b, c; z)$ be the Gaussian hypergeometric function defined by the series

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad z \in \mathcal{U} \tag{4}$$

where a, b, c are complex numbers with $c \neq 0, -1, -2, ...$ and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda+1)...(\lambda+n-1) \;\text{ for }\; n = 1, 2, 3, ... \;\text{ and }\; (\lambda)_0 = 1.$$

Since the hypergeometric series in (4) converges absolutely in $\mathcal{U}$, it follows that $F(a, b, c; z)$ defines a function which is analytic in $\mathcal{U}$, provided that c is neither zero nor a negative integer. The well-known Gauss's summation theorem: If $Re(c - a - b) > 0$, then

$$F(a, b, c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c \neq 0, -1, -2, \dots .$$

Throughout this paper, let $G(z) = \phi_1(z) + \overline{\phi_2(z)}$, where

$$\phi_1(z) = zF(a_1, b_1, c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} (b_1)_{n-1}}{(c_1)_{n-1} (1)_{n-1}} z^n, \tag{5}$$

$$\phi_2(z) = F(a_2, b_2, c_2; z) - 1 = \sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} z^n, \;\; |a_2 b_2| < |c_2|. \tag{6}$$

Harmonic mapping of hypergeometric functions plays a significant and attractive part in Geometric Function Theory (GFT). The famed author Ahuja together with Silverman [2] in 2004 have uncovered some interesting studies on the connections between the amazing of harmonic univalent functions and distinct hypergeometric functions. For example, the result involves the convolution multipliers $f * \left(\phi_1 + \bar{\phi}_2\right)$, where ϕ_1, ϕ_2 are as defined by (5) and (6) and f is an efficiently given harmonic starlike univalent (or harmonic convex univalent) function in the open unit disk. Numerous inclusion properties and other studies, including hypergeometric functions and harmonic univalent functions, have newly been investigated by prominent mathematician researcher Ahuja in (2007) [3], followed by various studies in (2008) [4] and (2009) [5]. Recently, in 2011 [6], certain inclusion results by involving uniformly harmonic starlike mappings and hypergeometric functions, have been studied. It should be remarked that some other important studies that bring out this connection have been done in [7–10].

Let $\mathcal{S}_{\mathcal{SH}}$ formulate the subclass of $\mathcal{SH}$ involving functions in $\mathcal{SH}$ that are starlike. Moreover, we suppose $\mathcal{S}_{\overline{\mathcal{SH}}}$ is the subclass of $\mathcal{S}_{\mathcal{SH}}$ including functions in $\mathcal{NH}$.

The families $\mathcal{S}_{\mathcal{SH}}$ and $\mathcal{S}_{\overline{\mathcal{SH}}}$, were firstly investigated by Avic and Zlotkiewicz [11]. Later Silverman [12] imposed the following necessary conditions:

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1 \tag{7}$$

for functions $f = h + \bar{g}$ to be in these families, and Silverman and Silvia [13] improved these outcomes of [12] to the case when the coefficient b_1 is not necessarily zero.

Ahuja and Jahangiri [14] introduced the class $\mathcal{N}_{\mathcal{SH}}(\beta)$ of functions in $\mathcal{SH}$ such that

$$\Re\left(\frac{\frac{\partial}{\partial\theta} f(z)}{\frac{\partial}{\partial z}}\right) \geq \beta, \quad \left(0 \leq \beta < 1, \ z = re^{i\theta} \in \mathcal{U}\right)$$

and they showed that the coefficient condition

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 2 - \beta \tag{8}$$

is sufficient for functions f in $\mathcal{N}_{\mathcal{SH}}(\beta)$.

In 2003 Yalcin et al. [15] investigated the class $\mathcal{HP}(\beta)$, with the subclass of $\mathcal{SH}$ satisfying the condition

$$Re\{h'(z) + g'(z)\} > \beta, \quad (0 \leq \beta < 1).$$

Yalcin et al. also studied the functions with negative coefficient that satisfies the above condition. Based on the class study in [15], Al-Khal and Al-Kharsani [16] investigated inequalities associating hypergeometric functions with planer harmonic mapping. In 2004 Yalcin and Oztork [17], introduced and studied the class $HP(\alpha)$, consisting of functions of the form (1), satisfying the condition

$$Re\{\alpha z(h''(z) + g''(z)) + (h'(z) + g'(z))\} > 0, \quad (\alpha \geq 0).$$

Moreover, they studied the above negative coefficient functions defined by (2). In 2010 Chandrashekar et al. [18], investigated a class $\mathcal{HP}(\alpha, \beta)$ consisting of functions of the form (1) satisfying the condition

$$Re\{\alpha z(h''(z) + g''(z)) + (h'(z) + g'(z))\} > \beta, \quad (\alpha \geq 0, 0 \leq \beta < 1),$$

based on the work of Yalcin and Ozturk in [17]. They have given some results that bring out the connections of hypergeometric functions with a class $\mathcal{HP}(\alpha, \beta)$ of harmonic univalent functions. Such a type of study on different subclasses was carried out by several researchers, such as Dixit et al. [19] Aouf et al. [20], El-Ashwah [21], Al-Khal and Al-Kharsani [22], S. Nagpal and Ravichandran [23], Ponnusamy et al. [24], Porwal and Dixit [25], Shelake et al. [26]. Pursuing this line of study and motivated by the each works of Yalcin et al. ([15, 17]), Ahuja and Silverman [2], Al-Khal and Al-Kharsani [16], Chandrashekar et al. [18] on the subject of harmonic functions, this paper presents and examines a geometric subclass $\mathcal{R}_{\mathcal{H}}(\alpha, \beta)$ of $\mathcal{SH}$. Let $\mathcal{H}(\alpha, \beta), (\alpha \geq 0, 0 \leq \beta < 1)$, denote the subclass of harmonic functions of the form (1) which satisfies the condition

$$Re\{(h'(z) + g'(z)) + 3\alpha z(h''(z) + g''(z)) + \alpha z^2(h'''(z) + g'''(z))\} > \beta. \tag{9}$$

Also, we define the class $\mathcal{NH}(\alpha, \beta)$ by

$$\mathcal{NH}(\alpha, \beta) = \mathcal{H}(\alpha, \beta) \cap \mathcal{NH}.$$

The coefficient conditions for the function in $\mathcal{H}(\alpha, \beta)$ are studied. Furthermore, there is a determination of the coefficient conditions, distortion bounds, extreme points, convolution, convex combinations and neighborhoods for the function in $\mathcal{NH}(\alpha, \beta)$. Moreover, the connections between harmonic univalent functions and hypergeometric functions are studied.

2 The class $\mathcal{NH}(\alpha, \beta)$

In our first theorem, we give a coefficient bound for harmonic functions in $\mathcal{H}(\alpha, \beta)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be such that h and g are given by* (1). *Assume that if*

$$\sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)](|a_n| + |b_n|) \leq 2 - \beta \tag{10}$$

where $a_1 = 1, \alpha \geq 0$ and $0 \leq \beta < 1$, then f is harmonic univalent.

Proof. Suppose that $z_1, z_2 \in \mathcal{U}$ such that $z_1 \neq z_2$, then by the condition (10), we obtain

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| \geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| = 1 - \left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}{(z_1 - z_2) - \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right|$$
$$> 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} \geq 1 - \frac{\sum_{n=1}^{\infty} \frac{n[1+\alpha(n^2-1)]}{1-\beta}|b_n|}{1 - \sum_{n=2}^{\infty} \frac{n[1+\alpha(n^2-1)]}{1-\beta}|a_n|} \geq 0.$$

Hence $|f(z_1) - f(z_2)| > 0$ and so f is univalent in $\mathcal{U}$. □

Theorem 2.2. *Let $f = h + \bar{g}$ be such that h and g are given by* (1) *and satisfies the condition* (10) *then f is sense-preserving in $\mathcal{U}$ and $f \in \mathcal{H}(\alpha, \beta)$*

Proof. Firstly, we show that f is locally univalent and sense-preserving in $\mathcal{U}$. It suffices to show that $|h'(z)| > |g'(z)|$ by using the condition (10). We have

$$|h'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \geq 1 - \beta - \sum_{n=2}^{\infty} n[1 + \alpha(n^2 - 1)]|a_n|$$
$$\geq \sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)]|b_n| > \sum_{n=1}^{\infty} n|b_n||z|^{n-1} = |g'(z)|.$$

Now, we show that $f \in H(\alpha, \beta)$. We only need to indicate that (10) is satisfied, using the fact that $Re(w) \geq \beta$ if and only if $|1 - \beta + w| \geq |1 + \beta - w|$. Thus, we obtain

$$\begin{aligned} &\left|(1 - \beta) + (h' + g') + 3\alpha z(h'' + g'') + \alpha z^2(h''' + g''')\right| \\ &- \left|(1 + \beta) - (h' + g') - 3\alpha z(h'' + g'') - \alpha z^2(h''' + g''')\right| \geq 0. \end{aligned} \tag{11}$$

Substituting for $h(z)$ and $g(z)$ in (11), we get

$$\left|(1 - \beta) + 1 + \sum_{n=2}^{\infty} n[1 + \alpha(n^2 - 1)]a_n z^{n-1} + \sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)]b_n \bar{z}^{n-1}\right|$$
$$- \left|(1 - \beta) - 1 - \sum_{n=2}^{\infty} n[1 + \alpha(n^2 - 1)]a_n z^{n-1} - \sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)]b_n \bar{z}^{n-1}\right|$$
$$\geq 2\left[(1 - \beta) - \left[\sum_{n=2}^{\infty} n[1 + \alpha(n^2 - 1)]|a_n| + \sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)]|b_n|\right]\right]$$
$$= 2\left[(2 - \beta) - \left[\sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)]\,(|a_n| + |b_n|)\right]\right] \geq 0,$$

by condition (10). The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{n[1 + \alpha(n^2 - 1)} x_n z^n + \sum_{n=1}^{\infty} \frac{1}{n[1 + \alpha(n^2 - 1)} \bar{y}_n \bar{z}^n, \tag{12}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$, shows that the coefficient bound given by (10) is sharp. The functions of the form (12) are in the class $\mathcal{H}(\alpha, \beta)$ because the condition (10) can be satisfied as follows:

$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]|a_n| + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]|b_n| = \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$$

This completes the proof of Theorem 2.2. □

Remark 2.3. *By specializing the parameter, we obtain the following interesting results analogous to Theorem 2.2, which have been efficiently studied by*

1. *Silverman [12] when* $\alpha = \beta = b_1 = 0$,
2. *Silverman and Silvia [13] when* $\alpha = \beta = 0$,
3. *Ahuja and Jahangiri [14] when* $\alpha = 0$,
4. *Yalcin et al. [15] when also* $\alpha = 0$.

We proceed to prove that the condition (10) is also necessary for functions $f = h + \bar{g}$, where h and g are of the form (2).

Theorem 2.4. *Let $f = h + \bar{g}$ be such that h and g are given by* (2). *Then $f \in \mathcal{NH}(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]|a_n| + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]|b_n| \le 1-\beta$$

where $a_1 = 1, \alpha \ge 0$ and $0 \le \beta < 1$.

Proof. Since $\mathcal{NH}(\alpha, \beta) \subset \mathcal{H}(\alpha, \beta)$. We only need to prove the "only if" part of this theorem. For functions $f(z)$ of the form (2), we have

$$Re\{(h'(z)+g'(z)) + 3\alpha z(h''(z)+g''(z)) + \alpha z^2(h'''(z)+g'''(z))\} > \beta.$$

Consequently, we obtain

$$Re\left\{(1-\beta) - \sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]|a_n|z^{n-1} - \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]|b_n|\bar{z}^{n-1}\right\} \ge 0$$

The above required condition must hold for all values of z in $\mathcal{U}$. Upon choosing the values of z on the positive real axis where $0 < |z| = r < 1$, we must have

$$(1-\beta) - \sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]|a_n|r^{n-1} - \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]|b_n|r^{n-1} \ge 0.$$

Letting $r \to 1^-$ through real values, it follows that

$$(1-\beta) - \left[\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]|a_n| + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]|b_n|\right] \ge 0.$$

Therefore, we have

$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]|a_n| + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]|b_n| \le 1-\beta.$$ □

Now we give distortion bounds for function in $\mathcal{NH}(\alpha, \beta)$.

Remark 2.5. *By specializing the parameter we obtain the following significant results analogous to the Theorem 2.4 which have been investigated by*

1. *Silverman [12] when* $\alpha = \beta = b_1 = 0$,

2. *Silverman and Silvia [13] when* $\alpha = \beta = 0$,
3. *Yalcin et al. [15] when* $\alpha = 0$.

Theorem 2.6. *Let* $f \in \mathcal{NH}(\alpha, \beta)$. *Then* $r = |z| < 1$

$$|f(z)| \le (1 + |b_1|)r + \left(\frac{(1-\beta)}{2[1+3\alpha]}\right)\left[1 - \frac{1}{1-\beta}|b_1|\right] r^2$$
$$|f(z)| \ge (1 + |b_1|)r - \left(\frac{(1-\beta)}{2[1+3\alpha]}\right)\left[1 - \frac{1}{1-\beta}|b_1|\right] r^2$$

Proof. Let $f \in \mathcal{NH}(\alpha, \beta)$. Taking the absolute value of f, we have

$$\begin{aligned}
|f(z)| &\le (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^n \\
&\le (1 + |b_1|)r + r^2 \sum_{n=2}^{\infty} \left(\frac{(1-\beta)}{2[1+3\alpha]}\right)\left(\frac{2[1+3\alpha]}{(1-\beta)}|a_n| + \frac{2[1+3\alpha]}{(1-\beta)}|b_n|\right) \\
&\le (1 + |b_1|)r + r^2 \sum_{n=2}^{\infty} \left(\frac{(1-\beta)}{2[1+3\alpha]}\right)\left(\frac{n[1+\alpha(n^2-1)]}{(1-\beta)}|a_n| + \frac{n[1+\alpha(n^2-1)]}{(1-\beta)}|b_n|\right) \\
&\le (1 + |b_1|)r + \left(\frac{(1-\beta)}{2[1+3\alpha]}\right)\left[1 - \frac{1}{1-\beta}|b_1|\right] r^2
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\ge (1 + |b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^n \\
&\ge (1 + |b_1|)r - r^2 \sum_{n=2}^{\infty} \left(\frac{(1-\beta)}{2[1+3\alpha]}\right)\left(\frac{2[1+3\alpha]}{(1-\beta)}|a_n| + \frac{2[1+3\alpha]}{(1-\beta)}|b_n|\right) \\
&\ge (1 + |b_1|)r - r^2 \left(\frac{(1-\beta)}{2[1+3\alpha]}\right) \sum_{n=2}^{\infty} \left(\frac{n[1+\alpha(n^2-1)]}{(1-\beta)}|a_n| + \frac{n[1+\alpha(n^2-1)]}{(1-\beta)}|b_n|\right) \\
&\ge (1 + |b_1|)r - \left(\frac{(1-\beta)}{2[1+3\alpha]}\right)\left[1 - \frac{1}{1-\beta}|b_1|\right] r^2.
\end{aligned}$$

□

In the next theorem, we determine the extreme points of closed convex hulls of $\mathcal{NH}(\alpha, \beta)$ denoted by $co\overline{\mathcal{NH}}(\alpha, \beta)$.

Theorem 2.7. *Let* $f = h + \bar{g}$ *be such that h and g are given by* (2). *If the harmonic function*

$$f(z) = \sum_{n=1}^{\infty} \left(x_n h_n(z) + y_n g_n(z)\right), \tag{13}$$

where

$$h_1(z) = z, \;\; h_n(z) = z - \frac{1-\beta}{n\left[1+\alpha(n^2-1)\right]} z^n \qquad (n = 2, 3, ...),$$
$$g_n(z) = z - \frac{1-\beta}{n\left[1+\alpha(n^2-1)\right]} \bar{z}^n \qquad (n = 1, 2, 3, ...),$$
$$\sum_{n=1}^{\infty} (x_n + y_n) = 1 \;\; x_n \ge 0 \;\; and \;\; y_n \ge 0.$$

Then $f \in co\overline{\mathcal{NH}}(\alpha, \beta)$.

Proof. A function f of the form (13) can be written as

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$$
$$= \sum_{n=1}^{\infty} (x_n + y_n)\, z - \sum_{n=2}^{\infty} \frac{1-\beta}{n\left[1+\alpha(n^2-1)\right]} x_n z^n - \sum_{n=1}^{\infty} \frac{1-\beta}{n\left[1+\alpha(n^2-1)\right]} y_n \bar{z}^n.$$

Thus, we obtain

$$\sum_{n=2}^{\infty} \frac{n[1+\alpha(n^2-1)]}{(1-\beta)} |a_n| + \sum_{n=1}^{\infty} \frac{n[1+\alpha(n^2-1)]}{(1-\beta)} |b_n| \le \sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = 1 - x_1 \le 1.$$

Hence, by Theorem 2.4, we have $f \in co\overline{\mathcal{NH}}(\alpha, \beta)$. □

Theorem 2.8. *Let $f = h + \bar{g}$ be such that h and g are given by* (2). *If $f \in co\overline{\mathcal{NH}}(\alpha, \beta)$ and achieves the condition* (10)*, then f satisfies the equation* (13). *In particular, the extreme points of $\mathcal{NH}(\alpha, \beta)$ are $\{h_n\}$ and $\{g_n\}$.*

Proof. Suppose that $f \in co\overline{\mathcal{NH}}(\alpha, \beta)$. Set

$$x_n = \frac{n[1+\alpha(n^2-1)]}{(1-\beta)} |a_n|, \qquad (n = 2, 3, ...),$$

and

$$y_n = \frac{n[1+\alpha(n^2-1)]}{(1-\beta)} |b_n|, \qquad (n = 1, 2, ...).$$

By (10), we note that $0 \le x_n$ $(n = 2, 3, ...)$ and $0 \le y_n$ $(n = 1, 2, ...)$. We define

$$x_1 = 1 - \sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

By Theorem 2.4, $x_1 \ge 0$, and

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$$

as required. □

Next, we can set the convex combination of the class $\mathcal{NH}(\alpha, \beta)$.

Theorem 2.9. *The class $\mathcal{NH}(\alpha, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2$, let $f_i \in \mathcal{NH}(\alpha, \beta)$ where

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}|\, z^n - \sum_{n=2}^{\infty} |b_{i,n}|\, \bar{z}^n.$$

Then, by Theorem 2.4, we have

$$\sum_{n=2}^{\infty} \frac{n[1+\alpha(n^2-1)]}{1-\beta} |a_{i,n}| + \sum_{n=1}^{\infty} \frac{n[1+\alpha(n^2-1)]}{1-\beta} |b_{i,n}| \le 1. \tag{14}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i,n}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i,n}| \right) \bar{z}^n.$$

Then by (2), we have

$$\sum_{n=2}^{\infty}\frac{n[1+\alpha(n^2-1)]}{1-\beta}\left(\sum_{i=1}^{\infty}t_i|a_{i,n}|\right)+\sum_{n=1}^{\infty}\frac{n[1+\alpha(n^2-1)]}{1-\beta}\left(\sum_{i=1}^{\infty}t_i|b_{i,n}|\right)$$
$$=\sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}\frac{n[1+\alpha(n^2-1)]}{1-\beta}|a_{i,n}|+\sum_{n=1}^{\infty}\frac{n[1+\alpha(n^2-1)]}{1-\beta}|b_{i,n}|\right)\le\sum_{i=1}^{\infty}t_i=1.$$

Therefore, $\sum_{i=1}^{\infty}t_i f_i\in\mathcal{NH}(\alpha,\beta)$ (see Theorem 2.4.) □

The class $\mathcal{NH}(\alpha,\beta)$ is closed under convolution as will be shown in the next theorem.

Theorem 2.10. *For $0\le\gamma\le\beta<1$, let $f\in\mathcal{NH}(\alpha,\beta)$, $F\in\mathcal{NH}(\alpha,\gamma)$ and*

$$\sum_{n=2}^{\infty}\frac{n[1+\alpha(n^2-1)]}{1-\beta}|A_n|<1. \tag{15}$$

Then $(f*F)\in\mathcal{NH}(\alpha,\beta)\subset\mathcal{NH}(\alpha,\gamma)$.

Proof. Let the harmonic function $f(z):=z-\sum_{n=2}^{\infty}|a_n|z^n-\sum_{n=1}^{\infty}|b_n|\bar{z}^n$ and $F(z):=z-\sum_{n=2}^{\infty}|A_n|z^n-\sum_{n=1}^{\infty}|B_n|\bar{z}^n$. Then the convolution of f and F is defined as follows:

$$(f*F)(z)=z-\sum_{n=2}^{\infty}|a_nA_n|z^n-\sum_{n=1}^{\infty}|b_nB_n|\bar{z}^n$$

By condition (15), Theorem 2.4 and since $F\in\mathcal{NH}(\alpha,\gamma)$, we conclude that $|A_n|\le 1$ and $|B_n|\le 1$. But $f\in\mathcal{NH}(\alpha,\beta)$, then we have

$$\sum_{n=2}^{\infty}n[1+\alpha(n^2-1)]|a_n||A_n|+\sum_{n=1}^{\infty}n[1+\alpha(n^2-1)]|b_n||B_n|$$
$$\le\sum_{n=2}^{\infty}n[1+\alpha(n^2-1)]|a_n|+\sum_{n=1}^{\infty}n[1+\alpha(n^2-1)]|b_n|\le 1-\beta\le 1-\gamma.$$

Thus $(f*F)\in\mathcal{NH}(\alpha,\beta)\subset\mathcal{NH}(\alpha,\gamma)$. □

Here, we look at a closure property of the class $\mathcal{NH}(\alpha,\beta)$ under the generalized Bernardi-Libera-Livingston integral operator $F(z)$ which is defined by (see [26])

$$F(z)=(\mu+1)\int_0^1 t^{\mu-1}f(tz)dt\qquad(\mu>-1).$$

Theorem 2.11. $f\in\mathcal{NH}(\alpha,\beta)\Rightarrow F\in\mathcal{NH}(\alpha,\beta)$.

Proof. Let

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n-\sum_{n=1}^{\infty}|b_n|\bar{z}^n$$

Then, we get

$$F(z)=(\mu+1)\int_0^1 t^{\mu-1}\left((tz)-\sum_{n=2}^{\infty}|a_n|(tz)^n-\sum_{n=1}^{\infty}|b_n|\left(\overline{tz}\right)^n\right)dt=z-\sum_{n=2}^{\infty}|A_n|z^n-\sum_{n=1}^{\infty}|B_n|\bar{z}^n,$$

where

$$A_n=\frac{\mu+1}{\mu+n}|a_n|\ \ and\ \ B_n=\frac{\mu+1}{\mu+n}|b_n|.$$

Thus, since $f \in \mathcal{NH}(\alpha, \beta)$,

$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]\left(\frac{\mu+1}{\mu+n}|a_n|\right) + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]\left(\frac{\mu+1}{\mu+n}|b_n|\right)$$
$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]|a_n| + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]|b_n| \leq 1-\beta.$$

In virtue of Theorem 2.4, we have $F \in \mathcal{NH}(\alpha, \beta)$. □

3 Hypergeometric functions

Here, we need the following result, which may be found in ([18, 24]).

Lemma 3.1. *If $a, b, c > 0$, then*

i. $\sum_{n=1}^{\infty} n \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{ab}{c-a-b-1} F(a,b;c;1)$ *if* $c > a+b+1$

ii. $\sum_{n=1}^{\infty} n^2 \frac{(a)_n (b)_n}{(c)_n (1)_n} = \left[\frac{(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right] F(a,b;c;1)$ *if* $c > a+b+2$

iii. $\sum_{n=1}^{\infty} n^3 \frac{(a)_n (b)_n}{(c)_n (1)_n} = \left[\frac{(a)_3 (b)_3}{(c-a-b-3)_3} + \frac{3(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right] F(a,b;c;1)$ *if* $c > a+b+3$.

In the following theorem, we obtain the coefficient condition for the Gaussian hypergeometric function:

Theorem 3.2. *If $a_j, b_j > 0$ and $c_j > a_j + b_j + 3$ for $j = 1, 2$, then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent and sense-preserving in $\mathcal{U}$ and $G \in \mathcal{H}(\alpha, \beta)$, is that*

$$\begin{aligned}&\left[\frac{\alpha\,(a_1)_3\,(b_1)_3}{(c_1-a_1-b_1-3)_3} + \frac{6\alpha\,(a_1)_2\,(b_1)_2}{(c_1-a_1-b_1-2)_2} + \frac{(1+6\alpha)\,a_1\,b_1}{c_1-a_1-b_1-1} + 1\right] F(a_1,b_1;c_1;1)\\ &+\left[\frac{\alpha\,(a_2)_3\,(b_2)_3}{(c_2-a_2-b_2-3)_3} + \frac{3\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2} + \frac{a_2\,b_2}{c_2-a_2-b_2-1}\right] F(a_2,b_2;c_2;1) \leq 2-\beta,\end{aligned} \tag{16}$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Proof. Let $G(z) = \phi_1(z) + \overline{\phi_2(z)}$

$$= z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} z^n + \overline{\sum_{n=1}^{\infty} \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} z^n}$$

Firstly, we want to show that G is locally univalent and sense-preserving in $\mathcal{U}$. It is enough to show that $|\phi_1'(z)| > |\phi_2'(z)|$

$$\begin{aligned}|\phi_1'(z)| &= \left|1 + \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} z^{n-1}\right| > 1 - \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} |z|^{n-1}\\
&> 1 - \sum_{n=2}^{\infty} (n-1) \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}}\\
&= 2 - \left[\frac{a_1\,b_1}{c_1-a_1-b_1-1} + 1\right] F(a_1,b_1;c_1;1) \quad \text{by part(i) of Lemma 3.1 and by Gauss summation formula}\\
&\geq 2-\beta - \left[\frac{\alpha\,(a_1)_3\,(b_1)_3}{(c_1-a_1-b_1-3)_3} + \frac{6\alpha\,(a_1)_2\,(b_1)_2}{(c_1-a_1-b_1-2)_2} + \frac{(1+6\alpha)\,a_1\,b_1}{c_1-a_1-b_1-1} + 1\right] F(a_1,b_1;c_1;1)\\
&\geq \left[\frac{\alpha\,(a_2)_3\,(b_2)_3}{(c_2-a_2-b_2-3)_3} + \frac{3\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2} + \frac{a_2\,b_2}{c_2-a_2-b_2-1}\right] F(a_2,b_2;c_2;1) \text{ by (15)}\\
&> \frac{a_2\,b_2}{c_2-a_2-b_2-1} F(a_2,b_2;c_2;1) > \sum_{n=1}^{\infty} n \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} |z|^{n-1} \geq \left|\sum_{n=1}^{\infty} n \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} z^{n-1}\right| = |\phi_2'(z)|.\end{aligned}$$

Now to show that G is univalent in $\mathcal{U}$, we suppose that $z_1,\ z_2 \in \mathcal{U}$ such that $z_1 \neq z_2$. Since $\mathcal{U}$ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in \mathcal{U}$, where $0 \le t \le 1$. Then we can write

$$G(z_1) - G(z_2) = \int_0^1 \left[(z_2 - z_1)\phi_1'(z(t)) + \overline{(z_2 - z_1)\phi_2'(z(t))} \right] dt$$

such that

$$\begin{aligned} Re\,\frac{G(z_1) - G(z_2)}{z_2 - z_1} &= \int_0^1 Re\left[\phi_1'(z(t)) + \frac{\overline{(z_2 - z_1)}}{z_2 - z_1}\,\overline{\phi_2'(z(t))} \right] dt \\ &> \int_0^1 \left[Re\,\phi_1'(z(t)) - \left|\phi_2'(z(t))\right| \right] dt. \end{aligned} \tag{17}$$

On the other hand, by condition (16), we conclude that

$$\begin{aligned} &Re\,\phi_1'(z) - \left|\phi_2'(z)\right| \\ &\ge 1 - \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}}\ |z|^{n-1} - \sum_{n=1}^{\infty} n \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}\ |z|^{n-1} \\ &> 1 - \sum_{n=2}^{\infty} (n-1+1) \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} - \sum_{n=1}^{\infty} n \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} \\ &= 2 - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-2}} - \sum_{n=0}^{\infty} \frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n} - \frac{a_2\,b_2}{c_2} \sum_{n=1}^{\infty} \frac{(a_2+1)_{n-1}\,(b_2+1)_{n-1}}{(c_2+1)_{n-1}\,(1)_{n-1}} \\ &= 2 - \left[\frac{a_1\,b_1}{c_1 - a_1 - b_1 - 1} + 1 \right] F(a_1, b_1; c_1; 1) - \frac{a_2\,b_2}{c_2 - a_2 - b_2 - 1}\, F(a_2, b_2; c_2; 1) \\ &\ge 2 - \left[\frac{\alpha\,(a_1)_3\,(b_1)_3}{(c_1 - a_1 - b_1 - 3)_3} + \frac{6\alpha\,(a_1)_2\,(b_1)_2}{(c_1 - a_1 - b_1 - 2)_2} + \frac{(1+6\alpha)\,a_1\,b_1}{c_1 - a_1 - b_1 - 1} + 1 \right] F(a_1, b_1; c_1; 1) \\ &+ \left[\frac{\alpha\,(a_2)_3\,(b_2)_3}{(c_2 - a_2 - b_2 - 3)_3} + \frac{3\alpha\,(a_2)_2\,(b_2)_2}{(c_2 - a_2 - b_2 - 2)_2} + \frac{a_2\,b_2}{c_2 - a_2 - b_2 - 1} \right] F(a_2, b_2; c_2; 1) \ge \beta \ge 0. \end{aligned}$$

Thus (17) by the above inequality, we receive that $G(z_1) \neq G(z_2)$ and hence G is univalent in $\mathcal{U}$. Finally, we proceed to prove that $G \in \mathcal{H}(\alpha, \beta)$. In view of Theorem 2.2, we need to prove that

$$\sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)] \left[\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} \right] \le 2 - \beta. \tag{18}$$

But,

$$\begin{aligned} &\sum_{n=1}^{\infty} n[1 + \alpha(n^2 - 1)] \left[\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} \right] \\ &= \sum_{n=1}^{\infty} (1-\alpha)(n-1) \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + \sum_{n=1}^{\infty} (1-\alpha) \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + \sum_{n=1}^{\infty} \alpha(n-1+1)^3 \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} \\ &+ \sum_{n=1}^{\infty} (1-\alpha)n \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} + \sum_{n=1}^{\infty} \alpha n^3 \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} \\ &= (1+2\alpha) \sum_{n=1}^{\infty} (n-1) \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + \alpha \sum_{n=1}^{\infty} (n-1)^3 \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} \\ &+ 3\alpha \sum_{n=1}^{\infty} (n-1)^2 \frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + (1-\alpha) \sum_{n=1}^{\infty} n \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} + \alpha \sum_{n=1}^{\infty} n^3 \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} \end{aligned}$$

$$= (1+2\alpha)\sum_{n=1}^{\infty} n\,\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n} + \sum_{n=0}^{\infty} \frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n} + \alpha\sum_{n=1}^{\infty} n^3\,\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n} + 3\alpha\sum_{n=1}^{\infty} n^2\,\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n}$$
$$+ (1-\alpha)\sum_{n=1}^{\infty} n\,\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} + \alpha\sum_{n=1}^{\infty} n^3\,\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}$$
$$= \left[\frac{\alpha\,(a_1)_3\,(b_1)_3}{(c_1-a_1-b_1-3)_3} + \frac{6\alpha\,(a_1)_2\,(b_1)_2}{(c_1-a_1-b_1-2)_2} + \frac{(1+6\alpha)\,a_1\,b_1}{c_1-a_1-b_1-1} + 1\right] F(a_1,b_1;c_1;1)$$
$$+ \left[\frac{\alpha\,(a_2)_3\,(b_2)_3}{(c_2-a_2-b_2-3)_3} + \frac{3\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2} + \frac{a_2\,b_2}{c_2-a_2-b_2-1}\right] F(a_2,b_2;c_2;1),$$

Thus, in view of Lemma 3.1, we get the inequality (18). This completes the proof. □

For our next theorem, we need to define the following function:

$$G_1(z) = z\left(2-\frac{\phi_1(z)}{z}\right) - \overline{\phi_2(z)} = z - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}}\,z^n - \overline{\sum_{n=1}^{\infty}\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}\,z^n}$$

on using (5) and (6). Clearly $G_1 \in \mathcal{NH}(\alpha,\beta)$, (see [18, 21]).

Theorem 3.3. *Let $\alpha \ge 0$, $0 \le \beta < 1$, $a_j, b_j > 0$, $c_j > a_j + b_j + 3$ for $j = 1,2$ and $a_2b_2 < c_2$. Then $G_1 \in \mathcal{NH}(\alpha,\beta)$ if and only if (16) holds.*

Proof. It is clear that $\mathcal{NH}(\alpha,\beta) \subset \mathcal{H}(\alpha,\beta)$. In view of Theorem 3.2, we need only to show the necessary condition for G_1 to be in $\mathcal{H}(\alpha,\beta)$. If $G_1 \in \mathcal{NH}(\alpha,\beta)$, then G_1 satisfies the inequality (18) by Theorem 2.4 and hence (16) holds. □

In the following theorem, we give the convolution $f * (\phi_1 + \phi_2)$, where ϕ_1 and ϕ_2, which are defined by (5) and (6).

Theorem 3.4. *Let $\alpha \ge 0$, $0 \le \beta < 1$, $a_j, b_j > 0$, $c_j > a_j + b_j + 3$ for $j = 1,2$ and $a_2b_2 < c_2$. Then a necessary and sufficient condition such that $f * (\phi_1+\phi_2) \in \mathcal{NH}(\alpha,\beta)$ for $f \in \mathcal{NH}(\alpha,\beta)$ is that*

$$F(a_1,b_1;c_1;1) + F(a_2,b_2;c_2;1) \le 3, \tag{19}$$

where ϕ_1, ϕ_2 are defined, respectively, by (5) and (6).

Proof. Let $f = h + \bar{g} \in \mathcal{NH}(\alpha,\beta)$, where h and g are given by (2). Then

$$f * (\phi_1+\phi_2)(z) = h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)}$$
$$= z - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}}\,a_n z^n - \overline{\sum_{n=1}^{\infty}\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}\,b_n z^n}.$$

In view of Theorem 2.4, we need to prove that $f * (\phi_1 + \phi_2)$ if and only if

$$\sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]\left[\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}}\,a_n + \frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}\,b_n\right] \le 2-\beta. \tag{20}$$

An application of Theorem 2.4, we get

$$\sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]\,(a_n+b_n) \le 2-\beta.$$

or

$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]\,a_n + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]\,b_n \le 1-\beta, \tag{21}$$

which implies that

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$$n[1+\alpha(n^2-1)]\,a_n \le 1-\beta \quad \text{and} \quad n[1+\alpha(n^2-1)]\,b_n \le 1-\beta.$$

Thus, we attain

$$a_n \le \frac{1-\beta}{n[1+\alpha(n^2-1)]} \qquad \text{and} \qquad b_n \le \frac{1-\beta}{n[1+\alpha(n^2-1)]}, \qquad (n \ge 1). \tag{22}$$

Rewriting (20), we get

$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]\,\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}}\,a_n + \sum_{n=1}^{\infty} n[1+\alpha(n^2-1)]\,\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}\,b_n \le 1-\beta. \tag{23}$$

By applying (22), the left hand side of (23) is bounded above by

$$\begin{aligned}
&\sum_{n=2}^{\infty}(1-\beta)\,\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}} + \sum_{n=1}^{\infty}(1-\beta)\,\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n} \\
&= (1-\beta)\left(\sum_{n=1}^{\infty}\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n} + \sum_{n=1}^{\infty}\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n}\right) \\
&= (1-\beta)\,\left(F(a_1,b_1;c_1;1) + F(a_2,b_2;c_2;1) - 2\right).
\end{aligned}$$

The last expression is bounded above by $(1-\beta)$ if and only if (19) is satisfied. This proves (20) and the result follows. □

The integral operator for the Gaussian hypergeometric function will be studied at the end of this section.

Theorem 3.5. *If $a_j, b_j > 0$ and $c_j > a_j + b_j + 2$ for $j = 1, 2$. Then a necessary and sufficient condition for a function*

$$G_2(z) = \int_0^z F(a_1,b_1;c_1;1)dt + \overline{\int_0^z F(a_2,b_2;c_2;1)dt}$$

to be in $\mathcal{H}(\alpha,\beta)$, is that

$$\begin{aligned}
&\left[1 + \frac{\alpha\,(a_1)_2\,(b_1)_2}{(c_1-a_1-b_1-2)_2} + \frac{3\alpha\,a_1\,b_1}{c_1-a_1-b_1-1}\right]F(a_1,b_1;c_1;1) \\
&+\left[1 + \frac{\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2} + \frac{3\alpha\,a_2\,b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1) \le 3-\beta.
\end{aligned}$$

where $\alpha \ge 0$, $0 \le \beta < 1$.

Proof. In view of Theorem 2.2, the function

$$G_2(z) = z - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{n\,(c_1)_{n-1}\,(1)_{n-1}}\,z^n - \overline{\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}\,(b_2)_{n-1}}{n\,(c_2)_{n-1}\,(1)_{n-1}}\,z^n}$$

is in $\mathcal{H}(\alpha,\beta)$ if

$$\sum_{n=2}^{\infty} n[1+\alpha(n^2-1)]\left[\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{n\,(c_1)_{n-1}\,(1)_{n-1}} + \frac{(a_2)_{n-1}\,(b_2)_{n-1}}{n\,(c_2)_{n-1}\,(1)_{n-1}}\right] \le 1-\beta. \tag{24}$$

The left side of (24) can be written as follows:

$$\sum_{n=2}^{\infty}[1+\alpha(n^2-1)]\left[\frac{(a_1)_{n-1}\,(b_1)_{n-1}}{(c_1)_{n-1}\,(1)_{n-1}}+\frac{(a_2)_{n-1}\,(b_2)_{n-1}}{(c_2)_{n-1}\,(1)_{n-1}}\right]$$
$$=\sum_{n=1}^{\infty}\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n}+\sum_{n=1}^{\infty}\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}+\alpha\sum_{n=1}^{\infty}n^2\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n}+\alpha\sum_{n=1}^{\infty}n^2\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}$$
$$+2\alpha\sum_{n=1}^{\infty}n\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n}+2\alpha\sum_{n=1}^{\infty}n\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}$$
$$=(F(a_1,b_1;c_1;1)-1)+(F(a_2,b_2;c_2;1)-1)+\left[\frac{\alpha\,(a_1)_2\,(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{\alpha\,a_1\,b_1}{c_1-a_1-b_1-1}\right]F(a_1,b_1;c_1;1)$$
$$+\left[\frac{\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{\alpha\,a_2\,b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1)+\left[\frac{2\alpha\,a_1\,b_1}{c_1-a_1-b_1-1}\right]F(a_1,b_1;c_1;1)$$
$$+\left[\frac{2\alpha\,a_2\,b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1)$$
$$=\left[1+\frac{\alpha\,(a_1)_2\,(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{3\,\alpha\,a_1\,b_1}{c_1-a_1-b_1-1}\right]F(a_1,b_1;c_1;1)$$
$$+\left[1+\frac{\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{3\,\alpha\,a_2\,b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1)-2.$$

The last expression is bounded above by $1-\beta$ and the result follows. □

Theorem 3.6. *If $a_1b_1 > -1$, $c_1 > 0$, $a_1b_1 < 0$, $a_2 > 0$, $b_2 > 0$ and $c_j > a_j + b_j + 2$ for $j = 1, 2$, then a necessary and sufficient condition for a function*

$$G_3(z)=\int_0^z F(a_1,b_1;c_1;1)dt-\overline{\int_0^z [F(a_2,b_2;c_2;1)-1]\,dt}$$

to be in $\mathcal{NH}(\alpha,\beta)$ is

$$\left[(1+3\alpha)+\frac{a_1\,b_1}{c_1-a_1-b_1-1}\right]F(a_1,b_1;c_1;1)$$
$$-\left[(1+\alpha)+\frac{\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{3\alpha\,a_2\,b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1)+1\geq\beta.$$

where $\alpha \geq 0$, $0 \leq \beta < 1$.

Proof. In view of Theorem 2.4, the function

$$G_3(z)=z-\frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}\,(b_1+1)_{n-2}}{n(c_1+1)_{n-2}\,(1)_{n-1}}z^n-\overline{\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}\,(b_2)_{n-1}}{n\,(c_2)_{n-1}\,(1)_{n-1}}z^n}$$

is in $\mathcal{NH}(\alpha,\beta)$ if

$$\sum_{n=2}^{\infty}\left[1+\alpha(n^2-1)\right]\left[\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-2}\,(b_1+1)_{n-2}}{(c_1+1)_{n-2}\,(1)_{n-1}}+\frac{(a_2)_{n-1}\,(b_2)_{n-1}}{(c_2)_{n-1}\,(1)_{n-1}}\right]\leq 1-\beta. \qquad (25)$$

After computation, inequality (25) can be written as

$$\sum_{n=2}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-2}\,(b_1+1)_{n-2}}{(c_1+1)_{n-2}\,(1)_{n-1}}+\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}\,(b_2)_{n-1}}{(c_2)_{n-1}\,(1)_{n-1}}$$
$$+\alpha\sum_{n=2}^{\infty}(n^2-1)\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-2}\,(b_1+1)_{n-2}}{(c_1+1)_{n-2}\,(1)_{n-1}}+\alpha\sum_{n=2}^{\infty}(n^2-1)\frac{(a_2)_{n-1}\,(b_2)_{n-1}}{(c_2)_{n-1}\,(1)_{n-1}}\leq 1-\beta.$$

Thus

$$\sum_{n=0}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_n\,(b_1+1)_n}{(c_1+1)_n\,(1)_{n+1}}+\alpha\sum_{n=0}^{\infty}n\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_n\,(b_1+1)_n}{(c_1+1)_n\,(1)_n}$$
$$+3\alpha\sum_{n=0}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_n\,(b_1+1)_n}{(c_1+1)_n\,(1)_n}+\alpha\sum_{n=1}^{\infty}n^2\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}$$
$$+2\alpha\sum_{n=1}^{\infty}n\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}+(1+\alpha)\sum_{n=1}^{\infty}\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}\le 1-\beta,$$

which implies

$$\frac{(1+3\alpha)c_1}{a_1b_1}\sum_{n=1}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n}+\frac{\alpha c_1}{a_1b_1}\sum_{n=1}^{\infty}n\frac{|a_1b_1|}{c_1}\frac{(a_1)_n\,(b_1)_n}{(c_1)_n\,(1)_n}+\alpha\sum_{n=1}^{\infty}n^2\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}$$
$$+2\alpha\sum_{n=1}^{\infty}n\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}+(1+\alpha)\sum_{n=1}^{\infty}\frac{(a_2)_n\,(b_2)_n}{(c_2)_n\,(1)_n}\le 1-\beta.$$

Therefore, we have

$$-\left[(1+3\alpha)+\frac{a_1\,b_1}{c_1-a_1-b_1-1}\right]F(a_1,b_1;c_1;1)$$
$$+\left[(1+\alpha)+\frac{\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{3\alpha\,a_2\,b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1)\le 1-\beta,$$

which yields

$$\left[(1+3\alpha)+\frac{a_1\,b_1}{c_1-a_1-b_1-1}\right]F(a_1,b_1;c_1;1)$$
$$-\left[(1+\alpha)+\frac{\alpha\,(a_2)_2\,(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{3\alpha\,a_2\,b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1)+1\ge\beta.$$

This completes the proof. □

In this paper, we have discussed a subclass of the class of functions that are harmonic univalent and sense-preserving in the open unit disc. Some results are gained by involving coefficient conditions and by showing the significance of these conditions for negative coefficient, distortion bounds, extreme points, convolution and convex combinations. Moreover, in this paper an investigation on some results is done to reveal some of the connections of hypergeometric functions with a subclass of harmonic univalent functions.

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