Semi-analytic valuation of European and Forward Starting Options within an affine framework

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, in fulfillment of the requirements of the degree of Master of Science.

Shamim Afshani

August 4, 2008

Declaration

I declare that this is my own, unaided work. It is being submitted for the Degree of Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

(Signature)

(Date)

Acknowledgements

Thanks to Dr. Graeme West for his supervision, Roelof Sheppard for the many hours spent discussing ideas, my parents for all their effort and Parastu for putting up with me.

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

- ALBERT EINSTEIN (1879-1955)

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Chapter 1

Introduction

We consider the time t_0 value of Forward Starting Call Options with payoffs of the form $\left[\frac{S_{T_2}}{S_{T_1}} - K\right]^+$ and $[S_{T_2} - KS_{T_1}]^+$ where $t_0 < T_1 < T_2$ with T_1 the determination date, T_2 the maturity date, S the underlying stock price and K the strike price. We refer to these as $\%$ and $\$$ type payoffs respectively, with $\%$ type options more commonly traded. We restrict our attention to a specific affine framework for the state variables S and V - the underlying process and the variance thereof. The affine framework is defined such that the natural logarithm of the conditional joint characteristic function for $X = \ln S$ and V is a linear function of these state variables. The characteristic function is the Fourier transform of the corresponding density function. Within this framework, we focus on the Stochastic Volatility Jump Jump (SVJJ) model, as presented in Duffie et al. [2000]. The model is driven by correlated Brownian motions and a single Poisson process which yields simultaneous jumps in X and V where correlated jump size distributions are assigned to the respective jumps. These dynamics are assumed under a specific risk-neutral measure.
The semi-analytic valuation of such options requires us to first consider the corresponding results for European Options, focussing on any technical issues that arise when numerically evaluating the pricing formulae.

From the efforts of Carr and Madan [1999], Lewis [2001] and Lee [2005], we have semi-analytic pricing formulae for European Options in terms of a damping parameter α. The role of this parameter is to (effectively) specify the contour of integration in the complex plane when obtaining option prices by means of complex Fourier inversion. The conditional characteristic function for X features in these formulae. In subsection 2.1.1 we derive the semi-analytic formula for a European Call Option. Making use of alternative versions of the option's payoff function and the value of such an option as presented in Bakshi and Madan [2000], we avoid having to directly introduce Residue Theory into the derivation. In subsection 2.1.2 we obtain the corresponding pricing formulae for $\%$ and $\$$ type Forward Starting Call Options, making use of the insight provided in Hong [2004]. These formulae feature what we refer to as the conditional forward $(\%, \$)$ characteristic functions for X. In subsection 2.1.3 we re-iterate the definition of this affine framework and point out how the conditional joint characteristic function may be used to determine the conditional characteristic and forward $(\%, \$)$ characteristic functions. In section

2.2 we derive the analytic form of the conditional joint characteristic function for X and V. In subsection 2.3.1 we highlight the fact that there is a discrepancy between forward implied volatilities obtained from $\%$ and $\$$ type Forward Starting Options. This discrepancy may be attributed to a shift between the risk-neutral and the stock price measure over the period $(t_0, T_1]$. In subsection 2.3.2 we make use of the above mentioned pricing formula from Bakshi and Madan [2000] to infer the effect of this shift in measure on the dynamics of the model. In section 2.4 we present the form of the conditional characteristic and forward $(\%, \$)$ characteristic functions allowing for piecewise constant, time-dependent parameters.

Numerical evaluation of these semi-analytic pricing formulae requires us to consider several technical issues. We begin section 3.1 by highlighting several useful results regarding the moment generating function for X. In subsection 3.1.1 we consider the issue of discontinuities arising from the complex logarithm featured in the conditional characteristic function for X. Within the context of the time-homogenous Heston model, several authors have considered proving the conjecture that for an appropriate representation of the conditional characteristic function, discontinuities cannot arise. The issue has been laid to rest in Lord and Kahl [2008]. In subsection 3.1.2 we address a potential discontinuity noted in Albrecher et al. [2007] and show that the issue may be ignored. In section 3.2 we consider the existence of these pricing formulae. This may be determined in terms of a valid range for the damping parameter α. For the time-homogenous case, the issue is considered in Lee [2005], Lord and Kahl [2007] and Lord and Kahl [2008] (excluding the case of jumps in the variance process). In subsections 3.2.1 we address the issue for the diffusion component of the SVJJ model where we allow for piecewise constant, time-dependent parameters. In subsections 3.2.2 and 3.2.3 we derive bounds for this valid range of α assuming time-homogenous parameters. This yields the result for the Heston model. In Lord and Kahl [2007] a result from Andersen and Piterbarg [2007] is used to determine this range. In subsection 3.2.4 we determine the valid range of α for the jump component of the SVJJ model allowing for piecewise constant, time-dependent parameters. It is specifically the presence of jumps in the variance process that requires us to consider the jump component in this context. In subsection 3.2.5 we determine the valid range of α for Forward Starting Options, making use of the preceding results from this section. Having obtained this valid range of α, we follow the novel approach of Lord and Kahl [2007] in section 3.3 to determine the optimal value of α for which the pricing integrand is neither too oscillatory nor too peaked. This approach is not complicated by the presence of piecewise constant parameters as long as the valid range of α has been determined appropriately. In section 3.4 we present the approach of Kahl and Jackel [2005] to avoid having to truncate the domain of integration and obtain the corresponding results allowing for piecewise constant parameters and for Forward Starting Options. In section 3.5 we return to the issue of complex discontinuities and address the issue in the context of the Heston model, for European and Forward Starting pricing formulae allowing for piecewise constant, time-dependent parameters. We prove that branch cutting is not an issue for $-1 \leq \alpha \leq 0$ (with parameter restrictions only for the case $\alpha = 0$).

Having obtained pricing formulae within this affine framework, we introduce the SABR model (or the SABR approximation) in section 4.1. In subsection 4.1.1 we motivate approximate semi-analytic pricing formulae for $\%$ type Forward Starting Options. In subsection 4.1.2 we highlight the complications that arise when attempting to obtain similar results for $\$$ type Forward Starting Options. In subsection 4.1.3 we briefly discuss the fact that consistent pricing of Forward Starting Options with determination date

T_1 and maturity date T_2, where the model is separately calibrated to the market prices of T_1 and T_2 maturity European Options, would require us to make use of the Dynamic SABR model as the SABR model yields maturity specific constant parameters. This complication highlights a merit of the affine framework - the analytic conditional joint characteristic function allows us to introduce piecewise constant, time-dependent parameters into the semi-analytic pricing formulae, a result that we find has also been documented in Mikhailov and Nogel [2005] and more recently in Elices [2007]. In section 4.2 we digress to obtain an analytic forward parameter for a special case of the square root CEV model. The term forward refers to the constant parameter value, as seen at time t_0, that should apply over the period $(T_1, T_2]$, for example.

Finally, in section 4.3 we present an application of the methods of this thesis. In Piterbarg [2005], approximate forward parameters are obtained for two time-dependent parameters in an (uncorrelated) stochastic volatility model. For one of these parameters, the result is specifically obtained for an at-the-money option. We show that for this parameter an exact result is available (assuming piecewise constant, time-dependence) and for an at-the-money option, can be evaluated efficiently. For alternative strike levels, an exact result is still available but we are required to make use of the methods described, to determine the valid range of α and the optimal value therein. Our approach illustrates that corresponding results may be obtained for the entire parameter set of the SVJJ model, as an example of an affine model.

Chapter 2

Semi-analytic pricing formulae

The Black-Scholes model yields a closed form solution for the value of a European Option at time t_0 with maturity T and strike K. This can be derived from the analytic form of the conditional density function, for the terminal value of the underlying asset S_T given S_{t_0}. As an alternative to this map from density function to option price, one can obtain the value, albeit in semi-analytic form, by mapping from the corresponding analytic conditional characteristic function where the characteristic function is the Fourier transform of the density function. The solution obtained from the latter approach is labeled as semi-analytic as the resulting formula must be evaluated numerically and so, in this case, is not preferable. However, when relaxing the assumptions of the model, specifically allowing for stochastic (correlated) volatility and jumps (in the now coupled stochastic process), the characteristic function approach remains valid (for judiciously specified dynamics) while the density function approach cannot hope to.

Assuming an analytic conditional characteristic function, we derive the form of this semi-analytic value for a European Call Option and apply the same approach to the valuation of Forward Starting Options.

2.1 Semi-analytic pricing formulae

To re-iterate, for $X_T = \ln S_T$ and $k = \ln K$, the value of an option at time t_0 whose payoff $\bar{\Pi}_{t_0,T}(X_T, k)$ depends only on the terminal value X_T, may be expressed as

$$\Pi_{t_0,T}(k) \quad = \quad e^{-r\tau} \int_{-\infty}^{\infty} \bar{\Pi}_{t_0,T}(x, k) f_{t_0,T}(x|\bar{x}_{t_0}) dx \tag{2.1}$$

where the value is an explicit function of the conditional risk neutral density function $f_{t_0,T}(x|\bar{x}_{t_0})$ with $\bar{x}_{t_0}$ the vector of state variables at t_0, $\tau = T - t_0$ and r the constant discount rate that applies over the period $(t_0, T]$.

Alternatively, we can write

$$\Pi_{t_0,T}(k) = \frac{1}{2\pi}\int_{-\infty-i\alpha}^{\infty-i\alpha} e^{-i\bar{z}k}\Psi_{t_0,T}(\bar{z})d\bar{z} \tag{2.2}$$

$$\Psi_{t_0,T}(\bar{z}) = \int_{-\infty}^{\infty} e^{i\bar{z}k}\Pi_{t_0,T}(k)dk \tag{2.3}$$

where $\Psi_{t_0,T}(\bar{z})$ is the complex Fourier transform of $\Pi_{t_0,T}(k)$ and $\bar{z} := u - i\alpha$ with $u \in \mathbb{R}$. The value $-\alpha$ specifies the contour of integration in the complex plane and must be chosen from within a valid range. Inserting equation (2.1) into equation (2.3), we attempt to express $\Psi_{t_0,T}(\bar{z})$ as an explicit, analytic function of $\Phi_{t_0,T}(\bar{z})$, the conditional characteristic function of X_T, i.e.

$$\Phi_{t_0,T}(\bar{z}) := \mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{i\bar{z}X_T}|\bar{x}_{t_0}\right] \tag{2.4}$$

$$= \int_{-\infty}^{\infty} e^{i\bar{z}x} f_{t_0,T}(x|\bar{x}_{t_0})dx \tag{2.5}$$

If an analytic expression is available for $\Phi_{t_0,T}(\bar{z})$ then inserting equation (2.3) into equation (2.2) yields a semi-analytic result for the option price as we need only to perform a one dimensional integration to obtain the final result. This approach is followed in subsections 2.1.1 and 2.1.2. The valid range of α is defined such that the moment generating function $\Phi_{t_0,T}(-i[\alpha+1])$ exists. We elaborate on this point in subsection 2.1.1.

Switching from the complex variable $\bar{z}$ to the real variable u, equation (2.2) may be simplified down to

$$\Pi_{t_0,T}(k) = \frac{1}{\pi}\int_0^{\infty} \text{Re}\left[e^{-i(u-i\alpha)k}\Psi_{t_0,T}(u,\alpha)\right] du \tag{2.6}$$

To see this, consider that

$$\begin{aligned}
\Psi_{t_0,T}(u,\alpha) &= \int_{-\infty}^{\infty} e^{i(u-i\alpha)k}\Pi_{t_0,T}(k)dk \\
&= \int_{-\infty}^{\infty} e^{\alpha k}\Pi_{t_0,T}(k)\cos(uk)dk + i\int_{-\infty}^{\infty} e^{\alpha k}\Pi_{t_0,T}(k)\sin(uk)dk
\end{aligned} \tag{2.7}$$

Since cosine and sine are even and odd functions respectively, we see that Re $\left[\Psi_{t_0,T}(u,\alpha)\right]$ is even in u, while Im $\left[\Psi_{t_0,T}(u,\alpha)\right]$ is odd in u where Re $[Z]$ and Im $[Z]$ refer to the real and imaginary parts of $Z \in \mathbb{C}$, respectively. Furthermore, we can write

$$\Pi_{t_0,T}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(u-i\alpha)k}\Psi_{t_0,T}(u,\alpha)du \tag{2.8}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha k}\left(\text{Re}\left[\Psi_{t_0,T}(u,\alpha)\right]\cos(-uk) - \text{Im}\left[\Psi_{t_0,T}(u,\alpha)\right]\sin(-uk)\right)du \tag{2.9}$$

$$+ \; i\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha k}\left(\text{Re}\left[\Psi_{t_0,T}(u,\alpha)\right]\sin(-uk) + \text{Im}\left[\Psi_{t_0,T}(u,\alpha)\right]\cos(-uk)\right)du \tag{2.10}$$

From equation (2.10), we see that the imaginary integrand is odd in u and since the domain of integration is symmetric about the point $u = 0$, the integral disappears. From equation (2.9), we see that the real integrand is even in u. This gives us equation (2.6).

2.1.1 European Call Options

From the work of Carr and Madan [1999], Lewis [2001] and Lee [2005], we have the semi-analytic option value

$$\Pi^{\mathrm{C}}_{t_0,T}(k) = e^{-r\tau} R^{\mathrm{C}}_{t_0,T}(\alpha) + \frac{e^{-r\tau}}{\pi} \int_0^\infty \mathrm{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right) \Phi_{t_0,T}(u-i[\alpha+1])\right] du \tag{2.11}$$

where the superscript C specifies the Call value and

$$R^{\mathrm{C}}_{t_0,T}(\alpha) \quad = \quad \mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha\leq 0]} - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha=0]} - e^k\mathbb{I}_{[\alpha\leq -1]} + \frac{1}{2}e^k\mathbb{I}_{[\alpha=-1]} \tag{2.12}$$

where $\mathbb{I}$ is an indicator function. The parameter α is chosen from within the interval $(\alpha^{\min}, \alpha^{\max})$ such that $\Phi_{t_0,T}(-i[\alpha+1])$ exists and hence the integrand exists, subject to points of singularity at $u=0$ and $\alpha=-1,0$.

Proof: Making use of alternative versions of a European Call's payoff function and a result of Bakshi and Madan [2000], we verify the pricing formulae for European Options, as presented in Lee [2005] Theorem 5, without having to explicitly appeal to Residue Theory.

We can express the payoff of a European Call Option $\bar{\Pi}^{\mathrm{C}}_{t_0,T}(X_T,k)$ in four distinct forms

$$\textbf{Case 1:}\quad \max\left[e^{X_T}-e^k,0\right] \tag{2.13}$$

$$\textbf{Case 2:}\quad e^{X_T}-e^k+\max\left[e^k-e^{X_T},0\right] \tag{2.14}$$

$$\textbf{Case 3:}\quad e^{X_T}-\min\left[e^{X_T},e^k\right] \tag{2.15}$$

$$\textbf{Case 4:}\quad e^{X_T}\mathbb{I}_{[X_T>k]}-e^k\mathbb{I}_{[X_T>k]} \tag{2.16}$$

From these cases, we derive the result.

Case 1: Working from $\bar{\Pi}^{\mathrm{C}}_{t_0,T}(X_T,k) = \max\left[e^{X_T}-e^k,0\right]$, we have

$$\begin{aligned}
\Psi^{\mathrm{C}}_{t_0,T}(\bar{z}) &= e^{-r\tau}\int_{-\infty}^{\infty} e^{i\bar{z}k}\int_{-\infty}^{\infty}\max\left[e^x-e^k,0\right] f_{t_0,T}(x|\bar{x}_{t_0})dxdk && (2.17)\\
&= e^{-r\tau}\int_{-\infty}^{\infty} f_{t_0,T}(x|\bar{x}_{t_0})\int_{-\infty}^{x}\left[e^{i\bar{z}k+x}-e^{i(\bar{z}-i)k}\right]dkdx && (2.18)\\
&= e^{-r\tau}\int_{-\infty}^{\infty} f_{t_0,T}(x|\bar{x}_{t_0})\left[\frac{e^{i\bar{z}k+x}}{i\bar{z}}\Big|_{-\infty}^{x} - \frac{e^{i(\bar{z}-i)k}}{i(\bar{z}-i)}\Big|_{-\infty}^{x}\right]dx && (2.19)\\
&= e^{-r\tau}\left(\frac{1}{i\bar{z}}-\frac{1}{i(\bar{z}-i)}\right)\int_{-\infty}^{\infty} e^{i(\bar{z}-i)x} f_{t_0,T}(x|\bar{x}_{t_0})dx && (2.20)\\
&= e^{-r\tau}\left(\frac{1}{i\bar{z}}-\frac{1}{i(\bar{z}-i)}\right)\Phi_{t_0,T}(\bar{z}-i) && (2.21)
\end{aligned}$$

where the integration leading up to equation (2.20) is valid only for $-\mathrm{Im}\left[\bar{z}\right] > 0$. Hence, for $\alpha > 0$, we have

$$\Pi^{\mathrm{C}}_{t_0,T}(k) \quad = \quad \frac{e^{-r\tau}}{\pi}\int_0^\infty \mathrm{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right)\Phi_{t_0,T}(u-i[\alpha+1])\right]du \tag{2.22}$$

Case 2: Working from $\bar{\Pi}^{\mathrm{C}}_{t_0,T}(X_T,k) = e^{X_T}-e^k+\max\left[e^k-e^{X_T},0\right]$, we consider the Fourier transform of a European Put (P) Option

$$\Psi^{\text{P}}_{t_0,T}(\bar{z}) = e^{-r\tau}\int_{-\infty}^{\infty} e^{i\bar{z}k}\int_{-\infty}^{\infty}\max\left[e^k - e^x, 0\right] f_{t_0,T}(x|\bar{x}_{t_0})dxdk \tag{2.23}$$

$$= e^{-r\tau}\int_{-\infty}^{\infty} f_{t_0,T}(x|\bar{x}_{t_0})\left[\left.\frac{e^{i(\bar{z}-i)k}}{i(\bar{z}-i)}\right|_x^{\infty} - \left.\frac{e^{i\bar{z}k+x}}{i\bar{z}}\right|_x^{\infty}\right]dx \tag{2.24}$$

$$= e^{-r\tau}\left(\frac{1}{i\bar{z}} - \frac{1}{i(\bar{z}-i)}\right)\Phi_{t_0,T}(\bar{z}-i) \tag{2.25}$$

where the integration leading up to equation (2.25) is valid only for $-\text{Im}\,[\bar{z}] < -1$. Hence, for $\alpha < -1$, we have

$$\Pi^{\text{C}}_{t_0,T}(k) = e^{-r\tau}\left(\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right] - e^k\right) + \frac{e^{-r\tau}}{\pi}\int_0^{\infty}\text{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right)\Phi_{t_0,T}(u-i[\alpha+1])\right]du$$

Case 3: Working from $\bar{\Pi}^{\text{C}}_{t_0,T}(X_T,k) = e^{X_T} - \min\left[e^{X_T}, e^k\right]$, we consider the Fourier transform of a Covered Call (CC) Option

$$\Psi^{\text{CC}}_{t_0,T}(\bar{z}) = e^{-r\tau}\int_{-\infty}^{\infty} e^{i\bar{z}k}\int_{-\infty}^{\infty}\min\left[e^x, e^k\right] f_{t_0,T}(x|\bar{x}_{t_0})dxdk \tag{2.26}$$

$$= e^{-r\tau}\int_{-\infty}^{\infty} f_{t_0,T}(x|\bar{x}_{t_0})\left[\left.\frac{e^{i\bar{z}k+x}}{i\bar{z}}\right|_x^{\infty} + \left.\frac{e^{i(\bar{z}-i)k}}{i(\bar{z}-i)}\right|_{-\infty}^{x}\right]dx \tag{2.27}$$

$$= -e^{-r\tau}\left(\frac{1}{i\bar{z}} - \frac{1}{i(\bar{z}-i)}\right)\Phi_{t_0,T}(\bar{z}-i) \tag{2.28}$$

where the integration leading up to equation (2.28) is valid only for $-1 < -\text{Im}\,[\bar{z}] < 0$. Hence, for $-1 < \alpha < 0$, we have

$$\Pi^{\text{C}}_{t_0,T}(k) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right] + \frac{e^{-r\tau}}{\pi}\int_0^{\infty}\text{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right)\Phi_{t_0,T}(u-i[\alpha+1])\right]du$$

Case 4: Working from $\bar{\Pi}^{\text{C}}_{t_0,T}(X_T,k) = e^{X_T}\mathbb{I}_{[X_T>k]} - e^k\mathbb{I}_{[X_T>k]}$, we must value an Asset or Nothing Call (AC) Option and a Cash or Nothing Call (BC) Option where the latter has a notional value of e^k. Regarding the Asset or Nothing Call Option, we choose to express the payoff as $e^{X_T}\left(1 - \mathbb{I}_{[X_T\leq k]}\right)$ and so we consider the Fourier transform of an Asset or Nothing Put (AP) Option

$$\Psi^{\text{AP}}_{t_0,T}(\bar{z}) = e^{-r\tau}\int_{-\infty}^{\infty} e^{i\bar{z}k}\int_{-\infty}^{\infty} e^x\mathbb{I}_{[x<k]} f_{t_0,T}(x|\bar{x}_{t_0})dxdk \tag{2.29}$$

$$= e^{-r\tau}\int_{-\infty}^{\infty} f_{t_0,T}(x|\bar{x}_{t_0})\left.\frac{e^{i\bar{z}k+x}}{i\bar{z}}\right|_x^{\infty}dx \tag{2.30}$$

$$= -e^{-r\tau}\frac{1}{i\bar{z}}\Phi_{t_0,T}(\bar{z}-i) \tag{2.31}$$

where the integration leading up to equation (2.31) is valid only for $-\text{Im}\,[\bar{z}] < 0$ i.e. for $\alpha < 0$.

Regarding the Cash or Nothing Call Option, we consider the Fourier transform specifically for a notional of e^k

$$\Psi^{\text{BC}}_{t_0,T}(\bar{z}) = e^{-r\tau}\int_{-\infty}^{\infty} e^{i\bar{z}k}\int_{-\infty}^{\infty} e^k \mathbb{I}_{[x>k]} f_{t_0,T}(x|\bar{x}_{t_0})dxdk \tag{2.32}$$

$$= e^{-r\tau}\int_{-\infty}^{\infty} f_{t_0,T}(x|\bar{x}_{t_0})\left.\frac{e^{i(\bar{z}-i)k}}{i(\bar{z}-i)}\right|_{-\infty}^{x} dx \tag{2.33}$$

$$= e^{-r\tau}\frac{1}{i(\bar{z}-i)}\Phi_{t_0,T}(\bar{z}-i) \tag{2.34}$$

where the integration leading up to equation (2.34) is valid only for $-\text{Im}\left[\bar{z}\right] > -1$ i.e. for $\alpha > -1$.

Returning to the Asset or Nothing Call Option, we specify $\alpha = -1$ and obtain

$$\Pi^{\text{AC}}_{t_0,T}(k) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right] - \frac{1}{\pi}\int_{-\infty-i\alpha}^{\infty-i\alpha}\text{Re}\left[e^{-i\bar{z}k}\Psi^{\text{AP}}_{t_0,T}(\bar{z})\right]d\bar{z} \tag{2.35}$$

$$= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right] + \frac{e^{-r\tau}}{\pi}\int_{0}^{\infty}\text{Re}\left[e^{-i(u+i)k}\frac{1}{i(u+i)}\Phi_{t_0,T}(u)\right]du \tag{2.36}$$

while for the Cash or Nothing Call Option with notional e^k, we specify $\alpha = 0$ and obtain

$$e^k\Pi^{\text{BC}}_{t_0,T}(k) = \frac{1}{\pi}\int_{-\infty-i\alpha}^{\infty-i\alpha}\text{Re}\left[e^{-i\bar{z}k}\Psi^{\overline{\text{BC}}}_{t_0,T}(\bar{z})\right]d\bar{z} \tag{2.37}$$

$$= \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\frac{1}{i(u-i)}\Phi_{t_0,T}(u-i)du \tag{2.38}$$

From Bakshi and Madan [2000] Theorem 1 and Case 2, we have the following formula for the value of a European Call Option

$$\Pi^{\text{C}}_{t_0,T}(k) = e^{-r\tau}\left(\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right]P_1 - e^k P_2\right) \tag{2.39}$$

where

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right]P_1 = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right] + \frac{1}{\pi}\int_{0}^{\infty}\text{Re}\left[e^{-iuk}\frac{1}{iu}\Phi_{t_0,T}(u-i)\right]du \tag{2.40}$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty}\text{Re}\left[e^{-iuk}\frac{1}{iu}\Phi_{t_0,T}(u)\right]du \tag{2.41}$$

Since $\Pi^{\text{AC}}_{t_0,T}(k) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right]P_1$, we can write

$$\Pi^{\text{C}}_{t_0,T}(k) = \Pi^{\text{AC}}_{t_0,T}(k) - e^{-r\tau}e^k P_2$$

$$= e^{-r\tau}\left(\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right] - \frac{1}{2}e^k\right) + \frac{e^{-r\tau}}{\pi}\int_{0}^{\infty}\text{Re}\left[\frac{e^{-i(u+i)k}}{-u(u+i)}\Phi_{t_0,T}(u)\right]du \tag{2.42}$$

for $\alpha = -1$ where we have used the fact that $\frac{1}{i(u+i)} = \frac{1}{iu} - \frac{1}{u(u+i)}$.

Since $e^k\Pi^{\text{BC}}_{t_0,T}(k) = e^{-r\tau}e^k P_2$, we can write

$$\Pi^{\text{C}}_{t_0,T}(k) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right]P_1 - e^k\Pi^{\text{BC}}_{t_0,T}(k) \tag{2.43}$$

$$= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_T}|\bar{x}_{t_0}\right] + \frac{e^{-r\tau}}{\pi}\int_{0}^{\infty}\text{Re}\left[\frac{e^{-iuk}}{-u(u-i)}\Phi_{t_0,T}(u-i)\right]du \tag{2.44}$$

for $\alpha = 0$ where we have used the fact that $\frac{1}{i(u-i)} = \frac{1}{iu} + \frac{1}{u(u-i)}$.

Hence, from these cases we obtain the option value in terms of the specified value of α.

To determine the valid range of α, we refer to the integrand in equation (2.11), for which we have

$$\begin{aligned} & \left| \mathrm{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])} \right) \Phi_{t_0,T}(u-i[\alpha+1]) \right] \right| \\ \leq & \left(\frac{\left|e^{-i(u-i\alpha)k}\right|}{|(u-i\alpha)(u-i[\alpha+1])|} \right) \left| \Phi_{t_0,T}(u-i[\alpha+1]) \right| \end{aligned} \tag{2.45}$$

The inequality results from the fact that $|Z| = \sqrt{\mathrm{Re}\,[Z]^2 + \mathrm{Im}\,[Z]^2}$, for $Z \in \mathbb{C}$. The fraction in equation (2.45) exists at all points except $\bar{z} = 0, i$ i.e. at $u = 0$ and $\alpha = -1, 0$. Regarding the characteristic function Φ, Jensen's inequality gives us

$$\begin{aligned} \left| \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{i(u-i[\alpha+1])X_T} | \bar{x}_{t_0} \right] \right| & \leq \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\left| e^{i(u-i[\alpha+1])X_T} \right| | \bar{x}_{t_0} \right] \\ & = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{[\alpha+1]X_T} | \bar{x}_{t_0} \right] & (2.46) \\ & = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[S_T^{\alpha+1} | \bar{x}_{t_0} \right] & (2.47) \end{aligned}$$

The moment generating function for X_T in equation (2.46) exists for α in an open interval about the point -1 i.e. the interval $(\alpha^{\min}, \alpha^{\max})$. $\square$

From Lee [2005] Appendix A.2, $\alpha^{\min}$ is the largest value in the range $(-\infty, -1)$ and $\alpha^{\max}$ is the smallest value in the range $(0, \infty)$ such that $\mathbb{E}_{t_0}^{\mathbb{Q}} \left[S_T^{\alpha+1} | \bar{x}_{t_0} \right]$ no longer exists i.e. the valid range of α is free of any moment explosions in S_T.

Considering the range $\alpha \in [-1, 0]$, we have

$$\mathbb{E}_{t_0}^{\mathbb{Q}} \left[S_T^{\alpha+1} | \bar{x}_{t_0} \right] \leq \max\left(\mathbb{E}_{t_0}^{\mathbb{Q}} \left[S_T | \bar{x}_{t_0} \right], 1 \right) \tag{2.48}$$

since $S^{\alpha+1} \leq S$ for $S > 1$ and $S^{\alpha+1} \leq 1$ for $S \leq 1$. So assuming the forward price exists, we have $\alpha^{\min} < -1$ and $0 < \alpha^{\max}$.

2.1.2 Forward Starting Call Options

Proposition 1. *The time t_0 value of a % type Forward Starting Call Option with determination date T_1 and maturity date T_2 is*

$$\begin{aligned} \Pi_{t_0,T_1,T_2}^{\%C}(k) & = e^{-r\tau} R_{t_0,T_1,T_2}^{\%C}(\alpha) \\ & + \frac{e^{-r\tau}}{\pi} \int_0^\infty Re \left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])} \right) \Phi_{t_0,T_1,T_2}(-u+i[\alpha+1], u-i[\alpha+1]) \right] du & (2.49) \\ R_{t_0,T_1,T_2}^{\%C}(\alpha) & = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{X_{T_2}-X_{T_1}} | \bar{x}_{t_0} \right] \mathbb{I}_{[\alpha \leq 0]} - \frac{1}{2} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{X_{T_2}-X_{T_1}} | \bar{x}_{t_0} \right] \mathbb{I}_{[\alpha=0]} - e^k \mathbb{I}_{[\alpha \leq -1]} + \frac{1}{2} e^k \mathbb{I}_{[\alpha=-1]} & (2.50) \end{aligned}$$

The time t_0 value of a \$ type Forward Starting Call Option with determination date T_1 and maturity date T_2 is

$$\begin{aligned}\Pi^{\$C}_{t_0,T_1,T_2}(k) &= e^{-r\tau}R^{\$C}_{t_0,T_1,T_2}(\alpha)\\ &+ \frac{e^{-r\tau}}{\pi}\int_0^{\infty} Re\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right)\Phi_{t_0,T_1,T_2}(-u+i\alpha,u-i[\alpha+1])\right]du\end{aligned} \tag{2.51}$$

$$\begin{aligned}R^{\$C}_{t_0,T_1,T_2}(\alpha) &= \mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{T_2}}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha\leq 0]} - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{T_2}}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha=0]} - e^{k}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{T_1}}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha\leq -1]}\\ &+ \frac{1}{2}e^{k}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{T_1}}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha=-1]}\end{aligned} \tag{2.52}$$

Regarding these pricing formulae, the subscripts t_0, T_1, T_2 refer to the valuation, determination and maturity dates respectively, $\tau = T_2 - t_0$ and r is the constant discount rate that applies over the period $(t_0, T_2]$. Furthermore, $\Phi_{t_0,T_1,T_2}(z_{(\%,\$)}, z) := \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_{(\%,\$)}X_{T_1} + izX_{T_2}\right)|\bar{x}_{t_0}\right]$, $\bar{x}_{t_0}$ is the vector of state variables at t_0, $z := u - i[\alpha+1]$, $z_{\%} := -z$ and $z_{\$} := -(z+i)$ with $z_{(\%,\$)} := z_{\%}$ for a $\%$ type option and $z_{(\%,\$)} := z_{\$}$ for a \$ type option. We refer to $\Phi_{t_0,T_1,T_2}(z_{\%}, z)$ as the conditional forward $\%$ characteristic function and $\Phi_{t_0,T_1,T_2}(z_{\$}, z)$ as the conditional forward \$ characteristic function.
The parameter α is chosen from within the interval $(\alpha^{\min,(\%,\$)}, \alpha^{\max,(\%,\$)})$ such that $\Phi_{t_0,T_1,T_2}(z_{(\%,\$)}, z)\big|_{u=0}$ exists and hence the respective integrand exists, subject to points of singularity at $u = 0$ and $\alpha = -1, 0$.

Proof: Having obtained pricing formulae for European Options, we can determine the corresponding results for Forward Starting Options by expressing the value of the latter as

$$\Pi^{(\%,\$)}_{t_0,T_1,T_2}(k) = e^{-r_1\tau_1}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{-r_2\tau_2}\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\bar{\Pi}^{(\%,\$)}_{t_0,T_1,T_2}(k)|\bar{x}_{T_1}\right]|\bar{x}_{t_0}\right] \tag{2.53}$$

where $\bar{\Pi}^{(\%,\$)}_{t_0,T_1,T_2}(k)$ is the appropriate payoff function, the superscript $(\%, \$)$ specifies whether the option is of a $\%$ or of a \$ type. The term is split into two increments $\tau_1 = T_1 - t_0$ and $\tau_2 = T_2 - T_1$, allowing for piecewise constant parameters. Specifically, r_1 is the constant discount rate that applies over the period $(t_0, T_1]$ and r_2 is the constant discount rate that applies over the forward period $(T_1, T_2]$.

For a $\%$ type Call Option, we have

$$\bar{\Pi}^{\%C}_{t_0,T_1,T_2}(k) = \max\left[\frac{S_{T_2}}{S_{T_1}} - K, 0\right] = \max\left[e^{X_{T_2}-X_{T_1}} - e^{k}, 0\right] \tag{2.54}$$

while for a \$ type Call Option, we have

$$\bar{\Pi}^{\$C}_{t_0,T_1,T_2}(k) = \max\left[S_{T_2} - KS_{T_1}, 0\right] = \max\left[e^{X_{T_2}} - e^{k+X_{T_1}}, 0\right] \tag{2.55}$$

Rather than obtaining the risk neutral t_0 value of the payoff at maturity directly, we first obtain the T_1 value of the payoff. At T_1, the payoff depends only on an uncertain outcome at the maturity date and so is European in nature. We can, therefore, make use of the form of the pricing formula for a European Call Option to obtain the option value at this point. We then obtain the t_0 value of this T_1 value.

For a $\%$ type option, we have the value at T_1

$$
\begin{aligned}
& \Pi^{\%C}_{T_1,T_1,T_2}(k) \\
= \; & e^{-r_2\tau_2}\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\max\left[e^{X_{T_2}-X_{T_1}}-e^k,0\right]|\bar{x}_{T_1}\right] \\
= \; & e^{-r_2\tau_2}\left(\mathbb{E}^{\mathbb{Q}}_{T_1}\left[e^{X_{T_2}-X_{T_1}}|\bar{x}_{T_1}\right]\mathbb{I}_{[\alpha\le 0]}-\frac{1}{2}\mathbb{E}^{\mathbb{Q}}_{T_1}\left[e^{X_{T_2}-X_{T_1}}|\bar{x}_{T_1}\right]\mathbb{I}_{[\alpha=0]}-e^k\mathbb{I}_{[\alpha\le -1]}+\frac{1}{2}e^k\mathbb{I}_{[\alpha=-1]}\right) \\
+ \; & \frac{e^{-r_2\tau_2}}{\pi}\int_0^\infty \text{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right)\mathbb{E}^{\mathbb{Q}}_{T_1}\left[e^{i(u-i[\alpha+1])(X_{T_2}-X_{T_1})}|\bar{x}_{T_1}\right]\right]du
\end{aligned}
\tag{2.56}
$$

and at t_0, the value is

$$
\begin{aligned}
& \Pi^{\%C}_{t_0,T_1,T_2}(k) \\
= \; & e^{-r_1\tau_1}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\Pi^{\%C}_{T_1,T_1,T_2}(k)|\bar{x}_{t_0}\right] \\
= \; & e^{-(r_1\tau_1+r_2\tau_2)}R^{\%C}_{t_0,T_1,T_2}(\alpha) \\
+ \; & \frac{e^{-(r_1\tau_1+r_2\tau_2)}}{\pi}\int_0^\infty \text{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right)\Phi_{t_0,T_1,T_2}(-u+i[\alpha+1],u-i[\alpha+1])\right]du
\end{aligned}
\tag{2.57}
$$

where

$$
R^{\%C}_{t_0,T_1,T_2}(\alpha) = \mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{T_2}-X_{T_1}}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha\le 0]}-\frac{1}{2}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{T_2}-X_{T_1}}|\bar{x}_{t_0}\right]\mathbb{I}_{[\alpha=0]}-e^k\mathbb{I}_{[\alpha\le -1]}+\frac{1}{2}e^k\mathbb{I}_{[\alpha=-1]}
\tag{2.58}
$$

For a \$ type option, we express the payoff in terms of the payoff of $\%$ type option

$$
\bar{\Pi}^{\$C}_{t_0,T_1,T_2}(k) = e^{X_{T_1}}\max\left[e^{X_{T_2}-X_{T_1}}-e^k,0\right]
\tag{2.59}
$$

and so we have the T_1 value

$$
\Pi^{\$C}_{T_1,T_1,T_2}(k) = e^{X_{T_1}}\Pi^{\%C}_{T_1,T_1,T_2}(k)
\tag{2.60}
$$

Making use of equations (2.56), (2.57), (2.58) and (2.60), we see that the t_0 value follows in a straightforward manner. For the piecewise constant, time-dependent discount rate arbitrage arguments yield $r_1\tau_1+r_2\tau_2=r\tau$.

Regarding the valid range of α for $\%$ and \$ type Forward Starting Options, we determine the values $\alpha^{\min,(\%,\$)}$ and $\alpha^{\max,(\%,\$)}$ from the respective conditional forward $(\%,\$)$ characteristic functions, in the same manner that $\alpha^{\min}$ and $\alpha^{\max}$ are determined from the conditional characteristic function. $\square$

It is worth noting that we have not required a change of measure argument to obtain the pricing formula for a \$ type option. Such an approach is followed in Kruse and Nogel [2005] where a 'Black-Scholes' type pricing formula of the form in equation (2.39) is obtained for a \$ type option, within the Heston model.

2.1.3 The affine framework

We define the affine framework as that where the conditional joint characteristic function, for the state variables X and V, has the form

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(izX_T + iz_vV_T\right)|X_{t_0}, V_{t_0}\right] = \exp\left[izX_{t_0} + D(\tau, iz, iz_v)V_{t_0} + C(\tau, iz, iz_v)\right] \tag{2.61}$$

where $z := u - i(\alpha+1)$ and we leave iz_v unspecified, for the moment. The analytic form of the functions C and D is derived in proposition 2 of section 2.2.

Regarding the arguments of the functions C and D, iz and iz_v refer to the coefficient of X and V respectively, at the terminal time T where we have the value $\exp\left(izX_T + iz_vV_T\right)$. Setting $z_v = 0$ yields the conditional characteristic function for X

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(izX_T\right)|X_{t_0}, V_{t_0}\right] = \exp\left[izX_{t_0} + D(\tau, iz, 0)V_{t_0} + C(\tau, iz, 0)\right] \tag{2.62}$$

while setting $z = 0$ yields the conditional characteristic function for V

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_vV_T\right)|X_{t_0}, V_{t_0}\right] = \exp\left[D(\tau, 0, iz_v)V_{t_0} + C(\tau, 0, iz_v)\right] \tag{2.63}$$

From equation (2.11), we see that an analytic expression for $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(izX_T)|X_{t_0}, V_{t_0}\right]$ allows us to value European Options in semi-closed form. Equation (2.62) lies at the heart of models presented in Heston [1993], Bates [1996], Scott [1997], Duffie et al. [2000] and Yan and Hanson [2006], for example.

From equation (2.49), we see that an analytic expression for the conditional forward $\%$ characteristic function $\Phi_{t_0,T_1,T_2}(z_{\%}, z)$ allows us to value $\%$ type Forward Starting Options in semi-closed form. Within the affine framework, we have

$$\begin{aligned}
\Phi_{t,T_1,T_2}(z_{\%}, z) &= \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_{\%}X_{T_1}\right)\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\exp\left(izX_{T_2}\right)|X_{T_1}, V_{T_1}\right]|X_{t_0}, V_{t_0}\right] && (2.64)\\
&= \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_{\%}X_{T_1} + izX_{T_1} + D_{2;2}(\tau_2, iz, 0)V_{T_1} + C_{2;2}(\tau_2, iz, 0)\right)|X_{t_0}, V_{t_0}\right] && (2.65)\\
&= \exp\left[C_{2;2}(\tau_2, iz, 0)\right]\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(D_{2;2}(\tau_2, iz, 0)V_{T_1}\right)|X_{t_0}, V_{t_0}\right] && (2.66)\\
&= \exp\left[D_{1;2}(\tau_1, 0, D_{2;2}(\tau_2, iz, 0))V_{t_0} + C_{1;2}(\tau_1, 0, D_{2;2}(\tau_2, iz, 0)) + C_{2;2}(\tau_2, iz, 0)\right] && (2.67)
\end{aligned}$$

where we have made use of the tower property to obtain equation (2.64), the analytic form of the conditional characteristic function for X to obtain equation (2.65), the fact that $z_{\%} = -z$ to obtain equation (2.66) and the analytic form of the conditional characteristic function for V to obtain equation (2.67). Regarding the subscripts of the functions $C_{m;n}$ and $D_{m;n}$, m specifies the increment currently considered while n specifies the total number of increments. We clarify the use of this subscripting in section 2.4.

From equation (2.51), we see that an analytic expression for the conditional forward \$ characteristic function $\Phi_{t_0,T_1,T_2}(z_{\$}, z)$ allows us to value \$ type Forward-Starting options in semi-closed form. Within the affine framework, we have

$$\begin{aligned}
\Phi_{t_0,T_1,T_2}(z_\$, z) &= \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_\$ X_{T_1}\right)\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\exp\left(izX_{T_2}\right)|X_{T_1}, V_{T_1}\right]|X_{t_0}, V_{t_0}\right] &(2.68)\\
&= \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_\$ X_{T_1} + izX_{T_1} + D_{2;2}(\tau_2, iz, 0)V_{T_1} + C_{2;2}(\tau_2, iz, 0)\right)|X_{t_0}, V_{t_0}\right] &(2.69)\\
&= \exp\left[C_{2;2}(\tau_2, iz, 0)\right]\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(X_{T_1} + D_{2;2}(\tau_2, iz, 0)V_{T_1}\right)|X_{t_0}, V_{t_0}\right] &(2.70)\\
&= \exp\left[X_{t_0} + D_{1;2}(\tau_1, 1, D_{2;2}(\tau_2, iz, 0))V_{t_0} + C_{1;2}(\tau_1, 1, D_{2;2}(\tau_2, iz, 0)) + C_{2;2}(\tau_2, iz, 0)\right] &(2.71)
\end{aligned}$$

where what differs from the corresponding result for $\%$ type options is that we have made use of the fact that $z_\$ = -(z+i)$ to obtain equation (2.70) and the analytic form of the joint conditional characteristic function for X and V to obtain equation (2.71).

We can now specify the conditional forward $(\%, \$)$ characteristic function in a more compact manner

$$\begin{aligned}
&\Phi_{t_0,T_1,T_2}(z_{(\%,\$)}, z)\\
&= \exp\left[\mathbb{I}X_{t_0} + D_{1;2}(\tau_1, \mathbb{I}, D_{2;2}(\tau_2, iz, 0))V_{t_0} + C_{1;2}(\tau_1, \mathbb{I}, D_{2;2}(\tau_2, iz, 0)) + C_{2;2}(\tau_2, iz, 0)\right] \quad (2.72)
\end{aligned}$$

where $z_{(\%,\$)} := z_\%$ for a $\%$ type option, $z_{(\%,\$)} := z_\$$ for a \$ type option and $\mathbb{I} := \mathbb{I}_{[z_{(\%,\$)} := z_\$]}$.

2.2 The analytic conditional joint characteristic function for the SVJJ model

From Duffie et al. [2000], we have the affine jump-diffusion stochastic volatility model

$$\begin{aligned}
dX_t &= \left(r - q - \lambda\omega - \frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^X + J_X dN_t &(2.73)\\
dV_t &= \kappa(\theta - V_t)dt + \nu\sqrt{V_t}dW_t^V + J_V dN_t &(2.74)\\
dW_t^X dW_t^V &= \rho dt &(2.75)
\end{aligned}$$

where

$$\begin{aligned}
J_V &\sim \exp\left(\frac{1}{\eta}\right) &(2.76)\\
J_X|J_V &\sim N\left(\mu + \rho^J J_V, \sigma^2\right) &(2.77)
\end{aligned}$$

Regarding the specified state variables X and V, each process has a drift, diffusion and jump component. The drift of X includes the parameter ω which provides a degree of freedom to ensure that the arbitrage condition $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[S_T|S_{t_0}\right] = S_{t_0}e^{(r-q)\tau}$ is not violated by the presence of jumps in the state variables. The drift of V mean reverts with the long term mean θ and rate of mean reversion κ. The Brownian motions driving the continuous component of each process are correlated by the parameter ρ. This may be specified in terms of the independent Brownian motions W^V and B where $W^X := \rho W^V + \sqrt{1-\rho^2}B$. The jump component of each process is driven by the same Poisson process N (with intensity parameter λ) which counts the number of jumps in a certain interval. The jump sizes J_V and J_X are correlated with an exponential distribution specified for J_V where $\mathbb{E}^{\mathbb{Q}}\left[J_V\right] = \eta$ and the conditional distribution of J_X

given J_V is specified to be normal with the correlation parameter ρ^J controlling the conditioning of J_X on J_V. Setting $\lambda = 0$ yields the dynamics of the Heston model.

Proposition 2. *For the period $(t_0, T]$ and $\tau = T - t_0$, the conditional joint characteristic function for X and V, within the SVJJ model, has the analytic form* $\exp\left[izX_{t_0} + D(\tau, iz, iz_v)V_{t_0} + C(\tau, iz, iz_v)\right]$ *where*

$$\begin{aligned}
D(\tau, iz, iz_v) &= \left(\frac{b(iz)-\gamma(iz)}{\nu^2}\right)\left[\frac{\bar{A}(iz, iz_v) - e^{-\gamma(iz)\tau}}{\bar{A}(iz, iz_v) - \left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)e^{-\gamma(iz)\tau}}\right] && (2.78)\\
C(\tau, iz, iz_v) &= (r-q)iz\tau + \bar{C}(\tau, iz, iz_v) + \lambda J(\tau, iz, iz_v) && (2.79)\\
\bar{C}(\tau, iz, iz_v) &= \frac{\kappa\theta}{\nu^2}\left[b(iz)-\gamma(iz)\right]\tau - \frac{2\kappa\theta}{\nu^2}\log\left[\psi(\tau, iz, iz_v)\right] && (2.80)\\
\psi(\tau, iz, iz_v) &= \frac{A^{-1}(iz, iz_v)e^{-\gamma(iz)\tau} - 1}{A^{-1}(iz, iz_v) - 1} && (2.81)\\
J(\tau, iz, iz_v) &= e^{iz\mu - \frac{1}{2}z^2\sigma^2}\bar{J}(\tau, iz, iz_v) - \left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J} - 1\right)iz\tau - \tau && (2.82)\\
\bar{J}(\tau, iz, iz_v) &= \frac{\bar{A}(iz, iz_v)\tau}{\vartheta(iz, iz_v)} + \frac{1}{\gamma(iz)}\left(\frac{\bar{A}(iz, iz_v)}{\vartheta(iz, iz_v)} + \frac{\left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)}{\bar{\vartheta}(iz)}\right)\log\left(\frac{\vartheta(iz, iz_v) + \bar{\vartheta}(iz)e^{-\gamma(iz)\tau}}{\vartheta(iz, iz_v) + \bar{\vartheta}(iz)}\right) && \\
& && (2.83)\\
\vartheta(iz, iz_v) &= \bar{A}(iz, iz_v)\left[1 - iz\eta\rho^J - \eta\left(\frac{b(iz)-\gamma(iz)}{\nu^2}\right)\right] && (2.84)\\
\bar{\vartheta}(iz) &= \left[\eta\left(\frac{b(iz)-\gamma(iz)}{\nu^2}\right) - (1 - iz\eta\rho^J)\left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)\right] && (2.85)\\
\bar{A}(iz, iz_v) &= A(iz, iz_v)\left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right) && (2.86)\\
A(iz, iz_v) &= \frac{\nu^2 iz_v - b(iz) - \gamma(iz)}{\nu^2 iz_v - b(iz) + \gamma(iz)} && (2.87)\\
\gamma(iz) &= \sqrt{b^2(iz) - 2\nu^2 c(iz)} && (2.88)\\
b(iz) &= \kappa - \rho\nu iz && (2.89)\\
c(iz) &= \frac{1}{2}iz(iz-1) && (2.90)\\
& && (2.91)
\end{aligned}$$

We assume the arguments z and z_v are specified such that the characteristic function exists. In particular, we have the restriction Re $\left[1 - iz\eta\rho^J - \eta D(s, iz, iz_v)\right] > 0$ for $0 \leq s \leq \tau$ and the parameter restriction $1 - \eta\rho^J > 0$.

Proof: As stated in Rockinger and Semenova [2005], the conditional joint characteristic function

$$\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(izX_T + iz_vV_T\right)|X_t, V_t\right] \quad =: \quad \phi_t(X_t, V_t; iz, iz_v)$$

must satisfy the PDE

$$\begin{aligned}
\frac{\partial\phi_t}{\partial t} + \left(r - q - \lambda\omega - \frac{1}{2}V_t\right)\frac{\partial\phi_t}{\partial X_t} + \frac{1}{2}V_t\frac{\partial^2\phi_t}{\partial X_t^2} + \kappa(\theta - V_t)\frac{\partial\phi_t}{\partial V_t} + \frac{1}{2}\nu^2V_t\frac{\partial^2\phi_t}{\partial V_t^2} + \rho\nu V_t\frac{\partial^2\phi_t}{\partial X_t\partial V_t} &+ \\
\lambda\mathbb{E}_t^{\mathbb{Q}}\left[\phi_t(X_t + J_X, V_t + J_V) - \phi_t(X_t, V_t)\right] &= 0
\end{aligned} \tag{2.92}$$

This follows from the appropriate version of Itô's formula and the Feynman-Kac theorem. We assume the solution $\phi_t(X_t, V_t; iz, iz_v)$ has the form

$$\exp\left[izX_t + D(T-t, iz, iz_v)V_t + C(T-t, iz, iz_v)\right] \tag{2.93}$$

and must now determine the analytic form of C and D subject to the terminal conditions

$$C(0, iz, iz_v) = 0 \tag{2.94}$$

$$D(0, iz, iz_v) = iz_v \tag{2.95}$$

The partial derivatives of ϕ_t are

$$\begin{aligned}
\frac{\partial \phi_t}{\partial t} &= \left(\frac{\partial C}{\partial t} + \frac{\partial D}{\partial t} V_t\right)\phi_t \\
\frac{\partial \phi_t}{\partial X_t} &= iz\phi_t \\
\frac{\partial^2 \phi_t}{\partial X_t^2} &= -z^2\phi_t \\
\frac{\partial \phi_t}{\partial V_t} &= D\phi_t \\
\frac{\partial^2 \phi_t}{\partial V_t^2} &= D^2\phi_t \\
\frac{\partial^2 \phi_t}{\partial X_t \partial V_t} &= izD\phi_t
\end{aligned} \tag{2.96}$$

Having specified the form of ϕ_t, we have

$$\phi_t(X_t + J_X, V_t + J_V; iz, iz_v) = \phi_t(X_t, V_t; iz, iz_v)\exp\left(izJ_X + DJ_V\right) \tag{2.97}$$

Inserting these partial derivatives and equation (2.97) into equation (2.92) yields

$$\begin{aligned}
\left(\frac{\partial C}{\partial t} + (r - q - \lambda\omega)iz + \kappa\theta D + \lambda\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(izJ_X + DJ_V\right) - 1\right]\right) &+ \\
\left(\frac{\partial D}{\partial t} + \frac{1}{2}iz(iz-1) - (\kappa - \rho\nu iz)D + \frac{1}{2}\nu^2 D^2\right)V &= 0
\end{aligned}$$

Switching variables from t to $s = T - t$, we must solve the ordinary differential equations

$$\frac{\partial D}{\partial s} = \frac{1}{2}\nu^2 D^2 - b(iz)D + c(iz) \tag{2.98}$$

$$\frac{\partial C}{\partial s} = (r - q - \lambda\omega)iz + \kappa\theta D + \lambda\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(izJ_X + DJ_V\right)\right] - \lambda \tag{2.99}$$

where

$$b(iz) = \kappa - \rho\nu iz \tag{2.100}$$

$$c(iz) = \frac{1}{2}iz(iz-1) \tag{2.101}$$

From Spiegel and Liu [1999] equation (17.12.1), we know

$$\int \frac{dx}{ax^2 - bx + c} = \frac{1}{\sqrt{b^2-4ac}} \ln\left(\frac{2ax - b - \sqrt{b^2-4ac}}{2ax - b + \sqrt{b^2-4ac}}\right) \tag{2.102}$$

for the coefficients a, b and c.

We choose to express this as

$$\int \frac{dx}{ax^2 - bx + c} = \frac{1}{-\sqrt{b^2-4ac}} \ln\left(\frac{2ax - b + \sqrt{b^2-4ac}}{2ax - b - \sqrt{b^2-4ac}}\right) \tag{2.103}$$

which can be viewed as working with the non principal square root function $-\sqrt{b^2-4ac}$. The significance of this choice (regarding the relevant literature) is discussed in subsection 3.1.1.

Using equation (2.95) (to obtain the integrating constant) and equation (2.103), for $\tau = T - t_0$, we obtain

$$D(\tau, iz, iz_v) = \left(\frac{b(iz) - \gamma(iz)}{\nu^2}\right)\left[\frac{\bar{A}(iz, iz_v) - e^{-\gamma(iz)\tau}}{\bar{A}(iz, iz_v) - \left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)e^{-\gamma(iz)\tau}}\right] \tag{2.104}$$

where

$$\begin{aligned}
\gamma(iz) &= \sqrt{b^2(iz) - 2\nu^2 c(iz)} && (2.105)\\
A(iz, iz_v) &= \frac{\nu^2 iz_v - b(iz) - \gamma(iz)}{\nu^2 iz_v - b(iz) + \gamma(iz)} && (2.106)\\
\bar{A}(iz, iz_v) &= A(iz, iz_v)\left(\frac{b(iz) - \gamma(iz)}{b(iz) + \gamma(iz)}\right) && (2.107)
\end{aligned}$$

We defer a discussion of the existence of the function $D(\tau, iz, iz_v)$ to section 3.2.

From Spiegel and Liu [1999] equation (17.1.4), we know

$$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left(\frac{x}{ax+b}\right) \tag{2.108}$$

for the coefficients a and b.

Using equation (2.104), we make the substitution $x = e^{-\gamma(iz)s}$ for $0 \le s \le \tau$ where $\frac{dx}{ds} = -\gamma(iz)x$ to obtain

$$\begin{aligned}
\int_0^\tau D(s, iz, iz_v)ds &= \int_{e^{-\gamma(iz)\tau}}^1 \left(\frac{b(iz) - \gamma(iz)}{\nu^2\gamma(iz)x}\right)\left[\frac{\bar{A}(iz, iz_v) - x}{\bar{A}(iz, iz_v) - \left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)x}\right]dx \\
&= \left(\frac{b(iz) - \gamma(iz)}{\nu^2\gamma(iz)}\right)\bar{A}(iz, iz_v)\int_{e^{-\gamma(iz)\tau}}^1 \left[\bar{A}(iz, iz_v) - \left(\frac{b(iz) - \gamma(iz)}{b(iz) + \gamma(iz)}\right)x\right]^{-1}\frac{1}{x}dx \\
&- \left(\frac{b(iz) - \gamma(iz)}{\nu^2\gamma(iz)}\right)\int_{e^{-\gamma(iz)\tau}}^1 \left[\bar{A}(iz, iz_v) - \left(\frac{b(iz) - \gamma(iz)}{b(iz) + \gamma(iz)}\right)x\right]^{-1}dx \\
&= \left(\frac{b(iz) - \gamma(iz)}{\nu^2\gamma(iz)}\right)\log\left(\frac{x}{\left[\bar{A}(iz, iz_v) - \left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)x\right]}\right)\Bigg|_{e^{-\gamma(iz)\tau}}^1 && (2.109)\\
&+ \left(\frac{b(iz) + \gamma(iz)}{\nu^2\gamma(iz)}\right)\log\left(\bar{A}(iz, iz_v) - \left(\frac{b(iz) - \gamma(iz)}{b(iz) + \gamma(iz)}\right)x\right)\Bigg|_{e^{-\gamma(iz)\tau}}^1 \\
&= \left(\frac{b(iz) - \gamma(iz)}{\nu^2}\right)\tau - \frac{2}{\nu^2}\log\left(\frac{A^{-1}(iz, iz_v)e^{-\gamma(iz)\tau} - 1}{A^{-1}(iz, iz_v) - 1}\right) && (2.110)
\end{aligned}$$

where we have made use of equation (2.108) to determine equation (2.109). Again, we defer a discussion of the existence of equation (2.110) to section 3.2.

Regarding the joint characteristic function for the jump sizes J_X and J_V, we have

$$
\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}}\left[e^{izJ_X + D(s,iz,iz_v)J_V}\right] && (2.111)\\
= & \int_0^\infty \int_{-\infty}^\infty e^{izj_X + D(s,iz,iz_v)j_V} f(j_X|j_V) f(j_V) dj_X dj_V \\
= & \int_0^\infty \int_{-\infty}^\infty e^{izj_X + D(s,iz,iz_v)j_V} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(j_X - (\mu + \rho^J j_V)\right)^2} \frac{1}{\eta} e^{-\frac{1}{\eta} j_V} dj_X dj_V \\
= & e^{iz\mu - \frac{1}{2}z^2\sigma^2} \frac{1}{\eta} \int_0^\infty e^{-j_V \frac{1}{\eta}\left(1 - iz\eta\rho^J - \eta D(s,iz,iz_v)\right)} dj_V && (2.112)\\
= & e^{iz\mu - \frac{1}{2}z^2\sigma^2} \left. \frac{e^{-j_V \frac{1}{\eta}\left(1 - iz\eta\rho^J - \eta D(s,iz,iz_v)\right)}}{-\left(1 - iz\eta\rho^J - \eta D(s,iz,iz_v)\right)} \right|_0^\infty && (2.113)\\
= & \frac{e^{iz\mu - \frac{1}{2}z^2\sigma^2}}{\left(1 - iz\eta\rho^J - \eta D(s,iz,iz_v)\right)} && (2.114)
\end{aligned}
$$

where equation (2.112) is obtained by completing the square in the previous equation and re-arranging terms. The result in equation (2.114) is valid only for Re $[1 - iz\eta\rho^J - \eta D(s,iz,iz_v)] > 0$.[1] We defer any further discussion of the existence of $\mathbb{E}_t^{\mathbb{Q}}\left[e^{izJ_X + D(s,iz,iz_v)J_V}\right]$ to subsection 3.2.4.

Furthermore, we have

$$
\begin{aligned}
& \bar{J}(\tau, iz, iz_v) \\
:= & \int_0^\tau \frac{1}{\left(1 - iz\eta\rho^J - \eta D(s,iz,iz_v)\right)} ds && (2.115)\\
= & \int_0^\tau \frac{\bar{A}(iz,iz_v) - \left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right) e^{-\gamma(iz)s}}{\vartheta(iz,iz_v) + \bar{\vartheta}(iz) e^{-\gamma(iz)s}} ds \\
= & \frac{1}{\gamma(iz)} \int_{e^{-\gamma(iz)\tau}}^1 \left[\left(\frac{\bar{A}(iz,iz_v)}{x\left[\vartheta(iz,iz_v) + \bar{\vartheta}(iz)x\right]}\right) - \left(\frac{\left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)}{\vartheta(iz,iz_v) + \bar{\vartheta}(iz)x}\right)\right] dx && (2.116)\\
= & \frac{1}{\gamma(iz)} \left[\frac{\bar{A}(iz,iz_v)}{\vartheta(iz,iz_v)} \log\left(\frac{x}{\vartheta(iz,iz_v) + \bar{\vartheta}(iz)x}\right) - \frac{\left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)}{\bar{\vartheta}(iz)} \log[\vartheta(iz,iz_v) + \bar{\vartheta}(iz)x] \Bigg|_{e^{-\gamma(iz)\tau}}^1 \right] && (2.117)\\
= & \frac{\bar{A}(iz,iz_v)\tau}{\vartheta(iz,iz_v)} + \frac{1}{\gamma(iz)} \left(\frac{\bar{A}(iz,iz_v)}{\vartheta(iz,iz_v)} + \frac{\left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)}{\bar{\vartheta}(iz)}\right) \log\left(\frac{\vartheta(iz,iz_v) + \bar{\vartheta}(iz)e^{-\gamma(iz)\tau}}{\vartheta(iz,iz_v) + \bar{\vartheta}(iz)}\right) && (2.118)
\end{aligned}
$$

where

$$
\begin{aligned}
\vartheta(iz,iz_v) &= \bar{A}(iz,iz_v)\left[1 - iz\eta\rho^J - \eta\left(\frac{b(iz)-\gamma(iz)}{\nu^2}\right)\right] && (2.119)\\
\bar{\vartheta}(iz) &= \left[\eta\left(\frac{b(iz)-\gamma(iz)}{\nu^2}\right) - \left(1 - iz\eta\rho^J\right)\left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)\right] && (2.120)
\end{aligned}
$$

We have switched variables from s to $x = e^{\gamma(iz)s}$ in equation (2.116) and used equation (2.108) to obtain equation (2.117). Again, we defer a discussion of the existence of $\bar{J}(\tau, iz, iz_v)$ to section 3.2.

[1] This point is not specifically made in Duffie et al. [2000] equation (4.5).

Using equation (2.94) (to obtain the integrating constant) and equations (2.110), (2.114) and (2.118), we solve equation (2.99) and obtain

$$C(\tau, iz, iz_v) = (r-q)iz\tau + \bar{C}(\tau, iz, iz_v) + \lambda J(\tau, iz, iz_v) \tag{2.121}$$

where

$$\bar{C}(\tau, iz, iz_v) = \frac{\kappa\theta}{\nu^2}\left[b(iz) - \gamma(iz)\right]\tau - \frac{2\kappa\theta}{\nu^2}\log\left(\frac{A^{-1}(iz, iz_v)e^{-\gamma(iz)\tau} - 1}{A^{-1}(iz, iz_v) - 1}\right) \tag{2.122}$$

$$J(\tau, iz, iz_v) = e^{iz\mu - \frac{1}{2}z^2\sigma^2}\bar{J}(\tau, iz, iz_v) - \tau - iz\int_0^\tau \omega ds \tag{2.123}$$

From Gatheral [2006], the compensating drift factor ω is specified such that the arbitrage condition $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[S_T|S_{t_0}\right] = S_{t_0}e^{(r-q)\tau}$ is satisfied. For $z = -i$ and $z_v = 0$, the conditional joint characteristic function yields $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[S_T|S_{t_0}\right]$ and we can show

$$D(\tau, 1, 0) = 0 \tag{2.124}$$

$$\bar{C}(\tau, 1, 0) = 0 \tag{2.125}$$

$$\bar{J}(\tau, 1, 0) = \left(\frac{1}{1 - \eta\rho^J}\right)\tau \tag{2.126}$$

since $\gamma(1) = |b(1)|$. We require $J(\tau, 1, 0) = 0$ so that $C(\tau, 1, 0) = (r-q)\tau$ and so we must have

$$\int_0^\tau \omega ds = \left(\frac{e^{\mu + \frac{1}{2}\sigma^2}}{1 - \eta\rho^J}\right)\tau - \tau \tag{2.127}$$

$$\omega = \frac{e^{\mu + \frac{1}{2}\sigma^2}}{1 - \eta\rho^J} - 1 \tag{2.128}$$

with the restriction

$$1 - \eta\rho^J - \eta D(s, 1, 0) = 1 - \eta\rho^J > 0$$

Finally, we have

$$J(\tau, iz, iz_v) = e^{iz\mu - \frac{1}{2}z^2\sigma^2}\bar{J}(\tau, iz, iz_v) - \left(\frac{e^{\mu + \frac{1}{2}\sigma^2}}{1 - \eta\rho^J} - 1\right)iz\tau - \tau \tag{2.129}$$

subject to the parameter restriction $1 - \eta\rho^J > 0$ and the restriction $1 - iz\eta\rho^J - \eta D(s, iz, iz_v) > 0$ for $0 \le s \le \tau$. $\square$

The dynamics of the model are specified under a risk neutral measure. The specified value of the market price of variance risk $\lambda_t^V(V_t)$ determines exactly which risk neutral measure we are working under. To determine the value of $\lambda_t^V(V_t)$ implied by the model, we consider the effect of shifting from the real world to the risk neutral measure, on the variance process (ignoring the jump component). Had we specified the square-root, mean-reverting dynamics in equation (2.74) with the parameters κ^* and θ^* under the real world measure, Girsanov's theorem would yield the risk neutral dynamics

$$dV_t = \kappa^*(\theta^* - V_t)dt - \nu\sqrt{V_t}\lambda_t^V(V_t)dt + \nu\sqrt{V_t}dW_t^V \tag{2.130}$$

For $\lambda_t^V(V_t) := \frac{\sqrt{V_t}}{\nu}\lambda^V$, we can write

$$dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t}dW_t^V \tag{2.131}$$

where $\kappa = \kappa^* + \lambda^V$ and $\theta = \frac{\kappa^*\theta^*}{\kappa^*+\lambda^V}$. This yields the result presented in Heston [1993]. By calibrating the parameters κ and θ to market prices, we implicitly specify λ^V and hence, the risk neutral measure.

2.2.1 The analytic conditional characteristic function

Working from proposition 2 and setting $z_v = 0$, we have

$$D(\tau, iz, 0) = \left(\frac{b(iz) - \gamma(iz)}{\nu^2}\right)\left[\frac{1 - e^{-\gamma(iz)\tau}}{1 - \left(\frac{b(iz)-\gamma(iz)}{b(iz)+\gamma(iz)}\right)e^{-\gamma(iz)\tau}}\right] \tag{2.132}$$

$$\bar{C}(\tau, iz, 0) = \frac{\kappa\theta}{\nu^2}\left[b(iz) - \gamma(iz)\right]\tau - \frac{2\kappa\theta}{\nu^2}\log\left(\frac{A^{-1}(iz,0)e^{-\gamma(iz)\tau} - 1}{A^{-1}(iz,0) - 1}\right) \tag{2.133}$$

$$\begin{aligned}\bar{J}(\tau, iz, 0) &= \frac{\left[b(iz) + \gamma(iz)\right]\tau}{\left[(1 - iz\eta\rho^J)\left[b(iz) + \gamma(iz)\right] - \frac{\eta}{\nu^2}\left[b^2(iz) - \gamma^2(iz)\right]\right]} \\ &+ \left(\frac{\frac{2\eta}{\nu^2}\left[b^2(iz) - \gamma^2(iz)\right]}{(1 - iz\eta\rho^J)^2\gamma^2(iz) - \left[(1 - iz\eta\rho^J)b(iz) - \left(\frac{\eta}{\nu^2}\right)\left[b^2(iz) - \gamma^2(iz)\right]\right]^2}\right) \\ &\times \log\left(1 - \frac{\left[\left(\frac{\eta}{\nu^2}\right)\left[b^2(iz) - \gamma^2(iz)\right] - (1 - iz\eta\rho^J)\left[b(iz) - \gamma(iz)\right]\right]\left(1 - e^{-\gamma(iz)\tau}\right)}{\frac{2(1-iz\eta\rho^J)[b(iz)-\gamma(iz)]\gamma(iz)}{[b(iz)-\gamma(iz)-\nu^2 iz_v]}}\right)\end{aligned} \tag{2.134}$$

$$C(\tau, iz, 0) = (r - q)iz\tau + \bar{C}(\tau, iz, 0) + \lambda e^{iz\mu - \frac{1}{2}z^2\sigma^2}\bar{J}(\tau, iz, 0) - \lambda\tau - \lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1 - \eta\rho^J} - 1\right)iz\tau \tag{2.135}$$

where $A(iz, 0) = \frac{b(iz)+\gamma(iz)}{b(iz)-\gamma(iz)}$ and using our notation, equation (2.134) confirms the result d obtained in Duffie et al. [2000] Pg 1362. The functions $D(\tau, iz, 0)$ and $C(\tau, iz, 0)$ yield the conditional characteristic function for X as can be seen from equation (2.62) of subsection 2.1.3. For $\lambda = 0$, we obtain the conditional characteristic function for the Heston model.

2.2.2 The analytic conditional forward $\%$ characteristic function

Working from proposition 2 and setting $z = 0$, we have $\gamma(0) = b(0) = \kappa > 0$ which yields

$$D(\tau, 0, iz_v) = \left(\frac{iz_v e^{-\kappa\tau}}{1 - \frac{iz_v}{\varpi_\kappa}}\right) \tag{2.136}$$

$$\begin{aligned}\bar{C}(\tau, 0, iz_v) &= -\frac{2\kappa\theta}{\nu^2}\log\left(\frac{A^{-1}(0, iz_v)e^{-\kappa\tau} - 1}{A^{-1}(0, iz_v) - 1}\right) && (2.137) \\ &= -\frac{2\kappa\theta}{\nu^2}\log\left(1 - \frac{iz_v}{\varpi_\kappa}\right) && (2.138)\end{aligned}$$

$$\bar{J}(\tau, 0, iz_v) = \tau + \left[\kappa - \frac{\nu^2}{2\eta}\right]^{-1}\log\left[1 - \frac{\left(1 - \frac{\nu^2}{2\kappa\eta}\right)(1 - e^{-\kappa\tau})}{\left(1 - \frac{1}{iz_v\eta}\right)}\right] \tag{2.139}$$

$$C(\tau, 0, iz_v) = \bar{C}(\tau, 0, iz_v) + \lambda\bar{J}(\tau, 0, iz_v) - \lambda\tau \tag{2.140}$$

where $\varpi_\kappa = \frac{2\kappa}{\nu^2(1-e^{-\kappa\tau})}$ and $A(0, iz_v) = 1 - \frac{2\kappa}{\nu^2 iz_v}$.

The analytic form of the functions $C(\tau, 0, iz_v)$ and $D(\tau, 0, iz_v)$ feature in the conditional forward % characteristic function as can be seen from equation (2.72) of subsection 2.1.3 (for $\mathbb{I} = 0$). The form of the functions $D(\tau, iz, 0)$ and $C(\tau, iz, 0)$ presented in subsection 2.2.1 also feature in the conditional forward % characteristic function. In particular, iz_v takes the form of $D(\tau, iz, 0)$. To clarify, we have

$$D_{2;2}(\tau_2, iz, 0) = D(\tau_2, iz, 0) \tag{2.141}$$

$$C_{2;2}(\tau_2, iz, 0) = C(\tau_2, iz, 0) \tag{2.142}$$

$$D_{1;2}(\tau_1, 0, D_{2;2}(\tau_2, iz, 0)) = D(\tau_1, 0, D(\tau_2, iz, 0)) \tag{2.143}$$

$$C_{1;2}(\tau_1, 0, D_{2;2}(\tau_2, iz, 0)) = C(\tau_1, 0, D(\tau_2, iz, 0)) \tag{2.144}$$

in equation (2.72) (for $\mathbb{I} = 0$) with a constant parameter set over the period $(t, T_2]$.

Setting $\lambda = 0$, we confirm the result presented in Hong [2004] for the Heston model.

2.2.3 The analytic conditional forward $ characteristic function

Working from proposition 2 and setting $z = -i$, we have $\gamma(1) = |b(1)| = |\kappa - \rho\nu|$. For $b(1) \neq 0$, we have

$$D(\tau, 1, iz_v) = \left(\frac{iz_v e^{-b(1)\tau}}{1 - \frac{iz_v}{\varpi_b}}\right) \tag{2.145}$$

$$\bar{C}(\tau, 1, iz_v) = -\frac{2\kappa\theta}{\nu^2}\log\left(\frac{A^{-1}(1, iz_v)e^{-|b(1)|\tau} - 1}{A^{-1}(1, iz_v) - 1}\right) \tag{2.146}$$

$$= -\frac{2\kappa\theta}{\nu^2}\log\left(1 - \frac{iz_v}{\varpi_b}\right) \tag{2.147}$$

$$\bar{J}(\tau, 1, iz_v) = \frac{1}{(1-\eta\rho^J)}\left(\tau + \left[b(1) - \frac{\nu^2}{2\bar{\eta}}\right]^{-1}\log\left[1 - \frac{\left(1 - \frac{\nu^2}{2b(1)\bar{\eta}}\right)\left(1 - e^{-b(1)\tau}\right)}{\left(1 - \frac{1}{iz_v\bar{\eta}}\right)}\right]\right) \tag{2.148}$$

$$C(\tau, 1, iz_v) = (r-q)\tau + \bar{C}(\tau, 1, iz_v) + \lambda e^{\mu + \frac{1}{2}\sigma^2}\bar{J}(\tau, 1, iz_v) - \lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J}\right)\tau \tag{2.149}$$

where $\bar{\eta} = \frac{\eta}{1-\eta\rho^J}$, $\varpi_b = \frac{2b(1)}{\nu^2(1-e^{-b(1)\tau})}$. For $b(1) = 0$, we apply l'H$\hat{\text{o}}$pital's rule to obtain

$$\lim_{b(1)\to 0} D(\tau, 1, iz_v) = \left(\frac{iz_v}{1 - iz_v\frac{1}{2}\nu^2\tau}\right) \tag{2.150}$$

$$\lim_{b(1)\to 0} \bar{C}(\tau, 1, iz_v) = -\frac{2\kappa\theta}{\nu^2}\log\left(1 - iz_v\frac{1}{2}\nu^2\tau\right) \tag{2.151}$$

$$\lim_{b(1)\to 0} \bar{J}(\tau, 1, iz_v) = \frac{1}{(1-\eta\rho^J)}\left(\tau - \frac{2\bar{\eta}}{\nu^2}\log\left[1 + \frac{\frac{\nu^2}{2\bar{\eta}}\tau}{\left(1 - \frac{1}{iz_v\bar{\eta}}\right)}\right]\right) \tag{2.152}$$

The analytic form of the functions $C(\tau, 1, iz_v)$ and $D(\tau, 1, iz_v)$ feature in the conditional forward $ characteristic function as can be seen from equation (2.72) of subsection 2.1.3 (for $\mathbb{I} = 1$). Again, the form of the functions $D(\tau, iz, 0)$ and $C(\tau, iz, 0)$ presented in subsection 2.2.1 also feature in the conditional forward \$ characteristic function with iz_v taking the form of $D(\tau, iz, 0)$. To clarify, we have

$$D_{2;2}(\tau_2, iz, 0) = D(\tau_2, iz, 0) \tag{2.153}$$

$$C_{2;2}(\tau_2, iz, 0) = C(\tau_2, iz, 0) \tag{2.154}$$

$$D_{1;2}(\tau_1, 1, D_{2;2}(\tau_2, iz, 0)) = D(\tau_1, 1, D(\tau_2, iz, 0)) \tag{2.155}$$

$$C_{1;2}(\tau_1, 1, D_{2;2}(\tau_2, iz, 0)) = C(\tau_1, 1, D(\tau_2, iz, 0)) \tag{2.156}$$

in equation (2.72) (for $\mathbb{I} = 1$) with a constant parameter set over the period $(t, T_2]$.

The analytic results obtained may be used to confirm simulation results for specific examples of \$ type Forward Starting Call Options in Broadie and Kaya [2006] table 8.

2.3 The discrepancy between $\%$ and $\$$ type Forward Starting Options

2.3.1 $\%$ and $\$$ type forward implied volatilities

We have analytic values for $\%$ and \$ type Forward Starting Options in the Black-Scholes world where we allow for a term structure of parameters - specifically, piecewise constant parameters for the increments $(t_0, T_1]$ and $(T_1, T_2]$.

In the Black-Scholes world, the value for a $\%$ type Call Option is

$$\text{BS}^{\%\,\text{C}}_{t_0,T_1,T_2} = e^{-r_1\tau_1}\text{BS}\left(1, K, r_2, q_2, \sigma_2, \tau_2\right) \tag{2.157}$$

and that for a \$ type Call Option is

$$\text{BS}^{\$\,\text{C}}_{t_0,T_1,T_2} = S_{t_0}e^{-q_1\tau_1}\text{BS}\left(1, K, r_2, q_2, \sigma_2, \tau_2\right) \tag{2.158}$$

where BS $(S, K, r_2, q_2, \sigma_2, \tau_2)$ refers to the Black-Scholes value with underlying S, strike K, risk-free rate r_2, dividend yield q_2, volatility σ_2 and period τ_2. Furthermore, $\tau_1 = T_1 - t_0$, $\tau_2 = T_2 - T_1$, the parameters r_1 and q_1 apply over $(t_0, T_1]$ and the parameters r_2, q_2 and σ_2 apply over $(T_1, T_2]$.

Proposition 3. *For a $\%$ type option, the forward implied volatility $\sigma_2^{\%}$ satisfies*

$$BS\left(1, K, r_2, q_2, \sigma_2^{\%}, \tau_2\right) = e^{-r_2\tau_2}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\left(\frac{S_{T_2}}{S_{T_1}} - K\right)^+\right]\right] \tag{2.159}$$

while for a \$ type option, the forward implied volatility $\sigma_2^{\$}$ satisfies

$$BS\left(1, K, r_2, q_2, \sigma_2^{\$}, \tau_2\right) = e^{-r_2\tau_2}\mathbb{E}^{\mathbb{Q}_S}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\left(\frac{S_{T_2}}{S_{T_1}} - K\right)^+\right]\right] \tag{2.160}$$

Proof:
In general, the value of a $\%$ type Call Option can be expressed as

$$\Pi^{\%\,\text{C}}_{t_0,T_1,T_2} = e^{-(r_1\tau_1 + r_2\tau_2)}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\left(\frac{S_{T_2}}{S_{T_1}} - K\right)^+\right]\right] \tag{2.161}$$

while that for a \$ type Call Option can be expressed as

$$\Pi^{\$C}_{t_0,T_1,T_2} = e^{-(r_1\tau_1+r_2\tau_2)}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\left(S_{T_2}-KS_{T_1}\right)^+\right]\right] \quad (2.162)$$

$$= S_{t_0}e^{-(q_1\tau_1+r_2\tau_2)}\mathbb{E}^{\mathbb{Q}_S}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\left(S_{T_2}-KS_{T_1}\right)^+\right]\frac{1}{S_{T_1}}\right] \quad (2.163)$$

$$= S_{t_0}e^{-(q_1\tau_1+r_2\tau_2)}\mathbb{E}^{\mathbb{Q}_S}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{T_1}\left[\left(\frac{S_{T_2}}{S_{T_1}}-K\right)^+\right]\right] \quad (2.164)$$

We have shifted from the risk neutral to the stock price measure in equation (2.163) where we specify that dividends are reinvested into the stock and so the value of the numeraire at time t_0 is $S_{t_0}e^{-q_1\tau_1}$. The usefulness of shifting to the stock price measure, in the context of pricing \$ type options, is pointed out in Kruse and Nogel [2005].
Setting the values of such options in the Black-Scholes world i.e. equations (2.157) and (2.158) equal to their respective general versions in equations (2.161) and (2.164) gives us the results. □

From proposition 3, we see that solving for the forward implied volatility depends on the form of the Forward Starting Option. The resulting difference between the forward implied volatilities can be attributed to the shift between the risk neutral and the stock price measures over the period $(t, T_1]$.

2.3.2 The effect of a shift from the risk-neutral to the stock price measure on the dynamics of the model

In Kruse and Nogel [2005], the authors confirm that Girsanov's theorem may be used to shift from the risk-neutral to the stock price measure, within the Heston model. Furthermore, it is shown that under the stock price measure, the dynamics are

$$dX_t = \left(r-q+\frac{1}{2}V_t\right)dt+\sqrt{V_t}dW_t^{X,\mathbb{Q}_S} \quad (2.165)$$

$$dV_t = \bar{\kappa}(\bar{\theta}-V_t)dt+\nu\sqrt{V_t}dW_t^{V,\mathbb{Q}_S} \quad (2.166)$$

$$dW_t^{X,\mathbb{Q}_S}dW_t^{V,\mathbb{Q}_S} = \rho dt \quad (2.167)$$

where $\bar{\kappa}=\kappa-\rho\nu$ and $\bar{\theta}=\frac{\kappa\theta}{\bar{\kappa}}$.

We now confirm the effect of this shift in measure on the dynamics of the Heston model (i.e. the continuous diffusion component of the SVJJ model) and show that this shift in measure affects the jump component by adjusting the jump rate intensity to $\lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J}\right)$ and the jump size distributions to

$$J_V \sim \exp\left(\frac{1}{\bar{\eta}}\right)$$

$$J_X|J_V \sim N\left(\mu+\sigma^2+\rho^J J_V, \sigma^2\right)$$

under the stock price measure with $\bar{\eta}=\frac{\eta}{1-\eta\rho^J}$, the parameter restriction $1-\eta\rho^J>0$ and the restriction $\mathrm{Re}\left[1-iz\bar{\eta}\rho^J-\bar{\eta}D(s,iz,iz_v)\right]>0$ for $0\le s\le\tau$.

Following the same methodology as that in section 2.2 (where we derive the analytic form of the conditional joint characteristic function) and focussing on the Heston model, we assume the dynamics under the stock price measure (where the drift of each process is adjusted) as specified in equations (2.165)-(2.167). We obtain the Ricatti equation

$$\frac{\partial D}{\partial s} = \frac{1}{2}\nu^2 D^2 - \bar{b}(iz)D + \bar{c}(iz) \tag{2.168}$$

where

$$\bar{b}(iz) = \bar{\kappa} - \rho\nu iz \tag{2.169}$$

$$\bar{c}(iz) = \frac{1}{2}iz(iz+1) \tag{2.170}$$

The coefficients $\bar{b}(iz)$ and $\bar{c}(iz)$ differ from the corresponding coefficients $b(iz)$ and $c(iz)$ for the Ricatti equation in (2.98) where the risk-neutral dynamics are assumed. Furthermore, we now have

$$\bar{\gamma}(iz) = \sqrt{\bar{b}(iz)^2 - \nu^2 iz(iz+1)} \tag{2.171}$$

To determine the effect of a shift from the risk-neutral to the stock price measure on the dynamics of the model, we refer to equations (2.40) and (2.41), where $P1 = \mathbb{E}_t^{\mathbb{Q}_S}\left[\mathbb{I}_{[X>k]}\right]$ and $P2 = \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{I}_{[X>k]}\right]$. Comparing $P2$ with $P1$ allows us to determine this effect where the two expressions differ as $P2$ is a function of

$$\Phi_{t_0,T}(u) = \exp\left[iu(X_t + r - q)\tau + D(\tau, iu, 0)V_t + \bar{C}(\tau, iu, 0) + \lambda J(\tau, iu, 0)\right] \tag{2.172}$$

while $P1$ is a function of

$$\begin{aligned} & \frac{\Phi_{t_0,T}(u-i)}{\Phi_{t_0,T}(-i)} \\ = & \frac{\exp\left[(X_t + r - q)\tau + iu(X_t + r - q)\tau + D(\tau, i[u-i], 0)V_t + \bar{C}(\tau, i[u-i], 0) + \lambda J(\tau, i[u-i], 0)\right]}{\exp\left[(X_t + r - q)\tau\right]} \end{aligned} \tag{2.173}$$

where $\Phi_{t_0,T}(-i)$ yields the forward price.

Setting $\lambda = 0$ in equations (2.172) and (2.173), we focus on the functions $P1$ and $P2$, within the Heston model. Regarding equation (2.172), $D(\tau, iu, 0)$ and $\bar{C}(\tau, iu, 0)$ are functions of $b(iu)$ and $\gamma(iu)$ while $\bar{C}(\tau, iu, 0)$ is also a function of $\kappa\theta$. Regarding equation (2.173), $D(\tau, i[u-i], 0)$ and $\bar{C}(\tau, i[u-i], 0)$ are functions of $\bar{b}(iu)$ and $\bar{\gamma}(iu)$ while $\bar{C}(\tau, iu, 0)$ is also a function of $\bar{\kappa}\bar{\theta} = \kappa\theta$. This confirms the effect of the shift in measure on the dynamics of the continuous diffusion component of each process.

The effect on the jump component i.e. the jump rate intensity and the jump size distributions can be determined by focusing on the functions $\lambda J(\tau, iu, 0)$ and $\lambda J(\tau, i[u-i], 0)$ in equations (2.172) and (2.173), respectively. To this end, we consider the risk neutral jump-rate intensity multiplied by the joint characteristic function for the jump sizes J_X and J_V, in equation (2.114) of section (2.2), in terms of the arguments u and $u - i$

$$\lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{iuJ_x + D(s,iu,0)J_v}\right] = \lambda\left(\frac{e^{iu\mu - \frac{1}{2}u^2\sigma^2}}{1 - iu\eta\rho^J - \eta D(s, iu, 0)}\right) \tag{2.174}$$

with the restriction Re $[1 - iu\eta\rho^J - \eta D(s, iu, 0)] > 0$ and

$$\begin{aligned}\lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{i[u-i]J_x + D(s,i[u-i],0)J_v}\right] &= \lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2} e^{iu\left(\mu+\sigma^2\right)-\frac{1}{2}u^2\sigma^2}}{1-\eta\rho^J - iu\eta\rho^J - \eta D(s,i[u-i],0)}\right) \\ &= \lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J}\right)\left(\frac{e^{iu\left(\mu+\sigma^2\right)-\frac{1}{2}u^2\sigma^2}}{1-iu\bar{\eta}\rho^J - \bar{\eta}D(s,i[u-i],0)}\right)\end{aligned}$$

with $\bar{\eta} = \frac{\eta}{1-\eta\rho^J}$ and the restriction

$$\text{Re}\,[1 - i[u-i]\eta\rho^J - \eta D(s,i[u-i],0)] = \frac{\text{Re}\,[1 - iu\bar{\eta}\rho^J - \bar{\eta}D(s,i[u-i],0)]}{1-\eta\rho^J} > 0 \tag{2.175}$$

so we must have Re $[1 - iu\bar{\eta}\rho^J - \bar{\eta}D(s, i[u-i], 0)] > 0$ since we already have the restriction $1 - \eta\rho^J > 0$. From this, we have

$$\lambda J(\tau, iu, 0) = \lambda\left[\int_0^\tau \frac{e^{iu\mu - \frac{1}{2}u^2\sigma^2}}{(1 - iu\eta\rho^J - \eta D(s, iu, 0))} ds - \tau\right] - \lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J} - 1\right) iu\tau \tag{2.176}$$

$$\lambda J(\tau, i[u-i], 0) = \lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J}\right)\left[\int_0^\tau \frac{e^{iu\left(\mu+\sigma^2\right)-\frac{1}{2}u^2\sigma^2}}{(1 - iu\bar{\eta}\rho^J - \bar{\eta}D(s,i[u-i],0))} ds - \tau\right] \tag{2.177}$$

$$- \lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J} - 1\right) iu\tau \tag{2.178}$$

Comparing $\lambda J(\tau, iu, 0)$ with $\lambda J(\tau, i[u-i], 0)$, we see that a shift, from the risk-neutral to the stock price measure, yields the jump rate intensity $\lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J}\right)$ and the jump size distributions

$$\begin{aligned} J_V &\sim \exp\left(\frac{1}{\bar{\eta}}\right) \\ J_X | J_V &\sim N\left(\mu + \sigma^2 + \rho^J J_V, \sigma^2\right)\end{aligned}$$

The constant term $\lambda\left(\frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\eta\rho^J} - 1\right) iu\tau$ ensures that $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[S_T | S_{t_0}\right] = S_{t_0} e^{(r-q)\tau}$ holds true (as already mentioned) and we see that this compensating term is a function of the difference between the jump rate intensities under the two measures.

Focusing on our pricing formulae for Forward Starting Options, we have

$$\frac{\Pi_{t_0,T_1,T_2}^{\%C}}{e^{-r_1\tau_1}} = e^{-r_2\tau_2}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\mathbb{E}_{T_1}^{\mathbb{Q}}\left[\left(\frac{S_{T_2}}{S_{T_1}} - K\right)^+\right]\right] \tag{2.179}$$

$$\frac{\Pi_{t_0,T_1,T_2}^{\$C}}{S_{t_0}e^{-q_1\tau_1}} = e^{-r_2\tau_2}\mathbb{E}_{t_0}^{\mathbb{Q}_S}\left[\mathbb{E}_{T_1}^{\mathbb{Q}}\left[\left(\frac{S_{T_2}}{S_{T_1}} - K\right)^+\right]\right] \tag{2.180}$$

from the proof of proposition 3.

Within the affine framework and regarding the state variables S and V, $\mathbb{E}_{T_1}^{\mathbb{Q}}\left[\left(\frac{S_{T_2}}{S_{T_1}} - K\right)^+\right]$ is a function of V_{T_1} only. Therefore, by comparing our semi-analytic formulae for $\frac{\Pi_{t_0,T_1,T_2}^{\%C}}{e^{-r_1\tau_1}}$ and $\frac{\Pi_{t_0,T_1,T_2}^{\$C}}{S_{t_0}e^{-q_1\tau_1}}$ we can confirm the effect of a shift, from the risk-neutral to the stock price measure, on the dynamics of the variance process as the shift in measure over the period $(t, T_1]$ is the only difference between the two expressions. Making use of the pricing formulae in equations (2.49) - (2.52) of section 2.1, we can show that $\frac{R^{\%\,\mathrm{C}}_{t_0,T_1,T_2}}{e^{-r_1\tau_1}} = \frac{R^{\$\,\mathrm{C}}_{t_0,T_1,T_2}}{S_{t_0}e^{-q_1\tau_1}}$ and so we need only to compare $\Phi_{t_0,T_1,T_2}(-u+i[\alpha+1], u-i[\alpha+1])$ with $\Phi_{t_0,T_1,T_2}(-u+i\alpha, u-i[\alpha+1])$ to confirm that the parameters of the mean reverting drift of V, the jump size distribution parameter of V and the jump rate intensity differ according to the discussion above.

2.4 The conditional characteristic and forward $(\%,\$)$ characteristic functions allowing for piecewise constant, time-dependent parameters

From equation (2.11), we know that an analytic expression for the conditional characteristic function for X allows us to price European Options in semi-closed form and so the formulae may be used to calibrate the model's time-homogenous parameter set[2]. Consider the period $\tau = t_n - t_0$ (where t_n is the maturity date and t_0 is the valuation date) divided into n increments with the mth increment $\tau_m = t_m - t_{m-1}$. Replacing $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(izX_{t_n}\right)|X_{t_0}, V_{t_0}\right]$ in the semi-analytic formula for a European Option with its iterated extension

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp\left(izX_{t_n}\right)|X_{t_{n-1}}, V_{t_{n-1}}\right]...|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \tag{2.181}$$

allows us to incorporate piecewise constant time-dependent parameters into the model by providing a practical approach with which to *calibrate* these piecewise constant parameters when an analytic expression is available for equation (2.181). To determine the form of equation (2.181), we must solve a time-homogenous PDE for each *increment* where from one increment to the next, the constant parameter set may differ. At t_{n-1}, we must solve the PDE presented in equation (2.92), assuming the solution $\phi_{t_{n-1}}(X_{t_{n-1}}, V_{t_{n-1}}; iz, 0) := \mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp\left(izX_{t_n}\right)|X_{t_{n-1}}, V_{t_{n-1}}\right]$ has the form

$$\exp\left[izX_{t_{n-1}} + D_{n;n}(\tau_n, iz, 0)V_{t_{n-1}} + C_{n;n}(\tau_n, iz, 0)\right] \tag{2.182}$$

subject to the terminal conditions

$$C_{n;n}(0, iz, 0) = 0 \tag{2.183}$$
$$D_{n;n}(0, iz, 0) = 0 \tag{2.184}$$

At this point, for the time-homogenous model, the task of solving for the analytic characteristic function (for the entire period $(t_0, t_n]$) would be complete. For the extended model, we move on to the preceding increment. At t_{n-2}, we must solve the same PDE, assuming the solution $\phi_{t_{n-2}}(X_{t_{n-2}}, V_{t_{n-2}}; iz, D_{n;n}(\tau_n, iz, 0)) := \mathbb{E}^{\mathbb{Q}}_{t_{n-2}}\left[\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp\left(izX_{t_n}\right)|X_{t_{n-1}}, V_{t_{n-1}}\right]|X_{t_{n-2}}, V_{t_{n-2}}\right]$ has the form

$$\begin{aligned}
& \exp\left[C_{n;n}(\tau_n, iz, 0)\right]\mathbb{E}^{\mathbb{Q}}_{t_{n-2}}\left[\exp\left(izX_{t_{n-1}} + D_{n;n}(\tau_n, iz, 0)V_{t_{n-1}}\right)|X_{t_{n-2}}, V_{t_{n-2}}\right] && (2.185)\\
= \; & \exp\left[C_{n;n}(\tau_n, iz, 0)\right] && (2.186)\\
\times \; & \exp\left[izX_{t_{n-2}} + D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))V_{t_{n-2}} + C_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))\right]
\end{aligned}$$

[2]Assuming these vanilla options represent the most liquid options on the underlying, the model must, at the very least, reproduce the corresponding market prices.

subject to the terminal conditions

$$C_{n-1;n}(0, iz, D_{n;n}(\tau_n, iz, 0)) = 0 \quad (2.187)$$

$$D_{n-1;n}(0, iz, D_{n;n}(\tau_n, iz, 0)) = D_{n;n}(\tau_n, iz, 0) \quad (2.188)$$

We continue in this manner until we reach t_0. Regarding the subscripts of the functions $C_{m;n}$ and $D_{m;n}$, m specifies the increment currently considered while n specifies the total number of increments.

The semi-analytic formulae for $\%$ and \$ type Forward Starting Option prices can also accommodate for piecewise constant, time-dependent parameters by replacing the respective conditional forward $(\%, \$)$ characteristic functions with their iterated extensions. We divide the period $\tau = (t_n - t_l) + (t_l - t_0)$ (where t_l is the determination date) into n increments where $1 \leq l \leq n-1$ and require an analytic expression for

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{l-1}}\left[\exp\left(iz_{(\%,\$)}X_{t_l}\right)\mathbb{E}^{\mathbb{Q}}_{t_l}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp\left(izX_{t_n}\right)|X_{t_{n-1}}, V_{t_{n-1}}\right] ...|X_{t_l}, V_{t_l}\right] |X_{t_{l-1}}, V_{t_{l-1}}\right] ...|X_{t_0}, V_{t_0}\right] \quad (2.189)$$

Consider the case $l = n-1$ i.e. only one increment separates the determination date t_l from the maturity date t_n. At t_l, we then have

$$\exp\left(iz_{(\%,\$)}X_{t_l}\right)\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp\left(izX_{t_n}\right)|X_{t_l}, V_{t_l}\right] \quad (2.190)$$

$$= \exp\left(i\left[z_{(\%,\$)} + z\right]X_{t_l} + D_{n;n}(\tau_n, iz, 0)V_{t_l} + C_{n;n}(\tau_n, iz, 0)\right) \quad (2.191)$$

For a $\%$ type option, $iz_{\%} + iz = 0$ while for a \$ type option, $iz_{\$} + iz = 1$. Hence, for $l = n-1$, we can re-express equation (2.189) as

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{l-1}}\left[\exp\left(\mathbb{I}X_{t_l} + D_{n;n}(\tau_n, iz, 0)V_{t_l} + C_{n;n}(\tau_n, iz, 0)\right)|X_{t_{l-1}}, V_{t_{l-1}}\right] ...|X_{t_0}, V_{t_0}\right]$$

where $\mathbb{I} := \mathbb{I}_{\left[z_{(\%,\$)} = z_{\$}\right]}$. At the terminal time t_l, the coefficient of X_{t_l}, to which the exponent is raised, is $\mathbb{I}$ rather than iz. So regarding the functions C and D, for each increment m where $m \leq l$, the second argument iz is replaced by $\mathbb{I}$.

We now present the form of the conditional characteristic and forward $(\%, \$)$ characteristic functions, allowing for piecewise constant, time-dependent parameters, in more detail.

Proposition 4. *Considering the period $\tau = (t_n - t_l) + (t_l - t_0)$ expressed in terms of n increments where $0 \leq l \leq n-1$ and the mth increment is $\tau_m = t_m - t_{m-1}$, we have*

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_{(\%,\$)}\hat{\mathbb{I}}X_{t_l} + izX_{t_n}\right)|X_{t_0}, V_{t_0}\right] \\ = &\exp\left[\left(\left[\mathbb{I} - iz\right]\hat{\mathbb{I}} + iz\right)X_{t_0} + D^{(\%,\$)}_{l;1;n}(\tau_1, \mathbb{I}, D^{(\%,\$)}_{l;2;n})V_{t_0}\right] \\ \times &\exp\left[\sum_{m=1}^{l} C^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}) + \sum_{m=l+1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right] \end{aligned} \quad (2.192)$$

where $\hat{\mathbb{I}} := \mathbb{I}_{[l \neq 0]}$, $z_{(\%,\$)} := z_{\%}$ for a $\%$ type option, $z_{(\%,\$)} := z_{\$}$ for a \$ type option and $\mathbb{I} := \mathbb{I}_{\left[z_{(\%,\$)} = z_{\$}\right]}$. Furthermore, we define the following:

For $m > l$,

$$\begin{aligned} D^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) &= D_{m;n}(\tau_m, iz, D_{m+1;n}) && (2.193)\\ D_{m;n}(\tau_m, iz, D_{m+1;n}) &:= D_{m;n}(\tau_m, iz, D_{m+1;n}(\tau_{m+1}, iz, ...D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))...)) && (2.194)\\ C^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) &= C_{m;n}(\tau_m, iz, D_{m+1;n}) && (2.195)\\ C_{m;n}(\tau_m, iz, D_{m+1;n}) &:= C_{m;n}(\tau_m, iz, D_{m+1;n}(\tau_{m+1}, iz, ...D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))...)) && (2.196) \end{aligned}$$

with $D_{n+1;n} := 0$.

For $m \leq l$,

$$\begin{aligned} D^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) &:= D^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}(\tau_{m+1}, \mathbb{I}, ...D^{(\%,\$)}_{l;l;n}(\tau_l, \mathbb{I}, D_{l+1;n}(\tau_{l+1}, iz, D_{l+2;n}))...)) && (2.197)\\ C^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) &:= C^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}(\tau_{m+1}, \mathbb{I}, ...D^{(\%,\$)}_{l;l;n}(\tau_l, \mathbb{I}, D_{l+1;n}(\tau_{l+1}, iz, D_{l+2;n}))...)) && (2.198) \end{aligned}$$

Regarding the subscripts of $C^{(\%,\$)}_{l;m;n}$ and $D^{(\%,\$)}_{l;m;n}$, l specifies the determination date t_l, m specifies the increment currently considered and n specifies the total number of increments.

Proof:
For $l = 0$, we have $\hat{\mathbb{I}} = 0$ and require an analytic expression for

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(izX_{t_n})\,|X_{t_0}, V_{t_0}\right] = \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp(izX_{t_n})\,|X_{t_{n-1}}, V_{t_{n-1}}\right]...|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right]$$

For $n = 1$, we have

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(izX_{t_1})\,|X_{t_0}, V_{t_0}\right] = \exp\left[izX_{t_0} + D_{1;1}(\tau_1, iz, 0)V_{t_0} + C_{1;1}(\tau_1, iz, 0)\right] \quad (2.199)$$

For $n = 2$, we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[\exp(izX_{t_2})\,|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right]\\ =\ &\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left[izX_{t_1} + D_{2;2}(\tau_2, iz, 0)V_{t_1} + C_{2;2}(\tau_2, iz, 0)\right]|X_{t_0}, V_{t_0}\right]\\ =\ &\exp\left[izX_{t_0} + D_{1;2}(\tau_1, iz, D_{2;2}(\tau_2, iz, 0))V_{t_0} + C_{1;2}(\tau_1, iz, D_{2;2}(\tau_2, iz, 0)) + C_{2;2}(\tau_2, iz, 0)\right] \end{aligned} \quad (2.200)$$

Continuing in this manner, for $n > 2$, we obtain

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp(izX_{t_n})\,|X_{t_{n-1}}, V_{t_{n-1}}\right]...|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right]\\ =\ &\exp\left[izX_{t_0} + D_{1;n}(\tau_1, iz, D_{2;n})V_{t_0} + \sum_{m=1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right] \end{aligned} \quad (2.201)$$

For $l > 0$, we have $\hat{\mathbb{I}} = 1$ and require an analytic expression for

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_{(\%,\$)}X_{t_l} + izX_{t_n}\right)|X_{t_0}, V_{t_0}\right] \\
= \;& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_{(\%,\$)}X_{t_l}\right)\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp\left(izX_{t_n}\right)|X_{t_l}, V_{t_l}\right]|X_{t_0}, V_{t_0}\right] \\
= \;& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{l-1}}\left[\exp\left(iz_{(\%,\$)}X_{t_l}\right)\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp\left(izX_{t_n}\right)|X_{t_l}, V_{t_l}\right]|X_{t_{l-1}}, V_{t_{l-1}}\right]...|X_{t_0}, V_{t_0}\right]
\end{aligned} \tag{2.202}$$

For a $\%$ type option, we have $z_{\%} = -z$ and so $iz_{\%} + iz = 0$ while for a \$ type option, we have $z_{\$} = -(z+i)$ and so $iz_{\$} + iz = 1$. Hence, we define $\mathbb{I} := \mathbb{I}_{\left[z_{(\%,\$)} = z_{\$}\right]} = iz_{(\%,\$)} + iz$. In what follows, we make use of the appropriate version of equation (2.201) to determine $\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp\left(izX_{t_n}\right)|X_{t_l}, V_{t_l}\right]$.

For $l = 1$, we have

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(iz_{(\%,\$)}X_{t_1}\right)\mathbb{E}^{\mathbb{Q}}_{t_1}\left[\exp\left(izX_{t_n}\right)|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \\
= \;& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left[\mathbb{I}X_{t_1} + D_{2;n}(\tau_2, iz, D_{3;n})V_{t_1} + \sum_{m=2}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right]|X_{t_0}, V_{t_0}\right] \\
= \;& \exp\left[\mathbb{I}X_{t_0} + D^{(\%,\$)}_{1;1;n}(\tau_1, \mathbb{I}, D_{2;n}(\tau_2, iz, D_{3;n}))V_{t_0}\right] \\
\times \;& \exp\left[C^{(\%,\$)}_{1;1;n}(\tau_1, \mathbb{I}, D_{2;n}(\tau_2, iz, D_{3;n})) + \sum_{m=2}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right]
\end{aligned} \tag{2.203}$$

For $l = 2$, we have

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[\exp\left(iz_{(\%,\$)}X_{t_2}\right)\mathbb{E}^{\mathbb{Q}}_{t_2}\left[\exp\left(izX_{t_n}\right)|X_{t_2}, V_{t_2}\right]|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \\
= \;& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[\exp\left[\mathbb{I}X_{t_2} + D_{3;n}(\tau_3, iz, D_{4;n})V_{t_2}\right]|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \\
\times \;& \exp\left[\sum_{m=3}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right] \\
= \;& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left[\mathbb{I}X_{t_1} + D^{(\%,\$)}_{2;2;n}(\tau_2, \mathbb{I}, D_{3;n}(\tau_3, iz, D_{4;n}))V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \\
\times \;& \exp\left[C^{(\%,\$)}_{2;2;n}(\tau_2, \mathbb{I}, D_{3;n}(\tau_3, iz, D_{4;n})) + \sum_{m=3}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right] \\
= \;& \exp\left[\mathbb{I}X_{t_0} + D^{(\%,\$)}_{2;1;n}(\tau_1, \mathbb{I}, D^{(\%,\$)}_{2;2;n}(\tau_2, \mathbb{I}, D_{3;n}(\tau_3, iz, D_{4;n})))V_{t_0}\right] \\
\times \;& \exp\left[C^{(\%,\$)}_{2;1;n}(\tau_1, \mathbb{I}, D^{(\%,\$)}_{2;2;n}(\tau_2, \mathbb{I}, D_{3;n}(\tau_3, iz, D_{4;n})))\right] \\
\times \;& \exp\left[C^{(\%,\$)}_{2;2;n}(\tau_2, \mathbb{I}, D_{3;n}(\tau_3, iz, D_{4;n})) + \sum_{m=3}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right]
\end{aligned} \tag{2.204}$$

Continuing in this manner for $l > 2$, we obtain

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{l-1}}\left[\exp\left(iz_{(\%,\$)}X_{t_l}\right)\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp\left(izX_{t_n}\right)|X_{t_l}, V_{t_l}\right] X_{t_{l-1}}, V_{t_{l-1}}\right]...|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \\
= \;& \exp\left[\mathbb{I}X_{t_0} + D^{(\%,\$)}_{l;1;n}(\tau_1, \mathbb{I}, D^{(\%,\$)}_{l;2;n})V_{t_0}\right] \\
\times \;& \exp\left[\sum_{m=1}^{l} C^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}) + \sum_{m=l+1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right]
\end{aligned} \tag{2.205}$$

□

The analytic expression for the conditional joint characteristic function for X and V, derived in section 2.2, is all we require to determine the functions $C^{(\%,\$)}_{l;m;n}(\,\cdot,\cdot,\cdot)$ and $D^{(\%,\$)}_{l;m;n}(\,\cdot,\cdot,\cdot)$ with the piecewise constant, time-dependent parameter set

$$(\kappa_t, \theta_t, \nu_t, \rho_t, \lambda_t, \eta_t, \mu_t, \sigma_t, \rho^J_t, r_t, q_t) \quad := \quad \sum_{m=1}^{n} (\kappa_m, \theta_m, \nu_m, \rho_m, \lambda_m, \eta_m, \mu_m, \sigma_m, \rho^J_m, r_m, q_m)\, \mathbb{I}_{[t_{m-1}<t\le t_m]} \tag{2.206}$$

where $x_m := x_{(t_0;t_{m-1},t_m]}$ for $m = 1, ..., n$ and all parameters x i.e. x_m is the constant parameter seen at t_0 that applies over the period $(t_{m-1}, t_m]$.

We have the iterative expressions

$$C_{m;n}(\tau_m, iz, D_{m+1;n}) \quad = \quad C(\tau_m, iz, D_{m+1;n}) \tag{2.207}$$

$$D_{m;n}(\tau_m, iz, D_{m+1;n}) \quad = \quad D(\tau_m, iz, D_{m+1;n}) \tag{2.208}$$

$$D_{m+1;n} \quad := \quad D_{m+1;n}(\tau_{m+1}, iz, D_{m+1;n}) \tag{2.209}$$

with $D_{n+1;n} := 0$. Regarding the subscripts of $C_{m;n}$ and $D_{m;n}$, the first subscript m specifies the current increment and so the constant parameter set $(\kappa_m, \theta_m, \nu_m, \rho_m, \lambda_m, \eta_m, \mu_m, \sigma_m, \rho^J_m, r_m, q_m)$ that applies over the period $(t_{m-1}, t_m]$ where $\tau_m = t_m - t_{m-1}$. The functions $D(\tau_m, iz, D_{m+1;n})$ and $C(\tau_m, iz, D_{m+1;n})$ are presented in equations (2.78) and (2.79) of proposition 2 respectively, where $iz_v = D_{m+1;n}$.

Chapter 3

Issues regarding the semi-analytic pricing formulae

3.1 The conditional characteristic function

For $z := u - i\zeta$ and $\zeta := \alpha + 1$, the characteristic function

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{izX_T}|X_{t_0}, V_{t_0}\right] = \int_{-\infty}^{\infty} e^{\zeta x}\cos(ux) f_{t_0,T}(x|\bar{x}_{t_0})dx + i\int_{-\infty}^{\infty} e^{\zeta x}\sin(ux) f_{t_0,T}(x|\bar{x}_{t_0})dx \quad (3.1)$$

is single-valued where $f_{t_0,T}$ is the corresponding density function and $\bar{x}_{t_0} = (X_{t_0}, V_{t_0})$. The restriction $\zeta \in (\zeta^{\min}, \zeta^{\max})$ ensures that the real integrals do, in fact, exist. Within the affine framework, we can work with $\pm\gamma(iz)$ (where $\gamma(iz)$ is defined in equation (2.88)) as seen from equations (2.102) and (2.103) and can express the conditional characteristic function as

$$\begin{aligned} & \exp\left[\zeta X_{t_0} + \text{Re}\left[D(\tau, iz, 0)\right] V_{t_0} + \text{Re}\left[C(\tau, iz, 0)\right]\right] \cos\left(uX_{t_0} + \text{Im}\left[D(\tau, iz, 0)\right] V_{t_0} + \text{Im}\left[C(\tau, iz, 0)\right]\right) \\ + \quad & i\exp\left[\zeta X_{t_0} + \text{Re}\left[D(\tau, iz, 0)\right] V_{t_0} + \text{Re}\left[C(\tau, iz, 0)\right]\right] \sin\left(uX_{t_0} + \text{Im}\left[D(\tau, iz, 0)\right] V_{t_0} + \text{Im}\left[C(\tau, iz, 0)\right]\right) \end{aligned} \quad (3.2)$$

We can show that the functions $D(\tau, iz, 0)$ and $\text{Re}\left[C(\tau, iz, 0)\right]$ are even in $\gamma(iz)$. Since $\text{Im}\left[C(\tau, iz, 0)\right]$ features only as part of the argument of the trigonometric functions sine and cosine, $\text{Im}\left[C(\tau, iz, 0)\right]$ must be even in $\gamma(iz)$ modulo a factor of 2π to ensure that the characteristic function is, in fact, even in $\gamma(iz)$ and so single-valued.

At $u = 0$ where $iz = \zeta$, we obtain the moment generating function $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}|X_{t_0}, V_{t_0}\right]$ which is both real and positive. Within the affine framework, $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}|X_{t_0}, V_{t_0}\right] = \exp\left[\zeta X_{t_0} + D(\tau, \zeta, 0)V_{t_0} + C(\tau, \zeta, 0)\right]$. We now highlight several results, regarding the moment generating function, which will be referred to within the chapter.

Proposition 5. *For $u = 0$ and $\zeta \in [\zeta^-, \zeta^+]$*

$$\gamma(\zeta) = \sqrt{(\kappa - \rho\nu\zeta)^2 - \nu^2[\zeta - 1]\zeta} \quad (3.3)$$

with $\gamma(\zeta) = 0$ for the roots $\zeta^{\pm}$ where $\zeta^{-} < 0$ and $\zeta^{+} \geq 1$. For $u = 0$ and $\zeta \in (-\infty, \zeta^{-}) \cup (\zeta^{+}, \infty)$

$$\begin{aligned} \gamma(\zeta) &= i\mathit{Im}\left[\gamma(\zeta)\right] && (3.4) \\ \mathit{Im}\left[\gamma(\zeta)\right] &:= \sqrt{-[(\kappa - \rho\nu\zeta)^2 - \nu^2[\zeta - 1]\zeta]} && (3.5) \end{aligned}$$

Proof: The quadratic function $\gamma^2(\zeta) = -\nu^2(1-\rho^2)\zeta^2 + \nu(\nu - 2\kappa\rho)\zeta + \kappa^2$ is concave in ζ since $-\nu^2(1-\rho^2) < 0$. At $\zeta = 0$, we have $\gamma^2(\zeta) = \kappa^2$ while at $\zeta = 1$, we have $\gamma^2(\zeta) = (\kappa - \rho\nu)^2$. Therefore, we have the roots $\zeta^{-} < 0$ and $\zeta^{+} \geq 1$ with $\gamma^2(\zeta) \geq 0$ for $\zeta \in [\zeta^{-}, \zeta^{+}]$. □

Proposition 6. *Within the affine framework, we have*

$$\mathit{Im}\left[D(\tau, \zeta, 0)\right] = 0 \tag{3.6}$$

and require that, for some $n \in \mathbb{Z}$,

$$\mathit{Im}\left[C(\tau, \zeta, 0)\right] = 2\pi n \tag{3.7}$$

Proof: We can express the moment generating function as

$$\begin{aligned} & \exp\left[\zeta X_{t_0} + \mathrm{Re}\left[D(\tau, \zeta, 0)\right] V_{t_0} + \mathrm{Re}\left[C(\tau, \zeta, 0)\right]\right] \cos\left(\mathrm{Im}\left[D(\tau, \zeta, 0)\right] V_{t_0} + \mathrm{Im}\left[C(\tau, \zeta, 0)\right]\right) \\ + \; & i \exp\left[\zeta X_{t_0} + \mathrm{Re}\left[D(\tau, \zeta, 0)\right] V_{t_0} + \mathrm{Re}\left[C(\tau, \zeta, 0)\right]\right] \sin\left(\mathrm{Im}\left[D(\tau, \zeta, 0)\right] V_{t_0} + \mathrm{Im}\left[C(\tau, \zeta, 0)\right]\right) \end{aligned} \tag{3.8}$$

Since the moment generating function is both real and positive, we must have

$$\mathrm{Im}\left[D(\tau, \zeta, 0)\right] V_{t_0} + \mathrm{Im}\left[C(\tau, \zeta, 0)\right] = 2\pi n \tag{3.9}$$

for some $n \in \mathbb{Z}$ to ensure that both

$$\begin{aligned} \sin\left(\mathrm{Im}\left[D(\tau, \zeta, 0)\right] V_{t_0} + \mathrm{Im}\left[C(\tau, \zeta, 0)\right]\right) &= 0 && (3.10) \\ \cos\left(\mathrm{Im}\left[D(\tau, \zeta, 0)\right] V_{t_0} + \mathrm{Im}\left[C(\tau, \zeta, 0)\right]\right) &> 0 && (3.11) \end{aligned}$$

Since equation (3.9) must hold for any positive V_{t_0}, we must have[1]

$$\mathrm{Im}\left[D(\tau, \zeta, 0)\right] = 0$$

leaving us with the requirement that for some $n \in \mathbb{Z}$

$$\mathrm{Im}\left[C(\tau, \zeta, 0)\right] = 2\pi n$$

□

From the proof of proposition 6 it follows that

$$D^{(\%,\$)}_{l;1;n}(\tau_1, \mathbb{I}, D^{(\%,\$)}_{l;2;n}|_{u=0}) \in \mathbb{R} \tag{3.12}$$

[1]We originally derived this result by tediously splitting the function into its real and imaginary parts and then made use of the properties of even and odd functions. Subsequently, it was pointed out by Roger Lord, in a personal communication, that the coefficient of the variance process can be shown to be real by simply appealing to the definition of the moment generating function.

for $0 \leq l < n$. Given n and l, $D^{(\%,\$)}_{l;2;n}(\tau_2, \mathbb{I}, D^{(\%,\$)}_{l;3;n}|_{u=0})$ has the same form (regarding the terminal conditions specified by the arguments $\mathbb{I}$ and $D^{(\%,\$)}_{l;3;n}|_{u=0}$) as $D^{(\%,\$)}_{l-1;1;n-1}(\tau_1, \mathbb{I}, D^{(\%,\$)}_{l-1;2;n-1}|_{u=0})$, $D^{(\%,\$)}_{l;3;n}(\tau_3, \mathbb{I}, D^{(\%,\$)}_{l;4;n}|_{u=0})$ has the same form as $D^{(\%,\$)}_{l-2;1;n-2}(\tau_1, \mathbb{I}, D^{(\%,\$)}_{l-2;2;n-2}|_{u=0})$ and so on. Hence, we have

$$D^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}|_{u=0}) \in \mathbb{R} \tag{3.13}$$

for $m = 1, ..., n$ and $0 \leq l < n$.

The logarithm of a complex number Z is multi-valued since its imaginary part is the argument or angle of Z. Specifically, we have

$$\log(Z) \quad := \quad \ln(|Z|) + i \arg(Z) \tag{3.14}$$

$$\arg(Z) \quad := \quad \mathrm{Arg}(Z) + 2\pi n \tag{3.15}$$

where $\ln(Z)$ is the natural logarithm of Z, $|Z|$ is the modulus or absolute value of Z, $\mathrm{Arg}(Z) \in (-\pi, \pi]$ is the principal branch of $\arg(Z)$, $n \in \mathbb{Z}$ where values of n other than zero shift the argument of Z to alternative branches with each branch specifying a single-valued portion of the argument of Z. Defined as such, $\arg(Z)$ has a branch cut along the negative real axis in the complex plane (including the origin). Working counter-clockwise, as Z crosses the branch cut, its argument increases from say, the value π along the branch cut to the value $\pi + \epsilon$ where these two points lie in different branches. When evaluated with a software package such as MatLab, the argument of a function is restricted to its principal branch. In this example, as the function crosses the branch cut it is forced to jump from the value π along the branch cut to the value $-\pi + \epsilon$.

Regarding the condition characteristic function for the Heston model, we work with the logarithm of the complex function $\psi(\tau, iz, 0)$. Focussing on the moment generating function, we have the following.

Proposition 7. *For $u = 0$ and $\zeta \in (-\infty, \zeta^-) \cup (\zeta^+, \infty)$, re-defining*

$$\arg[\psi(\tau, \zeta, 0)] \quad := \quad -\frac{1}{2} Im[\gamma(\zeta)]\, \tau \tag{3.16}$$

ensures equation (3.7) of proposition 6 is satisfied for any value of $\frac{2\kappa\theta}{\nu^2}$. In particular, we have $Im[C(\tau, \zeta, 0)] = 0$.

Proof: From proposition 5, we have $\gamma(\zeta) = i\mathrm{Im}[\gamma(\zeta)]$ for $u = 0$ and $\zeta \in (-\infty, \zeta^-) \cup (\zeta^+, \infty)$ and working from equation (2.103) of the proof of proposition 2 in section 2.2, we obtain

$$\mathrm{Im}[C(\tau, \zeta, 0)] \quad = \quad -\frac{2\kappa\theta}{\nu^2}\left(\frac{1}{2}\mathrm{Im}[\gamma(\zeta)]\, \tau + \arg[\psi(\tau, \zeta, 0)]\right) \tag{3.17}$$

Working from equation (2.102), we would obtain

$$\mathrm{Im}[C(\tau, \zeta, 0)] \quad = \quad \frac{2\kappa\theta}{\nu^2}\left(\frac{1}{2}\mathrm{Im}[\gamma(\zeta)]\, \tau - \arg\left[\psi(\tau, \zeta, 0)e^{\gamma(\zeta)\tau}\right]\right) \tag{3.18}$$

From equation (3.7) of proposition 6, we know that $\mathrm{Im}[C(\tau, \zeta, 0)]$ must be an integer multiple of 2π. Focussing on equation (3.17) and assuming that this requirement is satisfied for a particular branch of $\arg[\psi(\tau, \zeta, 0)]$, it does not necessarily follow that the requirement is satisfied for alternative branch choices because of the constant $\frac{2\kappa\theta}{\nu^2}$ attached to this multi-valued function. Re-defining

$$\arg[\psi(\tau, \zeta, 0)] \quad := \quad -\frac{1}{2}\mathrm{Im}[\gamma(\zeta)]\, \tau \tag{3.19}$$

ensures that the requirement is satisfied for any particular choice of the parameters κ, θ and ν i.e. $\text{Im}\left[C(\tau,\zeta,0)\right] = 0$. We can show that $\psi(\tau,\zeta,0)e^{\gamma(\zeta)\tau}$ is the complex conjugate of $\psi(\tau,\zeta,0)$[2] and so we obtain the same result if we work from equation (3.18) instead. □

From the proof of proposition 10 in subsection 3.2.2 (which follows) it will be clarified that for $\zeta \in (\zeta^{\min},\zeta^{-}) \cup (\zeta^{+},\zeta^{\max})$ we have

$$\frac{1}{2}\text{Im}\left[\gamma(\zeta)\right]\tau \quad \in \quad (0,\pi) \tag{3.20}$$

and so by making use of proposition 7 we are effectively restricting the multi-valued argument to its principal branch.

3.1.1 Discontinuities introduced by the complex logarithm in the Heston model

Regarding the conditional characteristic function for the Heston model and working from equation (2.103) of the proof of proposition 2 in section 2.2, we obtain

$$\text{Im}\left[C(\tau,iz,0)\right] \quad = \quad (r-q)u\tau + \frac{\kappa\theta}{\nu^2}\left[\text{Im}\left[b(iz)\right] - \text{Im}\left[\gamma(iz)\right]\right]\tau - \frac{2\kappa\theta}{\nu^2}\arg\left[\psi(\tau,iz,0)\right] \tag{3.21}$$

Had we started off working with equation (2.102) (where only the sign of the square root function differs from that in equation (2.103)), we would have obtained

$$\text{Im}\left[C(\tau,iz,0)\right] \quad = \quad (r-q)u\tau + \frac{\kappa\theta}{\nu^2}\left[\text{Im}\left[b(iz)\right] + \text{Im}\left[\gamma(iz)\right]\right]\tau - \frac{2\kappa\theta}{\nu^2}\arg\left[\psi(\tau,iz,0)e^{\gamma(iz)\tau}\right] \tag{3.22}$$

The conditional characteristic function as originally presented in Heston [1993] is expressed in terms of equation (3.22). In principle, equations (3.21) and (3.22) are the same (multi-valued) function. However, when evaluated within a software package, $\arg\left[\psi(\tau,iz,0)e^{\gamma(iz)\tau}\right]$ is restricted to its principal branch which introduces discontinuities when the function crosses the negative real axis. This problem was originally noted in Schobel and Zhu [1999] footnote 7 where the authors suggest keeping track of the complex logarithm along its integration path. Effectively, one must correct any discontinuities arising from the restriction to the principal branch of the multi-valued argument. In Kahl and Jackel [2005], an algorithm is presented to avoid having to track the function. The key step is to redefine the imaginary part of the complex logarithm as

$$\arg\left[\psi(\tau,iz,0)e^{\gamma(iz)\tau}\right] \quad := \quad \text{Arg}\left[A(iz,0)e^{\gamma(iz)\tau} - 1\right] - \text{Arg}\left[A(iz,0) - 1\right] + 2\pi(n-m) \tag{3.23}$$

[2]We can write

$$\begin{aligned}
\text{Re}\left[\psi(\tau,\zeta,0)\right] &= \frac{\text{Im}\left[\gamma(\zeta)\right](1+\cos[-\text{Im}\left[\gamma(\zeta)\right]\tau]) - b(\zeta)\sin[-\text{Im}\left[\gamma(\zeta)\right]\tau]}{2\text{Im}\left[\gamma(\zeta)\right]} \\
\text{Im}\left[\psi(\tau,\zeta,0)\right] &= \frac{b(\zeta)(\cos[-\text{Im}\left[\gamma(\zeta)\right]\tau] - 1) + \text{Im}\left[\gamma(\zeta)\right]\sin[-\text{Im}\left[\gamma(\zeta)\right]\tau]}{2\text{Im}\left[\gamma(\zeta)\right]}
\end{aligned}$$

For $\zeta \in (-\infty,\zeta^{-}) \cup (\zeta^{+},\infty)$, Euler's formula gives us $e^{\gamma(\zeta)\tau} = \cos[-\text{Im}\left[\gamma(\zeta)\right]\tau] - i\sin[-\text{Im}\left[\gamma(\zeta)\right]\tau]$. It follows that

$$\begin{aligned}
\text{Re}\left[\psi(\tau,\zeta,0)e^{\gamma(\zeta)\tau}\right] &= \text{Re}\left[\psi(\tau,\zeta,0)\right] \\
\text{Im}\left[\psi(\tau,\zeta,0)e^{\gamma(\zeta)\tau}\right] &= -\text{Im}\left[\psi(\tau,\zeta,0)\right]
\end{aligned}$$

where[3]

$$m = \text{int}\left[\frac{\text{Arg}[A(iz,0)]+\pi}{2\pi}\right] \tag{3.24}$$
$$n = \text{int}\left[\frac{\text{Arg}[A(iz,0)]+\pi+\text{Im}\left[\gamma(iz)\right]\tau}{2\pi}\right] \tag{3.25}$$

For a detailed discussion of equation (3.23), we refer the reader to the original article.

However, in Lord and Kahl [2007], the authors observe that an 'alternative' expression for the conditional characteristic function has emerged in the relevant literature since Heston's seminal work. The defining difference being that Im $[C(\tau, iz, 0)]$ is a function of $\arg\left[\psi(\tau, iz, 0)\right]$ rather than $\arg\left[\psi(\tau, iz, 0)e^{\gamma(iz)\tau}\right]$ i.e. the alternative arises from working with equation (3.21) rather than equation (3.22). It has been noted that using this alternative leads to results free of any complex discontinuities. In Gatheral [2006], it is conjectured that $\arg\left[\psi(\tau, iz, 0)\right]$ may be restricted to its principal branch without introducing any discontinuities (this claim is made at least for Heston's original representation of the option price which has the 'Black-Scholes' form of equation (2.39) and so makes use of the characteristic function for $\zeta = 0, 1$). Several attempts have been made to prove this result. In Lord and Kahl [2006] a proof is provided for $\zeta \in (\zeta^{\min}, \zeta^{\max})$ with the restriction $\rho < \frac{\kappa}{\nu}$ or $\zeta \leq \frac{\kappa}{\rho\nu}$ and $\frac{\kappa}{\nu} \leq \rho < \frac{2\kappa}{\nu}$. In Albrecher et al. [2007] a restriction free proof is provided for $\zeta > 1$. In Fahrner [2007], a proof is provided for the 'displaced-diffusion' extention of the Heston model (which we elaborate on in subsection 3.5.3), specifically for the case $\zeta = \frac{1}{2}$. Finally, for the (time-homogenous) Heston model, the issue is laid to rest in Lord and Kahl [2008] where a restriction free proof is provided for $\zeta \in (\zeta^{\min}, \zeta^{\max})$. Hence, we can restrict $\arg\left[\psi(\tau, iz, 0)\right]$ to any one branch, for all $u \in [0, \infty)$. We return to the issue of branch cutting in section 3.5, where we allow for piecewise constant, time-dependent parameters.

Discontinuities arise when the branch cut is crossed. However, this in itself is not the reason for the problem. Referring to the form of the conditional characteristic function as presented in equation (3.2), we see that Im $[C(\tau, iz, 0)]$ features only as part of the argument of the trigonometric sine and cosine functions. The problem arises specifically because of the constant coefficient of the multi-valued argument or angle, $\frac{2\kappa\theta}{\nu^2}$. The value Im $[C(\tau, iz, 0)]$ may be specified modulo a factor of $\frac{2\kappa\theta}{\nu^2}2\pi$. Hence, restricting the argument to a specific branch when the branch cut is attainable by the original function leads to discontinuities when the function crosses the branch cut, the argument is forced to jump by an integer multiple of 2π and Im $[C(\tau, iz, 0)]$ jumps by a non-integer multiple of 2π. When working with equation (3.21) where the range of $\psi(\tau, iz, 0)$ does not include the branch cut, one can specify any particular branch, for all $u \in [0, \infty)$. The principal branch avoids any complications introduced by the constant coefficient $\frac{2\kappa\theta}{\nu^2}$.

3.1.2 A second discontinuity

In Albrecher et al. [2007], a second discontinuity is noted at $u = 0$. As effectively stated by the authors, for $z = u - i\zeta$, to avoid a discontinuity in $\gamma(iz)$ at $u = 0$, let $\gamma(\zeta) := \lim_{u\to 0}\gamma(iz)$.

We analyse this issue in more detail to illustrate that it can, in fact, be ignored. From the definition of

[3] int(x) refers to the integer part of x

$\gamma(iz)$ in equation (2.88) of proposition 2 in section 2.2, we have

$$\gamma(iz) \quad = \quad \sqrt{[\kappa-\rho\nu\zeta]^2-\nu^2(\zeta-1)\zeta+u^2\nu^2(1-\rho^2)-iu\left[\nu^2(2\zeta-1)+2\rho\nu[\kappa-\rho\nu\zeta]\right]} \tag{3.26}$$

Restricting ourselves to the principal square root (whose real part is positive), we have the following result from Rabinowitz [1993]:

Theorem 1. *If a and b are real ($b \neq 0$), then $\sqrt{a+bi} = p+qi$ where p and q are real and are given by*

$$p \quad = \quad \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a} \tag{3.27}$$

$$q \quad = \quad \frac{sign(b)}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a} \tag{3.28}$$

Theorem 1 allows us to highlight the case within which a discontinuity arises. For $a<0$, we have

$$\begin{aligned} \lim_{b\uparrow 0}\sqrt{a+bi} \quad &= \quad \frac{1}{\sqrt{2}}\sqrt{|a|+a}+i\frac{-1}{\sqrt{2}}\sqrt{|a|-a} \\ &= \quad -i\sqrt{-a} \end{aligned} \tag{3.29}$$

since $\lim_{b\uparrow 0}$ sign$(b)=-1$ while for $b=0$, we have

$$\sqrt{a} \quad = \quad i\sqrt{-a} \tag{3.30}$$

From equation (3.26), we see that as $u\downarrow 0$, the problem arises for $\nu^2(2\zeta-1)+2\rho\nu[\kappa-\rho\nu\zeta] > 0$ and $[\kappa-\rho\nu\zeta]^2-\nu^2(\zeta-1)\zeta<0$. At $u=0$, we have $\gamma(\zeta)=i\text{Im}\left[\gamma(\zeta)\right]$ while

$$\lim_{u\downarrow 0}\gamma(iz) = -i\text{Im}\left[\gamma(\zeta)\right] \tag{3.31}$$

and so a discontinuity arises, at $u=0$, as the function switches sign from its limit at this point.

For $\gamma(\zeta)=i\text{Im}\left[\gamma(\zeta)\right]$, Matlab evaluates $\text{Im}\left[C(\tau,\zeta,0)\right]$ (from the representation in equation (3.17)) as

$$\text{Im}\left[C(\tau,\zeta,0)\right] \quad = \quad -\frac{2\kappa\theta}{\nu^2}\left(\frac{1}{2}\text{Im}\left[\gamma(\zeta)\right]\tau+\text{Arg}\left[\psi(\tau,\zeta,0)\right]\right) \tag{3.32}$$

For $\gamma(\zeta)=-i\text{Im}\left[\gamma(\zeta)\right]$, Matlab evaluates $\text{Im}\left[C(\tau,\zeta,0)\right]$ (from the representation in equation (3.18)) as

$$\text{Im}\left[C(\tau,\zeta,0)\right] \quad = \quad \frac{2\kappa\theta}{\nu^2}\left(\frac{1}{2}\text{Im}\left[\gamma(\zeta)\right]\tau-\text{Arg}\left[\psi(\tau,\zeta,0)e^{\gamma(\zeta)\tau}\right]\right) \tag{3.33}$$

However, as pointed out in the proof of proposition 10, $\psi(\tau,\zeta,0)e^{\gamma(\zeta)\tau}$ is the complex conjugate of $\psi(\tau,\zeta,0)$. Hence, $\text{Arg}\left[\psi(\tau,\zeta,0)e^{\gamma(\zeta)\tau}\right] = -\text{Arg}\left[\psi(\tau,\zeta,0)\right]$ and $\text{Im}\left[C(\tau,\zeta,0)\right]$ is evaluated as an odd function of $\text{Im}\left[\gamma(\zeta)\right]$.

Assuming $\text{Im}\left[C(\tau,\zeta,0)\right]$ satisfies equation (3.7) of proposition 6 for $-\text{Im}\left[\gamma(\zeta)\right]$ it then follows from the odd property that the requirement is also satisfied for $+\text{Im}\left[\gamma(\zeta)\right]$. Therefore, the sign change in $\text{Im}\left[\gamma(\zeta)\right]$ does not affect the moment generating function.

3.2 Strip of regularity

The range of $\zeta := \alpha + 1$ such that the moment generating function $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}|X_{t_0}, V_{t_0}\right]$ exists is referred to as the strip of regularity. For this strip, the option price obtained via a Fourier inversion in equation (2.11) exists subject to points of singularity at $u = 0$ and $\zeta = 0, 1$. From the discussion in subsections 2.1.1 and 2.1.2, we need only to consider the respective conditional characteristic and forward $(\%, \$)$ characteristic functions to determine these strips for European and Forward Starting Options where, for the latter, we refer to the strip as a $(\%, \$)$ strip of regularity.

In this section, we determine the strip of regularity for the SVJJ model (allowing for piecewise constant, time-dependent paremeters) with respect to European and Forward Starting Options. Making use of proposition 4 in section 2.4 and Jensen's inequality, we have

$$\begin{aligned}
& \left|\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp\left(iz_{(\%,\$)}\hat{\mathbb{I}}X_{t_l} + izX_{t_n}\right)|X_{t_0}, V_{t_0}\right]\right| \\
\leq \quad & \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left|\exp\left(iz_{(\%,\$)}\hat{\mathbb{I}}X_{t_l} + izX_{t_n}\right)\right||X_{t_0}, V_{t_0}\right] \\
= \quad & \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp\left(\mathrm{Re}\left[iz_{(\%,\$)}\right]\hat{\mathbb{I}}X_{t_l} + \zeta X_{t_n}\right)|X_{t_0}, V_{t_0}\right] \\
= \quad & \exp\left[\left(\left[\mathbb{I} - \zeta\right]\hat{\mathbb{I}} + \zeta\right)X_{t_0} + D_{l;1;n}^{(\%,\$)}\left(\tau_1, \mathbb{I}, D_{l;2;n}^{(\%,\$)}\big|_{u=0}\right)V_{t_0}\right] \\
\times \quad & \exp\left[\sum_{m=1}^{l} C_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\right) + \sum_{m=l+1}^{n} C_{m;n}\left(\tau_m, \zeta, D_{m+1;n}\right)\right]
\end{aligned} \tag{3.34}$$

To clarify our notation

$$D_{l;m+1;n}^{(\%,\$)}\big|_{u=0} \quad = \quad D_{l;m+1;n}^{(\%,\$)}(\tau_{m+1}, \mathbb{I}, D_{l;m+2;n}^{(\%,\$)}(\tau_{m+2}, \mathbb{I}, ...D_{l;l;n}^{(\%,\$)}(\tau_l, \mathbb{I}, D_{l+1;n}(\tau_{l+1}, \zeta, D_{l+2;n}))...))$$

i.e. C and D are functions of ζ rather than iz. The required strips of regularity may all be determined from equation (3.34). Focussing on the European case ($\hat{\mathbb{I}} = 0$) in subsections 3.2.1, 3.2.2, 3.2.3 and 3.2.4, we determine the effect of the continuous diffusion and jump components of the characteristic function on this strip separately and allow for piecewise constant, time-dependent parameters. In subsection 3.2.5 we determine the $\%$ and $\$$ strips for the corresponding Forward Starting Options ($\hat{\mathbb{I}} = 1$).

We begin by identifying useful properties of the cumulant generating function in the affine framework where the cumulant generating function is the natural logarithm of the moment generating function.

Proposition 8. *The cumulant generating function $K(\zeta) := \ln\left(\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}|X_{t_0}, V_{t_0}\right]\right)$ is convex in ζ. Within the affine framework, where $K(\zeta) = \zeta X_{t_0} + D_{1;n}(\tau_1, \zeta, D_{2;n})V_{t_0} + \sum_{m=1}^{n} C_{m;n}(\tau_m, \zeta, D_{m+1;n})$, the functions $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$ for $m = 1, ..., n$ and $\sum_{m=1}^{n} C_{m;n}(\tau_m, \zeta, D_{m+1;n})$ are also convex in ζ.*

Proof: We have

$$\frac{\partial^2}{\partial\zeta^2}K(\zeta) \quad = \quad \frac{1}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}|X_{t_0}, V_{t_0}\right]^2}\left[\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}|X_{t_0}, V_{t_0}\right]\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}X_T^2|X_{t_0}, V_{t_0}\right] - \mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{\zeta X_T}X_T|X_{t_0}, V_{t_0}\right]^2\right]$$

In Venezian [2005] the author points out that the convexity of $K(\zeta)$ may be determined from the Cauchy-Schwarz Inequality:

Theorem 2. *Let $y_1(x)$ and $y_2(x)$ be real, integrable functions in $[a,b]$ then*

$$\left[\int_a^b y_1(x)y_2(x)dx\right]^2 \quad \leq \quad \int_a^b [y_1(x)]^2\, dx \int_a^b [y_2(x)]^2\, dx \tag{3.35}$$

with equality if and only if $y_1(x) = \bar{k}y_2(x)$ where $\bar{k}$ is a constant.

Setting $y_1(x) = \sqrt{e^{\zeta x} f_{t_0,T}(x|\bar{x}_{t_0})}$ and $y_2(x) = \sqrt{e^{\zeta x} f_{t_0,T}(x|\bar{x}_{t_0})}x$ yields the inequality

$$\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{\zeta X_T}X_T|X_{t_0},V_{t_0}\right]^2 \quad < \quad \mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{\zeta X_T}|X_{t_0},V_{t_0}\right]\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{\zeta X_T}X_T^2|X_{t_0},V_{t_0}\right] \tag{3.36}$$

where the restricted range $(\zeta^{\min},\zeta^{\max})$ for ζ ensures that the integrability assumption is satisfied and so we have

$$\frac{\partial^2}{\partial\zeta^2}K(\zeta) \quad > \quad 0 \tag{3.37}$$

Furthermore, in the affine framework, we have $K(\zeta) = \zeta X_{t_0} + D(\tau,\zeta,0)V_{t_0} + C(\tau,\zeta,0)$ and so we can write

$$\frac{\partial^2}{\partial\zeta^2}K(\zeta) \quad = \quad \frac{\partial^2}{\partial\zeta^2}D(\tau,\zeta,0)V_{t_0} + \frac{\partial^2}{\partial\zeta^2}C(\tau,\zeta,0) \tag{3.38}$$

Equation (3.38) must be strictly positive for any positive V_{t_0} and so we must have

$$\frac{\partial^2}{\partial\zeta^2}D(\tau,\zeta,0) \quad > \quad 0 \tag{3.39}$$

$$\frac{\partial^2}{\partial\zeta^2}C(\tau,\zeta,0) \quad > \quad 0 \tag{3.40}$$

Considering the iterated extension for $K(\zeta)$

$$\ln\left(\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp\left(\zeta X_{t_n}\right)|X_{t_{n-1}},V_{t_{n-1}}\right]...|X_{t_1},V_{t_1}\right]|X_{t_0},V_{t_0}\right]\right) \tag{3.41}$$

we have

$$\zeta X_{t_0} + D_{1;n}(\tau_1,\zeta,D_{2;n})V_{t_0} + \sum_{m=1}^{n} C_{m;n}(\tau_m,\zeta,D_{m+1;n}) \tag{3.42}$$

at t_0 and so from the argument above, we must have

$$\frac{\partial^2}{\partial\zeta^2}D_{1;n}(\tau_1,\zeta,D_{2;n}) \quad > \quad 0 \tag{3.43}$$

$$\frac{\partial^2}{\partial\zeta^2}\sum_{m=1}^{n} C_{m;n}(\tau_m,\zeta,D_{m+1;n}) \quad > \quad 0 \tag{3.44}$$

This holds for any $n \geq 1$.

Given n, $D_{2;n}(\tau_2,\zeta,D_{3;n})$ has the same form (regarding the terminal conditions specified by the arguments ζ and $D_{3;n}$) as $D_{1;n-1}(\tau_1,\zeta,D_{2;n-1})$, $D_{3;n}(\tau_3,\zeta,D_{4;n})$ has the same form as $D_{1;n-2}(\tau_1,\zeta,D_{2;n-2})$ and so on. From this, we have

$$\frac{\partial^2}{\partial\zeta^2}D_{m;n}(\tau_m,\zeta,D_{m+1;n}) \quad > \quad 0 \tag{3.45}$$

for $m = 1,...,n$. □

3.2.1 Strip of regularity for the diffusion component

For the diffusion component of the SVJJ characteristic function i.e. the Heston characteristic function, we now determine appropriate conditions to identify the critical values $\zeta^{\min}$ and $\zeta^{\max}$ of the strip of regularity, allowing for piecewise constant parameters.

Proposition 9. *For $m = 1, ..., n$ we can write*

$$\psi_{m;n}(\tau_m, \zeta, D_{m+1;n}) = \frac{[b_m(\zeta) + \gamma_m(\zeta)] + [\gamma_m(\zeta) - b_m(\zeta)]e^{-\gamma_m(\zeta)\tau_m} - \nu_m^2 D_{m+1;n}[1 - e^{-\gamma_m(\zeta)\tau_m}]}{2\gamma_m(\zeta)}$$

with

$$\lim_{\zeta \to \zeta_m^\pm} \psi_{m;n}(\tau_m, \zeta, D_{m+1;n}) = \frac{[2 + b_m(\zeta_m^\pm)\tau_m] - \nu_m^2 D_{m+1;n}|_{\zeta=\zeta_m^\pm}\tau_m}{2}$$

where $\zeta_m^\pm$ are the roots of the quadratic function $\gamma_m^2(\zeta)$. The critical values $\zeta_m^{\min}$ and $\zeta_m^{\max}$ satisfy

$$\psi_{m;n}(\tau_m, \zeta, D_{m+1;n}) = 0 \tag{3.46}$$

with $\zeta_m^{\min}$ the largest value in the range $(-\infty, 0)$ and $\zeta_m^{\max}$ the smallest value in the range $(1, \infty)$.

Proof: Working from equations (2.78) of proposition 2 in section 2.2 (and bearing in mind the iterative expression in equation (2.208)), we can write

$$\begin{aligned} & D_{m;n}(\tau_m, \zeta, D_{m+1;n}) \\ = \; & \frac{\nu_m^2 D_{m+1;n}\left([b_m(\zeta) + \gamma_m(\zeta)]e^{-\gamma_m(\zeta)\tau_m} + [\gamma_m(\zeta) - b_m(\zeta)]\right) - [\gamma_m^2(\zeta) - b_m^2(\zeta)][1 - e^{-\gamma_m(\zeta)\tau_m}]}{\nu_m^2\left([b_m(\zeta) + \gamma_m(\zeta)] + [\gamma_m(\zeta) - b_m(\zeta)]e^{-\gamma_m(\zeta)\tau_m} - \nu_m^2 D_{m+1;n}[1 - e^{-\gamma_m(\zeta)\tau_m}]\right)} \end{aligned}$$

Working from equations (2.80) of proposition 2 (and bearing in mind the iterative expression in equation (2.207)), we can write

$$\begin{aligned} & \bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n}) \\ = \; & \frac{\kappa_m\theta_m}{\nu_m^2}[b_m(\zeta) - \gamma_m(\zeta)]\tau_m \\ - \; & \frac{2\kappa_m\theta_m}{\nu_m^2}\log\left(\frac{[b_m(\zeta) + \gamma_m(\zeta)] + [\gamma_m(\zeta) - b_m(\zeta)]e^{-\gamma_m(\zeta)\tau_m} - \nu_m^2 D_{m+1;n}[1 - e^{-\gamma_m(\zeta)\tau_m}]}{2\gamma_m(\zeta)}\right) \end{aligned}$$

For

$$[b_m(\zeta) + \gamma_m(\zeta)] + [\gamma_m(\zeta) - b_m(\zeta)]e^{-\gamma_m(\zeta)\tau_m} - \nu_m^2 D_{m+1;n}[1 - e^{-\gamma_m(\zeta)\tau_m}] = 0 \tag{3.47}$$

both $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$ and $\bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n})$ explode.

At $\zeta = \zeta_m^\pm$ we have $\gamma_m(\zeta) = 0$ with $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$ and $\bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n})$ both functions of the indeterminant $\frac{0}{0}$ form. Applying l'H$\hat{o}$pital's rule, we have

$$\begin{aligned} \lim_{\zeta \to \zeta_m^\pm} D_{m;n}(\tau_m, \zeta, D_{m+1;n}) &= \frac{\nu_m^2 D_{m+1;n}|_{\zeta=\zeta_m^\pm}[2 - b_m(\zeta_m^\pm)\tau_m] + b_m^2(\zeta_m^\pm)\tau_m}{\nu_m^2([2 + b_m(\zeta_m^\pm)\tau_m] - \nu_m^2 D_{m+1;n}|_{\zeta=\zeta_m^\pm}\tau_m)} \qquad (3.48) \\ \lim_{\zeta \to \zeta_m^\pm} \bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n}) &= \frac{\kappa_m\theta_m}{\nu_m^2}b_m(\zeta_m^\pm)\tau_m - \frac{2\kappa_m\theta_m}{\nu_m^2}\log\left(\frac{[2 + b_m(\zeta_m^\pm)\tau_m] - \nu_m^2 D_{m+1;n}|_{\zeta=\zeta_m^\pm}\tau_m}{2}\right) \end{aligned}$$

When the denominator in equation (3.48) is zero the numerator is $\frac{4}{\tau_m}$ so $D_{m;n}(\tau_m, \zeta_m^{\pm}, D_{m+1;n})$ explodes to positive infinity. At this point, $\bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n})$ also explodes to positive infinity.

When equation (3.47) is satisfied for $\gamma_m(\zeta) \neq 0$, $\bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n})$ explodes to positive infinity while the sign of $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$ is unclear. At this point, the numerator of $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$ is

$$\frac{4\gamma_m^2(\zeta)e^{-\gamma_m(\zeta)\tau_m}}{1 - e^{-\gamma_m(\zeta)\tau_m}} \tag{3.49}$$

and so for $\gamma_m(\zeta) \neq 0$ there are no further complications. The convexity of $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$, established in proposition 8, tells us that the sign of the explosion must be positive and so $\bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n})$ explodes to positive infinity only when $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$ (whose integral it is a function of) explodes to positive infinity.

At these points of explosion, we have the logarithm of zero in $\bar{C}_{m;n}(\tau_m, \zeta, D_{m+1;n})$ and so, for $\zeta \in (\zeta^{\min}, \zeta^{\max})$ and all u

$$\psi_{m;n}(\tau_m, iz, D_{m+1;n}) \quad \neq \quad 0 \tag{3.50}$$

We now show that the range $[0, 1]$ always lies in the strip of regularity. At $\zeta = 0, 1$, we have $\gamma(\zeta) = |b(\zeta)|$. For $b(\zeta) \neq 0$, this gives us

$$D_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad = \quad \frac{2D_{m+1;n}b_m(\zeta)e^{-b_m(\zeta)\tau_m}}{\left(2b_m(\zeta) - \nu_m^2 D_{m+1;n}[1 - e^{-b_m(\zeta)\tau_m}]\right)} \tag{3.51}$$

For $b(\zeta) = 0$, we have an indeterminant $\frac{0}{0}$ form. l'H$\hat{o}$pital's rule gives us

$$\lim_{b(\zeta)\to 0} D_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad = \quad \frac{2D_{m+1;n}}{2 - \nu_m^2 D_{m+1;n}\tau_m} \tag{3.52}$$

For $m = n$, we have $D_{n+1;n} := 0$ and so from equations (3.51) and (3.52), we have $D_{n;n}(\tau_m, \zeta, 0) = 0$. Working backwards, from $m = n - 1$ to $m = 1$ an inductive argument yields

$$D_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad = \quad 0 \tag{3.53}$$

for $m = 1, ..., n$ and $\zeta = 0, 1$ and from proposition 8 we know $D_{m;n}(\tau_m, \zeta, D_{m+1;n})$ is convex in ζ. Hence, $\zeta_m^{\min} < 0$ and $\zeta_m^{\max} > 1$. $\square$

For a piecewise constant parameter set, we must work backwards from the nth to the 1st increment to determine the effect of each increment on the strip of regularity. For the nth increment, we determine $\zeta_n^{\min} < 0$ and $\zeta_n^{\max} > 1$ from equation (3.46) for $m = n$. For each preceding increment $m < n$, we then determine whether equation (3.46) is satisfied for $\zeta \in (\zeta_{m+1}^{\min}, 0)$ and $\zeta \in (1, \zeta_{m+1}^{\max})$. If so, these critical values $\zeta_m^{\min}$ and $\zeta_m^{\max}$ replace $\zeta_{m+1}^{\min}$ and $\zeta_{m+1}^{\max}$, respectively.

3.2.2 Bounds for the strip of regularity within the Heston Model

We now derive bounds for the critical values $\zeta_n^{\min}$ and $\zeta_n^{\max}$. For $n = 1$, these are bounds for the strip of regularity $(\zeta^{\min}, \zeta^{\max})$ in the Heston model.

Proposition 10. *For $x \geq 0$ and $\tau > 0$, we define*

$$\zeta^{x,\pm} \quad := \quad \frac{(\nu - 2\kappa\rho) \pm \sqrt{(\nu - 2\kappa\rho)^2 + 4\left(\kappa^2 + \frac{x^2}{\tau^2}\right)(1-\rho^2)}}{2\nu(1-\rho^2)} \tag{3.54}$$

where $\zeta^{0,\pm} = \zeta^{\pm}$ (the roots of the quadratic function $\gamma^2(\zeta)$).

Case 1: $-1 < \rho \leq 0$

$$\zeta^{\min} \quad \in \quad (\zeta^{2\pi,-}, \zeta^{-}) \tag{3.55}$$

$$\zeta^{\max} \quad \in \quad (\zeta^{+}, \zeta^{2\pi,+}) \tag{3.56}$$

Furthermore, at $\rho = 0$, we have

$$\zeta^{\min} = 1 - \zeta^{\max} \tag{3.57}$$

Case 2: $0 < \rho < 1$

$$\zeta^{\min} \in (\zeta^{2\pi,-}, \zeta^{-}) \tag{3.58}$$

with subcases for $\zeta^{\max}$

Case 2a: $0 < \rho < 1$ *and* $\kappa - \rho\nu < 0$
We search for $\zeta^{\max}$ in the range $(1, \zeta^{+})$. If $\zeta^{\max}$ does not fall in this range, we must determine whether $\zeta^{+} = \frac{\kappa + \frac{2}{\tau}}{\rho\nu}$. If so $\zeta^{\max} = \zeta^{+}$. If not $\zeta^{\max} \in (\zeta^{+}, \zeta^{2\pi,+})$.
Case 2b: $0 < \rho < 1$, $\kappa - \rho\nu > 0$ *and* $\frac{\kappa}{\rho\nu} < \zeta^{+}$
We search for $\zeta^{\max}$ in the range $(\frac{\kappa}{\rho\nu}, \zeta^{+})$. If $\zeta^{\max}$ does not fall in this range, we must determine whether $\zeta^{+} = \frac{\kappa + \frac{2}{\tau}}{\rho\nu}$. If so $\zeta^{\max} = \zeta^{+}$. If not $\zeta^{\max} \in (\zeta^{+}, \zeta^{2\pi,+})$.
Case 2c: $0 < \rho < 1$, $\kappa - \rho\nu = 0$
We have $\zeta^{+} = 1$ and $\zeta^{\max} \in (\zeta^{+}, \zeta^{2\pi,+})$.

Proof: From proposition 9, we know that the moment generating function explodes to positive infinity only when $\psi(\tau, \zeta, 0) = 0$. We can write

$$\psi(\tau, \zeta, 0) \quad = \quad \frac{[b(\zeta) + \gamma(\zeta)] + [\gamma(\zeta) - b(\zeta)]e^{-\gamma(\zeta)\tau}}{2\gamma(\zeta)} \tag{3.59}$$

$$\lim_{\zeta \to \zeta^{\pm}} \psi(\tau, \zeta, 0) \quad = \quad \frac{2 + b(\zeta^{\pm})\tau}{2} \tag{3.60}$$

which yields the conditions

$$b(\zeta)\left(1 - e^{-\gamma(\zeta)\tau}\right) + \gamma(\zeta)\left(1 + e^{-\gamma(\zeta)\tau}\right) \quad = \quad 0 \tag{3.61}$$

$$b(\zeta^{\pm}) + \frac{2}{\tau} \quad = \quad 0 \tag{3.62}$$

from which we can determine $\zeta^{\min}$ and $\zeta^{\max}$.

We have $b(\zeta) = \kappa - \rho\nu\zeta \in \mathbb{R}$ and from proposition 9, we know that $\zeta^{\min} < 0$ and $\zeta^{\max} > 1$. We now consider the range $[\zeta^{-}, 0) \cup (1, \zeta^{+}]$. To satisfy equation (3.61) for some $\zeta \in (\zeta^{-}, 0) \cup (1, \zeta^{+})$ where, from proposition 5, we know $\gamma(\zeta) \in \mathbb{R}$, we require $b(\zeta) < 0$ since $\gamma(\zeta) > 0$. To satisfy equation (3.62) for $\zeta = \zeta^{\pm}$, we also require $b(\zeta^{\pm}) < 0$.

Case 1: $-1 < \rho \leq 0$
Making use of the inequality

$$(\nu - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2) \quad < \quad (\nu + 2\kappa)^2 \tag{3.63}$$

we obtain, for $-1 < \rho < 0$,

$$\begin{aligned} b(\zeta^-) &= \kappa - \rho\nu \left[\frac{(\nu - 2\kappa\rho) - \sqrt{(\nu-2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\nu(1-\rho^2)} \right] \\ &> \kappa \left(\frac{1+\rho}{1-\rho^2} \right) \\ &> 0 \end{aligned} \tag{3.64}$$

and since $\frac{\partial b(\zeta)}{\partial \zeta} = -\rho\nu > 0$, we have $b(\zeta) > 0$ for $\zeta \geq \zeta^-$. For $\rho = 0$, we have $b(\zeta) = \kappa > 0$. Hence, for $-1 < \rho \leq 0$, $\zeta^{\min} < \zeta^-$ and $\zeta^{\max} > \zeta^+$.

Case 2: $0 < \rho < 1$
Here $\frac{\partial b(\zeta)}{\partial \zeta} = -\rho\nu < 0$ and $b(0) = \kappa > 0$ and so we need only to consider the range $(1, \zeta^+]$. Since $b(1) = \kappa - \rho\nu$, we must consider the sign of $\kappa - \rho\nu$:

Case 2a: $0 < \rho < 1$ and $\kappa - \rho\nu < 0$
Here $b(1) < 0$ and so we must consider the entire range $(1, \zeta^+]$. At $\zeta = \zeta^+$, we have an explosion if $b(\zeta^+) = -\frac{2}{\tau}$ and so we need to confirm whether $\zeta^+ = \frac{\kappa + \frac{2}{\tau}}{\rho\nu}$ if $\zeta^{\max} \notin (1, \zeta^+)$.

Case 2b: $0 < \rho < 1$, $\kappa - \rho\nu > 0$ and $\frac{\kappa}{\rho\nu} < \zeta^+$
Here $b(1) > 0$ and for $\zeta = \frac{\kappa}{\rho\nu}$, $b(\zeta) = 0$ so we need to consider the range $\left(\frac{\kappa}{\rho\nu}, \zeta^+\right]$. Again, we need to confirm whether $\zeta^+ = \frac{\kappa + \frac{2}{\tau}}{\rho\nu}$ if $\zeta^{\max} \notin \left(\frac{\kappa}{\rho\nu}, \zeta^+\right)$.

Case 2c: $0 < \rho < 1$, $\kappa - \rho\nu = 0$
Here we have $\zeta^+ = 1$ and $b(\zeta^+) = 0$ and so $\zeta^{\max} > \zeta^+$.

Hence, for $0 < \rho < 1$, $\zeta^{\min} < \zeta^-$ and if $\zeta^{\max}$ does not fall within the intervals considered for cases 2a and 2b, we have $\zeta^{\max} > \zeta^+$.

For $\zeta^{\min} < \zeta^-$ or $\zeta^{\max} > \zeta^+$, we must consider the intervals $(-\infty, \zeta^-)$ and (ζ^+, ∞) respectively where, from proposition 5, we know $\gamma(\zeta) = i\mathrm{Im}\,[\gamma(\zeta)]$. We now derive bounds for $\zeta^{\min} < \zeta^-$ and $\zeta^{\max} > \zeta^+$.

Following proposition 7, we re-define

$$\arg\left[\psi(\tau, \zeta, 0)\right] \quad := \quad -\frac{1}{2}\mathrm{Im}\left[\gamma(\zeta)\right]\tau \tag{3.65}$$

For $x \geq 0$ and $\tau > 0$, we define $\zeta^{x,\pm}$ as that in equation (3.54) such that $\mathrm{Im}\left[\gamma(\zeta^{x,\pm})\right] = \frac{x}{\tau}$ and $\zeta^{0,\pm} = \zeta^{\pm}$ where $\gamma(\zeta^{\pm}) = 0$. We restrict our attention to $x \in [0, 2\pi]$ i.e. to the principal branch of $\arg\left[\psi(\tau, \zeta, 0)\right]$.

From the RHS of equation (3.65), we see that $\frac{\partial}{\partial \zeta}\mathrm{Arg}\left[\psi(\tau, \zeta, 0)\right] > 0$ for $\zeta \in (\zeta^{2\pi,-}, \zeta^-)$ and $\frac{\partial}{\partial \zeta}\mathrm{Arg}\left[\psi(\tau, \zeta, 0)\right] < 0$ for $\zeta \in (\zeta^+, \zeta^{2\pi,+})$. Furthermore, $\mathrm{Arg}\left[\psi(\tau, \zeta, 0)\right]$ is continuous in ζ. At $\zeta = \zeta^{\pm}$, we have $\psi(\tau, \zeta^{\pm}, 0) = \frac{2 + b(\zeta^{\pm})\tau}{2} \in \mathbb{R}$. From subsection 3.1.1, we know $\psi(\tau, \zeta^{\pm}, 0) > 0$ assuming $\zeta^{\max} \neq \zeta^+$

for cases 2a and 2b where $0 < \rho < 1$ (in which case we would not need to obtain bounds for $\zeta^{\max}$). From the fact that $\exp\left[i\mathrm{Im}\left[\gamma(\zeta^{2\pi,\pm})\right]\tau\right] = 1$ where $\mathrm{Im}\left[\gamma(\zeta^{2\pi,\pm})\right] = \frac{2\pi}{\tau}$ and equation (3.59), we can see that $\psi(\tau, \zeta^{2\pi,\pm}, 0) = 1$.

For $\zeta^{\max} > \zeta^{+}$, we start off on the positive real axis with $\mathrm{Arg}\left[\psi(\tau,\zeta^{+},0)\right] = 0$. Since $\mathrm{Arg}\left[\psi(\tau,\zeta,0)\right]$ is strictly decreasing in ζ for $\zeta \in (\zeta^{+}, \zeta^{2\pi,+})$, we rotate about the complex plane in a strictly clockwise manner, into the negative imaginary plane, where the positive real axis cannot be attained again from this plane. Furthermore, $\psi(\tau,\zeta,0)$ cannot attain the negative real axis (again, from subsection 3.1.1) and yet, we have $\psi(\tau,\zeta^{2\pi,+},0) = 1$. Since $\mathrm{Arg}\left[\psi(\tau,\zeta,0)\right]$ is continuous in ζ, for $\zeta \in (\zeta^{+}, \zeta^{2\pi,+})$, we must have $\zeta^{\max} < \zeta^{2\pi,+}$ where $\psi(\tau,\zeta,0)$ attains the origin at $\zeta^{\max}$. The same reasoning holds for $\zeta^{\min} < \zeta^{-}$ where $\mathrm{Arg}\left[\psi(\tau,\zeta^{-},0)\right] = 0$, the argument on this principal branch is strictly increasing and continuous in ζ for $\zeta \in (\zeta^{2\pi,-}, \zeta^{-})$ and $\psi(\tau,\zeta^{2\pi,-},0) = 1$. The only difference, in this case, is that we rotate about the complex plane in a strictly anti-clockwise manner, into the positive imaginary plane.

Finally, for $\rho = 0$, we have $\zeta^{\min} = 1 - \zeta^{\max}$. We see this from the following argument. For $\varepsilon := \zeta - \frac{1}{2}$, we have

$$\gamma(\zeta) \quad = \quad \sqrt{\kappa^2 + \frac{1}{4}\nu^2 - \nu^2\varepsilon^2} \tag{3.66}$$

Therefore, $\gamma(\zeta)$ is even in ε about the point $\varepsilon = 0$ and so even in ζ about the point $\zeta = \frac{1}{2}$.
This symmetry which arises for $\rho = 0$ is noted in Lord and Kahl [2007]. □

This result tells us that $\zeta^{\min} \to \zeta^{-}$ and $\zeta^{\max} \to \zeta^{+}$ as $\tau \to \infty$ and accommodates for Proposition 3.1 of Lord and Kahl [2007] which says that $\zeta^{\max} \approx \zeta^{2\pi,+}$ for $\rho \approx -1$.
The proof of proposition 10 illustrates that $\zeta^{2\pi,\pm}$ are bounds for the critical values of ζ. However, the proof does not determine whether these bounds are appropriate, meaning that it is not clear whether the conditions used to determine these critical values in proposition 9 (for $n = 1$) can be satisfied for more than one value of ζ within these bounds. We now address this issue for $\rho \leq 0$.

Proposition 11. *For $\rho \leq 0$, the explosion condition $\psi(\tau,\zeta,0) = 0$ can only be satisfied once in the respective ranges $(\zeta^{2\pi,-}, \zeta^{-})$ and $(\zeta^{+}, \zeta^{2\pi,+})$. Furthermore, we tighten the bounds for $\zeta^{\min}$ and $\zeta^{\max}$.*

For $\rho \leq 0$

$$\zeta^{\max} \quad \in \quad (\zeta^{\pi,+}, \zeta^{2\pi,+}) \tag{3.67}$$

where one can make use of the fact that for $\rho = 0$, $\zeta^{\min} = 1 - \zeta^{\max}$.

For $\rho < 0$

$$\zeta^{\min} \quad \in \quad (\zeta^{\pi,-}, \zeta^{x_0,-}) \qquad \textit{for} \quad x_0 < \pi \tag{3.68}$$

$$\zeta^{\min} \quad = \quad \zeta^{\pi,-} \qquad \textit{for} \quad x_0 = \pi \tag{3.69}$$

$$\zeta^{\min} \quad \in \quad (\zeta^{\min[2\pi,x_0],-}, \zeta^{\pi,-}) \qquad \textit{for} \quad x_0 > \pi \tag{3.70}$$

where

$$x_0 \quad := \quad \sqrt{\frac{\kappa}{\rho^2}\left(\kappa - \rho\nu\right)\tau^2} \tag{3.71}$$

Proof: For $\zeta \in (\zeta^{2\pi,-}, \zeta^{-}) \cup (\zeta^{+}, \zeta^{2\pi,+})$ where $\gamma(\zeta) = i\text{Im}\left[\gamma(\zeta)\right]$ and re-expressing the explosion condition $\psi(\tau, \zeta, 0) = 0$, both $\zeta^{\min}$ and $\zeta^{\max}$ satisfy the pair of equations

$$\sin(-\text{Im}\left[\gamma(\zeta)\right]\tau) = \frac{2b(\zeta)\text{Im}\left[\gamma(\zeta)\right]}{b(\zeta)^2 + \text{Im}\left[\gamma(\zeta)\right]^2} \tag{3.72}$$

$$\cos(-\text{Im}\left[\gamma(\zeta)\right]\tau) = \frac{b(\zeta)^2 - \text{Im}\left[\gamma(\zeta)\right]^2}{b(\zeta)^2 + \text{Im}\left[\gamma(\zeta)\right]^2} \tag{3.73}$$

For $\zeta = \zeta^{x,\pm}$ (where $\zeta^{x,\pm}$ is defined in equation (3.54) of proposition 10 with $\text{Im}\left[\gamma(\zeta^{x,\pm})\right]\tau = x$ and $\zeta^{0,\pm} = \zeta^{\pm}$), we can show

$$\frac{\partial}{\partial x}\left(\frac{2b(\zeta)\text{Im}\left[\gamma(\zeta)\right]}{b(\zeta)^2 + \text{Im}\left[\gamma(\zeta)\right]^2}\right) = -2\xi(\zeta)\left(\frac{b(\zeta)^2 - \text{Im}\left[\gamma(\zeta)\right]^2}{b(\zeta)^2 + \text{Im}\left[\gamma(\zeta)\right]^2}\right)\frac{\partial \zeta}{\partial x} \tag{3.74}$$

$$\frac{\partial}{\partial x}\left(\frac{b(\zeta)^2 - \text{Im}\left[\gamma(\zeta)\right]^2}{b(\zeta)^2 + \text{Im}\left[\gamma(\zeta)\right]^2}\right) = 2\xi(\zeta)\left(\frac{2b(\zeta)\text{Im}\left[\gamma(\zeta)\right]}{b(\zeta)^2 + \text{Im}\left[\gamma(\zeta)\right]^2}\right)\frac{\partial \zeta}{\partial x} \tag{3.75}$$

with

$$\frac{\partial \zeta}{\partial x} < 0 \qquad \text{for} \quad \zeta = \zeta^{x,-} \tag{3.76}$$

$$\frac{\partial \zeta}{\partial x} > 0 \qquad \text{for} \quad \zeta = \zeta^{x,+} \tag{3.77}$$

and for $\rho \leq 0$

$$\xi(\zeta) := \frac{\nu^2}{2\text{Im}\left[\gamma(\zeta)\right]}\left(\frac{\rho\nu\zeta + \kappa(1 - 2\zeta)}{b(\zeta)^2 + \text{Im}\left[\gamma(\zeta)\right]^2}\right) \tag{3.78}$$

$$< 0 \qquad \text{for} \quad \zeta = \zeta^{x,+} > 1 \tag{3.79}$$

$$> 0 \qquad \text{for} \quad \zeta = \zeta^{x,-} < 0 \tag{3.80}$$

With respect to equation (3.73), we have

$$\frac{\partial}{\partial x}\cos(-\text{Im}\left[\gamma(\zeta)\right]\tau) = \sin(-\text{Im}\left[\gamma(\zeta)\right]\tau)\frac{\partial \text{Im}\left[\gamma(\zeta)\right]\tau}{\partial x} = \sin(-\text{Im}\left[\gamma(\zeta)\right]\tau) \tag{3.81}$$

and we know

$$\sin(-\text{Im}\left[\gamma(\zeta)\right]\tau) < 0 \qquad \text{for} \quad x \in (0, \pi) \tag{3.82}$$

$$= 0 \qquad \text{for} \quad x = \pi \tag{3.83}$$

$$> 0 \qquad \text{for} \quad x \in (\pi, 2\pi) \tag{3.84}$$

For $\zeta^{\max}$, we consider the range $(\zeta^{+}, \zeta^{2\pi,+})$ where $\zeta^{+} > 1$, $\xi(\zeta) < 0$, $\frac{\partial \zeta}{\partial x} > 0$ and $b(\zeta) = \kappa - \rho\nu\zeta > 0$ for $\rho \leq 0$. The RHS of equation (3.72) is, therefore, positive and so, from equation (3.84), we need only to consider $x \in (\pi, 2\pi)$ to ensure that equation (3.72) is satisfied. To satisfy equation (3.73), in this range of x and for $b(\zeta) > 0$, consider that from equations (3.81) and (3.84), we have

$$\frac{\partial}{\partial x}\cos(-\text{Im}\left[\gamma(\zeta)\right]\tau) > 0 \tag{3.85}$$

and from equation (3.75), we have

$$\frac{\partial}{\partial x}\left(\frac{b(\zeta)^2 - \mathrm{Im}\left[\gamma(\zeta)\right]^2}{b(\zeta)^2 + \mathrm{Im}\left[\gamma(\zeta)\right]^2}\right) \quad < \quad 0 \tag{3.86}$$

Therefore, equation (3.73) can only be satisfied for one value of $\zeta \in (\zeta^{\pi,+}, \zeta^{2\pi,+})$.

For $\zeta^{\min}$, we consider the range $(\zeta^{2\pi,-}, \zeta^{-})$ where $\zeta^{-} < 0$, $\xi(\zeta) > 0$, $\frac{\partial\zeta}{\partial x} < 0$. The sign of $b(\zeta)$ is, however, unclear. We have three cases to consider.

Case 1: $b(\zeta) > 0$
The same reasoning followed when considering $\zeta^{\max}$ yields the result that equation (3.73) can only be satisfied for one value of $\zeta \in (\zeta^{2\pi,-}, \zeta^{\pi,-})$.

Case 2: $b(\zeta) < 0$
The RHS of equation (3.72) is negative and so, from equation (3.82), we need only to consider $x \in (0, \pi)$ to ensure that equation (3.72) is satisfied. To satisfy equation (3.73), in this range of x and for $b(\zeta) < 0$, consider that from equations (3.81) and (3.82), we have

$$\frac{\partial}{\partial x}\cos(-\mathrm{Im}\left[\gamma(\zeta)\right]\tau) \quad < \quad 0 \tag{3.87}$$

and from equation (3.75), we have

$$\frac{\partial}{\partial x}\left(\frac{b(\zeta)^2 - \mathrm{Im}\left[\gamma(\zeta)\right]^2}{b(\zeta)^2 + \mathrm{Im}\left[\gamma(\zeta)\right]^2}\right) \quad > \quad 0 \tag{3.88}$$

Therefore, equation (3.73) can only be satisfied for one value of $\zeta \in (\zeta^{\pi,-}, \zeta^{-})$.

Case 3: $b(\zeta) = 0$
The RHS of equation (3.72) is zero and from equation (3.83), we see that equation (3.72) can only be satisfied at $\zeta = \zeta^{\pi,-}$.

From these cases, we now show that the presented bounds are appropriate for $\zeta^{\min}$. Using equation (3.76), we have

$$\frac{\partial b(\zeta)}{\partial x} = -\rho\nu\frac{\partial\zeta}{\partial x} < 0 \tag{3.89}$$

Solving for the point $\zeta^{x_0,-}$ such that $b(\zeta^{x_0,-}) = 0$, we have

$$\kappa - \rho\nu\left[\frac{(\nu - 2\kappa\rho) - \sqrt{(\nu - 2\kappa\rho)^2 + 4\left(\kappa^2 + \frac{x_0^2}{\tau^2}\right)(1-\rho^2)}}{2\nu(1-\rho^2)}\right] \quad = \quad 0$$

Re-arranging terms, we obtain

$$x_0 \quad = \quad \sqrt{\frac{\kappa}{\rho^2}\left(\kappa - \rho\nu\right)\tau^2} \tag{3.90}$$

Since $b(\zeta)$ is decreasing in x we have $b(\zeta) > 0$ for $x < x_0$ and $b(\zeta) < 0$ for $x > x_0$.

If $x_0 = \pi$ then $b(\zeta) > 0$ for $x \in (0, \pi)$ and $b(\zeta) < 0$ for $x \in (\pi, 2\pi)$. Hence, from the cases above we see that $\zeta^{\min} = \zeta^{\pi,-}$.

If $x_0 < \pi$ then $b(\zeta) > 0$ for $x \in (0, x_0)$ and $b(\zeta) < 0$ for $x \in (x_0, \pi)$ and $x \in [\pi, 2\pi)$. From case 1 we see that $\zeta^{\min} \notin (0, x_0)$ and from case 2 we see that $\zeta^{\min} \notin [\pi, 2\pi)$. Hence, from case 2 we have $\zeta^{\min} \in (\zeta^{\pi,-}, \zeta^{x_0,-})$.

If $\pi < x_0 < 2\pi$ then $b(\zeta) > 0$ for $x \in (0, \pi]$ and $x \in (\pi, x_0)$ and $b(\zeta) < 0$ for $x \in (x_0, 2\pi)$. From case 1 we see that $\zeta^{\min} \notin (0, \pi]$ and from case 2 we see that $\zeta^{\min} \notin (x_0, 2\pi)$. If $x_0 > 2\pi$ then $b(\zeta) > 0$ for $x \in (0, 2\pi)$ and from case 1 we see that $\zeta^{\min} \notin (0, \pi]$. Hence, from case 1 we have $\zeta^{\min} \in (\zeta^{\min[2\pi, x_0], -}, \zeta^{\pi,-})$. $\square$

The results of proposition 11 are valid for the nth increment in a piecewise constant setting.

As noted in Lord and Kahl [2008] for the time-homogenous case ($n = 1$), solving for the critical values of ζ, in the manner presented in proposition 9, is not a well-posed problem as these values are not unique. The bounds we have derived in propositions 10 and 11 are useful for dealing with this problem. However, a slightly different approach is followed in Lord and Kahl [2007] where the critical values $\zeta^{\min}$ and $\zeta^{\max}$ are obtained by referring to Andersen and Piterbarg [2007] Proposition 3.1 (as presented in Lord and Kahl [2007]):

The ζ-th moment of S_T is finite for $T < T^*$ and infinite for $T \geq T^*$ where T^* is given by one of three possibilities:

1. *For $\gamma(\zeta)^2 \geq 0$, $b(\zeta) \geq 0$ or $\zeta \in [0, 1]$*

$$T^* = \infty \tag{3.91}$$

2. *For $\gamma(\zeta)^2 \geq 0$, $b(\zeta) < 0$*

$$T^* = \frac{1}{\bar{c}(\zeta)} \ln \left(\frac{b(\zeta) - \bar{c}(\zeta)}{b(\zeta) + \bar{c}(\zeta)} \right) \tag{3.92}$$

3. *For $\gamma(\zeta)^2 < 0$*

$$T^* = \frac{2}{\bar{c}(\zeta)} \left[\mathbb{I}_{[b(\zeta)>0]} \pi + \arctan \left(-\frac{\bar{c}(\zeta)}{b(\zeta)} \right) \right] \tag{3.93}$$

where $\bar{c}(\zeta) = |\gamma(\zeta)|$.[4]

The approach proves useful when searching for the critical values of ζ given T (or the period τ) particularly for case 3 where $\gamma(\zeta) = i\mathrm{Im}\,[\gamma(\zeta)]$ and our explosion condition $\psi(\tau, \zeta, 0) = 0$ is not unique for $\zeta \in (-\infty, 0)$ and $\zeta \in (1, \infty)$. Regarding this critical time approach, we now motivate equation (3.93) where $b(\zeta) \in \mathbb{R}$ and the explosion condition may be written as

$$\begin{aligned} -i\mathrm{Im}\,[\gamma(\zeta)]\,\tau &= \log \left(\frac{b(\zeta) + i\mathrm{Im}\,[\gamma(\zeta)]}{b(\zeta) - i\mathrm{Im}\,[\gamma(\zeta)]} \right) && (3.94) \\ &= \ln \left(\left| \frac{b(\zeta) + i\mathrm{Im}\,[\gamma(\zeta)]}{b(\zeta) - i\mathrm{Im}\,[\gamma(\zeta)]} \right| \right) + i \arg \left(\frac{b(\zeta) + i\mathrm{Im}\,[\gamma(\zeta)]}{b(\zeta) - i\mathrm{Im}\,[\gamma(\zeta)]} \right) && (3.95) \end{aligned}$$

We have $\left| \frac{b(\zeta)+i\mathrm{Im}[\gamma(\zeta)]}{b(\zeta)-i\mathrm{Im}[\gamma(\zeta)]} \right| = 1$ and $\arg \left(\frac{b(\zeta)+i\mathrm{Im}[\gamma(\zeta)]}{b(\zeta)-i\mathrm{Im}[\gamma(\zeta)]} \right) = 2 \arg \left(b(\zeta) + i\mathrm{Im}\,[\gamma(\zeta)] \right)$. Since $\mathrm{Im}\,[\gamma(\zeta)] > 0$ we know that $b(\zeta) + i\mathrm{Im}\,[\gamma(\zeta)]$ does not cross the branch cut and we have

$$\mathrm{Arg}\left(b(\zeta) + i\mathrm{Im}\,[\gamma(\zeta)] \right) \quad \in \quad (0, \pi) \tag{3.96}$$

[4]The function $\bar{c}(\zeta)$ was originally defined as $\frac{1}{2}|\gamma(\zeta)|$ in Lord and Kahl [2007], the typo was observed by Roger Lord as a result of a discrepancy between the outcome of our respective approaches.

From the proof of proposition 10 we know that for $\zeta \in (\zeta^{2\pi,-}, \zeta^{-}) \cup (\zeta^{+}, \zeta^{2\pi,+})$ we have

$$-\mathrm{Im}\left[\gamma(\zeta)\right]\tau \quad \in \quad (-2\pi, 0) \tag{3.97}$$

Hence, the critical period for a given value of ζ satisfies

$$\tau^{*} \quad = \quad -\frac{2}{\mathrm{Im}\left[\gamma(\zeta)\right]}\left[\mathrm{Arg}\left(b(\zeta) + i\mathrm{Im}\left[\gamma(\zeta)\right]\right) - \pi\right] \tag{3.98}$$

For a given τ equation (3.98) may be used to solve for $\zeta^{\min}$ and $\zeta^{\max}$ where one can conveniently make use of the bounds we have derived in this subsection.

3.2.3 Bounds for the strip of regularity within the Heston Model allowing for piece-wise constant parameters

Proposition 12. *For the increments $m = 1, ..., n-1$, $x \geq 0$ and $\tau_m > 0$, we define*

$$\zeta_m^{x,\pm} \quad := \quad \frac{(\nu_m - 2\kappa_m\rho_m) \pm \sqrt{(\nu_m - 2\kappa_m\rho_m)^2 + 4\left(\kappa_m^2 + \frac{x^2}{\tau_m^2}\right)(1-\rho_m^2)}}{2\nu_m(1-\rho_m^2)} \tag{3.99}$$

$$b_{m;n}(\zeta, D_{m+1;n}) \quad := \quad b_m(\zeta) - \nu_m^2 D_{m+1;n} \tag{3.100}$$

and assume $\psi_{m;n}(\tau_m, iz, D_{m+1;n})$ cannot lie on the branch cut $(-\infty, 0)$.

We search for $\zeta_m^{\min}$ in the range $(\zeta_m^-, 0)$. If $\zeta_m^{\min}$ does not fall in this range, we must determine whether $b_{m;n}(\zeta_m^-, D_{m+1;n}) = -\frac{2}{\tau_m}$. If so $\zeta_m^{\min} = \zeta_m^-$. If not $\zeta_m^{\min} \in (\zeta_m^{2\pi,-}, \zeta_m^-)$.

We search for $\zeta_m^{\max}$ in the range $(1, \zeta_m^+)$. If $\zeta_m^{\max}$ does not fall in this range, we must determine whether $b_{m;n}(\zeta_m^+, D_{m+1;n}) = -\frac{2}{\tau_m}$. If so $\zeta_m^{\max} = \zeta_m^+$. If not $\zeta_m^{\max} \in (\zeta_m^+, \zeta_m^{2\pi,+})$.

For $\rho_m = 0$, we have

$$\zeta_m^{\min} = 1 - \zeta_m^{\max} \tag{3.101}$$

Proof: From proposition 9, we know that $\psi_{m;n}(\tau_m, \zeta, D_{m+1;n}) = 0$ at $\zeta_m^{\min}$ and $\zeta_m^{\max}$. We can write

$$\psi_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad = \quad \frac{[b_{m;n}(\zeta, D_{m+1;n}) + \gamma_m(\zeta)] + [\gamma_m(\zeta) - b_{m;n}(\zeta, D_{m+1;n})]e^{-\gamma_m(\zeta)\tau}}{2\gamma_m(\zeta)} \tag{3.102}$$

$$\lim_{\zeta \to \zeta_m^{\pm}} \psi_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad = \quad \frac{2 + b_{m;n}(\zeta_m^{\pm}, D_{m+1;n})\tau}{2} \tag{3.103}$$

We follow the same approach as that used to prove proposition 10 as we have $b_{m;n}(\zeta, D_{m+1;n}) \in \mathbb{R}$ since $D_{m+1;n} = D_{m+1;n}(\tau_{m+1}, \zeta, D_{m+2;n}) \in \mathbb{R}$ from equation (3.13) with $\zeta_m^- < 0$ and $\zeta_m^+ \geq 1$. However approaching the problem in terms of the sign of $b_{m;n}(\zeta, D_{m+1;n})$ is not as convenient as the corresponding case in proposition 10 where we need only consider the sign of $b_m(\zeta)$.

Following the proof of proposition 7, we re-define

$$\arg\left[\psi_{m;n}(\tau_m, \zeta, D_{m+1;n})\right] \quad := \quad -\frac{1}{2}\mathrm{Im}\left[\gamma_m(\zeta)\right]\tau_m \tag{3.104}$$

For $x \geq 0$ and $\tau_m > 0$, we define $\zeta_m^{x,\pm}$ as that in equation (3.99) where $\text{Im}\left[\gamma\left(\zeta_m^{x,\pm}\right)\right] = \frac{x}{\tau_m}$ and $\zeta_m^{0,\pm} = \zeta_m^{\pm}$ with $\gamma(\zeta_m^{\pm}) = 0$. We restrict our attention to $x \in [0, 2\pi]$ i.e. to the principal branch of $\arg\left[\psi_{m;n}(\tau_m, \zeta, D_{m+1;n})\right]$.

From the RHS of equation (3.104), we see that $\frac{\partial}{\partial\zeta}\text{Arg}\left[\psi_{m;n}(\tau_m, \zeta, D_{m+1;n})\right] > 0$ for $\zeta \in (\zeta^{2\pi,-}, \zeta^-)$ and $\frac{\partial}{\partial\zeta}\text{Arg}\left[\psi_{m;n}(\tau_m, \zeta, D_{m+1;n})\right] < 0$ for $\zeta \in (\zeta^+, \zeta^{2\pi,+})$. Furthermore, for the intervals considered, $\text{Arg}\left[\psi_{m;n}(\tau_m, \zeta, D_{m+1;n})\right]$ is continuous in ζ.
At $\zeta = \zeta_m^{\pm}$, we have $\psi_{m;n}(\tau_m, \zeta_m^{\pm}, D_{m+1;n}) = \frac{2+b_{m;n}(\zeta_m^{\pm}, D_{m+1;n})\tau_m}{2} \in \mathbb{R}$. If $b_{m;n}(\zeta_m^{\pm}, D_{m+1;n}) = -\frac{2}{\tau_m}$ then $\zeta_m^{\max} = \zeta_m^+$ and $\zeta_m^{\min} = \zeta_m^-$ respectively. For $\zeta^{\min} < \zeta^-$ and/or $\zeta^{\max} > \zeta^+$, we consider that $\psi_{m;n}(\tau_m, \zeta, D_{m+1;n}) > 0$ (by assumption) and make use of the fact that $\psi_{m;n}\left(\tau_m, \zeta_m^{2\pi,\pm}, D_{m+1;n}\right) = 1$. The result follows from the same reasoning as that followed in the proof of proposition 10 as does the fact that $\zeta_m^{\min} = 1 - \zeta_m^{\max}$ for $\rho_m = 0$. □

Regarding the critical time approach and equation (3.98) in particular, we can accommodate for piecewise constant parameters by replacing $b(\zeta)$ with $b_{m;n}(\zeta, D_{m+1;n}) \in \mathbb{R}$ and so we have the critical period

$$\tau_m^* = -\frac{2}{\text{Im}\left[\gamma_m(\zeta)\right]}\left[\text{Arg}\left(b_{m;n}(\zeta, D_{m+1;n}) + i\text{Im}\left[\gamma_m(\zeta)\right]\right) - \pi\right] \tag{3.105}$$

For a given τ_m, equation (3.105) may be used to solve for $\zeta_m^{\min} < \zeta_m^-$ and $\zeta_m^{\max} > \zeta_m^+$ where the critical values lie in the respective intervals $(\zeta_m^{2\pi,-}, \zeta_m^-)$ and $(\zeta_m^+, \zeta_m^{2\pi,+})$.

3.2.4 Strip of regularity for the jump component

We now present an approach with which to determine the critical values ${}^J\zeta_m^{\min}$ and ${}^J\zeta_m^{\max}$ of the strip of regularity for the jump component of the SVJJ characteristic function, allowing for piecewise constant, time-dependent parameters.

Proposition 13. *For the increments $m = 1, ..., n$, the strip of regularity for the jump component $({}^J\zeta_m^{\min},\ {}^J\zeta_m^{\max})$ is specified such that*

$$\min\left[1 - \zeta\eta_m\rho_m^J - \eta_m D_{m;n}(\tau_m, \zeta, D_{m+1;n}), 1 - \zeta\eta_m\rho_m^J - \eta_m D_{m;n}(0, \zeta, D_{m+1;n})\right] \tag{3.106}$$

is positive. Subject to the parameter restriction $1 - \eta_m\rho_m^J > 0$, ${}^J\zeta_m^{\min} < 0$ and ${}^J\zeta_m^{\max} > 1$ and at these critical points equation (3.106) is zero.

Proof: Referring to the joint characteristic function for the jump sizes J_X and J_V, derived in the proof of proposition 2 in section 2.2 and making use of Jensen's inequality, we have

$$\begin{aligned}\left|\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(izJ_X + D(s, iz, iz_v)J_V\right)\right]\right| &\leq \mathbb{E}_t^{\mathbb{Q}}\left[\left|\exp\left(izJ_X + D(s, iz, iz_v)J_V\right)\right|\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\zeta J_X + \text{Re}\left[D(s, iz, iz_v)J_V\right]\right)\right] \\ &= \frac{e^{\zeta\mu + \frac{1}{2}\zeta^2\sigma^2}}{\left(1 - \zeta\eta\rho^J - \eta\text{Re}\left[D(s, iz, iz_v)\right]\right)}\end{aligned} \tag{3.107}$$

From the derivation of equation (2.114) in the proof of proposition 2, we know that equation (3.107) is valid only for $1 - \zeta\eta\rho^J - \eta\text{Re}\left[D(s, iz, iz_v)\right] > 0$.

From proposition 19 (which follows), we have $\text{Re}\left[D_{m;n}(s, iz, D_{m+1;n})\right] \leq D_{m;n}(s, \zeta, D_{m+1;n})$ so

$$1 - \zeta\eta_m\rho_m^J - \eta_m\text{Re}\left[D_{m;n}(s, iz, D_{m+1;n})\right] \geq 1 - \zeta\eta_m\rho_m^J - \eta_m D_{m;n}(s, \zeta, D_{m+1;n}) \tag{3.108}$$

From the inequality in equation (3.108), we need only to ensure that

$$1-\zeta\eta_m\rho_m^{J}-\eta_m D_{m;n}(s,\zeta,D_{m+1;n}) \quad > \quad 0 \tag{3.109}$$

to ensure that the joint characteristic function for the jump sizes J_X and J_V exists.

From proposition 2, we have the parameter restriction $1-\eta_m\rho_m^{J}>0$ and so for $\zeta\in[0,1]$, we have $1-\zeta\eta_m\rho_m^{J}>0$. From proposition 20 (which follows), we have $D_{m;n}(s,\zeta,D_{m+1;n})\leq 0$ for $\zeta\in[0,1]$. This yields

$$1-\zeta\eta_m\rho_m^{J}-\eta_m D_{m;n}(s,\zeta,D_{m+1;n})\geq 1-\zeta\eta_m\rho_m^{J}>0 \tag{3.110}$$

and so from equation (2.115) in the proof of proposition 2 we have

$$\bar{J}_{m;n}(\tau_m,\zeta,D_{m+1;n}) \quad = \quad \int_0^{\tau_m}\frac{1}{(1-\zeta\eta_m\rho_m^{J}-\eta_m D_{m;n}(s,\zeta,D_{m+1;n}))}ds \tag{3.111}$$

$$\leq \quad \frac{\tau_m}{1-\zeta\eta_m\rho_m^{J}} \tag{3.112}$$

Therefore, the range $[0,1]$ always lies in the strip of regularity for the jump component subject to the restriction $1-\eta_m\rho_m^{J}>0$.

For $\zeta\notin[0,1]$, we restate the defining ODE from the proof of proposition 2 in section 2.2,

$$\frac{\partial}{\partial s}D_{m;n}(s,\zeta,D_{m+1;n}) \quad = \quad \frac{1}{2}\nu_m^2 D^2(s,\zeta,D_{m+1;n})-b_m(\zeta)D(s,\zeta,D_{m+1;n})+c(\zeta) \tag{3.113}$$

where $b_m(\zeta)=\kappa_m-\rho_m\nu_m\zeta$ and $c(\zeta)=\frac{1}{2}\zeta(\zeta-1)$ and from equation (3.13) we know $D_{m;n}(s,\zeta,D_{m+1;n})\in\mathbb{R}$. The function $\frac{\partial}{\partial s}D_{m;n}(s,\zeta,D_{m+1;n})$ is quadratic in $D_{m;n}(s,\zeta,D_{m+1;n})$ and convex since $\frac{1}{2}\nu_m^2>0$, with roots at $D_{m;n}(s,\zeta,D_{m+1;n})=\frac{b_m(\zeta)\pm\gamma_m(\zeta)}{\nu_m^2}$.

For $\zeta\in(\zeta^-,0)\cup(1,\zeta^+)$, we have $b_m(\zeta)$, $\gamma_m(\zeta)$, $A_m(\zeta,D_{m+1;n})$, $\bar{A}_m(\zeta,D_{m+1;n})\in\mathbb{R}$ where the form of A_m and $\bar{A}_m$ is presented in proposition 2. We can write

$$D_{m;n}(s,\zeta,D_{m+1;n}) \quad = \quad \left(\frac{b_m(\zeta)-\gamma_m(\zeta)}{\nu_m^2}\right)\left[\frac{\bar{A}_m(\zeta,D_{m+1;n})-e^{-\gamma_m(\zeta)s}}{\bar{A}_m(\zeta,D_{m+1;n})-\left(\frac{b_m(\zeta)-\gamma_m(\zeta)}{b_m(\zeta)+\gamma_m(\zeta)}\right)e^{-\gamma_m(\zeta)s}}\right] \tag{3.114}$$

$$= \quad \left(\frac{b_m(\zeta)+\gamma_m(\zeta)}{\nu_m^2}\right)\left[\frac{\bar{A}_m(\zeta,D_{m+1;n})-e^{-\gamma_m(\zeta)s}}{A_m(\zeta,D_{m+1;n})-e^{-\gamma_m(\zeta)s}}\right] \tag{3.115}$$

From equations (3.114) and (3.115), we see that $D_{m;n}(s,\zeta,D_{m+1;n})\neq\frac{b_m(\zeta)\pm\gamma_m(\zeta)}{\nu_m^2}$ as this would require $\left(\frac{b_m(\zeta)-\gamma_m(\zeta)}{b_m(\zeta)+\gamma_m(\zeta)}\right)=1$ i.e. $\gamma_m(\zeta)=0$ which is not possible for the considered range of ζ since $\gamma_m(\zeta)=0$ for $\zeta=\zeta_m^{\pm}$. Furthermore, at $\zeta=\zeta_m^{\pm}$, the function $D_{m;n}(s,\zeta,D_{m+1;n})$ is of the indeterminant $\frac{0}{0}$ form. However, we need only point out that, at $\zeta=\zeta^{\pm}$, the function $\frac{\partial}{\partial s}D_{m;n}(s,\zeta,D_{m+1;n})$ has a single root at $D_{m;n}(s,\zeta,D_{m+1;n})=\frac{b_m(\zeta)}{\nu_m^2}$. For $\zeta\in(-\infty,\zeta^-)\cup(\zeta^+,\infty)$, we have $\gamma(\zeta)\in\mathbb{C}$ i.e. the quadratic function has no real roots. This analysis yields

$$\frac{\partial}{\partial s}D_{m;n}(s,\zeta,D_{m+1;n}) \quad \neq \quad 0 \quad \text{for} \quad \zeta\in(\zeta^-,\zeta^+) \tag{3.116}$$

$$\geq \quad 0 \quad \text{for} \quad \zeta=\zeta^{\pm} \tag{3.117}$$

$$> \quad 0 \quad \text{for} \quad \zeta\in(-\infty,\zeta^-)\cup(\zeta^+,\infty) \tag{3.118}$$

For $\frac{\partial}{\partial s}D_{m;n}(s,\zeta,D_{m+1;n}) \geq 0$ and $0 \leq s \leq \tau_m$, we have

$$D_{m;n}(0,\zeta,D_{m+1;n}) \leq D_{m;n}(s,\zeta,D_{m+1;n}) \leq D_{m;n}(\tau_m,\zeta,D_{m+1;n}) \tag{3.119}$$

and so

$$1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(s,\zeta,D_{m+1;n}) \geq 1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(\tau_m,\zeta,D_{m+1;n})$$

Therefore

$$\bar{J}_{m;n}(\tau_m,\zeta,D_{m+1;n}) = \int_0^{\tau_m} \frac{1}{(1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(s,\zeta,D_{m+1;n}))} ds \tag{3.120}$$

$$\leq \frac{\tau_m}{1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(\tau_m,\zeta,D_{m+1;n})} \tag{3.121}$$

for $1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(\tau_m,\zeta,D_{m+1;n}) > 0$. This ensures that $J_{m;n}(\tau_m,iz,D_{m+1;n})$ exists where $J(\tau,iz,\cdot)$ is defined in equation (2.82) of proposition 2 in section 2.2.

For $\frac{\partial}{\partial s}D_{m;n}(s,\zeta,D_{m+1;n}) < 0$ and $0 \leq s \leq \tau_m$, we have

$$D_{m;n}(0,\zeta,D_{m+1;n}) > D_{m;n}(s,\zeta,D_{m+1;n}) > D_{m;n}(\tau_m,\zeta,D_{m+1;n}) \tag{3.122}$$

This yields

$$1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(s,\zeta,D_{m+1;n}) > 1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(0,\zeta,D_{m+1;n}) \tag{3.123}$$

and

$$\bar{J}_{m;n}(\tau_m,\zeta,D_{m+1;n}) = \int_0^{\tau_m} \frac{1}{(1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(s,\zeta,D_{m+1;n}))} ds \tag{3.124}$$

$$< \frac{\tau_m}{1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(0,\zeta,D_{m+1;n})} \tag{3.125}$$

for $1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(0,\zeta,D_{m+1;n}) > 0$. This ensures that $J_{m;n}(\tau_m,iz,D_{m+1;n})$ exists. $\square$

From proposition 13, we see that setting $\eta_m = 0$ for all $m \leq n$, explosions cannot occur in the jump component. Hence, in the SVJJ model, only if we allow for jumps in the variance process do we need to consider the strip of regularity for the jump component.

Solving for the critical values of ζ, from proposition 13, is a well-posed problem. We see this since, from proposition 8, we know $D_{m;n}(\tau_m,\zeta,D_{m+1;n})$ is convex in ζ for $m = 1,...,n$ and $D_{m;n}(0,\zeta,D_{m+1;n}) = D_{m+1;n}(\tau_{m+1},\zeta,D_{m+2;n})$. Hence, the equations $1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(\tau_m,\zeta,D_{m+1;n})$ and $1-\zeta\eta_m\rho_m^J - \eta_m D_{m;n}(0,\zeta,D_{m+1;n})$ are concave in ζ and positive for $\zeta \in [0,1]$ (subject to the specified parameter restriction).

The strip of regularity for the characteristic function is given by $\left(\max[\zeta_m^{\min}, {}^J\zeta_m^{\min}], \min[\zeta_m^{\max}, {}^J\zeta_m^{\max}]\right)$ and so we may conveniently use $\zeta_m^{\min}$ and $\zeta_m^{\max}$ as bounds for ${}^J\zeta_m^{\min}$ and ${}^J\zeta_m^{\max}$, respectively.

3.2.5 Strip of regularity for Forward Starting Options

For $\%$ and \$ type Forward Starting Options where the period $t_n - t_l + t_l - t_0$ is split into n increments with the determination date t_l, we make use of the forward $(\%, \$)$ characteristic functions where $u = 0$ to determine the corresponding strip of regularity. The defining difference between the characteristic and the forward $(\%, \$)$ characteristic functions, expressed in terms of piecewise constant parameters for the same number of increments n, is that for the increments $m = 1, ..., l$, we replace iz (where iz is the second argument of the functions C and D) with $\mathbb{I} := \mathbb{I}_{[z_{(\%,\$)}=z_{\$}]}$. Hence, for the increments $m = l+1, ..., n$, we have the critical values $\zeta_m^{\min,(\%,\$)} = \zeta_m^{\min}$, $\zeta_m^{\max,(\%,\$)} = \zeta_m^{\max}$, ${}^J\zeta_m^{\min,(\%,\$)} = {}^J\zeta_m^{\min}$ and ${}^J\zeta_m^{\max,(\%,\$)} = {}^J\zeta_m^{\max}$ and we need only to address the effect of this argument specification on the strip of regularity for the diffusion and jump components of the characteristic function for the increments $m = 1, ..., l$ where we have

$$\begin{aligned} \gamma_m(\mathbb{I}) &= \sqrt{b_m(\mathbb{I})^2 - \nu_m^2 \mathbb{I}(\mathbb{I}-1)} && (3.126) \\ &= |b_m(\mathbb{I})| && (3.127) \end{aligned}$$

Proposition 14. *For $m = 1, ..., l$, we have*

$$\begin{aligned} &\psi_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\right) \\ = \;& \frac{2b_m(\mathbb{I}) - \nu_m^2 D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\left(1 - e^{-b_m(\mathbb{I})\tau_m}\right)}{2b_m(\mathbb{I})} && \text{for} \quad b_m(\mathbb{I}) > 0 && (3.128) \\ = \;& \left[\frac{2b_m(\mathbb{I}) - \nu_m^2 D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\left(1 - e^{-b_m(\mathbb{I})\tau_m}\right)}{2b_m(\mathbb{I})}\right] e^{b_m(\mathbb{I})\tau_m} && \text{for} \quad b_m(\mathbb{I}) < 0 && (3.129) \\ = \;& \frac{2 - \nu_m^2 D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\tau_m}{2} && \text{for} \quad b_m(\mathbb{I}) = 0 && (3.130) \end{aligned}$$

For a $\%$ type option, we have $b_m(0) = \kappa_m > 0$, while for a \$ type option, we have $b_m(1) = \kappa_m - \rho_m\nu_m$. $\zeta_m^{\min,(\%,\$)}$ and $\zeta_m^{\max,(\%,\$)}$ satisfy

$$\psi_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\right) = 0 \qquad (3.131)$$

with $\zeta_m^{\min,(\%,\$)} < 0$ and $\zeta_m^{\max,(\%,\$)} > 1$.

Proof: From proposition 23 (which follows) we have $D_{l;m+1;n}^{(\%,\$)}\big|_{u=0} \leq 0$ and so $\psi_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\right) > 0$ for $\zeta \in [0,1]$. Hence, $\zeta_m^{\min,(\%,\$)} < 0$ and $\zeta_m^{\max,(\%,\$)} > 1$. The remainder of the result follows from the comments above and the proof of proposition 9. To clarify our notation

$$D_{l;m+1;n}^{(\%,\$)}\big|_{u=0} = D_{l;m+1;n}^{(\%,\$)}(\tau_{m+1}, \mathbb{I}, D_{l;m+2;n}^{(\%,\$)}(\tau_{m+2}, \mathbb{I}, ...D_{l;l;n}^{(\%,\$)}(\tau_l, \mathbb{I}, D_{l+1;n}(\tau_{l+1}, \zeta, D_{l+2;n}))...)) \qquad (3.132)$$

□

Proposition 15. *For the increments $m = 1, ..., l$, the strip of regularity for the jump component $({}^J\zeta_m^{\min,(\%,\$)}, {}^J\zeta_m^{\max,(\%,\$)})$ is specified such that*

$$\min\left[1 - \mathbb{I}\eta_m\rho_m^J - \eta_m D_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\right), 1 - \mathbb{I}\eta_m\rho_m^J - \eta_m D_{l;m;n}^{(\%,\$)}\left(0, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\right)\right] \qquad (3.133)$$

is positive. We have ${}^J\zeta_m^{\min,(\%,\$)} < 0$ and ${}^J\zeta_m^{\max,(\%,\$)} > 1$ and at these critical points equation (3.133) is zero where *the parameter restriction $1 - \eta_m \rho_m^J > 0$ (for $m = 1, ..., l$) applies to the case of a \$ type option only and we have*

$$D_{l;m;n}^{(\%,\$)}(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}) = \frac{2D_{l;m+1;n}^{(\%,\$)}\big|_{u=0} b_m(\mathbb{I}) e^{-b_m(\mathbb{I})\tau_m}}{\Big(2b_m(\mathbb{I}) - \nu_m^2 D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}[1 - e^{-b_m(\mathbb{I})\tau_m}]\Big)} \quad \textit{for} \quad b_m(\mathbb{I}) \neq 0 \tag{3.134}$$

$$= \frac{2D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}}{2 - \nu_m^2 D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}\tau_m} \quad \textit{for} \quad b_m(\mathbb{I}) = 0 \tag{3.135}$$

where $b_m(\mathbb{I}) = 0$ can occur only for a \$ type option with $\kappa_m - \rho_m \nu_m = 0$.

Proof: The result follows from the comments above and proposition 13. □

Following the proof of proposition 8, we can see that $D_{l;m;n}^{(\%,\$)}(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0})$ is convex in ζ for $m = 1, ..., n$. From equation (3.53) of the proof of proposition 9 in subsection 3.2.1, we know $D_{m;n}(\tau_m, \zeta, D_{m+1;n}) = 0$ for $m = l + 1, ..., n$ and $\zeta = 0, 1$. From equations (3.134) and (3.135) it follows that $D_{l;m;n}^{(\%,\$)}(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\big|_{u=0}) = 0$ for $m = 1, ..., n$ and $\zeta = 0, 1$. Hence, from propositions 14 and 15, we see that identifying $\zeta_m^{\min,(\%,\$)}$, $\zeta_m^{\max,(\%,\$)}$ and ${}^J\zeta_m^{\min,(\%,\$)}$, ${}^J\zeta_m^{\max,(\%,\$)}$ are well-posed problems as the equations from which we determine these critical values are concave in ζ and positive for $\zeta \in [0, 1]$ (subject to any specified parameter restrictions).
Furthermore, the values $\zeta_{l+1}^{\min}$, $\zeta_{l+1}^{\max}$ and ${}^J\zeta_{l+1}^{\min}$, ${}^J\zeta_{l+1}^{\max}$ serve as bounds for the respective critical values.

3.3 The optimal contour of integration

An important point to consider is the chosen value of α where $-\alpha$ specifies the contour of integration in the complex plane. Starting with Carr and Madan [1999], who introduced this damping parameter to option pricing, it has been observed that the shape of the pricing integrand is sensitive to the value of α passed through it. In particular, for far out-the-money and short maturity options, the shape may become highly oscillatory. Several ad hoc suggestions have been made to deal with this significant problem - Carr and Madan [1999] suggest working with $\frac{\alpha_{\max}}{4}$, Schoutens et al. [2005] suggest working with $\alpha = 0.75$ while Lewis [2001] specifies $\alpha = -0.5$.

In Lee [2005], a bound is obtained for the truncation and discretization errors that arise for a discrete Fourier transform of the option price. A constant value of α is then chosen such that this bound is minimised. With this constant value, one then prices a set of options ranging in strike via the Fast Fourier Transform (FFT) method. It should be noted, however, that this approach relies on a bound for the truncation error that is decreasing in u. Obtaining such a bound is model specific and non-trivial. With respect to the Heston model, the author provides such a bound in Appendix A.2. Its derivation is presented in Lee [2006][5]. The issue of identifying an appropriate value of α is tackled in Lord and Kahl [2007] where the authors suggest working with the value α^* that minimises the total variation of the integrand

$$\alpha^* = \underset{\alpha \in (\alpha^{\min}, \alpha^{\max})}{\operatorname{argmin}} \int_0^\infty \Big| \frac{\partial}{\partial u} \tilde{\Psi}_{t_0,T}(u, \alpha) \Big| du \tag{3.136}$$

[5] Thanks to the author for providing this derivation on request http://www.math.uchicago.edu/∼rl/dftHestonBound.pdf

where

$$\tilde{\Psi}_{t_0,T}(u,\alpha) \quad := \quad \text{Re}\left[\left(\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\right)\Phi_{t_0,T}(u-i[\alpha+1])\right] \tag{3.137}$$

Practically, the authors suggest determining α^* by assuming that $\tilde{\Psi}_{t_0,T}(u,\alpha)$ is a monotone function of $u \in [0,\infty)$. This leads to

$$\begin{aligned}\int_0^\infty \left|\frac{\partial}{\partial u}\tilde{\Psi}_{t_0,T}(u,\alpha)\right| du &= \left|\tilde{\Psi}_{t_0,T}(0,\alpha) - \tilde{\Psi}_{t_0,T}(\infty,\alpha)\right| \\ &= \left|\tilde{\Psi}_{t_0,T}(0,\alpha)\right| \end{aligned} \tag{3.138}$$

since $\tilde{\Psi}_{t_0,T}(\infty,\alpha) = 0$. Therefore, we have

$$\alpha^*$$

$$:= \underset{\alpha\,\in\,(\alpha^{\min},\,\alpha^{\max})}{\text{argmin}} \left|\tilde{\Psi}_{t_0,T}(0,\alpha)\right| \tag{3.139}$$

$$= \underset{\alpha\,\in\,(\alpha^{\min},\,\alpha^{\max})}{\text{argmin}} \ln\left(\left|\tilde{\Psi}_{t_0,T}(0,\alpha)\right|\right) \tag{3.140}$$

$$= \underset{\alpha\,\in\,(\alpha^{\min},\,\alpha^{\max})}{\text{argmin}} \left[-\alpha k - \ln(|\alpha(\alpha+1)|) + \ln\left(\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{(\alpha+1)X_T}|X_{t_0},V_{t_0}\right]\right)\right] \tag{3.141}$$

$$= \underset{\alpha\,\in\,(\alpha^{\min},\,\alpha^{\max})}{\text{argmin}} \left[-\alpha k - \ln(|\alpha(\alpha+1)|) + (\alpha+1)X_{t_0} + \text{Re}\left[D(\tau,\alpha+1,0)\right]V_{t_0} + \text{Re}\left[C(\tau,\alpha+1,0)\right]\right] \tag{3.142}$$

We distinguish between equation (3.141), which is effectively the form of α^* specified in Lord and Kahl [2007], and equation (3.142) as we wish to avoid the evaluation of any exponents when solving for α^*. In MatLab, we have $e^x := \infty$ for values of $x > \ln(\text{realmax}) \approx 709$.[6] It is possible for $(\alpha+1)X_{t_0} + \text{Re}\left[D(\tau,\alpha+1,0)\right]V_{t_0} + \text{Re}\left[C(\tau,\alpha+1,0)\right] > 709$ for some parameter set. If this occurs, for a range of α containing α^*, then the approach may fail to determine α^* from equation (3.141).

In proposition 8, we confirm that the cumulant generating function $(\alpha+1)X_{t_0} + D(\tau,\alpha+1,0)V_{t_0} + C(\tau,\alpha+1,0)$ is convex in α and explodes to positive infinity at $\alpha^{\min}$ and $\alpha^{\max}$. The remaining term $-\alpha k - \ln(|\alpha(\alpha+1)|)$ is also convex in α and explodes to positive infinity at $\alpha = -1, 0$. The sum of two convex functions remains convex and since we have positive vertical asymptotes at $\alpha = \alpha^{\min}, -1, 0, \alpha^{\max}$, we must have local minima in the ranges $(\alpha^{\min}, -1)$, $(-1, 0)$ and $(0, \alpha^{\max})$ as stated in Lord and Kahl [2007]. Hence, we search for α^* in these three ranges. This approach avoids specifying $\alpha = -1, 0$ and so we avoid the points of singularity $u = 0$ and $\alpha = -1, 0$ mentioned in subsection 2.1.1.

Referring to equations (2.45) and (2.46), we can show that for the range of α considered

$$\left|\tilde{\Psi}_{t_0,T}(u,\alpha)\right| \quad \leq \quad \left(\frac{\left|e^{-i(u-i\alpha)k}\right|}{|(u-i\alpha)(u-i[\alpha+1])|}\right)\Phi_{t_0,T}(u-i[\alpha+1])|_{u=0} \tag{3.143}$$

$$\leq \quad \left|\tilde{\Psi}_{t_0,T}(0,\alpha)\right| \tag{3.144}$$

and so by following the α^* approach we are, in fact, minimizing the absolute value of the integrand at its maximum point - this minimization is also considered in Ng [2005].

[6] realmax is the largest value that can be represented as a double-precision floating-point number

In general, the simplifying assumption of monotonicity is not valid for values of α in the interval $(\alpha^{\min}, \alpha^{\max})$ as we are considering the issue of an oscillating integrand for $u \in [0, \infty)$. At the very least, however, this approach should prove to be effective for parameter sets where the assumption is valid over the bulk of the integrand (evaluated at the optimal value α^*) i.e. for $u \in [0, u^*)$ and some value u^* where the magnitude of $\tilde{\Psi}_{t_0,T}(u, \alpha^*)$ is insignificant for $u \geq u^*$. As an example, we consider the integrand $\tilde{\Psi}_{t_0,T}(u, \alpha)$ in figure 3.1 for a parameter set considered in figure 2(a) of Lord and Kahl [2007]. In figure 3.1(a) we present the integrand (scaled such that the value at $u = 0$ is 1) evaluated at α^* as well as at $\alpha = -0.5, 400$. We see that, as a function of α^*, the integrand appears to be monotone in u. Taking a closer look at this α^* integrand in figure 3.1(b), we see that its decay to zero is not monotone, though insignificantly so. In figures 3.1(c) and 3.1(d) we present the unscaled integrand evaluated at α^* and $\alpha = -0.5$ respectively. The latter is more peaked.
Following the approach suggested in Lord and Kahl [2007], we transform the domain of integration (as described in section 3.4) and make use of the adaptive Gauss-Lobatto quadrature algorithm of Gander and Gautschi [2000] to evaluate the call value. We, very briefly, discuss the implementation of this algorithm in subsection 4.3.2. Evaluated at α^*, we obtain the value 3.2521×10^{-126} while at $\alpha = -0.5$, we obtain -1.9984×10^{-15}. We refer the reader to Lord and Kahl [2007] sections 3.2, 3.3 and 4 for a more detailed discussion of the optimal contour of integration.

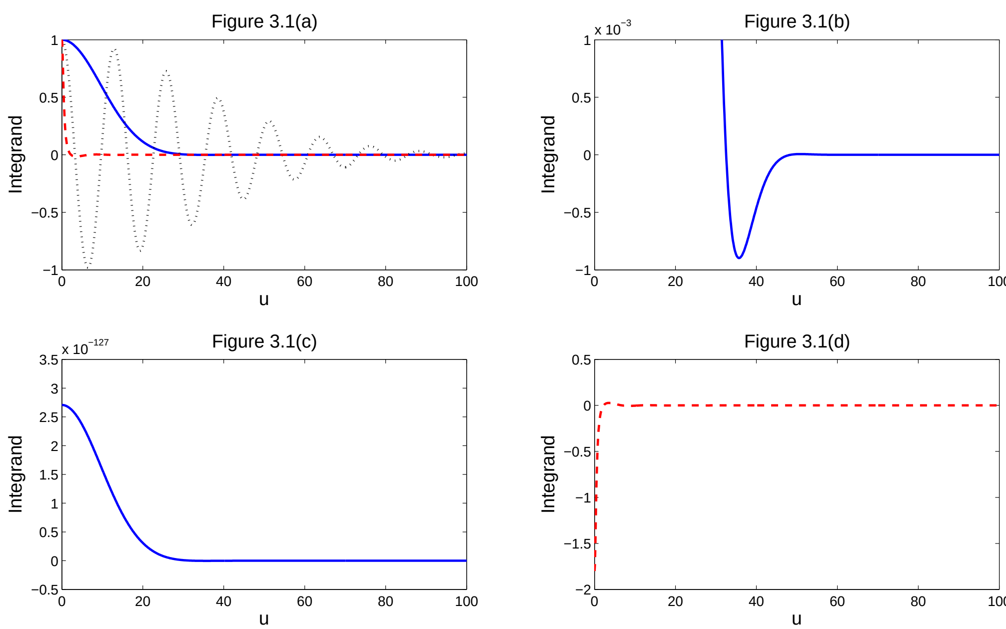

Figure 3.1: Heston integrand as a function of α for the parameter set: $S = 1$, $K = 2$, $r = 0$, $q = 0$, $\kappa = 1$, $\rho = -0.9$, $\nu = 1$, $\theta = 0.1$, $V_{t_0} = 0.1$, $\tau = \frac{1}{52}$ with $\alpha = \alpha^* = 541.93$ for ——, $\alpha = 400$ for $\cdots$ and $\alpha = -0.5$ for - - -.

3.4 Transforming the domain of integration

Numerical evaluation of the semi-analytic formulae for European and Forward Starting Options requires one to truncate the domain of integration $[0,\infty)$. Alternatively, as suggested in Kahl and Jackel [2005] and Lord and Kahl [2007], an appropriate transformation of the integration variable u yields the domain of integration $[0,1]$. To determine this transformation, we make use of the following result from Lord and Kahl [2007] Proposition 2.2 (assuming the dynamics of the Heston model), for the pricing integrand in equation (2.6) of section 2.1. We make use of the functions derived in proposition 2 of section 2.2 throughout this section.

Theorem 3. *Assuming $\kappa,\nu,\theta,\tau>0$ and $\rho\in(-1,1)$, we have*

$$\lim_{u\to\infty} e^{-iuk}\Psi^{C}_{t_0,T}(u,\alpha) \quad\approx\quad \Psi^{C}_{t_0,T}(0,\alpha)e^{-u\Re_\infty}\frac{\cos(u\Im_\infty)}{-u^2} \tag{3.145}$$

where

$$\Re_\infty \quad=\quad \frac{\sqrt{1-\rho^2}\,(V_{t_0}+\kappa\theta\tau)}{\nu} \tag{3.146}$$

$$\Im_\infty \quad=\quad X_{t_0}+(r-q)\tau-k-\frac{\rho\,(V_{t_0}+\kappa\theta\tau)}{\nu} \tag{3.147}$$

The proof follows from Kahl and Jackel [2005] Appendix A Proposition 3.1. We now work through the proof in order to obtain the corresponding results for the SVJJ model allowing for piecewise constant, time-dependent parameters in proposition 16 and for Forward Starting Options in proposition 17.

From equation (3.26) and equation (3.27) of theorem 1 in subsection 3.1.2, we see

$$\lim_{u\to\infty}\mathrm{Re}\left[\gamma(iz)\right] \quad=\quad \infty \tag{3.148}$$

and so

$$\lim_{u\to\infty} e^{-\gamma(iz)\tau} \quad=\quad 0 \tag{3.149}$$

Working from the authors' proof, we have

$$\lim_{u\to\infty}\frac{b(iz)}{u} \quad=\quad -i\rho\nu \tag{3.150}$$

and making use of equation (3.26) again, we have

$$\begin{aligned}
&\lim_{u\to\infty}\frac{\gamma(iz)}{u}\\
=\;&\lim_{u\to\infty}\sqrt{\frac{[\kappa-\rho\nu(\alpha+1)]^2-\nu^2\alpha(\alpha+1)+u^2\nu^2(1-\rho^2)-iu\left[\nu^2(2\alpha+1)+2\rho\nu[\kappa-\rho\nu(\alpha+1)]\right]}{u^2}}\\
=\;&\nu\sqrt{1-\rho^2}
\end{aligned} \tag{3.151}$$

Using equations (3.150) and (3.151), we have

$$\begin{aligned}
\lim_{u\to\infty}A^{-1}(iz,0) \quad&=\quad \lim_{u\to\infty}\frac{\frac{b(iz)-\gamma(iz)}{u}}{\frac{b(iz)+\gamma(iz)}{u}}\\
&=\quad \frac{-i\rho\nu-\nu\sqrt{1-\rho^2}}{-i\rho\nu+\nu\sqrt{1-\rho^2}}\\
&=\quad -1+2\rho^2-i2\rho\sqrt{1-\rho^2}
\end{aligned} \tag{3.152}$$

From equations (3.149) and (3.152), we have

$$\begin{aligned}
\lim_{u\to\infty}\frac{D(\tau,iz,0)}{u} &= \lim_{u\to\infty}\frac{1}{u}\left(\frac{b(iz)-\gamma(iz)}{\nu^2}\right)\left[\frac{1-e^{-\gamma(iz)\tau}}{1-A^{-1}(iz,0)e^{-\gamma(iz)\tau}}\right] && (3.153)\\
&= \lim_{u\to\infty}\frac{1}{\nu^2}\left(\frac{b(iz)-\gamma(iz)}{u}\right)\\
&= \frac{-\sqrt{1-\rho^2}-i\rho}{\nu} && (3.154)
\end{aligned}$$

Again, from equations (3.149) and (3.152), we have

$$\begin{aligned}
\lim_{u\to\infty}\frac{\bar{C}(\tau,iz,0)}{u} &= \lim_{u\to\infty}\frac{1}{u}\left[\frac{\kappa\theta}{\nu^2}\left[b(iz)-\gamma(iz)\right]\tau-\frac{2\kappa\theta}{\nu^2}\log\left(\frac{A^{-1}(iz,0)e^{-\gamma(iz)\tau}-1}{A^{-1}(iz,0)-1}\right)\right] && (3.155)\\
&= \lim_{u\to\infty}\frac{\kappa\theta}{\nu^2}\left(\frac{b(iz)-\gamma(iz)}{u}\right)\tau\\
&= \kappa\theta\left(\frac{-\sqrt{1-\rho^2}-i\rho}{\nu}\right)\tau && (3.156)
\end{aligned}$$

The authors state that this analysis leads to the result in theorem 3.

We elaborate by observing that Re $\left[e^{-iuk}\Psi^{C}_{t_0,\,T}(u,\alpha)\right]$ may be written as

$$e^{\Re}\left(\frac{[-u^2+\alpha(\alpha+1)]\cos(\Im)+u(2\alpha+1)\sin(\Im)}{[-u^2+\alpha(\alpha+1)]^2+u^2(2\alpha+1)^2}\right) \tag{3.157}$$

where

$$\begin{aligned}
\Re &= -r\tau+(\alpha+1)X_{t_0}+\text{Re}\left[D(\tau,iz,0)\right]V_{t_0}+\text{Re}\left[C(\tau,iz,0)\right] && (3.158)\\
\Im &= u(X_{t_0}-k)+\Im_{D(\tau,iz,0)}V_{t_0}+\Im_{C(\tau,iz,0)} && (3.159)
\end{aligned}$$

From equations (3.154) and (3.156), we have

$$\begin{aligned}
\lim_{u\to\infty}\frac{\Re}{u} &= -\Re_\infty && (3.160)\\
\lim_{u\to\infty}\frac{\Im}{u} &= \Im_\infty && (3.161)
\end{aligned}$$

with

$$\frac{[-u^2+\alpha(\alpha+1)]\cos(\Im)+u(2\alpha+1)\sin(\Im)}{[-u^2+\alpha(\alpha+1)]^2+u^2(2\alpha+1)^2}\in\mathcal{O}\left(\frac{1}{u^2}\right) \tag{3.162}$$

where $g_1(x)\in\mathcal{O}\left(g_2(x)\right)$ as $x\to\infty$ if and only if there is a constant M and a value x_0 such that $|g_1(x)|\leq M|g_2(x)|$ for $x>x_0$.

From their result in equation (3.145), the authors observe that the asymptotic decay of the integrand is at least exponential. Focussing on the term $e^{-u\Re_\infty}$, since $\Re_\infty>0$ the following transformation is valid

$$u(x) = -\frac{\ln(x)}{\Re_\infty} \tag{3.163}$$

where $x\in[0,1]$. In this range of x the pricing integrand is a function of

$$\frac{\text{Re}\left[e^{-iu(x)k}\Psi^{C}_{t_0,\,T}(u(x),\alpha)\right]}{x\Re_\infty} \tag{3.164}$$

which is undefined at $x=0$. Making use of equation (3.145) (as suggested in Kahl and Jackel [2005]), we see that the limit at this point is zero.

Proposition 16. *For the valuation of European Options with piecewise constant, time-dependent parameters*

$$u(x) = -\frac{\ln(x)}{\Re_\infty} \tag{3.165}$$

where

$$\Re_\infty = \frac{\sqrt{1-\rho_1^2}}{\nu_1} V_{t_0} + \sum_{m=1}^{n} \kappa_m \theta_m \left(\frac{\sqrt{1-\rho_m^2}}{\nu_m} \right) \tau_m \tag{3.166}$$

and $x \in [0,1]$ *is an appropriate transformation from* $u \in [0, \infty)$.

Proof:

Following exactly the same approach as that outlined above for the time-homogenous case, we write $\mathrm{Re}\left[e^{-iuk}\Psi^{\mathrm{C}}_{t_0,\,T}(u,\alpha)\right]$ as

$$e^{\Re} \left(\frac{[-u^2+\alpha(\alpha+1)]\cos(\Im) + u(2\alpha+1)\sin(\Im)}{[-u^2+\alpha(\alpha+1)]^2 + u^2(2\alpha+1)^2} \right) \tag{3.167}$$

where we now have

$$\Re = -r\tau + (\alpha+1)X_{t_0} + \mathrm{Re}\left[D_{1;n}(\tau_1, iz, D_{2;n})\right] V_{t_0} + \sum_{m=1}^{n} \mathrm{Re}\left[C_{m;n}(\tau_m, iz, D_{m+1;n})\right] \tag{3.168}$$

$$\Im = u(X_{t_0} - k) + \mathrm{Im}\left[D_{1;n}(\tau_1, iz, D_{2;n})\right] V_{t_0} + \sum_{m=1}^{n} \mathrm{Im}\left[C_{m;n}(\tau_m, iz, D_{m+1;n})\right] \tag{3.169}$$

with

$$\begin{aligned} C_{m;n}(\tau_m, iz, D_{m+1;n}) &= (r_m - q_m)\tau_m + \bar{C}_{m;n}(\tau_m, iz, D_{m+1;n}) \\ &+ \lambda_m \left[e^{iz\mu_m - \frac{1}{2}z^2\sigma_m^2} \bar{J}_{m;n}(\tau_m, iz, D_{m+1;n}) - \left(\frac{e^{\mu_m + \frac{1}{2}\sigma_m^2}}{1-\eta_m\rho_m^J} - 1 \right) iz\tau_m - \tau_m \right] \end{aligned}$$

Again, from equations (3.149) and (3.152), we have

$$\begin{aligned} &\lim_{u\to\infty} \frac{D_{m;n}(\tau_m, iz, D_{m+1;n})}{u} && (3.170) \\ &= \lim_{u\to\infty} \frac{1}{u} \left(\frac{b_m(iz) - \gamma_m(iz)}{\nu_m^2} \right) \left[\frac{\bar{A}_{m;n}(iz, D_{m+1;n}) - e^{-\gamma_m(iz)\tau_m}}{\bar{A}_{m;n}(iz, D_{m+1;n}) - \left(\frac{b_m(iz)-\gamma_m(iz)}{b_m(iz)+\gamma_m(iz)}\right) e^{-\gamma_m(iz)\tau_m}} \right] && (3.171) \\ &= \lim_{u\to\infty} \frac{1}{\nu_m^2} \left(\frac{b_m(iz) - \gamma_m(iz)}{u} \right) && (3.172) \\ &= \frac{-\sqrt{1-\rho_m^2} - i\rho_m}{\nu_m} && (3.173) \end{aligned}$$

From equations (3.152) and (3.173), we have

$$\begin{aligned} \lim_{u\to\infty} A^{-1}_{m;n}(iz, D_{m+1;n}) &= \lim_{u\to\infty} \frac{\frac{b_m(iz)-\gamma_m(iz)-\nu_m^2 D_{m+1;n}}{u}}{\frac{b_m(iz)+\gamma_m(iz)-\nu_m^2 D_{m+1;n}}{u}} && (3.174) \\ &= \frac{-i\rho_m\nu_m - \nu_m\sqrt{1-\rho_m^2} + \nu_m^2 \left[\frac{\sqrt{1-\rho_{m+1}^2} + i\rho_{m+1}}{\nu_{m+1}}\right]}{-i\rho_m\nu_m + \nu_m\sqrt{1-\rho_m^2} + \nu_m^2 \left[\frac{\sqrt{1-\rho_{m+1}^2} + i\rho_{m+1}}{\nu_{m+1}}\right]} \\ &:= A^{-1}_{m;n} && (3.175) \end{aligned}$$

From equations (3.149), (3.150), (3.151) and (3.175), we have

$$\lim_{u\to\infty} \frac{\bar{C}_{m;n}(\tau_m, iz, D_{m+1;n})}{u} \tag{3.176}$$

$$= \lim_{u\to\infty} \frac{1}{u}\left[\frac{\kappa_m\theta_m}{\nu_m^2}\left[b_m(iz)-\gamma_m(iz)\right]\tau_m - \frac{2\kappa_m\theta_m}{\nu_m^2}\ln\left(\frac{A_{m;n}^{-1}(iz, D_{m+1;n})e^{-\gamma_m(iz)\tau_m}-1}{A_{m;n}^{-1}(iz, D_{m+1;n})-1}\right)\right] \tag{3.177}$$

$$= \lim_{u\to\infty} \frac{\kappa_m\theta_m}{\nu_m^2}\left(\frac{b_m(iz)-\gamma_m(iz)}{u}\right)\tau_m \tag{3.178}$$

$$= \kappa_m\theta_m\left(\frac{-\sqrt{1-\rho_m^2}-i\rho_m}{\nu_m}\right)\tau_m \tag{3.179}$$

We can write

$$\bar{J}_{m;n}(\tau_m, iz, D_{m+1;n}) = \frac{\tau_m}{g_m(iz)} - \frac{2\eta_m}{\nu_m^2 g_m(iz)h_m(iz)}\log\left[\frac{A_{m;n}(iz, D_{m+1;n})g_m(iz)-h_m(iz)e^{-\gamma_m(iz)\tau_m}}{A_{m;n}(iz, D_{m+1;n})g_m(iz)-h_m(iz)}\right] \tag{3.180}$$

where

$$g_m(iz) = 1 - iz\eta_m\rho_m^{\mathrm{J}} - \eta_m\left(\frac{b_m(iz)-\gamma_m(iz)}{\nu_m^2}\right) \tag{3.181}$$

$$h_m(iz) = 1 - iz\eta_m\rho_m^{\mathrm{J}} - \eta_m\left(\frac{b_m(iz)+\gamma_m(iz)}{\nu_m^2}\right) \tag{3.182}$$

$$\lim_{u\to\infty}\frac{g_m(iz)}{u} = -i\eta_m\rho_m^{\mathrm{J}} + \eta_m\left(\frac{\sqrt{1-\rho_m^2}+i\rho_m}{\nu_m}\right) \tag{3.183}$$

$$:= g_m \tag{3.184}$$

$$\lim_{u\to\infty}\frac{h_m(iz)}{u} = -i\eta_m\rho_m^{\mathrm{J}} + \eta_m\left(\frac{-\sqrt{1-\rho_m^2}+i\rho_m}{\nu_m}\right) \tag{3.185}$$

$$:= h_m \tag{3.186}$$

and so from equations (3.149), (3.175), (3.184) and (3.186), we have

$$\lim_{u\to\infty}\bar{J}_{m;n}(\tau_m, iz, D_{m+1;n}) = \lim_{u\to\infty}\left(\frac{\tau_m}{ug_m} - \frac{2\eta_m}{\nu_m^2 u^2 g_m h_m}\log\left[\frac{A_{m;n}g_m}{A_{m;n}g_m - h_m}\right]\right)$$

$$= 0 \tag{3.187}$$

Therefore,

$$\lim_{u\to\infty}\frac{\Re}{u} = -\Re_\infty \tag{3.188}$$

$$\lim_{u\to\infty}\frac{\Im}{u} = \Im_\infty \tag{3.189}$$

where

$$\Re_\infty = \frac{\sqrt{1-\rho_1^2}}{\nu_1}V_{t_0} + \sum_{m=1}^{n}\kappa_m\theta_m\left(\frac{\sqrt{1-\rho_m^2}}{\nu_m}\right)\tau_m \tag{3.190}$$

$$\Im_\infty = X_{t_0} + \sum_{m=1}^{n}(r_m-q_m)\tau_m - k - \frac{\rho_1}{\nu_1}V_{t_0} - \sum_{m=1}^{n}\kappa_m\theta_m\left(\frac{\rho_m}{\nu_m}\right)\tau_m - \sum_{m=1}^{n}\lambda_m\left(\frac{e^{\mu_m+\frac{1}{2}\sigma_m^2}}{1-\eta_m\rho_m^{J}}-1\right)\tau_m$$

□

For a set of constant parameters, the result in proposition 16 reduces to that which follows from theorem 3.

Proposition 17. *For the valuation of $\%$ and $\$$ type Forward Starting Options with piecewise constant, time-dependent parameters*

$$u(x) = -\frac{\ln(x)}{\Re_\infty} \tag{3.191}$$

where

$$\Re_\infty = \sum_{m=l+1}^{n} \kappa_m\theta_m \left(\frac{\sqrt{1-\rho_m^2}}{\nu_m}\right)\tau_m \tag{3.192}$$

and $x \in [0,1]$ is an appropriate transformation from $u \in [0,\infty)$.

Proof:

The distinction from the European case (for the same number of increments n) is that for $\%$ and $\$$ type Forward Starting Options with determination date t_l, the conditional forward $(\%, \$)$ characteristic functions are functions of $\gamma_m(\mathbb{I})$ and $b_m(\mathbb{I})$ where $\gamma_m(\mathbb{I}) = |b_m(\mathbb{I})|$ and $b_m(\mathbb{I}) = \kappa_m - \rho_m\nu_m\mathbb{I}$ for the increments $m = 1, ..., l$ and functions of $\gamma_m(iz)$ and $b_m(iz)$ for the increments $m = l+1, ..., n$. The conditional characteristic function is a function of $\gamma_m(iz)$ and $b_m(iz)$ for all increments $m = 1, ..., n$. Since $b_m(\mathbb{I})$ is not a function of u, we see from equations (3.172) and (3.178) respectively that for $m = 1, ..., l$

$$\lim_{u\to\infty} \frac{D^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n})}{u} = 0 \tag{3.193}$$

$$\lim_{u\to\infty} \frac{\bar{C}^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n})}{u} = 0 \tag{3.194}$$

From equation (3.174) and equations (3.180)-(3.186) of proposition 16, we see that only $A_{m;n}(\mathbb{I}, D^{(\%,\$)}_{l;m+1;n})$ is a function of u and $\lim_{u\to\infty} A_{m;n}(\mathbb{I}, D^{(\%,\$)}_{l;m+1;n}) = 1$. It follows that for $m = 1, ..., l$

$$\lim_{u\to\infty} \frac{J^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n})}{u} = 0 \tag{3.195}$$

Hence, $\Re_\infty$ consists of the contributions made from $\sum_{m=l+1}^{n} \bar{C}_{m;n}(\tau_m, iz, D_{m+1;n})$ which we obtain from equation (3.179). $\Box$

From the proofs of propositions 16 and 17, it is clear that the presence of jumps does not influence the form of the transformations - this is pointed out in Lord and Kahl [2007].

3.5 Branch cutting in the presence of piecewise constant, time-dependent parameters

We now return to the issue of branch cutting raised in subsection 3.1.1 and make use of the notation of proposition 4 in section 2.4. Allowing for piecewise constant, time-dependent parameters in the Heston model, introduces a series of complex logarithms into the conditional characteristic function

$$-\sum_{m=1}^{n} \frac{2\kappa_m\theta_m}{\nu_m^2} \log\left[\psi_{m;n}(\tau_m, iz, D_{m+1;n})\right] \tag{3.196}$$

where

$$\psi_{m;n}(\tau_m, iz, D_{m+1;n}) \quad := \quad \psi_{m;n}(\tau_m, iz, D_{m+1;n}(\tau_{m+1}, iz, ...D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))...)) \tag{3.197}$$

for the increments $m = 1, ..., n$ with $D_{n+1;n} := 0$, $z := u - i\zeta$ and $\zeta := \alpha + 1$.

For the forward $(\%, \$)$ characteristic functions, we have the series

$$-\sum_{m=1}^{l} \frac{2\kappa_m \theta_m}{\nu_m^2} \log\left[\psi_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\right)\right] - \sum_{m=l+1}^{n} \frac{2\kappa_m \theta_m}{\nu_m^2} \log\left[\psi_{m;n}(\tau_m, iz, D_{m+1;n})\right] \tag{3.198}$$

where

$$\psi_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\right) \quad = \quad \psi_{m;n}(\tau_m, iz, D_{m+1;n}) \tag{3.199}$$

for the increments $m = l+1, ..., n$ and

$$\psi_{l;m;n}^{(\%,\$)}\left(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}\right) \quad := \quad \psi_{l;m;n}^{(\%,\$)}(\tau_m, \mathbb{I}, D_{l;m+1;n}^{(\%,\$)}(\tau_{m+1}, \mathbb{I}, ...D_{l;l;n}^{(\%,\$)}(\tau_l, \mathbb{I}, D_{l+1;n}(\tau_{l+1}, iz, D_{l+2;n}))...)) \tag{3.200}$$

for the increments $m = 1, ..., l$ with $\mathbb{I} := \mathbb{I}_{[z_{(\%,\$)} = z_\$]}$.

We aim to show that the range of the complex function $\psi^{(\%,\$)}$ does not include the negative real line for $\zeta \in [0, 1]$. We achieve this by first identifying a property that must be satisfied by the conditional characteristic and forward $(\%, \$)$ characteristic functions in this stochastic volatility setting. We then show that if the function ψ lies on the negative real line then this property is violated. To illustrate the approach, we consider the time-homogenous Heston model and the corresponding conditional characteristic function. We can express $D(\tau, iz, 0)$ as an explicit function of $\psi(\tau, iz, 0)$,

$$D(\tau, iz, 0) \quad = \quad \frac{[b(iz) + \gamma(iz)]}{\nu^2}\left(1 - \frac{1}{\psi(\tau, iz, 0)}\right) \tag{3.201}$$

Assuming that $\psi(\tau, iz, 0) = -\varphi \in \mathbb{R}$ where $\varphi > 0$ we then have

$$\text{Re}\left[D(\tau, iz, 0)\right] \quad = \quad \frac{[\text{Re}\left[b(iz)\right] + \text{Re}\left[\gamma(iz)\right]]}{\nu^2}\left(1 - \frac{1}{-\varphi}\right) \tag{3.202}$$

If we can show that $\text{Re}\left[D(\tau, iz, 0)\right] \le 0$ and $\text{Re}\left[b(iz)\right] + \text{Re}\left[\gamma(iz)\right] > 0$ then the assumption yields a contradiction.

We consider the issue separately for the conditional characteristic function and the conditional forward $(\%, \$)$ characteristic functions. In both cases, however, we will make use of the following proposition and the notation established in section 2.4.

Proposition 18. *For $z := u - i\zeta$, $\zeta \in [0, 1]$ and increments $m = 1, ..., n$, we have*

$$Re\left[\gamma_m(iz)\right] \ge \left|Re\left[b_m(iz)\right]\right| \tag{3.203}$$

with an equality only for $u = 0$ and $\zeta = 0, 1$. For $u = 0$ and $\zeta = 0$, we have $Re\left[b_m(0)\right] > 0$ while for $u = 0$ and $\zeta = 1$, the restriction $\rho_m < \frac{\kappa_m}{\nu_m}$ gives us $Re\left[b_m(1)\right] > 0$. Furthermore, for $u = 0$ and $\zeta \in [0, 1]$, we have

$$\gamma_m(\zeta), b_m(\zeta) \in \mathbb{R} \tag{3.204}$$

Proof: We have

$$b_m(iz) \quad = \quad [\kappa_m - \rho_m\nu_m\zeta] - i\rho_m\nu_m u \tag{3.205}$$

From equation (3.26) of subsection 3.1.2, we can write

$$\begin{aligned} \gamma_m(iz) \quad &= \quad \sqrt{\left[\text{Re}\,[b_m(iz)]^2 - \nu_m^2(\zeta-1)\zeta + u^2\nu_m^2(1-\rho_m^2)\right] - iu\left[\nu_m^2(2\zeta-1) + 2\rho_m\nu_m(\kappa_m - \rho_m\nu_m\zeta)\right]} \\ &=: \quad \sqrt{\text{Re}\,[\gamma_m^2(iz)] + i\text{Im}\,[\gamma_m^2(iz)]} \end{aligned} \tag{3.206}$$

From theorem 1 of subsection 3.1.2, we have

$$\text{Re}\,[\gamma_m(iz)] \quad = \quad \frac{1}{\sqrt{2}}\sqrt{\sqrt{\text{Re}\,[\gamma_m^2(iz)]^2 + \text{Im}\,[\gamma_m^2(iz)]^2} + \text{Re}\,[\gamma_m^2(iz)]} \tag{3.207}$$

$$\tag{3.208}$$

For $\zeta \in [0,1]$, we have $\text{Re}\left[\gamma_m^2(iz)\right] > 0$ and so

$$\begin{aligned} \text{Re}\,[\gamma_m(iz)] \quad &\geq \quad \sqrt{\text{Re}\,[\gamma_m^2(iz)]} \\ &= \quad \sqrt{\text{Re}\,[b_m(iz)]^2 - \nu_m^2(\zeta-1)\zeta + u^2\nu_m^2(1-\rho_m^2)} \\ &\geq \quad |\text{Re}\,[b_m(iz)]\,| \end{aligned} \tag{3.209}$$

For $u = 0$ and $\zeta = 0, 1$, we have $\text{Re}\,[\gamma_m(iz)] = |\text{Re}\,[b_m(iz)]\,|$. At $u = 0$, we have $\text{Re}\,[b_m(\zeta)] = \kappa_m - \rho_m\nu_m\zeta$. Specifically, $\text{Re}\,[b_m(0)] = \kappa_m > 0$ and $\text{Re}\,[b_m(1)] = \kappa_m - \rho_m\nu_m > 0$ where the latter holds for $\rho_m < \frac{\kappa_m}{\nu_m}$. We have $\text{Im}\,[b_m(iz)] = -\rho_m\nu_m u$ and so $b_m(\zeta) \in \mathbb{R}$ and from equation (3.206), we have $\gamma_m(\zeta) \in \mathbb{R}$ for $\zeta \in [0,1]$. □

3.5.1 European Options

The semi-analytic formula for a European Option features the conditional characteristic function for X as can be seen from equation (2.11) of subsection 2.1.1. We prove that $\psi_{m;n}(\tau_m, iz, D_{m+1;n})$ cannot lie on the negative real line for $\zeta \in [0,1]$ (subject to parameter restrictions for $\zeta = 1$). To achieve this, we focus on $D_{m;n}(\tau_m, iz, D_{m+1;n})$ for $m = 1, ..., n$, identifying properties of this function in propositions 19 and 20 which allow us to then prove the final result in proposition 21.

Proposition 19. *For the increments $m = 1, ..., n$, we have*

$$Re\left[D_{m;n}\left(\tau_m, iz, D_{m+1;n}\right)\right] \quad \leq \quad D_{m;n}\left(\tau_m, \zeta, D_{m+1;n}\right) \tag{3.210}$$

Proof: Jensen's inequality gives us

$$\begin{aligned} \left|\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp(izX_{t_n})|X_{t_0}, V_{t_0}\right]\right| \quad &\leq \quad \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\left|\exp(izX_{t_n})\right||X_{t_0}, V_{t_0}\right] \\ &= \quad \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp(\zeta X_{t_n})|X_{t_0}, V_{t_0}\right] \end{aligned} \tag{3.211}$$

Making use of proposition 4 in section 2.4, for $l = 0$, we have

$$\begin{aligned}
& \left|\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(izX_{t_n})|X_{t_0}, V_{t_0}\right]\right| \\
= \quad & \exp\left[\zeta X_{t_0} + \mathrm{Re}\left[D_{1;n}(\tau_1, iz, D_{2;n})\right] V_{t_0} + \mathrm{Re}\left[\sum_{m=1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right]\right]
\end{aligned} \tag{3.212}$$

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(\zeta X_{t_n})|X_{t_0}, V_{t_0}\right] \\
= \quad & \exp\left[\zeta X_{t_0} + D_{1;n}(\tau_1, \zeta, D_{2;n}) V_{t_0} + \mathrm{Re}\left[\sum_{m=1}^{n} C_{m;n}(\tau_m, \zeta, D_{m+1;n})\right]\right]
\end{aligned} \tag{3.213}$$

within the affine framework where we know $D_{1;n}(\tau_1, \zeta, D_{2;n}) \in \mathbb{R}$ from equation (3.13) and so

$$\begin{aligned}
0 \quad \leq \quad & \left[D_{1;n}(\tau_1, \zeta, D_{2;n}) - \mathrm{Re}\left[D_{1;n}(\tau_1, iz, D_{2;n})\right]\right] V_{t_0} \\
+ \quad & \left[\mathrm{Re}\left[\sum_{m=1}^{n} C_{m;n}(\tau_m, \zeta, D_{m+1;n})\right] - \mathrm{Re}\left[\sum_{m=1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right]\right]
\end{aligned} \tag{3.214}$$

Equation (3.214) must hold for any positive V_{t_0} and so we must have

$$\mathrm{Re}\left[D_{1;n}(\tau_1, iz, D_{2;n})\right] \quad \leq \quad D_{1;n}(\tau_1, \zeta, D_{2;n}) \tag{3.215}$$

Given n, $D_{2;n}(\tau_2, iz, D_{3;n})$ has the same form (regarding the terminal conditions specified by the arguments iz and $D_{3;n}$) as $D_{1;n-1}(\tau_1, iz, D_{2;n-1})$, $D_{3;n}(\tau_3, iz, D_{4;n})$ has the same form as $D_{1;n-2}(\tau_1, iz, D_{2;n-2})$ and so on. From this, we obtain the result, for $m = 1, ..., n$. $\square$

Proposition 20. *For the increments $m = 1, ..., n$, we have*

$$D_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad < \quad 0 \quad \textit{for} \quad \zeta \in (0, 1) \tag{3.216}$$

and

$$D_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad = \quad 0 \quad \textit{for} \quad \zeta = 0, 1 \tag{3.217}$$

Proof: From equation (2.78) of proposition 2 in section 2.2 and for $u = 0$, we can write

$$\begin{aligned}
D_{m;n}(\tau_m, \zeta, D_{m+1;n}) \quad = \quad & \frac{\frac{1}{\nu_m^2}\left[b_m(\zeta) - \gamma_m(\zeta)\right]\left[b_m(\zeta) + \gamma_m(\zeta)\right]\left(1 - e^{-\gamma_m(\zeta)\tau_m}\right)}{\left[b_m(\zeta) + \gamma_m(\zeta) - \nu_m^2 D_{m+1;n}\right]\left(1 - e^{-\gamma_m(\zeta)\tau_m}\right) + 2\gamma_m(\zeta)e^{-\gamma_m(\zeta)\tau_m}} \\
- \quad & \frac{\left(\left[b_m(\zeta) - \gamma_m(\zeta)\right] - \left[b_m(\zeta) + \gamma_m(\zeta)\right]e^{-\gamma_m(\zeta)\tau_m}\right) D_{m+1;n}}{\left[b_m(\zeta) + \gamma_m(\zeta) - \nu_m^2 D_{m+1;n}\right]\left(1 - e^{-\gamma_m(\zeta)\tau_m}\right) + 2\gamma_m(\zeta)e^{-\gamma_m(\zeta)\tau_m}}
\end{aligned} \tag{3.218}$$

From proposition 18 we have $b_m(\zeta), \gamma_m(\zeta) \in \mathbb{R}$, $b_m(\zeta) - \gamma_m(\zeta) < 0$ and $b_m(\zeta) + \gamma_m(\zeta) > 0$ for $u = 0$ and $\zeta \in (0, 1)$. For $m = n$ and $D_{n+1;n} := 0$, we have

$$\begin{aligned}
D_{n;n}(\tau_n, \zeta, 0) \quad = \quad & \frac{\frac{1}{\nu_n^2}\left[b_n(\zeta) - \gamma_n(\zeta)\right]\left[b_n(\zeta) + \gamma_n(\zeta)\right]\left(1 - e^{-\gamma_n(\zeta)\tau_n}\right)}{\left[b_n(\zeta) + \gamma_n(\zeta)\right]\left(1 - e^{-\gamma_n(\zeta)\tau_n}\right) + 2\gamma_n(\zeta)e^{-\gamma_n(\zeta)\tau_n}} \\
< \quad & 0
\end{aligned} \tag{3.219}$$

For $1 \leq m < n$, the above mentioned inequalities (which follow from proposition 18) and equation (3.219) may be used to induct the final result.

From proposition 18 we have $\gamma_m(\zeta) = |b_m(\zeta)|$ for $u = 0$ and $\zeta = 0, 1$. Hence, equation (3.218) reduces to

$$D_{m;n}(\tau_m, \zeta, D_{m+1;n}) = \frac{D_{m+1;n} b_m(\zeta) e^{-b_m(\zeta)\tau_m}}{b_m(\zeta) - \frac{1}{2}\nu_m^2 \left(1 - e^{-b_m(\zeta)\tau_m}\right) D_{m+1;n}} \tag{3.220}$$

$$\lim_{b_m(\zeta)\to 0} D_{m;n}(\tau_m, \zeta, D_{m+1;n}) = \frac{D_{m+1;n}}{1 - \frac{1}{2}\nu_m^2 D_{m+1;n}\tau_m} \tag{3.221}$$

For $m = n$ and $D_{n+1;n} := 0$, we have $D_{n;n}(\tau_n, \zeta, 0) = 0$ where, again, an inductive argument yields the result for $1 \leq m < n$. $\square$

Proposition 21. *For the increments $m = 1, ..., n$ and $\zeta \in [0, 1)$, $\psi_{m;n}(\tau_m, iz, D_{m;n})$ cannot lie on the branch cut $(-\infty, 0]$. Subject to the parameter restriction $\rho_m < \frac{\kappa_m}{\nu_m}$, the same is true for $\zeta = 1$.*

Proof: From the proof of proposition 9 in subsection 3.2.1, we know that for $\zeta \in (\zeta^{\min}, \zeta^{\max})$ and all u

$$\psi_{m;n}(\tau_m, iz, D_{m+1;n}) \neq 0 \tag{3.222}$$

As $\psi_{m;n}(\tau_m, iz, D_{m+1;n})$ hits the origin, the conditional characteristic function explodes to infinity. Hence, we can ignore the origin when considering the range of the function. Regarding the negative real line, we can express $D_{m;n}(\tau_m, iz, D_{m+1;n})$ in a far more enlightening form

$$D_{m;n}(\tau_m, iz, D_{m+1;n}) = \frac{D_{m+1;n}}{\psi_{m;n}(\tau, iz, D_{m+1;n})} + \frac{[b_m(iz) + \gamma_m(iz)]}{\nu_m^2}\left(1 - \frac{1}{\psi_{m;n}(\tau, iz, D_{m+1;n})}\right)$$

From propositions 19 and 20, we have

$$\mathrm{Re}\left[D_{m;n}(\tau_m, iz, D_{m+1;n})\right] \leq 0 \tag{3.223}$$

for $m = 1, ..., n$ and $\zeta \in [0, 1]$.

We now obtain the result by means of a contradiction. Assume $\psi_{m;n}\left(\tau_m, iz, D_{m+1;n}\right) = -\varphi \in \mathbb{R}$ and $\varphi > 0$. This gives us

$$\mathrm{Re}\left[D_{m;n}(\tau_m, iz, D_{m+1;n})\right] = \frac{\mathrm{Re}\left[D_{m+1;n}\right]}{-\varphi} + \frac{\left[\mathrm{Re}\left[b_m(iz)\right] + \mathrm{Re}\left[\gamma_m(iz)\right]\right]}{\nu_m^2}\left(1 - \frac{1}{-\varphi}\right) \tag{3.224}$$

From equation (3.223), we see that the RHS of equation (3.224) must be non-positive. From proposition 18, we have $\mathrm{Re}\left[b_m(iz)\right] + \mathrm{Re}\left[\gamma_m(iz)\right] > 0$ (where the parameter restriction $\rho_m < \frac{\kappa_m}{\nu_m}$ applies to the case $\zeta = 1$) and since the sign of $\mathrm{Re}\left[D_{m+1;n}\right]$ is the same as that of $\mathrm{Re}\left[D_{m;n}(\tau_m, iz, D_{m+1;n})\right]$, the assumption $\varphi > 0$ implies that the RHS of equation (3.224) is always positive. $\square$

The available literature only considers this issue of branch cutting for the time-homogenous case where $n = 1$. For this case, proposition 21 confirms the result proved in Lord and Kahl [2008] (granted only for $\zeta \in [0, 1)$). However, the major result of Fahrner [2007] - proving that branch cutting is not an issue for $\zeta = \frac{1}{2}$ in the 'displaced diffusion' extension of the Heston model - is confirmed without having to introduce any parameter restrictions (we elaborate on this in subsection 3.5.3). For $n > 1$, proposition 21 tells us that for the considered range of ζ (and the specified parameter restrictions) branch cutting is not an issue and hence, discontinuities will not arise when valuing European Options with piecewise constant, time-dependent parameters in terms of the functions presented in proposition 2 of section 2.2. However, we claim that the range of the function $\psi_{m;n}(\tau_m, iz, D_{m+1;n})$ does not include $(-\infty, 0]$ for all $\zeta \in (\zeta^{\min}, \zeta^{\max})$ where $\zeta^{\min} < 0$ and $\zeta^{\max} > 1$. Unfortunately, we must leave this claim as a conjecture.

3.5.2 Forward Starting Options

The semi-analytic formulae for $\%$ and $\$$ type Forward Starting Options feature the corresponding conditional forward $(\%, \$)$ characteristic functions as can be seen from equations (2.49) and (2.51) of proposition 1 in subsection 2.1.2. We prove that the function $\psi^{(\%,\$)}_{m;n}(\tau_m, iz, D^{(\%,\$)}_{m+1;n})$ cannot lie on the negative real line for $\zeta \in [0,1]$ (subject to parameter restrictions for $\zeta = 1$). For the period $\tau = (t_n - t_l) + (t_l - t_0)$ split into n increments with the determination date t_l and $1 \leq l < n$, the form of the conditional forward $(\%, \$)$ characteristic functions differ from the form of the conditional characteristic function for the same period (with n increments) only because the argument iz is replaced with $\mathbb{I} := \mathbb{I}_{\left[z_{(\%,\$)} = z_{\$}\right]}$, for the increments $1, ..., l$. The form of the respective functions are exactly the same for the increments $l+1, ..., n$. Hence, for increments $m > l$, the result follows from proposition 21. To prove the result for increments $m \leq l$ and $\zeta \in [0,1]$ (subject to parameter restrictions for $\zeta = 1$), we follow the same approach as that taken in subsection 3.5.1. Focussing on the function $D^{(\%,\$)}_{m;n}(\tau_m, iz, D_{m+1;n})$, we determine two of its properties in propositions 22 and 23 which then allow us to directly prove the final result in proposition 24.

Proposition 22. *For the increments $m = 1, ..., n$, we have*

$$Re\left[D^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right)\right] \leq D^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\big|_{u=0}\right) \tag{3.225}$$

where

$$D^{(\%,\$)}_{l;m+1;n}\big|_{u=0} = D^{(\%,\$)}_{l;m+1;n}(\tau_{m+1}, \mathbb{I}, D^{(\%,\$)}_{l;m+2;n}(\tau_{m+2}, \mathbb{I}, ...D^{(\%,\$)}_{l;l;n}(\tau_l, \mathbb{I}, D_{l+1;n}(\tau_{l+1}, \zeta, D_{l+2;n}))...)) \tag{3.226}$$

Proof: For $m > l$, we have

$$D^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) = D_{m;n}(\tau_m, iz, D_{m+1;n}) \tag{3.227}$$

and the result follows from proposition 19.

For $m \leq l$, we consider the forward $(\%, \$)$ characteristic function $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{iz_{(\%,\$)}X_{t_l} + izX_{t_n}}|X_{t_0}, V_{t_0}\right]$ for $n \geq 2$ and $1 \leq l < n$. Jensen's inequality gives us

$$\begin{aligned}
& \left|\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(iz_{(\%,\$)}X_{t_l})\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp(izX_{t_n})|X_{t_l}, V_{t_l}\right]|X_{t_0}, V_{t_0}\right]\right| \\
\leq \; & \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\left|\exp(iz_{(\%,\$)}X_{t_l})\right|\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\left|\exp(izX_{t_n})\right||X_{t_l}, V_{t_l}\right]|X_{t_0}, V_{t_0}\right] \\
= \; & \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(\mathrm{Re}\left[iz_{(\%,\$)}\right]X_{t_l})\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp\left(\zeta X_{t_n}\right)|X_{t_l}, V_{t_l}\right]|X_{t_0}, V_{t_0}\right]
\end{aligned}$$

Making use of proposition 4 in section 2.4, for $l > 0$, we have

$$\begin{aligned}
& \left|\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp(iz_{(\%,\$)}X_{t_l})\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp(izX_{t_n})|X_{t_l},V_{t_l}\right]|X_{t_0},V_{t_0}\right]\right| \\
= & \exp\left[\mathbb{I}X_{t_0} + \mathrm{Re}\left[D^{(\%,\$)}_{l;1;n}\left(\tau_1,\mathbb{I},D^{(\%,\$)}_{l;2;n}\right)\right]V_{t_0} + \sum_{m=1}^{l}\mathrm{Re}\left[C^{(\%,\$)}_{l;m;n}\left(\tau_m,\mathbb{I},D^{(\%,\$)}_{l;m+1;n}\right)\right]\right] \\
\times & \exp\left[\sum_{m=l+1}^{n}\mathrm{Re}\left[C_{m;n}\left(\tau_m,iz,D_{m+1;n}\right)\right]\right]
\end{aligned} \tag{3.228}$$

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\exp\left(\mathrm{Re}\left[iz_{(\%,\$)}\right]X_{t_l}\right)\mathbb{E}^{\mathbb{Q}}_{t_l}\left[\exp\left(\zeta X_{t_n}\right)|X_{t_l},V_{t_l}\right]|X_{t_0},V_{t_0}\right] \\
= & \exp\left[\mathbb{I}X_{t_0} + D^{(\%,\$)}_{l;1;n}\left(\tau_1,\mathbb{I},D^{(\%,\$)}_{l;2;n}|_{u=0}\right)V_{t_0} + \sum_{m=1}^{l}\mathrm{Re}\left[C^{(\%,\$)}_{l;m;n}\left(\tau_m,\mathbb{I},D^{(\%,\$)}_{l;m+1;n}|_{u=0}\right)\right]\right] \\
\times & \exp\left[\sum_{m=l+1}^{n}\mathrm{Re}\left[C_{m;n}\left(\tau_m,\zeta,D_{m+1;n}\right)\right]\right]
\end{aligned} \tag{3.229}$$

within the affine framework where we know $D^{(\%,\$)}_{l;1;n}\left(\tau_1,\mathbb{I},D^{(\%,\$)}_{l;2;n}|_{u=0}\right) \in \mathbb{R}$ from equation (3.13) and for $z_{(\%,\$)} = z_{\%}$, we have $\mathbb{I} = 0$ while for $z_{(\%,\$)} = z_{\$}$, we have $\mathbb{I} = 1$. This leads to

$$\mathrm{Re}\left[D^{(\%,\$)}_{l;1;n}\left(\tau_1,\mathbb{I},D^{(\%,\$)}_{l;2;n}\right)\right] \leq D^{(\%,\$)}_{l;1;n}\left(\tau_1,\mathbb{I},D^{(\%,\$)}_{l;2;n}|_{u=0}\right) \tag{3.230}$$

Given n and l, $D^{(\%,\$)}_{l;2;n}(\tau_2,\mathbb{I},D^{(\%,\$)}_{l;3;n})$ has the same form (regarding the terminal conditions specified by the arguments $\mathbb{I}$ and $D^{(\%,\$)}_{l;3;n}$) as $D^{(\%,\$)}_{l-1;1;n-1}(\tau_1,\mathbb{I},D^{(\%,\$)}_{l-1;2;n-1})$, $D^{(\%,\$)}_{l;3;n}(\tau_3,\mathbb{I},D^{(\%,\$)}_{4;n})$ has the same form as $D^{(\%,\$)}_{l-2;1;n-2}(\tau_1,\mathbb{I},D^{(\%,\$)}_{l-2;2;n-2})$ and so on. From this, we obtain the result. □

Proposition 23. *For the increments $m = 1,...,n$, we have*

$$D^{(\%,\$)}_{l;m;n}\left(\tau_m,\mathbb{I},D^{(\%,\$)}_{l;m+1;n}|_{u=0}\right) < 0 \quad \textit{for} \quad \zeta \in (0,1) \tag{3.231}$$

and

$$D^{(\%,\$)}_{l;m;n}\left(\tau_m,\mathbb{I},D^{(\%,\$)}_{l;m+1;n}|_{u=0}\right) = 0 \quad \textit{for} \quad \zeta = 0,1 \tag{3.232}$$

Proof: For $m > l$, we have

$$D^{(\%,\$)}_{l;m;n}\left(\tau_m,\mathbb{I},D^{(\%,\$)}_{l;m+1;n}|_{u=0}\right) = D_{m;n}(\tau_m,\zeta,D_{m+1;n}) \tag{3.233}$$

and the result follows from proposition 20.

For $m \leq l$, we have $\gamma_m(\mathbb{I}) = |b_m(\mathbb{I})|$ from proposition 18. For $z_{(\%,\$)} = z_{\%}$, we have $\mathbb{I} = 0$ while for $z_{(\%,\$)} = z_{\$}$, we have $\mathbb{I} = 1$. From equation (2.78) of proposition 2 in section 2.2 and for $\gamma_m(\mathbb{I}) = |b_m(\mathbb{I})|$, we have

$$D^{(\%,\$)}_{l;m;n}\left(\tau_m,\mathbb{I},D^{(\%,\$)}_{l;m+1;n}|_{u=0}\right) = \frac{D^{(\%,\$)}_{l;m+1;n}|_{u=0}b_m(\mathbb{I})e^{-b_m(\mathbb{I})\tau_m}}{b_m(\mathbb{I}) - \frac{1}{2}\nu_m^2\left(1-e^{-b_m(\mathbb{I})\tau_m}\right)D^{(\%,\$)}_{l;m+1;n}|_{u=0}} \tag{3.234}$$

$$\lim_{b_m(\mathbb{I})\to 0} D^{(\%,\$)}_{l;m;n}\left(\tau_m,\mathbb{I},D^{(\%,\$)}_{l;m+1;n}|_{u=0}\right) = \frac{D^{(\%,\$)}_{l;m+1;n}|_{u=0}}{1-\frac{1}{2}\nu_m^2 D^{(\%,\$)}_{l;m+1;n}|_{u=0}\tau_m} \tag{3.235}$$

To clarify our notation

$$D^{(\%,\$)}_{l;m+1;n}\big|_{u=0} = D^{(\%,\$)}_{l;m+1;n}(\tau_{m+1}, \mathbb{I}, D^{(\%,\$)}_{l;m+2;n}(\tau_{m+2}, \mathbb{I}, ...D^{(\%,\$)}_{l;l;n}(\tau_l, \mathbb{I}, D_{l+1;n}(\tau_{l+1}, \zeta, D_{l+2;n}))...)) \tag{3.236}$$

For $m = l$, $\zeta \in (0,1)$ and making use of equations (3.234) and (3.235), we have

$$D^{(\%,\$)}_{l;l;n}\left(\tau_l, \mathbb{I}, D_{l+1;n}\big|_{u=0}\right) < 0$$

since $D^{(\%,\$)}_{l;l+1;n}|_{u=0} = D_{l+1;n}(\tau_{l+1}, \zeta, D_{l+2;n}) < 0$ from equation (3.216) of proposition 20.

Similarly, for $m = l$ and $\zeta = 0, 1$, we have

$$D^{(\%,\$)}_{l;l;n}\left(\tau_l, \mathbb{I}, D_{l+1;n}\big|_{u=0}\right) = 0$$

since $D^{(\%,\$)}_{l;l+1;n}|_{u=0} = D_{l+1;n}(\tau_{l+1}, \zeta, D_{l+2;n}) = 0$ from equation (3.217) of proposition 20.

For $m < l$, an inductive argument yields the result. □

Proposition 24. *For the increments $m = 1, ..., n$ and $\zeta \in [0,1)$, $\psi^{(\%,\$)}_{l;m;n}(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m;n})$ cannot lie on the branch cut $(-\infty, 0]$. Subject to the parameter restriction $\rho_m < \frac{\kappa_m}{\nu_m}$ for $m > l$, the same is true for $\zeta = 1$.*

Proof: For $m > l$, we have

$$\psi^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) = \psi_{m;n}(\tau_m, iz, D_{m+1;n}) \tag{3.237}$$

and the result follows from proposition 21.

From equation (2.81) of proposition 2 in section 2.2 we have the following for the increments $m \le l$. For $b_m(\mathbb{I}) > 0$

$$\psi^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) = 1 - \frac{\frac{1}{2}\nu_m^2 D^{(\%,\$)}_{l;m+1;n}\left[1 - e^{-b_m(\mathbb{I})\tau_m}\right]}{b_m(\mathbb{I})} \tag{3.238}$$

For $b_m(\mathbb{I}) < 0$

$$\psi^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) = \left(1 - \frac{\frac{1}{2}\nu_m^2 D^{(\%,\$)}_{l;m+1;n}\left[1 - e^{-b_m(\mathbb{I})\tau_m}\right]}{b_m(\mathbb{I})}\right) e^{b_m(\mathbb{I})\tau_m}$$

For $b_m(\mathbb{I}) = 0$

$$\psi^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right) = 1 - \frac{1}{2}\nu_m^2 D^{(\%,\$)}_{l;m+1;n}\tau_m \tag{3.239}$$

Since $b_m(\mathbb{I}) = \kappa_m - \rho_m\nu_m\mathbb{I} \in \mathbb{R}$, Re $\left[\psi^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right)\right]$ is a function only of Re $\left[D^{(\%,\$)}_{l;m+1;n}\right]$ and not Im $\left[D^{(\%,\$)}_{l;m+1;n}\right]$. From propositions 22 and 23, we have

$$\text{Re}\left[D^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right)\right] \le 0 \tag{3.240}$$

for $m = 1, ..., n$ and so

$$\text{Re}\left[\psi^{(\%,\$)}_{l;m;n}\left(\tau_m, \mathbb{I}, D^{(\%,\$)}_{l;m+1;n}\right)\right] > 0 \tag{3.241}$$

□

3.5.3 An additional parameter

Regarding the Heston model, we can introduce the parameter $\bar{\sigma}$ to the underlying process such that

$$dS_t = rS_t dt + \bar{\sigma} S_t \sqrt{V_t} dW_t^X \tag{3.242}$$

without complicating the analytic tractability of the model. For $X = \ln S$, Itô's formula yields the dynamics

$$\begin{aligned} dX_t &= \left(r - q - \frac{1}{2}\bar{\sigma}^2 V_t\right) dt + \bar{\sigma}\sqrt{V_t} dW_t^X \\ dV_t &= \kappa(\theta - V_t)dt + \nu\sqrt{V_t} dW_t^V \\ dW_t^X dW_t^V &= \rho dt \end{aligned}$$

Solving for the analytic characteristic function, one need only replace the functions $b(iz)$ and $c(iz)$ in equations (2.100) and (2.101) of the proof of proposition 2 in section 2.2, respectively with

$$b(iz) = \kappa - \rho\nu\bar{\sigma} iz \tag{3.243}$$

$$c(iz) = \frac{1}{2}\bar{\sigma}^2 iz(iz-1) \tag{3.244}$$

and so

$$\gamma(iz) = \sqrt{(\kappa - \rho\nu\bar{\sigma} iz)^2 - \nu^2\bar{\sigma}^2 iz(iz-1)} \tag{3.245}$$

Focussing on the functions $b(iz)$ and $\gamma(iz)$, we see that the results derived in subsections 3.5.1 and 3.5.2 accommodate for this extension of the Heston model where $\bar{\sigma} > 0$ as we then have $\nu\bar{\sigma} > 0$ in the functions $b(iz)$ and $\gamma(iz)$.

In Fahrner [2007], the issue of branch cutting is considered regarding the dynamics

$$\begin{aligned} dS_t &= \sigma\left[\beta S_t + (1-\beta)L\right]\sqrt{V_t} dW_t^X \\ dV_t &= \kappa(\theta - V_t)dt + \nu\sqrt{V_t} dW_t^V \\ dW_t^X dW_t^V &= \rho dt \end{aligned}$$

where $0 < \beta \leq 1$, $\sigma > 0$ and $L > 0$. For $\bar{X} = \beta S + (1-\beta)L$ (as considered in Andersen and Brotherton-Ratcliffe [2005]) and $X = \ln \bar{X}$, Itô's formula yields the dynamics

$$dX_t = -\frac{1}{2}\sigma^2\beta^2 V_t dt + \sigma\beta\sqrt{V_t} dW_t^X \tag{3.246}$$

$$dV_t = \kappa(\theta - V_t) dt + \nu\sqrt{V_t} dW_t^V \tag{3.247}$$

$$dW_t^X dW_t^V = \rho dt \tag{3.248}$$

Since $\sigma > 0$ and $0 < \beta \leq 1$ we can treat this specification as that arising from equation (3.242) where $\bar{\sigma} > 0$ as we have

$$b(iz) = \kappa - \rho\nu\sigma\beta iz \tag{3.249}$$

$$\gamma(iz) = \sqrt{(\kappa - \rho\nu\sigma\beta iz)^2 - \nu^2\sigma^2\beta^2 iz(iz-1)} \tag{3.250}$$

However, we restrict the parameter β to a constant when allowing for piecewise constant, time-dependent parameters as specifying $\bar{X} = \beta S + (1-\beta)L$ complicates the evaluation of a European payoff. Specifically, we have the payoff

$$[S_T - K]^+ \quad = \quad \frac{1}{\beta}\left[\bar{X}_T - \bar{K}\right]^+ \tag{3.251}$$

where $\bar{K} = \beta K + (1-\beta)L$ and so it follows that within the valuation formulae for European Options, the parameter β is not restricted to feature only within the characteristic function.

For time-homogenous parameters, Fahrner [2007] specifically considers a proof for the case $\zeta = \frac{1}{2}$ assuming that Re $[b(iz)] > 0$ and states that, in this case, Im $[\gamma(iz)]$ and Re $[b(iz)]$ Im $[b(iz)]$ have the same sign. However, from this we know that Im $[\gamma(iz)]$ and Im $[b(iz)]$ have the same sign (and specifying Re $[\gamma(iz)] > 0)$[7], we then have

$$\text{Re}\,[b(iz)]\,\text{Re}\,[\gamma(iz)] + \text{Im}\,[b(iz)]\,\text{Im}\,[\gamma(iz)] \quad > \quad 0 \tag{3.252}$$

From Lord and Kahl [2008] Lemma 2, equation (A.17) and Theorem 2, we know that if the form of the inequality in equation (3.252) is satisfied then $\psi(\tau, iz, 0)$ cannot lie on the branch cut $(-\infty, 0]$.[8] Hence, for the assumption made, the result may be inferred from the work of Lord and Kahl [2008].

Furthermore, our method of proof accommodates for the specified dynamics and $\zeta = \frac{1}{2}$ within the context of both European and Forward Starting Options with piecewise constant, time-dependent parameters, without the need to introduce any parameter restrictions.

3.5.4 A more general problem

The jump component of the conditional characteristic function for the SVJJ model also features a complex logarithm. Working from equation (2.83) of proposition 2 in section 2.2, we can express this as

$$\log\left(\frac{\vartheta(iz, iz_v) + \bar{\vartheta}(iz)e^{-\gamma(iz)\tau}}{\vartheta(iz, iz_v) + \bar{\vartheta}(iz)}\right) \quad = \quad \log\left(\frac{\tilde{A}^{-1}(iz, iz_v)e^{-\gamma(iz)\tau} - 1}{\tilde{A}^{-1}(iz, iz_v) - 1}\right) \tag{3.253}$$

where

$$\tilde{A}(iz, iz_v) \quad = \quad A(iz, iz_v)\left[\frac{1 - iz\eta\rho^{J} - \eta\left(\frac{b(iz)-\gamma(iz)}{\nu^2}\right)}{1 - iz\eta\rho^{J} - \eta\left(\frac{b(iz)+\gamma(iz)}{\nu^2}\right)}\right] \tag{3.254}$$

and so for $\eta > 0$, we have a more general version of the complex logarithm that appears within the diffusion component of the conditional characteristic function. However, again it would seem that the function, whose logarithm we are considering in equation (3.253), cannot lie on the branch cut $(-\infty, 0]$. Having pointed out articles, in subsection 3.1.1, which prove that the branch cut is not crossed for the diffusion component of the SVJJ model for $\zeta \in (\zeta^{\min}, \zeta^{\max})$ and presented a proof, in this section, allowing for piecewise constant parameters where $\zeta \in [0, 1]$, it would seem that there is an underlying reason that has, as of yet, not been identified which ensures that, even for the jump component, branch cutting is not an issue.

[7] This choice represents the principal value for the complex square root.

[8] This appears in an earlier version of the result in Lord and Kahl [2007] Lemma 3, equation (A.19) and Theorem 3.

Chapter 4

Obtaining forward parameters from the semi-analytic pricing formulae

4.1 The SABR model and Forward Starting Options

From Hagan et al. [2002], we assume the following dynamics for the forward value F and volatility α

$$\begin{aligned} dF_t &= \alpha_t F_t^{\beta} dW_t^F \\ d\alpha_t &= \nu \alpha_t dW_t^{\alpha} \\ dW_t^F dW_t^{\alpha} &= \rho dt \end{aligned} \tag{4.1}$$

where F_{t_0} is the current forward price.

Had we first considered the dynamics of α under the real world measure, we would have obtained

$$d\alpha_t = \left[p(\alpha_t) - q(\alpha_t)\lambda_t^{\alpha}(\alpha_t)\right] dt + q(\alpha_t) dW_t^{\alpha} \tag{4.2}$$

under the forward $\lambda_t^{\alpha}(\alpha_t)$ measure. Hence, in the SABR model

$$\lambda_t^{\alpha}(\alpha_t) := \frac{p(\alpha_t)}{q(\alpha_t)} \tag{4.3}$$

where $q(\alpha_t) = \nu \alpha_t$ but $p(\alpha_t)$ is left unspecified.

In Hagan et al. [2002], an approximate solution is derived for the value of a European Option in terms of a skew in Black's model[1] where the implied volatility is given by

$$\sigma_{t_0,T,K}(F_{t_0},\alpha_{t_0},\beta,\rho,\nu) = \frac{\alpha_{t_0}\left(1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha_{t_0}^2}{(F_{t_0}K)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha_{t_0}}{(F_{t_0}K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]\tau\right)}{(F_{t_0}K)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{F_{t_0}}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{F_{t_0}}{K}\right)\right]}\left(\frac{y}{\xi(y)}\right) \quad (4.4)$$

with

$$y = \frac{\nu}{\alpha_{t_0}}(F_{t_0}K)^{(1-\beta)/2}\ln\left(\frac{F_{t_0}}{K}\right)$$

$$\xi(y) = \ln\left(\frac{\sqrt{1-2\rho y+y^2}+y-\rho}{1-\rho}\right)$$

where l'Hôpital's rule is used to deal with the indeterminant $\frac{0}{0}$ form that arises in equation (4.4) for $K = F_{t_0}$.

4.1.1 Forward Starting % Call Options

To obtain the time t_0 value of the Forward Starting % payoff $\left[\frac{S_{T_2}}{S_{T_1}} - K\right]^+$, we first obtain the time T_1 value. Assuming the specified dynamics, we have the approximate time T_1 value,

$$\Pi^{\%C}_{T_1,T_1,T_2} = B_{T_1,T_2}\left[\frac{F_{T_1}}{S_{T_1}}N(d_+) - KN(d_-)\right] \quad (4.5)$$

$$d_\pm = \frac{\ln\left(\frac{F_{T_1}}{S_{T_1}K}\right) \pm \frac{1}{2}\sigma_2^2\tau_2}{\sigma_2\sqrt{\tau_2}}$$

with $\tau_2 = T_2 - T_1$, $\sigma_2 = \sigma_{T_1,T_2,K}\left(\frac{F_{T_1}}{S_{T_1}}, \alpha_{T_1}, \beta_2, \rho_2, \nu_2\right)$ where the constant parameters β_2, ρ_2 and ν_2 are valid for the period $(T_1, T_2]$. Assuming a constant dividend yield q_2 for the period $(T_1, T_2]$, arbitrage arguments yield $\frac{F_{T_1}}{S_{T_1}} = \frac{e^{-q_2\tau_2}}{B_{T_1,T_2}}$. Regarding the state variables F and α, it is clear that the value of the option at T_1 is only a function of α_{T_1} and so the time t_0 expectation (under the forward-T_1 measure) of $\Pi^{\%C}_{T_1,T_1,T_2}$ takes into account only one source of randomness. Furthermore, for $\tau_1 = T_1 - t_0$ and ν_1 the constant parameter that applies over the period $(t_0, T_1]$, we have

$$\ln\alpha_{T_1} \sim N\left(\ln\alpha_{t_0} - \frac{\nu_1^2}{2}\tau_1, \nu_1^2\tau_1\right) \quad (4.6)$$

This follows from equation (4.1).

[1] Referring to Hull [2002], Black's Model gives us the value at t_0 for a European Call Option on the underlying S, maturing at T with strike K

$$\Pi^C_{t_0,T} = B_{t_0,T}\left[F_{t_0}N(d_1) - KN(d_2)\right]$$

$$d_\pm = \frac{\ln\left(\frac{F_{t_0}}{K}\right) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

where F is the forward price of S for a contract maturing at T, $B_{t_0,T}$ is the price at t_0 of a Zero Coupon Bond paying 1 at T, σ is the volatility of F and we explicitly assume that S_T is lognormally distributed where $\sigma\sqrt{\tau}$ is the standard deviation of $\ln S_T$, $\mathbb{E}^{\mathbb{Q}_T}_{t_0}[S_T] = F_{t_0}$ and $\mathbb{Q}_T$ refers to the forward-T measure where $B_{t_0,T}$ is our numeraire.

From the approximate value of the option at time T_1, in equation (4.5), and the fact that the marginal distribution of α_{T_1} is lognormal, we have the approximate value at time t_0

$$\begin{aligned}\Pi^{\%\mathrm{C}}_{t_0,T_1,T_2} &= B_{t_0,T_1}\mathbb{E}^{\mathbb{Q}^{T_1}}_{t_0}\left[\Pi^{\%\mathrm{C}}_{T_1,T_1,T_2}|\alpha_{t_0}\right] \\ &= B_{t_0,T_1}\int_0^\infty \left[e^{-q_2\tau_2}N(d_+) - B_{T_1,T_2}KN(d_-)\right] f_{(\alpha_{T_1}|\alpha_{t_0})}d\alpha_{T_1} \qquad (4.7)\end{aligned}$$

where $f_{(\alpha_{T_1}|\alpha_{t_0})}$ is a lognormal density function with parameters $\mu = \ln\alpha_{t_0} - \frac{\nu_1^2}{2}\tau_1$ and $\sigma = \nu_1^2\tau_1$.

Equation (4.7) must be evaluated numerically. The analytic form of the integrand simplifies this exercise as one must, effectively, carry out a one dimensional integration. A simple Gaussian or Adaptive Quadrature rule may be used to determine $\Pi^{\%\mathrm{C}}_{t_0,T_1,T_2}$. It may be useful to specify an upper bound for the integration variable in terms of the standard deviation $\sigma = \nu_1^2\tau_1$. Alternatively, one could make use of a one factor finite difference method, such as the Crank-Nicholson scheme, to determine $\Pi^{\%\mathrm{C}}_{t_0,T_1,T_2}$. The problem can be conveniently evaluated using this latter approach as the terminal condition $\Pi^{\%\mathrm{C}}_{T_1,T_1,T_2}$ is a smooth function of α_{T_1} with $\lim_{\alpha_{T_1}\to 0}\sigma_{T_1,T_2,K} = 0$ and $\lim_{\alpha_{T_1}\to\infty}\sigma_{T_1,T_2,K} = \infty$.

4.1.2 Forward Starting \$ Call Options

For the \$ type payoff $[S_{T_2} - KS_{T_1}]^+$, we have the approximate value at T_1,

$$\begin{aligned}\Pi^{\$\mathrm{C}}_{T_1,T_1,T_2} &= B_{T_1,T_2}\left[F_{T_1}N(d_+) - S_{T_1}KN(d_-)\right] \\ &= S_{T_1}\left[e^{-q_2\tau_2}N(d_+) - B_{T_1,T_2}KN(d_-)\right] \qquad (4.8)\end{aligned}$$

and $d_\pm$ is as specified for a $\%$ type option. Again, referring to Kruse and Nogel [2005], the valuation of \$ type options may be simplified by shifting to the stock price measure with numeraire S_{T_1} (where we assume that dividends are re-invested into the asset). The time t_0 expectation (under the S_{T_1} measure) of $\Pi^{\$\mathrm{C}}_{T_1,T_1,T_2}$ would then take into account only one source of randomness as the numeraire would remove S_{T_1} from $\Pi^{\$\mathrm{C}}_{T_1,T_1,T_2}$. This approach was specifically considered assuming the Heston dynamics. Unfortunately, we cannot make use of this approach when assuming the SABR dynamics.

Firstly, there is a non-zero probability of the underlying process hitting zero. We do not prove this claim[2] but simply motivate it by observing that the SABR model reduces to the form of the CEV (Constant Elasticity of Variance) model[3] when $\nu = 0$. Cox [1996] presents the non-zero probability of the underlying hitting zero, assuming the CEV dynamics. This implies that use of S_{T_1} as a numeraire is not technically valid. However, if one insists on making use of the underlying as a numeraire when the process can become zero, Boyle and Tian [1999] describe an approach to circumvent the problem. As it stands, when working with the CEV process, an absorbing barrier is specified at zero. Replacing this with a closely related but strictly positive process would allow the underlying to be used as a numeraire. A minimum value $\varepsilon > 0$ is specified and if this level is reached, one would effectively liquidate the position in the underlying and invest the proceeds in a money market account. For more details, we refer the reader to the original article.

Having addressed the validity of a shift in measure, one must then consider the effect of this shift on

[2] A proof is provided in Andersen and Piterbarg [2007] proposition 5.1

[3] For the underlying S, the CEV dynamics are $dS = rSdt + \sigma S^{\frac{\beta}{2}}dW$ where $0 \le \beta \le 2$

the dynamics of the model. Specifically, α_{T_1} is no longer lognormally distributed and so tractability of the density $f_{(\alpha_{T_1}|\alpha_{t_0})}$ would appear to have been lost, under the stock price measure. Hence, we are not able to provide an approximate and efficient pricing methodology for a \$ type option within the SABR model.

4.1.3 The SABR model and forward parameters

Equation (4.4) may be used to calibrate the SABR model directly to the implied volatilities of European Options for a range of strikes and a *specific* maturity. This yields maturity specific, constant parameter sets. As stated in Hagan et al. [2002], the derived approximation is not intended to provide an adequate fit to market prices when a single, constant parameter set is obtained from a calibration over a range of both strike and maturity. Fitting the implied volatility surface requires use of the dynamic SABR model Hagan et al. [2002] Appendix B with the specification

$$\begin{aligned} dF_t &= \gamma_t \alpha_t F_t^{\beta_t} dW_t^F \\ d\alpha_t &= \nu_t \alpha_t dW_t^\alpha \\ dW_t^F dW_t^\alpha &= \rho_t dt \end{aligned} \tag{4.9}$$

where the parameters γ, β, ν and ρ are all time-dependent. In a manner similar to that within the non-dynamic model, an approximate solution for European Options is derived. The result, however, is not easily interpreted.

In principle, an approximate formula for European Options incorporating time-dependence allows us to calibrate the model to specific maturities, building up a term structure for the respective parameters and so consistently price European and Forward Starting Options, for example. Elaborating on this point, we start off at time t_0 and assume the time-dependent parameters are all piecewise constant. For the first maturity T_1, the model is calibrated to the market prices of European Options with maturity T_1, ranging in strike. This yields a constant $(t_0, T_1]$ parameter set i.e. we have $\left(\gamma_{(t_0,T_1]}, \beta_{(t_0,T_1]}, \nu_{(t_0,T_1]}, \rho_{(t_0,T_1]}\right)$.[4] For the second maturity T_2, we use the already calibrated $(t_0, T_1]$ parameter set as an input into the calibration procedure to determine the forward $(t_0; T_1, T_2]$ parameter set that provides the most appropriate fit to the market prices of European Options with maturity T_2. For $t_0 < t \leq T_2$, this yields

$$\begin{aligned} (\gamma_t, \beta_t, \nu_t, \rho_t) &:= \left(\gamma_{(t_0,T_1]}, \beta_{(t_0,T_1]}, \nu_{(t_0,T_1]}, \rho_{(t_0,T_1]}\right) \mathbb{I}_{[t_0 < t \leq T_1]} \\ &+ \left(\gamma_{(t_0;T_1,T_2]}, \beta_{(t_0;T_1,T_2]}, \nu_{(t_0;T_1,T_2]}, \rho_{(t_0;T_1,T_2]}\right) \mathbb{I}_{[T_1 < t \leq T_2]} \end{aligned} \tag{4.10}$$

allowing us to price $\%$ type Forward Starting Options (as presented in equation (4.7) for the determination date T_1 and maturity date T_2) with parameters that will return the calibrated model prices of European options of maturities T_1 *and* T_2. Hence, the Forward Starting prices specified by the model are consistent with the prices specified by the model for T_1 and T_2 European Options. Without the dynamic model, we would be left calibrating the SABR model to the maturities T_1 and T_2 separately, resulting in constant $(t_0, T_1]$ and $(t_0, T_2]$ parameter sets, respectively. Which of these parameter sets would we then

[4] In fact, if we can assume that $\gamma_{(t_0,T_1]} = 1$ then the remaining $(t_0, T_1]$ parameters can be calibrated directly from the non-dynamic SABR approximation.

use to price the specified Forward Starting Option in equation (4.7)? Using the $(t_0, T_2]$ parameter set, for example, would yield Forward Starting prices that are inconsistent with the model prices of T_1 European Options. For a consistent price, we would have to simultaneously calibrate the model to both maturities and so obtain a single, constant parameter set. However, as already stated, the model's resulting fit to T_1 and T_2 European market prices may not be adequate.

4.2 A digression: forward parameters for a very special case

Proposition 25. *For the affine square root process*

$$dV_t = \kappa(\theta_t - V_t)dt + \nu_t\sqrt{V_t}dW_t^V$$

where we assume κ is constant while θ_t and ν_t are piecewise constant and

$$\frac{\theta_t}{\nu_t^2} = \bar{k} \tag{4.11}$$

for some constant $\bar{k}$, we have the forward parameter

$$\nu_2 = \sqrt{\frac{\nu^2\left(e^{\kappa\tau}-1\right)-\nu_1^2\left(e^{\kappa\tau_1}-1\right)}{e^{\kappa\tau}-e^{\kappa\tau_1}}} \tag{4.12}$$

for the periods $\tau = T_2 - t_0$, $\tau_1 = T_1 - t_0$ and $\tau_2 = T_2 - T_1$ where ν_1 is the constant parameter that applies over the period $(t_0, T_1]$, ν is the constant parameter that applies over the period $(t_0, T_2]$ and ν_2 is the constant parameter that applies over the forward period $(T_1, T_2]$.

Proof: To obtain formulae for forward parameters, we attempt to analytically satisfy

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{iz_vV_{T_2}}|V_{t_0}\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\mathbb{E}_{T_1}^{\mathbb{Q}}\left[e^{iz_vV_{T_2}}|V_{T_1}\right]|V_{t_0}\right] \tag{4.13}$$

Working from equation (4.13), we have

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{iz_vV_{T_2}}|V_{t_0}\right] = e^{D(\tau,0,iz_v)V_{t_0}+C(\tau,0,iz_v)} \tag{4.14}$$

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\mathbb{E}_{T_1}^{\mathbb{Q}}\left[e^{iz_vV_{T_2}}|V_{T_1}\right]|V_{t_0}\right] = e^{D_{1;2}(\tau_1,0,D_{2;2}(\tau_2,0,iz_v))V_{t_0}+C_{1;2}(\tau_1,0,D_{2;2}(\tau_2,0,iz_v))+C_{2;2}(\tau_2,0,iz_v)} \tag{4.15}$$

where from section 2.2, we have

$$D(\tau,0,iz_v) = \frac{\kappa iz_v}{\frac{1}{2}\nu^2iz_v(1-e^{\kappa\tau})+\kappa e^{\kappa\tau}} \tag{4.16}$$

$$C(\tau,0,iz_v) = -\frac{2\kappa\theta}{\nu^2}\log\left(1-\frac{iz_v\nu^2\left(1-e^{-\kappa\tau}\right)}{2\kappa}\right) \tag{4.17}$$

The forward parameters ν_2 and θ_2 must satisfy

$$D(\tau,0,iz_v) = D_{1;2}(\tau_1,0,D_{2;2}(\tau_2,0,iz_v)) \tag{4.18}$$

$$C(\tau,0,iz_v) = C_{1;2}(\tau_1,0,D_{2;2}(\tau_2,0,iz_v)) + C_{2;2}(\tau_2,0,iz_v) \tag{4.19}$$

Working from equation (4.18), we make use of equation (4.16) and the assumption that κ is constant

$$\begin{aligned} D_{1;2}(\tau_1, 0, D_{2;2}(\tau_2, 0, iz_v)) &= \frac{\kappa\left[\frac{\kappa iz_v}{\frac{1}{2}\nu_2^2 iz_v(1-e^{\kappa\tau_2})+\kappa e^{\kappa\tau_2}}\right]}{\frac{1}{2}\nu_1^2\left[\frac{\kappa iz_v}{\frac{1}{2}\nu_2^2 iz_v(1-e^{\kappa\tau_2})+\kappa e^{\kappa\tau_2}}\right](1-e^{\kappa\tau_1})+\kappa e^{\kappa\tau_1}} \\ &= \frac{\kappa iz_v}{\frac{1}{2}\nu_1^2 iz_v(1-e^{\kappa\tau_1})+\frac{1}{2}\nu_2^2 iz_v(e^{\kappa\tau_1}-e^{\kappa\tau})+\kappa e^{\kappa\tau}} \\ &= \frac{\kappa iz_v}{\frac{1}{2}\nu_2^2 iz_v(1-e^{\kappa\tau})+\kappa e^{\kappa\tau}} \qquad (4.20) \\ &= D(\tau, 0, iz_v) \end{aligned}$$

where equation (4.20) holds for ν_2 defined as that in equation (4.12). Working from equation (4.19), we make use of equation (4.17) and the assumptions that κ and the ratio $\frac{\theta_t}{\nu_t^2}$ are constant

$$\begin{aligned} & C_{1;2}(\tau_1, 0, D_{2;2}(\tau_2, 0, iz_v)) + C_{2;2}(\tau_2, 0, iz_v) \\ = \;& -2\kappa\bar{k}\log\left(1-\left[\frac{\kappa iz_v}{\frac{1}{2}iz_v\nu_2^2(1-e^{\kappa\tau_2})+\kappa e^{\kappa\tau_2}}\right]\frac{1}{2\kappa}\nu_1^2\left(1-e^{-\kappa\tau_1}\right)\right) - 2\kappa\bar{k}\log\left(1-\frac{1}{2\kappa}iz_v\nu_2^2\left(1-e^{-\kappa\tau_2}\right)\right) \\ = \;& -2\kappa\bar{k}\log\left(1-\left[\frac{iz_v e^{-\kappa\tau_2}}{1-\frac{1}{2\kappa}iz_v\nu_2^2(1-e^{-\kappa\tau_2})}\right]\frac{1}{2\kappa}\nu_1^2\left(1-e^{-\kappa\tau_1}\right)\right) - 2\kappa\bar{k}\log\left(1-\frac{1}{2\kappa}iz_v\nu_2^2\left(1-e^{-\kappa\tau_2}\right)\right) \\ = \;& -2\kappa\bar{k}\log\left(1-\frac{1}{2\kappa}iz_v\left[\nu_2^2(1-e^{-\kappa\tau_2})+e^{-\kappa\tau_2}\nu_1^2\left(1-e^{-\kappa\tau_1}\right)\right]\right) \\ = \;& -2\kappa\bar{k}\log\left(1-\frac{1}{2\kappa}iz_v e^{-\kappa\tau}\left[\nu_2^2(e^{\kappa\tau}-e^{\kappa\tau_1})+\nu_1^2\left(e^{\kappa\tau_1}-1\right)\right]\right) \\ = \;& -2\kappa\bar{k}\log\left(1-\frac{1}{2\kappa}iz_v\nu^2\left(1-e^{-\kappa\tau}\right)\right) \qquad (4.21) \\ = \;& C(\tau, 0, iz_v) \qquad (4.22) \end{aligned}$$

where, again, equation (4.21) holds for ν_2 defined as that in equation (4.12). $\square$

From proposition 25, setting $\theta_t = 0$, we can write

$$dY = \mu Y dt + \nu_t\sqrt{Y}dW^Y$$

for some process Y and constant drift parameter $\mu = -\kappa$ where the SDE for Y has the square root CEV form. For this process, we then have

$$\nu_2 = \sqrt{\frac{\nu^2\left(1-e^{-\mu\tau}\right)-\nu_1^2\left(1-e^{-\mu\tau_1}\right)}{e^{-\mu\tau_1}-e^{-\mu\tau}}} \qquad (4.23)$$

To obtain our analytic formulae for the considered forward parameters, we have matched the functions C and D, separately. This is a special case. The same approach immediately yields the forward parameter for a piecewise constant, time-dependent volatility σ in the extended Black-Scholes model (working from the corresponding conditional characteristic function for $X = \ln S$). In general, we must match the resulting European Option prices numerically.

4.3 *Stochastic Volatility Model with Time-dependent skew*: An affine approach for the "effective" volatility

Consider the process

$$dS_t = \sigma_t \left[\beta_t S_t + (1-\beta_t) S_{t_0}\right] \sqrt{V_t} dW_t^S \tag{4.24}$$
$$dV_t = \kappa \left(\theta - V_t\right) dt + \nu \sqrt{V_t} dW_t^V \tag{4.25}$$

where β and σ are time-dependent while κ, θ and ν are constant and the Brownian motions driving S and V are uncorrelated.[5] As stated in Andersen and Brotherton-Ratcliffe [2005][6] *"Piterbarg [2005] provides an approximative algorithm to reduce a time-dependent σ into a single representative constant; this is particularly useful in calibrations, as option prices can always be represented by an implied constant σ before the calibration algorithm is activated."* This single representative constant is referred to as the "effective" volatility. Furthermore, Piterbarg [2005] provides an analytic, approximate formula to reduce a time-dependent β into a single representative constant. The latter result may be determined from theorem 3.1 and corollary 3.3 of Piterbarg [2005]. For the period $(t_0, t_n]$, this yields

$$\beta = \int_{t_0}^{t_n} \beta_t w_t dt \tag{4.26}$$
$$w_t = \frac{v_t^2 \sigma_t^2}{\int_{t_0}^{t_n} v_t^2 \sigma_t^2 dt} \tag{4.27}$$
$$v_t^2 = \theta^2 \int_{t_0}^{t} \sigma_s^2 ds + \theta \nu^2 e^{-\kappa t} \int_{t_0}^{t} \sigma_s^2 \frac{\left(e^{\kappa s} - e^{-\kappa s}\right)}{2\kappa} ds \tag{4.28}$$

with $V_{t_0} = \theta$.

This result is strike independent and allows us to approximate the SDE in equations (4.24) and (4.25) by

$$dS_t = \sigma_t \left[\beta S_t + (1-\beta) S_{t_0}\right] \sqrt{V_t} dW_t^S \tag{4.29}$$
$$dV_t = \kappa \left(\theta - V_t\right) dt + \nu \sqrt{V_t} dW_t^V \tag{4.30}$$

where $\beta = \beta_{(t_0,t_n]}$ is a constant value valid for the entire period $(t_0, t_n]$.

We now focus on σ_t. Essentially, for the special case of an at-the-money ($K = S_{t_0}$) European Call Option, Piterbarg [2005] derives an approximate pricing formula that accommodates for a time-dependent σ. From Piterbarg [2005] Theorem 4.1, we present the approximative algorithm for a maturity of t_n.

Theorem 4. *Denote the Laplace transform of the integral of $\sigma_t^2 V_t$ by*

$$\mathcal{L}(u) := \mathbb{E}\left[\exp\left(-u \int_{t_0}^{t_n} \sigma_t^2 V_t dt\right)\right] \tag{4.31}$$

and the Laplace transform of the integral of V_t by

$$\bar{\mathcal{L}}(u) := \mathbb{E}\left[\exp\left(-u \int_{t_0}^{t_n} V_t dt\right)\right] \tag{4.32}$$

[5] As stated in Piterbarg [2005], the slope of the implied volatility smile is generated by the function $\beta_t S_t + (1-\beta_t) S_{t_0}$ and so one does not need to introduce correlation between the driving Brownian motions.

[6] regarding caplet pricing formulae

The second-order accurate "effective" volatility σ is given as a solution to the equation

$$\bar{\mathcal{L}}\left(-\frac{\frac{\partial^2}{\partial\xi^2}g(\xi)}{\frac{\partial}{\partial\xi}g(\xi)}\sigma^2\right) = \mathcal{L}\left(-\frac{\frac{\partial^2}{\partial\xi^2}g(\xi)}{\frac{\partial}{\partial\xi}g(\xi)}\right) \tag{4.33}$$

where

$$\xi = \theta\int_{t_0}^{t_n}\sigma_t^2 dt \tag{4.34}$$

$$g(x) = \frac{S_{t_0}}{\beta}\left[2\Phi\left(\frac{1}{2}\beta\sqrt{x}\right)-1\right] \tag{4.35}$$

$$\Phi(y) = P(Y<y) \tag{4.36}$$

$$Y \sim N(0,1) \tag{4.37}$$

In Piterbarg [2005] Appendix D, the application of this theorem is considered. From Andersen and Brotherton-Ratcliffe [2005] Lemma 1, the RHS of equation (4.33) may be determined by observing that $\mathcal{L}(u)$ has the affine form $\exp\left[A(t_0,t_n)-B(t_0,t_n)V_{t_0}\right]$ where the functions $A(t,t_n)$ and $B(t,t_n)$ satisfy the Ricatti system of ODEs

$$\frac{\partial}{\partial t}A(t,t_n) = \kappa\theta B(t,t_n) \tag{4.38}$$

$$\frac{\partial}{\partial t}B(t,t_n) = \kappa B(t,t_n)+\frac{1}{2}\nu^2B^2(t,t_n)-u\sigma_t^2 \tag{4.39}$$

with the terminal conditions

$$A(t_n,t_n) = 0 \tag{4.40}$$

$$B(t_n,t_n) = 0 \tag{4.41}$$

Furthermore, the LHS of equation (4.33) may be determined by observing that $\bar{\mathcal{L}}(u)$ satisfies the same system of equations where $\sigma_t = 1$ and so an analytic solution may be obtained. Specifically, we have

$$\bar{\mathcal{L}}(u) = \exp\left[\bar{A}(t_0,t_n)-\bar{B}(t_0,t_n)V_{t_0}\right] \tag{4.42}$$

where[7]

$$\bar{B}(t_0,t_n) = \frac{2u\left(1-e^{-\gamma\tau}\right)}{(\kappa+\gamma)\left(1-e^{-\gamma\tau}\right)+2\gamma e^{-\gamma\tau}} \tag{4.43}$$

$$\bar{A}(t_0,t_n) = \frac{2\kappa\theta}{\nu^2}\log\left[\frac{2\gamma}{(\kappa+\gamma)\left(1-e^{-\gamma\tau}\right)+2\gamma e^{-\gamma\tau}}\right]-2\kappa\theta\frac{u}{\kappa+\gamma}\tau \tag{4.44}$$

$$\gamma = \sqrt{\kappa^2+2\nu^2u} \tag{4.45}$$

This result allows us to approximate the SDE in equations (4.29) and (4.30) by the time-homogenous process

$$dS_t = \sigma\left[\beta S_t+(1-\beta)S_{t_0}\right]\sqrt{V_t}dW_t^S \tag{4.46}$$

$$dV_t = \kappa\left(\theta-V_t\right)dt+\nu\sqrt{V_t}dW_t^V \tag{4.47}$$

[7] Note the typo in Piterbarg [2005] equation (D3) for $\bar{A}(t_0,t_n)$ which appears within the denominator of the term whose logarithm is considered.

where $\sigma = \sigma_{(t_0,t_n]}$ is a constant value valid for the entire period $(t_0, t_n]$ while β, κ, θ and ν are the same constant parameters from the SDE in equations (4.29) and (4.30).

For the term $\tau = t_n - t_0$ split into n increments with the mth increment $\tau_m = t_m - t_{m-1}$, we restrict ourselves to a piecewise constant, time-dependent σ and a constant β. In particular, for $\sigma_m := \sigma_{(t_0;t_{m-1},t_m]}$ we have

$$\begin{aligned}\sigma_t &= \sum_{m=1}^{n} \sigma_m \mathbb{I}_{[t_{m-1}<t\leq t_m]} \\ \xi &= \theta \sum_{m=1}^{n} \sigma_m^2 \tau_m \end{aligned} \tag{4.48}$$

Consider the case $n = 2$ where we have the periods $(t_0, t_1]$ and $(t_0, t_2]$ with $t_1 < t_2$. From the quote above, one can represent option prices for the respective periods in terms of the implied constant values $\sigma_{(t_0,t_1]}$ and $\sigma_{(t_0,t_2]}$ which correspond to the SDE in equations (4.46) and (4.47) for the respective periods $(t_0, t_1]$ and $(t_0, t_2]$. Given $\sigma_{(t_0,t_1]}$ and $\sigma_{(t_0,t_2]}$ (as well as $\beta_{(t_0,t_2]}$, κ, θ and ν), theorem 4 may be used to approximate $\sigma_{(t_0;t_1,t_2]}$. One can then make use of the SDE in equations (4.29) and (4.30) for the entire period $(t_0, t_2]$ where $\sigma_t = \sigma_{(t_0,t_1]}\mathbb{I}_{[t_0<t\leq t_1]} + \sigma_{(t_0;t_1,t_2]}\mathbb{I}_{[t_1<t\leq t_2]}$. The approach remains valid when solving for $\sigma_{(t_0;t_{n-1},t_n]}$ with $n > 2$.

Regarding the implementation of theorem 4, it is not clear from Piterbarg [2005] Appendix D as to exactly how the author intends for the algorithm to be implemented. Specifically, evaluation of the ODEs in equations (4.38) and (4.39) requires some clarification. One can evaluate the simple system numerically with, for example, MatLab's *ode45* function. This, however, is not an efficient approach. The merit of theorem 4 lies in the fact that the RHS of equation (4.33) may *also* be determined analytically for a piecewise constant σ and hence, forward parameters may be determined almost instantaneously.[8]

For the specified model, an *exact* semi-analytic pricing formula is also available, for a piecewise constant, time-dependent σ and constant β, from which we can determine $\sigma_{(t_0;t_{n-1},t_n]}$ efficiently. In the following subsections we describe how approximate and exact forward parameters may both be determined.

4.3.1 The conditional joint characteristic function

Following Andersen and Brotherton-Ratcliffe [2005], we set $\bar{X} = \beta S + (1-\beta)S_{t_0}$ and make use of Itô's formula together with equations (4.29) and (4.30) to obtain

$$dX = -\frac{1}{2}\sigma_t^2\beta^2 V\,dt + \sigma_t\beta\sqrt{V}dW^S \tag{4.49}$$

$$dV = \kappa\,(\theta - V)\,dt + \nu\sqrt{V}dW^V \tag{4.50}$$

where $X = \ln(\bar{X})$ and so we have expressed the specified dynamics in the same affine form as that of Heston [1993], for example, where the natural logarithm of the corresponding characteristic function is linear in terms of the state variables (X and V).

The corresponding conditional joint characteristic function for X and V,

$$\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(izX_{t_n} + iz_vV_{t_n}\right)|X_t, V_t\right] \quad =: \quad \phi_t(X_t, V_t; iz, iz_v) \tag{4.51}$$

[8] The intention, to make use of available analytic results, was communicated to us by the author Vladimir V. Piterbarg.

must satisfy the PDE

$$\frac{\partial \phi_t}{\partial t} - \frac{1}{2}\sigma_t^2\beta^2 V_t \frac{\partial \phi_t}{\partial X_t} + \frac{1}{2}\sigma_t^2\beta^2 V_t \frac{\partial^2 \phi_t}{\partial X_t^2} + \kappa(\theta - V_t)\frac{\partial \phi_t}{\partial V_t} + \frac{1}{2}\nu^2 V_t \frac{\partial^2 \phi_t}{\partial V_t^2} = 0 \tag{4.52}$$

Switching variables from t to $\bar{\tau} = t_n - t$, we assume that the solution ϕ_t has the form

$$\exp\left[izX_t + D(\bar{\tau}, iz, iz_v)V_t + C(\bar{\tau}, iz, iz_v)\right] \tag{4.53}$$

reducing the problem to a system of ODEs

$$\begin{aligned} \frac{\partial}{\partial \bar{\tau}} D(\bar{\tau}, iz, iz_v) &= \frac{1}{2}\nu^2 D^2(\bar{\tau}, iz, iz_v) - bD(\bar{\tau}, iz, iz_v) + c && (4.54)\\ \frac{\partial}{\partial \bar{\tau}} C(\bar{\tau}, iz, iz_v) &= \kappa\theta D(\bar{\tau}, iz, iz_v) && (4.55) \end{aligned}$$

with

$$\begin{aligned} b &= \kappa && (4.56)\\ c(iz) &= \frac{1}{2}\sigma^2\beta^2 iz(iz-1) && (4.57) \end{aligned}$$

and the terminal conditions

$$\begin{aligned} C(0, iz, iz_v) &= 0 && (4.58)\\ D(0, iz, iz_v) &= iz_v && (4.59) \end{aligned}$$

Regarding the arguments of the functions C and D, iz and iz_v refer to the coefficients of X_{t_n} and V_{t_n} respectively to which the exponent is raised at the terminal time t_n. For $\tau = t_n - t_0$, this yields the analytic solution

$$\begin{aligned} D(\tau, iz, iz_v) &= \left(\frac{\kappa - \gamma(iz)}{\nu^2}\right)\left[\frac{\bar{A}(iz, iz_v) - e^{-\gamma(iz)\tau}}{\bar{A}(iz, iz_v) - \left(\frac{\kappa-\gamma(iz)}{\kappa+\gamma(iz)}\right)e^{-\gamma(iz)\tau}}\right] && (4.60)\\ C(\tau, iz, iz_v) &= \frac{\kappa\theta}{\nu^2}\left[\kappa - \gamma(iz)\right]\tau - \frac{2\kappa\theta}{\nu^2}\log\left(\frac{A^{-1}(iz, iz_v)e^{-\gamma(iz)\tau} - 1}{A^{-1}(iz, iz_v) - 1}\right) && (4.61)\\ \bar{A}(iz, iz_v) &= A(iz, iz_v)\left(\frac{\kappa - \gamma(iz)}{\kappa + \gamma(iz)}\right) && (4.62)\\ A(iz, iz_v) &= \frac{\nu^2 iz_v - \kappa - \gamma(iz)}{\nu^2 iz_v - \kappa + \gamma(iz)} && (4.63)\\ \gamma(iz) &= \sqrt{\kappa^2 - 2\nu^2 c(iz)} && (4.64) \end{aligned}$$

Setting $z_v = 0$ yields the characteristic function for X

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp(izX_{t_n})|X_{t_0}, V_{t_0}\right] = \exp\left[izX_{t_0} + D(\tau, iz, 0)V_{t_0} + C(\tau, iz, 0)\right] \tag{4.65}$$

We allow for a piecewise constant σ by making use of the tower property

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp(izX_{t_n})|X_{t_0}, V_{t_0}\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\mathbb{E}_{t_1}^{\mathbb{Q}}\left[...\mathbb{E}_{t_{n-1}}^{\mathbb{Q}}\left[\exp(izX_{t_n})|X_{t_{n-1}}, V_{t_{n-1}}\right]...|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \tag{4.66}$$

and must now determine an analytic expression for the RHS of equation (4.66). To achieve this, we make use of the analytic result for the conditional joint characteristic function presented in equations (4.60)-(4.64). Dividing the term $\tau = t_n - t_0$ into n increments with $\tau_m = t_m - t_{m-1}$, we solve a time-homogenous PDE for *each* increment where from one increment to the next, the constant parameter set may differ. At t_{n-1}, we must solve the PDE presented in equation (4.52) assuming the solution $\phi_{t_{n-1}}(X_{t_{n-1}}, V_{t_{n-1}}; iz, 0) := \mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp\left(izX_{t_n}\right)|X_{t_{n-1}}, V_{t_{n-1}}\right]$ has the form

$$\exp\left[izX_{t_{n-1}} + D_{n;n}(\tau_n, iz, 0)V_{t_{n-1}} + C_{n;n}(\tau_n, iz, 0)\right] \tag{4.67}$$

subject to the terminal conditions

$$\begin{aligned} C_{n;n}(0, iz, 0) &= 0 & (4.68)\\ D_{n;n}(0, iz, 0) &= 0 & (4.69)\end{aligned}$$

The functions $C_{n;n}(\tau_n, iz, 0)$ and $D_{n;n}(\tau_n, iz, 0)$ are determined from equations (4.60)-(4.64) where $iz_v := 0$ and $(\tau, \beta, \kappa, \theta, \nu) := (\tau_n, \beta_n, \kappa_n, \theta_n, \nu_n)$.

At t_{n-2}, we must solve the same PDE, assuming the solution $\phi_{t_{n-2}}(X_{t_{n-2}}, V_{t_{n-2}}; iz, D_{n;n}(\tau_n, iz, 0)) := \mathbb{E}^{\mathbb{Q}}_{t_{n-2}}\left[\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp\left(izX_{t_n}\right)|X_{t_{n-1}}, V_{t_{n-1}}\right]|X_{t_{n-2}}, V_{t_{n-2}}\right]$ has the form

$$\begin{aligned} & \exp\left[C_{n;n}(\tau_n, iz, 0)\right]\mathbb{E}^{\mathbb{Q}}_{t_{n-2}}\left[\exp\left(izX_{t_{n-1}} + D_{n;n}(\tau_n, iz, 0)V_{t_{n-1}}\right)|X_{t_{n-2}}, V_{t_{n-2}}\right] & (4.70)\\ = \;& \exp\left[C_{n;n}(\tau_n, iz, 0)\right] & (4.71)\\ \times \;& \exp\left[izX_{t_{n-2}} + D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))V_{t_{n-2}} + C_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))\right] & (4.72)\end{aligned}$$

subject to the terminal conditions

$$\begin{aligned} C_{n-1;n}(0, iz, D_{n;n}(\tau_n, iz, 0)) &= 0 & (4.73)\\ D_{n-1;n}(0, iz, D_{n;n}(\tau_n, iz, 0)) &= D_{n;n}(\tau_n, iz, 0) & (4.74)\end{aligned}$$

The functions $C_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))$ and $D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))$ are determined from equations (4.60)-(4.64) where $iz_v := D_{n;n}(\tau_n, iz, 0)$ and $(\tau, \beta, \kappa, \theta, \nu) := (\tau_{n-1}, \beta_{n-1}, \kappa_{n-1}, \theta_{n-1}, \nu_{n-1})$.

Continuing in this manner, until we reach t_0, yields the analytic result

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[\exp(izX_{t_n})|X_{t_{n-1}}, V_{t_{n-1}}\right]...|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right]\\ = \;& \exp\left[izX_{t_0} + D_{1;n}(\tau_1, iz, D_{2;n})V_{t_0} + \sum_{m=1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})\right] \end{aligned} \tag{4.75}$$

where

$$\begin{aligned} D_{1;n}(\tau_1, iz, D_{2;n}) &:= D_{1;n}(\tau_1, iz, D_{2;n}(\tau_2, iz, ...D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))...))\\ C_{m;n}(\tau_m, iz, D_{m+1;n}) &:= C_{m;n}(\tau_m, iz, D_{m+1;n}(\tau_{m+1}, iz, ...D_{n-1;n}(\tau_{n-1}, iz, D_{n;n}(\tau_n, iz, 0))...))\end{aligned}$$

with $D_{n+1,n} := 0$. Regarding the subscripts of the functions C and D, the first argument specifies the increment currently considered while the second specifies the total number of increments.

Returning to the implementation of theorem 4, we can make use of the analytic result in equations (4.60)-(4.64) to determine the RHS of equation (4.33). For convenience, we specify $\exp\left[A(t, t_n) + B(t, t_n)V_{t_0}\right]$ as the affine form of $\mathcal{L}(u)$ and set $\bar{\tau} := t_n - t$. This yields the Ricatti system of ODEs

$$\frac{\partial}{\partial \bar{\tau}} A(\bar{\tau}) = \kappa\theta B(\bar{\tau}) \tag{4.76}$$

$$\frac{\partial}{\partial \bar{\tau}} B(\bar{\tau}) = \frac{1}{2}\nu^2 B^2(\bar{\tau}) - \kappa B(\bar{\tau}) - u\sigma_t^2 \tag{4.77}$$

Setting $c(iz) = u\sigma_t^2$, in equation (4.64), where σ_t is piecewise constant and specifying the terminal conditions

$$A(0) = 0 \tag{4.78}$$

$$B(0) = iz_v \tag{4.79}$$

for an appropriate choice of the value iz_v, allows us to determine an analytic solution for $\mathcal{L}(u)$. For $\sigma_t = 1$ and $iz_v = 0$, we obtain the solution for $\bar{\mathcal{L}}(u)$ as presented in equations (4.43)-(4.45).

4.3.2 Solving for the forward parameter

Regarding the payoff of a European Call Option, we have

$$[S_{t_n} - K]^+ = \frac{1}{\beta}\left[\bar{X}_{t_n} - \bar{K}\right]^+ \tag{4.80}$$

where $\bar{X}_{t_n} = \beta S_{t_n} + (1-\beta)S_{t_0}$ and $\bar{K} = \beta K + (1-\beta)S_{t_0}$. For $X_{t_n} = \ln(\bar{X}_{t_n})$ and $k = \ln(\bar{K})$, we have the semi-analytic undiscounted value of the payoff $\left[e^{X_{t_n}} - e^k\right]^+$

$$\Pi^{\mathrm{C}}_{t_0,t_n}(\sigma_{(t_0,t_n]}) = R^{\mathrm{C}}_{t_0,t_n}(\alpha) + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i(u-i\alpha)k}}{-(u-i\alpha)(u-i[\alpha+1])}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{i(u-i[\alpha+1])X_{t_n}}|X_{t_0}, V_{t_0}\right]\right]du \tag{4.81}$$

$$R^{\mathrm{C}}_{t_0,t_n}(\alpha) = \mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{t_n}}|X_{t_0}\right]\mathbb{I}_{[\alpha\leq 0]} - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_{t_n}}|X_{t_0}\right]\mathbb{I}_{[\alpha=0]} - e^k\mathbb{I}_{[\alpha\leq -1]} + \frac{1}{2}e^k\mathbb{I}_{[\alpha=-1]} \tag{4.82}$$

The result is valid for $\alpha \in (\alpha^{\min}, \alpha^{\max})$ where $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{(\alpha+1)X_{t_n}}|X_{t_0}, V_{t_0}\right]$ exists.

Replacing $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{i(u-i[\alpha+1])X_{t_n}}|X_{t_0}, V_{t_0}\right]$ in equation (4.81) by its iterated extension yields the pricing formula $\Pi^{\mathrm{C}}_{t_0,t_n}(\sigma_{[t_0,\,t_1]}, \sigma_{(t_0;t_1,t_2]}, ..., \sigma_{(t_0;t_{n-1},t_n]})$. From

$$\Pi^{\mathrm{C}}_{t_0,t_n}(\sigma_{(t_0,t_n]}) = \Pi^{\mathrm{C}}_{t_0,t_n}(\sigma_{(t_0,t_1]}, \sigma_{(t_0;t_1,t_2]}, ..., \sigma_{(t_0;t_{n-1},t_n]}) \tag{4.83}$$

we can solve for one of the parameters $\sigma_{(t_0,t_1]}, \sigma_{(t_0;t_1,t_2]}, ..., \sigma_{(t_0;t_{n-1},t_n]}$ and $\sigma_{(t_0,t_n]}$ given that the rest are inputs. We focus on solving for $\sigma_{(t_0;t_{n-1},t_n]}$.

At this point, we must acknowledge that, regarding semi-analytic pricing formulae, it has been noted in Andersen and Andreasen [2002] that the specified dynamics can accommodate for a piecewise constant σ. Hence, we can claim only to have identified the usefulness of this result when working through the approach of Piterbarg [2005].

Regarding the chosen contour of integration i.e. the value $-\alpha$, we make use of the optimal α^* approach of Lord and Kahl [2007] where this optimal value is a function of *all* the parameters of the model. Regarding the numerical search for $\sigma_{(t_0;t_{n-1},t_n]}$, at each iteration within our preferred algorithm, $\Pi^{\mathrm{C}}_{t_0,t_n}(\sigma_{(t_0,t_1]}, \sigma_{(t_0;t_1,t_2]}, ..., \sigma_{(t_0;t_{n-1},t_n]})$ will be evaluated at a different value of $\sigma_{(t_0;t_{n-1},t_n]}$. Hence, the corresponding α^* will differ for each iteration. Furthermore, to obtain α^* one must first determine the range $(\alpha^{\min}, \alpha^{\max})$ numerically, as considered in section 3.2

When evaluating the option prices in equation (4.83), we specify the adaptive Gauss-Lobatto quadrature algorithm of Gander and Gautschi [2000] (as suggested in Kahl and Jackel [2005]) with the optimal α and the transformed domain of integration, as our benchmark approach.[9] As pointed out in Kahl and Jackel [2005], the algorithm evaluates the integrand at its boundary values. Having transformed the domain of integration from $u \in [0, \infty)$ to $x \in [0, 1]$, we must define the integrand to be zero at the boundary $x = 0$. The MatLab code for the adaptive algorithm is conveniently presented in Gander and Gautschi [2000], allowing us to specify the value of the integrand at this boundary. To work with the transformed domain, we must determine the appropriate transformation. Following the approach in section 3.4, we specify the transformation

$$u(x) = -\frac{\ln(x)}{\Re_\infty} \tag{4.84}$$

$$\Re_\infty = \frac{\beta}{\nu}\left[\sigma_1 V_{t_0} + \kappa\theta \sum_{m=1}^{n} \sigma_m \tau_m\right] \tag{4.85}$$

where $x \in [0, 1]$.

Piterbarg [2005] focusses on obtaining results for at-the-money options ($K = S_{t_0}$). Our affine approach accommodates for alternative strike levels. However, for at-the-money options, we can obtain results conveniently by reconsidering the contour of integration.

Proposition 26. *For $\alpha = -\frac{1}{2}$, we have the integrand*

$$\frac{e^{-i(u+i\frac{1}{2})k}}{-(u^2+\frac{1}{4})}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[e^{i(u-i\frac{1}{2})X_{t_n}}|X_{t_{n-1}}, V_{t_{n-1}}\right]...|X_{t_1}, V_{t_1}\right]|X_{t_0}, V_{t_0}\right] \tag{4.86}$$

which is at its minimum at $u = 0$. For $k = X_{t_0}$, the integrand is strictly increasing in u where $u \in (0, \infty)$.

Proof: We can write

$$D(\tau, iz, 0) = \frac{[\kappa + \gamma(iz)]}{\nu^2}\left(1 - \frac{1}{\psi(\tau, iz, 0)}\right) \tag{4.87}$$

$$C(\tau, iz, 0) = \frac{\kappa\theta}{\nu^2}[\kappa - \gamma(iz)]\,\tau - \frac{2\kappa\theta}{\nu^2}\ln[\psi(\tau, iz, 0)] \tag{4.88}$$

$$\psi(\tau, iz, 0) = \frac{A^{-1}(iz)e^{-\gamma(iz)\tau} - 1}{A^{-1}(iz) - 1} \tag{4.89}$$

$$A^{-1}(iz) = \frac{\kappa - \gamma(iz)}{\kappa + \gamma(iz)} \tag{4.90}$$

For $\alpha = -\frac{1}{2}$, we have

$$\gamma(iz) = \sqrt{\kappa^2 + \nu^2\sigma^2\beta^2\left(u^2 + \frac{1}{4}\right)} \in \mathbb{R} \tag{4.91}$$

and so $\gamma(iz) > \kappa$. This yields $-1 < A^{-1}(iz) < 0$ and $\frac{1}{2} < \psi(\tau, iz, 0) < 1$ with $C(\tau, iz, 0)$, $D(\tau, iz, 0) \in \mathbb{R}$.

[9] This is the preferred valuation methodology presented in Lord and Kahl [2007]

Regarding $D(\tau, iz, 0)$, we can show that

$$\frac{\partial}{\partial\gamma(iz)}D(\tau,iz,0) = \frac{1}{\nu^2}\left(1-\frac{1}{\psi(\tau,iz,0)}\right)+\frac{[\kappa+\gamma(iz)]}{\nu^2}\left(\frac{\frac{\partial}{\partial\gamma(iz)}\psi(\tau,iz,0)}{\psi^2(\tau,iz,0)}\right) \tag{4.92}$$

$$\frac{\partial}{\partial\gamma(iz)}\psi(\tau,iz,0) = \frac{\left(1-e^{-\gamma(iz)\tau}\right)\frac{\partial}{\partial\gamma(iz)}A^{-1}(iz)-A^{-1}(iz)\left[A^{-1}(iz)-1\right]e^{-\gamma(iz)\tau}\tau}{\left[A^{-1}(iz)-1\right]^2} \tag{4.93}$$

$$\frac{\partial}{\partial\gamma(iz)}A^{-1}(iz) = -\frac{2\kappa}{[\kappa+\gamma(iz)]^2} \tag{4.94}$$

where $\frac{\partial}{\partial\gamma(iz)}A^{-1}(iz)<0$, $\frac{\partial}{\partial\gamma(iz)}\psi(\tau,iz,0)<0$ and so

$$\frac{\partial}{\partial\gamma(iz)}D(\tau,iz,0) < 0 \tag{4.95}$$

Regarding $C(\tau, iz, 0)$, we can show that

$$\frac{\partial}{\partial\gamma(iz)}C(\tau,iz,0) = -\frac{2\kappa\theta}{\nu^2}\frac{1}{[A^{-1}(iz)-1]^2\psi(\tau,iz,0)}Y(u,\tau) \tag{4.96}$$

$$Y(u,\tau) = -\frac{1}{2}\tau[A^{-1}(iz)-1][1+A^{-1}(iz)e^{-\gamma(iz)\tau}]+[1-e^{-\gamma(iz)\tau}]\frac{\partial}{\partial\gamma(iz)}A^{-1}(iz)$$

The sign of $Y(u,\tau)$ is not immediately clear. However, we have

$$\begin{aligned}\frac{\partial}{\partial\tau}Y(u,\tau) &= \frac{\gamma(iz)}{[\kappa+\gamma(iz)]^2}\left([\kappa+\gamma(iz)][1-e^{-\gamma(iz)\tau}]+\gamma(iz)[\gamma(iz)-\kappa]e^{-\gamma(iz)\tau}\tau\right)\\ &> 0\end{aligned} \tag{4.97}$$

for $\tau>0$ and any u while $Y(u,0)=0$. Therefore, $Y(u,\tau)>0$ for $\tau>0$ and so

$$\frac{\partial}{\partial\gamma(iz)}C(\tau,iz,0) < 0 \tag{4.98}$$

We aim to show that the integrand (evaluated at $\alpha=-\frac{1}{2}$) in equation (4.86) is monotonic in u for $k=X_{t_0}$. Within the affine framework, we can write

$$\frac{e^{-i(u+i\frac{1}{2})k}}{-(u^2+\frac{1}{4})}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[e^{i(u-i\frac{1}{2})X_{t_n}}|X_{t_{n-1}},V_{t_{n-1}}\right]...|X_{t_1},V_{t_1}\right]|X_{t_0},V_{t_0}\right] \tag{4.99}$$

$$= \frac{1}{-(u^2+\frac{1}{4})}\exp\left[X_{t_0}+D_{1;n}\left(\tau_1,iu+\frac{1}{2},D_{2;n}\right)V_{t_0}+\sum_{m=1}^{n}C_{m;n}\left(\tau_m,iu+\frac{1}{2},D_{m+1;n}\right)\right] \tag{4.100}$$

for $k=X_{t_0}$, $\alpha=-\frac{1}{2}$ and making use of the notation established in section 2.4. From equation 4.100 and the fact that $D(\tau,iz,0)$, $C(\tau,iz,0)\in\mathbb{R}$, we can see that the integrand is real and negative for $z:=u-i(\alpha+1)$. Furthermore, from equation 4.99 and making use of Jensen's inequality, we can show that the integrand is at its minimum at $u=0$.

We have

$$\begin{aligned}&\frac{\partial}{\partial u}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[e^{i(u-i[\alpha+1])X_{t_n}}|X_{t_{n-1}},V_{t_{n-1}}\right]...|X_{t_1},V_{t_1}\right]|X_{t_0},V_{t_0}\right]\\ =\ &\frac{\partial}{\partial u}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{i(u-i[\alpha+1])X_{t_n}}|X_{t_0},V_{t_0}\right]\end{aligned}$$

and so it follows that for $k = X_{t_0}$, we have

$$= \frac{\partial}{\partial u}\left(e^{-i(u-i\alpha)k}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[\mathbb{E}^{\mathbb{Q}}_{t_1}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[e^{i(u-i[\alpha+1])X_{t_n}}|X_{t_{n-1}},V_{t_{n-1}}\right]...|X_{t_1},V_{t_1}\right]|X_{t_0},V_{t_0}\right]\right) \quad (4.101)$$

$$= \frac{\partial}{\partial u}\left(e^{-i(u-i\alpha)k}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{i(u-i[\alpha+1])X_{t_n}}|X_{t_0},V_{t_0}\right]\right) \quad (4.102)$$

$$= e^{X_{t_0}+D(\tau,iz,0)V_{t_0}+C(\tau,iz,0)}\left[\frac{\partial}{\partial u}D(\tau,iz,0)V_{t_0}+\frac{\partial}{\partial u}C(\tau,iz,0)\right] \quad (4.103)$$

From equation (4.91), we have

$$\frac{\partial}{\partial u}\gamma(iz) = \frac{\nu^2\sigma^2\beta^2 u}{\gamma(iz)} \geq 0 \quad (4.104)$$

for $u \geq 0$ with an equality only at $u = 0$.

Making use of equations (4.95), (4.98) and (4.104), we have

$$\frac{\partial}{\partial u}D(\tau,iz,0) = \frac{\partial}{\partial\gamma(iz)}D(\tau,iz,0)\frac{\partial}{\partial u}\gamma(iz) \leq 0 \quad (4.105)$$

$$\frac{\partial}{\partial u}C(\tau,iz,0) = \frac{\partial}{\partial\gamma(iz)}C(\tau,iz,0)\frac{\partial}{\partial u}\gamma(iz) \leq 0 \quad (4.106)$$

with an equality only for $u = 0$.

Therefore $\frac{\partial}{\partial u}\left(e^{-i(u-i\alpha)k}\mathbb{E}^{\mathbb{Q}}_{t_0}\left[...\mathbb{E}^{\mathbb{Q}}_{t_{n-1}}\left[e^{i(u-i[\alpha+1])X_{t_n}}|X_{t_{n-1}},V_{t_{n-1}}\right]...|X_{t_0},V_{t_0}\right]\right) < 0$ for $k = X_{t_0}$, $\alpha = -\frac{1}{2}$ and $u > 0$. It follows that the integrand is strictly increasing in u for $u > 0$. □

From proposition 26, we see that for $k = X_{t_0}$ specifying the contour $\alpha = -\frac{1}{2}$ allows us to avoid the issue of an oscillating integrand. Hence, rather than solving for the optimal α at each iteration of our numerical search, we suggest working with $\alpha = -\frac{1}{2}$ thoughout (which always lies in the strip of regularity). In terms of the optimal α approach, we can also confirm that the local minimum for $\alpha \in (-1, 0)$ is always $-\frac{1}{2}$ for $k = X_{t_0}$.

The monotonicity of the integrand in u allows us to effectively make use of a simple Gauss-Legendre Quadrature Rule to perform the required numerical integration. From Abramowitz and Stegun [1974] equation (25.4.30), we make use of a 32-point rule for the integrand $f(u)$

$$\int_a^b f(u)du \approx \left(\frac{b-a}{2}\right)\sum_{i=1}^{32} w_i f(u_i) \quad (4.107)$$

where

$$u_i = \left(\frac{b-a}{2}\right)x_i + \left(\frac{b+a}{2}\right) \quad (4.108)$$

with the abscissas x_i and weights w_i specified in Abramowitz and Stegun [1974] Table 25.4.

The final point we consider is the range of $\sigma_{(t_0,t_n]}$ that the model can accommodate for by simply varying $\sigma_{(t_0;t_{n-1},t_n]}$.

Proposition 27. *For $\alpha = -\frac{1}{2}$ and $k = X_{t_0}$, $\Pi^{c}_{t_0,t_n}(\sigma_{(t_0,t_n]})$ is strictly increasing in $\sigma_{(t_0,t_n]}$ and $\Pi^{c}_{t_0,t_n}(\sigma_{(t_0,t_1]}, \sigma_{(t_0;t_1,t_2]}, ..., \sigma_{(t_0;t_{n-1},t_n]})$ is strictly increasing in $\sigma_{(t_0;t_{n-1},t_n]}$.*

Proof: For the time-dependent case, we can write

$$
\begin{aligned}
D_{m;n}(\tau_m, iz, D_{m+1;n}) &= \frac{D_{m+1;n}(\tau_{m+1}, iz, D_{m+2;n})}{\psi_{m;n}(\tau_m, iz, D_{m+1;n})} && (4.109)\\
&+ \frac{[\kappa_m + \gamma_m(iz)]}{\nu_m^2}\left[1 - \frac{1}{\psi_{m;n}(\tau_m, iz, D_{m+1;n})}\right] && (4.110)\\
C_{m;n}(\tau_m, iz, D_{m+1;n}) &= \frac{\kappa_m\theta_m}{\nu_m^2}[\kappa_m - \gamma_m(iz)]\tau_m - \frac{2\kappa_m\theta_m}{\nu_m^2}\ln[\psi_{m;n}(\tau_m, iz, D_{m+1;n})] && (4.111)\\
\psi_{m;n}(\tau_m, iz, D_{m+1;n}) &= -\frac{\nu_m^2}{2\gamma_m(iz)}\left[1 - e^{-\gamma_m(iz)\tau_m}\right] D_{m+1;n}(\tau_{m+1}, iz, D_{m+2;n}) && (4.112)\\
&+ \frac{[\kappa_m + \gamma_m(iz)] + [\gamma_m(iz) - \kappa_m]e^{-\gamma_m(iz)\tau_m}}{2\gamma_m(iz)} && (4.113)
\end{aligned}
$$

We consider the case $n \geq 2$ with $\sigma_n := \sigma_{(t_0;t_{n-1},t_n)}$ and $\sigma := \sigma_{(t_0,t_n)}$. We can show

$$
\begin{aligned}
\frac{\partial}{\partial\sigma_n} D_{m;n}(\tau_m, iz, D_{m+1;n}) &= \left[\prod_{j=m}^{n-1} \frac{e^{-\gamma_j(iz)\tau_j}}{\psi_{j;n}(\tau_j, iz, D_{j+1;n})}\right] \frac{\partial}{\partial\sigma_n} D_{n;n}(\tau_n, iz, 0)\\
\frac{\partial}{\partial\sigma_n}\sum_{m=1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n}) &= \sum_{m=1}^{n-1} \frac{\kappa_m\theta_m}{\gamma_m(iz)\psi_{m;n}(\tau_m, iz, D_{m+1;n})}\left[1 - e^{-\gamma_m(iz)\tau_m}\right] && (4.114)\\
&\times \frac{\partial}{\partial\sigma_n} D_{m+1;n}(\tau_{m+1}, iz, D_{m+2;n}) + \frac{\partial}{\partial\sigma_n} C_{n;n}(\tau_n, iz, 0) && (4.115)
\end{aligned}
$$

and from equation (4.91), we have

$$
\frac{\partial}{\partial\sigma}\gamma(iz) = \frac{\nu^2\sigma\beta^2\left(u^2 + \frac{1}{4}\right)}{\gamma(iz)} > 0 \qquad (4.116)
$$

We know $D_{m+1;n}(\tau_{m+1}, iz, D_{m+2;n}) \in \mathbb{R}$ from equation (3.13) and from proposition 20 and the discussion in subsection 3.5.3 it follows that for $\alpha = -\frac{1}{2}$ and $1 \leq m \leq n$, we have $D_{m+1;n}(\tau_{m+1}, iz, D_{m+2;n}) < 0$. We also have $\gamma_m(iz) > \kappa_m$ and so

$$
\psi_{m;n}(\tau_m, iz, D_{m+1;n}) > 0 \qquad (4.117)
$$

Therefore, the sign of $\frac{\partial}{\partial\sigma_n} D_{m;n}(\tau_m, iz, D_{m+1;n})$ and $\frac{\partial}{\partial\sigma_n}\sum_{m=1}^{n} C_{m;n}(\tau_m, iz, D_{m+1;n})$ depend on the sign of $\frac{\partial}{\partial\sigma_n} D_{n;n}(\tau_n, iz, 0)$ and $\frac{\partial}{\partial\sigma_n} C_{n;n}(\tau_n, iz, 0)$.
The form of the functions $D_{n;n}(\tau_n, iz, 0)$ and $C_{n;n}(\tau_n, iz, 0)$ is the same as that of $D(\tau, iz, 0)$ and $C(\tau, iz, 0)$ since the terminal conditions, which determine these analytic functions, are the same. So making use of equations (4.95), (4.98) and the form of equation (4.116), we see that these partial derivatives are both negative, the integrand is strictly increasing in σ_n and so the same is true for the option price. Regarding σ, equations (4.95), (4.98) and (4.116) can be directly considered, to determine that the option price is also strictly increasing in σ. □

From proposition 27, we see that setting $\sigma_{(t_0;t_{n-1},t_n]} = 0$, we can determine $\sigma^*_{(t_0,t_n]}$ - the minimum value of $\sigma_{(t_0,t_n]}$ that the specified parameters can accommodate for.

4.3.3 Numerical results

To illustrate the efficiency of our 'exact' approach and the accuracy of the approximate approach, we consider the case $n = 4$, where we specify $\sigma_{(t_0,t_1]}$, $\sigma_{(t_0;t_1,t_2]}$, $\sigma_{(t_0;t_2,t_3]}$ and $\sigma_{(t_0,t_4]}$ and so must determine

$\sigma_{(t_0;t_3,t_4]}$ restricting our search to the range $[0, 10]$.

We first present results for the parameter set: $\sigma_{(t_0,t_1)} = 0.9$, $\sigma_{(t_0;t_1,t_2]} = 2$, $\sigma_{(t_0;t_2,t_3]} = 1.3$, $\kappa = 1$, $\nu = .2$, $\theta = 0.1$, $V_{t_0} = 0.1$, $\tau_m = 1$ for $1 \leq m \leq 4$ and $\beta = 1$ where $\sigma^*_{(t_0,t_4]} = 1.2689$:

	$\sigma_{(t_0;t_3,t_4]}$			Evaluation time			Discrepancy		
$\sigma_{(t_0,t_4]}$	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg
1.27	0.1033	0.1033	0.1024	2.547s	0.024s	0.005s	4×10^{-9}	4×10^{-7}	1×10^{-5}
1.3	0.5595	0.5594	0.5594	1.891s	0.022s	0.004s	1×10^{-7}	4×10^{-7}	2×10^{-6}
1.8	2.5554	2.5554	2.5553	1.093s	0.024s	0.005s	7×10^{-8}	6×10^{-8}	9×10^{-6}
1.9	2.8345	2.8345	2.8345	1.141s	0.029s	0.005s	2×10^{-7}	3×10^{-7}	9×10^{-6}
2.0	3.1026	3.1025	3.1025	1.125s	0.020s	0.005s	5×10^{-7}	5×10^{-7}	1×10^{-5}
2.5	4.3457	4.3457	4.3452	1.469s	0.021s	0.005s	1×10^{-7}	1×10^{-6}	6×10^{-5}

The 'Exact' $\sigma_{(t_0;t_3,t_4]}$ column refers to the solution obtained from our benchmark approach where we specify a relative error tolerance of 10^{-6}. The $\text{GQ}_{\alpha=-\frac{1}{2}}$ column refers to the solution obtained by means of the specified Gauss-Legendre Quadrature Rule. We have left the domain of integration untransformed, truncated the upper bound of integration to the point $u = 100$ and split the domain of integration into 2 equally sized pieces. The Piterbarg $\sigma_{(t_0;t_3,t_4]}$ column refers to the solution obtained by following the methodology of theorem 4.

The discrepancies refer to

$$\frac{|\Pi^{\text{C}}_{t_0,t_n}(\sigma_{(t_0,t_1]}, \sigma_{(t_0;t_1,t_2]}, \sigma_{[t_0;\, t_2,\, t_3]}, \sigma_{[t_0;\, t_3,\, t_4]}) - \Pi^{\text{C}}_{t_0,t_n}(\sigma_{[t_0,\, t_4]})|}{\Pi^{\text{C}}_{t_0,t_n}(\sigma_{[t_0,\, t_4]})} \tag{4.118}$$

where the option values are evaluated using the benchmark approach and the value of $\sigma_{(t_0;t_3,t_4]}$ is specified from the respective approaches.

Regarding the value of $\sigma_{(t_0;t_{n-1},t_n]}$ determined from our affine approach and equation (4.83), only the RHS of equation (4.83) needs to be repeatedly evaluated. At each iteration of the optimisation algorithm, the undiscounted option price must be determined by means of a numerical integration. However, the Gauss-Legendre Quadrature Rule used to perform this integration can be efficiently implemented in MatLab by making use of the software package's vectorization feature. We have also made use of Matlab's *fminbnd* function (with *TolX* set to 10^{-5}) when searching for α^* and $\sigma_{(t_0;t_{n-1},t_n]}$.

Furthermore, it is worth noting that for $\alpha = -\frac{1}{2}$ and $k = X_{t_0}$ the issue of branch cutting does not arise as the functions involved are all real. For alternative values of α, the functions involved may be complex. The 'Exact' approach makes use of the optimal value α^* which lies in the range $(\alpha^{\min}, \alpha^{\max})$ and, in particular, is not restricted to the range $(-1, 0)$ (for which we have provided a proof in section 3.5 to show that branch cutting cannot occur). The consistency of the 'Exact' and $\text{GQ}_{\alpha=-\frac{1}{2}}$ solutions serves to motivate the conjecture that branch cutting is not an issue for $\alpha \in (\alpha^{\min}, \alpha^{\max})$.

We now consider the same parameter set with β set to 0.1 rather than 1 with $\sigma^*_{(t_0,t_4]} = 1.2698$:

	$\sigma_{(t_0;t_3,t_4]}$			Evaluation time			Discrepancy		
$\sigma_{(t_0,t_4]}$	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg
1.27	0.0508	0.0508	0.0522	3.141s	0.020s	0.005s	8×10^{-9}	5×10^{-7}	1×10^{-5}
1.3	0.5526	0.5526	0.5529	2.187s	0.018s	0.005s	5×10^{-8}	3×10^{-7}	2×10^{-5}
1.8	2.5520	2.5520	2.5520	1.281s	0.021s	0.005s	2×10^{-7}	3×10^{-7}	1×10^{-5}
1.9	2.8300	2.8300	2.8302	1.282s	0.018s	0.004s	3×10^{-7}	1×10^{-7}	2×10^{-5}
2.0	3.0967	3.0967	3.0968	1.500s	0.018s	0.005s	4×10^{-8}	5×10^{-7}	3×10^{-5}
2.5	4.3284	4.3284	4.3286	1.094s	0.020s	0.004s	3×10^{-7}	1×10^{-6}	5×10^{-5}

For the value of $\sigma_{(t_0;t_3,t_4]}$ obtained from the $\text{GQ}_{\alpha=-\frac{1}{2}}$ approach, we have left the untransformed, truncated domain of integration as 1 piece.

In Piterbarg [2005], the author states that "*Test results indicate that the approximations are excellent, even for high volatility of variance/ low mean reversion of variance parameters.*" For the initial parameter set (with $\beta = 1$), we increase ν from 0.2 to 2 with $\sigma^*_{(t_0,t_4]} = 1.1355$:

	$\sigma_{(t_0;t_3,t_4]}$			Evaluation time			Discrepancy		
$\sigma_{(t_0,t_4]}$	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg
1.136	0.0455	0.0454	0.2827	2.266s	0.031s	0.005s	9×10^{-9}	9×10^{-7}	1×10^{-2}
1.3	1.1043	1.1043	1.1484	2.047s	0.030s	0.005s	2×10^{-7}	3×10^{-7}	7×10^{-3}
1.8	3.0959	3.0959	3.0805	2.078s	0.030s	0.005s	2×10^{-8}	6×10^{-7}	2×10^{-3}
1.9	3.4850	3.4850	3.4522	2.547s	0.038s	0.005s	6×10^{-8}	6×10^{-7}	4×10^{-3}
2.0	3.8762	3.8762	3.8227	1.953s	0.029s	0.004s	7×10^{-8}	7×10^{-7}	6×10^{-3}
2.5	5.8731	5.8731	5.6784	1.453s	0.030s	0.005s	8×10^{-8}	3×10^{-7}	1×10^{-2}

With regard to the benchmark approach, we specify a relative error tolerance of 10^{-8} instead of 10^{-6}. With regard to the $\text{GQ}_{\alpha=-\frac{1}{2}}$ approach, we split the truncated, untransformed domain into 4 equally sized pieces.

We now consider obtaining $\sigma_{(t_0;t_3,t_4]}$ for alternative strike levels. For the original parameter set and the case $\sigma_{(t_0,t_4]} = 1.9$ with $\sigma_{(t_0;t_3,t_4]} = 2.8345$ for an at-the-money option, we obtain the following:

	$\sigma_{(t_0;t_3,t_4]}$		Evaluation time		Discrepancy		
K	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	'Exact'	$\text{GQ}_{\alpha=-\frac{1}{2}}$	Piterbarg
25	2.8180	2.8180	2.344s	0.023s	2×10^{-8}	3×10^{-7}	4×10^{-4}
50	2.8302	2.8302	2.641s	0.027s	1×10^{-7}	5×10^{-8}	3×10^{-4}
75	2.8338	2.8338	2.218s	0.023s	4×10^{-7}	4×10^{-7}	9×10^{-5}
125	2.8341	2.8341	2.281s	0.025s	1×10^{-7}	3×10^{-7}	9×10^{-5}
150	2.8330	2.8330	2.297s	0.026s	6×10^{-8}	9×10^{-8}	4×10^{-4}
175	2.8317	2.8317	2.141s	0.024s	2×10^{-7}	4×10^{-7}	8×10^{-4}
200	2.8302	2.8302	2.656s	0.027s	6×10^{-7}	2×10^{-7}	2×10^{-3}

With regard to the $\text{GQ}_{\alpha=-\frac{1}{2}}$ approach, we split the truncated, untransformed domain into 2 equally sized pieces.

The Piterbarg discrepancy refers to the discrepancy that arises from the at-the-money approximation obtained from theorem 4.

For the original parameter set, we replace $\tau_m = 1$ with $\tau_m = \frac{1}{365}$ for $1 \leq m \leq 4$ and consider the case $\sigma_{(t_0,t_4]} = 1.3$ with $\sigma_{(t_0;t_3,t_4]} = 0.5098$ (obtained in 0.016s with $\alpha = -\frac{1}{2}$) for an at-the-money option. We obtain the following:

	$\sigma_{(t_0;t_3,t_4]}$		Evaluation time		Discrepancy		
K	'Exact'	GQ$_{\alpha^*}$	'Exact'	GQ$_{\alpha^*}$	'Exact'	GQ$_{\alpha^*}$	Piterbarg
25	0.5631	0.5631	9.390s	0.734s+0.016s	8×10^{-8}	1×10^{-7}	0.96
50	0.5277	0.5277	11.562s	0.718s+0.016s	3×10^{-8}	6×10^{-9}	0.28
75	0.5132	0.5132	10.094s	0.609s+0.016s	1×10^{-8}	5×10^{-8}	1×10^{-2}
125	0.5119	0.5119	9.000s	0.593s+0.016s	3×10^{-9}	5×10^{-8}	5×10^{-3}
150	0.5164	0.5164	12.109s	0.641s+0.016s	1×10^{-8}	4×10^{-7}	4×10^{-2}
175	0.5219	0.5219	11.922s	0.719s+0.016s	5×10^{-8}	2×10^{-7}	0.14
200	0.5277	0.5277	11.563s	0.718s+0.016s	3×10^{-8}	6×10^{-9}	0.28

The GQ$_{\alpha^*}$ columns refer to use of the 32-point Gauss-Legendre Quadrature Rule in conjunction with the optimal α (the GQ$_{\alpha=-\frac{1}{2}}$ approach does not provide adequate accuracy for this parameter set). We have left the truncated, untransformed domain as 1 piece. Furthermore, when searching for $\sigma_{(t_0;t_3,t_4]}$, we have reduced the search to the range $[0, 0.5098 + 1]$. This reduces the evaluation time to approximately two thirds of that taken to search the range $[0, 10]$.

The Piterbarg discrepancy refers to the discrepancy that arises from the at-the-money approximation $\sigma_{(t_0;t_3,t_4]} = 0.5098$ (obtained from theorem 4).

With regard to the parameter and discrepancy values obtained for the cases $K = 25, 50$ (and the discrepancy values for $K = 75$), we observed a problem that arises for these extremely short maturity examples. In MatLab, the value $1 + 1 \times 10^{-16}$ is reported as 1. From the semi-analytic pricing formulae in equations (4.81) and (4.82), we see that for $\alpha^* < 0$, we may have to add a constant (residue contribution) to the value obtained from the numerical integration of the pricing integrand. If the value to which we add this constant is less than or equal to 1×10^{-16} then we will not be able to determine an appropriate solution for $\sigma_{(t_0;t_3,t_4]}$. However, from Lord and Kahl [2007], we have the following rule of thumb: For $F < K$, $\alpha^* > 0$ while for $F > K$, $\alpha^* < -1$ where F is the forward price (this is not claimed to hold for all parameter sets). From this we see that the problem is more likely to arise for $F > K$ (as it has in our example). Confirming that $\alpha^* < -1$ for the option value as a function of $\sigma_{(t_0,t_4]}$ and as a function of $\sigma_{(t_0,t_1]}$, $\sigma_{(t_0;t_1,t_2]}$, $\sigma_{(t_0;t_2,t_3]}$ and $\sigma_{(t_0;t_3,t_4]}$, we can ignore the residue contribution and simply compare the values obtained from the numerical integration. For the case $K = 25$, numerical integration yields the value 4.69×10^{-183} where the integrand is a function of $\sigma_{(t_0,t_4]}$ while the corresponding value for $K = 50$ is 4.31×10^{-55}. Ignoring the residue contributions in these cases yields the presented results. A similar problem arises for the discrepancy which we address in the same manner.

To conclude, a simple Gauss-Legendre Quadrature Rule would seem to be an appropriate tool with which to determine the forward values for a piecewise constant, time-dependent σ. For $k = X_{t_0}$, $\alpha = -\frac{1}{2}$

specifies an appropriate contour of integration. For alternative strike levels, one may need to make use of the optimal value of α.

The technique presented provides an approach to determine forward parameters within the SVJJ model, as an example of an affine model.

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