

NEAR-RINGS AND THEIR MODULES

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I hereby declare that this is entirely my own work and has not been submitted to any other university

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ABSTRACT

After an introduction defining basic structural aspects of near-rings, this report examines how the ring-theoretic notions of generation and cogeneration can be extended from modules over a ring to modules over a near-ring. Chapter four examines matrix near-rings and connections between the J_2 and J_S radicals of the near-ring and the corresponding matrix near-ring.

By extending the concepts of generation and cogeneration from ring modules to near-ring modules we are investigating how important distributivity and an abelian additive structure are to these two concepts. The concept of generation faces the obstacle that the image of a near-ring module homomorphism is not necessarily a submodule of the image space but only a subgroup, while the sum of two subgroups need not even be a subgroup. In chapter two, generation, trace and socle are defined for near-ring modules and these ideas are linked with those of the essential and module-essential subgroups. Cogeneration, dealing with kernels which are always submodules proved easier to generalise. This is discussed in chapter three together with the concept of the reject, and these ideas are linked to the $J_{1/2}$ and J_2 radicals. The duality of the ring theory case is lost. The results are less simple than in the ring theory case due to the different types of near-ring module substructures which give rise to several Jacobson-type radicals.

A near-ring of matrices can be obtained from an arbitrary near-ring by regarding each $r \times r$ matrix as a mapping from N^r to N^r , where N is the near-ring from which entries are taken. The argument showing that the near-ring is 2-semisimple if and only if the associated near-ring of matrices is 2-semisimple is presented and investigated in the case of s -semisimplicity.

Questions arising from this report are presented in the final chapter.

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CHAPTER ONE

DEFINITIONS AND GENERAL RESULTS

A left near-ring N is a system with two binary operations, $+$ and \cdot such that

$(N,+)$ is a group

(N,\cdot) is a semigroup

for any $a,b,c \in N$, $a(b+c) = ab+ac$

A near-ring differs from a ring in that the additive group need not be abelian and only one distributive law holds. Left near-rings, that is, those in which the left distributive law holds, will be used throughout this work. Some authors, for example Pilz [13] use right near-rings in which the right distributive law holds, and analogous results hold for these.

The study of near-rings was initiated in 1905 by Dickson [4],[5] who showed the existence of near-fields, fields in which only one distributive rule holds. His main interest was to apply near-fields to geometry. They proved useful in co-ordinatising certain classes of geometric planes and today they are used in characterising doubly transitive groups, incidence groups and Fröbenius groups. Between 1930 and 1952 near-rings were studied by Wielandt. Much of his work was unpublished until his student G Betsch extended his results and started publishing them in the early 1960's. In the early 50's, D Blackett, working independently, researched along the same lines. Through his papers published between 1958 and 1962, A Fröhlich increased interest in distributively generated near-rings while H Neumann showed connections between reduced free groups and near-rings.

There are many natural examples of near-rings. Among these are the set of all mappings of an additive group into itself, with pointwise addition and composition of maps, the set of all zero-fixing maps with the same addition and multiplication and the set of all polynomials over a commutative ring with identity under addition and

substitution. Of course every ring is a near-ring.

In general it need not be true that $0 \cdot n = 0$ for every n in N , where 0 is the zero of the near-ring. Near-rings for which this does hold are called zero-symmetric. We will assume that the near-ring which we use is zero-symmetric. We will also assume that the near-ring possesses a multiplicative identity 1 , so that for any $n \in N$, $n1 = 1n = n$.

Although only left distribution is assumed to hold, it may happen that certain elements are also distributive from the right. These are called distributive elements of the near-ring. If the near-ring N is generated as an additive group by a multiplicative semigroup of distributive elements M , then N is called a distributively generated (*dg*) near-ring. We write (N, M) to denote such near-ring. *Dg* near-rings are in fact natural examples of zero-symmetric near-rings.

MODULES AND SUBSTRUCTURES

As in the ring theory case we can define modules over a near-ring N .

Definition 1.1: Let S be a (not necessarily abelian) additive group. S is called an N -module if there exists a mapping from $S \times N$ to S , $(s, n) \rightarrow sn$ such that for any $n_1, n_2 \in N$ and any $s \in S$,

$$a) s(n_1 + n_2) = sn_1 + sn_2$$

$$b) s(n_1 n_2) = (sn_1)n_2$$

We shall also assume that S is unitary, that is $s \cdot 1 = s$ for every s in S , where 1 is the multiplicative identity of N .

The additive group $(N, +)$ is an N -module, using the near-ring multiplication, and any additive group is a module over the near-ring of the set of zero-fixing maps from the group to itself.

N -modules are called N -groups by some authors.

Just as there are homomorphisms between modules over a ring, there are homomorphisms between modules over near-rings. These are important in defining module

substructures.

Definition 1.2: An additive group homomorphism f from the N -module S_1 to the N -module S_2 is called an N -homomorphism if for any $s_1 \in S_1$ and any $n \in N$, $(sn)f = (sf)n$.

The substructures of a module over a ring are the submodules and factor modules. Every submodule is itself a module over the ring and since the additive group is abelian, submodules are normal in the module. Every submodule T is also the kernel of the canonical homomorphism from the module S to the factor module S/T . In the case of near-ring modules the situation is less simple. A substructure that is itself a module need not be normal in the module, and those subsets that are normal are not always kernels of N -homomorphisms as the factor need not be an N -module because only one distributive rule holds. We define two different substructures, the N -subgroups and the N -submodules and the latter will yield factor modules.

Definition 1.3: An N -subgroup T of the N -module S is a subgroup of the group S such that $tn \in T$ for every $t \in T$ and every $n \in N$.

Definition 1.4: T is an N -submodule of the N -module S if T is the kernel of an N -homomorphism of S . This is equivalent to the conditions that T be both normal in S and satisfy $(t+s)n - sn \in T$ for each $t \in T$, $s \in S$ and $n \in N$.

N -submodules are also called N -kernels.

We will occasionally use the terms module, submodule and subgroup in place of N -module, N -submodule and N -subgroup.

Definition 1.5: A normal subgroup I of $(N, +)$ is called a *left ideal* of N if $NI \subseteq I$. A normal subgroup I is called a *right ideal* of N if $(i+n)n' - nn' \in I$ for each $i \in I$, $n, n' \in N$. So the right ideals of N are the N -submodules of $(N, +)$, viewed as an N -module. A normal subgroup that is both a left and a right ideal of N is called an *ideal* of N .

If M is a subgroup of $(N, +)$ such that $MM \subseteq M$, we call M a *subnear-ring* of N .

Definition 1.6: An N -module S is called *faithful* if whenever $n \in N$ and $sn = 0$ for all s in S , then $n = 0$.

The existence of N -subgroups which are not N -submodules also complicates the extension of the concept of simple modules in rings to near-rings. This leads to the notions of type-0, type-1 and type-2 N -modules. Consequently, several different Jacobson-type radicals may be defined.

Definition 1.7: An N -module S is called monogenic if there is some $s \in S$ such that $sN = S$. We call s a generator for S .

Definition 1.8: Let S be a non-zero monogenic N -module. Then S is said to be of

- a) type-0 if S has no proper non-zero N -submodules
- b) type-1 if S has no proper non-zero N -submodules and for each $s \in S$ either $sN = 0$ or $sN = S$. A monogenic N -module satisfying this latter condition is called strongly monogenic
- c) type-2 if S has no proper non-zero N -subgroups.

So S -modules of type-2 are of type-1 and those of type-1 are of type-0. Since we assume that N has a multiplicative identity, type-1 and type-2 modules are the same, and they will generally be called type-2. We note that since $1 \in N$ and S is unitary, $SN \neq (0)$ if $S \neq (0)$.

DIRECT SUMS

As in ring theory, we can define internal and external direct sums of submodules and modules respectively. Unlike the ring theory situation though, for a general near-ring these will not always be isomorphic. In our case however, where we assume all near-rings to be zero-symmetric, the internal and external direct sums are isomorphic, as will be shown.

Definition 1.9: Let $(S_i)_{i \in I}$ be an indexed collection of N -modules. Then the external direct sum of $(S_i)_{i \in I}$, denoted $\oplus_I S_i$, is defined by

$$\oplus_I S_i = \{ f: I \rightarrow \cup_I S_i : f(i) \in S_i \text{ for each } i \in I \text{ and } f(i) \neq 0 \text{ for only finitely many } i \in I \}$$

We will denote the elements of $\oplus_I S_i$ by $(s_i)_{i \in I}$ where $s_i = f(i)$ and it is understood that only finitely many s_i are non-zero.

$\oplus_I S_i$ becomes an N -module when operations are defined co-ordinate wise; that is, for $(s_j)_{j \in I}$ and $(s'_j)_{j \in I}$ in $\oplus_I S_i$ and any $n \in N$,

$$(s_j)_{j \in I} + (s'_j)_{j \in I} = (s_j + s'_j)_{j \in I} \quad \text{and} \quad (s_j)_{j \in I} n = (s_j n)_{j \in I}$$

If we remove the restriction that only finitely many s_j be non-zero, we have the external direct product of the S_i , denoted $\prod_I S_i$, also an N -module with the same co-ordinate wise addition and multiplication.

For any $i \in I$, ι_i will denote the i -th injection map from S_i to $\oplus_I S_i$; that is, for any $s_i \in S_i$, $s_i \iota_i$ is defined to be $(v_j)_{j \in I}$ where $v_j = \delta_{ij} s_i$ for each $j \in I$, with δ_{ij} the Kronecker delta. ι_i is an N -isomorphism from S_i to the set of all those members of $\oplus_I S_i$ who have all entries except the i -th co-ordinate zero, namely $S_i \iota_i$.

Lemma 1.10

Let $(S_i)_{i \in I}$ be an indexed collection of N -modules. Then for each $i \in I$, $S_i \iota_i$ is an N -submodule of $\oplus_I S_i$.

Proof

$S_i \iota_i$ is clearly normal in $\oplus_I S_i$ for if $(s_j)_{j \in I} \in \oplus_I S_i$ and $s'_i \in S_i$ then

$$(s_j)_{j \in I} + s'_i \iota_i - (s_j)_{j \in I} = (s_i + s'_i - s_i) \iota_i \in S_i \iota_i$$

Now let $n \in N$. It remains to show that $((s_j)_{j \in I} + s'_i \iota_i) n - (s_j)_{j \in I} n \in S_i \iota_i$. Now

$$((s_j)_{j \in I} + s'_i \iota_i) n - (s_j)_{j \in I} n = ((s_i + s'_i) n - s_i n) \iota_i \in S_i \iota_i \quad \text{and hence } S_i \iota_i \text{ is an } N\text{-submodule of } \oplus_I S_i.$$

Since each S_i is isomorphic to $S_i \iota_i$, each S_i is isomorphic to an N -submodule of $\oplus_I S_i$.

Let S be an N -module and let T be any non-empty subset of S . By the N -submodule (N -subgroup) of S generated by T , we mean the intersection of all the N -submodules (N -subgroups) of S containing T . An N -module S is called finitely generated if it contains a finite subset T such that S is the N -submodule of S generated by T .

If T and U are two subsets of an N -module S , by $T+U$ we will mean the set of all $t+u$ where $t \in T$ and $u \in U$. Inductively we then have, for any collection of subsets of S ,

$\{T_i\}_0^k$, where k is a natural number, $\sum_{j=0}^k T_j = (\sum_{j=0}^{k-1} T_j) + T_k$. If I is any index set, then by $\sum_{i \in I} T_i$ we will mean the set of all expressions of the form $t_1 + t_2 + \dots + t_n$ where $t_i \in T_i$ and n is finite.

Note that while the sum of two submodules of a particular module is again a submodule, the sum of two N -subgroups need no longer be an N -subgroup. However we do have:

Lemma 1.11

If T is an N -subgroup of the N -module S and U is an N -submodule of S , then $T+U$ is an N -subgroup of S .

Proof

This follows easily, since if $t \in T$, $u \in U$ and $n \in N$,

$$(t+u)n = (t+u)n - tn + tn \in U+T = T+U, \text{ since } U \text{ is an } N\text{-submodule of } S.$$

Note that any finite sum of submodules of an N -module is again a submodule.

Let S be an N -module and $\{S_i\}_{i \in I}$ a family of normal N -subgroups of S . If $S = \sum_{i \in I} S_i$ and

$(\sum_{i \in I} S_i) \cap S_j = (0)$ for each $j \in I$ then we say that S is the internal direct sum of the S_i and $i \neq j$

we write $S = \sum_{i \in I} S_i$. We note that the S_i need not be N -submodules of S . The elements from different S_i commute with respect to addition, as in the standard group theory result.

Definition 1.12: A direct sum $\sum_{i \in I} I_i = I$ of ideals of near-ring N is called *distributive* if for any $\sum_{i \in I} i_i$ and $\sum_{j \in I} j_j$ in I , $(\sum_{i \in I} i_i)(\sum_{j \in I} j_j) = \sum_{i, j \in I} i_i j_j$. A direct sum $\sum_{i \in I} S_i = T$ of N -submodules S_i of S is called *distributive* if for any $\sum_{i \in I} s_i \in T$ and for any $n \in N$, $(\sum_{i \in I} s_i)n = \sum_{i \in I} s_i n$, where the s_i come from different S_i .

We then have the following:

Proposition 1.13

If $(S_i)_{i \in I}$ is a family of normal N -subgroups of S with $\sum_I S_i = \sum_I S_i$, then if $\sum_I S_i$ is distributive, each S_i is a submodule of S . Conversely, if each S_i is an N -submodule of S , then the sum is distributive.

Proof

Suppose firstly that $\sum_{i \in I} S_i$ is a distributive sum of normal N -subgroups. Then clearly each S_i is an N -submodule of $\sum_I S_i$ since for any $s = (s_1 + s_2 + \dots + s_k)$ in $\sum_I S_i$, $(s + s_i)n - sn \in S_i$, since elements from different S_j commute.

To prove the converse, note first that since N is zero-symmetric, any N -submodule T of S is also an N -subgroup of S , since for any $t \in T$, $n \in N$ $tn = (0+t)n - 0n \in T$. Once this is known we can prove the result.

Let S be an N -module such that $S = \sum_I S_i$ where each S_i is an N -submodule of S . Let $s_1 + s_2 + \dots + s_r \in S$, $n \in N$. We prove the result by induction on r . Certainly with $r = 1$ the result is true.

Suppose that for any $1 \leq k < r$, $(s_1 + s_2 + \dots + s_k)n = s_1 n + s_2 n + \dots + s_k n$. Now S_{k+1} and $\sum_{i=1}^k S_i$ are N -submodules of S , and so,

$(s_1 + s_2 + \dots + s_k + s_{k+1})n - s_{k+1}n \in \sum_{i=1}^k S_i$ and $-(s_1 + \dots + s_k)n + (s_1 + \dots + s_k + s_{k+1})n \in S_{k+1}$. Hence

$-(s_1 + s_2 + \dots + s_k)n + (s_1 + s_2 + \dots + s_k + s_{k+1})n - s_{k+1}n \in (\sum_{i=1}^k S_i) \cap S_{k+1}$ and so

$-(s_1 + \dots + s_k)n + (s_1 + \dots + s_k + s_{k+1})n - s_{k+1}n = 0$.

Remarks 1.13

a) In the proof of this result we showed that if N is zero-symmetric, then every N -submodule of an N -module is an N -subgroup.

b) If N is zero-symmetric and S is an N -module with N -submodules $(S_i)_{i \in I}$ such that $S =$

$\Sigma_1 S_i$, then $\Sigma_1 S_i \cong \oplus_1 S_i$. Also any external direct sum of N -modules, $\oplus_1 S_i$ is the internal direct sum of its N -submodules $(S_i)_{i \in I}$. For these reasons the internal direct sum is also usually written using the symbol \oplus .

c) A result similar to 1.13 holds for ideals of a near-ring; i.e. if N is a zero-symmetric near-ring, then every direct sum of ideals of N is distributive.

JACOBSON-TYPE RADICALS

Several different Jacobson-type radicals have been defined in near-ring theory, and the most well-known will be described here. Recall that where R is a ring, $J(R)$ is defined to be the intersection of all the primitive ideals of R , an ideal I being primitive if R/I has a simple faithful module, and R itself being primitive if the zero ideal is primitive. $J(R)$ annihilates every simple R module and can also be characterised by any of the following:

- a) The intersection of all the maximal left (right) ideals of R
- b) The intersection of all left (right) primitive ideals of R
- c) $\{x \in R: rx \text{ is quasi-regular for all } r, s \in R\}$
- d) $\{x \in R: rx \text{ is quasi-regular for all } r \in R\}$
- e) $\{x \in R: xs \text{ is quasi-regular for all } s \in R\}$
- f) The union of all the quasi-regular left (right) ideals of R
- g) The union of all the quasi-regular ideals of R
- h) The unique largest superfluous left (right) ideal of R .

We now indicate how these notions were extended to near-rings.

Definition 1.15: A near-ring N is called v -primitive on S if S is a faithful N -module of type- v ($v \in \{0, 2\}$). N is called v -primitive if there exists an N -module S on which N is v -primitive. Ideal I of N is called v -primitive if N/I is a v -primitive near-ring. Equivalently, ideal I of N is v -primitive if and only if I is the annihilator in N of an N -module of type- v ; that is, $I = \text{Ann}_N(S)$ where S is of type- v .

We can now define the v -radicals of N , for $v \in \{0, 2\}$.

Definition 1.16: For $v \in \{0, 2\}$, $J_v(N) = \cap \{v\text{-primitive ideals of } N\} = \cap \{\text{Ann}_N(S) : S \text{ is an } N\text{-module of type-}v\}$. If there are no N -modules of type- v , put $J_v(N)$ equal to N . Since we are assuming that N has an identity, by Zorn's lemma N has a proper v -primitive ideal and so $J_v(N) \neq N$ in the near-rings we are considering.

We have the relationship $J_0(N) \subset J_2(N)$.

Unlike in rings, the intersection of all maximal right ideals of a near-ring is not in general a two sided ideal, but only a right ideal. In the near-ring case we denote this intersection $J_{1/2}(N)$ and call it the quasi-radical of N . $J_0(N) \subset J_{1/2}(N) \subset J_2(N)$, since any 0-primitive ideal is contained in a maximal right ideal. For if P is 0-primitive, then P is the annihilating ideal of an N -module $N^r - A$ where A is a maximal right ideal. In fact P is the largest 2-sided ideal contained in A . We now define another radical, the s -radical introduced by Hartney [6].

Definition 1.17: Let S be an N -module and let $s, s' \in S$. s is said to be equivalent to s' written $s \sim s'$, if

$$sN = s'N \text{ and}$$

$$\text{Ann}_N(s) = \text{Ann}_N(s'). \sim \text{ is an equivalence relation on } S.$$

Definition 1.18: An N -module S of type-0 is said to be of type- s if for each $s \in S$ such that $sN \neq 0$, we have

- $sN = \bigoplus_{i \in I} H_i$ where each H_i is an N -module of type-0 and a submodule of sN
- there exists an $s' \in sN$ such that $s \sim s'$.

Note that since N has a multiplicative identity, the second condition becomes redundant as we can take $s' = s.1$.

Lemma 1.19

If S is an N -module of type- s as in the previous definition, then the set I is finite.

Proof

By b) there exists $s' \in sN = \bigoplus_{i \in I} H_i$ such that $s \sim s'$. So there exists a set $\{i_1, i_2, \dots, i_n\}$ in I such that $s' \in H_{i_1} \oplus H_{i_2} \oplus \dots \oplus H_{i_n}$. Then $s'N \subseteq (H_{i_1} \oplus H_{i_2} \oplus \dots \oplus H_{i_n})N = H_{i_1}N \oplus H_{i_2}N \oplus \dots \oplus H_{i_n}N$ by lemma 1.13. But $s'N = sN = \bigoplus_{i \in I} H_i \supset H_{i_1} \oplus H_{i_2} \oplus \dots \oplus H_{i_n}$. Hence $sN = H_{i_1} \oplus H_{i_2} \oplus \dots \oplus H_{i_n}$.

By the definition of a type-2 N -module, any type-2 N -module is also of type- s .

Definition 1.20: A near-ring N is called s -primitive if it has a faithful type- s N -module. Ideal I of N is called s -primitive if N/I is an s -primitive near-ring; that is, if I annihilates an N -module of type- s .

Definition 1.21: The intersection of all the s -primitive ideals of a near-ring N is called the s -radical of N , $J_s(N)$. If N has no s -primitive ideals, we define $J_s(N) = N$.

In our case, $J_s(N) \neq N$. Also, we have the inclusion

$$J_2(N) \supset J_s(N) \supset J_{1/2}(N) \supset J_0(N)$$

In the ring theory case all of these coincide and $J_\nu(N) = J(N)$, $\nu = 0, 1, 2, s$.

CHAPTER TWO

GENERATION IN NEAR-RING MODULES

In this chapter we extend the concept of generation, from modules over an arbitrary ring with identity to modules over a near-ring with identity. We first briefly examine the ring-theory situation.

Let S be a module over a ring. A submodule T of S is called essential in S if for every non-zero submodule U of S , $U \cap T \neq (0)$. The symbol " \leq " will mean "is a submodule of", so that $T \leq S$ means that T is a submodule of S . Then several results are easily proved:

Let $T_1 \leq S_1 \leq S$ and $T_2 \leq S_2 \leq S$ where S is a module over R , a ring with identity. As S is a module over a ring, both T_1 and T_2 are also submodules of S . Then T_1 is essential in S if and only if T_1 is essential in S_1 and S_1 is essential in S , while $T_1 \cap T_2$ is essential in S if and only if both T_1 and T_2 are essential in S . If $S = S_1 \oplus S_2$, then $T_1 \oplus T_2$ is essential in S if and only if T_1 is essential in S_1 and T_2 is essential in S_2 .

Using Zorn's lemma it can be shown that if T is a submodule of S , there exists another submodule of S maximal with respect to having zero intersection with T , and the sum of this submodule and T is essential in S . (Recall that the restricted sum of submodules is always again a submodule in the ring theory case.)

Let \mathcal{Z} be any class of modules over the ring. S is said to be generated by \mathcal{Z} if there is an indexed set $(U_i)_{i \in I}$ in \mathcal{Z} and an epimorphism $h: \oplus_I U_i \rightarrow S$.

We define the trace of \mathcal{Z} in S , $\text{Tr}_S(\mathcal{Z})$ to be

$\text{Tr}_S(\mathcal{Z}) = \sum \{ \text{Im } h \mid h: U \rightarrow S \text{ and } U \in \mathcal{Z} \}$. Then $\text{Tr}_S(\mathcal{Z})$ is the unique largest submodule of S generated by \mathcal{Z} . We note that $\text{Tr}_S(\mathcal{Z})$ may be S itself.

These definitions cannot be directly applied to near-ring modules as the image of a homomorphism of a near-ring module is not necessarily an N -submodule of the image $\text{sp. } A$, but only an N -subgroup; whilst the sum of two N -subgroups is not always an

N -subgroup. Some restrictions to the definitions of generation and trace are therefore necessary.

With \mathcal{S} the class of all simple modules over ring R , we define the socle of S , $\text{Soc}(S)$, to be $\text{Tr}_{\mathcal{S}}(S)$. It is the largest semisimple submodule of S . It can be shown that $\text{Soc}(S)$ is the intersection of all the essential submodules of S . Since near-ring modules have several substructures, there are several generalisations of the "simple" concept, affecting the definition of a socle.

Let N denote a zero-symmetric left near-ring with identity. The concept of an essential submodule in ring theory must be adjusted to deal with the two main substructures occurring in modules over a near-ring.

Definition 2.1: Let S be an N -module. An N -subgroup T of S is called module essential in S if, whenever U is a non-zero N -submodule of S , then $U \cap T \neq (0)$. T is called essential in S if for every non-zero N -subgroup U of S , $U \cap T \neq (0)$.

The following lemma will be used often in what follows.

Lemma 2.2

If T is any N -subgroup of the N -module S and U is a submodule of S maximal with respect to $T \cap U = (0)$ [the existence of U is guaranteed by Zorn's lemma] then $T+U$ is module essential in S .

Proof

If A is a non-zero submodule of S such that $A \cap (T+U) = (0)$ then $(A+U) \cap T = (0)$ contradicting the maximality of U .

Clearly every essential N -subgroup is module-essential.

These definitions are due to Beidleman [2] and Oswald [12].

Definition 2.3: An N -subgroup T of S is semicomplemented if there exists an N -submodule

U of S such that $U \cap T = (0)$ and $U + T = S$. U is then called a semicomplement of T in S . If every element of a set of N -subgroups is semicomplemented, then the set is called semicomplemented.

An N -module S will be called minimal if it contains no proper non-zero N -subgroups, and irreducible if it contains no proper non-zero N -submodules. We note that every type-2 N -module is minimal, but the zero N -module is minimal but not of type-2, as type-2 N -modules are by definition non-zero. Similarly every type-0 N -module is irreducible but an irreducible N -module need not be of type-0.

Definition 2.4: An N -module S is called completely reducible if it is a direct sum of minimal N -submodules.

We now come to the definition of generation in near-ring modules.

Definition 2.5: Let N be a near-ring and let \mathcal{U} be any class of N -modules, and let S be an N -module. An N -submodule T of S is said to be generated by \mathcal{U} in S if there is a subset $(U_i)_{i \in I}$ in \mathcal{U} for some index set I and an epimorphism $h: \bigoplus_{i \in I} U_i \rightarrow T$ such that each image of U_i is an N -submodule of S , where $U_i h$ is the restriction of h to U_i . i_i is of course the injection map from U_i to $\bigoplus_{i \in I} U_i$.

Note that an N -submodule of S generated by \mathcal{U} in S need not be generated by \mathcal{U} in every N -module containing S . This is because if S is properly contained in N -module S' , an N -submodule of S need not be an N -submodule of S' .

Definition 2.6: We define the trace of \mathcal{U} in S , $\text{Tr}_S(\mathcal{U})$ to be

$$\text{Tr}_S(\mathcal{U}) = \sum \{ \text{Im } h \mid h: U \rightarrow S, U \in \mathcal{U} \text{ and } (U)h \text{ is an } N\text{-submodule of } S \}$$

This last condition is necessary since $(U)h$ is in general only an N -subgroup of S and the sum of N -subgroups need not be an N -subgroup. As in the ring-theory case, we have:

Proposition 2.7

$\text{Tr}_S(\mathcal{U})$ is the unique largest N -submodule of S generated by \mathcal{U} in S .

Proof

First we show that $\text{Tr}_S(\mathcal{U})$ is generated by \mathcal{U} in S . There exists $(U_i)_{i \in I}$ in \mathcal{U} and N -homomorphisms $(h_i)_{i \in I} : h_i : U_i \rightarrow S$ with $(U_i)h_i$ an N -submodule of S for each $i \in I$, such that $\text{Tr}_S(\mathcal{U}) = \sum_I \text{Im } U_i h_i$. Define $h : \bigoplus_I U_i \rightarrow S$ by $((u_j)_{j \in I}) \pi_j h = (u_j) h_j$ for all $j \in I$. Then $\pi_i h = h_i$ and so $(U_i) \pi_i h$ is an N -submodule of S for each $i \in I$. Clearly $\text{Tr}_S(\mathcal{U}) = \text{Im } h$, so that $\text{Tr}_S(\mathcal{U})$ is generated by \mathcal{U} in S .

Next suppose T is an N -submodule of S generated by \mathcal{U} in S . Then we have

$\bigoplus_I U_i \xrightarrow{h} T \rightarrow (0)$ exact for some family $(U_i)_{i \in I}$ in the class \mathcal{U} and each $(U_i) \pi_i h$ is an N -submodule of S . Then $T = \sum_I \text{Im } \pi_i h$ and so $T \subseteq \text{Tr}_S(\mathcal{U})$.

We now give some general results concerning generation and trace.

Proposition 2.8

a) If S is an N -module generated by a class of N -modules \mathcal{U} , then every epimorphic image of S is also generated by \mathcal{U} .

b) If $(S_i)_{i \in I}$ is an indexed set of N -modules each generated by \mathcal{U} , then $\bigoplus_I S_i$ is also generated by \mathcal{U} .

Proof

a) follows by taking the composition mapping

$$\bigoplus_{i \in I} U_i \rightarrow S \rightarrow T$$

b) If for each $i \in I$ we have

$$f_i : \bigoplus_{\beta \in B_i} U_{\beta} \rightarrow S_i \text{ epic, define } f : \bigoplus_{i \in I} \left(\bigoplus_{\beta \in B_i} U_{\beta} \right) \rightarrow \bigoplus_{i \in I} S_i \text{ by } ((u_{\beta})_{\beta \in B_i}) \pi_k = \sum_{j \in B_i} (\delta_{jk}) u_{\beta_j} f_j \text{ with } \delta$$

the kronecker delta.

As a corollary to this result we have the following.

Corollary 2.9

If \mathcal{S} is a class of N -modules each of which is generated by \mathcal{U} , then any N -module generated by \mathcal{S} is generated by \mathcal{U} .

Proof.

First we show that $\text{Tr}_S(\mathcal{U})$ is generated by \mathcal{U} in S . There exists $(U_i)_{i \in I}$ in \mathcal{U} and N -homomorphisms $(h_i)_{i \in I} : U_i \rightarrow S$ with $(U_i)h_i$ an N -submodule of S for each $i \in I$, such that $\text{Tr}_S(\mathcal{U}) = \sum_I \text{Im } U_i h_i$. Define $h : \bigoplus_I U_i \rightarrow S$ by $((u_i)_{i \in I}) \pi_j h = (u_j) h_j$ for all $j \in I$. Then $\iota_i h = h_i$ and so $(U_i) \iota_i h$ is an N -submodule of S for each $i \in I$. Clearly $\text{Tr}_S(\mathcal{U}) = \text{Im } h$, so that $\text{Tr}_S(\mathcal{U})$ is generated by \mathcal{U} in S .

Next suppose T is an N -submodule of S generated by \mathcal{U} in S . Then we have

$\bigoplus_I U_i \xrightarrow{h} T \rightarrow (0)$ exact for some family $(U_i)_{i \in I}$ in the class \mathcal{U} and each $(U_i) \iota_i h$ is an N -submodule of S . Then $T = \sum_I \text{Im } \iota_i h$ and so $T \subseteq \text{Tr}_S(\mathcal{U})$.

We now give some general results concerning generation and trace.

Proposition 2.8

- If S is an N -module generated by a class of N -modules \mathcal{U} , then every epimorphic image of S is also generated by \mathcal{U} .
- If $(S_i)_{i \in I}$ is an indexed set of N -modules each generated by \mathcal{U} , then $\bigoplus_I S_i$ is also generated by \mathcal{U} .

Proof.

- follows by taking the composition mapping

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- If for each $i \in I$ we have

$$f_i : \bigoplus_{B_i} U_{\beta_i} \rightarrow S_i \text{ epic, define } f : \bigoplus_{i \in I} (\bigoplus_{B_i} U_{\beta_i}) \rightarrow \bigoplus_{i \in I} S_i \text{ by } ((u_{\beta_j})_{j \in B_i}) \pi_k = \sum_{j \in B_i} (u_{\beta_j} \delta_{jk}) \pi_k \text{ with } \delta$$

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As a corollary to this result we have the following.

Corollary 2.9

If \mathcal{S} is a class of N -modules each of which is generated by \mathcal{U} , then any N -module generated by \mathcal{S} is generated by \mathcal{U} .

Proposition 2.10

Let T and S be N -modules. Then T generates S if and only if there exists a set of N -homomorphisms H from T to S with $S = \sum_H \text{Im } h$ and each $\text{Im } h$ is an N -submodule of S .

Proof

First suppose that $S = \sum_H \text{Im } h$ where each h is an N -homomorphism from T to S such that $\text{Im } h$ is a submodule of S . Then the mapping $f: T^{(H)} \rightarrow S$ with co-ordinatewise operations determined by H is epic and each $\text{Im } \iota_h f = \text{Im } h$ is a submodule of S .

Clearly, if S is generated by T and we have $T^{(A)} \xrightarrow{h} S \rightarrow (0)$ exact, then

$$S = \sum_A \text{Im } \iota_h.$$

Corollary 2.11

Let T and S be N -modules. Then T generates S if and only if for every non-zero N -homomorphism f with domain S there exists an N -homomorphism $h: T \rightarrow S$ with $\text{Im } h$ a submodule of S such that $hf \neq 0$.

Proof

Let H be the set of all N -homomorphisms h from T to S with $\text{Im } h$ a submodule of S , and let $X = \sum_H \text{Im } h$. If $f: S \rightarrow U$, then $hf = 0$ for each $h \in H$ if and only if $X \subseteq \ker f$.

Now suppose that T generates S . By proposition 2.7 $S = X = \sum_H \text{Im } h$ and so $hf = 0$

for all $h \in H$ implies that $S = X \subseteq \ker f$ i.e. f is the zero map. This is false and so there exists $h \in H$ such that $hf \neq 0$.

Suppose that for any non-zero $f: S \rightarrow U$ there exists $h \in H$ such that $hf \neq 0$. Let f be the natural map from S to S/X . Then $X \subseteq \ker f$ and so $hf = 0$ for all $h \in H$. This contradicts the hypothesis unless f is the zero map i.e. $S = X$.

While in general it is not true that if $f: S \rightarrow T$ then $(\text{Tr}_S(\mathcal{Z}))f \subseteq \text{Tr}_T(\mathcal{Z})$, this does hold if the

map f is an N -epimorphism since if $\text{Tr}_S(\mathcal{Z}) = \sum_I \text{Im } h_i$ where $h_i: U_i \rightarrow S$ then $(\text{Tr}_S(\mathcal{Z}))f = \sum_I \text{Im } h_i f$ and each $\text{Im } h_i f$ is a submodule of T .

Proposition 2.12

Let $(S_i)_{i \in I}$ be a collection of N -modules. Then for any class of N -modules \mathcal{Z} ,

$$\text{Tr}_{\bigoplus_I S_i}(\mathcal{Z}) = \bigoplus_{i \in I} \text{Tr}_{S_i}(\mathcal{Z}).$$

Proof

For each $i \in I$ there exists an epimorphism $h_i: \bigoplus_{\beta_i} U_{\beta_i} \rightarrow \text{Tr}_{S_i}(\mathcal{Z})$ for some subset $(U_{\beta_i})_{\beta_i}$ of \mathcal{Z} such that each $U_{\beta_i} h_i$ is a submodule of S_i . By letting C be the disjoint union of the β_i , we have an epimorphism from $\bigoplus_{\gamma \in C} U_\gamma$ to $\bigoplus_I \text{Tr}_{S_i}(\mathcal{Z})$ such that the image of each U_γ is a submodule of $\bigoplus_I S_i$. Hence $\bigoplus_I \text{Tr}_{S_i}(\mathcal{Z}) \subset \text{Tr}_{\bigoplus_I S_i}(\mathcal{Z})$.

$\text{Tr}_{\bigoplus_I S_i}(\mathcal{Z}) = \sum_I (\text{Tr}_{\bigoplus_I S_i}(\mathcal{Z})) \pi_i \iota_i \subset \sum_I \text{Tr}_{S_i}(\mathcal{Z}) \iota_i$, since π_i is epic from $\bigoplus_I S_i$ to S_i . $\sum_I (\text{Tr}_{S_i}(\mathcal{Z})) \iota_i = \bigoplus_I \text{Tr}_{S_i}(\mathcal{Z})$ and so $\text{Tr}_{\bigoplus_I S_i}(\mathcal{Z}) \subset \bigoplus_I \text{Tr}_{S_i}(\mathcal{Z})$ and the proof is complete.

Now let \mathcal{Z} be the class of all minimal N -modules and let S be an N -module. Then $\text{Tr}_S(\mathcal{Z})$, the largest N -submodule of S generated by \mathcal{Z} in S is just the sum of all the minimal N -submodules of S , the image of a minimal submodule being minimal again. We reiterate that the zero module is considered to be minimal.

Definition 2.13: For any N -module S , the socle of S , $\text{Soc } S$, is the sum of all the minimal N -submodules of S ; that is $\text{Soc } S = \text{Tr}_S(\mathcal{Z})$.

We now investigate the structure of the socle of a module further. The following lemma will be useful for this purpose.

Lemma 2.14 (Oswald [12])

If T is an N -subgroup of S and the set of all N -subgroups of S is semicomplemented in S , then so is the set of all N -subgroups of T . In particular, if T is an N -submodule of S and U is an N -subgroup of T , then there exists an N -submodule V of S such that $U \cap V = (0)$ and $T = U + V$.

Proof

This follows easily since if U is an N -subgroup of T then U is an N -subgroup of S and hence there is a submodule W of S such that $U \cap W = (0)$ and $U + W = S$. Putting $V = W \cap T$ we have $U \cap V = (0)$ and $T = U + V$.

Corollary 2.15

If T is a non-minimal N -submodule of S and the set of all N -subgroups of S is semicomplemented, then T is decomposable as a direct sum of non-zero submodules of S .

Theorem 2.16

Let S be an N -module. Then the following are equivalent:

- a) $S = \text{Soc } S$
- b) S is a direct sum of minimal N -submodules.
- c) S contains no proper module-essential N -subgroups.
- d) The set of all N -subgroups of S is semicomplemented in S .

Proof

We prove that b) \Rightarrow c) \Rightarrow d) \Rightarrow b) b) \Rightarrow a) and a) \Rightarrow c).

If b) holds then any N -submodule of S is a direct sum of some of the minimal submodules and so b) implies c).

Now suppose that S contains no proper module-essential N -subgroups and let T be an N -subgroup of S . We apply Zorn's lemma to \mathcal{S} , the set of all N -submodules of S having zero intersection with T . (0) is in \mathcal{S} and we order \mathcal{S} by set inclusion. The union of a chain in \mathcal{S} is again an element of \mathcal{S} and is an upper bound for the chain, and so there exists an

N -submodule U of S maximal with respect to $T \cap U = (0)$. By lemma 2.2 $T+U$ is module-essential in S . This contradicts the hypothesis unless $S = T+U$ and so d) is implied by c).

Suppose that the set of N -subgroups of S is semicomplemented. Let X be the sum of all the minimal N -submodules of S . X exists as (0) is a minimal N -submodule of S . If $X \neq S$ we can find an N -submodule K such that $X \cap K = (0)$ and $S = X+K$. Let $0 \neq k \in K$. Using Zorn's lemma we can find an N -submodule L of S maximal with respect to $X \subset L$ and $k \notin L$. By condition d) there is an N -submodule P such that $S = L+P$ and $L \cap P = (0)$. If P is not minimal then by corollary 2.15 $P = Y \oplus Z$ where Y and Z are non-zero submodules of S . Then L is properly contained in both $L+Y$ and $L+Z$, and so $k \in (L+Y) \cap (L+Z)$. Now $(L+Y) \cap (L+Z) = L$, since if $l_1 + y = l_2 + z \in (L+Y) \cap (L+Z)$ then $-l_2 + l_1 = z - y = 0$, since $L \cap P = (0)$. So $l_1 = l_2$ and $y = z = 0$, since $Y \cap Z = (0)$. Hence $l_1 + y = l_1 \in L$ and we have $(L+Y) \cap (L+Z) \subset L$. Clearly $L \subset (L+Y) \cap (L+Z)$ and so equality holds, giving $k \in L$. But this contradicts the choice of L . So P must be minimal, but this contradicts the fact that $P \cap X = (0)$. Hence $X = S$; that is, $S = \sum_{i \in I} S_i$ where each S_i is a

minimal N -submodule of S . Now let \mathcal{S} be the collection of all submodules of S which can be written as a direct sum of minimal submodules of S . $\mathcal{S} \neq \emptyset$. Partially order \mathcal{S} by set

inclusion. If $T_1 \subset T_2 \subset \dots$ is a chain in \mathcal{S} , then $\bigcup_{i=1}^{\infty} T_i \in \mathcal{S}$ since if each $T_i = \bigoplus_{\alpha \in A_i} S_\alpha$

then $\bigcup_{i=1}^{\infty} T_i = \bigoplus_{\alpha \in \bigcup A_i} S_\alpha$. Hence S has a submodule T maximal with respect to having a direct sum decomposition into minimal submodules of S . Since $S = \sum_{i \in I} S_i$ and all N -subgroups of S are semicomplemented in S , we have $T = S$. (ie b) holds.

If b) holds then a) clearly holds.

Now suppose that a) holds and $\text{Soc } S = S$. Then $S = \sum_{i \in I} S_i$ where each S_i is a minimal submodule of S . Suppose that S has a module-essential N -subgroup K . But then for any

non-zero S_i we have $K \cap S_i \neq (0)$ and so $S_i \subset K$ as the S_i are minimal. Then $\sum_{i \in I} S_i = K$. So S has no proper module-essential N -subgroups, i.e. c) holds.

Continue to let \mathcal{U} denote the class of minimal N -modules.

Proposition 2.17

Let T denote the intersection of all the module-essential N -subgroups of S . Then T is generated by \mathcal{U} ; that is T is completely reducible as an N -module.

Proof

If X is an N -subgroup of T , then X is an N -subgroup of S . By Zorn's lemma we can find a submodule Q of S maximal with respect to $Q \cap X = (0)$. Then $X+Q$ is module-essential in S by lemma 2.1 and so $T \subset X+Q$ and $T = X+(T \cap Q)$ where $T \cap Q$ is a submodule of T . T is completely reducible by theorem 2.16.

Notice that we have not proved that T is generated by \mathcal{U} in S , but only that T is generated by \mathcal{U} (in T). From this result we cannot conclude that T is a submodule of S . What we can deduce is that $\text{Soc } S \subset T$ since if U is a minimal submodule of S , $U \subset T$. In the ring theory case, $\text{Soc } S = T$. In our case, that is, the near-ring case, we have:

Proposition 2.18

Let T be the intersection of all the module-essential N -subgroups of S . If T is a submodule of S then $\text{Soc } S = T$.

Proof

Suppose there exists $0 \neq p \in T \setminus \text{Soc } S$. By Zorn's lemma there exists a submodule Q of S maximal with respect to $p \notin Q$ and $\text{Soc } S \subset Q$. Let $Q_1 = Q \cap T$. Again using Zorn's lemma there exists a submodule U of S maximal with respect to $Q_1 \cap U = (0)$, and by lemma 2.1

$Q_1 + U$ is module-essential in S and hence contains T . So $T = \mathfrak{M}(Q_1 + U) = Q_1 + (U \cap T) = Q_1 \oplus U_1$ where $U_1 = U \cap T$.

Let X be a non-zero N -subgroup. Then X is an N -subgroup of S and so there exists a submodule V of S maximal with respect to $\mathfrak{M}X = (0)$. By lemma 2.1 $X + V$ is module-essential in S and so $U_1 \subset T \subset X + V$. Hence $U_1 = U_1 \cap (X + V) = X + (\mathfrak{M}U_1)$. If $X \neq U_1$ then $V_1 = \mathfrak{M}U_1 \neq (0)$. Applying the same argument to V_1 as has just been applied to X we see that there exists a submodule W_1 of S such that $U_1 = V_1 + W_1$ and $V_1 \cap W_1 = (0)$.

$V_1 \neq (0)$ and since $X \neq (0)$ we must have $W_1 \neq (0)$ [Since if $W_1 = (0)$, we must have $U_1 \cap Q = (0)$, $V_1 \cap Q = W_1 \cap Q = (0)$ and $p \in (Q + V_1)(Q + W_1)$. Writing $p = q_1 + v = q_2 + w$, we have $-q_2 + q_1 \in (V_1 + W_1) \cap Q = U_1 \cap Q = (0)$ and so $v = w = 0$, contradicting the fact that $p \notin Q$.

Hence $X = U_1$ and U_1 is a minimal submodule of S , from which we have $U_1 \subset \text{Soc } S \subset Q$, $Q_1 \cap U_1 = (0)$ and so $U_1 = (0)$. Then $T = Q_1 = \mathfrak{M}Q$ so $T \subset Q$ contrary to $p \notin Q$. It follows that $T = \text{Soc } S$.

Proposition 2.19

Let T be the intersection of all the module-essential N -subgroups of S . If T is not a submodule of S then $\text{Soc } S$ is the largest submodule of S contained in T .

Proof

Suppose Q is a submodule of S such that $\text{Soc } S \subset Q \subset T$. Let $q \in Q \setminus \text{Soc } S$. By Zorn's lemma there exists a submodule U of S maximal with respect to $q \notin U$ and $\text{Soc } S \subset U$. Let $U_1 = U \cap Q$. Again using Zorn's lemma there exists a submodule V of S maximal with respect to $U_1 \cap V = (0)$ and by lemma 2.1 $U_1 + V$ is module-essential in S . Let $V_1 = \mathfrak{M}Q$. Then V_1 is zero by a proof very similar to that above, replacing the U_1 in proposition 2.18 by V_1 .

Since $U_1 + V$ is module-essential in S , $Q = Q \cap (U_1 + V) = U_1 + (Q \cap V) = U_1 + V_1$. $V_1 = (0)$, so $Q = U_1 = U \cap Q$, giving $Q \subset U$. This contradicts the fact that $q \notin U$ and hence $Q = \text{Soc } S$.

Proposition 2.20

Let S be an N -module such that $S = \oplus_I S_i$ where each S_i is a minimal (irreducible) N -submodule of S . Then any short exact sequence

$$(0) \rightarrow U \xrightarrow{f} S \xrightarrow{g} T \rightarrow (0)$$

splits and U and T are sums of their minimal (irreducible) submodules.

Proof

$\text{Im } f = \ker g$ so $\text{Im } f$ is a submodule of S . Since $S = \oplus_I S_i$ where each S_i is minimal (irreducible), $\text{Im } f = \oplus_J S_j$ for some $J \subset I$ and $\text{Im } f$ is a direct summand of S . For any $s = (u) f + s'$, define $h: S \rightarrow U$ by $(s)h = u$. Then $fh = 1_U$ and so the left hand side of the sequence splits.

Since $\text{Im } g \cong S/\ker g$, and g is epic, and $\text{Im } f = \ker g$, $T \cong S/\text{Im } f$ and each t in T is associated with a unique $s + \text{Im } f$ in $S/\text{Im } f$. Define $k: T \rightarrow S$ by $(t)k = s - (s)h$. Then $kg = 1_T$ and the right hand side splits.

$$U \cong \text{Im } f = \oplus_J S_j \text{ while } T \cong S/(\text{Im } f) \cong \oplus_{I \setminus J} S_i.$$

We now turn our attention to generation by \mathcal{I} , the class of all irreducible N -modules. A module is generated by \mathcal{I} if and only if it is the sum of submodules each irreducible as an N -module. Since a submodule of a direct summand of S is also a submodule of S , S is generated by \mathcal{I} if and only if it is the sum of its irreducible submodules. Submodule T of S is generated by \mathcal{I} in S if and only if it is the sum of submodules of S each irreducible as an N -module. The following proposition gives more equivalent conditions for generation by \mathcal{I} .

Proposition 2.18

Let S be an N -module. The following are equivalent:

- S is a direct sum of irreducible submodules.
- S is generated by the class of irreducible N -modules.
- S is the sum of some set of irreducible submodules.

- d) S is the sum of its irreducible submodules.
 e) Every submodule of S is a direct summand of S .
 f) Every short exact sequence $(0) \rightarrow U \rightarrow S \rightarrow T \rightarrow (0)$ splits.

Proof

We have a) \Rightarrow f) \Rightarrow e) ; b) \Rightarrow a) and b) \Leftrightarrow c) \Leftrightarrow d). We need only show that e) \Rightarrow d). Suppose then that every submodule of S is a direct summand of S . We first show that every non-zero submodule of S contains an irreducible submodule of S .

Let $0 \neq s \in S$ and let $[s]$ be the submodule of S generated by s . Then $[s]$ contains a submodule of S , U , maximal among submodules of S contained in $[s]$:

Let \mathcal{S} be the set of all submodules of S properly contained in $[s]$. Order \mathcal{S} by set inclusion. $(0) \in \mathcal{S}$. The union of any chain in \mathcal{S} is again a submodule of S and is properly contained in $[s]$, since if not there is some element in the chain containing s and hence $[s]$, a contradiction. Hence by Zorn's lemma, U exists.

U is a direct summand of S by hypothesis, so $S = U \oplus U'$ for some submodule of S , U' . Then $[s] = [s] \cap (U \oplus U') = U \oplus ([s] \cap U')$ and so $[s] \cap U' \simeq [s]/U$ is an irreducible submodule of S .

Let X be the union of all the irreducible submodules of S . If $X \neq S$, $S = X \oplus X'$ where $X' \neq (0)$. But then by the previous part, X' contains an irreducible submodule of S , contradicting the definition of X . So $X = S$.

CHAPTER THREE

COGENERATION IN NEAR-RING MODULES

Since the kernels of N -homomorphisms are submodules of the domains, the notion of cogeneration is extended to near-rings more easily than that of generation. Generation is concerned with the sums of images of N -homomorphisms. Images of N -homomorphisms are not necessarily N -submodules but only N -subgroups of the image space. Very briefly we now outline the concept of cogeneration in ring theory. Unless otherwise specified, for any module S over a ring R (over a near-ring N), $T \leq S$ will mean that T is an R (respectively N) submodule of S .

Let S be a module over the ring R . $T \leq S$ is called superfluous or small in S if whenever $U \leq S$ such that $T+U = S$, we have $U = S$. This notion is dual to that of the essential submodule.

If $T_1 \leq S_1 \leq S$ and $T_2 \leq S_2 \leq S$, then S_1 is small in S if and only if T_1 is small in S_1 and S_1/T_1 is small in S/T_1 , while T_1+T_2 is small in S if and only if both T_1 and T_2 are small in S .

If $f: S_1 \rightarrow S_2$ is an R -homomorphism and T_1 is small in S_1 , then $f(T_1)$ is small in S_2 .

If $S = S_1 \oplus S_2$, then $T_1 \oplus T_2$ is small in S if and only if T_1 is small in S_1 and T_2 is small in S_2 .

Let \mathcal{Z} be any class of modules over the ring R . S is said to be cogenerated by \mathcal{Z} if there exists an indexed set $(U_i)_{i \in I}$ in \mathcal{Z} and a monomorphism $S \rightarrow \prod_{i \in I} U_i$.

We define the reject of \mathcal{Z} in S , $\text{Rej}_S(\mathcal{Z})$, by

$$\text{Rej}_S(\mathcal{Z}) = \cap \{ \ker h \mid h: S \rightarrow U, U \in \mathcal{Z} \}.$$

Then $\text{Rej}_S(\mathcal{Z})$ is the unique smallest submodule T of S such that S/T is cogenerated by \mathcal{Z} .

With \mathcal{S} the class of all simple modules over ring R , we define the radical of S , $\text{Rad} S$, to be equal to $\text{Rej}_S(\mathcal{S})$. $\text{Rad} R$, the Jacobson radical of R , is also equal to:

a) The intersection of all the maximal left (right) ideals of R

- b) The intersection of all left (right) primitive ideals of R
- c) The sum of all quasi-regular left (right) ideals of R
- d) The sum of all quasi-regular ideals of R
- e) The unique largest superfluous left (right) ideal of R

The near-ring case

Here we investigate $\text{Rej}_S(\mathcal{U})$ and $\text{Rej}_S(\mathcal{V})$ where S is a near-ring module and \mathcal{U} and \mathcal{V} are the irreducible and minimal near-ring modules respectively.

Definition 3.1: Let \mathcal{U} be a class of N -modules. An N -module S is said to be cogenerated by \mathcal{U} if there exists an indexed set $(U_i)_{i \in I}$ in \mathcal{U} and a monomorphism

$$(0) \rightarrow S \rightarrow \prod_I U_i$$

If S is cogenerated by \mathcal{U} and there exists a monomorphism from S' to S , then S' is cogenerated by \mathcal{U} . If \mathcal{V} is a class of N -modules, each cogenerated by \mathcal{U} , then the class of modules cogenerated by \mathcal{V} is contained in the class of modules cogenerated by \mathcal{U} .

For any N -modules T and S , let $\text{Hom}_N(T, S)$ denote the class of all N -homomorphisms from T to S .

Proposition 3.2

Let T and S be N -modules. T cogenerates S if and only if there exists a subset

$$H \subset \text{Hom}_N(S, T) \text{ with } \bigcap_{h \in H} \ker h = (0).$$

Proof

Suppose T cogenerates S and $f: S \rightarrow T^I$ is monic. For each $i \in I$, let $f_i = \pi_i \circ f$. Then f_i maps S to T and is an N -homomorphism for each $i \in I$ and $\bigcap_{i \in I} \ker f_i = \ker f = (0)$.

Conversely, suppose $H \subset \text{Hom}_N(S, T)$ with $\bigcap_{h \in H} \ker h = (0)$. Define $f: S \rightarrow T^H$ by

$$(s)f = ((s)h)_{h \in H} \text{ for all } s \in S. \text{ Then } f \text{ is an } N\text{-homomorphism and } \ker f = \bigcap_{h \in H} \ker h = (0), \text{ i.e. } f$$

is monic.

As a corollary we have the following

Corollary 3.3

Let T and S be N -modules. Then T cogenerates S if and only if for every non-zero homomorphism $f: U \rightarrow S$, there exists $h \in \text{Hom}_N(S, T)$ such that $fh \neq (0)$.

Proof

Note first that $fh = (0)$ for all $h \in \text{Hom}_N(S, T)$ if and only if $\text{Im } f \subseteq \bigcap_{h \in \text{Hom}(S, T)} \ker h$.

Suppose that T cogenerates S . By proposition 3.2 there exists $H \subseteq \text{Hom}_N(S, T)$ such that $\bigcap_{h \in H} \ker h = (0)$. Then $\bigcap_{h \in \text{Hom}_N(S, T)} \ker h = (0)$, so if $fh = 0$ for every $h \in \text{Hom}_N(S, T)$ then

$\text{Im } f = (0)$. But f is non-zero and so for some $h \in \text{Hom}_N$

Now suppose that for any non-zero $f: U \rightarrow S$ there is $h \in \text{Hom}_N(S, T)$ such that $fh \neq 0$. Let $H = \text{Hom}_N(S, T)$ and let $A = \bigcap_{h \in H} \ker h$, with $f: A \rightarrow S$ the inclusion map. If $f \neq 0$ then

there exists $h \in H$ such that $fh \neq 0$. But $fh = 0$ for each $h \in H$, by the definition of A and f , so f is the zero map i.e. $\bigcap_{h \in H} \ker h = (0)$, and by the previous proposition, T cogenerates S .

Notice that while the previous proposition and corollary are in some ways dual to 2.10 and 2.11, since we are dealing here with kernels, which are N -submodules, we do not need the restrictions on H given in 2.10 and 2.11.

In some sense dual to the notion of trace, we have the reject:

Definition 3.4: Let \mathcal{U} be a class of N -modules and let S be an N -module. Then the reject of \mathcal{U} in S , $\text{Rej}_S(\mathcal{U}) := \bigcap \{ \ker h \mid h: S \rightarrow U \text{ and } U \in \mathcal{U} \}$.

Proposition 3.5

$\text{Rej}_S(\mathcal{U})$ is the unique smallest submodule T of S such that S/T is cogenerated by \mathcal{U} .

Proof

Let $(U_i)_{i \in I}$ be an indexed set in \mathcal{U} with $h: S \rightarrow \prod_i U_i$ an N -homomorphism. Let

$T = \ker h$. Then $\text{Rej}_S(\mathcal{Z}) \subset \bigcap_I \ker(h\pi_i) = \ker h = T$, and so if S/T is cogenerated by \mathcal{Z} , then $\text{Rej}_S(\mathcal{Z}) \subset T$.

Now there exists an indexed set $(U_i)_{i \in I}$ in \mathcal{Z} and N -homomorphisms $h_i: S \rightarrow U_i$ with $\text{Rej}_S(\mathcal{Z}) = \bigcap_{i \in I} \ker h_i$. Then $f: S \rightarrow \prod_I U_i$ defined by $(s)f = ((s)h_i)_{i \in I}$ has kernel $\text{Rej}_S(\mathcal{Z})$ and so $S/\text{Rej}_S(\mathcal{Z})$ is cogenerated by \mathcal{Z} .

From this result and the fact that $S \simeq S/(0)$, we can deduce that S is cogenerated by \mathcal{Z} if and only if $\text{Rej}_S(\mathcal{Z}) = (0)$. Also, if T is a submodule of S , then $T = \text{Rej}_S(\mathcal{Z})$ if and only if $T \leq \text{Rej}_S(\mathcal{Z})$ and $\text{Rej}_{S/T}(\mathcal{Z}) = (0)$.

Let S and T be N -modules, with $f: S \rightarrow T$ an N -homomorphism. We have shown that if f is epic, then $(\text{Tr}_S(\mathcal{Z}))f \subset \text{Tr}_T(\mathcal{Z})$. In the case of retracts however, f can be any N -homomorphism:

Proposition 3.6

If $f: S \rightarrow T$ is an N -homomorphism, then $(\text{Rej}_S(\mathcal{Z}))f \subset \text{Rej}_T(\mathcal{Z})$.

Proof

Let $x \in \text{Rej}_S(\mathcal{Z})$. Now $\text{Rej}_T(\mathcal{Z}) = \bigcap_I \ker h_i$ where each h_i is an N -homomorphism from T to some $U_i \in \mathcal{Z}$. Then $(x)h_i = 0$ for each $i \in I$ as $h_i \in \text{Hom}_N(S, U_i)$, and $x \in \text{Rej}_S(\mathcal{Z})$. Hence $(x)f \in \text{Rej}_T(\mathcal{Z})$.

Proposition 3.7

Let S be an N -module. Then $\text{Rej}_N(S) = \text{Ann}_N(S)$. So in particular, S is a faithful N -module if and only if S cogenerates N .

Proof

First note that if $f: N \rightarrow S$ is an N -homomorphism, then f just multiplies each $n \in N$ on the left by some fixed $s \in S$, for if $(1)f = s$, then $(n)f = (1)fn = sn$. We denote f by \bar{s} .

$$\text{So } \text{Rej}_N(S) = \bigcap \{ \ker f \mid f \in \text{Hom}_N(N, S) \} = \bigcap \{ \ker \bar{s} \mid s \in S \} = \bigcap_{s \in S} \text{Ann}_N(s) =$$

$\text{Ann}_N(S)$.

Motivated by this proposition, we can define, for any class \mathcal{Z} of N -modules, $\text{Ann}_N(\mathcal{Z})$ to be $\text{Rej}_N(\mathcal{Z})$.

We now take \mathcal{Z} to be the class of all irreducible N -modules and \mathcal{S} the class of all minimal N -modules. Since $\mathcal{S} \subset \mathcal{Z}$, for any N -module S we have $\text{Rej}_S(\mathcal{Z}) \subset \text{Rej}_S(\mathcal{S})$. In fact we have the following inequalities.

Proposition 3.8

$$\begin{aligned} \text{Rej}_S(\mathcal{Z}) \subset \Sigma \{\text{small submodules of } S\} = \cap \{\text{maximal submodules of } S\} \\ \subset \cap \{\text{strictly maximal submodules of } S\} = \text{Rej}_S(\mathcal{S}) \end{aligned}$$

Proof

If T is a maximal submodule of S , then S/T is an irreducible N -module; that is S/T is cogenerated by \mathcal{Z} . Hence $\text{Rej}_S(\mathcal{Z}) \subset \cap \{\text{maximal submodules of } S\}$.

Now let U be any small submodule of S and let T be any maximal submodule of S . If U is not contained in T , we would have $U+T = S$, implying that $T = S$. Hence $U \subset T$ and $\Sigma \{\text{small submodules of } S\} \subset \cap \{\text{maximal submodules of } S\}$.

On the other hand, let $0 \neq s \in \cap \{\text{maximal submodules of } S\}$, and let $[s]$ denote the submodule of S generated by s . Suppose that T is a submodule of S such that $[s]+T = S$. By Zorn's lemma there exists a submodule U of S maximal with respect to $s \notin U$ and $T \subset U$. This contradicts the choice of s , and so we must have $T = S$ i.e. $[s]$ is small and so $\cap \{\text{maximal submodules of } S\} = \Sigma \{\text{small submodules of } S\}$.

T is a strictly maximal submodule of S if and only if S/T is a minimal N -module if and only if S/T is cogenerated by \mathcal{S} , and consequently,

$$\cap \{\text{strictly maximal submodules of } S\} = \text{Rej}_S(\mathcal{Z}).$$

Beidleman [2] defines a radical for an N -module S to be the intersection of all the

strictly maximal submodules of S . He then shows that if this radical, $\text{Rej}_S(\mathcal{A})$ is small, then it is the intersection of all the maximal submodules of S , and contains all other small submodules of S . The preceding proposition gives the same results. We also have

Proposition 3.9

Let S be an N -module. If every proper submodule of S is contained in a strictly maximal submodule, then $\text{Rej}_S(\mathcal{A})$ is the unique largest small submodule of S .

Proof

$\text{Rej}_S(\mathcal{A})$ is small, since if T is a proper submodule of S such that $\text{Rej}_S(\mathcal{A}) + T = S$, then T is contained in a strictly maximal submodule of S , U . But $\text{Rej}_S(\mathcal{A}) \subset U$ and so $T = S$.

The uniqueness follows by proposition 3.8 which shows that all the small submodules of S are contained in $\text{Rej}_S(\mathcal{A})$.

Beidleman proved the following necessary and sufficient condition for $\text{Rej}_S(\mathcal{A})$ to be equal to the sum of all the small submodules of S .

Proposition 3.10

$\text{Rej}_S(\mathcal{A})$ is equal to the sum of all the small submodules of S if and only if every submodule of S generated by a finite subset of $\text{Rej}_S(\mathcal{A})$ is small.

Proof

Let $A = \Sigma$ {small submodules of S }. If every submodule generated by a finite subset of $\text{Rej}_S(\mathcal{A})$ is small, then $\text{Rej}_S(\mathcal{A}) \subset A$. Now $A \subset \text{Rej}_S(\mathcal{A})$ and so $\text{Rej}_S(\mathcal{A}) = \Sigma$ {small submodules of S }.

Conversely, suppose $\text{Rej}_S(\mathcal{A}) = \Sigma$ {small submodules of S }. Let T be any submodule of S generated by a finite subset of $\text{Rej}_S(\mathcal{A})$, say $\{x_1, x_2, \dots, x_k\}$. Then there exist small submodules of S , T_1, T_2, \dots, T_n such that $T \subset \sum_{i=1}^n T_i$. Since a finite sum of small N -submodules is again small, T is small in S .

Let S be any N -module.

Definition 3.11: S is called finitely generated if for any set Ω of submodules of S such that $S = \Sigma \Omega$, there exists a finite $\aleph \subset \Omega$ such that $S = \Sigma \aleph$.

S is called finitely cogenerated if for any set Ω of N -submodules of S such that $\cap \Omega = (0)$, there exists a finite $\aleph \subset \Omega$ such that $\cap \aleph = (0)$.

Proposition 3.12

Let S be an N -module and let \mathcal{Z} be the class of all irreducible N -modules. Then S is finitely generated if and only if $S/\text{Rej}_S(\mathcal{Z})$ is finitely generated and $\text{Rej}_S(\mathcal{Z})$ is small in S .

Proof

Suppose that S is finitely generated. Then since any factor module of S is also finitely generated, we need only show that $\text{Rej}_S(\mathcal{Z})$ is small in S . Suppose that $\text{Rej}_S(\mathcal{Z}) + T = S$ where T is an N -submodule of S . Then with $\Omega = \{\text{small } N\text{-submodules of } S\}$, $(\Sigma \Omega) + T = S$. As S is finitely generated, there exists some finite $\aleph \subset \Omega$ such that $(\Sigma \aleph) + T = S$. Now $\Sigma \aleph$ is small in S and hence $T = S$.

Conversely, suppose that $S/\text{Rej}_S(\mathcal{Z})$ is finitely generated and $\text{Rej}_S(\mathcal{Z})$ is small in S . Suppose that Ω is a set of N -submodules of S such that $\Sigma \Omega = S$. Then $\Sigma(\Omega + \text{Rej}_S(\mathcal{Z}))/\text{Rej}_S(\mathcal{Z}) = S/\text{Rej}_S(\mathcal{Z})$ and so there exists some finite $\aleph \subset \Omega$ such that $S = (\Sigma \aleph) + \text{Rej}_S(\mathcal{Z})$. $\text{Rej}_S(\mathcal{Z})$ is small in S and so $S = \Sigma \aleph$.

When S is the direct sum of minimal N -submodules, then S is finitely generated if and only if S is finitely cogenerated, as shown in the following proposition.

Proposition 3.13

Let S be the direct sum of minimal N -submodules. Then the following are equivalent:

- S is finitely cogenerated
- $S = T_1 + T_2 + \dots + T_n$ where each T_i is a minimal N -submodule of S
- S is finitely generated

Proof

a) \Rightarrow b): As S is a direct sum of minimal submodules, there exists a monomorphism

$$f: S \rightarrow \prod_I U_i \text{ where each } U_i \text{ is a minimal } N\text{-module, } \bigcap_I \ker f\pi_i = (0) \text{ and since a)}$$

holds, there exists a finite $F \subset I$ such that $\bigcap_F \ker f\pi_i = (0)$ and $S = \bigoplus_F T_i$ where each T_i is a minimal N -submodule of S .

c) \Rightarrow a): since S is finitely generated and a sum of minimal N -submodules,

$S = T_1 + T_2 + \dots + T_n$, where each T_i is a minimal N -submodule of S . We show that S is finitely cogenerated by induction on n .

If $n = 1$ then S is minimal and hence finitely cogenerated.

Suppose then that $n > 1$ and that any module that is the sum of less than n of its minimal submodules is finitely cogenerated. Suppose further that Ω is a set of N -submodules of S with $\bigcap \Omega = (0)$. Then for some $B \in \Omega$, $T_n \cap B = (0)$, otherwise $T_n \subset \bigcap \Omega = (0)$. So $B = S_1 + S_2 + \dots + S_m$ where each S_j is simple and $s < n$. Let $\Omega' = \{A \cap B \mid A \in \Omega\}$. Then Ω' is a set of N -submodules of B and $\bigcap \Omega' = \bigcap \Omega = (0)$. B is finitely cogenerated and so by the induction hypothesis there is a finite set $\mathbb{K} \subset \Omega'$ such that $\bigcap \mathbb{K} = (0)$. $(\bigcap \mathbb{K}) = B \cap A_1 \cap A_2 \dots \cap A_k$ for some $A_1, A_2, \dots, A_k \in \mathbb{K}$, so S is finitely cogenerated.

b) \Rightarrow c): Suppose that $S = T_1 \oplus T_2 \oplus \dots \oplus T_n$ where each T_i is a minimal N -submodule of S , and let Ω be a set of N -submodules of S such that $S = \sum \Omega$. As each T_i is minimal, for each i , $T_i \subset A_i$ for some $A_i \in \Omega$. Then $S = \sum_i A_i$.

Since $\text{Soc } S$ is defined to be the sum of all the minimal N -submodules of S , from this proposition we see that $\text{Soc } S$ is finitely generated if and only if it is finitely cogenerated.

When dealing with modules over a ring, where submodules and subgroups coincide, more results concerning finite generation and cogeneratedness can be proved. For example, there is a dual to proposition 3.12: when S is a module over a ring, then S is finitely cogenerated if and only if $\text{Soc } S$ is finitely cogenerated and $\text{Soc } S$ is essential in S . This result is not true in

modules over a near-ring, since submodules of $\text{Soc } S$ are not necessarily submodules of S . Both this result and that of proposition 3.12 will hold if we were to define finite generation and finite cogeneration in terms of N -subgroups rather than N -submodules but this is not practical as the sum of two N -subgroups need not be an N -subgroup again, and subgroup summation is not commutative.

In the case of modules over a ring, it can be easily shown that a module S is finitely generated if and only if every module generating S finitely generates S , while if S is finitely cogenerated it is true that every module cogenerating S finitely cogenerates S , but the converse of this is not true. $\oplus_{\mathbb{P}} \mathbb{Z}_{\mathbb{P}}$ is not finitely cogenerated, but every module that cogenerates $\oplus_{\mathbb{P}} \mathbb{Z}_{\mathbb{P}}$, finitely cogenerates it.

If S is a module over a near-ring, the implications in the case of both cogeneration and generation are one-sided; i.e.

- a) If S is finitely cogenerated, then every N -module cogenerating S finitely cogenerates S
- b) If S is finitely generated, then every N -module generating S finitely generates S .

$\oplus_{\mathbb{P}} \mathbb{Z}_{\mathbb{P}}$, seen as a module over the integers, shows that the converse of a) does not hold.

Since for any near-ring N , the additive group $(N, +)$ is an N -module with right ideals corresponding to N -submodules, the intersection of the maximal N -submodules of $(N, +)$ is equal to the intersection of all the maximal right ideals of N , $J_{1/2}(N)$. So $J_{1/2}(N)$ is the sum of all the small right ideals of N by proposition 3.8. Since N has an identity, we have a stronger result:

Proposition 3.14

If N is a near-ring with identity, the $J_{1/2}(N)$ is the unique largest small right ideal of N .

Proof

We must show that $J_{1/2}(N)$ is small in N . Suppose that M is a right ideal of N such that $M + J_{1/2}(N) = N$. Since N has an identity, by Zorn's lemma, M can be contained in a maximal right ideal of N , P say. Then $J_{1/2}(N) + P = N$. But $J_{1/2}(N) \subset P$, so $P = N$. This is a

contradiction and so $M = N$.

Note that when N does not have a multiplicative identity, it may not be true that $J_{1/2}(N)$ is small, since N may have no maximal right ideals, in which case we have $J_{1/2}(N) = N$, which is not small in N .

Definition 3.15: $z \in N$ is called (right) quasi-regular if the minimal right ideal containing all elements of the form $n-zn$, $n \in N$, also contains z . Equivalently, $z \in N$ is quasi-regular if the minimal right ideal containing all $n-zn$, $n \in N$ coincides with N . A subset of N is called quasi-regular if every element is quasi-regular.

It is easily seen that for any near-ring N , $J_{1/2}(N)$ is quasi-regular. For let $z \in J_{1/2}(N)$ and let R_z be the least right ideal of N containing $\{n-zn \mid n \in N\}$. If $z \notin R_z$, by Zorn's lemma there exists a right ideal R maximal with respect to $z \notin R$ and $R_z \subset R$. But R is maximal in N and so we must have $z \in R$ as $z \in J_{1/2}(N)$. Hence $z \in R_z$.

Proposition 3.16

Let N be a near-ring with identity. A right ideal M of N is quasi-regular if and only if M is small in N .

Proof

Let M be a quasi-regular right ideal of N and suppose that $M+P = N$ where P is a right ideal of N . Since N has a multiplicative identity, 1 , there exists some $m \in M$ and some $p \in P$ such that $1 = m+p$. Then for any $n \in N$,

$$n-mn = 1 \cdot n - mn = (m+p)n - mn \in P \quad \text{as } P \text{ is a right ideal of } N.$$

So the minimal right ideal containing all $n-mn$ is contained in P . This right ideal is N since m is quasi-regular and so $N = P$; that is, M is small in N .

Conversely, $J_{1/2}(N)$ is quasi-regular, and since every small right ideal of N is contained in $J_{1/2}(N)$ by proposition 3.15, every small right ideal of N is quasi-regular.

So if N has identity, $J_{1/2}(N)$ is the unique largest quasi-regular and the unique largest small right ideal of N .

Where N is a near-ring without identity, it does not follow that every quasi-regular right ideal of N is small, even in the case where N is finite. This is seen in the following example.

Example 3.17

Let N be the distributive, commutative, zero-symmetric near-ring with the following addition and multiplication tables:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	a	a

Then $A = \{0, a\}$ and $B = \{0, b\}$ are ideals of N , with $A+B = N$. B is a quasi-regular right ideal of N , but B is not small in N as $A \neq N$.

Every small ideal is however quasi-regular, since $J_{1/2}(N)$ is quasi-regular and contains all the small right ideals.

CHAPTER FOUR

MATRIX NEAR-RINGS

In what follows we use results due to Meldrum, van der Walt [11] and Meyer [14]. We try to extend these results to \mathfrak{S} -primitive matrix near-rings.

For any natural number n , the set of all $n \times n$ matrices with entries from a ring is again a ring. This is not generally true when dealing with near-rings, for the set of all $n \times n$ square arrays with entries from an arbitrary near-ring, is not necessarily, under normal matrix multiplication and addition, again a near-ring, since multiplication of the arrays is not associative. This is easily seen, for if N is a non-distributive left near-ring and if $a, b, c \in N$ such that $(a+b)c \neq ac+bc$, then

$$\begin{aligned} & \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (a+b)c & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} ac+bc & 0 \\ 0 & 0 \end{bmatrix} \\ & = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \right\}. \end{aligned}$$

Sligh [9] gives a necessary and sufficient condition for this set of square arrays to become a near-ring under the usual matrix addition and multiplication. Let $M_r(N)$ denote the set of all $n \times r$ square arrays with entries from the near-ring N .

Definition 4.1. A near-ring N is called n -distributive, n a positive integer, if for each $a, b, c, d, e, a_1, b_1 \in N$,

$$ab+cd = cd+ab \quad \text{and} \quad \left(\sum_{i=1}^n ab \right) e = \sum_{i=1}^n a b e$$

Theorem 4.2

Let N be any near-ring and let $r > 1$ be a positive integer. Then $M_r(N)$ is a near-ring if and only if N is r -distributive.

The proof is done by calculation. We give the following example for the case $r = 2$ by way of illustration.

Suppose that $M_r(N)$ is a near-ring.

Since $A(B+C) = AB+AC$ for any $A, B, C \in M_r(N)$, we put

$$A = \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \quad \text{to obtain } ab+cd = cd+ab.$$

Now let $A' = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}$, $B' = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \end{bmatrix}$ and $C' = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$. As $M_2(N)$ is a near-ring, $(A'B')C' = A'(B'C')$ and so $(a_1 b_1 + a_2 b_2)e = a_1 b_1 e + a_2 b_2 e$.

It was earlier thought (Heatherly [3]), that whenever $M_r(N)$ was a near-ring, N was distributive. The following example shows this to be untrue:

Let N be the near-ring with addition and multiplication tables as follows

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

Then $M_3(N)$ is a near-ring since N is 3-distributive by calculation, but N is not distributive as $a(a+a) = a$ while $a.a+a.a = 0$. Note that N is not zero-symmetric.

The class of near-rings giving rise to near-rings of matrices under the usual matrix multiplication and addition with the Ligh constraints is small. Consequently Meldrum and van der Walt [11] developed a different way of looking at matrices, allowing a near-ring of matrices to be formed from any near-ring and for any natural number r . $r \times r$ matrices over a near-ring N are regarded as functions from N^r to N^r .

For any near-ring N , let $M(N)$ denote the near-ring of all mappings from $(N, +)$ to $(N, +)$. We first define the functions $f_{ji}^n: N^r \rightarrow N^r$ where $n \in N$ and $1 \leq i, j \leq r$, which in the

ring case correspond to matrices with n in position (i, j) and zero elsewhere. Let $f_{ji}^n: N \rightarrow N$ be defined by $f_{ji}^n: s \rightarrow sn$ for all $s \in N$. Then define $f_{ji}^n := \pi_j f_{ji}^n$, $1 \leq i, j \leq r$, $n \in N$. f_{ji}^n multiplies the i -th entry of an element of N^r by n and then moves the result to position j , putting zero in all other positions.

Definition 4.3: The near-ring of $r \times r$ matrices over N , $\mathbb{R}_r(N)$, is the subnear-ring of $M(N^r)$ generated by $\{f_{ij}^n \mid n \in N \text{ and } 1 \leq i, j \leq r\}$. The elements of $\mathbb{R}_r(N)$ are called $r \times r$ matrices over N .

Then $\mathbb{R}_r(N)$ is a left near-ring with identity (since we assume that N has an identity) and if R is a ring with identity, then $\mathbb{R}_r(R)$ is the familiar ring of $r \times r$ matrices over R .

The matrix units are the matrices $E_{ij} := f_{ij}^1$, $1 \leq i, j \leq r$ and the identity matrix is $I = E_{11} + E_{22} + \dots + E_{rr}$.

The i -th column of the matrix A is the function $A\pi_i: N^r \rightarrow N$. Denote it by

$$A(i). \text{ Then } A = \sum_{i=1}^r A(i)\epsilon_i.$$

Let $A \in \mathbb{R}_r(N)$ and $n \in N$. Then $nA = (f_{11}^n + f_{22}^n + \dots + f_{rr}^n)A$. We have that $nE_{ij} = f_{ij}^n$.

$An = \sum_{i=1}^r A(i)f_{i1}^n$. $E_{ij}n = f_{ij}^n$ in our case, but note that if n is not zero-symmetric then this is not true in general.

A matrix of the form $n_1 E_{11} + n_2 E_{22} + \dots + n_r E_{rr}$ is called a diagonal matrix. If $n_1 = n_2 = \dots = n_r$, the matrix is called a scalar matrix as it is equal to $n_1 I$.

Theorem 4.4 Rules for matrix multiplication

Let N be a left near-ring with identity. Then

- $f_{ij}^n + f_{ij}^s = f_{ij}^{n+s}$
- $f_{ij}^n + f_{kl}^s = f_{kl}^s + f_{ij}^n$ if $k \neq i$
- $f_{ij}^n f_{kl}^s = \begin{cases} f_{kj}^{ns} & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$

$$d) E_{ij} E_{kl} = \begin{cases} E_{kj} & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$$

$$e) (f_{1k_1}^n + f_{2k_2}^n + \dots + f_{rk_r}^n) f_{ij}^n = f_{ik_j}^n$$

f) n is distributive in N if and only if f_{ij}^n is distributive in $\mathbb{R}_r(N)$.

Proof

a) for any $\underline{x} \in N^r$, $\underline{x}(f_{ij}^n + f_{ij}^s) = \underline{x}(\pi_j f_{ij}^n + \pi_j f_{ij}^s) = (0, 0, \dots, x_j n, 0, \dots, 0) + (0, 0, \dots, x_j s, 0, \dots, 0)$
 where both $x_j n$ and $x_j s$ appear as the i -th entry,

$$= (0, \dots, x_j n + x_j s, 0, \dots, 0) = (0, \dots, x_j (n+s), 0, \dots, 0)$$

$$= \underline{x} f_{ij}^{n+s}$$

b) to f) follow similarly.

As corollaries to these results we have:

1) N is distributively generated by D if and only if $\mathbb{R}_r(N)$ is distributively generated by

$$\{ f_{ij}^d \mid d \in D \ 1 \leq i, j \leq r \}.$$

2) If γ is a non-empty subset of $\{1, 2, \dots, r\}$, then $\{ \sum_{i \in \gamma} f_{ii}^n \mid n \in N \}$ is a subnear-ring of

$\mathbb{R}_r(N)$ which is isomorphic to N .

Since $\mathbb{R}_r(N)$ is a subnear-ring of $M(N^r)$ and by 2) just stated, $\mathbb{R}_r(N)$ has a subnear-ring isomorphic to N , N is abelian if and only if $\mathbb{R}_r(N)$ is abelian.

Remark 4.5

Our work deals with zero-symmetric near-rings. For an arbitrary, not necessarily zero-symmetric near-ring X we have:

g) n is zero-symmetric in X if and only if f_{ij}^n is zero-symmetric in $\mathbb{R}_r(X)$.

h) $\{ n \in X \mid 0n = 0 \}$ is called the zero-symmetric part of X and

$\{ n \in X \mid \forall n' \in X, nn' = n \}$ is called the constant part of X . The element n in X is constant if and only if f_{ij}^n is constant in $\mathbb{R}_r(X)$.

i) If $n = s + t$ is the standard decomposition of n into a zero-symmetric part s and a

constant part t , then $f_{ij}^A = f_{ij}^B + f_{ij}^C$ is the corresponding decomposition of f_{ij}^A .

From g) we can deduce that X is zero-symmetric if and only if $R_r(X)$ is zero-symmetric. We now return to our zero-symmetric near-rings.

By the definition of $R_r(N)$, any $A \in R_r(N)$ can be represented as an expression only in terms of f_{ij}^A . The length of such an expression is the number of f_{ij}^A that appear in it. The weight of A , $w(A)$ is the length of the expression of minimal length for A .

The weight is useful for proving results inductively. For if $w(A) \geq 2$, then $A = B+C$ or $A = BC$ where both $w(B)$ and $w(C)$ are less than $w(A)$. These facts are used in the following lemma.

Lemma 4.6

For any matrix $A \in R_r(N)$ and any $1 \leq k \leq r$ and any $x_1, x_2, \dots, x_r \in N$, there exist

$$a_1, a_2, \dots, a_r \in N \text{ such that } (f_{1k}^{x_1} + f_{2k}^{x_2} + \dots + f_{rk}^{x_r})A = f_{1k}^{a_1} + f_{2k}^{a_2} + \dots + f_{rk}^{a_r}$$

Proof

This follows by induction on the weight of A , $w(A)$. If $w(A) = 1$ then $A = f_{ij}^n$ for some $n \in N$ and by e) of theorem 4.4 the result is true.

Suppose then that the result holds for all matrices with weight less than m , $m \geq 2$. If $w(A) = m$, then $A = B+C$ or $A = BC$ where both $w(B)$ and $w(C)$ are less than m . By the induction hypothesis there exist a_1, a_2, \dots, a_r and b_1, b_2, \dots, b_r in N such that

$$\begin{aligned} (f_{1k}^{x_1} + f_{2k}^{x_2} + \dots + f_{rk}^{x_r})A &= (f_{1k}^{x_1} + f_{2k}^{x_2} + \dots + f_{rk}^{x_r})(B+C) \\ &= (f_{1k}^{a_1} + f_{2k}^{a_2} + \dots + f_{rk}^{a_r}) + (f_{1k}^{b_1} + f_{2k}^{b_2} + \dots + f_{rk}^{b_r}) = (f_{1k}^{a+b_1} + \dots + f_{rk}^{a+b_r}) \end{aligned}$$

or,

$$\begin{aligned} (f_{1k}^{x_1} + f_{2k}^{x_2} + \dots + f_{rk}^{x_r})A &= (f_{1k}^{a_1} + f_{2k}^{a_2} + \dots + f_{rk}^{a_r})BC \\ &= (f_{1k}^{a_1} + f_{2k}^{a_2} + \dots + f_{rk}^{a_r})C = (f_{1k}^{b_1} + f_{2k}^{b_2} + \dots + f_{rk}^{b_r}) \end{aligned}$$

completing the proof.

Corollary 4.7

For any matrix A and any $x_1, x_2, \dots, x_r \in N$, there exists $n \in N$ such that

$$(f_{11}^x + f_{21}^x + \dots + f_{r1}^x)AE_{11} = f_{11}^n.$$

Also, if $\alpha \in N^r$, there exists $\beta \in R_r(N)$ such that $\alpha = \epsilon_1 \beta$ (ϵ_1 is that element of N^r with 1 in the first position and zeros elsewhere.) This follows since if $\alpha = (a_1, a_2, \dots, a_r)$ we put

$$\beta = (f_{11}^a + f_{21}^a + \dots + f_{r1}^a).$$

Corollary 4.8

If $X \in R_r(N)$ is left invariant, then $N^r X \subset \epsilon_1 X$.

Proposition 4.9

If I is a right ideal of N , then I^r is a submodule of the $R_r(N)$ -module N^r .

Proof

Certainly I^r is a normal subgroup of N^r .

Now let $\alpha = (a_1, a_2, \dots, a_r) \in I^r$ and let $\rho = (n_1, n_2, \dots, n_r) \in N^r$. We must show that for any matrix $A \in R_r(N)$, $(\rho + \alpha)A = \alpha' + \rho A$ for some $\alpha' \in I^r$. We prove this by induction on $w(A)$.

|| If $A = f_{ij}^s$, then $(\rho + \alpha)A = ((a_j + n_j)s)_i = (a_j' + n_j s)_i$ for some $a_j' \in I$, since I is a right ideal of N . So $(\rho + \alpha)A = \alpha' + \rho A$.

The induction step follows by writing $A = B + C$ or $A = BC$, as in the proof of lemma 4.6. \square

Corollary 4.10

If I is a right ideal of N , then $(N^r : I^r) = \{A \in R_r(N) \mid \rho A \in I^r \forall \rho \in N^r\}$ is a two-sided ideal of $R_r(N)$.

The following four results enable us to connect the ideals of N with those of $\mathbb{R}_r(N)$.

Proposition 4.11

Let I_1 and I_2 be two-sided ideals of N . Then $I_1 \neq I_2$ if and only if $(N^r : I_1^r) \neq (N^r : I_2^r)$.

Proof

Suppose $I_1 \neq I_2$ and let $a \in I_2 \setminus I_1$. Then $f_{11}^a \in (N^r : I_2^r)$ as I_2 is a left ideal. However $f_{11}^a \notin (N^r : I_1^r)$ since $\epsilon_1 f_{11}^a = (a, 0, 0, \dots, 0) \in I_1^r$.

Conversely, suppose $(N^r : I_1^r) \neq (N^r : I_2^r)$ and let $A \in (N^r : I_1^r) \setminus (N^r : I_2^r)$. Let $\rho \in N^r$ such that $\rho A \notin I_2^r$. Suppose $\rho A = (a_1, a_2, \dots, a_r)$ with $a_k \notin I_2$. Then $\epsilon_1 (f_{11}^a + f_{21}^a + \dots + f_{r1}^a) A E_{1k} = (a_k, 0, \dots, 0)$; by corollary 4.7 the matrix on the left is f_{11}^a . Now $f_{11}^a \in (N^r : I_1^r)$ as E_{1k} is zero-symmetric, but f_{11}^a is not in $(N^r : I_2^r)$. Therefore $a_k \in I_1 \setminus I_2$ and the result holds.

Let $(\)^*$ be the function defined on the set of two-sided ideals of N by $(\mathcal{J})^* = (N^r : \mathcal{J})$. By the previous proposition this map is injective.

Let \mathcal{J} be a two-sided ideal in $\mathbb{R}_r(N)$. Define a map $(\)_*$ from the set of two-sided ideals of $\mathbb{R}_r(N)$ to the set of two-sided ideals of N by

$\mathcal{J}_* = \{n \in N \mid n \in \text{Im}(A \pi_j) \text{ for some } A \in \mathcal{J} \text{ and some } 1 \leq j \leq r\}$. By proposition 4.9 this is a surjection.

Lemma 4.12

If \mathcal{J} is a two-sided ideal of $\mathbb{R}_r(N)$, then $a \in \mathcal{J}_*$ if and only if $f_{11}^a \in \mathcal{J}$.

Proof

Let $a \in \mathcal{J}_*$. Then there exists some $A \in \mathcal{J}$, $\rho \in N$ and $j \in \{1, 2, \dots, r\}$ such that $(\rho A) \pi_j = a$.

Then by corollary 4.8 there exists $B \in \mathcal{J}$ such that $\rho A = \epsilon_1 B$. Now

$\epsilon_1 B = \epsilon_1 (f_{11}^a + f_{21}^a + \dots + f_{r1}^a) B$ and so by lemma 4.6 there exist a_1, a_2, \dots, a_r such that

$(f_{11}^a + f_{21}^a + \dots + f_{r1}^a) B = f_{11}^{a_1} + f_{21}^{a_2} + \dots + f_{r1}^{a_r}$. It follows that $a_j = a$. But then

$(f_{11}^a + f_{21}^a + \dots + f_{r1}^a) B E_{1j} = f_{11}^a \in \mathcal{J} E_{1j}$ being zero-symmetric.

The converse follows by the definition of \mathcal{I} .

Corollary 4.13

If \mathcal{I} is a two-sided ideal of $R_r(N)$, then $a \in \mathcal{I}$ if and only if $f_{ij}^a \in \mathcal{I}$ for all $1 \leq i, j \leq r$.

The proof follows since

$$f_{ij}^a = \begin{cases} f_{ij}^a & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{So } a \in \mathcal{I} \Rightarrow f_{11}^a \in \mathcal{I} \text{ is a two-sided ideal of } R_r(N) \text{ and}$$

so $f_{ij}^a f_{11}^a f_{11}^a = f_{ij}^a \in \mathcal{I}$

Proposition 4.14

If \mathcal{I} is a two-sided ideal of $R_r(N)$, then \mathcal{I} is a two-sided ideal of N .

Proof.

$N \cong \{ f_{11}^n \mid n \in N \}$, which is a subnear-ring of $R_r(N)$.

The following proposition gives the relationships between $(\mathcal{I})^*$ and \mathcal{I} and between $((\mathcal{I})^*)^*$ and \mathcal{I} .

Proposition 4.15

If \mathcal{I} is a two-sided ideal in $R_r(N)$ and \mathcal{J} is a two-sided ideal in N , then

a) $(\mathcal{I})^* \supset \mathcal{I}$

b) $((\mathcal{I})^*)^* = \mathcal{I}$

c) $((\mathcal{J})^*)^* = \mathcal{J}$

Proof

a) Suppose $A \in (\mathcal{I})^*$. Then there exists some $\rho \in N^*$ such that $\rho A \in (\mathcal{I})^*$, so there exists j , $1 \leq j \leq r$ with $(\rho A) \pi_j \in \mathcal{I}$. This implies $A \in \mathcal{I}$.

b) $a \in ((\mathcal{I})^*)^*$ if and only if $f_{11}^a \in (\mathcal{I})^*$ if and only if $a \in \mathcal{I}$.

c) By putting $\mathcal{J} = \mathcal{I}$ in b), we have c).

Definition 4.16: An ideal of $R_r(N)$ is called *full* if it is of the form $(\mathcal{J})^*$ for some ideal \mathcal{J} of N .

From this definition and preceding results we can state the following.

Proposition 4.17

There is a bijection between the set of ideals of N and the set of full ideals of $R_r(N)$ given by $\mathcal{J} \mapsto (\mathcal{J})^*$ and $\mathcal{I} \mapsto \mathcal{I}_*$, such that $((\mathcal{J})^*)_* = \mathcal{J}$ and $(\mathcal{I}_*)^* = \mathcal{I}$ for an ideal \mathcal{J} of N and a full ideal \mathcal{I} of $R_r(N)$. Also, (0) and $R_r(N)$ are full ideals of $R_r(N)$.

Proposition 4.18

N is irreducible if and only if $R_r(N)$ is irreducible.

Proof

First, suppose that N is irreducible and let \mathcal{I} be a non-zero ideal of $R_r(N)$. Then \mathcal{I} is non-zero and so $\mathcal{I} = N$. By corollary 4.13 then, $E_{ij} \in \mathcal{I}$ for all $1 \leq i, j \leq r$, and hence $I \in \mathcal{I}$ implying that $\mathcal{I} = R_r(N)$. So $R_r(N)$ is irreducible.

Conversely, suppose $R_r(N)$ is irreducible. Then for any non-zero ideal \mathcal{J} of N , $(\mathcal{J})^* = R_r(N) = N^*$ and so by proposition 4.11 $\mathcal{J} = N$ i.e. N is irreducible.

As a generalisation of the concept of a monogenic module, we have the following

Definition 4.19: An N -module S is called *connected* if for every $s_1, s_2 \in S$ there exist $n_1, n_2 \in N$ and $k \in S$ such that $s_1 = kn_1$ and $s_2 = kn_2$.

Clearly every monogenic N -module is connected. The converse is false, even in the ring case. However, if S is a connected N -module and satisfies the ascending chain condition for N -subgroups, then S is monogenic.

The following lemma shows that an N -module S is connected if and only if for every finite subset of elements of S , there exists $k \in S$ such that all the elements of the subset are contained in kS .

Lemma 4.20

Let S be a connected N -module. Then

a) For any $k \geq 2$ elements of S , s_1, s_2, \dots, s_k there are $n_1, n_2, \dots, n_k \in N$ and $s \in S$ such that $s_i = sn_i$.

b) If d is distributive in N , then $(s_1 + s_2)d = s_1 d + s_2 d$ for all $s_1, s_2 \in S$.

Proof

a) follows easily by induction on k : when $k = 2$ the result holds by the definition of connectedness. Suppose then that the result holds for any $t-1$ elements of S . Then given

$s_1, s_2, \dots, s_t \in S$, there exist $n_1, n_2, \dots, n_{t-1} \in N$ and $s \in S$ such that $s_i = sn_i$, $i = 1, 2, \dots, t-1$. Also there exist $n_t, n_0 \in N$ and $s_0 \in S$ such that $s = s_0 n_0$ and $s_t = s_0 n_t$. Substituting this expression for s into the expressions for s_i , $i = 1, 2, \dots, t-1$ gives the result.

b) holds since, given $s_1, s_2 \in S$ there exist $n_1, n_2 \in N$ and $s \in S$ such that $s_i = sn_i$. Then $(s_1 + s_2)d = s(n_1 + n_2)d = sn_1 d + sn_2 d = s_1 d + s_2 d$.

Every matrix has at least one expression in terms of f_{ij}^A . As some matrices in $R_r(N)$ can have several expressions, we form $E_r(N)$, the set of matrix expressions, a subset of the free semi-group over the symbols $\{f_{ij}^A\} \cup \{(\ , \) , +\}$, defined by:

$$f_{ij}^A \in E_r(N) \forall n \in N \ 1 \leq i, j \leq r$$

$$\text{If } A, B \in E_r(N) \text{ then } A+B \in E_r(N)$$

$$\text{If } A, B \in E_r(N) \text{ then } AB \in E_r(N).$$

We now show that if S is a connected N -module, then S^r can be regarded as a connected $R_r(N)$ module. Define the action of $R_r(N)$ on S^r as follows:

If $A \in R_r(N)$ and $(s_1, s_2, \dots, s_r) \in S^r$, let $n_1, n_2, \dots, n_r \in N$ and $s \in S$ such that $s_i = sn_i$, $i = 1, 2, \dots, r$. Then $(s_1, s_2, \dots, s_r)A = s[(n_1, n_2, \dots, n_r)A]$, where $s(t_1, t_2, \dots, t_r) = (st_1, st_2, \dots, st_r)$ for all $(t_1, t_2, \dots, t_r) \in N^r$.

This is a well-defined action since if $s_i = sn_i$ and $s_i = s^0 n_i^0$, $i = 1, 2, \dots, r$ as S is connected there exists $s' \in S$ and $n'_1, n'_2 \in N$ such that $s = s'n'_1$ and $s^0 = s'n'_2$ and so $s_i = s'n'_1 n_i = s'n'_2 n_i^0$. This means that $n'_1 n_i \in n'_2 n_i^0 + K$ where $K = \text{Ann}_N(s')$. But then

$$\begin{aligned}
s((n_1, n_2, \dots, n_r)A) &= s'n_1'((n_1, \dots, n_r)A) = s'((n_1'n_1, \dots, n_1'n_r)A) \\
&= s'((n_1'n_1^0, \dots, n_1'n_r^0)A + k) \text{ where } k \in K^x \\
&= s'n_2'((n_1^0, \dots, n_r^0)A) = s^0((n_1^0, \dots, n_r^0)A) \text{ and so we have:}
\end{aligned}$$

Proposition 4.21

If S is a connected N -module then S^x is a connected $R_r(N)$ module.

Proof

The action of $R_r(N)$ on S^x is well defined, as shown above.

S^x is connected: let $(s_1, s_2, \dots, s_r) \in S^x$ and $(t_1, t_2, \dots, t_r) \in S^x$. Then there exist

$$\begin{aligned}
n_1, n_2, \dots, n_r, n_{r+1}, \dots, n_{2r} \in N \text{ and } s \in S \text{ such that } s_i = sn_i \text{ and } t_i = sn_{r+i} \quad i = 1, 2, \dots, r. \text{ Put } A = \\
f_{11}^n + f_{21}^n + \dots + f_{r1}^n \text{ and } B = f_{11}^{n_{r+1}} + f_{21}^{n_{r+2}} + \dots + f_{r1}^{n_{2r}} \text{ and } \bar{s} = (s, 0, \dots, 0). \text{ Then} \\
(s_1, s_2, \dots, s_r) = \bar{s}A \text{ and } (t_1, t_2, \dots, t_r) = \bar{s}B, \text{ so } S^x \text{ is connected.}
\end{aligned}$$

Corollary 4.22

If S is a monogenic N -module, then S^x is a monogenic $R_r(N)$ module.

This follows by the above proof, showing that if S is generated by s , then S^x is generated by $(s, 0, \dots, 0)$. Notice that this generator for S^x is not unique, since we could have put $\bar{s} = (s, s, \dots, s)$ in the above proof.

If Γ is any connected $R_r(N)$ module, then we can form an N -module G from Γ by defining $G := \Gamma f_{11}^1 = \{\gamma f_{11}^1 \mid \gamma \in \Gamma\}$. Since f_{11}^1 is distributive in $R_r(N)$ and Γ is connected, G is isomorphic to a subgroup of Γ , by lemma 4.20 b). G becomes an N -module under $(\gamma f_{11}^1)n := (\gamma f_{11}^1 f_{11}^n) f_{11}^1 = \gamma f_{11}^n$.

Note that for any $1 \leq i \leq r$ and any $\gamma \in \Gamma$, $\gamma f_{11}^1 = \gamma f_{11}^1 f_{11}^1 \in G$. G is called the N -module derived from the connected $R_r(N)$ -module Γ .

Lemma 4.23

Let S be a connected N -module. Then any $\mathbb{R}_r(N)$ -submodule (subgroup) of S^r is of the form L^r , where L is an N -submodule (subgroup) of S .

Proof

If K is any $\mathbb{R}_r(N)$ -submodule of S^r , then $L_i = L_j$, $1 \leq i, j \leq r$ where $L_i := \{\gamma \pi_i \mid \gamma \in K\}$, since $K \pi_{i+1} \subset K$. It is easily shown that L_1 is an N -submodule of S and that $K = L_1^r$. Similarly for subgroups.

Corollary 4.24

If S is a connected N -module, then S is minimal (respectively irreducible) if and only if S^r is a minimal (irreducible) $\mathbb{R}_r(N)$ -module.

Proof

This follows by lemma 4.23 and the observation that if L is an N -submodule of S , then L^r is an $\mathbb{R}_r(N)$ -submodule of S^r .

Corollary 4.25

If N is v -primitive on S , then $\mathbb{R}_r(N)$ is v -primitive on S^r , $v = 0, 2$.

Proof

If the monogenic module S is faithful, then S^r is faithful. By corollaries 4.22 and 4.24, if S is of type- v , then S^r is of type- v over $\mathbb{R}_r(N)$, $v = 0, 2$.

Theorem 4.26

N is 2-primitive if and only if $\mathbb{R}_r(N)$ is 2-primitive.

Proof

Suppose that Γ is a faithful $\mathbb{R}_r(N)$ module of type-2, and let G be the N -module derived from Γ i.e. $G = \{\gamma \pi_{11} \mid \gamma \in \Gamma\}$. Since $n \in \text{Ann}_N(G)$ if and only if $\pi_{11} n \in \text{Ann}_{\mathbb{R}_r(N)}(\Gamma)$, G is a faithful N module.

To prove that G is a type-2 N -module we must show that every non-zero element of G generates G . Consequently, let γ_{11}^1 be a non-zero element of G . If $\delta_{11}^1 \in G$, then since Γ is faithful of type-2, there exists $A \in R_r(N)$ such that $(\gamma_{11}^1)A = \delta_{11}^1$. Now by lemma 4.6 there exist n_1, n_2, \dots, n_r such that $f_{11}^1 A = f_{11}^1 n_1 + f_{21}^2 n_2 + \dots + f_{r1}^r n_r$, so

$(\gamma_{11}^1)A f_{11}^1 = \gamma_{11}^1 n_1 = (\gamma_{11}^1) n_1 = \delta_{11}^1$. So every non-zero element of G generates G and G is of type-2.

Proposition 4.27

If N is s -primitive then $R_r(N)$ is s -primitive.

Proof

Suppose that N is s -primitive with faithful type- s N -module S . Then S^r is a faithful type-0 $R_r(N)$ -module by corollary 4.24 and the fact that $n \in \text{Ann}_N(S)$ if and only if $f_{11}^1 n \in \text{Ann}_{R_r(N)}(S^r)$. In fact S^r is a type- s $R_r(N)$ module:

Let $(s_1, s_2, \dots, s_r) \in S^r$ such that $(s_1, s_2, \dots, s_r) R_r(N) \neq (0)$. Then for at least one i , $s_i N \neq (0)$, since $R_r(N)$ is generated by $\{f_{ij}^n \mid n \in N\}$. For every i such that $s_i N \neq (0)$, $s_i N = \bigoplus_{j=1}^{k_i} L_j$ where each L_j is an N -module of type-0 and an N -submodule of $s_i N$. Let

$$m = \sum_{i: s_i N \neq 0} k_i. \text{ We claim that } (s_1, s_2, \dots, s_r) R_r(N) = \bigoplus_{j=1}^m L_j^r.$$

First we show that for any $A \in R_r(N)$, $(s_1, s_2, \dots, s_r) A \in \bigoplus_{j=1}^m L_j^r$, by induction on the weight of A .

If $w(A) = 1$ then $A = f_{ij}^n$ for some $n \in N$ and $(s_1, s_2, \dots, s_r) A = (0, \dots, s_j n, 0, \dots, 0)$ with $s_j n$ in the i -th position. $(0, \dots, s_j n, 0, \dots, 0) \in \bigoplus_{a=1}^k L_a^r \subset \bigoplus_{a=1}^m L_a^r$ and so the result is true with $w(A) = 1$.

Suppose then that $(s_1, s_2, \dots, s_r) B \in \bigoplus_{j=1}^m L_j^r$ for any $B \in R_r(N)$ with $w(B) < k$, where

$k \geq 2$. If $w(A) = k$, then $A = B + C$ or $A = BC$ where $w(B), w(C) < k$. So either

$(s_1, s_2, \dots, s_r)A = (s_1, s_2, \dots, s_r)(B+C) = (s_1, s_2, \dots, s_r)B + (s_1, s_2, \dots, s_r)C \in \bigoplus_{j=1}^m L_j^r$ by the induction hypothesis, or

$(s_1, s_2, \dots, s_r)A = (s_1, s_2, \dots, s_r)BC = ((s_1, s_2, \dots, s_r)B)C \in \bigcup_{j=1}^r L_j^r$, as by the induction hypothesis,

$(s_1, s_2, \dots, s_r)B \in \bigoplus_{j=1}^m L_j^r$ and this is an $\mathbb{R}_r(N)$ -module. So we always have $(s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$

$\subset \bigoplus_{j=1}^m L_j^r$.

For any $1 \leq j \leq m$, $L_j \subset s_i N$ for some $1 \leq i \leq r$. So $L_j^r \subset (s_i N)^r$. Now

$(s_i N)^r \subset (s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$ since if $(s_{i1} n_1, s_{i2} n_2, \dots, s_{ir} n_r) \in (s_i N)^r$, $(s_{i1} n_1, s_{i2} n_2, \dots, s_{ir} n_r) =$

$(s_1, s_2, \dots, s_r)(f_{11}^i + f_{21}^i + \dots + f_{r1}^i) \in (s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$. So $L_j^r \in (s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$ for every $1 \leq j \leq$

m and hence $\bigoplus_{j=1}^m L_j^r \subset (s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$.

Each L_j is an N -submodule of $s_i N$ and so each L_j^r is an $\mathbb{R}_r(N)$ submodule of $(s_i N)^r$.

Now $s_i N = \bigoplus_{j=1}^k L_j$ and so $(s_i N)^r = \bigoplus_{j=1}^k L_j^r$ giving $\bigoplus_1^m (s_i N)^r = \bigoplus_1^m L_j^r = (s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$. Since

each $(s_i N)^r$ is a direct summand of $(s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$, each L_j^r is a submodule of

$(s_1, s_2, \dots, s_r)\mathbb{R}_r(N)$. So S^r is a faithful type- s $\mathbb{R}_r(N)$ -module.

It is not known whether the converse of this proposition is true. In trying a standard proof, converting between matrix module substructures and substructures of the near-ring module, one frequently meets the questions: If Γ is connected (monogenic), is S , the N -module derived from Γ , connected (monogenic) and what is the relationship between Γ and S^r ? Also, are submodules and subgroups of connected modules always connected?

We now investigate the J_2 radicals of N and $\mathbb{R}_r(N)$ further, to find a connection between them. Recall that for an ideal I of N , $I^* \cong (N^r : I^r) = \{A \in \mathbb{R}_r(N) \mid \rho A \in I^r \forall \rho \in N^r\}$.

The following lemma was proved by Méyer [14]:

Lemma 4.28

If $\Gamma \neq (0)$ is a type-2 $\mathbb{R}_r(N)$ -module and $K = (\Gamma; 0)$, then there exists an ideal I in N such that $I^* = K$.

Proof

First note that if $A \in \mathbb{R}_r(N)$ and $n_1, n_2, \dots, n_r \in N$ then $(n_1, n_2, \dots, n_r)A = (m_1, m_2, \dots, m_r)$ if and only if $(f_{11}^1 + f_{21}^2 + \dots + f_{r1}^r)A = f_{11}^m + f_{21}^m + \dots + f_{r1}^m$, since $(n_1, n_2, \dots, n_r)A = (1, 0, \dots, 0)(f_{11}^1 + f_{21}^2 + \dots + f_{r1}^r)A$.

Now let G be the N -module derived from Γ , $G = \Gamma f_{11}^1$, and let $I = (G, 0) = \{n \in N \mid \gamma f_{11}^1 = 0 \forall \gamma \in \Gamma\}$. We show that $K = I^*$.

a) $I^* \subset K$: Suppose $\rho A \in I^*$ for all $\rho \in N^*$. We must show that for any $\gamma \in \Gamma$, $\gamma A = 0$. Let $\gamma \in \Gamma$. For any non-zero δf_{11}^1 there exists $B \in \mathbb{R}_r(N)$ such that $\gamma = (\delta f_{11}^1)B = (\delta f_{11}^1)(f_{11}^1 B)$ and so $\gamma A = (\delta f_{11}^1)(f_{11}^1 B)A$.

Now by lemma 4.6, $f_{11}^1 B = f_{11}^n + f_{21}^n + \dots + f_{r1}^n$ for some $n_1, n_2, \dots, n_r \in N$ and so if $(f_{11}^1 B)A = f_{11}^m + f_{21}^m + \dots + f_{r1}^m$, by the opening remark $(n_1, n_2, \dots, n_r)A = (m_1, m_2, \dots, m_r) \in I^*$ (since $\rho A \in I^*$ for all $\rho \in N^*$).

So $\gamma A = (\delta f_{11}^1)(f_{11}^m + f_{21}^m + \dots + f_{r1}^m)$ with each $m_i \in I$
 $= \delta(f_{11}^m + f_{21}^m + \dots + f_{r1}^m) = 0$ by definition of I , which proves that $A \in K$.

b) Conversely suppose that $\Gamma A = 0$ and let $(n_1, n_2, \dots, n_r)A = (m_1, m_2, \dots, m_r)$. We must show that $m_1, m_2, \dots, m_r \in I$. But $0 = \gamma f_{11}^1 (f_{11}^1 + f_{21}^2 + \dots + f_{r1}^r)A = \gamma f_{11}^1 (f_{11}^m + f_{21}^m + \dots + f_{r1}^m)$ for any $\gamma \in \Gamma$. This implies that $m_1, m_2, \dots, m_r \in I$ and so $K \subset I^*$.

Lemma 4.29

If A is an ideal of N , then $\mathbb{R}_r(N/A) \simeq \mathbb{R}_r(N)/(A^*)$.

Proof

We want an epimorphism from $\mathbb{R}_r(N)$ to $\mathbb{R}_r(N/A)$ with kernel equal to A^* . For every matrix in $\mathbb{R}_r(N)$, we want to change every f_{ij}^n to f_{ij}^{n+A} , to obtain a matrix in $\mathbb{R}_r(N/A)$.

However, since a matrix in $R_r(N)$ can have several different representations in terms of f_{ij}^n , we proceed as follows:

If N is any near-ring and $X \in E_r(N/A)$, let $X\mu$ denote the matrix in $R_r(N)$ represented by X .

Let $\theta: E_r(N) \rightarrow E_r(N/A)$ be the map defined by changing every f_{ij}^n in $E_r(N)$ to f_{ij}^{n+A} in $E_r(N/A)$. Then μ and θ are surjective and $(X+Y)\theta = X\theta + Y\theta$ and $(XY)\theta = (X\theta)(Y\theta)$.

Define $\varphi: R_r(N) \rightarrow R_r(N/A)$ by $B\varphi = (X\theta)\mu$, where $X = B\mu^{-1}$ for every $B \in R_r(N)$. To show that φ is well-defined, it is sufficient to prove that $(\rho+A^r)(X\theta)\mu = \rho(X\mu)+A^r$ for all $X \in E_r(N)$ and all $\rho \in N^*$, which is easily done, by recursively following the definition of $E_r(N)$. φ is a surjection and a homomorphism by the properties of μ and θ .

$\text{Ker } \varphi = \{B \in R_r(N) \mid B\varphi = 0 \text{ in } R_r(N/A)\} = \{B \in R_r(N) \mid \rho B \in A^r \text{ for all } \rho \in N^*\} = A^{*r}$.

Using lemmas 4.28 and 4.29 and theorems 4.26 and 4.27, we have:

Theorem 4.30

An ideal \mathcal{A} in $R_r(N)$ is 2-primitive if and only if $\mathcal{A} = I^{*r}$ and I is a 2-primitive ideal of N .

Proof

Suppose $\mathcal{A} = I^{*r}$ and I is a 2-primitive ideal of N . Then N/I is 2-primitive and hence $R_r(N/I) \simeq R_r(N)/I^{*r}$ is 2-primitive i.e. $I^{*r} = \mathcal{A}$ is a 2-primitive ideal of $R_r(N)$.

Conversely, suppose \mathcal{A} is a 2-primitive ideal of $R_r(N)$. Then $R_r(N)/\mathcal{A}$ has a faithful type-2 module Γ . Γ is a type-2 $R_r(N)$ -module and $\mathcal{A} = (\Gamma:0) = I^{*r}$ for an ideal I of N . $R_r(N)/\mathcal{A} = R_r(N)/I^{*r} \simeq R_r(N/I)$ is 2-primitive and hence N/I is 2-primitive, so I is a 2-primitive ideal of N .

Since $(\cap_{\lambda \in \Lambda} I_\lambda)^* = \cap_{\lambda \in \Lambda} I_\lambda^{*r}$, from the preceding theorem we can conclude

Theorem 4.31

$$J_2(\mathbb{R}_r(N)) = (J_2(N))^*$$

Corollary 4.32

N is 2-semisimple if and only if $\mathbb{R}_r(N)$ is 2-semisimple.

CHAPTER FIVE

SOME RELATED QUESTIONS

In this short chapter we mention some of the questions arising from concepts introduced in the previous chapters.

We showed in chapter three that when N is a zero-symmetric near-ring with identity, then the sum of all the small right ideals of N is equal to $J_{1/2}(N)$. $J_{1/2}(N)$ is in general not an ideal of N but only a right ideal. The question that arises is: what is the smallest ideal of N containing all the small right ideals? Certainly this ideal contains $J_{1/2}(N)$. Let C denote the least ideal of N containing all the small right ideals of N . Hartney [7] showed that in the case where N satisfies the descending chain condition for right ideals, then $C = J_s(N)$. What is the general characterisation of C and is C a Kurosch-Arnitsar type radical?

Hartney shows that when the descending chain condition for right ideals holds and $J_s(N) \neq (0)$, then there exists an ideal A of N contained in $J_s(N)$ such that A is uniquely minimal among all ideals B such that $J_s(N/B)$ is non-zero and nilpotent. Also, $J_s(N/A) = J_s(N)/A$. Then $J_s(N) = J_{1/2}(N) + A + B$ where $J_{1/2}(N)$ is nilpotent and B is an ideal of N . The structure of B is not known and knowing this would perhaps be useful for finding C .

From the matrix near-rings in chapter four we get the question: what types of near-rings are (isomorphic to) matrix near-rings? If R is a ring with $J(R) = (0)$ and R has a descending chain condition on ideals, then $R \cong M_n(\text{End}(R))$, the ring of matrices with entries from $\text{End}(R)$. If N is a near-ring with $J_s(N) = (0)$ and with a descending chain condition on right ideals, then we know that $N = \bigoplus_{i=1}^r K_i$ where each K_i is a right ideal of N and an N -module of type- s . Is N a matrix near-ring? We will now define the Neumann

near-rings, also candidates for matrix near-rings. The Neumann near-rings have not been extensively studied.

Definition 5.1: A subset X of an additive group G is called a relatively free basis for G if $G = \langle X \rangle$ and every map from X into G can be extended to an endomorphism of G .

Note that the extension of a map from X into G to an endomorphism must be unique since $G = \langle X \rangle$.

A group may have several relatively free bases. We can define a Neumann near-ring on any group having a relatively free basis.

Definition 5.2: Let G be a group with relatively free basis X . Let $N(G)$ be the set of all endomorphisms of G . Define a multiplication on $N(G)$ by composition of mappings. Let $\alpha, \beta \in N(G)$ and $x \in X$. Define $\alpha + \beta$ to be the endomorphism of G obtained by extending the map $x(\alpha + \beta) := x\alpha + x\beta$ from X into G . We call $(N(G), +, \cdot)$ the Neumann near-ring on G with respect to X .

Note that this addition depends on the choice of the relatively free basis X .

It can be shown that $(N(G), +, \cdot)$ is a right near-ring. Although all the elements of $N(G)$ are endomorphisms of G , they are not all distributive.

If X is a relatively free basis for a group G , and $y, z \in G$ define $\epsilon(y, z) \in N(G)$ by

$$x\epsilon(y, z) := \begin{cases} 0 & \text{if } x \neq y \\ x & \text{if } x = y \end{cases} \text{ . Then it can be shown that } \epsilon(y, z) \text{ is a distributive element}$$

of $N(G)$. Hence it can be shown that if $\{x_1, x_2, \dots, x_n\}$ is a finite relatively free basis for G then $N(G)$ is a *dg* near-ring with a distributive generating set given by

$$\{ \epsilon(x_i, x_j) \mid 1 \leq i, j \leq n \} \text{ .}$$

This result gives us a large collection of *dg* Neumann near-rings. If X is infinite then every element of $N(G)$ is not necessarily distributive.

Many other matrix related questions arise. In chapter four we query the relationship between the J_s radical of the near-ring and the associated matrix near-ring. We also ask (page 47) whether the N -module S derived from a connected near-ring module Γ is

connected and if so, what the relationship between S^+ and Γ is. *

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