ABSTRACT

A derivation is given of the constitutive equation for an incompressible transversely isotropic hyperelastic material in which the direction of the anisotropic director is unspecified. The field equations for a transversely isotropic incompressible hyperelastic material are obtained.

Nonlinear radial oscillations in transversely isotropic incompressible cylindrical tubes are investigated. A second order nonlinear ordinary differential equation, expressed in terms of the strain-energy function, is derived. It has the same form as for radial oscillations in an isotropic tube. A generalised Mooney-Rivlin strain-energy function is used.

Radial oscillations with a time dependent net applied surface pressure are first considered. For a radial transversely isotropic thin-walled tube the differential equation has a Lie point symmetry for a special form of the strain-energy function and a special time dependent applied surface pressure. The Lie point symmetry is used to transform the equation to an autonomous differential equation which is reduced to an Abel equation of the second kind. A similar analysis is done for radial oscillations in a tangential transversely isotropic tube but computer graphs show that the solution is unstable. Radial oscillations in a longitudinal transversely isotropic tube and an isotropic tube are the same. The Ermakov-Pinney equation is derived.

Radial oscillations in thick-walled and thin-walled cylindrical tubes with the Heaviside step loading boundary condition are next investigated. For radial, tangential and longitudinal transversely isotropic tubes a first integral is derived and effective potentials are defined. Using the effective potentials, conditions for bounded oscillations and the end points of the oscillations are obtained. Upper and lower bounds on the period are derived. Anisotropy reduces the amplitude of the oscillation making the tube stiffer and reduces the period.

Thirdly, free radial oscillations in a thin-walled cylindrical tube are investigated. Knowles(1960) has shown that for free radial oscillations in an isotropic tube, $ab = 1$ where a and b are the minimum and maximum values of the radial coordinate. It is shown that if the initial velocity v_0 vanishes or if $v_0 \neq 1$ but second order terms in the anisotropy are neglected then for free radial oscillations, $ab > 1$ in a radial transversely isotropic tube and $ab < 1$ in a tangential transversely isotropic tube.

Radial oscillations in transversely isotropic incompressible spherical shells are investigated. Only radial transversely isotropic shells are considered because it is found that the Cauchy stress tensor is not bounded everywhere in tangential and longitudinal transversely isotropic shells. For a thin-walled radial transversely isotropic spherical shell with generalised Mooney-Rivlin strain-energy function the differential equation for radial oscillations has no Lie point symmetries if the net applied surface pressure is time dependent.

The inflation of a thin-walled radial transversely isotropic spherical shell of generalised Mooney-Rivlin material is considered. It is assumed that the inflation proceeds sufficiently slowly that the inertia term in the equation for radial oscillations can be neglected. The conditions for snap buckling to occur, in which the pressure decreases before steadily increasing again, are investigated. The maximum value of the parameter for snap buckling to occur is increased by the anisotropy.

DECLARATION

I declare that this thesis is my own unaided work unless otherwise acknowledged. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any other degree or examination to any other institution.

G. H. Maluleke
March 2006

DEDICATION

To my parents, for giving me wings so that I can fly and explore the world.

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Contents

Chapter 1

Introduction

1.1 Introduction

Although a large amount of research has been done on nonlinear radial oscillations in incompressible isotropic cylindrical tubes and spherical shells, comparatively little research has been done on nonlinear radial oscillations in anisotropic cylindrical tubes and spherical shells.

In this thesis nonlinear radial oscillations in incompressible transversely isotropic cylindrical tubes and spherical shells are investigated. The inflation of a transversely isotropic spherical shell in which inertia effects can be neglected is also considered. An elastic material is transversely isotropic with respect to a direction $\boldsymbol{h}$ if its strain-energy function is invariant under rotations about $\boldsymbol{h}$ and reflections in any plane containing $\boldsymbol{h}$. The unit vector $\boldsymbol{h}$ is the anisotropic director. Effects due to the anisotropy of the cylindrical tube and spherical shell will be investigated.

1.2 Transversely isotropic cylindrical tube

Radial, tangential and longitudinal transversely isotropic cylindrical tubes will be considered. In a radial transversely isotropic tube the anisotropic director $\boldsymbol{h}$ is a unit vector in the radial direction in the undeformed tube. In a tangential transversely isotropic tube the anisotropic director is in the tangential direction in the undeformed tube. In a longitudinal transversely isotropic tube the anisotropic director is in the direction of the axis of the tube in the undeformed tube

Nonlinear radial oscillations in a longitudinal transversely isotropic cylindrical tube are the same as in an isotropic tube. Free radial oscillations in an incompressible isotropic tube were first considered by Knowles(1960). Knowles(1962) also considered radial oscillations with the Heaviside step loading boundary condition for the net applied surface pressure and derived upper and lower bounds for the period in a thick-walled tube. Shahinpoor and Nowinski(1971) considered a thin-walled tube and showed that the radial oscillations satisfy the Ermakov-Pinney equation. The solution of the Ermakov-Pinney equation can be expressed as a nonlinear superposition. Higher order effects in the thickness parameter of a thin-walled tube have been investigated by Mason and Roussos(2000). Nonlinear radial oscillations in a thin-walled double layer tube were considered by Roussos and Mason(1998).

References to further work on radial oscillations in an isotropic tube will be given later.

Nonlinear radial oscillations in an incompressible transversely isotropic cylindrical tube were first considered by Huigol(1967). Huigol considered a radial transversely isotropic tube and derived the conditions on the strain-energy function for the existence of periodic solutions. He also briefly considered radial oscillations in a tangential transverse isotropic tube and outlined the changes that have to be made. Shahinpoor(1974) kept the anisotropic director unspecified and derived the equations for radial oscillations in an incompressible transversely isotropic tube. He then considered a thin-walled longitudinal transversely isotropic tube. The equation reduced to the Ermakov-Pinney equation which is the same as for nonlinear radial oscillations in an isotropic tube.

1.3 Transversely isotropic spherical shell

Radial, tangential and longitudinal transversely isotropic spherical shells will also be considered. In a radial transversely isotropic spherical shell, the anisotropic director $\boldsymbol{h}$ is a unit vector in the radial direction of the undeformed shell. For a tangential transversely isotropic spherical shell we define the anisotropic director to be tangential to a line of latitude in the undeformed shell and for a longitudinal transversely isotropic spherical shell we define the anisotropic director to be tangential to a line of longitude in the unstrained shell.

A large amount of research has been done on radial oscillations in an isotropic spherical shell (Guo and Solecki 1963, Wang 1965, Roussos, Mason and Hill 2002). We find that for radial oscillations in tangential and longitudinal transversely isotropic shells the stress tensor is unbounded at the end points, $\theta = 0$ and $\theta = \pi$. We therefore consider only a radial transversely isotropic spherical shell.

Since the analysis of radial oscillations in a spherical shell is similar to radial oscillations in a cylindrical tube we consider instead of radial oscillations the related problem of inflation of a spherical shell. The inflation is assumed to proceed sufficiently slowly that the inertia term in the equation for radial oscillations can be neglected. The inflation of an isotropic spherical shell has been reviewed by Beatty(1987). Snap buckling(Holzaphel, 2000) can occur in which the pressure decreases during inflation before increasing. Anisotropic effects in the inflation of a radial transversely isotropic spherical shell will be investigated.

1.4 Outline of thesis

The notation of finite elasticity of Green and Zerna(1968) and Green and Adkins(1970) is used throughout the thesis.

In Chapter 2, an outline is given of the derivation of the field equations for an incompressible transversely isotropic hyperelastic material. A hyperelastic material is an elastic material with a strain-energy function. Green and Adkins(1970) only considered the case in which the anisotropic director is along the 3-axis. A derivation is given of the constitutive equation for an incompressible transversely isotropic material in which the direction of the anisotropic director is unspecified.

In Chapter 3, a second order nonlinear ordinary differential equation is derived for radial oscillations of a transversely isotropic incompressible hyperelastic cylindrical tube. The equation is expressed in terms of the strain-energy function and applies for radial, tangential and longitudinal transversely isotropic cylindrical tubes. It has the same form as the differential equation for radial oscillations in an isotropic tube. It is quite general and applies for a thick-walled tube and for any strain-energy function for a transverse isotropic material. The generalised Mooney-Rivlin strain-energy function for a transversely isotropic material is introduced.

In Chapter 4, nonlinear radial oscillations in transversely isotropic cylindrical tubes with time dependent net applied surface pressure are investigated. Thin-walled cylindrical tubes with a generalised Mooney-Rivlin strain-energy function are considered. The second order ordinary differential equation for radial oscillations in a radial transversely isotropic tube has a Lie point symmetry generator for a certain relation between the constants in the Mooney-Rivlin strain-energy function and provided the time dependent net applied surface pressure has a special form. The Lie point symmetry is used to transform the differential equation to an autonomous equation which is reduced further to an Abel equation of the second kind. A similar analysis is done with the differential equation for radial oscillations in a tangential transversely isotropic tube. The differential equation is transformed to another Abel equation of the second kind. The results are compared with those of Roussos and Mason(2005) for radial oscillations in an isotropic tube with time dependent net applied surface pressure. Radial oscillations in a longitudinal transversely isotropic tube are the same as in an isotropic tube.

In Chapter 5, nonlinear radial oscillations in a transversely isotropic cylindrical tube with Heaviside step loading boundary condition are investigated. Both thick-walled and thin-walled tubes are considered. A first integral of the second order differential equation for radial oscillations is derived. Effective potentials for radial oscillations in radial, tangential and longitudinal transversely isotropic tubes are obtained. The effective potential determines when the oscillation is bounded and the end points of the bounded oscillations. Upper and lower bounds on the period are derived using inequalities introduced by Knowles(1962) for the isotropic tube.

In Chapter 6, free nonlinear radial oscillations in transversely isotropic thin-walled cylindrical tubes are considered. Knowles(1960) has shown that for free radial oscillations in a thin-walled isotropic tube and for all strain-energy functions, $ab = 1$ where a and b are the minimum and maximum values of the radial coordinate. The extension of this result to free radial oscillations in radial and tangential transversely isotropic tubes with generalised Mooney-Rivlin strain-energy function is investigated. It is shown that if either the initial velocity v_0 vanishes or if $v_0 \neq 0$ but second order terms in the anisotropy can be neglected, then $ab > 1$ for a radial transversely isotropic tube and $ab < 1$ for a tangential transversely isotropic tube. The effect of the anisotropy on the period of free oscillations is also investigated.

In Chapter 7, radial oscillations in a transversely isotropic incompressible hyperelastic spherical shell are investigated. It is found that the Cauchy stress tensor is not bounded everywhere for radial oscillations in tangential and longitudinal transversely isotropic spherical shells. Only radial transversely isotropic spherical shells are therefore considered. It is found that the differential equation for radial oscillations in a thin-walled radial transversely isotropic spherical shell has no Lie point symmetries for a time dependent net applied surface pressure.

In Chapter 8 the inflation of a thin-walled spherical shell is investigated. The condition for snap buckling to occur in an isotropic spherical shell is reviewed. Snap buckling is then investigated in the inflation of a radial transversely isotropic spherical shell and the modifications to the results for an isotropic shell are obtained using perturbation methods. The maximum value of the parameter for which snap buckling can occur in an isotropic shell is increased for a radial transversely isotropic shell.

Finally, the conclusions are summarised in Chapter 9.

Chapter 2

The Equations of Finite Elasticity for Transversely Isotropic Incompressible Elastic Materials

2.1 Introduction

In this chapter the derivation of the field equations for transversely isotropic incompressible materials is outlined.

The strain tensor is first introduced and the three strain invariants are defined. The stress vector is defined and the Cauchy stress tensor is then introduced. Cauchy's formula relating the stress vector to the stress tensor is stated.

The principle of linear momentum for a continuum is stated. An outline is then given of the derivation of Cauchy's first law of motion by starting from the principle of linear momentum.

The principle of angular momentum for a continuum is stated and Cauchy's second law of motion is then considered.

The constitutive equation for transversely isotropic incompressible elastic materials is considered. Green and Adkins (1970) considered only the case in which the anisotropic director $\boldsymbol{h}$ is along the 3− axis. The constitutive equation for the general case in which the anisotropic director $\boldsymbol{h}$ is unspecified is derived. Constitutive equations for radial, tangential and longitudinal transversely isotropic materials can then be obtained as special cases. By substituting the constitutive equation into Cauchy's first law of motion, the field equations for transversely isotropic incompressible elastic materials are derived.

The strain energy function for a transversely isotropic incompressible material is considered. The generalised Mooney-Rivlin strain energy function introduced by Shahinpoor (1974) is discussed. A relation between the constants in the strain energy function is determined from the condition that the undeformed body B_0 must be stress free.

The notation of Green and Zerna (1968) and Green and Adkins (1970) will be used. A review of the equations of finite elasticity for isotropic elastic materials using the notation of Green and Zerna (1968) has been given by Mason (1996).

2.2 Coordinate systems and base vectors

An elastic body is defined as a deformable body which recovers its original shape when the forces causing the deformation are removed. If a body undergoes geometric changes, we need in addition to the reference rectangular cartesian coordinate system, a coordinate system that follows the deformed shape. Such a coordinate system is called a Lagrangian or material coordinate system.

The position vector relative to the origin of a typical material point P_0 of an undeformed body B_0 is defined by

$$\boldsymbol{r} = x^n \, \boldsymbol{i}_n \tag{2.1}$$

as shown in Figure 2.2.1. The fixed rectangular cartesian base vectors $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ satisfy

$$\boldsymbol{i}_n \cdot \boldsymbol{i}_m = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} . \tag{2.2}$$

At time t the material point P_0 in B_0 has moved to the point P in the deformed body B with position vector

$$\boldsymbol{R} = y^n \, \boldsymbol{i}_n. \tag{2.3}$$

The reference rectangular cartesian coordinate system is $\boldsymbol{R} = \left(y^1, y^2, y^3\right)$. The Lagrangian coordinate system is $\boldsymbol{r} = \left(x^1, x^2, x^3\right)$. In the Lagrangian coordinate system, $\boldsymbol{r} = \left(x^1, x^2, x^3\right)$ and t are regarded as independent variables and $\boldsymbol{r}$ is independent of time.

The material time derivative $\frac{D}{Dt}$ is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t}\bigg|_{\boldsymbol{r} \text{ kept fixed}} . \tag{2.4}$$

The partial derivative $\frac{D}{Dt}$ is the time rate-of-change following the motion of the material particle initially at position $\boldsymbol{r}$ in the undeformed body B_0 . Since $\boldsymbol{r}$ and t are independent in the Lagrangian coordinate system, $\frac{D}{Dt}$ commutes with $\frac{\partial}{\partial x^i}$.

The *velocity* $\boldsymbol{v}(\boldsymbol{r}, t)$ of a material particle is defined as

$$\boldsymbol{v}(\boldsymbol{r}, t) = \frac{D}{Dt} \boldsymbol{R}(\boldsymbol{r}, t). \tag{2.5}$$

The velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ is referred to as the *Lagrangian velocity*. Also

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{r}, t) &= \boldsymbol{v}(\boldsymbol{r}(\boldsymbol{R}, t), t) \\ &= \boldsymbol{V}(\boldsymbol{R}, t). \end{aligned} \tag{2.6}$$

The velocity $\boldsymbol{V}(\boldsymbol{R}, t)$ is referred to as the *Eulerian velocity*.

We assume that each material point P at time t is related to its original position P_0 at time t_0 by

$$\begin{aligned} y^i &= y^i\Big(x^1, x^2, x^3, t\Big), \\ x^i &= x^i\Big(y^1, y^2, y^3, t\Big), \end{aligned} \tag{2.7}$$

where y^i and x^i are single-valued and continuously differentiable with respect to each of their variables. For the deformation to be possible it is necessary that

$$\det\left[\frac{\partial y^i}{\partial x^j}\right] > 0. \tag{2.8}$$

Figure 2.2.1

Condition (2.8) ensures that the basis vectors at P in the deformed body form a right-handed system of axes.

The undeformed body B_0 may also be described by curvilinear coordinates $(\theta^1, \theta^2, \theta^3)$. Then

$$x^i = x^i\left(\theta^1, \theta^2, \theta^3\right), \; i = 1, 2, 3. \tag{2.9}$$

It is assumed that the transformation (2.9) can be inverted to give

$$\theta^i = \theta^i\left(x^1, x^2, x^3\right), \; i = 1, 2, 3. \tag{2.10}$$

The *coordinate surfaces* are given by the equations

$$\theta^i\left(x^1, x^2, x^3\right) = constant, \; i = 1, 2, 3. \tag{2.11}$$

The curves produced by the intersections of the three coordinate surfaces at a point P_0 are called the *coordinate curves*.

The *covariant base vectors*, $\boldsymbol{g}_i, \; i = 1, 2, 3$, at a point P_0 of the undeformed body B_0 are given by

$$\boldsymbol{g}_i = \frac{\partial x^n}{\partial \theta^i} \boldsymbol{i}_n = \boldsymbol{r}_{,i} \tag{2.12}$$

where a comma denotes ordinary partial differentiation. The base vector $\boldsymbol{g}_i$ is directed tangentially along the θ^i coordinate curve. The covariant base vectors of the undeformed body B_0 are shown in Figure 2.2.2.

The *contravariant base vectors*, $\boldsymbol{g}^i, \; i = 1, 2, 3$, are defined by

$$\boldsymbol{g}^i = \frac{\partial \theta^i}{\partial x^n} \boldsymbol{i}^n. \tag{2.13}$$

It is readily verified that

$$\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j. \tag{2.14}$$

The covariant and contravariant metric tensors of the undeformed body B_0 are given respectively by

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \tag{2.15}$$

and

$$g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j. \tag{2.16}$$

The curvilinear coordinate system $(\theta^1, \theta^2, \theta^3)$ moves continuously with the body as it deforms from the original configuration B_0 at time t_0 to the configuration B at time t. The values of $(\theta^1, \theta^2, \theta^3)$ which define the material point P_0 in B_0 remain fixed with the material point as it moves from position P_0 in B_0 to position P in B. Thus

$$y^i = y^i\left(x^1(\theta^1, \theta^2, \theta^3), x^2(\theta^1, \theta^2, \theta^3), x^3(\theta^1, \theta^2, \theta^3), t\right) \tag{2.17}$$

$$= y^i\left(\theta^1, \theta^2, \theta^3, t\right) \tag{2.18}$$

where the functional forms of y^i in (2.17) and (2.18) are generally different. It is assumed that the transformation (2.18) can be inverted to give

$$\theta^i = \theta^i\left(y^1, y^2, y^3, t\right), \; i = 1, 2, 3. \tag{2.19}$$

Figure 2.2.2

The *covariant base vectors* of the deformed body B are

$$\boldsymbol{G}_i = \frac{\partial y^n}{\partial \theta^i}\,\boldsymbol{i}_n = \boldsymbol{R}_{,i} \tag{2.20}$$

and they are directed along the coordinate curves at point P in the body B. The covariant base vectors of the deformed body B are shown in Figure 2.2.2.

The *contravariant base vectors* of the deformed body B are given by

$$\boldsymbol{G}^i = \frac{\partial \theta^i}{\partial y^n}\,\boldsymbol{i}^n. \tag{2.21}$$

It is readily shown that

$$\boldsymbol{G}_i \cdot \boldsymbol{G}^j = \delta_i^j. \tag{2.22}$$

The covariant and contravariant metric tensors of the deformed body B are respectively

$$G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j \tag{2.23}$$

and

$$G^{ij} = \boldsymbol{G}^i \cdot \boldsymbol{G}^j. \tag{2.24}$$

The Christoffel symbols calculated for the undeformed body B_0 from the metric tensors g_{ij} and g^{ij} are given respectively by

$${}_0\Gamma_{ijk} = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}) \tag{2.25}$$

and

$${}_0\Gamma^k_{ij} = \frac{1}{2}g^{ks}(g_{is,j} + g_{js,i} - g_{ij,s}), \tag{2.26}$$

with

$$\boldsymbol{g}_{i,j} = {}_0\Gamma^k_{ij}\boldsymbol{g}_k \tag{2.27}$$

and

$$\boldsymbol{g}^i{}_{,j} = -{}_0\Gamma^i_{jk}\boldsymbol{g}^k. \tag{2.28}$$

A single vertical line denotes covariant differentiation with respect to the metric tensor of the undeformed body B_0. Hence

$$A^i{}_{|j} = A^i{}_{,j} + {}_0\Gamma^i_{jk}A^k, \tag{2.29}$$

$$A_{i|j} = A_{i,j} - {}_0\Gamma^k_{ij}A_k. \tag{2.30}$$

The covariant derivative of all components of the metric tensor in B_0 vanish:

$$g_{ij|k} = 0, \quad g^i{}_{j|k} = 0, \quad g^{ij}{}_{|k} = 0. \tag{2.31}$$

Similarly, the Christoffel symbols calculated for the deformed body B from the metric tensors G_{ij} and G^{ij} are respectively

$$\Gamma_{ijk} = \frac{1}{2}(G_{ik,j} + G_{jk,i} - G_{ij,k}) \tag{2.32}$$

and

$$\Gamma^k_{ij} = \frac{1}{2} G^{ks}(G_{is,j} + G_{js,i} - G_{ij,s}), \tag{2.33}$$

where

$$\boldsymbol{G}_{i,j} = \Gamma^k_{ij} \boldsymbol{G}_k \tag{2.34}$$

and

$$\boldsymbol{G}^i_{,j} = -\Gamma^i_{jk} \boldsymbol{G}^k. \tag{2.35}$$

A double vertical line denotes covariant differentiation with respect to the metric tensor of the deformed body B. Thus

$$A^i{}_{||j} = A^i{}_{,j} + \Gamma^i_{jk} A^k, \tag{2.36}$$

$$A_{i||j} = A_{i,j} - \Gamma^k_{ij} A_k. \tag{2.37}$$

The covariant derivative of all components of the metric tensor in B vanish:

$$G_{ij||k} = 0, \quad G^i{}_{j||k} = 0, \quad G^{ij}{}_{||k} = 0. \tag{2.38}$$

2.3 Strain tensor and strain invariants

The *strain tensor* γ_{ij} is defined in terms of the covariant metric tensors of the deformed body B and undeformed body B_0 and is

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}). \tag{2.39}$$

We now show that the strain tensor determines the difference of the squares of the line elements in the bodies B and B_0. The line element at the point P_0 in the undeformed body B_0 at $\boldsymbol{r}\big(\theta^1, \theta^2, \theta^3\big)$ is given by

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial \theta^i} d\theta^i = \frac{\partial x^n}{\partial \theta^i} \boldsymbol{i}_n d\theta^i = \boldsymbol{g}_i \, d\theta^i. \tag{2.40}$$

Then

$$ds_0^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = g_{ij} \, d\theta^i \, d\theta^j. \tag{2.41}$$

In the deformed body B, the line element at the same material point P at $\boldsymbol{R}\big(\theta^1, \theta^2, \theta^3, t\big)$ is

$$d\boldsymbol{R} = \frac{\partial \boldsymbol{R}}{\partial \theta^i} d\theta^i = \frac{\partial y^n}{\partial \theta^i} \boldsymbol{i}_n d\theta^i = \boldsymbol{G}_i \, d\theta^i. \tag{2.42}$$

Then

$$ds^2 = d\boldsymbol{R} \cdot d\boldsymbol{R} = G_{ij} \, d\theta^i \, d\theta^j. \tag{2.43}$$

Hence

$$ds^2 - ds_0^2 = 2\gamma_{ij} \, d\theta^i \, d\theta^j. \tag{2.44}$$

The strain tensor is therefore explained as a measure of the difference of the squares of corresponding line elements in the deformed and undeformed bodies.

The strain tensor can be written in a more familiar form in terms of the displacement vector $\boldsymbol{u}$ defined by

$$\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{R}(\boldsymbol{r},t) - \boldsymbol{r}. \tag{2.45}$$

For

$$\boldsymbol{u}_{,i} = \boldsymbol{R}_{,i} - \boldsymbol{r}_{,i} = \boldsymbol{G}_i - \boldsymbol{g}_i, \tag{2.46}$$

and eliminating G_{ij} from (2.39) gives

$$\gamma_{ij} = \frac{1}{2}(\boldsymbol{g}_i \cdot \boldsymbol{u}_{,j} + \boldsymbol{g}_j \cdot \boldsymbol{u}_{,i} + \boldsymbol{u}_{,i} \cdot \boldsymbol{u}_{,j}). \tag{2.47}$$

But

$$\boldsymbol{u}_{,i} = u_{n|i}\boldsymbol{g}^n \tag{2.48}$$

and (2.47) can be written as

$$\gamma_{ij} = \frac{1}{2}\Big(u_{i|j} + u_{j|i} + u^m{}_{|i} u_{m|j}\Big). \tag{2.49}$$

Equation (2.49) gives the strain tensor γ_{ij} in terms of the displacement vector $\boldsymbol{u}$.

In order to introduce the strain invariants, consider first a general second order mixed tensor, A^i_j . Then the characteristic polynomial of A^i_j is of the form

$$\det[A^i_j - \lambda\delta^i_j] = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3, \tag{2.50}$$

where

$$\begin{aligned} I_1 &= A^i{}_i = \mathrm{trace}A, && (2.51)\\ I_2 &= \frac{1}{2}[A^i{}_i A^j{}_j - A^i{}_r A^r{}_i] \\ &= \frac{1}{2}[(\mathrm{trace}A)^2 - \mathrm{trace}(A^2)], && (2.52)\\ I_3 &= \det[A^i{}_j]. && (2.53) \end{aligned}$$

Further, I_1, I_2 and I_3 are invariants under transformations of curvilinear coordinates $\theta^i \to \bar{\theta}^i$ where

$$\bar{\theta}^i = \bar{\theta}^i(\theta^1, \theta^2, \theta^3). \tag{2.54}$$

Consider now the strain tensor. Mixed tensors can be formed from γ_{ij} in two ways dependng on whether the metric tensor g^{ik} in B_0 or the metric tensor G^{ik} in B is used to raise the index. Following Green and Zerna (1954) we use the metric tensor g^{ik}. Define

$$\gamma^i_j = g^{ik}\gamma_{kj} = \frac{1}{2}g^{ik}(G_{kj} - g_{kj}) = \frac{1}{2}(g^{ik}G_{kj} - \delta^i_j). \tag{2.55}$$

Thus

$$g^{ik}G_{kj} = 2\gamma^i_j + \delta^i_j. \tag{2.56}$$

We work with the invariants of $g^{ik}G_{kj}$ instead of the invariants of γ^i_j. The invariants of $g^{ik}G_{kj}$ are denoted by I_1, I_2, I_3 and are called the *strain invariants*. The strain invariants are

$$\begin{aligned} I_1 &= \text{trace}[g^{ik}\,G_{kj}], && (2.57)\\ I_2 &= \frac{1}{2}[(\text{trace}[g^{ik}\,G_{kj}])^2 - \text{trace}([g^{ik}\,G_{kj}]^2)], && (2.58)\\ I_3 &= \det[g^{ik}\,G_{kj}]. && (2.59) \end{aligned}$$

The strain invariants can be expressed as follows:

$$\begin{aligned} I_1 &= g^{ik}\,G_{ik}, && (2.60)\\ I_2 &= \frac{1}{2}g^{ik}\,g^{rs}[G_{ik}\,G_{rs} - G_{is}\,G_{kr}], && (2.61)\\ I_3 &= \frac{G}{g}, && (2.62) \end{aligned}$$

where

$$G = \det[G_{ik}], \quad g = \det[g_{ik}] = \frac{1}{\det[g^{ik}]}. \tag{2.63}$$

If the invariants of γ^i_j are denoted by J_1, J_2 and J_3, then

$$\begin{aligned} J_1 &= \frac{1}{2}(I_1 - 3), && (2.64)\\ J_2 &= \frac{1}{4}(I_2 - 2I_1 + 3), && (2.65)\\ J_3 &= \frac{1}{8}(I_3 - I_2 + I_1 - 1). && (2.66) \end{aligned}$$

Hence, once I_1 , I_2 and I_3 have been determined, the three invariants of γ^i_j can be obtained if required.

The following result is useful in calculating the strain invariant I_2 :

$$I_2 = G^{ik}\,g_{ik}\,I_3. \tag{2.67}$$

Because of its importance a brief outline is given of the derivation of (2.67). Let

$$A^i_j = g^{ik}\,G_{kj}. \tag{2.68}$$

The characteristic polynomial of A^i_j is given by (2.50) and therefore its characteristic equation is

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0. \tag{2.69}$$

By the Cayley-Hamilton theorem, a matrix satisfies its own characteristic equation. Thus

$$A^3 - I_1A^2 + I_2A - I_3 = 0. \tag{2.70}$$

Multiplying (2.70) by A^{-1} gives

$$A^2 - I_1A + I_2I - I_3A^{-1} = 0. \tag{2.71}$$

Thus, by taking the trace of (2.71), we obtain

$$A^i_{\ s}A^s_{\ i} - I_1 A^i_{\ i} + 3I_2 - I_3 A^{-1i}_{\ \ \ i} = 0. \tag{2.72}$$

But from (2.51) and (2.52),

$$A^i_{\ i} = I_1, \quad A^i_{\ s}A^s_{\ i} = I_1^2 - 2I_2, \tag{2.73}$$

and therefore (2.72) becomes

$$I_2 = I_3 A^{-1i}_{\ \ \ i}. \tag{2.74}$$

But

$$A^{-1i}_{\ \ \ j} = G^{ik} g_{kj} \tag{2.75}$$

and substituting (2.75) into (2.74) gives (2.67).

An *incompressible* elastic material is one for which the volume of all material elements is conserved during all deformations.

We now outline the derivation of the result that for an incompressible elastic material,

$$g = G, \tag{2.76}$$

where g and G are defined by (2.63). In order to establish (2.76), consider a volume element $d\tau_0$ at P_0 in the undeformed body B_0 with edges along the coordinate curves at P_0. Now, the material line elements along the coordinate curves are

$$d\boldsymbol{s}_1 = \boldsymbol{g}_1 d\theta^1, \quad d\boldsymbol{s}_2 = \boldsymbol{g}_2 d\theta^2, \quad d\boldsymbol{s}_3 = \boldsymbol{g}_3 d\theta^3 \tag{2.77}$$

and therefore

$$d\tau_0 = d\boldsymbol{s}_1 \cdot (d\boldsymbol{s}_2 \times d\boldsymbol{s}_3) = \boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3) d\theta^1 d\theta^2 \theta^3. \tag{2.78}$$

But

$$\boldsymbol{g}_2 \times \boldsymbol{g}_3 = \varepsilon_{231}\, \boldsymbol{g}^1 = \sqrt{g}\, e_{231}\, \boldsymbol{g}^1 = \sqrt{g}\, \boldsymbol{g}^1 \tag{2.79}$$

where ε_{ijk} is the alternating tensor and e_{ijk} is the permutation symbol :

$$\varepsilon_{ijk} = \sqrt{g}\, e_{ijk}. \tag{2.80}$$

Hence

$$d\tau_0 = \sqrt{g}\, \boldsymbol{g}_1 \cdot \boldsymbol{g}^1\, d\theta^1 d\theta^2 d\theta^3 = \sqrt{g}\, d\theta^1 d\theta^2 d\theta^3 \tag{2.81}$$

since

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta^j_{\ i}. \tag{2.82}$$

Now the material line elements $\boldsymbol{g}_1 d\theta^1$, $\boldsymbol{g}_2 d\theta^2$ and $\boldsymbol{g}_3 d\theta^3$ in B_0 deform into the material line elements $\boldsymbol{G}_1 d\theta^1$, $\boldsymbol{G}_2 d\theta^2$ and $\boldsymbol{G}_3 d\theta^3$ in B, consisting of the same material particles. Thus the material volume element $d\tau_0$ in B_0 deforms into the material volume element $d\tau$ in B where

$$d\tau = \boldsymbol{G}_1 \cdot (\boldsymbol{G}_2 \times \boldsymbol{G}_3)\, d\theta^1 d\theta^2 d\theta^3 = \sqrt{G}\, d\theta^1 d\theta^2 d\theta^3. \tag{2.83}$$

Thus from (2.81) and (2.83),

$$d\tau = \frac{\sqrt{G}}{\sqrt{g}} d\tau_0. \tag{2.84}$$

But for an incompressible material, material volume elements are conserved. Hence

$$d\tau = d\tau_0 \tag{2.85}$$

and therefore from (2.84)

$$\sqrt{G} = \sqrt{g}. \tag{2.86}$$

Hence $G = g$.

It therefore follows from (2.62) that for an incompressible elastic material,

$$I_3 = 1. \tag{2.87}$$

For an incompressible elastic material there are only two strain invariants which from (2.60) and (2.67) are

$$I_1 = g^{ik}\, G_{ik}, \tag{2.88}$$

$$I_2 = G^{ik}\, g_{ik}. \tag{2.89}$$

The strain invariants, I_1 and I_2, are in general not equal because the metric tensor g^{ik} cannot be used to raise or lower the indices of G_{ik} and similarly G^{ik} cannot be used to raised or lower the indices of g_{ik}.

2.4 Stress vector and stress tensor

The *stress vector*, $\boldsymbol{t}(\boldsymbol{R}, t, \boldsymbol{n})$, is defined as the force per unit area on a surface element in the current configuration at $\boldsymbol{R}$ at time t with unit normal $\boldsymbol{n}$.

We see that $\boldsymbol{t}$ depends on position $\boldsymbol{R}$, time t and orientation $\boldsymbol{n}$ of the surface element. The stress vector $\boldsymbol{t}(\boldsymbol{n})$ describes the stress exerted by the material on the side to which $\boldsymbol{n}$ points, on the material on the side $\boldsymbol{n}$ points away from. At any given point, the stress vector acting on one side of a surface balances that on the other side of the surface. Thus for any surface in the continuum, (Green and Zerna (1968))

$$\boldsymbol{t}(\boldsymbol{R}, t, -\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{R}, t, \boldsymbol{n}). \tag{2.90}$$

Cauchy's formula is (Green and Zerna (1968))

$$t^i \boldsymbol{G}_i = n_k \tau^{ki} \boldsymbol{G}_i, \tag{2.91}$$

where $t^i = t^i(\boldsymbol{R}, t, \boldsymbol{n})$ is the stress vector acting on a surface element at $\boldsymbol{R}$ at time t with unit normal $\boldsymbol{n}$ in the current configuration and the contravariant tensor $\tau^{ki}(\boldsymbol{R}, t)$ is called the *Cauchy stress tensor*. The Cauchy stress tensor is independent of the orientation, $\boldsymbol{n}$, of the surface.

The force per unit area, denoted by $\boldsymbol{P}$, applied to the surface of the continuum by an external agency is called the *surface traction*.

The surface traction occurs in the boundary condition. By making use of Cauchy's formula (2.91), the boundary condition at the surface is given by

$$P^i \boldsymbol{G}_i = n_k \tau^{ki} \boldsymbol{G}_i. \tag{2.92}$$

If the surface traction vanishes, the surface is said to be traction-free and such a surface is referred to as a free surface.

2.5 Balance laws and field equations

A *material volume* and a *material surface* consist at all times of the same material particles. They move with the continuum as the continuum evolves.

The linear momentum per unit volume of a continuum at position $\boldsymbol{R}$ at time t is $\rho\boldsymbol{V}$, where ρ is the density of the continuum and $\boldsymbol{V}(\boldsymbol{R},t)$ is the velocity of the continuum. The total linear momentum of a given part P of the continuum is

$$\int_\tau \rho\boldsymbol{V}d\tau,$$

where τ is the material volume consisting of part P.

The *principle of linear momentum* states that the rate of change of the linear momentum of an arbitrary part P of the continuum is equal to the resultant force on part P.

The principle of linear momentum can be written as

$$\frac{d}{dt}\int_\tau \rho\boldsymbol{V}d\tau = \int_S \boldsymbol{t}(\boldsymbol{R},t,\boldsymbol{n})dS + \int_\tau \rho\boldsymbol{F}(\boldsymbol{R},t)d\tau, \tag{2.93}$$

where S is the closed material surface bounding the material volume τ and $\boldsymbol{F}(\boldsymbol{R},t)$ is the body force per unit mass. The principle of linear momentum is motivated by the corresponding result for systems of particles in classical mechanics. However, in continuum mechanics, the principle of linear momentum is a fundamental principle. Its justification does not depend on particle mechanics. It depends entirely on the usefulness of the theories based on it.

The *divergence theorem* states that, if τ is a volume bounded by a closed surface S, then

$$\int_S Tn_i dS = \int_\tau \frac{1}{\sqrt{G}}(\sqrt{G}T)_{,i}\, d\tau, \tag{2.94}$$

where T is any tensor field and $\boldsymbol{n}$ is the unit normal vector along the outward normal to S.

We will also require the *Reynolds transport theorem* which states that if C is any tensor field and mass is conserved then

$$\frac{d}{dt}\int_\tau \rho C d\tau = \int_\tau \rho\frac{DC}{Dt}d\tau, \tag{2.95}$$

where τ is a material volume consisting at all times of the same material particles.

We now derive from the principle of linear momentum, (2.93), a partial differential equation which holds at each point of the continuum.

Suppose that part P of the continuum occupies volume τ and is bounded by the closed surface S at time t. Since it is assumed that mass is conserved, it follows from the Reynolds transport theorem that

$$\frac{d}{dt}\int_\tau \rho\boldsymbol{V}d\tau = \int_\tau \rho\frac{D\boldsymbol{V}}{Dt}d\tau. \tag{2.96}$$

Also using Cauchy's formula (2.91) and the divergence theorem (2.94) it follows that

$$\int_S \boldsymbol{t}(\boldsymbol{R},t,\boldsymbol{n})dS = \int_\tau n_k\tau^{ki}\,\boldsymbol{G}_i dS = \int_\tau \tau^{ki}{}_{||k}\,\boldsymbol{G}_i d\tau, \tag{2.97}$$

where $||k$ denotes the covariant derivative with respect to the metric tensor G_{ik}. Substituting equations (2.96) and (2.97) into the principle of linear momentum, equation (2.93), gives

$$\rho \frac{D\boldsymbol{V}}{Dt} = \tau^{ki}{}_{||k}\, \boldsymbol{G}_i + \rho \boldsymbol{F}. \tag{2.98}$$

The partial differential equation (2.98) is called the momentum balance equation. It is also known as *Cauchy's first law of motion.*

The angular momentum per unit volume of a continuum at position $\boldsymbol{R}$ at time t about a fixed point, which is taken to be the origin of the rectangular cartesian coordinates, is $\boldsymbol{R}\times\rho\boldsymbol{V}$. The total angular momentum of the part P of the continuum about the origin is

$$\int_\tau \boldsymbol{R} \times \rho \boldsymbol{V} d\tau.$$

The *principle of angular momentum* states that the rate of change of angular momentum of an arbitrary part P of a continuum about a fixed point, which is taken to be the origin of the rectangular cartesian coordinate system, equals the total torque on P.

The principle of angular momentum can be expressed as

$$\frac{d}{dt}\int_\tau \boldsymbol{R} \times \rho \boldsymbol{V} d\tau = \int_S \boldsymbol{R} \times \boldsymbol{t}(\boldsymbol{R}, t, \boldsymbol{n}) dS + \int_\tau \boldsymbol{R} \times \rho \boldsymbol{F}(\boldsymbol{R}, t) d\tau, \tag{2.99}$$

where S is the closed material surface bounding the material volume τ. Equation (2.99) is sometimes referred to as Cauchy's equation of moments. The principle of angular momentum can be used together with the Reynolds transport and divergence theorems to derive Cauchy's second law of motion, which states that if there is no distributed body or surface couples, and mass is conserved, then

$$\tau^{ik} = \tau^{ki}. \tag{2.100}$$

We will assume throughout this thesis that there are no distributed body or surface couples so that the Cauchy stress tensor is always symmetric.

2.6 Strain-energy function

We first introduce a strain-energy function and then derive an expression for the stress tensor in terms of the derivative of the strain-energy function with respect to the strain tensor.

Consider an arbitrary volume τ bounded by a closed surface S in the strained body. Then the rate of work of surface forces over S plus the rate of work of body forces throughout τ equals to the rate of increase of kinetic energy of the material in τ plus the rate of increase of the strain-energy of the material in τ. This can be written as

$$\int_S \boldsymbol{t}(\boldsymbol{n}) \cdot \boldsymbol{V} dS + \int_\tau \rho \boldsymbol{F} \cdot \boldsymbol{V} d\tau = \frac{d}{dt}\int_\tau \frac{1}{2}\rho \boldsymbol{V} \cdot \boldsymbol{V} d\tau + R, \tag{2.101}$$

where R is the rate of increase of the strain-energy of the material in τ. Equation (2.101) can be rewritten as

$$R = \int_S \boldsymbol{t}(\boldsymbol{n}) \cdot \boldsymbol{V} dS + \int_\tau \rho \boldsymbol{F} \cdot \boldsymbol{V} d\tau - \frac{d}{dt} \int_\tau \frac{1}{2} \rho \boldsymbol{V} \cdot \boldsymbol{V} d\tau. \tag{2.102}$$

But, by applying Cauchy's formula (2.91) and the divergence theorem (2.94), it can be shown that

$$\int_S \boldsymbol{t}(\boldsymbol{n}) \cdot \boldsymbol{V} dS = \int_\tau \big[\tau^{ki}{}_{||k} \boldsymbol{G}_i \cdot \boldsymbol{V} + \tau^{ki} \boldsymbol{G}_i \cdot \boldsymbol{V}_{,k} \big] d\tau. \tag{2.103}$$

Also, using the Reynolds transport theorem (2.95) which applies since mass is conserved, it follows that

$$\int_\tau \frac{1}{2} \rho \boldsymbol{V} \cdot \boldsymbol{V} d\tau = \int_\tau \boldsymbol{V} \cdot \rho \frac{D\boldsymbol{V}}{Dt} d\tau. \tag{2.104}$$

Substituting (2.103) and (2.104) into (2.102) gives

$$R = \int_\tau \big[\big(-\rho \frac{D\boldsymbol{v}}{Dt} + \tau^{ki}{}_{||k} \boldsymbol{G}_i + \rho \boldsymbol{F} \big) \cdot \boldsymbol{V} + \tau^{ki} \boldsymbol{G}_i \cdot \boldsymbol{V}_{,k} \big] d\tau \tag{2.105}$$

and hence, using Cauchy's first law of motion, we obtain

$$R = \int_\tau \tau^{ki} \boldsymbol{G}_i \cdot \boldsymbol{V}_{,k} \, d\tau. \tag{2.106}$$

But since $\frac{D}{Dt}$ and $\frac{\partial}{\partial \theta^k}$ commute,

$$\boldsymbol{V}_{,k} = \frac{\partial}{\partial \theta^k} \Big(\frac{D\boldsymbol{R}}{Dt} \Big) = \frac{D}{Dt} \Big(\frac{\partial \boldsymbol{R}}{\partial \theta^k} \Big). \tag{2.107}$$

Now

$$\boldsymbol{R}_{,k} = \boldsymbol{G}_k \tag{2.108}$$

and therefore

$$\boldsymbol{V}_{,k} = \frac{D}{Dt} \boldsymbol{G}_k. \tag{2.109}$$

Thus, since τ^{ik} is symmetric,

$$\tau^{ki} \boldsymbol{G}_i \cdot \boldsymbol{V}_{,k} = \frac{1}{2} \tau^{ik} \frac{D}{Dt} \Big(\boldsymbol{G}_i \cdot \boldsymbol{G}_k \Big) = \frac{1}{2} \tau^{ik} \frac{D}{Dt} G_{ik} \tag{2.110}$$

and by using the definition (2.39) for γ_{ik} and also

$$\frac{D}{Dt} g_{ik} = 0 \tag{2.111}$$

it follows that

$$\tau^{ki} \boldsymbol{G}_i \cdot \boldsymbol{V}_{,k} = \tau^{ik} \frac{D}{Dt} \gamma_{ik}. \tag{2.112}$$

Thus

$$R = \int_\tau \tau^{ik} \frac{D}{Dt} (\gamma_{ik}) d\tau. \tag{2.113}$$

In this thesis we consider only elastic bodies for which

$$R = \frac{d}{dt}\int_{\tau_0} W d\tau_0, \tag{2.114}$$

where τ_0 is the volume in the unstrained body occupied by the material in volume τ in the strained body at time t and W is the elastic $strain - energy\ function$ measured per unit volume of the unstrained body. An elastic material which possesses a strain-energy function is called a hyperelastic material. It is assumed that W depends only on the state of strain of the strained body at time t and therefore

$$W = W(\gamma_{ik}). \tag{2.115}$$

Equating (2.113) and (2.114) and using the Reynolds transport theorem (2.95) gives

$$\int_{\tau}\Big(\tau^{ik}\frac{D}{Dt}\gamma_{ik} - \frac{\rho}{\rho_0}\frac{DW}{Dt}\Big)d\tau = 0, \tag{2.116}$$

where ρ_0 is the density in the undeformed body B_0. Later we will consider only incompressible materials for which $\rho = \rho_0$. We assume that the integrand in (2.116) is continuous. Since τ is an arbitrary material volume in the deformed body B, it follows that the integrand must vanish. Hence

$$\tau^{ik}\frac{D}{Dt}(\gamma_{ik}) = \frac{\rho}{\rho_0}\frac{DW}{Dt}. \tag{2.117}$$

Since $W = W(\gamma_{ik})$, equation (2.117) becomes

$$\Big(\tau^{ik} - \frac{\rho}{\rho_0}\frac{\partial W}{\partial\gamma_{ik}}\Big)\frac{D}{Dt}\gamma_{ik} = 0. \tag{2.118}$$

When performing differentiation with respect to γ_{ik} it is understood that all other components of the strain tensor are held constant including γ_{ki} if $k \neq i$. Now

$$\frac{\partial W}{\partial\gamma_{ik}} = \frac{\partial W}{\partial\gamma_{ki}} \tag{2.119}$$

only if W is expressed symmetrically in the suffices of γ_{ik}. We do not assume that W is expressed symmetrically in the suffices of γ_{ik}. Hence we decompose $\frac{\partial W}{\partial\gamma_{ik}}$ into the sum of its symmetric and skew-symmetric parts

$$\frac{\partial W}{\partial\gamma_{ik}} = \frac{1}{2}\big(\frac{\partial W}{\partial\gamma_{ik}} + \frac{\partial W}{\partial\gamma_{ki}}\big) + \frac{1}{2}\big(\frac{\partial W}{\partial\gamma_{ik}} - \frac{\partial W}{\partial\gamma_{ki}}\big). \tag{2.120}$$

Since $\frac{D}{Dt}\gamma_{ik}$ is symmetric, equation (2.118) becomes

$$\Big[\tau^{ik} - \frac{1}{2}\frac{\rho}{\rho_0}\Big(\frac{\partial W}{\partial\gamma_{ik}} + \frac{\partial W}{\partial\gamma_{ki}}\Big)\Big]\frac{D}{Dt}\gamma_{ik} = 0. \tag{2.121}$$

We will introduce the notation that an overhead dot denotes the material time derivative $\frac{D}{Dt}$. Hence

$$\dot{\gamma}_{ik} = \frac{D}{Dt}(\gamma_{ik}). \tag{2.122}$$

Consider now an incompressible hyperelastic material. For an incompressible material, $\rho = \rho_0$. Further, the components $\dot{\gamma}_{ik}$ are not independent. For, if $d\tau$ is a material volume element

$$(d\tau)^{\cdot} = G^{ik}\dot{\gamma}_{ik}d\tau. \tag{2.123}$$

This result does not appear to have been given by Green and Zerna (1968). The derivation of (2.123) given by Mason (1996) is outlined in Appendix A. Now, if the material is incompressible, the volume of any material element remains constant as it moves with the material and therefore $(d\tau)^{\cdot} = 0$. Hence, for an incompressible material, the components $\dot{\gamma}_{ik}$ must satisfy the relation

$$G^{ik}\dot{\gamma}_{ik} = 0. \tag{2.124}$$

We therefore cannot deduce from (2.121) that the coefficient of $\dot{\gamma}_{ik}$ in (2.121) vanishes.

Now suppose that τ^{ik} satisfies (2.121) . Then

$$\hat{\tau}^{ik} = \tau^{ik} + pG^{ik}, \tag{2.125}$$

where p is an arbitrary scalar function, also satisfies (2.121). Thus τ^{ik} does not depend only on γ_{rs}. We therefore make the constitutive assumption

$$\tau^{ik} = pG^{ik} + F^{ik}(\gamma_{rs}), \tag{2.126}$$

where p is a scalar function undetermined by the local strain. Substitute equation (2.126) into (2.121). Thus

$$P^{ik}\dot{\gamma}_{ik} = 0, \tag{2.127}$$

where

$$P^{ik} = F^{ik}(\gamma_{rs}) - \frac{1}{2}\Big(\frac{\partial W}{\partial \gamma_{ik}} + \frac{\partial W}{\partial \gamma_{ki}}\Big). \tag{2.128}$$

In order to obtain P^{ik} from (2.127) the method of Lagrange undetermined multipliers is used as described by Atkin and Fox (1980) and Mason (1996). Green and Zerna (1968) did not follow this procedure. These authors took the incompressible limit of the constitutive equation for a compressible material. Multiply (2.124) by a scalar function $\bar{p}$ which is not yet specified and subtract (2.124) from (2.127) . This gives

$$\Big(P^{ik} - \bar{p}G^{ik}\Big)\dot{\gamma}_{ik} = 0. \tag{2.129}$$

The scalar $\bar{p}$ is the Lagrange undetermined multiplier. Choose $\bar{p}$ to satisfy

$$P^{33} - \bar{p}G^{33} = 0. \tag{2.130}$$

Then $\dot{\gamma}_{33}$ is absent from (2.129). The remaining components of $\dot{\gamma}_{ik}$ may be chosen arbitrarily subject to $\dot{\gamma}_{ik} = \dot{\gamma}_{ki}$ while the components $P^{ik} - \bar{p}G^{ik}$, which do not depend on $\dot{\gamma}_{ik}$, remain fixed. Thus since $P^{ik} - \bar{p}G^{ik}$ is symmetric, it follows that

$$P^{ik} - \bar{p}G^{ik} = 0 \tag{2.131}$$

for all values of i and k except $i = k = 3$. But from (2.130) it follows that (2.131) also holds for $i = k = 3$. Hence (2.131) is satisfied for all $1 \leq i \leq 3$, $1 \leq k \leq 3$. Thus

$$F^{ik}(\gamma_{rs}) = \frac{1}{2}\Big(\frac{\partial W}{\partial \gamma_{ik}} + \frac{\partial W}{\partial \gamma_{ki}}\Big) + \bar{p}G^{ik} \tag{2.132}$$

and therefore

$$\tau^{ik} = \frac{1}{2}\Big(\frac{\partial W}{\partial \gamma_{ik}} + \frac{\partial W}{\partial \gamma_{ki}}\Big) + (p+\bar{p})G^{ik}. \tag{2.133}$$

But $\bar{p}$ may be incorporated in p since p is an undetermined scalar function. Thus

$$\tau^{ik} = \frac{1}{2}\Big(\frac{\partial W}{\partial \gamma_{ik}} + \frac{\partial W}{\partial \gamma_{ki}}\Big) + pG^{ik}. \tag{2.134}$$

The constitutive equation (2.134) holds for all incompressible hyperelastic materials.

2.7 Constitutive equation for a transversely isotropic incompressible elastic material

Consider now transversely isotropic elastic materials (Ericksen and Rivlin (1954), Green and Adkins (1970)) .

A material is said to possess *transverse isotropy* with respect to the direction $\boldsymbol{h}$ if its strain-energy function is invariant under rotations about $\boldsymbol{h}$ and reflection in any plane containing $\boldsymbol{h}$.

The anisotropic director $\boldsymbol{h}$ is a unit vector. Green and Adkins (1970) derived the constitutive equation for a transversely isotropic elastic material for the special case when the anisotropic director $\boldsymbol{h}$ is along the $\boldsymbol{i}_3$ direction. We will derive the generalisation of the constitutive equation which applies when the direction of $\boldsymbol{h}$ is unspecified.

Let H^1, H^2, H^3 denote the components of the anisotropic director $\boldsymbol{h}$ when the curvilinear coordinates $\theta^1, \theta^2, \theta^3$ are the rectangular cartesian coordinates x^1, x^2, x^3 of the undeformed body B_0. Then

$$\boldsymbol{h} = H^n \boldsymbol{i}_n = h^i \boldsymbol{g}_i, \tag{2.135}$$

$$h^i = \frac{\partial \theta^i}{\partial x^n} H^n, \tag{2.136}$$

$$\boldsymbol{h}\cdot\boldsymbol{h} = \delta_{ik} H^i\, H^k = g_{ik} h^i\, h^k = 1. \tag{2.137}$$

Let e_{ik} denote the components of the strain tensor γ_{ik} when the curvilinear coordinates $\theta^1, \theta^2, \theta^3$ are the rectangular cartesian coordinates x^1, x^2, x^3 of the undeformed body B_0. Then

$$\gamma_{ik} = \frac{\partial x^r}{\partial \theta^i}\frac{\partial x^s}{\partial \theta^k} e_{rs}. \tag{2.138}$$

Green and Adkins (1970) have shown that if an incompressible material possesses transverse isotropy with respect to the direction with cartesian components,

$$H^1 = 0, \quad H^2 = 0, \quad H^3 = 1, \tag{2.139}$$

then the strain-energy function W is of the form

$$W^* = W(I_1, I_2, K_1^*, K_2^*), \tag{2.140}$$

where from (2.88) and (2.89)

$$I_1 = g^{ik}G_{ik}, \quad I_2 = g_{ik}G^{ik} \tag{2.141}$$

and

$$K_1^* = e_{33}, \quad K_2^* = e_{3\alpha}e_{3\alpha}, \tag{2.142}$$

where the Greek index α is summed over the values of 1 and 2. The third strain invariant I_3 does not occur in W because when the material is incompressible, $I_3 = 1$. Equation (2.140) is a generalisation of the result for an incompressible isotropic hyperelastic material that $W = W(I_1, I_2)$. When $H^n = \delta_3^n$, (2.142) for K_1^* and K_2^* can be written as

$$K_1^* = e_{rs}H^rH^s, \tag{2.143}$$
$$K_2^* = e_{ir}e_{js}H^iH^j\delta^{rs} - K_1^{*2}. \tag{2.144}$$

We now transform from cartesian coordinates (x^1, x^2, x^3) to the curvilinear coordinates $(\theta^1, \theta^2, \theta^3)$ by using the inverse transformations

$$e_{rs} = \frac{\partial\theta^i}{\partial x^r}\frac{\partial\theta^j}{\partial x^s}\gamma_{ij}, \tag{2.145}$$
$$H^r = \frac{\partial x^r}{\partial\theta^a}h^a, \tag{2.146}$$
$$\delta^{rs} = \frac{\partial x^r}{\partial\theta^i}\frac{\partial x^s}{\partial\theta^j}g^{ij}. \tag{2.147}$$

Then (2.143) and (2.144) become

$$K_1^* = \gamma_{ij}h^ih^j, \tag{2.148}$$
$$K_2^* = \gamma_{ab}\gamma_{cd}g^{bd}h^ah^c - \left(\gamma_{ab}h^ah^b\right)^2. \tag{2.149}$$

Since the strain invariants I_1 and I_2 are expressed in terms g_{ik} and G_{ik} instead of in terms of γ_{ik}, we express K_1^* and K_2^* in terms of g_{ik} and G_{ik}. Since

$$\gamma_{ik} = \frac{1}{2}(G_{ik} - g_{ik}), \tag{2.150}$$

(2.148) and (2.149) may be written as

$$K_1^* = \frac{1}{2}(K_1 - 1), \tag{2.151}$$

$$K_2^* = \frac{1}{4}(K_2 - K_1^2), \tag{2.152}$$

where

$$K_1 = G_{ab}h^ah^b, \tag{2.153}$$
$$K_2 = G_{ab}G_{cd}g^{bd}h^ah^c. \tag{2.154}$$

We will work with K_1 and K_2 instead of with K_1^* and K_2^*. The strain energy function becomes

$$W^*(I_1, I_2, K_1^*, K_2^*) = W(I_1, I_2, K_1, K_2). \tag{2.155}$$

In the undeformed body, B_0,

$$\begin{aligned} I_1 &= I_2 = 3, \\ K_1 &= K_2 = 1. \end{aligned} \tag{2.156}$$

Substitute (2.155) into the constitutive equation (2.134). This gives

$$\begin{aligned} \tau^{ik} = \frac{1}{2}\Big[&\frac{\partial W}{\partial I_1}\Big(\frac{\partial I_1}{\partial \gamma_{ik}} + \frac{\partial I_1}{\partial \gamma_{ki}}\Big) + \frac{\partial W}{\partial I_2}\Big(\frac{\partial I_2}{\partial \gamma_{ik}} + \frac{\partial I_2}{\partial \gamma_{ki}}\Big) \\ &+ \frac{\partial W}{\partial K_1}\Big(\frac{\partial K_1}{\partial \gamma_{ik}} + \frac{\partial K_1}{\partial \gamma_{ki}}\Big) + \frac{\partial W}{\partial K_2}\Big(\frac{\partial K_2}{\partial \gamma_{ik}} + \frac{\partial K_2}{\partial \gamma_{ki}}\Big)\Big] + pG^{ik}. \end{aligned} \tag{2.157}$$

Since g^{ik} is the metric tensor in the undeformed body B_0 and h^i is the anisotropic director in the undeformed body B_0, they are independent of the state of strain γ_{ik}. Hence

$$\frac{\partial g^{rs}}{\partial \gamma_{ik}} = 0, \quad \frac{\partial h^r}{\partial \gamma_{ik}} = 0. \tag{2.158}$$

Also, since

$$G_{rs} = 2\gamma_{rs} + g_{rs}, \tag{2.159}$$

it follows that

$$\frac{\partial G_{rs}}{\partial \gamma_{ik}} = 2\delta^i_r \delta^k_s. \tag{2.160}$$

Hence using (2.141) for I_1, (2.61) for I_2, (2.143) for K_1 and (2.144) for K_2 we obtain

$$\frac{\partial I_1}{\partial \gamma_{ik}} + \frac{\partial I_1}{\partial \gamma_{ki}} = 4g^{ik}, \tag{2.161}$$

$$\frac{\partial I_2}{\partial \gamma_{ik}} + \frac{\partial I_2}{\partial \gamma_{ki}} = 4B^{ik}, \tag{2.162}$$

$$\frac{\partial K_1}{\partial \gamma_{ik}} + \frac{\partial K_1}{\partial \gamma_{ki}} = 4M^{ik}, \tag{2.163}$$

$$\frac{\partial K_2}{\partial \gamma_{ik}} + \frac{\partial K_2}{\partial \gamma_{ki}} = 4N^{ik}, \tag{2.164}$$

where

$$B^{ik} = \Big(g^{ik}g^{rs} - g^{ir}g^{ks}\Big)G_{rs}, \tag{2.165}$$

$$M^{ik} = h^i h^k, \tag{2.166}$$

$$N^{ik} = \Big(h^i g^{kr} + h^k g^{ir}\Big)G_{rs}h^s. \tag{2.167}$$

Substituting (2.161) to (2.164) into (2.157) gives

$$\tau^{ik} = 2\frac{\partial W}{\partial I_1}g^{ik} + 2\frac{\partial W}{\partial I_2}B^{ik} + 2\frac{\partial W}{\partial K_1}M^{ik} + 2\frac{\partial W}{\partial K_2}N^{ik} + pG^{ik}, \tag{2.168}$$

which may be written alternatively as

$$\tau^{ik} = \Phi g^{ik} + \Psi B^{ik} + \Theta M^{ik} + \Lambda N^{ik} + pG^{ik}, \qquad (2.169)$$

where

$$\Phi = 2\frac{\partial W}{\partial I_1}, \quad \Psi = 2\frac{\partial W}{\partial I_2}, \quad \Theta = 2\frac{\partial W}{\partial K_1}, \quad \Lambda = 2\frac{\partial W}{\partial K_2}. \qquad (2.170)$$

The scalar p is not determined by the deformation. It is determined from the field equations and boundary conditions. Equation (2.168) is the constitutive equation for an incompressible hyperelastic material which possesses transverse isotropy with respect to the direction $\boldsymbol{h}$.

The field equations for a transversely isotropic hyperelastic material are obtained by substituting the constitutive equation (2.168) into the momentum balance equation

$$\rho\frac{D\boldsymbol{V}}{Dt} = \tau^{ki}{}_{||k}\boldsymbol{G}_i + \rho\boldsymbol{F}. \qquad (2.171)$$

2.8 Strain-energy function for a transversely isotropic material

Shahinpoor (1974) considered the following strain-energy function for an incompressible hyperelastic material that possesses transverse isotropy :

$$W(I_1, I_2, K_1, K_2) = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(K_1 - 1) + C_4(K_2 - 1), \qquad (2.172)$$

where C_1, C_2, C_3 and C_4 are constants. When $C_3 = C_4 = 0$, (2.172) reduces to the Mooney-Rivlin strain-energy function for an incompressible isotropic material. We will use the strain-energy function (2.172) when we need to specify a strain-energy function.

A strain-energy function must satisfy the following two conditions.

Firstly, the strain energy W must vanish in the undeformed body B_0. Now in B_0,

$$I_1 = I_2 = 3, \quad K_1 = K_2 = 1. \qquad (2.173)$$

Thus $W = 0$ in B_0 as required.

Secondly, the undeformed body B_0 must be stress free. Now, in B_0,

$$G^{ik} = g^{ik} \qquad (2.174)$$

and therefore in B_0,

$$B^{ik} = 2g^{ik}, \quad M^{ik} = h^i h^k, \quad N^{ik} = 2h^i h^k. \qquad (2.175)$$

Thus, in the undeformed body B_0, (2.168) reduces to

$$\tau^{ik} = \Big(2\frac{\partial W}{\partial I_1} + 4\frac{\partial W}{\partial I_2} + p_0\Big)g^{ik} + 2\Big(\frac{\partial W}{\partial K_1} + 2\frac{\partial W}{\partial K_2}\Big)h^i h^k, \qquad (2.176)$$

where p_0 is the value of p in B_0. Thus, for arbitrary anisotropic directors h^i, B_0 will be stress free provided

$$2\frac{\partial W}{\partial I_1} + 4\frac{\partial W}{\partial I_2} + p_0 = 0, \tag{2.177}$$

$$\frac{\partial W}{\partial K_1} + 2\frac{\partial W}{\partial K_2} = 0. \tag{2.178}$$

Using the strain-energy function (2.172), conditions (2.177) and (2.178) become

$$2C_1 + 4C_2 + p_0 = 0, \tag{2.179}$$

$$C_3 + 2C_4 = 0. \tag{2.180}$$

Equation (2.179) determines p_0. Equation (2.180) is a condition on the constants C_3 and C_4 :

$$C_3 = -2C_4. \tag{2.181}$$

2.9 Summary

In this section the results derived in Chapter 2 which will be used in the remaining chapters are listed.

The constitutive equation for an incompressible hyperelastic material with strain-energy function $W(I_1, I_2, K_1, K_2)$ which possesses transverse isotropy with respect to the direction $\boldsymbol{h}$ is

$$\tau^{ik} = \Phi g^{ik} + \Psi B^{ik} + \Theta M^{ik} + \Lambda N^{ik} + pG^{ik}, \tag{2.182}$$

where

$$\Phi = 2\frac{\partial W}{\partial I_1}, \quad \Psi = 2\frac{\partial W}{\partial I_2}, \quad \Theta = 2\frac{\partial W}{\partial K_1}, \quad \Lambda = 2\frac{\partial W}{\partial K_2}, \tag{2.183}$$

and

$$\begin{aligned} B^{ik} &= \left(g^{ik}g^{rs} - g^{ir}g^{ks}\right)G_{rs}, & (2.184)\\ M^{ik} &= h^i h^k, & (2.185)\\ N^{ik} &= \left(h^i g^{kr} + h^k g^{ir}\right)G_{rs}h^s, & (2.186)\\ I_1 &= g^{ik}G_{ik}, \quad I_2 = g_{ik}G^{ik}, & (2.187)\\ K_1 &= G_{ab}h^a h^b, \quad K_2 = G_{ab}G_{cd}g^{bd}h^a h^c. & (2.188) \end{aligned}$$

The anisotropic director $\boldsymbol{h}$ is the unit vector

$$\begin{aligned} \boldsymbol{h} &= H^n \boldsymbol{i}_n = h^i \boldsymbol{g}_i, & (2.189)\\ \boldsymbol{h} \cdot \boldsymbol{h} &= \delta_{ik}H^i H^k = g_{ik}h^i h^k = 1, & (2.190)\\ h^i &= \frac{\partial \theta^i}{\partial x^n}H^n. & (2.191) \end{aligned}$$

A specific strain-energy function is

$$W(I_1, I_2, K_1, K_2) = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(K_1 - 1) + C_4(K_2 - 1), \tag{2.192}$$

where

$$C_3 = -2C_4. \tag{2.193}$$

2.10 Conclusions

In this chapter we reviewed the formulation of the equations of finite elasticity as given by Green and Zerna (1968) and Green and Adkins (1970). We derived the constitutive equation for a hyperelastic material that possesses transverse isotropy with respect to a direction $\boldsymbol{h}$ where $\boldsymbol{h}$ is unspecified . Green and Adkins (1970) considered only the case in which the anisotropic director $\boldsymbol{h}$ is in the direction of the 3-axis. We also reviewed a strain-energy function which will be used in subsequent chapters when the strain-energy function needs to be specified.

Chapter 3

Nonlinear Radial Oscillations of a Transversely Isotropic Incompressible Cylindrical Tube : General Results

3.1 Introduction

In this chapter a second order ordinary differential equation is derived which describes radial oscillations of a transversely isotropic incompressible hyperelastic cylindrical tube. The differential equation is expressed in terms of the strain-energy function and applies for radial, tangential and longitudinal transversely isotropic cylindrical tubes.

Nonlinear radial oscillations of an isotropic incompressible hyperelastic cylindrical tube were first considered by Knowles (1960, 1962). Shahinpoor and Nowinski (1971) then considered nonlinear radial oscillations in a thin-walled cylindrical tube and showed that the radial oscillations were governed by the Ermakov-Pinney equation. Mason and Roussos (2000) investigated the solution to higher order in the small parameter defined in terms of the thickness of the wall of the cylindrical tube. Roussos and Mason (1998) also investigated nonlinear radial oscillations in a thin-walled double-layer isotropic cylindrical tube.

Nonlinear radial oscillations in a transversely isotropic cylindrical tube were first considered by Huigol (1967). Huigol derived the equation describing nonlinear radial oscillations in a cylindrical tube with radial transverse isotropy and investigated the conditions on the strain-energy function for periodic solutions to exist. Huigol also outlined the modifications that have to be made to the equations if the anisotropy is in the tangential direction. Shahinpoor (1974) derived the equations for nonlinear radial oscillations in a transversely isotropic cylindrical tube and kept the components of the anisotropic director general. He then considered a thin-walled tube with longitudinal transverse isotropy. The equation describing the oscillations reduces to the Ermakov-Pinney equation which is the same as for nonlinear radial oscillations in an isotropic cylindrical tube.

An outline of the chapter is as follows. The mathematical model is described in

Section 3.2. In Section 3.3 the base vectors and the metric tensors in the strained and in the unstrained bodies are derived. The condition that the elastic material is incompressible is imposed. In Section 3.4 the anisotropic directors for radial, tangential and longitudinal transversely isotropic cylindrical tubes are calculated. The strain invariants are then obtained in Section 3.5. In Section 3.6 the components of the Cauchy stress tensor for cylindrical tubes with radial, tangential and longitudinal transverse isotropy are calculated. The boundary conditions are considered in Section 3.7. In Section 3.8 Cauchy's first law of motion is applied for the three cases of radial, tangential and longitudinal transversely isotropic tubes. In Section 3.9 a second order ordinary differential equation for the dimensionless inner radius of the cylindrical tube is derived from Cauchy's first law of motion. It is shown that when expressed in terms of the strain-energy function this equation has the same form for radial, tangential and longitudinal transversely isotropic tubes.

The equations derived in this chapter are quite general. It is not assumed that the cylindrical tube is thin-walled.

3.2 Mathematical formulation

The radial oscillations of an infinitely long cylindrical tube of incompressible transversely isotropic hyperelastic material are considered. Three cases of anisotropy are studied : radial, tangential and longitudinal transverse isotropy.

The coordinate systems in the unstrained body B_0 and the strained body B are shown in Figure 3.2.1.

Rectangular cartesian base vectors $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ are chosen with origin at the centre of the cylinder and with $\boldsymbol{i}_3$ along the longitudinal axis of the cylinder. The cartesian coordinates of a material point P_0 in the unstrained cylinder are (x^1, x^2, x^3) and the position vector of P_0 is

$$\boldsymbol{r} = x^n \boldsymbol{i}_n. \tag{3.1}$$

Cylindrical polar coordinates (ρ, θ, z) in the unstrained cylinder are defined by

$$x^1 = \rho\cos\theta, \quad x^2 = \rho\sin\theta, \quad x^3 = z. \tag{3.2}$$

The inner radius of the unstrained cylinder is ρ_1 and the outer radius is ρ_2. Hence

$$\rho_1 \leq \rho \leq \rho_2. \tag{3.3}$$

The material particle at P_0 in the unstrained body B_0 is displaced to the point P in the strained body B with position vector

$$\boldsymbol{R} = y^n \boldsymbol{i}_n. \tag{3.4}$$

Cylindrical polar coordinates (r, θ, z) in the strained body B are defined by

$$y^1 = r(\rho, t)\cos\theta, \quad y^2 = r(\rho, t)\sin\theta, \quad y^3 = z. \tag{3.5}$$

At time t, the inner radius of the strained cylinder is $r_1(t)$ and the outer radius of the strained cylinder is $r_2(t)$. Hence

$$r_1(t) \leq r \leq r_2(t). \tag{3.6}$$

A pressure $\mathcal{P}_1(t)$ is applied to the inner surface $r = r_1(t)$ of the cylindrical tube and a pressure $\mathcal{P}_2(t)$ is applied to the outer surface, $r = r_2(t)$.

Figure 3.2.1

Consider now the ends $z = \pm\infty$ (Knowles , 1960). For the motion to be axisymmetric there must be vanishing radial shear stress τ_{zr} and vanishing tangential shear stress $\tau_{z\theta}$ on the ends $z = \pm\infty$. Also, for the motion to be independent of z, the displacement in the $z-$ direction must vanish on the ends $z = \pm\infty$. Thus a suitable axial stress τ_{zz} must be applied on the ends $z = \pm\infty$ such that the displacement vanishes on the ends.

We consider coordinates of points in the strained body B as reference points since the form of B is known (Green and Zerna, 1968, p83). Cauchy's first law of motion is simpler in these coordinates. The curvilinear coordinate system $(\theta^1, \theta^2, \theta^3)$ is therefore taken to be the cylindrical polar coordinate system in the strained body B (Rogers and Ames , 1989). Thus

$$\theta^1 = r, \quad \theta^2 = \theta, \quad \theta^3 = z. \tag{3.7}$$

We summarise the coordinate systems in B_0 and B. In the unstrained body B_0:

$$\begin{aligned} x^1 &= \rho(r,t)\cos\theta, \quad x^1 = \rho(\theta^1,t)\cos\theta^2, \\ x^2 &= \rho(r,t)\sin\theta, \quad x^2 = \rho(\theta^1,t)\sin\theta^2, \\ x^3 &= z, \qquad\qquad\quad x^3 = \theta^3. \end{aligned} \tag{3.8}$$

In the strained body B:

$$\begin{aligned} y^1 &= r\cos\theta, \quad y^1 = \theta^1\cos\theta^2, \\ y^2 &= r\sin\theta, \quad y^2 = \theta^1\sin\theta^2, \\ y^3 &= z, \qquad\quad y^3 = \theta^3. \end{aligned} \tag{3.9}$$

This completes the mathematical formulation of the problem.

3.3 Base vectors, metric tensors and incompressibility condition

Consider first the strained body B.

The covariant base vectors, $\boldsymbol{G}_i$, are defined by

$$\boldsymbol{G}_i = \frac{\partial y^n}{\partial \theta^i}\, \boldsymbol{i}_n. \tag{3.10}$$

From (3.9) it is readily shown that

$$\begin{aligned} \boldsymbol{G}_1 &= \cos\theta\, \boldsymbol{i}_1 + \sin\theta\, \boldsymbol{i}_2, \\ \boldsymbol{G}_2 &= r(-\sin\theta\, \boldsymbol{i}_1 + \cos\theta\, \boldsymbol{i}_2), \\ \boldsymbol{G}_3 &= \boldsymbol{i}_3. \end{aligned} \tag{3.11}$$

The base vectors $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are drawn in Figure 3.3.1. The covariant components of the metric tensor, G_{ik}, are

$$G_{ik} = \boldsymbol{G}_i \cdot \boldsymbol{G}_k \tag{3.12}$$

Figure 3.3.1.

and therefore

$$[G_{ik}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.13}$$

The contravariant base vectors, $\boldsymbol{G}^i$, are defined by

$$\boldsymbol{G}^i = \frac{\partial \theta^i}{\partial y^n} \, \boldsymbol{i}^n. \tag{3.14}$$

Hence, from (3.9),

$$\begin{aligned} \boldsymbol{G}^1 &= \cos\theta \, \boldsymbol{i}_1 + \sin\theta \, \boldsymbol{i}_2, \\ \boldsymbol{G}^2 &= \frac{1}{r}(-\sin\theta \, \boldsymbol{i}_1 + \cos\theta \, \boldsymbol{i}_2), \\ \boldsymbol{G}^3 &= \boldsymbol{i}_3. \end{aligned} \tag{3.15}$$

The base vectors $\boldsymbol{G}^1$ and $\boldsymbol{G}^2$ are drawn in Figure 3.3.1. The contravariant components of the metric tensor, G^{ik}, are

$$G^{ik} = \boldsymbol{G}^i \cdot \boldsymbol{G}^k \tag{3.16}$$

and therefore

$$[G^{ik}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.17}$$

Consider next the unstrained body B_0.

The covariant base vectors, $\boldsymbol{g}_i$, are defined by

$$\boldsymbol{g}_i = \frac{\partial x^n}{\partial \theta^i} \, \boldsymbol{i}_n. \tag{3.18}$$

From (3.8) we therefore obtain

$$\begin{aligned} \boldsymbol{g}_1 &= \frac{\partial \rho}{\partial r}(\cos\theta \, \boldsymbol{i}_1 + \sin\theta \, \boldsymbol{i}_2), \\ \boldsymbol{g}_2 &= \rho(-\sin\theta \, \boldsymbol{i}_1 + \cos\theta \, \boldsymbol{i}_2), \\ \boldsymbol{g}_3 &= \boldsymbol{i}_3. \end{aligned} \tag{3.19}$$

The covariant components of the metric tensor, g_{ik}, are

$$g_{ik} = \boldsymbol{g}_i \cdot \boldsymbol{g}_k. \tag{3.20}$$

Hence

$$[g_{ik}] = \begin{bmatrix} \left(\frac{\partial \rho}{\partial r}\right)^2 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.21}$$

The contravariant base vectors, $\boldsymbol{g}^i$, are defined by

$$\boldsymbol{g}^i = \frac{\partial \theta^i}{\partial x^n} \, \boldsymbol{i}_n. \tag{3.22}$$

From (3.8) it follows that

$$\begin{aligned} \boldsymbol{g}^1 &= \frac{1}{\frac{\partial \rho}{\partial r}}(\cos\theta\ \boldsymbol{i}_1 + \sin\theta\ \boldsymbol{i}_2), \\ \boldsymbol{g}^2 &= \frac{1}{\rho}(-\sin\theta\ \boldsymbol{i}_1 + \cos\theta\ \boldsymbol{i}_2), \\ \boldsymbol{g}^3 &= \boldsymbol{i}_3. \end{aligned} \tag{3.23}$$

The contravariant components of the metric tensor, g^{ik}, are

$$g^{ik} = \boldsymbol{g}^i \cdot \boldsymbol{g}^k \tag{3.24}$$

and therefore

$$[g^{ik}] = \begin{bmatrix} \frac{1}{\left(\frac{\partial \rho}{\partial r}\right)^2} & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.25}$$

Now, the elastic material is incompressible. Hence from (2.62) and (2.87),

$$I_3 = \frac{G}{g} = 1. \tag{3.26}$$

But

$$G = \det[G_{ik}] = r^2, \quad g = \det[g_{ik}] = \rho^2 \left(\frac{\partial \rho}{\partial r}\right)^2 \tag{3.27}$$

and condition (3.26) becomes

$$\frac{\partial \rho}{\partial r} = \frac{r}{\rho}. \tag{3.28}$$

Integration of (3.28) gives

$$\rho^2 - r^2 = f(t), \tag{3.29}$$

where $f(t)$ is an arbitrary function of t. But $r = r_1(t)$ when $\rho = \rho_1$ and therefore

$$\rho^2 - r^2 = \rho_1^2 - r_1^2(t). \tag{3.30}$$

Also, $r = r_2(t)$ when $\rho = \rho_2$ and hence

$$\rho^2 - r^2 = \rho_2^2 - r_2^2(t). \tag{3.31}$$

Thus, from (3.30) and (3.31) ,

$$\rho^2 - r^2 = \rho_1^2 - r_1^2(t) = \rho_2^2 - r_2^2(t). \tag{3.32}$$

Further, using (3.28), the base vectors and metric tensors in the unstrained body B_0 can be written as

$$\begin{aligned} \boldsymbol{g}_1 &= \frac{r}{\rho}(\cos\theta\ \boldsymbol{i}_1 + \sin\theta\ \boldsymbol{i}_2), & \boldsymbol{g}^1 &= \frac{\rho}{r}(\cos\theta\ \boldsymbol{i}_1 + \sin\theta\ \boldsymbol{i}_2), \\ \boldsymbol{g}_2 &= \rho(-\sin\theta\ \boldsymbol{i}_1 + \cos\theta\ \boldsymbol{i}_2), & \boldsymbol{g}^2 &= \frac{1}{\rho}(-\sin\theta\ \boldsymbol{i}_1 + \cos\theta\ \boldsymbol{i}_2), \\ \boldsymbol{g}_3 &= \boldsymbol{i}_3, & \boldsymbol{g}^3 &= \boldsymbol{i}_3\,, \end{aligned} \tag{3.33}$$

and

$$[g_{ik}] = \begin{bmatrix} \frac{r^2}{\rho^2} & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ik} = \begin{bmatrix} \frac{\rho^2}{r^2} & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.34}$$

3.4 Anisotropic directors

Consider now the anisotropic directors, $\boldsymbol{h}$, for cylindrical tubes that possess a radial, tangential or longitudinal transverse isotropy. The anisotropic director is a unit vector with cartesian components H^n and curvilinear components h^i in the unstrained body B_0 :

$$\boldsymbol{h} = H^n\,\boldsymbol{i}_n = h^i\,\boldsymbol{g}_i, \tag{3.35}$$

$$h^i = \frac{\partial \theta^i}{\partial x^n}\,H^n, \tag{3.36}$$

$$\boldsymbol{h}\cdot\boldsymbol{h} = g_{ik}\,h^i h^k = \delta_{ik} H^i H^k. \tag{3.37}$$

Consider first a radial transversely isotropic cylindrical tube. Then $\boldsymbol{h}$ is a unit vector in the radial direction in the unstrained body B_0 as shown in Figure 3.4.1 :

$$\boldsymbol{h} = \cos\theta\boldsymbol{i}_1 + \sin\theta\boldsymbol{i}_2. \tag{3.38}$$

The cartesian components of $\boldsymbol{h}$ in B_0 are therefore

$$H^1 = \cos\theta, \quad H^2 = \sin\theta, \quad H^3 = 0. \tag{3.39}$$

The curvilinear components, h^i, in B_0 are given by (3.36). Now, from (3.8) and using the incompressibility condition (3.28) it can be verified that

$$\begin{array}{lll}
\dfrac{\partial\theta^1}{\partial x^1} = \dfrac{\rho}{r}\cos\theta, & \dfrac{\partial\theta^1}{\partial x^2} = \dfrac{\rho}{r}\sin\theta, & \dfrac{\partial\theta^1}{\partial x^3} = 0, \\
\dfrac{\partial\theta^2}{\partial x^1} = -\dfrac{\sin\theta}{\rho}, & \dfrac{\partial\theta^2}{\partial x^2} = \dfrac{\cos\theta}{\rho}, & \dfrac{\partial\theta^2}{\partial x^3} = 0, \\
\dfrac{\partial\theta^3}{\partial x^1} = 0, & \dfrac{\partial\theta^3}{\partial x^2} = 0, & \dfrac{\partial\theta^3}{\partial x^3} = 1.
\end{array} \tag{3.40}$$

Hence, from (3.36), (3.39) and (3.40), the anisotropic director in the radial direction has components :

$$h^1 = \frac{\rho}{r}, \quad h^2 = 0, \quad h^3 = 0. \tag{3.41}$$

Consider next a cylindrical tube with tangential transverse isotropy. Then the anisotropic director is a unit vector in the tangential direction in the unstrained body B_0 as shown in Figure 3.4.1 :

$$\boldsymbol{h} = -\sin\theta\boldsymbol{i}_1 + \cos\theta\boldsymbol{i}_2. \tag{3.42}$$

Thus the cartesian components of $\boldsymbol{h}$ in B_0 are

$$H^1 = -\sin\theta, \quad H^2 = \cos\theta, \quad H^3 = 0. \tag{3.43}$$

The curvilinear components h^i in B_0 are given by (3.36) with (3.40) and (3.43) :

$$h^1 = 0, \quad h^2 = \frac{1}{\rho}, \quad h^3 = 0. \tag{3.44}$$

Figure 3.4.1

Consider finally a cylindrical tube with longitudinal transverse isotropy. The anisotropic director, $\boldsymbol{h}$, is a unit vector in the direction of the axis of the cylinder in the unstrained body B_0 as shown in Figure 3.4.1 :

$$\boldsymbol{h} = \boldsymbol{i}_3. \tag{3.45}$$

Hence the cartesian components of $\boldsymbol{h}$ in B_0 are

$$H^1 = 0, \quad H^2 = 0, \quad H^3 = 1. \tag{3.46}$$

The curvilinear components h^i in B_0 are obtained from (3.36), (3.40) and (3.46) :

$$h^1 = 0, \quad h^2 = 0, \quad h^3 = 1. \tag{3.47}$$

The anisotropic directors $\boldsymbol{h}$ for a cylindrical tube with a radial, tangential and longitudinal isotropy are therefore given by (3.41), (3.44) and (3.47), respectively. It is readily verified that the components h^i satisfy the condition (3.37) that $\boldsymbol{h}$ is a unit vector.

3.5 Strain invariants

Since the elastic material of the cylindrical tube is incompressible, $I_3 = 1$. There are therefore only four strain invariants given by (2.88), (2.89), (2.153) and (2.154):

$$I_1 = g^{ik}\, G_{ik}, \quad I_2 = G^{ik}\, g_{ik}, \tag{3.48}$$

$$K_1 = G_{ab}\, h^a h^b, \quad K_2 = G_{ab} G_{cd}\, g^{bd} h^a h^c. \tag{3.49}$$

Using (3.13), (3.17) and (3.34) the strain invariants (3.48) are

$$I_1 = I_2 = \frac{r^2}{\rho^2} + \frac{\rho^2}{r^2} + 1. \tag{3.50}$$

The remaining two strain invariants depend on the kind of anisotropy of the cylindrical tube. For a radial transversely isotropic tube, $\boldsymbol{h}$ is given by (3.41) and

$$K_1 = \frac{\rho^2}{r^2}, \quad K_2 = \frac{\rho^4}{r^4}. \tag{3.51}$$

For a tangential transversely isotropic tube, $\boldsymbol{h}$ is given by (3.44) and

$$K_1 = \frac{r^2}{\rho^2}, \quad K_2 = \frac{r^4}{\rho^4}. \tag{3.52}$$

For longitudinal transversely isotropic tube, $\boldsymbol{h}$ is given by (3.47) and

$$K_1 = 1, \quad K_2 = 1. \tag{3.53}$$

Except for (3.53), the strain invariants are not constant.

3.6 Cauchy stress tensor

From (2.169) the constitutive equation for a transversely isotropic incompressible material is

$$\tau^{ik} = \Phi g^{ik} + \Psi B^{ik} + \Theta M^{ik} + \Lambda N^{ik} + pG^{ik}, \tag{3.54}$$

where from (2.165) to (2.167),

$$B^{ik} = \Big(g^{ik}g^{rs} - g^{ir}g^{ks}\Big)G_{rs}, \tag{3.55}$$

$$M^{ik} = h^i h^k, \tag{3.56}$$

$$N^{ik} = \Big(h^i g^{kr} + h^k g^{ir}\Big)G_{rs}h^s \tag{3.57}$$

and from (2.170)

$$\Phi = 2\frac{\partial W}{\partial I_1}, \quad \Psi = 2\frac{\partial W}{\partial I_2}, \quad \Theta = 2\frac{\partial W}{\partial K_1}, \quad \Lambda = 2\frac{\partial W}{\partial K_2}. \tag{3.58}$$

The scalar $p(r, \theta, z)$ is determined from the field equations and boundary conditions.

The metric tensors G^{ik}, G_{ik} and g^{ik} are given by (3.13), (3.17) and (3.34). It follows that

$$[B^{ik}] = \text{diag}\Big[1 + \frac{\rho^2}{r^2}, \quad \frac{1}{r^2} + \frac{1}{\rho^2}, \quad \frac{r^2}{\rho^2} + \frac{\rho^2}{r^2}\Big]. \tag{3.59}$$

The tensors M^{ik} and N^{ik} depend on which anisotropic cylindrical tube is considered.

For a radial transversely isotropic tube, $\boldsymbol{h}$ is given by (3.41) and

$$[M^{ik}] = \text{diag}\Big[\frac{\rho^2}{r^2}, \quad 0, \quad 0\Big], \tag{3.60}$$

$$[N^{ik}] = \text{diag}\Big[2\frac{\rho^4}{r^4}, \quad 0, \quad 0\Big]. \tag{3.61}$$

The components of the Cauchy stress tensor are

$$\tau^{11} = \frac{\rho^2}{r^2}\Phi + \Big(1 + \frac{\rho^2}{r^2}\Big)\Psi + \frac{\rho^2}{r^2}\Theta + 2\frac{\rho^4}{r^4}\Lambda + p, \tag{3.62}$$

$$\tau^{22} = \frac{1}{r^2}\Big[\frac{r^2}{\rho^2}\Phi + \Big(1 + \frac{r^2}{\rho^2}\Big)\Psi + p\Big], \tag{3.63}$$

$$\tau^{33} = \Phi + \Big(\frac{r^2}{\rho^2} + \frac{\rho^2}{r^2}\Big)\Psi + p, \tag{3.64}$$

$$\tau^{ik} = 0, \quad i \neq k. \tag{3.65}$$

For a tangential transversely isotropic tube, $\boldsymbol{h}$ is given by (3.44) and

$$[M^{ik}] = \text{diag}\Big[0, \quad \frac{1}{\rho^2}, \quad 0\Big], \tag{3.66}$$

$$[N^{ik}] = \text{diag}\Big[0, \quad 2\frac{r^2}{\rho^4}, \quad 0\Big]. \tag{3.67}$$

The components of the Cauchy stress tensor are

$$\tau^{11} = \frac{\rho^2}{r^2}\Phi + \Big(1 + \frac{\rho^2}{r^2}\Big)\Psi + p, \tag{3.68}$$

$$\tau^{22} = \frac{1}{r^2}\Big[\frac{r^2}{\rho^2}\Phi + \Big(1 + \frac{r^2}{\rho^2}\Big)\Psi + \frac{r^2}{\rho^2}\Theta + 2\frac{r^4}{\rho^4}\Lambda + p\Big], \tag{3.69}$$

$$\tau^{33} = \Phi + \Big(\frac{r^2}{\rho^2} + \frac{\rho^2}{r^2}\Big)\Psi + p, \tag{3.70}$$

$$\tau^{ik} = 0, \;\; i \neq k. \tag{3.71}$$

Finally, for a longitudinal transversely isotropic tube, $\boldsymbol{h}$ is given by (3.47) and

$$[M^{ik}] = \mathrm{diag}\Big[0, \quad 0, \quad 1\Big], \tag{3.72}$$

$$[N^{ik}] = \mathrm{diag}\Big[0, \quad 0, \quad 2\Big]. \tag{3.73}$$

The components of the Cauchy stress tensor are

$$\tau^{11} = \frac{\rho^2}{r^2}\Phi + \Big(1 + \frac{\rho^2}{r^2}\Big)\Psi + p, \tag{3.74}$$

$$\tau^{22} = \frac{1}{r^2}\Big[\frac{r^2}{\rho^2}\Phi + \Big(1 + \frac{r^2}{\rho^2}\Big)\Psi + p\Big], \tag{3.75}$$

$$\tau^{33} = \Phi + \Big(\frac{r^2}{\rho^2} + \frac{\rho^2}{r^2}\Big)\Psi + \Theta + 2\Lambda + p, \tag{3.76}$$

$$\tau^{ik} = 0, \;\; i \neq k. \tag{3.77}$$

The Cauchy stress tensor for an incompressible isotropic elastic tube is obtained by setting $\Theta = \Lambda = 0$. We see that the Cauchy stress tensor for a radial transversely isotropic tube differs from that for an isotropic tube by two additional terms in τ^{11}, a tangential transversely isotropic tube by two additional terms in τ^{22} and a longitudinal transversely isotropic tube by two additional terms in τ^{33}.

3.7 Boundary conditions

A pressure $\mathcal{P}_1(t)$ is applied to the inner surface $r = r_1(t)$ and a pressure $\mathcal{P}_2(t)$ is applied to the outer surface $r = r_2(t)$ of the cylindrical tube as shown in Figure 3.7.1. A suitable axial stress must be applied to the ends $z = \pm\infty$ to ensure that the displacement in the $z-$direction vanishes. Since $\tau^{ik} = 0$ for $i \neq k$, the radial shear stress τ_{zr} and the tangential shear stress $\tau_{z\theta}$ vanish at the ends $z = \pm\infty$ as required.

The boundary conditions are obtained using Cauchy's formula (2.92) :

$$P^i \boldsymbol{G}_i = n_k \tau^{ki} \boldsymbol{G}_i \tag{3.78}$$

where $\boldsymbol{P}$ is the applied surface traction,

$$\boldsymbol{n} = n_k \boldsymbol{G}^k \tag{3.79}$$

Figure 3.7.1

is the unit outward normal vector to the surface and $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$ are covariant and contravariant base vectors. Consider first the inner surface of the cylindrical tube $r = r_1(t)$. The contravariant base vectors are given by (3.15). The base vector $\boldsymbol{G}^1$ is a unit vector in the radial direction. From Figure 3.7.1, the unit outward normal vector $\boldsymbol{n}$ is given by

$$\boldsymbol{n} = -\boldsymbol{G}^1 \tag{3.80}$$

and therefore

$$n_k = -\delta^1_k. \tag{3.81}$$

Since also $\tau^{ik} = 0$ for $i \neq k$, Cauchy's formula (3.78) becomes

$$P^i \boldsymbol{G}_i = -\tau^{11}(r_1(t), t)\boldsymbol{G}_1. \tag{3.82}$$

The covariant base vectors are given by (3.11). The base vector $\boldsymbol{G}_1$ is a unit vector in the radial direction. From Figure 3.7.1,

$$P^i \boldsymbol{G}_i = \mathcal{P}_1(t)\boldsymbol{G}_1 \tag{3.83}$$

and therefore

$$\mathcal{P}_1(t)\boldsymbol{G}_1 = -\tau^{11}(r_1(t), t)\boldsymbol{G}_1. \tag{3.84}$$

Hence

$$\tau^{11}(r_1(t), t) = -\mathcal{P}_1(t). \tag{3.85}$$

Consider next the outer surface of the cylindrical tube $r = r_2(t)$. The unit outward normal vector is

$$\boldsymbol{n} = \boldsymbol{G}^1 \tag{3.86}$$

and therefore

$$n_k = \delta^1_k. \tag{3.87}$$

Cauchy's formula (3.78) becomes

$$P^i \boldsymbol{G}_i = \tau^{11}(r_2(t), t)\boldsymbol{G}_1. \tag{3.88}$$

But from Figure 3.7.1,

$$P^i \boldsymbol{G}_i = -\mathcal{P}_2(t)\boldsymbol{G}_1 \tag{3.89}$$

and therefore

$$-\mathcal{P}_2(t)\boldsymbol{G}_1 = \tau^{11}(r_2(t), t)\boldsymbol{G}_1. \tag{3.90}$$

Thus

$$\tau^{11}(r_2(t), t) = -\mathcal{P}_2(t). \tag{3.91}$$

Consider now the end $z = +\infty$. The unit normal vector is

$$\boldsymbol{n} = \boldsymbol{G}^3, \quad n_k = \delta^3_k \tag{3.92}$$

and Cauchy's formula (3.78) becomes

$$\boldsymbol{P} = \tau^{33}(r, t)\,\boldsymbol{G}_3. \tag{3.93}$$

A surface traction $\boldsymbol{P}$ given by (3.93) must therefore be applied to the end $z = +\infty$.

Similarly, a surface traction

$$\boldsymbol{P} = -\tau^{33}(r, t)\,\boldsymbol{G}_3 \tag{3.94}$$

must be applied to the end $z = -\infty$.

3.8 Cauchy's first law of motion

Cauchy's first law of motion in the strained body, B, is given by (2.98) :

$$\rho^* \frac{D\boldsymbol{V}}{Dt} = \tau^{ki}{}_{\|k} \boldsymbol{G_i} + \rho^* \boldsymbol{F}, \tag{3.95}$$

where ρ^* is the density of the elastic material and $\boldsymbol{F}$ is the body force per unit mass. Since the material is incompressible, ρ^* is constant. The body force $\boldsymbol{F}$ is neglected.

Consider first the left hand side of (3.95). Now

$$\boldsymbol{V} = \frac{D\boldsymbol{R}}{Dt}, \tag{3.96}$$

where $\boldsymbol{R}$ is the position vector of the material particle in the strained body B and $\frac{D}{Dt}$ is the partial derivative with respect to t keeping the cylindrical polar coordinates (ρ, θ, z) fixed. But $\boldsymbol{G}_1$ and $\boldsymbol{G}_3$ are unit vectors in the radial and $z-$directions as shown in Figure 3.8.1 and therefore

$$\boldsymbol{R} = r(\rho, t)\boldsymbol{G}_1 + z\boldsymbol{G}_3. \tag{3.97}$$

Hence

$$\boldsymbol{V} = \left.\frac{\partial \boldsymbol{R}}{\partial t}\right|_{(\rho,\theta,z)} = \left.\frac{\partial r}{\partial t}\right|_{\rho} \boldsymbol{G}_1 \tag{3.98}$$

and

$$\frac{D\boldsymbol{V}}{Dt} = \left.\frac{\partial \boldsymbol{V}}{\partial t}\right|_{(\rho,\theta,z)} = \left.\frac{\partial^2 r}{\partial t^2}\right|_{\rho} \boldsymbol{G}_1. \tag{3.99}$$

Consider next the right hand side of (3.95). Now

$$\tau^{ki}{}_{\|k} = \tau^{ki}{}_{,k} + \Gamma^{k}_{ks}\,\tau^{si} + \Gamma^{i}_{ks}\,\tau^{ks}, \tag{3.100}$$

where Γ^{i}_{jk} is the Christoffel symbol of the second kind of the metric tensor G_{ik}. For cylindrical polar coordinates (r, θ, z), G_{ik} is given by (3.13) and

$$\Gamma^{1}_{22} = -r, \quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r}, \tag{3.101}$$

$$\Gamma^{i}_{jk} = 0, \text{ otherwise.} \tag{3.102}$$

Also,

$$\Gamma^{k}_{k1} = \frac{1}{r}, \quad \Gamma^{k}_{k2} = 0, \quad \Gamma^{k}_{k3} = 0. \tag{3.103}$$

Using (3.99) and (3.100), Cauchy's first law of motion (3.95) becomes

$$\rho^* \left.\frac{\partial^2 r}{\partial t^2}\right|_{\rho} \boldsymbol{G}_1 = \left(\tau^{ki}{}_{,k} + \Gamma^{k}_{ks}\,\tau^{si} + \Gamma^{i}_{ks}\,\tau^{ks}\right) \boldsymbol{G_i}. \tag{3.104}$$

But $\tau^{ik} = 0$ for $i \neq k$ for the three transversely isotropic stress tensors considered. Hence

$$i = 1: \quad \rho^* \frac{\partial^2 r}{\partial t^2} = \frac{\partial \tau^{11}}{\partial r} + \frac{1}{r}\tau^{11} - r\tau^{22}, \tag{3.105}$$

$$i = 2: \quad 0 = \frac{\partial \tau^{22}}{\partial \theta}, \tag{3.106}$$

$$i = 3: \quad 0 = \frac{\partial \tau^{33}}{\partial z}. \tag{3.107}$$

Figure 3.8.1

But for the three transversely isotropic tubes we are considering, the strain invariants I_1, I_2, K_1 and K_2, given by (3.50) and (3.51) to (3.53), depend on r and t only. Thus the strain energy function $W(I_1, I_2, K_1, K_2)$ depends on r and t only and therefore Φ, Ψ, Θ and Λ defined by (3.58) depend on r and t only. Hence the stress tensors τ^{ik} can depend on θ and z only through $p(r, \theta, z, t)$. Equations (3.106) and (3.107) become

$$\frac{\partial p}{\partial \theta} = 0, \qquad \frac{\partial p}{\partial z} = 0 \tag{3.108}$$

and therefore

$$p = p(r, t). \tag{3.109}$$

In order to evaluate $\left.\frac{\partial^2 r}{\partial t^2}\right|_\rho$ on the left hand side of (3.104) consider the incompressibility condition. From (3.32),

$$r^2 = \rho^2 - \rho_1^2 + r_1^2(t) \tag{3.110}$$

and therefore

$$\left.\frac{\partial r}{\partial t}\right|_\rho = \frac{r_1(t)\,\dot{r}_1(t)}{r} \tag{3.111}$$

and

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_\rho = \left(\dot{r}_1^2(t) + r_1(t)\ddot{r}_1(t)\right)\frac{1}{r} - r_1^2(t)\dot{r}_1^2(t)\frac{1}{r^3}. \tag{3.112}$$

Cauchy's first law of motion therefore reduces to (3.105) and (3.109) where the acceleration is given by (3.112). The results apply for the three transversely isotropic tubes we are considering. To proceed further it is necessary to specify τ^{ik} and therefore to consider each transversely isotropic tube separately, at least for part of the derivation.

3.9 Ordinary differential equation for dimensionless inner radius of the cylindrical tube

It was shown in Section 3.8 that for radial, tangential and longitudinal transversely isotropic cylindrical tubes,

$$\rho^* \frac{\partial^2 r}{\partial t^2} = \frac{\partial \tau^{11}}{\partial r} + \frac{1}{r}\tau^{11} - r\tau^{22}, \tag{3.113}$$

$$p = p(r, t), \tag{3.114}$$

where

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_\rho = \left(\dot{r}_1^2(t) + r_1(t)\ddot{r}_1(t)\right)\frac{1}{r} - r_1^2(t)\dot{r}_1^2(t)\frac{1}{r^3}. \tag{3.115}$$

Also, from (3.85) and (3.91) the boundary conditions are

$$\tau^{11}(r_1(t), t) = -\mathcal{P}_1(t), \tag{3.116}$$

$$\tau^{11}(r_2(t),t) = -\mathcal{P}_2(t). \tag{3.117}$$

We now derive an ordinary differential equation for the dimensionless inner radius of the cylindrical tube. Since τ^{11} and τ^{22} need to be specified in (3.113) the three cases of radial, tangential and longitudinal transversely isotropic tubes need to be treated separately in the first part of the derivation.

3.9.1 Radial transversely isotropic cylindrical tube

For a radial transversely isotropic tube, τ^{11} and τ^{22} are given by (3.62) and (3.63) where Φ, Ψ, Θ and Λ are given by (3.58). Hence

$$\frac{1}{r}\tau^{11} - r\tau^{22} = \frac{2}{r}\left[\left(\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) + \frac{\rho^2}{r^2}\frac{\partial W}{\partial K_1} + 2\frac{\rho^4}{r^4}\frac{\partial W}{\partial K_2}\right] \tag{3.118}$$

and equation (3.113) becomes

$$\begin{aligned}
&\rho^*\left[\left(\dot{r}_1^2(t) + r_1(t)\ddot{r}_1(t)\right)\frac{1}{r} - r_1^2(t)\dot{r}_1^2(t)\frac{1}{r^3}\right] \\
= \ & \frac{\partial \tau^{11}}{\partial r} + \frac{2}{r}\left[\left(\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\right. \\
+ \ & \left.\frac{\rho^2}{r^2}\frac{\partial W}{\partial K_1} + 2\frac{\rho^4}{r^4}\frac{\partial W}{\partial K_2}\right].
\end{aligned} \tag{3.119}$$

Integrate equation (3.119) with respect to r from $r = r_1(t)$ to $r = r_2(t)$. This gives

$$\begin{aligned}
&\rho^*\left[\left(\dot{r}_1^2(t) + r_1(t)\ddot{r}_1(t)\right)\ln\left(\frac{r_2(t)}{r_1(t)}\right) + \frac{1}{2}\dot{r}_1^2(t)\left(\frac{r_1^2(t)}{r_2^2(t)} - 1\right)\right] \\
= \ & \tau^{11}(r_2(t),t) - \tau^{11}(r_1(t),t) + U(t),
\end{aligned} \tag{3.120}$$

where

$$U(t) = 2\int_{r_1(t)}^{r_2(t)}\left[\left(\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) + \frac{\rho^2}{r^2}\frac{\partial W}{\partial K_1} + 2\frac{\rho^4}{r^4}\frac{\partial W}{\partial K_2}\right]\frac{dr}{r}. \tag{3.121}$$

Using the boundary conditions (3.115) and (3.116), equation (3.119) can be expressed as

$$\begin{aligned}
&\rho^*\left[\left(\dot{r}_1^2(t) + r_1(t)\ddot{r}_1(t)\right)\ln\left(\frac{r_2(t)}{r_1(t)}\right) + \frac{1}{2}\dot{r}_1^2(t)\left(\frac{r_1^2(t)}{r_2^2(t)} - 1\right)\right] \\
= \ & \mathcal{P}_1(t) - \mathcal{P}_2(t) + U(t).
\end{aligned} \tag{3.122}$$

The boundary conditions are therefore included in the final differential equation which will be obtained.

Consider $U(t)$ and let

$$u = \frac{r}{\rho}. \tag{3.123}$$

Then

$$du = \frac{dr}{\rho} - \frac{r}{\rho^2}\frac{\partial \rho}{\partial r}\,dr \tag{3.124}$$

and using the incompressibility condition in the form

$$\frac{\partial \rho}{\partial r} = \frac{r}{\rho}, \tag{3.125}$$

(3.124) becomes

$$\frac{dr}{r} = \frac{du}{u(1-u^2)}. \tag{3.126}$$

Thus, (3.121) can be written as

$$U(t) = 2\int_{\frac{r_1(t)}{\rho_1}}^{\frac{r_2(t)}{\rho_2}} \left[\left(\frac{1}{u^2} - u^2\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) + \frac{1}{u^2}\frac{\partial W}{\partial K_1} + \frac{2}{u^4}\frac{\partial W}{\partial K_2}\right]\frac{du}{u(1-u^2)}. \tag{3.127}$$

But, expressed in terms of u, the strain invariants I_1 and I_2 given by (3.50) become

$$I_1 = I_2 = u^2 + \frac{1}{u^2} + 1. \tag{3.128}$$

Also, for radial transverse isotropy, K_1 and K_2 are given by (3.51) which, expressed in terms of u, become

$$K_1 = \frac{1}{u^2}, \qquad K_2 = \frac{1}{u^4}. \tag{3.129}$$

Now, since $W = W(I_1, I_2, K_1, K_2)$, it follows that

$$\begin{aligned} \frac{dW}{du} &= \frac{\partial W}{\partial I_1}\frac{dI_1}{du} + \frac{\partial W}{\partial I_2}\frac{dI_2}{du} + \frac{\partial W}{\partial K_1}\frac{dK_1}{du} + \frac{\partial W}{\partial K_2}\frac{dK_2}{du} \\ &= -\frac{2}{u}\left[\left(\frac{1}{u^2} - u^2\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) + \frac{1}{u^2}\frac{\partial W}{\partial K_1} + \frac{2}{u^4}\frac{\partial W}{\partial K_2}\right]. \end{aligned} \tag{3.130}$$

Hence, (3.127) becomes

$$U(t) = -\int_{\frac{r_1(t)}{\rho_1}}^{\frac{r_2(t)}{\rho_2}} \frac{1}{(1-u^2)}\frac{dW}{du}\,du. \tag{3.131}$$

Equation (3.122) therefore becomes

$$\begin{aligned} &\rho^*\left[\left(\dot{r}_1^2(t) + r_1(t)\ddot{r}_1(t)\right)\ln\left(\frac{r_2(t)}{r_1(t)}\right) + \frac{1}{2}\dot{r}_1^2(t)\left(\frac{r_1^2(t)}{r_2^2(t)} - 1\right)\right] \\ &+ \int_{\frac{r_1(t)}{\rho_1}}^{\frac{r_2(t)}{\rho_2}} \frac{1}{(1-u^2)}\frac{dW}{du}\,du = \mathcal{P}_1(t) - \mathcal{P}_2(t). \end{aligned} \tag{3.132}$$

Equation (3.132) for a radial transversely isotropic cylindrical tube has the same form as the equation for an isotropic tube (Knowles , 1960 , 1962). We now show that equation (3.132) is also obtained for tangential and longitudinal transversely isotropic tubes.

3.9.2 Tangential transversely isotropic cylindrical tube

For a tangential transversely isotropic tube, τ^{11} and τ^{22} are given by (3.68) and (3.69) with Φ, Ψ, Θ and Λ given by (3.58). Thus

$$\frac{1}{r}\tau^{11} - r\tau^{22} = \frac{2}{r}\left[\left(\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) - \frac{r^2}{\rho^2}\frac{\partial W}{\partial K_1} - 2\frac{r^4}{\rho^4}\frac{\partial W}{\partial K_2}\right] \tag{3.133}$$

and equation (3.113) becomes

$$\begin{aligned} &\rho^*\left[\left(\dot{r}_1^2(t) + r_1(t)\ddot{r}_1(t)\right)\frac{1}{r} - r_1^2(t)\dot{r}_1^2(t)\frac{1}{r^3}\right] \\ = \; &\frac{\partial\tau^{11}}{\partial r} + \frac{2}{r}\left[\left(\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) - \frac{r^2}{\rho^2}\frac{\partial W}{\partial K_1} - 2\frac{r^4}{\rho^4}\frac{\partial W}{\partial K_2}\right]. \end{aligned} \tag{3.134}$$

If equation (3.134) is integrated with respect to r from $r = r_1(t)$ to $r = r_2(t)$ and the boundary conditions (3.116) and (3.117) are imposed then equation (3.122) is again obtained but with $U(t)$ given by

$$U(t) = 2\int_{r_1(t)}^{r_2(t)} \left[\left(\frac{\rho^2}{r^2} - \frac{r^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) - \frac{r^2}{\rho^2}\frac{\partial W}{\partial K_1} - 2\frac{r^4}{\rho^4}\frac{\partial W}{\partial K_2}\right]\frac{dr}{r}. \tag{3.135}$$

By making the change of variable (3.123), $U(t)$ becomes

$$\begin{aligned} U(t) = 2\int_{\frac{r_1(t)}{\rho_1}}^{\frac{r_2(t)}{\rho_2}} \quad &\left[\left(\frac{1}{u^2} - u^2\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\right. \\ &\left. -u^2\frac{\partial W}{\partial K_1} - 2u^4\frac{\partial W}{\partial K_2}\right]\frac{du}{u(1-u^2)}. \end{aligned} \tag{3.136}$$

But I_1 and I_2 when expressed in terms of u are given by (3.128). Also, for tangential transverse isotropic tubes, K_1 and K_2 are given by (3.52) which expressed in terms of u become

$$K_1 = u^2, \qquad K_2 = u^4. \tag{3.137}$$

Thus, since $W = W(I_1, I_2, K_1, K_2)$,

$$\frac{dW}{du} = -\frac{2}{u}\left[\left(\frac{1}{u^2} - u^2\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) - u^2\frac{\partial W}{\partial K_1} - 2u^4\frac{\partial W}{\partial K_2}\right]. \tag{3.138}$$

Hence (3.136) for $U(t)$ becomes (3.131) and equation (3.132) is again derived.

3.9.3 Longitudinal transversely isotropic cylindrical tube

For longitudinal transversely isotropic tubes, τ^{11} and τ^{22} are given by (3.74) and (3.75), which are independent of anisotropic terms. The anisotropic terms occur only in τ^{33}. Hence, equation (3.122) is again obtained but with the anisotropic terms absent from (3.127) for $U(t)$:

$$U(t) = 2\int_{\frac{r_1(t)}{\rho_1}}^{\frac{r_2(t)}{\rho_2}} \left[\left(\frac{1}{u^2} - u^2\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\frac{du}{u(1-u^2)}\right. \tag{3.139}$$

The strain invariants I_1 and I_2, expressed in terms of u, are given by (3.128). For a longitudinal transversely isotropic tube, K_1 and K_2 are given by (3.53):

$$K_1 = K_2 = 1. \tag{3.140}$$

Thus, although $W = W(I_1, I_2, K_1, K_2)$, since K_1 and K_2 are constants,

$$\frac{dW}{du} = -\frac{2}{u}\left[\left(\frac{1}{u^2} - u^2\right)\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\right]. \tag{3.141}$$

Equation (3.131) for $U(t)$ is again derived and therefore the differential equation (3.132) is again obtained.

3.9.4 Ordinary differential equation

In this subsection the three cases of radial, tangential and longitudinal transversely isotropic cylindrical tubes are treated together.

The differential equation (3.132) applies for radial, tangential and longitudinal transversely isotropic tubes. Let

$$x(t) = \frac{r_1(t)}{\rho_1}. \tag{3.142}$$

Then $x(t)$ is the dimensionless inner radius of the cylindrical tube. We derive from (3.132) a second order ordinary differential equation for $x(t)$.

It is first necessary to express $r_2(t)$ in terms of $x(t)$. From the incompressibility condition (3.32),

$$r_2^2(t) = r_1^2(t) + \rho_2^2 - \rho_1^2 \tag{3.143}$$

and therefore

$$r_2^2(t) = \rho_1^2\left[x^2 + \left(\frac{\rho_2}{\rho_1}\right)^2 - 1\right]. \tag{3.144}$$

Define

$$\mu = \left(\frac{\rho_2}{\rho_1}\right)^2 - 1. \tag{3.145}$$

Then

$$r_2(t) = \rho_1 x\left(1 + \frac{\mu}{x^2}\right)^{\frac{1}{2}}, \tag{3.146}$$

$$\rho_2 = \rho_1\big(1+\mu\big)^{\frac{1}{2}}. \tag{3.147}$$

When expressed in terms of $x(t)$, equation (3.132) becomes

$$\ddot{x}x \ln\left(1+\frac{\mu}{x^2}\right)+\left[\ln\left(1+\frac{\mu}{x^2}\right)-\frac{\mu}{x^2\left(1+\frac{\mu}{x^2}\right)}\right]\dot{x}^2$$

$$+\quad \frac{2}{\rho^*\rho_1^2}\int_x^{x\left(\frac{1+\frac{\mu}{x^2}}{1+\mu}\right)^{\frac{1}{2}}}\frac{1}{\left(1-u^2\right)}\frac{dW}{du}=\frac{2\left(\mathcal{P}_1(t)-\mathcal{P}_2(t)\right)}{\rho^*\rho_1^2}. \tag{3.148}$$

Define

$$W_0(u) \;=\; \frac{1}{\rho^*\rho_1^2}W(u), \tag{3.149}$$

$$\mathcal{P}(t) \;=\; \frac{\mathcal{P}_1(t)-\mathcal{P}_2(t)}{\rho^*\rho_1^2}. \tag{3.150}$$

Equation (3.148) becomes

$$\ddot{x}x \ln\left(1+\frac{\mu}{x^2}\right)+\left[\ln\left(1+\frac{\mu}{x^2}\right)-\frac{\mu}{x^2\left(1+\frac{\mu}{x^2}\right)}\right]\dot{x}^2$$

$$+\quad 2\int_x^{x\left(\frac{1+\frac{\mu}{x^2}}{1+\mu}\right)^{\frac{1}{2}}}\frac{1}{\left(1-u^2\right)}\frac{dW_0}{du}=2\mathcal{P}(t). \tag{3.151}$$

Equation (3.151) is the required ordinary differential equation for the dimensionless inner radius. It applies for radial, tangential and longitudinal transversely isotropic cylindrical tubes and it has the same form as the differential equation for radial oscillations of an isotropic cylindrical tube (Knowles, 1960, 1962). Equation (3.151) applies for a cylindrical tube of arbitrary thickness.

3.10 Strain-energy function

When the strain-energy function has to be specified the generalised Mooney-Rivlin strain-energy function (2.192) will be used :

$$W(I_1,I_2,K_1,K_2)=C_1(I_1-3)+C_2(I_2-3)+C_3(K_1-1)+C_4(K_2-1), \tag{3.152}$$

where, since the unstrained body B_0 must be stress free,

$$C_3=-2C_4. \tag{3.153}$$

In the unstrained body B_0 the strain-energy function W must also be a minimum. We now investigate the conditions this places on the constants in (3.152).

Consider $W=W(u)$. In B_0, $r=\rho$ and $u=\frac{r}{\rho}=1$. For $W(u)$ to be a minimum in B_0, it is necessary that

$$\frac{dW}{du}(1)=0, \tag{3.154}$$

$$\frac{d^2W}{du^2}(1)>0. \tag{3.155}$$

Now using (3.128),

$$W(u) = (C_1 + C_2)(u^2 + \frac{1}{u^2} - 2) + C_3(K_1 - 1) + C_4(K_2 - 1). \tag{3.156}$$

Since K_1 and K_2 are different for the three transversely isotropic tubes, each case must be treated separately.

For a radial transversely isotropic cylindrical tube, from (3.129),

$$K_1 = \frac{1}{u^2}, \quad K_2 = \frac{1}{u^4} \tag{3.157}$$

and (3.156) becomes

$$W(u) = (C_1 + C_2)(u^2 + \frac{1}{u^2} - 2) + C_3(\frac{1}{u^2} - 1) + C_4(\frac{1}{u^4} - 1). \tag{3.158}$$

Thus $W(1) = 0$ and

$$\frac{dW}{du}(1) = -2(C_3 + 2C_4) = 0, \tag{3.159}$$

by condition (3.153). This gives an alternative derivation of condition (3.153). Also

$$\frac{d^2W}{du^2}(1) = 8(C_1 + C_2 + C_4) > 0 \tag{3.160}$$

provided

$$C_1 + C_2 + C_4 > 0. \tag{3.161}$$

For a tangential transversely isotropic cylindrical tube

$$K_1 = u^2, \quad K_2 = u^4 \tag{3.162}$$

and (3.156) becomes

$$W(u) = (C_1 + C_2)(u^2 + \frac{1}{u^2} - 2) + C_3(u^2 - 1) + C_4(u^4 - 1). \tag{3.163}$$

It follows that $W(1) = 0$ and

$$\frac{dW}{du}(1) = 2(C_3 + 2C_4) = 0, \tag{3.164}$$

by (3.153) and

$$\frac{d^2W}{du^2}(1) = 8(C_1 + C_2 + C_4) > 0, \tag{3.165}$$

provided (3.161) is satisfied.

For a longitudinal transversely isotropic cylinder, by (3.140),

$$K_1 = 1, \quad K_2 = 1 \tag{3.166}$$

and (3.156) reduces to the Mooney-Rivlin strain-energy function

$$W(u) = (C_1 + C_2)(u^2 + \frac{1}{u^2} - 2). \tag{3.167}$$

We have again $W(1) = 0$ and

$$\frac{dW}{du}(1) = 0, \tag{3.168}$$

$$\frac{d^2W}{du^2}(1) = 8(C_1 + C_2) > 0 \tag{3.169}$$

provided

$$C_1 + C_2 > 0. \tag{3.170}$$

Since $W_0(u)$ defined by (3.149) occurs in (3.151) we express the results in terms of $W_0(u)$. Define

$$D_1 = \frac{2(C_1 + C_2)}{\rho^* \rho_1^2}, \quad D_2 = \frac{4C_4}{\rho^* \rho_1^2}. \tag{3.171}$$

Then, for the three transversely isotropic cylindrical tubes

$$\text{radial}: \ W_0(u) = \frac{D_1}{2}(u^2 + \frac{1}{u^2} - 2) + \frac{D_2}{4}(\frac{1}{u^4} - \frac{2}{u^2} + 1), \tag{3.172}$$

$$= \frac{D_1}{2}(u - \frac{1}{u})^2 + \frac{D_2}{4}(\frac{1}{u^2} - 1)^2, \tag{3.173}$$

$$2D_1 + D_2 > 0; \tag{3.174}$$

$$\text{tangential}: \ W_0(u) = \frac{D_1}{2}(u^2 + \frac{1}{u^2} - 2) + \frac{D_2}{4}(u^4 - 2u^2 + 1), \tag{3.175}$$

$$= \frac{D_1}{2}(u - \frac{1}{u})^2 + \frac{D_2}{4}(u^2 - 1)^2, \tag{3.176}$$

$$2D_1 + D_2 > 0; \tag{3.177}$$

$$\text{longitudinal}: \ W_0(u) = \frac{D_1}{2}(u^2 + \frac{1}{u^2} - 2), \tag{3.178}$$

$$= \frac{D_1}{2}(u - \frac{1}{u})^2, \tag{3.179}$$

$$D_1 > 0. \tag{3.180}$$

The strain-energy function (3.178) for a longitudinal transversely isotropic cylindrical tube is the same as for an isotropic tube. Radial oscillations in a longitudinal transversely isotropic cylindrical tube will therefore be the same as in an isotropic tube (Shahinpoor, 1974). We will compare nonlinear radial oscillations in radial and tangential transversely isotropic cylindrical tubes with corresponding oscillations in isotropic tubes and determine anisotropic effects.

It will be assumed that

$$D_1 \geq 0, \quad D_2 \geq 0 \text{ but not } D_1 = 0 \text{ and } D_2 = 0. \tag{3.181}$$

The conditions (3.181) are sufficient for (3.174), (3.177) and (3.180) to be satisfied. The forms (3.173), (3.176) and (3.179) show that $W(u)$ has a minimum value at $u = 1$ when (3.181) is satisfied. For an isotropic cylindrical tube, it is necessary that $D_1 > 0$. A solution with $D_1 = 0$, $D_2 > 0$ will be investigated in Chapter 4.

3.11 Conclusions

The second order nonlinear ordinary differential equations for radial oscillations in radial, tangential and longitudinal transversely isotropic cylindrical tubes have the same form when expressed in terms of the strain-energy function. This form is the same as the equation derived by Knowles (1960, 1962) for radial oscillations in an isotropic tube.

When the strain-energy function is specified, the differential equation will be different in general for different transversely isotropic tubes. However, since the strain invariants are constant for a longitudinal transversely isotropic tube, the differential equation for a longitudinal transversely isotropic tube will be the same as for an isotropic tube. This is because in a cross section of the tube the anisotropic director is orthogonal to the plane of the radial oscillations.

The differential equation (3.151) applies for cylindrical tubes of arbitrary thickness and for all strain-energy functions.

Chapter 4

Nonlinear Radial Oscillations of Transversely Isotropic Incompressible Cylindrical Tubes: Time Dependent Net Applied Surface Pressure

4.1 Introduction

In this chapter nonlinear radial oscillations of transversely isotropic incompressible cylindrical tubes for time dependent net applied surface pressure will be considered. For nonlinear radial oscillations in an isotropic thin-walled cylindrical tube the application of Lie symmetry methods for differential equations produced exact analytical solutions. We will investigate if Lie symmetry methods will also give exact analytical solutions for nonlinear radial oscillations in transversely isotropic cylindrical tubes. The nonlinear radial oscillations are described by the ordinary differential equation (3.151).

Nonlinear radial oscillations in a thick-walled isotropic cylindrical tube were investigated by Mason and Roussos (2000). They used the Money-Rivlin strain-energy function and showed that if the net applied surface pressure depends on time then the ordinary differential equation (3.151) admits no Lie point symmetries. Thus thin-walled cylindrical tubes will be considered.

For a thin-walled isotropic cylindrical tube with Mooney-Rivlin strain-energy function, (3.151) reduces to the Ermakov-Pinney equation. The Ermakov-Pinney equation has three Lie point symmetries and a nonlinear superposition principle can be derived (Shahinpoor and Nowinski, 1971 and Rogers and Ames, 1989). Therefore we consider a thin-walled cylindrical tube with the generalised Mooney-Rivlin strain-energy function (2.192). Roussos and Mason (2005) have shown that for a thin-walled isotropic cylindrical tube with time dependent applied surface pressure, the differential equation has a Lie point symmetry only for a special class of strain-energy functions. The special class includes the Mooney-Rivlin strain-energy function. Thus it is not too restrictive to use the generalised Mooney-Rivlin strain-energy function (2.192).

Since the strain invariants, K_1 and K_2, are different for radial, tangential and longitudinal transversely isotropic cylindrical tubes, equation (3.151) will be different for the three transversely isotropic tubes. The differential equation for a longitudinal transversely isotropic tube is the same as for an isotropic tube. The nonlinear radial oscillations in radial, tangential and longitudinal transversely isotropic tubes will be compared. The effect of the anisotropy on the nonlinear radial oscillations will be investigated.

An outline of the chapter is as follows. In Section 4.2 the differential equation describing nonlinear radial oscillations in a thin-walled cylindrical tube is derived from (3.151) for a general strain-energy function. The generalised Mooney-Rivlin strain-energy function is then applied and three nonlinear second order ordinary differential equations describing radial oscillations in radial, tangential and longitudinal transversely isotropic cylindrical tubes are obtained. In Section 4.3, general results are derived for the Lie point symmetry generators for second order ordinary differential equations of the form $\ddot{x} = F(t, x)$ without specifying $F(t, x)$. In Section 4.4, a Lie point symmetry generator for the differential equation describing radial oscillations in a radial tranversely isotropic tube is derived for a special time dependent applied surface pressure. The Lie point symmetry is then used to transform the second order ordinary differential equation to an autonomous second order ordinary differential equation which is then reduced to an Abel equation of the second kind. In Section 4.5 a similar investigation is done for nonlinear radial oscillations in a tangential tranversely isotropic cylindrical tube. A Lie point symmetry generator is again derived for a special time dependent applied surface pressure. The Lie point symmetry is then used to transform the differential equation to an autonomous equation which is reduced to another Abel equation of the second kind. In Section 4.6 nonlinear radial oscillations in a longitudinal transversely isotropic cylindrical tube are considered. The differential equation is the same as for radial oscillations in an isotropic tube. Roussos and Mason (2005) showed that the differential equation has a Lie point symmetry for a special time dependent net applied surface pressure and used it to reduce the differential equation to an integral from which exact analytical solutions are derived. These results are reviewed briefly for comparison. Finally, conclusions are summarised in Section 4.7.

4.2 Thin-walled cylindrical tube

The dimensionless inner radius of the cylindrical tube $x(t)$ satisfies the differential equation (3.151):

$$\ddot{x} x \ln\left(1 + \frac{\mu}{x^2}\right) + \left[\ln\left(1 + \frac{\mu}{x^2}\right) - \frac{\mu}{x^2\left(1 + \frac{\mu}{x^2}\right)}\right]\dot{x}^2 + I(x; \mu) = 2\mathcal{P}(t), \tag{4.1}$$

where

$$I(x; \mu) = 2 \int_x^{x\left(\frac{1+\frac{\mu}{x^2}}{1+\mu}\right)^{\frac{1}{2}}} \frac{1}{\left(1 - u^2\right)} \frac{dW_0}{du}\, du. \tag{4.2}$$

Consider a thin-walled cylindrical tube in the unstrained body B_0. Then $\rho_1 \doteqdot \rho_2$ where ρ_1 and ρ_2 are the inner and outer radii of the tube in B_0. Now (3.145) becomes

$$\mu = \left(\frac{\rho_2}{\rho_1}\right)^2 - 1 = \frac{(\rho_2 - \rho_1)(\rho_2 + \rho_1)}{\rho_1^2} \doteqdot \frac{2(\rho_2 - \rho_1)}{\rho_1} \tag{4.3}$$

and therefore $0 < \mu << 1$. Expand each term in (4.1) in powers of μ:

$$\ln\left(1 + \frac{\mu}{x^2}\right) = \frac{\mu}{x^2} + O(\mu^2), \tag{4.4}$$

$$x\left(\frac{1 + \frac{\mu}{x^2}}{1 + \mu}\right)^{\frac{1}{2}} = x + \frac{\mu}{2x}(1 - x^2) + O(\mu^2), \tag{4.5}$$

$$\mathcal{P}(t) = P_0(t) + \mu P_1(t) + O(\mu^2), \tag{4.6}$$

as $\mu \longrightarrow 0$. Consider next the integral (4.2). In order to evaluate $I(x;\mu)$ we use the First Integral Theorem of Mean Value (Gillespie, 1959) which states:

If $g(x)$ is continuous then

$$\int_a^b g(x)\,dx = (b - a)\,g(\xi), \tag{4.7}$$

where $a \leq \xi \leq b$.

Now, from (4.5)

$$x \leq \xi \leq x + \frac{\mu}{2x}(1 - x^2) + O(\mu^2) \tag{4.8}$$

and therefore

$$\xi = x + O(\mu), \tag{4.9}$$

as $\mu \longrightarrow 0$. The integral (4.2) becomes

$$\begin{aligned} I(x;\mu) &= 2\left(\frac{\mu}{2x}(1 - x^2) + O(\mu^2)\right)\frac{1}{(1 - \xi^2)}\frac{dW_0(\xi)}{d\xi} \\ &= \left(\frac{\mu}{x}(1 - x^2) + O(\mu^2)\right)\left(\frac{1}{(1 - x^2)}\frac{dW_0(x)}{dx} + O(\mu)\right) \\ &= \frac{\mu}{x}\frac{dW_0(x)}{dx} + O(\mu^2) \end{aligned} \tag{4.10}$$

as $\mu \longrightarrow 0$.

Expanding the differential equation (4.1) in powers of μ gives

$$\left[\frac{\mu}{x} + O(\mu^2)\right]\ddot{x} + \left[O(\mu^2)\right]\dot{x}^2 + \left[\frac{\mu}{x}\frac{dW_0(x)}{dx} + O(\mu^2)\right]$$

$$= 2P_0(t) + 2\mu P_1(t) + O(\mu^2), \tag{4.11}$$

which can be rewritten as

$$2xP_0(t) + \mu\left[\ddot{x} + \frac{dW_0(x)}{dx} - 2xP_1(t)\right] + O(\mu^2) = 0, \tag{4.12}$$

as $\mu \to 0$. Separate equation (4.12) in powers of μ:

$$\text{zero order } \mu: \qquad P_0(t) = 0, \tag{4.13}$$

$$\text{first order } \mu: \qquad \ddot{x} + \frac{dW_0(x)}{dx} = 2xP_1(t). \tag{4.14}$$

To explain the result (4.13) we note that to zero order in μ the tube wall has zero thickness. The net applied surface pressure must therefore vanish because otherwise a finite force would be applied to an interface with zero mass which would produce infinite acceleration, which is not acceptable. Equation (4.14) applies for radial, tangential and longitudinal transversely isotropic cylindrical tubes. It has the same form as for an isotropic tube (Roussos and Mason, 2005). It is valid for all strain-energy functions.

The strain-energy functions for radial, tangential and longitudinal transverse isotropic cylinders are given by (3.172), (3.175) and (3.178):

$$\text{radial}: \qquad W_0(u) = \frac{D_1}{2}(u^2 + \frac{1}{u^2} - 2) + \frac{D_2}{4}(\frac{1}{u^4} - \frac{2}{u^2} + 1), \tag{4.15}$$

$$\text{tangential}: \qquad W_0(u) = \frac{D_1}{2}(u^2 + \frac{1}{u^2} - 2) + \frac{D_2}{4}(u^4 - 2u^2 + 1), \tag{4.16}$$

$$\text{longitudinal}: \qquad W_0(u) = \frac{D_1}{2}(u^2 + \frac{1}{u^2} - 2). \tag{4.17}$$

Substitute (4.15) to (4.17) into the differential equation (4.14). This gives the following three nonlinear second order ordinary differential equations

$$\text{radial}: \qquad \ddot{x} + (D_1 - 2P_1(t))x = \frac{D_1 - D_2}{x^3} + \frac{D_2}{x^5}, \tag{4.18}$$

$$\text{tangential}: \qquad \ddot{x} + (D_1 - D_2 - 2P_1(t))x = \frac{D_1}{x^3} - D_2x^3, \tag{4.19}$$

$$\text{longitudinal}: \qquad \ddot{x} + (D_1 - 2P_1(t))x = \frac{D_1}{x^3}. \tag{4.20}$$

The constants D_1 and D_2 satisfy the inequality

$$2D_1 + D_2 > 0. \tag{4.21}$$

The ordinary differential equation (4.20) is the Ermakov-Pinney equation (Ermakov 1880, Pinney 1950). It also applies for radial oscillations in an incompressible isotropic cylindrical tube (Shahinpoor and Nowinski, 1971).

4.3 Lie point symmetry generators: general results

We now investigate the Lie point symmetry generators of the three differential equations (4.18), (4.19) and (4.20). They can be written in the form

$$\ddot{x} = F(t, x). \tag{4.22}$$

In this section general results for the Lie point symmetries of (4.22) will be derived without specifying $F(t, x)$. The theory of Lie point symmetries of differential equations is given by several authors (Olver 1986, Bluman and Kumei 1989, Ibragimov and Anderson 1994, Ibragimov 1999).

A Lie point symmetry generator of (4.22) is

$$X = \xi(t, x)\frac{\partial}{\partial t} + \eta(t, x)\frac{\partial}{\partial x}. \tag{4.23}$$

The ordinary differential equation (4.22) is invariant under the group of infinitesimal transformations with the infinitesimal generator X if and only if

$$X^{[2]}\Big(\ddot{x} - F(t, x)\Big)\Big|_{\ddot{x}=F} = 0, \tag{4.24}$$

where the symbol $\Big|_{\ddot{x}=F}$ means evaluated on the frame of the differential equation (4.22). The second prolongation $X^{[2]}$ of the generator (4.23) is

$$X^{[2]} = X + \zeta_1(t, x, \dot{x})\frac{\partial}{\partial \dot{x}} + \zeta_2(t, x, \dot{x}, \ddot{x})\frac{\partial}{\partial \ddot{x}}\,, \tag{4.25}$$

where ζ_1 and ζ_2 are

$$\zeta_1(t, x, \dot{x}) = D(\eta) - \dot{x}D(\xi), \tag{4.26}$$

$$\zeta_2(t, x, \dot{x}, \ddot{x}) = D(\zeta_1) - \ddot{x}D(\xi) \tag{4.27}$$

and total derivative D is

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + ... \tag{4.28}$$

The prolongation coefficients (4.26) and (4.27) are

$$\zeta_1(t, x, \dot{x}) = \frac{\partial \eta}{\partial t} + \left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial t}\right)\dot{x} - \frac{\partial \xi}{\partial x}\dot{x}^2, \tag{4.29}$$

$$\begin{aligned} \zeta_2(t, x, \dot{x}, \ddot{x}) &= \frac{\partial^2 \eta}{\partial t^2} + \left(2\frac{\partial^2 \eta}{\partial t \partial x} - \frac{\partial^2 \xi}{\partial t^2}\right)\dot{x} + \left(\frac{\partial^2 \eta}{\partial x^2} - 2\frac{\partial^2 \xi}{\partial t \partial x}\right)\dot{x}^2 \\ &- \frac{\partial^2 \xi}{\partial x^2}\dot{x}^3 + \left(\frac{\partial \eta}{\partial x} - 2\frac{\partial \xi}{\partial t}\right)\ddot{x} - 3\frac{\partial \xi}{\partial x}\dot{x}\ddot{x}. \end{aligned} \tag{4.30}$$

The determining equation (4.24) becomes

$$-\xi\frac{\partial F}{\partial t} - \eta\frac{\partial F}{\partial x} + \zeta_2(t, x, \dot{x}, \ddot{x})\Big|_{\ddot{x}=F} = 0. \tag{4.31}$$

Substituting (4.30) into (4.31) and replacing $\ddot{x}$ by F in ζ_2 gives

$$\begin{aligned} & - \quad \xi\frac{\partial F}{\partial t} - \eta\frac{\partial F}{\partial x} + \frac{\partial^2\eta}{\partial t^2} + \left(2\frac{\partial^2\eta}{\partial t\partial x} - \frac{\partial^2\xi}{\partial t^2}\right)\dot{x} + \left(\frac{\partial^2\eta}{\partial x^2} - 2\frac{\partial^2\xi}{\partial t\partial x}\right)\dot{x}^2 \\ & - \quad \frac{\partial^2\xi}{\partial x^2}\dot{x}^3 + \left(\frac{\partial\eta}{\partial x} - 2\frac{\partial\xi}{\partial t}\right)F - 3\frac{\partial\xi}{\partial x}\dot{x}F = 0. \end{aligned} \tag{4.32}$$

Since $F = F(t,x)$, split the determining equation (4.32) according to powers of $\dot{x}$.

$$\dot{x}^3: \qquad \frac{\partial^2\xi}{\partial x^2} = 0, \tag{4.33}$$

$$\dot{x}^2: \qquad \frac{\partial^2\eta}{\partial x^2} - 2\frac{\partial^2\xi}{\partial t\partial x} = 0, \tag{4.34}$$

$$\dot{x}: \qquad 2\frac{\partial^2\eta}{\partial t\partial x} - \frac{\partial^2\xi}{\partial t^2} - 3F(t,x)\frac{\partial\xi}{\partial x} = 0, \tag{4.35}$$

$$\dot{x}^0: \qquad \frac{\partial^2\eta}{\partial t^2} - \xi\frac{\partial F}{\partial t} - \eta\frac{\partial F}{\partial x} + \left(\frac{\partial\eta}{\partial x} - 2\frac{\partial\xi}{\partial t}\right)F(t,x) = 0. \tag{4.36}$$

Equations (4.33) and (4.34) are independent of $F(t,x)$ and can therefore be solved in general. From (4.33)

$$\xi(t,x) = xf_1(t) + f_2(t), \tag{4.37}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions of t. Substituting (4.37) into (4.34) and integrating gives

$$\eta(t,x) = x^2\dot{f}_1(t) + xf_3(t) + f_4(t), \tag{4.38}$$

where $f_3(t)$ and $f_4(t)$ are arbitrary functions of t. Finally, substitute (4.37) and (4.38) into (4.35) and (4.36). This gives

$$3x\ddot{f}_1(t) - \ddot{f}_2(t) + 2\dot{f}_3(t) - 3F(t,x)f_1(t) = 0, \tag{4.39}$$

$$\begin{aligned} & x^2\dddot{f}_1(t) + x\ddot{f}_3(t) + \ddot{f}_4(t) + \left[f_3(t) - 2\dot{f}_2(t)\right]F(t,x) \\ & -\left[xf_1(t) + f_2(t)\right]\frac{\partial F}{\partial t} - \left[x^2\dot{f}_1(t) + xf_3(t) + f_4(t)\right]\frac{\partial F}{\partial x} = 0. \end{aligned} \tag{4.40}$$

In summary, the Lie point symmetry of (4.22) is given by (4.23) where $\xi(t,x)$ and $\eta(t,x)$ are given by (4.37) and (4.38) and the function $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$ satisfy the conditions (4.39) and (4.40).

In order to proceed further $F(t,x)$ has to be specified. It is necessary to treat radial, tangential and longitudinal transversely isotropic tubes separately because $F(t,x)$ is different for the three cases.

4.4 Thin-walled radial transversely isotropic cylindrical tube

From (4.18), the dimensionless inner radius $x(t)$ satisfies the differential equation

$$\ddot{x} + (D_1 - 2P_1(t))x = \frac{D_1 - D_2}{x^3} + \frac{D_2}{x^5}\,. \tag{4.41}$$

If $D_2 = 0$, then the tube is isotropic and the differential equation (4.41) reduces to (4.21). We therefore suppose that

$$D_2 \neq 0. \tag{4.42}$$

Compare equation (4.41) with (4.22). Thus

$$F(t,x) = (2P_1(t) - D_1)x + \frac{D_1 - D_2}{x^3} + \frac{D_2}{x^5}\,. \tag{4.43}$$

Consider first (4.39). Substituting (4.43) into (4.39) gives

$$3x\ddot{f}_1(t) - \ddot{f}_2(t) + 2\dot{f}_3(t) - 3\left((2P_1(t) - D_1)x + \frac{D_1 - D_2}{x^3} + \frac{D_2}{x^5}\right) f_1(t) = 0. \tag{4.44}$$

Separate equation (4.44) according to powers of x:

$$\frac{1}{x^5}: \qquad D_2 f_1(t) = 0, \tag{4.45}$$

$$\frac{1}{x^3}: \qquad (D_1 - D_2) f_1(t) = 0, \tag{4.46}$$

$$x^0: \qquad 2\dot{f}_3(t) - \ddot{f}_2(t) = 0, \tag{4.47}$$

$$x: \qquad \ddot{f}_1(t) + (D_1 - 2P_1(t)) f_1(t) = 0. \tag{4.48}$$

Since $D_2 \neq 0$, it follows from (4.45) that

$$f_1(t) = 0. \tag{4.49}$$

Equations (4.46) and (4.48) are identically satisfied. Integration of (4.47) gives

$$f_3(t) = \frac{1}{2}\dot{f}_2(t) + c_3, \tag{4.50}$$

where c_3 is a constant. Thus (4.37) and (4.38) become

$$\xi(t,x) = f_2(t), \tag{4.51}$$

$$\eta(t,x) = \left(\frac{1}{2}\dot{f}_2(t) + c_3\right)x + f_4(t). \tag{4.52}$$

The final condition is given by (4.40). Using (4.43), (4.49) and (4.50), (4.40) becomes

$$\begin{aligned}
&\left(c_3 - \frac{3}{2}\dot{f}_2(t)\right)\left[(2P_1(t) - D_1)x + \frac{(D_1 - D_2)}{x^3} + \frac{D_2}{x^5}\right] - 2f_2(t)\,\frac{dP_1}{dt}\,x \\
&+\left[x\left(\frac{1}{2}\dot{f}_2(t) + c_3\right) + f_4(t)\right]\left[D_1 - 2P_1(t) + \frac{3(D_1 - D_2)}{x^4} + \frac{5D_2}{x^6}\right] \\
&+\frac{x}{2}\dddot{f}_2(t) + \ddot{f}_4(t) = 0.
\end{aligned} \tag{4.53}$$

Separating (4.53) according to powers of x gives

$$\frac{1}{x^6}: \qquad D_2 f_4(t) = 0, \tag{4.54}$$

$$\frac{1}{x^5}: \qquad D_2(\dot{f}_2(t) + 6c_3) = 0, \tag{4.55}$$

$$\frac{1}{x^4}: \qquad (D_1 - D_2) f_4(t) = 0, \tag{4.56}$$

$$\frac{1}{x^3}: \qquad (D_1 - D_2) c_3 = 0, \tag{4.57}$$

$$x^0: \qquad \ddot{f}_4(t) + (D_1 - 2P_1(t)) f_4(t) = 0, \tag{4.58}$$

$$x: \qquad \ddot{f}_4(t) + 4(D_1 - 2P_1(t))\dot{f}_2(t) - 4\,\frac{dP_1}{dt}\; f_2(t) = 0. \tag{4.59}$$

Since $D_2 \neq 0$, we deduce from (4.54) that

$$f_4(t) = 0. \tag{4.60}$$

Also, since $D_2 \neq 0$, it follows from (4.55) that

$$f_2(t) = -6c_3 t + c_2, \tag{4.61}$$

where c_2 is a constant. Thus equations (4.51) and (4.52) are now

$$\xi(t,x) \;=\; -6c_3 t + c_2, \tag{4.62}$$

$$\eta(t,x) \;=\; -2c_3 x. \tag{4.63}$$

Equations (4.56) and (4.58) are identically satisfied. We are left with only two equations, (4.57) and (4.59), to be satisfied. There are two cases depending on whether $D_1 \neq D_2$ or $D_1 = D_2$.

Case (i) $D_1 \neq D_2$

Then from (4.57)

$$c_3 = 0. \tag{4.64}$$

Equation (4.59) becomes

$$c_2 \frac{dP_1}{dt} = 0. \tag{4.65}$$

When $P_1(t) =$ constant, then $c_2 \neq 0$. Thus

$$\xi(t,x) \;=\; c_2, \tag{4.66}$$

$$\eta(t,x) \;=\; 0 \tag{4.67}$$

and the Lie point symmetry generator (4.23) is

$$X = \frac{\partial}{\partial t}\,. \tag{4.68}$$

There is one Lie point symmetry corresponding to time translational invariance. When $\frac{dP_1}{dt} \neq 0$, then $c_2 = 0$. Thus from (4.62) and (4.63),

$$\xi(t, x) = 0, \tag{4.69}$$

$$\eta(t, x) = 0 \tag{4.70}$$

and there is no Lie point symmetry generator.

Case (ii) $D_1 = D_2$

Equation (4.57) is identically satisfied and there is no condition on c_3. Suppose $c_3 \neq 0$, since $c_3 = 0$ reduces to Case (i). Using (4.61), equation (4.59) gives a first order ordinary differential equation for $P_1(t)$:

$$\frac{dP_1}{dt} + \frac{12c_3}{(6c_3t - c_2)}P_1(t) = \frac{6c_3D_1}{(6c_3t - c_2)}\,. \tag{4.71}$$

The integrating factor for (4.71) is $(6c_3t - c_2)^2$ and its general solution is

$$P_1(t) = \frac{D_1}{2} + \frac{k}{(6c_3t - c_2)^2}\,, \tag{4.72}$$

where k is a constant. Also, ξ and η are given by (4.62) and (4.63) and the Lie point symmetry generator (4.23) is

$$X = (6c_3t - c_2)\frac{\partial}{\partial t} + 2c_3x\frac{\partial}{\partial x}\,. \tag{4.73}$$

In summary, when $D_1 = D_2$ the differential equation (4.41) has one Lie point symmetry generator (4.73) provided $P_1(t)$ is of the form (4.72). The constant D_1 in (4.72) is the same constant as in the strain-energy function (4.15). Substituting (4.72) into (4.41) with $D_1 = D_2$ gives the following ordinary differential equation for $x(t)$:

$$\frac{d^2x}{dt^2} - \frac{2k}{(6c_3t - c_2)^2}x = \frac{D_1}{x^5}\,. \tag{4.74}$$

In order to develop this case further we redefine the constants. Let

$$A = -\frac{c_2}{6c_3}, \qquad B = -\frac{k}{(6c_3)^2}\,. \tag{4.75}$$

Thus if $D_1 = D_2$ and if the net applied pressure is

$$P_1(t) = \frac{D_1}{2} - \frac{B}{(t + A)^2} \tag{4.76}$$

then the dimensionless inner radius $x(t)$ of a radial transversely isotropic cylindrical tube satisfies the differential equation

$$\frac{d^2x}{dt^2} + \frac{2B}{(t+A)^2}x = \frac{D_1}{x^5} \tag{4.77}$$

which has one Lie point symmetry generator

$$X = (t+A)\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x}\,. \tag{4.78}$$

We take $A > 0$ to ensure that $P_1(t)$ is finite for $t \geq 0$.

The Lie point symmetry generator (4.78) is now used to transform the ordinary differential equation (4.77) to an autonomous equation which does not depend on time explicitly. We insist that under the transformation $(t, x) \longrightarrow (t^*, x^*)$ the transformed differential equation admits the time translational generator $\frac{\partial}{\partial t^*}$. Now, under the transformation $(t, x) \longrightarrow (t^*, x^*)$ the Lie point symmetry generator (4.23) transform as

$$\begin{aligned} X^* &= \xi(t,x)\Big(\frac{\partial t^*}{\partial t}\,\frac{\partial}{\partial t^*} + \frac{\partial x^*}{\partial t}\,\frac{\partial}{\partial x^*}\Big) + \eta(t,x)\Big(\frac{\partial t^*}{\partial x}\,\frac{\partial}{\partial t^*} + \frac{\partial x^*}{\partial x}\,\frac{\partial}{\partial x^*}\Big) \\ &= X(t^*)\frac{\partial}{\partial t^*} + X(x^*)\frac{\partial}{\partial x^*}\,. \end{aligned} \tag{4.79}$$

Thus

$$X^* = \frac{\partial}{\partial t^*} \tag{4.80}$$

provided

$$X(t^*) = 1, \qquad X(x^*) = 0\,, \tag{4.81}$$

that is, provided

$$(t+A)\frac{\partial t^*}{\partial t} + \frac{x}{3}\frac{\partial t^*}{\partial x} = 1\,, \tag{4.82}$$

$$(t+A)\frac{\partial x^*}{\partial t} + \frac{x}{3}\frac{\partial x^*}{\partial x} = 0\,. \tag{4.83}$$

Consider first (4.83). The differential equations of the characteristic curves are

$$\frac{dt}{t+A} = 3\frac{dx}{x} = \frac{dx^*}{0}\,. \tag{4.84}$$

The last term of (4.84) gives

$$dx^* = 0 \tag{4.85}$$

and hence

$$x^* = a_1\,, \tag{4.86}$$

where a_1 is a constant. Integration of the first pair of (4.84) gives

$$\frac{x}{(t+A)^{\frac{1}{3}}} = a_2\,, \tag{4.87}$$

where a_2 is a constant. The general solution of (4.83) is $a_1 = F(a_2)$ where F is an arbitrary function. A particular solution is $a_1 = a_2$, that is

$$x^* = \frac{x}{(t+A)^{\frac{1}{3}}} \,. \tag{4.88}$$

Consider next (4.82). The differential equations of the characteristic curves are

$$\frac{dt}{t+A} = 3\frac{dx}{x} = \frac{dt^*}{1} \,. \tag{4.89}$$

Integration of the first and last terms in (4.89) gives

$$t^* = \ln(t+A) + C, \tag{4.90}$$

where C is a constant. Choose C so that $t^* = 0$ when $t = 0$. Thus $C = -\ln A$ and

$$t^* = \ln\big(1 + \frac{t}{A}\big) \,. \tag{4.91}$$

Transform the ordinary differential equation (4.77) from (t, x) to (t^*, x^*) where x^* and t^* are defined by (4.88) and (4.90). Now

$$\frac{dx}{dt} = \Big(t+A\Big)^{-\frac{2}{3}}\Big(\frac{dx^*}{dt^*} + \frac{x^*}{3}\Big), \tag{4.92}$$

$$\frac{d^2x}{dt^2} = \Big(t+A\Big)^{-\frac{5}{3}}\Big(\frac{d^2x^*}{dt^{*2}} - \frac{1}{3}\frac{dx^*}{dt^*} - \frac{2}{9}x^*\Big). \tag{4.93}$$

Equation (4.77) transforms to the autonomous equation

$$\frac{d^2x^*}{dt^{*2}} - \frac{1}{3}\frac{dx^*}{dt^*} + 2\big(B - \frac{1}{9}\big)x^* = \frac{D_1}{x^{*5}} \,. \tag{4.94}$$

Equation (4.94) can be reduced to an Abel equation of second kind which has the general form (Polyanin and Zaitsev, 1995)

$$y\frac{dy}{dx} - y = f(x) \,. \tag{4.95}$$

To do this let

$$\frac{dx^*}{dt^*} = y^* \,. \tag{4.96}$$

Then

$$\frac{d^2x^*}{dt^{*2}} = \frac{d}{dx^*}\left(\frac{dx^*}{dt^*}\right)\frac{dx^*}{dt^*} = \frac{dy^*}{dx^*}y^* \tag{4.97}$$

and (4.94) becomes

$$y^*\frac{dy^*}{dx^*} - \frac{1}{3}y^* = 2\big(\frac{1}{9} - B\big)x^* + \frac{D_1}{x^{*5}} \,. \tag{4.98}$$

If we let

$$z^* = 3y^* = 3\frac{dx^*}{dt^*} \,, \tag{4.99}$$

then (4.98) becomes

$$z^* \frac{dz^*}{dx^*} - z^* = 2\big(1 - 9B\big)x^* + \frac{9D_1}{x^{*5}} \,. \tag{4.100}$$

The differential equation (4.100) is an Abel equation of second kind. It is not however one of the standard equations whose solution is given by Polyanin and Zaitsev (1995).

In order to compare radial oscillations in a radial transversely isotropic cylindrical tube with radial oscillations in an isotropic tube, return to equation (4.94). For an isotropic tube the differential equation is (4.20). Substitute (4.76) for $P_1(t)$ into (4.20) and transform from (t, x) to (t^*, x^*) defined by (4.88) and (4.91). This gives

$$\frac{d^2x^*}{dt^{*2}} - \frac{1}{3}\frac{dx^*}{dt^*} + 2\big(B - \frac{1}{9}\big)x^* = \frac{D_1 A^{\frac{2}{3}} \exp(\frac{2}{3}t^*)}{x^{*3}} \,. \tag{4.101}$$

Expressed in terms of t^*, the net applied surface pressure (4.76) becomes

$$P_1(t^*) = \frac{D_1}{2} - \frac{B}{A^2}\exp(-2t^*) \,. \tag{4.102}$$

The initial conditions for $x(t)$ are

$$x(0) = x_0, \qquad \frac{dx(0)}{dt} = v_0 \,. \tag{4.103}$$

Then, expressed in terms of (t^*, x^*) the initial conditions are

$$x^*(0) = \frac{x_0}{A^{\frac{1}{3}}}, \qquad \frac{dx^*(0)}{dt^*} = A^{\frac{2}{3}}\left(v_0 - \frac{x_0}{3A}\right) . \tag{4.104}$$

The differential equations (4.94) and (4.101) are solved numerically using a fourth order Runge-Kutta method. The values of the parameters are $A = 1$, $D_1 = 1$ and $B = \pm 1$ and the initial conditions are

$$x_0 = 1, \quad v_0 = -1. \tag{4.105}$$

Hence

$$x^*(0) = 1, \quad \frac{dx^*}{dt^*}(0) = -\frac{4}{3}. \tag{4.106}$$

In Figures 4.4.1 and 4.4.2, x^* is plotted against t^* for radial oscillations in a radial transversely isotropic cylindrical tube and in an isotropic tube for $B = +1$ and $B = -1$, respectively. The unstrained point $x = 1$ becomes the curve

$$x^* = \exp\big(-\frac{t^*}{3}\big) \tag{4.107}$$

in the (t^*, x^*) plane. In Figure 4.4.3 the net applied surface pressure $P_1(t^*)$ is plotted against the transformed time t^* for $B = \pm 1$. From (4.102),

$$\lim_{t^* \to \infty} P_1(t^*) = \frac{D_1}{2} \,, \tag{4.108}$$

Figure 4.4.1

Figure 4.4.2

Figure 4.4.3

Figure 4.4.4

Figure 4.4.5

Figure 4.4.6

where D_1 is the constant in the strain-energy function (4.15). When $B = +1$, $P_1(t^*)$ is tensile for small values of t^* and becomes compressive for large values of t^*. When $B = -1$, $P_1(t^*)$ is always compressive.

In Figures 4.4.4 to 4.4.6, corresponding graphs are plotted for x against t with $D_1 = D_2$. The net applied surface pressure is given by (4.76). For a radial transversely isotropic cylindrical tube with $D_1 = D_2$, $x(t)$ satisfies (4.77). For an isotropic tube with $P_1(t)$ given by (4.76), $x(t)$ satisfies

$$\frac{d^2x}{dt^2} + \frac{2B}{(t+A)^2}\, x = \frac{D_1}{x^3}\,. \tag{4.109}$$

Equations (4.77) and (4.109) are solved numerically using a fourth order Runge-Kutta method.

In Figure 4.4.1, $B = +1$. We see that for both a radial transversely isotropic cylindrical tube and an isotropic tube there are two time scales associated with the radial oscillations. The short time scale describes the oscillations and is of order of the period of the free oscillations. The long time scale describes the growth in amplitude of the oscillations. For small t^* the oscillation is about the unstrained state (4.107). However, for longer time the oscillations do not pass through the unstrained state and the cylinder is always in a state of extension. For the radial transverse isotropic cylinder the period of the oscillation, the amplitude of the oscillation and the departure from the unstrained state are *less* than for the isotropic cylinder. This agrees with Figure 4.4.4 where $x(t)$ is plotted against t. The radial transverse isotropy therefore has the effect of making the cylinder stiffer.

In Figure 4.4.2, $B = -1$. The solutions for a radial transverse isotropic cylinder and an isotropic cylinder are similar. The cylindrical tube is initially compressed and then steadily extended. This agrees with Figure 4.4.5 where $x(t)$ is plotted against t. Since the compression lasts for a slightly shorter time in the radial transversely isotropic tube we see again that the effect of the anisotropy is to make the cylinder stiffer.

The transformed variables (x^*, t^*) are more convenient to use to investigate the oscillations than the natural coordinates (x, t) for which large values have to be used, especially for t^* in Figure 4.4.4.

When we investigate Heaviside step loading for a thin-walled cylindrical tube in Chapter 5 we will see that $P_1 = \frac{D_1}{2}$ is a critical value for P_1. From (4.108), this critical value is the limiting value of $P_1(t)$ as $t \to \infty$, which is illustrated in Figures 4.4.3 and 4.4.6. The solutions for x^* and x presented here illustrate the behaviour of the radial oscillations as $P_1(t)$ tends to $\frac{D_1}{2}$ from below and above.

4.5 Thin-walled tangential transversely isotropic cylinder

From (4.19), the second order ordinary differential equation for the dimensionless inner radius, $x(t)$, for a tangential transversely isotropic cylindrical tube is

$$\ddot{x} + (D_1 - D_2 - 2P_1(t))x = \frac{D_1}{x^3} - D_2 x^3\,. \tag{4.110}$$

Assume that $D_2 \neq 0$, because $D_2 = 0$ describes an isotropic cylindrical tube.

Compare equation (4.110) with (4.22). Thus

$$F(t,x) = (2P_1(t) + D_2 - D_1)x + \frac{D_1}{x^3} - D_2 x^3 . \tag{4.111}$$

Consider first (4.39) and substitute (4.111) into (4.39). This gives

$$3x\ddot{f}_1(t) - \ddot{f}_2(t) + 2\dot{f}_3(t) - 3\left(2(P_1(t) + D_2 - D_1)x + \frac{D_1}{x^3} - D_2 x^3\right) f_1(t) = 0. \tag{4.112}$$

Equation (4.112) is split according to powers of x. Thus

$$\frac{1}{x^3}: \qquad D_1 f_1(t) = 0, \tag{4.113}$$

$$x^0: \qquad 2\dot{f}_3(t) - \ddot{f}_2(t) = 0, \tag{4.114}$$

$$x: \qquad \ddot{f}_1(t) + (D_1 - D_2 - 2P_1(t))f_1(t) = 0, \tag{4.115}$$

$$x^3: \qquad D_2 f_1(t) = 0. \tag{4.116}$$

Since $D_2 \neq 0$, it follows from (4.116) that

$$f_1(t) = 0. \tag{4.117}$$

Equations (4.113) and (4.115) are identically satisfied. Integrate (4.114) to obtain

$$f_3(t) = \frac{1}{2}\dot{f}_2(t) + c_3, \tag{4.118}$$

where c_3 is a constant. Hence (4.37) and (4.38) become

$$\xi(t,x) = f_2(t), \tag{4.119}$$

$$\eta(t,x) = \left(\frac{1}{2}\dot{f}_2(t) + c_3\right)x + f_4(t). \tag{4.120}$$

The remaining condition is (4.40). Substituting (4.111), (4.117) and (4.118) into (4.40) gives

$$\begin{aligned}
&\frac{x}{2}\dddot{f}_2(t) + \ddot{f}_4(t) \\
&+\left(c_3 - \frac{3}{2}\dot{f}_2(t)\right)\left[(2P_1(t) + D_2 - D_1)x + \frac{D_1}{x^3} - D_2 x^3\right] \\
&-f_2(t)\,\frac{dP}{dt}\,x + \left[x\left(\frac{1}{2}\dot{f}_2(t) + c_3\right)\right. \\
&\left.+f_4(t)\right]\left[D_1 - D_2 - 2P_1(t) + \frac{3D_1}{x^4} + 3D_2 x^2\right] = 0.
\end{aligned} \tag{4.121}$$

Equation (4.121) is separated according to powers of x.

$$\frac{1}{x^4}: \qquad D_1 f_4(t) = 0, \tag{4.122}$$

$$\frac{1}{x^3}: \qquad D_1 c_3 = 0, \tag{4.123}$$

$$x^0: \qquad \ddot{f}_4(t) + (D_1 - D_2 - 2P_1(t))f_4(t) = 0, \tag{4.124}$$

$$x: \qquad \dddot{f}_2(t) + 4(D_1 - D_2 - 2P_1(t))\dot{f}_2(t) - 4\frac{dP_1}{dt}\, f_2(t) = 0, \tag{4.125}$$

$$x^2: \qquad D_2 f_4(t) = 0, \tag{4.126}$$

$$x^3: \qquad D_2(3\dot{f}_2(t) + 2c_3) = 0. \tag{4.127}$$

Since $D_2 \neq 0$, it follows from (4.126) and (4.127) that

$$f_4(t) = 0, \tag{4.128}$$

$$f_2(t) = -\frac{2}{3}c_3 t + c_2, \tag{4.129}$$

where c_2 is a constant. Hence equations (4.119) and (4.120) become

$$\xi(t,x) = -\frac{2}{3}\, c_3\, t + c_2, \tag{4.130}$$

$$\eta(t,x) = \frac{2}{3}\, c_3\, x. \tag{4.131}$$

Equations (4.122) and (4.124) are identically satisfied. There are two equations, (4.123) and (4.125), that remain to be satisfied. There are two cases depending on whether $D_1 \neq 0$ or $D_1 = 0$. From the inequality (4.21), a necessary condition is $2D_1 + D_2 > 0$. We can therefore consider $D_1 = 0$ provided $D_2 > 0$.

Case (i) $D_1 \neq 0$

Then from (4.123)

$$c_3 = 0. \tag{4.132}$$

Substituting (4.129) and (4.132) into (4.125) gives

$$c_2\, \frac{dP_1}{dt} = 0. \tag{4.133}$$

If $P_1(t) = \text{constant}$, then $c_2 \neq 0$ and (4.130) and (4.131) become

$$\xi(t,x) = c_2, \quad \eta(t,x) = 0. \tag{4.134}$$

The Lie point symmetry generator (4.23) becomes

$$X = \frac{\partial}{\partial t}\,. \tag{4.135}$$

If $\frac{dP_1}{dt} \neq 0$, then $c_2 = 0$ and therefore (4.130) and (4.131) become

$$\xi(t,x) = 0, \quad \eta(t,x) = 0. \tag{4.136}$$

Thus $X = 0$ and there is no Lie point symmetry generator admitted by the differential equation (4.110) for this case.

Case (ii) $D_1 = 0$

Equation (4.123) is identically satisfied and there is no restriction on c_3. Assume that $c_3 \neq 0$ because otherwise the results of Case (i) are again obtained. Substituting (4.129) into (4.125) and setting $D_1 = 0$ gives a first order ordinary differential equation for $P_1(t)$:

$$\frac{dP_1}{dt} + \frac{4c_3}{(2c_3 t - 3c_2)} P_1(t) = -\frac{2c_3 D_2}{(2c_3 t - 3c_2)}\,. \tag{4.137}$$

The integrating factor for the differential equation (4.137) is $(2c_3 t - 3c_2)^2$ and its general solution is

$$P_1(t) = \frac{k}{(2c_3 t - 3c_2)^2} - \frac{D_2}{2}\,, \tag{4.138}$$

where k is a constant. The Lie point symmetry generator of the differential equation (4.110) is given by (4.130) and (4.131) :

$$X = (3c_2 - 2c_3\, t)\frac{\partial}{\partial t} + 2c_3\, x\frac{\partial}{\partial x}\,. \tag{4.139}$$

In summary, when $D_1 = 0$ and $D_2 > 0$ to satisfy (4.21), the differential equation (4.110) has one Lie point symmetry given by (4.139) provided $P_1(t)$ is of the form (4.138). The constant D_2 in (4.138) is one of the constants in the strain-energy function (4.16). When $D_1 = 0$ and $P_1(t)$ is given by (4.138) the differential equation (4.110) becomes

$$\frac{d^2x}{dt^2} - \frac{2k}{(2c_3 t - 3c_2)^2}\, x = -D_2\, x^3\,. \tag{4.140}$$

Redefine the constants and let

$$A = -\frac{3c_2}{2c_3}, \qquad B = -\frac{k}{4c_3^2}\,. \tag{4.141}$$

Then the net applied surface pressure is

$$P_1(t) = -\frac{D_2}{2} - \frac{B}{(t+A)^2}\,. \tag{4.142}$$

The ordinary differential equation (4.140) becomes

$$\frac{d^2x}{dt^2} + \frac{2B}{(t+A)^2} x = -D_2 x^3 \tag{4.143}$$

which has one Lie point symmetry generator

$$X = (t+A)\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\,. \tag{4.144}$$

Assume $A > 0$ so that $P_1(t)$ is finite for $t \geq 0$.

We proceed in the same way as for the radial transversely isotropic cylindrical tube and use the Lie point symmetry generator (4.144) to transform the ordinary differential equation (4.143) to an autonomous differential equation. Consider the transformation from (t, x) to (t^*, x^*) such that the transformed differential equation has the Lie point symmetry

$$X^* = \frac{\partial}{\partial t^*} \,. \tag{4.145}$$

Then from (4.79)

$$X^* = X(t^*)\frac{\partial}{\partial t^*} + X(x^*)\frac{\partial}{\partial x^*} = \frac{\partial}{\partial t^*} \tag{4.146}$$

provided

$$X(t^*) = 1: \quad (t + A)\frac{\partial t^*}{\partial t} - x\frac{\partial t^*}{\partial x} = 1\,, \tag{4.147}$$

$$X(x^*) = 0: \quad (t + A)\frac{\partial x^*}{\partial t} - x\frac{\partial x^*}{\partial x} = 0\,. \tag{4.148}$$

The differential equations of the characteristic curves of (4.148) are

$$\frac{dt}{t + A} = -\frac{dx}{x} = \frac{dx^*}{0}\,. \tag{4.149}$$

The last term gives

$$x^* = a_1\,, \tag{4.150}$$

where a_1 is a constant and integration of the first pair gives

$$x(t + A) = a_2\,, \tag{4.151}$$

where a_2 is a constant. A particular solution of (4.148) is $a_1 = a_2$ which gives

$$x^* = x(t + A)\,. \tag{4.152}$$

The differential equations of the characteristic curves of (4.147) are

$$\frac{dt}{t + A} = -\frac{dx}{x} = dt^*\,. \tag{4.153}$$

Integration of first and last terms in (4.153) gives

$$t^* = \ln(t + A) + C, \tag{4.154}$$

where C is a constant. Choosing $t^* = 0$ when $t = 0$ gives

$$t^* = \ln\left(1 + \frac{t}{A}\right), \tag{4.155}$$

which is the same as for the radial transversely isotropic cylindrical tube. Transform the ordinary differential equation (4.143) from (t, x) to (t^*, x^*) where t^* and x^* are defined by (4.155) and (4.152). Now

$$\frac{dx}{dt} = \frac{1}{(t + A)^2}\left(\frac{dx^*}{dt^*} - x^*\right), \tag{4.156}$$

$$\frac{d^2x}{dt^2} = \frac{1}{(t + A)^3}\left(\frac{d^2x^*}{dt^{*2}} - 3\frac{dx^*}{dt^*} + 2x^*\right). \tag{4.157}$$

Equation (4.143) transforms to

$$\frac{d^2x^*}{dt^{*2}} - 3\frac{dx^*}{dt^*} + 2\big(1+B\big)x^* = -D_2x^{*3}\,, \tag{4.158}$$

which is an autonomous differential equation.

Equation (4.158) compares with (4.94) for a radial transverse isotropic cylinder. As with (4.94), equation (4.158) can be reduced to an Abel equation of second kind. Using (4.96) and (4.97), equation (4.158) becomes

$$y^*\frac{dy^*}{dx^*} - \frac{1}{3}y^* = -2\big(1+B\big)x^* - D_2x^{*3} \tag{4.159}$$

and by letting

$$z^* = \frac{1}{3}y^* = \frac{1}{3}\frac{dx^*}{dt^*} \tag{4.160}$$

the differential equation (4.159) becomes

$$z^*\frac{dz^*}{dx^*} - z^* = -\frac{2}{9}\big(1+B\big)x^* - \frac{D_2}{9}\,x^{*3}\,. \tag{4.161}$$

The differential equation (4.161) is an Abel equation of second kind but its solution is not readily obtained (Polyanin and Zaitsev , 1995). Equation (4.161) for a tangential transversely isotropic cylindrical tube compares with (4.100) for a radial transversely isotropic cylindrical tube.

In order to investigate the radial oscillations for the net applied surface pressure (4.142) when $D_1 = 0$, consider equation (4.158). Radial oscillations in an isotropic cylindrical tube are described by (4.20). If we set $D_1 = 0$, use (4.142) for $P_1(t)$ and transform to variables (t^*, x^*) defined by (4.152) and (4.155), then (4.20) becomes

$$\frac{d^2x^*}{dt^{*2}} - 3\frac{dx^*}{dt^*} + \big[2(1+B) + D_2A^2\exp(2t^*)\big]x^* = 0\,. \tag{4.162}$$

Since we have set $D_1 = 0$, the condition that $D_1 > 0$ for the strain-energy to be a minimum in the unstrained body B_0 in an isotropic cylindrical tube, is not satisfied. Expressed in terms of t^* the net applied pressure $P_1(t)$, given by (4.142), becomes

$$P_1(t^*) = -\frac{D_2}{2} - \frac{B}{A^2}\,\exp(-2t^*)\,. \tag{4.163}$$

The initial conditions for $x(t)$ are given by (4.103). Expressed in terms of (t^*, x^*) the initial conditions become

$$x^*(0) = A\,x_0, \qquad \frac{dx^*(0)}{dt^*} = A\left(A\,v_0 + x_0\right). \tag{4.164}$$

The differential equations (4.158) and (4.162) are solved numerically using a fourth order Runge-Kutta method. The parameter values are $A = 1$, $D_2 = 1$ and $B = \pm 1$. The initial conditions are again (4.105), that is, $x_0 = 1$, $v_0 = -1$. Thus

$$x^*(0) = 1, \qquad \frac{dx^*}{dt^*}(0) = 0\,. \tag{4.165}$$

Figure 4.5.1

Figure 4.5.2

Figure 4.5.3

Figure 4.5.4

In Figures 4.5.1 and 4.5.2, x^* is plotted against t^* for radial oscillations in a tangential transversely isotropic cylindrical tube and in an isotropic tube for $B = +1$ and $B = -1$. The unstrained point $x = 1$ is the curve

$$x^* = \exp(t^*)\,. \tag{4.166}$$

In Figure 4.5.3, the net applied surface pressure $P_1(t)$ is plotted against t^* for $B \pm 1$. From (4.163),

$$\lim_{t^* \to \infty} P_1(t^*) = -\frac{D_2}{2}\,, \tag{4.167}$$

where D_2 is the constant in the strain-energy function (4.16). When $B = +1$, $P_1(t^*)$ is compressive for all t^*. When $B = -1$, $P_1(t^*)$ is tensile for small t^* but compressive for large t^*.

When $D_1 = 0$, radial oscillations in an isotropic cylindrical tube are unstable because condition (3.180) is not satisfied. This is clearly seen in Figures 4.5.1(b) and 4.5.2(b) where x^* becomes negative which is not physical and the amplitude of the negative oscillations increases as t^* increases. We also see from Figures 4.5.1(a) and 4.5.2(a) that x^* also becomes negative and the amplitude of the negative oscillations increases for radial oscillations in a tangential transversely isotropic tube. Radial oscillations in a tangential transversely isotropic tube with net applied surface pressure (4.142) are therefore unstable when $D_1 = 0$. The condition (4.21) is therefore a necessary but not sufficient condition for stability.

In order to investigate radial oscillations in a tangential transversely isotropic tube with D_1 close to zero, substitute (4.142) for $P_1(t)$ into the differential equation (4.19):

$$\frac{d^2x}{dt^2} + \Big(D_1 + \frac{2B}{(t+A)^2}\Big)x = \frac{D_1}{x^3} - D_2 x^3\,. \tag{4.168}$$

Equation (4.168) with the initial conditions (4.105), is solved by a fourth order Runge-Kutta method. In Figure 4.5.4 the numerical solution of (4.168) for x is plotted against t for $D_1 = 0.01$. It is seen that the radial oscillations in a tangential transversely isotropic tube are stable for $D_1 = 0.01$.

The special time dependent pressure (4.142) with $D_1 = 0$ gives an unstable solution in a tangential transversely isotropic tube.

4.6 Thin-walled longitudinal transversely isotropic cylindrical tube

The dimensionless inner radius $x(t)$ satisfies the Ermakov-Pinney equation (4.20):

$$\ddot{x} + (D_1 - 2P_1(t))x = \frac{D_1}{x^3}\,. \tag{4.169}$$

Comparing equation (4.169) with (4.22) gives

$$F(t,x) = (2P_1(t) - D_1)x + \frac{D_1}{x^3}\,. \tag{4.170}$$

Assume that $D_1 > 0$, in order to satisfy condition (4.21).

The differential equation (4.169) is the same as the one derived for radial oscillations in an isotropic cylindrical tube.

Nonlinear radial oscillations in an isotropic cylindrical tube have been investigated by Shahinpoor and Nowinski (1971) and Rogers and Ames (1989) using the nonlinear superposition principle for the Ermakov-Pinney equation. Mason and Roussos (2000) and Roussos and Mason (2005) have investigated solutions using the Lie point symmetries of (4.169). We outline here the derivation of the Lie point symmetries of (4.169) in order to compare the results with those obtained for radial and tangential transversely isotropic cylindrical tubes.

Consider first equation (4.39). Substituting (4.170) into (4.39) gives

$$3x\ddot{f}_1(t) - \ddot{f}_2(t) + 2\dot{f}_3(t) - 3\left((2P_1(t) - D_1)x + \frac{D_1}{x^3}\right)f_1(t) = 0. \tag{4.171}$$

Equation (4.171) is separated according to powers of x:

$$\frac{1}{x^3}: \qquad D_1 f_1(t) = 0, \tag{4.172}$$

$$x^0: \qquad 2\dot{f}_3(t) - \ddot{f}_2(t) = 0, \tag{4.173}$$

$$x: \qquad \ddot{f}_1(t) + (D_1 - 2P_1(t))f_1(t) = 0\,. \tag{4.174}$$

Since $D_1 > 0$, it follows from (4.172) that

$$f_1(t) = 0\,. \tag{4.175}$$

Equation (4.174) is identically satisfied. Integrating (4.173) gives

$$f_3(t) = \frac{1}{2}\dot{f}_2(t) + c_3, \tag{4.176}$$

where c_3 is a constant. Thus (4.37) and (4.38) become

$$\xi(t,x) \;=\; f_2(t), \tag{4.177}$$

$$\eta(t,x) \;=\; \left(\frac{1}{2}\dot{f}_2(t) + c_3\right)x + f_4(t), \tag{4.178}$$

which is the same as (4.51) and (4.52) for radial and (4.119) and (4.120) for tangential transversely isotropic tubes. Substitute (4.170) into the remaining condition (4.40) and use (4.170), (4.175) and (4.176). This gives

$$\begin{aligned}&\frac{x}{2}\dddot{f}_2(t) + \ddot{f}_4(t)\\ &+\left(c_3 - \frac{3}{2}\dot{f}_2(t)\right)\left[-(D_1 - 2P_1(t))x + \frac{D_1}{x^3}\right] - 2f_2(t)\,\frac{dP_1}{dt}\,x\\ &+\left[x\left(\frac{1}{2}\dot{f}_2(t) + c_3\right) + f_4(t)\right]\left[D_1 - 2P_1(t) + \frac{3D_1}{x^4}\right] = 0.\end{aligned} \tag{4.179}$$

Separate (4.179) according to powers of x :

$$\frac{1}{x^4}: \qquad D_1 f_4(t) = 0, \tag{4.180}$$

$$\frac{1}{x^3}: \qquad D_1 c_3 = 0, \tag{4.181}$$

$$x^0: \qquad \ddot{f}_4(t) + (D_1 - 2P_1(t)) f_4(t) = 0, \tag{4.182}$$

$$x: \qquad \dddot{f}_2(t) + 4(D_1 - 2P_1(t))\dot{f}_2(t) - 4\frac{dP_1}{dt} f_2(t) = 0. \tag{4.183}$$

Since $D_1 > 0$, it follows from (4.180) and (4.181) that

$$f_4(t) = 0, \qquad c_3 = 0. \tag{4.184}$$

Equation (4.182) is identically satisfied. Equations (4.177) and (4.178) become

$$\xi(t,x) = f_2(t), \tag{4.185}$$

$$\eta(t,x) = \frac{1}{2}\dot{f}_2(t)\, x. \tag{4.186}$$

Therefore the Lie point symmetry generators of the differential equation (4.169) are of the form

$$X = f_2(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{f}_2(t) x \frac{\partial}{\partial x} \tag{4.187}$$

where $f_2(t)$ and $P_1(t)$ satisfy (4.183).

For radial and tangential transversely isotropic tubes, $f_2(t)$ is obtained as a linear function of time by (4.61) and (4.129) and equation (4.183) becomes a first order ordinary differential equation for $P_1(t)$. For an isotropic cylindrical tube, $f_2(t)$ and $P_1(t)$ are not determined separately but satisfy (4.183). This is an important difference between the solutions for the two transversely isotropic cylindrical tubes and the isotropic tube.

Equation (4.183) can be interpreted in two ways.

Firstly, $P_1(t)$ is prescribed and equation (4.183) is a third order ordinary differential equation for $f_2(t)$. Three linearly independent solutions for $f_2(t)$ are obtained from (4.183) and the ordinary differential equation (4.169) therefore has three Lie point symmetries. A nonlinear superposition principle can be derived using the three Lie point symmetries (Rogers and Ames 1989, Ibragimov and Mahomed 1996 and Roussos and Mason 1998). This is the conventional approach. An example of a time dependent net applied surface pressure is a blast load with linear decay with time (Shahinpoor and Nowinski, 1971 and Rogers and Ames, 1989) :

$$P_1(t) = \begin{cases} P_0\left(1 - \frac{t}{T}\right), & 0 \le t \le T \\ 0, & t > T \end{cases}. \tag{4.188}$$

This was also considered by Shahinpoor (1974) for a longitudinal transversely isotropic cylindrical tube but the results are exactly the same as for an isotropic tube. Other forms of $P_1(t)$ that have been considered are harmonic load and periodic step pulse load (Shahinpoor and Nowinski, 1971).

In the second interpretation of (4.183), $f_2(t)$ is prescribed and equation (4.183) becomes a first order ordinary differential equation for $P_1(t)$. This interpretation was considered by Roussos and Mason (2005) who prescribed

$$f_2(t) = t + A, \tag{4.189}$$

where A is a constant. This choice gave an analytical solution. From (4.61) and (4.75) for a radial transversely isotropic cylinder

$$\text{radial :} \quad f_2(t) = -6c_3(t + A), \tag{4.190}$$

and from (4.129) and (4.141) for a tangential transversely isotropic cylinder

$$\text{tangential :} \quad f_2(t) = -\frac{2}{3}c_3(t + A). \tag{4.191}$$

Since (4.183) is linear and homogeneous, the constant factors in (4.190) and (4.191) play no part. Since (4.190) and (4.191) have the same form as (4.189) we outline briefly for comparison the solution using (4.189) for an isotropic cylindrical tube.

If (4.189) is substituted into (4.183), equation (4.183) becomes

$$\frac{dP_1}{dt} + \frac{2}{(t + A)} P_1(t) = \frac{D_1}{(t + A)} \tag{4.192}$$

and its general solution is

$$P_1(t) = \frac{D_1}{2} - \frac{B}{(t + A)^2} \tag{4.193}$$

where B is a constant which is the same as (4.76) for a radial transversely isotropic cylinder. It is assumed that $A > 0$ to keep $P_1(t)$ finite for $t \geq 0$. The Ermakov-Pinney equation (4.169) becomes

$$\frac{d^2x}{dt^2} + \frac{2B}{(t + A)^2}\, x = \frac{D_1}{x^3}\,. \tag{4.194}$$

From (4.187) and (4.189), the differential equation (4.194) has the Lie point symmetry

$$X = (t + A)\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}\,. \tag{4.195}$$

Use the Lie point symmetry generator (4.195) to transform the ordinary differential equation (4.194) to an autonomous differential equation. We insist that in terms of the transformed variables (t^*, x^*) the ordinary differential equation for $x^*(t^*)$ has the Lie point symmetry

$$X^* = X(t^*)\frac{\partial}{\partial t^*} + X(x^*)\frac{\partial}{\partial x^*} = \frac{\partial}{\partial t^*}\,. \tag{4.196}$$

Hence

$$X(t^*) = 1: \qquad (t+A)\frac{\partial t^*}{\partial t} + \frac{x}{2}\frac{\partial t^*}{\partial x} = 1\,, \tag{4.197}$$

$$X(x^*) = 0: \qquad (t+A)\frac{\partial x^*}{\partial t} + \frac{x}{2}\frac{\partial x^*}{\partial x} = 0\,. \tag{4.198}$$

The differential equations of the characteristic curves of (4.198) are

$$\frac{dt}{t+A} = \frac{2dx}{x} = \frac{dx^*}{0}\,. \tag{4.199}$$

Thus

$$x^* = a_1\,, \qquad \frac{x}{(t+A)^{\frac{1}{2}}} = a_2\,, \tag{4.200}$$

where a_1 and a_2 are constants. A particular solution of (4.198) is $a_1 = a_2$, that is

$$x^* = \frac{x}{(t+A)^{\frac{1}{2}}}\,. \tag{4.201}$$

The differential equations of the characteristic curves of (4.197) are

$$\frac{dt}{t+A} = 2\frac{dx}{x} = \frac{dt^*}{1} \tag{4.202}$$

and the first and the last terms give, choosing $t^* = 0$ when $t = 0$,

$$t^* = \ln\Big(1 + \frac{t}{A}\Big). \tag{4.203}$$

Under the transformation (4.201) and (4.203) the ordinary differential equation (4.194) becomes the autonomous Ermakov-Pinney equation

$$\frac{d^2x^*}{dt^{*2}} + 2\big(B - \frac{1}{8}\big)x^* = \frac{D_1}{x^{*3}}\,. \tag{4.204}$$

By defining

$$\frac{dx^*}{dt^*} = y^* \tag{4.205}$$

and using (4.97), equation (4.204) reduces to

$$y^*\frac{dy^*}{dx^*} = 2\big(\frac{1}{8} - B\big)x^* + \frac{D_1}{x^{*3}}\,. \tag{4.206}$$

Unlike (4.100) and (4.161) the differential equation (4.206) is not an Abel equation of the second kind because (4.204) does not have the term, $\frac{dx^*}{dt^*}$, as do (4.94) and (4.158) . Integrating (4.206) once gives

$$\frac{1}{2}\left(\frac{dx^*}{dt^*}\right)^2 = \big(\frac{1}{8} - B\big)x^{*2} - \frac{D_1}{2x^{*2}} + J\,, \tag{4.207}$$

where J is a constant. The differential equation (4.207) can be reduced to an integral and exact analytical solutions derived (Roussos and Mason, 2005). We do not consider analytical solutions further here. Instead we will consider the numerical solution of (4.204) for the transformed inner radius $x^*(t^*)$ as was done for radial and tangential transversely isotropic cylindrical tubes. The numerical solution of (4.194) for $x(t)$ was presented in Figures 4.4.4(b) and 4.4.5(b) for $B = +1$ and $B = -1$, respectively.

The net applied surface pressure (4.193), expressed in terms of t^*, is

$$P_1(t^*) = \frac{D_1}{2} - \frac{B}{A^2}\exp(-2t^*) \tag{4.208}$$

which is the same as (4.102) for a radial transversely isotropic tube. The initial conditions for $x(t)$ are (4.103) and, expressed in terms of (t^*, x^*), they are

$$x^*(0) = \frac{x_0}{A^{\frac{1}{2}}}, \qquad \frac{dx^*}{dt^*}(0) = A^{\frac{1}{2}}\left(v_0 - \frac{x_0}{2A}\right). \tag{4.209}$$

The differential equation (4.204) is solved numerically using a fourth order Runge-Kutta method.

The values of the parameters are $A = 1$, $D_1 = 1$ and $B = \pm 1$ and the initial conditions are again (4.105), that is $x_0 = 1\,, \quad v_0 = -1$. Hence

$$x^*(0) = 1\,, \qquad \frac{dx^*}{dt^*}(0) = -\frac{3}{2}\,. \tag{4.210}$$

Since P_1 for an isotropic cylindrical tube is the same as P_1 for a radial transversely isotropic tube, the graphs of $P_1(t^*)$ against t^* and $P_1(t)$ against t are the same as in Figures 4.4.3 and 4.4.6. Also

$$\lim_{t^*\to\infty} P_1(t^*) = \frac{D_1}{2}\,, \qquad \lim_{t\to\infty} P_1(t) = \frac{D_1}{2}\,, \tag{4.211}$$

where D_1 is the constant in the strain-energy function (4.17).

In Figure 4.6.1, x^* is plotted against t^* where x^* satisfies (4.204) for $B = +1$ and $B = -1$. The unstrained point $x = 1$ is the curve

$$x^* = \exp\left(-\frac{t^*}{2}\right) \tag{4.212}$$

in the (t^*, x^*) plane. When $B = +1$, the oscillations have constant amplitude in the (x^*, t^*) coordinates. Except for small values of t^* the oscillations are not about the unstrained state (4.212) and the tube is always extended. When $B = -1$, except for small values of t^*, the inner radius x^* increases steadily as t^* increases.

In Figure 4.6.2 the net applied surface pressure $P_1(t^*)$ given by (4.208) is plotted against t^* for $B = +1$ and $B = -1$. Except for small t^* when $B = +1$, $P_1(t^*)$ is compressive.

Figure 4.6.1

Figure 4.6.2

4.7 Comparison of results

In Table 4.6.1, corresponding results for radial, tangential and longitudinal transversely isotropic cylindrical tubes are presented for comparison. The results for longitudinal transversely isotropic tubes are the same as for isotropic tubes. For radial and tangential transversely isotropic cylindrical tubes the determining equation gives $f_2(t) = t + A$ while for the isotropic tube we choose $f_2(t) = t + A$. The transformed time t^* is the same for all three cases while x^* is different for each case. For radial and tangential transversely isotropic cylindrical tubes, Abel's equation of the second kind is derived for $y^*(t^*)$, from which it is difficult to proceed further, while for the isotropic tube an autonomous Ermakov-Pinney equation is derived which can be integrated to give analytical solutions.

For the isotropic cylindrical tube, other choices of $f_2(t)$ besides $f_2(t) = t + A$ can be considered. For example $f_2(t) = t^2 + At + B$ could be considered. For each choice of $f_2(t)$, $P_1(t)$ satisfies the first order ordinary differential equation

$$\frac{dP_1}{dt} + 2\frac{\dot{f}_2(t)}{f_2(t)}\, P_1(t) = 2D_1\frac{\dot{f}_2(t)}{f_2(t)} + \frac{\dddot{f}_2(t)}{2f_2(t)}\,, \tag{4.213}$$

which is readily solved. The integrating factor is $f_2^2(t)$.

Table 4.7.1

Table 4.7.1

4.8 Conclusions

In this chapter the derivation of analytical solutions for radial oscillations in radial, tangential and longitudinal transversely isotropic cylindrical tubes when the net applied surface pressure depends on time was investigated. The Lie point symmetry generators of the governing second order ordinary differential equations were used. For radial and tangential isotropic tubes we were able to reduce the differential equation to an Abel equation of the second kind for special net applied surface pressures but we were not able to proceed further and obtain an analytical solution. The application of recent advances by Adam and Mahomed (2002) on the solution of Abel equations of the second kind using non-local symmetries may give analytical solutions for special cases. In comparison, for the isotropic cylindrical tube, exact analytical solutions can be derived for special time dependent net applied surface pressures.

The results that were obtained for radial and tangential transversely isotropic cylindrical tubes apply only for special strain-energy functions for which either $D_1 = D_2$ or $D_1 = 0$. They also apply only for special net applied surface pressures, $P_1(t)$, which depend on the constants D_1 and D_2 in the strain-energy function.

We found that the effect of radial transverse isotropy was to reduce the amplitude and period of the oscillations and also to reduce the departure from the unstrained state. The effect of the anisotropy was therefore to make the cylindrical tube stiffer.

For radial oscillations in a tangential transversely isotropic cylindrical tube it was found that when $D_1 = 0$ and $D_2 > 0$ unstable solutions exist for which $x < 0$ which is not physical. The condition $2D_1 + D_2 > 0$ is therefore a necessary condition for stability but not a sufficient condition.

Nonlinear radial oscillations in a longitudinal transverse isotropic cylindrical tube are the same as in an isotropic cylindrical tube because the plane of oscillation is perpendicular to the anisotropic director.

The graphs show how the amplitude of the oscillation increases as $P_1(t)$ tends to $\frac{D_1}{2}$. In Chapter 5, constant net applied surface pressure is considered and it is found that $\frac{D_1}{2}$ is a critical value for $P_1(t)$ for a thin-walled tube.

Chapter 5

Nonlinear Radial Oscillations of Transversely Isotropic Incompressible Cylindrical Tubes: Heaviside Step Loading

5.1 Introduction

This chapter is concerned with nonlinear radial oscillations in transversely isotropic cylindrical tubes subject to constant net applied surface pressure. The constant pressure that will be considered is Heaviside step loading in which the net applied pressure is zero for time $t < 0$ and a constant value for $t > 0$. Both thick-walled and thin-walled cylindrical tubes will be considered.

The nonlinear radial oscillations are described by the ordinary differential equation (3.151). The strain-energy function W will be specified because it is difficult to derive results from (3.151) for general strain-energy functions. The strain-energy function which will be used is the generalised Mooney-Rivlin strain energy function (2.192). Since the strain invariants, K_1 and K_2, are different for radial, tangential and longitudinal transversely isotropic tubes, equation (3.151) will be different for the three cases. As in Chapter 4 radial, tangential and longitudinal transversely isotropic cylinders will be treated separately.

The nonlinear radial oscillations in radial, tangential and longitudinal transversely isotropic cylindrical tubes will be compared. Nonlinear radial oscillations in a longitudinal transversely isotropic cylindrical tube are the same as in an isotropic cylindrical tube. The effect of the anisotropy on the nonlinear radial oscillations will be investigated.

The period and the amplitude of the radial oscillations will be compared. Upper and lower bounds will be derived for the period. This was first done by Knowles (1962) for an isotropic cylindrical tube. The method was developed further by Mason and Roussos (2000) for an isotropic cylindrical tube and by Roussos and Mason (2002) for an isotropic spherical shell. The amplitude of the oscillation will be investigated by considering the effective potential for the oscillation.

An outline of the Chapter is as follows. In Section 5.2 first integrals of equation

(3.151) for radial, tangential and longitudinal transversely isotropic cylindrical tubes subject to constant net applied surface pressure are derived. In Sections 5.3, 5.4 and 5.5, the Heaviside step loading boundary condition is considered. In Section 5.3, the results for radial oscillations in a (longitudinal transversely) isotropic cylindrical tube subjected to Heaviside step loading are reviewed. An effective potential for radial oscillations in a thick-walled cylindrical tube is derived. Upper and lower bounds for the period of the oscillation are obtained and the oscillations corresponding to the upper and lower bounds on the period are derived. The results for the limit of a thin-walled cylindrical tube are obtained. In Sections 5.4 and 5.5, a similar analysis is performed for radial oscillations in radial and tangential transversely isotropic cylindrical tubes subjected to Heaviside step loading. The results are compared with those of Section 5.3 for an isotropic cylindrical tube. Finally, conclusions are summarised in Section 5.6.

5.2 First integral of the differential equation

The dimensionless inner radius of the cylindrical tube satisfies the second order differential equation (3.151):

$$\ddot{x}x \ln\left(1+\frac{\mu}{x^2}\right) + \left[\ln\left(1+\frac{\mu}{x^2}\right) - \frac{\mu}{x^2\left(1+\frac{\mu}{x^2}\right)}\right]\dot{x}^2 + I(x) = 2\mathcal{P}(t) \tag{5.1}$$

where

$$I(x) = 2\int_x^{x\left(\frac{1+\frac{\mu}{x^2}}{1+\mu}\right)^{\frac{1}{2}}} \frac{1}{\left(1-u^2\right)} \frac{dW_0}{du}\, du \tag{5.2}$$

and by (3.149),

$$W_0(u) = \frac{1}{\rho^*\rho_1^2}\, W(u). \tag{5.3}$$

Consider the strain-energy function (2.192) and (2.193):

$$W(I_1, I_2, K_1, K_2) = C_1(I_1-3) + C_2(I_2-3) + C_3(K_1-1) + C_4(K_2-1), \tag{5.4}$$

where $C_3 = -2C_4$. For radial, tangential and longitudinal transverse isotropic cylinders, (5.3) with (5.4) become (3.172), (3.175) and (3.178):

$$\text{radial :} \qquad W_0(u) = \frac{D_1}{2}(u^2+\frac{1}{u^2}-2) + \frac{D_2}{4}(\frac{1}{u^4}-\frac{2}{u^2}+1), \tag{5.5}$$

$$\text{tangential :} \quad W_0(u) = \frac{D_1}{2}(u^2+\frac{1}{u^2}-2) + \frac{D_2}{4}(u^4-2u^2+1), \tag{5.6}$$

$$\text{longitudinal :} \; W_0(u) = \frac{D_1}{2}(u^2+\frac{1}{u^2}-2). \tag{5.7}$$

The constants D_1 and D_2 are given by (3.171) and they satisfy the inequality

$$2D_1 + D_2 > 0. \tag{5.8}$$

Since W_0 is different for radial, tangential and longitudinal transversely isotropic cylindrical tubes the three cases are treated separately.

5.2.1 Radial transversely isotropic cylindrical tube

Consider first a radial transversely isotropic cylindrical tube. Substituting (5.5) into (5.2) gives

$$I(x) = -2\int_{x}^{x\left(\frac{1+\frac{\mu}{x^2}}{1+\mu}\right)^{\frac{1}{2}}} \Big[D_1\Big(\frac{1}{u}+\frac{1}{u^3}\Big)+\frac{D_2}{u^5}\Big]\,du. \tag{5.9}$$

The integral (5.9) is readily evaluated and (5.1) becomes

$$\begin{aligned} & \ddot{x}x\ln\left(1+\frac{\mu}{x^2}\right)+\left[\ln\left(1+\frac{\mu}{x^2}\right)-\frac{\mu}{x^2\big(1+\frac{\mu}{x^2}\big)}\right]\dot{x}^2 \\ + & D_1\left[\ln(1+\mu)-\ln\left(1+\frac{\mu}{x^2}\right)+\frac{\mu(x^2-1)}{x^4\big(1+\frac{\mu}{x^2}\big)}\right] \\ + & \frac{D_2}{2x^4}\left[\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right)^2-1\right]=2\mathcal{P}(t). \end{aligned} \tag{5.10}$$

Consider now

$$\mathcal{P}(t) = \text{constant} = P_0. \tag{5.11}$$

Multiply (5.10) by x. Since

$$\frac{d}{dx}(\dot{x}^2) = 2\ddot{x}, \tag{5.12}$$

it follows that

$$\begin{aligned} & \frac{d}{dx}\left[\frac{\dot{x}^2\,x^2}{2}\ln\left(1+\frac{\mu}{x^2}\right)\right] \\ = & \ddot{x}x^2\ln\left(1+\frac{\mu}{x^2}\right)+\left[\ln\left(1+\frac{\mu}{x^2}\right)-\frac{\mu}{x^2\big(1+\frac{\mu}{x^2}\big)}\right]x\dot{x}^2. \end{aligned} \tag{5.13}$$

Equation (5.10) becomes

$$\begin{aligned} & \frac{d}{dx}\left[\frac{\dot{x}^2\,x^2}{2}\ln\left(1+\frac{\mu}{x^2}\right)\right] \\ + & D_1\left[x\ln(1+\mu)-x\ln\left(1+\frac{\mu}{x^2}\right)+\frac{\mu(x^2-1)}{x^3\big(1+\frac{\mu}{x^2}\big)}\right] \\ + & \frac{D_2}{2}\left[\frac{(1+\mu)^2}{x^3(1+\frac{\mu}{x^2})^2}-\frac{1}{x^3}\right]=2xP_0. \end{aligned} \tag{5.14}$$

Integrate equation (5.14) with respect to x. Now

$$\begin{aligned} & \int\left[-x\ln\left(1+\frac{\mu}{x^2}\right)+\frac{\mu(x^2-1)}{x^3\big(1+\frac{\mu}{x^2}\big)}\right]dx \\ = & \frac{1}{2}(1-x^2)\ln\left(1+\frac{\mu}{x^2}\right)+\text{constant} \end{aligned} \tag{5.15}$$

and

$$\int \frac{dx}{x^3(1+\frac{\mu}{x^2})^2} = \frac{1}{2\mu(1+\frac{\mu}{x^2})} + \text{constant}. \tag{5.16}$$

Equation (5.14) becomes

$$\begin{aligned} & \dot{x}^2 x^2 \ln\left(1+\frac{\mu}{x^2}\right) + D_1\left[x^2\ln(1+\mu) + (1-x^2)\ln\left(1+\frac{\mu}{x^2}\right)\right] \\ + & \frac{D_2}{2}\left[\frac{(1+\mu)^2}{\mu(1+\frac{\mu}{x^2})} + \frac{1}{x^2}\right] - 2P_0x^2 = 2I, \end{aligned} \tag{5.17}$$

where I is a constant. Equation (5.17) is the first integral of (5.10).

5.2.2 Tangential transversely isotropic cylindrical tube

Consider next a tangential transversely isotropic cylindrical tube. Substituting (5.6) into (5.2), gives

$$I(x) = -2\int_x^{x\left(\frac{1+\frac{\mu}{x^2}}{1+\mu}\right)^{\frac{1}{2}}} \left[D_1\Big(\frac{1}{u}+\frac{1}{u^3}\Big) + D_2u\right] du. \tag{5.18}$$

The integral in (5.18) is easily evaluated and (5.1) becomes

$$\begin{aligned} & \ddot{x}x\ln\left(1+\frac{\mu}{x^2}\right) + \left[\ln\left(1+\frac{\mu}{x^2}\right) - \frac{\mu}{x^2\big(1+\frac{\mu}{x^2}\big)}\right]\dot{x}^2 \\ + & D_1\left[\ln(1+\mu) - \ln\left(1+\frac{\mu}{x^2}\right) + \frac{\mu(x^2-1)}{x^4\big(1+\frac{\mu}{x^2}\big)}\right] \\ - & D_2\frac{\mu}{(1+\mu)}(1-x^2) = 2\mathcal{P}(t). \end{aligned} \tag{5.19}$$

Consider now the first integral of (5.19) when $\mathcal{P} = P_0$ where P_0 is a constant. Follow the same procedure as used to derive the first integral of (5.10). Multiply (5.19) by x and use (5.13). Then integrate the resulting equation with respect to x and use (5.15). This gives the first integral

$$\begin{aligned} & \dot{x}^2 x^2 \ln\left(1+\frac{\mu}{x^2}\right) + D_1\left[x^2\ln(1+\mu) + (1-x^2)\ln\left(1+\frac{\mu}{x^2}\right)\right] \\ - & D_2\frac{\mu}{2(1+\mu)}x^2(2-x^2) - 2P_0x^2 = 2I, \end{aligned} \tag{5.20}$$

where I is a constant.

5.2.3 Longitudinal transversely isotropic cylindrical tube

Finally, consider a longitudinal transversely isotropic cylindrical tube. The strain-energy function (5.7) is obtained by setting $D_2 = 0$ in (5.5) and (5.6) for radial and tangential transversely isotropic tubes. Hence, from (5.10) with $D_2 = 0$, $x(t)$ satisfies the second order differential equation

$$\ddot{x}x \ln\left(1 + \frac{\mu}{x^2}\right) + \left[\ln\left(1 + \frac{\mu}{x^2}\right) - \frac{\mu}{x^2\left(1 + \frac{\mu}{x^2}\right)}\right]\dot{x}^2$$
$$+ \quad D_1\left[\ln(1+\mu) - \ln\left(1 + \frac{\mu}{x^2}\right) + \frac{\mu(x^2-1)}{x^4\left(1 + \frac{\mu}{x^2}\right)}\right] = 2\mathcal{P}(t). \tag{5.21}$$

From (5.17) with $D_2 = 0$, the first integral when $\mathcal{P}(t) = \text{constant} = P_0$ is

$$\dot{x}^2x^2 \ln\left(1 + \frac{\mu}{x^2}\right) + D_1\left[x^2\ln(1+\mu) + (1-x^2)\ln\left(1 + \frac{\mu}{x^2}\right)\right]$$
$$- \quad 2P_0x^2 = 2I. \tag{5.22}$$

In summary, for radial, tangential and longitudinal transversely isotropic cylindrical tubes the second order ordinary differential equation for $x(t)$ is given by (5.10), (5.19) and (5.21) and the first integral is given by (5.17), (5.20), and (5.22) respectively.

The second order differential equation and first integral when $\mathcal{P}(t)$ is constant for radial oscillations in a longitudinal transverse isotropic cylindrical tube are the same as the equations derived by Knowles (1960, 1962) for an isotropic cylinder. Radial oscillations in a (longitudinal transversely) isotropic cylinder will first be considered. The anisotropic effects in radial and tangential transversely isotropic cylindrical tubes will then be clear.

5.3 Longitudinal transversely isotropic cylindrical tube

Consider nonlinear radial oscillations in a longitudinal transversely isotropic cylindrical tube subjected to Heaviside step loading,

$$\mathcal{P}(t) = \begin{cases} 0, & t < 0 \\ P_0, & t \geq 0 \end{cases} \tag{5.23}$$

where P_0 is a constant and subject to the initial conditions

$$x(0) = 1, \;\; \dot{x}(0) = 0. \tag{5.24}$$

Imposing the initial conditions (5.24) on (5.22) gives

$$I = \frac{D_1}{2}\ln(1+\mu) - P_0. \tag{5.25}$$

Equation (5.22) becomes

$$\dot{x}^2x^2 \ln\left(1 + \frac{\mu}{x^2}\right) = 2(x^2-1)\left[P_0 - \frac{D_1}{2}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right)\right]. \tag{5.26}$$

The results obtained by Knowles (1962) are briefly reviewed. The effective pontential is introduced which was not considered by Knowles. The effective potential gives qualitative information about the oscillation.

5.3.1 Critical net applied surface pressure

The condition which determines the greatest or least value of $x(t)$ is $\dot{x} = 0$. This occurs at $x = 1$ and at x which satisfies

$$P_0 = \frac{D_1}{2} \ln\Big(\frac{1+\mu}{1+\frac{\mu}{x^2}}\Big). \tag{5.27}$$

Define

$$P(x) = \frac{D_1}{2} \ln\Big(\frac{1+\mu}{1+\frac{\mu}{x^2}}\Big). \tag{5.28}$$

The graph of $P(x)$ against x for $x > 0$ is shown in Figure 5.3.1. The function is a monotone increasing function of x. It increases from $P(0) = -\infty$ through negative values, it vanishes at $x = 1$ and tends to P_c as $x \to \infty$ where

$$P_c = \frac{D_1}{2} \ln(1+\mu). \tag{5.29}$$

Thus for $-\infty < P_0 < P_c$, a unique positive root $x = a$ of (5.27) exists. It can be shown that

$$a = \left[\frac{\mu}{(1+\mu)\exp\Big(-\frac{2P_0}{D_1}\Big) - 1}\right]^{\frac{1}{2}}. \tag{5.30}$$

If $P_0 < 0$, the cylindrical tube undergoes periodic oscillations with $a \leq x \leq 1$. If $0 < P_0 < P_c$ then it undergoes periodic oscillations with $1 \leq x \leq a$. If

$$P_0 \geq P_c = \frac{D_1}{2} \ln(1+\mu), \tag{5.31}$$

then periodic motion does not exist. The oscillation is unbounded with $1 \leq x < \infty$.

For a thin-walled cylindrical tube, terms of order μ^2 can be neglected and (5.29) reduces to

$$P_c = \frac{D_1}{2}\mu. \tag{5.32}$$

In Chapter 4, $P_1(t)$ is defined by (4.6). It is different from P_c by a factor μ. The special values (4.76) and (4.193) for $P_1(t)$ satisfy

$$\lim_{t\to\infty} \mu P_1(t) = \frac{D_1}{2}\mu = P_c. \tag{5.33}$$

The results of Chapter 4 show how the radial oscillations change as the net surface pressure tends to P_c.

5.3.2 Effective potential

Equation (5.26) can be written as

$$\frac{1}{2}\dot{x}^2 + V(x) = 0, \tag{5.34}$$

Figure 5.3.1

where

$$V(x) = \frac{(1-x^2)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)}\Big[P_0 - P_c + \frac{D_1}{2}\ln\left(1+\frac{\mu}{x^2}\right)\Big]. \tag{5.35}$$

The function $V(x)$ is the effective potential. Since $\dot{x}^2 \geq 0$, the oscillations exist for the range of values of x for which $V(x) \leq 0$. The end points of the oscillation, where $\dot{x} = 0$, occur for $V(x) = 0$.

Consider the asymptotic behaviour of $V(x)$ for large and small values of x. Now,

$$V(x) = \frac{1}{\mu}(P_c - P_0)x^2 - \frac{D_1}{2} + \frac{1}{\mu}\left(1 - \frac{\mu}{2}\right)(P_0 - P_c) + O\left(\frac{1}{x^2}\right), \tag{5.36}$$

as $x \to \infty$ and therefore

$$P_0 \neq P_c, \qquad V(x) = \frac{1}{\mu}(P_c - P_0)x^2 + O(1) \quad \text{as } x \to \infty, \tag{5.37}$$

$$P_0 = P_c, \qquad V(x) = -\frac{D_1}{2} + O\left(\frac{1}{x^2}\right) \quad \text{as } x \to \infty. \tag{5.38}$$

Also, for small x,

$$V(x) \sim \frac{1}{2}\Big[\frac{D_1}{x^2} + \frac{(P_0 - P_c)}{x^2(-\ln x)} - D_1 - \frac{(P_0 - P_c)}{(-\ln x)}\Big] \tag{5.39}$$

and therefore

$$P_0 \neq P_c, \qquad V(x) = \frac{D_1}{2x^2} + O\left(\frac{1}{x^2 \ln x}\right) \quad \text{as } x \to 0, \tag{5.40}$$

$$P_0 = P_c, \qquad V(x) = \frac{D_1}{2x^2} + O(1) \quad \text{as } x \to 0. \tag{5.41}$$

Also, $V(1) = 0$ and from Figure 5.3.1, $V(a) = 0$ where $a > 1$ if $0 < P_0 < P_c$ and $0 < a < 1$ if $P_0 < 0$.

In Figure 5.3.2 the effective potential $V(x)$ is plotted against x for $P_0 < 0$, $0 < P_0 < P_c$, $P_0 = P_c$ and $P_0 > P_c$. Oscillations exist for $P_0 < P_c$ and the tube is compressed for $P_0 < 0$ and extended for $0 < P_0 < P_c$. Bounded oscillations do not exist for $P_0 \geq P_c$.

Consider now a thin-walled cylindrical tube. Expand $V(x)$ in powers of μ and neglect terms of order μ^2. This gives

$$V(x) = \frac{D_1}{2}(1-x^2)\Big[\frac{1}{x^2} - \left(1 - \frac{P_0}{P_c}\right)\Big], \tag{5.42}$$

where P_c is given by (5.32). When $P_0 < P_c$, the end points of the oscillations are

$$x = 1, \quad x = a = \left(1 - \frac{P_0}{P_c}\right)^{-\frac{1}{2}} \tag{5.43}$$

and $V(x)$ takes its minimum value when

$$\frac{dV}{dx} = 0: \quad x = \left(1 - \frac{P_0}{P_c}\right)^{-\frac{1}{4}}. \tag{5.44}$$

Figure 5.3.2

Figure 5.3.3

In Figure 5.3.3, graphs of $V(x)$ against x are plotted for $D_1 = 1$, $\mu = 0.1$, $P_c = 0.05$ and $P_0 < 0$, $0 < P_0 < P_c$, $P_0 = P_c$ and $P_0 > P_c$. The effective potentials for a thick-walled tube in Figure 5.3.2 and for a thin-walled tube in Figure 5.3.3 have the same general properties.

5.3.3 Period of the oscillation

Consider periodic oscillations so that $P_0 < P_c$. The derivation of upper and lower bounds for the period of the oscillation for a thick-walled cylindrical tube will be outlined (Knowles, 1962).

Now, $\dot{x} = 0$ at $x = 1$ and $x = a$. From (5.26), since $\dot{x} = 0$ at $x = a$,

$$P_0 = \frac{D_1}{2} \ln\left(\frac{1+\mu}{1+\frac{\mu}{a^2}}\right). \tag{5.45}$$

Thus (5.26) becomes

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1(x^2-1)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right). \tag{5.46}$$

Consider first $a > 1$. Then $1 \le x \le a$ and from (5.45), $P_0 > 0$. The right hand side of (5.46) is positive and

$$\frac{dx}{dt} = \pm \frac{\sqrt{D_1}}{x}\left[(x^2-1)\frac{\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right)}{\ln\left(1+\frac{\mu}{x^2}\right)}\right]^{\frac{1}{2}}. \tag{5.47}$$

The plus sign describes motion from 1 to a and the minus sign motion from a to 1. The period of the oscillation is

$$T = \frac{2}{\sqrt{D_1}} \int_1^a x\left[\frac{\ln\left(1+\frac{\mu}{x^2}\right)}{(x^2-1)\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right)}\right]^{\frac{1}{2}} dx. \tag{5.48}$$

Let $z = x^2$. Then

$$T = \frac{1}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{\ln\left(1+\frac{\mu}{z}\right)}{(z-1)\ln\left(\frac{1+\frac{\mu}{z}}{1+\frac{\mu}{a^2}}\right)}\right]^{\frac{1}{2}} dz. \tag{5.49}$$

Consider next $0 < a < 1$. Then $a \le x \le 1$ and from (5.45), $P_0 < 0$. Equation (5.46) becomes

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1(1-x^2)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{x^2}}\right). \tag{5.50}$$

The period of the oscillation is

$$T = \frac{1}{\sqrt{D_1}} \int_{a^2}^{1} \left[\frac{\ln\left(1+\frac{\mu}{z}\right)}{(1-z)\ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{z}}\right)}\right]^{\frac{1}{2}} dz. \tag{5.51}$$

The upper and lower bounds on the period are obtained using the following inequalities for logarithms (Knowles , 1962)

$$x \geq y: \quad \frac{x-y}{1+x} \leq \ln\left(\frac{1+x}{1+y}\right) \leq \frac{x-y}{1+y}, \tag{5.52}$$

$$x \geq 0: \quad \frac{x}{1+x} \leq \ln(1+x) \leq x. \tag{5.53}$$

There are four cases.

Case (i) Upper bound for the period when $a > 1$

The period is given by (5.49). From (5.53) with $x = \frac{\mu}{z}$,

$$\ln\left(1+\frac{\mu}{z}\right) \leq \frac{\mu}{z}. \tag{5.54}$$

Also, if $x = \frac{\mu}{z}$ and $y = \frac{\mu}{a^2}$, then $x \geq y$ and from (5.52),

$$\ln\left(\frac{1+\frac{\mu}{z}}{1+\frac{\mu}{a^2}}\right) \geq \frac{\frac{\mu}{z}-\frac{\mu}{a^2}}{1+\frac{\mu}{z}} = \frac{\mu(a^2-z)}{a^2(z+\mu)}. \tag{5.55}$$

Thus from (5.49),

$$T \leq \frac{a}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{z+\mu}{z(z-1)(a^2-z)}\right]^{\frac{1}{2}} dz. \tag{5.56}$$

Let

$$f(z) = \frac{z+\mu}{z}. \tag{5.57}$$

Then

$$\frac{df}{dz} = -\frac{\mu}{z^2} < 0 \tag{5.58}$$

and $f(z)$ is therefore a decreasing function of z. Hence for $1 \leq z \leq a^2$,

$$f(z) \leq f(1) = 1+\mu. \tag{5.59}$$

Thus

$$T \leq \frac{a}{\sqrt{D_1}}(1+\mu)^{\frac{1}{2}} \int_1^{a^2} \frac{dz}{[(z-1)(a^2-z)]^{\frac{1}{2}}}. \tag{5.60}$$

But

$$I_1(a^2) = \int_1^{a^2} \frac{dz}{[(z-1)(a^2-z)]^{\frac{1}{2}}} = \pi \tag{5.61}$$

and hence

$$T \leq a(1+\mu)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}. \tag{5.62}$$

Case (ii) Lower bound for the period when $a > 1$

The period is again given by (5.49). From (5.53) with $x = \frac{\mu}{z}$,

$$\ln\left(1+\frac{\mu}{z}\right) \geq \frac{\frac{\mu}{z}}{1+\frac{\mu}{z}} = \frac{\mu}{z+\mu}. \tag{5.63}$$

Also, if $x = \frac{\mu}{z}$ and $y = \frac{\mu}{a^2}$, then $x \geq y$ and from (5.52),

$$\ln\left(\frac{1+\frac{\mu}{z}}{1+\frac{\mu}{a^2}}\right) \leq \frac{\frac{\mu}{z}-\frac{\mu}{a^2}}{1+\frac{\mu}{a^2}} = \frac{\mu(a^2-z)}{z(a^2+\mu)}\,. \tag{5.64}$$

Thus from (5.49),

$$T \geq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{z}{(z+\mu)(z-1)(a^2-z)}\right]^{\frac{1}{2}} dz\,. \tag{5.65}$$

Let

$$f(z) = \frac{z}{z+\mu}\,. \tag{5.66}$$

But

$$\frac{df}{dz} = \frac{\mu}{(z+\mu)^2} > 0 \tag{5.67}$$

and hence $f(z)$ is an increasing function of z. Thus for $1 \leq z \leq a^2$,

$$f(z) \geq f(1) = \frac{1}{1+\mu}\,. \tag{5.68}$$

Then (5.65) becomes

$$T \geq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}(1+\mu)^{\frac{1}{2}}}\, I_1 \tag{5.69}$$

where I_1 is given by (5.61). Hence

$$T \geq \left(\frac{a^2+\mu}{1+\mu}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}\,. \tag{5.70}$$

Combining (5.62) and (5.70) gives

$$T_L \leq T \leq T_U \tag{5.71}$$

where

$$T_L = \left(\frac{a^2+\mu}{1+\mu}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}, \qquad T_U = a(1+\mu)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} \tag{5.72}$$

and a is given by (5.30). For a thin-walled cylindrical tube in which terms of order μ are neglected,

$$T_0 = a\frac{\pi}{\sqrt{D_1}}, \tag{5.73}$$

where a is given by (5.43).

Case (iii) Upper bound for the period when $0 < a < 1$

The period is given by (5.51). The inequality (5.54) again applies. Also if $x = \frac{\mu}{a^2}$ and $y = \frac{\mu}{z}$, then $x \geq y$ and from (5.52)

$$\ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{z}}\right) \geq \frac{\frac{\mu}{a^2}-\frac{\mu}{z}}{1+\frac{\mu}{a^2}} = \frac{\mu(z-a^2)}{z(a^2+\mu)}\,. \tag{5.74}$$

Thus from (5.51)

$$T \leq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}} \int_{a^2}^{1} \frac{dz}{[(1-z)(z-a^2)]^{\frac{1}{2}}}\,. \tag{5.75}$$

But

$$I_2(a^2) = \int_{a^2}^{1} \frac{dz}{[(1-z)(z-a^2)]^{\frac{1}{2}}} = \pi \tag{5.76}$$

and therefore

$$T \leq (a^2+\mu)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}\,. \tag{5.77}$$

Case (iv) Lower bound for the period when $0 < a < 1$

The period T is again given by (5.51). The inequality (5.63) again applies. Also, if $x = \frac{\mu}{a^2}$ and $y = \frac{\mu}{z}$, then $x \geq y$ and

$$\ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{z}}\right) \leq \frac{\frac{\mu}{a^2}-\frac{\mu}{z}}{1+\frac{\mu}{z}} = \frac{\mu(z-a^2)}{a^2(z+\mu)}\,. \tag{5.78}$$

Thus (5.51) becomes

$$T \geq \frac{a}{\sqrt{D_1}} \int_{a^2}^{1} \frac{dz}{[(1-z)(z-a^2)]^{\frac{1}{2}}} \tag{5.79}$$

and using (5.76) it follows that

$$T \geq a\frac{\pi}{\sqrt{D_1}}\,. \tag{5.80}$$

By combining (5.77) and (5.80),

$$T_L \leq T \leq T_U \tag{5.81}$$

where

$$T_L = a\frac{\pi}{\sqrt{D_1}}, \quad T_U = (a^2+\mu)^{\frac{1}{2}}\frac{\pi}{\sqrt{D_1}} \tag{5.82}$$

and a is given by (5.30). For a thin-walled cylindrical tube in which terms of order μ are neglected, the period T_0 is again given by (5.73) where a is given by (5.43). The lower bound, T_L, has the same form as T_0 although a is different.

5.3.4 Amplitude of the oscillation

Consider now the oscillation. The oscillations which correspond to the upper and lower bounds on the period will be derived by performing the same approximations on the oscillation as where made on the period.

Consider first $a > 1$ and x in the range $1 \leq x \leq a$. Let t be the time for the inner radius to oscillate from $x = 1$ to x. In this range $\dot{x} > 0$. Then from (5.47) with $z = x^2$,

$$t = \frac{1}{2\sqrt{D_1}} \int_{1}^{z} \left[\frac{\ln\left(1+\frac{\mu}{z}\right)}{(z-1)\ln\left(\frac{1+\frac{\mu}{z}}{1+\frac{\mu}{a^2}}\right)}\right]^{\frac{1}{2}} dz. \tag{5.83}$$

Consider next $0 < a < 1$ and x in the range $a \leq x \leq 1$. Let t be the time for the inner radius to oscillate from $x = 1$ to x. During this oscillation $\dot{x} < 0$. From (5.50) with $z = x^2$,

$$t = \frac{1}{2\sqrt{D_1}} \int_z^1 \left[\frac{\ln\left(1 + \frac{\mu}{z}\right)}{(1-z)\ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{z}}\right)} \right]^{\frac{1}{2}} dz. \tag{5.84}$$

There are four cases.

Case (i) Oscillation corresponding to upper bound for the period when $a > 1$

Making the same approximations in (5.83) as used to derive the upper bound for the period, (5.60), gives

$$t \leq \frac{a(1+\mu)^{\frac{1}{2}}}{2\sqrt{D_1}}\, I_1(z) \tag{5.85}$$

where

$$I_1(z) = \int_1^z \frac{dz}{[(z-1)(a^2-z)]^{\frac{1}{2}}}. \tag{5.86}$$

But

$$I_1(z) = \int_1^z \frac{1}{(z-1)} \left(\frac{z-1}{a^2-z}\right)^{\frac{1}{2}} dz \;=\; 2\int_0^y \frac{dy}{1+y^2} \;=\; 2\tan^{-1} y \tag{5.87}$$

where

$$y^2 = \frac{z-1}{a^2-z}. \tag{5.88}$$

Hence

$$I_1(z) = 2\tan^{-1}\left[\left(\frac{z-1}{a^2-z}\right)^{\frac{1}{2}}\right]. \tag{5.89}$$

Equation (5.85) becomes

$$t \leq \frac{a(1+\mu)^{\frac{1}{2}}}{\sqrt{D_1}} \tan^{-1}\left[\left(\frac{x^2-1}{a^2-x^2}\right)^{\frac{1}{2}}\right] \tag{5.90}$$

and therefore

$$\left(\frac{x^2-1}{a^2-x^2}\right)^{\frac{1}{2}} \geq \tan\left(\frac{\sqrt{D_1}}{a(1+\mu)^{\frac{1}{2}}}\, t\right). \tag{5.91}$$

Thus if

$$0 \leq t < \frac{a(1+\mu)^{\frac{1}{2}}}{2\sqrt{D_1}}\pi, \tag{5.92}$$

then

$$x \geq \left[\cos^2\left(\frac{\sqrt{D_1}\, t}{a(1+\mu)^{\frac{1}{2}}}\right) + a^2 \sin^2\left(\frac{\sqrt{D_1}\, t}{a(1+\mu)^{\frac{1}{2}}}\right)\right]^{\frac{1}{2}}. \tag{5.93}$$

The limiting oscillation is therefore

$$x_U = \left[\cos^2\left(\frac{\sqrt{D_1}\, t}{a(1+\mu)^{\frac{1}{2}}}\right) + a^2 \sin^2\left(\frac{\sqrt{D_1}\, t}{a(1+\mu)^{\frac{1}{2}}}\right)\right]^{\frac{1}{2}} \tag{5.94}$$

which can be written as

$$x_U = \Big[\frac{a^2+1}{2} - \frac{(a^2-1)}{2}\cos\Big(\frac{2\sqrt{D_1}\,t}{a(1+\mu)^{\frac{1}{2}}}\Big)\Big]^{\frac{1}{2}}. \tag{5.95}$$

The period of the limiting oscillation is T_U, given by (5.72) and its amplitude is a, given by (5.30), which is the amplitude of the exact solution. The limiting oscillation has the form of a nonlinear superposition (Shahinpoor and Nowinski, 1971).

Case (ii) Oscillation corresponding to lower bound for the period when $a > 1$

Making the same approximations in (5.83) as used to derive the lower bound for the period, (5.69), gives

$$t \geq \frac{1}{\sqrt{D_1}}\Big(\frac{a^2+\mu}{1+\mu}\Big)^{\frac{1}{2}} I_1(z) \tag{5.96}$$

where $I_1(z)$ is defined by (5.86). Using (5.89) and proceeding as before it is found that if

$$0 \leq t < \frac{\pi}{2\sqrt{D_1}}\left(\frac{a^2+\mu}{1+\mu}\right)^{\frac{1}{2}} \tag{5.97}$$

then

$$x \leq \Big[\cos^2\Big(\sqrt{D_1}\Big(\frac{1+\mu}{a^2+\mu}\Big)^{\frac{1}{2}} t\Big) + a^2\sin^2\Big(\sqrt{D_1}\Big(\frac{1+\mu}{a^2+\mu}\Big)^{\frac{1}{2}} t\Big)\Big]^{\frac{1}{2}}. \tag{5.98}$$

The limiting oscillation is

$$x_L = \Big[\cos^2\Big(\sqrt{D_1}\Big(\frac{1+\mu}{a^2+\mu}\Big)^{\frac{1}{2}} t\Big) + a^2\sin^2\Big(\sqrt{D_1}\Big(\frac{1+\mu}{a^2+\mu}\Big)^{\frac{1}{2}} t\Big)\Big]^{\frac{1}{2}}, \tag{5.99}$$

which may be written as

$$x_L = \Big[\frac{a^2+1}{2} - \frac{(a^2-1)}{2}\cos\Big(2\sqrt{D_1}\Big(\frac{1+\mu}{a^2+\mu}\Big)^{\frac{1}{2}} t\Big)\Big]^{\frac{1}{2}}. \tag{5.100}$$

The period of the limiting oscillation is T_L given by (5.72) and its amplitude is the amplitude of the exact solution, (5.30).

For a thin-walled cylindrical tube in which terms of order μ are neglected (Shahinpoor and Nowinski, 1971),

$$x = \Big[\cos^2\Big(\frac{\sqrt{D_1}}{a}\,t\Big) + a^2\sin^2\Big(\frac{\sqrt{D_1}}{a}\,t\Big)\Big]^{\frac{1}{2}}, \tag{5.101}$$

where a is given by (5.43).

Case (iii) Oscillation corresponding to upper bound for the period when $0 < a < 1$

Making the same approximations in (5.84) as used to derive the upper bound for the period, (5.75), gives

$$t \leq \frac{(a^2+\mu)^{\frac{1}{2}}}{2\sqrt{D_1}}\, I_2(z) \tag{5.102}$$

where

$$I_2(z) = \int_z^1 \frac{dz}{[(1-z)(z-a^2)]^{\frac{1}{2}}} = 2\tan^{-1}\Big[\Big(\frac{1-z}{z-a^2}\Big)^{\frac{1}{2}}\Big]. \tag{5.103}$$

Thus (5.102) becomes

$$t \le \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}} \tan^{-1}\Big(\Big(\frac{1-x^2}{x^2-a^2}\Big)^{\frac{1}{2}}\Big) \tag{5.104}$$

and therefore

$$\Big(\frac{1-x^2}{x^2-a^2}\Big)^{\frac{1}{2}} \ge \tan\Big(\frac{\sqrt{D_1}\,t}{(a^2+\mu)^{\frac{1}{2}}}\Big). \tag{5.105}$$

Hence if

$$0 \le t < (a^2+\mu)^{\frac{1}{2}} \frac{\pi}{2\sqrt{D_1}} \tag{5.106}$$

then

$$x \ge \Big[\cos^2\Big(\frac{\sqrt{D_1}}{(a^2+\mu)^{\frac{1}{2}}}\,t\Big) + a^2\sin^2\Big(\frac{\sqrt{D_1}}{(a^2+\mu)^{\frac{1}{2}}}\,t\Big)\Big]^{\frac{1}{2}}. \tag{5.107}$$

The limiting oscillation is

$$x_U = \Big[\cos^2\Big(\frac{\sqrt{D_1}}{(a^2+\mu)^{\frac{1}{2}}}\,t\Big) + a^2\sin^2\Big(\frac{\sqrt{D_1}}{(a^2+\mu)^{\frac{1}{2}}}\,t\Big)\Big]^{\frac{1}{2}}, \tag{5.108}$$

which can be written as

$$x_U = \Big[\frac{a^2+1}{2} - \frac{(a^2-1)}{2}\cos\Big(\frac{2\sqrt{D_1}}{(a^2+\mu)^{\frac{1}{2}}}\,t\Big)\Big]^{\frac{1}{2}}. \tag{5.109}$$

The period of the limiting oscillation is T_U given by (5.82) and its amplitude is the exact amplitude, a, given by (5.30).

Case (iv) Oscillation corresponding to lower bound for the period when $0 < a < 1$

Making the same approximations in (5.84) as used to derive the lower bound for the period, (5.79), gives

$$t \ge \frac{a}{2\sqrt{D_1}} \int_z^1 \frac{dz}{[(1-z)(z-a^2)]^{\frac{1}{2}}} = \frac{a}{2\sqrt{D_1}} I_2(z). \tag{5.110}$$

Using (5.103) and proceeding as before it is found that if

$$0 \le t < \frac{\pi\, a}{2\sqrt{D_1}}, \tag{5.111}$$

then

$$x \le \Big[\cos^2\Big(\frac{\sqrt{D_1}}{a}\,t\Big) + a^2\sin^2\Big(\frac{\sqrt{D_1}}{a}\,t\Big)\Big]^{\frac{1}{2}}. \tag{5.112}$$

The limiting oscillation is

$$x_L = \Big[\cos^2\Big(\frac{\sqrt{D_1}}{a}\,t\Big) + a^2\sin^2\Big(\frac{\sqrt{D_1}}{a}\,t\Big)\Big]^{\frac{1}{2}}, \tag{5.113}$$

Table 5.3.1

which can be written as

$$x_L = \left[\frac{a^2+1}{2} - \frac{(a^2-1)}{2}\cos\left(\frac{2\sqrt{D_1}}{a}t\right)\right]^{\frac{1}{2}}. \tag{5.114}$$

The period of the limiting oscillation, T_L, is (5.82) and the amplitude is the exact amplitude a given by (5.30).

For a thin-walled cylindrical tube in which terms of order μ are neglected, $x(t)$ is again (5.101). The limiting oscillation, $x_L(t)$, given by (5.113) has the same form as (5.101) but in (5.113), a is (5.30) while in (5.101), a is (5.43).

The results of this section are summarised in Table 5.3.1.

5.4 Radial transversely isotropic cylindrical tube

Consider nonlinear radial oscillations in a radial transversely isotropic cylindrical tube subjected to the Heaviside step loading boundary condition (5.23) and to the initial conditions (5.24). Equation (5.17) becomes

$$\begin{aligned}\dot{x}^2 x^2 \ln\left(1+\frac{\mu}{x^2}\right) &= 2(x^2-1)\left[P_0 - \frac{D_1}{2}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right)\right.\\ &- \left.\frac{D_2}{4}\mu\frac{(x^2-1)}{x^4\left(1+\frac{\mu}{x^2}\right)}\right].\end{aligned} \tag{5.115}$$

The effects of the anisotropy on the radial oscillations will be investigated.

5.4.1 Critical net applied surface pressure

The greatest and least values of $x(t)$ are obtained when $\dot{x}=0$. Now, $\dot{x}=0$ when $x=1$ and when

$$P_0 = \frac{D_1}{2}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) + \frac{D_2}{4}\mu\frac{(x^2-1)}{x^4\left(1+\frac{\mu}{x^2}\right)}. \tag{5.116}$$

Let

$$\varepsilon = \frac{D_2}{D_1}. \tag{5.117}$$

Then $\varepsilon \geq 0$ and describes the anisotropy. Also, ε is not necessarily small. Define

$$P(x) = \frac{D_1}{2}\mu\left[\frac{1}{\mu}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) + \frac{\varepsilon(x^2-1)}{2x^4\left(1+\frac{\mu}{x^2}\right)}\right]. \tag{5.118}$$

Then

$$\frac{dP}{dx} = \frac{D_1\mu}{2x^3(x^2+\mu)^2}\left[(2-\varepsilon)x^4 + 2(\mu+\varepsilon)x^2 + \varepsilon\mu\right]. \tag{5.119}$$

Consider first $0 \leq \varepsilon \leq 2$. Then $P(x)$ is an increasing function of x and $P(x)$ therefore has no turning points. Also,

$$\lim_{x\to\infty} P(x) = \frac{D_1}{2}\ln(1+\mu) = P_c \tag{5.120}$$

which is the critical maximum net applied pressure for bounded radial oscillations in an isotropic cylindrical tube. For $0 \leq \varepsilon \leq 2$ and $P_0 \geq P_c$ there are no bounded oscillations as for an isotropic cylindrical tube. In Figure 5.4.1, $P(x)$ given by (5.118) is plotted against x for $\varepsilon = 1.5$ and $\varepsilon = 0$. For each value of $P_0 < P_c$ there corresponds a unique value of x. If $x = a$ and $x = a_I$ are the points at which $\dot{x} = 0$ for a radial transversely isotropic cylindrical tube and for an isotropic cylindrical tube then

$$P_0 > 0: \qquad a < a_I, \qquad a - 1 < a_I - 1, \tag{5.121}$$

$$P_0 < 0: \qquad a > a_I, \qquad 1 - a < 1 - a_I. \tag{5.122}$$

In both cases the amplitude of the oscillation in a radial transversely isotropic tube is less than in an isotropic tube. The amplitude is reduced by the anisotropy. The anisotropy makes the tube stiffer.

Consider next strong anisotropy for which $\varepsilon > 2$. Descartes' rule of signs states that the polynomial equation $Q(x) = 0$ cannot have more positive roots than there are changes of sign from + to − and from − to + in the coefficients of $Q(x)$ (Barnard and Child, 1936). The polynomial equation

$$Q(x) = -(\varepsilon - 2)x^4 + 2(\mu + \varepsilon)x^2 + \varepsilon\mu = 0 \tag{5.123}$$

has one change of sign and hence there is at most one positive root. Since $Q(0) = \mu\varepsilon > 0$ and $Q(\infty) = -\infty$, there is exactly one positive root. Denote this root by x_{max}. Then

$$x_{max} = \left[\frac{\mu + \varepsilon + ((\mu + \varepsilon)^2 + \varepsilon\mu(\varepsilon - 2))^{\frac{1}{2}}}{\varepsilon - 2}\right]^{\frac{1}{2}} \tag{5.124}$$

and

$$P_{max} = P(x_{max}). \tag{5.125}$$

Since $\frac{dP}{dx} < 0$ for $x > x_{max}$ and the limit (5.120) again applies, it follows that

$$P(x_{max}) > \frac{D_1}{2}\ln(1 + \mu) = P_c\,. \tag{5.126}$$

For $P_0 \geq P_{max}$ there are no bounded oscillations. For $-\infty < P_0 < P_{max}$ the oscillations are bounded. The maximum value of P_0 for bounded oscillations is therefore increased from P_c to P_{max} which is an anisotropic effect. In Figure 5.4.2, $P(x)$ given by (5.118) is plotted against x for $\varepsilon = 5 (\varepsilon > 2)$ and $\varepsilon = 0$ for comparison. For $P_c < P_0 < P_{max}$ there are two values of x corresponding to each value of P_0. The value which applies will be made clear from the effective potential. For $-\infty < P_0 < P_c$ there is one value of x corresponding to each value of P_0. Figure 5.4.2 shows again that the amplitude of the oscillation in a radial transversely isotropic tube is less than the amplitude in an isotropic tube.

Consider now a thin-walled cylindrical tube in which terms of order μ can be neglected. Then (5.118) and (5.119) become

$$P(x) = \frac{P_c}{2x^4}(x^2 - 1)(2x^2 + \varepsilon)\,, \tag{5.127}$$

$$\frac{dP}{dx} = \frac{P_c}{x^5}[(2 - \varepsilon)x^2 + 2\varepsilon]\,, \tag{5.128}$$

Figure 5.4.1

Figure 5.4.2

where

$$P_c = \frac{D_1 \mu}{2} \tag{5.129}$$

which is the critical maximum net applied pressure for bounded oscillations in an isotropic tube. If $0 \le \varepsilon \le 2$ then $P(x)$ is an increasing function of x and

$$\lim_{x \to \infty} P(x) = P_c \,. \tag{5.130}$$

If $\varepsilon > 2$ then $P(x)$ has a maximum value at

$$x_{max} = \left(\frac{2\varepsilon}{\varepsilon - 2} \right)^{\frac{1}{2}} \tag{5.131}$$

and

$$P_{max} = P(x_{max}) = P_c[1 + \frac{1}{8\varepsilon}(\varepsilon - 2)^2] \,. \tag{5.132}$$

For $P_0 \ge P_{max}$ there are no bounded radial oscillations. For $-\infty \le P_0 < P_{max}$ the oscillations are bounded. The bounded oscillations for $P_c \le P_0 < P_{max}$ is an anisotropic effect.

The end point $x = a$ of the oscillation is obtained by putting $P(x) = P_0$ and solving (5.127) for x. This gives for all $\varepsilon > 0$,

$$-\infty < P_0 < P_c : \qquad a = \frac{1}{2} \left[\frac{-(\varepsilon - 2) + [(\varepsilon - 2)^2 + 8\left(1 - \frac{P_0}{P_c}\right)\varepsilon]^{\frac{1}{2}}}{\left(1 - \frac{P_0}{P_c}\right)} \right]^{\frac{1}{2}} \tag{5.133}$$

and for $\varepsilon > 2$,

$$P_0 = P_c : \qquad a = \left(\frac{\varepsilon}{\varepsilon - 2} \right)^{\frac{1}{2}}, \tag{5.134}$$

$$P_c < P_0 \le P_{max} : \; a_1 = \frac{1}{2} \left[\frac{\varepsilon - 2 - [(\varepsilon - 2)^2 - 8\left(\frac{P_0}{P_c} - 1\right)]^{\frac{1}{2}}}{\left(\frac{P_0}{P_c} - 1\right)} \right]^{\frac{1}{2}}, \tag{5.135}$$

$$a_2 = \frac{1}{2} \left[\frac{\varepsilon - 2 + [(\varepsilon - 2)^2 - 8\left(\frac{P_0}{P_c} - 1\right)]^{\frac{1}{2}}}{\left(\frac{P_0}{P_c} - 1\right)} \right]^{\frac{1}{2}}. \tag{5.136}$$

When we consider the effective potential it will be seen that a_1 and not a_2 must be used. Equations (5.133) and (5.135) are the same.

In Figures 5.4.3 and 5.4.4, $P(x)$ is plotted against x for $\mu = 0.2$ and $0 \le \varepsilon \le 2$ and $\varepsilon > 2$. These graphs for a thin-walled tube have the same general properties as in Figures 5.3.1 and 5.3.2 for a thick-walled tube.

5.4.2 Effective potential

Equation (5.115) can be written as

$$\frac{1}{2}\dot{x}^2 + V(x) = 0 \,, \tag{5.137}$$

Figure 5.4.3

Figure 5.4.4

where

$$V(x) = \frac{(1-x^2)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \left[P_0 - \frac{D_1}{2} \ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) - \frac{D_2}{4} \frac{\mu(x^2-1)}{x^4\left(1+\frac{\mu}{x^2}\right)} \right]. \tag{5.138}$$

The effective potential $V(x)$ can be expressed in terms of ε as

$$V(x) = \frac{D_1\mu}{2} \frac{(1-x^2)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \left[\frac{2P_0}{D_1\mu} - \frac{1}{\mu} \ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) - \frac{\varepsilon}{2} \frac{(x^2-1)}{x^4\left(1+\frac{\mu}{x^2}\right)} \right]. \tag{5.139}$$

Consider the asymptotic behaviour of $V(x)$ for large and small x. For large x,

$$V(x) = \frac{1}{\mu}(P_c - P_0)x^2 + \frac{D_1}{4}(\varepsilon - 2) + \frac{(1-\mu)}{\mu}(P_0 - P_c) + O\Big(\frac{1}{x^2}\Big), \tag{5.140}$$

as $x \to \infty$, where P_c is defined by (5.120). Thus,

$$P_0 \neq P_c: \quad V(x) = \frac{1}{\mu}(P_c - P_0)x^2 + O(1) \quad \text{as}\, x \to \infty, \tag{5.141}$$

$$P_0 = P_c: \quad V(x) = \frac{D_1}{4}(\varepsilon - 2) + O\Big(\frac{1}{x^2}\Big) \quad \text{as}\, x \to \infty. \tag{5.142}$$

For small x,

$$V(x) = \frac{D_2}{8x^4(-\ln x)} + \frac{D_1}{2x^2} + O\Big(\frac{1}{x^2(-\ln x)}\Big) \quad \text{as}\, x \to 0 \tag{5.143}$$

and hence

$$V(x) \to +\infty \quad \text{as}\, x \to 0. \tag{5.144}$$

In Figures 5.4.5 to 5.4.7, $V(x)$ are plotted against x for $\varepsilon > 2$, $\varepsilon = 2$ and $0 < \varepsilon < 2$. Figure 5.4.5 shows that for $P_0 > P_{max}$ the radial motion is unbounded. The velocity $\dot{x}$ first increases, then decreases before it increases again and tends to infinity. Bounded oscillations exist for $P_0 < P_{max}$. When $0 < P_0 < P_{max}$,

$$\text{maximum displacement} = a - 1 < a_{max} - 1, \tag{5.145}$$

where $a_{max} = x_{max}$ defined by (5.124). In comparison, for radial oscillations in an isotropic cylindrical tube, there is no upper bound on the maximum displacement when $P_0 < P_c$ and bounded oscillations do not exist for $P_0 \geq P_c$. The upper bound (5.145) is an anisotropic effect. When $P_c < P_0 < P_{max}$ Figure 5.4.5 shows that there are two values of x at which $V(x) = 0$ when $x > 1$. The value to take is the smaller value because the oscillation must include the initial condition $x = 1$. For $\varepsilon = 2$ Figure 5.4.6 shows that when $P_0 = P_c$ the motion is unbounded and $\dot{x} \to 0$ as $x \to \infty$.

Figure 5.4.5

Figure 5.4.6

Figure 5.4.7

For $0 < \varepsilon < 2$, Figure 5.4.7 shows that when $P_0 = P_c$ the velocity tends to a limiting value as $x \to \infty$ which from (5.137) and (5.142) is

$$\dot{x} = \left(\frac{D_1(2-\varepsilon)}{2}\right)^{\frac{1}{2}}. \tag{5.146}$$

Consider now a thin-walled cylindrical tube in which terms of order μ^2 are neglected. Analytical results are easier to obtain for a thin-walled tube. Equation (5.139) becomes

$$V(x) = \frac{D_1}{4}\frac{(1-x^2)}{x^4}\left[2\left(\frac{P_0}{P_c}-1\right)x^4 + (2-\varepsilon)x^2 + \varepsilon\right] \tag{5.147}$$

$$= \frac{D_1}{2}\left[\left(1-\frac{P_0}{P_c}\right)x^2 + \left(\frac{P_0}{P_c} - 2 + \frac{\varepsilon}{2}\right) + (1-\varepsilon)\frac{1}{x^2} + \frac{\varepsilon}{2x^4}\right], \tag{5.148}$$

where P_c is defined by (5.129). Thus

$$\frac{dV}{dx} = \frac{D_1}{x^5}\left[\left(1-\frac{P_0}{P_c}\right)x^6 + (\varepsilon-1)x^2 - \varepsilon\right]. \tag{5.149}$$

The asymptotic behaviour of $V(x)$ for large x is again given by (5.141) and (5.142). For small x,

$$V(x) = \frac{D_2}{4x^4} + O\left(\frac{1}{x^2}\right) \quad \text{as } x \to 0 \tag{5.150}$$

and (5.144) is always satisfied since $D_2 > 0$.

There are four cases, $\varepsilon > 2$, $\varepsilon = 2$, $1 < \varepsilon < 2$ and $0 < \epsilon \leq 1$. The properties of $V(x)$ for the four cases are summarised in Table 5.4.1. Descartes' rule of signs (Barnard and Child, 1936) was used to obtain an upper bound on the number of zeros of $\frac{dV}{dx}$ and therefore on the number of turning points of $V(x)$.

Graphs of the effective potential $V(x)$ against x for $\mu = 0.2$ and $\varepsilon > 2$, $\varepsilon = 2$, $1 < \varepsilon < 2$ and $0 < \varepsilon \leq 1$ are plotted in Figures 5.4.8 to 5.4.11. The oscillations take place for $V(x) \leq 0$. The velocity $\dot{x}$ increases as $V(x)$ decreases and has a local maximum value at local minimum points of $V(x)$ and a local minimum at local maximum points of $V(x)$. Since the asymptotic behaviour of $V(x)$ for large x for the thin-walled cylindrical tube is the same as for a thick-walled tube, the limiting values of $V(x)$ for large x on the graphs for thin and thick-walled cylindrical tubes are the same. The end points of the oscillation occur for $V(x) = 0$ and are given by $x = 1$ and by $x = a$ defined by (5.133) and (5.135). Figure 5.4.8 shows that when $\varepsilon > 2$, a_1 and not a_2 is the upper limit of the oscillation because the oscillation must include $x = 1$.

For $\varepsilon > 2$, bounded oscillations exist for $P_0 \leq P_{max}$. The bounded oscillations for $P_c \leq P_0 \leq P_{max}$ is the main anisotropic effect. When $P_0 = P_{max}$ the end point of the oscillation, $a_1 = a_{max}$, is

$$a_{max} = \left(\frac{2\varepsilon}{\varepsilon - 2}\right)^{\frac{1}{2}} \tag{5.151}$$

Table 5.4.1

Table 5.4.1

Figure 5.4.8

Figure 5.4.9

Figure 5.4.10

Figure 5.4.11

and for oscillations with $0 < P_0 < P_{max}$ and $\varepsilon > 2$,

$$\text{maximum displacement} = a_1 - 1 < \left(\frac{2\varepsilon}{\varepsilon - 2}\right)^{\frac{1}{2}} - 1\,. \tag{5.152}$$

When $\varepsilon > 2$ and $P_0 = P_c$ the oscillation is still bounded while for an isotropic tube with $\varepsilon = 0$ the oscillation is unbounded. The end point of the oscillation when $P_0 = P_c$ is

$$a = \left(\frac{\varepsilon}{\varepsilon - 2}\right)^{\frac{1}{2}} \tag{5.153}$$

and $V(x)$ has a minimum value at

$$x = \left(\frac{\varepsilon}{\varepsilon - 1}\right)^{\frac{1}{2}}. \tag{5.154}$$

The maximum velocity when $P_0 = P_c$ is

$$|\dot{x}| = \left(\frac{D_1}{2\varepsilon}\right)^{\frac{1}{2}}. \tag{5.155}$$

When $\varepsilon = 2$ and $P_0 \geq P_c$ the oscillation is unbounded. When $P_0 = P_c$, $V(x)$ has a minimum value at $x = \sqrt{2}$ and at this point $\dot{x}$ has a maximum value

$$|\dot{x}| = \frac{1}{2} D_1^{\frac{1}{2}}\,. \tag{5.156}$$

When $0 < P_0 < P_c$ the range of oscillation is

$$1 \leq x \leq \left(1 - \frac{P_0}{P_c}\right)^{-\frac{1}{4}} \tag{5.157}$$

while if $P_0 < 0$ the range of oscillation is

$$\left(1 - \frac{P_0}{P_c}\right)^{-\frac{1}{4}} \leq x \leq 1\,. \tag{5.158}$$

In comparison when $\varepsilon = 0$, the exponent in (5.157) and (5.158) is $-\frac{1}{2}$. The result for $\varepsilon = 0$ is (5.43).

When $1 < \varepsilon < 2$ and $P_0 > P_c$, the effective potential $V(x)$ is not monotonically decreasing but has a local minimum and a local maximum before decreasing to $-\infty$ as $x \to \infty$. When $P_0 = P_c$, $V(x)$ has its minimum value at

$$x = \left(\frac{\varepsilon}{\varepsilon - 1}\right)^{\frac{1}{2}} \tag{5.159}$$

before increasing and tending to

$$V(\infty) = -\frac{D_1}{4}(2 - \varepsilon)\,, \tag{5.160}$$

as $x \to \infty$. The oscillation is unbounded for $P_0 \geq P_c$ and bounded for $P_0 < P_c$.

When $0 < \varepsilon \leq 1$, $V(x)$ is monotonic decreasing for $P_0 \geq P_c$ and the oscillation is unbounded for $P_0 \geq P_c$ and bounded for $P_0 < P_c$.

5.4.3 Period of the oscillation

Consider periodic oscillations so that $P_0 < P_c$ when $0 \le \varepsilon \le 2$ and $P_0 < P_{max}$ when $\varepsilon > 2$.

Now, $\dot{x} = 0$ at $x = 1$ and $x = a$. Since $\dot{x} = 0$ at $x = a$, it follows from (5.115) that

$$P_0 = \frac{D_1}{2} \ln\left(\frac{1+\mu}{1+\frac{\mu}{a^2}}\right) + \frac{D_2}{4}\mu \frac{(a^2-1)}{a^4\left(1+\frac{\mu}{a^2}\right)} \,. \tag{5.161}$$

Substituting (5.161) into (5.115) gives

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1(x^2-1)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \left[\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon\mu}{2} \frac{(a^2-x^2)(a^2+x^2-a^2x^2+\mu)}{a^2(a^2+\mu)x^2(x^2+\mu)}\right] . \tag{5.162}$$

The derivation of the period of the oscillations from (5.162) is similar to that for oscillations in an isotropic cylinder outlined in Section 5.3.3.

Consider first $a > 1$. The range of the oscillation is $1 \le x \le a$ and $P_0 > 0$. For oscillations the right hand side of (5.162) is positive. Thus

$$\frac{dx}{dt} = \pm \frac{\sqrt{D_1}(x^2-1)^{\frac{1}{2}}}{x\left[\ln\left(1+\frac{\mu}{x^2}\right)\right]^{\frac{1}{2}}} \left[\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon\mu}{2} \frac{(a^2-x^2)(a^2+x^2-a^2x^2+\mu)}{a^2(a^2+\mu)x^2(x^2+\mu)}\right]^{\frac{1}{2}} . \tag{5.163}$$

The plus and minus signs describe motion from 1 to a and a to 1, respectively. The period T of the oscillation is

$$T = \frac{2}{\sqrt{D_1}} \int_1^a \frac{x}{(x^2-1)^{\frac{1}{2}}} \left[\frac{\ln\left(1+\frac{\mu}{x^2}\right)}{\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon\mu}{2}\frac{(a^2-x^2)(a^2+x^2-a^2x^2+\mu)}{a^2(a^2+\mu)x^2(x^2+\mu)}}\right]^{\frac{1}{2}} dx \,. \tag{5.164}$$

On letting $z = x^2$, (5.164) becomes

$$T = \frac{1}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{\ln\left(1+\frac{\mu}{z}\right)}{(z-1)g(z)}\right]^{\frac{1}{2}} dz \tag{5.165}$$

where

$$g(z) = \ln\left(\frac{1+\frac{\mu}{z}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon\mu}{2} \frac{(a^2-z)(a^2+z-a^2z+\mu)}{a^2(a^2+\mu)z(z+\mu)} \,. \tag{5.166}$$

Consider next $0 < a < 1$ so that $a \le x \le 1$. From (5.161), $P_0 < 0$. Equation (5.162) becomes

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1(1-x^2)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \left[\ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{x^2}}\right) + \frac{\varepsilon\mu}{2} \frac{(x^2-a^2)(a^2+x^2-a^2x^2+\mu)}{a^2(a^2+\mu)x^2(x^2+\mu)}\right] . \tag{5.167}$$

The right hand side of (5.167) is always positive. The period of oscillation is

$$T = \frac{1}{\sqrt{D_1}} \int_{a^2}^{1} \left[\frac{\ln\left(1+\frac{\mu}{z}\right)}{(1-z)h(z)}\right]^{\frac{1}{2}} dz \tag{5.168}$$

where

$$h(z) = -g(z) = \ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{z}}\right) + \frac{\varepsilon\mu}{2}\frac{(z-a^2)(a^2+z-a^2z+\mu)}{a^2(a^2+\mu)z(z+\mu)} \ . \tag{5.169}$$

The upper and lower bounds on the period are obtained using the inequalities (5.52) and (5.53) for logarithms (Knowles , 1962). There are four cases.

Case (i) Upper bound for the period when $a > 1$

The period is given by (5.165). The inequality (5.54) again applies. Using the inequality (5.55) in (5.165) gives

$$g(z) \geq \frac{\mu(a^2-z)}{a^2(\mu+z)}\left[1 - \frac{\varepsilon}{2}\left(\frac{a^2-1}{a^2+\mu}\right) + \frac{\varepsilon}{2z}\right] . \tag{5.170}$$

Now the right hand side of (5.170) is positive. For, the term inside square brackets takes its minimum value at $z = a^2$ and

$$1 - \tfrac{\varepsilon}{2}\left(\tfrac{a^2-1}{a^2+\mu}\right) + \tfrac{\varepsilon}{2a^2} > 0 \ ,$$

if and only if

$$(2-\varepsilon)a^4 + 2(\mu+\varepsilon)a^2 + \mu\varepsilon > 0. \tag{5.171}$$

Condition (5.171) is clearly satisfied for $\varepsilon \leq 2$. For $\varepsilon > 2$ consider (5.119). Now $\frac{dP}{dx} > 0$ for $a < a_{max} = x_{max}$ and therefore (5.171) is satisfied for all $a < a_{max}$. There are no bounded oscillations for $a > a_{max}$ when $\varepsilon > 2$. Using (5.54) and (5.170), (5.165) becomes

$$T \leq \frac{a}{\sqrt{D_1}}\int_1^{a^2}\left[\frac{f(z)}{(z-1)(a^2-z)}\right]^{\frac{1}{2}} dz, \tag{5.172}$$

where

$$f(z) = \frac{z+\mu}{A+Bz} \tag{5.173}$$

and

$$A = \frac{\varepsilon}{2}, \quad B = 1 - \frac{\varepsilon}{2}\frac{(a^2-1)}{(a^2+\mu)} \ . \tag{5.174}$$

Now

$$\frac{df}{dz} = \left[\varepsilon - \frac{2\mu}{(1+\mu)}\frac{a^2}{(a^2+\mu)}\right]\frac{(1+\mu)a^2}{2(a^2+\mu)(A+Bz)^2} \ . \tag{5.175}$$

There are two cases.

Firstly, if

$$\varepsilon < \frac{2\mu}{(1+\mu)}\frac{(a^2+\mu)}{a^2} \tag{5.176}$$

then $f(z)$ is a decreasing function of z and

$$f(z) \leq f(1) = \frac{1+\mu}{\left(1+\frac{\varepsilon}{2}\left(\frac{1+\mu}{a^2+\mu}\right)\right)} \ . \tag{5.177}$$

Thus

$$T \leq \frac{a}{\sqrt{D_1}}[f(1)]^{\frac{1}{2}} I_1(a^2) \tag{5.178}$$

where $I_1(a^2)$ is defined by (5.61). Hence

$$T \leq a\left(\frac{1+\mu}{1+\frac{\varepsilon}{2}\left(\frac{1+\mu}{a^2+\mu}\right)}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}\ . \tag{5.179}$$

Secondly, if

$$\varepsilon > \frac{2\mu}{(1+\mu)}\frac{(a^2+\mu)}{a^2} \tag{5.180}$$

then $f(z)$ is an increasing function of z and

$$f(z) \leq f(a^2) = \frac{1+\frac{\mu}{a^2}}{\left(1+\frac{\varepsilon}{2a^2}\left(1-\frac{a^2(a^2-1)}{a^2+\mu}\right)\right)}\ . \tag{5.181}$$

Thus

$$T \leq \frac{a}{\sqrt{D_1}}[f(a^2)]^{\frac{1}{2}} I_1(a^2) \tag{5.182}$$

and using (5.61),

$$T \leq a\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\varepsilon}{2a^2}\left(1-\frac{a^2(a^2-1)}{a^2+\mu}\right)}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}\ . \tag{5.183}$$

The denominator in (5.183) is positive because of (5.171).

Case (ii) Lower bound for the period when $a > 1$

The period is again given by (5.165). The inequality (5.63) applies. Using the inequality (5.64), (5.165) becomes

$$g(z) \leq \frac{\mu(a^2-z)}{z(a^2+\mu)}\left[1+\frac{\varepsilon}{2a^2}\frac{(a^2+\mu-(a^2-1)z)}{z+\mu}\right]. \tag{5.184}$$

The right hand side of (5.184) is positive because it is greater than the right hand side of (5.166). Using the inequalities (5.63) and (5.184), (5.165) becomes

$$T \geq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{f(z)}{(z-1)(a^2-z)}\right]^{\frac{1}{2}} dz \tag{5.185}$$

where

$$f(z) = \frac{z}{A+Bz} \tag{5.186}$$

and

$$A = \mu + \frac{\varepsilon}{2a^2}(a^2+\mu), \quad B = 1-\frac{\varepsilon}{2a^2}(a^2-1)\ . \tag{5.187}$$

Now

$$\frac{df}{dz} = \frac{A}{(A+Bz)^2} > 0 \tag{5.188}$$

and $f(z)$ is therefore an increasing function of z. Hence

$$f(z) \geq f(1) = \frac{1}{(1+\mu)\left(1+\frac{\varepsilon}{2a^2}\right)} \tag{5.189}$$

and (5.184) becomes

$$T \geq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}}[f(1)]^{\frac{1}{2}}\, I_1(a^2). \tag{5.190}$$

Finally, using (5.61) for $I_1(a^2)$,

$$T \geq a\left(\frac{1+\frac{\mu}{a^2}}{(1+\mu)\left(1+\frac{\varepsilon}{2a^2}\right)}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}\ . \tag{5.191}$$

Case (iii) Upper bound on the period when $0 < a < 1$

The period is given by (5.168). The inequality (5.54) again applies. Using the inequality (5.74), (5.169) gives

$$h(z) \geq \frac{\mu(z-a^2)}{z(a^2+\mu)}\left[1+\frac{\varepsilon}{2a^2}\frac{(a^2+\mu+(1-a^2)z)}{z+\mu}\right]. \tag{5.192}$$

Since $0 < a^2 < 1$, the term inside the square bracket is positive. Using (5.54) and (5.192), (5.168) becomes

$$T \leq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}}\int_{a^2}^{1}\left[\frac{f(z)}{(1-z)(z-a^2)}\right]^{\frac{1}{2}} dz \tag{5.193}$$

where

$$f(z) = \frac{z+\mu}{A+Bz} \tag{5.194}$$

and

$$A = \mu + \frac{\varepsilon}{2a^2}(a^2+\mu), \quad B = 1+\frac{\varepsilon}{2a^2}(1-a^2)\ . \tag{5.195}$$

But

$$\frac{df}{dz} = \frac{\varepsilon(1+\mu)}{2(A+Bz)^2} > 0. \tag{5.196}$$

Thus $f(z)$ is an increasing function of z and since $a^2 \leq z \leq 1$,

$$f(z) \leq f(1) = \frac{1}{\left(1+\frac{\varepsilon}{2a^2}\right)}. \tag{5.197}$$

Hence

$$T \leq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}}[f(1)]^{\frac{1}{2}}\, I_2(a^2) \tag{5.198}$$

where $I_2(a^2)$ is defined by (5.76) and therefore

$$T \leq a\left(\frac{1+\frac{\mu}{a^2}}{\left(1+\frac{\varepsilon}{2a^2}\right)}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}\ . \tag{5.199}$$

Case (iv) Lower bound on the period when $0 < a < 1$

The period is again given by (5.168). The inequality (5.63) applies. Also, using the inequality (5.78), (5.169) becomes

$$h(z) \leq \frac{\mu(z-a^2)}{a^2(z+\mu)z}\left[\frac{\varepsilon}{2} + \left(1 + \frac{\varepsilon(1-a^2)}{2(a^2+\mu)}\right)z\right] . \tag{5.200}$$

The term inside the square brackets is positive because $0 < a < 1$. Applying (5.63) and (5.200), (5.168) becomes

$$T \geq \frac{a}{\sqrt{D_1}} \int_{a^2}^{1} \left[\frac{f(z)}{(1-z)(z-a^2)}\right]^{\frac{1}{2}} dz \tag{5.201}$$

where

$$f(z) = \frac{z}{A+Bz} \tag{5.202}$$

and

$$A = \frac{\varepsilon}{2}, \quad B = 1 + \frac{\varepsilon(1-a^2)}{2(a^2+\mu)} . \tag{5.203}$$

But

$$\frac{df}{dz} = \frac{A}{(A+Bz)^2} > 0 \tag{5.204}$$

and $f(z)$ is therefore an increasing function of z. Thus since $a^2 \leq z \leq 1$,

$$f(z) \geq f(a^2) = \frac{1}{1 + \frac{\varepsilon}{2a^2}\left(1 + \frac{a^2(1-a^2)}{a^2+\mu}\right)} . \tag{5.205}$$

From (5.201),

$$T \geq \frac{a}{\sqrt{D_1}} [f(a^2)]^{\frac{1}{2}} \, I_2(a^2) \tag{5.206}$$

where $I_2(a^2)$ is given by (5.76). Thus

$$T \geq a\left(\frac{1}{1 + \frac{\varepsilon}{2a^2}\left(1 + \frac{a^2(1-a^2)}{a^2+\mu}\right)}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} . \tag{5.207}$$

The lower and upper bounds for the period are summarised in Table 5.4.2. They can be compared with the results for an isotropic cylindrical tube in Table 5.3.1. For $0 < a < 1$ the lower and upper bounds for the period are decreased by the anisotropy. When $a > 1$, the lower bound is decreased. The upper bound which can be compared with the isotropic result is (5.179) because then (5.176) is satisfied. This upper bound is decreased by the anisotropy. These results suggest that the effect of the anisotropy is to decrease the period of oscillation.

The lower and upper bounds for the period of oscillation for a thin walled cylindrical tube are obtained by setting $\mu = 0$. When $a > 1$ the upper bound for $\mu = 0$ is obtained from (5.183) because from (5.180), $\varepsilon > 0$.

Table 5.4.2

5.4.4 Amplitude of the oscillation

The oscillations which correspond to the upper and lower bounds on the period are derived by performing the same approximations on the oscillations as were made on the period. The derivation is the same as in Section 5.3.4 for an isotropic cylindrical tube. The period of the oscillation is the same as that of the upper and lower bounds and the amplitude is the amplitude of the exact solution. The limiting oscillation has the form of a nonlinear superposition. The results which are obtained are listed in Table 5.4.2. They compare with the results for an isotropic cylindrical tube in Table 5.3.1.

The corresponding oscillations for a thin-walled cylindrical tube are obtained by setting $\mu = 0$. For $P_0 > 0$ and $a > 1$, to obtain the limit $\mu \to 0$, the solution which satisfies (5.180) is used.

5.5 Tangential transversely isotropic cylindrical tube

Finally, consider nonlinear radial oscillations in a tangential transversely isotropic cylindrical tube subjected to the Heaviside step loading boundary condition (5.23) and to the initial conditions (5.24). Equation (5.17) becomes

$$\begin{aligned} \dot{x}^2 x^2 \ln\left(1+\frac{\mu}{x^2}\right) &= 2(x^2-1)\Bigg[P_0 - \frac{D_1}{2}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) \\ &- \frac{D_2}{4}\frac{\mu}{(1+\mu)}(x^2-1)\Bigg]. \end{aligned} \qquad (5.208)$$

The effect of tangential transversely isotropy on the oscillations will now be investigated.

5.5.1 Critical net applied surface pressure

The maximum and minimum values of $x(t)$ are given by $\dot{x} = 0$. Now, $\dot{x} = 0$ when $x = 1$ and when

$$P_0 = \frac{D_1}{2}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) + \frac{D_2}{4}\frac{\mu}{(1+\mu)}(x^2-1). \qquad (5.209)$$

Define

$$P(x) = P_c\left[\frac{1}{\mu}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) + \frac{\varepsilon(x^2-1)}{2(1+\mu)}\right], \qquad (5.210)$$

where P_c is given by (5.129) and ε by (5.117). Now

$$\frac{dP}{dx} = P_c\left[\frac{2}{x^3\left(1+\frac{\mu}{x^2}\right)} + \frac{\varepsilon x}{1+\mu}\right] > 0 \qquad (5.211)$$

for $x > 0$. Thus $P(x)$ is a monotonic increasing function of x with $P(0) = -\infty$, $P(1) = 0$ and $P(\infty) = \infty$. In Figure 5.5.1, $P(x)$ is plotted against x for $\varepsilon = 2$ and for an isotropic tube with $\varepsilon = 0$. For all P_0 in the range $-\infty < P_0 < \infty$ there is a unique solution $x = a$ of (5.209). There are therefore bounded oscillations for

Figure 5.5.1

$-\infty < P_0 < \infty$. This compares with radial oscillations in an isotropic tube for which bounded oscillations exist only for

$$-\infty < P_0 < \frac{D_1}{2}\ln(1+\mu). \qquad (5.212)$$

The bounded oscillations for

$$\frac{D_1}{2}\ln(1+\mu) \leq P_0 < \infty \qquad (5.213)$$

are therefore an effect of a tangential transversely isotropic tube.

Let $x = a$ and $x = a_I$ be the points at which $\dot{x} = 0$ for a given net applied surface pressure P_0 for a tangential transversely isotropic cylindrical tube and an isotropic cylindrical tube. Then from Figure 5.5.1,

$$0 < P_0 \leq \frac{D_1}{2}\ln(1+\mu): \qquad a < a_I, \qquad a-1 < a_I - 1, \qquad (5.214)$$

$$-\infty < P_0 < 0: \qquad a > a_I, \qquad 1-a < 1-a_I. \qquad (5.215)$$

In both cases the amplitude of oscillation in a tangential transversely isotropic tube is less than in an isotropic tube. The amplitude is reduced by the anisotropy which has the effect of making the tube stiffer.

Consider a thin-walled cylindrical tube in which terms of order μ^2 are neglected. Then (5.210) and (5.211) become

$$P(x) = P_c(x^2-1)\left[\frac{1}{x^2} + \frac{\varepsilon}{2}\right], \qquad (5.216)$$

$$\frac{dP}{dx} = P_c\left(\frac{2}{x^3} + \varepsilon x\right) > 0. \qquad (5.217)$$

The general properties for a thin-walled tube are the same as for a thick-walled tube. There are bounded oscillations for $-\infty < P_0 < \infty$ and the range $P_c \leq P_0 < \infty$ is an effect of the tangential transversely isotropic tube. The end point of the oscillation for a given $P(x) = P_0$ where $-\infty < P_0 < \infty$ is obtained by solving (5.216) for x:

$$a = \left[-\frac{1}{\varepsilon}\left(1 - \frac{P_0}{P_c} - \frac{\varepsilon}{2}\right) + \frac{1}{\varepsilon}\left[\left(1 - \frac{P_0}{P_c} - \frac{\varepsilon}{2}\right)^2 + 2\varepsilon\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (5.218)$$

$$= \frac{1}{\left(1-\frac{P_0}{P_c}\right)^{\frac{1}{2}}} - \frac{\varepsilon}{4\left(1-\frac{P_0}{P_c}\right)^{\frac{5}{2}}}\frac{P_0}{P_c} + O(\varepsilon^2), \qquad (5.219)$$

as $\varepsilon \to 0$. The expansion (5.219) agrees with (5.43) for $\varepsilon = 0$ and shows again that the anisotropy decreases the amplitude which is $a - 1$ when $P_0 > 0$ and $1 - a$ when $P_0 < 0$.

5.5.2 Effective potential

Equation (5.208) can be written as

$$\frac{1}{2}\dot{x}^2 + V(x) = 0 \tag{5.220}$$

where

$$V(x) = \frac{(1-x^2)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)}\left[P_0 - \frac{D_1}{2}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right) - \frac{D_2}{4}\frac{\mu}{(1+\mu)}(x^2-1)\right]. \tag{5.221}$$

The effective potential $V(x)$ can be written in terms of ε as

$$\begin{aligned} V(x) &= \frac{(1-x^2)P_c}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)}\left[\frac{P_0}{P_c} - \frac{1}{\mu}\ln\left(\frac{1+\mu}{1+\frac{\mu}{x^2}}\right)\right. \\ &- \left.\frac{\varepsilon}{2}\frac{(x^2-1)}{(1+\mu)}\right], \end{aligned} \tag{5.222}$$

where P_c is defined by (5.129).

Consider now the asymptotic behaviour of $V(x)$ for large and small x. For large x,

$$V(x) = \frac{\varepsilon\,\mu\,P_c}{2(1+\mu)}x^4 + O(x^2) \quad \text{as} \quad x \to \infty \tag{5.223}$$

and hence

$$V(x) \to +\infty \quad \text{as} \quad x \to \infty. \tag{5.224}$$

For small x,

$$V(x) = \frac{P_c}{\mu x^2} + O(1) \quad \text{as} \quad x \to 0+ \tag{5.225}$$

and hence

$$V(x) \to +\infty \quad \text{as} \quad x \to 0+. \tag{5.226}$$

Also, because of the factor $(1-x^2)$ in $V(x)$, it can be shown that

$$\frac{dV}{dx}(1) = -\frac{2P_0}{\ln(1+\mu)} \tag{5.227}$$

and therefore

$$\begin{aligned} &\text{if } P_0 > 0 \text{ then } \frac{dV}{dx}(1) < 0 \\ &\text{if } P_0 < 0 \text{ then } \frac{dV}{dx}(1) > 0. \end{aligned} \tag{5.228}$$

In Figure 5.5.2 the effective potential $V(x)$ is plotted against x for $\mu = 5$ for $P_0 < 0$ and $P_0 > 0$. The oscillation exists for $V(x) < 0$ and the end points are at

Figure 5.5.2

$V(x) = 0$. For $P_0 < 0$ the tube is compressed while for $P_0 > 0$ the tube is extended. Bounded oscillations exist for $-\infty < P_0 < \infty$. The bounded oscillations for

$$\frac{D_1}{2}\ln(1+\mu) \leq P_0 < \infty \tag{5.229}$$

is an anisotropic effect of the tangential transversely isotropic tube.

Consider now a thin-walled cylindrical tube. Expanding (5.222) in powers of μ, using (5.129) for P_c and neglecting terms of order μ^2 gives

$$V(x) = \frac{D_1(1-x^2)}{4x^2}\Big[-\varepsilon x^4 - 2\Big(1 - \frac{P_0}{P_c} - \frac{\varepsilon}{2}\Big)x^2 + 2\Big]. \tag{5.230}$$

Also,

$$\frac{dV}{dx} = \frac{D_1}{x^3}\Big[\varepsilon x^6 + \Big(1 - \frac{P_0}{P_c} - \varepsilon\Big)x^4 - 1\Big]. \tag{5.231}$$

The turning points of $V(x)$ are determined from

$$Q(x) = \varepsilon x^6 + \Big(1 - \frac{P_0}{P_c} - \varepsilon\Big)x^4 - 1 = 0. \tag{5.232}$$

By Descartes rule of signs (Barnard and Child, 1936), (5.232) cannot have more than one positive root. This is true whether the coefficient of x^4 in (5.232) is positive or negative. Since $Q(0) = -1$ and $Q(x) \to +\infty$ as $x \to +\infty$, (5.232) has exactly one positive root. Thus $V(x)$ has exactly one turning point and since $V(x) \to \infty$ as $x \to 0$ and $x \to \infty$, the turning point is a minimum.

The end points of the oscillation satisfy $V(x) = 0$. They are $x = 1$ and the real root, a, of

$$\varepsilon x^4 + 2\Big(1 - \frac{P_0}{P_c} - \frac{\varepsilon}{2}\Big)x^2 - 2 = 0, \tag{5.233}$$

which is given by (5.218). We show that if $P_0 > 0$ then $a > 1$ while if $P_0 < 0$ then $0 < a < 1$. Suppose that $P_0 > 0$. Then $a > 1$ provided

$$\Big[\Big(1 - \frac{P_0}{P_c} - \frac{\varepsilon}{2}\Big)^2 + 2\varepsilon\Big]^{\frac{1}{2}} > 1 - \frac{P_0}{P_c} + \frac{\varepsilon}{2}, \tag{5.234}$$

which can be rewritten as

$$\Big[\Big(1 - \frac{P_0}{P_c} + \frac{\varepsilon}{2}\Big)^2 + 2\varepsilon\frac{P_0}{P_c}\Big]^{\frac{1}{2}} > 1 - \frac{P_0}{P_c} + \frac{\varepsilon}{2}. \tag{5.235}$$

Condition (5.235) is satisfied when $P_0 > 0$. Similarly $a < 1$ provided

$$\Big[\Big(1 - \frac{P_0}{P_c} + \frac{\varepsilon}{2}\Big)^2 + 2\varepsilon\frac{P_0}{P_c}\Big]^{\frac{1}{2}} < 1 - \frac{P_0}{P_c} + \frac{\varepsilon}{2} \tag{5.236}$$

which is satisfied when $P_0 < 0$.

In Figure 5.5.3, $V(x)$ given by (5.230) is plotted against x for $\mu = 0.2$ and for $P_0 < 0$ and $P_0 > 0$. Bounded oscillations exist for $-\infty < P_0 < \infty$. The bounded oscillations for

$$P_c \leq P_0 < \infty \tag{5.237}$$

is an anisotropic effect of the thin-walled tangential transversely isotropic tube. The general properties of Figures 5.5.2 and 5.5.3 are the same.

Figure 5.5.3

5.5.3 Period of the oscillation

Consider the period of the oscillations. Now $\dot{x} = 0$ at $x = 1$ and $x = a$. From (5.208), since $\dot{x} = 0$ at $x = a$,

$$P_0 = \frac{D_1}{2} \ln\left(\frac{1+\mu}{1+\frac{\mu}{a^2}}\right) + \frac{D_2}{4} \frac{\mu}{(1+\mu)}(a^2 - 1). \tag{5.238}$$

Substituting (5.238) into (5.208) gives

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1(x^2-1)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \left[\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon}{2}\frac{\mu}{(1+\mu)}(a^2 - x^2)\right]. \tag{5.239}$$

Consider first $a > 1$. The range of oscillation is $1 \leq x \leq a$ and from (5.238), $P_0 > 0$. The right hand side of (5.238) is positive. Thus

$$\frac{dx}{dt} = \pm \frac{\sqrt{D_1}\,(x^2-1)^{\frac{1}{2}}}{x\left[\ln\left(1+\frac{\mu}{x^2}\right)\right]^{\frac{1}{2}}} \left[\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon}{2}\frac{\mu}{(1+\mu)}(a^2 - x^2)\right]^{\frac{1}{2}} \tag{5.240}$$

where the plus sign describes the motion from 1 to a and the minus sign from a to 1. The period T of the oscillation is

$$T = \frac{2}{\sqrt{D_1}} \int_1^a \frac{x}{(x^2-1)^{\frac{1}{2}}} \left[\frac{\ln\left(1+\frac{\mu}{x^2}\right)}{\ln\left(\frac{1+\frac{\mu}{x^2}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon}{2}\frac{\mu}{(1+\mu)}(a^2 - x^2)}\right]^{\frac{1}{2}} dx. \tag{5.241}$$

Let $z = x^2$. Then (5.241) becomes

$$T = \frac{1}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{\ln\left(1+\frac{\mu}{z}\right)}{(z-1)g(z)}\right]^{\frac{1}{2}} dz, \tag{5.242}$$

where

$$g(z) = \ln\left(\frac{1+\frac{\mu}{z}}{1+\frac{\mu}{a^2}}\right) + \frac{\varepsilon}{2}\frac{\mu}{(1+\mu)}(a^2 - z). \tag{5.243}$$

Consider next $0 < a < 1$ so that $a \leq x \leq 1$ and from (5.238), $P_0 < 0$. Equation (5.239) becomes

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1(1-x^2)}{x^2 \ln\left(1+\frac{\mu}{x^2}\right)} \left[\ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{x^2}}\right) + \frac{\varepsilon}{2}\frac{\mu}{(1+\mu)}(x^2 - a^2)\right]. \tag{5.244}$$

The right hand side of (5.244) is always positive. The period of oscillation is

$$T = \frac{1}{\sqrt{D_1}} \int_{a^2}^{1} \left[\frac{\ln\left(1+\frac{\mu}{z}\right)}{(1-z)h(z)}\right]^{\frac{1}{2}} dz, \tag{5.245}$$

where

$$h(z) = -g(z) = \ln\left(\frac{1+\frac{\mu}{a^2}}{1+\frac{\mu}{z}}\right) + \frac{\varepsilon}{2}\frac{\mu}{(1+\mu)}(z - a^2). \tag{5.246}$$

The upper and lower bounds for the period are obtained using the inequalities (5.52) and (5.53) (Knowles , 1962). As with isotropic and radial transversely isotropic tubes there are four cases.

Case (i) Upper bound for the period when $a > 1$

The period is given by (5.242). The inequalities (5.54) and (5.55) again apply. Using (5.55), (5.243) becomes

$$g(z) \geq \frac{\mu(a^2 - z)}{a^2(\mu + z)}\left[1 + \frac{\varepsilon}{2}\, a^2\, \frac{(\mu + z)}{(1 + \mu)}\right]. \tag{5.247}$$

Using (5.54) and (5.247), (5.242) becomes

$$T \leq \frac{a}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{f(z)}{(z - 1)(a^2 - z)}\right]^{\frac{1}{2}} dz \tag{5.248}$$

where

$$f(z) = \frac{z + \mu}{(A + Bz)z} \tag{5.249}$$

and

$$A = 1 + \frac{\varepsilon}{2}\frac{\mu}{(1 + \mu)}a^2, \quad B = \frac{\varepsilon}{2}\frac{a^2}{(1 + \mu)}\ . \tag{5.250}$$

Now

$$\frac{df}{dz} = -\left(\frac{B(\mu + z)^2 + \mu}{z^2(A + Bz)}\right) < 0. \tag{5.251}$$

Thus $f(z)$ is a decreasing function of z and therefore

$$f(z) \leq f(1) = \frac{1 + \mu}{\left(1 + \frac{1}{2}\,\varepsilon\, a^2\right)}\ . \tag{5.252}$$

Hence

$$T \leq \frac{a}{\sqrt{D_1}}[f(1)]^{\frac{1}{2}} I_1[a^2], \tag{5.253}$$

where $I_1[a^2]$ is defined by (5.61). Thus

$$T \leq a\left(\frac{1 + \mu}{1 + \frac{1}{2}\,\varepsilon a^2}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}}\ . \tag{5.254}$$

Case (ii) Lower bound for the period when $a > 1$

The period is again given by (5.242). The inequalities (5.63) and (5.64) apply. Using (5.64), (5.243) becomes

$$g(z) \leq \frac{\mu(a^2 - z)}{z(a^2 + \mu)}\left[1 + \frac{\varepsilon}{2}\frac{(a^2 + \mu)}{(1 + \mu)} z\right]. \tag{5.255}$$

Using (5.63) and (5.255), (5.242) becomes

$$T \geq \frac{(a^2 + \mu)^{\frac{1}{2}}}{\sqrt{D_1}} \int_1^{a^2} \left[\frac{f(z)}{(z - 1)(a^2 - z)}\right]^{\frac{1}{2}} dz \tag{5.256}$$

where

$$f(z)=\frac{z}{(z+\mu)(1+Az)} \tag{5.257}$$

and

$$A=\frac{\varepsilon}{2}\frac{(a^2+\mu)}{(1+\mu)}\ . \tag{5.258}$$

Now,

$$\frac{df}{dz}=\frac{\mu-Az^2}{(z+\mu)^2(1+Az)^2}\ . \tag{5.259}$$

Thus $f(z)$ has one turning point in the range $z>0$ at $z=z_{max}$ where

$$z_{max}=\left(\frac{\mu}{A}\right)^{\frac{1}{2}}=\left(\frac{2\mu(1+\mu)}{\varepsilon(a^2+\mu)}\right)^{\frac{1}{2}}\ . \tag{5.260}$$

It can be shown that

$$\frac{d^2f}{dz^2}(z_{max})=-\frac{2A^{\frac{3}{2}}}{\mu^{\frac{1}{2}}(1+(\mu A)^{\frac{1}{2}})^2}<0 \tag{5.261}$$

and therefore the turning point is a maximum. As z increases from $z=0$, $f(z)$ increases from $f(0)=0$ to a maximum value $f(z_{max})$ then decreases and $f(z)\to 0$ as $z\to\infty$. The minimum value of $f(z)$ in the range $1\le z\le a^2$ is the minimum of $f(1)$ and $f(a^2)$. It can be shown that

$$f(1)>f(a^2) \text{ if } z_{max}<a, \text{ that is, if } \varepsilon>\tfrac{2\mu(1+\mu)}{a^2(a^2+\mu)}\ ,$$

$$f(1)=f(a^2) \text{ if } z_{max}=a, \text{ that is, if } \varepsilon=\tfrac{2\mu(1+\mu)}{a^2(a^2+\mu)}\ ,$$

$$f(1)<f(a^2) \text{ if } z_{max}>a, \text{ that is, if } \varepsilon<\tfrac{2\mu(1+\mu)}{a^2(a^2+\mu)}\ .$$

Consider first

$$\varepsilon\le\frac{2\mu}{a^2}\frac{(1+\mu)}{(a^2+\mu)}\ . \tag{5.262}$$

Then $f(1)$ is the minimum value of $f(z)$ in the range $1\le z\le a^2$ and therefore from (5.256),

$$T\ge\frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}}[f(1)]^{\frac{1}{2}}\,I_1[a^2] \tag{5.263}$$

and using (5.61) and

$$f(1)=\frac{1}{(1+\mu)\left(1+\frac{\varepsilon}{2}\frac{(a^2+\mu)}{(1+\mu)}\right)}\ , \tag{5.264}$$

(5.263) becomes

$$T\ge a\left(\frac{1+\frac{\mu}{a^2}}{1+\mu+\frac{\varepsilon a^2}{2}\left(1+\frac{\mu}{a^2}\right)}\right)^{\frac{1}{2}}\frac{\pi}{\sqrt{D_1}}\ . \tag{5.265}$$

Consider next

$$\varepsilon \geq \frac{2\mu}{a^2}\frac{(1+\mu)}{(a^2+\mu)} \ . \tag{5.266}$$

Then $f(a^2)$ is the minimum value of $f(z)$ in the range $1 \leq z \leq a^2$ and therefore from (5.256),

$$T \geq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}}[f(a^2)]^{\frac{1}{2}}\, I_1[a^2]. \tag{5.267}$$

Using (5.61) and

$$f(a^2) = \frac{a^2}{(a^2+\mu)\left(1+\frac{\varepsilon a^2}{2}\frac{(a^2+\mu)}{(1+\mu)}\right)} \ , \tag{5.268}$$

(5.267) becomes

$$T \geq \frac{a}{\left(1+\frac{\varepsilon a^2}{2}\frac{(a^2+\mu)}{(1+\mu)}\right)}\frac{\pi}{\sqrt{D_1}} \ . \tag{5.269}$$

Case (iii) Upper bound on the period when $0 < a < 1$

The period is given by (5.245). The inequalities (5.54) and (5.74) again apply. Using (5.74), (5.246) becomes

$$h(z) \geq \frac{\mu(z-a^2)}{z(a^2+\mu)}\left[1+\frac{\varepsilon}{2}\frac{(a^2+\mu)}{(1+\mu)}z\right]. \tag{5.270}$$

Using (5.54) and (5.270), (5.245) becomes

$$T \leq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}}\int_{a^2}^{1}\left[\frac{f(z)}{(1-z)(z-a^2)}\right]^{\frac{1}{2}} dz \tag{5.271}$$

where

$$f(z) = \frac{1}{1+Az} \ , \quad A = \frac{\varepsilon}{2}\frac{(a^2+\mu)}{(1+\mu)} \ . \tag{5.272}$$

But

$$\frac{df}{dz} = -\frac{A}{(1+Az)^2} < 0 \tag{5.273}$$

and $f(z)$ is therefore a decreasing function of z. Hence, for $a^2 \leq z \leq 1$,

$$f(z) \leq f(a^2) = \frac{1}{\left(1+\frac{\varepsilon a^2}{2}\frac{(a^2+\mu)}{(1+\mu)}\right)} \ . \tag{5.274}$$

Equation (5.271) becomes

$$T \leq \frac{(a^2+\mu)^{\frac{1}{2}}}{\sqrt{D_1}}[f(a^2)]^{\frac{1}{2}}\, I_2(a^2) \tag{5.275}$$

where $I_2(a^2)$ is defined by (5.76). Thus

$$T \leq a \left(\frac{1 + \frac{\mu}{a^2}}{\left(1 + \frac{\varepsilon a^2}{2} \frac{(a^2+\mu)}{(1+\mu)}\right)} \right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} \ . \tag{5.276}$$

Case (iv) Lower bound on the period when $0 < a < 1$

The period is again given by (5.245). The inequalities (5.63) and (5.78) again apply. Using (5.78), equation (5.246) becomes

$$h(z) \leq \frac{\mu(z - a^2)}{a^2(z + \mu)}(A + Bz) \tag{5.277}$$

where

$$A = 1 + \frac{\varepsilon}{2} a^2 \frac{\mu}{(1 + \mu)}, \quad B = \frac{\varepsilon a^2}{2(1 + \mu)} \ . \tag{5.278}$$

Using (5.63) and (5.277), (5.245) becomes

$$T \geq \frac{a}{\sqrt{D_1}} \int_{a^2}^{1} \left[\frac{f(z)}{(1 - z)(z - a^2)} \right]^{\frac{1}{2}} dz \tag{5.279}$$

where

$$f(z) = \frac{1}{A + Bz} \ . \tag{5.280}$$

But

$$\frac{df}{dz} = -\frac{B}{(A + Bz)^2} < 0 \tag{5.281}$$

and $f(z)$ is a decreasing function of z. Thus for $a^2 \leq z \leq 1$,

$$f(z) \geq f(1) = \frac{1}{\left(1 + \frac{\varepsilon a^2}{2}\right)} \tag{5.282}$$

and (5.279) becomes

$$T \geq \frac{a}{\left(1 + \frac{\varepsilon a^2}{2}\right)^{\frac{1}{2}}} \frac{\pi}{\sqrt{D_1}} \ . \tag{5.283}$$

The results for the lower and upper bounds for the period of radial oscillations in a tangential transversely isotropic cylindrical tube are summarised in Table 5.5.1. They can be compared with the results for an isotropic cylindrical tube in Table 5.3.1 and for a radial transversely isotropic cylindrical tube in Table 5.4.1. For $0 < a < 1$ the lower and upper bounds for the period are again decreased by the anisotropy. When $a > 1$ the result for one of the cases, this time the lower bound, splits into two subcases depending on the value of ε. The lower bound (5.261) is the one to compare with (5.70) for an isotropic tube because the limit $\varepsilon \to 0$ can be taken in (5.262). This lower bound for the period is decreased by the anisotropy. The upper bound (5.254) for the period is decreased by the anisotropy.

Table 5.5.1

These approximate analytical results suggest that the effect of the anisotropy is to decrease the period of the oscillation.

The lower and upper bounds for the period of oscillation in a thin-walled transversely isotropic cylindrical tube are obtained by setting $\mu = 0$. When $a > 1$ the lower bound for $\mu = 0$ is obtained from (5.269) because we can take the limit $\mu \to 0$ in (5.266) and keep $\varepsilon > 0$.

5.5.4 Amplitude of the oscillation

The oscillations in a tangential transversely isotropic cylindrical tube which correspond to the upper and lower bounds for the period are listed in Table 5.5.1. They are derived as in Section 5.3.4 for an isotropic cylindrical tube by performing the same approximations on the oscillations as were made to derive the upper and lower bounds on the periods. The amplitude is the same as the amplitude of the exact solution and the period is the upper and lower bounds for the period. The oscillations have the same form as in Tables 5.3.1 and 5.4.1 for an isotropic and radial transversely isotropic cylindrical tube but the angular frequency is defferent for the three cases.

The limiting oscillations for a thin-walled tangential transversely isotropic cylindrical tube are obtained in the limit $\mu \to 0$ in the period and amplitude. For $P_0 > 0$ and $a > 1$ the oscillation corresponding to the lower bound in the period is the one which satisfies (5.266).

5.6 Conclusions

Radial oscillations in a thick-walled cylindrical tube subjected to Heaviside step loading were investigated. When the tube is thick-walled exact analytical results are difficult to obtain. However, useful conclusions could be obtained by introducing the effective potential and by deriving upper and lower bounds for the period.

Three effects due to radial and tangential transverse isotropy were found. These effects depend on the parameters $\varepsilon = \frac{D_2}{D_1}$ and μ for the transverse isotropy and thickness, respectively.

Firstly, the transverse isotropy extended the range of the net applied surface pressure, P_0, for which bounded oscillations exist. For an isotropic cylindrical tube, bounded oscillations exist only for

$$P_0 < P_c = \frac{D_1}{2\mu} \ln(1 + \mu). \tag{5.284}$$

For a radial transversely isotropic cylindrical tube with $\varepsilon \leq 2$ bounded oscillations exist only if P_0 satisfies (5.284). However, if $\varepsilon > 2$ then bounded oscillations exist for $P_0 < P_{max}$ where $P_{max} > P_c$ and depends on ε and μ. For a tangential transversely isotropic tube, there is no upper bound on P_0 and bounded oscillations exist for $-\infty \leq P_0 \leq \infty$.

Secondly, the transverse isotropy reduced the amplitude of the radial oscillations. The amplitude was reduced for both compression and extension of the tube and for both radial and tangential transversely isotropic tubes. The effect of the transverse isotropy was to make the tube stiffer.

Thirdly, the upper and lower bounds on the period suggest that the effect of the transverse isotropy is to reduce the period of the oscillations. For both radial and tangential transversely isotropic tubes the upper and lower bounds on the period are reduced by the anisotropy. The oscillations corresponding to the upper and lower bounds on the period have the form of a nonlinear superposition in all three cases although the frequencies are different.

Chapter 6

Nonlinear Radial Oscillations of Transversely Isotropic Incompressible Cylindrical Tubes: Free Oscillations

6.1 Introduction

In this chapter free radial oscillations in transversely isotropic cylindrical tubes will be considered. The net applied surface pressure is zero.

Knowles (1960) showed that for free radial oscillations of a thin-walled isotropic cylindrical tube and for all strain-energy functions,

$$ab = 1 \tag{6.1}$$

where $0 < a < 1$ is the minimum value and $b > 1$ is the maximum value of the inner radius of the cylindrical tube during an oscillation. The result (6.1) depends on the property

$$W_0\left(\frac{1}{u}\right) = W_0(u) \tag{6.2}$$

which holds for all strain-energy functions because of the form of the strain invariants.

For radial oscillations in radial and tangential transversely isotropic cylindrical tubes, (6.2) is not satisfied because of the extra strain invariants. We will investigate how the result (6.1) is changed for free radial oscillations in radial and tangential transversely isotropic cylindrical tubes of generalised Mooney-Rivlin material.

We will also investigate how the period of free radial oscillations is changed for radial and tangential transversely isotropic cylindrical tubes.

Free radial oscillations in a longitudinal transversely isotropic cylindrical tube are the same as in an isotropic tube.

An outline of the chapter is as follows. Only thin-walled cylindrical tubes are considered. In Section 6.2, results for free radial oscillations for general strain-energy functions are derived. In Section 6.3 the results for free radial oscillations in an isotropic cylindrical tube are reviewed. In Section 6.4, radial oscillations in a radial transversely isotropic cylindrical tube are considered. The generalisation of (6.1) for the minimum and maximum values of the inner radius in a free oscillation and the period of free oscillations are investigated for two special cases, firstly when the initial velocity $v_0 = 0$ and secondly when $v_0 \neq 0$ but the anisotropy is weak. A similar investigation is performed in Section 6.5 for free radial oscillations in a tangential transversely isotropic cylindrical tube. The conclusions are summarised in Section 6.6.

6.2 General results for free radial oscillations

When the net applied surface pressure is zero, (4.14) for radial oscillations in a thin-walled cylindrical tube reduces to

$$\frac{d^2x}{dt^2} + \frac{dW_0(x)}{dx} = 0. \tag{6.3}$$

Integrate with respect to x. Equation (6.3) becomes

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + W_0(x) = I, \tag{6.4}$$

where I is a constant. Consider the initial conditions

$$t = 0: \quad x = x_0, \quad \frac{dx}{dt} = v_0. \tag{6.5}$$

Then (6.4) becomes

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + W_0(x) = \frac{1}{2}v_0^2 + W_0(x_0). \tag{6.6}$$

Let a and b be the end points of the oscillation such that $0 < a < 1$ and $b > 1$. At the end points,

$$\frac{dx(a)}{dt} = 0, \quad \frac{dx(b)}{dt} = 0. \tag{6.7}$$

Thus a and b are the roots of

$$W_0(x) = \frac{1}{2}v_0^2 + W_0(x_0). \tag{6.8}$$

6.3 Free radial oscillations of an isotropic thin-walled cylindrical tube

For an isotropic cylindrical tube,

$$W_0 = W_0(I_1, I_2) \tag{6.9}$$

where from (3.128)

$$I_1 = I_2 = x^2 + \frac{1}{x^2} + 1. \tag{6.10}$$

Equation (6.8) becomes

$$W_0\Big(x^2+\frac{1}{x^2}+1,\ x^2+\frac{1}{x^2}+1\Big)=\frac{1}{2}v_0^2+W_0\Big(x_0^2+\frac{1}{x_0^2}+1,\ x_0^2+\frac{1}{x_0^2}+1\Big). \quad (6.11)$$

Now, if $x=a$ is a solution of (6.11) then

$$W_0\Big(a^2+\frac{1}{a^2}+1,\ a^2+\frac{1}{a^2}+1\Big)=\frac{1}{2}v_0^2+W_0\Big(x_0^2+\frac{1}{x_0^2}+1,\ x_0^2+\frac{1}{x_0^2}+1\Big). \quad (6.12)$$

It follows immediately from (6.12) that $x=\frac{1}{a}$ is also a solution of (6.11). Thus

$$b=\frac{1}{a} \quad (6.13)$$

and therefore

$$ab=1. \quad (6.14)$$

Equation (6.14) is satisfied for all strain-energy functions W_0.

When radial and tangential transversely isotropic cylindrical tubes are considered, the strain-energy function is prescribed as a generalised Mooney-Rivlin strain-energy function. We therefore derive (6.14) for a Mooney-Rivlin strain-energy function for comparison. From (3.178),

$$W_0(x)=\frac{D_1}{2}\Big(x^2+\frac{1}{x^2}-2\Big), \quad (6.15)$$

where D_1 is given by (3.171). Using (6.15), equation (6.8) becomes

$$x^4-\Big(\frac{v_0^2}{D_1}+x_0^2+\frac{1}{x_0^2}\Big)x^2+1=0, \quad (6.16)$$

which is a quadratic equation for x^2. But the product of the roots of (6.16) is

$$a^2b^2=1 \quad (6.17)$$

and since $a>0$ and $b>0$, equation (6.14) is again derived.

It follows from (6.16) that

$$a=\Big[\frac{1}{2}\Big(\frac{v_0^2}{D_1}+x_0^2+\frac{1}{x_0^2}\Big)-\frac{1}{2}\Big[\Big(\frac{v_0^2}{D_1}+x_0^2+\frac{1}{x_0^2}\Big)^2-4\Big]^{\frac{1}{2}}\Big]^{\frac{1}{2}} \quad (6.18)$$

$$b=\Big[\frac{1}{2}\Big(\frac{v_0^2}{D_1}+x_0^2+\frac{1}{x_0^2}\Big)+\frac{1}{2}\Big[\Big(\frac{v_0^2}{D_1}+x_0^2+\frac{1}{x_0^2}\Big)^2-4\Big]^{\frac{1}{2}}\Big]^{\frac{1}{2}}. \quad (6.19)$$

Both a and b are real because

$$\Big(\frac{v_0^2}{D_1}+x_0^2+\frac{1}{x_0^2}\Big)^2-4\geq\Big(x_0^2-\frac{1}{x_0^2}\Big)^2 \quad (6.20)$$

and it can be shown that $0<a<1$ and $b>1$. When $v_0=0$, $a=x_0$ and $b=\frac{1}{x_0}$ if $0<x_0<1$ and $a=\frac{1}{x_0}$ and $b=x_0$ if $x_0>1$.

Finally, consider the period, T, for free oscillations of a Mooney-Rivlin cylindrical tube. Using (6.15), equation (6.6) becomes

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1}{x^2}\Big[-x^4 + \Big(\frac{v_0^2}{D_1} + x_0^2 + \frac{1}{x_0^2}\Big)x^2 - 1\Big] \tag{6.21}$$

and since the end points of the oscillations are $0 < a < 1$ and $b > 1$,

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1}{x^2}\Big[(b^2 - x^2)(x^2 - a^2)\Big]. \tag{6.22}$$

Hence

$$T = \frac{2}{\sqrt{D_1}} \int_a^b \frac{x\,dx}{[(b^2 - x^2)(x^2 - a^2)]^{\frac{1}{2}}} \tag{6.23}$$

and letting $u = x^2$,

$$T = \frac{1}{\sqrt{D_1}} \int_{a^2}^{b^2} \frac{du}{[(b^2 - u)(u - a^2)]^{\frac{1}{2}}} = \frac{\pi}{\sqrt{D_1}}. \tag{6.24}$$

The way the period is changed for radial and tangential transversely isotropic cylindrical tubes will be investigated.

6.4 Free radial oscillations of a radial transversely isotropic thin-walled cylindrical tube

The result corresponding to (6.1) for the end points, a and b, will be considered and then the period of the oscillation will be investigated. There are two cases. In the first case $v_0 = 0$. In the second case $v_0 \neq 0$ but the anisotropy is weak.

The end points, a and b, are obtained from (6.8). For radial oscillations of a radial transverse isotropic cylindrical tube, $W_0(x)$ is given by (3.172):

$$W_0(x) = \frac{D_1}{2}\Big(x^2 + \frac{1}{x^2} - 2\Big) + \frac{D_2}{4}\Big(\frac{1}{x^4} - \frac{2}{x^2} + 1\Big). \tag{6.25}$$

Define

$$\varepsilon = \frac{D_2}{D_1} > 0. \tag{6.26}$$

Substituting (6.25) into (6.8) gives

$$x^6 - \Big(\frac{v_0^2}{D_1} + \frac{(1-\varepsilon)}{x_0^2} + x_0^2 + \frac{\varepsilon}{2x_0^4}\Big)x^4 + (1-\varepsilon)x^2 + \frac{\varepsilon}{2} = 0. \tag{6.27}$$

Equation (6.27) is a cubic equation for x^2.

The period T is obtained from (6.6). Substituting (6.25) into (6.6) gives

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1}{x^4}\Big[-x^6 + \Big(\frac{v_0^2}{D_1} + \frac{(1-\varepsilon)}{x_0^2} + x_0^2 + \frac{\varepsilon}{2x_0^4}\Big)x^4 - (1-\varepsilon)x^2 - \frac{\varepsilon}{2}\Big]. \tag{6.28}$$

6.4.1 Initial velocity $v_0 = 0$

When $v_0 = 0$, (6.27) reduces to

$$x^6 - \Big(\frac{(1-\varepsilon)}{x_0^2} + x_0^2 + \frac{\varepsilon}{2x_0^4}\Big)x^4 + (1-\varepsilon)x^2 + \frac{\varepsilon}{2} = 0. \tag{6.29}$$

Equation (6.29) can be factorised as

$$(x^2 - x_0^2)\Big(x^4 - \Big(\frac{(1-\varepsilon)}{x_0^2} + \frac{\varepsilon}{2x_0^4}\Big)x^2 - \frac{\varepsilon}{2x_0^2}\Big) = 0. \tag{6.30}$$

The roots of (6.30) are

$$x^2 = x_0^2, \tag{6.31}$$

$$x^2 = x_1^2 = \frac{(1-\varepsilon)}{2x_0^2} + \frac{\varepsilon}{4x_0^4} + \frac{1}{2}\Big[\Big(\frac{(1-\varepsilon)}{x_0^2} + \frac{\varepsilon}{2x_0^4}\Big)^2 + \frac{2\varepsilon}{x_0^2}\Big]^{\frac{1}{2}}, \tag{6.32}$$

$$x^2 = x_2^2 = \frac{(1-\varepsilon)}{2x_0^2} + \frac{\varepsilon}{4x_0^4} - \frac{1}{2}\Big[\Big(\frac{(1-\varepsilon)}{x_0^2} + \frac{\varepsilon}{2x_0^4}\Big)^2 + \frac{2\varepsilon}{x_0^2}\Big]^{\frac{1}{2}}. \tag{6.33}$$

It can be shown that if $x_0 > 1$ then $0 < x_1 < 1$ while if $0 < x_0 < 1$ then $x_1 > 1$. We note that

$$\lim_{\varepsilon\to 0} x_1 = \frac{1}{x_0}. \tag{6.34}$$

Also $x_2^2 < 0$. Only roots for which $x^2 > 0$ are considered. The end points of the oscillation are therefore x_0 and x_1.

We show that

$$x_0 x_1 > 1. \tag{6.35}$$

Using (6.32), condition (6.35) is satisfied provided

$$x_0^2\Big[\Big(\frac{1-\varepsilon}{x_0^2} + \frac{\varepsilon}{2x_0^4}\Big)^2 + \frac{2\varepsilon}{x_0^2}\Big]^{\frac{1}{2}} > 1 + \varepsilon - \frac{\varepsilon}{2x_0^2}\,. \tag{6.36}$$

If the right hand side of (6.36) is negative then the inequality is satisfied. Suppose that the right hand side of (6.36) is non-negative. Then both sides of (6.36) can be squared and the inequality will remain valid. Squaring both sides of (6.36) reduces the condition to

$$\varepsilon\Big(x_0 - \frac{1}{x_0}\Big)^2 > 0, \tag{6.37}$$

which is satisfied since $\varepsilon > 0$. Hence for radial oscillations in a radial transversely isotropic thin walled cylindrical tube, if the initial velocity $v_0 = 0$, then

$$ab > 1 \tag{6.38}$$

where a is the minimum end point and b is the maximum end point.

Consider now the period of the oscillations. When $v_0 = 0$, (6.28) becomes

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1}{x^4}\left[(x_0^2 - x^2)(x^2 - x_1^2)(x^2 - x_2^2)\right]^{\frac{1}{2}} \tag{6.39}$$

if $x_0 > 1$ and

$$\left(\frac{dx}{dt}\right)^2 = \frac{D_1}{x^4}\left[(x^2 - x_0^2)(x_1^2 - x^2)(x^2 - x_2^2)\right]^{\frac{1}{2}} \tag{6.40}$$

if $0 < x_0 < 1$. Suppose first that $x_0 > 1$. Then the range of oscillation is $x_1 \leq x \leq x_0$ and the period T is

$$T = \frac{2}{\sqrt{D_1}} \int_{x_1}^{x_0} \frac{x^2\, dx}{[(x_0^2 - x^2)(x^2 - x_1^2)(x^2 - x_2^2)]^{\frac{1}{2}}} \ . \tag{6.41}$$

Let $u = x^2$. Then

$$T = \frac{1}{\sqrt{D_1}} \int_{x_1^2}^{x_0^2} \left[\frac{u}{(x_0^2 - u)(u - x_1^2)(u - x_2^2)}\right]^{\frac{1}{2}} du \ . \tag{6.42}$$

We derive upper and lower bounds for T. Let

$$f(u) = \frac{u}{u - x_2^2} \ . \tag{6.43}$$

Then

$$\frac{df}{du} = -\frac{x_2^2}{(u - x_2^2)^2} > 0 \tag{6.44}$$

since $x_2^2 < 0$. Thus $f(u)$ is an increasing function of u. Hence since

$$\int_{x_1^2}^{x_0^2} \frac{du}{[(x_0^2 - u)(u - x_1^2)]^{\frac{1}{2}}} = \pi \tag{6.45}$$

it follows that

$$\left(\frac{x_1^2}{x_1^2 - x_2^2}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} < T < \left(\frac{x_0^2}{x_0^2 - x_2^2}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} \ . \tag{6.46}$$

If $0 < x_0 < 1$, the range of oscillation is $x_0 \leq x \leq x_1$ and it can be shown similarly that

$$\left(\frac{x_0^2}{x_0^2 - x_2^2}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} < T < \left(\frac{x_1^2}{x_1^2 - x_2^2}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} \ . \tag{6.47}$$

Since $x_2^2 < 0$ it follows from (6.46) and (6.47) that

$$T < \frac{\pi}{\sqrt{D_1}} \ . \tag{6.48}$$

When $v_0 = 0$ the period of free radial oscillations in a radial transversely isotropic cylindrical tube is less than the period in an isotropic tube.

6.4.2 Terms of order ε^2 neglected

Suppose now that $v_0 \neq 0$ but neglect terms of order ε^2. Let $u = x^2$. Then (6.27) becomes

$$G(u) = 0, \tag{6.49}$$

where

$$G(u) = u^3 - \Big(\frac{v_0^2}{D_1} + \frac{(1-\varepsilon)}{x_0^2} + x_0^2 + \frac{\varepsilon}{2x_0^4}\Big)u^2 + (1-\varepsilon)u + \frac{\varepsilon}{2}\ . \tag{6.50}$$

Consider the straight forward perturbation expansion

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2), \tag{6.51}$$

as $\varepsilon \to 0$. Substitute (6.51) into (6.49) and equate the coefficients of like powers of ε:

$$\varepsilon^0 : \quad u_0\Big[u_0^2 - \Big(\frac{v_0^2}{D_1} + \frac{1}{x_0^2} + x_0^2\Big)u_0 + 1\Big] = 0, \tag{6.52}$$

$$\begin{aligned} \varepsilon : \quad & u_1\Big[3u_0^2 - 2u_0\Big(\frac{v_0^2}{D_1} + \frac{1}{x_0^2} + x_0^2\Big) + 1\Big] \\ & + \Big(\frac{1}{x_0^2} - \frac{1}{2x_0^4}\Big)u_0^2 - u_0 + \frac{1}{2} = 0. \end{aligned} \tag{6.53}$$

From (6.52), either

$$u_0 = 0 \tag{6.54}$$

or

$$u_0^2 - \Big(\frac{v_0^2}{D_1} + \frac{1}{x_0^2} + x_0^2\Big)u_0 + 1 = 0. \tag{6.55}$$

If $u_0 = 0$ then

$$u_1 = -\frac{1}{2} \tag{6.56}$$

and therefore

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2) = -\frac{\varepsilon}{2} + O(\varepsilon^2). \tag{6.57}$$

This root is negative. It is therefore not one of the end points of the oscillation.

The other two roots to zero order of the cubic (6.49) are given by (6.55). Thus

$$u_0 = a_0^2, \quad u_0 = b_0^2 \tag{6.58}$$

where a_0 and b_0 are given by (6.18) and (6.19) . They satisfy $0 < a_0 < 1$ and

$$b_0 = \frac{1}{a_0}\ . \tag{6.59}$$

Using (6.55) in (6.53) gives

$$u_1 = \frac{1}{(1-u_0^2)}\Big[\Big(\frac{1}{x_0^2} - \frac{1}{2x_0^4}\Big)u_0^2 - u_0 + \frac{1}{2}\Big]\ . \tag{6.60}$$

Thus, correct to order ε, two roots of (6.49) are

$$u = a^2 = a_0^2 + \frac{\varepsilon}{(1-a_0^4)}\Big[\Big(\frac{1}{x_0^2} - \frac{1}{2x_0^4}\Big)a_0^4 - a_0^2 + \frac{1}{2}\Big], \tag{6.61}$$

$$u = b^2 = \frac{1}{a_0^2} - \frac{\varepsilon}{(1-a_0^4)}\Big[\Big(\frac{1}{x_0^2} - \frac{1}{2x_0^4}\Big) - a_0^2 + \frac{a_0^4}{2}\Big]. \tag{6.62}$$

The range of the oscillation is $a^2 \leq u \leq b^2$ and $G(u)$ given by (6.50) can be factorised, to first order in ε, as

$$G(u) = -\Big(u + \frac{\varepsilon}{2}\Big)(b^2 - u)(u - a^2)\ . \tag{6.63}$$

Consider now ab. From (6.61) and (6.62) ,

$$a = a_0\Big[1 + \frac{\varepsilon}{2a_0^2(1-a_0^4)}\Big(\Big(\frac{1}{x_0^2} - \frac{1}{2x_0^4}\Big)a_0^4 - a_0^2 + \frac{1}{2}\Big) + O(\varepsilon^2)\Big], \tag{6.64}$$

$$b = \frac{1}{a_0}\Big[1 - \frac{\varepsilon a_0^2}{2(1-a_0^4)}\Big(\Big(\frac{1}{x_0^2} - \frac{1}{2x_0^4}\Big) - a_0^2 + \frac{a_0^4}{2}\Big) + O(\varepsilon^2)\Big]. \tag{6.65}$$

Hence

$$ab = 1 + \frac{\varepsilon(1-a_0^2)^2}{4a_0^2} + O(\varepsilon^2), \tag{6.66}$$

as $\varepsilon \to 0$. Equation (6.66) can be expressed in terms of v_0 and x_0 using

$$\frac{(1-a_0^2)^2}{a_0^2} = \frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2. \tag{6.67}$$

Equation (6.67) can be derived from (6.18) and (6.19) using $b_0 = \frac{1}{a_0}$. Thus (6.66) can be written alternatively as

$$ab = 1 + \frac{\varepsilon}{4}\Big(\frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2\Big) + O(\varepsilon^2), \tag{6.68}$$

as $\varepsilon \to 0$. Thus, correct to order ε,

$$ab > 1. \tag{6.69}$$

We have shown that for free radial oscillations in a radial transversely isotropic thin-walled cylindrical tube, if either $v_0 = 0$ or when $v_0 \neq 0$ but terms of order ε^2 are neglected, then (6.69) holds where a is the minimum end point and b is the maximum end point of the oscillation.

Consider now the period T of the oscillation. From (6.28)

$$T = \frac{2}{\sqrt{D_1}}\int_a^b \frac{x^2\,dx}{\Big[-x^6 + \Big(\frac{v_0^2}{D_1} + \frac{(1-\varepsilon)}{x_0^2} + x_0^2 + \frac{\varepsilon}{2x_0^4}\Big)x^4 - (1-\varepsilon)x^2 - \frac{\varepsilon}{2}\Big]^{\frac{1}{2}}}\ . \tag{6.70}$$

Let $u = x^2$. Then

$$T = \frac{1}{\sqrt{D_1}}\int_{a^2}^{b^2}\Big[\frac{u}{-G(u)}\Big]\ du \tag{6.71}$$

where $G(u)$ is defined by (6.50). Using (6.63), (6.71) becomes

$$T = \frac{1}{\sqrt{D_1}} \int_{a^2}^{b^2} \left[\frac{u}{\left(u+\frac{\varepsilon}{2}\right)(b^2-u)(u-a^2)} \right]^{\frac{1}{2}} du. \tag{6.72}$$

Let

$$f(u) = \frac{u}{u+\frac{\varepsilon}{2}} \ . \tag{6.73}$$

Then

$$\frac{df}{du} = \frac{\varepsilon}{2\left(u+\frac{\varepsilon}{2}\right)^2} > 0 \tag{6.74}$$

and therefore $f(u)$ is an increasing function of u. Thus using again (6.45),

$$\left(\frac{a^2}{a^2+\frac{\varepsilon}{2}}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} < T < \left(\frac{b^2}{b^2+\frac{\varepsilon}{2}}\right)^{\frac{1}{2}} \frac{\pi}{\sqrt{D_1}} \ . \tag{6.75}$$

But

$$\left(\frac{a^2}{a^2+\frac{\varepsilon}{2}}\right)^{\frac{1}{2}} = \left(1+\frac{\varepsilon}{2a^2}\right)^{-\frac{1}{2}} = 1 - \frac{\varepsilon}{4a_0^2} + O(\varepsilon^2) \tag{6.76}$$

and similarly, using also $b_0 = \frac{1}{a_0}$,

$$\left(\frac{b^2}{b^2+\frac{\varepsilon}{2}}\right)^{\frac{1}{2}} = 1 - \frac{\varepsilon a_0^2}{4} + O(\varepsilon^2) \ . \tag{6.77}$$

Thus (6.75) becomes

$$\left(1 - \frac{\varepsilon}{4a_0^2} + O(\varepsilon^2)\right)\frac{\pi}{\sqrt{D_1}} < T < \left(1 - \frac{\varepsilon a_0^2}{4} + O(\varepsilon^2)\right)\frac{\pi}{\sqrt{D_1}} \ . \tag{6.78}$$

The period of the oscillation is therefore reduced by the anisotropy.

We have shown that for free radial oscillations in a radial transversely isotropic thin-walled cylindrical tube, if either $v_0 = 0$ or $v_0 \neq 0$ but terms of order ε^2 can be neglected then the period of oscillation is less than in an isotropic cylindrical tube.

6.5 Free radial oscillations of a tangential transversely isotropic thin-walled cylindrical tube

A similar investigation is now done for a tangential transversely isotropic thin-walled cylindrical tube.

The end points a and b are obtained from (6.8). For radial oscillations of a tangential transversely isotropic cylindrical tube, $W_0(x)$ is given by (3.175):

$$W_0(x) = \frac{D_1}{2}\left(x^2 + \frac{1}{x^2} - 2\right) + \frac{D_2}{4}\left(x^4 - 2x^2 + 1\right). \tag{6.79}$$

Define ε again by (6.26). Substituting (6.79) into (6.8) gives

$$\varepsilon x^6 + 2(1-\varepsilon)x^4 - 2\Big(\frac{v_0^2}{D_1} + (1-\varepsilon)x_0^2 + \frac{1}{x_0^2} + \frac{\varepsilon}{2}x_0^4\Big)x^2 + 2 = 0. \tag{6.80}$$

Equation (6.80) is a cubic equation for x^2 in which the small parameter ε multiplies the highest order term .

The period T is obtained from (6.6). Substituting (6.79) into (6.6) gives

$$\Big(\frac{dx}{dt}\Big)^2 = \frac{D_1}{2x^2}\Big[-\varepsilon x^6 - 2(1-\varepsilon)x^4 + 2\Big(\frac{v_0^2}{D_1} + (1-\varepsilon)x_0^2 + \frac{1}{x_0^2} + \frac{\varepsilon}{2}x_0^4\Big)x^2 - 2\Big]. \tag{6.81}$$

6.5.1 Initial velocity $v_0 = 0$

When $v_0 = 0$, (6.80) reduces to

$$\varepsilon x^6 + 2(1-\varepsilon)x^4 - 2\Big((1-\varepsilon)x_0^2 + \frac{1}{x_0^2} + \frac{\varepsilon}{2}x_0^4\Big)x^2 + 2 = 0. \tag{6.82}$$

Equation (6.82) can be factorised as

$$(x^2 - x_0^2)\Big[\varepsilon x^4 + 2\Big(1 - \varepsilon + \frac{\varepsilon}{2}x_0^2\Big)x^2 - \frac{2}{x_0^2}\Big] = 0. \tag{6.83}$$

The roots of (6.83) are

$$x^2 = x_0^2, \tag{6.84}$$

$$x^2 = x_1^2 = \frac{1}{\varepsilon}\Big[-\Big(1-\varepsilon+\frac{\varepsilon}{2}x_0^2\Big) + \Big(\Big(1-\varepsilon+\frac{\varepsilon}{2}x_0^2\Big)^2 + \frac{2\varepsilon}{x_0^2}\Big)^{\frac{1}{2}}\Big], \tag{6.85}$$

$$x^2 = x_1^2 = \frac{1}{\varepsilon}\Big[-\Big(1-\varepsilon+\frac{\varepsilon}{2}x_0^2\Big) - \Big(\Big(1-\varepsilon+\frac{\varepsilon}{2}x_0^2\Big)^2 + \frac{2\varepsilon}{x_0^2}\Big)^{\frac{1}{2}}\Big]. \tag{6.86}$$

It can be shown that if $x_0 > 1$ then $0 < x_1 < 1$ while if $0 < x_0 < 1$ then $x_1 > 1$. Also, using L'Hopital's rule,

$$\lim_{\varepsilon \to 0} x_1^2 = \frac{1}{x_0^2} \ . \tag{6.87}$$

From (6.86), $x_2^2 < 0$. Only roots for which $x^2 > 0$ are considered. The end points of the oscillation are x_0 and x_1.

We now show that

$$x_0 x_1 < 1. \tag{6.88}$$

From (6.85), the inequality (6.88) is satisfied provided

$$\Big[\Big(1-\varepsilon+\frac{\varepsilon}{2}x_0^2\Big)^2 + \frac{2\varepsilon}{x_0^2}\Big]^{\frac{1}{2}} < \frac{\varepsilon}{2}\Big(\frac{1}{x_0} - x_0\Big)^2 + 1 + \frac{\varepsilon}{2x_0^2} \ . \tag{6.89}$$

Since the right hand side of (6.89) is positive, both sides of (6.89) can be squared without destroying the inequality. It can be shown that (6.89) reduces to the condition

$$\varepsilon^2\Big(\frac{1}{x_0^2}-1\Big)^2 > 0 \tag{6.90}$$

which is satisfied. Hence, for radial oscillations in a tangential transversely isotropic thin-walled cylindrical tube, if the initial velocity $v_0 = 0$, then

$$ab < 1 \tag{6.91}$$

where a is the minimum end point and b is the maximum end point.

Consider now the period of the oscillation. When $v_0 = 0$, (6.81) becomes, if $x_0 > 1$,

$$\Big(\frac{dx}{dt}\Big)^2 = \frac{\varepsilon}{2}\frac{D_1}{x^2}\Big[(x_0^2-x^2)(x^2-x_1^2)(x^2-x_2^2)\Big]^{\frac{1}{2}} \tag{6.92}$$

and if $0 < x_0 < 1$,

$$\Big(\frac{dx}{dt}\Big)^2 = \frac{\varepsilon}{2}\frac{D_1}{x^2}\Big[(x^2-x_0^2)(x_1^2-x^2)(x^2-x_2^2)\Big]^{\frac{1}{2}}\ . \tag{6.93}$$

When $x_0 > 1$, the range of oscillation is $x_1 \le x \le x_0$ and the period T is

$$T = \frac{2}{\sqrt{D_1}}\Big(\frac{2}{\varepsilon}\Big)^{\frac{1}{2}}\int_{x_1}^{x_0}\frac{x\,dx}{[(x_0^2-x^2)(x^2-x_1^2)(x^2-x_2^2)]^{\frac{1}{2}}}\ . \tag{6.94}$$

On letting $u = x^2$, (6.94) becomes

$$T = \frac{1}{\sqrt{D_1}}\Big(\frac{2}{\varepsilon}\Big)^{\frac{1}{2}}\int_{x_1^2}^{x_0^2}\frac{du}{[(x_0^2-u)(u-x_1^2)(u-x_2^2)]^{\frac{1}{2}}}\ . \tag{6.95}$$

When $0 < x_0 < 1$ the range of oscillation is $x_0 \le x \le x_1$ and

$$T = \frac{1}{\sqrt{D_1}}\Big(\frac{2}{\varepsilon}\Big)^{\frac{1}{2}}\int_{x_0^2}^{x_1^2}\frac{du}{[(u-x_0^2)(x_1^2-u)(u-x_2^2)]^{\frac{1}{2}}}\ . \tag{6.96}$$

Unlike (6.42) for the period of free oscillations of a radial transversely isotropic tube, useful upper and lower bounds for (6.95) and (6.96) are not readily obtained. An approximation to the period will be derived in the next subsection for small ε.

A summary of the results for free radial oscillations when $v_0 = 0$ is given in Table 6.5.1.

6.5.2 Terms of order ε^2 neglected

Suppose that $v_0 \neq 0$ but neglect terms of order ε^2. Let $u = x^2$. Then (6.80) becomes

$$G(u) = 0 \tag{6.97}$$

where

$$G(u) = \varepsilon u^3 + 2(1-\varepsilon)u^2 - 2\Big(\frac{v_0^2}{D_1} + (1-\varepsilon)x_0^2 + \frac{1}{x_0^2} + \frac{\varepsilon}{2}x_0^4\Big)u + 2. \tag{6.98}$$

Table 6.5.1

First consider a straightforward perturbation expansion

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2), \tag{6.99}$$

as $\varepsilon \to 0$. Substitute (6.99) into (6.97) and equate the coefficients of like powers of ε.

$$\varepsilon^0 : \quad u_0^2 - \Big(\frac{v_0^2}{D_1} + x_0^2 + \frac{1}{x_0^2}\Big)u_0 + 1 = 0, \tag{6.100}$$

$$\varepsilon : \quad u_1\Big[4u_0 - 2\Big(\frac{v_0^2}{D_1} + x_0^2 + \frac{1}{x_0^2}\Big)\Big] + u_0^3 - 2u_0^2 + (2x_0^2 - x_0^4)u_0 = 0. \tag{6.101}$$

From (6.100),

$$u_0 = a_0^2, \quad u_0 = b_0^2, \tag{6.102}$$

where a_0 and b_0 are given by (6.18) and (6.19) . They again satisfy

$$b_0 = \frac{1}{a_0} \ . \tag{6.103}$$

Using (6.100), equation (6.101) becomes

$$u_1 = \frac{u_0^2}{2(1-u_0^2)}[u_0^2 - 2u_0 + 2x_0^2 - x_0^4]. \tag{6.104}$$

Thus, correct to first order in ε, two roots of the cubic equation (6.97) are

$$u = a^2 = a_0^2 + \frac{\varepsilon a_0^4}{2(1-a_0^4)}[a_0^4 - 2a_0^2 + 2x_0^2 - x_0^4], \tag{6.105}$$

$$u = b^2 = \frac{1}{a_0^2} - \frac{\varepsilon}{2a_0^4(1-a_0^4)}[1 - 2a_0^2 + (2x_0^2 - x_0^4)a_0^4]. \tag{6.106}$$

Only two roots, (6.105) and (6.106), are obtained to zero order. Unlike the cubic equation (6.49), the root $u_0 = 0$ does not occur at zero order. This is because the small parameter ε multiplies the highest degree term in (6.97) which makes the derivation of the roots a singular perturbation problem (Nayfeh, 1981).

To find the third root of (6.97) consider the scaling transformation (Nayfeh, 1981)

$$u = \frac{y}{\varepsilon^\nu}, \quad \nu > 0. \tag{6.107}$$

Equation (6.97) becomes

$$\varepsilon^{1-3\nu}y^3 + 2\varepsilon^{-2\nu}y^2 - 2\varepsilon^{1-2\nu}y^2 - 2\Big(\frac{v_0^2}{D_1} + x_0^2 + \frac{1}{x_0^2}\Big)\varepsilon^{-\nu}y$$

$$+(2x_0^2 - x_0^4)\varepsilon^{1-\nu}y + 2 = 0. \tag{6.108}$$

To derive the third root the first term in (6.108) must be retained. Since $\nu > 0$, the dominant terms are the first two terms. The first two terms in (6.108) will balance provided

$$1 - 3\nu = -2\nu \ \text{ or } \ \nu = 1. \tag{6.109}$$

Equation (6.108) becomes

$$y^3 + 2(1-\varepsilon)y^2 - 2\Big(\frac{v_0^2}{D_1} + x_0^2 + \frac{1}{x_0^2}\Big)\varepsilon y + (2x_0^2 - x_0^4)\varepsilon^2 y + 2\varepsilon^2 = 0. \tag{6.110}$$

Expand

$$y = y_0 + \varepsilon y_1 + O(\varepsilon^2), \tag{6.111}$$

as $\varepsilon \to 0$. Substitute (6.111) into (6.110) and equate the coefficients of like powers of ε:

$$\varepsilon^0: \quad y_0^2(y_0+2) = 0, \tag{6.112}$$

$$\varepsilon: \quad y_0(3y_0+4)y_1 = 2y_0\Big[y_0 + \frac{v_0^2}{D_1} + x_0^2 + \frac{1}{x_0^2}\Big]. \tag{6.113}$$

From (6.112),

$$y_0 = 0, \quad y_0 = 0, \quad y_0 = -2. \tag{6.114}$$

The two roots, $y_0 = 0$, correspond to the two roots, (6.105) and (6.106), already found because these expansions have no terms of order ε^{-1}. Consider $y_0 = -2$. Equation (6.113) gives

$$y_1 = -\Big(\frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2\Big). \tag{6.115}$$

Thus

$$y = -2 - \Big(\frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2\Big)\varepsilon + O(\varepsilon^2) \tag{6.116}$$

and hence by (6.107) with $\nu = 1$,

$$u = u_3 = -\frac{2}{\varepsilon} - \Big(\frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2\Big) + O(\varepsilon), \tag{6.117}$$

as $\varepsilon \to 0$. Thus $u_3 < 0$ and the range of oscillation is $a^2 \le u \le b^2$ where a and b are given by (6.105) and (6.106). The cubic polynomial (6.98) can be factorised, correct to order ε, as

$$G(u) = -\varepsilon(u - u_3)(b^2 - u)(u - a^2). \tag{6.118}$$

Consider now ab. From (6.105) and (6.106),

$$a = a_0\Big[1 + \frac{\varepsilon a_0^2}{4(1-a_0^4)}\Big(a_0^4 - 2a_0^2 + 2x_0^2 - x_0^4\Big) + O(\varepsilon^2)\Big], \tag{6.119}$$

$$b = \frac{1}{a_0}\Big[1 - \frac{\varepsilon}{4a_0^2(1-a_0^4)}\Big(1 - 2a_0^2 + (2x_0^2 - x_0^4)a_0^4\Big) + O(\varepsilon^2)\Big]. \tag{6.120}$$

Thus

$$ab = 1 - \frac{\varepsilon(1-a_0^2)^2}{4a_0^2} + O(\varepsilon^2), \tag{6.121}$$

as $\varepsilon \to 0$. Using (6.67), (6.121) can be expressed in terms of v_0 and x_0 as

$$ab = 1 - \frac{\varepsilon}{4}\Big(\frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2\Big) + O(\varepsilon^2), \tag{6.122}$$

as $\varepsilon \to 0$. Hence, correct to order ε,

$$ab < 1. \tag{6.123}$$

We have shown that for free oscillations in a tangential transversely isotropic thin-walled cylindrical tube, if either $v_0 = 0$ or $v_0 \neq 0$ but terms of order ε^2 are neglected, then (6.123) is satisfied where a is the minimum end point and b is the maximum end point of the oscillation.

Consider now the period T of the free oscillations. From (6.81),

$$T = 2\Big(\frac{2}{D_1}\Big)^{\frac{1}{2}} \int_a^b \frac{x\,dx}{\Big[-\varepsilon x^6 - 2(1-\varepsilon)x^4 + 2\Big(\frac{v_0^2}{D_1} + (1-\varepsilon)x_0^2 + \frac{1}{x_0^2} + \frac{\varepsilon}{2}x_0^4\Big)x^2 - 2\Big]^{\frac{1}{2}}} \,. \tag{6.124}$$

Let $u = x^2$. Then (6.124) becomes

$$T = \Big(\frac{2}{D_1}\Big)^{\frac{1}{2}} \int_{a^2}^{b^2} \frac{du}{\Big[-G(u)\Big]^{\frac{1}{2}}} \tag{6.125}$$

where $G(u)$ is defined by (6.98). Using (6.118) for $G(u)$, (6.125) becomes

$$T = \Big(\frac{2}{D_1}\Big)^{\frac{1}{2}} \int_{a^2}^{b^2} \frac{du}{[\varepsilon(u-u_3)(b^2-u)(u-a^2)]^{\frac{1}{2}}} \,. \tag{6.126}$$

Using (6.117) for u_3, (6.126) becomes

$$T = \frac{1}{\sqrt{D_1}} \int_{a^2}^{b^2} \frac{1}{[(b^2-u)(u-a^2)]^{\frac{1}{2}}} \Big[1 + \frac{\varepsilon}{2}\Big(u + \frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2\Big)\Big]^{-\frac{1}{2}} du \tag{6.127}$$

and using the binomial expansion, (6.127) becomes

$$T = \Big[1 - \frac{\varepsilon}{4}\Big(\frac{v_0^2}{D_1} + \Big(x_0 - \frac{1}{x_0}\Big)^2\Big)\Big] \frac{I_1}{\sqrt{D_1}} - \frac{\varepsilon}{4}\frac{I_2}{\sqrt{D_1}} + O(\varepsilon^2), \tag{6.128}$$

where

$$I_1 = \int_{a^2}^{b^2} \frac{du}{[(b^2-u)(u-a^2)]^{\frac{1}{2}}} = \pi, \tag{6.129}$$

$$I_2 = \int_{a^2}^{b^2} \frac{u\,du}{[(b^2-u)(u-a^2)]^{\frac{1}{2}}} \,. \tag{6.130}$$

Now

$$\int_{a^2}^{b^2} \frac{[u - \frac{1}{2}(a^2+b^2)]\;du}{[(b^2-u)(u-a^2)]^{\frac{1}{2}}} = 0 \tag{6.131}$$

since the integrand is an odd function with respect to the mid-point $\frac{1}{2}(a^2+b^2)$ of the range of integration. Thus

$$\int_{a^2}^{b^2} \frac{u\,du}{[(b^2-u)(u-a^2)]^{\frac{1}{2}}} = \frac{1}{2}(a^2+b^2)I_1 = \frac{1}{2}(a^2+b^2)\pi. \tag{6.132}$$

Table 6.5.2

Now in (6.128), (a^2+b^2) is required only to zero order in ε. Thus using (6.58), (6.18) and (6.19),

$$a^2+b^2 = a_0^2+b_0^2+O(\varepsilon) = \frac{v_0^2}{D_1}+\Big(x_0-\frac{1}{x_0}\Big)^2+2+O(\varepsilon) \tag{6.133}$$

and therefore

$$I_2 = \frac{1}{2}\Big(\frac{v_0^2}{D_1}+\Big(x_0-\frac{1}{x_0}\Big)^2+2\Big)\pi+O(\varepsilon). \tag{6.134}$$

Substituting (6.129) and (6.134) into (6.128) gives

$$T = \frac{\pi}{\sqrt{D_1}}\Big[1-\frac{3}{8}\Big(\frac{v_0^2}{D_1}+\Big(x_0-\frac{1}{x_0}\Big)^2+\frac{2}{3}\Big)\varepsilon+O(\varepsilon^2)\Big]. \tag{6.135}$$

Using (6.67), T can be written alternatively in terms of a_0 as

$$T = \frac{\pi}{\sqrt{D_1}}\Big[1-\frac{3}{8}\Big(\frac{(1-a_0^2)^2}{a_0^2}+\frac{2}{3}\Big)\varepsilon+O(\varepsilon^2)\Big]. \tag{6.136}$$

Thus if terms of order ε^2 can be neglected,

$$T < \frac{\pi}{\sqrt{D_1}}\ . \tag{6.137}$$

The period of free oscillations is therefore reduced by the anisotropy.

We have shown that for free radial oscillations in a tangential transversely isotropic thin-walled cylindrical tube, correct to order ε and for any initial velocity v_0, the period of oscillation is less than in an isotropic cylindrical tube.

A summary of the results for free radial oscillations when $v_0 \neq 0$ but terms of order ε^2 can be neglected is given in Table 6.5.2.

6.6 Conclusions

For free radial oscillations in a cylindrical tube with generalised Mooney-Rivlin strain-energy function it was found that if either the initial velocity $v_0 = 0$ or when $v_0 \neq 0$, if terms of order ε^2 can be neglected, then the end points, a and b, of the oscillation satisfy :

isotropic or longitudinal transversely isotropic tube	$ab = 1$
radial transversely isotropic tube	$ab > 1$
tangential transversely isotropic tube	$ab < 1$

This is a distinguishing property of free radial oscillations in isotropic and radial and tangential transversely isotropic cylindrical tubes.

It was found that when $v_0 = 0$ or when $v_0 \neq 0$ but terms of order ε^2 can be neglected, the period of free radial oscillations in radial and tangential transversely isotropic cylindrical tubes is less than in a (longitudinal transversely) isotropic tube. The transverse isotropy reduces the period of oscillation. For the case $v_0 = 0$

in a tangential transversely isotropic cylindrical tube we were not able to derive useful bounds on the period.

The period largely depends on the third root of the cubic equation. The first two roots give the end points a and b. For a radial transversely isotropic tube the solution of the cubic is a regular perturbation problem while for a tangential transversely isotropic tube it is a singular perturbation problem.

It is an open question if the results hold when $v_0 \neq 0$ for all values of ε and not only if terms of order ε^2 are neglected.

Chapter 7

Nonlinear Radial Oscillations of a Transversely Isotropic Incompressible Spherical Shell: General Results

7.1 Introduction

In this chapter radial oscillations of a transversely isotropic incompressible hyperelastic spherical shell are investigated. There are both similarities and differences between radial oscillations in a spherical shell and a cylindrical tube.

Radial, tangential and longitudinal transversely isotropic spherical shells are considered. For a radial transversely isotropic spherical shell the anisotropic director is a unit vector in the radial direction in the undeformed spherical shell. We will define tangential and longitudinal transversely isotropic spherical shells as shells for which the anisotropic director is a unit vector tangential to a line of latitude and a line of longitude, respectively, in the undeformed spherical shell.

Radial oscillations of an incompressible isotropic spherical shell have been investigated by several authors. Guo and Solecki (1963) considered a thick-walled spherical shell and investigated both free oscillations and forced oscillations due to Heaviside step loading. Wang (1965) considered radial oscillations in a thin-walled spherical shell and investigated Heaviside step loading and also oscillations of a sealed shell. Calderer (1983) considered thick-walled and thin-walled spherical shells and did a phase plane analysis for time independent applied surface pressure. Roussos, Mason and Hill (2002) considered a thin-walled spherical shell and showed that for a Mooney-Rivlin material the differential equation describing radial oscillations had no Lie point symmetries if the net applied surface pressure is time dependent. They also derived approximate solutions for a neo-Hookean material. Roussos and Mason (2005) later investigated the existence of Lie point symmetries for arbitrary strain-energy functions and showed that they exist when the net applied surface pressure is time dependent only if a very restrictive condition on the strain-energy function is satisfied.

An outline of the chapter is as follows. In Section 7.2 the problem of radial oscillations in a transversely isotropic spherical shell is formulated mathematically and the spherical polar coordinate systems in the undeformed and deformed shell are defined. In Section 7.3 the base vectors and metric tensors in the undeformed and deformed shells are derived. The anisotropic directors for radial, tangential and longitudinal transversely isotropic spherical shells are derived in Section 7.4. In Section 7.5 the strain invariants are derived. Unlike a cylindrical tube the strain invariants, I_1 and I_2, are unequal. The components of the Cauchy stress tensor are derived in Section 7.6 and the the boundary conditions are derived in Section 7.7. Cauchy's first law of motion is derived in Section 7.8. In Section 7.9 an ordinary differential equation for the dimensionless inner radius of a radial transversely isotropic spherical shell is obtained. It is found that the components of the Cauchy stress tensor are not bounded everywhere for tangential and longitudinal transversely isotropic spherical shells and these cases are not considered further. The remainder of the chapter is concerned with radial oscillations of a radial transversely isotropic spherical shell. In Section 7.10 the limit of a thin-walled spherical shell is considered. In Section 7.11 the generalised Mooney-Rivlin strain-energy function is applied to a spherical shell. In Section 7.12 the Lie point symmetry generators of the differential equation describing radial oscillations of a thin-walled radial transversely isotropic spherical shell are investigated. Finally, conclusions are summarised in Section 7.13.

7.2 Mathematical formulation

Consider radial oscillations of a transversely isotropic incompressible hyperelastic spherical shell.

The coordinate systems in the unstrained spherical shell B_0 and in the strained spherical shell B are shown in Figure 7.2.1.

Rectangular cartesian base vectors $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ are chosen with origin at the centre of the sphere. The cartesian coordinates of a material point P_0 in the unstrained spherical shell are (x^1, x^2, x^3) and the position vector of P_0 is

$$\boldsymbol{r} = x^n \boldsymbol{i}_n. \tag{7.1}$$

Spherical polar coordinates (ρ, θ, ϕ) in the unstrained spherical shell are defined by

$$x^1 = \rho \sin\theta \cos\phi, \quad x^2 = \rho \sin\theta \sin\phi, \quad x^3 = \rho \cos\theta. \tag{7.2}$$

The inner radius of the unstrained spherical shell is ρ_1 and the outer radius is ρ_2. Hence

$$\rho_1 \le \rho \le \rho_2. \tag{7.3}$$

The material particle at P_0 in the unstrained spherical shell B_0 is displaced to the point P in the strained spherical shell B with position vector

$$\boldsymbol{R} = y^n \boldsymbol{i}_n. \tag{7.4}$$

Spherical polar coordinates (r, θ, ϕ) in the strained spherical shell B are defined by

$$y^1 = r(\rho, t) \sin\theta \cos\phi, \quad y^2 = r(\rho, t) \sin\theta \sin\phi, \quad y^3 = r(\rho, t) \cos\theta. \tag{7.5}$$

Figure 7.2.1

At time t, the inner radius of the strained spherical shell is $r_1(t)$ and the outer radius of the strained spherical shell is $r_2(t)$. Hence

$$r_1(t) \leq r \leq r_2(t). \tag{7.6}$$

A pressure $\mathcal{P}_1(t)$ is applied to the inner surface $r = r_1(t)$ of the spherical shell and a pressure $\mathcal{P}_2(t)$ is applied to the outer surface, $r = r_2(t)$.

As with radial oscillations in a cylindrical tube coordinates of points in the strained body B will be considered as reference points since the form of B is known. The curvilinear coordinate system $(\theta^1, \theta^2, \theta^3)$ is therefore taken to be the spherical polar coordinate system in the strained spherical shell B. Hence

$$\theta^1 = r, \quad \theta^2 = \theta, \quad \theta^3 = \phi. \tag{7.7}$$

In summary, the coordinate system in the unstrained spherical shell B_0 is

$$\begin{aligned}
x^1 &= \rho(r,t)\sin\theta\cos\phi, \quad x^1 = \rho(\theta^1,t)\sin\theta^2\cos\theta^3, \\
x^2 &= \rho(r,t)\sin\theta\sin\phi, \quad x^2 = \rho(\theta^1,t)\sin\theta^2\sin\theta^3, \\
x^3 &= \rho(r,t)\cos\theta, \qquad\quad x^3 = \rho(\theta^1,t)\cos\theta^2
\end{aligned} \tag{7.8}$$

and in the strained spherical shell B the coordinate system is

$$\begin{aligned}
y^1 &= r\sin\theta\cos\phi, \quad y^1 = \theta^1\sin\theta^2\cos\theta^3, \\
y^2 &= r\sin\theta\sin\phi, \quad y^2 = \theta^1\sin\theta^2\sin\theta^3, \\
y^3 &= r\cos\theta, \qquad\quad y^3 = \theta^1\cos\theta^2.
\end{aligned} \tag{7.9}$$

This completes the mathematical formulation of the problem.

7.3 Base vectors, metric tensors and incompressibility condition

Consider first the deformed spherical shell B.

The covariant base vectors, $\boldsymbol{G}_i$, are defined by

$$\boldsymbol{G}_i = \frac{\partial y^n}{\partial \theta^i}\, \boldsymbol{i}_n. \tag{7.10}$$

From (7.9) it follows that

$$\begin{aligned}
\boldsymbol{G}_1 &= \sin\theta\cos\phi\ \boldsymbol{i}_1 + \sin\theta\sin\phi\ \boldsymbol{i}_2 + \cos\theta\ \boldsymbol{i}_3, \\
\boldsymbol{G}_2 &= r(\cos\theta\cos\phi\ \boldsymbol{i}_1 + \cos\theta\sin\phi\ \boldsymbol{i}_2 - \sin\theta\ \boldsymbol{i}_3), \\
\boldsymbol{G}_3 &= r\sin\theta(-\sin\phi\boldsymbol{i}_1 + \cos\phi\boldsymbol{i}_2).
\end{aligned} \tag{7.11}$$

The covariant components of the metric tensor, G_{ik}, are

$$G_{ik} = \boldsymbol{G}_i \cdot \boldsymbol{G}_k \tag{7.12}$$

and hence

$$[G_{ik}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{7.13}$$

The contravariant metric tensor, G^{ik}, is therefore

$$[G^{ik}] = [G_{ik}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}. \tag{7.14}$$

The contravariant base vectors, $\boldsymbol{G}^i$, are calculated most easily using

$$\boldsymbol{G}^i = G^{ik}\ \boldsymbol{G}_k. \tag{7.15}$$

This gives

$$\begin{aligned} \boldsymbol{G}^1 &= \sin\theta\cos\phi\ \boldsymbol{i}_1 + \sin\theta\sin\phi\ \boldsymbol{i}_2 + \cos\theta\ \boldsymbol{i}_3, \\ \boldsymbol{G}^2 &= \frac{1}{r}(\cos\theta\cos\phi\ \boldsymbol{i}_1 + \cos\theta\sin\phi\ \boldsymbol{i}_2 - \sin\theta\ \boldsymbol{i}_3), \\ \boldsymbol{G}^3 &= \frac{1}{r\sin\theta}(-\sin\phi\boldsymbol{i}_1 + \cos\phi\boldsymbol{i}_2). \end{aligned} \tag{7.16}$$

It can be shown that the contravariant metric tensor G^{ik} calculated using

$$G^{ik} = \boldsymbol{G}^i \cdot \boldsymbol{G}^k \tag{7.17}$$

agrees with (7.14). The covariant and contravariant base vectors, $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$, are shown in Figure 7.3.2.

Consider next the unstrained spherical shell B_0.

The covariant base vectors, $\boldsymbol{g}_i$, are defined by

$$\boldsymbol{g}_i = \frac{\partial x^n}{\partial \theta^i}\ \boldsymbol{i}_n. \tag{7.18}$$

Thus from (7.8)

$$\begin{aligned} \boldsymbol{g}_1 &= \frac{\partial\rho}{\partial r}(\sin\theta\cos\phi\ \boldsymbol{i}_1 + \sin\theta\sin\phi\ \boldsymbol{i}_2 + \cos\theta\ \boldsymbol{i}_3) = \frac{\partial\rho}{\partial r}\ \boldsymbol{G}_1, \\ \boldsymbol{g}_2 &= \rho(\cos\theta\cos\phi\ \boldsymbol{i}_1 + \cos\theta\sin\phi\ \boldsymbol{i}_2 - \sin\theta\ \boldsymbol{i}_3) = \frac{\rho}{r}\ \boldsymbol{G}_2, \\ \boldsymbol{g}_3 &= \rho\sin\theta(-\sin\phi\ \boldsymbol{i}_1 + \cos\phi\ \boldsymbol{i}_2) = \frac{\rho}{r}\ \boldsymbol{G}_3. \end{aligned} \tag{7.19}$$

The covariant components of the metric tensor, g_{ik}, are

$$g_{ik} = \boldsymbol{g}_i \cdot \boldsymbol{g}_k \tag{7.20}$$

and therefore

$$[g_{ik}] = \begin{bmatrix} \left(\frac{\partial\rho}{\partial r}\right)^2 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2\sin^2\theta \end{bmatrix}. \tag{7.21}$$

The contravariant metric tensor, g^{ik}, is

$$[g^{ik}] = [g_{ik}^{-1}] = \begin{bmatrix} \frac{1}{\left(\frac{\partial\rho}{\partial r}\right)^2} & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2\sin^2\theta} \end{bmatrix}. \tag{7.22}$$

The contravariant base vectors, $\boldsymbol{g}^i$, are most easily obtained using

$$\boldsymbol{g}^i = g^{ik}\ \boldsymbol{g}_k. \tag{7.23}$$

Figure 7.3.2

Thus from (7.19) and (7.22),

$$\begin{aligned}
\boldsymbol{g}^1 &= \frac{1}{\frac{\partial\rho}{\partial r}}(\sin\theta\cos\phi\ \boldsymbol{i}_1 + \sin\theta\sin\phi\ \boldsymbol{i}_2 + \cos\theta\ \boldsymbol{i}_3) = \frac{1}{\frac{\partial\rho}{\partial r}}\ \boldsymbol{G}^1, \\
\boldsymbol{g}^2 &= \frac{1}{\rho}(\cos\theta\cos\phi\ \boldsymbol{i}_1 + \cos\theta\sin\phi\ \boldsymbol{i}_2 - \sin\theta\ \boldsymbol{i}_3) = \frac{r}{\rho}\ \boldsymbol{G}^2, \\
\boldsymbol{g}^3 &= \frac{1}{\rho\sin\theta}(-\sin\phi\ \boldsymbol{i}_1 + \cos\phi\ \boldsymbol{i}_2) = \frac{r}{\rho}\ \boldsymbol{G}^3.
\end{aligned} \tag{7.24}$$

It can be shown using (7.14), that the metric tensor g^{ik} calculated from

$$g^{ik} = \boldsymbol{g}^i \cdot \boldsymbol{g}^k \tag{7.25}$$

agrees with (7.22). The drawing for the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ is the same as that for $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$ in Figure 7.3.2 because $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ are parallel to $\boldsymbol{G}_i$ and $\boldsymbol{G}^i$, respectively.

Finally consider the incompressibility condition which from (2.62) and (2.87) is

$$I_3 = \frac{G}{g} = 1. \tag{7.26}$$

But

$$G = \det[G_{ik}] = r^4\sin^2\theta, \quad g = \det[g_{ik}] = \rho^4\left(\frac{\partial\rho}{\partial r}\right)^2\sin^2\theta \tag{7.27}$$

and (7.26) becomes

$$\frac{\partial\rho}{\partial r} = \frac{r^2}{\rho^2}. \tag{7.28}$$

Integrating (7.28) gives

$$\rho^3 - r^3 = f(t), \tag{7.29}$$

where $f(t)$ is an arbitrary function of t. But since $r = r_1(t)$ when $\rho = \rho_1$,

$$\rho^3 - r^3 = \rho_1^3 - r_1^3(t) \tag{7.30}$$

and since $r = r_2(t)$ when $\rho = \rho_2$,

$$\rho^3 - r^3 = \rho_2^3 - r_2^3(t). \tag{7.31}$$

Thus, from (7.30) and (7.31) ,

$$\rho^3 - r^3 = \rho_1^3 - r_1^3(t) = \rho_2^3 - r_2^3(t). \tag{7.32}$$

By using (7.28) the base vectors and metric tensors in the unstrained spherical shell B_0 can be simplified :

$$\begin{aligned}
\boldsymbol{g}_1 &= \frac{r^2}{\rho^2}(\sin\theta\cos\phi\ \boldsymbol{i}_1 + \sin\theta\sin\phi\ \boldsymbol{i}_2 + \cos\theta\ \boldsymbol{i}_3) = \frac{r^2}{\rho^2}\boldsymbol{G}_1, \\
\boldsymbol{g}_2 &= \rho(\cos\theta\cos\phi\ \boldsymbol{i}_1 + \cos\theta\sin\phi\ \boldsymbol{i}_2 - \sin\theta\ \boldsymbol{i}_3) = \frac{\rho}{r}\boldsymbol{G}_2, \\
\boldsymbol{g}_3 &= \rho\sin\theta(-\sin\phi\ \boldsymbol{i}_1 + \cos\phi\ \boldsymbol{i}_2) = \frac{\rho}{r}\boldsymbol{G}_3, \\
\boldsymbol{g}^1 &= \frac{\rho^2}{r^2}(\sin\theta\cos\phi\ \boldsymbol{i}_1 + \sin\theta\sin\phi\ \boldsymbol{i}_2 + \cos\theta\ \boldsymbol{i}_3) = \frac{\rho^2}{r^2}\boldsymbol{G}^1, \\
\boldsymbol{g}^2 &= \frac{1}{\rho}(\cos\theta\cos\phi\ \boldsymbol{i}_1 + \cos\theta\sin\phi\ \boldsymbol{i}_2 - \sin\theta\ \boldsymbol{i}_3) = \frac{r}{\rho}\boldsymbol{G}^2, \\
\boldsymbol{g}^3 &= \frac{1}{\rho\sin\theta}(-\sin\phi\ \boldsymbol{i}_1 + \cos\phi\ \boldsymbol{i}_2) = \frac{r}{\rho}\boldsymbol{G}^3
\end{aligned} \tag{7.33}$$

and

$$
[g_{ik}] = \begin{bmatrix} \frac{r^4}{\rho^4} & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2\theta \end{bmatrix}, \quad [g^{ik}] = \begin{bmatrix} \frac{\rho^4}{r^4} & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2 \sin^2\theta} \end{bmatrix}. \tag{7.34}
$$

7.4 Anisotropic directors

Consider now the anisotropic directors $\boldsymbol{h}$ for radial, tangential and longitudinal transversely isotropic spherical shells. The anisotropic director is a unit vector with cartesian components H^n and curvilinear components h^i in the unstrained shell B_0 :

$$
\boldsymbol{h} = H^n \, \boldsymbol{i}_n = h^i \, \boldsymbol{g}_i, \tag{7.35}
$$

$$
h^i = \frac{\partial \theta^i}{\partial x^n} \, H^n, \tag{7.36}
$$

$$
\boldsymbol{h} \cdot \boldsymbol{h} = g_{ik} \, h^i h^k = \delta_{ik} H^i H^k. \tag{7.37}
$$

Consider first a radial transversely isotropic spherical shell. Then $\boldsymbol{h}$ is a unit vector in the radial direction of the unstrained spherical shell B_0 :

$$
\boldsymbol{h} = \cos\phi \sin\theta \; \boldsymbol{i}_1 + \sin\phi \sin\theta \; \boldsymbol{i}_2 + \cos\theta \; \boldsymbol{i}_3 = H^n \boldsymbol{i}_n. \tag{7.38}
$$

The anisotropic director is shown in Figure 7.4.1. The cartesian components of $\boldsymbol{h}$ in B_0 are

$$
H^1 = \cos\phi \sin\theta, \quad H^2 = \sin\phi \sin\theta, \quad H^3 = \cos\theta. \tag{7.39}
$$

The curvilinear components of $\boldsymbol{h}$ in B_0 are obtained from (7.36). Now by inverting the transformations (7.8) to obtain

$$
\rho^2 = {x^1}^2 + {x^2}^2 + {x^3}^2, \quad \tan\theta = \frac{\left({x^1}^2 + {x^2}^2\right)^{\frac{1}{2}}}{x^3}, \quad \tan\phi = \frac{x^2}{x^1} \tag{7.40}
$$

and by using the incompressibility condition (7.28) it can be shown that

$$
\begin{aligned}
\frac{\partial \theta^1}{\partial x^1} &= \frac{\rho^2}{r^2} \sin\theta \cos\phi, & \frac{\partial \theta^1}{\partial x^2} &= \frac{\rho^2}{r^2} \sin\theta \sin\phi, & \frac{\partial \theta^1}{\partial x^3} &= \frac{\rho^2}{r^2} \cos\theta, \\
\frac{\partial \theta^2}{\partial x^1} &= \frac{\cos\theta \cos\phi}{\rho}, & \frac{\partial \theta^2}{\partial x^2} &= \frac{\cos\theta \sin\phi}{\rho}, & \frac{\partial \theta^2}{\partial x^3} &= -\frac{\sin\theta}{\rho}, \\
\frac{\partial \theta^3}{\partial x^1} &= -\frac{\sin\phi}{\rho \sin\theta}, & \frac{\partial \theta^3}{\partial x^2} &= \frac{\cos\phi}{\rho \sin\theta}, & \frac{\partial \theta^3}{\partial x^3} &= 0.
\end{aligned} \tag{7.41}
$$

Hence, from (7.36), (7.39) and (7.41), the curvilinear components of $\boldsymbol{h}$ in the unstrained spherical shell B_0 are

$$
h^1 = \frac{\rho^2}{r^2}, \quad h^2 = 0, \quad h^3 = 0. \tag{7.42}
$$

Figure 7.4.1

Consider now a tangential transversely isotropic spherical shell. We have defined this to be a spherical shell with the anisotropic director $\boldsymbol{h}$ tangential to a line of latitude in the unstrained spherical shell B_0 as shown in Figure 7.4.1. Hence

$$\boldsymbol{h} = -\sin\phi\ \boldsymbol{i}_1 + \cos\phi\ \boldsymbol{i}_2 = H^n \boldsymbol{i}_n \tag{7.43}$$

and the cartesian components of $\boldsymbol{h}$ in B_0 therefore are

$$H^1 = -\sin\phi, \quad H^2 = \cos\phi, \quad H^3 = 0. \tag{7.44}$$

Thus from (7.36), (7.41) and (7.44) the curvilinear components of $\boldsymbol{h}$ in B_0 are

$$h^1 = 0, \quad h^2 = 0, \quad h^3 = \frac{1}{\rho\sin\theta}\ . \tag{7.45}$$

Consider finally a longitudinal transversely isotropic spherical shell. We have defined this to be a spherical shell with the anisotropic director $\boldsymbol{h}$ tangential to a line of longitude in the unstrained spherical shell B_0 as shown in Figure 7.4.1. Thus

$$\boldsymbol{h} = \cos\theta\cos\phi\ \boldsymbol{i}_1 + \cos\theta\sin\phi\ \boldsymbol{i}_2 - \sin\theta\ \boldsymbol{i}_3 = H^n \boldsymbol{i}_n \tag{7.46}$$

and the cartesian components of $\boldsymbol{h}$ in B_0 are

$$H^1 = \cos\theta\cos\phi, \quad H^2 = \cos\theta\sin\phi, \quad H^3 = -\sin\theta. \tag{7.47}$$

Thus from (7.36), (7.41) and (7.47) the curvilinear components of $\boldsymbol{h}$ in B_0 are

$$h^1 = 0, \quad h^2 = \frac{1}{\rho}, \quad h^3 = 0. \tag{7.48}$$

In summary, the anisotropic directors $\boldsymbol{h}$ in terms of curvilinear coordinates in the unstrained spherical shell B_0 are

$$\text{radial transverse isotropy} : h^1 = \frac{\rho^2}{r^2}, \quad h^2 = 0, \quad h^3 = 0\,, \tag{7.49}$$

$$\text{tangential transverse isotropy} : h^1 = 0,\ h^2 = 0,\ h^3 = \frac{1}{\rho\sin\theta}\ , \tag{7.50}$$

$$\text{longitudinal transverse isotropy} : h^1 = 0,\ \ h^2 = \frac{1}{\rho},\ \ h^3 = 0. \tag{7.51}$$

It can be verified that the anisotropic directors are unit vectors in B_0, that is, that

$$\boldsymbol{h}\cdot\boldsymbol{h} = g_{ik}h^i h^k = 1 \tag{7.52}$$

where g_{ik} is given by (7.34).

7.5 Strain invariants

The elastic material is incompressible. Thus $I_3 = 1$ and the remaining four strain invariants are

$$I_1 = g^{ik} G_{ik}, \quad I_2 = G^{ik} g_{ik}, \tag{7.53}$$

$$K_1 = G_{ab} h^a h^b, \quad K_2 = G_{ab} G_{cd} g^{bd} h^a h^c. \tag{7.54}$$

By using (7.13), (7.14) and (7.34), it can be shown from (7.53) that

$$I_1 = \frac{\rho^4}{r^4} + 2\frac{r^2}{\rho^2}, \quad I_2 = \frac{r^4}{\rho^4} + 2\frac{\rho^2}{r^2}. \tag{7.55}$$

The remaining two strain invariants depend on the kind of transverse isotropy. For a radial transversely isotropic spherical shell, $\boldsymbol{h}$ is given by (7.49) and

$$K_1 = \frac{\rho^4}{r^4}, \quad K_2 = \frac{\rho^8}{r^8}. \tag{7.56}$$

For a tangential transversely isotropic spherical shell, $\boldsymbol{h}$ is given by (7.50) and

$$K_1 = \frac{r^2}{\rho^2}, \quad K_2 = \frac{r^4}{\rho^4}. \tag{7.57}$$

For a longitudinal transversely isotropic spherical shell, $\boldsymbol{h}$ is given by (7.51) and

$$K_1 = \frac{r^2}{\rho^2}, \quad K_2 = \frac{r^4}{\rho^4}. \tag{7.58}$$

All the strain invariants are not constant and unlike the strain invariants for an cylindrical tube, $I_1 \neq I_2$.

7.6 Cauchy stress tensor

The constitutive equation for a transversely isotropic incompressible elastic material is given by (2.169) :

$$\tau^{ik} = \Phi g^{ik} + \Psi B^{ik} + \Theta M^{ik} + \Lambda N^{ik} + p G^{ik}, \tag{7.59}$$

where by (2.165) to (2.167),

$$B^{ik} = \left(g^{ik} g^{rs} - g^{ir} g^{ks}\right) G_{rs}, \tag{7.60}$$

$$M^{ik} = h^i h^k, \tag{7.61}$$

$$N^{ik} = \left(h^i g^{kr} + h^k g^{ir}\right) G_{rs} h^s \tag{7.62}$$

and from (2.170),

$$\Phi = 2\frac{\partial W}{\partial I_1}, \ \Psi = 2\frac{\partial W}{\partial I_2}, \ \Theta = 2\frac{\partial W}{\partial K_1}, \ \Lambda = 2\frac{\partial W}{\partial K_2} \tag{7.63}$$

and $p(r,\theta,\phi)$ is a scalar which is obtained from the field equations and boundary conditions.

Consider first B^{ik} which is the same for the three transversely isotropic spherical shells. Using (7.13), (7.14) and (7.34) for the metric tensors G_{ik}, G^{ik} and g^{ik}, it can be shown that

$$[B^{ik}] = \operatorname{diag}\Big[2\frac{\rho^2}{r^2},\quad \frac{\rho^2}{r^4}+\frac{r^2}{\rho^4},\quad \frac{1}{\sin^2\theta}\Big(\frac{\rho^2}{r^4}+\frac{r^2}{\rho^4}\Big)\Big]. \tag{7.64}$$

The remaining tensors, M^{ik} and N^{ik}, depend on which transversely isotropic spherical shell is considered.

Consider first a radial transversely isotropic spherical shell with $\boldsymbol{h}$ given by (7.49). Then

$$[M^{ik}] = \operatorname{diag}\Big[\frac{\rho^4}{r^4},\quad 0,\quad 0\Big], \tag{7.65}$$

$$[N^{ik}] = \operatorname{diag}\Big[2\frac{\rho^8}{r^8},\quad 0,\quad 0\Big]. \tag{7.66}$$

The components of the Cauchy stress tensor are

$$\tau^{11} = \frac{\rho^4}{r^4}\Phi + 2\frac{\rho^2}{r^2}\Psi + \frac{\rho^4}{r^4}\Theta + 2\frac{\rho^8}{r^8}\Lambda + p, \tag{7.67}$$

$$\tau^{22} = \frac{1}{r^2}\Big[\frac{r^2}{\rho^2}\Phi + \Big(\frac{\rho^2}{r^2}+\frac{r^4}{\rho^4}\Big)\Psi + p\Big], \tag{7.68}$$

$$\tau^{33} = \frac{\tau^{22}}{\sin^2\theta}, \tag{7.69}$$

$$\tau^{ik} = 0,\ \ i\neq k. \tag{7.70}$$

Consider next a tangential transversely isotropic spherical shell. Then $\boldsymbol{h}$ is given by (7.50) and

$$[M^{ik}] = \operatorname{diag}\Big[0,\quad 0,\quad \frac{1}{\rho^2\sin^2\theta}\Big], \tag{7.71}$$

$$[N^{ik}] = \operatorname{diag}\Big[0,\quad 0,\quad 2\frac{r^2}{\rho^4\sin^2\theta}\Big]. \tag{7.72}$$

The components of the Cauchy stress tensor are

$$\tau^{11} = \frac{\rho^4}{r^4}\Phi + 2\frac{\rho^2}{r^2}\Psi + p, \tag{7.73}$$

$$\tau^{22} = \frac{1}{r^2}\Big[\frac{r^2}{\rho^2}\Phi + \Big(\frac{\rho^2}{r^2}+\frac{r^4}{\rho^4}\Big)\Psi + p\Big], \tag{7.74}$$

$$\tau^{33} = \frac{1}{r^2\sin^2\theta}\Big[\frac{r^2}{\rho^2}\Phi + \Big(\frac{\rho^2}{r^2}+\frac{r^4}{\rho^4}\Big)\Psi + p + \frac{r^2}{\rho^2}\Theta + 2\frac{r^4}{\rho^4}\Lambda\Big], \tag{7.75}$$

$$\tau^{ik} = 0,\ \ i\neq k. \tag{7.76}$$

Consider finally a longitudinal transversely isotropic spherical shell with $\boldsymbol{h}$ given by (7.51). Then

$$[M^{ik}] = \text{diag}\Big[0, \quad \frac{1}{\rho^2}, \quad 0\Big], \tag{7.77}$$

$$[N^{ik}] = \text{diag}\Big[0, \quad 2\frac{r^2}{\rho^4}, \quad 0\Big]. \tag{7.78}$$

The components of the Cauchy stress tensor are

$$\tau^{11} = \frac{\rho^4}{r^4}\Phi + 2\frac{\rho^2}{r^2}\Psi + p, \tag{7.79}$$

$$\tau^{22} = \frac{1}{r^2}\Big[\frac{r^2}{\rho^2}\Phi + \Big(\frac{\rho^2}{r^2} + \frac{r^4}{\rho^4}\Big)\Psi + p + \frac{r^2}{\rho^2}\Theta + 2\frac{r^4}{\rho^4}\Lambda\Big], \tag{7.80}$$

$$\tau^{33} = \frac{1}{r^2\sin^2\theta}\Big[\frac{r^2}{\rho^2}\Phi + \Big(\frac{\rho^2}{r^2} + \frac{r^4}{\rho^4}\Big)\Psi + p\Big], \tag{7.81}$$

$$\tau^{ik} = 0, \ \ i \neq k. \tag{7.82}$$

For an isotropic spherical shell the Cauchy stress tensor is obtained by putting $\Theta = \Lambda = 0$. The additional terms due to the transverse isotropy occur in τ^{11} for a radial transversely isotropic sphere, in τ^{33} for a tangential transversely isotropic sphere and in τ^{22} for a longitudinal transversely isotropic sphere.

7.7 Boundary conditions

The pressure applied to the inner surface of the spherical shell, $r = r_1(t)$, is $\mathcal{P}_1(t)$ and the pressure applied to the outer surface of the shell, $r = r_2(t)$, is $\mathcal{P}_2(t)$. The unit outward normal vector $\boldsymbol{n}$ to the inner and outer surfaces and the base vectors $\boldsymbol{G}_1$ and $\boldsymbol{G}^1$ are shown in Figure 7.7.1. Cauchy's formula for the applied surface traction $\boldsymbol{P}$ to a surface with unit outward normal $\boldsymbol{n}$ is

$$P^i\boldsymbol{G}_i = n_k\tau^{ki}\boldsymbol{G}_i\,. \tag{7.83}$$

Consider first the inner surface $r = r_1(t)$ of the spherical shell. The base vector $\boldsymbol{G}^1$ given by (7.16) is a unit vector in the radial direction. From Figure 7.7.1, the unit outward normal $\boldsymbol{n}$ to the inner surface is

$$\boldsymbol{n} = n_k\boldsymbol{G}^k = -\boldsymbol{G}^1\,. \tag{7.84}$$

Hence

$$n_k = -\delta_k^1 \tag{7.85}$$

and since $\tau^{ki} = 0$ for $k \neq i$, Cauchy's formula (7.83) becomes

$$r = r_1(t): \quad P^i\boldsymbol{G}_i = -\tau^{11}(r_1(t), t)\boldsymbol{G}_1. \tag{7.86}$$

The base vector $\boldsymbol{G}_1$ is a unit vector in the radial direction. From Figure 7.7.1,

$$P^i\boldsymbol{G}_i = +\mathcal{P}_1(t)\boldsymbol{G}_1. \tag{7.87}$$

Figure 7.7.1

Substituting (7.87) into (7.86) gives

$$r = r_1(t): \quad \mathcal{P}_1(t)\boldsymbol{G}_1 = -\tau^{11}(r_1(t), t)\boldsymbol{G}_1 \tag{7.88}$$

and hence

$$\tau^{11}(r_1(t), t) = -\mathcal{P}_1(t). \tag{7.89}$$

Consider next the outer surface of the spherical shell, $r = r_2(t)$. The unit outward normal vector $\boldsymbol{n}$ is

$$\boldsymbol{n} = n_k \boldsymbol{G}^k = +\boldsymbol{G}^1 \tag{7.90}$$

and therefore

$$n_k = \delta_k^1. \tag{7.91}$$

Cauchy's formula (7.83) becomes

$$r = r_2(t): \quad P^i \boldsymbol{G}_i = \tau^{11}(r_2(t), t)\boldsymbol{G}_1. \tag{7.92}$$

But from Figure 7.7.1,

$$P^i \boldsymbol{G}_i = -\mathcal{P}_2(t)\boldsymbol{G}_1 \tag{7.93}$$

and therefore

$$r = r_2(t): \quad -\mathcal{P}_2(t)\boldsymbol{G}_1 = \tau^{11}(r_2(t), t)\boldsymbol{G}_1. \tag{7.94}$$

Hence

$$\tau^{11}(r_2(t), t) = -\mathcal{P}_2(t). \tag{7.95}$$

In summary, the boundary conditions at the inner and outer surfaces of the spherical shell are (7.89) and (7.95).

7.8 Cauchy's first law of motion

The body force due to gravity is neglected. From (2.98), Cauchy's first law of motion with zero body force is

$$\rho^* \frac{D\boldsymbol{V}}{Dt} = \tau^{ki}\|_k \boldsymbol{G}_i \tag{7.96}$$

where ρ^* is the density of the spherical shell which is constant since the spherical shell is incompressible.

Now

$$\boldsymbol{V} = \frac{D\boldsymbol{R}}{Dt} \tag{7.97}$$

where $\boldsymbol{R}$ is the position vector of a material particle in the strained spherical shell B and $\frac{D}{Dt}$ is the partial derivative with respect to t keeping the spherical polar coordinates (ρ, θ, ϕ) fixed. Now, since $\boldsymbol{G}_1$, given by (7.11), is a unit vector in the radial direction,

$$\boldsymbol{R} = r(\rho, t)\boldsymbol{G}_1 \tag{7.98}$$

and therefore

$$\boldsymbol{V} = \left.\frac{\partial \boldsymbol{R}}{\partial t}\right|_{(\rho,\theta,\phi)} = \left.\frac{\partial r}{\partial t}\right|_{\rho} \boldsymbol{G}_1\,, \tag{7.99}$$

$$\frac{D\boldsymbol{V}}{Dt} = \left.\frac{\partial \boldsymbol{V}}{\partial t}\right|_{(\rho,\theta,\phi)} = \left.\frac{\partial^2 r}{\partial t^2}\right|_{\rho} \boldsymbol{G}_1. \tag{7.100}$$

Thus, using (7.100) and expanding the covariant derivative, (7.96) becomes

$$\rho^* \frac{\partial^2 r}{\partial t^2}\bigg|_\rho \boldsymbol{G}_1 = \Big[\tau^{ki}{}_{,k} + \Gamma^k_{ks}\,\tau^{si} + \Gamma^i_{ks}\,\tau^{ks}\Big]\boldsymbol{G}_i \tag{7.101}$$

where Γ^i_{jk} is the Christoffel symbol of the second kind of the metric tensor G_{ik}. For spherical polar coordinates (r, θ, ϕ), G_{ik} is given by (7.13) and it can be shown that

$$\begin{aligned}
\Gamma^1_{22} &= -r, & \Gamma^1_{33} &= -r\sin^2\theta,\\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r}, & \Gamma^2_{33} &= -\sin\theta\cos\theta,\\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r}, & \Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta,\\
\Gamma^i_{jk} &= 0 \text{ otherwise}
\end{aligned} \tag{7.102}$$

and

$$\Gamma^k_{k1} = \frac{2}{r}, \qquad \Gamma^k_{k2} = \cot\theta, \qquad \Gamma^k_{k3} = 0. \tag{7.103}$$

Now, $\tau^{ik} = 0$ for $i \neq k$ for the three transversely isotropic spherical shells. Equation (7.101) gives the following three equations :

$$i = 1: \qquad \rho^* \frac{\partial^2 r}{\partial t^2}\bigg|_\rho = \frac{\partial \tau^{11}}{\partial r} + \frac{2}{r}\tau^{11} - r\tau^{22} - r\sin^2\theta\tau^{33}, \tag{7.104}$$

$$i = 2: \qquad 0 = \frac{\partial \tau^{22}}{\partial \theta} + \cot\theta\tau^{22} - \sin\theta\cos\theta\tau^{33}, \tag{7.105}$$

$$i = 3: \qquad 0 = \frac{\partial \tau^{33}}{\partial \phi}\,. \tag{7.106}$$

The strain invariants I_1, I_2, K_1 and K_2, which are given by (7.55) and (7.56) to (7.58), depend only on r and $\rho(r,t)$ and hence the strain-energy function $W(I_1, I_2, K_1, K_2)$ and therefore Φ, Ψ, Θ and Λ depend on r and t only. The stress tensor τ^{ik} can therefore depend on θ and ϕ only through $p(r, \theta, \phi, t)$. Equations (7.105) and (7.106) therefore become

$$\frac{\partial p}{\partial \theta} = -\cot\theta\tau^{22} + \sin\theta\cos\theta\tau^{33}, \tag{7.107}$$

$$\frac{\partial p}{\partial \phi} = 0 \tag{7.108}$$

and hence

$$p = p(r, \theta, t). \tag{7.109}$$

The incompressibility condition (7.32) is used to evaluate the left hand side of (7.104) :

$$r^3 - \rho^3 = r_1^3(t) - \rho_1^3\,. \tag{7.110}$$

Thus

$$\frac{\partial r}{\partial t}\bigg|_\rho = \frac{r_1^2(t)\,\dot{r}_1(t)}{r^2} \tag{7.111}$$

and

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_\rho = \left[2r_1(t)\dot{r}_1^2(t) + r_1^2(t)\ddot{r}_1(t)\right]\frac{1}{r^2} - 2r_1^4(t)\dot{r}_1^2(t)\frac{1}{r^5}\,, \tag{7.112}$$

which expresses the left hand side of (7.104) in terms of the radius of the inner surface of the spherical shell, $r_1(t)$ and the radial coordinate, r.

Cauchy's first law of motion therefore gives equations (7.104), (7.107) and (7.109) where the acceleration is given by (7.112). For the three transversely isotropic cylindrical tubes, $p = p(r,\theta,t)$. The right hand side of (7.107) depends on τ^{22} and τ^{33} and is not identically zero.

7.9 Ordinary differential equation for dimensionless inner radius of spherical shell

The equations describing radial oscillations in radial, tangential and longitudinal transversely isotropic spherical shells were derived in Section 7.8 :

$$\rho^*\left.\frac{\partial^2 r}{\partial t^2}\right|_\rho = \frac{\partial \tau^{11}}{\partial r} + \frac{2}{r}\tau^{11} - r\tau^{22} - r\sin^2\theta\tau^{33}\,, \tag{7.113}$$

$$\frac{\partial p}{\partial \theta} = -\cot\theta\tau^{22} + \sin\theta\cos\theta\tau^{33}\,, \tag{7.114}$$

$$p = p(r,\theta,t), \tag{7.115}$$

where

$$\left.\frac{\partial^2 r}{\partial t^2}\right|_\rho = \left[2r_1(t)\dot{r}_1^2(t) + r_1^2(t)\ddot{r}_1(t)\right]\frac{1}{r^2} - 2r_1^4(t)\dot{r}_1^2(t)\frac{1}{r^5}\,. \tag{7.116}$$

The boundary conditions are given by (7.89) and (7.95) :

$$\tau^{11}(r_1(t),t) = -\mathcal{P}_1(t), \tag{7.117}$$

$$\tau^{11}(r_2(t),t) = -\mathcal{P}_2(t). \tag{7.118}$$

We now investigate if an ordinary differential equation can be derived for the dimensionless inner radius of the spherical shell similar to the differential equation for a cylindrical tube. Since τ^{11}, τ^{22} and τ^{33} have to be specified, the three cases of radial, tangential and longitudinal transversely isotropic spherical shells have to be treated separately.

7.9.1 Radial transversely isotropic spherical shells

Consider first (7.114). From (7.69),

$$\tau^{33} = \frac{\tau^{22}}{\sin^2\theta} \tag{7.119}$$

and (7.114) reduces to

$$\frac{\partial p}{\partial \theta} = 0. \tag{7.120}$$

Hence, from (7.115),

$$p = p(r,t). \tag{7.121}$$

Consider next (7.113), which using (7.119) again, becomes

$$\rho^* \frac{\partial^2 r}{\partial t^2}\bigg|_\rho = \frac{\partial \tau^{11}}{\partial r} + \frac{2}{r}\left(\tau^{11} - r^2\tau^{22}\right). \tag{7.122}$$

Substituting (7.67) and (7.68) into (7.122) gives

$$\begin{aligned} \rho^* \frac{\partial^2 r}{\partial t^2}\bigg|_\rho &= \frac{\partial \tau^{11}}{\partial r} + \frac{2}{r}\left[\left(\frac{\rho^4}{r^4} - \frac{r^2}{\rho^2}\right)\Phi \right. \\ &+ \left.\left(\frac{\rho^2}{r^2} - \frac{r^4}{\rho^4}\right)\Psi + \frac{\rho^4}{r^4}\Theta + 2\frac{\rho^8}{r^8}\Lambda\right]. \end{aligned} \tag{7.123}$$

Integrate (7.123) with respect to r from $r = r_1(t)$ to $r = r_2(t)$. Then

$$\rho^* \int_{r_1(t)}^{r_2(t)} \frac{\partial^2 r}{\partial t^2}\bigg|_\rho dr = \tau^{11}\Big[r_2(t),t) - \tau^{11}(r_1(t),t\Big] + U(t) \tag{7.124}$$

where

$$U(t) = 2\int_{r_1(t)}^{r_2(t)} \left[\left(\frac{\rho^4}{r^4} - \frac{r^2}{\rho^2}\right)\Phi + \left(\frac{\rho^2}{r^2} - \frac{r^4}{\rho^4}\right)\Psi + \frac{\rho^4}{r^4}\Theta + 2\frac{\rho^8}{r^8}\Lambda\right]\frac{dr}{r}. \tag{7.125}$$

But using (7.116) it can be shown that

$$\int_{r_1(t)}^{r_2(t)} \frac{\partial^2 r}{\partial t^2}\bigg|_\rho dr = \frac{r_1}{r_2}(r_2 - r_1)\ddot{r}_1 + \frac{1}{2r_2^4}(r_2 - r_1)^2(3r_2^2 + 2r_1r_2 + r_1^2)\dot{r}_1^2 \tag{7.126}$$

and imposing the boundary conditions (7.117) and (7.118) gives

$$\tau^{11}(r_2(t),t) - \tau^{11}(r_1(t),t) = \mathcal{P}_1(t) - \mathcal{P}_2(t). \tag{7.127}$$

Consider next $U(t)$. Using (7.63), $U(t)$ may be expressed in terms of W as

$$\begin{aligned} U(t) &= 4\int_{r_1(t)}^{r_2(t)} \left[\left(\frac{\rho^4}{r^4} - \frac{r^2}{\rho^2}\right)\frac{\partial W}{\partial I_1} + \left(\frac{\rho^2}{r^2} - \frac{r^4}{\rho^4}\right)\frac{\partial W}{\partial I_2} \right. \\ &+ \left.\frac{\rho^4}{r^4}\frac{\partial W}{\partial K_1} + 2\frac{\rho^8}{r^8}\frac{\partial W}{\partial K_2}\right]\frac{dr}{r}. \end{aligned} \tag{7.128}$$

Make the change of variable

$$u = \frac{r}{\rho}. \tag{7.129}$$

But the incompressibility condition (7.28) is

$$\frac{\partial \rho}{\partial r} = \frac{r^2}{\rho^2} \tag{7.130}$$

and therefore

$$\frac{dr}{r} = \frac{du}{u(1-u^3)}. \tag{7.131}$$

Equation (7.128) becomes

$$
\begin{aligned}
U(t) &= 4\int_{\frac{r_1}{\rho_1}}^{\frac{r_2}{\rho_2}} \left[\left(\frac{1}{u^4}-u^2\right)\frac{\partial W}{\partial I_1}+\left(\frac{1}{u^2}-u^4\right)\frac{\partial W}{\partial I_2}\right. \\
&+ \left.\frac{1}{u^4}\frac{\partial W}{\partial K_1}+\frac{2}{u^8}\frac{\partial W}{\partial K_2}\right]\frac{du}{u(1-u^3)}\,. \qquad (7.132)
\end{aligned}
$$

But, expressed in terms of u the strain invariants (7.55) and (7.56) are

$$
I_1=\frac{1}{u^4}+2u^2, \qquad I_2=u^4+\frac{2}{u^2}, \qquad K_1=\frac{1}{u^4}, \qquad K_2=\frac{1}{u^8} \qquad (7.133)
$$

and therefore, since $W=W(I_1,I_2,K_1,K_2)$,

$$
\begin{aligned}
\frac{dW}{du} &= -\frac{4}{u}\left[\left(\frac{1}{u^4}-u^2\right)\frac{\partial W}{\partial I_1}+\left(\frac{1}{u^2}-u^4\right)\frac{\partial W}{\partial I_2}\right. \\
&+ \left.\frac{1}{u^4}\frac{\partial W}{\partial K_1}+\frac{2}{u^8}\frac{\partial W}{\partial K_2}\right]. \qquad (7.134)
\end{aligned}
$$

Hence (7.132) becomes

$$
U(t)=-\int_{\frac{r_1}{\rho_1}}^{\frac{r_2}{\rho_2}} \frac{1}{(1-u^3)}\frac{dW}{du}\,du\,. \qquad (7.135)
$$

Substituting (7.126), (7.127) and (7.135) into (7.124) gives

$$
\begin{aligned}
&\rho^*\left[\frac{r_1}{r_2}(r_2-r_1)\ddot{r}_1+\frac{1}{2r_2^4}(r_2-r_1)^2(3r_2^2+2r_1r_2+r_1^2)\dot{r}_1^2\right] \\
&+\int_{\frac{r_1}{\rho_1}}^{\frac{r_2}{\rho_2}} \frac{1}{(1-u^3)}\frac{dW}{du}\,du=\mathcal{P}_1(t)-\mathcal{P}_2(t)\,. \qquad (7.136)
\end{aligned}
$$

Finally, rewrite (7.136) as an ordinary differential equation for the dimensionless inner radius, $x(t)$, defined by

$$
x(t)=\frac{r_1(t)}{\rho_1}\,. \qquad (7.137)
$$

First express $r_2(t)$ in terms of $x(t)$ using the incompressibility condition

$$
r_2^3(t)-\rho_2^3=r_1^3(t)-\rho_1^3\,. \qquad (7.138)
$$

Then

$$
r_2(t)=\rho_1 x_1\left[1+\left(\left(\frac{\rho_2}{\rho_1}\right)^3-1\right)\frac{1}{x^3}\right]^{\frac{1}{3}} \qquad (7.139)
$$

and define

$$
\mu=\left(\frac{\rho_2}{\rho_1}\right)^3-1\,. \qquad (7.140)
$$

Hence

$$
r_2(t)=\rho_1 x\left(1+\frac{\mu}{x^3}\right)^{\frac{1}{3}}, \qquad (7.141)
$$

$$\rho_2 = \rho_1\big(1+\mu\big)^{\frac{1}{3}}\,. \tag{7.142}$$

When expressed in terms of $x(t)$, equation (7.136) becomes

$$\begin{aligned}
&\left[1-\left(1+\frac{\mu}{x^3}\right)^{-\frac{1}{3}}\right]x\ddot{x}\\
&+\frac{1}{2}\left[1-\left(1+\frac{\mu}{x^3}\right)^{-\frac{1}{3}}\right]^2\left[3+2\left(1+\frac{\mu}{x^3}\right)^{-\frac{1}{3}}+\left(1+\frac{\mu}{x^3}\right)^{-\frac{2}{3}}\right]\dot{x}^2\\
&+\frac{1}{\rho^*\rho_1^2}\int_x^{x\left(\frac{1+\frac{\mu}{x^3}}{1+\mu}\right)^{\frac{1}{3}}}\frac{1}{\big(1-u^3\big)}\frac{dW}{du}\,du = \frac{\mathcal{P}_1(t)-\mathcal{P}_2(t)}{\rho^*\rho_1^2}.
\end{aligned} \tag{7.143}$$

Define

$$W_0(u) = \frac{1}{\rho^*\rho_1^2}\,W(u), \qquad \mathcal{P}(t) = \frac{\mathcal{P}_1(t)-\mathcal{P}_2(t)}{\rho^*\rho_1^2}. \tag{7.144}$$

Equation (7.143) becomes

$$\begin{aligned}
&\left[1-\left(1+\frac{\mu}{x^3}\right)^{-\frac{1}{3}}\right]x\ddot{x}\\
&+\frac{1}{2}\left[1-\left(1+\frac{\mu}{x^3}\right)^{-\frac{1}{3}}\right]^2\left[3+2\left(1+\frac{\mu}{x^3}\right)^{-\frac{1}{3}}+\left(1+\frac{\mu}{x^3}\right)^{-\frac{2}{3}}\right]\dot{x}^2\\
&+\int_x^{x\left(\frac{1+\frac{\mu}{x^3}}{1+\mu}\right)^{\frac{1}{3}}}\frac{1}{\big(1-u^3\big)}\frac{dW_0}{du}\,du = \mathcal{P}(t).
\end{aligned} \tag{7.145}$$

Equation (7.145) is the required ordinary differential equation for the dimensionless inner radius $x(t)$. It has the same form as the equation describing radial oscillations in an isotropic spherical shell (Guo and Solecki 1963, Roussos, Mason and Hill 2002). Equation (7.145) corresponds to (3.151) for radial oscillations in a transversely isotropic cylindrical tube.

7.9.2 Tangential transversely isotropic spherical shells

Consider first (7.114). Substituting (7.74) and (7.75) into (7.114) gives

$$\frac{\partial p}{\partial \theta} = \frac{1}{r^2}\left[\frac{r^2}{\rho^2}\Theta + 2\frac{r^4}{\rho^4}\Lambda\right]\cot\theta\,. \tag{7.146}$$

Now Θ and Λ, defined by (7.63), depend only on the strain invariants, (7.55) and (7.57), and therefore are functions of r and t only. Thus integrating (7.146) with respect to θ gives

$$p(r,\theta,t) = \frac{1}{r^2}\left[\frac{r^2}{\rho^2}\Theta + 2\frac{r^4}{\rho^4}\Lambda\right]\ln(\sin\theta) + C(r,t), \tag{7.147}$$

where $C(r,t)$ is an arbitrary function of r and t. Thus $p(r,\theta,t)\to-\infty$ as $\theta\to 0$ and $\theta\to\pi$. The components of the stress tensor, (7.73) to (7.75), depend on p and are therefore unbounded at $\theta=0$ and $\theta=\pi$. Since this is not physically acceptable, we do not consider this solution further.

7.9.3 Longitudinal transversely isotropic materials

Consider again (7.114). Substituting (7.80) and (7.81) into (7.114) gives

$$\frac{\partial p}{\partial \theta} = -\frac{1}{r^2}\left[\frac{r^2}{\rho^2}\Theta + 2\frac{r^4}{\rho^4}\Lambda\right]\cot\theta \tag{7.148}$$

and therefore, integrating with respect to θ,

$$p(r,\theta,t) = -\frac{1}{r^2}\left[\frac{r^2}{\rho^2}\Theta + 2\frac{r^4}{\rho^4}\Lambda\right]\ln(\sin\theta) + C(r,t), \tag{7.149}$$

where $C(r,t)$ is an arbitrary function of r and t. Thus $p(r,\theta,t) \to +\infty$ as $\theta \to 0$ and $\theta \to \pi$, and the components of the stress tensor, (7.79) to (7.81), are therefore unbounded at $\theta = 0$ and $\theta = \pi$. Again, this is not physically acceptable. We do not consider this solution further.

7.10 Thin-walled radial transversely isotropic spherical shell

The approximation of (7.145) for radial oscillations in a thin-walled radial transversely isotropic spherical shell will now be derived. The parameter μ defined by (7.140) is a measure of the thickness of the wall of the spherical shell. It corresponds to μ defined by (3.145) for a cylindrical tube.

Consider a spherical shell that is thin-walled in the unstrained state B_0. In B_0, the inner radius is ρ_1 and the outer radius is ρ_2. Since the shell is thin-walled, $\rho_2 \doteqdot \rho_1$. Thus from (7.140),

$$\mu = \frac{\rho_2^3 - \rho_1^3}{\rho_1^3} = \frac{(\rho_2-\rho_1)(\rho_2^2+\rho_1\rho_2+\rho_1^2)}{\rho_1^3} \doteqdot \frac{3(\rho_2-\rho_1)}{\rho_1} \tag{7.150}$$

and hence $0 < \mu << 1$. Expand each term in (7.145) in powers of μ. Then

$$\left(1+\frac{\mu}{x^3}\right)^{-\frac{1}{3}} = 1 - \frac{\mu}{3x^3} + O(\mu^2), \tag{7.151}$$

$$\left(1+\frac{\mu}{x^3}\right)^{-\frac{2}{3}} = 1 - \frac{2\mu}{3x^3} + O(\mu^2), \tag{7.152}$$

$$x\left(\frac{1+\frac{\mu}{x^3}}{1+\mu}\right)^{\frac{1}{3}} = x + \frac{\mu}{3x^2}(1-x^3) + O(\mu^2), \tag{7.153}$$

$$\mathcal{P}(t) = P_0(t) + \mu P_1(t) + O(\mu^2), \tag{7.154}$$

as $\mu \to 0$. Define

$$I(x;\mu) = \int_x^{x\left(\frac{1+\frac{\mu}{x^3}}{1+\mu}\right)^{\frac{1}{3}}} \frac{1}{(1-u^3)}\frac{dW}{du}\,du. \tag{7.155}$$

The integral $I(x;\mu)$ is evaluated approximately with the aid of the First Integral Theorem of Mean Value (Gillespie, 1959) which is stated in Section 4.2. Then

$$I(x;\mu) = \left[\frac{\mu}{3x^2}(1-x^3) + O(\mu^2)\right]\left[\frac{1}{(1-\xi^3)}\frac{dW_0}{d\xi}(\xi)\right], \tag{7.156}$$

where from (7.153)

$$x \le \xi \le x + \frac{\mu}{3x^2}(1-x^3) + O(\mu^2)\,. \tag{7.157}$$

Thus

$$\xi = x + O(\mu) \tag{7.158}$$

as $\mu \to 0$ and (7.156) becomes

$$\begin{aligned} I(x;\mu) &= \left[\frac{\mu}{3x^2}(1-x^3) + O(\mu^2)\right]\left[\frac{1}{(1-x^3)}\frac{dW_0}{dx}(x) + O(\mu)\right] \\ &= \frac{\mu}{3x^2}\frac{dW_0}{dx}(x) + O(\mu^2), \end{aligned} \tag{7.159}$$

as $\mu \to 0$. Thus equation (7.145) becomes

$$\begin{aligned} &\left[\frac{\mu}{3x^2} + O(\mu^2)\right]\ddot{x} + \left[O(\mu^2)\right]\dot{x}^2 + \left[\frac{\mu}{3x^2}\frac{dW_0}{dx}(x) + O(\mu^2)\right] \\ &= P_0(t) + \mu P_1(t) + O(\mu^2) \end{aligned} \tag{7.160}$$

and hence

$$3x^2 P_0(t) + \mu\left[\ddot{x} + \frac{dW_0}{dx} - 3x^2 P_1(t)\right] + O(\mu^2) = 0, \tag{7.161}$$

as $\mu \to 0$. Equate the coefficients of the same powers of μ in (7.161):

$$\text{zero order in } \mu: \qquad P_0(t) = 0, \tag{7.162}$$

$$\text{first order in } \mu: \qquad \ddot{x} + \frac{dW_0}{dx}(x) = 3x^2 P_1(t). \tag{7.163}$$

Equations (7.162) and (7.163) for a spherical shell correspond to (4.13) and (4.14) for a cylindrical tube. To zero order in μ the spherical shell has zero thickness and therefore the net surface pressure, $P_0(t)$, must vanish because otherwise a finite force would be applied to an interface of zero mass causing an infinite acceleration which would destroy the shell. Equation (7.163) has the same form as the equation describing radial oscillations in an isotropic spherical shell (Calderer 1983, Roussos, Mason and Hill 2002).

Equations (7.145) and (7.163) apply for all strain-energy functions. To proceed further the generalised Mooney-Rivlin strain-energy function will be considered.

7.11 Strain-energy function

Consider the generalised Mooney-Rivlin strain-energy function (2.192):

$$W(I_1, I_2, K_1, K_2) = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(K_1 - 1) + C_4(K_2 - 1). \quad (7.164)$$

Using the strain invariants (7.133) for a spherical shell, (7.164) becomes

$$\begin{aligned} W(x) &= C_1\Big(\frac{1}{x^4} + 2x^2 - 3\Big) + C_2\Big(x^4 + \frac{2}{x^2} - 3\Big) \\ &+ C_3\Big(\frac{1}{x^4} - 1\Big) + C_4\Big(\frac{1}{x^8} - 1\Big). \end{aligned} \quad (7.165)$$

In the unstrained body B_0, $W(x)$ must be a minimum. Now, in B_0, $r = \rho$ and therefore $x = 1$. Thus $W(x)$ must satisfy

$$\frac{dW}{dx}(1) = 0, \quad (7.166)$$

$$\frac{d^2W}{dx^2}(1) > 0. \quad (7.167)$$

But

$$\frac{dW}{dx}(1) = -4(C_3 + 2C_4) \quad (7.168)$$

and (7.166) is therefore satisfied provided

$$C_3 = -2C_4. \quad (7.169)$$

Condition (7.169) is the same as condition (2.181) for an unstrained cylindrical tube to be stress-free. Using (7.169), the strain-energy function (7.165) becomes

$$W(x) = C_1\Big(\frac{1}{x^4} + 2x^2 - 3\Big) + C_2\Big(x^4 + \frac{2}{x^2} - 3\Big) + C_4\Big(\frac{1}{x^8} - \frac{2}{x^4} + 1\Big). \quad (7.170)$$

Thus

$$\frac{d^2W}{dx^2}(1) = 8[3(C_1 + C_2) + 4C_4]. \quad (7.171)$$

and condition (7.167)is satisfied provided

$$3(C_1 + C_2) + 4C_4 > 0. \quad (7.172)$$

Consider finally $W_0(x)$ defined by (7.144) and let

$$D_1 = \frac{4C_1}{\rho^* \rho_1^2}, \quad D_2 = \frac{4C_2}{\rho^* \rho_1^2}, \quad D_4 = \frac{8C_4}{\rho^* \rho_1^2}. \quad (7.173)$$

Then (7.170) becomes

$$W_0(x) = \frac{D_1}{4}\Big(\frac{1}{x^4} + 2x^2 - 3\Big) + \frac{D_2}{4}\Big(x^4 + \frac{2}{x^2} - 3\Big) + \frac{D_4}{8}\Big(\frac{1}{x^8} - \frac{2}{x^4} + 1\Big) \quad (7.174)$$

and condition (7.172) expressed in terms of D_1, D_2 and D_4 is

$$3(D_1 + D_2) + 2D_4 > 0. \quad (7.175)$$

7.12 Lie point symmetry generators

Equation (7.145) for a thick-walled spherical shell is difficult to solve when the net applied surface pressure depends on time. Equation (7.163),

$$\ddot{x} + \frac{dW_0}{dx}(x) = 3x^2 P_1(t), \tag{7.176}$$

for a thin-walled spherical shell will therefore first be considered. The Lie point symmetry generators of (7.176) with strain-energy function (7.174) will be investigated.

If (7.174) is substituted into (7.176), then (7.176) becomes

$$\ddot{x} + D_1 x = \frac{D_2}{x^3} + \frac{D_1 - D_4}{x^5} + \frac{D_4}{x^9} + 3x^2 P_1(t) - D_2 x^3. \tag{7.177}$$

For an isotropic spherical shell for which $D_4 = 0$, Roussos, Mason and Hill (2002) showed that

$$\begin{cases} \text{if} \quad \dot{P}(t) = 0, \quad \text{there is one Lie point symmetry generator } X = \frac{\partial}{\partial t} \\ \text{if} \quad \dot{P}(t) \neq 0, \quad \text{there are no Lie point symmetry generators} \end{cases} \tag{7.178}$$

Equation (7.177) is of the form

$$\ddot{x} = F(t, x) \tag{7.179}$$

where

$$F(t,x) = -D_1 x + \frac{D_2}{x^3} + \frac{D_1 - D_4}{x^5} + \frac{D_4}{x^9} + 3P_1(t)x^2 - D_2 x^3\,. \tag{7.180}$$

General results for the Lie point symmetry generator

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} \tag{7.181}$$

of (7.179) were derived in Section 4.3. It was shown that

$$\xi(t,x) \;=\; x f_1(t) + f_2(t), \tag{7.182}$$

$$\eta(t,x) \;=\; x^2 \dot{f}_1(t) + x f_3(t) + f_4(t) \tag{7.183}$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$ satisfy the two equations

$$3x\ddot{f}_1(t) - \ddot{f}_2(t) + 2\dot{f}_3(t) - 3F(t,x)f_1(t) = 0, \tag{7.184}$$

$$x^2 \dddot{f}_1(t) + x\ddot{f}_3(t) + \ddot{f}_4(t) + [f_3(t) - 2\dot{f}_2(t)]F(t,x)$$

$$-[x f_1(t) + f_2(t)]\frac{\partial F}{\partial t} - [x^2 \dot{f}_1(t) + x f_3(t) + f_4(t)]\frac{\partial F}{\partial x} = 0. \tag{7.185}$$

First substitute $F(t,x)$ given by (7.180) into (7.184). This gives

$$3x\ddot{f}_1(t) - \ddot{f}_2(t) + 2\dot{f}_3(t) - 3f_1(t)\Big[- D_1 x + \frac{D_2}{x^3}$$

$$+\frac{(D_1 - D_4)}{x^5} + \frac{D_4}{x^9} + 3P_1(t)x^2 - D_2 x^3\Big] = 0. \tag{7.186}$$

Separate equation (7.186) according to powers of x:

$$x^3 : \qquad D_2 f_1(t) = 0, \tag{7.187}$$

$$x^2 : \qquad P_1(t) f_1(t) = 0, \tag{7.188}$$

$$x : \qquad \ddot{f}_1(t) - D_1 f_1(t) = 0, \tag{7.189}$$

$$x^0 : \qquad \ddot{f}_2(t) - 2\dot{f}_3(t) = 0, \tag{7.190}$$

$$x^{-3} : \qquad D_2 f_1(t) = 0, \tag{7.191}$$

$$x^{-5} : \qquad (D_1 - D_4) f_1(t) = 0, \tag{7.192}$$

$$x^{-9} : \qquad D_4 f_1(t) = 0. \tag{7.193}$$

Case 1 $D_4 \neq 0$

The spherical shell is radial transversely isotropic. From (7.193) ,

$$f_1(t) = 0 \tag{7.194}$$

since $D_4 \neq 0$. From (7.190),

$$\dot{f}_3(t) = \frac{1}{2}\ddot{f}_2(t) \tag{7.195}$$

and therefore

$$f_3(t) = \frac{1}{2}\dot{f}_2(t) + c_3, \tag{7.196}$$

where c_3 is a constant. Since $f_1(t) = 0$, the remaining equations are identically satisfied. Equations (7.183) and (7.184) become

$$\xi(t,x) = f_2(t), \tag{7.197}$$

$$\eta(t,x) = (\frac{1}{2}\dot{f}_2(t) + c_3)x + f_4(t). \tag{7.198}$$

Next, substitute (7.180), (7.194) and (7.196) into (7.185). This gives

$$
\begin{aligned}
& \frac{x}{2}\dddot{f}_2(t) + \dddot{f}_4(t) + \Big(c_3 - \frac{3}{2}\dot{f}_2(t)\Big)\Big[- D_1 x + \frac{D_2}{x^3} \\
+ \quad & \frac{(D_1 - D_4)}{x^5} + \frac{D_4}{x^9} + 3P_1(t)x^2 - D_2 x^3\Big] \\
- \quad & 3f_2(t)\frac{dP}{dt}x^2 - \Big(\frac{1}{2}\dot{f}_2(t) + c_3\Big)\Big[- D_1 x - \frac{3D_2}{x^3} \\
- \quad & \frac{5(D_1 - D_4)}{x^5} - \frac{9D_4}{x^9} + 6P_1(t)x^2 - 3D_2 x^3\Big] \\
- \quad & f_4(t)\Big[- D_1 - \frac{3D_2}{x^4} - \frac{5(D_1 - D_4)}{x^6} \\
- \quad & \frac{9D_4}{x^{10}} + 6P_1(t)x - 3D_2 x^2\Big] = 0.
\end{aligned} \tag{7.199}
$$

Separating equation (7.199) according to powers of x yields

$$x^3: \qquad D_2(3\dot{f}_2(t) + 2c_3) = 0, \tag{7.200}$$

$$x^2: \qquad D_2 f_4(t) - \Big(c_3 + \frac{5}{2}\dot{f}_2(t)\Big)P_1(t) - \frac{dP_1}{dt}f_2(t) = 0, \tag{7.201}$$

$$x: \qquad \dddot{f}_2(t) + 4D_1\dot{f}_2(t) - 12P_1(t)f_4(t) = 0, \tag{7.202}$$

$$x^0: \qquad \ddot{f}_4(t) + D_1 f_4(t) = 0, \tag{7.203}$$

$$\frac{1}{x^3}: \qquad c_3 D_2 = 0, \tag{7.204}$$

$$\frac{1}{x^4}: \qquad D_2 f_4(t) = 0, \tag{7.205}$$

$$\frac{1}{x^5}: \qquad (D_1 - D_4)[6c_3 + \dot{f}_2(t)] = 0, \tag{7.206}$$

$$\frac{1}{x^6}: \qquad (D_1 - D_4)f_4(t) = 0, \tag{7.207}$$

$$\frac{1}{x^9}: \qquad D_4(10c_3 + 3\dot{f}_2(t)) = 0, \tag{7.208}$$

$$\frac{1}{x^{10}}: \qquad D_4 f_4(t) = 0. \tag{7.209}$$

Since $D_4 \neq 0$ it follows from (7.209) that

$$f_4(t) = 0 \tag{7.210}$$

and from (7.208) that

$$\dot{f}_2(t) = -\frac{10}{3}c_3 \tag{7.211}$$

and therefore

$$f_2(t) = -\frac{10}{3}c_3 t + c_2, \tag{7.212}$$

where c_2 is a constant. Equations (7.200) to (7.209) now simplify to

$$x^3 \text{ and } \frac{1}{x^3}: \qquad c_3 D_2 = 0, \tag{7.213}$$

$$x^2: \qquad 22c_3 P_1(t) + (10c_3 t - 3c_2)\frac{dP_1}{dt} = 0, \tag{7.214}$$

$$x: \qquad c_3 D_1 = 0, \tag{7.215}$$

$$\frac{1}{x^5}: \qquad c_3(D_1 - D_4) = 0 \tag{7.216}$$

and (7.197) and (7.198) reduce to

$$\xi(t,x) = -\frac{10}{3}c_3 t + c_2, \tag{7.217}$$

$$\eta(t,x) = -\frac{2}{3}c_3 x. \tag{7.218}$$

There are three cases.

Case 1.1 $D_1 \neq 0$

Then from (7.215), $c_3 = 0$, and from (7.214)

$$c_2 \frac{dP_1}{dt} = 0. \tag{7.219}$$

The other equations are identically satisfied and (7.217) and (7.218) become

$$\xi(t,x) = c_2, \qquad \eta(t,x) = 0, \tag{7.220}$$

so that

$$X = c_2 \frac{\partial}{\partial t}. \tag{7.221}$$

Thus if $\frac{dP_1}{dt} = 0$ then c_2 is arbitrary and there is one Lie point symmetry generator

$$X = \frac{\partial}{\partial t}. \tag{7.222}$$

If $\frac{dP_1}{dt} \neq 0$, then $c_2 = 0$ and $X = 0$. There is no Lie point symmetry generator. Thus the result (7.178) holds.

Case 1.2 $D_2 \neq 0$

Then from (7.213), $c_3 = 0$ and (7.219) is again valid. The other equations are identically satisfied. Thus the result (7.178) again applies.

Case 1.3 $D_1 = 0$ **and** $D_2 = 0$

Then from (7.216), $c_3 = 0$ since $D_4 \neq 0$ and (7.219) again holds. The other equations are identically satisfied. Thus the result (7.178) is again valid.

Thus in all three cases the result (7.178) applies. There are no special time dependent net applied surface pressures for which equation (7.177) has a Lie point symmetry generator which could be used to integrate the equation.

7.13 Conclusions

The differential equations for radial oscillations in thick-walled and thin-walled radial transversely isotropic spherical shells, when expressed in terms of the strain-energy function, have the same form as for an isotropic spherical shell.

The differential equation (7.177) for radial oscillations in a thin-walled radial transversely isotropic spherical shell with generalised Mooney-Rivlin strain-energy function has no Lie point symmetries when the net applied surface pressure depends on time. It can be expected that the differential equation (7.145) for a thick-walled spherical shell with generalised Mooney-Rivlin strain-energy function will have no Lie point symmetries when $\mathcal{P}$ depends on time. Equations (7.145) and (7.177) may have to be solved numerically when $\mathcal{P}$ depends on time. The solution for radial oscillations in a sphere is more difficult than in a cylinder because in a sphere the principal invariants, I_1 and I_2, are not equal.

When the net applied surface pressure is constant and the generalised Mooney-Rivlin strain-energy function is used, the differential equations (7.145) and (7.177) can be integrated once and the solution can be expressed as an integral. Results similar to those obtained in Chapter 5 for Heaviside step loading and in Chapter 6 for free oscillations could be investigated. This will not be done because the analysis would be similar to that in Chapters 5 and 6. Instead, the problem of inflation of a radial transversely isotropic thin-walled spherical shell will be considered in Chapter 8 and the results will be compared with the inflation of a spherical shell.

Chapter 8

Inflation of a Radial Transversely Isotropic Spherical Shell

8.1 Introduction

In this chapter the inflation of a radial transversely isotropic thin-walled spherical shell is considered. It will be compared with the inflation of an isotropic spherical shell.

The inflation of an isotropic spherical shell has been investigated by several authors including Needleman(1977), Ogden(1984), Beatty(1987) and Holzapfel(2000). Alexander(1971) performed experiments on the inflation of spherical balloons. The inflation of a spherical shell has application, for example, to meteorological balloons used for measurements at high altitudes.

The inflation is assumed to proceed sufficiently slowly that the inertia can be neglected. Snap buckling (Holzapfel, 2000) can occur in which the pressure decreases during inflation before increasing again. The effect of the radial transverse isotropy on the snap buckling is investigated.

An outline of the chapter is as follows. In Section 8.2 the inflation pressure is obtained in terms of the extension of a radial transversely isotropic thin-walled spherical shell. In Section 8.3, snap buckling in the inflation of an isotropic spherical shell is reviewed. New perturbation results are derived. In Section 7.4 the inflation of a radial transversely isotropic thin-walled spherical shell is investigated. The conclusions are summarised in Section 8.5.

8.2 Inflation pressure

Consider the inflation of a radial transversely isotropic incompressible thin-walled spherical shell. The inflation is assumed to proceed sufficiently slowly that the inertia can be neglected.

The inflation pressure is obtained from (7.176),

$$\ddot{x} + \frac{dW_0}{dx} = 3x^2 P_1(t), \tag{8.1}$$

where from (7.144)

$$W_0(x) = \frac{1}{\rho^* \rho_1^2} W(x)\,. \tag{8.2}$$

But from (7.144) and (7.154),

$$\mathcal{P}(t) = \frac{\mathcal{P}_1(t) - \mathcal{P}_2(t)}{\rho^* \rho_1^2} = P_0(t) + \mu P_1(t) + O(\mu^2) \tag{8.3}$$

and since by (7.162), $P_0(t) = 0$, it follows that

$$P_1(t) = \frac{\mathcal{P}_1(t) - \mathcal{P}_2(t)}{\mu \rho^* \rho_1^2} + O(\mu). \tag{8.4}$$

Thus (8.1) becomes

$$\rho^* \ddot{x} + \frac{1}{\rho_1^2}\frac{dW}{dx} = \frac{3x^2(\mathcal{P}_1(t) - \mathcal{P}_2(t))}{\mu \rho_1^2} + O(\mu). \tag{8.5}$$

Define

$$P = \mathcal{P}_1 - \mathcal{P}_2, \qquad \varepsilon = \frac{\mu}{3} \doteqdot \frac{(\rho_2 - \rho_1)}{\rho_1} \tag{8.6}$$

where μ is given by (7.150). Make the approximation that the inflation occurs sufficiently slowly that the inertia, $\rho^* \ddot{x}$, can be neglected. Also neglect terms of order ε^2 since the shell is thin-walled. Then (8.5) becomes

$$P = \frac{\varepsilon}{x^2}\frac{dW}{dx}\,. \tag{8.7}$$

Equation (8.7) has the same form as for an isotropic shell. It gives the inflation pressure P in terms of the stretch x for general W.

Consider an incompressible hyperelastic material with generalised Mooney-Rivlin strain-energy function. Then for a radial transversely isotropic spherical shell, from (7.170),

$$W(x) = C_1\Big(\frac{1}{x^4} + 2x^2 - 3\Big) + C_2\Big(x^4 + \frac{2}{x^2} - 3\Big) + C_4\Big(\frac{1}{x^8} - \frac{2}{x^4} + 1\Big). \tag{8.8}$$

Substituting (8.8) into (8.7) gives

$$P(x) = 4\varepsilon\Big[C_1\Big(\frac{1}{x} - \frac{1}{x^7}\Big) + C_2\Big(x - \frac{1}{x^5}\Big) + 2C_4\Big(\frac{1}{x^7} - \frac{1}{x^{11}}\Big)\Big]. \tag{8.9}$$

Assume that $C_1 \neq 0$ and define

$$\Gamma = \frac{C_2}{C_1}\,, \qquad \lambda = \frac{C_4}{C_1}\,. \tag{8.10}$$

It will be assumed that $\Gamma \geq 0$ and $\lambda \geq 0$. Equation (8.9) becomes

$$P(x) = 4\varepsilon C_1\Big[\frac{1}{x} - \frac{1}{x^7} + \Big(x - \frac{1}{x^5}\Big)\Gamma + 2\Big(\frac{1}{x^7} - \frac{1}{x^{11}}\Big)\lambda\Big]. \tag{8.11}$$

Also

$$\frac{dP}{dx} = \frac{4\varepsilon C_1}{x^{12}}[\Gamma x^{12} - x^{10} + 5\Gamma x^6 + 7(1 - 2\lambda)x^4 + 22\lambda]. \tag{8.12}$$

Equation (8.11) gives the inflation pressure $P(x)$ in terms of the stretch x for a radial transversely isotropic spherical shell. The inflation of an isotropic spherical shell for which $\lambda = 0$ will first be considered. The radial transverse isotropic effects given by $\lambda \neq 0$ will then be investigated.

8.3 Inflation of an isotropic thin-walled spherical shell

Consider now the inflation of an isotropic spherical shell(Needleman 1977, Ogden 1984, Beatty 1987, Holzapfel 2000). When $\lambda = 0$, (8.11) and (8.12) become

$$P(x) = 4\varepsilon C_1\Big[\frac{1}{x} - \frac{1}{x^7} + \Big(x - \frac{1}{x^5}\Big)\Gamma\Big], \tag{8.13}$$

$$\frac{dP}{dx} = \frac{4\varepsilon C_1}{x^8}[\Gamma x^8 - x^6 + 5\Gamma x^2 + 7]. \tag{8.14}$$

Consider first a neo-Hookean material for which $C_2 = 0$ and therefore $\Gamma = 0$. Then

$$P(x) = 4\varepsilon C_1\Big(\frac{1}{x} - \frac{1}{x^7}\Big), \tag{8.15}$$

$$\frac{dP}{dx} = \frac{4\varepsilon C_1}{x^8}(7 - x^6). \tag{8.16}$$

The inflation pressure has a maximum value at $x = 7^{\frac{1}{6}} = 1.383$. The pressure rises steeply from $-\infty$ as x increases from zero, attains a maximum value at $x = 1.383$ and then tends to zero as $x \to \infty$. Since $P(x) \to 0$ as $x \to \infty$ is not physical, the neo-Hookean strain-energy function does not describe accurately the inflation for large inflation.

Consider now a Mooney-Rivlin material with $C_2 > 0$ and therefore $\Gamma > 0$. Then

$$\frac{dP}{dx} = 0 \tag{8.17}$$

when

$$\Gamma x^8 - x^6 + 5\Gamma x^2 + 7 = 0. \tag{8.18}$$

By Descartes rule of signs, $\frac{dP}{dx} = 0$ cannot have more than two positive roots. It may have no positive roots. The range of values of Γ for which (8.18) has two positive roots may be determined as follows (Beatty, 1987). From (8.14)

$$\frac{dP}{dx} > 0 \text{ for } 0 < x \leq 1 \tag{8.19}$$

and

$$\frac{dP}{dx}(\infty) = 4\varepsilon C_2 > 0. \tag{8.20}$$

Thus if the pressure has a local maximum it will be in the range $x > 1$ and the maximum must be followed by a local minimum. The pressure will have a minimum value only if the slope, $\frac{dP}{dx}$, is negative for some range of x with $x > 1$. From (8.14),

$$\frac{dP}{dx} < 0 \text{ if } \Gamma < g(x) \tag{8.21}$$

where

$$g(x) = \frac{x^6 - 7}{x^8 + 5x^2}. \tag{8.22}$$

The greatest value which Γ can have for two stationary values to exist is therefore the maximum value g. Now

$$\frac{dg}{dx} = \frac{2x[-x^{12} + 38x^6 + 35]}{(x^8 + 5x^2)^2} = 0 \tag{8.23}$$

when $x = 0$ or

$$x^{12} - 38x^6 - 35 = 0\,. \tag{8.24}$$

When $x = 0$, $g = -\infty$ which is not a maximum value. Equation (8.24) is a quadratic equation for x^6 which has two real roots,

$$x^6 = 38.9 \ \text{ and } \ x^6 = -0.9\,.$$

Only real positive values of x are considered. Hence $x = 1.84$ and therefore

$$g_{max} = \Gamma_{max} = 0.214\,. \tag{8.25}$$

Thus if the spherical shell has a Mooney-Rivlin strain-energy function with $0 < \Gamma < 0.214$ an initial rise in pressure will be followed by a fall as inflation proceeds before the pressure starts to rise again at high values of inflation.

Graphs of $P(x)$ against x for $\Gamma = 0$ (neo-Hookean material), $0 < \Gamma < 0.214$ and $\Gamma > 0.214$ are shown in Figure 8.3.1. Initially the pressure gradient is high. A large increase in pressure is needed to produce a small change in stretch. The spherical shell is therefore stiff initially. After the pressure has reached a maximum value there is inflation with a decrease in pressure. The spherical shell is said to suddenly *snap through* and the decrease in pressure will allow it to *snap back* (Holzapfel, 2000). The effect is known as *snap buckling*. It is caused by the net applied surface pressure being dependent on the stretch x. It is a dynamic effect.

Consider now the stretches, x_{max} and x_{min}, at which $P(x)$ has its local maximum and minimum values. A numerical solution for x_{max} and x_{min} and the local maximum and minimum pressures, P_{max} and P_{min}, is given by Beatty (1987). A perturbation solution for x_{max}, x_{min}, P_{max} and P_{min} will now be derived and compared with the numerical results. The stretches x_{max} and x_{min} occur for $0 < \Gamma < 0.214$ and are the roots of (8.18). They depend only on the ratio $\Gamma = \frac{C_2}{C_1}$ and not on C_1 and C_2 separately or on the thickness ε of the spherical shell. Let

$$y = x^2. \tag{8.26}$$

Then y satisfies the quartic equation

$$\Gamma y^4 - y^3 + 5\Gamma y + 7 = 0. \tag{8.27}$$

Since Γ is small ($0 < \Gamma < 0.214$), a perturbation solution should be accurate. Consider first the straight forward perturbation expansion

$$y = y_0 + \Gamma y_1 + O(\Gamma^2), \tag{8.28}$$

as $\Gamma \to 0$. Substitute (8.28)into (8.27):

$$\Gamma y_0^4 - y_0^3 - 3\Gamma y_0^2 y_1 + 5\Gamma y_0 + 7 + O(\Gamma^2) = 0. \tag{8.29}$$

Figure 8.3.1

Equate the coefficients of like powers of Γ:

$$\Gamma^0: \qquad y_0^3 = 7$$

$$y_0 = 7^{\frac{1}{3}} = 1.9129 \tag{8.30}$$

$$\Gamma^1: \qquad y_1 = \frac{y_0^3 + 5}{3y_0} . \tag{8.31}$$

Hence

$$\begin{aligned} y &= y_0 + \Gamma y_1 + O(\Gamma^2) \\ &= y_0 \left[1 + \frac{(y_0^3 + 5)}{3y_0^2} \Gamma + O(\Gamma^2) \right], \end{aligned} \tag{8.32}$$

as $\Gamma \to 0$ and therefore

$$x = y_0^{\frac{1}{2}} \left[1 + \frac{(y_0^3 + 5)}{6y_0^2} \Gamma + O(\Gamma^2) \right], \tag{8.33}$$

as $\Gamma \to 0$. Thus, using (8.30), (8.33) becomes

$$x_{max} = 1.383[1 + 0.547\Gamma + O(\Gamma^2)], \tag{8.34}$$

as $\Gamma \to 0$. When $\Gamma = 0$, (8.34) reduces to the stretch at the maximum value of the inflation pressure in a neo-Hookean material. For $\Gamma > 0$, it gives the first order in Γ correction to the stretch at the local maximum of the inflation pressure.

The perturbation solution (8.34) is compared with the numerical solution of (8.18) for x_{max} in Table 8.3.1 for $0 \leq \Gamma \leq 0.214$. When $\Gamma = 0.214$, x_{max} is a point of inflexion and it was calculated using

$$\tfrac{d^2P}{dx^2} = 0 \ .$$

When comparing theory with experiment for a Mooney-Rivlin balloon, Beatty (1987) found $\Gamma = 0.055$. There is good agreement between the perturbation solution and the numerical solution for Γ in the range of $\Gamma = 0.055$.

To obtain a perturbation solution for P_{max}, the inflation pressure at the local maximum x_{max}, substitute (8.34) into (8.13). This gives

$$P_{max} = P(x_{max}) = 4\varepsilon C_1[0.620 + 1.185\Gamma + O(\Gamma^2)], \tag{8.35}$$

as $\Gamma \to 0$. A comparison of the perturbation solution (8.35) with the numerical solution for P_{max} is given in Table 8.3.2. The numerical results of Tables 8.3.1 and 8.3.2 agree with the numerical results of Beatty (1987) for the inflation of a Mooney-Rivlin balloon.

Consider next the second root of (8.27). The small parameter, Γ, multiplies the highest order term. It is therefore a singular perturbation problem (Nayfeh, 1981). Make the transformation

$$y = \frac{w}{\Gamma^\nu} \quad , \nu > 0. \tag{8.36}$$

Tables 8.3.1 and 8.3.2

Equation (8.27) becomes

$$\Gamma^{1-4\nu}w^4 - \Gamma^{-3\nu}w^3 + 5\Gamma^{1-\nu}w + 7 = 0. \tag{8.37}$$

To obtain the second root the first term in (8.37) must be retained. Since $\nu > 0$, the dominant terms are the first two terms. For the first two terms to balance each other it is necessary that

$$1 - 4\nu = -3\nu \ \text{ or } \ \nu = 1. \tag{8.38}$$

Equation (8.37) becomes

$$w^4 - w^3 + 5\Gamma^3 w + 7\Gamma^3 = 0. \tag{8.39}$$

Expand

$$w = w_0 + \Gamma w_1 + \Gamma^2 w_2 + \Gamma^3 w_3 + O(\Gamma^4). \tag{8.40}$$

Consider first the solution to first order in Γ. Substituting (8.40) into (8.39) gives

$$w_0^4 + 4\Gamma w_0^3 w_1 - w_0^3 - 3\Gamma w_0^2 w_1 + O(\Gamma^2) = 0. \tag{8.41}$$

Equate the coefficients of like powers of Γ:

$$\Gamma^0: \quad w_0^3(w_0 - 1) = 0 \quad \text{or} \quad w_0 = 0,\ w_0 = 1. \tag{8.42}$$

The root $w_0 = 0$ corresponds to x_{max} which has already been found because terms of order Γ^{-1} in y vanish. We therefore take $w_0 = 1$.

$$\Gamma^1: \quad w_1 = 0. \tag{8.43}$$

Thus (8.40) becomes

$$w = 1 + \Gamma^2 w_2 + \Gamma^3 w_3 + O(\Gamma^4). \tag{8.44}$$

Consider next the solution to third order in Γ. Substitute (8.44) into (8.39):

$$w_2\Gamma^2 + w_3\Gamma^3 + 12\Gamma^3 + O(\Gamma^4) = 0. \tag{8.45}$$

Equate the coefficients of corresponding powers of Γ:

$$\Gamma^2: \quad w_2 = 0, \tag{8.46}$$

$$\Gamma^3: \quad w_3 = -12. \tag{8.47}$$

Hence

$$w = 1 - 12\Gamma^3 + O(\Gamma^4) \tag{8.48}$$

and from (8.36) with $\nu = 1$,

$$y = \frac{1}{\Gamma}\Big(1 - 12\Gamma^3 + O(\Gamma^4)\Big). \tag{8.49}$$

Thus

$$x = y^{\frac{1}{2}} = \frac{1}{\sqrt{\Gamma}}\Big(1 - 12\Gamma^3 + O(\Gamma^4)\Big)^{\frac{1}{2}} \tag{8.50}$$

Tables 8.3.3 and 8.3.4

and therefore

$$x_{min} = \frac{1}{\sqrt{\Gamma}}\Big(1 - 6\Gamma^3 + O(\Gamma^4)\Big). \tag{8.51}$$

The lowest order perturbation solution is

$$x_{min} = \frac{1}{\sqrt{\Gamma}}\ . \tag{8.52}$$

Equation (8.52) should be accurate because $0 < \Gamma < 0.214$ and the next term is of order $\Gamma^{\frac{5}{2}}$. In Table 8.3.3, the zero order and first order perturbation solutions for x_{min} are compared with the numerical solution. When $\Gamma = 0.214$, $x_{min} = x_{max}$. It is a point of inflexion and it was calculated using

$$\tfrac{d^2P}{dx^2} = 0\ .$$

By substituting (8.51) for x_{min} into (8.13) for $P(x)$ it can be shown that

$$P_{min} = 8\varepsilon C_1\sqrt{\Gamma}\Big(1 - \Gamma^3 + O(\Gamma^6)\Big). \tag{8.53}$$

The lowest order perturbation solution is

$$P_{min} = 8\varepsilon C_1\sqrt{\Gamma}\ . \tag{8.54}$$

Equation (8.54) should be accurate because the next term is of order $\Gamma^{\frac{7}{2}}$. In Table 8.3.4, the zero order and first order perturbation solutions for P_{min} are compared with the numerical solution.

The numerical results in Tables 8.3.3 and 8.3.4 agree with the numerical results of Beatty (1987). Beatty used $\Gamma = 0.055$ for a Mooney-Rivlin balloon. There is good agreement between the zero order perturbation solutions (8.52) and (8.54) and the numerical solutions for Γ in the range of $\Gamma = 0.055$.

The perturbation solutions for x_{max}, x_{min}, P_{max} and P_{min} are not given in the review by Beatty (1987) and may be new results. The approximate solutions (8.52) and (8.54) relate x_{min} and P_{min} in a simple way to Γ.

Since (8.27) is a quartic equation the condition $\Gamma < 0.214$ for two turning points in the pressure can be derived alternatively by using standard results for the roots of a quartic equation (Barnard and Child, 1936). This approach was not used because an equation of degree six is obtained for the inflation of a radial transversely isotropic spherical shell. The results for a quartic equation cannot be extended to an equation of degree six. The approach used can be extended.

8.4 Inflation of a radial transversely isotropic thin-walled spherical shell

The inflation pressure $P(x)$ is given by (8.11) and $\frac{dP}{dx}$ by (8.12). Now

$$\frac{dP}{dx} = 0 \tag{8.55}$$

when

$$\Gamma x^{12} - x^{10} + 5\Gamma x^6 + 7(1-2\lambda)x^4 + 22\lambda = 0. \tag{8.56}$$

By Descartes' rule of signs:

$0 < \lambda \leq \frac{1}{2}$: $\frac{dP}{dx}$ cannot have more than two zeros

$\lambda > \frac{1}{2}$: $\frac{dP}{dx}$ cannot have more than four zeros.

If the anisotropy is not strong ($0 < \lambda \leq \frac{1}{2}$) then the behaviour of the inflation pressure will be similar to that for an isotropic spherical shell although the range of Γ for two turning points in the pressure to exist will depend on λ. For strong anisotropy ($\lambda > \frac{1}{2}$) there may be four turning points in the pressure for some ranges of values of Γ and λ. Large values of λ may be difficult to obtain physically.

We first consider $0 < \lambda \leq \frac{1}{2}$ and by using a perturbation expansion in λ we determine how the range, $0 < \Gamma < 0.214$, for two turning points in $P(x)$ to exist, is changed. We will then consider a perturbation expansion in Γ and determine the minimum value of λ for two turning points in the inflation pressure to exist.

Equation (8.12) can be written as

$$\frac{dP}{dx} = \frac{4\varepsilon C_1}{x^{12}}\Big[\Gamma x^{12} + 5\Gamma x^6 + x^4(7 - x^6) + (22 - 14x^4)\lambda\Big]. \tag{8.57}$$

When $0 < x \leq 1$ then $x^6 \leq 1$ and $x^4 \leq 1$. Hence $7 - x^6 > 0$ and $22 - 14x^4 > 0$ and therefore

$$\frac{dP}{dx} > 0 \quad \text{for} \quad 0 < x \leq 1. \tag{8.58}$$

Also

$$\frac{dP}{dx}(\infty) = 4\varepsilon C_2 > 0. \tag{8.59}$$

Thus, as with $\lambda = 0$, when $\lambda > 0$, if the pressure has a local maximum it will be in the range $x > 1$ and the maximum will be followed by a local minimum.

Let $y = x^2$. Then (8.12) becomes

$$\frac{dP}{dx} = \frac{4\varepsilon C_1}{y^6}\Big[\Gamma y^6 - y^5 + 5\Gamma y^3 + 7(1-2\lambda)y^2 + 22\lambda\Big]. \tag{8.60}$$

The pressure will have a minimum value only if the slope, $\frac{dP}{dx}$, is negative for some range of $y > 1$. Now

$$\frac{dP}{dx} < 0 \quad \text{if} \quad \Gamma < g(y;\lambda), \tag{8.61}$$

where

$$g(y;\lambda) = \frac{y^5 - 7(1-2\lambda)y^2 - 22\lambda}{y^3(y^3+5)}. \tag{8.62}$$

The maximum value of Γ for at least two stationary values in the pressure to exist is the maximum value of $g(y;\lambda)$. Now

$$\frac{dg}{dy}(y;\lambda) = \frac{[-y^8 + 2(19 - 28\lambda)y^5 + 132\lambda y^3 + 35(1-2\lambda)y^2 + 330\lambda]}{y^4(y^3+5)^2} = 0 \tag{8.63}$$

when

$$Q(y) = y^8 - 2(19 - 28\lambda)y^5 - 132\lambda y^3 - 35(1 - 2\lambda)y^2 - 330\lambda = 0. \tag{8.64}$$

From Descartes's rule of signs if $0 < \lambda \leq \frac{1}{2}$ then the maximum number of real positive roots of equation (8.64) is one and since $Q(0) = -330\lambda < 0$ and $Q(\infty) = +\infty$, equation (8.64) has exactly one real positive root. To determine the root when $\lambda > 0$, consider a straightforward perturbation expansion with λ as the small perturbation parameter. Let

$$y = y_0 + \lambda y_1 + O(\lambda^2)\,, \tag{8.65}$$

as $\lambda \to 0$. Substitute (8.65) into (8.64). Then

$$\begin{aligned} &y_0^8 + 8\lambda y_0^7 y_1 - 38y_0^5 - 190\lambda y_0^4 y_1 + 56\lambda y_0^5 - 132\lambda y_0^3 \\ &-35y_0^2 - 70\lambda y_0 y_1 + 70\lambda y_0^2 - 330\lambda = O(\lambda^2). \end{aligned} \tag{8.66}$$

Equate the coefficients of the same power of λ:

$$\lambda^0 \quad : \qquad y_0^2(y_0^6 - 38y_0^3 - 35) = 0, \tag{8.67}$$

$$\lambda \quad : \qquad y_1 = -\frac{\left[56y_0^5 - 132y_0^3 + 70y_0^2 - 330\right]}{2y_0\left[4y_0^6 - 95y_0^3 - 35\right]}\,. \tag{8.68}$$

Consider first (8.67). When $y_0 = 0$, $g = -\infty$ which is not a maximum value. Thus

$$y_0^6 - 38y_0^3 - 35 = 0, \tag{8.69}$$

which is the same as (8.24) with $y = x^2$. The real positive root of (8.69) is therefore

$$y_0 = \left(38.9\right)^{\frac{1}{3}} = 3.388. \tag{8.70}$$

Thus from (8.68),

$$y_1 = -1.294. \tag{8.71}$$

Hence $g(y;\lambda)$ has a stationary value, which is a maximum, at

$$y = y_0 + \lambda y_1 + O(\lambda^2) = 3.388(1 - 0.382\lambda + O(\lambda^2)). \tag{8.72}$$

Substitute $y = y_0 + \lambda y_1$ into (8.62) and expand in powers of λ. This gives

$$\begin{aligned} g(y_0 + \lambda y_1;\lambda) \quad &= \quad \frac{\left(y_0^3 - 7\right)}{y_0\left(y_0^2 + 5\right)}\left[1 + \frac{\left(5y_0^4 y_1 - 14y_0 y_1 + 14y_0^2 - 22\right)\lambda}{y_0^2\left(y_0^3 - 7\right)}\right. \\ &- \quad \left.\frac{3\left(2y_0^3 + 5\right)\lambda}{y_0\left(y_0^3 + 5\right)} + O(\lambda^2)\right], \end{aligned} \tag{8.73}$$

as $\lambda \to 0$. Hence using (8.70) and (8.71),

$$\Gamma_{max}(\lambda) = g_{max} = 0.214(1 + 0.379\lambda + O(\lambda^2)), \tag{8.74}$$

as $\lambda \to 0$. If

$$0 < \Gamma < 0.214(1 + 0.379\lambda + O(\lambda^2)), \tag{8.75}$$

then an initial rise in pressure will be followed by a fall as inflation proceeds before the pressure starts to rise again. The effect of radial transverse isotropy is to increase Γ_{max} and therefore to increase the range of values of Γ for which *snap buckling* occurs. In Table 8.4.1, $\Gamma_{max}(\lambda)$ is calculated using (8.74) to first order in λ for $0 \leq \lambda \leq \frac{1}{2}$.

In Figure 8.4.1, $P(x)$ is plotted against x for a range of values of Γ for $\lambda = 0.3$. When $\lambda = 0.3$, then (8.74) gives $\Gamma_{max} = 0.238$. In Figure 8.4.1 the inflation pressure curve for $\Gamma = 0.238$ approximately divides the curves into those for which $P(x)$ increases steadily ($\Gamma > \Gamma_{max}$) and those for which a pressure rise is followed by a fall as inflation proceeds ($\Gamma < \Gamma_{max}$). Unlike an isotropic spherical shell, when $\Gamma = 0.214$ the pressure rise is followed by a fall before it starts to rise again.

In the preceeding analysis, λ was given and the maximum value of Γ for the inflation pressure to fall and rise again was found to first order in λ. Suppose now that Γ is given and it is required to determine λ_{min} such that for $\lambda > \lambda_{min}$ snap buckling will occur. It is known that for $\Gamma \leq 0.214$, $\lambda_{min} = 0$.

From (8.60)

$$\frac{dP}{dx} < 0 \quad \text{if} \quad \lambda > h(y;\Gamma), \tag{8.76}$$

where

$$h(y;\Gamma) = \frac{y^2\Big(\Gamma y^4 - y^3 + 5\Gamma y + 7\Big)}{2(7y^2 - 11)}\ , \tag{8.77}$$

provided $7y^2 - 11 > 0$, that is, provided $y > 1.254$. The minimum value of λ, $\lambda_{min}(\Gamma)$, for the pressure to fall before rising again is the minimum value of $h(y;\Gamma)$ in the range $y > 1.254$. Now

$$\frac{dh}{dy}(y;\Gamma) = \frac{y\Big[28\Gamma y^6 - 21y^5 - 66\Gamma y^4 + (55 + 35\Gamma)y^3 - 165\Gamma y - 154\Big]}{2(7y^2 - 11)^2} = 0 \tag{8.78}$$

when

$$R(y) = 28\Gamma y^6 - 21y^5 - 66\Gamma y^4 + (55 + 35\Gamma)y^3 - 165\Gamma y - 154 = 0. \tag{8.79}$$

By Descartes' rule of signs the maximum number of real positive roots of equation (8.79) is three. Since $R(0) = -154 < 0$ and $R(\infty) = +\infty$, there is always at least one real positive root. We are interested in the roots which exist for small Γ since $\Gamma \gtrsim 0.214$. These are found from a singular perturbation solution because Γ multiplies y^6, the highest power of y. Make the transformation

$$y = \frac{w}{\Gamma^\nu}, \quad \nu > 0. \tag{8.80}$$

Equation (8.79) becomes

$$\begin{aligned} &28\Gamma^{1-6\nu}w^6 - 21\Gamma^{-5\nu}w^5 - 66\Gamma^{1-4\nu}w^4 + 55\Gamma^{-3\nu}w^3 \\ &+35\Gamma^{1-3\nu}w^3 - 165\Gamma^{1-\nu}w - 154 = 0. \end{aligned} \tag{8.81}$$

Table 8.4.1

Figure 8.4.1

Since $\nu > 0$, the dominant terms are the first two terms. The dominant terms will balance each other provided

$$1 - 6\nu = -5\nu \text{ or } \nu = 1. \tag{8.82}$$

Equation (8.81) becomes

$$28w^6 - 21w^5 - 66\Gamma^2 w^4 + 55\Gamma^2 w^3 + 35\Gamma^3 w^3 - 165\Gamma^5 w - 154\Gamma^5 = 0. \tag{8.83}$$

Expand

$$w = w_0 + w_1\Gamma + w_2\Gamma^2 + w_3\Gamma^3 + O(\Gamma^4). \tag{8.84}$$

Consider first the solution to order Γ. Substituting (8.84) into (8.83) gives

$$4w_0^6 + 24w_0^5 w_1\Gamma - 3w_0^5 - 15w_0^4 w_1\Gamma + O(\Gamma^2) = 0. \tag{8.85}$$

Equate the coefficients of the same power of Γ.

$$\Gamma^0: \quad w_0^5(w_0 - \frac{3}{4}) = 0. \tag{8.86}$$

The root $w_0 = 0$ corresponds to a straightforward perturbation expansion of y because there is no term of order Γ^{-1}. We will not consider that solution here. Thus $w_0 = \frac{3}{4}$.

$$\Gamma^1: \quad w_0^4(w_0 - \frac{5}{8})w_1 = 0. \tag{8.87}$$

Thus $w_1 = 0$.

Next, substitute (8.84) into (8.83) and keep terms to order Γ^3. This gives

$$168w_0^5 w_2\Gamma^2 + 168w_0^5 w_3\Gamma^3 - 105w_0^4 w_2\Gamma^2 - 105w_0^4 w_3\Gamma^3$$

$$-66w_0^4\Gamma^2 + 55w_0^3\Gamma^2 + 35w_0^3\Gamma^3 = O(\Gamma^4). \tag{8.88}$$

Equate the coefficients of the same power of Γ.

$$\Gamma^2: \quad w_2 = \frac{11(6w_0 - 5)}{21w_0(8w_0 - 5)} = -\frac{22}{63}, \tag{8.89}$$

$$\Gamma^3: \quad w_3 = -\frac{5}{3w_0(8w_0 - 5)} = -\frac{20}{9} . \tag{8.90}$$

Thus

$$y = \frac{w}{\Gamma} = \frac{w_0}{\Gamma} + w_2\Gamma + w_3\Gamma^2 + O(\Gamma^3) \tag{8.91}$$

where

$$w_0 = \frac{3}{4}, \quad w_2 = -\frac{22}{63}, \quad w_3 = -\frac{20}{9}. \tag{8.92}$$

Next substitute (8.91) into (8.77). This gives

$\lambda_{min} = h_{min}$

$$= \frac{[w_0^2 + 2w_0w_2\Gamma^2 + 2w_0w_3\Gamma^3 + O(\Gamma^4)][-(1 - w_0)w_0^3 + (5w_0 + 7)\Gamma^3 + O(\Gamma^4)]}{2\Gamma^3[7w_0^2 + (14w_0w_2 - 11)\Gamma^2 + 14w_0w_3\Gamma^3 + O(\Gamma^4)]}. \tag{8.93}$$

Using the values given by (8.92), equation (8.93) becomes

$$\lambda_{min} = h_{min} = \frac{43}{56}\frac{[\Gamma^3 + 0.157\Gamma^2 - 0.169][\Gamma^3 - 0.00981]}{\Gamma^3[\Gamma^3 + 0.629\Gamma^2 - 0.169]}. \tag{8.94}$$

The three cubic polynomials in (8.94) each have one and only one real positive zero:

$$\Gamma^3 + 0.157\Gamma^2 - 0.169 = 0 \quad \text{at } \Gamma = 0.505, \tag{8.95}$$

$$\Gamma^3 - 0.00981 = 0 \quad \text{at } \Gamma = 0.214, \tag{8.96}$$

$$\Gamma^3 + 0.629\Gamma^2 - 0.169 = 0 \quad \text{at } \Gamma = 0.404. \tag{8.97}$$

The graph of λ_{min} against Γ is drawn in Figure 8.4.2. We are assuming that $\lambda > 0$ and $\Gamma > 0$. Figure 8.4.2 shows that for $\Gamma < 0.214$ the pressure will fall before rising steadily for $\lambda \geq 0$, in agreement with the result for an isotropic spherical shell. For $0.214 \leq \Gamma < 0.4044$ the pressure will fall before rising if $\lambda > \lambda_{min}$. Also as $\Gamma \to 0.4044$, $\lambda_{min} \to \infty$. Figure 8.4.2 therefore gives a lower bound on λ for snap buckling to occur for a given value of $\Gamma < 0.4044$.

The effect of radial transverse isotropy is to extend the range of Γ for snap buckling to occur. However, snap buckling will only occur if λ is sufficiently large, that is if $\lambda > \lambda_{min}$. If $\Gamma > 0.2141$, snap buckling will not occur if $\lambda < \lambda_{min}$. This is illustrated in Figure 8.4.3 for $\Gamma = 0.35$.

Consider now x_{max} and x_{min}, the stretches at the local maximum and local minimum of the inflation pressure. We investigate how (8.34) for x_{max} and (8.51) for x_{min} for an isotropic spherical shell are changed for a transversely isotropic spherical shell.

Suppose that $0 < \lambda \leq \frac{1}{2}$. The maximum and minimum turning points of $P(x)$ are at the roots of (8.60). Let $y = x^2$. Then (8.60) becomes

$$\Gamma y^6 - y^5 + 5\Gamma y^3 + 7(1 - 2\lambda)y^2 + 22\lambda = 0. \tag{8.98}$$

Consider a straight foward perturbation expansion. For an isotropic spherical shell it gave x_{max}. Expand

$$y(\lambda, \Gamma) = y_0(\lambda) + \Gamma y_1(\lambda) + O(\Gamma^2), \tag{8.99}$$

as $\Gamma \to 0$. Equation (8.98) becomes

$$\Gamma(y_0^6 + 6y_0^5 y_1\Gamma) - (y_0^5 + 5y_0^4 y_1\Gamma) + 5\Gamma y_0^3$$

$$+7(1 - 2\lambda)(y_0^2 + 2y_0 y_1\Gamma) + 22\lambda = O(\Gamma^2). \tag{8.100}$$

Equate the coefficients of like powers of Γ.

$$\Gamma^0: \quad y_0^5 - 7y_0^2 + 14\lambda y_0^2 - 22\lambda = 0. \tag{8.101}$$

Expand $y_0(\lambda)$ in powers of λ:

$$y_0(\lambda) = y_{00} + \lambda y_{01} + O(\lambda^2), \tag{8.102}$$

Figures 8.4.2 and 8.4.3

as $\lambda \to 0$. Equation (8.101) becomes

$$y_{00}^5 + 5y_{00}^4 y_{01}\lambda - 7y_{00}^2 - 14y_{00}y_{01}\lambda + 14\lambda y_{00}^2 - 22\lambda = 0. \tag{8.103}$$

Equate the coefficients of like powers of λ.

$$\lambda^0 : \quad y_{00} = 7^{\frac{1}{3}} = 1.913 \tag{8.104}$$

$$\lambda^1 : \quad y_{01} = \frac{22 - 14y_{00}^2}{y_{00}(5y_{00}^3 - 14)} = -0.728 \tag{8.105}$$

Consider now first order in Γ. Equating the coefficients of Γ in (8.101) gives

$$\Gamma^1 : \quad y_0^6 - 5y_0^4 y_1 + 5y_0^3 + 14y_0 y_1 - 28\lambda y_0 y_1 = 0. \tag{8.106}$$

Expand $y_1(\lambda)$ in powers of λ:

$$y_1(\lambda) = y_{10} + \lambda y_{11} + O(\lambda^2), \tag{8.107}$$

as $\lambda \to 0$. Substituting (8.102) and (8.107) into (8.106) gives

$$y_{00}^6 + 6y_{00}^5 y_{01}\lambda - 5(y_{00}^4 + 4y_{00}^3 y_{01}\lambda)(y_{10} + \lambda y_{11}) + 5(y_{00}^3 + 3y_{00}^2 y_{01}\lambda)$$

$$+14(y_{00} + \lambda y_{01})(y_{10} + \lambda y_{11}) - 28\lambda y_{00} y_{10} = O(\lambda^2). \tag{8.108}$$

Equate the coefficients of like powers of λ.

$$\lambda^0 : \quad y_{10} = \frac{y_{00}^2(y_{00}^3 + 5)}{(5y_{00}^3 - 14)} = 2.091, \tag{8.109}$$

$$\lambda^1 : \quad y_{11} = \frac{57y_{00}^2 y_{01} - 126y_{01}y_{10} - 28y_{00}y_{10}}{21y_{00}}, \tag{8.110}$$

where $y_{00}^3 = 7$ was used to simplify (8.110). Thus

$$y_{11} = -1.794. \tag{8.111}$$

Now

$$\begin{aligned} y(\lambda, \Gamma) \quad &= y_{00}\Bigg[1 + \frac{y_{01}}{y_{00}}\lambda + O(\lambda^2) \\ &+ \frac{y_{10}}{y_{00}}\Gamma\bigg(1 + \frac{y_{11}}{y_{10}}\lambda + O(\lambda^2)\bigg) + O(\Gamma^2)\Bigg], \end{aligned} \tag{8.112}$$

as $\lambda \to 0$ and $\Gamma \to 0$. Thus, since $y = x^2$,

$$\begin{aligned} x_{max} \quad &= y_{00}^{\frac{1}{2}}\Bigg[1 + \frac{y_{01}}{2y_{00}}\lambda + O(\lambda^2) \\ &+ \frac{y_{10}}{2y_{00}}\Gamma\bigg(1 + \Big(\frac{y_{11}}{y_{10}} - \frac{y_{01}}{2y_{00}}\Big)\lambda + O(\lambda^2)\bigg) + O(\Gamma^2)\Bigg], \end{aligned} \tag{8.113}$$

as $\lambda \to 0$ and $\Gamma \to 0$. But

$$y_{00} = 1.913, \quad y_{01} = -0.728, \quad y_{10} = 2.091, \quad y_{11} = -1.794$$

and therefore

$$\begin{aligned} x_{max} &= 1.383\Big[1 - 0.190\,\lambda + O(\lambda^2) \\ &+0.547\,\Gamma\Big(1 - 0.668\,\lambda + O(\lambda^2)\Big) + O(\Gamma^2)\Big], \end{aligned} \tag{8.114}$$

as $\lambda \to 0$ and $\Gamma \to 0$. The dual perturbation expansion in Γ and λ, (8.114), shows that for a radial transversely isotropic spherical shell the local maximum of the inflation pressure occurs at a smaller stretch than in an isotropic spherical shell. When $\lambda = 0$, the expansion (8.114) reduces to (8.34) for an isotropic spherical shell.

The inflation pressure at the local maximum, P_{max}, is obtained by substituting (8.114) into (8.11):

$$\begin{aligned} P_{max} &= P(x_{max}) \\ &= 4\varepsilon C_1\Big[0.620\Big(1 + 0.242\,\lambda + O(\lambda^2)\Big) \\ &+1.185\,\Gamma\Big(1 - 0.765\,\lambda + O(\lambda^2)\Big) + O(\Gamma^2)\Big], \end{aligned} \tag{8.115}$$

as $\lambda \to 0$ and $\Gamma \to 0$. Equation (8.115) reduces to (8.35) when $\lambda = 0$. In $\Gamma < 0.165$, the effect of radial transversely isotropy is to increase P_{max} while if $\Gamma > 0.165$ its effect is to decrease P_{max}.

Consider now x_{min}, the stretch at the minimum value of the inflation pressure. The second root of (8.98) is obtained using a singular perturbation method. The small parameter Γ multiplies the highest order term. Let

$$y = \frac{w}{\Gamma^\nu}, \quad \nu > 0. \tag{8.116}$$

Then (8.98) becomes

$$\Gamma^{1-6\nu}w^6 - \Gamma^{-5\nu}w^5 + 5\Gamma^{1-3\nu}w^3 + 7(1-2\lambda)\Gamma^{-2}w^2 + 22\lambda = 0. \tag{8.117}$$

To obtain the second root the highest degree term must be retained. Since $\nu > 0$, the dominant terms are the first two terms. The dominant terms balance provided

$$1 - 6\nu = -5\nu \text{ or } \nu = 1. \tag{8.118}$$

Equation (8.117) becomes

$$w^6 - w^5 + 5\Gamma^3 w^3 + 7(1-2\lambda)\Gamma^3 w^2 + 22\lambda\Gamma^5 = 0. \tag{8.119}$$

Expand $w(\lambda, \Gamma)$ in powers of Γ:

$$\begin{aligned} w(\lambda,\Gamma) &= w_0(\lambda) + \Gamma w_1(\lambda) + \Gamma^2 w_2(\lambda) + \Gamma^3 w_3(\lambda) \\ &+\Gamma^4 w_4(\lambda) + \Gamma^5 w_5(\lambda) + O(\Gamma^6). \end{aligned} \tag{8.120}$$

The anisotropic parameter, λ, enters (8.119) only at third order in Γ. Hence, as for an isotropic spherical shell,

$$w_0 = 1, \quad w_1 = 0, \quad w_2 = 0. \tag{8.121}$$

Thus (8.120) reduces to

$$w(\lambda, \Gamma) = 1 + \Gamma^3 w_3(\lambda) + \Gamma^4 w_4(\lambda) + \Gamma^5 w_5(\lambda) + O(\Gamma^6). \tag{8.122}$$

Substitute (8.122) into (8.119) :

$$\begin{aligned} &6(\Gamma^3 w_3 + \Gamma^4 w_4 + \Gamma^5 w_5) - 5(\Gamma^3 w_3 + \Gamma^4 w_4 + \Gamma^5 w_5) \\ &+5\Gamma^3 + 7(1-2\lambda)\Gamma^3 + 22\lambda\Gamma^5 + O(\Gamma^6) = 0. \end{aligned} \tag{8.123}$$

Equate the coefficients of the same power of Γ.

$$\Gamma^3 : \qquad w_3 = -2(6 - 7\lambda) \tag{8.124}$$

$$\Gamma^4 : \qquad w_4 = 0 \tag{8.125}$$

$$\Gamma^5 : \qquad w_5 = -22\lambda. \tag{8.126}$$

Hence

$$w(\lambda, \Gamma) = 1 - 2(6 - 7\lambda)\Gamma^3 - 22\lambda\Gamma^5 + O(\Gamma^6) \tag{8.127}$$

and therefore

$$y = \frac{1}{\Gamma} w = \frac{1}{\Gamma}\Big[1 - 2(6 - 7\lambda)\Gamma^3 - 22\lambda\Gamma^5 + O(\Gamma^6)\Big]. \tag{8.128}$$

Thus, since $x = y^{\frac{1}{2}}$,

$$x_{min} = \frac{1}{\sqrt{\Gamma}}\Big[1 - (6 - 7\lambda)\Gamma^3 - 11\lambda\Gamma^5 + O(\Gamma^6)\Big]. \tag{8.129}$$

Equation (8.129) reduces to (8.51) when $\lambda = 0$.

Equation (8.129) can be written as

$$x_{min} = \frac{1}{\sqrt{\Gamma}}\Big[1 - 6\Gamma^3 + \lambda(7 - 11\Gamma^2)\Gamma^3 + O(\Gamma^6)\Big]. \tag{8.130}$$

Thus if

$$\Gamma < \left(\frac{7}{11}\right)^{\frac{1}{2}} = 0.798 \tag{8.131}$$

then the anisotropy increases x_{min}. In general, (8.131) will be satisfied when snap buckling occurs because of condition (8.75). Since the anisotropy also decreases x_{max}, the range of the stretch x during which snap buckling is taking place is increased by the anisotropy.

By substituting x_{min} given by (8.130) into $P(x)$ given by (8.11) it can be shown that

$$P_{min} = 8\,\varepsilon\, C_1 \sqrt{\Gamma}\Big[1 - (1-\lambda)\,\Gamma^3 - \lambda\,\Gamma^5 + O(\Gamma^6)\Big], \tag{8.132}$$

as $\Gamma \to 0$. Equation (8.132) reduces to (8.53) for an isotropic spherical shell on setting $\lambda = 0$. It can be rewritten as

$$P_{min} = 8\,\varepsilon\, C_1 \sqrt{\Gamma}\Big[1 - \Gamma^3 + \lambda(1-\Gamma^2)\,\Gamma^3 + O(\Gamma^6)\Big] \tag{8.133}$$

and therefore if $\Gamma < 1$, which in general will be satisfied for snap buckling from condition (8.75), then the effect of the anisotropy is to increase the local minimum pressure in the snap buckling.

Equations (8.129) and (8.132) for x_{min} and P_{min} are perturbation expansions in Γ only. They are exact in λ unlike (8.114) and (8.115) which are dual perturbation expansions in Γ and λ.

8.5 Conclusions

Several radial transversely isotropic effects were found in the inflation of a thin-walled spherical shell. The maximum value of Γ for snap buckling to occur is increased and for given $\Gamma < 0.404$, snap buckling occurs provided λ is sufficiently large. The stretch at the local maximum of the inflation pressure is decreased and the stretch at the local minimum is increased. The range of stretch during which snap buckling occurs is therefore increased. The local maximum in the inflation pressure, P_{max}, is increased if $\Gamma < 0.165$ and decreased if $\Gamma > 0.165$. The local minimum of the inflation pressure, P_{min}, is increased. Thus if $\Gamma > 0.165$ the change in pressure during snap buckling is decreased.

The lowest order perturbation solutions

$$x_{min} = \frac{1}{\sqrt{\Gamma}} \ , \quad P_{min} = 8\,\varepsilon\, C_1 \sqrt{\Gamma}, \tag{8.134}$$

remain valid for a radial transversely isotropic spherical shell. The next terms are of order $\Gamma^{\frac{5}{2}}$ and order $\Gamma^{\frac{7}{2}}$ as for an isotropic spherical shell. The approximate solutions (8.134) are simple results and may be useful.

The radial transversely isotropic effects for the inflation of a thin-walled spherical shell are independent of the thickness parameter ε and depend only on λ and Γ. Radial transversely isotropic effects can exist in thin-walled spherical shells. They do not exist only in thick-walled shells.

Chapter 9

Conclusions

The extension of the constitutive equation for a transversely isotropic hyperelastic material from the special case in which the anisotropic director $\boldsymbol{h}$ is along the $\boldsymbol{i}_3$-direction to the general case in which $\boldsymbol{h}$ is unspecified makes the formulation of problems more systematic. The problems of radial oscillations in radial and tangential transversely isotropic cylindrical tubes and spherical shells were formulated in a unified way.

It was possible to make some progress analytically on the problem of radial oscillations in a thick-walled cylindrical tube. This was done by considering the effective potential which determined when the oscillation is bounded and the end points of the oscillation. Approximate analytical results were also obtained in the form of upper and lower bounds on the period. The corresponding oscillations which are in the form of a nonlinear superposition were obtained.

More analytical progress could be made on radial oscillations in a thin-walled cylindrical tube or spherical shell. For radial and tangential transversely isotropic cylindrical tubes a Lie point symmetry exists provided the net applied surface pressure is of a certain form. For inflation of a thin-walled spherical shell, polynomials of degree four, six and eight occured. Analytical progress could be made using singular perturbation methods.

Anisotropic effects on the amplitude of radial oscillations in a cylindrical tube were found. For the special time dependent net applied surface pressure acting on a thin-walled radial transversely isotropic tube the amplitude of the oscillations was reduced and the departure from the unstrained state when there was steady growth was also reduced. The effect of the radial transverse isotropy was to make the tube stiffer.

For Heaviside step loading, in which the net applied surface pressure is constant, the amplitude of the oscillation, in both compression and extension, was reduced in radial and tangential transversely isotropic cylindrical tubes. The effect of the anisotropy, both radial and tangential, was again to make the tube stiffer. For free oscillations in a thin-walled cylindrical tube, results for the amplitude were obtained provided either the initial velocity $v_0 = 0$ or, when $v_0 \neq 0$, terms of order ε^2 in the anisotropy can be neglected. For an isotropic tube, $ab = 1$, where a and b are the minimum and maximum values of the radial coordinate, respectively. For a radial transversely isotropic tube, $ab > 1$, while for a tangential transversely isotropic tube, $ab < 1$. Thus if the three tubes have the same compression a, for an isotropic tube,

$b = \frac{1}{a}$, for a radial transversely isotropic tube $b > \frac{1}{a}$ and the range of oscillation $a \leq x \leq b$ is increased while for a tangential transversely isotropic tube, $b < \frac{1}{a}$ and the range of oscillation is decreased. In comparison if the three tubes have the same extension b, for an isotropic tube $a = \frac{1}{b}$, for a radial transversely isotropic tube $a > \frac{1}{b}$ and the range of oscillation is reduced while for a tangential transversely isotropic tube $a < \frac{1}{b}$ and the range of oscillation is increased. It remains an open question whether these results are valid when $v_0 \neq 0$ for all values of ε.

Results were obtained for the effect of the anisotropy on the period of the oscillations. For the special time dependent net surface pressure applied to a thin-walled radial transversely isotropic cylindrical tube the effect of the anisotropy was to reduce the period of the oscillation. For Heaviside step loading of thick-walled radial and tangential transversely isotropic cylindrical tubes the upper and lower bounds on the period were reduced. This suggests that the period of oscillation is reduced by the anisotropy. For free oscillations in thin-walled radial and tangential transversely isotropic tubes with $v_0 \neq 0$ but terms of order ε^2 neglected, the period of oscillation is less than in an isotropic tube. This was also established for a radial transversely isotropic tube for all values of ε if $v_0 = 0$. These results all indicate that in general the effect of radial and tangential transverse isotropy is to reduce the period of radial oscillations.

It is more difficult to derive analytical results for radial oscillations in a spherical shell than in a cylindrical tube because the principal invariants, I_1 and I_2, are unequal in a spherical shell. However, anisotropic effects were found for the inflation of a radial transversely isotropic spherical shell. The range of values of Γ, the ratio of the Mooney-Rivlin constants, for which snap buckling can occur is increased. The range of stretch over which snap buckling takes place is increased and if $\Gamma > 0.165$ the change in pressure during snap buckling is decreased.

Anisotropic effects in radial transverse isotropic cylindrical tubes and spherical shells are not confined to thick-walled tubes and shells as may be expected. They occur in thin-walled tubes and spherical shells. The differential equation for radial oscillations in a thin-walled cylindrical tube was derived by an expansion in powers of the thickness of the tube but the differential equation obtained did not depend on the thickness. The special pressure solution satisfied the condition, $D_1 = D_2$, which is independent of the thickness. For inflation of a spherical shell, the magnitude of the inflation pressure is proportional to the thickness of the shell. However, the nonlinear effects in snap buckling are determined by the parameter Γ and the anisotropic effects by the parameter λ both of which are independent of the thickness of the shell.

Appendix A

Time derivative of a material volume element

In this appendix the derivation given by Mason(1996) of the result

$$\big(d\tau\big)^{\cdot} = G^{ik}\dot{\gamma_{ik}}d\tau \tag{A.1}$$

is outlined. The overhead dot denotes the material time derivative $\frac{D}{Dt}$ and γ_{ik} is the strain tensor,

$$\gamma_{ik} = \frac{1}{2}(G_{ik} - g_{ik}). \tag{A.2}$$

From (2.84),

$$d\tau = \frac{\sqrt{G}}{\sqrt{g}}d\tau_0, \tag{A.3}$$

where

$$g = \det[g_{ik}], \quad G = \det[G_{ik}]. \tag{A.4}$$

But since g_{ik} is the metric tensor in the undeformed body B_0 and $d\tau_0$ is the volume element in B_0 ,

$$\dot{g} = 0, \quad (d\tau_0)^{\cdot} = 0. \tag{A.5}$$

Hence

$$(d\tau)^{\cdot} = \frac{1}{2\sqrt{Gg}}\,\dot{G}\,d\tau_0. \tag{A.6}$$

But

$$G = e^{rst}G_{r1}G_{s2}G_{t3}, \tag{A.7}$$

where e^{rst} is the permutation symbol and therefore

$$\dot{G} = e^{rst}\dot{G}_{r1}G_{s2}G_{t3} + e^{rst}G_{r1}\dot{G}_{s2}G_{t3} + e^{rst}G_{r1}G_{s2}\dot{G}_{t3}. \tag{A.8}$$

But

$$\dot{G}_{r1} = 2\dot{\gamma}_{r1} = 2\delta^a_r\dot{\gamma}_{a1} = 2G_{rc}G^{ca}\dot{\gamma}_{a1}. \tag{A.9}$$

Thus

$$\begin{aligned} e^{rst}\dot{G}_{r1}G_{s2}G_{t3} &= 2G^{ca}\dot{\gamma}_{a1}e^{rst}G_{rc}G_{s2}G_{t3} \\ &= 2G^{ca}\dot{\gamma}_{a1}e_{c23}G \\ &= 2G^{1a}\dot{\gamma}_{a1}G, \end{aligned} \tag{A.10}$$

where we used

$$e^{rst}G_{ra}G_{sb}G_{tc} = e_{abc}G. \tag{A.11}$$

Also,

$$e^{rst}G_{r1}\dot{G}_{s2}G_{t3} = 2G^{2a}\dot{\gamma}_{a2}G, \tag{A.12}$$

$$e^{rst}G_{r1}G_{s2}\dot{G}_{t3} = 2G^{3a}\dot{\gamma}_{a3}G. \tag{A.13}$$

By substituting (A.11), (A.12) and (A.13) into (A.8) we obtain

$$\dot{G} = 2G^{ab}\dot{\gamma}_{ab}G. \tag{A.14}$$

Using (A.14) for $\dot{G}$, equation (A.6) becomes

$$(d\tau)^{\cdot} = G^{ab}\dot{\gamma}_{ab}\frac{\sqrt{G}}{\sqrt{g}}d\tau_0 \tag{A.15}$$

and finally using (A.3), the result (A.1) is obtained.

References

Adam, A.A. and Mahomed, F.M., 2002. *Integration of ordinary differential equations via nonlocal symmetries*, Nonlinear Dynamics, **30**, 267-275.

Alexander, H., 1971. *Tensile instability of initially spherical balloons*, Inter. J. Engineering Science, **9**, 151-162.

Atkin, R.J. and Fox, N., 1980. *An Introduction to the Theory of Elasticity*, Longman: London, Ch. 1.

Barnard, S. and Child, J.M., 1936. *Higher Algebra*, MacMillan and Co. : London, pp. 88-89 and 186-197.

Beatty, M.F., 1987. *Topics in finite elasticity : hyperelasticity of rubber, elastomers and biological tissues with examples*, Appl. Mech. Rev, **40**, 1699-1734.

Bluman, G.W. and Kumei, S., 1989. *Symmetries of Differential Equations*, Springer-Verlag : New York.

Calderer, C., 1983. *The dynamical behaviour of nonlinear elastic spherical shells*, J. Elasticity, **13**, 17-47.

Ericksen, J.L. and Rivlin, R.S., 1954. *Large elastic deformations of homogeneous anisotropic materials*, J. Rat. Mech. and Analysis, **3**, 281-301.

Ermakov, V., 1880. *Second order equations. Conditions of complete integrability.* Univ. Isz. Kiev, Series III, **9**, 1. Translated by Hann, A.O.

Gillespie, R.P., 1959. *Integration*, Oliver and Boyd : Edinburgh, p. 106.

Green, A.E. and Zerna, W., 1968. *Theoretical Elasticity*, Clarendon Press : Oxford, Chs. 1,2 and 3.

Green, A.E. and Adkins, J.E., 1970. *Large Elastic Deformations*, Clarendon Press : Oxford, Chs. 1,2 and 7.

Guo, Z.H. and Solecki, R., 1963. *Free and forced finite-amplitude oscillations of an elastic thick-walled hollow sphere made of incompressible material*, Arch. Mech. Stos., **15**, 427-433.

Holzapfel, G.A., 2000. *Nonlinear Solid Mechanics*, John Wiley and Sons : New York, pp. 239-242.

Huilgol, R.R., 1967. *Finite amplitude oscillations in curvilinearly aeolotropic elastic cylinders*, Quart. App. Math., **25**, 293-298.

Ibragimov, N.H., 1999. *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley and Sons Ltd : Chichester, Ch. 5.

Ibragimov, N.H. and Anderson, R.L., 1994. *Lie theory of differential equations*, In : Lie Group Analysis of Differential Equations, Vol. 1, Symmetries, Exact Solutions and Conservation Laws, ed. Ibragimov, N.H., CRC Press, Boca Raton, pp 7-14.

Ibragimov, N.H. and Mahomed, F.M., 1996. *Ordinary differential equations*, In : Lie Group Analysis of Differential Equations, Vol. 3, New Trends in Theoretical Developments and Computational Methods, ed. Ibragimov, N.H., CRC Press, Boca Raton, pp 205-207.

Knowles, J.K., 1960. *Large amplitude oscillations of a tube of incompressible elastic material*, Quart. App. Math., **18**, 71-77.

Knowles, J.K., 1962. *On a class of oscillations in the finite deformation theory of elasticity*, J. App. Mech., **29**, 283-286.

Mason, D.P., 1996. *The equations of finite elasticity*, In : Proceedings of the Differential Equations and Chaos Workshop, New Age International : New Delhi, pp. 305-351.

Mason, D.P. and Roussos, N., 2000. *Lie symmetry analysis and approximate solutions for nonlinear radial oscillations of an incompressible Mooney-Rivlin cylindrical tube*, J. Math. Anal. Appl., **245**, 346-392.

Nayfeh, A.H., 1981. *Introduction to Perturbation Techniques*, John Wiley and Sons: New York, Ch. 2.

Needleman, A., 1977. *Inflation of spherical rubber balloons*, Inter. J. Solids Structures, **13**, 409-421.

Ogden, R.W., 1984. *Non-linear Elastic Deformations*, John Wiley and Sons : New York, pp. 283-287.

Olver, P.J., 1986. *Applications of Lie Groups to Differential Equations*, Springer-Verlag : New York.

Pinney, E., 1950. *The non-linear differential equation* $y'' + p(x)y + cy^{-3} = 0$, Proc. Am. Math. Soc., **1**, 681.

Polyanin, A.D. and Zaitsev, V.F., 1995. *Handbook of Exact Solutions of Ordinary Differential Equations*, CRC Press : Boca Raton, pp. 29-57.

Rogers, C. and Ames, W.F., 1989. *Nonlinear Boundary Value Problems in Science and Engineering*, Academic Press : New York, pp. 147-169 and 387-389.

Roussos, N. and Mason, D.P., 1998. *Non-linear radial oscillations of a thin-walled double-layer hyperelastic cylindrical tube*, Int. J. Non-Linear Mech., **33**, 507-530.

Roussos, N., Mason D.P. and Hill, D.L., 2002. *On non-linear radial oscillations of an incompressible hyperelastic spherical shell*, Math. and Mech. of Solids, **7**, 67-85.

Roussos, N. and Mason D.P., 2005. *Radial oscillations of thin cylindrical tubes and spherical shells: investigation of Lie point symmetries for arbitrary strain-energy functions*, Commun. in Nonlinear Sci. and Numerical Simulation, **10**, 139-150.

Shahinpoor, M., 1974. *Exact solution to finite amplitude oscillations of an anisotropic thin rubber tube*, J. Acoust. Soc. Am., **56**, 477-480.

Shahinpoor, M. and Nowinski, J.L., 1971. *Exact solution to the problem of forced large amplitude radial oscillations of a thin hyperelastic tube*, Int. J. Non-Linear Mech., **6**, 193-207.

Wang, C.C., 1965-1966. *On the radial oscillations of a spherical thin shell in finite elasticity theory*, Quart. App. Math., **23**, 270-274.