

A thesis submitted to the Wits School of Education, Faculty of Humanities, University of the Witwatersrand in fulfilment of the requirements for the degree of Doctor of Philosophy

By
Vasantha Moodley
Student Number: 510832

Supervisors
Professors Craig Pournara and Mike Askew

July 2022

@Copyright Vasantha Moodley 2022

An investigation into the nature and coherence of mathematical instructional explanations

ABSTRACT

This study investigated the nature of instructional explanations (IEs) through their features which were used to develop a framework of coherence of IEs. The IEs of six teachers who attended a professional development course were studied. All grade 10 lessons on two topics taught by the six teachers were observed and video recorded. The theory of IEs as elaborated by Leinhardt (2001) was used to examine the features that contribute to the coherence of IEs within the context of mathematics classrooms in South Africa. This is especially relevant in light of literature that illuminates that the IEs of teachers within some South African classrooms, as well as more globally, are characterised by a lack of coherence.

This thesis is structured in two parts. The first part focuses on the features of IEs. A preliminary analysis of the data suggested a heavy emphasis on the use of examples and the use of the chalkboard in the IEs presented by teachers. These preliminary findings were used to re-organise the features of IEs, as highlighted by Leinhardt (2001), to indicate the relationships between them. Teachers' use of examples and teachers' use of inscriptions (e.g. on the chalkboard) and talk were used as lenses to examine the features of IE's. In addition, a third lens was identified through the analysis of the data i.e. a summoning back lens where content used in previous IE's were used again in subsequent IE's that illustrate the connections between them.

The second part of the thesis is a focus on the coherence of IE's. The features of IE's illuminated through the use of the lenses in the first part of the study, together with literature on coherence of IE's informed the features used to develop a framework of coherence of IE's.

The study makes three main contributions to the theory of IEs. First, a re-organisation of the features of Leinhardt's theory of IE's is proposed to illustrate relationships between them. Second, an expansion of the definition of IEs is proposed. Instead of IEs being those that are offered at moments in time, many different IEs may combine to form a composite IE that extends over an entire lesson. Third, a framework is offered that can be used to analyse the coherence of IE's offered by teachers.

DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.



Vasantha Moodley

4th day of July in the year 2022

DEDICATION

To my parents, who taught me the value of learning and the need to expand
my horizons

PUBLICATIONS EMANATING FROM THIS RESEARCH

Conference Proceedings

Long Paper:

Moodley, V., Pournara, C., & Askew, M. (2021). Mathematical Instructional Explanations with a spatial component. In Qhobela, M., Ntsohi, M.M.E., & Mohafa, L.G. (Eds.). *Proceedings of the 29th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education*, p 67. Lesotho: SAARMSTE.

Short Paper:

Moodley, V. (2019). *Tensions in the take-up from a professional development programme: The case of one secondary mathematics teacher*. Paper presented at the 27th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education, Durban, South Africa.

Oral Communication:

Moodley, V (to appear). Re-visiting instructional explanations: how might the organisation of a lesson contribute to an explanation. In *Proceedings of the 14th International Congress on Mathematical Education*. Shanghai, China.

Poster presentation:

Moodley, V; Pournara, C. (2019). One teacher's tensions in the contrasting introductions to two algebra topics. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). (2019). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education*, (vol 4, pp 4-166). Pretoria, South Africa: PME.

ACKNOWLEDGEMENTS

An undertaking of this nature cannot be accomplished in isolation. My heartfelt thank you goes to the following people, who supported me in some way at different points.

First and foremost, my supervisors, Professors Craig Pournara and Mike Askew, deserve a medal for getting me to this stage. I could never have completed this mammoth task without the accumulated experience and wisdom of both. They constantly urged me to find my own voice so that I could present my views “with hands on hips and finger wagging”. Although at many times I felt like giving up, it was your perseverance more than mine that kept me going. No words can express my gratitude to you both, so I will make do with a simple but heartfelt “Thank You”.

To Professor Jill Adler, who although I had two supervisors, acted as my unofficial third. A big thank you for always making time available to “chat” about my work, knowing that they were much more than just “chats”. Your feedback on so many occasions provided me with the much-needed impetus to keep going.

To the members of the WMCS team, who provided me with support (work-related or not) at all times. Wanda and Yvonne, thank you for keeping me sane throughout this journey.

To the teachers who so readily and graciously agreed to participate in this study, a huge thank you. The sacrifices in terms of giving of your time and effort without once complaining, are things that I can never repay you for. I only hope that I have done justice to your professionalism, enthusiasm and passion for your work.

My support base: my entire extended family who supported me in so many ways, I cannot repay you for all you have done to make coping with life during phd easier. Thank you Vanitha, Mervin, Juvaine, Tylin, Kantha, Keith, Preshailen, Esailen and Kathy.

List of acronyms used in this thesis

Abbreviation	Description
ALD	Attending to Learner Difficulties
APNK	Attaching Prior to New Knowledge
CoMIE	Coherence of Mathematical Instructional Explanation
DP	Describing a Procedure
HKI	Highlighting Key Ideas
IdCU	Identifying Conditions of Use
IE	Mathematical Instructional Explanation
OoL	Object of Learning of a composite IE
ooL	Object of Learning of an IE
PJ	Providing Justifications

Table of Contents

Chapter 1 : Introduction.....	14
1.1. Introduction	14
1.2. Study background and problem statement	15
1.3. Gaps in literature on explanations.....	17
1.3.1. A core practice but not enough attention in literature	17
1.3.2. IEs in literature focus on pre-service teachers.....	18
1.3.3. IEs in different contexts	19
1.3.4. No relationships between the features of Leinhardt’s (2001) theory of IE’s	19
1.3.5. No differentiation between quality and features of IEs	19
1.3.6. Verbal and written as separate IEs	20
1.4. A note on the Research Design	20
1.5. Research aim questions	22
1.6. Study context	23
1.7. Significance of the study	24
1.8. The thesis outline.....	25
1.9. Conclusion.....	28
Chapter 2 : Theoretical framing of explanations	31
2.1. Introduction	31
2.2. What is an explanation?.....	32
2.3. Acts of explaining.....	33
2.4. Explanations in literature	33
2.4.1. Leinhardt’s theory of explanations	34
2.4.2. Features of IEs.....	39
2.5. Discussion and conclusion	40
Chapter 3 : How I use Leinhardt’s theory of IEs in this study	42
3.1. Introduction	42
3.2. Examples in literature	43
3.2.1. What are examples, their different types and their purpose?	44
3.2.2. Different types of examples.....	45
3.2.3. Variation Theory in literature	47
3.2.4. Key tenets of variation theory	48
3.3. How I use the different types of examples in this study.....	51
3.4. A useful set of representations as inscriptions.....	52
3.4.1. Different ways of using the chalkboard and their benefits	53
3.5. Teacher talk in IEs	56
3.5.1. Teacher talk as purposeful.....	57
3.6. Discussion and conclusion	59
Chapter 4 : Methodology.....	60
4.1. Introduction	60
4.2. Research design and methodology.....	60
4.3. Comparison of two main paradigms.....	61
4.3.1. The post-positivist paradigm	61
4.3.2. The interpretivist paradigm	61

4.3.3. Limitations of the interpretivist paradigm and my stance in this study	63
4.4. My research approach	64
4.5. Research methods	64
4.5.1. Sample Selection	65
4.5.2. Data sources.....	67
4.5.3. Data Capturing	68
4.6. Conclusion.....	69
Chapter 5 : Data analysis methods	70
5.1. Introduction	70
5.2. The use of grounded theory in this research.....	70
5.2.1. A description of grounded theory.....	70
5.2.2. The two different versions of grounded theory	71
5.2.3. Why grounded theory?.....	72
5.2.4. Why the Straussian approach?	74
5.3. Classroom observation analysis procedure	74
5.3.1. Analysis of classroom observations	74
5.3.2. The use of posters as an analytical tool.....	76
5.3.3. The use of open coding in this study	77
5.3.4. The use of axial coding in this study	78
5.3.5. Theoretical sensitivity in my data analysis.....	82
5.4. Ethical considerations	82
5.5. Trustworthiness of the study	84
5.5.1. Credibility	84
5.5.2. Generativity of the findings	84
5.5.3. Dependability of the results.....	85
5.5.4. Confirmability of the results	86
5.6. Discussion and conclusion	86
Interlude 1: Viewing IEs through three lenses.....	88
IEs viewed through the lens of teachers' use of examples.....	89
IEs through the lens of teachers' use of inscriptions and talk	89
IEs through the lens of summoning-back	90
Conclusion.....	90
Chapter 6 : Viewing IEs through the lens of using examples.....	91
6.1. Introduction	91
6.2. IE 1 Tsebo's use of examples in revealing the four features of IEs.....	91
6.2.1. Description of the IE.....	91
6.2.2. Examples used.....	92
6.2.3. Features of IEs seen through the lens of using examples.....	95
6.3. IE2 Tsebo's Use of examples in looking at features of IEs	95
6.3.1. Description of the IE.....	95
6.3.2. Using similar examples.....	96
6.3.3. Using examples with variation.....	96
6.4. The features of IEs revealed through the use of variation	97
6.4.1. Attaching prior knowledge to new knowledge.....	97
6.4.2. Addressing common learner errors	98

6.4.3. Identifying conditions of use of a procedure	98
6.5. Discussion and Conclusion	100
Chapter 7 : Teachers' use of Inscriptions and Talk in IEs	102
7.1. Introduction	102
7.2. IE1: Ndivho's Use of inscriptions and talk in identifying the conditions of use of a mathematical idea	103
7.2.1. Description of the IE.....	103
7.2.2. Ndivho's use of inscriptions and talk in revealing the feature identifying the conditions of use of the zero product law	106
7.3. IE2: The lens of inscriptions and teacher talk in highlighting connections between closely related questions and the core mathematical principles in a topic	107
7.3.1. Description of the IE.....	107
7.3.2. Inscriptions in the IE.....	108
7.3.3. Inscriptions and teacher talk in the IE.....	109
7.3.4. Features of IEs revealed by looking through the lens of inscriptions and teacher talk.....	113
7.4. IE3: Inscriptions and talk in highlighting the different solution strategies to a question and addressing common learner errors	114
7.4.1. Description of the IE.....	114
7.4.2. Inscriptions on the chalkboard in the IE	116
7.4.3. Teacher talk in the IE.....	116
7.5. Conclusion and discussion	117
Chapter 8 : IEs through the lens of summoning- back.....	119
8.1. Introduction	119
8.2. IE1 Tsebo's IE through the lens of summoning-back other IEs.....	119
8.2.1. Description of the IE.....	120
8.2.2. Summoning-back previous IEs	123
8.2.3. Using the same examples	123
8.2.4. The role of inscriptions and talk through the lens of summoning-back	124
8.2.5. Features of IEs in focus through the summoning-back lens.....	125
8.3. IE through the lens of summoning-back a procedure	126
8.3.1. Description of the IE.....	126
8.3.2. Using examples in the IE	128
8.3.3. Using Inscriptions and talk in the IE	128
8.3.4. Features of IEs in focus through summoning-back in this IE	130
8.4. Discussion and Conclusion	131
Chapter 9 : Towards a definition of coherence in relation to instructional explanations (IEs)	136
9.1. Introduction	136
9.2. The need to develop a framework on coherence of IEs	137
9.3. Classification of features of coherence identified in literature on mathematics education	138
9.3.1. Coherence as a focus on connections.....	139
9.3.2 Coherence as a focus on mathematical content	142
9.3.3. Coherence as a focus on learners	142
9.4. My definitions of 'coherent', 'partially coherent' and 'not coherent'	142
9.5. Conclusion.....	145
Chapter 10 : Coherence of mathematical instructional explanations (IEs)	147

10.1. Introduction	147
10.2. My adaptation of Leinhardt’s features to the coherence of IEs	148
10.3. My framework of coherence of IEs	150
10.4. Analysis of coherence of an IE using the CoMIE framework	153
10.4.1. Overview of Ndivho’s IE within a lesson on the solving of quadratic equations	153
10.4.2. Using the CoMIE framework to analyse the coherence of Ndivho’s IE	155
10.4.3. The coherence of the IE	159
10.5. Summary of findings using the CoMIE Framework.....	161
10.6. Conclusion.....	162
Chapter 11 : Discussion and conclusion.....	164
11.1. Introduction	164
11.2. Sample and analytic methods used	165
11.3. Main findings of this study	166
11.3.1. Viewing IEs through teachers’ use of examples	168
11.3.2. Viewing IEs through inscriptions and talk.....	169
11.3.3. Viewing IEs through the lens of summoning-back.....	170
11.3.4. Framework of Coherence (CoMIE framework).....	171
11.4. Contributions of this study.....	173
11.4.1. Contribution to the theory of IEs	173
11.4.2. Contribution to pedagogy	173
11.4.3. Contribution to grounded theory methodology.....	175
11.5. Significance of the study and recommendations	176
11.6. Limitations of this study.....	177
11.7. Conclusion.....	178
References	179

List of Figures

FIGURE 1.1: THE MTF.....	16
FIGURE 2.1: LIST OF LEINHARDT’S (2001) FEATURES OF IES	39
FIGURE 5.1:PICTURE OF A POSTER	77
FIGURE 7.1: PICTURE OF BOARD-WORK OF CONTRASTING EXAMPLES	104
FIGURE 7.2: AN INCORRECT SOLUTION PROVIDED BY LEARNER 1.....	104
FIGURE 7.3: TSEBO’S QUESTIONS BASED ON THE GIVEN SEQUENCE.....	108
FIGURE 7.4: TSEBO’S LAYOUT OF SOLUTIONS ON THE BOARD	108
FIGURE 7.5: ULWAZI’S ALTERNATIVE SOLUTIONS TO A QUESTION	114
FIGURE 7.6: PICTURE OF CHALKBOARD WITH ALTERNATE SOLUTIONS PLACED NEXT TO EACH OTHER	115
FIGURE 8.1: CONNECTIONS BETWEEN SUB-EXPLANATIONS.....	122

FIGURE 8.2: PICTURE OF TSEBO'S CHALKBOARD, HIGHLIGHTING THE ZERO PRODUCT LAW	125
FIGURE 8.3: PICTURE OF TSEBO'S LAYOUT OF WORK ON THE CHALKBOARD	127
FIGURE 8.4: STEPS OF THE PROCEDURE TO SOLVE A QUADRATIC EQUATION	128
FIGURE 8.5: TSEBO'S TALK IN CONNECTING THE EXAMPLES AND THE STEPS OF PROCEDURE	129
FIGURE 8.6: SUMMONING- BACK STEPS OF PROCEDURE INSCRIBED ON CHALKBOARD	130
FIGURE 9.1: FEATURES OF IES USED IN THIS STUDY	136
TABLE 10.1: LEINHARDT'S FEATURES OF IES	FIGURE 10.1: MY ARRANGEMENT OF LEINHARDT'S
FEATURES OF IES.....	149
FIGURE 10.2: COMIE FRAMEWORK	151
FIGURE 10.3 : A PICTURE OF THE CHALKBOARD WITH BOTH RESPONSES INSCRIBED	154
FIGURE 10.4: PICTURE OF THE CHALKBOARD WITH CORRECTED FIRST STEP OF THE SOLUTION BY LEARNER 1	154
FIGURE 10.5: PICTURE OF THE CHALKBOARD WITH CONTRASTING EXAMPLE	155
FIGURE 10.6: FEATURES PRESENT IN A FOCUS ON CONTENT	156
FIGURE 10.7: FEATURES PRESENT IN A FOCUS ON LEARNERS	157
FIGURE 10.8: FEATURES PRESENT IN A FOCUS ON CONNECTIONS	158
FIGURE 10.9: SUMMARY OF FEATURES OF COHERENCE IN NDIVHO'S IE	160
TABLE 11.1: LEINHARDT'S FEATURES OF IES	FIGURE 11.1 : MY RE-ARRANGEMENT OF
LEINHARDT'S FEATURES OF IE'S.....	167
FIGURE 11.2: RELATIONSHIP BETWEEN LENSES AND FEATURES OF IE'S	168
FIGURE 11.3: EXAMPLE DEMONSTRATING AN IE THROUGH THE LENS OF SUMMONING-BACK	171
FIGURE 11.4: THE COMIE FRAMEWORK	172
FIGURE 11.5: PART OF MTF	FIGURE 11.6: PART OF MTF AS I SEE IT
AFTER THIS STUDY	175
FIGURE 11.7: EXAMPLE OF A POSTER USED IN MY ANALYSIS USING GROUNDED THEORY	176

List of Tables

TABLE 1.1: PART OF A CONVERSATION BETWEEN THE RESEARCHER AND A FAMILY MEMBER	14
TABLE 3.1: EXAMPLE SET TO IDENTIFY A QUADRATIC EQUATION USING VARIATION	50
TABLE 3.2: MY REARRANGEMENT OF LEINHARDT'S FEATURES OF IES	58
TABLE 4.1: SUMMARY OF PARTICIPANTS' INFORMATION	66
TABLE 4.2: TEACHERS OBSERVED AND INTERVIEWED IN THE TWO YEARS.....	68
TABLE 4.3: DATA FROM CLASSROOM OBSERVATIONS.....	68
TABLE 5.1: EXAMPLE TO DEMONSTRATE OPEN CODING	78
TABLE 5.2: EXAMPLE OF AXIAL CODING	81
TABLE 6.1: OVERVIEW OF THE TSEBO'S LESSON	92
TABLE 6.2: EXAMPLES USED BY TSEBO	92
TABLE 6.3: EXCERPT INDICATING TSEBO'S UTTERANCES	93
TABLE 6.4: EXAMPLES USED BY TSEBO TO ILLUSTRATE THE PROCEDURE TO SOLVE QUADRATIC EQUATIONS	96
TABLE 6.5: THE VARIATION USED IN TSEBO'S EXAMPLES AND THE FEATURES OF IES	100
TABLE 6.6: FEATURES OF IES IN FOCUS THROUGH VARIATION IN EXAMPLES.	101

TABLE 7.1: DIAGRAM ILLUSTRATING THE IE OF DIFFERENCES BETWEEN CONTRASTING EXAMPLES	105
TABLE 7.2: EXCERPT INDICATING TSEBO'S UTTERANCES RELATING TO DIFFERENCES BETWEEN CONTRASTING EXAMPLES	105
TABLE 7.3: DIAGRAM ILLUSTRATING INTER-RELATIONSHIPS BETWEEN SOLUTIONS.....	110
TABLE 7.4: EXCERPT OF TSEBO'S UTTERANCES ILLUSTRATING RELATIONSHIPS BETWEEN QUESTIONS	111
TABLE 8.1: OVERVIEW OF TSEBO'S LESSON	120
TABLE 8.2: SUMMARY OF FEATURES IDENTIFIED THROUGH THE THREE LENSES	132
TABLE 10.1: LEINHARDT'S FEATURES OF IES FIGURE 10.1: MY ARRANGEMENT OF LEINHARDT'S FEATURES OF IES.....	149
TABLE 10.2: SUMMARY OF FEATURES OF COHERENCE OF IES.....	150
TABLE 10.3: SUMMARY OF COHERENCE USING THE COMIE FRAMEWORK	162
TABLE 11.1: LEINHARDT'S FEATURES OF IES FIGURE 11.1 : MY RE-ARRANGEMENT OF LEINHARDT'S FEATURES OF IE'S.....	167
TABLE 11.2: EXAMPLE DEMONSTRATING AN IE THROUGH THE LENS OF INSCRIPTIONS AND TALK	169

Chapter 1 : Introduction

1.1. Introduction

V: So are you coping with maths at school?
K: No (placing hands over her face).
V: What's the problem?
K: I'm not enjoying it cos I find it very difficult.
V: Are you not asking your teacher for help when you are struggling with a topic?
K: What's the point? When I ask her to explain how she did an example again, she explains it in the same way with the same example as the first time.
Does she not realise that if I didn't understand it the first time, no matter how many times she explains in the same way, I will still not understand.
She also refers to something she says she has explained previously but I have no idea what she is talking about. How does she expect me to remember something she explained previously when she doesn't remind me of it?

Table 1.1: Part of a conversation between the researcher and a family member

Table 1.1. is a vignette of an actual conversation I had with a family member (K) at an informal family gathering. K is a young family member who was in Grade 9 at the time of our conversation. K's statement, "I have no idea what she is talking about" after her teacher provided an explanation, suggests that the explanation did not make sense to K. Her problem with mathematics is not uncommon. There have been many other conversations with family members describing their view of the problems they experience with the explanations provided by their respective teachers.

The difficulties experienced by K suggests that teachers' explanations are central to the struggles or success experienced by learners in mathematics. A teacher explaining something in the same way as previously explained, using the same example/s and referring to some mathematical idea done previously which cannot be remembered or was not understood by a learner initially, are some of the reasons for learner difficulties with teachers' explanations. A teacher's attention to a

careful choice of examples and how they are sequenced can make it possible for learners to perceive the goal of the explanation and so is an important consideration in an explanation. How an explanation provided by a teacher includes mathematical ideas used previously is also an aspect of a teacher's explanation that requires consideration.

Another issue to be considered by teachers when providing explanations is that visual cues, i.e. what learners "see", is foregrounded by the learner, suggesting a lack of connection between what is written and the utterances of a teacher in the explanation provided. All of the above issues may make the teacher's explanation not coherent to the learner. While explanations may be central to teaching, K's experience of her teacher's explanation suggests that there are many aspects that should be considered in providing an explanation to make it possible for learners to attend to the particular mathematical idea in focus. How to provide explanations is therefore a complex feature of teacher instruction that requires further investigation. There are very few studies providing guidance on the features that can lead to coherent teacher explanations. My study aims to fill this gap.

This dissertation is a grounded study of the nature of teacher explanations and their resultant coherence, as illuminated by teachers who have attended a professional development course. This study was based on the direct observation of six teachers (who attended the course) from "poor"¹ schools providing explanations to learners in two different topics. In this first chapter I present the background to this study, outlining the problem that initiated the study, the subsequent refinement together with the significance of the study, and present an overview of the methodology used to identify the findings. I conclude by providing an outline of the main findings and their significance, while also noting the limitations of the study.

1.2. Study background and problem statement

The focus of this study is to examine the nature and quality of teachers' mathematical explanations. The study had its genesis in a project that I joined at the beginning of 2017, which influenced the focus of this study.

The Wits Maths Connect Secondary project (WMCS) is a research and development project at the University of the Witwatersrand. The members of the WMCS project designed and implemented a professional development programme. This programme was offered through a course, Transition Maths 1 (TM1), to focus on teachers teaching Grades 8, 9 and 10 mathematics, with the goal of assisting them in their teaching of learners making the transition from Grade 9 to Grade 10 mathematics. A focus of the course was the introduction of the Mathematics Teaching Framework (MTF) focusing on four components of mathematics

¹ I use poor schools in the sense of schools lacking many basic facilities like clean running water and with a shortage of classrooms, furniture etc.

teaching. One of the four components of the MTF is explanatory communication, which focuses on teachers' language use and strategies to consider when choosing and using examples to highlight some mathematical idea (Adler & Ronda, 2015). Teachers in the course were involved in activities on the use of mathematical and non-mathematical language in explanations, and how to justify mathematical ideas to help promote the generalisation of these ideas. My participation in the implementation of the course, together with my years of teaching experience, contributed to my making teachers' explanations the focus of this study.

While I joined the project with more than 25 years of teaching experience, I realised that my knowledge on how to offer mathematical explanations was very limited. The introduction of a Mathematics Teaching Framework (MTF) in the course to guide teachers in their teaching of mathematics was a crucial factor in deciding on the particular focus of my study. One of the components of the MTF is *explanatory communication* as depicted in fig. 1.1.

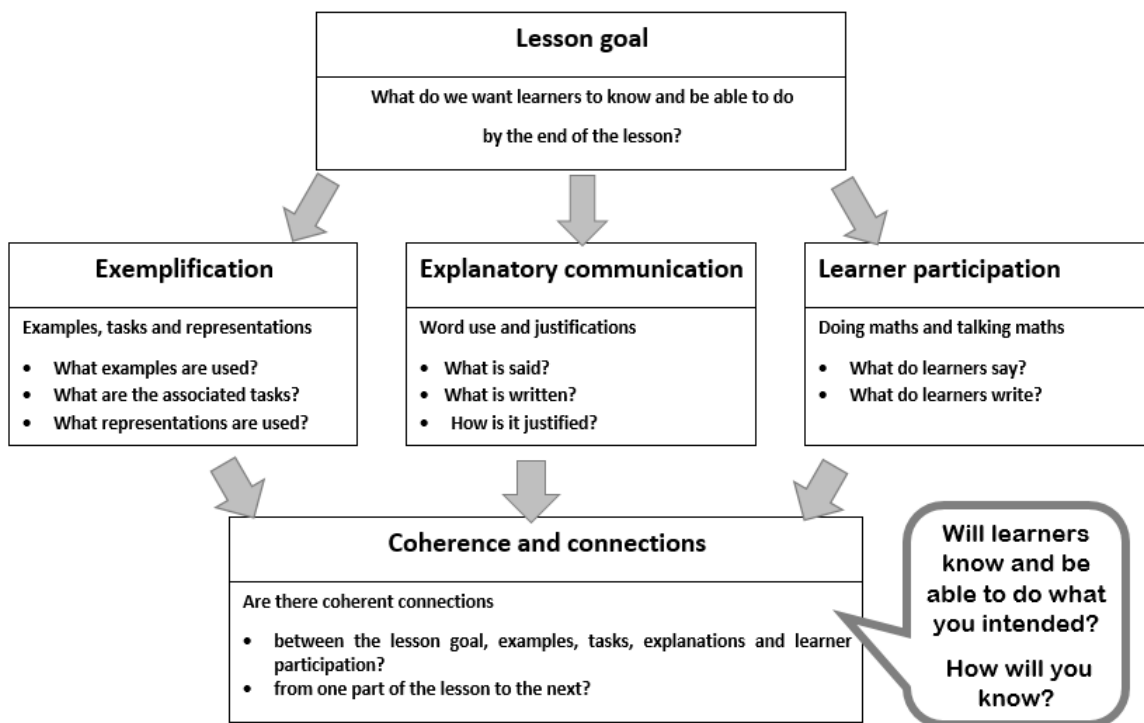


Figure 1.1: The MTF

While the *explanatory communication* component provided some guidelines on how teachers could develop and offer explanations that are coherent, my work with teachers in the course indicates that this component of the MTF still requires more elaboration. For instance, one of the features of the explanatory communication component of the framework was on how to use the chalkboard effectively. The focus in the MTF on the chalkboard aspect of explanations is a focus on *what is written* and *how it is written* on the chalkboard. The focus of chalkboard work in this component of the framework is on the words/phrases, solutions to a question or illustrations like

graphs and diagrams and whether these are written accurately and in a structured way. I refer to work on the chalkboard in this study as the use of inscriptions. While these aspects of working on the chalkboard are important, what is meant by structured is limited to whether or not words or statements are written haphazardly or randomly. It is, however, not only how inscriptions are made on the chalkboard that requires more elaboration, but also how the use of the chalkboard together with a teacher's talk contributes to what mathematics is made possible to learners. The MTF does not focus on how the verbal and the written combine in an explanation. The chalkboard feature, together with how it is used in combination with teacher talk in contributing to a coherent explanation thus requires further elaboration.

In order to be able to elaborate on the explanatory communication component, it was first necessary to examine how teacher explanations are elaborated on in literature. Examining literature indicates that there are still many gaps in research on teacher explanations.

1.3. Gaps in literature on explanations

I have identified six main gaps in my reading of literature on teacher explanations which informed the focus of my study. First is that there is not much literature on explanations offered by teachers that I could find. Second is that of literature focusing on explanations offered by pre-service teachers and not on experienced teachers² in the classroom. Third, where literature on explanations of experienced teachers is present, the focus is on those offered by teachers in the global north as opposed to literature in classrooms in developing countries like South Africa, where the cultural backgrounds and the contexts are not taken into consideration. Fourth is that Leinhardt (2001) presents a list of discrete features of IEs without indicating the relationships between them. Fifth is a lack of a disaggregation between the nature and quality of teacher explanations. Finally, literature on explanations focuses on those that are verbal and those that are written separately. These gaps are elaborated on below.

1.3.1. A core practice but not enough attention in literature

The lack of literature on explanations provided specifically by teachers belies its importance. I use "instructional explanations" (IEs) in this study to refer to mathematical explanations that are provided by teachers to learners in the classroom in alignment with Leinhardt's (2001) use. There is agreement in literature that teacher explanations are a core element of any mathematics classroom (Charalambous, Hill, & Ball, 2011; Esmonde, 2009; Venkat & Adler, 2012) and acknowledgement that offering mathematical explanations of high quality is a key skill in instruction (Eisenhart et al., 1993). Offering explanations is, however, a complex task, with many teachers offering explanations that lack coherence (e.g. Venkat & Adler, 2012) in both primary and high school classrooms in South Africa. While the study by Venkat and Adler

² I use the term "experienced teachers" to refer to those teachers with more than five years of teaching experience.

(2012) points to difficulties experienced by teachers, there are many studies pointing to difficulties experienced by pre-service teachers in offering explanations (Inoue, 2009; Kinach, 2002; Leinhardt, 1989). There is very little research on IEs offered by experienced teachers as far as I have been able to ascertain. One of the few studies on explanations provided by experienced teachers is that conducted by Leinhardt (2001), which therefore informs this study. My research focuses on the IEs offered by teachers who have been teaching for more than five years as they will have had some experience in the classroom. In conducting this research, I aim to reduce this gap in the literature by illuminating the nature of IEs offered by experienced teachers whose knowledge of content and teaching were supported by a professional development (PD) course.

1.3.2. IEs in literature focus on pre-service teachers

A further issue is that many studies focus on IEs provided by pre-service or beginning teachers and their problems in developing and offering IEs (e.g. Charalambous et al., 2011; Erath, Prediger, Quasthoff, & Heller, 2018; Inoue, 2009; Kinach, 2002). These studies focus on providing strategies to pre-service teachers in developing IEs. While the focus on pre-service teachers is understandable as it is an important part of their training as future teachers, it cannot be assumed that experienced teachers do not need assistance in developing IEs. Research conducted in some South African classrooms demonstrates the difficulties that teachers experience in developing and offering IEs. For example, Venkat and Adler (2012) show in their study that IEs provided by teachers in some classrooms in South Africa are largely incoherent; they are characterised by ambiguity and a range of disconnections between teacher talk and the examples provided. In a further study, similar findings were reported by (Adler & Venkat, 2014). They provide empirical proof that teachers' lessons are often incoherent and lack connections to the object of learning (OoL), which is a focus on the content and how this content can be used by learners. My study sheds light on the nature of IEs which can inform mathematics-focused PD programmes that can be of benefit to teachers.

Another issue with the focus on IEs provided by pre-service teachers is that some of these studies are conducted without the presence of learners. While these studies are beneficial in guiding pre-service teachers on offering IEs, with guidance from a more experienced other, they would be even more beneficial if provided to a target audience, i.e. learners. Pre-service teachers have no experience of what learners find difficult in different topics. While the features of explanations that they may be exposed to during their pre-service training may be informed by research, this type of training is not as effective as actually offering explanations to learners who react to teachers' explanations in different ways. Learning how to offer explanations without the presence of learners is therefore a difficult task. The benefit of my study is that it reports on teachers who offer IEs to learners in the classroom. Observing teachers in practice takes into consideration explanations offered to a particular audience, which in this instance is the learners.

1.3.3. IEs in different contexts

One of the few studies on IEs is that conducted by Leinhardt, which consists of a series of studies spanning two decades (1989-2001). She provides a detailed exposition of IEs and offers many features that should be present in IEs. However, her work focused on teachers providing explanations within a context different from that of a South African classroom. It is important to account for social and cultural contexts in developing and offering IEs. For example, teaching in many South African classrooms is characterised by teacher-centred teaching methods, where the teacher does much of the talking while learner interaction is limited to chorus answers to teacher questions (Adler & Venkat, 2014). Leinhardt's (2001) work is conducted in classrooms where a more learner-centred approach is followed, i.e. learners work in groups with the teacher facilitating instruction. It thus has limited use within the South African context. So, while in some other countries, for instance, the chalkboard is a useless tool that collects dust (Ermeling, 2015), for many teachers in South Africa it is the only aid they have in the classroom. My research aims to shed light on the features of IEs that can be used to examine the coherence of teacher explanation. While this is conducted within a South African context, the findings will be beneficial more generally to classrooms where teacher-centred teaching is the predominant pedagogy.

1.3.4. No relationships between the features of Leinhardt's (2001) theory of IE's

Leinhardt provides features of IEs that are a combination of teacher actions and what the actions make possible. She, however, presents these as discrete features without illustrating the relationships between them. For instance, she mentions that IEs should consist of **a useful set of examples** while also **attaching learner prior knowledge to new knowledge**. The manner in which examples can be chosen and sequenced by a teacher in an IE can make it possible to attach learners' prior knowledge to new knowledge. Choosing examples suggest a teacher action while attaching prior to new knowledge is what the action of choosing examples makes possible. The features of IEs as presented by Leinhardt therefore requires rearrangement to illustrate the relationships between them.

1.3.5. No differentiation between quality and features of IEs

Many studies on IEs do not differentiate between the features of IEs and their quality. For example, Leinhardt (2001) outlines the features of "good" IEs by specifying that they should be those that use different representations, a range of examples, etc. She adds that IEs should be coherent. However, neither her work nor other literature on explanations elaborates on what coherence of an IE entails. While there are studies that outline what coherent lessons should consist of, there is a dearth of literature on the coherence of IEs as far as I could determine. For example, Chen and Li (2010) consider coherence of a lesson as following a logical structure by proceeding from the simple to the more complex, while Shimizu (2007) points to coherence of a lesson as highlighting the key ideas within a lesson as well as the linking of a previous lesson to the next. Coherence as relating to instructional explanations is not in focus in these studies.

The two studies that focus on the coherence of IEs that I consider in this study both highlight coherence as connections. A study by Venkat and Adler (2012) reports on the coherence of IEs as relating to the connections between different stages of working through examples. For instance, they show connections or breakdowns in different steps between the initial representation of an example and the final answer. Another study by Askew, Venkat, and Mathews (2012), considers coherence of IEs as teachers making it possible for learners to connect between their prior and new knowledge. While it is important for teachers to make it possible for learners to connect between prior and new knowledge, the suggestion from these studies is that coherence relates to connections only. I draw on the findings of studies on coherence of *lessons* as well as those on coherence of IEs to develop a framework on the coherence of IEs that is intended to make some inroads into the complex issue of coherence of IEs.

1.3.6. Verbal and written as separate IEs

Literature on IEs focuses on either written or verbal forms, but not on a combination of both. For example, Morgan (1998) focuses on written explanations in texts with attention on how words are written to illustrate the “progression of the story” (p. 87). However, the role of the teacher’s verbal utterances in supporting the written form (which I refer to as inscriptions) of an explanation is not in focus. Leinhardt (2001) provides a description of teachers’ verbal explanations. For example, she considers an IE to consist of examples, discussions that connect these examples to particular rules, and discussions that limit the applicability of the rules. While she does not specify what the word *discussions* entails, it seems to me that the focus is on the verbal aspects of teachers’ IEs. There seems to be no theorization of how the verbal and written connect together in mathematical IEs. How the verbal and the written forms of IEs are woven together to form an explanation does not seem to be a focus in Leinhardt’s work nor in literature on instructional explanations in general. This is an explicit focus in my study.

1.4. A note on the Research Design

Before presenting my research aims, I need to point out that the aims of this study changed from what I originally intended to focus on. This study was initially designed with a different focus in mind.

My study was originally designed based on my interest in finding out how teachers used the MTF outside of the context of the TM1 course. The course focused on some topics within the Grade 9 and 10 curricula while others were not in focus. For example, solving equations was in focus whereas working with exponents was not. I intended to compare how teachers used the MTF in a topic that was covered in the course with a topic that was not in focus in order to investigate how teachers used the MTF in their classrooms.

I designed my study to collect data on the teaching of two different topics. I observed the lessons of four teachers on two topics: the solving of quadratic equations (covered in the course) and linear number patterns (not covered in the course). In addition, I conducted interviews with these teachers after the classroom observations so that they could talk about their use of the MTF in different parts of their lessons. Both the classroom observations as well as the interviews were intended to enable me to find out whether teachers were able to transfer their knowledge of the framework in a topic that was covered in the course to a topic that was not. I could then compare and contrast the features of the MTF that were in focus for the teachers.

The design of my study was also based on research regarding the delayed effects of PD. The findings of some studies show that no matter how effective PD programmes are, the expected teacher change in adopting or adapting innovative approaches is gradual. Taking this factor into consideration, I designed my study to include collecting data over two years. The intention was to observe the same teachers while teaching the same topics in two different years. The classroom observations as well as interviews conducted in two different years with the same teachers would have enabled me to examine if there were changes in the adoption of the MTF in the different years, and the nature of these changes.

I however refined my focus during the course of the study for two reasons. First, after the first year of data collection (from classroom observations and teacher interviews) two of the four chosen teachers were not teaching Grade 10 mathematics in the second year of data collection for various reasons. For example, the managers of the school required one of the teachers to teach mathematical literacy at the Grade 10 level. This posed a dilemma after the first year of data collection and had a direct bearing on the aim of my study. Not having data on two teachers teaching the same topics over two years meant that I could only compare how two teachers used the framework in the different years. To enable me to still consider the delayed effects of PD, I subsequently selected another two teachers who attended the same course in the same year as the other chosen teachers. This meant that I collected data from two teachers over two years, another two teachers in the first year and a different set of two teachers in the next year. I therefore did not have sufficient data to compare teachers use of the MTF over two years.

The second reason arose after my initial analysis of the data using the MDI framework (see appendix D2), which is a framework out of which the MTF developed (Adler & Ronda, 2015). Using the MDI to analyse my data proved to be a challenge. I found that teachers were not using the MTF in any structured way in the classroom. For example, many of the procedures used by teachers were not justified. The language they used was also mainly colloquial rather than mathematical. My initial analysis using the MDI framework to examine the use of the MTF indicated a deficit analysis of teachers' explanations and use of examples. I wanted to illustrate what teachers could do after their attendance at the TM1 course rather than what they could not do. While using another framework e.g. the Mathematical Quality of Instruction Framework

(Hill et al., 2008) could have been an option, frameworks developed in other countries, and more especially the global north, may not take into account the specific cultures within classrooms in other countries and more especially the global south. These frameworks were therefore not considered in this study. I therefore found that there was not an appropriate framework that could be used to analyse my data.

I refined the focus of my study because of the reasons indicated. Instead of using an existing framework, I opted to use grounded theory in my analysis of classroom observations. The use of grounded theory enabled me to observe some of the features of explanations provided by the teachers in my study that would not have been identified had I used the MDI or another framework to analyse my data. The use of grounded theory allowed me to make a contribution to much more than the MTF. It enabled me to add to the theory of IEs.

One of the positive outcomes from the use of grounded theory was that analysing the IEs of teachers helped me to illustrate how features of IEs, as outlined by Leinhardt (2001), can be linked together to enable the development of a framework to examine their coherence. My analysis therefore resulted in findings that allowed me to contribute to the body of literature on IEs.

1.5. Research aim questions

My aim is to examine the nature and coherence of IEs provided by teachers in South African mathematics classrooms. While there may be other measures of quality of IEs, until they are coherent there is no point in examining other aspects relating to the quality of IEs. I would like to see whether there are ways of examining and identifying coherence in mathematical IEs provided by teachers in South African classrooms. The value of finding a way to examine the coherence of IEs is underscored by studies that have indicated that there is a lack of coherence of IEs within South African mathematics classrooms. For instance, a study by Adler and Ronda (2015) has shown that while the examples used were appropriate, as in those recommended by curriculum documents, the verbal IEs offered by teachers were limited. They found that the explanations provided by teachers in their study did not support learners' independent solving of quadratic inequalities. Learners would have had to rely on the visual appearance of symbols and expressions used rather than on their meaning. There are several other studies suggesting the lack of coherence of teacher explanations in classrooms (primary and secondary) in South Africa (e.g. Askew et al., 2012; Venkat & Adler, 2012).

What is lacking is a framework that can be used to examine the coherence of teacher IEs within the context of teacher-centred instruction. Applying frameworks that originate from the global north paints a picture of lessons that are not coherent. It is, however, not clear whether this is a result of the particular framework that is applied or whether it is a measure of the IEs offered by teachers in countries like South Africa, where the predominant pedagogy is teacher centred. I

therefore examine the IEs provided by teachers who have attended TM1 and who have been provided with guidelines on how to offer IEs. A study of IEs offered by these teachers will provide more insight into the features as a way to examine the nature of IEs. The features can be used to develop a framework to analyse the quality of IEs through their coherence. The IEs provided by the selected teachers may not necessarily be perfect, but they have strong elements of coherence in them in that they illustrate how the verbal and written parts of IEs connect to each other, how examples are used to bring into focus the goal of the lesson and how IEs build up and connect to composite IEs. I use composite IEs to refer to those built from more than one IE.

In order to examine the coherence of IEs it was first necessary to identify what makes up an IE. More specifically, the features of IEs need to be identified in order to examine their coherence. I use literature as well as data from this study to examine the features of IEs which culminated in my development of a framework to examine their coherence.

It must be noted that this study is a focus on teachers' IEs. While the IEs by teachers are provided to learners, with the intention that learners understand or learn the mathematical idea that is the focus of an explanation, what learners actually gather from the explanation is beyond the scope of this study. All that is possible for me to note is what is made possible for learners to learn in the IEs provided by teachers.

The broad aims outlined have been condensed into the following research questions, which I seek to address in this study:

1. What features of mathematical instructional explanations are revealed through:
 - a) teachers' use of examples
 - b) teachers' use of inscriptions and talk
 - c) teachers' summoning back prior explanations
2. What makes a mathematical instructional explanation coherent?

1.6. Study context

This study was conducted within the context of teacher-centred pedagogy. The general view in mathematics education is that a traditional style of teaching is inferior to a reform-oriented or learner-centred style of teaching. There are many studies however that counter this view. Teacher-centred teaching refers to a style of teaching where the teacher generally does the talking while learners listen. When learners do contribute it is generally to ask or to answer a question. This is contrasted with a learner- or reform-oriented teaching style, which is characterised by learners generally working in groups on tasks provided by a teacher. Schwerdt and Wuppermann (2011) contend that a teacher-centred approach to instruction is frowned upon because of the many disadvantages. Some of them are that the traditional approach assumes that

all learners learn at the same pace, that learners' attention wanes if they are only listening, and that it ignores those learners who prefer other teaching styles. Schwerdt and Wuppermann (2011) argue that these criticisms of the traditional style of teaching presume that the other teaching styles do not suffer from the same disadvantages. The findings of their study show that there are no detrimental effects of teacher-centred teaching styles to learner outcomes, but it is rather how instruction is implemented that has an effect. This study is therefore a focus on how instruction and more specifically IEs are implemented.

This study is also conducted within the context of poor schools in South Africa. These are schools that lack many of the rudimentary resources, including basic amenities like flushing toilets and furniture as well as computers, text books and the financial resources to print worksheets (Sedibe, 2011). In many of these schools, learners share text books if they are at all available.

It is within this context that I observe and analyse teachers' IE. I show how teachers in this study enact explanations that have the potential to improve teacher-centred instruction using the resources that they have available. I illustrate how my framework of coherence takes into consideration the use of the resources that are readily accessible i.e. the chalkboard, within a teacher-centred style of instruction that make it possible for the OoL to come into focus for learners.

1.7. Significance of the study

The findings of this study contribute to literature on the coherence of IEs. In this regard this study makes a contribution to the research community in general as well as to professional development practitioners and mathematics teachers.

I used the findings of my analysis to develop a framework that can be used to examine the coherence of IEs. The framework on coherence is relevant to contexts where teacher-centred pedagogy dominates mathematics classrooms. It is particularly relevant to the South African context, where studies indicate that the IEs of teachers in many disadvantaged schools lack coherence.

My study makes a contribution to the field of coherence by situating it within the context of IEs within mathematics lessons. Teachers' IEs should be coherent in order for the IE to make sense to learners. A definition of coherence of IEs can guide teachers in providing a coherent IE to learners. However, this is not sufficient for the mathematics education research community. A framework is necessary to analyse whether a teacher's IE is coherent or not. Pinning down a meaning for coherence within instruction is a difficult endeavour, but finding a framework to analyse IEs is even more elusive because the literature focuses on the level of coherence of lessons and not on IEs. My analytic focus on coherence within IEs extends the current literature

on coherence, which I hope will stimulate further debate and study on this complex issue. Studies on coherence by other authors as well as my own data have contributed to offering a framework to analyse the coherence of IEs. While this framework may not be perfect or the only way to examine the coherence of IEs, it offers a starting point for other studies.

Apart from a framework to examine the coherence of IEs, I also highlight in this study the features that contribute to the coherence of IEs. While Leinhardt (2001) provides seven features of IEs, they are presented as being discrete, without an indication of the relationships between them. I shed more light on how a teacher develops a useful set of examples and a useful set of representations, which are two features that Leinhardt (2001) includes in her theory of explanations, that make it possible for learners to attend to particular mathematical ideas. While these may be practices that teachers often use in their classrooms, they may be taken for granted and so not be a deliberate focus of teachers when developing explanations. This study is meant to highlight what the deliberate attention to these practices can make possible for learners to attend to.

Using grounded theory to analyse classroom observations presented an enormous challenge to me as a novice researcher. I had a large volume of data with which to work and as a result I could not “see the woods for the trees” as expressed by Vollstedt (2015) in reference to differentiating between what was important in the study from that which was not. Literature on grounded theory most often focus on the analysis of interviews. In this research I used grounded theory to analyse the IEs offered by teachers through classroom observations. Finding literature on grounded theory to assist me in analysing classroom teaching proved to be a challenge. Documenting the main ideas of a lesson on a poster proved to be the most valuable tool in conducting my analysis. Providing a description of how my analysis of classroom observations was conducted using grounded theory and my use of posters will be of benefit to other (especially beginning) researchers using grounded theory to analyse their data.

1.8. The thesis outline

At this point, it must be noted that in writing this thesis I have chosen not to follow the traditional model of providing a literature review followed by the methods used and the findings. As noted in section 1.4. the process of conducting this research is more iterative in that my findings have informed my use of the literature and vice versa. Although in chapter 2, I do provide a review of the literature but the choice of literature examined was done in the light of later findings. I return to literature in chapter 9 in order to frame the later work on coherence of IEs. While, like in any writing, the study is presented in a linear fashion, it was actually more circular and iterative than my writing portrays.

This thesis is structured in two parts. The first part of the study relates to the features of IEs, where I answer the first research question, while in the second part I focus on the coherence of

IEs. In the first part of the thesis, I provide a review of the literature on features of IEs but more importantly, how I use these features in this study. In part two, I provide a review of the literature on coherence of lessons and IEs to illustrate how I use literature as well as the findings on the features in the first part of the study that concluded with me developing a framework to examine the coherence of IEs.

Part 1 (Chapters 2-9)

Chapter 2

I follow this introductory chapter with chapter 2, where I provide a review of the literature on IEs with a particular focus on Leinhardt's (1989-2010) theory of IEs. I focus on Leinhardt's theory of IEs because other studies use her theory to study pre-service teachers' IEs, written IEs and those used in online lessons. My focus is on the IEs of experienced teachers that are not in focus in many of the studies on IEs. In this chapter, I illustrate how I have adapted and used Leinhardt's definitions of IEs in this study. I also provide a list of features of IEs provided by Leinhardt (2001), and show how I adapt them in the next chapter.

Chapter 3

As mentioned, in this chapter, I illustrate how I see the relationships between Leinhardt's features of IEs and offer my re-arrangement of these features. The features provided by Leinhardt include:

1. Establishing a significant query
2. Having a useful set of examples
3. Having appropriate representations
4. Attaching learners' prior knowledge to new knowledge
5. Identifying core principles of mathematics
6. Identifying conditions of use, and
7. Resolving the nature of errors.

I illustrate in this chapter my rearrangement of these features to show how the features of a useful set of examples and a useful set of representations are used to bring into focus the four features (4-7) while the feature of establishing a significant query is used in part two when addressing the coherence of IEs.

Chapter 4

I provide details of the research methods that I used to collect my data. Details on my sample selection and classroom observations is provided. I include in this chapter, the details of the number of lessons observed from each of my sample of teachers. How I addressed the trustworthiness and credibility of this study, are also attended to.

Chapter 5

In this chapter, I provide details of how I analysed the data gathered. This includes how I used the version of grounded theory described by Corbin and Strauss (2014) to analyse my data. While the examples of grounded theory provided by Corbin and Strauss (2014) focus on data gathered through interviews, I provide the details of my method of a grounded analysis of classroom observations. A description of my use of posters in the analytic process that illuminated my findings is provided in this chapter.

Interlude 1

I provide an overview of how I examine teachers' **use of examples** and their **use of inscriptions and talk** as lenses. I provide details of the power of these lenses in bringing into focus the features of IEs. In addition, I also provide a brief outline of a third lens that I used i.e. the lens of *summoning-back* IEs or parts of IEs which is one of the findings resulting from the grounded analysis of my data.

Chapter 6

In this chapter, I illustrate how I used the first lens, i.e. **using examples** to examine IEs of teachers used in this study. Two example sets of a teacher were used to foreground what Leinhardt (2001) means by a “useful set of examples” in an IE. By examining IEs through this lens, I illustrate how the choice and sequencing of the examples in sets of examples reveal the four other features of IEs.

Chapter 7

The focus of this chapter is to show how examining IEs through the lens of **using inscriptions and talk**, can also bring into focus four of the features of IEs as outlined by Leinhardt (2001). More importantly, I show how this lens of IEs reveals the power of both the written and verbal representations of IEs together in bringing into focus the other features. The written representations are contextual in that my data indicates the prevalent use of the chalkboard in the written representations of IEs. I therefore consider the use of inscriptions as written representations on the chalkboard while I consider the use of teacher talk as a verbal representation.

Chapter 8

This chapter presents how I used the lens of **summoning-back** to reveal further features of IEs. This was a lens that was identified through a grounded analysis of my data and indicates how IEs used in a previous part of a lesson are brought back in a subsequent IE to foreground a mathematical idea. I provide examples to illustrate how looking through this lens foregrounded features of IEs.

Part two (chapters 9-11)

Interlude 2

I provide an overview of the section on coherence of IEs in this chapter. I first summarise my findings on the lenses through which I examined IE and the features they brought into focus. I then provide a brief synopsis of the use of these features in examining the coherence of IEs.

Chapter 9

I provide a review of literature on coherence of lessons and IEs. I use the findings from other studies to arrive at a definition of coherence that I make use of in this study. I also use the literature on coherence of lessons to extract those features of coherence of lessons that are relevant to IEs and my adaptation of the features of coherence of IEs

Chapter 10

In this chapter, I provide details of how I used the features of coherence from literature, as well as my findings on the features of IEs in developing a framework that can be used to analyse the coherence of IEs. I also provide an example of how I used the framework to analyse the coherence of an IE from my data to illustrate the usefulness of the framework.

Chapter 11

I conclude this study by highlighting the main findings of this study. The first research question was answered by illustrating the power of the three lenses used to examine IEs in bringing into focus the features of IEs. This adds to literature on IEs by providing insights into how the features can be seen. I also include how the features of IEs as well as literature on coherence of lessons provided the foundation in developing a framework to examine the coherence of IEs which is lacking in literature that I consulted. In this way, I show how my findings contribute to literature on IEs and so to the mathematics education community. I also present some of the limitations of the study.

1.9. Conclusion

In the introduction of this chapter, I reconstructed a part of a conversation I had with a family member that provides an example of the challenges faced by learners from their teachers' explanations. This highlights the importance of teachers offering coherent IEs. The example provided, however, is not the only example of an IE that is not coherent to learners. Studies, mostly conducted in classrooms in poor schools, suggest that there is a lack of coherence of IEs offered by teachers in these classrooms.

Assessing the coherence of IEs, however, is a difficult task because of the lack of a suitable framework to achieve this. Applying frameworks to assess the coherence of IEs from other

countries, and more especially the global north, does not take into consideration the particular context of classrooms where teacher-centred pedagogy is prevalent. In addition, the background and culture of classrooms differ in different countries. Applying these frameworks to assess the coherence of IEs within this context results in painting a deficit picture of the coherence of IEs within these classrooms. This may be the case because of the framework applied rather than the lack of coherence of the IEs themselves. It is therefore necessary to use a framework to examine the coherence of IEs that is more suitable to the context of teacher-centred pedagogy. Developing a framework that can be used to examine the coherence of IEs is a specific aim of this study.

In order to develop a framework to examine the coherence of IEs, it is first necessary to identify what an IE consists of. In the next chapter, I use literature on IEs and specifically draw on Leinhardt's theory of IEs to illustrate the difficulties with using the definitions and the features of IEs as provided by Leinhardt. I show how and why I adapt Leinhardt's (2001) definition of IEs, together with why I adapt the features of IEs that I use in this study.

Part 1

In this part of the thesis, I address the research question:

What features of mathematical instructional explanations are revealed through:

- a) teachers' use of examples
- b) teachers' use of inscriptions and talk
- c) teachers' summoning back prior explanations

Chapter 2 : Theoretical framing of explanations

2.1. Introduction

The question “What is an explanation?” is a simple but deceptive one that has occupied the minds of philosophers as well as social scientists (Hargie, 2010). Most people will agree with the statement that the main focus of a mathematics teacher in the classroom is to explain some mathematical idea to learners. It is, however, difficult to pin down a definition of an explanation or even to point to what makes an explanation an explanation. It is little wonder then that it is not easy to identify when an explanation is organised and presented in such a way as to make it possible for learners to perceive the particular mathematical idea.

In this chapter, I first examine the literature on the definition and features of teacher explanations. I look to the literature in order to find a suitable definition that I can make use of in this study. While my aim in this study is to examine how the coherence of teachers’ mathematical explanations can be examined, it is first necessary to examine the definition and features of teacher explanations in the literature. It is only once the definition and features of teacher explanations are highlighted that both can be used to examine the coherence of teachers’ mathematical explanations. After examining literature on the definition and features of teachers’ mathematical explanations, in part 2 of the thesis I turn my focus on the coherence of teachers’ mathematical explanations (see chapter 9 page 137).

Literature on the theory of explanations as provided by teachers in the classroom, i.e. instructional explanations (IEs), while providing guidelines, does not provide much insight into IEs. For instance, no detail is provided on the role that the selection and sequencing of examples plays in teachers offering of IEs (e.g. Charalambous et al., 2011; Gonzalez & Magally, 2013; Leinhardt, 2001). Neither is there an indication, in the literature I consulted, on the role of the interplay between what is written (e.g. solutions, definitions and steps of a procedure) and how it is written (whether on posters, the chalkboard etc.) and the verbal dimension of an IE. The literature suggests that IEs are small, occur at specific moments of time during a lesson, and are focused on a mathematical idea (Leinhardt, 2001). It is, however, possible that IEs can be arranged in such a way that several of them can be connected to each other. In this way, while each IE may focus on a different mathematical idea, their arrangement can contribute to connecting the different mathematical ideas together. Finally, the literature conflates the nature and quality of IEs without disaggregating them. A focus in this part of the study is to elaborate on the literature on IEs to shed light on all of these details.

I provide a review of the literature on IEs before showing how I use these ideas in this study that culminates in developing a framework to examine the coherence of IEs.

2.2. What is an explanation?

There are two broad views that I have identified in the literature on explanations. These relate to whether the explanation takes into consideration the recipient of the explanation. One view is that of explanations as being arguments, propositions or sentences. The belief is that if an explanation is provided with the goal of the explanation in focus and the argument is logical, then an individual will learn what it is that is being explained. This view is attributed to philosophers like Hempel and Aristotle (Keil & Wilson, 2000). This view, however, does not take into consideration the person to whom the explanation is being provided, or the prior experiences of the person listening to the explanation.

A contrasting view is that there are many factors that influence what is being learnt and what is being explained. For instance, an individual's prior knowledge, and whether the individual paid any attention to what was being explained, will affect whether and what they learn from the explanation. It is not only these factors that may influence what is learnt from the explanation but also how it is interpreted by individuals with the same prior knowledge, for instance. How the explanation is presented may also affect how the explanation is interpreted (Achinstein, 1988). For instance, the particular inflections used by the presenter with different emphases may determine how what is being explained is interpreted.

The stance that I take in this study is aligned with the latter view. In the classroom, the prior knowledge of learners, the knowledge base of the teacher, and the context within which learning occurs are all factors that affect learning. For example, a teacher who has insight into the zero product law underpinning the solution of quadratic equations may illustrate the law before or when explaining how a quadratic equation can be solved. A teacher, however, with little or no knowledge of the zero product law may not make use of this law when explaining how to solve a quadratic equation. In the latter case, the teacher may only explain the procedure to solve quadratic equations. The teachers' own knowledge therefore influences what is made possible for learners in these two classrooms. Furthermore, even though two teachers may have knowledge of the zero product law, both teachers may make possible different learning of the solving of quadratic equations to different learners, depending on whether or not teachers take into consideration the prior knowledge of learners. For instance, a teacher may use numerical values to explain that if the product of two numbers is zero then either or both factors is zero while another may explain the zero factor law using the generalisation, i.e. if $a \cdot b = 0$ then either $a = 0$ or $b = 0$ or $a = 0$ and $b = 0$. An explanation taking into consideration learners' prior knowledge may therefore influence what is made possible to learners in the IE. In this study I focus on what is made possible to learn and not on what learners have learnt from the IEs offered. It is necessary to examine what is made possible before any investigation into what learners have actually learnt from an IE.

2.3. Acts of explaining

By acts of explaining I mean not only the content of IEs and but also how that content is delivered, as each of these impact on what mathematics is made possible to learn. While it is important to consider what learners may or may not have learnt from an IE, this is not a focus of my study. My attention is on how teachers make possible the learning of particular mathematical ideas through their explaining acts. Studying these explaining acts foregrounds what makes a coherent explanation. Only once an IE is coherent does it make sense to study whether and what learners have learned.

While an act of explaining is seen to be an “illocutionary act performed by uttering words in certain contexts” (Achinstein, 1988, p. 200), I consider acts of explaining to not only consist of words but also how examples are chosen and sequenced, how they are written and the use of gestures. For instance, examples that may be inscribed and arranged in particular ways (on a worksheet, chalkboard, projector etc.) and gestures which may be used to point at something that is being referred to are acts of explaining that support utterances.

How words are uttered may also affect their meanings. A change in emphasis when talking, for instance, can alter how a listener interprets the literal words. For example, a teacher may state that “if two numbers are multiplied to produce zero then at least one of them should be zero”. If the emphasis by the teacher is on the latter part of the statement, i.e. “at least one of them should be zero”, then learners may interpret those words as being true for numbers that are multiplied to give a non-zero answer. In this example, the emphasis should be on both parts, i.e. “to produce zero” as well as on “at least one of them should be zero”. It is therefore necessary to consider other ways that support the utterances of a teacher. While a change in emphasis is one such way, in this study I also consider what is written as well as the gestures used by teachers when offering explanations.

A comprehensive study on teacher explanations was conducted by Leinhardt, spanning two decades (1986-2010) and as such provides a sound foundation for other research. Over this body of research, she focuses on the explaining acts of teachers. In this study I consider teacher explanations, like Leinhardt, to refer to the explaining acts of teachers.

2.4. Explanations in literature

In her comprehensive body of work Leinhardt outlines the development over time of her theory on teacher explanations, and many researchers who have conducted more recent studies draw on her work (e.g. Andrews, 2009; Charalambous et al., 2011; Gonzalez & Magally, 2013; Windschilt, Thompson, Braaten, & Stroupe, 2012). This body of literature focuses, however, on a wide variety of school subjects, and on explanations provided by prospective teachers, learners and text books as well as practicing teachers. For example Gonzalez (2013) studied explanations provided by prospective teachers and Windschilt et al. (2012) focused on science

teachers, while the work by Andrews (2009) has a focus on general teaching strategies of which instructional explanations is a part. Other literature on instructional explanations focus on those provided in textbooks (e.g. Morgan, 1998) and explanations provided by learners (e.g. Evens & Houssart, 2004).

As my focus is on those instructional explanations provided by mathematics teachers much of this research is tangential to my study. The above studies either do not focus on instructional explanations as provided by teachers or they focus on the training of prospective teachers and not specifically on instructional explanations. For instance, while the focus of a study by Charalambous et al. (2011) is on instructional explanations provided by prospective teachers, the study explored whether providing instructional explanations is a skill that can be learnt and if so what contributes to this learning. Their study therefore use the features of instructional explanations as highlighted by Leinhardt (2001) in an effort to examine if and how prospective teachers can learn to provide explanations. Similarly, a study by Andrews (2009) was conducted on teacher explanations and draws on Leinhardt's (2001) theory of teacher explanations. The focus, however, was on the different types of explanations and not on the features of teacher explanations. For instance, Andrews (2009) classified teacher explanations as explaining with coaching, explaining with motivating, explaining with questioning and so on. My focus in this study is on the features of teachers' mathematical explanations. I, therefore, look to Leinhardt's (2001) theory of explanations as she focuses specifically on features of explanations provided by mathematics teachers.

2.4.1. Leinhardt's theory of explanations

Leinhardt's theory of explanations encompasses the different types of explanations, their definitions and features.

2.4.1.1. Leinhardt's four different types of explanations

Leinhardt (2001) differentiates between four types of explanations, i.e. common, disciplinary, instructional and self-explanations. Common explanations are responses to direct and simple questions. Self-explanations are short and provided to the self and are usually in response to a query. Disciplinary explanations are those which abide by sets of conventions determined by members of a particular discipline where repetition and redundancy are minimal because much is taken as shared. An explanation that is developed specifically for learners in the classroom is called an instructional explanation.

Instructional explanations (IEs) are used specifically to teach subject matter to others. They can be offered by teachers, text books, a computer or a learner; however, the focus in this study is on those offered by the teacher. Leinhardt offers many definitions of IEs through her years of work in an attempt to provide a more precise definition from her previous definitions.

2.4.1.2. Leinhardt's definitions of IEs and their contribution to this study

Leinhardt's work developed over the years to encompass different definitions of IEs. She started by first defining an IE as:

“a system of goals and actions in the actual transmission of subject matter content” (Leinhardt, 1990, p. 56).

The definition is supported by the statement that the three main goals of an explanation are “to clarify, to learn and to understand” (Leinhardt, 1990, p. 57). Each of these goals in turn consists of actions; some of which are **verbal demonstrations, identifying conditions of use and demonstrations across representations.**

The word “actions” suggest an emphasis on how the IE is offered while “goal” suggests an end result of the actions. The goals listed are a combination of features that relate to both learners and teachers. “To clarify” is a goal of teachers in providing an explanation whereas “to learn” and “to understand” are goal states of learners. Similarly, the actions of “identifying conditions of use” suggest learner actions whereas “verbal demonstrations” and “demonstrate across representations” suggest teacher actions. In this study I focus on what is provided by teachers, i.e. that which is enacted by teachers. While the IEs by teachers are provided to learners with the intention that learners understand or learn the mathematical idea that is the focus of an explanation, what learners actually gather from the explanation is beyond the scope of this study. All that is possible for me to note is what is made possible for learners to learn in the IEs provided by teachers.

In later work, Leinhardt described an explanation as:

“a unique communicative form that supports the learning and understanding of others” (Leinhardt, 1997, p. 231).

The emphasis in this second definition of Leinhardt (1997) is on something being communicated to others. What is not clear in this definition is whether the communications Leinhardt refers to are verbal, written or other forms.

The first two broad definitions became more detailed in subsequent work, where Leinhardt described a typical instructional explanation as:

containing an instance of something to be explained, an example of it, a set of discussions that connect what is being explained to particular rules or principles; and, finally, a set of discussions that bound it or limit its applicability, thus distinguishing it from other closely allied ideas or practices (Leinhardt, 2001, p. 341).

All three of the definitions indicate more detail from one to the other, culminating in the above more elaborated third definition.

The lack of clarity on what communicative forms in an IE represent is still unclear in the third definition i.e. Leinhardt (2001). What is meant by the words “a set of discussions that connect what is being explained to particular rules” as well as “a set of discussions that bound it or limit its applicability” is not clear. It is unclear whether they refer to utterances in the interaction between a teacher and a learner or whether it is more than just the utterances only. However, she later describes “classroom talk as the means by which we can see explanations being developed” (Leinhardt, 2001, p. 342). This suggests a focus on utterances (by a teacher or learners) in IEs. I agree that utterances are an important component of IEs. They are the means by which the teacher makes explicit the ooL of an IE but I do not consider it to be the only means through which the ooL can be brought into focus. Utterances can be supported by other strategies like the careful choice and sequencing of examples, together with how they are written (e.g. on chalkboards, posters etc.), all of which I consider to be key elements of an IE in this study.

In the third definition, I take the “something to be explained” as the goal of an IE. The goal of a mathematical IE is to answer an explicit or implicit query so that learners are able to know and to do something after the explanation has been provided (Leinhardt, 2001). The “something” to be explained is a query in Leinhardt’s (2001) description and corresponds to *the what* that is being explained. Marton and Pang (2006) describe this as the object of learning (OoL), which represents the skills, insights or capabilities that learners are expected to develop during a lesson. My position in this study is that an IE is an explanation that has deliberate and focused attention by the teacher on the OoL of a lesson. I differentiate between the object of learning of a lesson, which I refer to as OoL (with an uppercase O) and the object of learning of an IE, which I refer to as the ooL (with a lowercase o).

The act of explaining and the ooL of an explanation both seem to be in focus in Leinhardt’s definitions of IEs. An IE is described as both “a means and an end” (Leinhardt, 2001, p. 338). The word “means” suggest actions while the “end” suggests a focus on the ooL. While Leinhardt’s earlier definition of an IE as a “system of goals and actions” (Leinhardt, 1990) makes specific mention of the actions used in an explanation, the enactment of these in an IE, however, requires further illumination even in her later studies. As Luxomo (2019) notes in her study, Leinhardt focuses on the explaining acts as the selection of examples, problems and tasks that are not observable in the lessons Leinhardt describes. These are actions that are part of the planning completed before the lesson is enacted. What requires further elaboration are aspects like how a teacher uses the examples she has selected as well as how examples that are unplanned are used during the enactment of an IE.

The use of examples seems to be the only action that is seen as part of an explaining act together with the verbal. Other actions like what is written (on a chalkboard, posters etc), how it is written, and decisions made on what to preserve on the chalkboard for use in another IE are not

sufficiently theorised by Leinhardt as explaining acts. My position is that both the explaining act as well as the ooL are important in any study of teacher IEs. I shed more light on explaining acts like the use of the chalkboard, together with choice and sequencing of examples and utterances, which are an explicit focus in this study. In part 2 of this study I focus on the explaining acts of an IE, while in part 3 I focus on developing a framework of coherence of IEs that can be used to determine what is made possible to learners from these explaining acts.

Leinhardt (2001) suggests that explanations are episodic. She mentions that IEs occur at particular moments in time in the classroom, either at temporal or intellectual moments. IEs can occur either before, after or in the middle of the enactment of a task or during what are referred to as intellectual moments. The three types of intellectual moments that prompt IEs, according to Leinhardt (2001), are the actions around operations or functions, the use of mathematical principles that constrain or promote some actions, and the metasystems of mathematics.

First, the actions around operations or functions are actions that are referred to by Leinhardt as “transformers of mathematics” (Leinhardt, 2001, p. 343). These correspond to how something can be done. For instance, there are specific actions that can be carried out to perform the procedure in calculating a R10 increase in the cost of a bag that costs R200. A different calculation may be necessary to calculate the cost of the bag if it was increased by 10%. The actions to be carried out when performing the calculations together with a justification of these actions is what prompts an IE.

The second intellectual moment that precipitates IEs is the use of mathematical principles that constrain certain actions while promoting others. For example, the commutative law holds for the operations addition and multiplication but not for division and subtraction.

The third intellectual moment that prompt IEs is the metasystems of mathematics, known as the “tools of mathematical reasoning” (Leinhardt, 2001, p. 343). Some of these tools include the discovery of short methods to simplify an expression, identifying extreme cases of a mathematical idea, whether particular mathematical actions make sense and are of mathematical value. IEs need to be offered at all of these moments to clarify something.

In this study, I extend the idea of IEs as occurring during particular moments (whether temporal or intellectual) by proposing an elaboration of IEs to sometimes encompass an entire lesson. A mathematics lesson may comprise many small IEs that may occur at specific moments to highlight a particular purpose, as Leinhardt proposes. Sometimes the OoL of a lesson may be so large that it is necessary to break it up into smaller parts or explanations. These explanations highlight different mathematical ideas. They have different foci, i.e. they address a particular ooL. These explanations are, however, used in conjunction with each other to form a larger IE,

which I call a composite IE, that contributes to the OoL of a lesson. This is best illustrated with the following example:

A lesson's overall objective of learning may be *to solve quadratic equations by factorisation*. The lesson, however, may contain more than one IE, each with different oOLs, for example, the *introduction of the zero product law*, or *the use of different factorisation methods*, both of which would contribute to the OoL of the lesson.

2.4.1.3. My expansion of Leinhardt's definition of an IE

I use Leinhardt's definitions to build a definition of IEs that I make use of in this study. As indicated earlier, Leinhardt refers to the use of examples to highlight what is being explained. She however does not offer a theorisation of how examples can be used within an IE. It is also unclear whether a "set of discussions", as contained in the definition of an IE by Leinhardt (2001), refers to verbal utterances by a teacher or to something more. Leinhardt (2001) indicates that IEs are episodic and so occur at particular moments in time. I use all of these ideas in her definition to illustrate how I expand on them for use in this study.

The definition of IEs that I propose and use in this study is that of an IE as consisting of explaining acts that aid in making it possible for learners to perceive the oOL of an IE. It is through viewing the explaining acts of an IE that the coherence of an IE can be examined with respect to the oOL. I consider this to be an expansion of Leinhardt's definition of IEs.

As in the definition provided by Leinhardt, I place examples as playing a central role in an IE. I, however, consider how the use of examples through variation (Marton & Pang, 2006), together with the utterances of a teacher, aid in highlighting the oOL of an IE. I expand her definition of an IE by considering the utterances of a teacher, the role of examples *and* also to the role of inscriptions, i.e. what is written and how it is written that aids in highlighting the oOL of an IE. I elaborate on Leinhardt's definition of an IE as a "set of discussions". I do this by considering how the examples are chosen and sequenced, how inscriptions are made, and how the utterances of a teacher combine to make it possible for learners to perceive the oOL of the IE. I consider an IE to consist of both *what* is being explained as well as *how* the something is being explained. The something that is being explained is the OoL. I further propose an expansion of Leinhardt's definition of IEs to not only occur at specific moments in time but to sometimes encompass an entire lesson broken up into smaller IEs which are used altogether to highlight the OoL of a lesson.

I next look at the features of IEs as provided by Leinhardt (2001) in order to examine what they offer in terms of the explaining acts that are part of IEs.

2.4.2. Features of IEs

Leinhardt provides the following seven features of IEs, which she describes as “goals that when met produce an IE” (Leinhardt, 2001, p. 344).

Fig. 2.1 is a list of the seven features of an IE, with a brief outline of each as provided by Leinhardt (2001).

Establishing a query	<ul style="list-style-type: none"> •The key goal of an IE is to answer an implicit or explicit query •The query is located within problems requiring linking of mathematical principles to specific parts of the task. •If a query is not established the explanation may be internally coherent but disconnected from a central issue
A useful set of examples	<ul style="list-style-type: none"> •IEs are built around examples and tasks •Examples are particular cases that represent a larger class of mathematical ideas
A useful set of representations in IEs	<ul style="list-style-type: none"> •Representations can be concrete, e.g. Cuisenaire rods; diagrammatic, e.g. drawings of abstract figures or verbal, e.g. analogies •IEs should connect between representations through talk
Identifying core principles of mathematics	<ul style="list-style-type: none"> •The main principles of mathematics should be addressed together with evidence i.e. justifications of the principles •validity of a solution can be illustrated in an IE by the use of multiple solution strategies or alternate solutions to a question.
Attaching learners' prior knowledge to new knowledge	<ul style="list-style-type: none"> •IEs should illustrate similarities and differences between old and new knowledge
Identifying conditions of use of a new mathematical idea	<ul style="list-style-type: none"> •IEs should consist of conditions under which a new mathematical idea is applicable •IEs should set the boundaries around a particular mathematical idea
Resolving the nature of errors	<ul style="list-style-type: none"> •Errors can be resolved in an IE by: •directly appealing to a rule •producing a sequence of contradictory evidence •intentionally providing a set of examples that provoke errors

Figure 2.1: List of Leinhardt's (2001) features of IEs

The seven features of IEs offered by Leinhardt are provided as separate features that do not seem to be linked together. For instance, resolving the nature of learner errors and IEs as having a useful set of examples are presented as separate features of an IE. Learner errors however, can be resolved by a useful set of examples as well as by a useful set of representations. Similarly, the

conditions of use of a mathematical idea, attaching the prior knowledge of learners to new knowledge, identifying core mathematical principals and identifying the conditions of use of a mathematical idea can all be addressed by an IE having a useful set of examples and representations. The latter four features of IEs can therefore be subsumed within the features having *a useful set of examples* and having *a useful set of representations*.

The feature *identifying core principles of mathematics* is unclear. What Leinhardt does mention is that the validity of a solution can be illustrated in an IE by the use of multiple solution strategies or alternate solutions to a question. In the absence of an elaboration of core mathematical principles by Leinhardt, I use the term mathematical ideas instead to refer to mathematical laws like the distributive law and zero product law, as well as mathematical methods or processes like factorisation and quadratic equations. Instead of Leinhardt's feature *identify core mathematical principles* I use *highlighting key mathematical ideas*. I use this term because it is a feature that is highlighted in features of coherence, as illustrated in chapter 9.

In the next chapter, I demonstrate the relationships between the features as outlined by Leinhardt (2001) that contribute to my re-arrangement of these features of IEs.

2.5. Discussion and conclusion

While Leinhardt's two decades of work on IEs has been extensive, it is not exhaustive. I therefore use her work as a foundation for my definition and features of IEs that I use in this study.

The following four key issues arise from Leinhardt's theory of IEs, which I focus on in this study.

1. The definition of IEs provided illustrates that examples are central to IEs. I concur. However, Leinhardt's theory on IEs does not focus on how examples can be used within IEs. In particular, how examples can be used to highlight the oOL of an IE or to *establish a query* (in Leinhardt's language) is not elaborated on.
2. While Leinhardt (2001) does consider the written representations within IE's my reading of her work suggests that the emphasis is on the verbal aspects. The utterances of a teacher are important in an IE, I, however, consider the written aspects to be just as important. The literature on IEs, however, focuses on these two aspects separately. While examples of IEs provided in the literature do suggest that IEs consist of both written and verbal aspects, how both of these aspects combine in an IE is not given much emphasis.
3. The suggestion that IEs are episodic and so occur at a particular moment in time, needs further expansion. While I agree that there are many IEs within a lesson, each focusing on a different mathematical idea, these IEs and their content can be arranged in such a way that they contribute to a composite IE that focuses on the OoL of a lesson.
4. The seven features of IEs provided by Leinhardt (2001) contain a mix of a focus on the content of IEs while others focus on the purpose of IEs. These features are presented as isolated features without illustrating the relationship between them. In this study I consider

the features *a useful set of examples* and *a useful set of representations* as explaining acts of IEs that can be used to address the other four features. Having *a useful set of examples* and *a useful set of representations* makes it possible to *attach learners' prior knowledge to new knowledge, highlight core mathematical ideas, identify conditions of use of a mathematical idea* and *resolve the nature of errors*.

A rearrangement of the features of IEs will provide more insight into what makes up an IE. More especially, it will provide insight into what constitutes *a useful set of examples* and *a useful set of representations* which can be considered to be explaining acts that brings into focus the other four features. These insights will inform how the coherence of IEs can be examined, which is a focus in part three of this study.

In the next chapter I illustrate how the relationships between the features contribute to my rearrangement of Leinhardt's (2001) features of IEs .

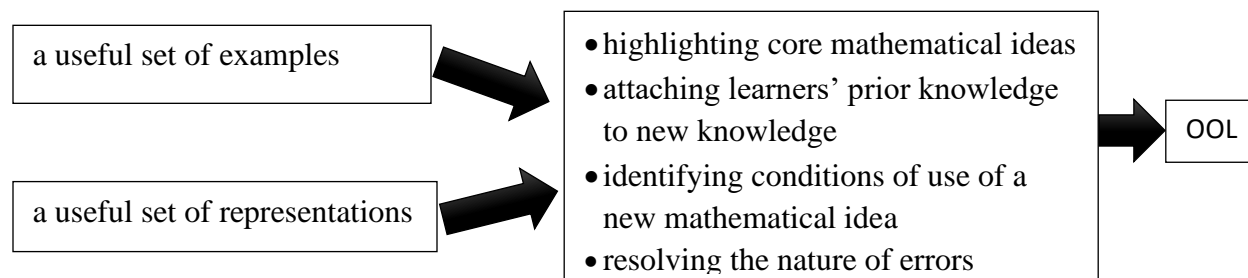
Chapter 3 : How I use Leinhardt’s theory of IEs in this study

3.1. Introduction

In this chapter I illustrate how I have made use of Leinhardt’s theory of IEs in this study. I propose a rearrangement of the features of IEs to illustrate how they can be linked together, which in turn will enable me to examine the coherence of IEs. As mentioned in the previous chapter, two of the seven features of IEs are related to the content of IEs, four relate to the purpose of IEs while I use the remaining feature in considering the coherence of IEs in the second part of this study. Leinhardt (2001), however, presents these as seven discrete features, without suggesting how they are linked together.

In this study I consider two features as the means through which four of the other five features can be examined. These two features are: IEs as consisting of *a useful set of examples* and *a useful set of representations*. The four features of IEs are: *highlighting core mathematical ideas*, *attaching learners’ prior knowledge to new knowledge*, *identifying conditions of use of a new mathematical idea* and *resolving the nature of errors* can all be examined through the two features *a useful set of examples* and *a useful set of representations* to explore how these are used in foregrounding the ool.

The above features can be represented as:



It must be noted that I although I use the term “attaching prior to new knowledge”, I cannot claim to know anything about learners’ knowledge as this is a study about teachers IEs. While I consider attaching prior knowledge of learners to the intended (by a teacher) new knowledge, for the sake of brevity I will refer to *attaching prior to new knowledge*.

In addition, Leinhardt (2001) mentions one more feature, i.e. establishing *a query*. I consider this feature to be the main purpose of the IE. I use the words *the main purpose of an IE* because Leinhardt mentions that “*a central goal of an explanation is to answer a query*” (Leinhardt, 2001, p. 344). This statement suggests that an IE develops around a query that is to be answered.

This is, however, one of the features that Leinhardt (2001) mentions *within* her list of features. Even though this can be considered to be a *purpose* of an IE, I consider this feature to be at a level above the other features.

I use the feature of establishing a query as relating to the ooL of an IE. I consider the purpose of an IE as addressing the OoL which I consider to be more specific than establishing a query. It is through establishing a query that learners are provided with the possibility of developing specific insights and capabilities at the end of the IE. This is however not encapsulated in the words “establishing a query”. I use ooL to refer to the purpose of an IE, since ooL specifically refers to the capabilities and insights into specific mathematical ideas that learners are meant to develop.

The feature of an IE as addressing its ooL allows me to examine the coherence of an IE. If the central goal of an IE is to address its ooL, then how the IE is developed in illuminating the ooL will help to determine whether the IE is coherent in terms of the ooL. This is a specific focus in part three of this study.

In examining the literature on the use of examples I focus largely on the use of variation theory. My analysis of the data suggests a strong emphasis on the use of variation in the example selection used in the IEs of teachers used in this study. Consequently, I review literature on the use of variation of examples within an IE.

Similarly, my data indicates the prominent use of written inscriptions made on the chalkboard. I therefore consider a useful set of representations as inscriptions that are made on a publicly available platform with a particular focus on the chalkboard. I therefore focus on literature on the use of the chalkboard in classrooms.

I review the literature on examples and inscriptions to delve into ways that these two features are used in my study to examine the four other features of IEs as outlined by Leinhardt. I first look to the literature to foreground how examples can be chosen and sequenced in order to highlight some mathematical idea/s before looking at the ways in which examples and other parts of an IE can be represented.

3.2. Examples in literature

The use of examples as a strategy to foreground mathematical ideas is a common one in the mathematics classroom (Hill et al., 2008). The importance of the use of examples is encapsulated by the comment “people learn mathematics principally through engagement with examples, rather than through formal definitions and techniques” (Watson & Mason, 2002, p. 2) thus underscoring the centrality and the importance of using examples. Examples chosen with no thought given to their purpose is, however, unlikely to be useful to learners and may in fact obscure rather than bring into focus a mathematical idea. Teachers therefore need to pay attention to how they select and use examples to make it possible for learners to perceive

mathematical ideas. Hill et al. (2008) also point to a teacher's need to possess knowledge of how to offer IEs to enhance the teaching of mathematics.

A teacher's IE generally develops around examples. Indeed, this is suggested in Leinhardt's (Leinhardt, 1990, 2001) definition of IEs, as mentioned previously. While I concur with Leinhardt's (2001) view on the centrality of example use in IEs, she does not provide details on how the selection and choice of examples support an IE. For instance, she states that

“A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non-or counter-cases” (Leinhardt, 1990, p. 6)

and later describes IEs as

“containing an instance of something to be explained, an example of it...” (Leinhardt, 2001, p. 341)

Yet, apart from mentioning *non- or counter-examples*, she does not make explicit particular strategies that can be used in the selection and sequencing of examples that help in highlighting the OoL of an IE.

The other issue arising from literature is that the use of examples are addressed independently of IEs. The use of examples and the development of IEs are seen as separate from each other in the literature. For instance, Ball, Thames, and Phelps (2008) describe the tasks of teachers in the classroom, two of which are *offering explanations* and the *choice and sequencing of examples*. While this may not suggest that they are two separate practices that teachers engage with in the classroom, how examples are used within an IE is not sufficiently elaborated on.

In this study I consider the examples as being a part of IEs with their use in not only helping to foreground the ooL but other features of an IE as well. Before examining how examples can be used in IEs I first address the questions: What are examples, what are the different types of examples, and what is their purpose?

3.2.1. What are examples, their different types and their purpose?

Examples are defined in many different ways in literature, depending on the role they play. Such roles include: to aid in generalisation (i.e. examples as a particular case of a larger class of ideas) of a mathematical idea, to suggest relationships, to illustrate mathematical principles, to expose possible variation, and to practice technique (Bills et al., 2006; Watson & Mason, 2002). It is important that these different purposes are addressed within the mathematics classrooms, in this study I focus on how they are used in an IE to highlight the ooL, which may be to illustrate mathematical principles, to suggest relationships etc. by their selection and sequencing. If one of

the purposes of an IE is to bring into focus the ooL of IEs, then it becomes necessary to consider how the examples used are selected and sequenced to bring this into focus. The importance of the choice of examples in illuminating the ooL is suggested by Rowland and Zaslavsky (2005). They argue that the teachers' choice of examples should be a deliberate and informed one as opposed to choosing examples randomly and without purpose. Examples should therefore be selected through a process of reflection, taking into account the particular outcome to be achieved by the teacher. In this regard, Bills et al. (2006) describe examples as anything that is used as raw material that aids in learners generalising the mathematical idea that is in focus. However, in order to make possible generalisation of the mathematical idea, this idea must first come to the fore for learners. It is for this reason that I use examples in this study as the raw material to bring into focus the ooL of an IE.

3.2.2. Different types of examples

Using a single or a few examples to highlight something is insufficient and may result in an inadequate view of the mathematical idea, as is argued by Watson and Mason (2002). For instance, a learner may not be able to arrive at a conclusion of the zero product law by a single example, such as $2 \times __ = 0$. A range of examples may be necessary to see common elements of the examples as well as to see the similarities between examples (Bills et al., 2006; Watson & Mason, 2002). The different types of examples are characterised according to their purpose.

Rowland (2008) offers two categorisations of examples. *Examples of* are those that are designed to be instances of a previously defined class, while *examples for* are those examples that are used to identify a still-to-be-defined class. Examples can be used to motivate specific mathematical ideas, in which case they are *examples for*. A range of examples can be considered as *examples for* if their purpose is to make it possible for learners to, for instance, identify a quadratic equation. In contrast, a range of examples can be designed to illustrate how an already developed mathematical idea is instantiated, which are *examples of* something. For instance, a range of examples can be generated to show how to solve a quadratic equation given in different forms. These are considered to be *examples of* how to solve a quadratic equation.

The two categories of examples, as described by Rowland (2008), can, however, be considered to be broader categories of the four types of examples offered by Watson and Mason (2002). The four types of examples are described according to their different roles, depending on the mathematical idea that teachers want to foreground. The four types are start-up examples, reference examples, model examples and contrasting examples.

1. *Start-up examples* are used to motivate definitions or mathematical ideas. They are used to arrive at a definition. For instance, in order to arrive at a definition of the zero product law, a range of numerical examples can be used to make this possible for learners.

A possible set of examples is:

- a) $3 \times 0 =$
- b) $0 \times 3 =$
- c) $5 \times 0 =$
- d) $___ \times 5 = 0$
- e) $-5 \times 0 =$
- f) $1432 \times ___ = 0$

All of these examples are designed to highlight that if two numbers are multiplied to produce zero, then at least one of them is zero.

2. *Reference examples* may be referred to at different times during the course of a lesson to link different mathematical ideas. For instance, the expression $x^2 + 5x + 6$ can serve as a reference example for learners when solving quadratic equations. The example helps learners link the factorisation of a trinomial to its use in solving a quadratic equation. It can also serve as an example used to identify a quadratic equation as well as to illustrate the general form of a quadratic equation (as in $x^2 + 5x + 6 = 0$).

3. *Model examples* refer to generic cases, which are used to generate specific instances. For instance, an example like

$$a \cdot b = 0$$

$$\therefore a = 0 \text{ or } b = 0$$

can be used to illustrate that in $(x - 1)(x + 2) = 0$ either $x - 1 = 0$ or $x + 2 = 0$ or both $x - 1$ and $x + 2$ can equal zero. It is therefore a model example used to generate specific instances of the use of the zero product law.

4. *Contrasting examples* are used to illustrate distinctions between mathematical ideas. For instance, while $x^2 + 5x + 6 = 0$ is a quadratic equation, providing an example like $x + 5x + 6 = 0$ is a contrasting example to illustrate how to identify a quadratic equation and differentiate it from a linear. It can also be used to show the difference in the solution strategies between a linear and a quadratic equation.

In this study I use the four categories by Watson and Mason as they are more specific and inclusive of the broader categories by (Rowland, 2008). A set of *start-up examples* can be used to show how a conclusion can be arrived at and are therefore *examples for* in Rowland's categories. A *reference example* can be regarded as an *example of* because it illustrates how to, for instance, solve a quadratic equation. Both the *model example* and *contrasting examples* can be considered to be *examples for* as they indicate a particular instance of a general case and to illustrate distinctions between mathematical ideas respectively. I therefore find Watson and

Mason's categories to be more useful in this study in specifying ways in which examples can be used to highlight the four features of IEs (as mentioned in section 3.1).

The use of variation theory in selecting and sequencing examples is another way that examples can be used to communicate teacher intentions. Variation theory is premised on the view that in order to see something in a certain way, a learner must be able to focus on some features that are critical in a range of examples. Put in another way, the range of examples must make it possible for learners to focus on what the teacher intends to highlight, i.e. the OoL. The value of variation theory in the classroom have been documented by many researchers within mathematics education.

3.2.3. Variation Theory in literature

Many studies illustrate the value of the use of variation in the teaching of mathematics. Some benefits of using variation theory include: making certain aspects of mathematical content more noticeable to learners (Kullberg, Runesson Kempe, & Marton, 2017); helping to promote generalisation of a mathematical idea (Pang, Bao, & Ki, 2017); advancing learner understanding of mathematical concepts (Huang & Leung, 2017).

Kullberg et al. (2017) in their study found that variations in examples used in two lessons on equations made it possible for learners to discern different mathematical ideas in each lesson. For instance, in the first lesson, the teacher used numerical values in equations (equal sign was invariant) but varied the numbers used, their order and the number of factors used. This variation made it possible for learners to discern the idea of equality in equations since the equations did not require solving (values were already known). In the second lesson, a letter (x) was introduced in equations with the same numeric values as used previously. This had the effect of making it possible for learners to discern the structure of an equation and how the different operations affected an equation.

Pang et al. (2017) in their study found that the use of variation theory in adding 3 digit numbers helped learners to not only perform the correct algorithm for adding 2 and 3 digit numbers but to also extend the general principles of adding to digit numbers to the addition of 3 digit numbers. For instance, variation in the placing of the digits and the value of the digits helped learners to look at the different ways of splitting and regrouping numbers and also how the placing of digits affected the carry over of a value. The results of this study showed that the use of variation in the examples promoted learning beyond that of the use of a particular algorithm. The use of variation helped to promote generalisation of mathematical principles by bridging the gap between what learners already knew to some new mathematical idea.

Huang and Leung (2017) studied the use of variation in the teaching of geometry showing that varying the orientation, sizes of geometric diagrams promoted learner understanding of

corresponding angles. The use of contrasting examples was used by the teacher to check if learners had mastered the identification of corresponding angles and when corresponding angles would be equal. The results showed that the variation of the orientation and size of the diagrams together with the different representations (use gestures, drawings and verbal) learners were guided into developing and consolidating the idea of corresponding angles being equal only when the lines forming them are parallel.

All of the above studies indicate the different benefits of the use of variation theory in the mathematics classroom. The focus in this study is on the coherence of teacher explanations and more specifically on how the purpose of the explanation is brought to the fore for learners. The use of variation theory in the selection and sequencing of examples is therefore pertinent to this study.

3.2.4. Key tenets of variation theory

I focus on three key tenets of variation theory that are useful for the purposes of this study. These are: a focus on the object of learning, variance amid invariance, and contrast. I elaborate on each in the following sections.

3.2.4.1. The object of learning

A key idea in variation theory is the object of learning (OoL). Variation theory is based on the premise that learning is about something. Any learning is directed towards something that is the focus of attention. That which is the focus of attention is called the OoL (Marton & Pang, 2006). The OoL is described as a capability that has a general and a specific aspect. The capability of a learner refers to what learners should know and do at the end of an IE. For instance, learners should develop the ability to define terms, or the ability to provide examples. The general aspect concerns the nature of the capability, for example, remembering, discerning and interpreting. In other words, these are the actions that learners should be able to carry out at the end of an IE. The specific capability relates to the thing or the object on which the action is carried out, for example, symbols, formulae or equations. Learners should know how to carry out actions on specific symbols, formulae or equations. While learners focus on what is acted upon, the teacher should focus on both. The teacher should focus on what it is that learners are trying to learn. They also need to focus on how learners are trying to learn this something. Teachers thus focus on an OoL.

The OoL may sometimes be more or less of a conscious thought by the teacher or may be more or less elaborated. A particular set of examples may be intended by the teacher for learners to focus on a particular OoL of an IE. The OoL refers to what is made possible to learn within a particular context and is what learners encounter in the classroom. Marton, Tsui, Chik, Ko, and Lo (2004) emphasise that the OoL is the researcher's interpretation of whether and to what

extent the OoL of an IE comes to the fore for learners. The OoL is therefore described as *what is made possible* to learners depending on the researcher's interpretation based on their research interest and theoretical orientation. The teacher's role in how she uses the examples to support her IEs is paramount in what is made possible to learn from the examples used. This study is a focus on teachers and their explanations. I therefore focus on what is made possible to learn and do not examine what learners actually learn from the IE as it falls outside the scope of this study.

3.2.4.2. *Variance amid invariance of critical features*

A set of examples may be designed by a teacher to highlight the ooL (I use lowercase "o" to refer to the ooL of an IE) of an IE which may be obvious to a teacher but not so for learners. The ooL of an IE may sometimes be largely left tacit or unsaid and so must be brought into focus for learners in some way. There are some aspects that are necessary to focus on in a set of examples in order for the ooL of the IE to come into focus for learners. These are known as critical features, which all teachers should be aware of (Lo & Marton, 2011). An enlightening example to describe how different features of something can be focused on in different ways by different people is provided below:

The different ways of seeing the person by different people are not wrong. Rather, they are incomplete ways of seeing him. If we want others to see an object in exactly the same way as we do, then they must also be able to focus on the same features that we do. To see an object in a particular way, we must focus on certain features that are critical to a certain way of seeing known as 'critical features' (Lo, 2012, p. 27).

It is the teacher's job in the classroom to make possible the discernment of the ooL by learners. One of the ways to make this possible is through the use of examples in an IE. It is necessary for a teacher to design examples as well as to sequence them in a way that would make it possible for learners to take note of the critical features.

A key idea in variation theory is to keep the critical features constant while varying other features. A set of examples can be developed with small changes from one to the other while keeping other parts invariant. If for instance the ooL of an IE is that learners should be able to identify a quadratic equation, then what is critical is that learners be able to identify that a quadratic equation is an equation with a degree of two. A range of examples can be provided to highlight that an equation having a degree of two is a necessary condition where the number of terms, their signs, their order or the values on the right hand side does not affect it being quadratic. The second-degree term is then kept invariant while the other aspects are varied to highlight that a quadratic equation is a second-degree equation. Adler and Pournara (2019) illustrate how a carefully constructed set of examples using variance amid invariance can bring into focus persistent learner errors (a feature of an IE as outlined by Leinhardt) and attend to connections between representations.

3.2.4.3. *Contrasting examples*

Marton and Pang (2013) caution against using “sameness” to point out what something is. If, for instance, a teacher wants to highlight what a quadratic equation is, it may be insufficient to develop a series of equations that are all quadratic to highlight what a quadratic equation is. According to variation theory, learners are able to discern the critical features of some mathematical idea if they are able to compare it with something it is not, i.e. a quadratic equation should be contrasted with an equation that is not quadratic (e.g. a linear equation). This is known as a contrasting example (Marton & Pang, 2013).. The contrasting example, while drawing attention to the critical as well as non-critical features of a mathematical idea, helps to draw boundaries around the idea (Zaslavsky, 2010). In other words, using contrasting examples can help to identify the conditions of use of a mathematical idea. Furthermore, Zazkis and Chernoff (2008) point to the power of contrasting examples to attend to learner errors. A contrasting example, when presented in the light of a learner error, can make it possible for the error to be foregrounded for the learner.

3.2.4.4. *Sequencing examples in a set of examples*

How examples are sequenced is an important consideration for what is made possible to learn in the variation of the examples. For instance, if the contrasting example is the first example of an example set, it may not have the desired purpose of distinguishing between what something is and what it is not. In addition, if there are many changes between two consecutive examples, what is intended by the set of examples may not be apparent to learners. The examples within the example set therefore need to be sequenced with small changes from one to the next in order to bring the ooL into focus for the learners. How the examples are sequenced will therefore determine what is made possible to learn from the set of examples.

3.2.4.5. *Example of the use of variation theory*

An example set illustrating the key tenets of variation theory used in this study is provided in table 3.1.

1) $x^2 + 5x + 6 = 0$
2) $x^2 + 5x = -6$
3) $5x - x^2 + 6 = 0$
4) $x^2 + 5x = 0$
5) $5x + 6 = 0$
6) $x^2 - 4 = 0$
7) $x^2 = 4$

Table 3.1: Example set to identify a quadratic equation using variation

This set of examples consists of varying some things while keeping others invariant, and the use of contrasting examples.

- **Invariance:** In all equations (except equation 5) x^2 is kept constant or invariant to highlight the condition that makes an equation quadratic, i.e. the degree 2. The coefficient of x^2 remains 1 or -1 in all examples. In addition, only the numerals 1, 4, 5 and 6 are used. The use of the same numbers and the same coefficient of x^2 helps to focus learners' attention on the degree of the expression. Using the same numbers, 1, 5 and 6, in the first three examples also makes it possible for learners to note that it is only the position of the terms and the signs that are varying while the x^2 is invariant.
- **Variance:** The number of terms, their signs, order and the values on the right-hand side vary to highlight that they have no effect on an equation being quadratic. Providing quadratic equations with two as well as three terms makes it possible for learners to note that it is the degree of the expression and not the number of terms that distinguishes a quadratic equation from a linear.
- **Contrasting example:** The fifth equation can be regarded as a contrasting example since it is a linear equation and not quadratic. Experiencing an equation that is linear makes it possible for learners to distinguish a quadratic from a linear equation. Although the last four examples consist of two terms, the difference between example five and the other three equations with two terms is that example 5 has a degree of one, unlike the other three.
- **Sequencing of examples:** The first three examples consist of three terms in different order. Having them in sequence with small changes from one to the other makes it possible for learners to note that the order of the terms does not affect an equation being quadratic. Having the next four equations, all with two terms and again a different order of terms, has the effect of drawing attention to the number of terms being irrelevant to an equation being quadratic. Sequencing them in this way rather than having equations with three and two terms in random order has the potential to bring into focus the oOL of the IE.

An example set can therefore be developed to make it possible to bring into focus the oOL of an IE.

3.3. How I use the different types of examples in this study

I use two of the four types of examples, as provided by Watson and Mason, in this study to examine the features of IEs and so to the oOL that they bring into focus. I use both *reference* and *model examples* in this study to look at the features of IEs that examples foreground. While *reference examples* highlight the relationships between different mathematical ideas, *model examples* are used to indicate a generalisation of a mathematical idea. I consider both to be

important features of IEs which are not highlighted as features in Leinhardt's theory. These types of examples are also not catered for within variation theory.

I do not use *start-up examples* from Watson and Mason's categories. These examples make it possible to arrive at a definition or a mathematical idea, which is an important consideration in IEs. The use of variation in a set of examples, however, can also be used for the same purpose. Varying the type of values (numerical and letters), changes in signs or varying the order of values used in a set of examples, for instance, can all make it possible for the definition to be arrived at. I therefore use the idea of variance in examples rather than *start-up examples* in this study.

Apart from the choice and sequencing of examples, to draw attention to a mathematical idea, it is also important that a teacher foregrounds what she wants to draw attention to. One of the ways that this can be achieved is through a useful set of representations, which Leinhardt (2001) mentions as a feature of IEs.

In this study I consider how the ooL of an IE are brought into focus by what is written and how it is written by the teacher, whether on a chalkboard (or whiteboard), using posters or in any other way. I refer to these as inscriptions. While examples are generally written, how the verbal and written forms combine to form an IE are not in focus in the literature on IEs. In fact, Leinhardt (2001) acknowledges this limitation in her work by specifying that examples are backgrounded while emphasising that discussions around tasks are important. The strategic arrangement of inscriptions made can help to highlight the ooL of an IE. In this study I show how the examples as well as solutions to questions and summaries are preserved and arranged that contribute to "*a set of discussions*", which therefore encompass more than the utterances of a teacher in an IE.

3.4. A useful set of representations as inscriptions

One of the ways to bring into focus the ooL of an IE is through a useful set of representations (Leinhardt, 2001). In this study, I focus on inscriptions made in an IE as a useful set of representations. I use the word inscriptions to refer to the written representations of mathematical ideas, which include examples written on the chalkboard, diagrams drawn on the chalkboard, written language, markings made on the chalkboard to emphasise something (e.g. underlining something) and written mathematical symbols (Venkat & Askew, 2018). I do not however include written forms as those found, for instance, in text books because my focus is on teacher-generated inscriptions that occur while an IE is being offered. I focus on the use of a publicly available space to write on which includes, but is not restricted to, a teacher writing on poster paper, an overhead projector, a whiteboard or by means of projecting what is written on a tablet onto a screen. While these are all spaces to write on, I refer to writing on the chalkboard in this study to encompass any or all of the above-mentioned ways to display information. I use the chalkboard to represent all of the different public displays of information. While recognising that there are other means of public displays of information in the classroom I concentrate on the use

of the chalkboard because my data indicates the heavy reliance, by teachers in my sample, on the chalkboard.

While the chalkboard is commonly used within South African classrooms, it may not always be used effectively. Although there is a commonly held view that the chalkboard is an old-fashioned tool that has no impact on student learning, its innovative use can have a profound effect on learning (Yoshida, 2005). I consider in this study some of the ways that teachers can use the chalkboard in an effective manner to highlight the ooL of an IE.

The use of the chalkboard has evolved through the years from being a means to display information to its current effective use (especially in Japan) in supporting teaching. (For a fascinating commentary on the evolution of the chalkboard, which is beyond the scope of this study, refer to Tan, Fukaya, and Nozaki (2018)).

3.4.1. Different ways of using the chalkboard and their benefits

Apart from displaying information, the chalkboard is used to build coherence during a lesson in the Japanese classroom (Ermeling, 2015), to provide opportunities for learner engagement during lessons, and to compare and discuss different ideas and thinking (Greenhalf, 2015; Yoshida, 2005). The literature highlights three main ways in which the chalkboard can be used to bring into focus mathematical ideas. They are:

1. What is written
2. What is preserved, and
3. What is highlighted and how it is highlighted.

I elaborate on each of these below.

3.4.1.1. What is written

Yoshida notes that the chalkboard is used to keep a record of the lesson. This includes the problem that is being investigated, questions posed by the teacher that provoked learners' thinking about the problem, learners' different solutions, and important mathematical ideas generated by discussions. Ermeling (2015) notes that the trajectory of a lesson is illustrated on the chalkboard. This makes it possible for learners to follow the progression of a lesson and so to see how the sequence of mathematical ideas fits together to arrive at particular conclusions. The inscriptions written in sequence promote the coherence of a lesson Yoshida (2005). Fig. 3.1 is an example of a teacher's plan of the content to be inscribed on the chalkboard in a particular sequence.

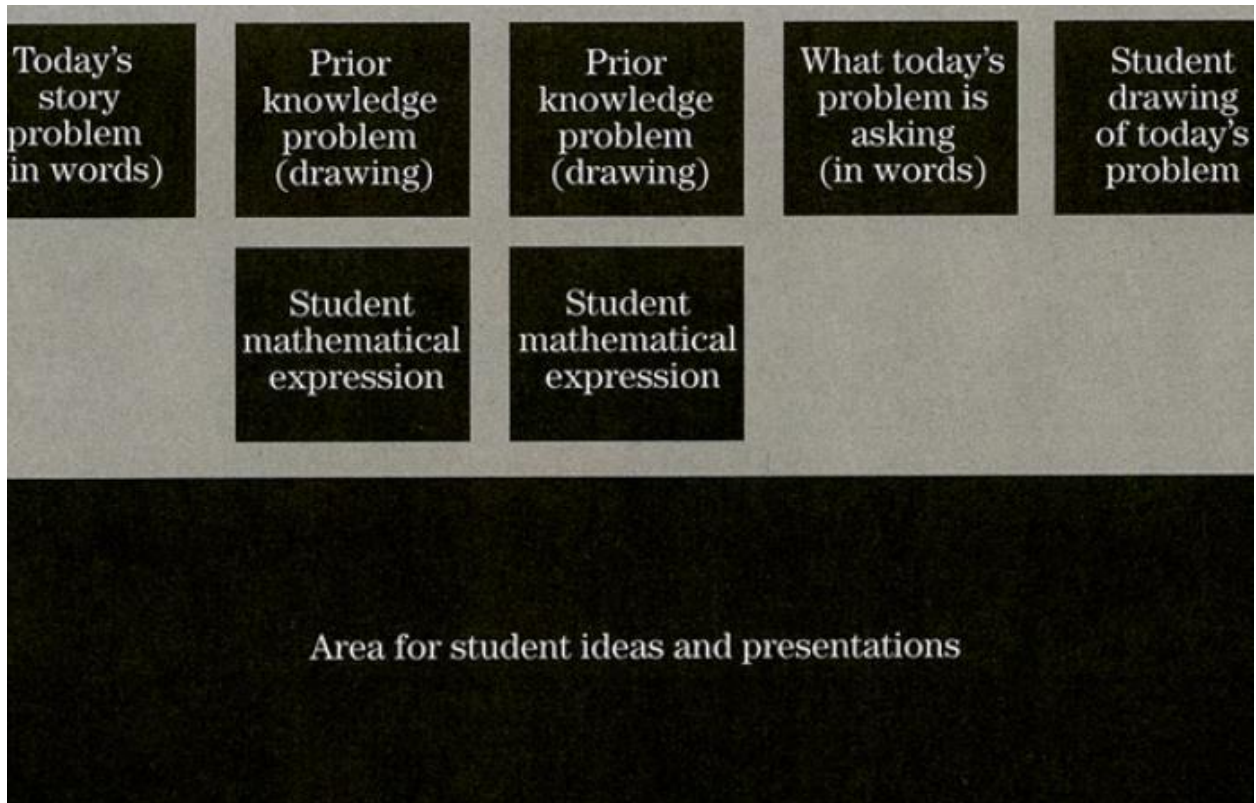


Figure 3.1: Example of content to be arranged on the chalkboard (Yoshida, 2005, p. 96)

As is evident in the plan, space on the chalkboard is allocated for the problem to be worked on, the content on the prior knowledge required by learners, inscription of an elaboration of the problem and learners' inscriptions of their thinking. The inscriptions of the content of the lesson written in sequence promotes the coherence of a lesson (Yoshida, 2005). It is not only what and how it is written on the chalkboard that promotes coherence but also preserving what is written for referring to at different points of the lesson.

3.4.1.2. *Preservation of content*

An important consideration in working on the chalkboard in Japanese classrooms is that inscriptions made on the chalkboard were seldom erased by teachers. Preserving content on the chalkboard helps teachers and learners to refer to content written on the chalkboard at an earlier point in the lesson. It also helps learners to see logical connections between different parts of the lesson (Greenhalf, 2015). It facilitates learner engagement in the discussion of key mathematical principles. Learners' work that remains on the chalkboard is used to compare, contrast and discuss the different learner ideas, their similarities and differences. The merits of ideas are discussed, which helps learners to make connections by discovering new relationships between different mathematical ideas. This use of the chalkboard ensures dynamic mathematics lessons. Yoshida (2005) refers to this part of chalkboard use as a "collective think pad" because of the whole-class discussion it generates based on the ideas written on the chalkboard. Learners however may not be able to see these connections themselves. It requires the teacher to make

these possible to perceive. The teacher's role in making the connections explicit is however not highlighted in the literature on chalkboard use.

3.4.1.3. Highlighting key ideas

Learner solutions are distinguished from teacher solutions by the use of different coloured chalk or markers. This is an important consideration so that the content written by teachers can be easily read by learners. Different coloured chalk can also be used to highlight different types of content written on the chalkboard. For instance, new terminology, questions for further elaboration, equivalent expressions or alternate solutions can all be colour-coded to distinguish different ideas (Ontario, 2011).

The literature does not focus on the teacher's role in highlighting the key ideas inscribed on the chalkboard. The literature on chalkboard work documents the planning prior to the lesson but does not document its enactment. Research on the use of the chalkboard in Japan stems mainly from pedagogy that is considered to be learner-centred. A large part of the lesson in a Japanese classroom centres around learners noting their ideas on the chalkboard, with a resultant discussion on all of the ideas between teachers and learners (Yoshida, 2005). While the limited literature on chalkboard use highlights the benefits of preserving work on the chalkboard, Tan et al. (2018) acknowledge that the teacher's role in what is written on the chalkboard, what is erased and what is not erased from the chalkboard, and how this promotes learners discovering connections between different mathematical ideas is not in focus in the research on chalkboard use.

I consult the literature on chalkboard use in Japan to illustrate how this teaching aid is used within the context of South African classrooms. The different uses of the chalkboard highlighted in the literature arise from a more learner-centred approach. As mentioned previously (in chapter 1), this study is conducted within the context of teacher-centred instruction. As a result, the strategies on how to use the chalkboard may be different from that used in Japanese classrooms. For example, in this study it is mostly the teacher who writes on the chalkboard while sometimes noting learner suggestions. It is the teacher who decides where to write on the chalkboard and how to place the content on the chalkboard. The teacher makes these decisions based on what she wants to foreground in an IE. For instance, a teacher may deliberately arrange examples with closely related ideas next to each other on the chalkboard to suggest close connections between them and make these connections explicit through her utterances. I illustrate how I have taken all the ideas of chalkboard use in Japanese classrooms to analyse my data on teacher IEs.

All of the research on the use of the chalkboard mentioned indicates its benefits during a lesson. I extend the strategies of chalkboard use highlighted in the literature to illustrate how the chalkboard can be used in IEs. For example, Leinhardt (2001) highlights that one of the purposes of an IE is to address learner errors. One of the ways that this can be done is through the strategic

use of the chalkboard. There are many ways in which the chalkboard can be used to highlight learner errors. A learner's incorrect solution to a question can be placed alongside the correct solution so that the differences in the solutions can be brought into focus by the teacher and other learners. Another strategy would be to place a contrasting question next to the given question to show the differences in the solution strategies.

The use of inscriptions on the chalkboard can provide insight on IEs. They can be used to highlight learner errors as well as the ooL of an IE, but these need to be made explicit to learners. How a teacher communicates these to learners will determine whether the ooL, as well as other mathematical ideas, is highlighted for learners.

3.5. Teacher talk in IEs

The most prevalent means of communication in the classroom is through teacher talk. This is suggested by Leinhardt (2001) when she refers to IEs as consisting of “*a set of discussions*”. While it may be implicit in her definition that IEs may be more than the verbal, she does state that “*classroom talk [is] ... the means by which we can see explanations being developed*” (Leinhardt, 2001, p. 342). Classroom talk in this sense includes both learner and teacher talk but the point is that in my reading of Leinhardt's work, it seems to me that she foregrounds the verbal as the principal means of communication in the classroom. This resonates with studies by Erath et al. (2018) and Esmonde (2009). Other studies, however, point to other means of communication besides the verbal. For instance Kress, Charalampos, Jewitt, and Ogborn (2006) treat teacher talk as one of several different modes of communication in the classroom. They argue that communication like gestures, written texts, pictures etc. are other ways, together with talk, that are used to communicate ideas in the classroom.

In this study I examine teacher talk as one of several other ways of examining the communication by a teacher in the classroom, as argued by Kress et al. (2006). There may be different ways in which a teacher's intentions are conveyed in an IE. For instance, the use of variation in examples can be used to highlight a mathematical idea/s, as illustrated in section 3.2.3.2. Similarly, inscriptions on the chalkboard (section 3.4.1.) can be made in particular ways to highlight learner errors or to indicate relationships between different mathematical ideas. While the use of variation in examples and the inscriptions on the chalkboard may be used to make it possible for learners to attend to some mathematical idea, these need to be made explicit. Teachers must explicitly guide learners into perceiving what it is that the examples are intended to show, as well as what is inscribed on the chalkboard (Planas, 2021). This suggests that a teacher's talk is used in the mathematics classroom for different purposes.

3.5.1. Teacher talk as purposeful

Teacher talk is used to communicate a teacher's intention in providing the IE. There may be different purposes that particular teacher talk is meant to highlight. For instance, Venkat and Askew (2018) identified three different purposes for teacher talk, as listed below:

1. Teacher talk for generating solutions that should be coherent to learners: a focus on whether the solutions to tasks were presented in a coherent manner.
2. Teacher talk to focus on mathematical connections between topics: a focus on whether teacher talk makes it possible for learners to be able to use a mathematical idea in another similar situation.
3. Teacher talk to help learners to make connections between their prior knowledge and new learning.

These are some of the purposes of teachers' talk in the mathematics classroom. The purpose of teacher talk in the classroom can be examined through many lenses depending on the purpose of the talk. Whatever the purpose of teacher talk is, though, Planas (2020) emphasises the need for a teacher's talk to make explicit to learners whatever is meant to be communicated.

In this study I focus on teacher talk in making explicit what the use of examples and the use of inscriptions and talk (mentioned above) foreground. I therefore examine whether a teacher makes explicit what he foregrounds through the use of variation in his examples and through the inscriptions made on the chalkboard. In addition, I examine whether a teacher's talk makes explicit links between different IEs that combine to form a composite IE. I use the two features of IEs, i.e. using examples and using representations (I use the terms 'using inscriptions' and 'talk' instead of 'using representations' in this study) to examine how the other features are enacted in an IE. I therefore examine how a teacher's talk in using the two features of IEs makes it possible for learners to:

1. Identify the conditions of use of a mathematical idea,
2. Highlight core mathematical ideas,
3. Resolve their errors, and
4. Attach prior knowledge to new knowledge.

While these are highlighted by Leinhardt (2001) in her theory of IEs, they may not be the only features that my study may highlight. There may be other features foregrounded through *using examples* and *using inscriptions and teacher talk*. I refer to these two features as lenses henceforth to differentiate between them and the four features (as listed above).

I use the lenses of the use of examples and inscriptions and talk to examine IEs in this study. Using these lenses to examine IEs can bring into focus the other features of IEs. I therefore propose the rearrangement of the features of IEs, as illustrated in Figure 3.2.

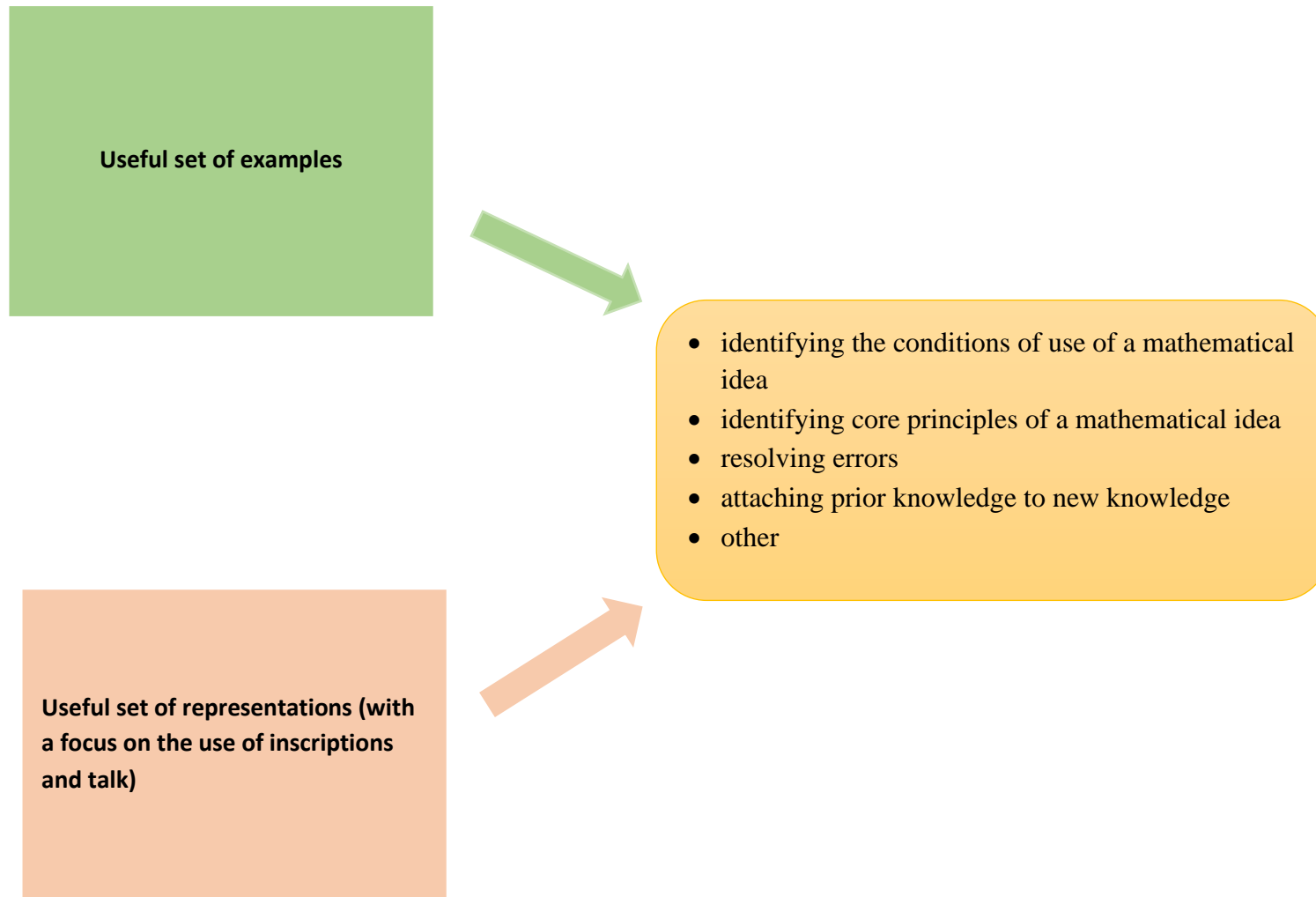


Table 3.2: My rearrangement of Leinhardt's features of IEs

3.6. Discussion and conclusion

I highlighted in this chapter how I have made use of Leinhardt's theory of IEs in this study. In the previous chapter, I provided a review of Leinhardt's theory of IEs. The review highlighted the need to elaborate on a definition of IEs but also pointed to a need to link together the different features of IEs which are presented as discrete features by (Leinhardt, 2001). While in the previous chapter I focused on an elaboration of Leinhardt's definition of an IE, in this chapter I have illustrated how I use the two features of IEs, which I refer to as lenses, in this study. I use the lenses of the use of examples and inscriptions and talk to examine IEs in order to illustrate their power in observing the four features of IEs, as provided by Leinhardt (2001).

Reviewing the literature on the use of examples and inscriptions and talk reveals, for instance, that it is through these that learner errors can be addressed or that the conditions of use of a mathematical idea can be identified. In addition, viewing IEs through the two lenses has the potential to foreground other features. For instance, preserving content on the chalkboard can make it possible for it to be referred to again in another IE, and this can make it possible for learners to connect between different mathematical ideas. Connections between different mathematical ideas are therefore an additional feature that the use of inscriptions can bring into focus.

It is through the rearrangement of the features of IEs that the links between *a useful set of examples* and *a useful set of representations* and other features of an IE can be highlighted. In highlighting these links, the rearrangement of the features makes it possible to arrive at a way to examine the coherence of IEs in bringing into focus the oOL of an IE.

In the next two chapters I elaborate on the methods used to collect data and the analytic process involved in illuminating the findings I report on in chapters 6, 7, 8 and 9. These findings were identified not only through the literature but also through a grounded analysis of teachers' IEs.

Chapter 4 : Methodology

4.1. Introduction

The focus of my study is to examine the nature and quality through the features and the coherence of IEs. I do this through addressing the following research questions:

1. What features of mathematical instructional explanations are revealed through:
 - a) teachers' use of examples
 - b) teachers' use of inscriptions and talk
 - c) teachers' summoning back prior explanations

3. What makes a mathematical instructional explanation coherent?

This is a qualitative study conducted within the interpretivist paradigm. Data was collected over two years through classroom observations of the lessons taught by six teachers. While teachers were interviewed using excerpts from their video-recorded lessons, I did not use the results of the interviews as this falls outside the scope of this study. The aim of this study is to examine the features of IEs which could be used to analyse their coherence. It was therefore not necessary to interview teachers understand their intentions as this was not an aim of the study. The analysis of the classroom observation was conducted using grounded theory. I elaborate on my research approach, design and methods in this chapter, together with my philosophical orientation that underpins these issues. I also outline how I addressed ethical issues during the study as well as why I consider the study to be trustworthy and credible.

4.2. Research design and methodology

Methodology is a way of thinking about and studying phenomena, including social phenomena, (Corbin & Strauss, 2008) and refers to the underlying principles of inquiry (Wolcott, 2008). Social phenomena can be studied by conducting research and while there are other ways to understand the social, e.g. through experience and reasoning, research provides a systematic and controlled way to do this. Research is described by Cohen, Manion, and Morrison (2013) as a systematic, controlled and critical investigation of assumptions held by an individual about supposed relationships between phenomena. It is systematic in that it is based on inductive or deductive methods of inquiry, and is self-correcting through mechanisms built in to minimise errors. However, this is but one view of research; another is a view that emphasises the differences between humans and inanimate, between the social and the physical. Such differing views originate from the varying ways in which researchers view reality and social behaviour. So research is not just a mechanical task to be carried out, but is influenced by how one views the world and the nature of knowledge generation. What an individual values determines to a large extent how she views the world (Cohen et al., 2013). Each researcher has different views about

truth and knowledge which guide the beliefs, assumptions and thoughts researchers have about the world they live in and while it may be largely tacit it nevertheless shapes the way in which they conceptualise and conduct their research (Yazan, 2015). The beliefs, assumptions and thoughts of a researcher influences any paradigm choice in investigating a particular topic. A paradigm is described as a shared world view which are represented by the beliefs and values by individuals within a discipline (Schwandt, 2001) and how these views are organised into a coherent whole (Cohen et al., 2013). The choice of paradigm in turn guides how a particular research topic is investigated by the researcher. This determines the way in which data is collected and what instruments are used in collecting this data. i.e. the research approach. I provide a brief summary of the different views of reality before providing the details of how I conducted this research. Through these I rationalise my data collection instruments, the analysis of the data collected, and how I test the trustworthiness of the collected data.

4.3. Comparison of two main paradigms

While different researchers consider many different paradigms including critical theory, post-positivism, emancipatory, postcolonialism etc., I focus on two paradigms - post-positivist and interpretivist - as they stand in contrast to each other, enabling me to make distinct my view of the world and how this view guided my study which ultimately I conducted within the interpretivist paradigm.

4.3.1. The post-positivist paradigm

Post-positivists believe that there is a true reality that exists but this cannot be fully apprehended (Morrow, 2007). They reject the positivist assertion that a researcher can be an independent observer of an event without being affected by personal bias. The post-positivist researcher does however see objectivity as an ideal to strive towards. Whilst the post-positivist researcher recognises and acknowledges biases in her work and so accept that her own ideas on a particular topic influences what she concludes, the post-positivist researcher accepts that there could objective answers to question by taking steps to minimise the effects of these beliefs makes it more likely to move closer to objectivity (Panhwar, Ansari, & Shah, 2017).

4.3.2. The interpretivist paradigm

Interpretivist researchers recognise that objectivity is impossible and therefore do not strive towards objectivity like the post-positivist researcher. Instead research conducted within the interpretivist paradigm acknowledges that there are as many realities as there are research participants (including the researcher) (Guba & Lincoln, 1994). The focus of researchers within the interpretivist paradigm is to work with subjective knowledge of a phenomenon. They report on events using their personal views, beliefs, experiences to provide a plausible reason or many

plausible reasons for the occurrence of phenomena. They acknowledge that people differ from each other and that people behave in ways that are context-bound.

There are two layers of subjectivity within the interpretivist paradigm. First the participants themselves react in certain ways because of their own views, past experiences, beliefs and contexts. The researcher thus seeks to understand individuals' interpretation of the world around them and interprets findings in light of how participants make sense of their world taking into consideration the participants' prejudices, biases, views and experiences (Cohen et al., 2013). Instead of merely perceiving social circumstances, individuals continually make sense of them within socially constructed and shared meanings and these interpretations influence behaviour. However, people not only react to or interpret circumstances, they co-construct them. The interpretivist paradigm presents a view of people as acting with intention; they are deliberate and creative in their actions. Knowledge is gained by individuals who actively co-construct the world in which they live and who are not merely passive reactors to their social world. Individuals construct models or concepts to make sense of some experience but continually change or adapt these in light of new experiences (Hughes, 2010). Research conducted within this view is conducted with the aim of seeking to understand how individuals create, modify and interpret the world in which they live.

My aim in this study was to understand how teachers used their knowledge and other resources to provide mathematical explanations to their learners. For instance, my findings show how teachers used a readily available resource i.e. the chalkboard in creative ways to provide explanations. One of the ways that they did this was to arrange content on the chalkboard in particular ways which together with their talk helped to connect questions to each other (see section 7.3).

While research participants are actively involved in constructing their worlds by interpreting and making meaning through their interaction with others, the researcher is also an active interpreter of the same social world which the participants inhabit (Fox, 2011). Researchers therefore have to acknowledge their own interpretations of the social world of the participants. Therefore, when researchers report on their findings they must acknowledge that they are imposing their own interpretations as well. Their own interpretations result from the way in which they have coded their data.

I conducted this study within the interpretivist paradigm. My coding and analysis of teachers' explanations resulted from my interpretations of these explanations. My interpretations were, however, based on my reading of literature on explanations and more specifically on Leinhardt's theory of teacher explanations. My prior teaching experience was also one of the factors that contributed to how I analysed teacher explanations. For instance, while I have more than 25 years of teaching experience, how some of the teachers in my study arranged work on the

chalkboard and how they used this arrangement to make connections between questions made me realise that my own teaching could be enhanced by focusing more on how content can be arranged on the chalkboard. This therefore influenced my decision to further analyse this aspect of teacher explanations.

4.3.3. Limitations of the interpretivist paradigm and my stance in this study

One of the limitations of using interpretivist approaches in research concerns the generalisation of findings. While positivists actively seek to generalise their findings, Corbin and Strauss (2008) caution against generalising the findings within the interpretivist paradigm (Corbin & Strauss, 2008). However, making generalisations is a human quality, as Stake (2010) reminds us:

Every thinking moment has its generalizations. No two experiences are entirely different. As soon as we have two, we start to expect something in a third. We generalise. Although we sometimes overgeneralise, we sometimes under-generalise. Keeping the balance is important. Particularisation and generalisation in balance (p. 196).

People interpret events within their particular contexts and situations and act accordingly.. While it is not always possible to refrain from generalising, the interpretive researcher sometimes generalises beyond particular situations. It is, therefore, the decision of the user to determine how and to what extent the findings of a research study can be used within other similar contexts by taking into consideration other intervening factors.

In conducting this research, I have developed a framework (see section 10.3) that can be used to analyse teacher explanations in mathematics classrooms. I therefore consider the results of this study to be generalisable to other mathematics classrooms. While the framework is a tool that can be used by researchers and teachers in other classrooms, I have provided the details of the classrooms within which this study was conducted. For instance, this study was conducted within the context of under-resourced schools and teacher-centred instruction. It is therefore left to the readers to determine the extent to which this framework can be used within their particular contexts.

One of the other limitations of the interpretivist paradigm is that it may be biased in that the researcher chooses what to select and report on. Stake (2010) argues that all researchers have bias but it is important that researchers learn how to minimise the effects on the study. What is to be reported on depends on the researcher, who decides what will deepen understanding of the phenomena being studied. A similar idea is echoed by Bogdan and Biklen (2007), who offer that it is impossible for a researcher to pursue everything because it will result in reporting on data that may be inappropriate for the study. The suggestion is that the focus of the research will dictate what data should be pursued in the particular study.

As the researcher, I chose what to investigate and further pursue then what to write about. I chose to focus on particular aspects of IEs and ignored others. For example, I did not focus on the terms used by a teacher in her IE e.g. mathematical versus non-mathematical terminology. I chose to do this for two reasons. First is that the use of mathematical and non-mathematical language has been extensively studied (e.g. Adler & Venkat, 2014; Planas, 2020; Raiker, 2002). My intention in this study was to add something new to the theory of explanations. The second reason is that I considered it important to study the interaction between what was spoken by a teacher and how this was intertwined with what was written. My focus in this study was therefore how the spoken and written words were used to support each other in an explanation. There is a paucity of literature on this aspect of teacher explanations.

While some consider this my bias as a researcher, my rationale for the selection of what results from my analysis to focus and write on was based on the purpose of my study. I chose to focus on those parts of teacher explanations that would help to shed more light on the theory of teacher explanations. While the teacher explanations that I analysed contained many features, I focused on those features that were either not extensively covered in the literature or that could add to the theory of teacher explanations. These features were selected to develop a framework that could be used to analyse the coherence of teacher explanations.

All IEs were initially analysed to explore what they offered. I selected to further analyse and write on particular IEs of teachers based on the selected features (as explained in the previous paragraph). For instance, all teachers in my study used the chalkboard to support their utterances. I only provided examples of those explanations that best illustrated the movement between the verbal and the written. I then used these features to eventually develop a framework of coherence.

4.4. My research approach

My aim in this study is to gain a deeper understanding of the nature and coherence of teacher explanations. I therefore conducted this research using an interpretive approach. qualitative approach. Guba and Lincoln (1994) and Sharan (2009) describe two main approaches to social science research. One way is to describe facts and characteristics of a phenomenon or relationships between events and phenomena that can be measured in terms of numbers. While the other approach also describes characteristics of phenomena, the focus is on understanding how people interpret their experiences or how they come to construct and make sense of the world in which they live. It is not meant to find causal relationships but rather to understand phenomena under investigation within a particular setting. I outline the research methods I used within this study.

4.5. RESEARCH METHODS

Methods are the techniques and procedures for gathering and analysing data (Corbin & Strauss, 2008). There are no particular methods for data collection and analysis in qualitative research;

however, if the research is organised by methods alone it will result in a mishmash of findings (Ragin & Becker, 1992). Stake (2010) therefore suggests that the methods used should be guided by the research questions. Once a researcher knows what she is looking for, she can determine how to go about searching for it. Sharan (2009) mentions that observations, interviews and document analysis are the main data collection methods related to qualitative research. The choice of one or more of these will depend on what the purpose of the research is.

In line with the literature, I collected data through classroom observations of six teachers, which were video recorded, after which interviews were conducted and audio-recorded. Since my focus in this study was on studying the nature and quality of explanations examined through their features, which were finally incorporated in developing a framework of coherence of IEs, classroom observation was the most suitable method. While interviews were conducted these were not analysed because of the change in focus of this study. Interviews were conducted in order to examine how teachers made use of their learning from a professional development course (TM1 course as discussed in section 1.2) which was the original focus of this study. The change in focus to that of examining the nature and quality of IEs, meant that data from interviews was not within the scope of the study and so were not analysed. The selection of the six teachers was based on their participation in TM1, where they were provided with guidelines on the offering of mathematical IEs during the course.

4.5.1. Sample Selection

The selection of participants in this study was done purposively. The participants of this study were six teachers (four in 2018 and two in 2019, as explained below) who were purposively chosen from five different schools, in and around Johannesburg, Gauteng, with two teachers from the same school. They were chosen on the basis of having participated in the WMCS course and were therefore considered to be well informed on offering IEs. All selected teachers had a minimum of five years' experience in teaching mathematics (at the time of the study) and some form of tertiary mathematics qualification.

Purposive sampling is used in interpretivist studies where the aim is to obtain insights to increase understanding of the phenomenon being studied (Onwuegbuzie & Leech, 2007). The researcher deliberately chooses participants on the basis of the qualities that they possess. The researcher sets out what it is that he is researching and looks for people who can or are willing to provide the information by virtue of their particular knowledge or experience (Etikan, Musa, & Alkassim, 2016). The purpose of a grounded study is to build theory. It is therefore suggested by Crowe et al. (2011) that participants who can contribute to the development of theory be selected.

Table 4.1. provides a summary of the qualifications, teaching experience and other relevant information of the six selected teachers.

Teacher name (Pseudonym)	School taught at during study (Pseudonym)	No of years teaching experience	Years teaching Gr 10 maths	Qualifications	Subjects taught during study	Age range	Other useful information
Tsebo	Hades High School	5	3	<ul style="list-style-type: none"> • B.Sc³ (economics and mathematics) • PGCE⁴ maths 	Maths	31-35	Presents workshops to other maths teachers Provides extra maths lessons to learners.
Ulwazi	Apollo High School	7	4	<ul style="list-style-type: none"> • Mechanical Engineering (Maths 3) • PGCE maths 	Maths	31-35	Provides extra maths lessons to learners
Vutivi	Apollo High School	5	4	<ul style="list-style-type: none"> • B Ed⁵ EMS, mathematics and accounting. 	Maths	31-35	Cluster leader, presents workshops for maths teachers, provides maths extra lessons to learners
Tshedi	Poseidon High School	15	4	<ul style="list-style-type: none"> • B. Ed (hon) (maths education) • ACE⁶ maths education 	Maths/Maths lit ⁷	36-49	
Ndivho	Hercules High School	15	14	<ul style="list-style-type: none"> • B. Ed (Hon) (maths education) 	Maths	36-49	
Elimu	Zeus High School	18	12	<ul style="list-style-type: none"> • ACE (Maths education) 	Maths	36-49	

Table 4.1: Summary of participants' information

³ Bachelor of Science is an undergraduate science-related degree

⁴ Post Graduate Certificate in Education (PGCE) is a one-year certificate course for those individuals who have completed an undergraduate degree and want to become teachers

⁵ Bachelor of Education is a three- or four-year degree in education, specialising in a specific subject

⁶ Advanced Certificate in Education is offered as a part-time course specifically for in-service teachers who have a teaching qualification but would like to upskill themselves

⁷ Maths literacy deals with the application of basic mathematical skills to everyday practical problems like budgeting or working out personal income tax. It is offered to learners who cannot cope with abstract mathematics.

I consider five of the six teachers to have strong mathematical content knowledge. Five have degrees specialising in mathematics with one possessing a teaching diploma. Two of the teachers had initially studied other degrees besides their one-year teaching certificate after the undergraduate degree. One of the two has a degree in economics and mathematics while the other studied mechanical engineering with mathematics as one of the majors. Three teachers have degrees in mathematics education, with two of them possessing an honours degree (in mathematics).

Further to their academic qualifications, all five obtained scores above 65% in the final WMCS course test. The test, which was written at the end of the course, assessed both content and pedagogical content knowledge of teachers. While the test scores may not accurately reflect each aspect of teachers' mathematical knowledge, i.e. content and pedagogy, all selected teachers were actively engaged for the duration of the course and maintained consistent above-average scores in short tests and assignments written during the course.

The sixth teacher was chosen to replace one of the two teachers who was not used in the second year of data collection. As mentioned in chapter 1, two teachers from the first year of study were no longer teaching maths at the Grade 10 level in the second year of data collection. The sixth teacher was not as mathematically strong as the other five, as evidenced from his performance in the end-of-year course tests as well as his academic qualifications. He obtained below 65% in the course test. In addition, he was a science teacher with a one-year certificate in mathematics education completed in an in-service course.

4.5.2. Data sources

The data used in this study was collected through classroom observation, which constituted my main source of data collection because of the benefits it offers, as noted in the literature. Observation entails systematically noting and looking at people, events, settings, behaviours, routines etc. (Cohen et al., 2013) and provides the researcher with an authoritative gaze into interpretation of the data (Denzin & Lincoln, 2011). One of the distinct features of observation is the opportunity to collect “live data from naturally occurring social situations” (Cohen et al., 2013, p. 456). It presents the opportunity to provide insights into the behaviour of people within a specific context in keeping with the interpretivist paradigm. The strength of direct observation is that the researcher observes what is taking place directly rather than having to rely on second-hand accounts. This contributes to unmediated and authentic data, which adds to its validity. A further attraction of observation is that people may do things differently from what they say they do. Observation therefore provides a reality check (Cohen et al., 2013).

The two topics I observed were “number patterns” and “the solving of quadratic equations”. Data was collected over two years although I do not compare findings over the two years as the focus of my study changed during the course of the study (as elaborated on in section 1.6). Having data

from two years, however, provided more data that could be used to provide more insights into IEs, enabling me to use the findings from their analysis to examine the nature and quality of IEs and to eventually contribute a framework to analyse the coherence of IEs. Table 4.2. indicates the six teachers observed during the two years.

Year data collected	Teachers					
2018	Tsebo	Ulwazi	Vutivi	Tshedi		
2019	Tsebo	Ulwazi			Ndivho	Elimu

Table 4.2: Teachers observed and interviewed in the two years

4.5.3. Data Capturing

As mentioned in section 4.4. data from all six teachers were captured through video recordings of their lessons on the two topics mentioned. The video camera was focused on teachers and the chalkboard and not on learners since my focus was on teachers' explanations. A checklist of each recording was completed where I noted the teacher's name, the topic, the time taken for each lesson and the year of the lesson. A summary is provided in table 4.3.

Teacher	Year	Number Patterns	Quadratic Equations	Duration
Tsebo	2018	3	3	35min/lesson
	2019	2	2	
Ulwazi	2018	3	2	45 min/lesson
	2019	2	2	
Vutivi	2018	2	2	45 min/lesson
Tshedi	2018	3	2	45 min/lesson
Ndivho	2019	3	3	1 hour/lesson
Elimu	2019	No data ⁸	3	45 min/lesson
Total		21	16	

Table 4.3: Data from classroom observations

⁸ No data was collected in number patterns for this teacher because this lesson was conducted at short notice over a weekend because of a lack of water at the school on the scheduled day of data collection. Learners were sent off early on the day and the teacher used time over the weekend to complete the topic without informing me.

The video recording of each lesson was transcribed. The transcriptions provided details (as in table 4.4 of what was said, by whom (learner/s or teacher), the time, what was done (actions) and pictures of the chalkboard work (see Appendix E for a complete example of a transcript).


Who	What was said	Actions	Boardwork
Teacher	1bc? So now, listen, listen that means it ...[intervened]		
Learners	b plus c.		
Teacher	b plus c? Why b plus c?		
Learners	Operation sign.		
Teacher	Because of the operation sign. So we can't multiply b and c. That means we multiply 1 times b, it will be b and then plus c. So b plus c, that is our first term in that sequence. Are we all together?	Teacher writes on the board. Teacher points to the equation.	 4:48 4:55
Learners	Yes.		
Teacher	Let's go to the second term. The second term is T2. Will you please help me out on that one?		
Learners	B2		

Table 4.4: Example of a transcript

The transcripts together with the video recordings were used to analyse the data.

4.6. Conclusion

The purpose of this chapter was to provide a synopsis of the different approaches to research in the literature. By providing an elaboration of the different approaches, I provided a rationale for the approach used in conducting this research. My use of the interpretivist paradigm in conducting this research informed my research design and the data collection process. The purpose of this research informed the type of data collected, i.e. classroom observation as well as the participants used in this study.

In the next chapter I describe the procedure I used in my analysis of the data collected through classroom observations. I provide details of my use of grounded theory in analysing my data together with an example of how this analysis informed my findings.

Chapter 5 : Data analysis methods

5.1. Introduction

In this chapter I outline the methods I used to analyse my data. I begin by discussing grounded theory, its development, its main features and how I used it to identify the lenses through which I viewed the features of the IEs offered by teachers in my study. I include here the analytical process I followed in analysing my data. I illustrate by means of an example how I used grounded theory to examine the features of IEs provided by teachers. I conclude with the ethical and trustworthiness issues I considered during the analysis of data.

5.2. The use of grounded theory in this research

As mentioned, I used grounded theory to analyse my data on classroom observations. While there are different versions, I use grounded theory as developed by Strauss and Corbin (1990). I first describe grounded theory and its features before explaining its development and why I use their version.

5.2.1. A description of grounded theory

Grounded theory, as the name suggests, is grounded in the data collected, which is used to develop theory. It is described as “a general methodology for developing theory that is grounded in data systematically gathered and analysed” (Strauss & Corbin, 1994). The main purpose of grounded theory is to increase knowledge by generating new theories instead of using existing ones. The two main streams of how grounded theory is applied are reflected by the seminal work done by Glaser in 1967 and a revised version of grounded theory by Strauss and Corbin in the 1990s. However, whichever stream one chooses to follow, there are common principles to a general grounded approach to the analytic process. Different versions of grounded theory exist but the following are some of the features they all have in common:

1. The main purpose is to generate theory that emerges from the data and not prior to it. The researcher must be cognisant of his prior knowledge, which may lead to pre-determined hypotheses.
2. Theory generation is a consequence of systematic data collection and analysis, where each unit of data is compared with other units to check for a new category or one that may be added to an existing category.
3. Patterns and theories are present but implicit in the data. The researcher needs to discover them.
4. The approach can be both inductive and deductive, is close to the data and is iterative.

(Cohen, Manion, & Morrison, 2017, pp. 716-717)

According to (Strauss & Corbin, 1994) many researchers claim to use grounded theory but do not understand important aspects of it. The above features serve to delineate the scope of

grounded theory, with the main idea being that sense-making derives from the data. This data is what directs relationships between categories instead of categories being theoretically derived. However, for my purposes, it was a mix of theoretically derived categories together with those derived from the data. I built on Leinhardt's theory with new categories emerging from my data within her theoretical framework. The theory developed in this study is therefore a hybrid of using existing theory and deriving new categories grounded in my data.

5.2.2. The two different versions of grounded theory

Grounded theory was developed by Glaser and Strauss (1967), who demonstrated that qualitative research had a logic of its own, contrary to assertions by positivists that qualitative research is unsystematic and biased. They revealed the potential of grounded theory to generate rather than to verify, replicate or refine existing theory. The theory was developed in response to researchers who built theory without the accompanying empirical research. Grounded theory was intended to bridge the gap between theory and empirical research by building a theory from raw data.

Grounded theory, according to Glaser and Strauss (1967), focuses on a method of constant comparison. This involves the constant comparison of incidents or units of analysis of any size to each other. The researcher starts by coding each unit of analysis and these are compared to each other to form categories. These are then integrated to form a story about the whole category. The resulting theory is delimited by clarifying the properties and removing unnecessary information. Any new incident is inspected to determine if it leads to a new aspect. Once all the incidents are examined and no new aspects are discovered, it leads to theoretical saturation. The focus of constant comparison is to generate theory. The process of constant comparison allows for a vagueness and flexibility of data to enable the creative generation of new theory. Grounded theory has since moved in divergent directions from the original theory by Glaser and Strauss (1967), leading to two main strands of grounded theory.

My use of grounded theory in this study, however, is in alignment with that of Corbin and Strauss (2014). While I believe that studying literature before analysing the data may get in the way of or influence the interpretation of the data, the purpose of this study is to examine the nature and quality of IEs. In order to do this, it was necessary for me to consult the literature to identify gaps on IEs before analysing the data. In this regard, I align my use of grounded theory to that of Corbin and Strauss (2014) because, while they kept to the basic ideas of grounded theory, they diverged from the theory as expounded by Glaser and Strauss (1967) in terms of the role of the literature prior to a study. For Glaser and Strauss (1967), the researcher should not engage with the literature prior to analysis as prior knowledge of the literature may contaminate the emergence of categories from the data and stifles creativity in the development of theory. Reading of the literature should only be done when the theory has sufficiently developed to avoid the bias of the researcher, who may favour one theory over another. Corbin and Strauss (2014), however, consider it a necessary step in the analytic process to be aware of literature on

the topic. They contend that no researcher enters the research process with an empty mind. Research is very often influenced by researchers' previous experiences as well as their own knowledge of theories that is brought to bear on their analysis of data. In support of this, Adler (1996, p. 108) states that "analysis is neither one of unspoiled or unframed grounded theory. Nor is it an attempt to fit data into pre-existing categories. It is rather a dialectical process that involves both induction and deduction, theory informed theory generation." This research can therefore be described as a hybrid involving both an inductive and deductive approach to the generation of theory on IE's.

In order to extract information that is relevant to the study, Corbin and Strauss (2014) maintain that theoretical sensitivity is required, i.e. some knowledge of the topic is necessary. They do, however, recommend that the researcher should read widely around the topic in order to avoid examining the data through one particular lens. In this study I did read widely through literature on IEs but focused on that outlined by Leinhardt (2001) because other studies made use of her theory. Therefore, as explained above and in the previous section (5.2.1.), my approach is a hybrid of using theory and raw data in my analysis.

In the next sections, I outline why I use a grounded approach to analyse my data and the stream of grounded theory that I align with in this study.

5.2.3. Why grounded theory?

The original plan in this study was to explore how teachers who had attended the TM1 course had used the explanatory communication and exemplification components of the MTF introduced to them during the course, as described in chapter 1. I therefore started the analytical process by using the MDI framework as a lens to analyse teachers' explanations as well as their selection of examples, but opted to rather use grounded theory for the five reasons outlined below.

First, grounded theory is suited to an interpretivist paradigm. Theory within the interpretivist paradigm arises out of the particular situations within which humans live. It is therefore important that theory is grounded in the data generated during the research study. Theory grounded in data offers the means through which to interpret events through an analysis of data grounded in participants' real-life contexts (Cohen, Manion, & Morrison, 2011).

Second, I initially started my analysis using the MDI framework, as previously mentioned. However, during my initial analysis I found that because teachers were not using the framework in any structured way, it resulted in a deficit analysis of teachers' lessons. For example, in most cases teachers did not justify the procedures they used in solving particular problems. There were many other categories within the explanatory communication component of the MDI framework that were not present in teachers' lessons. While using another framework, for example, the

Mathematical Quality of Instruction (MQI) was an option, but it would also not have been helpful since MQI was developed in another country with a vastly different instructional context. Using an international framework may therefore not be suited to the South African context. This may also result in a deficit interpretation of teachers' IEs, which may be a result of the framework used rather than the IEs provided by teachers. Grounded theory being emergent from data offers a way to explore what is present rather than absences in teachers' IEs. It therefore provides an opportunity to add to the literature on IEs as well as to the MDI framework.

Third, not only would using a framework to analyse IEs from another country not be suitable in other contexts, but findings on the features of IEs also emanate from the global north. The prior knowledge of learners and the common errors of learners may be localised to specific contexts. In addition, they may differ between countries and especially between the so-called developed countries and the developing countries. Findings from the bulk of research conducted in the global north may have limited applicability to the particular culture, circumstances and history of teaching in South Africa, which calls for more locally grounded research.

Fourth, while the focus of this research is to examine the coherence of IEs through their features, there is a paucity of literature on how the IE of a teacher is enacted in the classroom. For example, IEs should take into account the prior knowledge of learners, common errors should be addressed, and they should be unambiguous, clear and precise (Leinhardt, 2001). It is through the enactment of a teacher's IE that the criteria that should be present can be brought into focus. This is therefore something that I decided to highlight in this study. Grounded theory offers a way to highlight the features of an IE through the exploration of how the IE is enacted.

Fifth, the initial analysis of data from one of the teachers in my sample, revealed parts of IEs that to my knowledge have not been highlighted in the literature on IEs. For example, a teacher divided the chalkboard in a very specific way when explaining how to find the solutions to a few questions relating to a given pattern of numbers. This piqued my curiosity about how the chalkboard can be used in providing an IE. I decided to explore this further. Grounded theory provided the means to foreground whether and how the chalkboard can be used within an IE. This proved to be a prominent theme emerging from my data contributing to building on Leinhardt's theory of IEs. A grounded analysis of my data assisted in highlighting data that otherwise may have been missed if an existing framework was used. This is especially important in the light of IEs within the South African context where, for instance the chalkboard plays a big role in the classroom. Grounded theory therefore enabled theory generation. In this way the main contribution of this study would be to add to the theory of IEs, especially within the particular context of teacher-centred instruction.

5.2.4. Why the Straussian approach?

My grounded theory analysis of data is in alignment with that of Corbin and Strauss (2014) for two reasons. As outlined earlier, I started this study with a different focus from my final interest in examining the nature and quality of IEs through their coherence. It was therefore necessary for me to widely consult the literature on explanations. I used Leinhardt's theory since other studies on IEs make use of her considerable work in this field. Since Glaser and Strauss (1967) specifically caution against consulting literature on the topic until after the theory is developed, I rather use the Straussian approach since they recommend that literature around a topic is consulted to identify theory that is already in existence. Second, Strauss and Corbin (1990) developed a set of technical procedures to guide the researcher in conducting a grounded analysis of data. Charmaz (2006), on the other hand, emphasises that there is no prescribed set of rules to follow when conducting a grounded analysis of data but rather flexible guidelines for researchers to follow. As a beginner researcher, I found the detailed guidelines outlined by Strauss and Corbin (1990) and their later work (Corbin & Strauss, 2014) helpful for me to follow in conducting my analysis.

5.3. Classroom observation analysis procedure

Analysis, according to Corbin and Strauss (2014) involves the examination of some phenomena (which I am referring to as data) together with its components. The purpose is to determine the properties and functions of the phenomena, which are used to make inferences about the whole. In short, analysis is the act of giving meaning to data.

According to Chenail (2012), one of the challenges in analysing data is identifying what pieces of the data would be a meaningful unit to analyse. A unit of analysis is deemed to be a single, undivided entity to which analysis is directed and from which specific qualities are perceived by the researcher. Some researchers consider conducting a line-by-line or word-by-word analysis. This may, however, lead to under- or over-sizing the units to be analysed, which does not reveal the essential features or qualities of what is being studied (Chenail, 2012) and the researcher losing sight of the bigger picture present in the data (Leatham, 2019).

5.3.1. Analysis of classroom observations

The videos of each lesson were first used to transcribe the classroom observations. Transcripts included teacher and learner utterances and actions. The utterances captured all words spoken while the actions were anything that was done by the teacher. For instance, a teacher writing something on the chalkboard, pointing to something written on the chalkboard, or walking around the classroom checking on learners' written work were all captured in the transcripts. Furthermore, pictures of all chalkboard work were also included in the transcripts.

I started my analysis by originally looking at each line of the transcripts of lessons. However, I found that the "bigger picture" in terms of the IE offered by the teacher, was missing. For

example, I started by writing codes like “justifying”, “addressing prior knowledge” etc. These, however, did not help in providing insight into the IEs provided by a teacher. I therefore opted to use episodes as my units of analysis of the IEs within them. Episodes were identified by shifts in the ooL and therefore time considerations were not in focus in identifying the episodes. The demarcation of episodes was based on my interpretation of the ooL in a particular lesson. If there was a change in ooL this indicated a different episode. While there may be other ways that episodes could be demarcated I chose to do this according to the different ooLs that I identified within a lesson. The IEs provided by the teacher in each episode were then analysed in a grounded way.

All lesson transcripts were divided into episodes but not all episodes were analysed. (Corbin & Strauss, 2014) recommend perusing data that have already been collected for relevant concepts and analysing these. I selected particular episodes to analyse based on the following criteria:

- My focus in this study was to gain more insight into the nature and quality of IEs. As mentioned previously, IEs are specifically those provided by the teacher. Hence, I only considered those episodes in which the teacher provided an IE and so not on learner IEs. However, it must be emphasised that all teachers in this study interacted with learners during their IEs. These interactions took the form of, for example, the teacher asking learners what he should do next or asking “why do we do this?” when solving an example. While my focus was on the IEs offered by teachers, learners’ responses were included in these IEs.
- An initial analysis was conducted on the episodes selected. I chose those episodes that brought into focus features of IEs whether from the literature or not but that needed further elaboration, for an in-depth analysis. This is in line with Strauss and Corbin’s (1994) guidelines that codes can be derived directly from the data or from the existing literature that refers to particular theories.

Corbin and Strauss (2014) liken the selection of particular episodes to funneling from the wide to the narrow, where data is sifted, sorted, reviewed and reflected on to highlight only those features that are the focus of the study. While this results in raw data being reduced, Gibbs (2007) maintains that the notes, summaries, reflexive insights, memos etc. serve to add bulk, density and complexity to the data. This is, however, a necessary part of contributing to a rich, thick description of data that describe not only the events but also participants’ strategies, intentions and agency. Corbin and Strauss (2014) emphasise that the use of memos allows the researcher to look at the data from different angles. Memos involve the writing of ideas, notes on surprising ideas, hunches, suggestions for further development of ideas, etc. that occur to the researcher during the analysis of data. Memos allow for the researcher to move back and forth through constant comparison between categories to find relationships between them. These memos not only describe the phenomena but allow the researcher to view the data at a higher conceptual and analytical level (Corbin & Strauss, 2014) and so to theorise the general development of the analytical framework (Gibbs, 2007).

5.3.2. The use of posters as an analytical tool

My version of a memo was a poster that I used as an analytical tool. I created posters of each lesson by every teacher in my sample, with each episode reflected in blocks on the poster. I recorded ideas of interest, notes on my thoughts, questions to think about and reflections in these blocks. The use of posters enabled me to examine the “big picture” of the lesson. In other words, it enabled me to look at what the lesson as a whole made possible for learners. However, it also enabled me to zoom into particular parts of lessons by looking at each episode and picking out anything in these episodes that was interesting and required further examination. The creation of posters therefore enabled me to zoom into particular parts of lessons while also being able to keep in sight how one portion of the lesson fits into the bigger purpose of the lesson overall. It was this process of zooming into the detail of smaller parts of a lesson and the zooming out to examine its relation to the whole lesson that helped identify the features of IEs.

Creating posters with all episodes in a lesson helped me to identify those episodes that satisfy the selection criteria mentioned previously. For example, Leinhardt (2001) posits that an IE should link together various mathematical ideas but does not elaborate on how this is enacted in the classroom. Creating posters highlighted those episodes which attended to the linking of mathematical ideas. Once these were identified, I examined in detail how the teacher attended to the pedagogy of linking these mathematical ideas. For example, one of the ways a teacher addressed the linking of ideas was attending to how the lesson was organised to focus on its oOL.

In examining the detail of each IE, I first noted the examples used in each since examples were central to the explanations provided by teachers. I wrote a brief summary of the main idea within each IE and how the teacher conveyed this idea. For example, I made note of relevant questions asked by the teacher and the key ideas of his IE when working through the examples. I examined in detail each IE, noting down reflections as well as questions to myself. I then used sticky notes to note particular features of each IE. For example, if in an episode the teacher dealt with learners’ prior knowledge or justifying a concept, these were noted on the sticky notes. I also stuck pieces of transcripts onto these blocks to support my notes.

The writing of notes was not done in a linear fashion. While my initial ideas, questions etc. were recorded episode by episode, I went back to each after the initial exploration of ideas. I moved back and forth between the episodes to look for relationships between them. It was while exploring the relationships between the episodes that the first analytical theme emerged. I noted that the same examples were used in the first and third episodes in the first lesson on the topic of quadratic equations. This led to me noting on the poster the question, “What is the purpose of using the same questions?” and noting (on the poster) a few possibilities. Another question that arose which I noted was, “What is the purpose of focusing on factorising before introducing a quadratic equation?” In my experience, most teachers deal with factorisation during the solving of quadratic equations. What was interesting in the episode was that the first two episodes

seemed to have no relationship to each other. Episodes one and two were used to illustrate the factorising of expressions and the zero product law respectively. However, all the episodes taken together foregrounded one of the findings of this study, i.e. of an IE as encompassing an entire lesson through the organisation of the content to focus on the OoL of the lesson. Figure 5.1 is a picture of a poster where my thoughts on a lesson were noted.

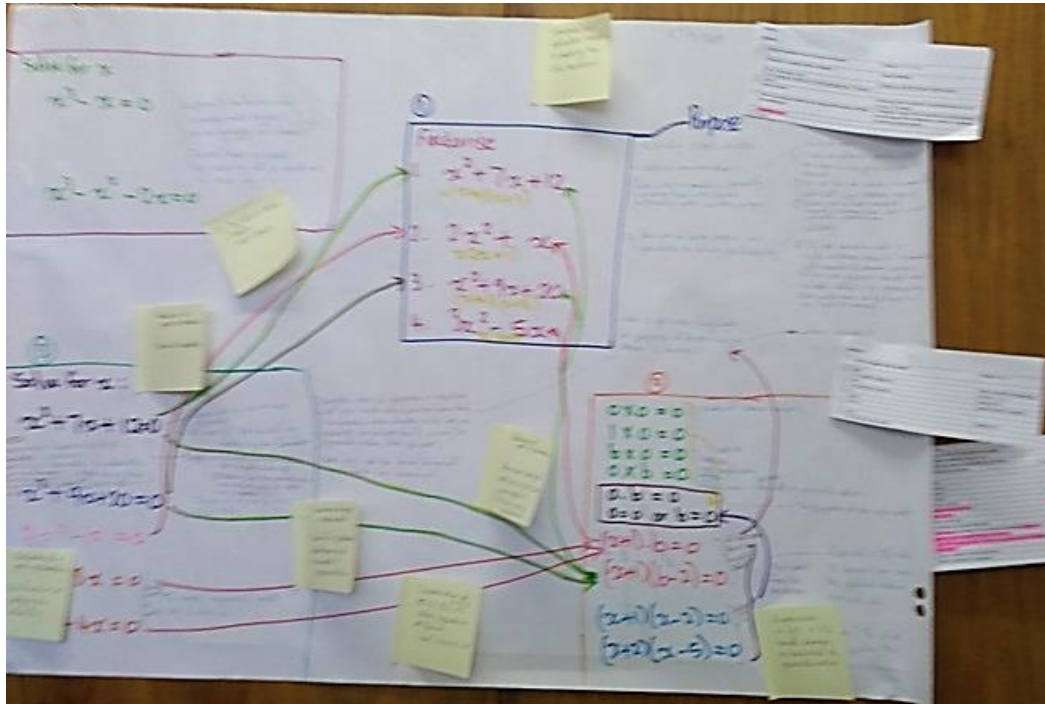


Figure 5.1: Picture of a poster

In addition to posters, I used the guide provided by Corbin and Strauss (2014) on the three types of coding to conduct a detailed analysis of the episodes in all lessons. I provide details of how I used the three types of coding, i.e. open, axial and selective coding, in my analysis of the data.

5.3.3. The use of open coding in this study

Open coding involves identifying units of analysis so that the coding reflects meanings, events, actions etc. The data is provided with new codes, categories and sub-categories and integrating them until the coding is complete. In alignment with Corbin and Strauss's (2014) theory of analysing data by breaking it up into smaller parts identified by their core idea, I divided the transcriptions of the observed lessons into episodes. I identified episodes by what was perceived to be shifts in the object of learning (oOL), i.e. learner skills and capabilities (Marton & Pang, 2006) to be developed in each episode. The oOL in each episode was not explicitly stated by teachers but was my interpretation of what was being explained by the teacher in that part of the lesson. During this process, I explored how the teacher went about trying to achieve the oOL of the particular episode. For example, I found that a teacher (Tsebo) organised his lessons into

different parts (episodes) that initially seemed unrelated to each other. However, in episode three, he linked together episodes one and two by repeating examples in different parts of the lesson.

I provide an example of how I used open coding in Table 5.1.

Episode and ooL	Examples	Description	Notes
1: Factorisation by removal of a common factor and trinomials	<ol style="list-style-type: none"> 1. $x^2 + 7x + 12$ 2. $2x^2 + x$ 3. $x^2 + 9x + 20$ 4. $3x^2 - 5x$ 	Tsebo <u>factorised</u> all expressions while discussing the methods to do so.	<ul style="list-style-type: none"> • ooL of each episode determined by what was done in the episode. • Examples 1 and 3 repeated in ep 3. • Similar example. to number.2 done in ep. 3 (underlined in column 2)
2: A focus on the development of the zero product law	<ol style="list-style-type: none"> 1. $0 \times 0 =$ 2. $1 \times 0 =$ 	Tsebo introduced the zero product law by first introducing <u>numeric examples</u> with 0 as one factor in first two examples	<ul style="list-style-type: none"> • Numeric values first • Zero in different positions
	<ol style="list-style-type: none"> 3. $b \times 0 =$ 4. $0 \times b =$ 	He introduced a <u>letter with 0 as a factor</u> in different positions	<ul style="list-style-type: none"> • Letter as 1 factor and zero as other • From known to unknown • <u>Zero in different positions</u>
	<ol style="list-style-type: none"> 5. $a \cdot b = 0$ 	He then asked for <u>values of a and b</u> that make the equation true.	<ul style="list-style-type: none"> • Zero factor law introduced • Showed that either a or b could be zero
	<ol style="list-style-type: none"> 6. $(x + 1) \cdot b = 0$ 7. $(x + 1)(b - 2) = 0$ 	He used the zero product law from previous step to show <u>why each factor was equated to 0</u>	<ul style="list-style-type: none"> • Use of zero factor law
3: How to use the zero product law to solve quadratic equations	<ol style="list-style-type: none"> 1. $(x + 1)(x - 2) = 0$ 2. $(x + 2)(x - 5) = 0$ 3. $x^2 + 7x + 12 = 0$ 4. $x^2 + 9x + 20 = 0$ 5. $2x^2 - x = 0$ 	<p>Tsebo demonstrated that <u>each factor equals 0 as in the last two</u> examples in episode 2.</p> <p>He asked learners to <u>remember how to factorise no.3 as done earlier</u>. So didn't discuss factorisation again.</p>	<ul style="list-style-type: none"> • Examples with two factors shown as in last example in ep. 2. • Examples from ep 1 repeated here (underlined) • Factorisation from ep 1 and zero prod law used in combination to solve each of these eg's.

Table 5.1: Example to demonstrate open coding

Key:

- In column 2 the underlined examples illustrate those that were repeated in a different episode.
- Examples in different colour indicate a similarity to other examples.
- Underlined words in column 3 (under description) indicate key points to be noted.

5.3.4. The use of axial coding in this study

Corbin and Strauss (2014) describe axial coding as the relationships between codes and categories developed in open coding. This is done through a process of constant comparison between the codes in open coding until a relationship between them is uncovered. In order to highlight these relationships, the researcher needs to take into consideration the contexts, i.e the

specific set of characteristics within which the phenomenon is rooted. The word context is a broad term that includes the actions-interactions of the teachers as well as the consequences of the actions elaborated on below.

- Action/interactions describe the responses of people to particular situations. In the example mentioned, the action of the teacher in offering an IE of solving quadratic equations was his organisation of his lesson into three different parts/episodes, taking into consideration the prior knowledge of learners (factorisation of expressions and developing the zero product law). He also used the same examples in episodes one (factorisation of expressions) and three (solving a quadratic equation).
- Consequences are the actual or anticipated outcomes of the actions. My analysis was based on classroom observations of teachers and not what learners gained from the IEs offered by teachers. As a result, I could not make findings on the actual outcomes of the actions by teachers. For example, my interpretation of the use of the same examples as well as organising the lesson into three parts resulted in a lesson that did not deviate from the OoL, which was solving quadratic equations by factorisation. This led to me providing a code that linked together the codes “using the same examples” and “organisation of lesson” and thus providing a second level or axial code of “A focus on the OoL of the lesson”.

I provide an example of how the axial coding on instructional explanations was developed in Table 5.2.

Episode	ooL	Evidence	Action	Consequence	Code
1	Factorisation of expressions	Factorise: 1. $x^2 + 7x + 12$ 2. $2x^2 + x$ 3. $x^2 + 9x + 20$ 4. $3x^2 - 5x$	The same examples were used in episode 1 and episode 3	The focus in episode 3 was on solving quadratic equations without deviation to methods of factorisation	Maintaining a focus on the OoL: The use of the same examples and having done the factorising of expressions and the introduction of the zero product law before the solving of quadratic equations prevented any deviation of the OoL when solving equations.
3	Solving of quadratic equations using factorisation	Solve: 1. $(x + 1)(x - 2) = 0$ 2. $(x + 2)(x - 5) = 0$ 3. $x^2 + 7x + 12 = 0$ 4. $x^2 + 9x + 20 = 0$ 5. $2x^2 - x = 0$			
1	Factorisation of expressions	Factorise: Factorise: 1. $x^2 + 7x + 12$ 2. $2x^2 + x$ 3. $x^2 + 9x + 20$ 4. $3x^2 - 5x$	Lesson was organised in three parts. First factorisation, then zero product law, and then solving quadratic equations.	No deviations were made to either factorisation methods or the zero product law when solving quadratic equations in episode 3.	
2	Introduction to the zero product law	Find: 1. $0 \times 0 =$ 2. $1 \times 0 =$ 3. $b \times 0 =$ 4. $0 \times b =$ What are the values of a and b? 5. $a \cdot b = 0$ Solve for x 6. $(x + 1) \cdot b = 0$			

Episode	ooL	Evidence	Action	Consequence	Code
		7. $(x + 1)(b - 2) = 0$			
3	Solving of quadratic equations by factorisation	Solve: 1. $(x + 1)(x - 2) = 0$ 2. $(x + 2)(x - 5) = 0$ 3. $x^2 + 7x + 12 = 0$ 4. $x^2 + 9x + 20 = 0$ 5. $2x^2 - x = 0$			
2	Introduction of the zero product law	Find: $0 \times 0 =$ $1 \times 0 =$ $b \times 0 =$ $0 \times b =$ What are the values of a and b? $a \cdot b = 0$ Solve for x: $(x + 1) \cdot b = 0$ $(x + 1)(b - 2) = 0$	Numerical factors in two examples Introduction of factors with a letter Introduction of a binomial factor Introduction of two binomial factors	ooL of the zero product law in focus because of the selection and sequence of examples used	Maintaining a focus on the ooL: The choice and sequencing of the examples focused on the ooL-its introduction and its use

Table 5.2: Example of axial coding

5.3.5. Theoretical sensitivity in my data analysis

Theoretical sensitivity or integration is the final step of the analytical process refers to the shift from description to conceptualisation. In order to build theory, concepts need to be linked and filled with detail. This occurs as the researcher becomes more in tune with the meanings of data, which occurs when the researcher is immersed in the data for a while (Corbin & Strauss, 2015). The goal is to integrate the categories formed during axial coding into a core category that answers the question: “What is the research all about?” (Strauss & Corbin, 1990, p. 116). Once this is addressed, it enables the research questions to be answered. Corbin and Strauss (2015) suggest techniques to facilitate the identification of a core category, e.g. making use of diagrams, writing the storyline or sorting through memos.

5.4. Ethical considerations

There are many ethical issues surrounding classroom observations. Some ethical concerns when conducting qualitative research are issues of confidentiality, anonymity and informed consent (Cohen et al., 2011).

Informed consent is an integral part of the ethics of any qualitative research. It relates to the researcher’s responsibility to disclose the nature and objective of the study, the potential role of the participants, the identity of the researcher, what data will be collected, and how they will be used. All of this information should be clarified in easily understood language. Classroom observation requires informed consent of participants, the right not to be observed, and permission from parents and the school management team (Cohen et al., 2011). In this research, informed consent was sought by providing all teachers, learners and parents of learners with a participant information sheet as well as a signed consent form. The participant information sheet contained information on the details of the study, the role of the participants, the assurance of anonymity in the reporting of findings and the option of voluntary participation as well as the option to withdraw from the study at any stage. The role of teachers and learners in this study differed. Teachers were the main participants while learners were involved only as far as their presence in the classroom, taught by the specific teacher, was required. As a result, different information and consent sheets were provided to the teachers (appendices C2 and C3) and learners (appendices C4 and C5). For example, the information sheet of the learners specified that their role in the study involved their presence in the classroom taught by a specific teacher and that they were not expected to do anything else besides being present in the particular lesson. They were informed, however, that they could be captured on camera if they were seated in a position that aligned with the camera position. They were also told that if they were uncomfortable with this, they could be positioned away from the line of the camera. Five learners in one class chose not to be in line with the camera and so their seating positions were accordingly changed. In addition, the ethics committee of the institution granted approval for the conducting of this study.

While letters of consent were given, ethical conduct, however, depends on the conduct of the researcher. The code of ethics, according to Orb, Eisenhauer, and Wynaden (2001) indicates the rights of subjects but does not specify how to respond in the case of unethical behavior. In social research, the dangers to participants are not physical but mental. It is possible that participants can be exposed to humiliation, embarrassment, and loss of respect by others as well as loss of self-respect (Stake, 2010). Any observation of classrooms produces the dilemma of handling the ethical issue of witnessing a teacher employing incorrect instructional methods. Merriam (1988) posits that the dilemma of knowing when to intervene is perplexing; however, actually intervening is the other half of a complex problem. Reporting on incorrect methods employed may embarrass and damage the reputation of the teacher involved; however, if ignored, this may have an impact on learners, who are the indirect participants in the research. When faced with this dilemma in my study, I felt a moral obligation to intervene but did so after the lesson so as not to cause embarrassment to the teacher/s involved. For example, one of the teachers made an error when factorising an expression. I mentioned this to him after the lesson and he committed to correcting the mistake the next time he met with the particular group of learners.

Ethical conduct also relates to ethical ways of reporting the findings of a study. In this regard it is the researcher's responsibility to ensure that the findings are reported truthfully from the data gathered. In this study, my supervisors viewed some of the data of classroom observations that were captured on video. However, it was not possible for them to view all of the data. It is therefore incumbent on me to mention that I reported on my findings truthfully based on the gathered data.

In terms of confidentiality, the researcher should clarify how the data will be used and who has access to the data, how it will be stored to prevent access by others, how long it will be kept for, and how it will be destroyed afterwards. In my study, information sheets were provided to teachers as well as learners that specified the purposes for which the data collected would be used. It was also mentioned that nobody else would be granted access to the data and that it would be safely stored and then destroyed between three and five years after completion of the study.

Protecting participants' privacy is another consideration in any research. Orb et al. (2001) provide many strategies to protect personal information. They suggest the removal of biographical details, the use of pseudonyms for participants, and secure data storage. I have used pseudonyms for all teachers used in this study. Data will be stored in a secure password protected server only accessed by the members of the WMCS Secondary project. Although the research proposal was subjected to scrutiny and approved by the ethics committee of the university (Protocol number: H17/11/35; Appendix A) and the Gauteng Department of Education (Appendix B), the researcher is ultimately responsible for the protection of the participants.

5.5. Trustworthiness of the study

According to Creswell and Miller (2000), there is general consensus that qualitative research needs to be credible as the analysis of data is subjective. Trustworthiness in qualitative research refers to the degree of confidence in data collection, its interpretation and the methods that are used to maintain the quality of a study (Polit & Beck, 2013). Although there is consensus among most experts that trustworthiness is necessary, there is widespread debate in the literature on what it constitutes (Leung, 2015). However, Polit and Beck (2013) contend that criteria on trustworthiness as outlined by Lincoln and Guba (1986) are accepted by many qualitative researchers. Lincoln and Guba (1986) consider four criteria that promote the trustworthiness of a study, i.e. *credibility*, *transferability of the findings*, *dependability of the results* and *confirmability of the result*. While I focus on the other three criteria, I have chosen to focus on generativity instead of transferability of findings as discussed below (5.5.2)

5.5.1. Credibility

Credibility refers to the extent to which the results of a study are regarded as stable (Opie & Sikes, 2004) to ensure that a true picture of the phenomenon under study is formed (Shenton, 2004). Member checks, triangulation, prolonged and persistent observation and negative case analysis are some of the methods suggested by Lincoln and Guba (1986) to promote the credibility of a study. To add to the credibility of this research, some of the videos of classroom observations were viewed by my supervisors, although, due to the large volume of data, it was not possible to view them all. However, they did check my analysis of the IEs from the videos that they did watch and provided feedback on them. I used this feedback with further analysis of other IE. According to Stake (2010) triangulation also involves making and maintaining good personal relationships. In this study, participants were involved in a year-long course with the researcher being present. This allowed for close interactions with all participants during which time a comfortable relationship was built up. While it may not have totally removed a feeling of discomfort by teachers with the researcher being present in the classroom, this was mitigated as they became more familiar with the researcher.

5.5.2. Generativity of the findings

In this study I refer to the generativity of my findings rather than transferability. Transferability of findings relate to the degree of applicability of the findings of a study to other settings (Lincoln & Guba, 1986). However, some researchers argue that applying the findings to other settings is not appropriate for qualitative research (e.g. Adler, 1996). In this study for instance, teacher's mathematical explanations are offered within particular contexts e.g. teacher centred direct instruction, schools with learners from poor and disadvantaged socio-economic backgrounds and lacking teaching resources. It is for this reason that I consider generativity of findings. I refer to generativity as the "potential to produce more enduring expansive and transformative consequences with regards to the development of ideas..." (Carlsen & Dutton,

2011, p. 15). I believe this study is generative in that it provides the “seeds” for stimulating conversations around the components of IEs and a framework to evaluate their coherence. Further research questions can be raised that has the potential to further expand on the theory of IEs. The framework of coherence of IEs that I propose in this study will have relevance to many other classrooms within similar contexts in South Africa and other developing countries. I have provided rich descriptions of teachers’ explanations in the classrooms through transcript evidence as well as detailed descriptions of what transpired before and after the reported episodes. These are intended to provide the reader with sufficient detail to determine whether or not the findings can be used in other similar contexts.

5.5.3. Dependability of the results

The dependability of the results of a study relate to whether the findings will be replicated if the same study is conducted in a similar context and/or by a different researcher (Lincoln & Guba, 1986). I concur with Sandelowski (1993), who argues that within the interpretivist paradigm, multiple rather than singular realities are constructed. It is therefore incumbent on the researcher to provide as much detail of the analytical process and the data from which the findings were drawn to demonstrate to other researchers whether the findings are likely to be replicated in another context or not. I take the view that, since there can be multiple interpretations of events, instead of referring to replication of results, the detail provided by a researcher is sufficient for another researcher to determine whether the findings can be replicated or adapted for use in another context. It is more a matter of practicing good science than about claiming to be right about a particular phenomenon. The view that I take is that if the findings of a study are useful in a particular field, then dependability of the results should not be a major focus. I am obliged to demonstrate that I have practiced “good science” so I need to provide an audit trail of the details, from data collection methods to how my analysis was conducted, to reveal the emergent themes.

In this regard, I consider it important in qualitative research that my interpretations of the data were challenged. One of the ways that I did this was to use peer debriefings. Barber and Walczak (2009) argue that the purpose of debriefing is not only to check if different researchers read the data in the same way, but it offers three benefits: a way for others to challenge the researcher’s assumptions of the data, ways to manage the researcher’s subjectivities, and to provide alternate interpretations of the data so that more robust knowledge is generated than when the researcher works alone.

During this study there were many opportunities for me to consult with and present my findings to various other researchers so as to provide an opportunity for my interpretation of my findings to be challenged. The following is a list of the different presentations made to either the WMCS team or to a wider community of mathematics education researchers and how they promoted the

- Presentation of my proposed research at a research symposium at my university. This provided an opportunity for me to focus my study.,

-
- Presentation of a poster of my first theme, together with excerpts from transcripts, to my supervisors and then to other members of the WMCS team, at a research degree symposium at my university. I used the feedback from this presentation to not only add to my coding of episodes but to also go back to my data to further analyse some episodes of a teacher. Feedback from this presentation also helped to create similar but more focused posters for all teachers in my study.
 - Presentation of a second theme in an international conference i.e. International Congress on Mathematica Education (ICME). The positive feedback from this presentation spurred me on to further pursue the theme of including the organisation of the content within a lesson as part of teacher explanations.
 - Presentation of a third theme i.e. relationship between verbal and written parts of teacher explanations in a conference organised by the Southern African Association for Research in Maths, Science and Technology Education. The feedback provided at this presentation helped me to further pursue this theme as something that adds to the theory of teacher explanations.

Presenting my findings to other researchers at these different events gave me frequent opportunities to discuss my emerging findings at different stages of the study, which helped me to refine or discard some of my ideas or to add new ideas.

5.5.4. Confirmability of the results

Confirmability of the results relates to the steps taken by a researcher to ensure that findings emanate from the data and not from the researcher's preconceived dispositions (Lincoln & Guba, 1986). They recommend detailed memos and the maintenance of an audit trail of ongoing analysis. During the course of this study, my detailed notes provided the basis of discussions with my supervisors on the analysis of my data. There were also opportunities during different stages of my study to present my memos to colleagues in the project I work in as well as to other researchers in my institution.

5.6. Discussion and conclusion

The analysis of data from classroom observations using grounded theory proved a challenging exercise. While the benefit of using grounded theory was that it enabled me to identify components of IEs that may not have been possible through the use of an existing framework, the piles of data that needed to be sifted through would be insurmountable if there was no structured way to "mine" the data. Using the more structured approach to grounded theory, as elaborated by Corbin and Strauss (2014), helped in some way to uncover my findings. The use of *open* and *axial coding* helped to reach the point of theoretical sensitivity and so to provide a story of what my research is about.

The use of open and axial coding, however, was made possible through the use of a poster as an analytical tool that proved to be invaluable in making sense of the data. My description of how I used the poster to highlight themes in my data will be of benefit to other researchers using grounded theory, especially in analysing data resulting from videos of classroom observations. This is an area that is not in focus in the literature on grounded theory, which mainly focuses on interview data.

The use of posters not only helped in illuminating the findings of this study but also added to the trustworthiness and credibility of this study. It provided some of the data from the classroom observations which resulted in the findings. Each of the episodes within lessons was accompanied by what was done by a teacher, how it was done, my questions asked from the data as well as my interpretations. Above all, my interpretations were accompanied by pieces of transcripts, which were used to substantiate my interpretations.

While the findings of the study are one of the most important part of a study, it is important that they are arrived at in an ethical manner. Measures taken to protect the anonymity and confidentiality of participants should be reported on to make clear to others that the study did not in any way cause harm to any of the participants. In this regard, the precautions taken were that names of participants were changed to protect their anonymity, and children who were uncomfortable to be in the camera's view were moved to other positions in the classroom. Where it was necessary to correct teachers' errors in terms of the mathematical content, this was done after a lesson, with learners not present, so as to prevent any potential embarrassment caused to teachers.

I have provided a detailed exposition of how and why I used grounded theory to analyse data collected from my primary source of data i.e. classroom observations to foreground the IE's offered by teachers in this study. I concluded with how I addressed issues of ethics and trustworthiness. The next part of this study focuses on an elaboration of my findings of the use of three different lenses to examine the features of IEs.

Interlude 1: Viewing IEs through three lenses

I illustrate how IEs can be viewed through three lenses in the next three chapters. These three lenses arose not only from my examination of Leinhardt's theory of IEs but are also a product of the analysis of my data. More importantly, viewing IEs through these three lenses provides insights that were eventually used to develop a framework to examine their coherence.

I identified the two lenses through an examination of Leinhardt's theory on IEs and identified the third through an analysis of my data. I view IEs through the three lenses of teachers' *use of examples* in IEs (this includes example choice and their organisation), teachers' *use of inscriptions and teacher talk* and a *summoning-back* lens.

I focus on these lenses for two reasons: the gaps in literature on IEs, and what is present in my data. The focus on *using examples* and *inscriptions and talk* is not new in literature, but how they interact together to form an IE is not in focus in the literature on IEs. In addition, I also consider a further lens of summoning-back IEs or parts of IEs that were used in other sections of a lesson and which are used again to form what I call a composite IE that may extend over an entire lesson. A composite IE is one that is made up of other IE's with each of these having a different oOL but which accumulate towards the OoL of the composite IE. This third lens is not in focus in the literature on IEs. These foci also resulted from what was present in my data. For instance, I do not focus on the use of digital technologies, which is a focus in literature, because it is not present in my data. By looking at IEs through the three lenses, I do not suggest that the other lenses are not present in IEs. They are merely backgrounded in this study.

I first view IEs through these lenses separately to see what can be learnt about IEs through them. Viewing them separately before looking at them all together over time reveals how the different features of IEs are enacted. For instance, they can reveal how learner error is attended to in an IE by the use of inscriptions and teacher talk or how different topics can be connected through the strategic selection and sequencing of examples.

The metaphor of looking through the lens of a camera is useful in explaining why I look through these lenses separately before viewing them all together. The zoom lens of a camera allows one to see images from different viewpoints. When too many parts of a picture are in focus at the same time, it is hard to focus on something in particular. The 'zooming in' lens allows a photographer to capture some parts of a scene in minute detail while other parts fade into the background. There is nothing else that distracts the viewer from what the photographer wants the viewer to see; however, it means that the surrounding context is not in focus. In order to see the "bigger picture", zooming out is required. Zooming out helps to see the wider environment, i.e. a "bird's eye view" (Kanter, 2011). While zooming in helps to examine the minute details of a

picture, these details may not make sense until the zooming out facility is used to see how each segment fits into the bigger story (Kanter, 2011).

IEs can be viewed through different lenses but if I examine IEs through all of these lenses together, the detail of each lens may not be in focus. I therefore zoom through one lens at a time to avoid the distraction of the other lenses. This does not mean that by looking through a single lens the others are not present in an IE, but rather that they are not in focus. However, I thereafter zoom out to see through all of the lenses to examine how they are used in combination to avoid losing sight of how they contribute to the bigger picture of what is brought into focus in the IE.

I provide a brief outline of the three different lenses that I use to view IEs.

IEs viewed through the lens of teachers' use of examples

I first examine how examples are used, i.e. chosen and organised, because examples form the core of an IE. In the remainder of this study, I use the term *using examples* to refer to how examples are chosen and sequenced. Everything else, whether it is talk or what is written, revolves around the examples used. By examining an IE through this lens, it is possible to see the enactment of some of the features of IEs as outlined in the literature (see chapter 2). For instance, I look at how examples are used to take into consideration the prior knowledge of learners, which provides the foundation for new learning (Sidney & Alibali, 2015). The strategic use of examples can also address common learner errors. Leinhardt (2001) points to the need for IEs to resolve learner error. I illustrate how these features are brought into focus by viewing two IEs through the lens of teachers' use of examples.

IEs through the lens of teachers' use of inscriptions and talk

While inscriptions and talk are in focus, this lens also includes how examples are used. I look at how what is written and how it is written, which mostly comprises how examples, and a teacher's talk interact to make it possible for learners to attend to the mathematical idea in focus. This is a focus on teacher talk in relation to what is written and so how the verbal and the written interact in an IE. More particularly, I examine how what is written on the chalkboard is laid out. The focus in this lens is to examine how examples are arranged to foreground something in particular. For instance, placing an example next to a contrasting example on the chalkboard helps to make it possible for learners to attend to the difference in the two examples and hence the different strategies that can be used to solve them. How the inscriptions are made on the chalkboard can also highlight the conditions of use of a mathematical idea. I use two illustrative IEs that show how looking through this lens reveals how some of the features of IEs as outlined in the literature are enacted.

IEs through the lens of summoning-back

The focus on IEs through the lens of *summoning-back* enables me to see how the previous two lenses are used together to form an IE that extends over time. I use the term *summoning-back* to refer to how some part/s of an IE are preserved for a period of time, to be used again in a different IE to highlight some mathematical idea. Viewing an IE through the lens of *summoning-back* reveals that using examples, how they are written on the chalkboard, and how the teacher interacts with these through his talk, all occur over time. An IE does not occur at a single moment in time but is connected to an IE or parts of an IE from another point during a lesson. For instance, an example used at a specific time within an IE may be used again at another point in time when a teacher summons-back the same example for a different purpose. The teacher's talk makes explicit references to examples to remind learners of what was done previously as well as how the same example is used to connect different mathematical ideas together. A teacher may also refer to something that was previously written on the chalkboard and deliberately not erased so that it could be used again in another IE. I illustrate by means of examples how looking through the summoning-back lens reveals that an IE may extend over an entire lesson.

Conclusion

By viewing IEs separately through these lenses, it allowed me to examine how the features of IEs were enacted but which looking through all the lenses simultaneously may not have shown. It was necessary for me to determine how the features of IEs, as outlined by Leinhardt (2001), were enacted through these lenses. Viewing IE's through the separate lenses allowed me to gain more insight into the features of IE's. These insights in turn were used to inform how the coherence of IE's could be examined.

Chapter 6 : Viewing IEs through the lens of using examples

6.1. Introduction

In this chapter, I examine IEs through the lens of using examples to examine what is revealed about IEs. I examine them through this lens to see how examples work on their own without the distraction of looking through other lenses as well. As mentioned previously (in the interlude), looking through this lens reveals how and what features of IEs are brought into focus through the use of variation within and between examples in a set of examples. These features were then used to examine the coherence of IEs.

The use of elements of variation theory is useful in bringing into focus some mathematical idea. For instance, keeping some parts of examples the same while varying others, can highlight a mathematical idea that a teacher would like learners to focus on depending on the ooL of the IE. Furthermore, small variations from one example to the next reduces gaps between examples potentially making it easier for learners to follow. The use of contrasting examples can make it possible to address learner errors as well.

A *useful set of examples* is regarded as a key feature of IEs as is suggested in Leinhardt's (2001) definition of IEs. What is meant by a *useful set of examples* is, however, unclear. Apart from outlining that examples should be used to make clear the ooL of an IE, there is no clear indication of how examples can be chosen or sequenced to bring into focus the ooL of an IE. By examining IEs through the lens of teachers' *use of examples*, more insight can be gained on what is meant by a *useful set of examples*. More importantly, looking through this lens assists in gaining more insight into how the features of IE's are enacted.

I illustrate, by means of two different IEs, how Leinhardt's (2001) features of IE's can be examined through the lens of teachers' use of examples. The four features of IEs that are present in the IE's are: *attaching learner prior knowledge to new knowledge; attending to learner common errors; identifying conditions of use of a mathematical idea; and highlighting core mathematical ideas.*

6.2. IE 1 Tsebo's use of examples in revealing the four features of IEs

6.2.1. Description of the IE

This IE is part of the introductory lesson on quadratic equations taught by a teacher, Tsebo (pseudonym). I divided the lesson into three IEs based on what was perceived to be the shifts in the ooL in each IE. An overview of the lesson, the examples used and a brief description of each IE is provided in Table 6.1:

IE and ooL	Examples	Description
1: Factorisation by removal of a common factor and trinomials	$x^2 + 7x + 12$ $2x^2 + x$ $x^2 + 9x + 20$ $3x^2 - 5x$	Tsebo discussed and factorised each expression, asking learners to remember how to factorise from previous lessons.
2: A focus on the development of the zero product law	$0 \times 0 =$ $1 \times 0 =$ $b \times 0 =$ $0 \times b =$ $a \cdot b = 0$ $a = 0 \text{ or } b = 0$ $(x + 1) \cdot b = 0$ $(x + 1)(b - 2) = 0$	<p>Learners contributed in answering the first four questions i.e. providing the answer of 0 for each.</p> <p>Learners stated that one of the numbers is 0 in $a \cdot b = 0$.</p> <p>Tsebo mentioned that $x + 1 = 0$ or $b = 0$ and $x + 1 = 0$ or $b - 2 = 0$ respectively in the last two examples.</p>
3: How to use the zero product law to solve quadratic equations	$(x + 1)(x - 2) = 0$ $(x + 2)(x - 5) = 0$ $x^2 + 7x + 12 = 0$ $x^2 + 9x + 20 = 0$ $2x^2 - x = 0$	Tsebo solved the first two equations using the conclusion from episode 2 i.e. each factor can be equated to zero if the product of factors is zero. He factorised the left hand side of each of the next three equations and again used the conclusion to solve each one

Table 6.1: Overview of the Tsebo's lesson

Zooming into how the examples were used in this IE reveals learners' prior knowledge was used to introduce a new mathematical idea by the strategic choice and sequencing of examples. This IE highlights how the examples were used to attach learners prior knowledge to new knowledge which is one of the features of IE's mentioned by Leinhardt (2001).

6.2.2. Examples used

Table. 6.2 illustrates the examples used:

1. $0 \times 0 =$
2. $1 \times 0 =$
3. $b \times 0 =$
4. $0 \times b =$
5. $a \cdot b = 0$ $a = 0 \text{ or } b = 0$
6. $(x + 1) \cdot b = 0$
7. $(x + 1)(b - 2) = 0$

Table 6.2: Examples used by Tsebo

The use of variation in the examples make it possible to highlight the generalisation, i.e. the zero product law. The variation in the order of the terms as well as their values, i.e. numerical and non-numerical values, made possible the illumination of the zero product law.

6.2.2.1. The use of variation in the examples used

The variation in the set of examples consisted of varying the type of values used as well as the order of the terms, both of which made it possible to arrive at the generalisation of the zero product law.

Variation in the type of values used

The first two examples contained numerical values only, while the next two contained a letter (b) together with the number zero. Working with numerical values would have been familiar to learners at the Grade 10 level and may seem very basic. So too with multiplication of numbers by zero. It is unlikely that learners at this level do not know that any number multiplied by zero is zero, but using the particular examples makes it possible to bring this fact to the forefront of learners' awareness. Using numerical values with one of the factors being zero, makes it possible for learners to see a pattern that they may have previously not consciously thought about. The use of the first four examples makes it possible for some learners at least to arrive at the generalisation that $a \cdot b = 0$ means that either value could be zero. After learners provided the answers to the first four questions, some of them were able to state the generalisation in the next example, i.e. in $a \cdot b = 0$. They mentioned that either a equals zero or b equals zero, as is illustrated by excerpt (table 6.3).

Line	Speaker	Utterances, gestures, actions
1	Tsebo	<i>Writes $a \cdot b = 0$ beneath the fourth example</i>
2	Tsebo	So which one is zero there? (<i>pointing to what he had written</i>)
3	Learners	1. Either way. a. <i>(Three different learners shout out different answers simultaneously)</i>
4	Tsebo	It is a?
5	Learners	Or b.
6	Tsebo	a or b?
7	Learners	It is either way.
8	Tsebo	<i>Writes $a = 0$ or $b = 0$ underneath $a \cdot b = 0$</i>

Table 6.3: Excerpt indicating Tsebo's utterances

When Tsebo asked “which one is zero there?” learners shouted out different answers (line 3). When he repeated the question, a group of learners shouted out “either way” (line 7). This suggests that some of the learners at least were able to arrive at the generalisation.

The fifth example indicates the generalisation of the zero product law, i.e. $a \cdot b = 0$ with its associated implication of either $a = 0$ or $b = 0$. This generalisation was referred to while Tsebo

solved quadratic equations in the next IE and so can be referred to as a model example. This generalisation was used to illustrate how the zero product law could be used in the solving of quadratic equations.

The use of numerical values before the use of letters made possible the generalisation of the zero product law. The varying of the values used in the first four examples highlighted the zero product law, which is one of the principles of mathematics. Variation in the values used in examples make it possible to foreground one of Leinhardt's features of IEs, i.e. *highlighting a key mathematical idea*, i.e. generalisation of a law.

Sequencing of examples using variation

While the first two examples were numerical, they were replaced by the next two non-numerical examples (examples 3 and 4), containing a letter (b). The letter b as one of the factors was kept constant or unvarying throughout the next five examples. Keeping the “ b ” invariant made it possible for learners to connect one example to the next. In addition, keeping the “ b ” invariant also helped to focus attention on zero as one of the factors.

The letter b was kept constant while the position of the zero was varied in all of the first four examples. Zero was positioned to be either the multiplier or the multiplicand in all of the examples preceding the generalisation. This assisted in illustrating that the position of the zero does not affect the product being zero.

In the last two examples, the letter b was kept constant. In addition, in the example after the generalization, i.e. $a \cdot b = 0$, the position of the letter b was also kept constant (as the second factor) while the first factor was changed to include a factor with more than one term, i.e. $x + 1$. Keeping the b constant helped to connect the example to the previous one, i.e. the generalisation of the zero product law. Changing the first factor to include more than one term made it possible to focus on how the generalisation arrived at could be used to equate factors with more than one term to zero.

The first factor $x + 1$ in the last example, i.e. $(x + 1) \cdot b = 0$ was the same as the previous example. Neither the factor nor its position was varied. Keeping both the factor and its position the same helps to focus attention on the second factor of the last two equations. In the first of the two, the factor consisted of one term while the second consisted of two terms.

The last two examples make it possible to bring into focus two mathematical ideas about the use of the zero product law. First, they illustrate that the factors can be either single-term factors or factors with two terms. The generalisation of $a \cdot b = 0$, while visually portraying single-term factors, represents factors with any number of terms. This generalisation may not be obvious to learners. Keeping the first factor the same while changing the last factor to a single-term factor in $(x + 1) \cdot b = 0$ and a factor with two terms in $(x + 1) \cdot b = 0$ makes it possible for learners to take note that irrespective of the number of terms in the factors, each factor is equated to zero in the zero product law. Second, these last two examples illustrate how the zero product law could

be used to equate either factor to 0. These examples were not numerical and did not require learners to solve for x . They were used to illustrate how each factor could be equated to zero while still not solving for x . They therefore provided the basis for the use of the zero product law in solving quadratic equations in the next IE. The sequencing of the examples in this IE illustrates an attention to one of the features of IEs highlighted by Leinhardt (2001), i.e. *attaching learners' prior knowledge to new knowledge*.

6.2.3. Features of IEs seen through the lens of using examples

Zooming into the use of examples in this IE foregrounds how examples can be chosen and sequenced to make it possible to *attach learners' prior knowledge to new knowledge* and *highlight a key mathematical idea*. The small changes from one example to the next, while keeping others constant, helped to focus attention on the parts of the equations that were changing in order to see something new, makes it possible for learners to proceed from what they know to something new. This is one of the features of IEs, as mentioned by Leinhardt (2001), i.e. *attaching the prior knowledge of learners to new knowledge*.

Viewing this IE through the lens of using examples also shows how a core principle of mathematics is highlighted. The use of numerical values before the use of letters, together with a variation in the order of the values used in each example, made it possible to bring into focus the generalisation of the zero product law, which is represented by a model example. The role of the choice and sequencing of examples in illuminating one of the features of IEs, i.e. *highlighting key mathematical ideas*, is foregrounded.

6.3. IE2 Tsebo's Use of examples in looking at features of IEs

In the next example, I illustrate how zooming into the use of examples reveals not only their role in *attaching learner prior knowledge to new knowledge* and *highlighting key mathematical ideas*, but also how they can be used to *attend to learner errors* and *identify the conditions of use of a mathematical idea*. These are four features that should be addressed by IEs, as outlined by Leinhardt (2001). My grounded analysis of data illustrates how these features are foregrounded through the enactment of an IE.

6.3.1. Description of the IE

The IE is a focus on the use of the zero product law in solving quadratic equations. Prior to this IE, the teacher, Tsebo, had used the example $x^2 + 7x + 12 = 0$ to demonstrate the steps of a procedure to solve a quadratic equation. In the IE in focus, Tsebo used a set of examples to illustrate how the procedure could be used to solve different quadratic equations. The examples chosen are illustrated in table 6.4.

1. $x^2 + 8x + 12 = 0$
2. $11x + x^2 - 12 = 0$
3. $11x - x^2 + 12 = 0$
4. $4x^2 = 9$
5. $x - 9 = \frac{-20}{x}$

Table 6.4: Examples used by Tsebo to illustrate the procedure to solve quadratic equations

All of the examples were used to illustrate how the procedure could be used and could therefore be classified as *examples of* the procedure.

6.3.2. Using similar examples

The first example in the IE in focus was similar to the one used to show how each step of the procedure to solve a quadratic equation was executed, i.e. the example $x^2 + 7x + 12 = 0$. I use the word ‘similar’ because both were trinomials provided in the general form $ax^2 + bx + c = 0$. In both equations, the order of the terms was the same, with only the coefficient of x differing. While this resulted in different factors in both equations, both resulted in factors consisting of two terms. Using a similar example as that used to demonstrate the different steps in the procedure provided the possibility for learners to connect between the procedure and the solving of the equation $x^2 + 8x + 12 = 0$. With the use of the first example being similar to the one used to demonstrate the procedure reveals the feature attaching learners’ prior knowledge to new knowledge.

6.3.3. Using examples with variation

The variation in the examples provided by Tsebo makes it possible to help focus learner attention on the different forms in which quadratic equations are given. The range of examples focuses on quadratic equations with three terms, two terms and rational terms. In addition, the examples include those equations where the second degree is not immediately visible but requires rewriting to determine that the equation is quadratic. All of these variations make it possible for learners to note the shift in each solution strategy in the use of the procedure. For instance, from the example used to demonstrate the procedure to the first example, the only difference was the change in the middle term (the coefficient of x). The only difference from examples 1 to 2 was that the order of the first two terms was changed, while between examples 2 and 3, the difference was in the sign of the coefficient of x^2 . The first three examples illustrate one change from one example to the next. The change from examples 3 to 4 was the number of terms given (from three terms to two) while the change from 4 to 5 was the type of terms given, in that example 5 contained rational terms.

6.4. The features of IEs revealed through the use of variation

The use of variation in the choice and sequencing of examples reveals that this IE attends to some of the features of IEs as mentioned in the literature. The features of IEs as outlined in the literature and attended to in this IE by the use of variation in the examples are *attaching prior knowledge to new knowledge*, *addressing common learner error*, and *identifying the conditions of use* of a mathematical idea.

6.4.1. Attaching prior knowledge to new knowledge

Varying some parts of the examples while keeping others the same from one example to the next made it possible to attach the prior knowledge of learners to new knowledge. In the first three examples, the number of terms was kept constant and the same as the number of terms in the example used to demonstrate the procedure of solving a quadratic equation. They were all trinomials, as was the example used to illustrate the procedure to solve quadratic equations. Keeping the number of terms constant makes it possible for learners to focus on the procedure to solve a quadratic equation and not be distracted by the different methods of factorisation, such as the difference of two squares required in some equations with two factors.

The second equation, $11x + x^2 - 12 = 0$, was similar to the previous example, i.e. $x^2 + 8x + 12 = 0$, in that it was a trinomial with the coefficient of x^2 being 1 and the last term 12. The order of the terms and the sign of the constant term were however changed. Keeping the number of terms and the coefficient of x^2 the same, even though the order of the terms was different, provided learners with the possibility of noting the similarity to the previous equation, i.e. it was still a trinomial. Some learners were thus able to state the order of the terms in order to rewrite the equation into the standard form and call out the factors of the equation. The small changes from the previous equation to the new provided the possibility for learners to connect between the previous and the new equation. This is evidenced by some learners being able to provide the correct standard form as well as the correct factors.

The only change in the third equation, i.e. $11x - x^2 + 12 = 0$, from the previous equation, $11x + x^2 - 12 = 0$, was that the coefficient of x^2 was negative. The order of the terms, the number of terms and the other signs were the same. Some learners were thus able to state the order of the terms to write the equation into its standard form since the order of the given terms was not different from the previous example. Keeping the order of the terms the same, even though it was not in standard form, provided the possibility for learners to focus on the negative first term. It was therefore possible for Tsebo to focus his explanation on the procedure to enable the change required to make the first term positive and so to factorise the trinomial without the added challenge of taking into consideration the negative first term. The use of the same example as the previous one, with only a change in sign, provided the possibility for Tsebo to focus his IE to proceed from the known to the unknown.

Making small changes from the equation used in the procedure to the equation used to demonstrate the application of the procedure and to the subsequent equations while keeping the number of terms the same in all three equations made it possible for learners to connect the steps of the procedure in solving a quadratic equation to its application in solving other equations. The use of variation in the examples reveals how examples can be used to *attach learners' prior knowledge to new knowledge*.

6.4.2. Addressing common learner errors

Working with a negative coefficient of x^2 provides the teacher with the possibility to address the issue of a strategy to prevent factorising a trinomial with a negative first term. The third equation, i.e. $11x - x^2 + 12 = 0$, contains a negative coefficient of x^2 . Some of the learner errors in solving quadratic equations by factorisation are caused by learners' inability to work with negative signs (Zakaria, 2010). Including this example points to the teacher providing the possibility for learners to learn how to factorise an expression with a negative coefficient of the first term in a quadratic equation.

6.4.3. Identifying conditions of use of a procedure

The presence of a contrasting example helps to identify the conditions under which the procedure of solving quadratic equations by factorisation can be used. The fourth example, $4x^2 = 9$, can be regarded as a contrasting example in that it consists of two terms. All the other examples, including the example used to illustrate the steps of the procedure to solve a quadratic equation, were trinomials, including those provided in the general form $ax^2 + bx + c = 0$ and those with terms given in a different order. The example $4x^2 = 9$ is in contrast with the other examples. While it is still a quadratic equation, indicated by the presence of the second-degree term x^2 , it has only two terms.

The use of the example $4x^2 = 9$ provides the possibility for learners to note two important ideas in the topic of quadratic equations. First is that a quadratic equation may not necessarily consist of three terms only, as is suggested by the general form $ax^2 + bx + c = 0$. A quadratic equation is therefore identified by the degree of the expression being two and not the number of terms in the equation. Second, the contrasting equation also provides the possibility for learners to note that the method used in factorising (the different methods were already addressed in previous lessons, according to the curriculum documents) a quadratic equation is immaterial to the method used in solving. One of the ways to solve quadratic equations is by using the zero product law. The important idea in solving quadratic equations is that the equation can be written as factors with a product of zero. The method of factorisation used is therefore immaterial. The use of this example therefore makes it possible for learners to *identify the conditions of use of the procedure* in solving a quadratic equation.

The fifth example also makes it possible for learners to identify the conditions of use of the procedure outlined by Tsebo. While the equation consists of three terms, like the first three examples, it contains a rational term. It can also be regarded as a contrasting example because the degree of the equation is not immediately apparent. While it may look like a linear equation, as it is given, the degree of the equation is only “visible” to learners once it is rewritten in the general form $ax^2 + bx + c = 0$. The presence of this example helps to focus learner attention on first manipulating the equation to rewrite it without the rational term to determine the degree of the equation. This in turn would indicate to learners that the procedure provided by Tsebo could in fact be used to solve the given equation. This contrasting example, while different from the previous contrasting example, provides the possibility for learners to note that it can be solved using the given procedure because it is a second-degree equation written in the general form.

The choice of the last two examples in this example set reveals one of the features of IEs, i.e. *identifying the conditions of use of a mathematical idea*. In this IE, the last two examples make it possible to note that neither the number of terms in the equation nor the form in which the equation is given determine the use of the procedure. It is rather the degree of the expression, once the equation is rewritten in the general form, that indicates the conditions under which the procedure can be used.

Table 6.5. provides a summary of how the variation of examples used by Tsebo brings into focus the features of IEs.

Examples used	Type of variation used	Features of IEs revealed
1. $x^2 + 8x + 12 = 0$	<p>What stays the same: Form same as reference example, i.e.</p> $x^2 + 7x + 12 = 0$ <p>Three terms, same order,</p> <p>What is different: coefficient of middle terms different.</p>	Attaching learner prior knowledge to new knowledge
2. $11x + x^2 - 12 = 0$	<p>What stays the same: Three terms, constant value the same.</p> <p>What is different: Order changed from previous; different sign of constant term.</p>	
3. $11x - x^2 + 12 = 0$	<p>What stays the same: Order same as previous, three terms the same, constant term the same,</p> <p>What is different: Negative coefficient of x^2</p>	Attaching learner prior knowledge to new knowledge. Attending to learner common error.
4. $4x^2 = 9$	Contrasting example: Two terms, RHS not zero, method of factorisation	Conditions of use of the method of solving quadratic equations by

Examples used	Type of variation used	Features of IEs revealed
	different.	factorisation.
5. $x - 9 = \frac{-20}{x}$	Contrasting example: Rational term, second degree, i.e. x^2 not obvious.	

Table 6.5: The variation used in Tsebo's examples and the features of IEs .

Viewing IEs through the lens of using examples sheds light on how using variation in examples can be used to reveal features of IEs. Using a similar example to the one used to demonstrate the procedure helps to highlight the steps of the procedure to be used in solving a quadratic equation. A small change in each subsequent example from the previous one, gave learners the chance to flexibly apply the procedure of solving quad equations by factorisation. These changes highlights that the steps of the procedure demonstrated can only be applied once the equation is written in the general form. The small changes from one example to the next illustrate how the equations could be rewritten in the standard form before the use of the procedure. Cheng, Leong, and Toh (2021) use the term “advance one more” to describe the small changes from one example to the next. This term is used to refer to example sets where each subsequent example extends the use of a particular technique or procedure towards a more comprehensive application of the technique or procedure. Using variation in the examples therefore makes it possible for learners to attach their prior knowledge to something new.

Varying the examples chosen can also be used to attend to common learner errors. Examples can be varied by teachers according to common difficulties that their learners experience. Finally, the choice of contrasting examples can identify the conditions under which a particular mathematical idea can be used.

6.5. Discussion and Conclusion

While Leinhardt provides seven features of IEs, these features do not shed light on how they link together in an IE. Using the lens of a useful set of examples brings into focus the role of examples in the enactment of the four features of IEs as outlined by Leinhardt. The IEs examined through the lens of using examples, in this chapter, thus provides insight into what constitutes a useful set of examples.

The example sets used in each of the two IEs in this chapter demonstrate how variation can be used to reveal each of the different features of IEs. While varying examples by making small changes from one example to the next illustrate how learners' prior knowledge can be attached to new knowledge, the use of contrasting examples sheds light on how examples make it possible to identify the conditions of use of a mathematical idea. In addition, varying the signs, the type of values used and the order of terms are some of the variations that highlight how common learner errors can be addressed. The particular error, to be addressed based on a teacher's awareness of common errors made by his learners, determines what is varied in a set of examples in an IE.

A summary of the features of IEs that are brought into focus through the lens of using examples is provided in table 6.6.

Variation used	Features in focus
Variance amongst invariance	Attach prior knowledge to new knowledge Address common learner error Highlight core mathematical ideas
Contrasting examples	Identify conditions of use of a mathematical idea

Table 6.6: Features of IEs in focus through variation in examples.

The features of IEs revealed through the use of examples in the examples of IEs provided in this chapter are by no means the only features that the choice and sequencing of examples can foreground. Other sets of examples may reveal other features of IEs than those highlighted in this chapter. The purpose of providing these IEs is to illustrate the power of looking through the separate lenses, and in this chapter through the use of examples, in the enactment of the features of IEs that may or may not be outlined in the literature on the theory of IEs. Looking at the use of examples separately, together with other lenses, may not have shown how these features are brought into focus in an IE.

The use of examples themselves is inadequate in bringing into focus the features or the ooL of an IE (Essien, 2021). This is but one of the lenses of IEs through which the features of IEs can be examined. There are other lenses through which the four features of IEs, as outlined by Leinhardt, can be examined. I next look through the lens of inscriptions and teacher talk, i.e. role of what is written and how it is written, and the accompanying talk, to examine the features of IEs revealed through this lens. I do this to support my view that looking at IEs through separate lenses can bring into focus features of IEs that can be used to examine their coherence.

Chapter 7 : Teachers' use of Inscriptions and Talk in IEs

7.1. Introduction

In this chapter, I examine IEs through the lens of inscriptions and talk. By inscriptions I mean the written representation of IEs. In this study, most of what is written by a teacher in an IE is inscribed on the chalkboard. I therefore examine how and what is written on the chalkboard. I also examine through this lens how a teacher's talk is used, together with what is inscribed on the chalkboard. Through the lens of the interaction between inscriptions and the talk, i.e. between the written and the verbal, I examine the features of IEs. These features are used ultimately to develop a framework that can be used to examine the coherence of IEs.

An IE in literature as a “set of discussions” (Leinhardt, 2001, p. 341) suggests that IEs are verbal. While there is also consideration in literature on written IEs, how the verbal and the written are woven together in an IE is not in focus. These two components of an IE, while viewed in isolation to each other, may reveal some features of an IE, can reveal even more features of IEs when viewed together. While there are many ways that inscriptions are made, e.g. in text books or on a computer, I focus on those that are written on the chalkboard. I do this because my data reveals the heavy reliance on the chalkboard in the IEs of teachers I have observed in this study.

There is much that a teacher writes on the chalkboard. For instance, a common use of the chalkboard in the mathematics classroom is to inscribe the solutions to a question or task. A teacher may also note definitions of particular mathematical terminology that may not be familiar to the learners. At other times, teachers may write down particular words they use as they speak because they are new words or because of a learner's reaction to particular words used. For example, learners may ask for clarification or elaboration that may motivate teachers to write down something in particular.

Some inscriptions made by teachers on the chalkboard may be *random* while others are more deliberate. I use the word *random* to refer to those inscriptions made on the chalkboard with no attention to where they are placed on the chalkboard, or what mathematical purpose they serve. For instance, teachers may sometimes include calculations that are considered to be rough work that may be necessary to arrive at a solution but do not follow a logical series of steps in the solution.

The chalkboard, however, can be used in deliberate ways to highlight different things. Teachers may pay particular attention to how work on the chalkboard is laid out to bring into focus a mathematical idea or to the content that needs to be preserved on the chalkboard to be used again in another IE. Looking at IEs through the lens of inscriptions can reveal how the features of an IE are enacted and which a teacher's talk can make explicit. I therefore examine how the

content inscribed on the chalkboard, supported by a teacher's utterances, sheds light on the features of an IE.

At this point, I must emphasise that viewing an IE through the lens of inscriptions and talk does not suggest that examples are not used in these IEs. I have mentioned previously that examples are at the core of IEs, which is also underscored in the literature (e.g. Leinhardt, 2001). I am rather foregrounding the inscriptions and talk in an IE that revolve around examples. The choice or the sequencing of examples is not in focus, but I shine a light on the interaction between inscriptions and talk in an IE.

I illustrate the different features of IEs revealed by the enactment of three IEs through the lens of teachers using inscriptions and talk in their IEs.

7.2. IE1: Ndivho's Use of inscriptions and talk in identifying the conditions of use of a mathematical idea

Placing contrasting examples next to each other on the chalkboard helps focus learner attention on these in a way that may not happen had the examples been inscribed on different parts of the chalkboard or with other examples written in between them. I provide an example to illustrate that looking at an IE through the lens of the interaction between inscriptions and teacher talk sheds light on how different strategies can be used in solving two contrasting examples of quadratic equations.

7.2.1. Description of the IE

The IE reported on here was part of the third lesson on solving quadratic equations by factorisation. Ndivho (pseudonym) had already illustrated to learners in the first lesson on quadratic equations how to solve examples of the form $(x - a)(x - b) = 0$. He had also illustrated how to solve equations of the form $ax^2 + bx + c = 0$ (with $a = 1$) using the zero product law in the same lesson. In lesson 2, after reviewing homework learners were given at the end of lesson 1, he explained how to solve equations of the form $ax^2 + bx + c = 0$ with $a \neq 1$. In lesson 3, he provided a range of examples in different forms. One of the examples he provided was $(x - 7)(x - 5) = 3$. He asked learners how they would solve this equation. I illustrate how the IE was enacted around the solution to this equation.

A learner provided the correct solution to the equation, which Ndivho wrote on the chalkboard. He provided an alternate first step to the solution, which he wrote to the right of the correct solution. Below this he wrote the equation $(x - 7)(x - 5) = 0$, as shown in Figure 7.1.

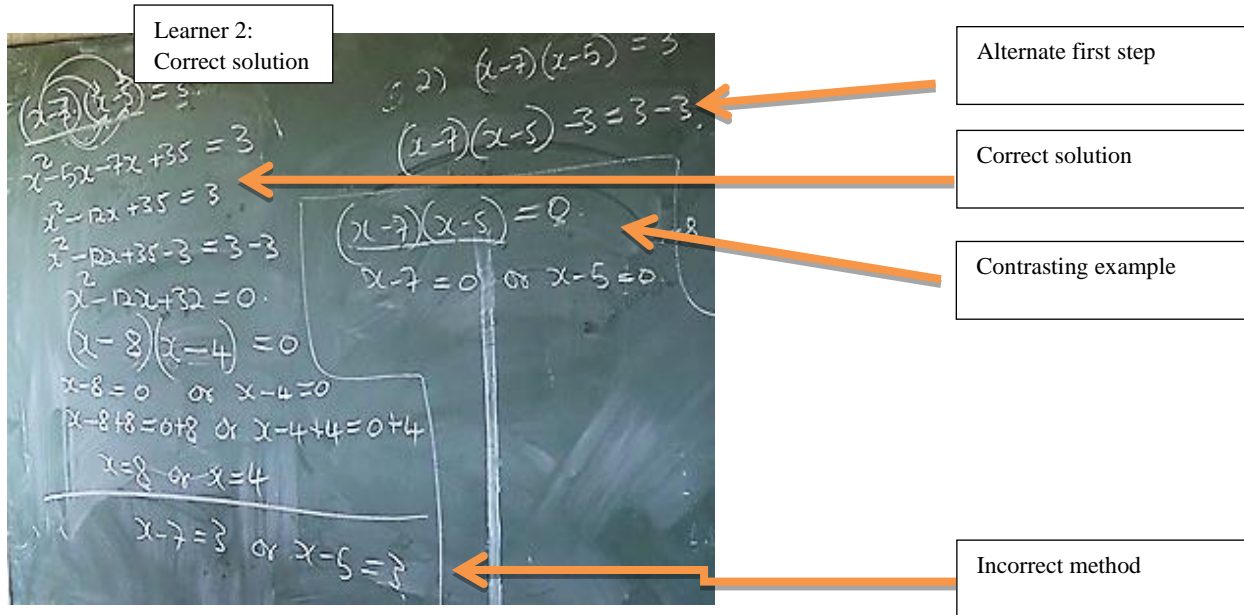


Figure 7.1: Picture of board-work of contrasting examples

The two examples can be regarded as contrasting examples. The left-hand-side of both equations was kept the same while the right-hand-side was different. While visually these examples may seem similar, their algebraic solution strategies⁹ are very different. While the zero product law could be applied to solving the equation $(x - 7)(x - 5) = 0$, the same strategy cannot be applied to the equation $(x - 7)(x - 5) = 3$. This, however, may not be obvious to learners as illustrated by a learner's (learner 1) incorrect solution to the equation $(x - 7)(x - 5) = 3$ (provided prior to a learner's correct solution) in fig. 7.2.

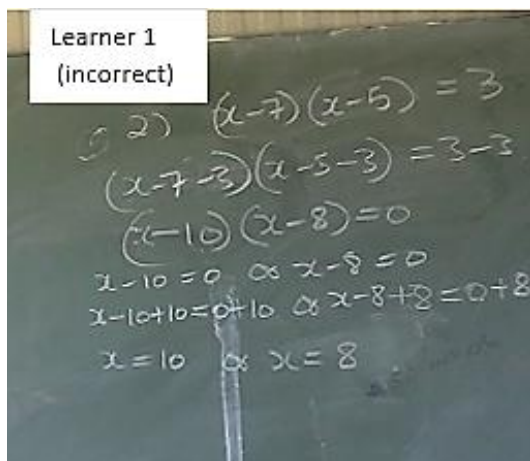


Figure 7.2: An incorrect solution provided by learner 1

⁹ The solutions can be found graphically but his method is not in focus in this IE and lesson.

Placing the two contrasting examples (and their solutions) next to each other on the chalkboard makes it possible to draw the attention of the learners to the differences in their solution strategies. Ndivho’s verbal utterances, however, made their differences explicit. Table 7.1 illustrates how he developed an IE that “moved” between the inscriptions made, his talk and the two examples.

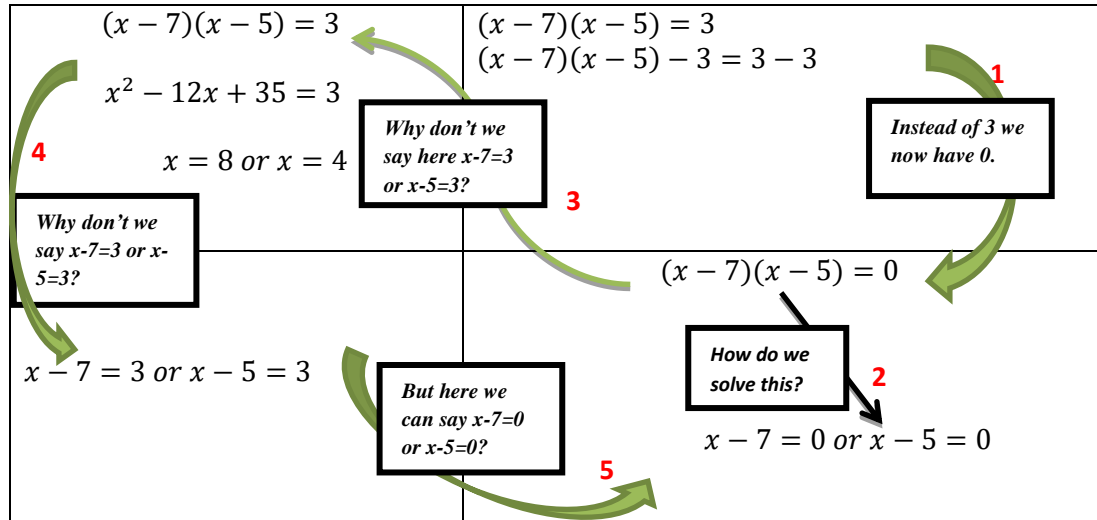


Table 7.1: Diagram illustrating the IE of differences between contrasting examples

Ndhivo asked learners how they would solve the second equation i.e. $(x - 7)(x - 5) = 0$ (arrow 2 in table 7.1). I provide an excerpt (table 7.2) of this part of the lesson to illustrate Ndivho’s talk in distinguishing between the two solution strategies.

Line	Speaker	Utterances and gestures
1	T	Instead of 3 we now have zero, can we solve for x here? (arrow 1 in table 7.1)
2	L1	x minus 7 is equal to zero.
3	T	x minus 7 is equal to zero.
4	Ls	Or x minus 5 is equal to zero.
5	T	Or x minus 5 is equal to zero. Do you see the difference here? (pointing to $(x-7)(x-5)=0$)
6	Ls	Yes.
7	T	Why don't we say here (pointing to first example), x minus 7 equals to 3? Or x minus 5 equals to 3? Why don't we say that? x minus 7 equals to 3, or x minus 5 equals to 3 (while writing this on the board)? (arrow 2 in table 7.1)
8		
9		
10		Why, in this case, don't we say, we do this (pointing to what he had just written)? Why don't
11		we do that? (arrows 3 and 4 in table 7.1)
12		But here (pointing to the 2 nd example) we say we can just say, it is either this is zero, or this is
13		zero? Why is that so?
Learners are silent		
14	T	It is because when we multiply two things and the answer is zero, it means one of those two is
15		a zero? It is either the first one is zero or the second is zero (pointing to the factors in the
16		second example). But in this case we cannot say this is equal to 3 or this equals 3 because we
17		need the side to be zero. Okay. We need the other side... we can only do that if the other side
18		is zero.

Table 7.2: Excerpt indicating Tsebo’s utterances relating to differences between contrasting examples

N.B. In all excerpts, I use
T: for teacher utterances/actions
L: for learner utterances
Ls: for groups of learners speaking in chorus

Ndivho used the close proximity of the two equations inscribed on the chalkboard to draw learners' attention to the differences between them when he pointed to each (arrow 1 in Table 7.1.) and said "instead of 3 we now have 0" (line 1). Lines 7 to 13 indicate his "movement" between the two examples to draw attention to the different solution strategies for each. He pointed to the first example, i.e. $(x - 7)(x - 5) = 3$ and asked why they couldn't solve by equating each factor to three, like they did with the second example. His utterances were accompanied by his pointing to the relevant examples, i.e. to the first example and then to the factors in the second (arrows 3 and 4 in Table 7.1). In lines 14-18, he attempted to draw attention to why their solutions differed by saying that each factor can equal to 0 only when two numbers ("things") are multiplied to give 0. He then referred to the first example again and said that in this case, the right hand side was three, unlike in the second example, where the right hand side (pointing to the right hand side of the second equation) was zero.

7.2.2. Ndivho's use of inscriptions and talk in revealing the feature identifying the conditions of use of the zero product law

This IE illustrates how the verbal and written components of an IE interact with each other to *identify the conditions of use* of the zero product law in solving a quadratic equation. As mentioned previously, this is a feature of IEs that is highlighted by Leinhardt (2001). The IE developed around the two contrasting examples $(x - 7)(x - 5) = 0$ and $(x - 7)(x - 5) = 3$ to highlight the differences in the solutions strategies to questions that look very similar. Ndivho kept the left hand side of both the examples the same while changing the value on the right hand side of the equation. Inscribing these two examples next to each other on the chalkboard makes it possible for learners to take note of the differences in the questions and therefore how the non-zero value on the right hand side of the equation $(x - 7)(x - 5) = 3$ affected the solution strategy.

The solutions, however, were not inscribed on the chalkboard in isolation of Ndivho's talk. His utterances were used to support his inscriptions by making these differences explicit. First, Ndivho drew attention to the differences between the right-hand sides of each equation by his utterances together with his gestures, "moving" between the two. He said "*Instead of 3 (pointing to the 3 in the equation written on the left) we now have zero*" (line 1, table 7.1). His words and gesture of pointing drew attention to the two different questions, even though they were placed next to each other. He then focused on explaining their different solution strategies by again

moving from one to the other verbally as well as by pointing to the relevant equation while talking (lines 12-15 in table 7.2).

While inscribing closely related examples next to each other on the chalkboard helps draw attention to their similarities and differences, it is not sufficient on its own. Learner attention may not necessarily be drawn to the differences merely by looking at the different questions. Teacher talk is also not sufficient on its own in drawing attention to differences. Learners may not be able to attend to the differences simply by listening to a teacher talk about them. Both the writing on the chalkboard together with the teacher's talk helps to foreground the differences and thus the conditions of use of a mathematical idea. This illustrates an IE that is not merely linear but flows back and forth between the physical space of the written examples on the chalkboard and the metaphorical space between the verbal and written. By the metaphorical space between the verbal and the written I mean the gap between what is written and what was said. While the close proximity of what was inscribed on the chalkboard was meant to highlight the boundaries around the use of the zero product law, this was made explicit by the teacher's talk. The teacher's talk was thus closely aligned with what was inscribed on the chalkboard.

Viewing an IE through the lens of inscriptions and talk helps to foreground how the verbal and the written are woven together to make it possible to reveal the conditions of use of the zero product law, which can be inferred to be the ooL of the IE. While the close proximity of the solutions on the chalkboard makes it possible for learners to note the different strategies used, the teacher's talk made explicit that equating each factor to the value on the right hand side can only be done if the value on the right is zero. The example choice, as well as how the inscriptions and talk interact with each other, help to make it possible to *identify the conditions of use* of the zero product law in solving a quadratic equation.

Inscriptions and talk in an IE can also be used to highlight the different solution strategies to a question. I next illustrate the enactment of such an IE.

7.3. IE2: The lens of inscriptions and teacher talk in highlighting connections between closely related questions and the core mathematical principles in a topic

7.3.1. Description of the IE

A teacher's (Tsebo's) arrangement of solutions to closely related questions contributed to developing an IE that helped draw attention to the relationship between the solutions to different questions and how they connected to the given number pattern as well as to the core mathematical principles of linear number patterns. The IE formed part of Tsebo's second lesson on linear number patterns. In the first lesson, he focused on defining the necessary terminology, finding the next few terms in a given number pattern, verbally describing number patterns, and

finding the general term without the use of a specific formula. I provide an illustration of an episode in the second lesson that followed a review of homework from the previous lesson. In this part of the lesson, Tsebo provided solutions to the three questions in fig. 7.3.

Given the sequence 23; 19; 15;

1. Find the general term if it is arithmetic.
2. Find the 15th term.
3. Which term in the sequence is equal to -53?

Figure 7.3: Tsebo's questions based on the given sequence

N.B. The first question requires confirming that the given number pattern is arithmetic (which I refer to as 1(a)) before finding the general term (referred to as 1 (b)).

7.3.2. Inscriptions in the IE

While the solutions were inscribed on the chalkboard, the focus of the IE, inferred by how the IE was enacted, was on connecting all three solutions to each other as well as to the given pattern of numbers.

The ooL of this IE, as is suggested by what was made possible to learners, was to make explicit the connections between all questions relating to the given pattern. Tsebo provided an IE that consisted of a specific layout of the inscriptions on the chalkboard, supported by his talk. The inscriptions were laid out spatially to suggest connections between the questions. However, this spatial arrangement does not only relate to the physical arrangement of the questions but also represents the metaphorical space connecting the written (physical) and the verbal space between them to highlight their connections. I first show how Tsebo laid out what was written on the chalkboard before a discussion of how his utterances and gestures contributed to the explanation. He wrote the questions and their solutions on the chalkboard, as shown in Figure 7.4.

Is the sequence arithmetic?

Solution to find T_n

Solution to find general term

Solution to which term is -53

Figure 7.4: Tsebo's layout of solutions on the board

How Tsebo went about making the inscriptions on the chalkboard suggests that his arrangement of the content was deliberate. As seen in Figure 7.4, Tsebo wrote the solutions with 1(b) written below 1(a). He then ruled a vertical line to the right of both before arranging questions 2 and 3 to the right of 1(a) and 1(b), with questions 2 and 3 arranged one below the other as shown in Figure 7.4. The result was that all four solutions were eventually laid out side by side on the chalkboard. A small portion of the chalkboard was used despite there being more chalkboard space available. Drawing a line to the right of question 1 to write the solutions to questions 2 and 3, even though the rest of the chalkboard was available, suggests that this layout was deliberate. The way in which he went about making the inscriptions on the chalkboard suggests that he deliberately laid out the solutions side by side for a particular purpose. The connections between them and to the given pattern of numbers was made explicit by how his IE was enacted.

7.3.3. Inscriptions and teacher talk in the IE

The close physical arrangement suggests that all four questions and solutions were connected to each other. However, the layout only provided a visual representation of the solutions. Tsebo's very specific arrangement of the solutions was accompanied by his utterances, which complemented the physical layout of the solutions on the chalkboard in making these connections explicit. The verbal and the written elements were connected to each other by his gestures. The resulting explanation was thus the interplay between the close proximity of the solutions, Tsebo's utterances as well as his gestures. Table 7.3 is a diagrammatic representation of his IE with his inscriptions, which were spatial in nature, and his utterances.

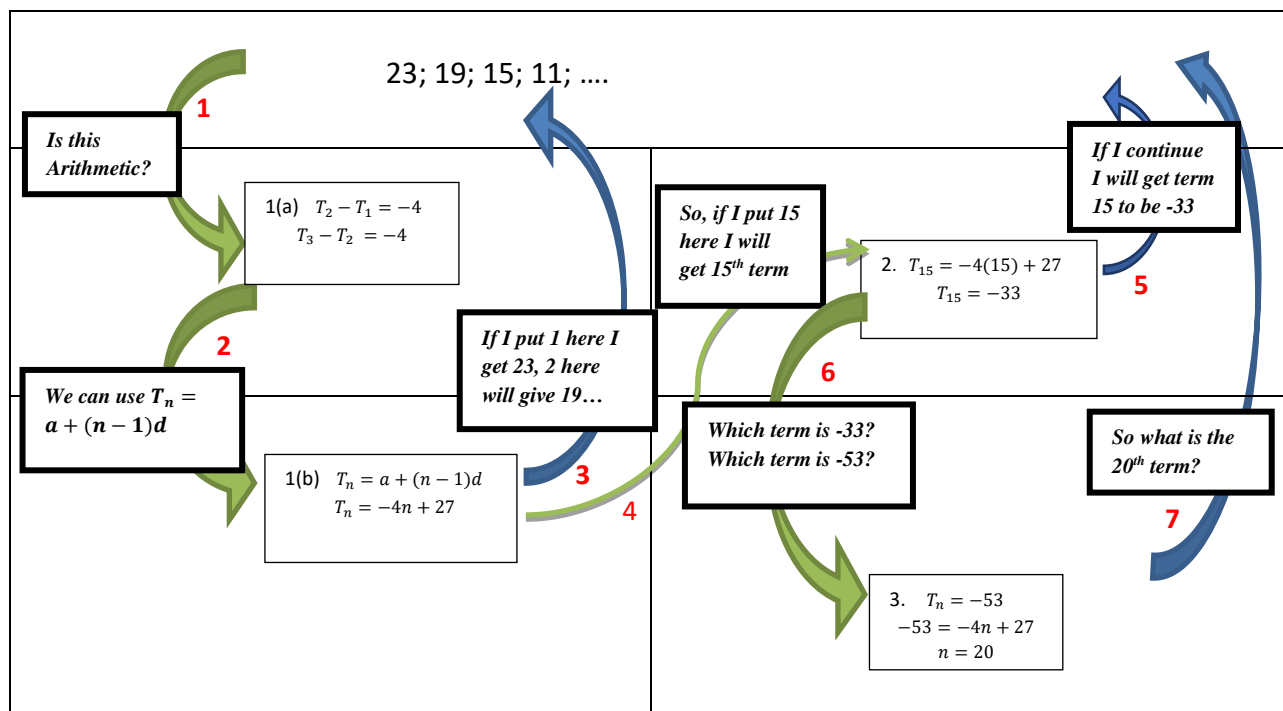


Table 7.3: Diagram illustrating inter-relationships between solutions

The arrows labelled 1-7 indicate the non-linear back and forth movement between the different solutions to illustrate how they are connected to each other, which cannot adequately be described by words alone. For instance, arrows 4, 5 and 6 illustrate how Tsebo used the relationship between the solution to question 1 (b) and the terms in the sequence to find the solution to question 2. The answer to question 2 was then connected to the number pattern, which was used to find the solution to question 3.

I next provide details of how Tsebo “moved” between the solutions inscribed on the chalkboard to make explicit how these questions were related to each other by his utterances, illustrated in table 7.4.

Line	Speaker	Utterances and gestures
1	T	If you put a 1 there (<i>pointing to n in the general term formula</i>) you get that number (<i>pointing to the first term in the sequence written in the top left corner of the grid</i>).
2		
3		If you put a 2 there (<i>pointing to the n in the general term formula worked out in previous question</i>), you
4		get the second term. If you put a 3 there, you get the third term (<i>arrow 3 in table 7.4.</i>) ...If you put 100 in
5		there, you get the 100 th term.
6		So, find the 15 th term? you put a 15 where the n is and it tells you, what the 15 th term is
7		(<i>arrow 4 in fig. 2</i>). So minus 4n plus 27 is the general term. Where there is n I put 15. Negative 16
8	T	plus 27 is negative 33. This negative 33 is the 15 th term. This means if we were to do this (<i>pointing to</i>
9		<i>terms in the sequence written in the top left corner of the grid</i>) all the way to 15 (<i>arrow 5 in table 7.4</i>),
10		the 15 th number would be negative 33.

Line	Speaker	Utterances and gestures
11 12 13 14 15 16 17	T	If we do this for two terms, the second term is 19, it means if we put a 2 in there (<i>pointing to the n in the general term formula</i>), we get 19 (<i>pointing to the second term in the sequence</i>). If we put a 3 where n is we get 15 (<i>pointing to third term in the sequence</i>). If you put a 1 where n is you get 23 (<i>pointing to the first term in the sequence</i>). So if you put a 15 where n is you get the 15 th term. We don't know what it looks like, we haven't expanded that sequence yet. But this one just makes us jump straight to the thing, and the answer is negative 33. Then, if I ask you negative 33 is which term? (<i>learners are silent</i>) Okay, 23 is which term?
18	L's	The first.
19	T	The first term. 19 is which term?
20	L's	The second.
21	T	The second term. Negative 33 is which term?
22	L's	The 15 th term.
23 24 25 26 27	T	The 15 th term. So the next question, which term is equal is to negative 53? So, we want the term that is equal to negative 53, and we say negative 53 is equal to and we equate it to the general term. We say negative 53 is equal to that general term (<i>pointing to the general term formula</i>). All we need to do is calculate the n , the position by solving for n , so we say negative 53 minus $4n$ plus 27 ... and we get 20 equals n . So, if I ask you what is the 20 th term of that sequence (<i>arrow 7 in table 7.4</i>)?

Table 7.4: Excerpt of Tsebo's utterances illustrating relationships between questions

After finding the formula for the general term of the number pattern, Tsebo mentioned that it was used to generate the number pattern. He clarified that the phrase “*used to generate the sequence*” meant that if he substituted 1 for n it would produce 23 and continued this for the values 2 and 3 (lines 1-4). By using the values for n as 1, 2 and 3, he provided the opportunity for learners to note that the formula could be used to generate the number pattern (arrow 3 in table 7.5). It also provided the basis for using the formula to find the 15th term (the next question), i.e. if the values of 1, 2 and 3 provided the first three terms, then the 15th term could be calculated by replacing n by 15 (lines 6-7). Having the solutions inscribed close to each other helps in focusing the attention of learners on the questions, their solutions and the given number pattern.

The layout of the inscriptions was accompanied by his utterances moving back and forth between the three questions (and their solutions). He used the general term to show how the first three terms of the number pattern could be found (arrow 3 in table 7.5), and used the same idea to show how the 15th term was found (arrow 4). He stated that if the number pattern was continued to 15 terms (arrow 5), then the 15th term would be -33. The answer of -33 as the 15th term was used to illustrate its salience to the number pattern (lines 8-10).

He did not only focus on the solutions to the questions but used each solution to show how it connected to the next one as well as to the given sequence of numbers. He used the solution to question 1(b), the general term, to illustrate how it could be used to find and so confirm the first three terms of the given pattern of numbers. This was then used as a justification for answering the next question, i.e. finding the 15th term. Tsebo's IE illustrates that finding the solution to each

question was not an end in itself. These solutions were connected to how the next question could be answered.

I propose that the close proximity of the solutions is intended to suggest the relationships between the questions but on its own is insufficient to bring this into learners' awareness. Tsebo's layout of the inscriptions made on the chalkboard was accompanied by his utterances to make these relationships explicit. His gestures consisted of pointing to the relevant number/letter connected the verbal and written parts of his IE. For instance, when he said, "*if you put a 1 in there*" (line 1), he also pointed to the n in the formula for the general term. When he said "*you get that number*" (lines 1-2) he pointed to term 1.

Tsebo's utterances continued to revolve around the closely laid-out solutions inscribed on the chalkboard to show how they were connected by moving back and forth between the solutions to questions 2 and 3 and the given number pattern. Having the solutions arranged one below the other suggests an intention to focus learners' attention on both the solutions together. His utterances supported this. He used the answer to the question on finding the 15th term and connected it to the next question (which term is -53? Arrow 6). He asked learners which terms in the sequence (written above all four questions) were 23, 19 and 15 (lines 17-21), drawing attention to how the question connected to the given sequence. He used learners' answers (1st, 2nd and 3rd) to then ask "*which term is -33?*" Having the questions arranged one below the other with the sequence written above draws attention to the value of term 15 being -33 and how this connected to the number pattern. He thus "moved" from question 2 to the number pattern and back to question 3 (arrows 3 and 4). This "movement" between them highlights that the next question was the converse of the previous question i.e. the value of the term was now given while the position of the term required to be found. This was made explicit by Tsebo's utterances. After illustrating the solution on the board, he reinforced the difference between the position and value of a term when he asked learners what the 20th term was (the answer to which term was -53) while pointing to the number pattern (arrow 7) provided above the questions. The close proximity of the questions inscribed on the chalkboard was accompanied by his utterances moving between the answer to the second question, how it connected to the third question and how both these solutions connected to the terms of the given number pattern.

The layout of the inscriptions on the chalkboard on its own is static and does not communicate anything besides the solutions to the questions. The utterances of the teacher on their own also do not communicate the key mathematical ideas. The layout of the written work on the chalkboard therefore "came to life" through the teacher's utterances. He used gestures to connect the verbal and written forms. His utterances complemented the physical layout of the solutions on the board. His utterances were aligned to what was written on the chalkboard. The close relationship between the solutions and the given number pattern suggested by their close proximity on the chalkboard was matched by Tsebo's utterances. It was this movement between

what was written and Tsebo's utterances that made the connections between the different questions and their solutions and how they were connected to the given number pattern explicit. His utterances involved shifting back and forth between the solutions and the given number pattern to offer an IE involving both inscriptions and talk.

7.3.4. Features of IEs revealed by looking through the lens of inscriptions and teacher talk

The close proximity of the inscriptions made on the chalkboard, together with the teacher talk in this IE, sheds light on how the inscriptions made and the talk can make it possible to connect closely related questions in a topic and to make explicit the core mathematical principles of linear number patterns.

7.3.4.1. Connections between closely related questions

The close physical arrangement of the solutions inscribed on the chalkboard was accompanied by the teacher's talk moving between all the questions and making explicit the connections between them. Finding that the terms of the sequence had a common difference led to using the appropriate general formula to find T_n . The formula was used to illustrate how the given terms of the sequence could be generated. Tsebo then used the same idea to find the 15th term. He used the answer to this question to demonstrate how the next question (which term is -53) could be answered. In other words, he did not simply answer each question independently of the others. He used each answer to explain how it related to the next question as well as its relevance to the given sequence of numbers. This suggests that his aim was not simply to find the solutions to each question; it was the connections between them that was highlighted.

7.3.4.2. Highlighting key mathematical ideas

Apart from illustrating the connections between the questions, what was also revealed by viewing this IE through the lens of inscriptions and talk was *highlighting the key mathematical ideas* around number patterns. Having all of the questions and solutions inscribed close together in one part of the chalkboard provides a detailed view of all the main ideas around number patterns required at the Grade 10 level i.e. generalising a rule, how to use the rule to find the position of a term and to find the value of a term given its position. Generalising a rule of a pattern is a key idea in number patterns. This rule can be written in the form of an expression that can be used to find other terms in a more efficient manner rather than trying to find a particular term by finding the preceding terms. The IE provided by Tsebo made explicit how to find the rule of a given pattern and how this rule could be used to find any term or the position of a particular term without having to first generate the other terms. Having all of the questions and solutions inscribed close together in one part of the chalkboard provides a detailed overview of all the key ideas in number patterns required by learners at the Grade 10 level. How the

inscriptions were made on the chalkboard, supported by teacher talk, was key in making these ideas explicit to learners.

I use another example to illustrate how viewing an IE through the lens of inscriptions and talk sheds light on the different solution strategies between two similar questions in a topic.

7.4. IE3: Inscriptions and talk in highlighting the different solution strategies to a question and addressing common learner errors

7.4.1. Description of the IE

Inscribing the different solutions to a question side by side on the chalkboard highlights that even though an equation may be solved in different ways, it results in the same answer and addresses learners' common errors. The IE that I use to illustrate this occurred in Ulwazi's first lesson on solving quadratic equations using the zero product law. Ulwazi explained how to use the zero product law to solve quadratic equations during the first half hour of his first lesson on quadratic equations. He used a range of examples to demonstrate this but did not include an example with two terms or one that required learners to factorise the difference of two squares. He then gave learners a classwork exercise that they completed individually. One of the examples in the exercise was to solve for x in $x^2 = 25$. The IE developed around the two different strategies that could be used to solve this equation.

Ulwazi began by asking learners what the next step was in solving the equation $x^2 = 25$, which he wrote on the chalkboard. A few learners called out the next few steps, which Ulwazi wrote on the chalkboard. The solution provided by the learners involved the use of the zero product law in solving the equation, which is shown on the left hand side in Figure 7.5.

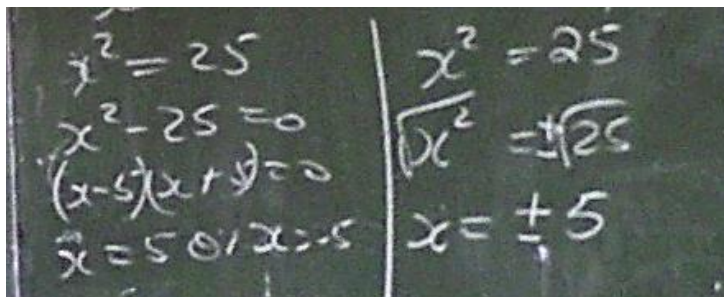


Figure 7.5: Ulwazi's alternative solutions to a question

After writing a learner's solution on the chalkboard, Ulwazi explained a different solution to the same equation. He drew a vertical line to the right of the first solution and wrote a different solution, thus placing both solutions next to each other. This was different to the way he used the chalkboard in writing the previous solutions. When Ulwazi reached the bottom of the chalkboard

in his previous writing on the it, he drew a vertical line and started writing at the top of the chalkboard to the right of the previous writings. This was the case even when continuing a solution to an example. In this case, however, Ulwazi reached the bottom of the chalkboard but instead of writing the alternate solution at the top right, where there was space (Figure 7.6), he chose to write this solution next to the first solution i.e. juxtaposing them. Figure 7.6 is a picture of the chalkboard that illustrates the two solutions placed next to each other while the right hand side of the chalkboard was available to write on.

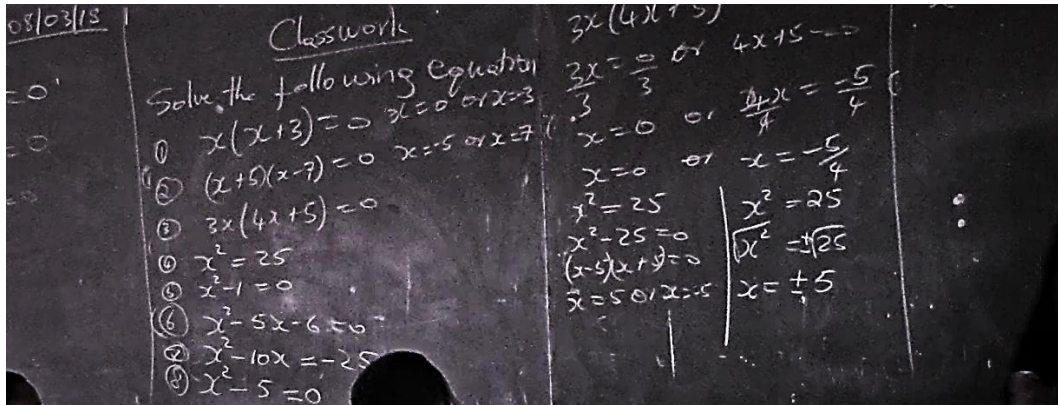


Figure 7.6: Picture of chalkboard with alternate solutions placed next to each other

Juxtaposing the alternate strategies helps to draw the learners' attention to the fact that the solutions were the same irrespective of the strategy used. This was, however, not made explicit through his utterances. He did not point out that the solutions were the same, but they were left on the chalkboard for learners to take note of. He did, however, mention that the solutions to the equation in the second strategy were either 5 or negative 5 while writing ± 5 on the chalkboard. It was left to learners to note that the solutions were the same in both strategies. Ulwazi did, however, through his utterances, draw attention to a common error made by learners when using the second strategy. The alternate strategy consisted of finding the square root on both sides of the equation when solving. Table 7.6 indicates his utterances while providing the solution.

Line	Speaker	Utterances and actions
1	Ulwazi	x is equal to 5 or x is equal to minus 5.
2	Ulwazi	Now, one thing ... you need to be very careful if you are, if you ... [inaudible]
3		so your x is equal to plus or minus 5, which means x is equal to five or x is
4		equal to negative 5 ... [inaudible] so you need to have plus or minus (<i>while</i>
5		<i>writing the solution on the chalkboard</i>).
6		Because if you do not put that plus or minus you have only one solution and
7	that solution will be positive. Are we all together?	
8	Ulwazi	So when you use another method to remove this one (<i>pointing to the square of</i>
9		x), that means we need to find the square root on both sides yes? So we find
10		the square root... [inaudible]...

Table 7.6: Excerpt indicating Ulwazi's utterances

Ulwazi wrote the alternate solution on the chalkboard while pointing out in lines 2-5 that the square root of a number can be either positive or negative. His utterances highlighted the common error made by learners when using the alternate strategy to solve the equation, i.e. omitting that the square root of a number can be either positive or negative. In lines 9-12, Ulwazi tried to draw attention to the common pitfall of learners using this method to solve: that of finding only one value for x . However, he did not draw attention to how this would have contradicted the answer of having two values for x when using the zero product law. The differences in solving using both strategies are therefore implicit in his utterances.

7.4.2. Inscriptions on the chalkboard in the IE

This IE highlights how the inscriptions made on the chalkboard were used to highlight the different solution strategies in solving an equation. Juxtaposing the different solution strategies on the chalkboard made it possible for learners to focus on both solutions together. This may not have been made possible if the two solutions were placed far apart from each other. The juxtaposition of the two solutions draws attention to one of the *core mathematical ideas*, i.e. that the strategies may be different but they result in the same answer. This serves to validate the answer found by using two different strategies to solve an equation.

While placing the solutions next to each other may have drawn the attention of some learners to the fact that using different strategies to find the solution of a question produces the same results, this may not be obvious to other learners. This needs to be made explicit through the teacher's talk.

7.4.3. Teacher talk in the IE

Ulwazi's utterances suggest that the example chosen not only addressed the feature of an IE, i.e. different solution strategies to a single question, but also revealed another feature of an IE, i.e. attending to learner common error. Ulwazi's utterances drew attention to the different solution strategies that can be used. He mentioned: "***When you use another method, which means to remove the square, you need to find the square root on both sides.***" His words "***when you use another method***" drew attention to another strategy to solve the same equation. The words signaled that what was to follow was another way to solve the equation.

While Ulwazi's utterances made explicit mention of another strategy to solve the same equation, the same cannot be said in drawing attention to the alternate strategy resulting in the same answer. While placing the solutions next to each other may have drawn attention to both having the same answer, this may not have been obvious to all learners. It required the teacher to be explicit about this with his utterances which he did not do.

Ulwazi’s talk did, however, make explicit an error learners made when solving the equation using the second strategy. His utterances in lines 2-5 (Table 7.5.) made explicit that learners should be careful when using this method to include a positive or negative sign when finding the square root, otherwise they will only arrive at a single solution to the equation, i.e. 5. His words “*should be very careful*” alluded to learners omitting the positive and negative signs when finding the square root of 25 as being a common error.

Making inscriptions and teacher talk in an IE are not new. These teacher actions have been written about extensively. However, examining IEs through this lens helps to see how these two interact with each other in an IE. Viewing an IE through the lens of inscriptions and talk highlight the ways in which having mathematical content laid out in a specific way, supported by a teacher’s talk, helps in highlighting features of an IE.

7.5. Conclusion and discussion

Viewing IEs through the lens of inscriptions and talk illustrates how the verbal and written components of an IE are used in combination to foreground the enactment of five features of IEs, as indicated in Table 7.7.

Inscriptions and talk	Leinhardt’s features in focus	Other features in focus
Juxtaposition of contrasting examples and talk	Identify the conditions of use of a mathematical idea	
Talk emphasising common error	Attend to learner errors	
Four questions arranged close to each other and talk making explicit the relationships	Highlighting key ideas of a topic	Connections between closely related mathematical ideas in a topic
Juxtaposition of alternate strategies to a question	Highlighting key mathematical ideas	

Table 7.7: Features of IEs foregrounded through the lens of inscriptions and talk

While the four features of IEs as outlined by Leinhardt were observable, an additional feature was also brought into focus through the lens of inscriptions and talk, i.e. connections between closely related mathematical ideas. The observation of other features from those highlighted by Leinhardt (2001) underscores the power of viewing IEs through this lens.

All of the IEs discussed in this chapter highlight features that can be brought into focus through the lens of inscriptions and teacher talk but are not by any means model IEs. There are

limitations to these IEs. For instance, placing different solution strategies next to each other can make it possible for learners to note that the solution strategies do not change the answer to the question. The strategies may be different but they result in the same answer drawing attention to how answers can be validated. However, the differences in the solution strategies and the fact that they result in the same answer may not be made explicit by a teacher's talk. If they are not made explicit, then some learners may take note of the different strategies, resulting in the same answer but other learners may not.

These IEs are merely examples to illustrate how looking through a particular lens provides the means through which some of the features are observable. I do not make any evaluations on the coherence of the IEs. Looking through this and other lenses, however, helps in illuminating features of IEs that help to develop a framework on the coherence of IEs, which is in focus in part three of this study.

While the use of examples and the role of inscriptions (useful set of representations) are identified by Leinhardt, the other lens that I use to examine IEs is *summoning-back* IEs or parts of IEs. This lens was identified through an analysis of my data and illustrates that IEs do not occur at specific moments within a lesson but can extend through an entire lesson. In the next chapter, I illustrate how viewing IEs through the lens of *summoning-back* also makes it possible to foreground the features of IEs.

Chapter 8 : IEs through the lens of summoning- back

8.1. Introduction

In this chapter, I illustrate how IEs can sometimes extend over time by summoning-back IEs or parts of IEs. I use the term summoning-back as mentioned previously to show how some part/s of an IE may be preserved for a period of time so that they can be used again in another IE to highlight a different ooL. Viewing an IE through the lens of summoning-back reveals how using examples, what is transcribed and left on the chalkboard, and how the teacher interacts with these through his talk, extend over time during the course of a lesson. In this sense I consider the lens of summoning-back to show how the previous two lenses are brought together to form a composite IE. This lens therefore incorporates both of the previous two lenses by adding another layer i.e. how these two contribute to a larger IE that extends over a period of time throughout a lesson. I use the features of IEs revealed through the lens of summoning-back to examine how they can be used to develop a framework of coherence of IEs.

Leinhardt (2001) mentions that IEs occur at specific moments during the course of a lesson. I propose that an IE may sometimes encompass an entire lesson. The OoL of a lesson may sometimes be so large that it is helpful to separate them into smaller IEs with different ooLs. These IEs, however, do not exist in isolation of each other but can be summoned back to be used in conjunction with each other to form a larger IE that contributes to making it possible to bring into focus the OoL of a lesson.

The lens of summoning back IEs sheds further light on the role of examples, inscriptions and talk in connecting smaller IEs together, which makes it possible to see how different mathematical ideas are connected. I illustrate by using IEs from lessons, how looking through the lens of summoning back parts of an IE not only reveals the connection between mathematical ideas but also a feature of IEs, i.e. *attaching learners' prior knowledge to new knowledge*, as mentioned by Leinhardt (2001).

8.2. IE1 Tsebo's IE through the lens of summoning-back other IEs

Examining IEs through the lens of summoning-back reveals that they can sometimes be organised in such way that they accumulate over time to form a composite IE. How a teacher organises the IEs offered within a lesson can make it possible to bring into focus the OoL of a lesson. While each IE focuses on a specific but different ooL, these IEs can be organised in a way that makes it possible to foreground a lesson's OoL. I illustrate the enactment of such an IE to shed light on how examples, inscriptions and talk are used over time and the features of IEs that they reveal.

8.2.1. Description of the IE

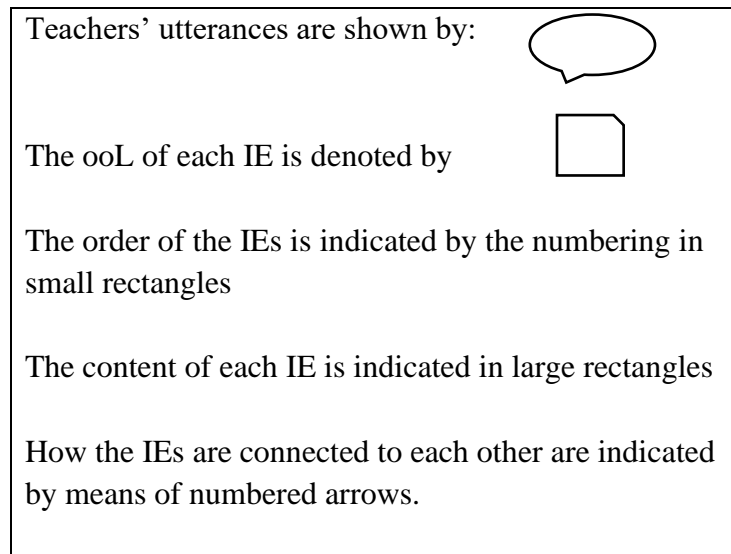
Tsebo's introductory lesson on quadratic equations was divided into IEs based on the different ool's inferred by what was done in each IE. An overview of the lesson, the examples used and a brief description of each IE, is provided in Table 8.1:

IE and ool	Examples	Description
1: Factorisation by removal of a common factor and trinomials	<ol style="list-style-type: none"> 1. $x^2 + 7x + 12$ 2. $2x^2 + x$ 3. $x^2 + 9x + 20$ 4. $3x^2 - 5x$ 	Tsebo discussed and factorised each expression, asking learners to remember how to factorise from previous lessons.
2: A focus on the development of the zero product law	<ol style="list-style-type: none"> 1. $0 \times 0 =$ 2. $1 \times 0 =$ 3. $b \times 0 =$ 4. $0 \times b =$ 5. $a \cdot b = 0$ $a = 0 \text{ or } b = 0$ 6. $(x + 1) \cdot b = 0$ 7. $(x + 1)(b - 2) = 0$ 	<p>Learners contributed in answering the first four questions, i.e. providing the answer of 0 for each.</p> <p>Learners stated that one of the numbers was 0 in $a \cdot b = 0$.</p> <p>Tsebo mentioned that $x + 1 = 0$ or $b = 0$ and $x + 1 = 0$ or $b - 2 = 0$ respectively in the last two examples.</p>
3: How to use the zero product law to solve quadratic equations	<ol style="list-style-type: none"> 1. $(x + 1)(x - 2) = 0$ 2. $(x + 2)(x - 5) = 0$ 3. $x^2 + 7x + 12 = 0$ 4. $x^2 + 9x + 20 = 0$ 5. $2x^2 - x = 0$ 	Tsebo solved the first two examples using the conclusion from IE2. He factorised the next three examples and again used the conclusion to solve each, with learners contributing in chorus.

Table 8.1: Overview of Tsebo's lesson

N.B. Each of the three IEs contribute to a composite IE that extends over an entire lesson. The ool may indicate the focus of a specific IE but may not necessarily connect to the OoL of a composite IE. For instance, the first IE in this example subscribes to Leinhardt (2001) definition of an IE as consisting of an instance of something (factorisation of expressions) together with examples of this instance (expressions with trinomials and those with common factors). The second IE also adheres to the definition. It is an IE of the zero product law, which is the mathematical principle of either factor being 0 in the product of two numbers equating to zero. Viewed separately, these IEs do not directly show any connection to the solving of quadratic equations. However, they were organised such that the first two IEs are summoned back in the third sub-explanation, both of which together contributed to highlight the OoL of the entire lesson, which I regard as a composite IE. I show how Tsebo organised these IEs (in episodes 1 and 2) to connect to the IE in episode 3 in highlighting the OoL of the lesson, i.e. the solving of quadratic equations. I first illustrate this diagrammatically in Figure 8.1.

N.B. In Figure 8.1:



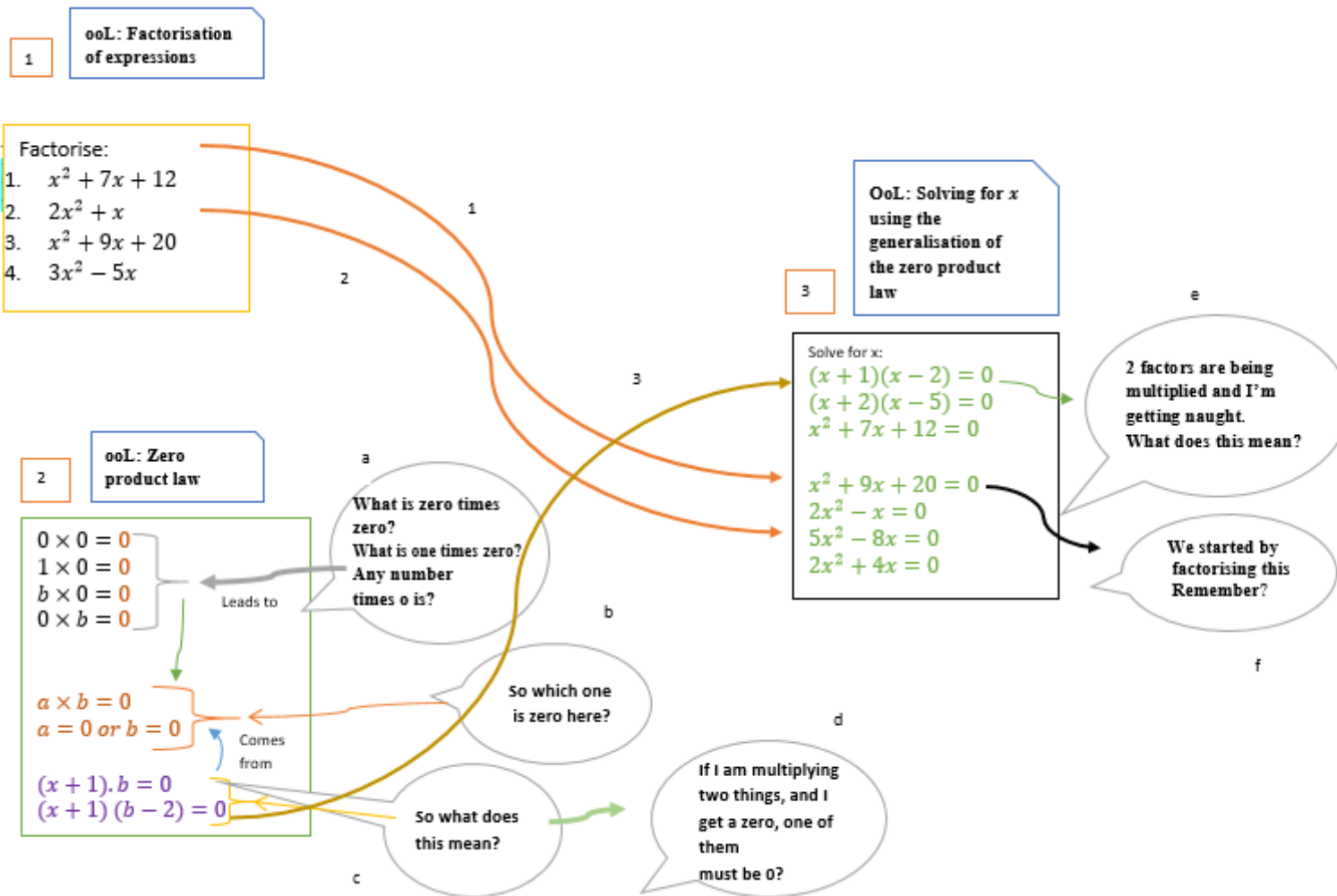


Figure 8.1: Connections between sub-explanations

8.2.2. Summoning-back previous IEs

IEs within a lesson may seem isolated from each other but can be arranged so that their content can be used again in another IE. In this instance, the first IE focused on the factorisation of expressions while the second focused on the zero product law. In the third IE, these two foci were brought together to illustrate both their uses in the solving of quadratic equations.

Tsebo revised all the necessary prerequisites for solving quadratic equations before its introduction. The first two IEs had different foci, i.e. factorising expressions and the zero product law respectively. However, they were organised in a way that connected to IE3. The factorising of expressions in IE1 provided the basis for the use of the zero product law in solving quadratic equations. The illustration of the zero product law in IE2 provided the mathematical idea underpinning the solving of quadratic equations. Arranging the content in this way foregrounds the solution strategy in solving quadratic equations in IE3 because the two mathematical ideas used in the solution strategy were already explained in previous IEs. This allowed for a focus on the strategy without the distractions that may be caused by explaining factorisation of expressions and the zero product law during the solving of equations. This organisation of smaller sub-explanations of factorisation and the zero product law made it possible to connect to the larger explanation of their uses in solving quadratic equations (the OoL).

Viewing an IE through the lens of summoning-back indicates the key role of example use, inscriptions and talk in IEs that contribute to a composite IE.

The summoning-back lens reveals how IEs can be arranged so that learners can attend to one mathematical idea at a time. Arranging the two IEs on revising factorisation methods and the zero product law before both their uses in the solving of quadratic equations makes it possible for learners to attend to one mathematical idea at a time before they are used together to highlight their use in another mathematical idea.

Examining IEs through the lens of summoning-back also reveals how examples are *summoned-back* to highlight the OoL of a composite IE. Inscriptions previously made on the chalkboard are also referred back to while the teacher's talk makes explicit references to the examples and inscriptions that are used again.

8.2.3. Using the same examples

Summoning-back examples from a previous IE to be used again in another IE are key elements in contributing to the composite IE. Using the same examples makes it possible to connect mathematical ideas and to maintain a focus on the particular ool of an IE. The use of the same examples, albeit in different forms, makes it possible for connections to be made between topics. In this case, the factorisation of expressions is connected to its use in solving quadratic equations. In IE3, Tsebo used the same trinomials when solving quadratic equations as used in

the factorisation of expressions (column 2 in table 8.1), i.e. $x^2 + 7x + 12$ and $x^2 + 9x + 20$ are used as indicated by arrows (1 and 2 in fig. 8.1) in IEs 1 and 3.

The use of the same examples in different IEs also helps to focus on the ooL of an IE. IE1 focused on the ooL of the factorisation of expressions. The use of the same expressions in IE3 when solving quadratic equations helped in foregrounding the strategy used to solve quadratic equations while the different methods of factorisation were backgrounded because the factors were previously determined. When factorising the trinomial of the equation $x^2 + 7x + 12 = 0$ in IE3, learners were thus able to provide the factors, with no further discussion required on how to obtain the signs within the brackets.

While in some cases the same example is summoned-back, in other cases it may be parts of examples that are used again. Parts of examples may be used again in another IE so as to form a bridge between the relevant IEs. For instance, at the end of IE2, the last two examples were not only used to indicate the use of the zero product law in equating factors to zero but were also used to connect to IE3. In IE2, Tsebo demonstrated the use of the zero product law in equating either factor to zero without solving the equations. In both the examples $(x + 1) \cdot b = 0$ and $(x + 1)(b - 2) = 0$, he stopped at equating each factor to zero, i.e. $x + 1 = 0$ or $b = 0$ and $x + 1 = 0$ or $b - 2 = 0$. This provided the basis for using the zero product law in solving quadratic equations in episode 3. In IE3 he used the same factor as used in these examples, i.e. $x + 1$. His first example in IE3 (indicated by arrow 3 in fig. 8.1) was $(x + 1)(x - 2) = 0$. Using the same factors in IE2 and IE3 while only replacing the letter “ b ” with an “ x ” makes it possible for learners to note the connections between the zero product law and its use in solving quadratic equations. *Summoning-back* parts of the same examples make it possible to connect IEs with different ooLs and in this way to *connect between different mathematical ideas*.

8.2.4. The role of inscriptions and talk through the lens of summoning-back

While the content within IE’s was arranged to promote connections between different mathematical ideas, this on its own may not help learners to see the connections between these ideas. Tsebo’s talk helped make these connections explicit. For instance, when solving the equation $x^2 + 7x + 12 = 0$, Tsebo reminded learners that they had factorised the expression $x^2 + 7x + 12$ earlier. He said “**We started by factorising this, remember**” (indicated by letter a in Figure 8.1.). His words were a reminder to learners that they had factorised the trinomial previously and so the factors were already known. These factors were then used to solve the quadratic equation $x^2 + 7x + 12 = 0$, thus making it possible for learners to perceive the connection between factorisation and solving quadratic equations.

Tsebo constantly reminded learners of what they had done earlier in the lesson, and this was summoned-back in another IE. Another example is when he explained the zero product law in IE2, and said: **“Okay, this means people, so if I am multiplying two things, and I get a zero, one of them must be (zero)?”** (indicated by the letter d in Figure 8.1). He reminded learners of this again in IE3 when solving the equation $(x + 1)(x - 2) = 0$. He said: **“Two factors are being multiplied and I’m getting naught. What does this mean?”** (indicated by the letter ‘e’ in Figure 8.1). He used the same or similar words in various IEs to remind learners of what had already been done in a previous IE.

It was not only his utterances but also his inscriptions on the chalkboard that drew attention to something that had previously been done. For instance, his inscriptions when explaining the zero product law in IE2 made it possible for learners to be reminded of the law in the next IE. While asking the question: **“Two factors are being multiplied and I’m getting naught. What does this mean?”** Tsebo drew a rectangle around the zero product law as depicted in Figure 8.2.

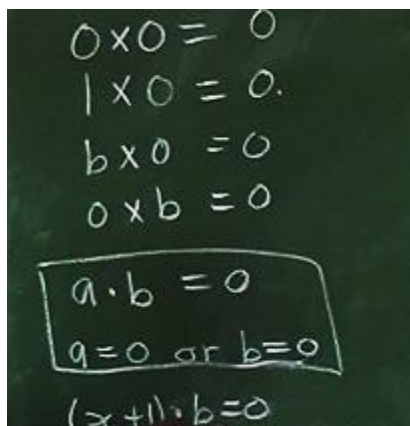


Figure 8.2: Picture of Tsebo’s chalkboard, highlighting the zero product law

Drawing a rectangle around the zero product law on the chalkboard makes it possible to draw the attention of learners to the key mathematical idea of the zero product law from the previous IE.

8.2.5. Features of IEs in focus through the summoning-back lens

The *summoning-back* lens illustrates the power of using different lenses in viewing IEs to observe not only their features but also a part of the definition of IEs provided by Leinhardt (2001). Viewing IEs through the lens of *summoning-back* reveals a part of the definition of IEs i.e. a *set of discussions that connect what is being explained to particular rules or principles* (Leinhardt, 2001, p. 341). Viewing an IE through the lens of summoning back reveals how IEs can make it possible for learners to connect between mathematical ideas. The arrangement of the IEs makes it possible for the mathematical idea of factorisation of expressions to be connected to

its use in solving quadratic equations. The use of the same example in different IEs is central to how different mathematical ideas are connected to each other while teacher talk makes this explicit. Making and keeping inscriptions on the chalkboard are also important in that they can be referred to in subsequent IEs.

Not only does this lens allow us to see how mathematical ideas are connected, but also how the arrangement of IEs with different content can make it possible to maintain a focus on the OoL of a lesson. Arranging IEs so that the prerequisites needed for a particular topic are all in focus in separate IEs before their use in the topic makes it possible to maintain a focus on the OoL of a lesson. The prerequisites are explained in previous IEs before they are brought together in a subsequent IE, so they do not detract from the focus of the lesson, i.e. the OoL. The use of the same examples, even if they are in different forms, is central to *maintaining a focus on the OoL*. The use of inscriptions is also an important consideration. Keeping examples as well as the central mathematical idea written on the chalkboard is important for referring back to them in a subsequent IE to maintain a focus on the OoL.

Viewing IEs through the lens of summoning back helps to shed light on how the content within IEs can be arranged to develop over time. Both the organisation of IEs as well as the content within the IEs can be arranged to illustrate their connections to each other.

I illustrate by means of another IE how looking through the lens of *summoning-back* reveals how a mathematical procedure is connected to its application.

8.3. IE through the lens of summoning-back a procedure

The steps of a procedure, explained and inscribed on the chalkboard in an IE, can be left on the chalkboard to be used again in another IE. Having the steps of a procedure left on the chalkboard makes it possible for learners to connect these steps to how it could be applied to solving a range of quadratic equations. I illustrate how this was enacted by a teacher, Tsebo, through his use of examples, inscriptions made on the chalkboard, and how his talk explicitly connected the IEs.

8.3.1. Description of the IE

Tsebo had introduced and used the zero product law in the first lesson on solving quadratic equations by factorisation. In his second lesson on the same topic, he focused on the procedure to be used when solving quadratic equations. The ool of the IE, which was not articulated by Tsebo but could be inferred by what was done in this part of the lesson, was a *description of the steps of the procedure used in solving a quadratic equation*. He first explained what a quadratic equation was and how it could be identified, i.e. it was an equation of the second degree with an elaboration that second degree meant that the highest power of x was 2. He wrote these on the

top left hand side of the chalkboard. He left the space, to the right of this, vacant and wrote the steps of the procedure to solve a quadratic equation on the far right hand side of the board, as depicted in Figure 8.3.



Figure 8.3: Picture of Tsebo's layout of work on the chalkboard

The space left on the chalkboard between the brief notes on how to identify a quadratic equation and the steps of the procedure was later used to demonstrate the application of the steps of the procedure to solve a range of examples on quadratic equations.

Tsebo wrote the steps of the procedure on the chalkboard, as illustrated on the left hand side of Figure 8.4. For the sake of legibility, I have reproduced what he wrote on the chalkboard on the right hand side in figure 8.4.

	<p>To solve</p> <ol style="list-style-type: none"> 1. Get the equation into standard form *descending powers of x eg. $x^2 + 7x + 12 = 0$ 2. factorise LHS $(x + 4)(x + 3)$ 3. equate the factors to zero $x + 4 = 0$ or $(x + 3) = 0$ 4. Solve the linear equations for x $x = -4$ or $x = -3$
<p>Procedure to solve quadratic equation written on chalkboard</p>	<p>A copy of what was written on the chalkboard</p>

Figure 8.4: Steps of the procedure to solve a quadratic equation

Tsebo included an example of a quadratic equation when the steps of the procedure were explained. He used the equation $x^2 + 7x + 12 = 0$ and illustrated the different steps in the solving of this equation below the written description of each step. For instance, step three was written as “equate each factor to zero”, below which he wrote: $(x + 4) = 0$ or $(x + 3) = 0$.

The written steps of the procedure were left on the chalkboard during the course of his working through the range of examples as well as during the classwork exercise, which learners worked with on their own. In addition, the examples discussed by Tsebo were all inscribed on the chalkboard in the space that was left. When he ran out of chalkboard space to solve further equations, he erased some of the first examples that had been done rather than the steps of the procedure written on the board. Writing on the far right of the board with space in between and erasing the content done in this space when he ran out of chalkboard space, suggests that he deliberately set out to leave the procedure on the chalkboard during the course of the lesson. His utterances made reference to the written procedure left on the board. This IE illustrates the role of examples, inscriptions and talk used altogether in summoning-back parts of a previous IE in a subsequent IE.

8.3.2. Using examples in the IE

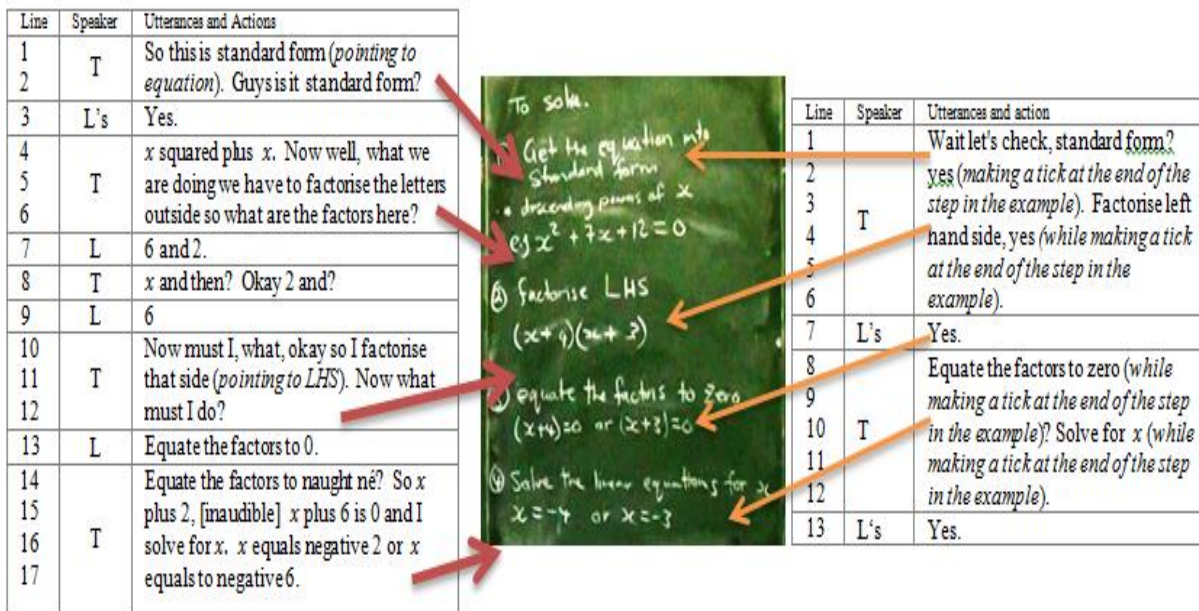
The example used by Tsebo i.e. $x^2 + 7x + 12 = 0$ can be regarded as a *reference example* that was used to connect the factorisation of trinomials to its use in solving a quadratic equation. Tsebo had initially (in the first lesson) provided the expression $x^2 + 7x + 12$ to be factorised. The same expression was now provided in the form of an equation. The use of the same expression made it possible for learners to connect the factorisation of expressions to its use in the solving of quadratic equations. The use of the same expression when factorising expressions, as well as in the solving of a quadratic equation, served to maintain a focus on the method used to solve a quadratic equation. If a different expression was used, learners’ focus may have been diverted in trying to factorise the expression rather than focusing on the method used to solve the equation. In addition, providing the equation in the general form, i.e. $ax^2 + bx + c = 0$, made it possible for learners to refer to the same procedure in solving the other equations that followed.

8.3.3. Using Inscriptions and talk in the IE

Leaving the steps of the procedure on the chalkboard from a previous IE makes it possible for learners to observe the written steps of the procedure, as Tsebo solved a range of equations. Learners could therefore follow the steps being executed in the solution of various equations through what was inscribed on the chalkboard. The written procedure was explicitly referred to by Tsebo’s talk as he explained how to complete each of a range of examples. He also referred to the procedure after each solution was completed by checking that each step followed the steps of

the procedure and in some cases ticking off the steps of the procedure after he had completed the solution to an equation.

Tsebo's questions, when solving each of the quadratic equations, were aligned to what he had written in his notes. Figure 8.5 is a diagrammatic version of how his talk connected between the examples and the steps of the procedure.



Excerpt 1: Connecting steps to procedure when completing examples

Steps of procedure written on the chalkboard

Excerpt 2: Connecting steps to procedure after completing an example

Figure 8.5: Tsebo's talk in connecting the examples and the steps of procedure

For instance, he first asked learners whether the given equation, i.e. $x^2 + 8x + 12 = 0$, was in standard form (excerpt 1, line 1). This matched step 1 of his description on the board, which was written as: "Get the equation into standard form". Below he had written (after an asterisk) "Descending powers of x ". His next comment was that the left hand side must be factorised (excerpt 1, lines 10-12,) in reference to step 2, which he had written as "factorise LHS". In addition to his words matching what had been written, he also pointed to the right hand side of the board as he spoke.

His actions and utterances indicate that the steps of the procedure written on the chalkboard were not just used for learners to write into their books. By leaving the written procedure on the chalkboard, Tsebo provided learners with a visual presentation of the procedure to be used when faced with solving a quadratic equation. He made specific mention of these notes and referred to the different steps when solving a range of different examples on quadratic equations. He not only referred to the steps written while solving different equations but also referred to them after solving. This is evidenced in Figure 8.6, which includes the steps of the procedure, the solution to the question $4x^2 = 9$, and his words while checking the solution.

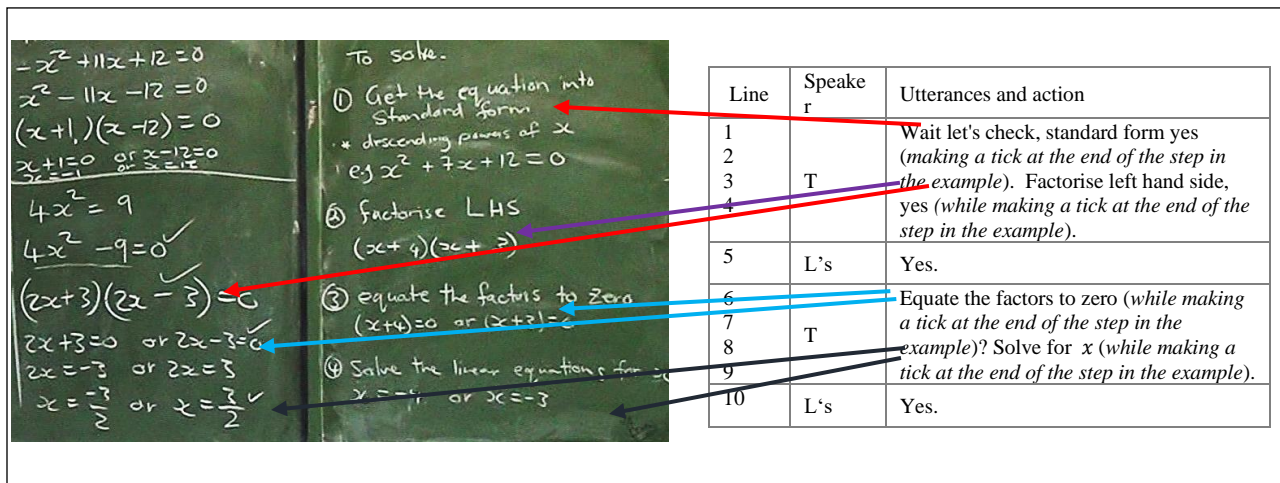


Figure 8.6: Summoning- back steps of procedure inscribed on chalkboard

After solving the equation $4x^2 = 9$, Tsebo went through each step in his solution and checked this against the steps he had written on the board. His words “standard form Yes” (excerpt 2, line 1) corresponded to step 1 of his written procedure; “factorise left hand side” (excerpt 2, line 3) corresponded to step 2; equate the factors to zero” (excerpt 2, line 6) corresponded to step 3; and “solve for x” (excerpt 2, line 8) to step 4. His words, together with ticking to indicate he had addressed each of the steps in the procedure, illustrated his use of the written procedure when solving the equations that he had provided. He used the steps of the written procedure to make explicit how they could be used when solving quadratic equations.

8.3.4. Features of IEs in focus through summoning-back in this IE

What is revealed in the IE through the lens of *summoning-back* is the connections between different topics. The topic of factorisation of expressions was connected to its use in solving quadratic equations, thus illustrating the interconnectedness of topics. This was made possible through the use of the same example used in an IE in lesson 1, i.e. the factorisation of expressions. Factorisation of expressions can therefore be seen as a topic that is not an end in

itself but rather a skill that is required to perform not only mathematical operations in solving quadratic equations (in this IE), but also to finding the x-intercepts of a function, among others.

8.4. Discussion and Conclusion

Looking through the lens of *summoning-back* parts of or entire IEs from different points in a lesson, reveals how the other two lenses of IEs interact with each other to bring into focus their features. How examples are chosen and organised, how the inscriptions and talk are used to connect IEs together reveals that IEs may not always occur at a specific moment in time but can extend over time to form a composite IE that comprises an entire lesson.

The choice of examples is central to *summoning-back* parts of previous IEs. The same example used in various forms in different IEs may be used for different purposes. They can be used to connect a procedure explained in an IE to the application of the procedure in another IE. They can also be used to connect together different mathematical ideas, for instance factorisation and its use in solving quadratic equations. These are just two ways in which the same examples can be used to connect different IEs. It is not only the same example but it may be that parts of an example are used again as a bridge connecting IEs to each other. In any case, no matter how they are used, the point is that looking at IEs through the lens of *summoning-back* reveals how examples are chosen and used in IEs so that they accumulate to form a single, composite IE that extends through an entire lesson.

Together with the choice, use and organisation of examples, the *summoning-back* lens also reveals how inscriptions can be used to form IEs that accumulate over time. Inscriptions made on the chalkboard may be deliberately left there to be used again in another IE. In addition, there may be some parts of the content written on the chalkboard that can be highlighted in specific ways (e.g. drawing a circle around an important mathematical idea) to emphasise a key mathematical idea required in a topic. The key idea is central to a topic and so used in more than one IE.

Both examples and inscriptions made on the chalkboard do not speak for themselves. It is a teacher's talk that makes explicit what the examples and inscriptions are meant to highlight. Repeating words that were used in a prior IE help to draw the attention of the learners to something in particular. For instance, the words "I am multiplying two factors to get zero; what does this mean?" are used in an IE explaining the zero product law. The same words are used again in another IE when explaining how to solve quadratic equations. The use of the same words helps to draw the attention of learners to the zero product law explained in a previous IE to another IE.

The *summoning-back* lens illustrates the interaction between the choice, use and organisation of examples as well as the inscriptions made by the teacher on the chalkboard and teacher talk in an IE. These, put together, illustrate how IEs extend over time and so are connected to each other to form a composite IE.

The three lenses through which IEs were examined provide the means through which the features of IEs can be observed. The four features of IEs as outlined by Leinhardt are that they should:

- a) Resolve the nature of errors
- b) Identify conditions of use, and
- c) Attach learners' prior knowledge to new knowledge.

Viewing IEs through the lens of the choice, use and organisation of examples, reveals how a teacher's use of examples makes it possible to *address learner errors*, to *identify the conditions of use* of a particular mathematical idea, and to *attach learners' prior knowledge to new knowledge*. The lens of inscriptions and talk, which is the interaction between what is written and the verbal, reveals how inscriptions can be made on the chalkboard to *connect between closely related ideas*, *highlight key mathematical ideas* in a topic, and *identify the conditions of use* of a mathematical idea. Finally, examining IEs through the lens of summoning-back shows how IEs extend over time to reveal features such as *connections between mathematical ideas* and *maintaining a focus on the OoL* of a lesson. A summary of the features foregrounded through each of the lenses is provided in table 8.2.

Lenses	Features of IEs in focus through lenses
Using example	Attaching prior knowledge to new knowledge Identifying conditions of use of the procedure Addressing learner common error
Using inscriptions and talk (interaction between written and verbal)	Connecting between closely related ideas Highlighting key mathematical ideas Identifying conditions of use
Summoning-back (interaction between the other two lenses)	Connecting between mathematical ideas Maintaining a focus on the OoL of a lesson

Table 8.2: Summary of features identified through the three lenses

My analysis of IEs through the three lenses enabled me to observe the features of IEs as indicated in the literature on IEs. There were some features though that are not present in the literature, like *connections between mathematical ideas*, *highlighting the key mathematical ideas* in a topic and *maintaining a focus on the OoL* of a lesson.

What is not clear from the analysis of IEs through the different lenses is the quality of the IEs presented. All of the examples of IEs used illustrate their power in illuminating the features of IEs. They were by no means model IEs. There were many limitations to the examples of the IEs used. Some of them were highlighted. For instance, one of the limitations was that some of the mathematical ideas were left implicit by a lack of teacher's talk in making them explicit despite the use of inscriptions and examples in highlighting them. The point is that the quality of the IEs mentioned have been absent from the analysis thus far. The analysis however revealed features of IEs that can be considered in gauging their quality. This is the subject of part 2 of this thesis. The next chapter is a focus on a theoretical framing of the coherence of IEs, while I present one possible way of determining the coherence of IEs in the chapter after that.

In the next two chapters, I illustrate how I have used the features of IEs revealed by looking at them through the different lenses as well as the literature of coherence in lessons to develop a framework that will help assess the quality of IEs.

In the next chapter, I illustrate how viewing IEs through the three lenses enabled me to develop a framework to examine the quality of IEs in terms of their coherence to the oOL of an IE.

Part 2

In this second part of the thesis, I focus on the coherence of instructional explanations (IEs) in order to answer the second research question:

What makes a mathematical instructional explanation coherent?

Interlude: Introduction to part two

I examine the coherence of IEs as a measure of the quality of IEs. A study of the coherence of IEs will not only benefit teachers in South Africa, but also globally. Studies (e.g. Venkat & Adler, 2012) indicate that the IEs offered by teachers in South African mathematics classrooms are not coherent. This problem is, however, not unique to South African classrooms. For instance, Stigler and Perry (1988) note the lack of coherence in mathematics lessons in many countries. Studies, however, focus on the coherence of lessons and not on IEs. There is therefore a lack of a framework that can be used to analyse the coherence of teachers' IEs. My aim is therefore to find a way to analyse the coherence of IEs by developing a framework that will be suitable for analysing the coherence of IEs. Since this framework is developed within the context of a teacher-centred pedagogy, it will be especially relevant to other, similar contexts.

In order to develop a framework, I first needed to identify features of coherent IEs and then to explore how these features contribute to their coherence. In the first part of this study I highlighted two lenses, i.e. teachers' use of examples and teacher's use of inscriptions and talk. Through these lenses I was able to look at how Leinhardt's (2001) features of IEs were implemented.

What has not been highlighted thus far is whether and how these features relate to the coherence of IEs. There may also be other features specifically relating to the coherence of IEs that may not have been highlighted by Leinhardt (2001). Once additional features that contribute to IEs coherence have been highlighted, these can then be used to develop a framework to examine the coherence of IEs.

In chapter 9 I review literature on coherence of lessons and IEs and I will show that there are features of coherence in lessons that can be adapted to coherence of IEs. Furthermore, these features can be categorised into a focus on mathematical connections and mathematical content, and a focus on learners. I highlight the different features of coherence within these categories and conclude the chapter with a definition of coherence of IEs which I use in this study.

In chapter 10, I illustrate how I use the features of coherence identified from the literature to develop a framework that can be used to analyse the coherence of IEs. I also show how I use the framework in analysing the coherence of an IE from my data.

Chapter 9 : Towards a definition of coherence in relation to instructional explanations (IEs)

9.1. Introduction

In trying to gain more insight into the nature of instructional explanations (IEs), I became aware of Leinhardt's (2001) conflation between their nature and quality. For instance, she mentions “good” IEs rather than IEs more generally. I fell into the same trap when initially analysing my data. There is thus a need to disaggregate the features that make up an IE from its quality. In this study I consider the quality of an IE through its coherence. In part 1 of this study I rearranged the list of the features of IEs provided by Leinhardt (2001) to indicate their relationships. Figure 9.1 illustrates my arrangement of the features of IEs used in this study.

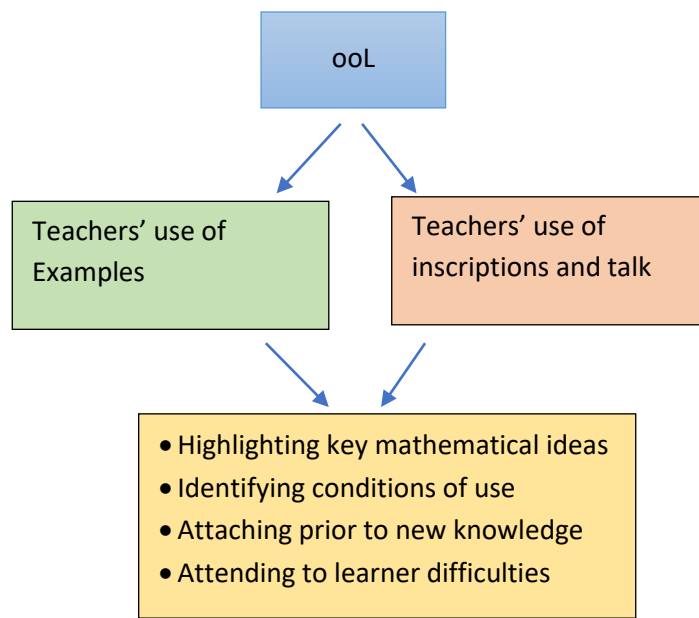


Figure 9.1: Features of IEs used in this study

My rearrangement of features of IEs allowed me to identify two lenses of IEs, i.e. *teachers' use of examples* and *teachers' use of inscriptions and talk*. Zooming in to examine IEs through these two lenses shed light on how the features identified by Leinhardt (2001) are enacted. How examples are varied, the arrangement of these and other content on the chalkboard, together with teacher talk determine what features are revealed in an IE. How these features contribute to the coherence of IEs has not yet been in focus.

In the past two decades, studies have highlighted many features of coherence but have not produced any consistent definition of ‘coherence’ in mathematics instruction (Shimizu, 2007). For instance coherence is seen as relating to constraint satisfaction (Thagard, 1989), following a consistent approach in all lessons (Shimizu, 2007), connectedness between mathematical topics

(Wang & Murphy, 2004) and making explicit clear lesson objectives (Chen & Li, 2010). A definition of coherence within mathematical lessons is therefore one of the aims of this study.

While a definition of coherence may be sufficient to guide teachers in providing coherent IEs in their daily practice, it is inadequate for analytical purposes and therefore insufficient for the mathematics education research community. Further to a definition of coherence, a framework will be helpful to analyse whether or not a teacher's IE is coherent. Pinning down a meaning for coherence within instruction is a difficult endeavour but finding a framework to analyse IEs is even more difficult because the literature focuses on coherence of lessons (e.g. Chen & Li, 2010; Shimizu, 2007; Wang & Murphy, 2004) and not on IEs. While these studies provide some features of coherence these relate to those that promote the coherence between lessons and thus cannot be used to develop a framework of the coherence of an IE. For instance, Shimizu (2007) considers it necessary for all lessons to have a beginning, middle and end i.e. introduction, the main content for the topic and a conclusion. This will help in establishing a coherent structure. This however does not necessarily relate to an IE that may consist of a part of a lesson. Therefore following a consistent lesson structure to promote coherence is not a feature that I consider in this study.

My aim in this study is to offer a definition of coherence of an IE which will help in developing a framework to analyse the coherence of IEs. In developing a framework to examine the coherence of IEs, I must emphasise that my focus is not on examining whether or not an IE is coherent *to learners*. My attention is rather on whether a teacher provides an IE that can be considered to be coherent as informed by the literature on coherence. Teachers' IEs should first be considered to be coherent before any claim can be made about what sense they make to learners. There is, however, no clear definition of coherence of IEs that can be used as a guide by teachers in providing coherent IEs to learners.

In this chapter, I examine literature to identify what is meant by *coherence* and specifically the coherence of IEs. Thereafter I examine the features that promote coherent IEs in order to develop a framework. I review literature on coherence because it will inform this part of the study in two ways: first, it will enable me to look at whether the features of IEs as outlined in chapter 2 contribute to the coherence of IEs and if so, how they can be used to develop a framework of coherence of IEs. Second, examining literature on coherence will allow me to add other features that are supported by my data and so contribute to the coherence of IEs.

9.2. The need to develop a framework on coherence of IEs

There are many studies focusing on coherence of lessons with different features that promote coherence. These studies, however, do not synthesise these features into a framework that can be used to analyse the coherence of IEs.

Studies focusing on the coherence of mathematics lessons gained prominence after the findings from the TIMSS 1999 video study were reported by Hiebert et al. (2003). These prompted the suggestion that mathematics instruction in classrooms in East Asian countries (Hong Kong and Japan¹⁰) were distinguished by higher coherence than those from other countries. They defined coherence as “the (implicit and explicit) interrelation of all mathematical components of the lesson” (Hiebert et al., 2003, p. 196) but did not, however, provide a distinction between implicit and explicit interrelations.

Many studies have since emanated on the coherence of lessons, particularly in Asian countries (e.g. Cai, Ding, & Wang, 2014; Sekiguchi, 2006; Shimizu, 2007). While many of these studies use the definition by Hiebert et al. (2003), studies have since evolved to include a wide range of other features to characterise coherence. These features relate to the coherence of lessons as a whole but the features have not been aggregated to develop a framework. Nevertheless, these studies provide the basis for my development of a framework on the coherence of IEs.

While there are some features relating to coherence of lessons that I consider in this study, there are others that I do not consider in developing a framework for the coherence of IEs. For instance, I do not consider the use of a consistent approach across lessons (Chen & Li, 2010) as this does not relate to IEs.

My focus in this chapter is therefore to offer a definition of coherence of IEs which I then use in developing a framework that can be used to analyse the coherence of IEs.

9.3. Classification of features of coherence identified in literature on mathematics education

My review of the literature indicates that a teacher’s use of examples (e.g. Sekiguchi, 2006) and the use of a teacher’s talk (e.g. Wang & Murphy, 2004) contribute to the coherence of lessons. The use of examples by a teacher as well as a teacher’s talk in IEs was highlighted in my analysis of IEs. In chapter 7, I highlighted the use of examples through which I examined the features of IEs. In chapter 8, I illustrated how a teacher’s talk can be used to examine IEs and their features. My data suggests that the arrangement of content on the chalkboard also contributes to an IE. Content can be arranged on the chalkboard by a teacher to highlight something which his talk makes explicit. I therefore consider a teacher’s use of examples as well as a teacher’s use of inscriptions and talk as the lenses through which I examine the features of coherence of IEs.

¹⁰ Although Japan did not participate in the 1999 TIMSS study, results from Japan’s participation in the 1995 study were used in the report.

My examination of literature on coherence of lessons suggests that the features can be classified into three foci which I call *a focus on connections*, *a focus on mathematical content* and *a focus on learners*. I provide an elaboration of the features identified in the literature within these three foci before illustrating how I have used the foci to develop a framework of the key features and their relationships.

9.3.1. Coherence as a focus on connections

Coherence of a lesson as one in which connections are made is a central idea in the literature on mathematics education. There are, however, three different views on connection-making in the literature, i.e. relationships between mathematical ideas, connections as occurring in the minds of learners, and activities that promote connection-making. I discuss each of these below.

Connections as relationships between mathematical ideas

Mathematical connections are considered to be relationships between mathematical ideas (e.g. Zazkis, 2000) considers. In this view, the emphasis is on what is gained by learners from the connections made. Ma (2010) asserts that connecting a mathematical idea to another makes it possible for a learner to increase their understanding of the original mathematical idea. A similar argument for the importance of mathematical connections is made by Bossé (2003), who asserts that most mathematics curricula address mathematical topics in a disjointed manner. This promotes the assumption by learners that previously learnt mathematical ideas have no significance to future topics. As a result, learners forget previously learnt mathematical ideas in order to accommodate new ideas. This view points to a feature of connection-making, i.e. connecting between mathematical ideas.

Connections as the activities that promote connection-making

According to (Evitts, 2004) the activities provided by a teacher will determine whether connections are made or not. For instance, Businskas (2008) describes an activity where learners worked with concrete representations to explore mathematical relationships. The aim of the activity was to describe geometric shapes using algebraic expressions, thus connecting between different representations. In this view the value is in the act of a teacher making it possible for learners to make connections (Businskas, 2008). I see this view of connections as relating to pedagogy rather than the type of connections being made, as the other two views highlight.

Connections in the minds of learners

Learners try to expand on existing mathematical ideas in the face of new mathematical ideas they are presented with (Noss, Healy, & Hoyles, 1997). Scott, Mortimer, and Ametller (2011) maintain that when learners make links between their existing knowledge and new ideas, it results in a *deep processing* of the mathematical idea as opposed to the memorisation of discrete

facts. This view highlights another feature of connection-making, i.e. learners attaching their prior knowledge to new knowledge.

While I consider all three views in this study, my position is that the activities are the means through which the other two types of connections are made possible, i.e. for learners to make connections between mathematical ideas and to attach learners' prior knowledge to new knowledge. As mentioned previously, this study is a focus on teachers' IEs and as such, the data does not allow for me to ascertain what learners have learnt from teachers IEs. The most that can be done is to offer what is made possible for learners to learn. Whatever the view of connections is, however, it is the learners in the classroom who need to make these connections. Learners may make connections spontaneously but it cannot be assumed that they will make these connections without any interventions (Weinberg, 2001). It is generally the teacher in the classroom who needs to make it possible for learners to make these connections. Teachers therefore require strategies to make these connections explicit. In this study I consider the teachers' use of examples and how what is made available for public viewing (e.g. through written inscriptions on the chalkboard) together with a teacher's talk, aid in connecting mathematical ideas.

One of the issues I encountered in my examination of literature on connections was the use of different words to refer to connections. The words "relationships", "linking" and "connections" are all used synonymously in the literature when describing coherence. For example, Cai et al. (2014) describe coherence as "connectedness" to a topic, Shimizu (2007) uses the word "linking" between different lessons as well as within the same lesson, and Hiebert et al. (2003) describe coherence as the "interrelation of all mathematical components of the lesson" (p 196). While different words are used by the different authors they are all used to denote a "joining" together or connection between parts. I therefore consider the words *relationship*, *linking* and *connections* to be used synonymously. In this study I use connections to avoid confusion between the different words.

The word "connections" is used in two different ways in literature. One relates to the organisation of a lesson and how the teacher connects different parts of a lesson, while the other relates to the connection of mathematical ideas. I focus on developing a framework for IEs that are limited in time and range. I therefore look at smaller IEs and not on those extending through an entire lesson. As a result, I do not consider connections between different parts of a lesson.

There are two features of a focus on connections identified from the literature:

- Providing justifications for a mathematical idea or a procedure is necessary for a procedure to be coherent (Chen & Li, 2010; Ma, 2010). Knowing why a procedure makes sense provides the basis for a deep understanding of a mathematical idea (Ma, 2010). One

of the ways that procedure can be justified is through the use of multiple strategies to a solution.

- Attaching prior knowledge to new knowledge: mathematical ideas must be connected to previously known ideas for them to be coherent for learners (Askew et al., 2012; Scott et al., 2011). Some of the ways that this can be done is through the use of a teacher's talk (Scott et al., 2011) and how examples can be chosen and sequenced.

The nature of the relationship between coherence and connections is unclear from the literature reviewed. While some studies suggest that making connections is a subset of coherence, others consider them to be parallel features of lessons. For Sekiguchi (2006), Shimizu (2007) and Wang and Murphy (2004), connections are a necessary part of coherence. Ma (2010), however, presents a different view.

Ma (2010) suggests that coherence and connections are separate features, both of which should be present in lessons. She presents four features that should be in focus within lessons, i.e. connectedness, multiple perspectives, basic ideas and longitudinal coherence. Illuminating underlying connections between mathematical ideas presents mathematics as a unified body of knowledge rather than a series of isolated topics. Multiple perspectives relate to the different ways of approaching a solution to a problem together with their advantages and disadvantages. Teachers should also reinforce the basic ideas of mathematical principles like the four basic operations and their relationships. Finally, longitudinal coherence relates to how topics are sequenced so that they link to each other.

The four features of mathematics lessons presented by Ma (2010) suggest that making connections is separate from and not necessarily related to coherence but that both features i.e. coherence and connections co-exist in all lessons. While presenting connections and coherence as separate to each other, a seeming contradiction is, however, visible in her work when she mentions that connections between mathematical ideas is necessary in order for these mathematical ideas to be coherent. She maintains that in order for mathematical procedures to be coherent, they must be connected to mathematical justifications. Furthermore, she mentions the linking of different topics to achieve what she terms *longitudinal coherence*. While she presents coherence and connections as two separate features of lessons, she suggests that connecting topics is necessary to achieve coherence. The contradiction may however be a result of her use of longitudinal coherence rather than coherence. The differentiation between coherence and longitudinal coherence is however not addressed in her study.

My position in this study is that connections within IEs is a necessary but insufficient component for the coherence of IEs. I consider it to be insufficient because there are other features, such as highlighting the key ideas of a topic or attending to learner difficulties which can contribute to

the coherence of IEs. Other features relate to a focus on mathematical content as well as a focus on learners.

9.3.2 Coherence as a focus on mathematical content

There are two features of coherence as relating to mathematical content the literature refers to. These are:

- Highlighting the key mathematical ideas of a topic: this feature involves teachers providing summaries (Shimizu, 2007) to highlight the main ideas, whether verbal or written, and written annotations like circling, ticking or underlining a key point. The main mathematical ideas of a topic include, for instance, the zero product law when solving quadratic equations or that a linear number pattern is characterised by a common difference between the terms in the pattern.
- Providing justifications of mathematical ideas: justification of a procedure can take the form of a formal proof, or can be addressed through the use of multiple solution strategies (Ma, 2010) or the use of different representations (Chen & Li, 2010).

The feature of *providing justifications* is categorised as a focus on content and connections. I show how I take this into consideration when developing my framework of coherence in section 10.3.

9.3.3. Coherence as a focus on learners

Coherence as paying attention to learners in the classroom is a feature of coherence that is either explicitly or implicitly addressed in the literature. I have identified two features within this category in literature. These are:

- Attaching prior knowledge to new knowledge: this relates to teachers laying a proper foundation for knowledge that is required by learners at the later stages (Ma, 2010). This can be done through a careful sequencing of activities to help learners perceive the connection between previous knowledge and new knowledge (Cai et al., 2014; Chen & Li, 2010; Leinhardt, 2001; Wang & Murphy, 2004).
- *Addressing learner difficulties*: this is a key element of coherence to be considered when planning lessons, as mentioned by teachers in a study by Cai et al. (2014), but does not address how to deal with learner difficulties. This feature includes learner *possible difficulties* that teachers may be aware of not only through their experience but also from their attendance in professional development programmes and from research. Also included in this feature are errors made by learners during a lesson (Leinhardt, 2001).

I use the categories of coherence to form a definition of coherence.

9.4. My definitions of 'coherent', 'partially coherent' and 'not coherent'

I consider IEs in this study to be either *coherent*, *partially coherent* or *not coherent*.

The Merriam Webster dictionary offers two general definitions of coherence. The first is that coherence refers to “systematic or logical connection or consistency”. The second is that it is an “integration of diverse elements, relationships, or values”. I take from the first definition that an IE must be mathematically logical, where one statement follows another in a “step-by-step” manner. This definition points to an IE being coherent if it is presented in a way that makes it possible for learners to follow each step. For instance, if a teacher provides the solution to a problem, then each step of the solution must follow from the previous step mathematically. In this regard, I not only refer to what is written but also how it matches with what is said (teachers’ utterances).

The second definition points to coherence as the integration of parts or relationships that form a whole, which I also take into consideration in my definition of coherence of IEs that I use in this study. In section 2.4.1.3, I illustrated my elaboration of Leinhardt’s (2001) definition of IEs to include how examples are chosen and sequenced, how inscriptions are made, and how the utterances of a teacher combine together. In this chapter I elaborated on a teacher’s use of examples, inscriptions and talk to focus on mathematical connections, mathematical content and learners. My elaboration of a definition of an IE and the features of coherence identified from the literature enables me to arrive at a definition of coherence of IEs that I use in this study.

I therefore consider the coherence of IEs to be the relationships between the different parts of an IE as well as the unity of these different parts in **relation to the ooL of the IE**. By different parts I mean how a teacher’s *use of examples* and *use of inscriptions and talk* focus on learners, mathematical connections and mathematical content to foreground the ooL of the IE. Note that when I refer to the coherence of IEs, henceforth I mean coherence in relation to the ooL of an IE.

I consider ‘not coherent’ to be when the ooL of the IE is not brought into focus. This happens when the IE does not address any of the three categories in bringing into focus the ooL.

I provide an example of an IE that is not coherent:

The ooL of the IE is *how to find the y-intercept of a quadratic function*.

Consider the following scenario:

A teacher intends to explain that the y-intercept of a parabolic graph of the form $y = ax^2 + bx + c$, is found by substituting 0 for x and is therefore indicated by the value of c . His IE, including the examples used, focuses on illustrating to learners how to write a given equation in the general form ($y = ax^2 + bx + c$) without addressing the issue of the y-intercept being the c -value. For instance, all examples used are given in the non-standard form, which he uses

to explain how to rewrite them in the general form. This IE does not explain how to find the y-intercept but is rather a focus on how to rewrite a given equation in the general form. I therefore consider this IE to be *not coherent* in relation to the ooL.

The IE may follow a logical order with each statement following from the previous and therefore may be fit the dictionary definition of coherent but it is not coherent in relation to the ooL of the IE. The IE may be coherent in terms of how to write a given equation in the standard form but it does not explain how the standard form will help in finding the y-intercept of the resulting graph from the equation. The IE is therefore *not coherent* because it does not address the ooL of the IE.

IEs that are partially coherent are those that do not address all three of the foci i.e. a focus on connections, mathematical content and learners. In other words, the IE may address one or two of the categories but not all three. An example of an IE that is partially coherent is as follows:

The ooL of an IE is *how to find the y-intercept of a quadratic function*.

Consider the following scenario:

A teacher may begin the IE by explaining how to rewrite a given equation in the general form i.e. $y = ax^2 + bx + c$. She may then explain that the y-intercept is the point where the graph intersects the y-axis and conclude that this means that the y-intercept of a graph is given by the c-value in the general form.

The statement that the y-intercept is given by the c-value of the general form does not follow from the previous statement, i.e. the y-intercept is the point at which the graph intersects the y-axis. The IE does not address how rewriting the equation in the general form $y = ax^2 + bx + c$ is necessary to find the y-intercept. The value of c being the y-intercept of the function does not follow from the previous statement. What is missing in the IE is that the y-intercept is given by the c-value because the value of x on the y-axis is zero and so the y-intercept is found by replacing x with zero, which results in the value of c being the y-intercept. This IE is therefore partially coherent.

It is partially coherent because there is not sufficient logical progression from one step to the other with key links missing. The writing of the equation in the general form and the y-intercept being c seem to have no relationship to each other in this IE. What is missing in this IE is the lack of a justification for the c-value being the y-intercept of the function. While the procedure of rewriting the equation into the general form is described and probably can be

followed by a learner, this procedure is not connected to a justification, i.e. a focus on connections is not present in this IE.

9.5. Conclusion

In this chapter I have defined coherence as a teacher's *use of examples* and *use of inscriptions and talk* focus on learners, mathematical connections and mathematical content to foreground the ooL of the IE. This definition emerged from my data while using literature on coherence as the foundation. It builds on the existing literature on coherence. My definition of coherence of IEs not only adds to the literature on coherence but also to Leinhardt's (2001) theory of IEs. Leinhardt mentions *coherent IEs* in her theory but I could not find an elaboration on what is meant by the word *coherent* in relation to IEs in my reading of Leinhardt's work. As discussed in section 3.1, I referred to Leinhardt's feature of establishing a query as the ooL of an IE. My definition of coherence of an IE relates to how an IE addresses the ooL of an IE and so adds to Leinhardt's theory of IEs.

While I provided a definition of coherence of IEs that I made use of in this study, it must be emphasised that my intention was not to offer this as the only way to view the coherence of IEs. The definition of coherence of IEs is but one of the ways that the coherence of IEs can be examined.

One of the main findings in the literature is the major contribution of connection-making to the coherence of IEs. The benefits of connecting between mathematical ideas and learners connecting their prior knowledge to new knowledge cannot be ignored. However, I do not consider it to be sufficient. For example, highlighting the key ideas of a topic does not necessarily relate to making connections. Highlighting a key idea may take the form of a summary of the main ideas in a topic and does not necessarily connect to another mathematical idea. However, it is a feature that can make it possible for an IE to be coherent to learners. The literature on coherence of lessons provides other features (e.g. attending to possible learner difficulties in a topic) that can be considered in tandem with connections to make it possible for a teacher to offer a coherent IE.

An aim of this study is to examine the coherence of IEs offered by teachers. I reviewed the literature on coherence of lessons and IEs in order to highlight features that can be used to develop a framework of coherence.

Studies have highlighted the following features that I make use of in developing a framework to gauge the coherence of IEs:

1. Making connections between mathematical ideas
2. Justifying procedures
3. Highlighting key mathematical ideas
4. Addressing learner difficulties

5. Attaching prior knowledge to new knowledge

This is, however, a list of features. How they combine to form a framework of coherence and how they are observable in an IE is the subject of the next chapter.

Chapter 10 : Coherence of mathematical instructional explanations (IEs)

10.1. Introduction

In this chapter, I illustrate how I used the definition, my rearrangement of Leinhardt's (2001) features of IEs, and the features of coherence within the three foci to develop a framework to examine the coherence of IEs. I call this framework the **Coherence of Mathematical Instructional Explanations Framework** (CoMIE framework). I illustrate how the different foci and the features within them are combined within the CoMIE framework. I conclude this chapter by an illustrative example of how I use the framework to analyse an IE from my data.

An examination of the literature on coherence of IEs suggests that there is a lack of research in this area. While there are many studies relating to coherence of lessons, this is not the case for the coherence of IEs. It is therefore not surprising that what is meant by coherence of IEs is difficult to pinpoint in literature. In the absence of a definition, I consulted various studies on coherence of IEs to offer a possible way to define the coherence of IEs. I used the features of coherence of lessons and placed them into three categories, i.e. a *focus on learners*, a *focus on mathematical content* and a *focus on connections*. I also used the feature of IEs as establishing a *query* (Leinhardt, 2001) but have referred to it in this study as the object of learning (ooL) of an IE (as argued in chapter two). The ooL of an IE, together with the three foci, together provided a definition of coherence of IEs in this study.

I defined coherence a teacher's *use of examples* and *use of inscriptions and talk* focus on learners, mathematical connections and mathematical content to foreground the ooL of the IE. This definition helped in developing my framework of coherence of IEs.

I elaborated on Leinhardt's definition of IEs (section 2.4.1.3.) to incorporate how the verbal and the written interact in an IE. I also showed how an IE may sometimes encompass an entire lesson (a composite IE), which is made up of other IEs that can be summoned back at different points. I elaborated on how examples can be used in illuminating the ooL of an IE. The use of examples, use of inscriptions and talk and the summoning-back of previous IEs are three lenses through which the features of IEs can be examined. I use the two lenses, i.e. a *teacher's use of examples* and a *teacher's use of inscriptions and talk* (and not the summoning-back lens as discussed below) to now examine the features that lead to the coherence of IEs. These lenses, together with the features within the three foci of coherence, are used to develop my framework of coherence of IEs.

The summoning-back lens of an IE is used in my expanded definition of an IE. A lesson is made up of many IEs that combine to form an IE that may encompass an entire lesson (a composite IE). A lesson may also consist of many IEs, as elaborated on in chapters 7 and 8. All IEs

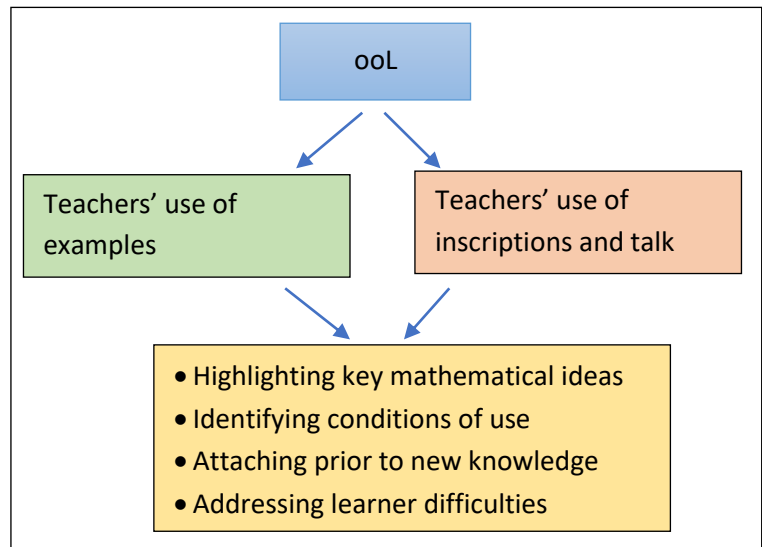
(composite and others) focus on an Object of Learning (*OoL* in a composite IE and *ooL* in other IEs) which refer to the purpose of an IE. In developing a framework of coherence of mathematical IEs, I concentrate on those IEs that are limited in time and scope and focused on a particular *ooL* because they are typically what are referred to as IEs in the literature. I therefore do not consider composite IEs and so do not use the lens of summoning-back in my framework.

In this chapter, I illustrate how I used the definition, my rearrangement of Leinhardt’s (2001) features of IEs, and the features of coherence within the three foci to develop a framework to examine the coherence of IEs. I call this framework the *Coherence of Mathematical Instructional Explanations Framework* (CoMIE framework). I illustrate how the different foci and the features are combined within the CoMIE framework. I conclude this chapter by an illustrative example of how I use the framework to analyse an IE from my data.

10.2. My adaptation of Leinhardt’s features to the coherence of IEs

In the conclusion of chapter 2, I illustrated my rearrangement of Leinhardt’s features of IEs. I now show how this rearrangement, as well as the features of coherence identified in chapter 10, informed the development of the CoMIE framework. Table 10.1 indicates Leinhardt’s features while in Figure 10.1. I re-arranged the features to indicate their relationships.

Leinhardt’s features of IEs
1. Answering a query
2. Use of examples
3. Use of representations
4. Highlighting key mathematical ideas
5. Identifying conditions of use
6. Attaching prior knowledge to new knowledge
7. Resolving the nature of errors



As mentioned, teachers' use of examples and teachers' use of inscriptions and talk are the lenses through which I examined the features in an IE. These lenses are also mentioned in studies on coherence. While studies on coherence mention the use of specific language used by a teacher, I use 'teacher talk' (a key term in this study) because it encompasses language use as well as how spoken words are used to connect to what is written. I incorporate a teacher using examples and how she uses inscriptions and talk as two lenses in my CoMIE framework because they foreground the features of coherence: for instance, how examples are used is linked to how learner difficulties are addressed.

While the rearrangement of the features of IEs assisted with analysing IEs through the lenses of a *teacher's use of examples* and *inscriptions and talk*, it does not illustrate how these features relate to the coherence of IEs. In the next section, I illustrate how I used these features, together with features of coherence highlighted in chapter 9 in developing the CoMIE framework.

The features *resolve learner errors* and *attaching prior knowledge to new knowledge* have been highlighted as features of both IEs and coherence, although different terminology is used. I use the term *attaching prior to new knowledge* instead of *proceeding from the known to the unknown* so that it is consistent with features of IEs. Instead of *resolve learner errors*, as used in features of IEs, I use the term *address learner difficulties* because the word *resolve* suggests that learners have already attended to the errors or overcome the errors, which is not necessarily the case. As mentioned previously, I consider what is made possible for learners to learn. The design of the study does enable me to consider what learners actually learnt. I use the word *difficulties* instead of a specific focus on learners' errors, to refer to potential common problems that learners may experience in a particular topic.

The feature of *highlighting key mathematical ideas* of a topic has been identified as a feature of coherence in literature. As mentioned in chapter 2, I consider this feature instead of identifying core principles of mathematics as identified by Leinhardt (2001). I include this feature of highlighting key mathematical ideas in the CoMIE framework as it can be used to bring into focus the oOL of an IE.

The other feature of IEs not mentioned in the literature on coherence is that of identifying *conditions of use of a mathematical idea*. If a particular mathematical idea is in focus in an IE, then it is necessary to specify the boundaries around its use in order to highlight how and when it can be used. I therefore consider this feature as one that contributes to highlighting the oOL of an IE, and thus to its coherence, and therefore include it in my CoMIE framework.

Finally, I separate the feature of *connecting a procedure to its justification* into two separate features i.e. *describing a procedure* and *justifying a procedure*. I do this because the description of a procedure can be coherent in terms of the ooL even if it is not connected to a justification.

Table 10.2 is a summary of the features together with the acronym that I use to refer to each in my framework of coherence.

Feature	Acronym
Addressing Learner Difficulties	ALD
Attaching Prior to New Knowledge	APNK
Describing a Procedure	DP
Providing Justification	JP
Identifying Conditions of Use	ICU
Highlighting Key Ideas	HKI

Table 10.2: Summary of features of coherence of IEs

I next illustrate how I use the features of coherence to develop my CoMIE framework, as well as how each contributes to the coherence of IEs.

10.3. My framework of coherence of IEs

I arranged all of the features within each of the foci: *a focus on learners*, *a focus on mathematical connections* and *a focus on mathematical content* (although I refer to this as *focus on content*) to develop the CoMIE framework. Figure 10.2 is a diagrammatic representation of the CoMIE framework.

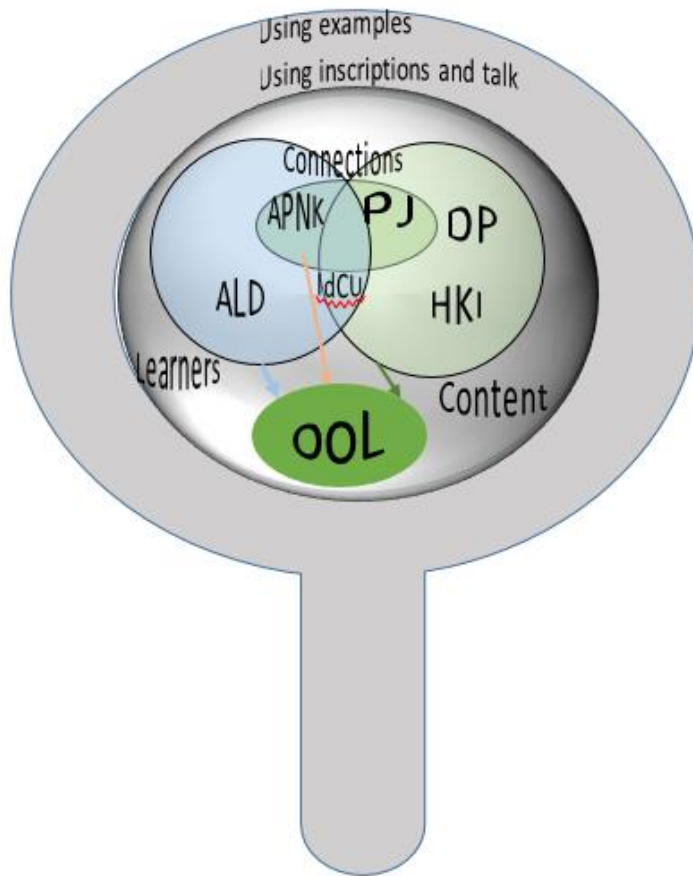


Figure 10.2: CoMIE framework

In section 9.3. I provided a definition of coherence of IEs as used in this study. The definition included features of IEs found within a *focus on content*, a *focus on connections* and a *focus on learners* which help to foreground the ooL of an IE. This encompasses the six features that can be placed within each foci or at the intersection of each. I next explain why the framework is designed as it is.

10.3.1. A focus on content

An IE must attend to the mathematical idea as well as related mathematical ideas in order to make it possible for learners to perceive the ooL. This can be done through the use of examples, inscriptions and teacher talk in *Highlighting Key Ideas* (HKI) of a topic, *Identifying Conditions of Use of a mathematical idea* (IdCU¹¹), *Describing a Mathematical Idea* (DMI) and *Providing Justifications* (PJ), all of which describe actions taken by teachers to *focus on the mathematical content* of an IE.

¹¹ I use the acronym IdCU to differentiate it from ICU (Intensive Care Unit).

The feature HKI consists of emphasising the important ideas of a particular ooL. I have placed this feature within a focus on content because it requires that a teacher be familiar with the content of a particular topic and therefore the main ideas that must be attended to in order to bring into focus the ooL of a particular IE. These ideas are specifically referred to and emphasised in some way during the IE. The teacher does this by verbally emphasising key words or by writing important ideas on the chalkboard.

The feature IdCU is a focus on the mathematical content of a topic; however, it is necessary for learners to be able to identify when to use a particular mathematical idea and when it cannot be used. For instance, if the ooL is the use of the zero product rule in solving quadratic equations then each factor can be equated to zero in an equation of the form $a \cdot b = 0$ but each factor cannot be equated to a constant value in $a \cdot b = k$ (where k is some constant value). The IE must make it possible for learners to identify the conditions of use of a mathematical idea in order to use it appropriately. I therefore place IdCU at the intersection of a focus on learners and a focus on mathematical content.

10.3.2. A focus on learners

Some IEs focus on the mathematical content without considering how the IE provided may potentially lead to errors. An IE will not be coherent to the learners, if their difficulties are not taken into consideration. I refer to this as *Addressing Learner Difficulties* (ALD), and it is an explicit *focus on learners*. It is through addressing learner difficulties that the ooL can be brought into focus for learners. In section 3.2.3.1 I described the ooL as something that learners should be able to know and do at the end of an IE. For instance, learners should develop the ability to define terms or interpret a particular problem. Learner difficulties can be addressed in an IE through the choice and sequencing of examples and how these are placed strategically on the chalkboard to draw attention to the ooL.

10.3.3. A focus on connections

A focus on connections relates to both connecting between mathematical ideas and how new mathematical ideas are built from learners' prior mathematical knowledge. For instance, the factorisation of expressions is the means through which a quadratic equation can be solved. Factorisation of expressions is not an isolated topic but is connected to other topics. Emphasising connections between mathematical ideas in an IE if brought to the fore makes it possible for learners to see mathematics as a network of connected ideas. The focus on connections is therefore subsumed within a focus on content and a focus on learners because it relates to both.

Attaching prior to new knowledge (APNK) relates to a *focus on learners* as well as to a *focus on connections*. APNK centres around how a teacher organises the content of a particular IE so that learners' previous knowledge is connected to the new. This may also involve organising the examples inscribed on the chalkboard so that a gradual progression is made from the easy to the more difficult, with small changes made from one example to the next. Organising the content of

an IE to proceed from the prior knowledge of learners to new knowledge seeks to involve learners in the IE. It is a deliberate action by a teacher to activate appropriate prior knowledge of learners so that they are included in the IE. By “included” I mean that learners are able to not only explicitly or verbally participate in the IE but are able to follow the IE of the teacher. These IEs may consist of steps of a procedure to be executed in solving a particular problem (e.g. solving a quadratic equation) or following a sequence of steps in arriving at a particular conclusion (e.g. zero product law).

Providing Justifications (PJ) is placed between a *focus on connections* and a *focus on content*. A mathematical idea or a procedure (I use mathematical idea to include a procedure) needs to be both described and justified to show that mathematics does not consist of a set of random rules. The justification should be connected to the description of a mathematical idea for it to make sense to learners so that it can be used in a similar way in another problem. For instance, the description of steps to follow in solving a quadratic equation may make it possible for learners to solve a quadratic equation. These steps, however, may not make it possible for learners to solve a quadratic equation given in a different form if learners were not provided with a justification for following the steps (see example in section 10.4.2.3 below for an elaboration of this). Earlier, in (section 9.4.), I defined coherence of an IE as how the use of examples, inscriptions and talk focus on learners, mathematical connections and mathematical content to foreground the ooL of the IE. Therefore, if an IE focuses on only the learners and a description of the procedure without connecting the steps of a procedure to its justification, then the IE is only partially coherent.

10.4. Analysis of coherence of an IE using the CoMIE framework

I now illustrate the interaction of all three categories by applying the CoMIE framework to an IE offered by Ndivho. Through this example I show how the framework can be used to examine the coherence of an IE.

10.4.1. Overview of Ndivho’s IE within a lesson on the solving of quadratic equations

I provide an overview of the IE before showing how I analyse it using the CoMIE framework. The ooL of the IE (as inferred by the teacher’s IE) was to illustrate the difference between solving an equation of the form $a \cdot b = 0$ and $a \cdot b = k$ (where k is a constant). This is an IE discussed in section 7.2 page 104 previously. I make use of the same IE that was used to demonstrate how the lens of the interaction between the verbal and written help to connect IE’s. What was not in focus in the previous discussion was the coherence or not of the IE. In this section I use the same example of an IE to demonstrate how the CoMIE framework can be used to analyse its coherence.

Ndivho's IE was provided in response to a learner's incorrect response in solving the quadratic equation $(x - 7)(x - 5) = 3$. His IE took the form of placing correct (Learner 2) and incorrect (by Learner 1) responses side by side (juxtaposing) on the chalkboard and comparing them. Figure 10.3 is a picture of the chalkboard with both responses inscribed.

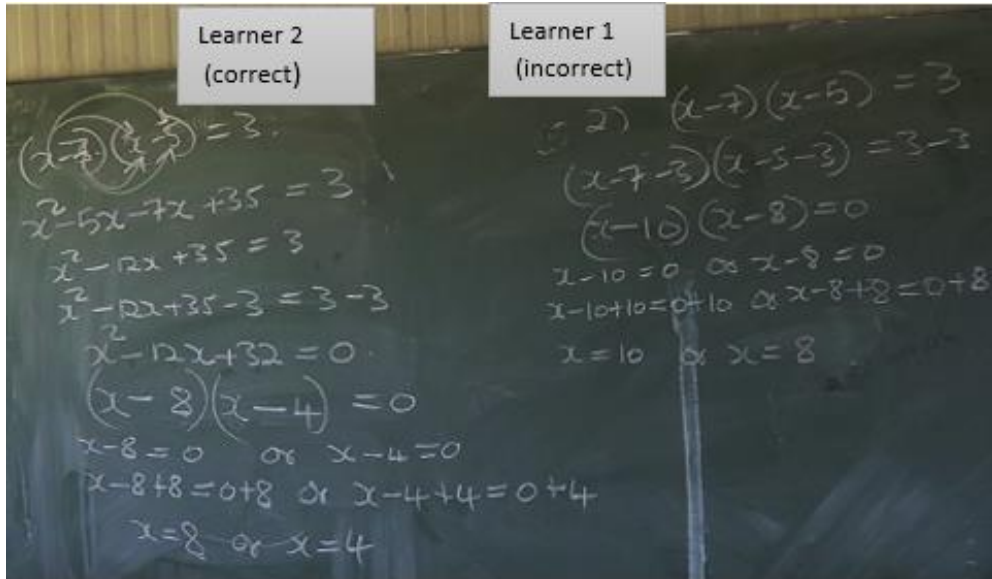


Figure 10.3 : A picture of the chalkboard with both responses inscribed

The first step of Learner 1's incorrect response was $(x - 7 - 3)(x - 5 - 3) = 0$. Apart from this error, the procedure was correct. Ndivho indicated that the learner had subtracted three twice. After writing Learner 2's (correct) response to the equation, Ndivho asked learners which response they thought was correct. Learner 1 said hers was incorrect. Ndivho demonstrated what she should have done by erasing her solution and writing $(x - 7)(x - 3) - 3 = 0$, as shown in Figure 10.4.

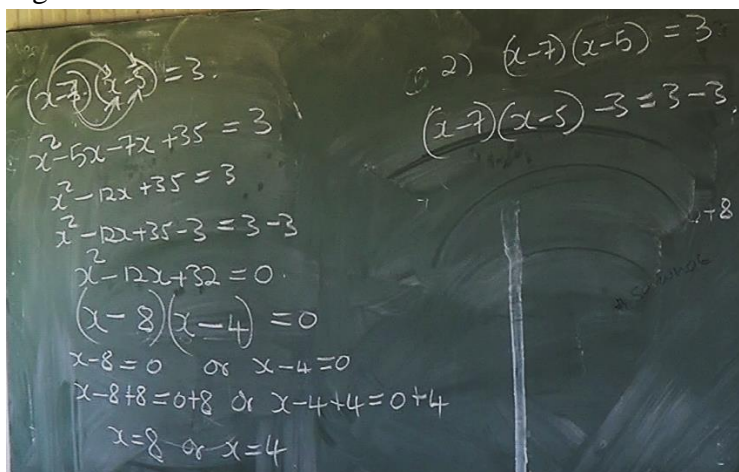


Figure 10.4: Picture of the chalkboard with corrected first step of the solution by Learner 1

Ndivho mentioned that this meant 3 was only being subtracted once. He also demonstrated the difference between the question $(x - 7)(x - 5) = 0$, which he wrote below $(x - 7)(x - 5) - 3 = 3 - 3$ as indicated in Figure 10.5.

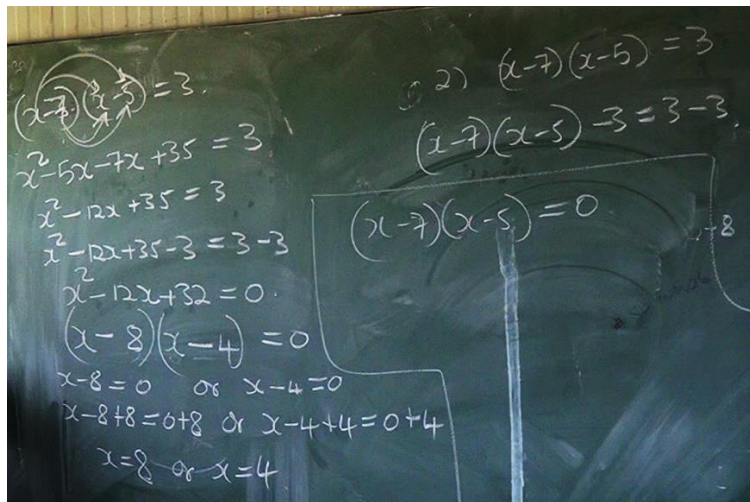


Figure 10.5: Picture of the chalkboard with contrasting example

The teacher stated that each factor could be equated to 0 in $(x - 7)(x - 5) = 0$ but not in $(x - 7)(x - 5) = 3$ because the right-hand side was 0 in the first case, whereas it was not zero in the second.

10.4.2. Using the CoMIE framework to analyse the coherence of Ndivho's IE

I now analyse this IE using the CoMIE framework through the lenses of Ndivho's use of examples and inscriptions and talk, to determine how the IE focuses on learners, mathematical content and mathematical connections in foregrounding the ooL. I look at the three foci separately to show which of these are present and those that are lacking. This leads me to conclude that Ndivho's IE is partially coherent because of insufficient attention to features in all three foci.

10.4.2.1. Focus on content

The features within a focus on content are Describing a Procedure (DP) and Highlighting Key Ideas (HKI) with the feature Identifying Conditions of use within the intersection of a focus on content and a focus on learners. The features DP and IdCU were in focus in the IE while HKI was not as is indicated in fig. 10.6.

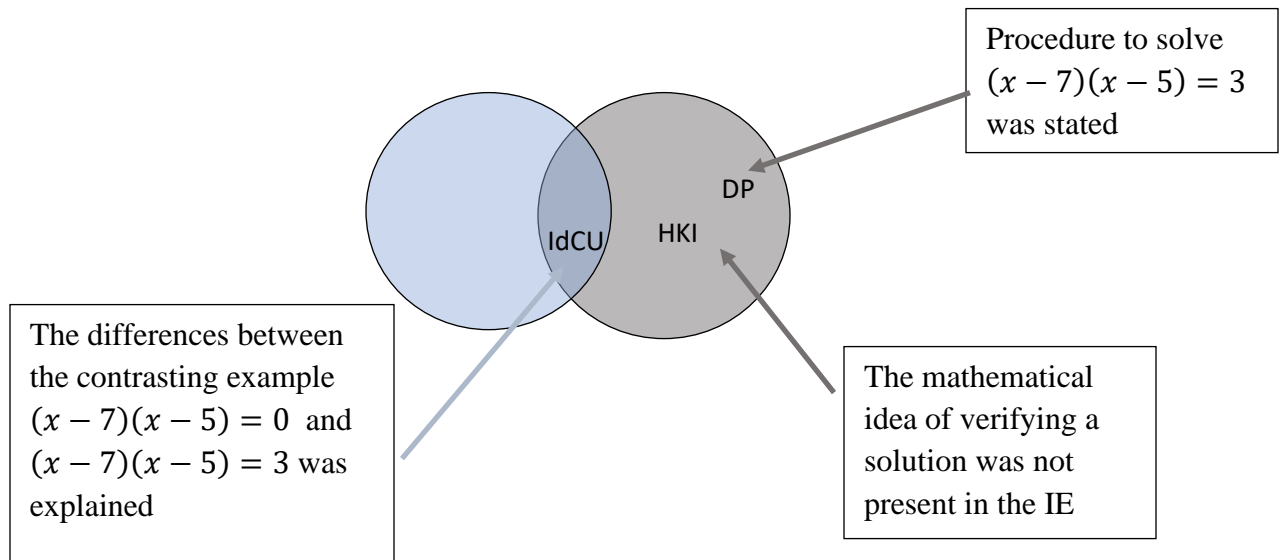


Figure 10.6: Features present in a focus on content

The procedure to solve the equation $(x - 7)(x - 5) = 3$ was correctly inscribed on the chalkboard. All relevant steps in solving were indicated. The description of the procedure (DP) was thus in focus in this IE.

An important mathematical idea when solving equations is the need to verify that the solution to an equation is valid. This is important because not all solutions to some equations result in valid values. It is therefore important when solving an equation that the solution to equations needs to be verified. The idea of verifying solutions can be extended to all mathematical problems more generally. In this IE, while Ndivho mentioned that learner 1 had subtracted three twice, he did not explain that her response to the equation was incorrect, i.e. that the two values for x would not have satisfied the equation. His IE did not include a motivation for learners to check their solutions after solving an equation. Highlighting the key idea (HKI) of verifying a solution was therefore missing in this IE. It must be noted that I placed this feature under a *focus on content* because it was the mathematical content that was foregrounded and not a learner difficulty. Ndivho focused on the use of a correct procedure in solving the equation and not on the error that the learner had made.

The IE offered by Ndivho made it possible to *identify the conditions of use* (IdCU) of the zero product law. This brought into focus the value on the right-hand side of the equal sign and the special properties of zero. Ndivho's words, as well as the differences in the solutions to the two

questions, made it possible for learners to attend to *when* the zero product law could be used and when it could not. Ndivho's talk made explicit that the zero product law could be used in the example $(x - 7)(x - 5) = 0$ but that it could not be used in the case of $(x - 7)(x - 5) = 3$ because the right-hand side was not zero. While I discussed this feature (IdCU) in relation to the mathematical content, it also relates to learners. The conditions of use of the zero product law must come into focus for learners in order for them to be able to generalise the law. This feature thus indicates the link between the two foci *focus on content* and *a focus on learners*.

10.4.2.2. A focus on learners

The features within a focus on content are Addressing Learner Difficulties (ALD) with Attaching Prior to New Knowledge (APNK) found at the intersection with a focus on connections. ALD was not present in the IE while APNK was addressed as indicated in fig. 10.7.

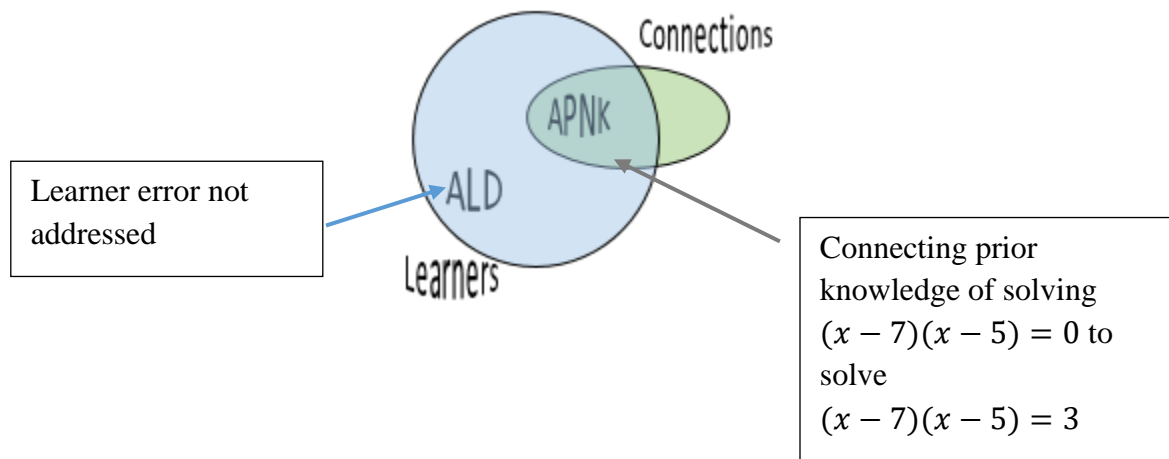


Figure 10.7: Features present in a focus on learners

This IE did not address a common error made by learners when solving an equation of the form $a \cdot b = k$. Anecdotal evidence suggests that an error made by learners when solving quadratic equations by factorisation occurs when they equate each factor to a non-zero value. For instance, one of the ways in which learners solve the equation $(x - 5)(x - 7) = 3$ is by equating each factor to 3, i.e. $x - 5 = 3$ or $x - 7 = 3$. The IE does not address why the zero product law could be used in $(x - 7)(x - 5) = 0$ but not in $(x - 7)(x - 5) = 3$. The feature ALD was therefore not in focus.

The other reason for my conclusion that the feature ALD was not in focus was that the error made by Learner 1 was not addressed. Ndivho's IE focused on providing the correct solution (as provided by Learner 2) as well as pointing out what learner 1 should have done, i.e. $(x - 7)(x - 5) - 3 = 3 - 3$. Both of these point to Ndivho emphasising the procedure to solve the

equation correctly rather than what was wrong with the response by Learner 2. The feature ALD was therefore missing in this IE.

The use of the contrasting example $(x - 7)(x - 5) = 0$ and its placement next to the question $(x - 7)(x - 5) = 3$ on the chalkboard make it possible for learners to notice the similarities and differences in the two questions and in their solutions. The use of the contrasting example made it possible for learners to connect what they had encountered previously to something new. Learners had previously solved equations of the form $a \cdot b = 0$ but not of the form $a \cdot b = k$ (k a constant value). Ndivho's use of the example $(x - 7)(x - 5) = 0$ thus indicated an attempt to *connect learners' prior mathematical knowledge to new knowledge*. The APNK feature was therefore in focus in this IE, demonstrating both a focus on learners and on mathematical connections.

10.4.2.3. A focus on connections

Providing a Justification (PJ) and Attaching Prior to New Knowledge (APNK) are the features of IEs placed within a focus on connections. Ndivho did not provide a justification for why the learner's solution was incorrect therefore PJ was not present in this IE while APNK was in focus (as discussed above). This is illustrated in fig. 10.8.

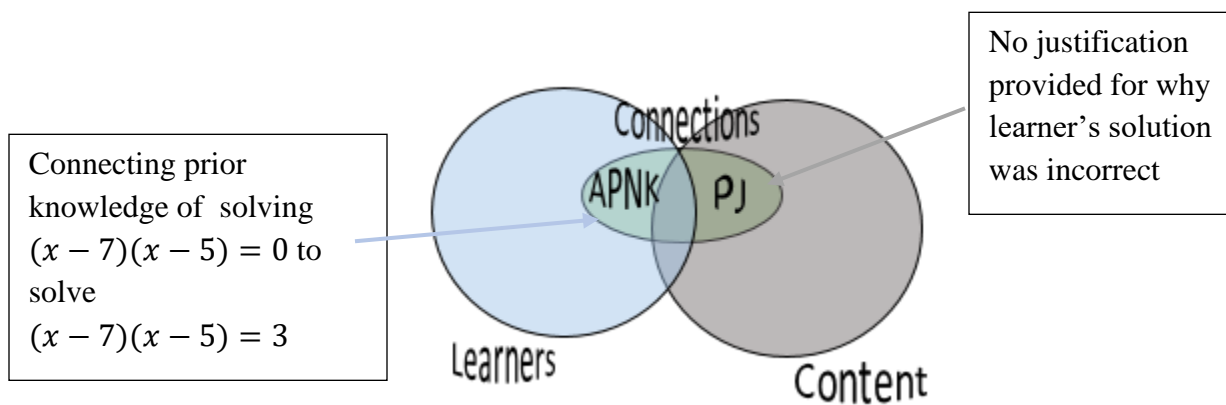


Figure 10.8: Features present in a focus on connections

Ndivho did not provide a justification for why the response by Learner 1 was incorrect. The justification that he provided for the learner's solution being incorrect was that she had subtracted 3 twice. This was not entirely correct, since she did not merely subtract 3 twice but subtracted it *from each factor*. In addition, what was missing from the IE was a justification for why the solution to the equation provided by Learner 2 was correct. The feature PJ was therefore not present in this IE. This indicates a missing link between *a focus on connections* and *a focus on mathematical content*.

APNK, however, was present in the IE as discussed above.

10.4.3. The coherence of the IE

My use of the framework to analyse the IE provided by Ndivho indicates that it was partially coherent. The ooL of the IE (as inferred by what was done) was to illustrate the difference between solving an equation of the form $a \cdot b = 0$ and $a \cdot b = k$ (where k is a constant). The link between a focus on learners and a focus on connections was present, as indicated by attention to the feature APNK. The feature DP indicated a focus on mathematical content while the IE showed that there was a link between the mathematical content and learners by addressing the feature IdCU. The IE, however, does not attend to providing a justification for the procedure in solving an equation of the form $a \cdot b = k$. The link between a focus on content and a focus on connections was therefore missing. The IE also did not attend to an important mathematical idea of verifying solutions in order to check for their correctness or not. Thus, this part of a focus on content was not present in the IE. The framework enabled me to see what was present and absent in the IE and suggested that the ooL, i.e. drawing attention to the differences between solving an equation of the form $a \cdot b = 0$ and $a \cdot b = k$, was therefore partially in focus.

I provide a summary of my analysis, indicating the features that were present and those that were not present in the IE in figure 10.9.

ooL: To illustrate the difference between solving an equation of the form $a \cdot b = 0$ and $a \cdot b = k$ (where k is a constant).

APNK present
 Use of equation
 $(x - 7)(x - 5) = 0$ was used to solve new equation
 $(x - 7)(x - 5) = 3$

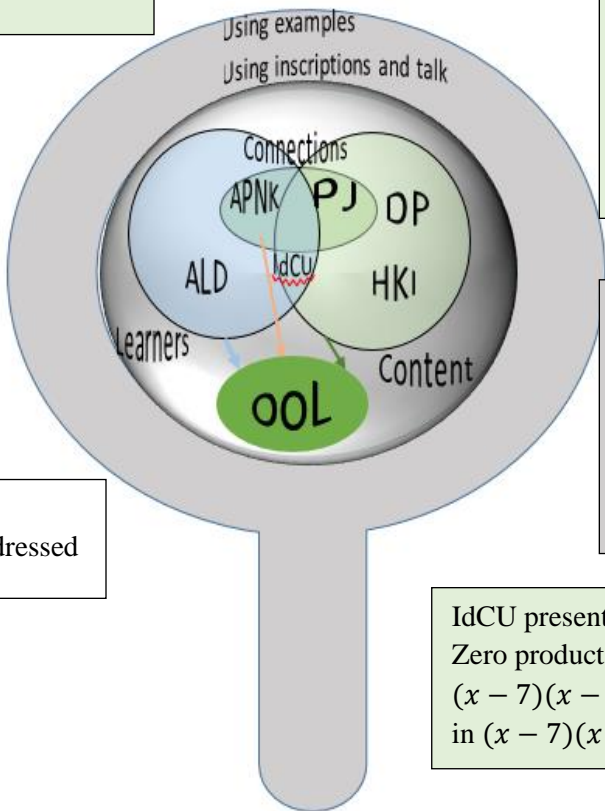
PJ not present
 No justification on why zero product law could be used in solving
 $(x - 7)(x - 5) = 0$ but not immediately in solving example
 $(x - 7)(x - 5) = 3$

DP present
 Steps used in solving
 $(x - 7)(x - 5) = 3$ was explained with inscriptions on

HKI partially present
 • Special property of zero when the left-hand side is factorised in focus.
 • Verifying solution of equation not addressed

ALD not present
 Learner error not addressed

IdCU present
 Zero product law used in
 $(x - 7)(x - 5) = 0$ but not in
 $(x - 7)(x - 5) = 3$



ooL: To illustrate the difference between solving an equation of the form $a \cdot b = 0$ and $a \cdot b = k$ (where k is a constant) is **partially** in focus

Figure 10.9: Summary of features of coherence in Ndivho's IE

The gaps in the IE, as indicated by the use of the CoMIE framework, leads me to conclude that the IE is partially coherent. I say this because Ndivho could have included examples where the equivalence of a mathematically correct equation, like $2 \times 5 = 10$, would not be maintained if 10 was subtracted from each factor. Subtracting 10 from each factor would result in $(2 - 10) \times (5 - 10) = 2.5 - 2.10 - 5.10 + 100 \neq 10 - 10$. This would have drawn attention to the fact that subtracting a value from each factor is incorrect and is more complex than subtracting the value twice.

The inclusion of the same numerical example would have also served to address another common difficulty of learners, i.e. equating each factor to a non-zero value on the right-hand side of the equation. The numerical example, $2 \times 5 = 10$, would have served to illustrate that equating each factor to 10 will not make sense.

Addressing the mathematics that underpins the error of the learner may, however, be complicated for learners to grasp. It may be the reason that Ndivho did not address the learner's error. Since teachers' decisions on what they choose to address in the classroom cannot be accessed through classroom observations the reasons for not addressing the error cannot be ascertained. It may also be that thinking about an example to address the learner's error requires a teacher to think about this on the spur of the moment. This may be a complex task for a teacher.

The analysis of the IE provided by Ndivho using the CoMIE framework indicates that the IE was partially coherent. I must, however, emphasise that this research was conducted within the interpretivist paradigm. My use of the framework indicates that this IE was partially coherent because of the gaps in the IE. This IE may nevertheless be considered to be adequate within the particular lesson. I used my framework to produce three foci to gauge the coherence of IEs. What it does not do is take into account such factors as what comes before and after this IE, topics done previously and the time factor. For instance, Ndivho may not have focused on the incorrectness of the mathematics involved in subtracting 3 from each factor due to time constraints of the lesson. It may also be that this was in focus in later lessons. I therefore mention that this IE can be considered to be appropriate given these other factors and constraints although my use of the framework suggests that it is partially coherent.

10.5. Summary of findings using the CoMIE Framework

I used the CoMIE framework to analyse the IEs of all 6 teachers in the topic "The solving of Quadratic Equations by factorisation". A total of 21 IEs within episodes were analysed. As mentioned previously (see section 5.3.1 on page 74), I only analysed those episodes where the teacher explained to the entire class. There were many episodes where learners worked individually and the teacher walked around helping and explaining to individual learners. The

IEs within these episodes were not analysed. Out of a total of 21 episodes analysed, I interpreted nine IE's to be coherent, nine partially coherent and 3 not coherent using the CoMIE framework while a more detailed summary is provided in Appendix F. A summary of the IEs of each of the teachers in this study is provided in table 10.3.

Teacher	Number of IEs within episodes analysed	Total number of episodes	Coherent	Partially coherent	Not coherent
Tsebo	5	7	5		
Ndivho	6	9	4	2	
Ulwazi	2	3		2	
Tshedi	3	4		1	2
Vutivi	3	4		3	
Elimu	2	4		1	1
Total	21	31	9	9	3

Table 10.3: Summary of Coherence using the CoMIE framework

As mentioned in section 10.4, these IE's were analysed using the framework. It is important to consider these results by taking into consideration the different contexts within which teaching occurs e.g. time constraints and order of the topics in the curriculum.

10.6. Conclusion

The definition and the different foci of coherence helped me to develop a framework for analysing the coherence of IEs, especially within the context of teacher-centred pedagogy. I call this the CoMIE framework.

My data reveals teachers' heavy reliance on the chalkboard while the use of examples is key in any mathematical IE. These, together with the support of teachers' talk, are lenses through which the features of coherence of IEs can be examined. These lenses make it possible to see how the features of coherence within the three foci are addressed.

The lenses, together with the features within the foci, informed the development of the CoMIE framework. The use of this framework has the potential to highlight how the strategic use of readily available resources, such the chalkboard, can make it possible to offer IEs that are coherent or partially coherent instead of a label of 'not coherent'.

While the CoMIE framework was developed within the context of South African classrooms, it also offers the mathematics education research community a tool to examine the coherence of

mathematical IEs within other similar contexts. In the next chapter, I conclude with the significance of not only the framework but also this study in general. I also note the limitations of this study, which can be used as a starting point for future studies on IEs in general, and the coherence of IEs in particular.

Chapter 11 : Discussion and conclusion

11.1. Introduction

Mathematics teachers providing mathematical instructional explanations (IEs) to learners is a central but “taken for granted” practice of teachers. While it may be central to teaching, developing IEs when planning a lesson may not be given sufficient attention by teachers, nor by pre-service or in-service teacher education. The lack of attention to developing IEs may therefore result in IEs that may not be coherent. The lack of coherence of IEs is indicated by studies, especially within the context of classrooms in disadvantaged schools in South Africa, where it was frequently found that the object of learning (ooL) was out of focus (e.g. Venkat & Adler, 2012).

The main aim in this study was to examine the nature and quality of IEs which culminated in the development of a framework used to analyse the coherence of IEs within South African mathematics classrooms and other similar contexts. This was, however, not my only aim. I also investigated the nature of IEs by examining Leinhardt’s (2001) theory of IEs. Interrogating the nature of IEs led to me finding relationships between the features that led to a logical re-organisation of them. Apart from the re-organisation of features I also expanded on Leinhardt’s definition of IEs to include an entire lesson and not just what occurs in shorter periods in time. This study culminated in my development of the Coherence of Mathematical Instructional Explanation (CoMIE) framework, presented in chapter 10, which was guided by the research questions:

1. What features of mathematical instructional explanations are revealed through:
 - a) Teachers’ use of examples
 - b) Teachers’ use of inscriptions and talk
 - c) Teachers’ summoning-back prior explanations.

2. What makes a mathematical instructional explanation coherent?

My journey started with investigating how teachers use the Mathematics Teaching Framework (Adler & Ronda, 2015), as introduced to teachers in a professional development course, in their classrooms. The aim was to contribute towards enhancing the MTF. I subsequently refined my focus to investigate the nature of IEs as outlined in chapter 1. At the end of this study, my findings suggest ways that the explanatory communication component of the MTF can be enhanced. This study has thus brought me full circle, taking me back to the MTF.

In this chapter, I briefly summarise my journey of how I proceeded in answering the research questions. I begin with a brief description of the teacher sample that participated in my study, the data analysis procedure, and my main findings. I also comment on the conclusions drawn from the findings and a discussion on how they add to the literature on IEs. I follow this with recommendations for teachers and researchers in the mathematics education community, and my own reflections.

11.2. Sample and analytic methods used

The findings of this study were illuminated by a grounded analysis of the IEs offered by my sample of teachers. My sample of six teachers was chosen purposively: all had attended a professional development course where they were exposed to a mathematics teaching framework developed by Adler and Ronda (2015). Teachers attending the course were provided with guidelines on how to offer IEs as outlined by the explanatory communication component of the framework. The chosen teachers were deemed to be reasonably proficient in offering IEs, not only because they were guided on how to offer them, but also because of their performance during the course. All teachers chosen obtained above 60% in their end-of-the-year course tests, which were based on mathematical content as well as pedagogy. They also performed well in the assessments conducted during the course (e.g. quizzes, assignments and short tests). Apart from their performance in assessments, they also engaged actively in all course activities (e.g. working in groups and presenting their work to others, asking questions and contributing to answering questions posed by others).

Data was collected through classroom observations and video recordings of the lessons of the six teachers. All lessons in the grade 10 topics, *solving quadratic equations* and *number patterns*, were observed. All teachers either used two or a maximum of three lessons in each topic. The number of lessons differed because of the duration of a lesson within different schools. For instance, at some schools, lessons were an hour long whereas in other schools, lessons were 35 or 45 minutes in length. A total of 21 lessons on number patterns and 16 on quadratic equations were recorded from all teachers over two years. (See section 4.4.3. for more detail of the duration and number of lessons for each teacher over the two years of data collection).

Video recordings of the lessons were used to transcribe each lesson, with notes taken of all words spoken and teachers' actions, and pictures taken of the chalkboard (see Appendix E for an example of a transcript of a lesson). These transcripts were divided into episodes, which were identified according to differences in the ooL of a particular part of a lesson. An episode therefore consisted of a part of a lesson with a specific ooL. Not all episodes were, however, analysed. An initial analysis of episodes was conducted to identify those episodes that were considered interesting and required further elaboration. These episodes were identified through the use of posters of each lesson (see section 5.3.2. for further details, together with an example

of a poster). The posters not only helped to bring into focus features from Leinhardt's theory but was key in my proposal of an IE to span an entire lesson instead of only occurring in moments of time within a lesson.

I used grounded theory to analyse the IEs of teachers. While grounded theory suggests that the findings emanate from the data collected, the version of grounded theory I used was in alignment with that of Corbin and Strauss (2014). They maintain that it is necessary for the researcher to consult literature on the topic in focus in order to be aware of information that is relevant to the study. I used literature on IEs, with a focus on Leinhardt's theory because of her decades of work on the topic and the use of her theory by other researchers. While I referred to the literature to guide the study, I conducted a grounded analysis of my data using posters of lessons (see section 5.3.2 for more detail on my use of posters). My use of posters was key in conducting my analysis. Although literature on the use of grounded theory indicate the use of memo's in analysis, they focus on analysis of interviews with not much guidance on how they can be used in analysing data from classroom observations. My use of posters shed more light on how memos can be used in the analysis of classroom observations. It is through the use of posters that I was able to identify my main findings.

11.3. Main findings of this study

Based on my analysis I was able to contribute to Leinhardt's (2001) theory of IEs in two ways. First, I illustrated how Leinhardt's features of IEs were linked together to underscore their relationships to each other. This resulted in my re-arrangement of the features. I illustrated how two of the seven features could be used to examine the enactment of the other four features of IEs while the seventh feature was included in the framework of coherence. The development of the CoMIE framework to analyse the coherence of IEs, was informed by the features of IEs as outlined by Leinhardt (2001), as well as from my data analysis.

Leinhardt provided a list of seven features of IEs but did not illustrate the relationship between them. I rearranged these features to illustrate their relationship to each other. I used the two features, i.e. *the use of examples* and *the use of representations*, as lenses in viewing IEs through which I examined how the four features were used. Table 11.1. illustrates Leinhardt's features, which I have rearranged, as illustrated in Figure 11.1.

Leinhardt's features of IEs
1. Answering a query
2. Use of examples
3. Use of representations
4. Highlighting key mathematical ideas
5. Identifying conditions of use
6. Attach prior to new knowledge
7. Resolve the nature of errors

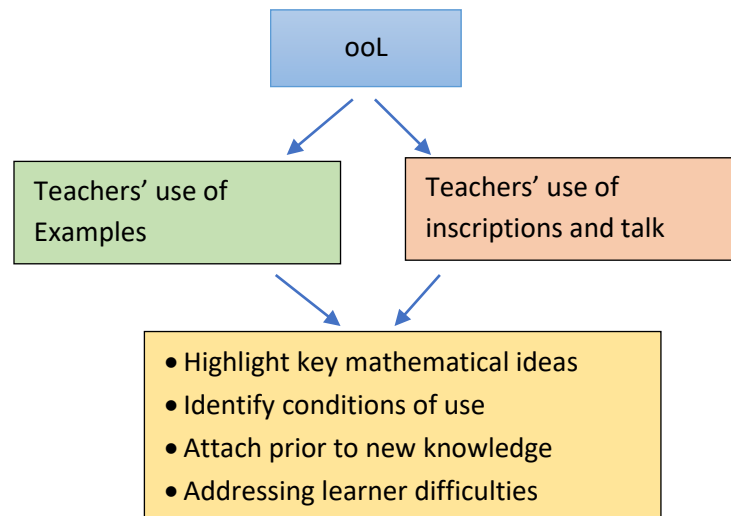


Table 11.1: Leinhardt's features of IEs

In my arrangement, Leinhardt's features of *examples* and teachers' use of *inscriptions and talk* feature, i.e. establishing a query as the *ooL* framework of coherence. In chapters 7, 8 features of IEs were examined through the *inscriptions and talk* while in the second how I used the *ooL* of an IE in the CoMI

While Leinhardt (2001) makes mention of that an IE may extend over an entire lesson contributing to the *OoL* of a lesson. The

organisation of content within the IEs, as well as the organisation of these IEs within a lesson, makes it possible for the features of IEs, including the *OoL* of a lesson, to be brought into focus. The IEs are organised so that they can be used again by a teacher in another IE. I called this *summoning-back*, which I used as an additional lens through which the features of IEs were examined. As a result of viewing IEs through the three lenses, all of Leinhardt's (2001) features of IEs could be observed, as illustrated in Figure 11.2.

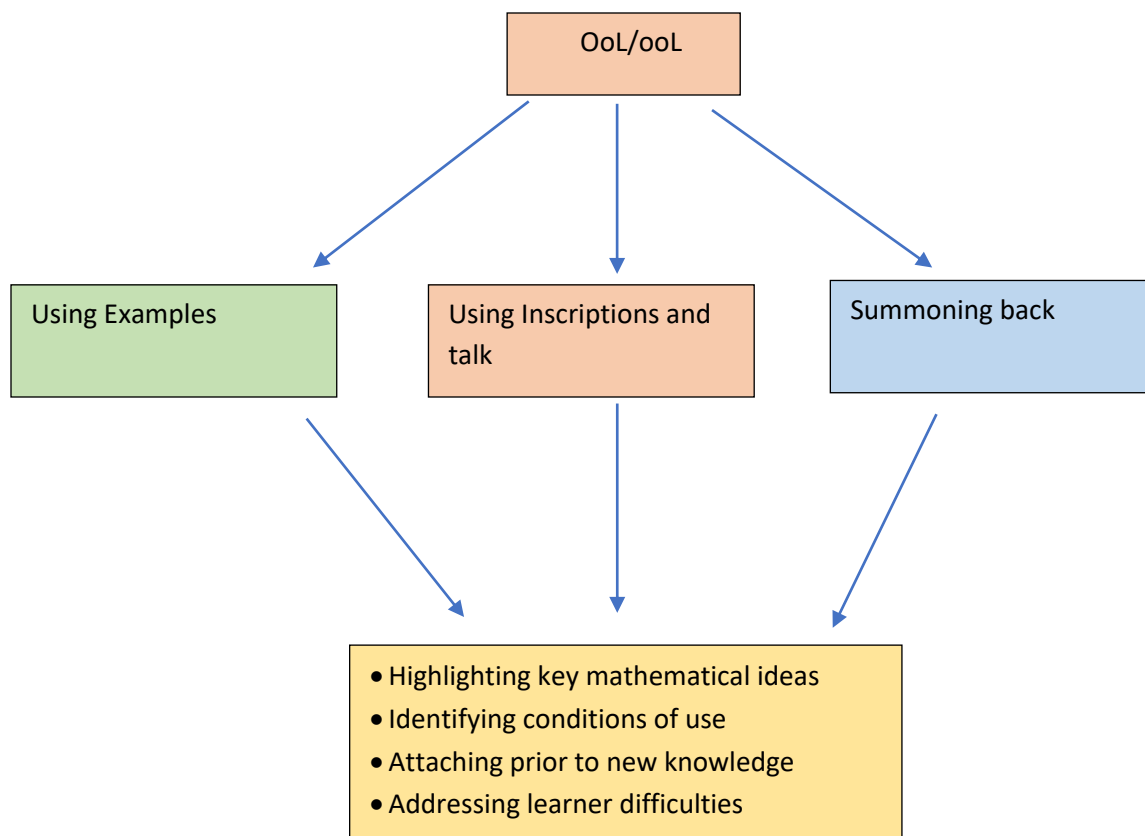


Figure 11.2: Relationship between lenses and features of IE's

Viewing IEs through the three lenses illustrated the power of these lenses in examining how the features were enacted within IEs. Zooming into each lens separately allowed me to observe features of IEs that may not have been visible if viewed through the lenses simultaneously. For example, varying some parts of examples while changing others showed how an IE could be organised to make it possible for learners to attach their prior knowledge to new knowledge. Arranging correct and incorrect learner responses to questions inscribed on the chalkboard side by side made it possible for learners to address common learner difficulties.

11.3.1. Viewing IEs through teachers' use of examples

Viewing IEs through the lens of *using examples* provided more insight into what Leinhardt (2001) means by an IE containing *a useful set of examples* in her definition of IEs but which she does not elaborate on. My data revealed the use of different types of examples as well as the use of variation in the examples to address the different features of IEs. The different types of examples used were:

- contrasting examples
- model examples, and
- reference examples.

Sets of examples demonstrating variance amid invariance, i.e. keeping some parts of examples the same while changing other parts, and the sequencing of examples with small changes from one to the next, illustrated how four features were enacted. In chapter 6, I illustrated how Tsebo constructed his examples in *highlighting key mathematical ideas*, *identifying conditions of use*, *attaching prior to new knowledge* and *addressing learner difficulties* were enacted

11.3.2. Viewing IEs through inscriptions and talk

Viewing IEs through the lens of inscriptions and talk suggests that an IE is much more than a teacher’s utterances. This lens challenges the assumption that an IE is either verbal or written. It illustrates how the verbal and written representations of an IE interact and can be used to examine how the features of an IE are enacted.

This feature of an IE was foregrounded by a teacher in this study, who arranged solutions to questions on a given number pattern on the chalkboard. Through this arrangement, he focused his IE on making connections between the solution to a question and the next question, together with connecting each answer to how it related to the given sequence, as illustrated in Table 11.2.

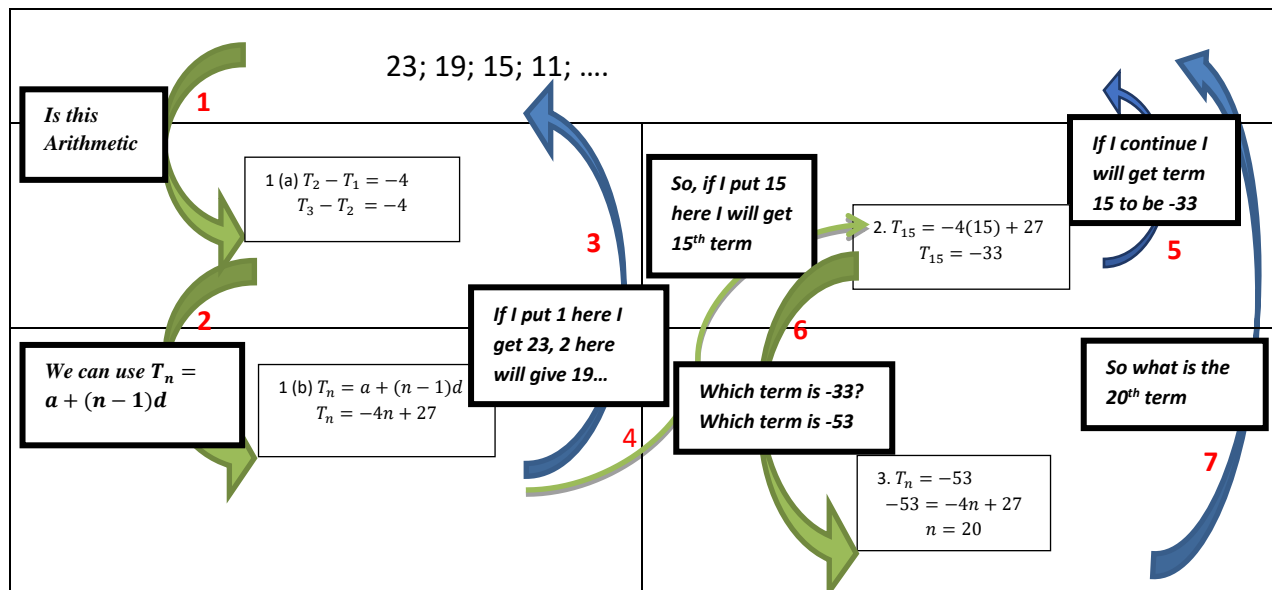


Table 11.2: Example demonstrating an IE through the lens of inscriptions and talk

This lens illustrated how three of the features of IEs, as identified by Leinhardt (2001), could be enacted in an IE. In chapter 7 I showed how Tsebo, Ndivho and Ulwazi arranged content on the chalkboard in *identifying conditions of use* (of a procedure), *addressing learner difficulties* and *highlighting key mathematical ideas*. While Tsebo and Ndhivo’s utterances served to support his inscriptions on the chalkboard, the utterances of Ulwazi was not explicit enough in supporting their inscriptions (as discussed in section 7.4.3).

11.3.3. Viewing IEs through the lens of summoning-back

This lens was identified through the analysis of my data and highlighted by my use of posters of lessons because of the same examples used in different parts of a lesson. It challenges Leinhardt's (2001) view of IEs as occurring in specific moments of time. A conclusion drawn from this finding is that an IE may extend to more than an IE focused on just one mathematical idea. What this lens highlights are the use of examples as well as inscriptions and talk in connecting between IEs to contribute to a composite IE. A teacher may use the same example in different forms to connect between different mathematical ideas as shown in the example in fig. 11.3. Similarly, what a teacher chooses to preserve on the chalkboard may be a deliberate act to connect what remains on the chalkboard to another mathematical idea that is anticipated in another IE later in the lesson. It cannot, however, be assumed that learners will make the connections on their own between mathematical ideas in different IEs. Teachers need to make explicit these connections through their utterances. It is through the use of examples and inscriptions and talk that IEs can be connected to each other.

This lens was highlighted in my study by one of the teachers, who organised the content of his lesson in such a way that IEs connected to each other, thus cumulatively highlighting the OoL of the composite IE. This example is illustrated in Figure 11.3.

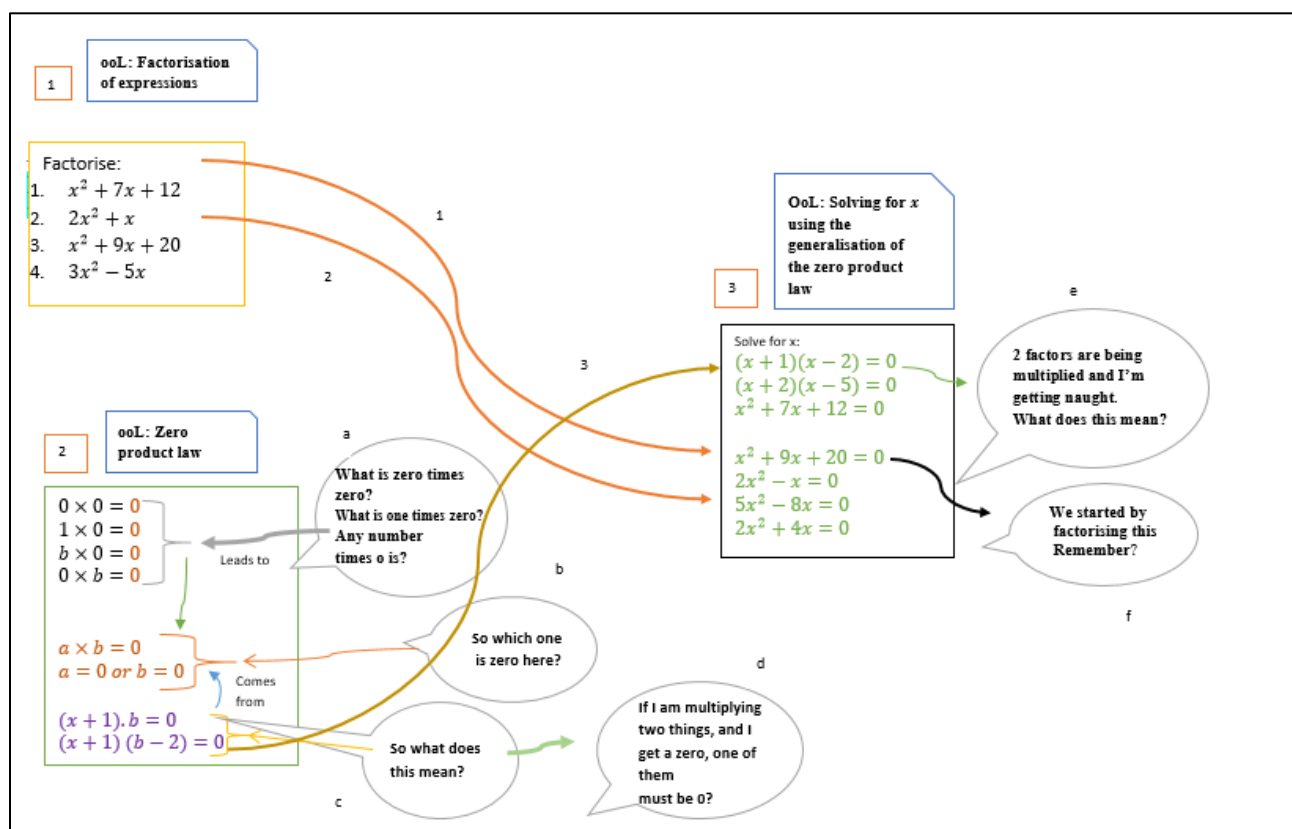


Figure 11.3: Example demonstrating an IE through the lens of summoning-back

My expansion of IEs to encompass an entire lesson, which I call a composite IE, made up of other IEs focusing on a mathematical idea, provides a different perspective into how IEs can be viewed. The IEs within a lesson may be organised in a way that paves the way for their contribution to another IE. In this way, IEs contribute to bring into focus the OoL of a composite IE. Viewing IEs through the lens of summoning-back prior IEs enabled me to see how the features *connections between different mathematical ideas* and *maintaining a focus on the OoL* (which is not a feature of IEs mentioned in literature) were enacted in an IE as discussed in section 8.4.

11.3.4. Framework of Coherence (CoMIE framework)

The three lenses demonstrated how the features of IEs can be examined, but did not address the issue of their quality. I examined the quality of an IE as its coherence in relation to the ooL of the IE. I did this because there was no framework that could be used to analyse the coherence of IEs, especially in light of studies that reveal that the ooL is often out of focus in the IEs provided by teachers (e.g. Venkat & Adler, 2012).

I used the features seen through the three lenses (as illustrated in chapters 7,8 and 9) as well as features of coherence identified in studies (chapter 11) to develop a framework, called the

Coherence of Mathematical Instructional Explanations (CoMIE framework). The features are arranged within foci on learners, mathematical connections and the mathematical content, as illustrated in Figure 11.4.

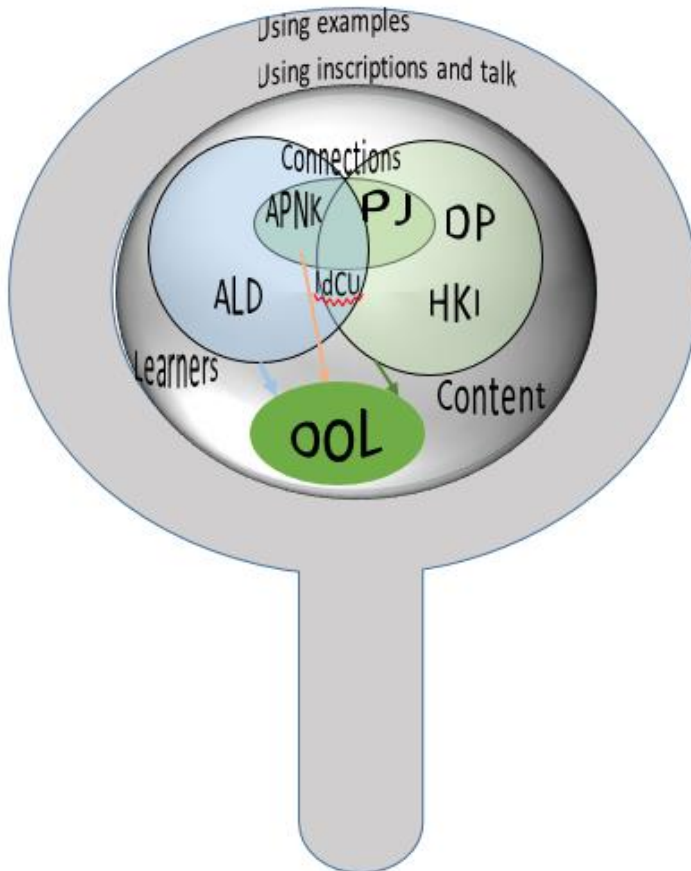


Figure 11.4: The CoMIE Framework

An IE is considered to be coherent if it focuses on learners, mathematical content and mathematical connections in foregrounding the ooL. An IE focuses on learners through addressing learner difficulties (ALD), attaching the prior knowledge of learners to new knowledge (APNK), and through making it possible for learners to identify the conditions of use of a mathematical idea (IdCU). An IE is also considered coherent if it focuses on the mathematical content of an IE. This focus includes the features highlighting key mathematical ideas (HKI) and describing a procedure (DP). The feature IdCU relates to both the mathematical content and learners, so is found at the intersection of these two foci. The feature providing a justification (PJ) connects a procedure to its justification and is therefore placed at the intersection of a focus on content and a focus on connections. A major contribution to coherence of IEs is a focus on the connections between mathematical ideas because it promotes a deep

understanding of the mathematical content (see section 10.3.1. for greater detail on the contribution of connecting mathematical ideas).

11.4. Contributions of this study

There are three main contributions of this study. The findings add to Leinhardt's (2001) theory of IEs, to grounded theory, and finally to pedagogy. I address each of these in more detail below.

11.4.1. Contribution to the theory of IEs

The findings of this study contribute to the theory of IEs. While Leinhardt (2001) considers IEs to be "*discussions that connect what is being explained to particular rules or principles; and, finally, a set of discussions that bound it or limit its applicability...*" she also considers examples to be part of IEs. However, she does not elaborate on what *discussions* entail. The word "discussions" implies teachers' utterances. The elaboration of IEs in this study includes the interweaving of examples. The chalkboard, together with teacher utterances, points to IEs as being more than the utterances of a teacher.

The second contribution of this study is to offer a definition of coherence of an IE and to elaborate on theory on the coherence of IEs. My definition of coherence of IEs as a teacher's *use of examples* and *use of inscriptions and talk* focus on learners, mathematical connections and mathematical content in foregrounding the ooL, shows the importance of the three foci in highlighting the ooL of an IE. As such this definition of coherence adds to literature on the theory of IEs as well as that of coherence.

Literature on coherence typically focuses on coherence of lessons (e.g. Cai et al., 2014; Chen & Li, 2010; Shimizu, 2007). Studies that do address coherence of IEs either do not elaborate on features of coherent IEs (e.g. Leinhardt, 2001) or restrict the notion of coherence in IEs to mathematical connections (Venkat & Adler, 2012). The CoMIE framework examines the coherence of IEs through the lenses of a teacher's use of examples and a teacher's use of inscriptions and talk to highlight features relating to the ooL of an IE. The framework emerges from contexts of teacher-centred pedagogies and hence is particularly suitable to examine the coherence of IEs within such contexts.

11.4.2. Contribution to pedagogy

The use of examples was a key element in all three of the foci of IEs identified in this study, i.e. a focus on connections, focus on mathematical content, and a focus on learners. This is not a new finding. Explicit mention is made of the use of examples as a feature of IEs in Leinhardt's (2001) work. She deems it necessary for teachers to have a wide range of examples to connect prior knowledge to new knowledge, to demonstrate the validity of a procedure and to correct learner errors. She, however, does not point to specific ways in which examples can be used to

highlight a mathematical idea. The use of examples as a lens through which to view IEs provided more insight into how examples can be selected and sequenced to highlight the features of IEs. The use of the key ideas of variation theory, i.e. variance amid invariance, the use of contrasting examples as well as attention to how they are sequenced, with small changes from one to the other, provides guidance on how using examples can foreground the purpose of IEs.

How teachers use the chalkboard, which is a widely used resource to develop their IEs, is one of the contributions of this study. The use of inscriptions and teacher talk are, however, not new. For instance, Venkat and Askew (2018) examine the coherence of IEs through inscriptions and talk, but their focus is on mathematical connections being made. The findings of this study are different from theirs in two ways: first, I take a broader view of coherence in this study as relating to how IEs focus on learners, connections and mathematical content, rather than a narrower focus on connections only. Second, I not only looked at what inscriptions were made on the chalkboard, but also how they were made. My data illustrated a focus on content arranged in specific ways on the chalkboard that suggested close relationships between them. The arrangement on the chalkboard was accompanied by teacher talk that made explicit these connections. Furthermore, content may be preserved on the chalkboard to be used again in another IE. In this way, what is preserved on the chalkboard can be connected to other mathematical ideas in a subsequent IE. The teacher's talk can make these connections explicit. Hence, the contribution of this study is that it sheds light on the specific uses of the chalkboard in arranging content in particular ways as well as preserving content that can be used to connect to other mathematical ideas.

The findings of this study can help enhance the MTF that can be used in PD courses with teachers. The MTF provides teachers with guidance on how to select and sequence examples (exemplification component of MTF); how to develop and provide an explanation (explanatory communication component of MTF) and the coherence and connections between them. The MTF however does not provide guidelines on how to achieve coherence and connections between the exemplification and explanatory communication components. Furthermore, these components are regarded as separate from each other in the MTF. In this study I have considered IEs to be much more than the utterances of a teacher, and so include the selection and sequencing of examples as part and parcel of IEs. I consider IEs as comprising the strategic selection, sequencing and arranging of examples on the chalkboard, together with the utterances of teachers. The MTF can be enhanced to take the findings into consideration. For example, these components of the MTF in Figure 11.5 can be adapted to how I see them in Figure 11.6.

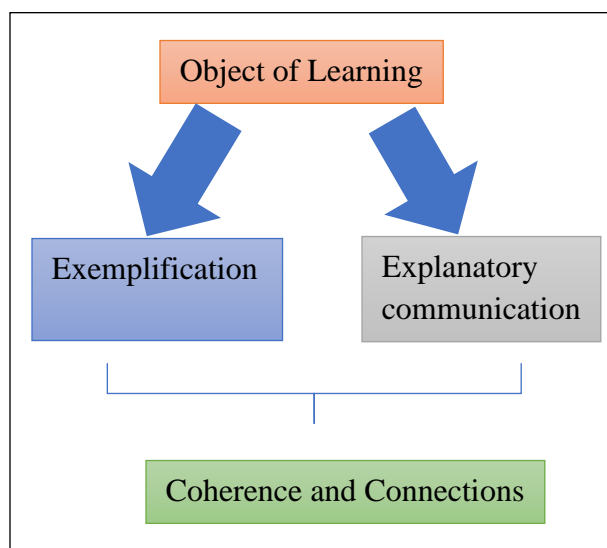


Figure 11.5: Part of MTF

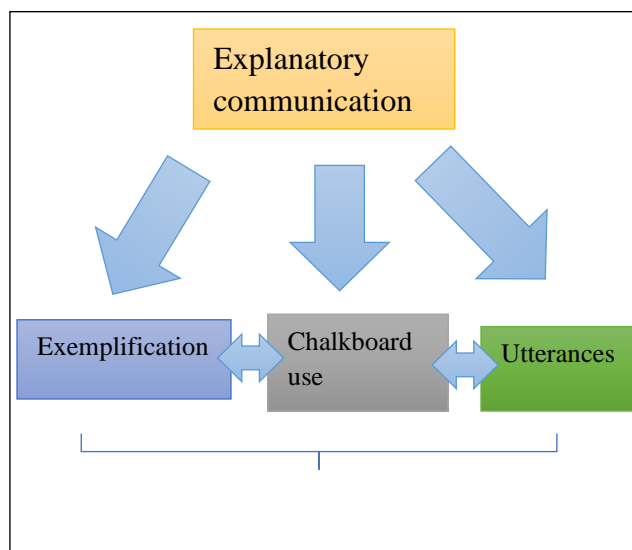


Figure 11.6: Part of MTF as I see it after this study

Instead of the MTF containing the two separate components, exemplification and explanatory communication, as relating to the OoL of the lesson with coherence and connections between them, I see the explanatory communication component as the interweaving of examples, use of the chalkboard and the utterances of a teacher.

11.4.3. Contribution to grounded theory methodology

The use of a poster as a memo (see section 5.3.2) in analysing classroom observations proved invaluable in illuminating my findings. Most literature outlining the use of grounded theory focuses on the analysis of interviews. Literature on the use of grounded theory to analyse classroom observation is sparse. While the use of memos is encouraged, how memos can be used to analyse classroom observation is difficult to find. The memos I used comprised creating posters for each lesson. Figure 11.7 is a picture of a poster I used to analyse IEs provided by my sample of teachers.

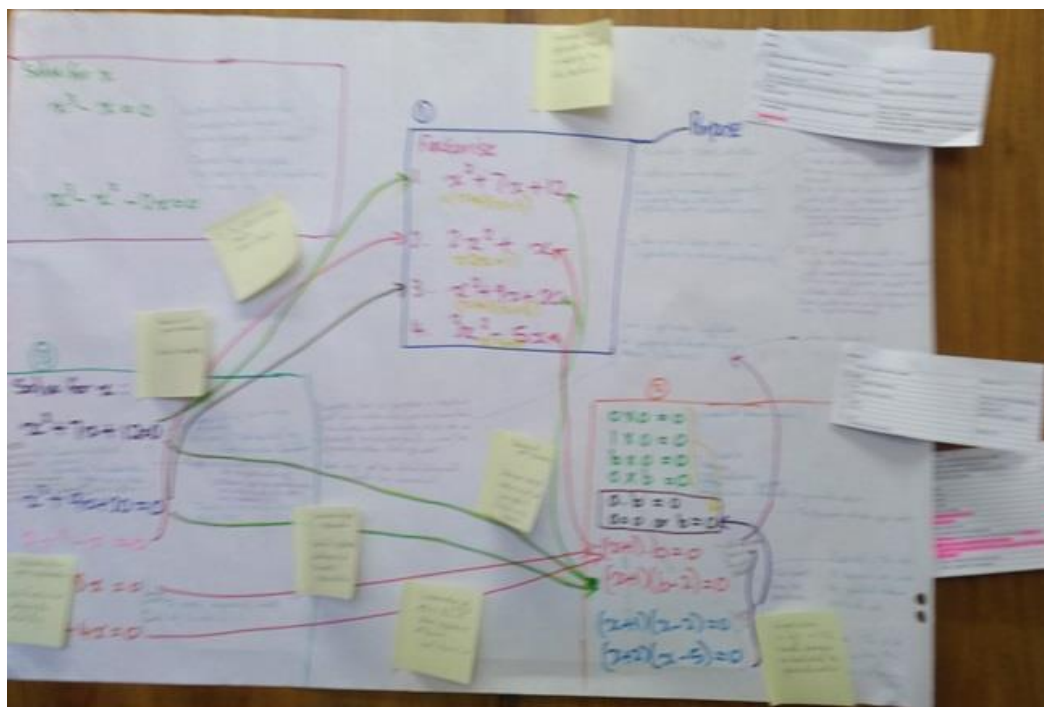


Figure 11.7: Example of a poster used in my analysis using grounded theory

The posters proved to be a vital analytical tool. The detail of how I used a poster in analysing data from classroom observations will be of benefit to other researchers using grounded theory to analyse classroom observation.

11.5. Significance of the study and recommendations

This research comes at a time when the performance of South African learners in the latest Trends in International Mathematics and Science Study (TIMSS) reflects that the majority of learners in Grade 9 performed below the level of 400 TIMSS points (Reddy et al., 2020). This means that learners have not acquired the basic mathematical knowledge in the particular grade. One of the findings was that 51% of all learners (irrespective of the kind of schools they attend) were found to not have acquired the basic mathematical skills in the grade they were in. One of the recommendations, made at the release of the TIMSS results, was that more needed to be learnt about what goes on inside the classroom, and that there should be a renewed call for the study of teachers' practices as enacted in the classroom.

Studying the enactment of teachers' IEs can ultimately help other teachers enhance their IEs. My recommendation is that the findings of this study be communicated by policymakers, PD providers and other stakeholders to teachers. The findings can be arranged into packages to be conveyed to teachers in a form that can be easily understood and used by teachers in the

classroom. In this regard it is vital that teachers have access to examples of how they can enhance the IEs they offer. Hiebert, Gallimore, and Stigler (2002) call for a *knowledge base* of the practice of teaching. They use the words *knowledge base* to refer to the accumulation of insights, knowledge and experiences of teachers in their practices to enhance instruction. The findings of this study can add to the knowledge base of teachers to help teachers develop more effective strategies to improve their IEs.

This call is echoed by Jansen (2021), who argues that one of the reasons for the persistence of poor teaching in South Africa is that some teachers have never been exposed to good mathematics lessons. He maintains that “the majority of our teachers have never seen a truly outstanding lesson in their lives as pupils, student teachers or professionals. In other words, they do not know what outstanding teaching looks like” (Jansen, 2021). He adds that teachers can benefit from the modelling of good teaching by offering resources to upscale teaching.

My recommendation is therefore that the findings of this study can be used to add to a repository that could offer such resources to teachers. It is envisaged that teachers use some of the strategies not as recipes but strategies that can be modified to their contexts and used for a particular purpose; strategies that are meaningful and practical. This repository can be used not only by teachers but also other stakeholders like subject advisors, professional development practitioners and other researchers.

11.6. Limitations of this study

There are three notable limitations to this study that I outline below.

While my research focuses on how the use of examples, the chalkboard as well as the utterances of a teacher, combine to make it possible for learners to perceive the ooL of an IE, the particular language used by a teacher in offering IEs is not in focus. Specific word use, whether mathematical or non-mathematical, was not in focus and is a limitation in this study. My focus was on how the utterances of teachers supported what was written and how it was written on the chalkboard to make the mathematical idea in focus explicit.

The study was a focus on teachers. I therefore did not consider how what was intended by the teachers IEs was perceived by learners. Neither learners’ classwork nor their performance in tests or examinations were in focus during the study. My focus was on what teachers made possible for learners to learn as a necessary first step before looking at their performance. The findings of this study can be used to inform future studies on the performance of learners.

The CoMIE framework that was developed offers a way to analyse the coherence of teachers’ IEs. I provided an example of the use of the framework to analyse the coherence of an IE provided by a teacher in this study. This was, however, just one example. The robustness of the

framework in evaluating the coherence of IEs requires its testing through other examples of IEs. This is a necessary next step in order to adapt and improve on the framework.

11.7. Conclusion

My initial thoughts after the analysis of the teachers' IEs was that the findings were banal. The use of examples, the chalkboard and teacher talk are commonplace in all mathematics classrooms. It is, however, precisely because they are so commonplace that they are not given the necessary consideration. I have always taken these lenses, through which I examined IEs, for granted. The findings of this study provoked my own desire to rethink how I have previously used these lenses of an IE. For instance, my own writing on the chalkboard could be regarded as very haphazard, i.e. writing wherever there was space, with no thought given to arranging closely related ideas to suggest their connections. Similarly, a deliberate arrangement of the content to promote connections between different IEs or even within an IE, was not a deliberate focus in my lesson planning. In addition, the presentation of my findings at conferences attracted a high level of engagement. This indicated that other researchers were also connecting with teacher explanations. The findings of this study therefore provided me with renewed impetus to pay more attention to these features of IEs in the classroom.

In chapter 1, I made mention of my desire to return to the classroom after being a part of the Transition Maths 1 (TM1) course and introduction to the Mathematics Teaching Framework (MTF). The TM1 course together with the MTF brought into focus for me how to deliberately develop sets of examples so that the ooL is in focus. It also made me aware of how the language used in the classroom can promote or confuse learners. I feel even more strongly about returning to the classroom at the conclusion of this study.

This research journey had the unintended effect of imbuing me with a renewed passion to return to the classroom in order to determine whether these findings have the desired effect on learners that my findings suggest they will. Although I cannot make assumptions about the correlation between what is made possible to learn and what learners actually learn, the findings of this study does suggest that IEs can be enhanced by teachers paying particular attention to example selection and sequencing, their inscriptions and talk. The findings of this study can be used as a starting point for further studies on IEs and their coherence and how this impacts learning.

References

- Achinstein, P. (1988). The Illocutionary Theory of Explanation. In J. Pitt (Ed.), *Theories of Explanation* (pp. 199-222). Oxford: Oxford University Press.
- Adler, J., & Pournara, C. (2019). Exemplifying with Variation and Its Development in Mathematics Teacher Education. In D. Potari & O. Chapman (Eds.), *International Handbook of Mathematics Teacher Education: Volume 1* (pp. 329-353). Netherlands: Brill Sense.
- Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education*, 19(3), 237-254.
- Adler, J., & Venkat, H. (2014). Teachers' mathematical discourse in instruction: Focus on examples and explanations. *Exploring Mathematics and Science Teachers' Knowledge* (pp. 154-168). Routledge.
- Adler, J. B. (1996). *Secondary school teachers' knowledge of the dynamics of teaching and learning mathematics in multilingual classrooms*. University of the Witwatersrand.
- Andrews, P. (2009). Mathematics teachers' didactic strategies: Examining the comparative potential of low inference generic descriptors. *Comparative Education Review*, 53(4), 559-581.
- Askew, M., Venkat, H., & Mathews, C. (2012). *Coherence and consistency in South African primary mathematics lessons*. In *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, No. 1, pp. 27-34). Taipei, Taiwan: PME.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of teacher education*, 59(5), 389-407.
- Barber, J. P., & Walczak, K. K. (2009). *Conscience and critic: Peer debriefing strategies in grounded theory research*. In *Annual Meeting of the American Educational Research Association* (pp. 13-17), San Diego, CA.
- Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). *Exemplification in mathematics education*. In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 126-154). Prague: PME.
- Bogdan, R. C., & Biklen, S. K. (2007). *Qualitative research for education: An introduction to theories and methods* (5th ed.). Boston, MA.
- Bossé, M. J. (2003). The Beauty of "and" and "or": Connections Within Mathematics for Students with Learning. *Mathematics & Computer Education*, 37(1), 105-114.
- Businskas, A. M. (2008). *Conversations about connections: How secondary mathematics teachers conceptualize and contend with mathematical connections*. (Doctoral dissertation, Faculty of Education-Simon Fraser University).
- Cai, J., Ding, M., & Wang, T. (2014). How do exemplary Chinese and U.S. mathematics teachers view instructional coherence? *Educational Studies in Mathematics*, 85(2), 265-280. doi: 10.1007/s10649-013-9513-3
- Carlsen, A., & Dutton, J. E. (2011). Research alive: The call for generativity. In A. Carlsen & J. E. Dutton (Eds.), *Research alive: Exploring generative moments in doing qualitative research* (pp. 12-24). Copenhagen: Copenhagen Business School Press.
- Charalambous, Y., Hill, H. C., & Ball, D. L. (2011). Prospective teachers' learning to provide instructional explanations: how does it look and what might it take? *Journal of mathematics teacher education*, 14(6), 441-463.
- Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*: Sage.
- Chen, X., & Li, Y. (2010). Instructional coherence in Chinese mathematics classroom—A case study of lessons on fraction division. *International Journal of Science and Mathematics Education*, 8(4), 711-735.

-
- Chenail, R. J. (2012). Conducting Qualitative Data Analysis: Reading Line-by-Line, but Analyzing by Meaningful Qualitative Units. *Qualitative Report*, 17(1), 266-269.
- Cheng, L. P., Leong, Y. H., & Toh, W. Y. K. (2021). Sequencing of Practice Examples for Mathematical Reasoning: A Case of a Singapore Secondary School Teacher's Practice. *Mathematics Instructional Practices in Singapore Secondary Schools* (pp. 249-277). Singapore: Springer.
- Cohen, L., Manion, L., & Morrison, K. (2011). Planning educational research. *Research methods in education*. New York: Routledge Editors.
- Cohen, L., Manion, L., & Morrison, K. (2013). *Research methods in education*. Routledge.
- Cohen, L., Manion, L., & Morrison, K. (2017). *Research methods in education*. Routledge.
- Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: SAGE Publications.
- Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Sage publications.
- Corbin, J., & Strauss, A. (2015). *Basics of qualitative research* (4th ed.). Sage.
- Creswell, J. W., & Miller, D. L. (2000). Determining Validity in Qualitative Inquiry. *Theory Into Practice*, 39(3), 124-130.
- Crowe, S., Cresswell, K., Robertson, A., Huby, G., Avery, A., & Sheikh, A. (2011). The case study approach. *BMC Medical Research Methodology*, 11(1), 1-9.
- Denzin, N. K., & Lincoln, Y. S. (2011). *The Sage handbook of qualitative research*. Sage.
- Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, 24(1), 8-40.
- Erath, K., Prediger, S., Quasthoff, U., & Heller, V. (2018). Discourse competence as important part of academic language proficiency in mathematics classrooms: the case of explaining to learn and learning to explain. *Educational Studies in Mathematics*, 99(2), 161-179. doi: 10.1007/s10649-018-9830-7
- Ermeling, B. (2015). Lessons Learned from a Chalkboard: Slow and Steady Technology Integration. *Teachers College Record*.
- Esmonde, I. (2009). Explanations in Mathematics Classrooms: A Discourse Analysis. *Canadian Journal of Science, Mathematics and Technology Education*, 9(2), 86-99. doi: 10.1080/14926150902942072
- Essien, A. A. (2021). *Rethinking exemplification in mathematics teacher education multilingual classrooms*. Exploring new ways to connect: Proceedings of the Eleventh International Mathematics Education and Society Conference, 11, 56-60.
- Etikan, I., Musa, S. A., & Alkassim, R. S. (2016). Comparison of convenience sampling and purposive sampling. *American journal of theoretical and applied statistics*, 5(1), 1-4.
- Evens, H., & Houssart, J. (2004). Categorizing pupils' written answers to a mathematics test question: 'I know but I can't explain'. *Educational Research*, 46(3), 269-282. doi: 10.1080/0013188042000277331
- Evitts, T. A. (2004). *Investigating the mathematical connections that preservice teachers use and develop while solving problems from reform curricula*. The Pennsylvania State University.
- Gibbs, G. (2007). *Analysing qualitative data*. London: Sage.
- Glaser, B. G., & Strauss, A. L. (1967). *Discovery of grounded theory: Strategies for qualitative research*. Routledge.
- Gonzalez, C., & Magally, V. (2013). *Developing skills to explain scientific concepts during initial teacher education: the role of peer assessment*. PhD thesis, University of Dundee.
- Greenhalf, K. (2015). The rise of the classroom blackboard. Retrieved from <https://blogs.ubc.ca/etec540sept13/2013/10/27/the-rise-of-the-classroom-blackboard/>
-

-
- Guba, E. G., & Lincoln, Y. S. (1994). Competing paradigms in qualitative research. *Handbook of qualitative research*, 2(163-194), 105.
- Hargie, O. (2010). *Skilled interpersonal communication: Research, theory and practice*: Routledge.
- Hiebert, J., Gallimore, R., Garnier, H., Givvin, K., Hollingsworth, H., Jacobs, J., . . . Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study*: DiaNe Publishing.
- Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A Knowledge Base for the Teaching Profession: What Would It Look Like and How Can We Get One? *Educational Researcher*, 31(5), 3-15. doi: 10.3102/0013189x031005003
- Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and instruction*, 26(4), 430-511.
- Huang, R., & Leung, F. K. (2017). Teaching geometrical concepts through variation: A case study of a Shanghai lesson. *Teaching and learning mathematics through variation* (pp. 151-168). Brill.
- Hughes, P. (2010). Paradigms, methods and knowledge. *Doing early childhood research: International perspectives on theory and practice*, 2, 35-61.
- Inoue, N. (2009). Rehearsing to teach: content-specific deconstruction of instructional explanations in pre-service teacher training. *Journal of Education for Teaching*, 35(1), 47-60. doi: 10.1080/02607470802587137
- Jansen, J. (2021, 10 February 2021). What SA's kids need are top-notch teachers. This is how to get them, *Sunday Times Daily*. Retrieved from <https://www.timeslive.co.za/sunday-times-daily/opinion-and-analysis/2021-02-10-jonathan-jansen--what-sas-kids-need-are-top-notch-teachers-this-is-how-to-get-them/>
- Kanter, R. M. (2011). Managing yourself: Zoom in, zoom out. *Harvard business review*, 89(3), 112-116.
- Keil, F. C., & Wilson, R. A. (2000). Explaining explanation. *Explanation and cognition*, 1-18.
- Kinach, B. M. (2002). A cognitive strategy for developing pedagogical content knowledge in the secondary mathematics methods course: Toward a model of effective practice. *Teaching and Teacher Education*, 18(1), 51-71.
- Kress, G., Charalampos, T., Jewitt, C., & Ogborn, J. (2006). *Multimodal teaching and learning: The rhetorics of the science classroom*. Bloomsbury Publishing.
- Kullberg, A., Runesson Kempe, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM*, 49(4), 559-569. doi: 10.1007/s11858-017-0858-4
- Leatham, K. R. (2019). *Designing, conducting, and publishing quality research in mathematics education*. Springer.
- Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal for research in Mathematics Education*, 20(1), 52-75.
- Leinhardt, G. (1990). *Towards Understanding Instructional Explanations*. Pittsburgh, PA: University of Pittsburgh.
- Leinhardt, G. (1997). Instructional explanations in history. *International Journal of Educational Research*, 27(3), 221-232.
- Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 333-357). American Educational Research Association.
- Lincoln, Y. S., & Guba, E. G. (1986). But is it rigorous? Trustworthiness and authenticity in naturalistic evaluation. *New Directions for Program Evaluation*, 1986(30), 73-84. doi: 10.1002/ev.1427
- Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*. Göteborg: Acta Universitatis Gothoburgensis.
-

-
- Lo, M. L., & Marton, F. (2011). Towards a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal for Lesson and Learning Studies*, 1(1), 7-22.
- Luxomo, N. N. (2019). *A study of the constitutive criteria for algebraic explanations and acts of explaining from a professional development course and in teachers' practices*. PhD, University of the Witwatersrand, Unpublished.
- Ma, L. (2010). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Routledge: New York.
- Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning Sciences*, 15(2), 193-220.
- Marton, F., & Pang, M. F. (2013). Meanings Are Acquired from Experiencing Differences against a Background of Sameness, Rather than from Experiencing Sameness against a Background of Difference: Putting a Conjecture to the Test by Embedding It in a Pedagogical Tool. *Frontline Learning Research*, 1(1), 24-41.
- Marton, F., Tsui, A. B., Chik, P. P., Ko, P. Y., & Lo, M. L. (2004). *Classroom discourse and the space of learning*. Routledge.
- Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. Jossey-Bass.
- Morgan, C. (1998). *Writing mathematically: The discourse of investigation*. Falmer Press.
- Morrow, S. L. (2007). Qualitative research in counseling psychology: Conceptual foundations. *The counseling psychologist*, 35(2), 209-235.
- Noss, R., Healy, L., & Hoyles, C. (1997). The Construction of Mathematical Meanings: Connecting the Visual with the Symbolic. *Educational Studies in Mathematics*, 33(2), 203-233. doi: 10.1023/a:1002943821419
- Ontario, M. o. E. (2011). *Bansho (Board Writing)*. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_bansho.pdf.
- Onwuegbuzie, A. J., & Leech, N. L. (2007). Sampling designs in qualitative research: Making the sampling process more public. *The qualitative report*, 12(2), 238-254.
- Opie, C., & Sikes, P. J. (2004). *Doing educational research*. Sage.
- Orb, A., Eisenhauer, L., & Wynaden, D. (2001). Ethics in qualitative research. *Journal of nursing scholarship*, 33(1), 93-96.
- Pang, M. F., Bao, J., & Ki, W. W. (2017). 'Bianshi' and the variation theory of learning: Illustrating two frameworks of variation and invariance in the teaching of mathematics. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation* (pp. 43-67). Rotterdam: Sense Publishers.
- Panhwar, A. H., Ansari, S., & Shah, A. A. (2017). Post-positivism: An effective paradigm for social and educational research. *International Research Journal of Arts & Humanities (IRJAH)*, 45(45).
- Planas, N. (2020). Study of and work on mathematics teacher talk in classroom teaching. *EUIN Working Paper*.
- Polit, D. F., & Beck, C. T. (2013). *Study guide for essentials of nursing research: appraising evidence for nursing practice*. Lippincott Williams & Wilkins.
- Ragin, C. C., & Becker, H. S. (1992). *What is a case?: exploring the foundations of social inquiry*. Cambridge university press.
- Raiker, A. (2002). Spoken Language and Mathematics. *Cambridge journal of education*, 32(1), 45-60. doi: 10.1080/03057640220116427
- Reddy, V., Winnaar, L., Juan, A., Arends, F., Harvey, J., Hannan, S., . . . Zulu, N. (2020). *TIMSS 2019: Highlights of South African Grade 9 results in mathematics and science. Achievement and achievement gaps*. Pretoria.

-
- Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics*, 69(2), 149-163.
- Rowland, T., & Zaslavsky, O. (2005). Pedagogical example-spaces. Paper presented at the *Miniconference on exemplification in mathematics*. Oxford University, PME.
- Sandelowski, M. (1993). Rigor or rigor mortis: the problem of rigor in qualitative research. *Advances in nursing science*, 16(2), 1-8.
- Schwandt, T. A. (2001). *Dictionary of qualitative inquiry (2nd Edition)*. Thousand Oaks: Sage.
- Schwerdt, G., & Wuppermann, A. C. (2011). Is traditional teaching really all that bad? A within-student between-subject approach. *Economics of Education Review*, 30(2), 365-379. doi: <https://doi.org/10.1016/j.econedurev.2010.11.005>
- Scott, P., Mortimer, E., & Ametller, J. (2011). Pedagogical link-making: a fundamental aspect of teaching and learning scientific conceptual knowledge. *Studies in Science Education*, 47(1), 3-36.
- Sedibe, M. (2011). Inequality of Access to Resources in Previously Disadvantaged South African High Schools. *Journal of Social Sciences*, 28(2), 129-135. doi: 10.1080/09718923.2011.11892937
- Sekiguchi, Y. (2006). Coherence of mathematics lessons in Japanese eighth-grade classrooms. In *Proceedings of 30 th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 81-88).
- Sharan, B. (2009). *Qualitative research: A guide to design and implementation*. Wiley.
- Shenton, A. K. (2004). Strategies for ensuring trustworthiness in qualitative research projects. *Education for Information*, 22(2), 63-75. doi: 10.3233/efi-2004-22201
- Shimizu, Y. (2007). Explicit linking in the sequence of consecutive lessons in mathematics classroom in Japan. Paper presented at the *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Seoul: PME
- Sidney, P. G., & Alibali, M. W. (2015). Making Connections in Math: Activating a Prior Knowledge Analogue Matters for Learning. *Journal of Cognition and Development*, 16(1), 160-185. doi: 10.1080/15248372.2013.792091
- Stake, R. E. (2010). *Qualitative research: Studying how things work*. Guilford Press.
- Stigler, J. W., & Perry, M. (1988). Mathematics learning in Japanese, Chinese, and American classrooms. *New Directions for Child and Adolescent Development*, 1988(41), 27-54.
- Strauss, A., & Corbin, J. (1990). *Basics of qualitative research*. Sage publications.
- Strauss, A., & Corbin, J. (1994). Grounded theory methodology. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 273-285). Sage Publications.
- Tan, S., Fukaya, K., & Nozaki, S. (2018). Development of bansho (board writing) analysis as a research method to improve observation and analysis of instruction in lesson study. *International Journal for Lesson and Learning Studies*, 7, 230-247.
- Thagard, P. (1989). Explanatory coherence. *Behavioral and brain sciences*, 12(3), 435-502.
- Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction. *Pythagoras*, 33(3), 1-8.
- Venkat, H., & Askew, M. (2018). Mediating primary mathematics: theory, concepts, and a framework for studying practice. *Educational Studies in Mathematics*, 97(1), 71-92.
- Vollstedt, M. (2015). To see the wood for the trees: The development of theory from empirical interview data using grounded theory *Approaches to qualitative research in mathematics education* (pp. 23-48). Springer.
- Wang, T., & Murphy, J. (2004). An examination of coherence in a Chinese mathematics classroom. In L. Fan, N. Y. Wong, J. Cai & S. Li (Eds.), *How Chinese learn mathematics* (pp. 107-123). New Jersey: World Scientific.
-

-
- Watson, A., & Mason, J. (2002). *Extending example spaces as a learning/teaching strategy in mathematics*. Paper presented at the Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education, Norwich: UK.
- Weinberg, S. L. (2001). *Is There a Connection between Fractions and Division? Students' Inconsistent Responses*. Paper presented at the Annual Meeting of the American Educational Research Association, Seattle: WA.
- Windschilt, M., Thompson, J., Braaten, M., & Stroupe, D. (2012). Proposing a core set of instructional practices and tools for teachers of science. *Science Education*, 96(5), 878-903.
- Wolcott, H. F. (2008). *Writing up qualitative research*: Sage Publications.
- Yazan, B. (2015). Three approaches to case study methods in education: Yin, Merriam, and Stake. *The qualitative report*, 20(2), 134-152.
- Yoshida, M. (2005). Using lesson study to develop effective blackboard practice. In W. Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 93-100). Research for Better Schools.
- Zakaria, E. (2010). Analysis of Students' Error in Learning of Quadratic Equations. *International Education Studies*, 3(3), 105-110.
- Zaslavsky, O. (2010). The explanatory power of examples in mathematics: Challenges for teaching. In M. K. Stein & L. Kucan (Eds.), *Instructional explanations in the disciplines* (pp. 107-128). New York: Springer.
- Zazkis, R. (2000). Factors, divisors and multiples: Exploring the web of students' connections. *Research in collegiate mathematics education*, 4, 210-238.
- Zazkis, R., & Chernoff, E. J. (2008). What makes a counterexample exemplary? *Educational Studies in Mathematics*, 68(3), 195-208.

Appendices

Appendix A

Wits Ethics approval



Research Office

HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)

R14/49 Moodley

CLEARANCE CERTIFICATE

PROTOCOL NUMBER: H17/1/35

PROJECT TITLE

Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics

INVESTIGATOR(S)

Ms V Moodley

SCHOOL/DEPARTMENT

Wits School of Education/

DATE CONSIDERED

17 November 2017

DECISION OF THE COMMITTEE

Approved
Permission letters are required before data collection can commence.

EXPIRY DATE

11 January 2021

DATE

12 January 2018

CHAIRPERSON

(Professor J Knight)

cc: Supervisor : Dr C Pournara

DECLARATION OF INVESTIGATOR(S)

To be completed in duplicate and **ONE COPY** returned to the Secretary at Room 10004, 10th Floor, Senate House, University. Unreported changes to the application may invalidate the clearance given by the HREC (Non-Medical)

I/we fully understand the conditions under which I am/we are authorized to carry out the abovementioned research and I/we guarantee to ensure compliance with these conditions. Should any departure to be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee. **I agree to completion of a yearly progress report.**

Signature _____

Date _____

PLEASE QUOTE THE PROTOCOL NUMBER ON ALL ENQUIRIES

Appendix B

GDE approval letter



GAUTENG PROVINCE

Department: Education
REPUBLIC OF SOUTH AFRICA

8/4/4/1/2

GDE RESEARCH APPROVAL LETTER

Date:	04 December 2017
Validity of Research Approval:	05 February 2018 – 28 September 2018 2017/338
Name of Researcher:	Moodley V
Address of Researcher:	8 Jordaan Drive Eastleigh Edenvale 1609
Telephone Number:	011 452 8914 084 694 4128
Email address:	Vasanth.Moodley@wits.ac.za
Research Topic:	Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.
Number and type of schools:	Three Secondary Schools
District/s/HO	Johannesburg South

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

F. Tshabalala 04/12/2017

1

Making education a societal priority

Office of the Director: Education Research and Knowledge Management

7th Floor, 17 Strimonda Street, Johannesburg, 2001
Tel: 011 355 0488
Email: Faith.Tshabalala@gauteng.gov.za
Website: www.education.gov.za

Appendix C1

Letter to the Principal and SGB

Date: 04/02/2019

Dear Sir/Madam

My name is Vasantha Moodley and I am a PhD student in Mathematics Education at Wits University in Johannesburg. As part of my studies I have to undertake a research project, and I am **Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.**

As you know, some of your teachers have participated in the Wits Maths Connect Transition Mathematics (TM1) professional development course focused on enhancing their mathematics and classroom practices. My study is related to the mathematics teaching framework, a professional development resource discussed in the course. My aim is to find out how teachers construct their own example sets or how they adapt existing example sets as well as how these examples are mediated in the form of explanations in a topic that was covered in the TM1 course as well as a topic that was not a focus in the course.

As part of this project I would like to invite your school to participate in this research to understand how teachers use the mathematics teaching framework that was a focus during the TM1 course. This will involve the video recording of at least 3 consecutive Grade 10 lessons in February and March 2019. In these observations the video recorder will be focused on the teacher and not the learners although the learners will be present. The learners will not be expected to do anything else but attend to the lesson in the normal way. I will thereafter be inviting the teacher to take part in an interview based on the observation of the lesson. This interview will take around 40 minutes and be conducted at the school after school hours or a time and place that is convenient for the teacher. With the teacher's permission, I would like to record the interview using a digital device. There are no direct benefits to you or the school for participating in this study, and equally there are no disadvantages or penalties for not participating. This study will be written up as a research report which will be available online through the university library website. Your school name will not be mentioned in any part of the report. If you wish to receive a summary of this report, I will be happy to send it to you upon request. All research data will be destroyed between 3-5 years after completion of the project.

If you have any questions afterwards about this research, feel free to contact me on the details listed below.

If you have any queries, concerns or complaints regarding the ethical procedures of this study, you are welcome to contact the University Human Research Ethics Committee (non-medical), telephone + 27(0)11 717 1408, email hrec-medical.researchoffice@wits.ac.za/ Shaun.Schoeman@wits.ac.za

Yours sincerely,



Vasantha Moodley

Supervisor: Dr. Craig Pournara

Email: Vasantha.Moodley@wits.ac.za

Email: Craig.Pournara@wits.ac.za

Tel: 011 717 3108

Tel: 011 717 3253

Appendix C2

Parent/Guardian Information Sheet

Date: 04/02/2019

Dear Parent/Guardian

My name is Vasantha Moodley and I am a PhD student in Mathematics Education at Wits University in Johannesburg. As part of my studies I have to undertake a research project, and I am **Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.**

My investigation involves the video recording of three consecutive lessons in which your child/ward will be present, in February and March 2019 (Term 1). I would like to request permission to video record lessons that your child/ward will be present in. Your child/ward will not be expected to do anything else other than to attend to the lesson in the way that s/he would normally do.

I want to assure you that participation of your child/ward in this study is entirely up to you and that you do not have to agree for her/him to be videotaped. If you decide that your child/ward should not be videotaped, I will position her/him in the class where s/he will not be in the range of the camera. S/he will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child/ward will not be paid to participate in this study.

Video recordings will be used only for the analysis of the results by myself and will not be used in any publications and presentations at conferences. Your child/ward will remain anonymous in all writings resulting from this study. This means that your child/ward's name will not be used when referring to anything s/he mentioned in class and her/his individual privacy will be maintained in all published and written data resulting from the study. However, this cannot be guaranteed as the classroom observations will be video recorded. The video recording will be done by the researcher as well as an assistant and therefore anonymity cannot be guaranteed.

This study will be written up as a research report which will be available online through the university library website. If you wish to receive a summary of this report, I will be happy to send it to you upon request. All research data will be destroyed between 3-5 years after completion of the project.

If you have any queries, concerns or complaints regarding the ethical procedures of this study, you are welcome to contact the University Human Research Ethics Committee (non-medical), telephone + 27(0)11 717 1408, email hrec-medical.researchoffice@wits.ac.za/ Shaun.Schoeman@wits.ac.za

Yours sincerely,



Vasantha Moodley

Supervisor: Dr. Craig Pournara

Email: Vasantha.Moodley@wits.ac.za

E-mail: Craig.Pournara@wits.ac.za

Tel: 011 717 3108

Tel: 011 717 3253

Appendix C3

Parent/Guardian Consent Form

Title: Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.

Name of researcher: Vasantha Moodley

Please fill in the reply slip below which should be returned to your child/ward's teacher by

_____.

I agree that my child/ward can participate in this research project.
I understand what my child's/ward's participation will involve. I give consent for the following:

Circle one

I agree that the researcher may use anonymous quotes *
in her research report

YES/NO

I agree for my child/ward to be observed in class.

YES/NO

I agree for my child/ward to be videotaped in class.

YES/NO

I understand that:

- my child's/ward's name and information will be kept confidential and safe and that my child's/ward's name will not be revealed.
- my child's/ward's face may be visible in the video recordings of the lessons but will only be used by the researcher and assistants during analysis of the results and not during presentations and in publications
- all the data collected during this study will be destroyed within 3-5 years after completion of the project.

YES/NO

YES/NO

YES/NO

*Anonymous quotes: Words/sentences that your child/ward may use during the lessons that I may use directly in all presentations or writings resulting from this study but will not contain the name of your child/ward.

Name: _____

Sign: _____

Date: _____

Appendix C4

INFORMATION SHEET LEARNERS

DATE: 04/02/2019

Dear Learner

My name is Vasantha Moodley and I am a doctoral student in Mathematics Education in the School of Education at the University of the Witwatersrand.

I am doing research on how teachers select examples and how they explain these examples during their teaching. My topic is **Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.**

My investigation involves the video recording of three lessons that follow each other in February and March 2019 (Term1). I would like to invite you to participate in this study.

I will be video recording lessons in which you will be present. You will not be expected to do anything else other than to behave in the way you would normally do in the lessons. Your face may be visible in the recording. The video recording will be done by the researcher as well as an assistant and therefore may be seen by at least one other person apart from the researcher.

You do not have to agree to be a part of this study, which means that you do not have to do it. If you do not wish to be video recorded you will be allowed to sit away from the camera. Also, if you decide halfway through that you would like to stop, you are free to not participate and this will not affect you in any way.

You will remain anonymous in all writings related to this study. This means that I will not be using your name or the names of any other learners in your class. If I have to use names, for example, to write about something you said during the lesson, the name will be made up so that you cannot be identified. All collected information will be stored safely and destroyed between 3-5 years after I have completed my project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to be part of this study. You will be required to fill in an assent form where you can or not give consent to participate in this study.

Please feel free to contact me or my supervisor if you have any questions. If you have any further complaints or concerns regarding the study you may phone Shaun Schoeman, telephone 011 717 1408 or ask your parent to email her, Shaun.Schoeman@wits.ac.za.

Thank you



Vasantha Moodley

Email: Vasantha.Moodley@wits.ac.za

Tel: 011 717 3108

Supervisor: Dr. Craig Pournara

E-mail: Craig.Pournara@wits.ac.za

Tel: 011 717 3253

Appendix C5

Learner Assent Form

Title: Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.

Name of researcher: Vasantha Moodley

Please fill in the reply slip below which should be returned to your teacher by

_____.

I agree to take part in this research project. I understand what my participation will involve and I agree to the following:

- | | |
|--|-----------------------------|
| I agree that my participation will remain anonymous | Circle one
YES/NO |
| I agree that the researcher may use anonymous quotes *
in her research report | YES/NO |
| I agree to be observed in class. | YES/NO |
| I agree to be videotaped in class. | YES/NO |
| I understand that: | |
| my name and information will not be used. | YES/NO |
| • my face may be visible in the video recordings of the lessons but
will only be used by the researcher and assistants during analysis of
the results and not during presentations and in publications | YES/NO |
| • all the data collected during this study will be destroyed
3-5 years after the project is completed. | YES/NO |

*Anonymous quotes: Words/sentences that you use during the lessons that I may use directly in all presentations or writings resulting from this study but will not contain your name.

Name: _____

Sign: _____

Date: _____

Appendix C 6

Teacher Information Sheet

Date: 20 February 2018

Dear

My name is Vasantha Moodley and I am a PhD student in Mathematics Education at Wits University in Johannesburg. As part of my studies I have to undertake a research project, and I am **Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.**

As you have participated in the professional development, Transition Maths 1, (TM1) course I would like to invite you to participate in this study. My study is related to the mathematics teaching framework, a professional development resource discussed in the course. My aim is to find out how teachers construct their own example sets or how they adapt existing example sets as well as how these examples are mediated in the form of explanations in a topic that was covered in the TM1 course as well as a topic that was not a focus in the course.

As part of this project I would like to invite you to participate in this research to understand how teachers use the mathematics teaching framework that was a focus during the TM1 course. This will involve the video recording of at least 3 consecutive Grade 10 lessons in late February and again in early March in 2018 (term 1) and repeated in February and March 2019. In these observations the video recorder will be focused on you, the teacher and not the learners, although the learners will be present. The learners will not be expected to do anything else but attend to the lesson in the normal way. I will thereafter be inviting you to take part in a video stimulated recall interview based on the observation of the lessons. This means that you will have an opportunity to watch videos of the observed lessons before the interview is conducted to assist in recalling aspects of the observed lessons. The interview will be conducted at a time and place (possibly after school hours at your school) that is convenient for you and will take around 40 minutes. With your permission, I would also like to record the interview using a digital device.

You will remain anonymous at all times and in all academic writing about the study. This means that your name and identity will be kept confidential and your individual privacy will be maintained. Every effort will be made to try to ensure anonymity and confidentiality, however these cannot be guaranteed as the classroom observations will be video recorded. The video recording will be done by the researcher as well as an assistant and therefore anonymity cannot be guaranteed. I will be using a pseudonym (false name) to represent you and your participation in my final research report. Video recordings will only be used for the analysis of the results by myself and will not be used in any publications and presentations at conferences

You will receive direct benefit from participating in this study as the interviews will assist you in reflecting on your classroom practices. There are no disadvantages or penalties for not participating. You may withdraw at any time or not answer any question if you do not want to. If you experience any distress or discomfort, we will stop the interview or resume at another time. If you have any questions afterwards about this research, feel free to contact me on the details listed below.

This study will be written up as a research report which will be available online through the university library website. If you wish to receive a summary of this report, I will be happy to send it to you upon request. All research data will be destroyed between 3-5 years after completion of the project.

If you have any queries, concerns or complaints regarding the ethical procedures of this study, you are welcome to contact the University Human Research Ethics Committee (non-medical), telephone + 27(0)11 717 1408, email hrec-medical.researchoffice@wits.ac.za/ Shaun.Schoeman@wits.ac.za

Yours sincerely,



Vasantha Moodley

Supervisor: Dr. Craig Pournara

Email: Vasantha.Moodley@wits.ac.za

E-mail: Craig.Pournara@wits.ac.za

Tel: 011 717 3108

Tel: 011 717 3253

Appendix C7

Teacher Consent Form

Title: Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.

Name of researcher: Vasantha Moodley

Please fill in the reply slip below. I will collect it from you by _____ .

I agree to participate in this research project. The research has been explained to me and I have had a chance to ask any questions regarding the study. I understand what my participation will involve. I give my consent for the following:

Circle one

I agree that the researcher may use anonymous quotes*
in her research report

YES/NO

I agree to be observed in class.

YES/NO

I agree to be videotaped in class

YES/NO

I agree to be audiotaped during the interview

YES/NO

I know that the audiotapes will be used for this project only

YES/NO

I agree to be interviewed for this study.

YES/NO

I know that I can stop the interview at any time and don't have to answer all the questions asked.

YES/NO

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- my face will be visible in the video recordings of the lessons but will only be used by the researcher and assistants during analysis of the results and not during presentations and in publications
- all the data collected during this study will be destroyed within 3-5 years after completion of the project.

YES/NO

YES/NO

YES/NO

YES/NO

*Anonymous quotes: Words/sentences that you use during the interview or lessons that I may use directly in all presentations or writings resulting from this study but will not contain your name.

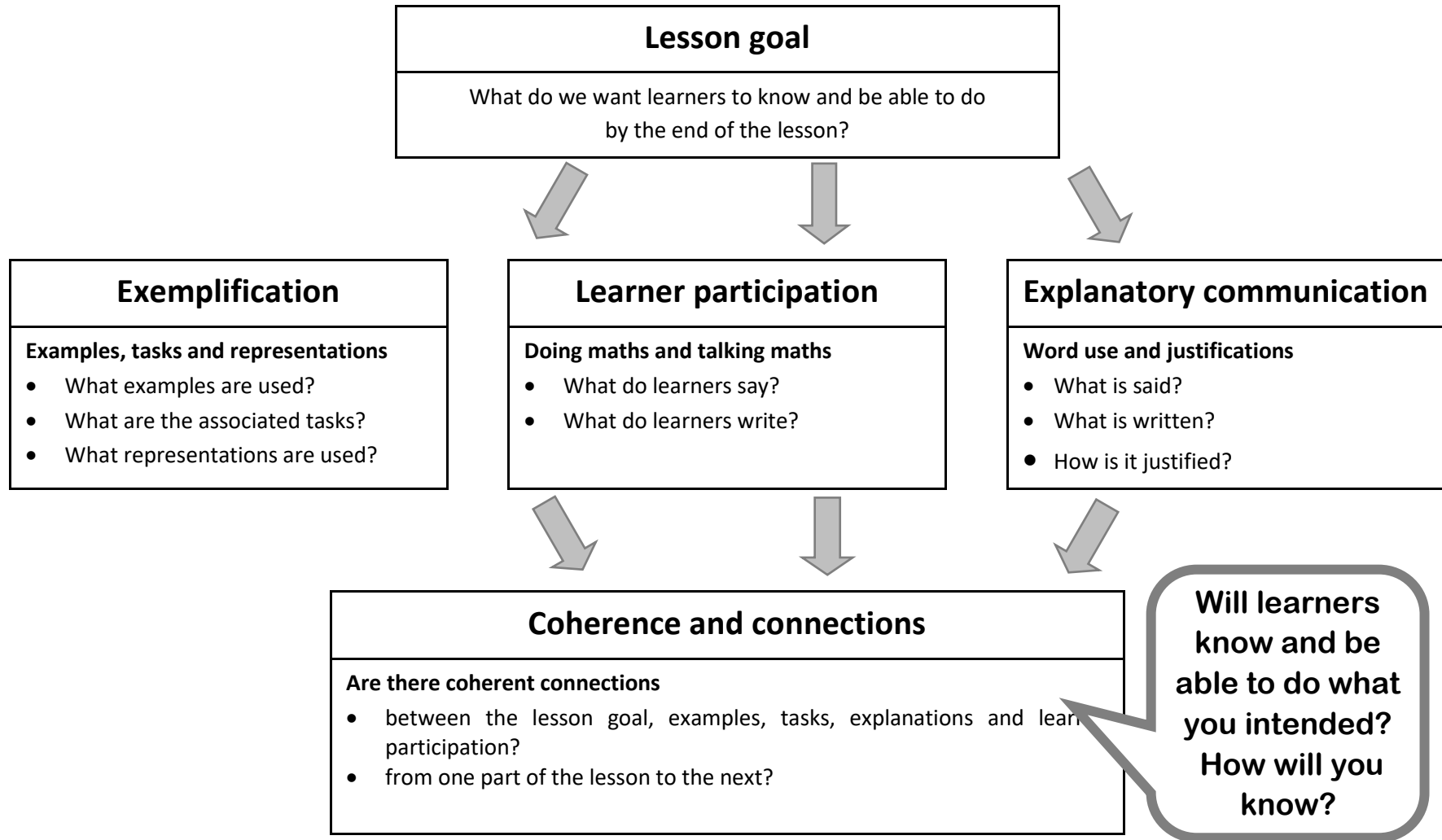
Name: _____

Sign: _____

Date: _____

Appendix D1

Mathematics Teaching Framework (MTF) – Overview








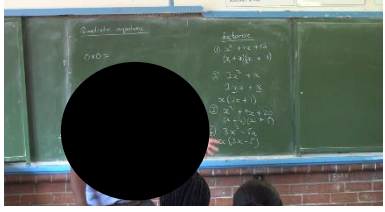

Appendix D2

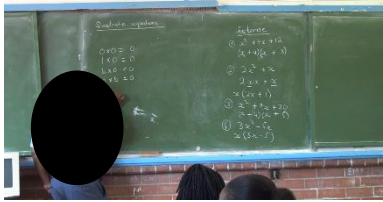

MDI Analytical Tool

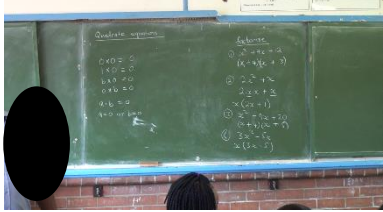


Examples	Exemplification Tasks	Object of learning		Learner participation
		Naming	Explanatory talk Legitimizing criteria	
<p>Examples provide opportunities within an episode or across episodes in a lesson for learners to experience variation in terms of similarity (S), contrast (C), fusion (F)</p>	<p>Across the lesson, learners are required to: <i>carry out known operations and procedures (K)</i> e.g. solve for x, multiply, factorise; <i>apply known skills, and/or decide on operation and/or procedure to use (A)</i>, e.g. compare/classify/ match representations; <i>use multiple concepts and make multiple connections (C/PS)</i>, e.g. solve problems in different ways, use multiple representations, pose problems, prove, reason, etc.</p>	<p>Within and across episodes word use is: <i>colloquial (NM)</i>, e.g. everyday language and/or ambiguous pronouns such as this, that, thing, to refer to objects in focus; <i>math words used as name only (Ms)</i>, e.g. to read string of symbols; <i>mathematical language used appropriately (Ma)</i> to refer to other words, symbols, images, procedures, etc.</p>	<p>Legitimizing criteria: <i>non-mathematical (NM) visual (V)</i>, e.g. cues are iconic or mnemonic; <i>positional (P)</i>, e.g. a statement or assertion, typically by the teacher, as if 'fact'; <i>everyday (E)</i>. <i>Mathematical criteria:</i> <i>local (L)</i>, e.g. a specific or single case (real-life or math), established shortcut, or convention; <i>general (G)</i> equivalent representation, definition, previously established generalization, principles, structures, properties, which can be partial (GP) or 'full' (GF)</p>	<p>Learners answer <i>yes/no questions</i> or <i>offer single words</i> to the teacher's unfinished sentence Y/N. Learners answer (what/how) questions in phrases/sentences (P/S). Learners answer why questions; present ideas in discussion; teacher revoices/ confirms/asks questions (D)</p>
<p>The set of examples provide opportunities in the lesson for learners to experience: Level 1—one form of variation, i.e. S or C; Level 2—at least two forms of variation, S and S OR S and C; Level 3—simultaneous variation (fusion) of more than one aspect of the object of learning and connected with similarity and contrast within the example set (S, C, F). Level 0: simultaneous variation with no attention to similarity and/or contrast</p>	<p>Tasks provide opportunities for: Level 1—carrying out known procedures only (K); Level 2—K and/or some application A; Level 3—K and/or A and C/PS L2 → L1: A → K or C/PS → K is assigned to tasks set up at level 2 or 3 but then reduced to 1 when it unfolds</p>	<p>Use of colloquial and mathematical words: Level 1—NM, there is no focused math talk, all colloquial/everyday; Level 2—movement between NM and Ms, some Ma; Level 3—movement between colloquial NM and formal math talk Ma</p>	<p>Criteria for what counts as mathematics that emerge over time in a lesson and provide opportunity for learning geared towards scientific concepts. Level 0—all criteria are NM, i.e. V, P, E; Level 1—criteria include L, e.g. single case; Level 2—criteria extend beyond NM and L to include generality, but this is partial GP; Level 3—GF math legitimation of a concept or procedure is principled and/or derived/ proved</p>	<p>Opportunity for learners to speak and so use math discourse is at: Level 1—Y/N only (single words only); Level 2—at least some P/S in more than one episode (phrases and sentences); Level 3—P/S and at least some D (discussion) in more than one episode</p>


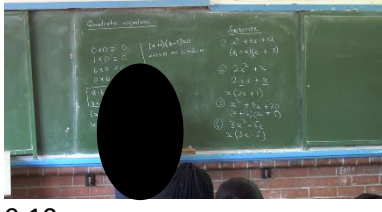
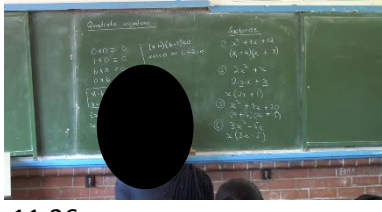

Appendix E: Example of a transcript

Time	Who	What is said	What is done	Chalkboard work
2:07	Teacher	Alright is everyone ready?		
	Learners	No, yes.		
	Teacher	Ok you write today's date. And then let's start by factorising. Okay do you remember how to factorise this?	Writing on the board	 2:15
	Learners	Yes.		
	Teacher	Obviously, I am going to have two brackets. The x squared means I have got an x and 12, so I am looking for factors of 12, who can give me some of them.		
	Learners	4 and 3		
	Teacher	4 and 3 correct. 4 and 3, then my last sign is positive, the signs are the same and then the middle sign. Do you remember this?		
	Learners	Yes sir.		
	Teacher	Okay. Okay factorising this one also. Highest common, can you all see it? Guys are we together? Your book is on the floor (pointing to the floor)		 2:50  3:02
	Learner	Sir it's just ...[inaudible 00:03:07] not important-about book on floor		
	Teacher	Guys do remember how to factorise this?		
	Learner	Yes.		

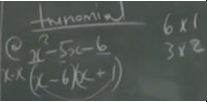
Time	Who	What is said	What is done	Chalkboard work
	Teacher	Honestly I can write it as 2 and expand it, x time x plus x . Then I look for the highest common factor between the terms.		 3:16
	Learners	x		
	Teacher	x . I add my x . Plus what am I left with? What am I going to put there?		
	Learners	1		
	Teacher	1		
	T	All together? Ok try this one yourself (writes $x^2 + 9x + 20$ as number 2. Then writes no.4 as $3x^2 - 5x$) (gives L's time to work on their own: 3:56-6:03)		
6:03	Teacher	What is 0 time 0?		 6:07
	Learners	0. Undefined.		
	Teacher	Undefined?		
	Learners	0.		
	Teacher	0? Guys b times zero?	Writes $b \cdot 0$ on board	
	Learners	Zero.		
	Teacher	Are you sure?		 6:36
	Learner	Any number multiplied by zero is zero. (A learner called out as Tsebo wrote 0 to question $0 \times b$ on the board)		
	Learners	Yes we know that. (shouted out by another learner)	Then writes $0 \cdot b = 0$	

Time	Who	What is said	What is done	Chalkboard work
	Teacher	Okay what do you notice there?	Pointing to all 4 examples on the board	
	Learners	Zero.		
	Teacher	Can you notice zero. What about it?		
	Learner	It's zero. (called out by a learner)		
	Another learner	Any number multiplied by 0 is 0		
	Teacher	(Repeats) Any number multiplied by zero is?		
	Learners	Zero.		
	Teacher	(Interrupted by L who walks in late) Okay this means people, so if I am multiplying two things, and I get a zero, one of them must be?		 7:12
	Learners	Zero.		
	Teacher	So which one is zero there?	Pointing to $a \cdot b = 0$ as next e.g.	
	Learners	1. Either way. A.		
	Teacher	It is A?		
	Learners	Or B.		
	Teacher	A or B?		 7:37
	Learners	It is either way. It's A.		
	T	(Writes $a=0$ or $b=0$ underneath $a \cdot b = 0$)		

Time	Who	What is said	What is done	Chalkboard work
	Teacher	Okay which, right wait, wait, Michela says I am wrong just explain.		 <p>7:47</p>
	Learners	No sir,.		
	T	(Writes $(x+1).b=0$)		
	T	Now what does that mean?		
	L	the B is a zero		
	Teacher	The B is the zero?		
	Learners	Yes sir. Because you take it to the other side.		
	Teacher	Okay look here? Grade 7's look here? We said it is A time B is equal to zero (while drawing a rectangle around $a.b=0$ and below it $a=0$ or $b=0$).		 <p>8:17</p>
	Learners	So it is x time 1 is ...[intervened]		
	Teacher	It means that A is zero or B is Zero, so now I am saying x plus 1 times B is equals to 0. So now I'm saying x plus 1 times b equals zero so what does that mean?		 <p>8:26</p>
	Learners	x plus 1 is zero or, B is equal to zero.	As T writes $x+1=0$ or $b=0$ on the board below the e.g	
	Teacher	Are you sure?		

Time	Who	What is said	What is done	Chalkboard work
	Learners	Yes.		
	Teacher	Okay let's do one more. x plus 1.	Writes $(x+1)(b-2)=0$	 8:53
	Learners	Equals zero.		
	Teacher	Or?		
	Learners	B minus 2.		 9:10
	Teacher	Equals to zero. Write this down	He writes this on board	
		Gives them time to write from 9:12-1:35. Then erases work on board and writes solve for x and below $(x+1)(x-2)=0$		
11:30	Teacher	Okay that one is, now the question is going to say solve for x . Now I have given you two brackets plus two factors. 2 factors are being multiplied and I'm getting naught. What does this mean?		 11:36
	Learners	x plus ...[intervened]		
	Teacher	x plus 1 ...[intervened]		
	Learners	Equals zero or x minus 2 equals zero.		
	Teacher	Now guys can you see I have got two linear equations there?		 11:53

Appendix F: Example of analysis using the CoMIE Framework

Quadratic Equations										
Teacher, lesson number and Episodes	ooL	DP	HKI	PJ	APNK	ALD	IdCU	Comment	Result	
Vutivi Lesson 1	E p 1	Factorisation of expressions i.e. trinomials, removal of a common factor, difference of 2 squares	Procedure for each method explained verbally with examples.	A focus on how to obtain the correct signs and factors when factorising trinomials Difference between an expression and equation was emphasised verbally How to check if factors are correct by multiplying out the factors	<ul style="list-style-type: none"> Explanation included why the relevant signs in each factor were used Focused on why x was a common factor in $x^2 - 2x$ by breaking up x^2 into its factors i.e. $x \cdot x$ and breaking up $2x$ into $3 \cdot 4 \cdot x \cdot x$ in 	This was a topic completed previously but Vutivi did not assume that learners remembered how to factorise. He attended to an example of each of the different methods of factorisation.	<p>Focus on not only the signs of factors when factorising trinomials but also which set of factors to use when there are many sets of factors e.g. in</p>  <p>$x^2 - 5x - 6$, he explained why the factors of 6 were 6 and 1 and not 3 and 2.</p>	By explaining the difference between an expression and an equation, he made it possible for learners to perceive that the expression does not need to be solved i.e. no need to solve for the unknown letter.	addresses 3 foci i.e. learners, mathematical content and connections- all worked together in bringing into focus the ooL of the IE. ooL in focus- all three methods of factorisation explained in detail i.e. signs (of each factor in trinomials) and how to obtain factors not only when factorising trinomials but also when removing a common factor e.g. $x^2 - 3x$ can be written as $x \cdot x - 3 \cdot x$	Coherent-

Quadratic Equations

Teacher, lesson number and Episodes	ooL	DP	HKI	PJ	APNK	ALD	IdCU	Comment	Result
E p 2	How to use the zero product law when solving quadratic equations by factorisation	Procedure to use the zero product law in equating each factor to zero explained	Difference between equation and expression was again explained Each factor is equal to zero if the prod of factors is zero Mentioned that learners should know what the standard form is but did not explain what standard form meant or how to recognise the standard form A quadratic equation has 2 solutions Zero product law mentioned as Zero factorial Verbal and written do not correspond	No justification provided as to why each factor is equated to zero	Did use method of factorisation used before Did not however draw attention to the use of the same expressions which were factorised earlier in lesson.	The use of the word "or" was explained. However this was not clear since he mentioned that only one of them would be correct and which would be done in the next year.	Did not differentiate between a linear and quadratic and why the method of solving changes How to identify a quadratic eq was not in focus	Some of the features within the 3 focii were partially in focus because they were not verbalised or written. The ooL may therefore not be perceived by learners. Learners had to recognise by themselves that a quadratic equation is a second degree polynomial - it may not come to the fore for learners who may not be able to recognise this from all the examples done. ooL in focus	Partially coherent



Appendix G1 Amendments to Enrolment Undergraduate/Postgraduate

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

Please return this form to the Faculty of Humanities

SECTION 1: CURRENT DETAILS (All applicants must complete this section)

Last Name/Surname (as per Identity Document)	Moodley
First Name	Vasantha
Level of Study	PhD

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

Requests for:

i) Extension of time for submission of research proposal	From	To
ii) Extension of time for submission of research for examination	From	To
iii) Give details of any previous extensions	From	To
iv)* Registration to be put in abeyance	From	To
v) Give details if your registration has been put in abeyance before	From	To

vi) Title of research to be changed

From: Investigating Secondary Mathematics Teachers' use of exemplification and explanatory communication and the affordances these provide for learning mathematics.

To : An investigation into the nature and coherence of mathematical instructional explanations

Appendix G2

Subject: PhD Title Amendment Form Vasantha Moodley.doc

Hi Madile

Pls find attached a request for change of title for Vasantha Moodley.

Thanks

Craig

Madile Ntebe <madile.ntebe@wits.ac.za>

Sep 30, 2021, 12:00 PM

to Craig, me, Mike ▾

Dear Craig

Receipt acknowledged and the title has been amended accordingly.

You are *welcome to check the Wits and Humanities Webpages* for further info.

<http://www.wits.ac.za/humanities> Your patience is appreciated.

Regards,

Madile Ntebe (Mrs)

**Faculty Officer
Wits School of Education**