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Masters Dissertation

Laminar wake flow behind a hump on a wall

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Declaration of Authorship

I, Jonathan Julyan, declare that this dissertation titled, 'Laminar wake flow behind a hump on a wall' and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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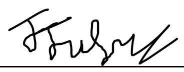
Signed: 

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University of the Witwatersrand,
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Abstract

Faculty of Science
Computer Science and Applied Mathematics

Master of Science

Laminar wake flow behind a hump on a wall

by Jonathan Julyan

The laminar wake flow behind a hump on a solid wall boundary is investigated. A Blasius boundary layer flow is perturbed by the hump and a wake forms directly downstream. Triple deck theory is applied to the wake region and the flow is divided into three decks. The governing equations are derived for each deck for both the near and the far wake. Particular attention is paid to the role of the boundary layer displacement effect. The conservation laws and conserved quantities for the governing equations are derived. The multiplier method is applied to the linearised governing equations for small humps and a basis of conserved vectors is constructed. Since, in general, the problem contains an unknown non-homogeneous boundary condition, each conserved vector needs to be carefully chosen and additional restrictions need to be applied to ensure that each conserved quantity, which is obtained by integrating the corresponding conservation law across the wake and imposing the relevant boundary conditions, has a finite value. Four non-trivial conserved quantities are found; three of which have only now been identified. The four conserved quantities relate to the conservation of mass, drag and the first and second moments of the momentum deficit. For each case the existence of a solution that satisfies the governing equations, boundary conditions and a finite valued conserved quantity is discussed. The solution corresponding to the near wall-wake flow is further discussed. Although the far wall-wake does not satisfy a conserved quantity, for completeness, it is included in this work.

Keywords: Triple deck theory; laminar flow; near wake; far wake; conserved quantity; conservation law; multiplier method; wall-wake; boundary layer

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Chapter 1

Introduction

The two-dimensional laminar wake flow behind a hump situated on a solid wall boundary, is also known as the laminar 'wall-wake'. Here a boundary layer is perturbed by a small hump on an otherwise flat plate. The first study conducted on wall-wake flows was by Hunt [1]. The main motivation for Hunt's research is to better understand the flow behind boundary layer trip wires. Although these flows are generally turbulent, an understanding of the laminar counterpart is necessary. Hunt's approach [1] was to divide the flow behind the hump into two regions: an inner viscous flow region near to the wall and an intermediate inviscid region that matches to the unperturbed boundary layer flow. It is assumed that the hump is small enough so that the unperturbed boundary layer is not displaced by its presence. At the time of the study, it was believed that Hunt had solved for the far wake flow.

A different approach to solving for the laminar wall-wake flow was provided by Smith [2]. Smith [2] applied triple deck theory [3, 4], to the problem of the wall-wake. The formulation of this theory is largely accredited to Stewartson [3, 5] and Messiter [4]. In this theory the flow behind the trailing edge of a flat plate is divided into three regions known as decks. Each deck has its own flow properties. Many extensions to triple deck theory have been developed [6–13]. Numerical approaches have been presented in these papers. A review on triple deck theory is provided by Nayfeh [14]. Triple deck theory has proved to be very successful in describing perturbed boundary layer flows. For the wall-wake, in addition to the two main regions or decks of flow that Hunt [1] defined, Smith [2] identified a third deck of inviscid flow outside of the boundary layer. This third deck is required because the flow outside of the boundary layer is displaced by the presence of the hump [2]. This is known as the boundary layer displacement effect.

At first appearance the results by Hunt [1] and Smith [2] are contradictory. Upon further investigation however, the results can be reconciled by applying triple deck theory which considers three main regions or decks of flow [15]. It was argued that Hunt's approach [1] solved for the near wake on the triple deck scale where only two decks are needed because the boundary layer displacement effect is negligible in this case [15]. Smith's solution [2] described the far wall-wake on the triple deck scale where all three decks are needed in order to include the boundary layer displacement effect [15]. For both the near and far wall-wakes, the wake is confined to the lower deck which is bounded on one side by the flat plate. The governing equations for the wake are solved subject to the no-slip condition at the solid wall interface, the matching conditions between the lower and intermediate decks which differ for near and far wakes, and if applicable, a conserved quantity. Inclusion of the boundary layer displacement effect results in a non-homogeneous boundary condition at the interface between the lower and intermediate decks. In the case of the near wake where the function describing the boundary layer displacement effect is set to zero, the boundary conditions between the lower and intermediate decks are homogeneous.

The governing equations for the wall-wake are non-linear. However, for very small humps, the governing equations can be linearised [2, 15]. When the boundary layer displacement effect is included, the governing equations and the boundary conditions are, in general, not homogeneous [2]. For the far wall-wake, the boundary layer displacement effect is specified which determines the non-homogeneous boundary condition [2]. Since the governing equations and boundary conditions are not homogeneous, a conserved quantity is not needed to complete the solution [2, 15]. For the near wall-wake, because the boundary layer displacement effect is negligible, the governing equations and boundary conditions are homogeneous and a conserved quantity is required to complete the solution [1]. For both the near and far wall-wakes, the boundary layer displacement effect is specified. If, however, the boundary layer displacement effect is not known then the governing equations need to be solved subject to an unknown non-homogeneous boundary condition. There is insufficient knowledge on this problem in the current literature to ascertain as to whether a conserved quantity is required to complete the solution when the boundary layer displacement effect is unknown.

In [16], various approaches to finding the conservation laws for partial differential equations are discussed. Once a conserved vector has been obtained, the Lie symmetry associated with this conserved vector can be calculated and then used to generate the invariant solution [17, 18]. For problems where a conserved quantity is required to complete the solution, the double reduction theorem can be used [19]. Other works on symmetries and conservation laws for differential equations are given in [20–32]. In this work the multiplier method [20, 33] is used to calculate a basis of conserved vectors for the governing equations of the wall-wake when expressed in terms of the velocity components and when expressed in terms of the stream function. This method has been used to calculate the conservation laws for the radial and two dimensional free jets [34] and for the classical wake and the wake of a self-propelled body [35]. For the governing equations pertaining to the wall-wake problem, four conservation laws are obtained. One of these corresponds to the near wall-wake whilst the rest are newly discovered. Each conservation law is then integrated across the wake and the relevant boundary conditions are imposed in order to generate the required conserved quantity. Much consideration needs to be taken when deriving the conserved vectors. As there is a possibility of an unknown non-homogeneous boundary condition, convergence of the integrals arising from integrating a conservation law across the wake is not guaranteed. However, it is shown how this issue can be overcome. The conserved quantity for the near wall-wake, which is the moment of momentum deficit, is re-derived in a systematic way. It is discovered that each of the three remaining conserved quantities correspond to the conservation of mass, drag and the second moment of the axial momentum deficit.

In this dissertation, the governing equations and boundary conditions for the wall-wake are derived using triple deck theory. Existing theory on conservation laws is adapted and applied to the governing equations of the wall-wake in order to derive a basis for the conserved vectors and to determine the conditions for which a finite conserved quantity corresponding to each conservation law exists. If the boundary layer displacement effect is not specified which then results in a non-homogeneous boundary condition, it is shown that under certain conditions finite conserved quantities can be found and that the boundary layer displacement effect can be determined. One of the conserved quantities corresponds to the near wall-wake flow. Subsequently, the solution for the near wall-wake flow is derived and compared to the solution obtained by Hunt [1] whose approach is also discussed. Although the far wall-wake does not require a conserved quantity to complete the solution, a brief review of Smith's work [2] on this problem is also provided.

This thesis is outlined as follows. In Chapter 2, a detailed description of the mathematical model is provided. The governing equations and boundary conditions for small humps are derived using triple deck theory. Chapter 3 investigates the conservation laws and conserved quantities associated with the governing equations of the wall-wake problem. In Section 3.1 the general theory for the multiplier method is presented. It is discussed how conserved vectors are chosen for problems with unknown non-homogeneous boundary conditions. The conservation laws for the governing equations for the wall-wake are derived in terms of the velocity components in Section 3.2.1 and in terms of the stream function in Section 3.2.2. In Section 3.3 the conservation laws are integrated across the wake to obtain the conserved quantities. Additional conditions that need to be imposed in order to generate finite conserved quantities are discussed. The conserved quantities in terms of the velocity components are given in Section 3.3.1 and in terms of the stream function in Section 3.3.2. In Section 3.3.3 a summary of the findings on conserved vectors is given including the requirements for the corresponding conserved quantity to exist. The physical significance of each conserved quantity is analysed in Section 3.3.4. In Section 3.4 similarity solutions of the governing equations are studied. Invariance of each conserved quantity enables the form of the similarity solution to be identified. The similarity solutions are then solved and it is shown that finite conserved quantities can be obtained. In Chapter 4 the near wake solution for the laminar wake flow behind a hump on a wall is discussed. Section 4.2 examines the approach used by Hunt [1], while the triple deck approach is considered in Section 4.3. The solutions obtained from each approach are shown to be equivalent. In Chapter 5 the far wake is discussed and conclusions are given in Chapter 6.

A large portion of Chapters 2 and 3 can be found in [36].

Chapter 2

Mathematical model

2.1 Description

Consider a laminar stream of viscous incompressible fluid flowing past a small symmetric hump on an otherwise smooth boundary. A Cartesian coordinate system (x^*, y^*) is used. The line $y^* = 0$ lies along the solid wall boundary and the line $x^* = 0$ lies along the axis of symmetry of the hump. The constant mainstream speed, density and kinematic viscosity of the fluid are given by u^*_∞, ρ, and $\nu = \mu/\rho$ respectively, where μ is the dynamic viscosity. The flow transitions through four different stages as shown in Figure 2.1. In stage A, the far upstream Blasius boundary layer flow is unaffected by the presence of the hump. Stage B represents the flow over the hump and the flow near to the leading and trailing edges of the hump. Once the boundary layer flow comes into contact with the hump, it is perturbed and a wake forms directly downstream of the hump as shown in stage C. Sufficiently far downstream, the flow reverts to its upstream configuration as shown in stage D.

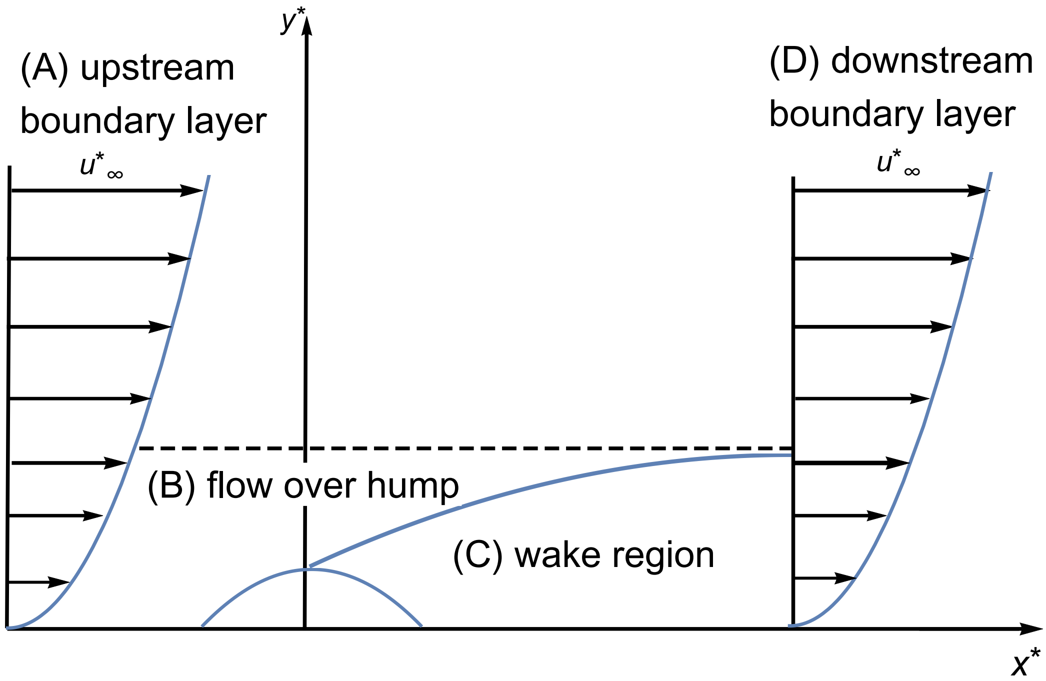

FIGURE 2.1: Stages in a wall-wake flow

Triple deck theory can be used to derive the governing equations for the wall-wake flow for both the near and far wall-wakes which satisfy the same governing equations, but different boundary conditions [2, 15]. The near wake flow applies for small x^* and the far wake flow is relevant for large x^*. The x^*- and y^*- velocity components and the fluid pressure in the wake are denoted by $u^*(x^*, y^*)$, $v^*(x^*, y^*)$ and $p^*(x^*, y^*)$ respectively. The Reynolds number Re for the flow is defined in terms of the upstream boundary layer flow [2]:

$$Re = \frac{u^*_\infty L}{\nu}, \tag{2.1.1}$$

where L is the development length of the oncoming boundary layer which determines the boundary layer thickness $\delta = LRe^{-\frac{1}{2}}$. The implementation of triple deck theory to this problem relies on the assumption that the parameter, ϵ, where [3]

$$\epsilon = Re^{-\frac{1}{8}}, \tag{2.1.2}$$

is small which is true for very large Reynolds numbers. The flow is further divided into three sub-regions or decks as shown in Figure 2.2.

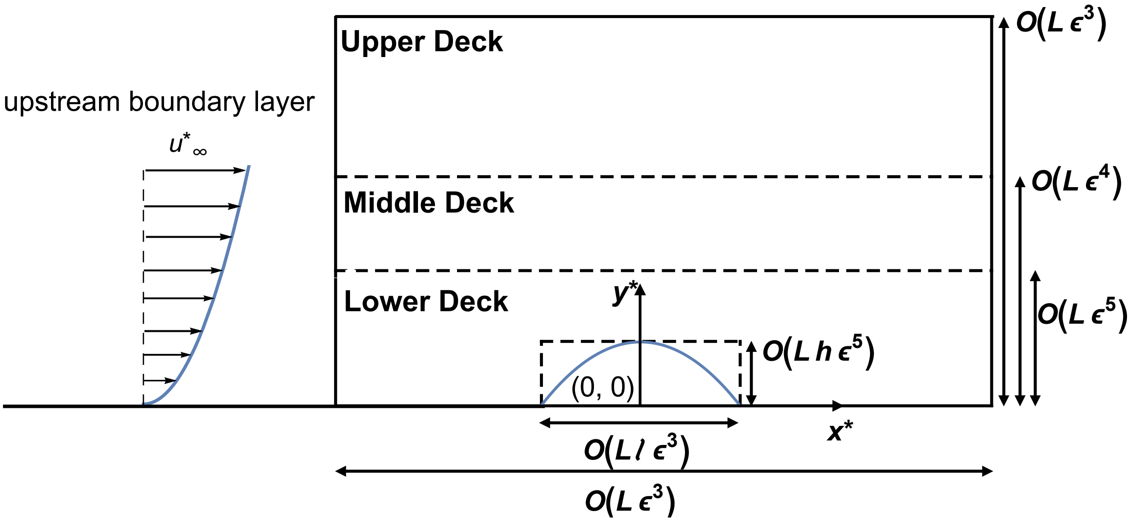

FIGURE 2.2: The length scales used for each deck

The x^* and y^* components of the Navier-Stokes equation and the continuity equation are, respectively,

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{1}{\rho}\frac{\partial p^*}{\partial x^*}+\nu\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right), \tag{2.1.3}$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-\frac{1}{\rho}\frac{\partial p^*}{\partial y^*}+\nu\left(\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right), \tag{2.1.4}$$

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0. \tag{2.1.5}$$

The dimensionless variables

$$x^*=\epsilon^n Lx, \qquad y^*=\epsilon^m Ly,$$

$$u^*=u^*_\infty u, \qquad v^*=u^*_\infty v, \qquad p^*=p^*_\infty+\rho u^{*2}_\infty p, \tag{2.1.6}$$

are defined where n and m are positive integers. Here $p^*_\infty = p^*(x,\infty)$. Substituting (2.1.6) into equations (2.1.3)-(2.1.5) results in

$$\epsilon^{m-n}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\epsilon^{m-n}\frac{\partial p}{\partial x}+\epsilon^{8+m-2n}\frac{\partial^2 u}{\partial x^2}+\epsilon^{8-m}\frac{\partial^2 u}{\partial y^2}, \tag{2.1.7}$$

$$\epsilon^{m-n}u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\epsilon^{8+m-2n}\frac{\partial^2 v}{\partial x^2}+\epsilon^{8-m}\frac{\partial^2 v}{\partial y^2}, \tag{2.1.8}$$

$$\epsilon^{m-n}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.1.9}$$

In the triple deck approach the horizontal length scale is $O(L\epsilon^3)$ which gives $n = 3$ [2, 3, 15] and therefore $x^* = \epsilon^3 Lx$. For the vertical length scale, m can take on the values $3, 4$ and 5 depending on the deck under investigation. In the lower deck $m = 5$. The middle deck requires $m = 4$ and for the upper deck $m = 3$. The choice of length scales ensures that the flow in the lower deck is dominated by viscous forces and the flow in the middle deck is inviscid. A full justification of the choices for the length scales can be found in [2–4].

2.2 Blasius boundary layer flow and boundary conditions

The classical boundary layer coordinates in the context of the triple deck scale are given by

$$x = \frac{x^*}{\epsilon^3 L}, \qquad y = \frac{y^*}{\epsilon^4 L}, \tag{2.2.1}$$

and therefore $n = 3$ and $m = 4$. As $x \to \pm\infty$, the wake flow must match with the Blasius boundary layer flow. Equations (2.1.7)-(2.1.9) become

$$\epsilon u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\epsilon \frac{\partial p}{\partial x} + \epsilon^6 \frac{\partial^2 u}{\partial x^2} + \epsilon^4 \frac{\partial^2 u}{\partial y^2}, \tag{2.2.2}$$

$$\epsilon u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \epsilon^6 \frac{\partial^2 v}{\partial x^2} + \epsilon^4 \frac{\partial^2 v}{\partial y^2}, \tag{2.2.3}$$

$$\epsilon \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.2.4}$$

The perturbation expansions in terms of the dimensionless classical boundary layer coordinates are, as $x \to \pm\infty$ [3]

$$u(x,y) = u_B(y) + \epsilon^3 u_1(x,y) + \epsilon^4 u_2(x,y) + \dots\ , \tag{2.2.5}$$

$$v(x,y) = \epsilon^4 v_1(x,y) + \epsilon^5 v_2(x,y) + \dots\ , \tag{2.2.6}$$

$$p(x,y) = \epsilon^3 p_1(x,y) + \epsilon^4 p_2(x,y) + \dots\ . \tag{2.2.7}$$

These expansions are required to merge the triple deck regions with the upstream Blasius flow when expressed in terms of the classical boundary layer coordinates [3]. Here, $u_B(y)$ is the dimensionless x-component of the velocity in the Blasius flow region. Substituting (2.2.5)-(2.2.7) into (2.2.3) gives

$$p_1 = p_1(x). \tag{2.2.8}$$

Substituting (2.2.5)-(2.2.7) and (2.2.8) into equation (2.2.2) gives

$$u_B \frac{\partial u_1}{\partial x} + v_1 \frac{du_B}{dy} = -\frac{dp_1}{dx} + \frac{d^2 u_B}{dy^2}. \tag{2.2.9}$$

However, at $y = 0$ the no-slip and no cavity conditions give

$$u_B(0) = 0, \qquad u_1(x,0) = 0, \qquad v_1(x,0) = 0. \tag{2.2.10}$$

For small y, u_B, u_1, $v_1 \to 0$ and equation (2.2.9) reduces to

$$\frac{d^2 u_B}{dy^2} = \frac{dp_1}{dx}. \tag{2.2.11}$$

Using separation of variables,

$$\frac{d^2 u_B}{dy^2} = b, \tag{2.2.12}$$

$$\frac{dp_1}{dx} = b, \tag{2.2.13}$$

where b is a constant. Solving for $p_1(x)$ in (2.2.13) gives

$$p_1(x) = bx + c, \tag{2.2.14}$$

where c is a constant. Since $p_1(x)$ is finite as $x \to \infty$

$$b = 0, \tag{2.2.15}$$

and therefore,

$$p_1(x) = c. \tag{2.2.16}$$

Solving equation (2.2.12) for $u_B(y)$ with $b = 0$, subject to (2.2.16) and the condition in (2.2.10) gives

$$u_B(y) = \lambda y, \tag{2.2.17}$$

where λ is the scaled skin friction. Thus u_B is defined as

$$u_B(y) = \begin{cases} \lambda y & \text{as } y \to 0, \\ 1 & \text{as } y \to \infty. \end{cases} \tag{2.2.18}$$

Stewartson argued that $p_1 = 0$. Since $p_1(x)$ is a constant its value can be obtained by matching the flow in the triple deck regions with the upstream Blasius boundary layer flow. This gives $p_1(x) = 0$ [3]. As $y \to \infty$, $u_B(y) \to 1$ slowly. Hence

$$\frac{du_B}{dy} = 0 \quad \text{as } y \to \infty, \tag{2.2.19}$$

$$\frac{d^2u_B}{dy^2} = 0 \quad \text{as } y \to \infty. \tag{2.2.20}$$

Consider now the limit $\epsilon y \to \infty$. It is assumed that in this region outside of the boundary layer, the following is true [3]:

$$u(x, y) = 1 + O(\epsilon^4), \tag{2.2.21}$$

$$v(x, y) = O(\epsilon^4), \tag{2.2.22}$$

$$p(x, y) = O(\epsilon^4). \tag{2.2.23}$$

The terms of $O(\epsilon^4)$ are as a result of the hump displacing the fluid outside of the boundary layer.

It is also important to note that at the solid wall boundary, the no slip condition and the no cavity condition

$$u(x, 0) = 0, \tag{2.2.24}$$

$$v(x, 0) = 0, \tag{2.2.25}$$

must be satisfied.

2.3 Shape of the hump

The size and shape of the hump need to be examined as these factors have an influence on the resulting wake flow. A smooth symmetric hump is considered. The hump is initially chosen to have a horizontal scale of $O(L\ell\epsilon^3)$ where ℓ is the dimensionless length factor and a vertical scale of $O(Lh\epsilon^5)$ to consider a range of hump sizes. First let

$$x = \frac{x^*}{\epsilon^3 L}. \tag{2.3.1}$$

A general equation for the surface of the hump is given by [2].

$$\frac{y^*}{L} = h\epsilon^5 F(x), \qquad y^* > 0, \tag{2.3.2}$$

where h is the dimensionless height factor and F is dimensionless with $F(0) = 1$, which is where the height of the hump is maximum. The cross-sectional area of the hump is finite and therefore

$$f_0 \equiv \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} F(x)dx < \infty. \tag{2.3.3}$$

From (2.3.2), the hump is confined to the lower deck. The vertical coordinate used in the lower deck is

$$z = \frac{y^*}{\epsilon^5 L}. \tag{2.3.4}$$

Substituting (2.3.4) into equation (2.3.2) gives

$$z = hF(x). \tag{2.3.5}$$

2.4 Middle deck

In the middle deck, the classical boundary layer coordinates given by (2.2.1) are used. Letting $n = 3$ and $m = 4$ in equations (2.1.7)-(2.1.9) and neglecting terms of $O(\epsilon^k)$, $k \geq 4$, results in

$$\epsilon u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\epsilon \frac{\partial p}{\partial x}, \tag{2.4.1}$$

$$\epsilon u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}, \tag{2.4.2}$$

$$\epsilon \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.4.3}$$

From (2.4.1) and (2.4.2) it is clear that the viscous terms play no role in the middle deck equations. The expansions for the middle deck which results in the flow being inviscid are [3]

$$u(x, y) = u_B(y) + \epsilon u_1(x, y) + \epsilon^2 u_2(x, y) + \dots , \tag{2.4.4}$$

$$v(x, y) = \epsilon^2 v_1(x, y) + \epsilon^3 v_2(x, y) + \dots , \tag{2.4.5}$$

$$p(x, y) = \epsilon p_1(x, y) + \epsilon^2 p_2(x, y) + \dots , \tag{2.4.6}$$

where u_B is defined in (2.2.18). Substituting expansions (2.4.4)-(2.4.6) into equation (2.4.2) and ignoring terms of $O(\epsilon^3)$ results in

$$\epsilon \frac{\partial p_1}{\partial y} + \epsilon^2 \frac{\partial p_2}{\partial y} = 0. \tag{2.4.7}$$

From equation (2.4.7) it is clear that

$$p_1 = p_1(x), \tag{2.4.8}$$

$$p_2 = p_2(x). \tag{2.4.9}$$

Stewartson's argument that $p_1 = 0$ [3] is adopted in this paper.

Substituting expansions (2.4.4)-(2.4.6) into equations (2.4.1) and (2.4.3) and ignoring terms of $O(\epsilon^3)$ results in

$$u_B \frac{\partial u_1}{\partial x} + v_1 \frac{du_B}{dy} = 0, \tag{2.4.10}$$

$$\frac{\partial u_1}{\partial x} = -\frac{\partial v_1}{\partial y}, \tag{2.4.11}$$

respectively. Now substituting equation (2.4.11) into equation (2.4.10) gives

$$-u_B \frac{\partial v_1}{\partial y} + v_1 \frac{du_B}{dy} = 0. \tag{2.4.12}$$

Solving equation (2.4.12) and then using (2.4.11) yields

$$v_1(x, y) = -u_B(y) A_1'(x), \tag{2.4.13}$$

$$u_1(x, y) = u_B'(y) A_1(x) + B(y), \tag{2.4.14}$$

where $A_1(x)$ is an arbitrary function of x and $B(y)$ is an arbitrary function of y. As $x \to \pm\infty$, the wake flow merges with the Blasius boundary layer flow, the expansions for which are given in (2.2.5)-(2.2.7). Therefore,

$$u_1(\pm\infty, y) = 0, \qquad v_1(\pm\infty, y) = 0. \tag{2.4.15}$$

From (2.4.15), it is seen that

$$B(y) = -u_B'(y) A_1(\infty). \tag{2.4.16}$$

Letting

$$A(x) = A_1(x) - A_1(\infty), \tag{2.4.17}$$

where

$$A(\pm\infty) = 0, \qquad A'(\pm\infty) = 0, \tag{2.4.18}$$

gives

$$v_1(x,y) = -u_B(y)A'(x), \tag{2.4.19}$$

$$u_1(x,y) = u_B'(y)A(x). \tag{2.4.20}$$

The expression for u_1 in (2.4.20) does not satisfy the no-slip condition given in (2.2.24). Therefore, a lower deck region must be inserted. Also, for $A'(x) \neq 0$, matching with the expansions in (2.2.21)-(2.2.23) is not possible and an upper deck needs to be added. For $A'(x) = 0$, only two decks are required.

For the upper and lower deck expansions, equations for u_2, v_2 and p_2 are required. Substituting the expansions (2.4.4)-(2.4.6) and the result (2.4.9) into equation (2.4.1) and excluding terms of $O(\epsilon^4)$ results in

$$u_B\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} + v_2\frac{du_B}{dy} + \frac{dp_2}{dx} = 0. \tag{2.4.21}$$

Introducing the expansions (2.4.4) and (2.4.5) into equation (2.4.3) and excluding terms of $O(\epsilon^4)$ gives

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0. \tag{2.4.22}$$

As $x \to \pm\infty$, the wake flow merges with the Blasius boundary layer flow, the expansions for which are given in (2.2.5)-(2.2.7). Therefore,

$$u_2(\pm\infty, y) = 0, \qquad v_2(\pm\infty, y) = 0, \qquad p_2(\pm\infty) = 0. \tag{2.4.23}$$

2.5 Upper deck

The upper deck is used to determine p in the cases where the other decks where not sufficient. Since p is continuous across the decks solving for p in the upper deck will give the result for p_2 in the middle deck.

In order to obtain the required perturbation expansions for the upper deck, the middle deck expansions must be evaluated as $y \to \infty$.

From (2.2.18), the solutions for u_1 and v_1 in (2.4.19) and (2.4.20) reduce to

$$v_1 = -A'(x), \tag{2.5.1}$$

$$u_1 = 0, \tag{2.5.2}$$

as $y \to \infty$. It is also necessary to derive the solutions for u_2 and v_2 in the limit $y \to \infty$. Using (2.2.18), (2.5.1) and (2.5.2), equation (2.4.21) becomes

$$\frac{\partial u_2}{\partial x} + \frac{dp_2}{dx} = 0, \tag{2.5.3}$$

which gives

$$u_2(x, y) = -p_2(x) + S(y), \tag{2.5.4}$$

where $S(y)$ is an arbitrary function of y. As $x \to \pm\infty$, from (2.4.23), u_2, $p_2 \to 0$ and therefore $S(y) = 0$. Thus

$$u_2(x, y) = -p_2(x). \tag{2.5.5}$$

From the continuity equation (2.4.22), the solution for v_2 can be calculated. It is given by

$$v_2(x, y) = \frac{dp_2}{dx} y + b(x). \tag{2.5.6}$$

Therefore, the expansions in the main deck as $y \to \infty$ are

$$u(x, y) = 1 - \epsilon^2 p_2(x) + O(\epsilon^3), \tag{2.5.7}$$

$$v(x, y) = -\epsilon^2 A'(x) + O(\epsilon^3), \tag{2.5.8}$$

$$p(x, y) = \epsilon^2 p_2(x) + O(\epsilon^3). \tag{2.5.9}$$

In the upper deck the required coordinates are

$$x = \frac{x^*}{L\epsilon^3}, \qquad \bar{y} = \frac{y^*}{L\epsilon^3}. \tag{2.5.10}$$

With $n = 3$ and $m = 3$, equations (2.1.7)-(2.1.9) become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial \bar{y}} = -\frac{\partial p}{\partial x} + \epsilon^5 \frac{\partial^2 u}{\partial x^2} + \epsilon^5 \frac{\partial^2 u}{\partial \bar{y}^2}, \tag{2.5.11}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial \bar{y}} = -\frac{\partial p}{\partial \bar{y}} + \epsilon^5 \frac{\partial^2 v}{\partial x^2} + \epsilon^5 \frac{\partial^2 v}{\partial \bar{y}^2}, \tag{2.5.12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial \bar{y}} = 0. \tag{2.5.13}$$

The expansions used in the upper deck must match with the expansions given in (2.5.7)-(2.5.9). Therefore

$$u(x,\bar{y}) = 1 + \epsilon^2 U_2(x,\bar{y}) + \epsilon^3 U_3(x,\bar{y}) + \dots \,, \tag{2.5.14}$$
$$v(x,\bar{y}) = \epsilon^2 V_2(x,\bar{y}) + \epsilon^3 V_3(x,\bar{y}) + \dots \,, \tag{2.5.15}$$
$$p(x,\bar{y}) = \epsilon^2 P_2(x,\bar{y}) + \epsilon^3 P_3(x,\bar{y}) + \dots \,. \tag{2.5.16}$$

From (2.5.7)-(2.5.9) and (2.5.14)-(2.5.16), for $\bar{y} = 0$, the matching conditions

$$P_2(x,0) = p_2(x), \qquad V_2(x,0) = -A'(x), \qquad U_2(x,0) = -p_2(x), \tag{2.5.17}$$

must be satisfied. Also from (2.2.21)-(2.2.23), as $\bar{y} \to \infty$, the following must hold:

$$P_2(x,\infty) = 0, \qquad V_2(x,\infty) = 0, \qquad U_2(x,\infty) = 0. \tag{2.5.18}$$

Matching with the Blasius boundary layer flow (2.4.23) gives

$$P_2(\pm\infty,\bar{y}) = 0, \qquad V_2(\pm\infty,\bar{y}) = 0, \qquad U_2(\pm\infty,\bar{y}) = 0. \tag{2.5.19}$$

Substituting the expansions (2.5.14)-(2.5.16) into equations (2.5.11)-(2.5.13), gives

$$\frac{\partial U_2}{\partial x} = -\frac{\partial P_2}{\partial x}, \tag{2.5.20}$$
$$\frac{\partial V_2}{\partial x} = -\frac{\partial P_2}{\partial \bar{y}}, \tag{2.5.21}$$
$$\frac{\partial U_2}{\partial x} = -\frac{\partial V_2}{\partial \bar{y}}. \tag{2.5.22}$$

Equations (2.5.20)-(2.5.22) can be written as the single equation

$$\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial \bar{y}^2} = 0, \tag{2.5.23}$$

which is Laplace's equation. Solving equation (2.5.23) gives [3]

$$V_2(x,\bar{y}) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\bar{y}A'(x_1)}{(x-x_1)^2+\bar{y}^2}dx_1. \tag{2.5.24}$$

Equation (2.5.21) gives

$$\begin{aligned}\frac{\partial P_2}{\partial y} &= -\frac{\partial V_2}{\partial x}, \\ &= -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\bar{y}(x-x_1)A'(x_1)}{[(x-x_1)^2+\bar{y}^2]^2}dx_1.\end{aligned} \tag{2.5.25}$$

Solving (2.5.25) gives

$$P_2(x,\bar{y}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x-x_1)A'(x_1)}{(x-x_1)^2+\bar{y}^2}dx_1 + f_1(x). \tag{2.5.26}$$

The condition in (2.5.18), $P_2(x,\infty) = 0$ gives

$$f_1(x) = 0. \tag{2.5.27}$$

The solution for $P_2(x,\bar{y})$ is therefore

$$P_2(x,\bar{y}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x-x_1)A'(x_1)}{(x-x_1)^2+\bar{y}^2}dx_1. \tag{2.5.28}$$

From (2.5.17), $P_2(x,0) = p_2(x)$. Therefore

$$P_2(x,0) = p_2(x) = \frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{A'(x_1)}{x-x_1}dx_1. \tag{2.5.29}$$

Since there is a singularity at $x = x_1$ the principal value of the integral needs to be taken [15]. This is defined as

$$\frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{A'(x_1)}{x-x_1}dx_1 = \lim_{\epsilon\to 0}\frac{1}{\pi}\left[\int_{-\infty}^{x_1-\epsilon}\frac{A'(x_1)}{x-x_1}dx_1 + \int_{x_1+\epsilon}^{\infty}\frac{A'(x_1)}{x-x_1}dx_1\right]. \tag{2.5.30}$$

2.6 Lower deck

Consider the middle deck expansions given in (2.4.4)-(2.4.6). In the limit $y \to 0$ these expansions are from (2.2.18), (2.4.19) and (2.4.20)

$$u(x,y) = \lambda y + \epsilon\lambda A(x) + O(\epsilon^2), \tag{2.6.1}$$
$$v(x,y) = -\epsilon^2\lambda y A'(x) + O(\epsilon^3), \tag{2.6.2}$$
$$p(x,y) = \epsilon^2 p_2(x) + O(\epsilon^3). \tag{2.6.3}$$

In the lower deck $m = 5$. The required independent variables are

$$x = \frac{x^*}{L\epsilon^3}, \qquad z = \frac{y^*}{L\epsilon^5}. \tag{2.6.4}$$

In terms of (2.6.4), the expansions (2.6.1)-(2.6.3) can be written as

$$u(x,z) = \epsilon\lambda z + \epsilon\lambda A(x) + O(\epsilon^2), \tag{2.6.5}$$
$$v(x,z) = -\epsilon^3(\lambda z A'(x) - v_2(x,z)) + O(\epsilon^4), \tag{2.6.6}$$
$$p(x,z) = \epsilon^2 p_2(x) + O(\epsilon^3). \tag{2.6.7}$$

It is crucial that v_2 is calculated as $y \to 0$ since in (2.6.6) ϵ^3 appears in front of v_2. Equation (2.4.21) as $y \to 0$ gives

$$v_2(x,0) = -\frac{1}{\lambda}\frac{dp_2}{dx} - \lambda A(x)A'(x). \tag{2.6.8}$$

Guided by the expansions in (2.6.5)-(2.6.7), the lower deck expansions are

$$u(x,z) = \epsilon \bar{u}_1(x,z) + \epsilon^2 \bar{u}_2(x,z) + \dots\ , \tag{2.6.9}$$
$$v(x,z) = \epsilon^3 \bar{v}_1(x,z) + \epsilon^4 \bar{v}_2(x,z) + \dots\ , \tag{2.6.10}$$
$$p(x,z) = \epsilon^2 \bar{p}_1(x,z) + \epsilon^3 \bar{p}_2(x,z) + \dots\ . \tag{2.6.11}$$

The conditions as $z \to \infty$ are obtained from (2.6.5)-(2.6.7). These are

$$\bar{u}_1(x,z) \to \lambda z + \lambda A(x), \qquad z \to \infty, \tag{2.6.12}$$
$$\bar{v}_1(x,z) \to -\lambda z A'(x) - \frac{1}{\lambda}\frac{dp_2}{dx} - \lambda A(x)A'(x), \qquad z \to \infty, \tag{2.6.13}$$
$$\bar{p}_1(x,z) \to p_2(x), \qquad z \to \infty. \tag{2.6.14}$$

The no slip condition and no cavity condition must be satisfied. Hence

$$\bar{u}_1 = \bar{v}_1 = 0 \qquad \text{on } z = hF(x). \tag{2.6.15}$$

Using (2.4.15), the conditions as $x \to \pm\infty$ in the lower deck are given by

$$\bar{u}_1(\pm\infty, z) = \lambda z, \qquad \bar{v}_1(\pm\infty, z) = 0. \tag{2.6.16}$$

Substituting expansions (2.6.9)-(2.6.11) into equations (2.1.7)-(2.1.9) with $m = 5$ and $n = 3$ and neglecting terms of $O(\epsilon)$ gives

$$\bar{u}_1\frac{\partial \bar{u}_1}{\partial x} + \bar{v}_1\frac{\partial \bar{u}_1}{\partial z} = -\frac{\partial \bar{p}_1}{\partial x} + \frac{\partial^2 \bar{u}_1}{\partial z^2}, \tag{2.6.17}$$
$$\frac{\partial \bar{p}_1}{\partial z} = 0, \tag{2.6.18}$$
$$\frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{v}_1}{\partial z} = 0. \tag{2.6.19}$$

From (2.6.18)

$$\bar{p}_1(x,z) = \bar{p}_1(x). \tag{2.6.20}$$

From (2.6.14) and (2.6.20)

$$\bar{p}_1(x) = p_2(x). \tag{2.6.21}$$

Using (2.6.21), equation (2.6.17) is given by

$$\bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{u}_1}{\partial z} = -\frac{dp_2}{dx} + \frac{\partial^2 \bar{u}_1}{\partial z^2}. \tag{2.6.22}$$

2.6.1 Linearised lower deck

The lower deck equations can be linearised for small humps where $h \ll 1$. The following expansions are introduced

$$\bar{u}_1(x,z) = \lambda z + h\tilde{u}_1(x,z) + \ldots\ , \tag{2.6.23}$$

$$\bar{v}_1(x,z) = h\tilde{v}_1(x,z) + \ldots\ , \tag{2.6.24}$$

$$p_2(x) = h\tilde{p}_2(x) + \ldots\ , \tag{2.6.25}$$

$$A(x) = h\tilde{A}(x) + \ldots\ . \tag{2.6.26}$$

Substituting expanisions (2.6.23)-(2.6.26) into equations (2.6.22) and (2.6.19) gives

$$\lambda z \frac{\partial \tilde{u}_1}{\partial x} + \lambda \tilde{v}_1 = -\frac{d\tilde{p}_2}{dx} + \frac{\partial^2 \tilde{u}_1}{\partial z^2}, \tag{2.6.27}$$

$$\frac{\partial \tilde{u}_1}{\partial x} + \frac{\partial \tilde{v}_1}{\partial z} = 0. \tag{2.6.28}$$

The matching conditions obtained from (2.6.12) and (2.6.13) using the expansions (2.6.23)-(2.6.26) are

$$\tilde{u}_1(x,z) \to \lambda \tilde{A}(x) \qquad \text{as } z \to \infty, \tag{2.6.29}$$

$$\tilde{v}_1(x,z) \to -\lambda z \tilde{A}'(x) - \frac{1}{\lambda}\frac{d\tilde{p}_2}{dx} \qquad \text{as } z \to \infty. \tag{2.6.30}$$

If the boundary condition (2.6.29) is satisfied, then (2.6.30) is automatically satisfied [3]. These conditions are imposed at the interface between the middle and lower deck and must be satisfied in order for the solution to be consistent across both decks. Since the displacement effect does not play a significant role in the near wake $A(x) = 0$. In this case equation (2.6.30) is used to calculate the pressure gradient $p'(x)$ which is independent of z [1]:

$$\frac{d\tilde{p}_2}{dx} = -\lambda \tilde{v}_1(x,\infty). \tag{2.6.31}$$

The conditions at the wall boundary must now be considered. The no slip and no cavity conditions (2.6.15), lead to [2, 15]

$$\tilde{u}_1(x,0) = -\lambda F(x), \tag{2.6.32}$$

$$\tilde{v}_1(x,0) = 0. \tag{2.6.33}$$

Sufficiently far downstream from the object the flow reverts to its upstream configuration. This condition gives rise to the requirements [2, 15]

$$(\tilde{u}_1, \tilde{v}_1, \tilde{p}_2, \tilde{A}) \to (0,0,0,0) \qquad \text{as } x \to \infty, \tag{2.6.34}$$

which are obtained from (2.4.15), (2.4.18) and (2.4.23). Lastly, as $z \to \infty$ the change in u is gradual and hence

$$\frac{\partial \tilde{u}_1}{\partial z}(x,z) \to 0 \qquad \text{as } z \to \infty. \tag{2.6.35}$$

In the work that follows, the stronger conditions

$$z^n(\tilde{u}_1(x,z) - \lambda \tilde{A}(x)) \to 0 \qquad \text{as } z \to \infty, \tag{2.6.36}$$

$$z^n\left(\tilde{v}_1(x,z) + \lambda z \tilde{A}'(x) + \frac{1}{\lambda}\tilde{p}_2'(x)\right) \to 0 \qquad \text{as } z \to \infty, \tag{2.6.37}$$

$$z^n\left(\frac{\partial \tilde{u}_1}{\partial z}(x,z)\right) \to 0 \qquad \text{as } z \to \infty, \tag{2.6.38}$$

will be required for $n = 0, 1, 2, 3$, in order to derive the conserved quantities.

Chapter 3

Conservation laws and conserved quantities of the governing equations for the wall-wake

In this chapter the conservation laws and conserved quantities of the governing equations are investigated. In Section 3.1 the general theory for the multiplier method, which is used to derive the conservation laws for the system, is discussed in detail. The conservation laws for the governing equations of the wall-wake are derived using the multiplier method in Section 3.2. The conservation laws are given in terms of the velocity components and in terms of the stream function. In Section 3.3 the conserved quantities are calculated from the conservation laws obtained in Section 3.2. The physical significance of each conserved quantity is then examined. In Section 3.4 similarity solutions of the governing equations are studied. Invariance of each conserved quantity enables the form of the similarity solution to be identified. It is shown that finite conserved quantities can be derived. For convenience $\tilde{u}_1$, $\tilde{v}_1$, $\tilde{p}_2$ and $\tilde{A}$ are replaced by u, v, p and A respectively.

3.1 The multiplier method

Consider an r-th order system of partial differential equations (PDEs) of two independent variables, x and z, and two dependent variables, $u(x, z)$ and $v(x, z)$:

$$F_j(x, z, u, v, u_{(1)}, v_{(1)}, ..., u_{(r)}, v_{(r)}) = 0, \qquad j = 1, 2, \tag{3.1.1}$$

where $u_{(i)}$ and $v_{(i)}$ denote the collection of i-th order partial derivatives of the dependent variables u and v.

The multiplier method [20, 33] can be used to calculate conservation laws for a system of PDEs. The suffix notation u_x, u_z, v_x, v_z, u_{xx}, u_{xz}, u_{zz}, ..., is used to denote partial derivatives of u and v when x, z, u, v and all partial derivatives of u and v are regarded as independent variables. The notation $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial z}$, $\frac{\partial^2 u}{\partial x^2}$, ..., is implemented when u and v and the partial derivatives of u and v are regarded as dependent variables which are functions of the independent variables x and z.

Consider a multiplier of the form $\Lambda = (\Lambda_1, \Lambda_2)$ where Λ_i, $i = 1, 2$, can depend on x, z, u, v and all partial derivatives of u and v of at most up to r-th order. The multiplier $\Lambda = (\Lambda_1, \Lambda_2)$ satisfies the equation

$$\Lambda^j F_j(x, z, u, v, u_{(1)}, v_{(1)}, ..., u_{(r)}, v_{(r)}) = D_j T^j, \tag{3.1.2}$$

for all functions u and v. The vector $T = (T^1, T^2)$ is known as a conserved vector. The components of the conserved vector T^i, $i = 1, 2$, can depend on x, z, u, v and all partial derivatives of u and v of at most up to r-th order. The total derivative operators, D_1 and D_2, are given by

$$D_1 = D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{xz} \frac{\partial}{\partial u_z} + v_{xz} \frac{\partial}{\partial v_z} + ... , \tag{3.1.3}$$

$$D_2 = D_z = \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + v_z \frac{\partial}{\partial v} + u_{zz} \frac{\partial}{\partial u_z} + v_{zz} \frac{\partial}{\partial v_z} + u_{zx} \frac{\partial}{\partial u_x} + v_{zx} \frac{\partial}{\partial v_x} + ... \tag{3.1.4}$$

The Euler operators, E_u and E_v, where

$$E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_z \frac{\partial}{\partial u_z} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_z \frac{\partial}{\partial u_{xz}} + D_z^2 \frac{\partial}{\partial u_{zz}} - ... , \tag{3.1.5}$$

$$E_v = \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v_x} - D_z \frac{\partial}{\partial v_z} + D_x^2 \frac{\partial}{\partial v_{xx}} + D_x D_z \frac{\partial}{\partial v_{xz}} + D_z^2 \frac{\partial}{\partial v_{zz}} - ... , \tag{3.1.6}$$

which annihilate divergence expressions, are applied to equation (3.1.2) in order to obtain the determining equations for the multiplier $\Lambda = (\Lambda_1, \Lambda_2)$. The resulting equations are

$$E_u \left[\Lambda^k F_k \right] = 0, \tag{3.1.7}$$

$$E_v \left[\Lambda^k F_k \right] = 0. \tag{3.1.8}$$

Once a multiplier $\Lambda = (\Lambda_1, \Lambda_2)$ has been found, equation (3.1.2) is used to calculate the conserved vector $T = (T^1, T^2)$ corresponding to this multiplier. If u and v are solutions to equations (3.1.1), equation (3.1.2) becomes

$$D_x T^1 + D_z T^2 = 0, \tag{3.1.9}$$

and $T = (T^1, T^2)$ is known as a local conserved vector.

For the problem of the wall-wake, F_1 and F_2 are defined as

$$F_1 = \lambda z \frac{\partial u}{\partial x} + \lambda v + \frac{dp}{dx} - \frac{\partial^2 u}{\partial z^2} = 0, \tag{3.1.10}$$

$$F_2 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \tag{3.1.11}$$

which is equivalent to the system in (2.6.27) and (2.6.28). This work focuses on (3.1.10) and (3.1.11). In order to derive the conservation laws, equations (3.1.10) and (3.1.11) are written as

$$F_1 = \lambda z u_x + \lambda v + p'(x) - u_{zz} = 0, \tag{3.1.12}$$

$$F_2 = u_x + v_z = 0. \tag{3.1.13}$$

For the particular fluid flow problem governed by equations (3.1.10) and (3.1.11), the domain of interest is $0 \leq z < \infty$. The conserved quantity corresponding to the conservation law $T = (T^1, T^2)$ is obtained by integrating (3.1.9) from $z = 0$ to $z = \infty$:

$$\frac{d}{dx} \int_0^\infty T^1 dz + T^2 \Big|_0^\infty = 0. \tag{3.1.14}$$

A conserved quantity exists provided $\int_0^\infty T^1 dz$ converges and $T^2|_{z=0}^{z=\infty} = 0$. If these two conditions hold, then from equation (3.1.14)

$$\int_0^\infty T^1 dz = c, \tag{3.1.15}$$

where c is a finite valued constant.

The boundary conditions as $z \to \infty$ play a crucial role in determining whether these two conditions hold. In the case where $T^1 \to f(x) \neq 0$ as $z \to \infty$, the integral will diverge. If $T^1 \to 0$ as $z \to \infty$, there is no guarantee that the integral $\int_0^\infty T^1 dz$ converges. However, if this condition does not hold, the integral $\int_0^\infty T^1 dz$ will certainly diverge. In the case where $T^1 \not\to 0$ and/or $T^2 \not\to 0$ as $z \to \infty$, the choice for the conserved vector $T = (T^1, T^2)$ can be modified so that $T^1 \to 0$ and $T^2 \to 0$ as $z \to \infty$. Each conserved vector $T = (T^1, T^2)$, is only defined up to an arbitrary vector, say $(f(x, z, u, v, u_x, u_z, v_x, ...), g(x, z, u, v, u_x, u_z, v_x, ...))$ that satisfies

$$D_x f + D_z g = 0, \tag{3.1.16}$$

identically without imposing the partial differential equations (3.1.12) and (3.1.13). The functions f and g can be chosen so that $T^1 \to 0$ and $T^2 \to 0$ as $z \to \infty$. An illustration of this is given in Section 3.2 for the governing equation for the wall-wake flow.

3.2 Conservation laws

In this section the multiplier method is used to obtain the conserved vectors for the governing equations when expressed in terms of the velocity components and when expressed in terms of the steam function. The conserved quantities are then derived by integrating each conservation law from $z = 0$ to $z = \infty$.

3.2.1 Conservation laws in terms of the velocity components

Consider multipliers of the form $\Lambda_1 = \Lambda_1(x, z, u, v)$ and $\Lambda_2 = \Lambda_2(x, z, u, v)$. Using (3.1.12) and (3.1.13), equation (3.1.2) becomes

$$\Lambda_1 \left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \Lambda_2 \left(u_x + v_z\right) = D_1 T^1 + D_2 T^2, \tag{3.2.1}$$

for all functions u and v. Once a conserved vector has been obtained, equation (3.2.1) is evaluated for u and v that are solutions to (3.1.12) and (3.1.13). The left-hand-side is then zero and $T = (T^1, T^2)$ is now a local conservation law. The governing equations for Λ_1 and Λ_2 are obtained by applying the Euler operators E_u and E_v given by (3.1.5) and (3.1.6) respectively, which annihilate divergence expressions, to equation (3.2.1) which results in the two equations

$$E_u \left[\Lambda_1 \left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \Lambda_2 \left(u_x + v_z\right)\right] = 0, \tag{3.2.2}$$

$$E_v \left[\Lambda_1 \left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \Lambda_2 \left(u_x + v_z\right)\right] = 0. \tag{3.2.3}$$

Equation (3.2.2) becomes

$$\frac{\partial \Lambda_1}{\partial u}\left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \frac{\partial \Lambda_2}{\partial u}\left(u_x + v_z\right) - D_x\left(\lambda z \Lambda_1 + \Lambda_2\right) - D_z^2 \Lambda_1 = 0,$$

which leads to

$$\begin{aligned} 0 =& \frac{\partial \Lambda_1}{\partial u}\left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \frac{\Lambda_2}{\partial u}\left(u_x + v_z\right) - \lambda z \frac{\partial \Lambda_1}{\partial x} - \frac{\partial \Lambda_2}{\partial x} - \lambda z u_x \frac{\partial \Lambda_1}{\partial u} \\ & - u_x \frac{\partial \Lambda_2}{\partial u} - \lambda z v_x \frac{\partial \Lambda_1}{\partial v} - v_x \frac{\partial \Lambda_2}{\partial v} - D_z\left(\frac{\partial \Lambda_1}{\partial z} + u_z \frac{\partial \Lambda_1}{\partial u} + v_z \frac{\partial \Lambda_1}{\partial v}\right), \end{aligned}$$

and after expanding the final term, the result is

$$\begin{aligned}0 =&\frac{\partial \Lambda_1}{\partial u}\left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \frac{\Lambda_2}{\partial u}(u_x + v_z) - \lambda z\frac{\partial \Lambda_1}{\partial x} - \frac{\partial \Lambda_2}{\partial x} - \lambda z u_x\frac{\partial \Lambda_1}{\partial u}\\ &- u_x\frac{\partial \Lambda_2}{\partial u} - \lambda z v_x\frac{\partial \Lambda_1}{\partial v} - v_x\frac{\partial \Lambda_2}{\partial v} - \frac{\partial^2 \Lambda_1}{\partial z^2} - 2u_z\frac{\partial^2 \Lambda_1}{\partial z\partial u} - 2v_z\frac{\partial^2 \Lambda_1}{\partial z\partial v} - u_z^2\frac{\partial^2 \Lambda_1}{\partial u^2}\\ &- v_z^2\frac{\partial^2 \Lambda_1}{\partial v^2} - 2u_z v_z\frac{\partial^2 \Lambda_1}{\partial u\partial v} - u_{zz}\frac{\partial \Lambda_1}{\partial u} - v_{zz}\frac{\partial \Lambda_1}{\partial v}.\end{aligned}$$

Simplifying gives

$$\begin{aligned}0 =& - 2\frac{\partial \Lambda_1}{\partial u}u_{zz} - \frac{\partial \Lambda_1}{\partial v}v_{zz} - \frac{\partial^2 \Lambda_1}{\partial u^2}u_z^2 - \frac{\partial^2 \Lambda_1}{\partial v^2}v_z^2 - 2\frac{\partial^2 \Lambda_1}{\partial u\partial v}u_z v_z - 2\frac{\partial^2 \Lambda_1}{\partial z\partial u}u_z\\ &+ \left(\frac{\partial \Lambda_2}{\partial u} - 2\frac{\partial^2 \Lambda_1}{\partial z\partial v}\right)v_z - \left(\frac{\partial \Lambda_2}{\partial v} + \lambda z\frac{\partial \Lambda_1}{\partial v}\right)v_x + \lambda\frac{\partial \Lambda_1}{\partial u}v\\ &+ \left(p'(x)\frac{\partial \Lambda_1}{\partial u} - \lambda z\frac{\partial \Lambda_1}{\partial x} - \frac{\partial \Lambda_2}{\partial x} - \frac{\partial^2 \Lambda_1}{\partial z^2}\right).\end{aligned} \tag{3.2.4}$$

Setting the coefficients of u_{zz} and v_{zz} to zero gives

$$\frac{\partial \Lambda_1}{\partial u} = 0, \qquad \frac{\partial \Lambda_1}{\partial v} = 0, \tag{3.2.5}$$

and therefore

$$\Lambda_1 = \Lambda_1(x, z). \tag{3.2.6}$$

Using (3.2.6), equation (3.2.4) reduces to

$$0 =\frac{\partial \Lambda_2}{\partial u}v_z - \frac{\partial \Lambda_2}{\partial v}v_x - \lambda z\frac{\partial \Lambda_1}{\partial x} - \frac{\partial \Lambda_2}{\partial x} - \frac{\partial^2 \Lambda_1}{\partial z^2}. \tag{3.2.7}$$

Setting the coefficients of v_x and v_z to zero in equation (3.2.7) results in

$$\frac{\partial \Lambda_2}{\partial v} = 0, \qquad \frac{\partial \Lambda_2}{\partial u} = 0, \tag{3.2.8}$$

and therefore

$$\Lambda_2 = \Lambda_2(x, z). \tag{3.2.9}$$

Equation (3.2.7) simplifies to

$$\frac{\partial^2 \Lambda_1}{\partial z^2} + \lambda z\frac{\partial \Lambda_1}{\partial x} + \frac{\partial \Lambda_2}{\partial x} = 0. \tag{3.2.10}$$

Now consider equation (3.2.3). Using (3.2.6) and (3.2.9), it is simply

$$\lambda\Lambda_1 - \frac{\partial \Lambda_2}{\partial z} = 0. \tag{3.2.11}$$

Differentiating equation (3.2.11) with respect to x gives

$$\frac{\partial^2 \Lambda_2}{\partial x \partial z} = \lambda \frac{\partial \Lambda_1}{\partial x}, \tag{3.2.12}$$

and differentiating equation (3.2.10) with respect to z gives

$$\frac{\partial^3 \Lambda_1}{\partial z^3} + \lambda \frac{\partial \Lambda_1}{\partial x} + \lambda z \frac{\partial^2 \Lambda_1}{\partial x \partial z} + \frac{\partial^2 \Lambda_2}{\partial x \partial z} = 0. \tag{3.2.13}$$

Substituting (3.2.12) into (3.2.13) enables the two equations to be combined into one equation:

$$\frac{\partial^3 \Lambda_1}{\partial z^3} + \lambda z \frac{\partial^2 \Lambda_1}{\partial x \partial z} + 2\lambda \frac{\partial \Lambda_1}{\partial x} = 0. \tag{3.2.14}$$

It is difficult to derive definite results when Λ_1 depends on x. In order to proceed, assume $\Lambda_1 = \Lambda_1(z)$. It will be shown that the conserved quantity for the near wall-wake can be obtained by considering a multiplier of the form $\Lambda = (\Lambda_1(z), \Lambda_2(x, z))$.

With $\Lambda_1 = \Lambda_1(z)$, equation (3.2.14) reduces to

$$\frac{\partial^3 \Lambda_1}{\partial z^3} = 0. \tag{3.2.15}$$

The solution to equation (3.2.15) is

$$\Lambda_1(z) = c_1 z^2 + c_2 z + c_3, \tag{3.2.16}$$

where c_1, c_2 and c_3 are arbitrary constants.

The solution for Λ_2 is obtained from equation (3.2.11):

$$\frac{\partial \Lambda_2}{\partial z} = \lambda(c_1 z^2 + c_2 z + c_3), \tag{3.2.17}$$

which upon integrating with respect to z gives

$$\Lambda_2(x, z) = \frac{1}{3}\lambda c_1 z^3 + \frac{1}{2}\lambda c_2 z^2 + \lambda c_3 z + f(x), \tag{3.2.18}$$

where $f(x)$ is a function of x. Equation (3.2.10) yields

$$\frac{\partial \Lambda_2}{\partial x} = -\frac{\partial^2 \Lambda_1}{\partial z^2},$$

and therefore

$$f'(x) = -2c_1, \tag{3.2.19}$$

which is used to solve for $f(x)$:

$$f(x) = -2c_1 x + c_4, \tag{3.2.20}$$

where c_4 is an arbitrary constant. Hence the solution for Λ_2 is given by

$$\Lambda_2(x,z) = \frac{1}{3}\lambda c_1 z^3 + \frac{1}{2}\lambda c_2 z^2 + \lambda c_3 z - 2c_1 x + c_4. \tag{3.2.21}$$

Equation (3.2.1) becomes

$$\begin{aligned} &\left(c_1 z^2 + c_2 z + c_3\right)\left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) \\ &+ \left(\frac{1}{3}\lambda c_1 z^3 + \frac{1}{2}\lambda c_2 z^2 + \lambda c_3 z - 2c_1 x + c_4\right)(u_x + v_z) = D_x T^1 + D_z T^2. \end{aligned} \tag{3.2.22}$$

Since there are four arbitrary constants, four cases arise. With $c_1 = 1$, $c_2 = c_3 = c_4 = 0$ equation (3.2.22) gives

$$z^2\left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \left(\frac{1}{3}\lambda z^3 - 2x\right)(u_x + v_z) = D_x T^1 + D_z T^2. \tag{3.2.23}$$

One seemingly obvious choice for T^1 and T^2 is

$$T^1 = \left(\frac{4}{3}\lambda z^3 - 2x\right) u, \tag{3.2.24}$$

$$T^2 = \left(\frac{1}{3}\lambda z^3 - 2x\right) v - z^2 u_z + 2zu + \frac{1}{3}z^3 p'(x). \tag{3.2.25}$$

With $T = (T^1, T^2)$ defined by (3.2.24) and (3.2.25), as $z \to \infty$, it is clear that for non-zero boundary conditions on u and v, $T^1 \not\to 0$ and $T^2 \not\to 0$. Therefore, from (3.1.14) it is seen that a conserved quantity cannot be obtained because $\int_0^\infty T^1 dz$ diverges and $T^2|_{z=0}^{z=\infty} \neq 0$. In order to address this issue, define an equivalent conserved vector $T^* = (T^{1*}, T^{2*})$ as follows:

$$T^{1*} = T^1 + f(x,z), \tag{3.2.26}$$

$$T^{2*} = T^2 + g(x,z), \tag{3.2.27}$$

where T^1 and T^2 are given by (3.2.24) and (3.2.25) and $f(x,z)$ and $g(x,z)$ identically satisfy

$$D_x f + D_z g = 0. \tag{3.2.28}$$

Consider the functions

$$f = -\left(\frac{4}{3}\lambda z^3 - 2x\right)\lambda A(x), \qquad g = \left(\frac{1}{3}\lambda z^3 - 2x\right)\lambda z A'(x) - \frac{2x}{\lambda}p'(x) - 2z\lambda A(x), \tag{3.2.29}$$

which satisfy (3.2.28). The new conserved vector $T^* = (T^{1*}, T^{2*})$ is

$$T^1 = \left(\frac{4}{3}\lambda z^3 - 2x\right)(u - \lambda A(x)), \tag{3.2.30}$$

$$T^2 = \left(\frac{1}{3}\lambda z^3 - 2x\right)\left(v + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right) - z^2 u_z + 2z\,(u - \lambda A(x)), \tag{3.2.31}$$

where the $*$ has been omitted for convenience. From (2.6.36)-(2.6.38) $T^1 \to 0$ and $T^2 \to 0$ as $z \to \infty$. Now as $z \to 0$, $T^2 \to -(2/\lambda)xp'(x)$ and so for $T^2|_{z=0}^{z=\infty} = 0$, the condition $p'(x) = 0$ must hold. For the remainder of this section, the conserved vectors will be constructed so that $T^1 \to 0$ and $T^2 \to 0$ as $z \to \infty$. The additional restrictions imposed for $T^2 \to 0$ as $z \to 0$ will be derived and discussed in Section 3.3. In Section 3.4 it is verified that solutions for u and v exist and that for these solutions the integral $\int_0^\infty T^1 dz$ converges.

A similar analysis to the case provided above is used for the remaining cases. It is seen that for $c_1 = 0$, $c_2 = 1$, $c_3 = c_4 = 0$, equation (3.2.22) gives

$$z\left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \frac{1}{2}\lambda z^2 (u_x + v_z) = D_x T^1 + D_z T^2. \tag{3.2.32}$$

A simple choice for T^1 and T^2 is

$$T^1 = \frac{3}{2}\lambda z^2 u, \tag{3.2.33}$$

$$T^2 = \frac{1}{2}\lambda z^2\left(v + \frac{1}{\lambda}p'(x)\right) - z u_z + u. \tag{3.2.34}$$

The functions f and g which are chosen to satisfy (3.2.28) are

$$f = -\frac{3}{2}\lambda^2 z^2 A(x), \qquad g = \frac{1}{2}\lambda^2 z^3 A'(x) - \lambda A(x). \tag{3.2.35}$$

From (3.2.26) and (3.2.27) the equivalent conserved vector $T^* = (T^{1*}, T^{2*})$ is

$$T^1 = \frac{3}{2}\lambda z^2 \left(u - \lambda A(x)\right), \tag{3.2.36}$$

$$T^2 = \frac{1}{2}\lambda z^2 \left(v + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right) - z u_z + \left(u - \lambda A(x)\right), \tag{3.2.37}$$

where the $*$ has been omitted for simplicity.

For $c_1 = c_2 = 0$, $c_3 = 1$, $c_4 = 0$, equation (3.2.22) gives

$$\left(\lambda z u_x + \lambda v + p'(x) - u_{zz}\right) + \lambda z \left(u_x + v_z\right) = D_x T^1 + D_z T^2. \tag{3.2.38}$$

An obvious choice for the conserved vector is

$$T^1 = 2\lambda z u, \tag{3.2.39}$$

$$T^2 = \lambda z \left(v + \frac{1}{\lambda}p'(x)\right) - u_z. \tag{3.2.40}$$

Defining f and g as

$$f = -2\lambda^2 z A(x), \qquad g = \lambda^2 z^2 A'(x), \tag{3.2.41}$$

gives the equivalent conserved vector $T^* = (T^{1*}, T^{2*})$:

$$T^1 = 2\lambda z \left(u - \lambda A(x)\right), \tag{3.2.42}$$

$$T^2 = \lambda z \left(v + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right) - u_z, \tag{3.2.43}$$

where the $*$ has been omitted.

Finally, for $c_1 = c_2 = c_3 = 0$, $c_4 = 1$, equation (3.2.22) gives

$$\left(u_x + v_z\right) = D_x T^1 + D_z T^2, \tag{3.2.44}$$

and a likely choice for the conserved vector is

$$T^1 = u, \tag{3.2.45}$$

$$T^2 = v. \tag{3.2.46}$$

The functions f and g are chosen as

$$f = -\lambda A(x), \qquad g = \lambda z A'(x) + \frac{1}{\lambda}p'(x), \tag{3.2.47}$$

which clearly satisfies (3.2.28). The equivalent conserved vector $T^* = (T^{1*}, T^{2*})$ is

$$T^1 = u - \lambda A(x), \tag{3.2.48}$$

$$T^2 = v + \lambda z A'(x) + \frac{1}{\lambda} p'(x), \tag{3.2.49}$$

where the $*$ has been omitted for notational convenience. This conserved vector was derived by using only equation (3.1.13).

3.2.2 Conservation laws in terms of the stream function

The governing equations can be formulated in terms of the stream function ψ defined by

$$u(x,z) = \frac{\partial \psi}{\partial z}, \qquad v(x,z) = -\frac{\partial \psi}{\partial x}. \tag{3.2.50}$$

When x, z, ψ and the partial derivatives of ψ are regarded as independent variables, equation (2.6.27) can be written as

$$\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz} = 0, \tag{3.2.51}$$

and the continuity equation, (2.6.28), is identically satisfied. Therefore, by introducing the stream function ψ the two governing equations, (2.6.27) and (2.6.28), can be expressed as the single third order equation, (3.2.51). The boundary conditions (2.6.29), (2.6.30) and (2.6.32)-(2.6.35), are in terms of the stream function,

$$\psi_x(x,0) = 0, \tag{3.2.52}$$

$$\psi_z(x,0) = -\lambda F(x), \tag{3.2.53}$$

$$\psi_x(x,z) \to \lambda z A'(x) + \frac{1}{\lambda} p'(x) \qquad \text{as } z \to \infty, \tag{3.2.54}$$

$$\psi_z(x,z) \to \lambda A(x) \qquad \text{as } z \to \infty, \tag{3.2.55}$$

$$(\psi_z, \psi_x, p, A) \to (0,0,0,0) \qquad \text{as } x \to \infty, \tag{3.2.56}$$

$$\psi_{zz}(x,z) \to 0 \qquad \text{as } z \to \infty. \tag{3.2.57}$$

The stronger conditions (2.6.36)-(2.6.38) are given by

$$z^n \left(\psi_z(x,z) - \lambda A(x)\right) \to 0 \qquad \text{as } z \to \infty, \tag{3.2.58}$$

$$z^n \left(\psi_x(x,z) - \lambda z A'(x) - \frac{1}{\lambda} p'(x)\right) \to 0 \qquad \text{as } z \to \infty, \tag{3.2.59}$$

$$z^n \left(\psi_{zz}(x,z)\right) \to 0 \qquad \text{as } z \to \infty, \tag{3.2.60}$$

where $n = 0, 1, 2, 3$.

Consider a multiplier of the form $\Lambda = \Lambda(x, z, \psi)$. The conserved form of equation (3.2.51) is

$$\Lambda \left(\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz}\right) = D_1 T^1 + D_2 T^2, \tag{3.2.61}$$

where $T = (T^1, T^2)$ is the conserved vector with components T^1 and T^2. The total derivative operators are defined by

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{zx} \frac{\partial}{\partial \psi_z} + \dots\,, \tag{3.2.62}$$

$$D_2 = D_z = \frac{\partial}{\partial z} + \psi_z \frac{\partial}{\partial \psi} + \psi_{zz} \frac{\partial}{\partial \psi_z} + \psi_{xz} \frac{\partial}{\partial \psi_x} + \dots\,. \tag{3.2.63}$$

The determining equation of the multiplier Λ is obtained by applying the standard Euler-operator E_ψ, where

$$E_\psi = \frac{\partial}{\partial \psi} - D_x \frac{\partial}{\partial \psi_x} - D_z \frac{\partial}{\partial \psi_z} + D_x^2 \frac{\partial}{\partial \psi_{xx}} + D_x D_z \frac{\partial}{\partial \psi_{xz}} + D_z^2 \frac{\partial}{\partial \psi_{zz}} - \dots\,, \tag{3.2.64}$$

to equation (3.2.61). The resulting equation is

$$E_\psi \left[\Lambda \left(\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz}\right)\right] = 0, \tag{3.2.65}$$

which yields

$$0 = \frac{\partial \Lambda}{\partial \psi} \left(\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz}\right) + D_x \left(\lambda \Lambda\right) + D_x D_z \left(\lambda z \Lambda\right) + D_z^3 \left(\Lambda\right).$$

Expanding the last three terms gives

$$\begin{aligned} 0 = & \frac{\partial \Lambda}{\partial \psi} \left(\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz}\right) + \lambda \frac{\partial \Lambda}{\partial x} + \lambda \psi_x \frac{\partial \Lambda}{\partial \psi} + D_z \left(\lambda z \frac{\partial \Lambda}{\partial x} + \lambda z \psi_x \frac{\partial \Lambda}{\partial \psi}\right) \\ & + D_z^2 \left(\frac{\partial \Lambda}{\partial z} + \psi_z \frac{\partial \Lambda}{\partial \psi}\right), \end{aligned}$$

and after expanding the last two terms, the result is

$$\begin{aligned} 0 = & \frac{\partial \Lambda}{\partial \psi} \left(\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz}\right) + \lambda \frac{\partial \Lambda}{\partial x} + \lambda \psi_x \frac{\partial \Lambda}{\partial \psi} + \lambda \frac{\partial \Lambda}{\partial x} + \lambda z \frac{\partial^2 \Lambda}{\partial x \partial z} \\ & + \lambda \psi_x \frac{\partial \Lambda}{\partial \psi} + \lambda z \psi_x \frac{\partial^2 \Lambda}{\partial z \partial \psi} + \lambda z \psi_z \frac{\partial^2 \Lambda}{\partial x \partial \psi} + \lambda z \psi_x \psi_z \frac{\partial^2 \Lambda}{\partial \psi^2} + \lambda z \psi_{xz} \frac{\partial \Lambda}{\partial \psi} \\ & + D_z \left(\frac{\partial^2 \Lambda}{\partial z^2} + 2\psi_z \frac{\partial^2 \Lambda}{\partial z \partial \psi} + \psi_z^2 \frac{\partial^2 \Lambda}{\partial \psi^2} + \psi_{zz} \frac{\partial \Lambda}{\partial \psi}\right). \end{aligned}$$

Finally,

$$
\begin{aligned}
0 =& \frac{\partial \Lambda}{\partial \psi}\left(\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz}\right) + 2\lambda \frac{\partial \Lambda}{\partial x} + 2\lambda \psi_x \frac{\partial \Lambda}{\partial \psi} + \lambda z \frac{\partial^2 \Lambda}{\partial x \partial z} \\
&+ \lambda z \psi_x \frac{\partial^2 \Lambda}{\partial z \partial \psi} + \lambda z \psi_z \frac{\partial^2 \Lambda}{\partial x \partial \psi} + \lambda z \psi_x \psi_z \frac{\partial^2 \Lambda}{\partial \psi^2} + \lambda z \psi_{xz} \frac{\partial \Lambda}{\partial \psi} + \frac{\partial^3 \Lambda}{\partial z^3} \\
&+ 3\psi_z \frac{\partial^3 \Lambda}{\partial z^2 \partial \psi} + 3\psi_z^2 \frac{\partial^3 \Lambda}{\partial z \partial \psi^2} + 3\psi_{zz} \frac{\partial^2 \Lambda}{\partial z \partial \psi} + \psi_z^3 \frac{\partial^3 \Lambda}{\partial \psi^3} + 3\psi_z \psi_{zz} \frac{\partial^2 \Lambda}{\partial \psi^2} + \psi_{zzz} \frac{\partial \Lambda}{\partial \psi}.
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
0 =& 3\frac{\partial^2 \Lambda}{\partial \psi^2}\psi_z \psi_{zz} + 3\frac{\partial^2 \Lambda}{\partial z \partial \psi}\psi_{zz} + 2\lambda z \frac{\partial \Lambda}{\partial \psi}\psi_{xz} + \frac{\partial^3 \Lambda}{\partial \psi^3}\psi_z^3 + 3\frac{\partial^3 \Lambda}{\partial z \partial \psi^2}\psi_z^2 \\
&+ \lambda z \frac{\partial^2 \Lambda}{\partial \psi^2}\psi_x \psi_z + \left(3\frac{\partial^3 \Lambda}{\partial z^2 \partial \psi} + \lambda z \frac{\partial^2 \Lambda}{\partial x \partial \psi}\right)\psi_z + \left(\lambda \frac{\partial \Lambda}{\partial \psi} + \lambda z \frac{\partial^2 \Lambda}{\partial z \partial \psi}\right)\psi_x \\
&+ \left(p'(x)\frac{\partial \Lambda}{\partial \psi} + 2\lambda \frac{\partial \Lambda}{\partial x} + \lambda z \frac{\partial^2 \Lambda}{\partial x \partial z} + \frac{\partial^3 \Lambda}{\partial z^3}\right). \qquad (3.2.66)
\end{aligned}
$$

Setting the coefficient of ψ_{xz} to zero gives

$$
\frac{\partial \Lambda}{\partial \psi} = 0, \qquad (3.2.67)
$$

and therefore $\Lambda = \Lambda(x, z)$. Equation (3.2.66) reduces to

$$
\frac{\partial^3 \Lambda}{\partial z^3} + \lambda z \frac{\partial^2 \Lambda}{\partial x \partial z} + 2\lambda \frac{\partial \Lambda}{\partial x} = 0. \qquad (3.2.68)
$$

Equation (3.2.68) is the same as equation (3.2.14). In order to derive definite results, assume that $\Lambda = \Lambda(z)$. Then equation (3.2.68) reduces to

$$
\frac{\partial^3 \Lambda}{\partial z^3} = 0, \qquad (3.2.69)
$$

which has the solution

$$
\Lambda(z) = c_5 z^2 + c_6 z + c_7, \qquad (3.2.70)
$$

where c_5, c_6 and c_7 are arbitrary constants. Equation (3.2.61) becomes

$$
\left(c_5 z^2 + c_6 z + c_7\right)\left(\lambda z \psi_{xz} - \lambda \psi_x + p'(x) - \psi_{zzz}\right) = D_1 T^1 + D_2 T^2. \qquad (3.2.71)
$$

A similar analysis to that in Section 3.2.1 is used for the stream function approach. Since there are three arbitrary constants, three cases arise. Setting $c_5 = 1$, $c_6 = c_7 = 0$

in equation (3.2.71) gives

$$z^2\left(\lambda z\psi_{xz}-\lambda\psi_x+p'(x)-\psi_{zzz}\right)=D_xT^1+D_zT^2. \tag{3.2.72}$$

A simple choice for T^1 and T^2 is

$$T^1=\left(\frac{4}{3}\lambda z^3-2x\right)\psi_z, \tag{3.2.73}$$

$$T^2=-\left(\frac{1}{3}\lambda z^3-2x\right)\left(\psi_x-\frac{1}{\lambda}p'(x)\right)+2z\psi_z-z^2\psi_{zz}. \tag{3.2.74}$$

The functions f and g are chosen as

$$f=-\lambda\left(\frac{4}{3}\lambda z^3-2x\right)A(x),\qquad g=\lambda z\left(\frac{1}{3}\lambda z^3-2x\right)A'(x)-2\lambda zA(x), \tag{3.2.75}$$

which clearly satisfies (3.2.28). The equivalent conserved vector $T^*=(T^{1*},T^{2*})$ with the $*$ omitted for convenience is

$$T^1=\left(\frac{4}{3}\lambda z^3-2x\right)\left(\psi_z-\lambda A(x)\right), \tag{3.2.76}$$

$$T^2=-\left(\frac{1}{3}\lambda z^3-2x\right)\left(\psi_x-\lambda zA'(x)-\frac{1}{\lambda}p'(x)\right)+2z\left(\psi_z-\lambda A(x)\right)-z^2\psi_{zz}. \tag{3.2.77}$$

From (3.2.58)-(3.2.60) $T^1\to 0$ and $T^2\to 0$ as $z\to\infty$.

It is seen that for $c_5=0$, $c_6=1$, $c_7=0$ equation (3.2.71) gives

$$z\left(\lambda z\psi_{xz}-\lambda\psi_x+p'(x)-\psi_{zzz}\right)=D_xT^1+D_zT^2. \tag{3.2.78}$$

An obvious choice for T^1 and T^2 is

$$T^1=\frac{3}{2}\lambda z^2\psi_z, \tag{3.2.79}$$

$$T^2=-\frac{1}{2}\lambda z^2\left(\psi_x-\frac{1}{\lambda}p'(x)\right)-z\psi_{zz}+\psi_z. \tag{3.2.80}$$

The functions f and g which satisfy (3.2.28) are chosen as

$$f=-\frac{3}{2}\lambda^2z^2A(x),\qquad g=\frac{1}{2}\lambda^2z^3A'(x)-\lambda A(x). \tag{3.2.81}$$

The equivalent conserved vector $T^*=(T^{1*},T^{2*})$ is

$$T^1=\frac{3}{2}\lambda z^2\left(\psi_z-\lambda A(x)\right), \tag{3.2.82}$$

$$T^2=-\frac{1}{2}\lambda z^2\left(\psi_x-\lambda zA'(x)-\frac{1}{\lambda}p'(x)\right)-z\psi_{zz}+\left(\psi_z-\lambda A(x)\right), \tag{3.2.83}$$

where the $*$ is omitted for convenience.

For $c_5 = c_6 = 0$, $c_7 = 1$ equation (3.2.71) gives

$$\left(\lambda z\psi_{xz} - \lambda\psi_x + p'(x) - \psi_{zzz}\right) = D_xT^1 + D_zT^2, \tag{3.2.84}$$

and a likely choice for T^1 and T^2 is

$$T^1 = 2\lambda z\psi_z, \tag{3.2.85}$$

$$T^2 = -\lambda z\left(\psi_x - \frac{1}{\lambda}p'(x)\right) - \psi_{zz}. \tag{3.2.86}$$

The functions f and g which satisfy (3.2.28) are chosen as

$$f = -2\lambda^2 zA(x), \qquad g = \lambda^2 z^2 A'(x). \tag{3.2.87}$$

The equivalent conserved vector $T^* = (T^{1*}, T^{2*})$ is

$$T^1 = 2\lambda z\left(\psi_z - \lambda A(x)\right), \tag{3.2.88}$$

$$T^2 = -\lambda z\left(\psi_x - \lambda zA'(x) - \frac{1}{\lambda}p'(x)\right) - \psi_{zz}. \tag{3.2.89}$$

Once each conserved vector is obtained, equation (3.2.71) is evaluated for ψ that solves (3.2.51). The left-hand-side of equation (3.2.71) is then zero and $T = (T^1, T^2)$ is now known as a conserved vector.

3.3 Conserved quantities

In this section the conserved quantities corresponding to the conservation laws of the governing equations for the laminar wall-wake are derived. When x and z are regarded as the only independent variables, equation (3.1.9) can be written as

$$D_xT^1 + D_zT^2 = \frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial z}. \tag{3.3.1}$$

When $T = (T^1, T^2)$ is a conserved vector, the left-hand-side of equation (3.3.1) is zero which gives

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial z} = 0. \tag{3.3.2}$$

3.3.1 Conserved quantities in terms of the velocity components

By making use of the results obtained in Section 3.2.1 the conserved quantities can be derived in terms of the velocity components. In order to achieve this the boundary conditions (2.6.32), (2.6.33) and (2.6.36)-(2.6.38) are required.

3.3.1.1 Case 1

In order to derive the conserved quantity corresponding to the conserved vector $T = (T^1, T^2)$ where T^1 and T^2 are given by (3.2.30) and (3.2.31) respectively, equations (3.2.30) and (3.2.31) are substituted into (3.3.2) which is then integrated with respect to z from 0 to ∞. This leads to

$$\begin{aligned} 0 = &\int_0^\infty \frac{\partial}{\partial x}\left[\left(\frac{4}{3}\lambda z^3 - 2x\right)(u(x,z) - \lambda A(x))\right] dz \\ &+ \int_0^\infty \frac{\partial}{\partial z}\left[\left(\frac{1}{3}\lambda z^3 - 2x\right)\left(v(x,z) + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right)\right. \\ &\left. \qquad - z^2 u_z(x,z) + 2z\left(u(x,z) - \lambda A(x)\right)\right] dz, \end{aligned}$$

which can be simplified to give

$$\begin{aligned} 0 = &\frac{d}{dx}\int_0^\infty \left(\frac{4}{3}\lambda z^3 - 2x\right)(u(x,z) - \lambda A(x))\, dz \\ &+ \left[\left(\frac{1}{3}\lambda z^3 - 2x\right)\left(v(x,z) + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right)\right]_0^\infty \\ &- \left[z^2 u_z(x,z)\right]_0^\infty + \left[2z\left(u(x,z) - \lambda A(x)\right)\right]_0^\infty . \end{aligned} \tag{3.3.3}$$

Using the boundary conditions given in (2.6.33) and (2.6.36)-(2.6.38), equation (3.3.3) can be written as

$$\frac{d}{dx}\int_0^\infty \left(\frac{4}{3}\lambda z^3 - 2x\right)(u(x,z) - \lambda A(x))\, dz = -\frac{2x}{\lambda}p'(x). \tag{3.3.4}$$

From (3.3.4) it is clear that in order for there to be a conserved quantity it is required that the pressure $p(x)$ be constant and that the integral $\int_0^\infty T^1 dz$ converge.

3.3.1.2 Case 2

For this case, the components of the conserved vector $T = (T^1, T^2)$ are defined by (3.2.36) and (3.2.37). Integrating (3.3.2) with respect to z from 0 to ∞ gives

$$\begin{aligned} 0 = &\int_0^\infty \frac{\partial}{\partial x}\left[\frac{3}{2}\lambda z^2\left(u(x,z) - \lambda A(x)\right)\right] dz \\ &+ \int_0^\infty \frac{\partial}{\partial z}\left[\frac{1}{2}\lambda z^2\left(v(x,z) + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right)\right. \\ &\qquad\left. - z u_z(x,z) + u(x,z) - \lambda A(x)\right] dz, \end{aligned}$$

which, after simplifying, results in

$$\begin{aligned} 0 =& \frac{d}{dx}\int_0^\infty \frac{3}{2}\lambda z^2\left(u(x,z) - \lambda A(x)\right)\, dz \\ &+ \left[\frac{1}{2}\lambda z^2\left(v(x,z) + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right)\right]_0^\infty \\ &- \left[z u_z(x,z)\right]_0^\infty + \left[u(x,z) - \lambda A(x)\right]_0^\infty, \end{aligned} \tag{3.3.5}$$

and imposing the boundary conditions (2.6.32) and (2.6.36)-(2.6.38) yields

$$\frac{d}{dx}\int_0^\infty \frac{3}{2}\lambda z^2\left(u(x,z) - \lambda A(x)\right)\, dz = -\lambda\left(A(x) + F(x)\right). \tag{3.3.6}$$

A conserved quantity exists provided $A(x) = -F(x)$ and that the integral $\int_0^\infty T^1 dz$ converges.

3.3.1.3 Case 3

The components of the conserved vector $T = (T^1, T^2)$ where T^1 and T^2 are defined by (3.2.42) and (3.2.43), are substituted into (3.3.2) which is integrated with respect to z from 0 to ∞. This results in

$$\begin{aligned} 0 = &\int_0^\infty \frac{\partial}{\partial x}\left[2\lambda z\left(u(x,z) - \lambda A(x)\right)\right] dz \\ &+ \int_0^\infty \frac{\partial}{\partial z}\left[\lambda z\left(v(x,z) + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right) - u_z(x,z)\right] dz, \end{aligned}$$

which after simplifying gives

$$\begin{aligned} 0 =& \frac{d}{dx}\int_0^\infty 2\lambda z\left(u(x,z) - \lambda A(x)\right)\, dz \\ &+ \left[\lambda z\left(v(x,z) + \lambda z A'(x) + \frac{1}{\lambda}p'(x)\right)\right]_0^\infty - \left[u_z(x,z)\right]_0^\infty. \end{aligned} \tag{3.3.7}$$

Using the boundary conditions given in (2.6.37) and (2.6.38), equation (3.3.7) can be written as

$$\frac{d}{dx}\int_0^\infty 2\lambda z\left(u(x,z)-\lambda A(x)\right)\ dz = -u_z(x,0). \tag{3.3.8}$$

A conserved quantity is obtained if $u_z(x,0)=0$ and $\int_0^\infty T^1 dz$ converges.

3.3.1.4 Case 4

Lastly, substituting the components of the conserved vector $T=(T^1,T^2)$ defined by (3.2.48) and (3.2.49) into (3.3.2) and integrating with respect to z from 0 to ∞ results in

$$0=\int_0^\infty \frac{\partial}{\partial x}[u(x,z)-\lambda A(x)]dz+\int_0^\infty \frac{\partial}{\partial z}\left[v(x,z)+\lambda zA'(x)+\frac{1}{\lambda}p'(x)\right]dz,$$

which leads to

$$0=\frac{d}{dx}\int_0^\infty \left(u(x,z)-\lambda A(x)\right)\ dz+\left[v(x,z)+\lambda zA'(x)+\frac{1}{\lambda}p'(x)\right]_0^\infty, \tag{3.3.9}$$

which after using the boundary conditions (2.6.33) and (2.6.37) yields

$$\frac{d}{dx}\int_0^\infty \left(u(x,z)-\lambda A(x)\right)dz=\frac{1}{\lambda}p'(x). \tag{3.3.10}$$

From (3.3.10) it is clear that in order for there to be a conserved quantity it is required that the pressure $p(x)$ is constant and $\int_0^\infty T^1 dz$ converges.

3.3.2 Conserved quantities in terms of the stream function

By making use of the results calculated in Section 3.2.2 the conserved quantities can be obtained in terms of the stream function. In order to do so, the boundary conditions (3.2.52), (3.2.53) and (3.2.58)-(3.2.60) are needed.

3.3.2.1 Case 1

The components of the conserved vector $T = (T^1, T^2)$ are defined by (3.2.76) and (3.2.77). Now integrating equation (3.3.2) with respect to z from 0 to ∞ gives

$$
\begin{aligned}
0 = &\int_0^\infty \frac{\partial}{\partial x}\left[\left(\frac{4}{3}\lambda z^3 - 2x\right)(\psi_z(x,z) - \lambda A(x))\right] dz - \\
&\int_0^\infty \frac{\partial}{\partial z}\Bigg[\left(\frac{1}{3}\lambda z^3 - 2x\right)\left(\psi_x(x,z) - \lambda z A'(x) - \frac{1}{\lambda}p'(x)\right) \\
&\qquad + 2z\left(\psi_z(x,z) - \lambda A(x)\right) - z^2\psi_{zz}(x,z)\Bigg] dz,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
0 =& \frac{d}{dx}\int_0^\infty \left(\frac{4}{3}\lambda z^3 - 2x\right)(\psi_z(x,z) - \lambda A(x))\ dz \\
&- \left[\left(\frac{1}{3}\lambda z^3 - 2x\right)\left(\psi_x(x,z) - \lambda z A'(x) - \frac{1}{\lambda}p'(x)\right)\right]_0^\infty \\
&+ \left[2z\left(\psi_z(x,z) - \lambda A(x)\right)\right]_0^\infty - \left[z^2\psi_{zz}(x,z)\right]_0^\infty,
\end{aligned}
\tag{3.3.11}
$$

and using the boundary conditions (3.2.52) and (3.2.58)-(3.2.60) yields

$$
\frac{d}{dx}\int_0^\infty \left(\frac{4}{3}\lambda z^3 - 2x\right)(\psi_z(x,z) - \lambda A(x))\ dz = -\frac{2x}{\lambda}p'(x). \tag{3.3.12}
$$

For $p'(x) = 0$, the conserved quantity calculated from (3.3.12) matches that of (3.3.4) provided that $\int_0^\infty T^1 dz$ converges.

3.3.2.2 Case 2

For this case, the components of the conserved vector $T = (T^1, T^2)$ are defined by (3.2.82) and (3.2.83). Substituting (3.2.82) and (3.2.83) into (3.3.2) and integrating with respect to z from 0 to ∞ results in

$$
\begin{aligned}
0 = &\int_0^\infty \frac{\partial}{\partial x}\left[\frac{3}{2}\lambda z^2\left(\psi_z(x,z) - \lambda A(x)\right)\right] dz \\
&+ \int_0^\infty \frac{\partial}{\partial z}\Bigg[-\frac{1}{2}\lambda z^2\left(\psi_x(x,z) - \lambda z A'(x) - \frac{1}{\lambda}p'(x)\right) \\
&\qquad - z\psi_{zz}(x,z) + \psi_z(x,z) - \lambda A(x)\Bigg] dz,
\end{aligned}
$$

which gives

$$\begin{aligned}0 =& \frac{d}{dx}\int_0^\infty \frac{3}{2}\lambda z^2 \left(\psi_z(x,z) - \lambda A(x)\right)\, dz \\ &- \left[\frac{1}{2}\lambda z^2 \left(\psi_x(x,z) - \lambda z A'(x) - \frac{1}{\lambda}p'(x)\right)\right]_0^\infty \\ &- \left[z\psi_{zz}(x,z)\right]_0^\infty + \left[\psi_z(x,z) - \lambda A(x)\right]_0^\infty . \end{aligned} \tag{3.3.13}$$

Using the boundary conditions in (3.2.53) and (3.2.58)-(3.2.60), equation (3.3.13) can be written as

$$\frac{d}{dx}\int_0^\infty \frac{3}{2}\lambda z^2 \left(\psi_z(x,z) - \lambda A(x)\right)\, dz = -\lambda \left(A(x) + F(x)\right). \tag{3.3.14}$$

For $A(x) = -F(x)$, it can be seen that the conserved quantity derived from (3.3.14) is equivalent to that of (3.3.6) on condition that $\int_0^\infty T^1 dz$ converges.

3.3.2.3 Case 3

Substituting the components of the conserved vector $T = (T^1, T^2)$ where T^1 and T^2 are defined by (3.2.88) and (3.2.89), into (3.3.2) and integrating with respect to z from 0 to ∞ gives

$$\begin{aligned}0 = &\int_0^\infty \frac{\partial}{\partial x}\left[2\lambda z \left(\psi_z(x,z) - \lambda A(x)\right)\right] dz \\ &+ \int_0^\infty \frac{\partial}{\partial z}\left[-\lambda z \left(\psi_x(x,z) - \lambda z A'(x) - \frac{1}{\lambda}p'(x)\right) - \psi_{zz}(x,z)\right] dz,\end{aligned}$$

which can be simplified, resulting in

$$\begin{aligned}0 =& \frac{d}{dx}\int_0^\infty 2\lambda z \left(\psi_z(x,z) - \lambda A(x)\right)\, dz - \left[\lambda z \left(\psi_x(x,z) - \lambda z A'(x) - \frac{1}{\lambda}p'(x)\right)\right]_0^\infty \\ &- \left[\psi_{zz}(x,z)\right]_0^\infty , \end{aligned} \tag{3.3.15}$$

and using the boundary conditions (3.2.59) and (3.2.60) yields

$$\frac{d}{dx}\int_0^\infty 2\lambda z \left(\psi_z(x,z) - \lambda A(x)\right)\, dz = -\psi_{zz}(x,0). \tag{3.3.16}$$

For $\psi_{zz}(x,0) = 0$, the conserved quantity evaluated from (3.3.16) is the same as that from (3.3.8) provided $\int_0^\infty T^1 dz$ converges.

3.3.3 Summary

The Table below shows the multiplier along with the corresponding conserved vector for each case in terms of both the velocity components and the stream function. The conditions required other than convergence of the integral $\int_0^\infty T^1 dz$ to obtain the conserved quantity associated with each conserved vector are also provided.

$\Lambda = (\Lambda_1, \Lambda_2)$	$T = (T^1, T^2)$	Condition	Conserved Quantity
	Velocity Components		
Case 1 $\Lambda_1 = z^2$, $\Lambda_2 = \frac{1}{3}\lambda z^3 - 2x$	$T^1 = \left(\frac{4}{3}\lambda z^3 - 2x\right)(u - \lambda A)$, $T^2 = \left(\frac{1}{3}\lambda z^3 - 2x\right)\left(v + \lambda z A' + \frac{1}{\lambda}p'\right)$ $-z^2 u_z + 2z(u - \lambda A)$	$p'(x) = 0$	$\int_0^\infty \left(\frac{4}{3}\lambda z^3 - 2x\right)(u - \lambda A)\, dz = c$
Case 2 $\Lambda_1 = z$, $\Lambda_2 = \frac{1}{2}\lambda z^2$	$T^1 = \frac{3}{2}\lambda z^2 (u - \lambda A)$, $T^2 = \frac{1}{2}\lambda z^2 \left(v + \lambda z A' + \frac{1}{\lambda}p'\right)$ $-z u_z + (u - \lambda A)$	$A(x) = -F(x)$	$\int_0^\infty \frac{3}{2}\lambda z^2 (u - \lambda A)\, dz = c$
Case 3 $\Lambda_1 = 1$, $\Lambda_2 = \lambda z$	$T^1 = 2\lambda z (u - \lambda A)$, $T^2 = \lambda z \left(v + \lambda z A' + \frac{1}{\lambda}p'\right) - u_z$	$u_z(x, 0) = 0$	$\int_0^\infty 2\lambda z (u - \lambda A)\, dz = c$
Case 4 $\Lambda_1 = 0$, $\Lambda_2 = 1$	$T^1 = u - \lambda A$, $T^2 = v + \lambda z A' + \frac{1}{\lambda}p'$	$p'(x) = 0$	$\int_0^\infty (u - \lambda A)\, dz = c$
	Stream Function		
Case 1 $\Lambda = z^2$	$T^1 = \left(\frac{4}{3}\lambda z^3 - 2x\right)(\psi_z - \lambda A)$, $T^2 = -\left(\frac{1}{3}\lambda z^3 - 2x\right)\left(\psi_x - \lambda z A' - \frac{1}{\lambda}p'\right)$ $+2z(\psi_z - \lambda A) - z^2 \psi_{zz}$	$p'(x) = 0$	$\int_0^\infty \left(\frac{4}{3}\lambda z^3 - 2x\right)(\psi_z - \lambda A)\, dz = c$
Case 2 $\Lambda = z$	$T^1 = \frac{3}{2}\lambda z^2 (\psi_z - \lambda A)$, $T^2 = -\frac{1}{2}\lambda z^2 \left(\psi_x - \lambda z A' - \frac{1}{\lambda}p'\right)$ $-z\psi_{zz} + (\psi_z - \lambda A)$	$A(x) = -F(x)$	$\int_0^\infty \frac{3}{2}\lambda z^2 (\psi_z - \lambda A)\, dz = c$
Case 3 $\Lambda = 1$	$T^1 = 2\lambda z (\psi_z - \lambda A)$, $T^2 = -\lambda z \left(\psi_x - \lambda z A' - \frac{1}{\lambda}p'\right) - \psi_{zz}$	$\psi_{zz}(x, 0) = 0$	$\int_0^\infty 2\lambda z (\psi_z - \lambda A)\, dz = c$

TABLE 3.1: Multipliers, conserved vectors and conserved quantities

Each conserved quantity that was obtained when the stream function formulation was used relates directly to a case when the velocity components were used. An additional conserved quantity was obtained when the governing equations were expressed in terms of the velocity components. This conserved quantity arose from the continuity equation which is identically satisfied when the stream function is used. In terms of both the velocity components and the stream function, Case 2 applies to the near wall-wake. Interestingly, for Cases 1 and 4 in terms of the velocity components, the same condition $p'(x) = 0$ must hold in order for a conserved quantity to exist.

3.3.4 Physical significance of the conserved quantities

In this section the physical meaning of each conserved quantity is analysed. Consider first a wake behind a body in a uniform flow. Let U and u be the dimensionless mainstream speed and velocity deficit respectively. For the classical far wake the drag is conserved. The conserved quantity is given by [37]

$$D = \int_{-\infty}^{\infty} Uu \ dy, \tag{3.3.17}$$

where D is a dimensionless constant proportional to the drag on the body. For the far wake of a self propelled body, the second moment of the axial momentum deficit is conserved. The conserved quantity is [38]

$$K = \int_{-\infty}^{\infty} y^2 Uu \ dy, \tag{3.3.18}$$

where K is a dimensionless constant proportional to the second moment of the axial momentum deficit.

Now consider the conserved quantities pertaining to the governing equations of the wall-wake when expressed in terms of the velocity components. The domain of interest is $0 \leq z < \infty$. The mainstream speed, U, is given by $U = \lambda z$ and the velocity deficit is $u(x,z) - \lambda A(x)$. Comparing the conserved quantity obtained in Case 3 in Table 3.1 to equation (3.3.17) with $U = \lambda z$ and $u(x,z) - \lambda A(x)$ in place of u, it is seen that the drag is conserved. The conserved quantity in Case 1 is of a similar form to the conserved quantity in (3.3.18). It may be that this conserved quantity also corresponds to the second moment of axial momentum deficit. The conserved quantity in Case 2 is required to complete the solution to the near wall-wake. Here, the moment of the momentum deficit is conserved. The last case, namely Case 4, represents the conservation of mass.

3.4 Similarity solutions

The existence of a finite conserved quantity relies on whether the integral $\int_0^\infty T^1 dz$ converges and if the condition $T^2|_{z=0}^{z=\infty} = 0$ holds. In Section 3.2, the conserved vectors were constructed in order to ensure that as $z \to \infty$ both $T^1 \to 0$ and $T^2 \to 0$. In most cases, $T^2 \not\to 0$ as $z \to 0$ and additional restrictions were imposed so that $T^2 \to 0$ as $z \to 0$. Although $T^1 \to 0$ as $z \to \infty$, convergence of the integral $\int_0^\infty T^1 dz$ is still not guaranteed. Convergence can be verified by solving the governing equations subject to the boundary conditions, and then evaluating the integral $\int_0^\infty T^1 dz$. In this section it is shown that similarity solutions that satisfy the relevant governing equations, boundary conditions and finite valued conserved quantities can be found. The conservation laws in terms of the velocity components will be used.

3.4.1 Similarity solutions for cases 1 and 4

For cases 1 and 4, the restriction $p'(x) = 0$ is imposed. The governing equations (2.6.27) and (2.6.28) are

$$\lambda z \frac{\partial u}{\partial x} + \lambda v = \frac{\partial^2 u}{\partial z^2}, \tag{3.4.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0. \tag{3.4.2}$$

Introduce scaling transformations as follows:

$$x = \kappa^a \bar{x}, \qquad z = \kappa^b \bar{z}, \qquad u = \kappa^c \bar{u}, \qquad v = \kappa^f \bar{v}. \tag{3.4.3}$$

Substituting (3.4.3) into equation (3.4.2) gives

$$\kappa^{b+c-a-f} \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{z}} = 0. \tag{3.4.4}$$

For invariance, it is required that

$$a = b + c - f. \tag{3.4.5}$$

Substituting (3.4.3) into equation (3.4.1) and using (3.4.5) gives

$$\lambda \bar{z} \frac{\partial \bar{u}}{\partial \bar{x}} + \lambda \bar{v} = \kappa^{c-2b-f} \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}. \tag{3.4.6}$$

For invariance

$$c = 2b + f, \tag{3.4.7}$$

must hold. Simplifying (3.4.5) using (3.4.7) gives

$$a = 3b. \tag{3.4.8}$$

Hence the scaling transformations in (3.4.3) can be written as

$$x = \kappa^{3b} \bar{x}, \qquad z = \kappa^b \bar{z}, \qquad u = \kappa^{2b+f} \bar{u}, \qquad v = \kappa^f \bar{v}. \tag{3.4.9}$$

Now let

$$u = G_1(x, z), \tag{3.4.10}$$

$$\bar{u} = G_1(\bar{x}, \bar{z}). \tag{3.4.11}$$

There is no bar on G_1 in (3.4.11) since the solutions (3.4.10) and (3.4.11) need to be of the same functional form. The only difference is that the variables are replaced by barred variables. This gives

$$\kappa^{-2b-f} G_1(x, z) = G_1(\kappa^{-3b} x, \kappa^{-b} z). \tag{3.4.12}$$

Differentiating (3.4.12) with respect to κ gives

$$(2b + f)\kappa^{-2b-f-1} G_1(x, z) = 3b\kappa^{-3b-1} x \frac{\partial G_1}{\partial \bar{x}} + b\kappa^{-b-1} z \frac{\partial G_1}{\partial \bar{z}}. \tag{3.4.13}$$

Setting $\kappa = 1$ gives

$$\bar{x} = x, \qquad\qquad \bar{z} = z, \tag{3.4.14}$$

and (3.4.13) becomes

$$(2b + f) G_1(x, z) = 3bx \frac{\partial G_1}{\partial x} + bz \frac{\partial G_1}{\partial z}. \tag{3.4.15}$$

From (3.4.15), the general case in which $2b + f \neq 0$ is considered and the differential equations of the characteristic curves are

$$\frac{dx}{3bx} = \frac{dz}{bz} = \frac{dG_1}{(2b + f)G_1}. \tag{3.4.16}$$

Solving the first pair of equations in (3.4.16) gives

$$\xi = \frac{z}{x^{\frac{1}{3}}}, \tag{3.4.17}$$

and solving for the first and last pair of terms results in

$$u(x, z) = x^{\frac{2}{3}+\alpha} F(\xi), \tag{3.4.18}$$

where

$$\alpha = \frac{f}{3b}. \tag{3.4.19}$$

Now let

$$v = h_1(x, z),$$
$$\bar{v} = h_1(\bar{x}, \bar{z}).$$

This results in

$$\kappa^{-f} h_1(x, z) = h_1(\kappa^{-3b}x, \kappa^{-b}z). \tag{3.4.20}$$

Differentiating (3.4.20) with respect to κ results in

$$f\kappa^{-f-1}h_1(x, z) = 3b\kappa^{-3b-1}x\frac{\partial h_1}{\partial \bar{x}} + b\kappa^{-b-1}z\frac{\partial h_1}{\partial \bar{z}}, \tag{3.4.21}$$

and letting $\kappa = 1$ gives

$$fh_1(x, z) = 3bx\frac{\partial h_1}{\partial x} + bz\frac{\partial h_1}{\partial z}. \tag{3.4.22}$$

The characteristic curves obtained from (3.4.22) are

$$\frac{dx}{3bx} = \frac{dz}{bz} = \frac{dh_1}{fh_1}. \tag{3.4.23}$$

Solving the first pair of equations in (3.4.23) results in (3.4.17) and solving for the first and last pair of terms gives

$$v(x, z) = x^{\alpha}G(\xi), \tag{3.4.24}$$

where α is given by (3.4.19).

Substituting (3.4.18) and (3.4.24) into equations (3.4.1) and (3.4.2) gives

$$\frac{d^2F}{d\xi^2} + \frac{1}{3}\lambda\xi^2\frac{dF}{d\xi} - \lambda\left(\frac{2}{3} + \alpha\right)\xi F(\xi) - \lambda G(\xi) = 0, \tag{3.4.25}$$

$$\frac{dG}{d\xi} - \frac{1}{3}\xi\frac{dF}{d\xi} + \left(\frac{2}{3} + \alpha\right)F(\xi) = 0. \tag{3.4.26}$$

Equation (3.4.26) results in

$$G(\xi) = \frac{1}{3}\xi\frac{dM}{d\xi} - (1 + \alpha)M(\xi), \tag{3.4.27}$$

where

$$M(\xi) = \int_0^{\xi} F(\alpha)d\alpha, \tag{3.4.28}$$

and $G(0) = 0$ was used. Equation (3.4.25) becomes

$$\frac{d^3M}{d\xi^3} + \frac{1}{3}\lambda\xi^2\frac{d^2M}{d\xi^2} - \lambda(1+\alpha)\xi\frac{dM}{d\xi} + \lambda(1+\alpha)M(\xi) = 0. \tag{3.4.29}$$

Introducing the transformation

$$\xi = B\bar{\xi}, \qquad M = \bar{M}, \tag{3.4.30}$$

into equation (3.4.29) gives

$$\frac{d^3\bar{M}}{d\bar{\xi}^3} + \frac{1}{3}\lambda B^3\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2} - \lambda(1+\alpha)B^3\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}} + \lambda(1+\alpha)B^3\bar{M}(\bar{\xi}) = 0. \tag{3.4.31}$$

Hence B is given by

$$B = \left(\frac{1}{\lambda}\right)^{\frac{1}{3}}. \tag{3.4.32}$$

The similarity solutions are

$$u(x,z) = \lambda^{\frac{1}{3}}\frac{d\bar{M}}{d\bar{\xi}}x^{\frac{2}{3}+\alpha}, \tag{3.4.33}$$

$$v(x,z) = \left[\frac{1}{3}\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}} - (1+\alpha)\bar{M}\right]x^{\alpha}, \tag{3.4.34}$$

$$p(x) = 0, \tag{3.4.35}$$

where α is a constant,

$$\bar{\xi} = \frac{z\lambda^{\frac{1}{3}}}{x^{\frac{1}{3}}}, \tag{3.4.36}$$

and $\bar{M}$ satisfies

$$\frac{d^3\bar{M}}{d\bar{\xi}^3} + \frac{1}{3}\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2} - (1+\alpha)\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}} + (1+\alpha)\bar{M} = 0. \tag{3.4.37}$$

In this section a very small hump with a length of, say, 2ϵ is examined. Small humps are considered in order to obtain the conserved quantity for the near wall-wake [15]. Since the axis of symmetry of the hump is the line $x = 0$, for $x > \epsilon$, equation (2.6.32) becomes

$$u(x,0) = 0. \tag{3.4.38}$$

In the analysis that follows, the region $x > \epsilon$ will be examined and equation (3.4.38) will be used in place of (2.6.32).

Now using (2.4.19) and (2.4.20), the boundary conditions (3.4.38), (2.6.35) and (2.6.33) become

$$\bar{M}'(0) = 0, \tag{3.4.39}$$
$$\bar{M}''(\infty) = 0, \tag{3.4.40}$$

and if $\alpha \neq -1$

$$\bar{M}(0) = 0. \tag{3.4.41}$$

Equation (3.4.37) and the boundary conditions (3.4.39)-(3.4.41) are homogeneous. Therefore, although three boundary conditions are specified for an ordinary differential equation of order three, because the governing equation and all the boundary conditions are homogeneous, a conserved quantity is required to complete the solution. Thus far no mention of the boundary conditions (2.6.34) and (2.6.29) has been made. For (2.6.34) to hold, α must satisfy $\alpha < -\frac{2}{3}$. The solution for $\bar{M}$, and hence $u(x,z)$, can be obtained using the three boundary conditions (3.4.39)-(3.4.41) and a conserved quantity. No other information needs to be specified to find $u(x,z)$. Equation (2.6.29) is important because once the solution for $u(x,z)$ has been found, equation (2.6.29) can be used to determine $A(x)$:

$$A(x) = \lambda^{-\frac{2}{3}}\bar{M}'(\infty)x^{\alpha+\frac{2}{3}}, \tag{3.4.42}$$

and the boundary layer displacement effect $A(x)$ can now be obtained from this result.

3.4.1.1 Case 1

In terms of the similarity variables defined in (3.4.33)-(3.4.35) the conserved quantity calculated from (3.3.4) with $p'(x) = 0$ is

$$\int_0^\infty \left(\frac{4}{3}\lambda z^3 - 2x\right)(u(x,z) - \lambda A(x))\,dz = x^{2+\alpha}\int_0^\infty \left(\frac{4}{3}\bar{\xi}^3 - 2\right)\left(\bar{M}'(\bar{\xi}) - \bar{M}'(\infty)\right)d\bar{\xi}$$
$$= B_1. \tag{3.4.43}$$

Since B_1 is a constant independent of x, $\alpha = -2$. The similarity solutions (3.4.33) and (3.4.34) with $\alpha = -2$ are

$$u(x,z) = \lambda^{\frac{1}{3}}\frac{d\bar{M}}{d\bar{\xi}}x^{-\frac{4}{3}}, \tag{3.4.44}$$
$$v(x,z) = \left[\frac{1}{3}\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}} + \bar{M}\right]x^{-2}. \tag{3.4.45}$$

Equation (3.4.37) becomes

$$\frac{d^3\bar{M}}{d\bar{\xi}^3}+\frac{1}{3}\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2}+\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}-\bar{M}=0. \tag{3.4.46}$$

Solving equation (3.4.46) subject to (3.4.39)-(3.4.41), using Mathematica, gives for $\bar{M}$

$$\bar{M}(\bar{\xi})=\frac{a\,\bar{\xi}}{3^{\frac{5}{3}}}\left(\Gamma\left[\frac{1}{3}\right]-\Gamma\left[\frac{1}{3},\frac{\bar{\xi}^3}{9}\right]\right). \tag{3.4.47}$$

Here, $\Gamma[c,z]$ is the upper incomplete Gamma function

$$\Gamma[c,z]=\int_z^{\infty}t^{c-1}e^{-t}dt. \tag{3.4.48}$$

Also

$$\gamma[c,z]+\Gamma[c,z]=\Gamma[c], \tag{3.4.49}$$

where $\gamma[c,z]$ and $\Gamma[c]$ are the lower incomplete Gamma function and the Gamma function respectively

$$\gamma[c,z]=\int_0^{z}t^{c-1}e^{-t}dt, \tag{3.4.50}$$

$$\Gamma[c]=\int_0^{\infty}t^{c-1}e^{-t}dt, \tag{3.4.51}$$

and $c,z\in\mathbb{R}$. The constant $a>0$ is determined from the conserved quantity (3.4.43):

$$a=\frac{B_1}{2\Gamma\left[\frac{2}{3}\right]}, \tag{3.4.52}$$

which proves that a finite valued conserved quantity is obtained. Substituting (3.4.52) into (3.4.47) gives

$$\bar{M}(\bar{\xi})=\frac{B_1\bar{\xi}\left(\Gamma\left[\frac{1}{3}\right]-\Gamma\left[\frac{1}{3},\frac{\bar{\xi}^3}{9}\right]\right)}{6\;3^{\frac{2}{3}}\Gamma\left[\frac{2}{3}\right]}. \tag{3.4.53}$$

Differentiating (3.4.53) with respect to $\bar{\xi}$ gives

$$\bar{M}'(\bar{\xi})=\frac{B_1\left(3^{\frac{1}{3}}e^{-\frac{\bar{\xi}^3}{9}}\bar{\xi}+\Gamma\left[\frac{1}{3}\right]-\Gamma\left[\frac{1}{3},\frac{\bar{\xi}^3}{9}\right]\right)}{6\;3^{\frac{2}{3}}\Gamma\left[\frac{2}{3}\right]}. \tag{3.4.54}$$

Figure 3.1 displays the solution for $\bar{M}'(\bar{\xi})$ with $B_1=1$.

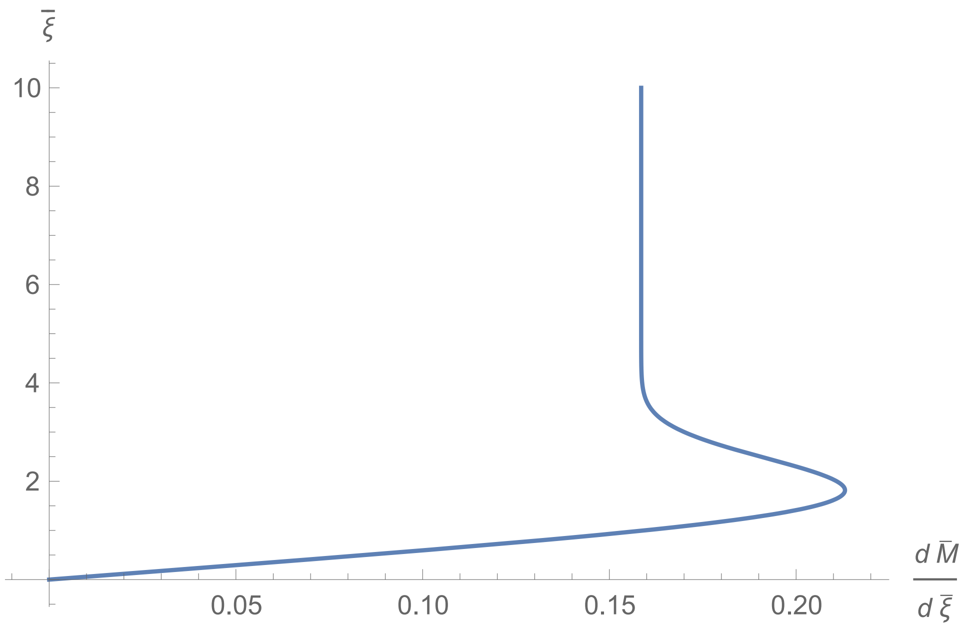

FIGURE 3.1: Solution for case 1

The similarity solutions (3.4.44) and (3.4.45) become

$$u(x,z) = \frac{\lambda^{\frac{1}{3}} B_1 \left(3^{\frac{1}{3}} e^{-\frac{\lambda z^3}{9x}} \left(\frac{\lambda^{\frac{1}{3}} z}{x^{\frac{1}{3}}}\right) + \Gamma\left[\frac{1}{3}\right] - \Gamma\left[\frac{1}{3}, \frac{\lambda z^3}{9x}\right]\right)}{6\ 3^{\frac{2}{3}} \Gamma\left[\frac{2}{3}\right]} x^{-\frac{4}{3}}, \tag{3.4.55}$$

$$v(x,z) = \frac{\lambda^{\frac{1}{3}} B_1 z \left(3^{\frac{1}{3}} e^{-\frac{\lambda z^3}{9x}} \left(\frac{\lambda^{\frac{1}{3}} z}{x^{\frac{1}{3}}}\right) + 4\Gamma\left[\frac{1}{3}\right] - 4\Gamma\left[\frac{1}{3}, \frac{\lambda z^3}{9x}\right]\right)}{6\ 3^{\frac{5}{3}} \Gamma\left[\frac{2}{3}\right]} x^{-\frac{7}{3}}. \tag{3.4.56}$$

3.4.1.2 Case 4

With $p'(x) = 0$, (3.3.10) gives a conserved quantity, which when written in terms of the similarity variables defined in (3.4.33)-(3.4.35) is

$$\int_0^\infty (u(x,z) - \lambda A(x))\, dz = x^{1+\alpha} \int_0^\infty \left(\bar{M}'(\bar{\xi}) - \bar{M}'(\infty)\right) d\bar{\xi}$$
$$= B_4. \tag{3.4.57}$$

Since B_4 is a constant independent of x, $\alpha = -1$. With $\alpha = -1$, the similarity solutions reduce to

$$u(x,z) = \lambda^{\frac{1}{3}} \frac{d\bar{M}}{d\bar{\xi}} x^{-\frac{1}{3}}, \tag{3.4.58}$$

$$v(x,z) = \frac{1}{3} \bar{\xi} \frac{d\bar{M}}{d\bar{\xi}} x^{-1}, \tag{3.4.59}$$

and a condition on $\bar{M}$ is not required. Equation (3.4.37) becomes

$$\frac{d^3\bar{M}}{d\bar{\xi}^3} + \frac{1}{3}\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2} = 0. \tag{3.4.60}$$

Solving equation (3.4.60) subject to (3.4.39) and (3.4.40), using Mathematica, gives for $\bar{M}'$

$$\bar{M}'(\bar{\xi}) = \frac{b}{3^{\frac{1}{3}}}\left(\Gamma\left[\frac{1}{3}\right] - \Gamma\left[\frac{1}{3}, \frac{\bar{\xi}^3}{9}\right]\right). \tag{3.4.61}$$

The constant b is determined from the conserved quantity (3.4.57)

$$b = -\frac{B_4}{3^{\frac{1}{3}}\Gamma\left[\frac{2}{3}\right]}, \tag{3.4.62}$$

and again, a finite valued conserved quantity is calculated. Substituting (3.4.62) into (3.4.61) gives

$$\bar{M}'(\bar{\xi}) = -\frac{B_4\left(\Gamma\left[\frac{1}{3}\right] - \Gamma\left[\frac{1}{3}, \frac{\bar{\xi}^3}{9}\right]\right)}{3^{\frac{2}{3}}\Gamma\left[\frac{2}{3}\right]}. \tag{3.4.63}$$

Figure 3.2 displays the solution for $\bar{M}'(\bar{\xi})$ with $B_4 = 1$.

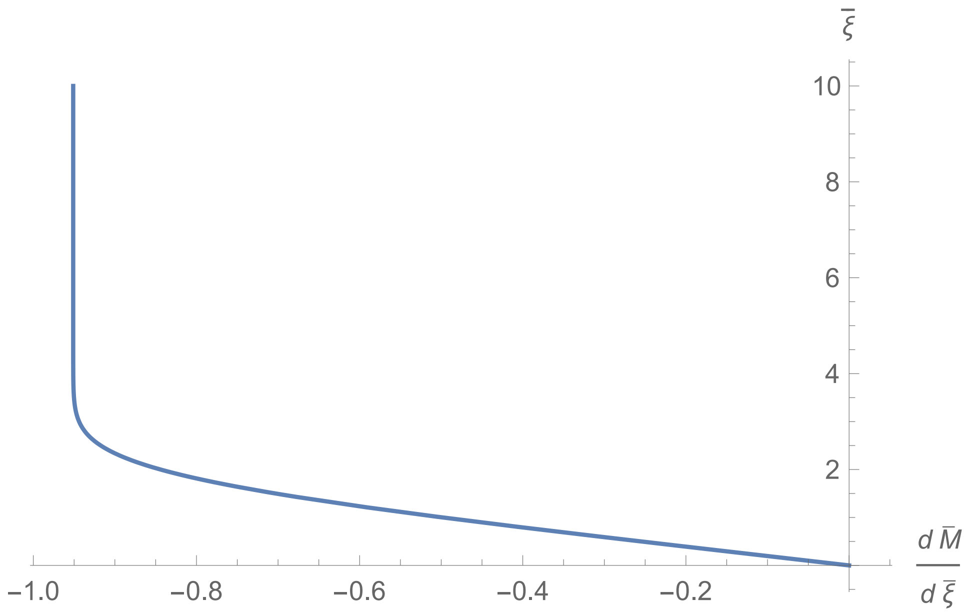

FIGURE 3.2: Solution for case 4

The similarity solutions can now be written as

$$u(x,z) = -\frac{\lambda^{\frac{1}{3}} B_4 \left(\Gamma\left[\frac{1}{3}\right] - \Gamma\left[\frac{1}{3}, \frac{\lambda z^3}{9x}\right]\right)}{3^{\frac{2}{3}}\Gamma\left[\frac{2}{3}\right]} x^{-\frac{1}{3}}, \tag{3.4.64}$$

$$v(x,z) = -\frac{\lambda^{\frac{1}{3}} B_4 z \left(\Gamma\left[\frac{1}{3}\right] - \Gamma\left[\frac{1}{3}, \frac{\lambda z^3}{9x}\right]\right)}{3^{\frac{5}{3}}\Gamma\left[\frac{2}{3}\right]} x^{-\frac{4}{3}}. \tag{3.4.65}$$

3.4.2 Similarity solution for case 3

Since $u_z(x,0) = 0$ for $T^2 \to 0$ as $z \to 0$, this case is not studied as this condition alters the fluid flow problem. The condition $u_z(x,0) = 0$ implies that the wall is frictionless. Contextually this problem may correspond to the behaviour of the fluid flow behind a hump on a solid frictionless wall boundary. However, when linearising the governing equations it is assumed that the wall is not frictionless when λz is used to approximate the boundary layer or Blasius plate flow. Further investigation is required in order to ascertain as to whether this particular problem has any physical usefulness.

3.4.3 Similarity solution for case 2

In order for $T^2 \to 0$ as $z \to 0$, the condition $A(x) = -F(x)$ must hold. For a small hump of length 2ϵ, $F(x) = 0$ for $x > \epsilon$ and therefore $A(x) = 0$. This case describes the physical fluid flow problem known as the near wall-wake. The pressure gradient $p'(x)$ obtained from (2.6.31), is substituted into equation (2.6.27) to give

$$\lambda z \frac{\partial u}{\partial x} + \lambda(v - v(x,\infty)) = \frac{\partial^2 u}{\partial z^2}. \tag{3.4.66}$$

The governing equation (2.6.28) is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0. \tag{3.4.67}$$

Introduce the scaling transformations

$$x = \kappa^a \bar{x}, \qquad z = \kappa^b \bar{z}, \qquad u = \kappa^c \bar{u}, \qquad v = \kappa^f \bar{v}. \tag{3.4.68}$$

Substituting (3.4.68) into equation (3.4.67), for invariance, (3.4.5) must hold. Substituting (3.4.68) into equation (3.4.66) and using (3.4.5) gives

$$\lambda z \frac{\partial u}{\partial x} + \lambda(v - v(x,\infty)) = \kappa^{c-2b-f} \frac{\partial^2 u}{\partial z^2}. \tag{3.4.69}$$

For invariance, the condition

$$c = 2b + f, \tag{3.4.70}$$

must hold. Solving (3.4.5) and (3.4.70) gives

$$a = 3b, \qquad c = 2b + f. \tag{3.4.71}$$

Hence the scaling transformations in (3.4.68) can be written as

$$x = \kappa^{3b}\bar{x}, \qquad z = \kappa^{b}\bar{z}, \qquad u = \kappa^{2b+f}\bar{u}, \qquad v = \kappa^{f}\bar{v}. \tag{3.4.72}$$

From the scaling transformations (3.4.72), the results obtained in (3.4.18) and (3.4.24) are re-derived. From (2.6.31),

$$\begin{aligned} \frac{dp}{dx} &= -\lambda v(x,\infty), \\ &= -\lambda x^{\alpha} G(\infty). \end{aligned} \tag{3.4.73}$$

However equation (3.4.27) gives

$$G(\xi) = \frac{1}{3}\xi\frac{dM}{d\xi} - (1+\alpha)M(\xi), \tag{3.4.74}$$

where

$$M(\xi) = \int_0^{\xi} F(\alpha)d\alpha. \tag{3.4.75}$$

However since $A(x) = 0$, equation (3.4.42) results in the boundary condition

$$M'(\infty) = 0. \tag{3.4.76}$$

Given condition (3.4.76) it is clear that (3.4.74) gives

$$G(\infty) = -(1+\alpha)M(\infty). \tag{3.4.77}$$

Therefore, (3.4.73) becomes

$$\frac{dp}{dx} = \lambda(1+\alpha)x^{\alpha}M(\infty). \tag{3.4.78}$$

Integrating (3.4.78) with respect to x gives

$$p(x) = \lambda M(\infty)x^{1+\alpha}. \tag{3.4.79}$$

Substituting (3.4.18) and (3.4.24) into (3.4.67) gives (3.4.26). Substituting (3.4.18), (3.4.24) into (3.4.66) gives

$$\frac{d^2F}{d\xi^2}+\frac{1}{3}\lambda\xi^2\frac{dF}{d\xi}-\lambda\left(\frac{2}{3}+\alpha\right)\xi F-\lambda G+\lambda G(\infty)=0. \tag{3.4.80}$$

Substituting (3.4.74), (3.4.75) and (3.4.77) into equation (3.4.80) gives

$$\frac{d^3M}{d\xi^3}+\frac{1}{3}\lambda\xi^2\frac{d^2M}{d\xi^2}-\lambda(1+\alpha)\xi\frac{dM}{d\xi}+\lambda(1+\alpha)M(\xi)-\lambda(1+\alpha)M(\infty)=0. \tag{3.4.81}$$

Introducing the transformation (3.4.30) into (3.4.81) gives

$$\frac{d^3\bar{M}}{d\bar{\xi}^3}+\frac{1}{3}\lambda B^3\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2}-\lambda(1+\alpha)B^3\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}+\lambda(1+\alpha)B^3\bar{M}(\bar{\xi})-\lambda(1+\alpha)B^3\bar{M}(\infty)=0. \tag{3.4.82}$$

From the transformation given by (3.4.30), B is chosen as in (3.4.32). The similarity solutions are

$$u(x,z)=\lambda^{\frac{1}{3}}\frac{d\bar{M}}{d\bar{\xi}}x^{\frac{2}{3}+\alpha}, \tag{3.4.83}$$

$$v(x,z)=\left[\frac{1}{3}\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}-(1+\alpha)\bar{M}\right]x^{\alpha}, \tag{3.4.84}$$

$$p(x)=\lambda\bar{M}(\infty)x^{1+\alpha}, \tag{3.4.85}$$

where α is a constant, $\bar{\xi}$ is given by (3.4.36) and $\bar{M}$ satisfies

$$\frac{d^3\bar{M}}{d\bar{\xi}^3}+\frac{1}{3}\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2}-(1+\alpha)\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}+(1+\alpha)\bar{M}=(1+\alpha)\bar{M}(\infty). \tag{3.4.86}$$

With $A(x)=-F(x)$ the conserved quantity calculated from (3.3.6) in terms of the similarity variables is

$$\begin{aligned}\int_0^\infty\frac{3}{2}\lambda z^2\left(u(x,z)\right)dz&=\frac{3}{2}\lambda^{\frac{1}{3}}x^{\frac{5}{3}+\alpha}\int_0^\infty\bar{\xi}^2\bar{M}'(\bar{\xi})d\bar{\xi}\\&=B_2.\end{aligned} \tag{3.4.87}$$

Since B_2 is a constant independent of x, $\alpha=-\frac{5}{3}$. The similarity solutions become

$$u(x,z)=\lambda^{\frac{1}{3}}\frac{d\bar{M}}{d\bar{\xi}}x^{-1}, \tag{3.4.88}$$

$$v(x,z)=\left[\frac{1}{3}\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}+\frac{2}{3}\bar{M}\right]x^{-\frac{5}{3}}, \tag{3.4.89}$$

$$p(x)=\lambda\bar{M}(\infty)x^{-\frac{2}{3}}, \tag{3.4.90}$$

where from equation (3.4.86), $\bar{M}$ satisfies

$$\frac{d^3\bar{M}}{d\bar{\xi}^3} + \frac{1}{3}\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2} + \frac{2}{3}\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}} - \frac{2}{3}\bar{M} = -\frac{2}{3}\bar{M}(\infty). \tag{3.4.91}$$

Solving equation (3.4.91) subject to (3.4.39), (3.4.41) and (3.4.76) (noting that (3.4.40) is redundant), using Mathematica, gives for $\bar{M}'(\bar{\xi})$:

$$\bar{M}'(\bar{\xi}) = \frac{c\ \left(\frac{2}{3}\right)^{\frac{1}{3}} e^{-\frac{\bar{\xi}^3}{18}}\bar{\xi}^{\frac{3}{2}} K\left[\frac{1}{6}, \frac{\bar{\xi}^3}{18}\right]}{\Gamma\left[\frac{5}{6}\right]}, \tag{3.4.92}$$

where $K[\nu, \eta]$ is the modified Bessel function of the second kind which satisfies the differential equation

$$\eta^2 y'' + \eta y' - (\eta^2 + \nu^2)y = 0, \tag{3.4.93}$$

and is given by

$$K[\nu, \eta] = \frac{\Gamma[\nu + \frac{1}{2}](2\eta)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos(t)}{(t^2 + \eta^2)^{\nu+\frac{1}{2}}} dt, \tag{3.4.94}$$

where $\nu, \eta \in \mathbb{R}$. The constant c is solved by substituting (3.4.92) into the conserved quantity. The solution is

$$c = \frac{\Gamma\left[\frac{5}{6}\right] B_2}{2^{\frac{4}{3}} 3^{\frac{1}{6}} \pi^{\frac{3}{2}} \lambda^{\frac{1}{3}}}. \tag{3.4.95}$$

Substituting (3.4.95) into (3.4.92) gives

$$\bar{M}'(\bar{\xi}) = \frac{B_2\ e^{-\frac{\bar{\xi}^3}{18}}\bar{\xi}^{\frac{3}{2}} K\left[\frac{1}{6}, \frac{\bar{\xi}^3}{18}\right]}{2\ 3^{\frac{1}{2}} \pi^{\frac{3}{2}} \lambda^{\frac{1}{3}}}. \tag{3.4.96}$$

This solution is investigated further in Chapter 4 as it is the solution to the near wake problem.

3.5 Conclusions

The conserved quantities in terms of both the velocity components and the stream function have been derived. The conserved vectors, derived using the multiplier method, are chosen carefully to ensure the conserved quantities relating to them are finite valued. The three conserved quantities obtained in terms of the stream function each correspond to one of the conserved quantities obtained in terms of the velocity components. However, when using the stream function the continuity equation is identically satisfied whilst when using the velocity components, an extra conserved quantity arises.

An ODE can be integrated at least once if the related PDE has a conserved vector associated with it. The integration of equations (3.4.46), (3.4.60) and (3.4.91) is an application of the double reduction theorem [19]. Similarity solutions that satisfied the relevant governing equations and boundary conditions are obtained for each case.

Chapter 4

The near wake

4.1 Hunt's approach

Hunt [1] was the first researcher to tackle the problem of a laminar flow past a hump situated on a solid wall boundary. In his initial study, he considered small humps. At the time of the study, it was believed that Hunt had solved for the laminar far wall-wake. Later studies [2, 15] used triple deck theory to solve for the wall-wake flow. It was shown that Hunt's approach solved for the near wake on the triple deck scale and Smith's [2] approach solved for the far wake on the triple deck scale.

This chapter is devoted to Hunt's initial approach. A discussion on how Hunt's research fits in with the triple deck approach is also provided.

4.2 Hunt's derivation of the wall-wake

Hunt [1] studied small humps and neglected the boundary layer displacement effect due to the presence of the hump and therefore only saw need for two decks as opposed to the triple deck structure developed by Stewartson [3, 5] and Messiter [4] which was implemented by Smith [2]. The flow variables that Hunt [1] used are defined as follows: The undisturbed boundary layer flow variables upstream of the hump are given by U and V for the x and y velocity components respectively, and P denotes the fluid pressure. In the wake behind the hump, u_1 and v_1 denote the x and y velocity components respectively and p_1 denotes the fluid pressure. The wake flow variables can be expressed in terms of perturbation flow variables u, v and p as follows:

$$u_1 = u + U, \qquad v_1 = v + V, \qquad p_1 = P + p. \tag{4.2.1}$$

Small humps are considered and it is assumed that the wake occupies the lower part of the boundary layer close to the wall. It can be shown that, for small y,

$$U = \alpha y, \tag{4.2.2}$$

where α is the shear rate.

Hunt [1] used two different sets of dimensionless variables. For the lower deck

$$x^* = \frac{x}{\ell}, \qquad y^* = Re^{\frac{1}{3}}\frac{y}{\ell},$$
$$u^* = Re^{\frac{1}{3}}\frac{u}{\epsilon\alpha\ell}, \qquad v^* = Re^{\frac{2}{3}}\frac{v}{\epsilon\alpha\ell}, \qquad p^* = Re^{\frac{2}{3}}\frac{p}{\epsilon\rho\alpha^2\ell^2}, \tag{4.2.3}$$

and for the middle deck

$$x^{**} = \frac{x}{\ell}, \qquad y^{**} = \frac{y}{\ell},$$
$$u^{**} = Re^{\frac{2}{3}}\frac{u}{\epsilon\alpha\ell}, \qquad v^{**} = Re^{\frac{2}{3}}\frac{v}{\epsilon\alpha\ell}, \qquad p^{**} = Re^{\frac{2}{3}}\frac{p}{\epsilon\rho\alpha^2\ell^2}, \tag{4.2.4}$$

where the Reynolds number Re is given by

$$Re = \frac{\alpha\ell^2}{\nu}. \tag{4.2.5}$$

Here $\epsilon \ll 1$, ℓ is the distance in which the wake decays and ν is the kinematic viscosity.

The dimensionless variables in the middle deck were chosen so that the y-component of the velocity is continuous across the two decks. Neglecting terms of $O(Re^{-\frac{2}{3}})$ and $O(\epsilon)$, the x and y components of the Navier-Stokes equation and the continuity equation in the lower deck become

$$y^*\frac{\partial u^*}{\partial x^*} + v^* = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{4.2.6}$$

$$0 = -\frac{\partial p^*}{\partial y^*}, \tag{4.2.7}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0. \tag{4.2.8}$$

The no-slip and no-cavity conditions are given by

$$u^*(x, 0) = 0, \qquad v^*(x, 0) = 0. \tag{4.2.9}$$

As $x \to \infty$ the flow must match with the Blasius flow resulting in the conditions

$$u^*(\infty, y) = 0, \qquad v^*(\infty, y) = 0. \tag{4.2.10}$$

As $y \to \infty$, $u^* \to 0$ slowly, which gives

$$u^*(x, \infty) = 0, \tag{4.2.11}$$

$$\frac{\partial u^*}{\partial y^*}(x, \infty) = 0. \tag{4.2.12}$$

In the middle deck, the x and y components of the Navier-Stokes equation and the continuity equation become

$$y^{**}\frac{\partial u^{**}}{\partial x^{**}} + v^{**} = -\frac{\partial p^{**}}{\partial x^{**}}, \tag{4.2.13}$$

$$y^{**}\frac{\partial v^{**}}{\partial x^{**}} = -\frac{\partial p^{**}}{\partial y^{**}}, \tag{4.2.14}$$

$$\frac{\partial u^{**}}{\partial x^{**}} + \frac{\partial v^{**}}{\partial y^{**}} = 0, \tag{4.2.15}$$

which must be solved subject to the boundary conditions

$$v^{**}(x, 0) = v^*_\infty(x), \tag{4.2.16}$$

$$u^{**}, v^{**} \to 0 \qquad \text{as } (x^{**2} + y^{**2}) \to \infty. \tag{4.2.17}$$

It is clear that upon comparison of the middle and lower decks, the viscous flow is contained within the lower deck while the middle deck is largely inviscid.

Equation (4.2.7) gives

$$p^* = p^*(x). \tag{4.2.18}$$

As $y^{**} \to 0$, equation (4.2.13) subject to the boundary condition (4.2.16) becomes

$$v^*_\infty(x) = -\frac{\partial p^{**}}{\partial x^{**}}(x, 0). \tag{4.2.19}$$

However, p is continuous across the middle and lower deck, therefore

$$\frac{\partial p^{**}}{\partial x^{**}}(x, 0) = \frac{dp^*}{dx^*}. \tag{4.2.20}$$

From equations (4.2.19) and (4.2.20) it is clear that

$$\frac{dp^*}{dx^*} = -v^*_\infty(x). \tag{4.2.21}$$

Upon substitution of (4.2.21) into equation (4.2.6) the following equation is obtained:

$$y^*\frac{\partial u^*}{\partial x^*} + v^* - v^*_\infty(x) = \frac{\partial^2 u^*}{\partial y^{*2}}. \tag{4.2.22}$$

Differentiating equation (4.2.22) with respect to y^* and using the continuity equation results in

$$\frac{\partial^3 u^*}{\partial y^{*3}} - y^*\frac{\partial^2 u^*}{\partial x^*\partial y^*} = 0. \tag{4.2.23}$$

Multiplying equation (4.2.23) by y^{*2} and integrating with respect to y^* from 0 to ∞ gives

$$\int_0^\infty y^{*2}\frac{\partial^3 u^*}{\partial y^{*3}}dy^* - \int_0^\infty y^{*3}\frac{\partial^2 u^*}{\partial x^*\partial y^*}dy^* = 0. \tag{4.2.24}$$

Using integration by parts, subject to the boundary conditions (4.2.9), (4.2.11) and (4.2.12), (4.2.24) reduces to

$$\int_0^\infty \frac{\partial u^*}{\partial y^*}dy^* + \frac{3}{2}\frac{d}{dx}\int_0^\infty y^{*2}u^*dy^* = 0. \tag{4.2.25}$$

Since the first term subject to the boundary conditions (4.2.9) and (4.2.11) is zero, (4.2.25) simplifies to

$$\frac{3}{2}\frac{d}{dx}\int_0^\infty y^{*2}u^*dy^* = 0, \tag{4.2.26}$$

which results in the conserved quantity

$$\frac{3}{2}\int_0^\infty y^{*2}u^*(x^*, y^*)dy^* = I^*, \tag{4.2.27}$$

where I^* is a constant. Hunt [1] instead integrated (4.2.22) with respect to y^* from y^* to ∞, and then again with respect to y^* from 0 to ∞. However, both approaches result in equation (4.2.27).

Introducing the scaling transformations

$$x^* = \lambda^a x, \qquad y^* = \lambda^b y, \qquad u^* = \lambda^c u, \tag{4.2.28}$$

into equation (4.2.23) gives

$$\lambda^{c-3b}\frac{\partial^3 u}{\partial y^3} - \lambda^{c-a}y\frac{\partial^2 u}{\partial x\partial y} = 0. \tag{4.2.29}$$

For invariance it is clear from equation (4.2.29) that $a = 3b$. The scaling transformations in (4.2.28) become

$$x^* = \lambda^{3b} x, \qquad y^* = \lambda^b y, \qquad u^* = \lambda^c u. \tag{4.2.30}$$

Let

$$u^* = f(x^*, y^*),$$
$$u = f(x, y).$$

This gives

$$\lambda^{-c} u^* = f(\lambda^{-3b} x^*, \lambda^{-b} y^*). \tag{4.2.31}$$

Differentiating (4.2.31) with respect to λ leads to

$$c\lambda^{-c-1} f(x^*, y^*) = 3be^{-3b-1} x^* \frac{\partial f}{\partial x} + b\lambda^{-b-1} y^* \frac{\partial f}{\partial y},$$

and setting $\lambda = 1$ gives $x^* = x$ and $y^* = y$, which results in

$$cf(x^*, y^*) = 3bx^* \frac{\partial f}{\partial x^*} + by^* \frac{\partial f}{\partial y^*}. \tag{4.2.32}$$

The characteristic curves obtained from (4.2.32) are

$$\frac{dx^*}{3bx^*} = \frac{dy^*}{by^*} = \frac{df}{cf}. \tag{4.2.33}$$

Solving the first pair of equations in (4.2.33) gives

$$\eta = \frac{y^*}{x^{*\frac{1}{3}}}. \tag{4.2.34}$$

Hunt [1] defined η as y^{*3}/x^*, which results in the same solution. However with the choice of η in (4.2.34) it is simpler to compare Hunt's approach with the triple deck approach.

Solving for the first and last pair of terms in (4.2.33) results in

$$u^* = x^{*k} F(\eta), \tag{4.2.35}$$

where

$$k = \frac{c}{3b}. \tag{4.2.36}$$

Equation (4.2.23), written in terms of η and $F(\eta)$ is given by

$$\frac{d^3F}{d\eta^3} + \frac{1}{3}\eta^2\frac{d^2F}{d\eta^2} + \eta\left(\frac{1}{3} - k\right)\frac{dF}{d\eta} = 0. \tag{4.2.37}$$

In order to solve for k the conserved quantity in equation (4.2.27) is written in terms of η and $F(\eta)$:

$$\frac{3}{2}x^{*k+1}\int_0^\infty \eta^2 F(\eta)d\eta = I^*, \tag{4.2.38}$$

which differs from Hunt's [1] result due to the choice of η. However from equation (4.2.38), $k = -1$ which is equivalent to Hunt's value. Equation (4.2.37) now becomes

$$\frac{d^3F}{d\eta^3} + \frac{1}{3}\eta^2\frac{d^2F}{d\eta^2} + \frac{4}{3}\eta\frac{dF}{d\eta} = 0,$$

which is again different from Hunt's result due to the choice of η in (4.2.34), but yields the same results. This is multiplied by $3\eta^2$ to give

$$3\eta^2\frac{d^3F}{d\eta^3} + \eta^4\frac{d^2F}{d\eta^2} + 4\eta^3\frac{dF}{d\eta} = 0. \tag{4.2.39}$$

The aim is to write equation (4.2.39) in a form which is directly integrable in terms of η. From (4.2.39) the last two terms can be combined to give

$$3\eta^2\frac{d^3F}{d\eta^3} + \frac{d}{d\eta}\left(\eta^4\frac{dF}{d\eta}\right) = 0.$$

Using the chain rule on the first term gives

$$3\left(\frac{d}{d\eta}\left(\eta^2\frac{d^2F}{d\eta^2}\right) - 2\eta\frac{d^2F}{d\eta^2}\right) + \frac{d}{d\eta}\left(\eta^4\frac{dF}{d\eta}\right) = 0. \tag{4.2.40}$$

Equation (4.2.40) is not yet in the form required. By again making use of the chain rule (4.2.40) becomes

$$3\left(\frac{d}{d\eta}\left(\eta^2\frac{d^2F}{d\eta^2}\right) - 2\left(\frac{d}{d\eta}\left(\eta\frac{dF}{d\eta}\right) - \frac{dF}{d\eta}\right)\right) + \frac{d}{d\eta}\left(\eta^4\frac{dF}{d\eta}\right) = 0,$$

which can be simplified to give

$$3\frac{d}{d\eta}\left(\eta^2\frac{d^2F}{d\eta^2}\right) - 6\frac{d}{d\eta}\left(\eta\frac{dF}{d\eta}\right) + 6\frac{dF}{d\eta} + \frac{d}{d\eta}\left(\eta^4\frac{dF}{d\eta}\right) = 0. \tag{4.2.41}$$

Integrating (4.2.41) with respect to η gives

$$3\eta^2\frac{d^2F}{d\eta^2} - 6\eta\frac{dF}{d\eta} + 6F(\eta) + \eta^4\frac{dF}{d\eta} = c. \tag{4.2.42}$$

The boundary conditions (4.2.9)-(4.2.12) in terms of η and $F(\eta)$ give

$$F(0) = 0, \tag{4.2.43}$$

$$F(\infty) = 0, \tag{4.2.44}$$

$$F'(\infty) = 0. \tag{4.2.45}$$

However as $\eta \to 0$ the boundary condition in (4.2.43) is applied and $c = 0$. Therefore,

$$3\eta^2\frac{d^2F}{d\eta^2} - 6\eta\frac{dF}{d\eta} + 6F(\eta) + \eta^4\frac{dF}{d\eta} = 0. \tag{4.2.46}$$

The conserved quantity is given by

$$\int_0^\infty \eta^2 F(\eta)d\eta = \frac{2}{3}I^*. \tag{4.2.47}$$

Equation (4.2.46) is solved subject to the boundary conditions (4.2.43)-(4.2.45) and the conserved quantity (4.2.47) using Mathematica, which results in

$$F(\eta) = \frac{Ae^{-\frac{\eta^3}{18}}\eta^{\frac{3}{2}}K[\frac{1}{6},\frac{\eta^3}{18}]\;\Gamma\left[\frac{5}{6}\right]}{3\;2^{\frac{1}{3}}\pi}, \tag{4.2.48}$$

where $K[n, z]$ is the modified Bessel function of the second kind. In order to solve for A in (4.2.48), the conserved quantity (4.2.47) is used. The solution for A is

$$A = \frac{3^{\frac{1}{2}}\;I^*}{2^{\frac{2}{3}}\pi^{\frac{1}{2}}\;\Gamma\left[\frac{5}{6}\right]}. \tag{4.2.49}$$

Substituting (4.2.49) into (4.2.48) results in

$$F(\eta) = \frac{I^*\;e^{-\frac{\eta^3}{18}}\eta^{\frac{3}{2}}K[\frac{1}{6},\frac{\eta^3}{18}]}{2\;3^{\frac{1}{2}}\pi^{\frac{3}{2}}}. \tag{4.2.50}$$

From the result in (4.2.50) the solution to u^* is

$$u^*(x^*, y^*) = \frac{I^*e^{-\frac{y^{*3}}{18x^*}}y^{*\frac{3}{2}}K[\frac{1}{6},\frac{y^{*3}}{18x^*}]}{2\;3^{\frac{1}{2}}\pi^{\frac{3}{2}}x^{*\frac{3}{2}}}. \tag{4.2.51}$$

Figure 4.1 displays the solution (4.2.51) at $x^* = 1$ with $I^* = 1$ and is useful in identifying the behaviour of the wake near to the hump.

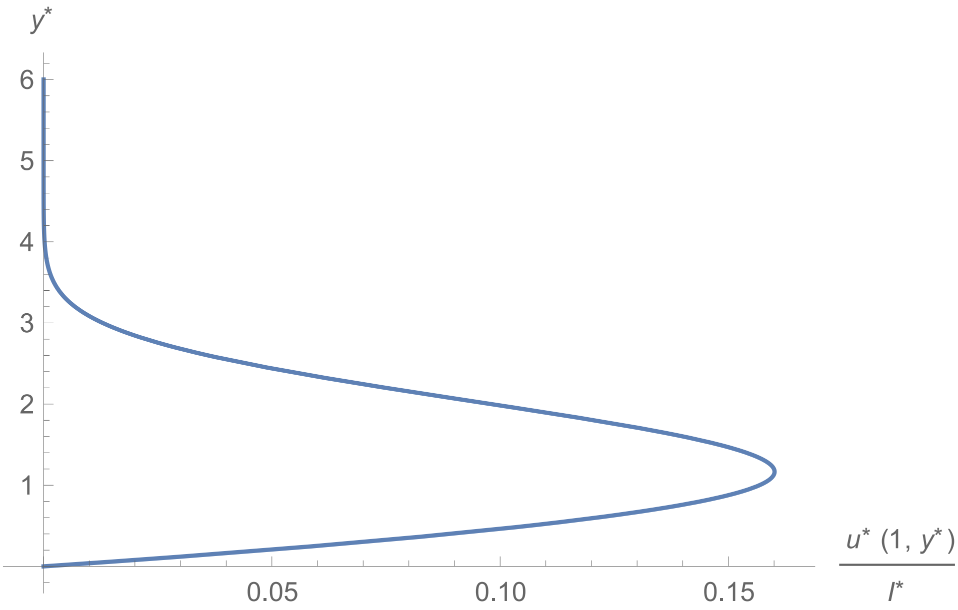

FIGURE 4.1: Near wake solution obtained by Hunt

In Figure 4.1 the presence of the hump effects the flow near to the wall. The x-component of the wake velocity is at a maximum at $y^* = 1.16615$. The width of the wake is approximately 3.5 with the velocity deficit tending to zero for y^* greater than 3.5.

The problem with Hunt's approach is that the velocity u^{**} cannot be specified from u^* because $u^* \to 0$ as $y^* \to \infty$. Therefore the x-component of the velocity is not continuous across the two decks. This problem is resolved in the triple deck approach.

4.3 Triple deck approach

The displacement effect $A(x)$ does not play a significant role in the developement of the near wake. The near wake has an effective width of the same order as the height of the obstruction, which is much less than the height of the lower deck. Hence to first order,

$$A(x) = 0. \tag{4.3.1}$$

In this case there is no need for an upper deck. It is then required that the result for p needs to be calculated from (3.4.90).

From Section 3.4.3 the solution for the near wake problem is given in (3.4.96) as

$$\bar{M}'(\xi) = \frac{B_2\, e^{-\frac{\bar{\xi}^3}{18}} \bar{\xi}^{\frac{3}{2}} K\left[\frac{1}{6}, \frac{\bar{\xi}^3}{18}\right]}{2\ 3^{\frac{1}{2}} \pi^{\frac{3}{2}} \lambda^{\frac{1}{3}}}. \tag{4.3.2}$$

Therefore the solution for u given in (3.4.88) becomes

$$u(x,z) = \frac{B_2\, \lambda^{\frac{1}{2}} z^{\frac{3}{2}} e^{-\frac{\lambda z^3}{18x}} K\left[\frac{1}{6}, \frac{\lambda z^3}{18x}\right]}{2\ 3^{\frac{1}{2}} \pi^{\frac{3}{2}} x^{\frac{3}{2}}}. \tag{4.3.3}$$

Figure 4.2 displays the solution with $x = 1$, $\lambda = 1$ and $B_2 = 1$.

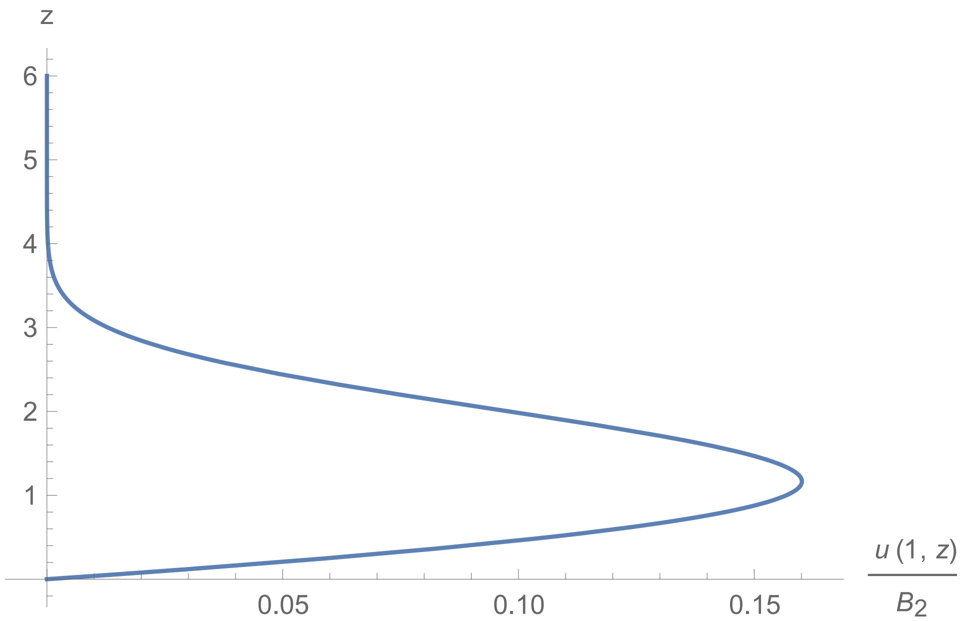

FIGURE 4.2: Triple deck theory solution for the near wake

4.4 Conclusions

The solution displayed in Figure 4.2 matches the solution obtained by Hunt [1] given in Figure 4.1. The near wake is situated well within the lower deck. Due to this, the development of the wake is not dependent on the boundary layer displacement effect. The near wake has a conserved quantity given in (3.4.87). There was no need for an upper deck.

It is clear from this comparison of Hunt's [1] approach to the triple deck approach that Hunt solved for the near wake on the triple deck scale. Smith [2, 15] gives the solution for the far wake, which is discussed in Chapter 5.

Chapter 5

The far wake

5.1 Smith's approach

Smith [2] considered the problem of the wall wake proposed by Hunt[1] and used triple deck theory to solve for the far wake flow. As mentioned previously, further work on wall-wake flows was conducted [15]. It was shown that Smith's solution corresponds to the far wake flow on the triple deck scale. In this chapter, the work done by Smith on the far wall wake is discussed.

5.2 Smith's derivation of the far wall-wake

The similarity solution for the far wake is considered. The governing equations (2.6.27) and (2.6.28) are repeated here for convenience:

$$\lambda z \frac{\partial u}{\partial x} + \lambda v = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial z^2}, \tag{5.2.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0. \tag{5.2.2}$$

Introduce the scaling transformations:

$$x = \kappa^a \bar{x}, \qquad z = \kappa^b \bar{z}, \tag{5.2.3}$$
$$u = \kappa^c \bar{u}, \qquad v = \kappa^f \bar{v}, \qquad p = \kappa^g \bar{p}.$$

Similarity solutions were calculated in Section 3.4.1 for the case $p'(x) = 0$. It is easily shown that the scaling transformations in (5.2.3) must have the properties

$$a = 3b, \qquad c = 2b + f, \qquad g = 3b + f, \tag{5.2.4}$$

for equations (5.2.1) and (5.2.2) to be invariant. Hence the scaling transformations in (5.2.3) can be written as

$$x = \kappa^{3b}\bar{x}, \qquad z = \kappa^{b}\bar{z}, \tag{5.2.5}$$
$$u = \kappa^{2b+f}\bar{u}, \qquad v = \kappa^{f}\bar{v}. \qquad p = \kappa^{3b+f}\bar{p}.$$

The results from Section 3.4.1 can be used to show that

$$u(x,z) = x^{\frac{2}{3}+\alpha}F(\xi) = x^{\frac{2}{3}+\alpha}\frac{dM}{d\xi}, \tag{5.2.6}$$

$$v(x,z) = x^{\alpha}G(\xi) = x^{\alpha}\left[\frac{1}{3}\xi\frac{dM}{d\xi} - (1+\alpha)M(\xi)\right]. \tag{5.2.7}$$

The solution for p now needs to be calculated. Let

$$p = K(x),$$
$$\bar{p} = K(\bar{x}).$$

This results in

$$\kappa^{-3b-f}K(x) = K(\kappa^{-3b}x). \tag{5.2.8}$$

Differentiating (5.2.8) with respect to κ gives

$$(3b+f)\kappa^{3b+f-1}K(x) = 3b\kappa^{-3b-1}\frac{dK}{d\bar{x}},$$

and letting $\kappa = 1$ gives

$$(3b+f)K(x) = 3b\frac{dK}{dx}. \tag{5.2.9}$$

Simplifying equation (5.2.9) gives

$$(1+\alpha)\frac{dx}{x} = \frac{dK}{K}, \tag{5.2.10}$$

where

$$\alpha = \frac{f}{3b}. \tag{5.2.11}$$

From the equation in (5.2.10)

$$p(x) = x^{1+\alpha}H_0. \tag{5.2.12}$$

Substituting (5.2.6), (5.2.7) and (5.2.12) into equation (5.2.1) gives

$$\frac{d^3M}{d\xi^3}+\frac{1}{3}\lambda\xi^2\frac{d^2M}{d\xi^2}-\lambda(1+\alpha)\xi\frac{dM}{d\xi}+\lambda(1+\alpha)M(\xi)-(1+\alpha)H_0=0. \tag{5.2.13}$$

Introducing the transformations (3.4.30) into equation (5.2.13) gives

$$\frac{d^3\bar{M}}{d\bar{\xi}^3}+\frac{1}{3}\lambda B^3\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2}-\lambda(1+\alpha)B^3\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}+\lambda(1+\alpha)B^3\bar{M}(\bar{\xi})-(1+\alpha)\frac{B^3}{A}H_0=0. \tag{5.2.14}$$

Hence

$$B=\left(\frac{1}{\lambda}\right)^{\frac{1}{3}}, \qquad A=\frac{H_0}{\lambda}. \tag{5.2.15}$$

The similarity solutions are

$$u(x,z)=H_0\lambda^{-\frac{2}{3}}\frac{d\bar{M}}{d\bar{\xi}}x^{\frac{2}{3}+\alpha}, \tag{5.2.16}$$

$$v(x,z)=H_0\lambda^{-1}\left[\frac{1}{3}\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}-(1+\alpha)\bar{M}\right]x^{\alpha}, \tag{5.2.17}$$

$$p(x)=H_0x^{1+\alpha}, \tag{5.2.18}$$

where H_0 and α are constants, $\bar{\xi}$ is defined as in (3.4.36) and $\bar{M}$ satisfies

$$\frac{d^3\bar{M}}{d\bar{\xi}^3}+\frac{1}{3}\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2}-(1+\alpha)\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}}+(1+\alpha)\bar{M}=(1+\alpha). \tag{5.2.19}$$

In order to obtain the result for the pressure and solve for the constant α, the upper deck needs to be considered. Linearising the upper deck solution (2.5.29) based on the expansions (2.6.25) and (2.6.26) gives

$$p(x)=\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{A'(x_1)}{x-x_1}dx_1. \tag{5.2.20}$$

Assuming that in the far wake region the hump can be described as a point disturbance,

$$A(x)=-F_0\delta(x), \tag{5.2.21}$$

which allows (5.2.20) to be solved. Substituting (5.2.21) into (5.2.20) gives

$$p(x)=-\frac{F_0}{\pi}\fint_{-\infty}^{\infty}\frac{\delta'(x_1)}{x-x_1}dx_1. \tag{5.2.22}$$

Integration by parts leads to

$$\begin{aligned} p(x) &= -\frac{F_0}{\pi}\left(\left[\frac{\delta(x_1)}{x-x_1}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\delta(x_1)}{(x-x_1)^2}dx_1\right), \\ &= \frac{F_0}{\pi}\int_{-\infty}^{\infty}\frac{\delta(x_1)}{(x-x_1)^2}dx_1, \\ &= \frac{F_0}{\pi}\frac{1}{x^2}. \end{aligned} \tag{5.2.23}$$

The solution for the pressure given by (5.2.23) must match the similarity solution for the pressure in the linearised lower deck given by (5.2.18). From (5.2.18) it is clear that

$$\alpha = -3, \qquad H_0 = \frac{F_0}{\pi}. \tag{5.2.24}$$

Therefore, the similarity solutions in the lower deck given by (5.2.16)-(5.2.18) are

$$u(x,z) = \frac{F_0\lambda^{-\frac{2}{3}}}{\pi}\frac{d\bar{M}}{d\bar{\xi}}x^{-\frac{7}{3}}, \tag{5.2.25}$$

$$v(x,z) = \frac{F_0\lambda^{-1}}{\pi}\left[\frac{1}{3}\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}} + 2\bar{M}\right]x^{-3}, \tag{5.2.26}$$

$$p(x) = \frac{F_0}{\pi}x^{-2}. \tag{5.2.27}$$

Using (5.2.24), equation (5.2.19) becomes

$$\frac{d^3\bar{M}}{d\bar{\xi}^3} + \frac{1}{3}\bar{\xi}^2\frac{d^2\bar{M}}{d\bar{\xi}^2} + 2\bar{\xi}\frac{d\bar{M}}{d\bar{\xi}} - 2\bar{M} = -2. \tag{5.2.28}$$

The boundary conditions

$$\bar{M}'(0) = 0, \tag{5.2.29}$$

$$\bar{M}''(\infty) = 0, \tag{5.2.30}$$

$$\bar{M}(0) = 0, \tag{5.2.31}$$

derived in Section 3.4.1 still hold for the far wake problem. However, the condition $\bar{M}''(\infty)$ is identically satisfied and a solution for $\bar{M}$ cannot be obtained from the three boundary conditions given in (5.2.29)-(5.2.31). In this work, the following approach is adopted: Consider the boundary condition

$$u(x,z) \to \lambda A(x), \qquad z \to \infty. \tag{5.2.32}$$

For $A(x) = -F_0\delta(x)$, condition (5.2.32) becomes

$$u(x,\infty) = 0, \qquad x > 0. \tag{5.2.33}$$

Because the far wake flow holds for large x, the boundary condition (5.2.33) can be used. In terms of the similarity variables this is

$$\bar{M}'(\infty) = 0, \tag{5.2.34}$$

Smith [2] did not consider this approach. A detailed analysis of his approach is given in [2]. The solution for $\bar{M}'(\bar{\xi})$ derived using Mathematica is

$$\begin{aligned}\bar{M}'(\bar{\xi}) = &-\frac{\Gamma\left[\frac{4}{3}\right]\left(4\;3^{\frac{2}{3}}\;\bar{\xi}\;\Gamma\left[\frac{1}{3},\frac{\bar{\xi}^3}{9}\right]+e^{-\frac{\bar{\xi}^3}{9}}\left(\bar{\xi}^5-\left(-\bar{\xi}^3\right)^{\frac{5}{3}}-15\left(-\bar{\xi}^3\right)^{\frac{2}{3}}-18\bar{\xi}^2\right)\right)}{9\;3^{\frac{1}{3}}\;\bar{\xi}}\\ &-\frac{2\left(9e^{-\frac{\bar{\xi}^3}{9}}\left(\bar{\xi}^3-15\right)E\left[\frac{5}{3},-\frac{\bar{\xi}^3}{9}\right]-8\;3^{\frac{1}{3}}\;\bar{\xi}\;\Gamma\left[-\frac{2}{3}\right]^2+81\right)}{243\;\bar{\xi}}\\ &-\frac{{}_2F_2\left[\left\{-\frac{1}{3},2\right\};\left\{\frac{1}{3},\frac{2}{3}\right\};-\frac{\bar{\xi}^3}{9}\right]}{\bar{\xi}},\end{aligned} \tag{5.2.35}$$

where $E\left[n,z\right]$ is the exponential integral function defined as

$$E\left[n,z\right] = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt, \tag{5.2.36}$$

where $n, z \in \mathbb{R}$ and ${}_pF_q[a;b;c]$ is the generalized hypergeometric function and has the series expansion

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{c^k}{k!}, \tag{5.2.37}$$

where $(a)_k$ is the Pochhammer symbol given by

$$(a)_k \equiv \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{5.2.38}$$

The results for the far wake for $\bar{M}'(\bar{\xi})$ are displayed in Figure 5.1

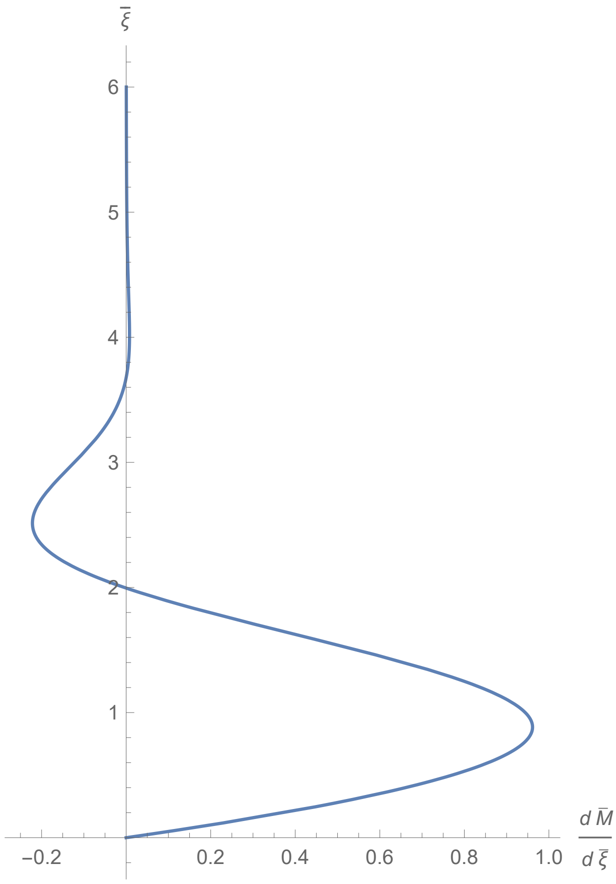

FIGURE 5.1: Triple deck theory solution for the far wake

In Figure 5.1 the region close to the wall where the wake velocity is a maximum is at approximately $\bar{\xi} = 0.883758$. In the far wake there is a region of back flow with a maximum at approximately $\bar{\xi} = 2.51393$. The velocity deficit approaches zero for $\bar{\xi}$ greater than approximately 3.5.

5.3 Concerns

The assumption taken for $A(x)$ in (5.2.21) is obtained by considering the hump as a point disturbance and in the far wake this gives $F(x) = F_0\delta(x)$. If this is the case then $A(x) = -F(x)$ and from Section 3.3.1.2 a conserved quantity does exist. Further work would need to be conducted in order to resolve this apparent contradiction.

5.4 Conclusions

Smith's approach solved for the far wake on the triple deck scale where the boundary layer displacement effect is of central importance. Because the boundary layer displacement effect is negligible for the near wake, it can be concluded that Hunt's approach is applicable to near wall-wake flows. A clearer understanding of the boundary layer displacement effect may allow for the relationship between the near and far wake to be found and a possible intermediate wake could be identified.

Chapter 6

Conclusions

A complete and accurate description of the laminar flow behind a hump on an otherwise smooth boundary, can be achieved by using triple deck theory. In Chapter 2, triple deck theory was used to derive the governing equations and the boundary conditions. The governing equations for small humps were obtained. The governing equations were shown to be the same for both near and far wakes. However, the boundary conditions were different and a conserved quantity was required to complete the solution for the near wall wake. It was shown that for the near wake, the boundary layer displacement effect has no influence on the near wake solution, whilst for the far wake it cannot be neglected.

In Chapter 3 an investigation into the conserved quantities pertaining to the governing equations describing a wall-wake flow has been undertaken. The multiplier method was used to derive the conserved vectors when the governing equations were expressed in terms of both the velocity components and the stream function. The conservation laws corresponding to each conserved vector were then integrated across the wake in order to obtain the conserved quantities. Because, in general, the boundary conditions were non-homogeneous, careful consideration when choosing the conserved vectors was required in order for the conserved quantities to be finite valued. To achieve this, additional restrictions needed to be imposed.

In terms of the velocity components, four conserved quantities were found. One of the conserved quantities corresponded to the moment of momentum deficit which was required to complete the solution for the near wall-wake flow for very small humps. Two of the conserved quantities could only be obtained by assuming that the fluid pressure gradient is zero. These conserved quantities corresponded to the conservation of mass and the second moment of the axial momentum deficit. The last conserved quantity, namely the conservation of drag, existed provided that there is zero shear at the solid wall. The condition of zero shear at the solid wall boundary completely alters the fluid flow problem which relies on the interpretation that a boundary layer is perturbed and thus this case was not investigated further.

In terms of the stream function, only three conserved quantities were found. Each of these corresponded to one of the conserved quantities that were obtained in terms of the velocity components. When using the stream function, the continuity equation is identically satisfied and one of the conserved quantities is not accounted for and can only be obtained when using the velocity components.

Similarity solutions that satisfied the relevant governing equations and boundary conditions and that admitted a finite valued conserved quantity were found for each case that had possibly plausible physical significance. Although an in-depth physical interpretation of the results was not conducted, it was proved that the solutions do in fact exist. It was also shown that for the two cases which had a zero pressure gradient, the unknown non-homogeneous boundary condition could be evaluated.

In Chapter 4, the near wake solution has been investigated in detail. It was shown that the solutions obtained from the triple deck approach [2] and from Hunt's approach [1] were equivalent. Since the boundary layer displacement effect is negligible for near wall-wake flows, only two decks were required. The governing equations for the lower deck were identical for both approaches. However, the governing equations and boundary conditions for the middle deck were different. The dimensionless variables used by Hunt [1] where chosen to ensure the y-component of the velocity is continuous across the middle and lower deck. The problem with Hunt's approach is that the x-component of the velocity in the middle deck cannot be specified from the x-component of the velocity in the lower deck. This problem does not occur using the triple deck approach and consistency between the decks is maintained.

The approach taken by Smith [2] is investigated in Chapter 5 and the solution for the far wall-wake is obtained. Here the hump is assumed to be a point disturbance which allows for the boundary layer displacement effect to be defined by a delta function. The pressure in the upper deck is solved and compared to the similarity solution in the lower deck, allowing for a solution to be obtained. From Smith [2] a conserved quantity was not required and does not exist. However by assuming that the hump is a point disturbance the result $A(x) = -F(x)$ is obtained which was the condition from Section 3.3.1.2 for a conserved quantity to exist. Further investigation is required to fully understand the far wake and the wall-wake problem.

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