fractal and fractional

MDPI

Article

Lie Symmetries and the Invariant Solutions of the Fractional Black–Scholes Equation under Time-Dependent Parameters

Sameerah Jamal [1,2,*], Reginald Champala [1] and Suhail Khan [3]

[1] School of Mathematics, University of the Witwatersrand, Johannesburg 2001, South Africa; 1114409@students.wits.ac.za
[2] DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Johannesburg 2001, South Africa
[3] Department of Mathematics, University of Peshawar, Peshawar 25120, Khyber Pakhtoonkhwa, Pakistan; suhail@uop.edu.pk
* Correspondence: sameerah.jamal@wits.ac.za

Abstract: In this paper, we consider the time-fractional Black–Scholes model with deterministic, time-varying coefficients. These time parametric constituents produce a model with greater flexibility that may capture empirical results from financial markets and their time-series datasets. We make use of transformations to reduce the underlying model to the classical heat transfer equation. We show that this transformation procedure is possible for a specific risk-free interest rate and volatility of stock function. Furthermore, we reverse these transformations and apply one-dimensional optimal subalgebras of the infinitesimal symmetry generators to establish invariant solutions.

Keywords: Black–Scholes model; time-dependent volatility; Riemann–Liouville

check for updates

Citation: Jamal, S.; Champala, R.; Khan, S. Lie Symmetries and the Invariant Solutions of the Fractional Black–Scholes Equation under Time-Dependent Parameters. *Fractal Fract.* **2024**, *8*, 269. https://doi.org/10.3390/fractalfract8050269

Academic Editors: Ricardo Almeida and Riccardo Caponetto

Received: 25 March 2024
Revised: 22 April 2024
Accepted: 27 April 2024
Published: 29 April 2024

1. Introduction

The Black–Scholes (B-S) equation commonly models the fair price for a put or call option based on strike price, underlying stock price, volatility, type of option, time, and risk-free rate [1,2]. Mathematically, the model originates from a stochastic differential equation [3]; however, this model is often criticized for the assumption that the underlying volatility and the risk-free interest rate are constant over time, as well as other deficiencies [4]. The obvious way to counter this problem is to introduce an improved model with time parameters to capture the time-sensitivity of price dynamics. In particular, the B-S model with time-dependent parameters plays a crucial role in quantitative finance, since time-dependent volatility influences investment expectations [5]. Several authors, for example, [6,7], have dealt with implementing a time-dependent volatility function.

Fractional Brownian motion (FBM)-based models are the backbone of recent modelling trends. These models are preferred as they exhibit self-similarity characteristics and long-range dependence [8–11]. Moreover, arguments by [12–15] convey that the time-fractional B-S equation is well defined as the limit tends to 1, in cohesion with the integer-order B-S model.

Taking into account the above discussion, the B-S model with time-dependent parameters may be expressed as a fractional version, viz.,

$$\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\sigma^2(t)S^2}{2}\frac{\partial^2 V}{\partial S^2} = r(t)\left(V - \frac{\partial V}{\partial S}S\right) \quad 0 < \alpha < 1. \tag{1}$$

where $V(S,t)$ is the value of the option, S is the stock price, t is time, $r(t)$ is the risk-free interest rate, and $\sigma(t)$ is the volatility of the stock, with the assumption that the European option on an underlying stock pays no dividends [1,3,16]. The parameter α is the order of the fractional derivative and $\alpha = 1$ represents the traditional local integer derivative.

Such a model contains differentiable functions of time and has the potential to yield greater insights into the underlying instrument for a market's viewpoint.

This study is the first of its kind. In [17], a general fractional bond equation was considered that had no connections to the B-S model. Moreover, the model in [17] had *constant* coefficients. In this study, the model possesses functions dependent on *time* and it is a famous version of the B-S equation. Therefore, we have designed new transformations to tackle this problem.

Furthermore, in [17] we chose to transform the model to a fractional-order equation for further analysis, whereas here, we showcase how it is possible to transform a different model into an integer-order model. Hence, this paper has a novel approach with original transformations for a model that has not been studied before under these contexts, and of course, all solutions presented are new.

In the ancient era, human civilizations possessed far-reaching ideas about geometry and transformations. One of the first geometric transformations was that of the superposition principle, or in layman's terms, placing one figure on another—this later became an axiom of Euclid. With fractional differential equations (FDEs), solution techniques are analogous to classical differential equations. A range of methods exist, and the mathematical structures required in investigations relating to FDEs are endless, but many methods involve transformations of some kind, either those that reduce to an integer-order equation or a reduction to one of fractional order [18–22]. Several notable and recent applications of FDEs can be seen in the literature: the method of separation of variables for some FDEs [23], a Zoomeron equation's solutions [24], and families of second-order numerical results were investigated [25].

The techniques applied centre around Lie symmetries, which it is now possible to use for FDE analysis [26–29], with fascinating applications [30,31]. Secondly, we use a lesser-known technique, from the renowned work of Lie's classification of second-order partial differential equations (PDEs) [32], that is, Lie's approach to deriving equivalence transformations.

FDEs are essential components in mathematics, and hence, the field of its study is rapidly expanding. The main reasons for its growing importance is that it models intermediate evolutionary behaviour, especially that of a time-fractional nature. FDEs are known to be free from memory effects that plague integer-order equations and many processes are now capable of inculcating not just instantaneous time but also historical time.

Since FDEs are non-local, their exact solutions are difficult to find. This has posed new challenges to researchers whereby new approaches must be conceived. In particular, there is a great need for understanding transformations of FDEs. The inceptions of many transformations in FDE analysis are an attempt to change derivatives to the integer type, and the reason is obvious—there are numerous well-defined analytic methods, that include the conventional Leibnitz and chain rule, that classical integer-order derivatives obey. This investigation is, therefore, of major significance where we have successfully developed an effective approach to study FDEs of type (1), by finding new transformations to simplify a complicated model with time dependency parameters.

The focus of this paper is categorized into two parts: firstly, to derive and prove the existence of new equivalence formulae that render Equation (1) into a classical heat transfer equation (HTE). Our procedure requires a careful construction of transformations. The second part invokes the establishment of detailed proofs, whereby exact solutions to the model are presented. To enhance our theory, graphical plots are given that exemplify a highly recommended approach to studying financial models.

In Section 2, we introduce the basic theory behind this study. Section 3 contains the variable changes required to alter the fractional B-S model with time-dependent parameters, into the HTE. Some symmetry properties of the HTE and the main results appear in Section 4, whereby we provide symmetry-generated solutions of the time-dependent model and an insightful discussion of the obtained solutions. Finally, we conclude in Section 5.

2. Preliminaries

It should be noted that we have the underlying assumption that the fractional derivative is a Riemann–Liouville derivative.

$$^{R}D_t^{\alpha}u(t,x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-\tau)^{n-\alpha-1}u(\tau,x)d\tau, \tag{2}$$

for α as any positive real number and n a natural number such that $n-1 \leq \alpha < n$.

Point Symmetry Calculations

In general, a symmetry is a transformation, mapping all solutions of a given equation to another solution of the very same equation. Consider the variables u^a, $\alpha = 1,\dots,q$ dependent on x^i, $i = 1,\dots,p$. Suppose that

$$M_\alpha(x, u^{(k)}) = 0, \tag{3}$$

is a system of nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x. A one-parameter Lie group of transformations with ε as the group parameter, invariant under (3), is defined as

$$\bar{x} = \Xi(x, u; \varepsilon) \quad \bar{u} = \Phi(x, u; \varepsilon). \tag{4}$$

Invariance of (3) under the transformation (4) dictates that any solution $u = \Theta(x)$ of (3) maps into another solution $v = \Psi(x;\varepsilon)$ of (3). The expansion of (4) around the identity $\varepsilon = 0$, generates the following transformations:

$$\bar{x}^i = x^i + \varepsilon\xi^i(x,u) + O(\varepsilon^2), \tag{5}$$
$$\bar{u}^\alpha = u^\alpha + \varepsilon\eta^\alpha(x,u) + O(\varepsilon^2). \tag{6}$$

The Lie group action is collected from the infinitesimal generators acting on the space of variables (dependent and independent). Thus, we assume the following symmetry generator:

$$Z = \xi^i(x,u)\partial_{x^i} + \eta^\alpha(x,u)\partial_{u^\alpha}. \tag{7}$$

The invariance criterion reads as follows:

$$Z[M_\alpha(x,u^{(k)})] = 0, \quad \text{when} \quad M_\alpha(x,u^{(k)}) = 0, \tag{8}$$

where Z is an appropriate prolonged expression on all derivatives of the equation. The operator in Equation (7) defines a Lagrange system or characteristic equation

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha},$$

whereby we find the zero-order invariant functions

$$W^{[0]}(x^i, u^\alpha). \tag{9}$$

3. New Equivalence Transformations

We provide consecutive transformations that render the time-dependent model (1) into the famous and classical HTE. It turns out that this is possible for particular functions of the time-dependent parameters and not in general for arbitrary functions [33]. The specifications of the time-varying parameters r and σ are

$$r(t) = \frac{1}{2t^2}, \tag{10}$$

$$\sigma(t) = \frac{1}{t}. \tag{11}$$

In [34], the authors considered $r(t) = (1/2)\sigma^2$ with $\sigma = \frac{1}{t}$ (see [34] pg. 2, Equation (2)). This is precisely our choice for $r(t)$ and $\sigma(t)$ in (10) and (11), and we confirm that it is a special case of interest. However, it may be possible to first transform the original model to another form that would allow for different choices of $r(t)$ and $\sigma(t)$. For the purpose of this study, we opt to consider only the original B-S model with specific time coefficients, and other modifications of (1) may be presented elsewhere in the future.

Theorem 1. *The model Equation (1) is reducible to the classical HTE:*

$$\frac{\partial \omega}{\partial \hat{t}} - \frac{\partial^2 \omega}{\partial y^2} = 0, \quad \omega(y, \hat{t}). \tag{12}$$

under the parameter settings (10) *and* (11), *and where*

$$\hat{t} = -\frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{13}$$

$$y = -\sqrt{2}\ln(S)\hat{t}. \tag{14}$$

Proof. The independent variables are subjected to the above invertible change of variables, and if the latter is a fractional complex transform [35], Equation (1) reduces to the form

$$a(y, \hat{t})\frac{\partial V}{\partial y} + c(y, \hat{t})V + \frac{\partial V}{\partial \hat{t}} - \frac{\partial^2 V}{\partial y^2} = 0, \tag{15}$$

where $V = V(y, \hat{t})$, $a(y, \hat{t}) = \frac{y}{\hat{t}}$, and $c(y, \hat{t}) = \frac{1}{2\hat{t}^2}$.

Thereafter, under a second transformation,

$$V(y, \hat{t}) = \omega(y, \hat{t}) \cdot e^{-\phi(y, \hat{t})}, \tag{16}$$

where

$$\phi(y, \hat{t}) = -\frac{y^2}{4\hat{t}} - \frac{1}{2\hat{t}} - \frac{\ln|\hat{t}|}{2}, \tag{17}$$

Equation (1) is converted to the classical HTE. □

These transformations are unique to the specific fractional B-S model with time-dependent coefficients. However, it is possible to extend our approach to other differential equations, fractional or integer, under certain prerequisites. That is, the equation must be a parabolic equation, as is Equation (1), and secondly, the transformations must exist. We find that not all parabolic equations are transformable to a simple model like the heat equation—it often requires some ingenuity to conceive a transformation, and thereafter, test that it works.

4. Solutions

This section has two parts to it: firstly, we list the symmetry features that are needed for the classical HTE, in order to proceed with our analysis. Finally, we prove theorems about the solutions of the time-dependent model (1).

4.1. The Heat Transfer Equation

The HTE admits a fundamental solution in terms of the Gaussian function and many studies of parabolic PDEs involve the HTE [36,37]. The HTE offers many benefits, but primarily, we focus on its symmetry properties.

The Lie point symmetries of (12) are well known. They are

$$Z_1 = \frac{\partial}{\partial y}, \quad Z_2 = \frac{\partial}{\partial \hat{t}}, \quad Z_3 = \omega\frac{\partial}{\partial \omega}, \quad Z_4 = y\frac{\partial}{\partial y} + 2\hat{t}\frac{\partial}{\partial \hat{t}},$$
$$Z_5 = 2\hat{t}\frac{\partial}{\partial y} - y\omega\frac{\partial}{\partial \omega}, \quad Z_6 = 4\hat{t}y\frac{\partial}{\partial y} + 4\hat{t}^2\frac{\partial}{\partial \hat{t}} - (y^2 + 2\hat{t})\omega\frac{\partial}{\partial \omega},$$
$$Z_\beta = \beta(y,\hat{t})\frac{\partial}{\partial \omega}, \tag{18}$$

where β is an arbitrary solution of the HTE, i.e., $\beta_{\hat{t}} = \beta_{yy}$.

The symmetries $Z_1 - Z_6$ form a 6-dimensional Lie algebra, with Z_β spanning an infinite-dimensional subalgebra. The Lie brackets, given by

$$[Z_i, Z_j] = Z_i Z_j - Z_j Z_i,$$

are presented in Table 1.

Table 1. Lie brackets for the symmetries of (12).

$[,]$	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_1	0	0	0	Z_1	$-Z_3$	Z_5
Z_2	0	0	0	$2Z_2$	$2Z_1$	$4Z_4 - 2Z_3$
Z_3	0	0	0	0	0	0
Z_4	$-Z_1$	$-2Z_2$	0	0	Z_5	$2Z_6$
Z_5	Z_3	$-2Z_1$	0	$-Z_5$	0	0
Z_6	$-Z_5$	$2Z_3 - 4Z_4$	0	$-2Z_6$	0	0

The optimal system of one-dimensional subalgebras is the symmetry combinations that provide a systematic procedure for constructing nonequivalent classes of invariant solutions.

A discussion for obtaining a one-dimensional optimal system is provided in [38]. One takes a general element from the Lie algebra, and then, reduces it to its simplest form by applying carefully chosen adjoint transformations (Table 2):

$$\mathrm{Ad}(\exp(\varepsilon Z_i))Z_j = Z_j - \varepsilon[Z_i, Z_j] + \frac{1}{2}\varepsilon^2[Z_i,[Z_i,Z_j]] - \cdots . \tag{19}$$

Table 2. Adjoint representation of subalgebras of (18).

Ad	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_1	Z_1	Z_2	Z_3	$Z_4 - \varepsilon Z_1$	$Z_5 + \varepsilon Z_3$	$-2\varepsilon Z_5 + Z_6 - \varepsilon^2 Z_3$
Z_2	Z_1	Z_2	Z_3	$Z_4 - 2\varepsilon Z_2$	$Z_5 - 2\varepsilon Z_1$	$Z_6 + 2\varepsilon Z_3 + 4\varepsilon^2 Z_2 - 4\varepsilon Z_4$
Z_3	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Z_4	$e^{\varepsilon} Z_1$	$e^{2\varepsilon} Z_2$	Z_3	Z_4	$e^{-\varepsilon} Z_5$	$e^{-2\varepsilon} Z_6$
Z_5	$Z_1 - \varepsilon Z_3$	$2\varepsilon Z_1 + Z_2 - \varepsilon^2 Z_3$	Z_3	$Z_4 + \varepsilon Z_5$	Z_5	Z_6
Z_6	$Z_1 + 2\varepsilon Z_5$	$Z_2 - 2\varepsilon Z_3 + 4\varepsilon Z_4 + 4\varepsilon^2 Z_6$	Z_3	$Z_4 + 2\varepsilon Z_6$	Z_5	Z_6

The optimal system for the HTE is given by the following five cases ($b \in \mathbb{R}$):

$$Z_4 + 2bZ_3,\ Z_1,\ Z_2 + bZ_3,\ Z_2 - Z_5,\ Z_2 + Z_6 + bZ_3.$$

The invariant solutions pertaining to these cases, labelled A–E, are discussed below.

4.1.1. Solution A

Under the symmetry $Z_4 + 2bZ_3$, we prove the underlying theorem.

Theorem 2. *The time-dependent fractional B-S Equation (1) admits the exact solution*

$$V(S,t) = \sqrt{2}\,\ln(S)t^{\alpha}\sqrt{\frac{t^{\alpha}}{\Gamma(1+\alpha)}}\,\mathrm{e}^{-\frac{t^{-\alpha}\Gamma(1+\alpha)}{2}}\left(-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}+b} \frac{\left(M\left(1+b,\frac{3}{2},-\frac{t^{\alpha}\ln(S)^2}{2\Gamma(1+\alpha)}\right)C_1 + U\left(1+b,\frac{3}{2},-\frac{t^{\alpha}\ln(S)^2}{2\Gamma(1+\alpha)}\right)C_2\right)}{\Gamma(1+\alpha)}. \tag{20}$$

Proof. The symmetries of this case yield invariants $m = y\frac{1}{\sqrt{\hat{t}}}$ and $\omega(y,\hat{t}) = z(m)\hat{t}^{b}$. Hence, the HTE is reduced to

$$\frac{z'(m)m}{2} - z(m)b + z''(m) = 0.$$

We solve the above to determine that the exact solution of (12) is

$$\omega(y,\hat{t}) = \mathrm{e}^{-\frac{y^2}{4\hat{t}}}\left(\frac{\hat{t}}{y^2}\right)^{b-\frac{1}{2}}\left(U\left(b+1,\frac{3}{2},\frac{y^2}{4\,\hat{t}}\right)C_2 + M\left(b+1,\frac{3}{2},\frac{y^2}{4\,\hat{t}}\right)C_1\right)y^{2\,b}, \tag{21}$$

where $U(\mu,\nu,z)$ and $M(\mu,\nu,z)$ are Kummer functions.

Hence, the proof concludes upon the application of Theorem 1's reversible transformations, viz., (13), (14), (16), and (17). □

4.1.2. Solution B

By applying the symmetry Z_1, we prove the underlying theorem.

Theorem 3. *The time-dependent fractional Black–Scholes Equation (1) admits the invariant solution*

$$V(S,t) = C_1\sqrt{\frac{t^{\alpha}}{\Gamma(1+\alpha)}}\,\mathrm{e}^{-\frac{t^{-\alpha}\Gamma(1+\alpha)^2 + t^{\alpha}\ln(S)^2}{2\Gamma(1+\alpha)}}. \tag{22}$$

Proof. Now, the symmetries of this case yield invariants $m = t$ and $\omega(y,\hat{t}) = z(m)$. Hence, the HTE is reduced to

$$z'(m) = 0.$$

We solve the reduced equation and this case provides the solution of (12) as

$$\omega(y,\hat{t}) = C_1, \tag{23}$$

where C_1 is an arbitrary constant. Thereafter, analogous to the previous case, the proof concludes upon application of Theorem 1's reversible transformations, viz., (13), (14), (16), and (17). □

4.1.3. Solution C

By applying the symmetry $Z_2 + bZ_3$, we prove the underlying theorem.

Theorem 4. *The time-dependent fractional Black–Scholes Equation (1) admits the invariant solution*

$$\begin{aligned} V(S,t) &= \left(e^{\frac{-t^{-\alpha}\Gamma(1+\alpha)^2 - 2t^{\alpha}\left(\sqrt{b}\sqrt{2}\ln(S) + \frac{\ln(S)^2}{2} + b\right)}{2\Gamma(1+\alpha)}} C_2 \right. \\ &\quad \left. + e^{\frac{-t^{-\alpha}\Gamma(1+\alpha)^2 - 2\left(-\sqrt{b}\sqrt{2}\ln(S) + \frac{\ln(S)^2}{2} + b\right)t^{\alpha}}{2\Gamma(1+\alpha)}} C_1 \right) \\ &\quad \times \sqrt{\frac{t^{\alpha}}{\Gamma(1+\alpha)}}, \end{aligned} \tag{24}$$

for $b \neq 0$, *and*

$$V(S,t) = \frac{\sqrt{\frac{t^{\alpha}}{\Gamma(1+\alpha)}}\, e^{-\frac{\ln(S)^2 t^{\alpha} + t^{-\alpha}\Gamma(1+\alpha)^2}{2\Gamma(1+\alpha)}} \left(C_1\sqrt{2}\, t^{\alpha}\ln(S) + C_2\Gamma(1+\alpha)\right)}{\Gamma(1+\alpha)}, \tag{25}$$

for $b = 0$.

Proof. Here, we have the invariants $m = y$ and $\omega(y,\hat{t}) = z(m)e^{bt}$. Hence, the HTE is reduced to

$$z(m)b - z''(m) = 0.$$

This linear combination of symmetries yields an exact solution of the HTE (12) as

$$\omega(y,\hat{t}) = \left(C_1 e^{\sqrt{b}y} + C_2 e^{-\sqrt{b}y}\right)e^{b\hat{t}}. \tag{26}$$

For $b \neq 0$, with $C_1 = C_2 = \frac{1}{2}$, we have

$$\omega(y,\hat{t}) = e^{b\hat{t}}\cosh(\sqrt{b}y), \tag{27}$$

and for $b = 0$, we have

$$\omega(y,\hat{t}) = C_1\, y + C_2. \tag{28}$$

Thus, the proof concludes upon application of Theorem 1's reversible transformations, (13), (14), (16), and (17). □

4.1.4. Solution D

Under the symmetry combination $Z_2 - Z_5$, we prove the underlying theorem.

Theorem 5. *The time-dependent fractional Black–Scholes Equation (1) admits the invariant solution*

$$\begin{aligned} V(S,t) = S^{-\frac{t^{2\alpha}\sqrt{2}}{\Gamma(1+\alpha)^2}} \sqrt{\frac{t^{\alpha}}{\Gamma(1+\alpha)}}\, e^{-\frac{3t^{-\alpha}\Gamma(1+\alpha)^4 + 3\ln(S)^2\Gamma(1+\alpha)^2 t^{\alpha} + 4t^{3\alpha}}{6\Gamma(1+\alpha)^3}} \times \\ \left(C_1 Ai\left(\frac{t^{\alpha}\sqrt{2}\ln(S)\Gamma(1+\alpha) + t^{2\alpha}}{\Gamma(1+\alpha)^2}\right) + C_2 Bi\left(\frac{t^{\alpha}\sqrt{2}\ln(S)\Gamma(1+\alpha) + t^{2\alpha}}{\Gamma(1+\alpha)^2}\right)\right). \end{aligned} \tag{29}$$

Proof. Now, the symmetries of this case yield invariants $m = \hat{t}^2 + y$ and

$$\omega(y,\hat{t}) = z(m)e^{y\hat{t} + \frac{2\hat{t}^3}{3}}.$$

Hence, the HTE is reduced to

$$m\,z(m) - z''(m) = 0.$$

That is, we solve the reduced equation so that an application of the invariants of this subalgebra produces the exact solution of Equation (12) as

$$\omega(y,\hat{t}) = \left(C_1\,\mathrm{Ai}\left(\hat{t}^2+y\right) + C_2\,\mathrm{Bi}\left(\hat{t}^2+y\right)\right)\mathrm{e}^{y\hat{t}+\frac{2\hat{t}^3}{3}}, \tag{30}$$

where $Ai(z)$ and $Bi(z)$ are the *Airy Ai* and *Airy Bi* functions, respectively.

Next, the proof concludes upon application of Theorem 1's reversible transformations, viz., (13), (14), (16), and (17). □

4.1.5. Solution E

The final case, $Z_2 + Z_6 + bZ_3$, yields the following result.

Theorem 6. *The time-dependent fractional Black–Scholes Equation (1) admits the invariant solution*

$$\begin{aligned} V(S,t) = {} & 2^{\frac{3}{4}}\sqrt{\frac{t^{\alpha}}{\Gamma(1+\alpha)}}\,\mathrm{e}^{\frac{4c\left(\frac{\Gamma(1+\alpha)^2}{4}+t^{2\alpha}\right)\arctan\left(\frac{t^{-\alpha}\Gamma(1+\alpha)}{2}\right)-\left(t^{-\alpha}\Gamma(1+\alpha)^2+t^{\alpha}\left(\ln(S)^2+4\right)\right)\Gamma(1+\alpha)}{8t^{2\alpha}+2\Gamma(1+\alpha)^2}} \\ & \times\frac{\left(C_1\bar{\mathrm{M}}_{\frac{\mathrm{I}}{4}c,\frac{1}{4}}\left(\frac{2\,\mathrm{I}t^{2\alpha}\ln(S)^2}{4t^{2\alpha}+\Gamma(1+\alpha)^2}\right)+C_2\bar{\mathrm{W}}_{\frac{\mathrm{I}}{4}\mathrm{c},\frac{1}{4}}\left(x\frac{2\,\mathrm{I}t^{2\alpha}\ln(S)^2}{4t^{2\alpha}+\Gamma(1+\alpha)^2}\right)\right)}{2\sqrt{\frac{t^{\alpha}\ln(S)}{\Gamma(1+\alpha)}}}. \end{aligned} \tag{31}$$

for $b \neq 0$, and

$$\begin{aligned} V(S,t) = {} & \frac{2^{\frac{5}{4}}\sqrt{\frac{t^{\alpha}}{\Gamma(1+\alpha)}}\,\mathrm{e}^{\frac{-\Gamma(1+\alpha)^3t^{-\alpha}-\Gamma(1+\alpha)t^{\alpha}\left(\ln(S)^2+4\right)}{2\Gamma(1+\alpha)^2+8t^{2\alpha}}}}{\sqrt{\frac{t^{-2\alpha}\Gamma(1+\alpha)^2+4}{\ln(S)^2}}\sqrt{\frac{t^{\alpha}\ln(S)}{\Gamma(1+\alpha)}}} \\ & \times\left(\mathrm{J}\left(-\frac{1}{4},\frac{t^{2\alpha}\ln(S)^2}{\Gamma(1+\alpha)^2+4t^{2\alpha}}\right)C_1+\mathrm{Y}\left(-\frac{1}{4},\frac{t^{2\alpha}\ln(S)^2}{\Gamma(1+\alpha)^2+4t^{2\alpha}}\right)C_2\right), \end{aligned} \tag{32}$$

for $b = 0$.

Proof. The symmetries of this case yield invariants $m = \frac{4\hat{t}^2+1}{4y^2}$ and

$$\omega(y,\hat{t}) = z(m)\mathrm{e}^{\frac{y^2}{8\hat{t}^2+2}\left(-\frac{b\left(4\hat{t}^2+1\right)}{y^2}\arctan\left(\frac{1}{2\hat{t}}\right)-2\hat{t}\right)}\frac{1}{\sqrt{y}}.$$

Hence, the HTE is reduced to

$$\left(-\frac{1}{4}+mb-3m^2\right)z(m) - 16\,m^4z''(m) - 32\,m^3z'(m) = 0.$$

We may solve the above ordinary differential equation, and hence, this subalgebra gives us the exact solution of Equation (12), that is,

$$\omega(y,\hat{t}) = \frac{\left(C_1\,\bar{M}_{\frac{i}{4}b,\frac{1}{4}}\left(\frac{iy^2}{4\hat{t}^2+1}\right)+C_2\,\bar{W}_{\frac{i}{4}b,\frac{1}{4}}\left(\frac{iy^2}{4\hat{t}^2+1}\right)\right)}{\sqrt{y}}\mathrm{e}^{\frac{1}{8\hat{t}^2+2}\left(\left(-4\hat{t}^2-1\right)b\arctan\left(\frac{1}{2\hat{t}}\right)-2\hat{t}y^2\right)}, \tag{33}$$

where $\bar{W}_{\mu,\nu}(z)$ and $\bar{M}_{\mu,\nu}(z)$ are Whittaker functions, for $b \neq 0$.

When $b = 0$, we have

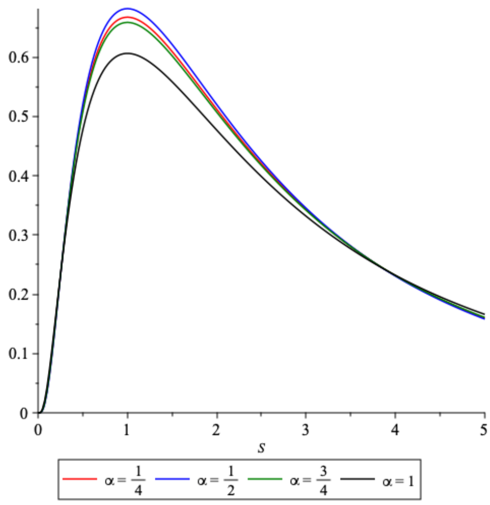
$$\omega(y,\hat{t}) = \left(\frac{C_1 \cdot \mathrm{J}\left(-\frac{1}{4}, \frac{y^2}{2\cdot(4\cdot\tau^2+1)}\right)}{\sqrt{\frac{4\cdot\tau^2+1}{4\cdot y^2}}} + \frac{C_2 \cdot \mathrm{Y}\left(-\frac{1}{4}, \frac{y^2}{2\cdot(4\cdot\tau^2+1)}\right)}{\sqrt{\frac{4\cdot\tau^2+1}{4\cdot y^2}}} \right) \cdot \frac{e^{-\frac{2\cdot y^2\cdot\tau}{8\cdot\tau^2+2}}}{\sqrt{y}}, \tag{34}$$

where $J(\mu, z)$ and $Y(\mu, z)$ are Bessel functions.

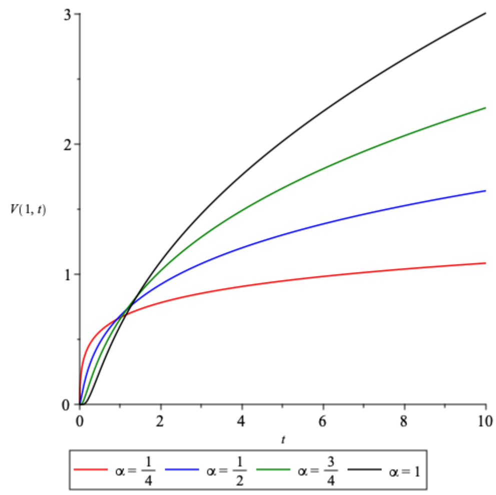
Once again, the proof concludes upon application of Theorem 1's reversible transformations, viz., (13), (14), (16), and (17). □

4.2. Graphical Analysis of Solutions

Symmetry combinations B and D, Figures 1 and 2 , display similar behaviour to each other across both asset price, S, and time, t. For small values of fixed asset price S, the solutions for option values display growth. As α approaches 1, these solutions experience accelerated growth rates. However, for sufficiently large values of asset price S, all functions decay and tend to zero over time. Regarding fixed-time behaviour, as α approaches 1, the solutions rapidly approach a minimum, reflecting the nature of options that typically mature within a year or two, with drastic stock price increases being uncommon. On the other hand, the solutions in symmetry combinations A and C, Figures 3–5, exhibit similar behaviour. As α approaches 1, these solutions tend to zero for fixed-time cases. Overall, symmetry combinations A and C exhibit solutions with global maximums. Notably, for both fractional and integer α, solutions in symmetry combination A become negative when the asset price is less than 1, followed by an increase until becoming positive. Symmetry combination E, when the parameter $b = 0$ (Figure 6) displays positive and negative option values for varying asset price ranges. Despite being unexpected, negative values are prevalent in some markets, such as the U.S. Treasury markets [39], and may indicate market frictions and imperfections. Remarkably, higher orders of α result in higher option values over time.

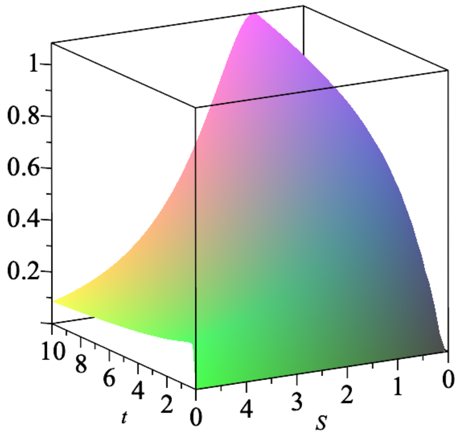


(**a**)



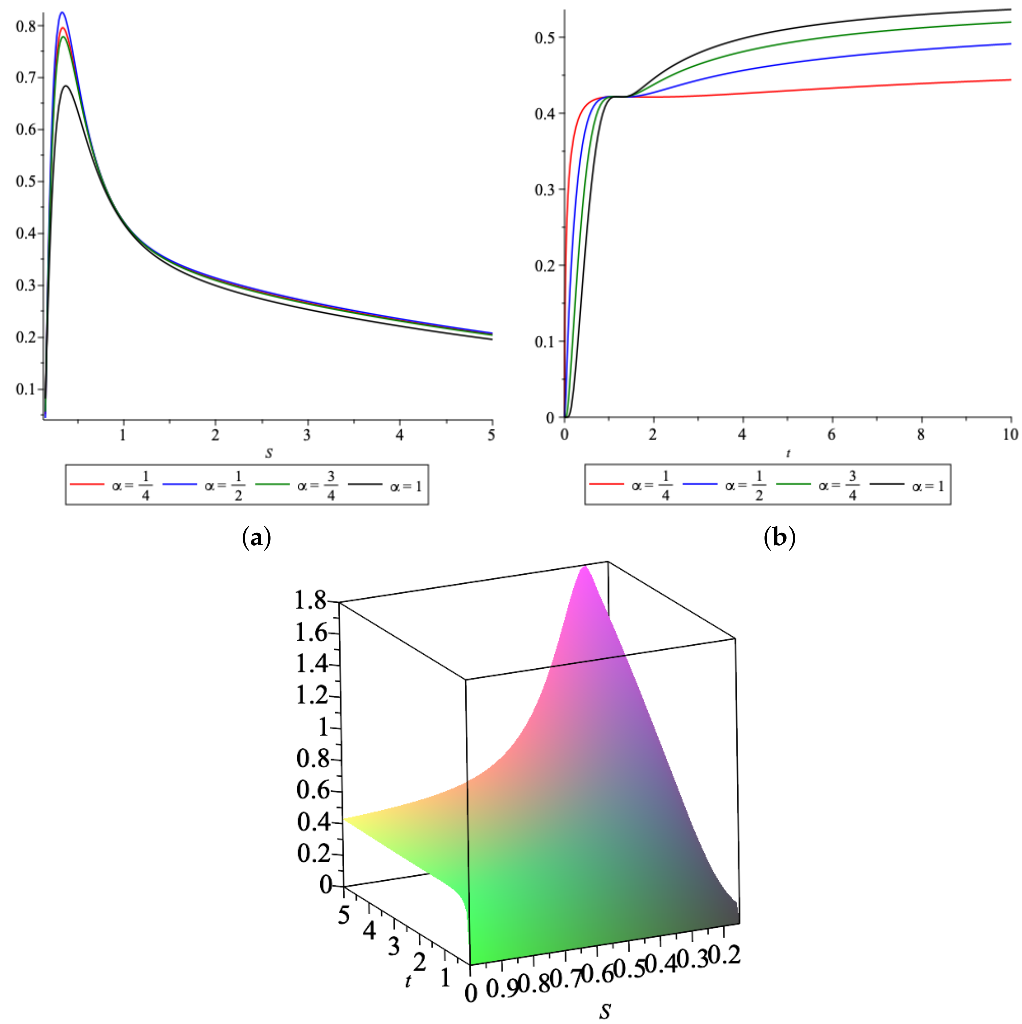
(**b**)

Figure 1. *Cont.*



(**c**)

Figure 1. In (**a**), the graph of Equation (22), when $C_1 = 1$ and $t = 1$. In (**b**), the 2D contours corresponding to (22), when $C_1 = 1$ and $S = 1$. Lastly, in (**c**), the 3D plot of (22), when $C_1 = 1$ and $\alpha = \frac{1}{4}$.



(**a**) (**b**)

(**c**)

Figure 2. In (**a**), the graph of Equation (29), when $C_1 = C_2 = 1$ and $t = 1$. In (**b**), the 2D contours corresponding to (29), when $C_1 = C_2 = 1$ and $S = 1$. Lastly, in (**c**), the 3D plot of (29), when $C_1 = C_2 = 1$ and $\alpha = \frac{1}{4}$.

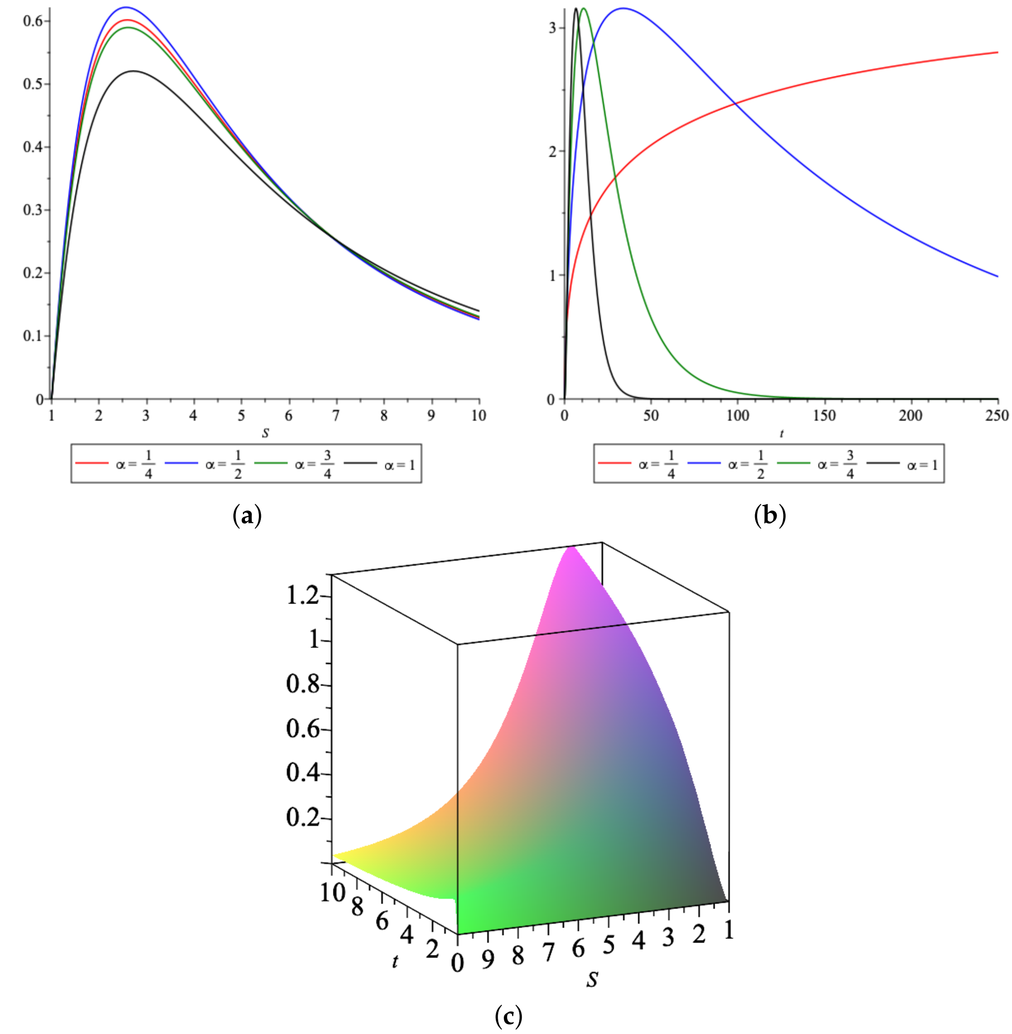


Figure 3. In (**a**), the graph of Equation (20), when $C_1 = 1$, $C_2 = 0$, $t = 1$, and $b = \frac{1}{2}$. In (**b**), the 2D contours corresponding to (20), when $C_1 = 1$, $C_2 = 0$, $S = 1$, and $b = \frac{1}{2}$. Lastly, in (**c**), the 3D plot of (20), when $C_1 = 1$, $C_2 = 0$, $\alpha = \frac{1}{4}$, and $b = \frac{1}{2}$.

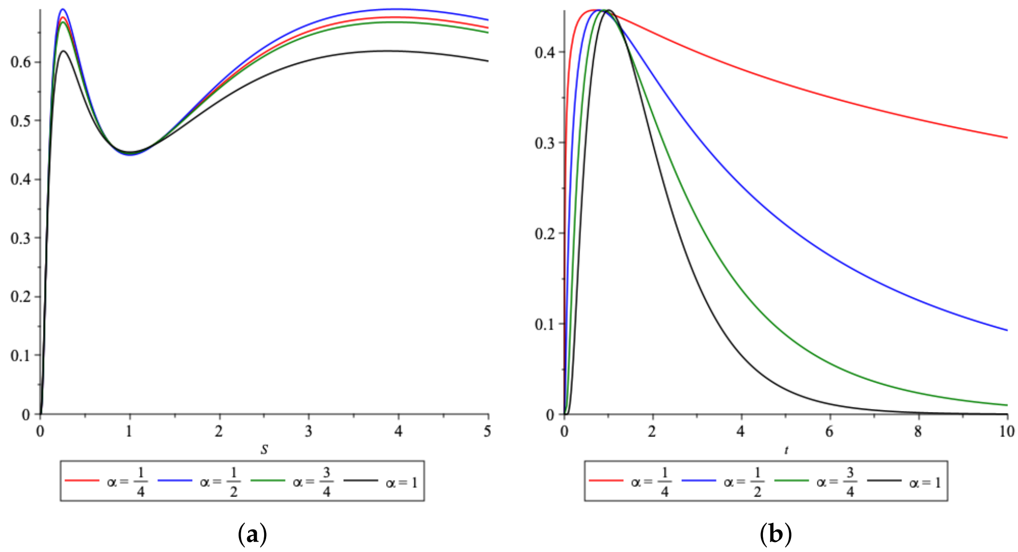
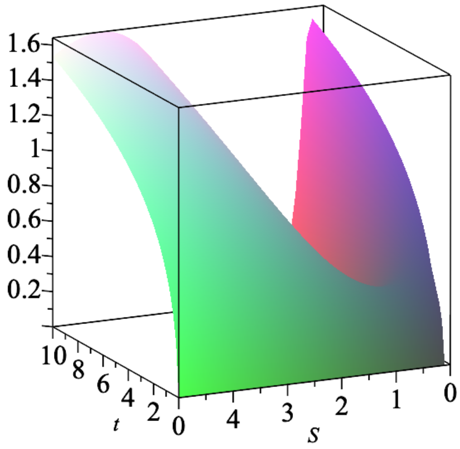
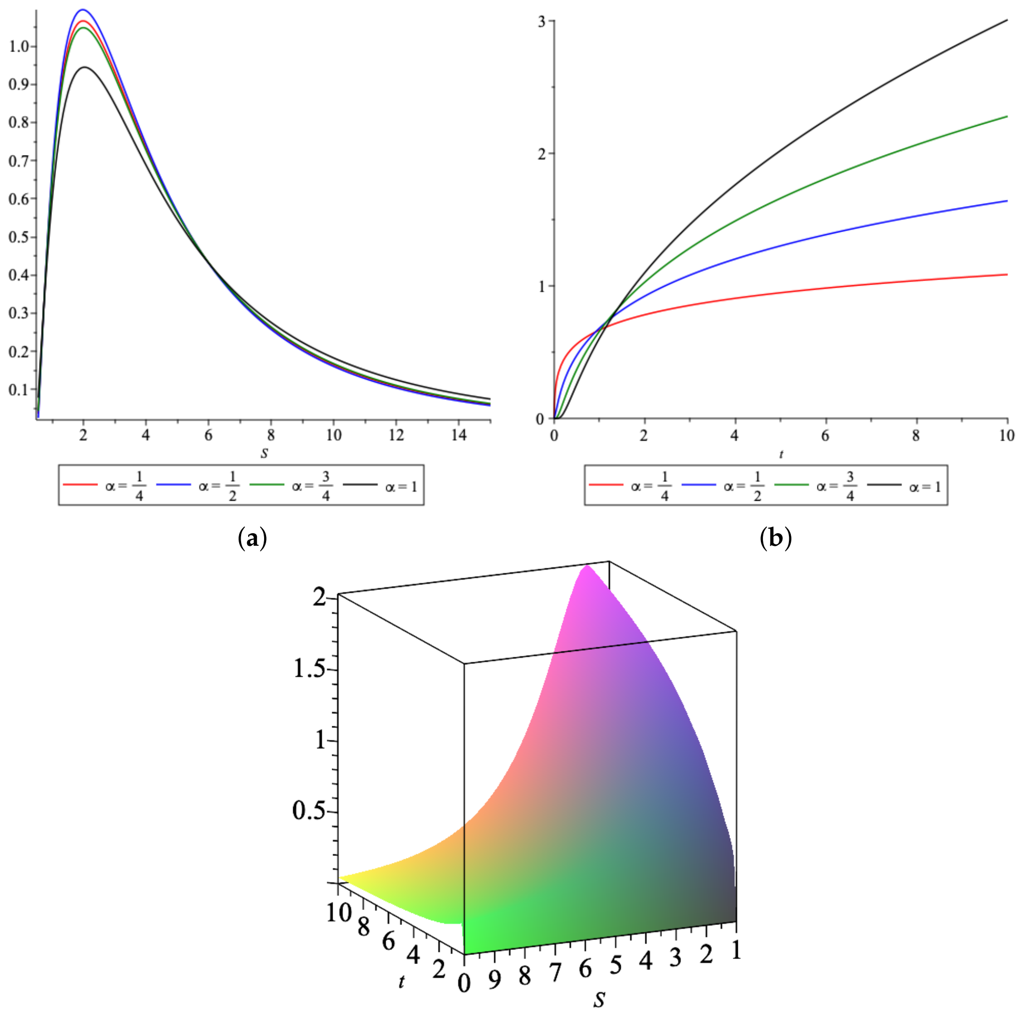


Figure 4. *Cont.*



(**c**)

Figure 4. In (**a**), the graph of Equation (24), when $C_1 = C_2 = 1, t = 1$, and $b = 1$. In (**b**), the 2D contours corresponding to (24), when $C_1 = C_2 = 1, S = 1$, and $b = 1$. Lastly, in (**c**), the 3D plot of (24), when $C_1 = C_2 = 1, \alpha = \frac{1}{4}$, and $b = 1$.



(**a**) (**b**)

(**c**)

Figure 5. In (**a**), the graph of Equation (25), when $C_1 = C_2 = 1, t = 1$, and $b = 0$. In (**b**), the 2D contours corresponding to (25), when $C_1 = C_2 = 1, S = 1$, and $b = 0$. Lastly, in (**c**), the 3D plot of (25), when $C_1 = C_2 = 1$, and $\alpha = \frac{1}{4}$.

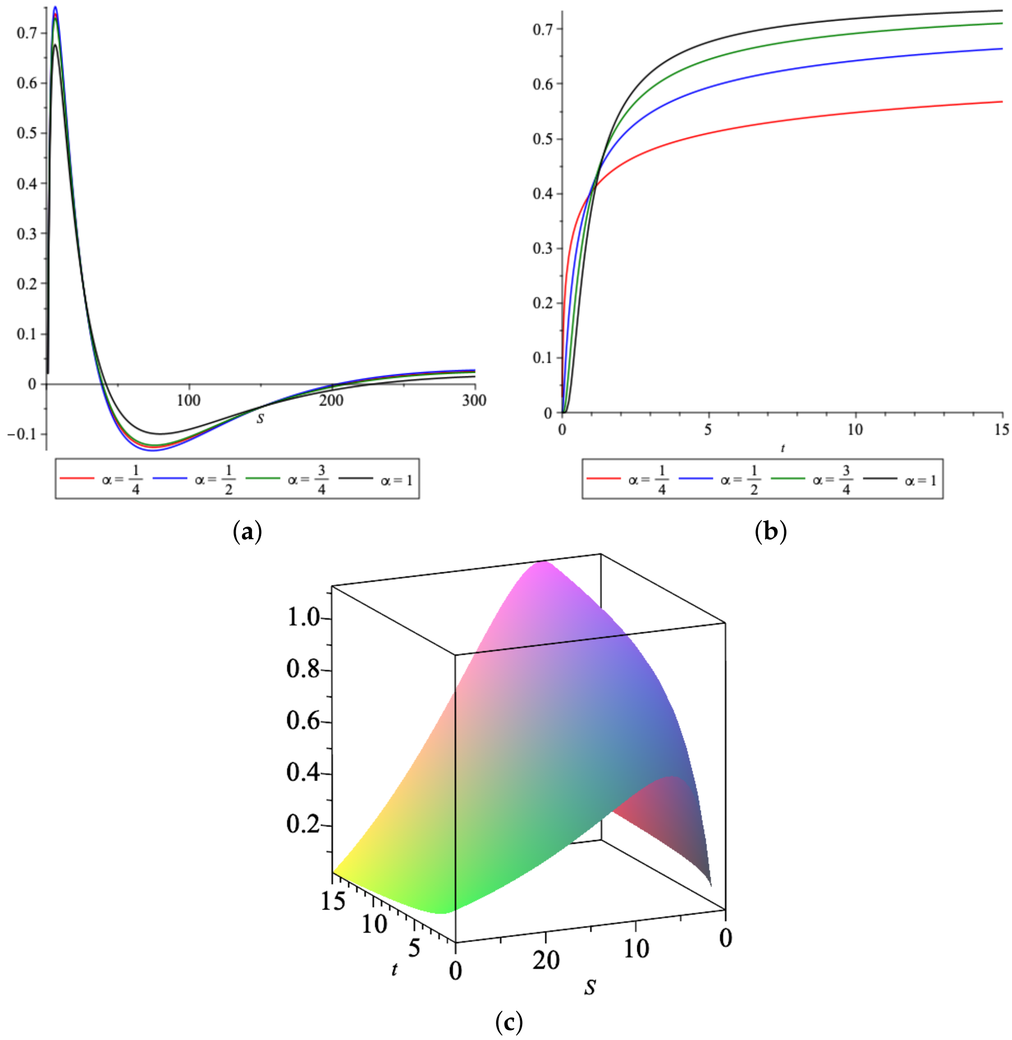


Figure 6. In (**a**), the graph of Equation (32), when $C_1 = C_2 = 1$ and $t = 1$. In (**b**), the 2D contours corresponding to (32), when $b = 0$, $C_1 = C_2 = 1$, and $S = 2$. Lastly, in (**c**), the 3D plot of (32), when $b = 0$, $C_1 = C_2 = 1$, and $\alpha = \frac{1}{4}$.

4.3. Remarks about the Transformations

Nearly every paper on FDEs encodes some type of transformation in its method. It is true that transformations, either to an integer equation or a fractional one, may result in a potential loss of information or altered behaviours. Conversely, some transformations, especially from symmetries, often lead to a gain in information via the emergence of new solutions—a major and powerful advantage.

Not all transformations are equal. The least desirable transformations are approximate ones, where a loss of information is guaranteed. Transformations that also cause a loss of differentiability or invertibility often obscure many features of the original equation. Transformations that affect stability properties are also unwanted. Bearing this in mind, our chosen transformations are of the best type to obtain accurate and meaningful results:

- The transformations are exact and provide equivalence relations.
- Theorem 1 describes a reversible change of variables that by design mandates the preservation of many dynamics of the original equation.
- By this invertibility, we can uniquely recover the original equation from the transformed one.
- In demonstrating the appropriateness of our transformation, we know that several analytical properties of the original FDE, such as linearity and causality, are retained

by (12), thereby ensuring that important characteristics of the model are captured in the transformed equation.
- The transformations leveraged here are consistent with established fractional calculus modus operandi.

Like with any mathematical approach, potential limitations exist. The change of variables can and has altered the appearance of solutions often expressed in terms of different functions, and the solutions may have different geometric interpretations. The chosen transformations will induce various changes in the behaviour of solutions. Any singular behaviour in the original form of the model, due to non-smoothness, may be lost by a conversion to an integer model. It is possible that converting to integer-order derivatives may overlook some fractional properties of the model.

By careful consideration, many of the above problems are mitigated by the following factors.

- It can be shown numerically that (12) is consistent with its fractional version via a limiting process, where, as the fractional order approaches an integer value, the FDE converges to integer order. Notably, the same behaviour is accurately displayed by the solutions to the original model, viz., as the fractional order approaches an integer value, the FDE (1) converges to its integer-order counterpart. This can also be shown via an Itô process [12].
- Approximate methods cause a deficiency in preserving non-local effects. On the contrary, exact methods (like ours) contain essential information about long-range dependencies.
- The numerical simulations conducted exhibit solutions, confirming intricate temporal dependencies which strongly imply the memory effects are accounted for.
- One can compare the solutions for the integer model versus the fractional and look for essential differences in behaviour. The slower decay and expansion rates observed for the option values are indicative of the usual fractional dynamics, thereby conveying the validity and accuracy of the resulting solutions

We can tailor a transformational approach to pursue a desired analytical requirement for a given model. That is, we could have transformed (1) to a fractional version of (12) without any difficulty. This would result in fewer solutions and some different ones, because the symmetry properties of a fractional heat equation are only four dimensional. This flexibility in the choice of transformations is to be celebrated as the advantages of such a construction far outweigh any potential drawbacks.

5. Conclusions

Traditional B-S models have been cast aside in favour of generalized models permitting parameters with differentiable functions of time. The latter display more accurate pricing in mature markets linked to financial instruments with futures and options. We have demonstrated that transformations exist to render the parametric coefficient model to the classical HTE, which is rich in symmetry. The main idea for such an approach is to exploit the convenient relationships connected to the heat equation. Namely, we have solved the B-S equation under the specification of time-varying parameters, $r(t)$ the risk-free interest and $\sigma(t)$ the volatility of the stock, given by (10) and (11).

The underlying assumption of the model is that the interest rate and volatility decline with time. In fact, as $t \to \infty$, both factors tend to zero, with the interest rate declining faster than volatility. Studying Equation (1) resulted in five variants of solutions. These solutions are created from the chosen symmetry combinations given by an optimal system of one-dimensional Lie algebras. This analysis shows that transformations provide an effective tool to simplify complicated models, and thus, enhance the solution process.

Equation (1) has critical applications in the mathematics of finance and in particular from a symmetry perspective [34,40]. This equation is widely accepted as being more realistic than the original B-S model with a constant risk-free rate and volatility, as these parameters, in real-world scenarios, are influenced by market forces and do not remain static for long periods.

Conventionally, local (integer) derivatives capture many memory effects and the discovery of a fractional derivative regime allows for measures of extreme asset price movements and repeated patterns in financial structures. Equation (1) is based on FBM, and hence, describes the dynamic characteristics of financial markets more accurately than a classical equation based on standard Brownian motion. In fact, extensive research evidence has shown [10] that integer derivative models do not reflect the reality of the financial market. According to empirical evidence, most traded assets' returns exhibit memory structures; see [9] and references therein. Memory problems in the valuation of derivative instruments have long been documented in data such as stock prices, forex rates, equity prices, etc. [8]. Today, the need for fractional models is imperative.

Equation (1)'s main application is to evaluate when to hedge an option by selling or buying some asset at an opportune time where the risk is minimized. This opportune time cannot be influenced by memory effects, therefore, solutions to Equation (1) offer an improved prediction of the mathematical evolution of option pricing and asset valuation.

There is a marked difference in all our solutions for the integer case $\alpha = 1$ versus the fractional solutions $0 < \alpha < 1$, showing more precision for the latter fractional case. We find that the option's price predicted by the fractional model is closely approximated to the observed price in the integer case for fixed time. A major deviation is seen in option values over time when stock price is fixed in the fractional model as compared to the integer case.

In further applications, the solutions found here may be used to explain the dynamics of complex price progressions and irregular increments in variables of financial instruments.

Author Contributions: Conceptualization, S.J.; formal analysis, R.C.; writing—original draft preparation, R.C.; writing—review and editing, S.K.; writing—review and editing, S.J.; visualization, R.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Shinde, A.S.; Takale K.C. Study of Black-Scholes model and its applications. *Procedia Eng.* **2012**, *38*, 270–279. [CrossRef]
2. Black, F.; Scholes, M. The pricing of options and corporate liabilities. In *World Scientific Reference on Contingent Claims Analysis in Corporate Finance: Volume 1: Foundations of CCA and Equity Valuation*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2019.
3. Oksendal, B. *Stochastic Differential Equations: An Introduction with Applications*; Springer: Berlin/Heidelberg, Germany, 2013.
4. Wu, C.L. The Finite Moment Log Stable Process and Option Pricing. *J. Financ.* **2003**, *58*, 753–777.
5. El-Nabulsi, R.A.; Golmankhaneh, A.K. Generalized heat diffusion equations with variable coefficients and their fractalization from the Black-Scholes equation. *Commun. Theor. Phys.* **2021**, *73*, 055002. [CrossRef]
6. Deng, Z.C.; Hon, Y.C.; Isakov, V. Recovery of time-dependent volatility in option pricing model. *Inverse Probl.* **2016**, *32*, 115010. [CrossRef]
7. Wróblewski, M. Nonlinear Schrödinger approach to European option pricing. *Open Phys.* **2017**, *15*, 280–291. [CrossRef]
8. Zhang, W.-G.; Xiao, W.-L.; He, C.-X. Equity warrants pricing model under fractional Brownian motion and an empirical study. *Expert Syst. Appl.* **2009**, *36*, 3056–3065. [CrossRef]
9. Garzarelli, F.; Cristelli, M.; Pompa, G.; Zaccaria, A.; Pietronero, L. Memory effects in stock price dynamics: evidences of technical trading. *Sci. Rep.* **2014**, *4*, 4487. [CrossRef] [PubMed]
10. Panas, E. Long memory and chaotic models of prices on the London metal exchange. *Resour. Policy* **2001**, *4*, 485–490. [CrossRef]
11. Jumarie, G. Derivation and solutions of some fractional Black–Scholes equations in coarse-grained space and time. Application to Merton's optimal portfolio. *Comput. Math. Appl.* **2010**, *59*, 1142–1164. [CrossRef]
12. Liang, J.R.; Wang, J.; Zhang, W.J.; Qiu, W.Y.; Ren, F.Y. The solutions to a bi-fractional black-scholes-merton differential equation. *Int. J. Pure Appl. Math* **2010**, *128*, 99–112.
13. Fall, A.N.; Ndiaye, S.N.; Sene, N. Black–Scholes option pricing equations described by the Caputo generalized fractional derivative. *Chaos Solit. Fractals* **2019**, *125*, 108–118. [CrossRef]
14. Giona, M.; Roman, H.E. Fractional diffusion equation on fractals: One-dimensional case and asymptotic behaviour. *J. Phys. A Math. Gen.* **1999**, *25*, 2093. [CrossRef]

15. Yavuz, M.; Özdemir, N. European vanilla option pricing model of fractional order without singular kernel. *Fractal Fract.* **2018**, *2*, 3. [CrossRef]
16. Hull, J.C. *Options Futures and Other Derivatives*; Pearson Education: Bangalore, India, 2003.
17. Champala, R.; Jamal, S.; Khan, S. Fractional Pricing Models: Transformations to a Heat Equation and Lie Symmetries. *Fractal Fract.* **2023**, *7*, 632. [CrossRef]
18. Raslan, K.R.; Ali, K.; Shallal, M.A. Solving the space-time fractional RLW and MRLW equations using modified extended tanh method with the Riccati equation. *Brit. J. Math. Comput. Sci.* **2017**, *21*, 1–15. [CrossRef]
19. Wang, G.; Shen, B.; He, M.; Guan, F.; Zhang, L. Symmetry Analysis and PT-Symmetric Extension of the Fifth-Order Korteweg-de Vries-Like Equation. *Fractal Fract.* **2022**, *6*, 468. [CrossRef]
20. Mnguni, N.; Jamal, S. Invariant solutions of fractional-order spatio-temporal partial differential equations. *Int. J. Nonlinear Sci. Numer. Simul.* **2021**, *22*, 1011–1022. [CrossRef]
21. El-Ajou, A.; Abu Arqub, O.; Al Zhour, Z.; Momani, S. New results on fractional power series: Theories and applications. *Entropy* **2013**, *15*, 5305–5323. [CrossRef]
22. Jamal, S.; Mnguni, N. Moving front solutions of a time-fractional power-law fluid under gravity. *Quaest. Math.* **2021**, *44*, 1295–1304. [CrossRef]
23. Maheswari, C.U.; Sahadevan, R.; Yogeshwaran, M. Method of separation of variables and exact solution of time fractional nonlinear partial differential and differential-difference equations. *Fract. Calc. Appl. Anal.* **2023**, *26*, 2421–2438. [CrossRef]
24. Alshammari, M.; Moaddy, K.; Naeem, M.; Alsheekhhussain, Z.; Alshammari, S.; Al-sawalha, M.M. Solitary and Periodic Wave Solutions of Fractional Zoomeron Equation. *Fractal Fract.* **2024**, *8*, 222. [CrossRef]
25. Yin, B.L.; Liu, Y.; Li, H.; Zhong, Z.M. Two families of second-order fractional numerical formulas and applications to fractional differential equations. *Fract. Calc. Appl. Anal.* **2023**, *26*, 1842–1867. [CrossRef]
26. Buckwar, E.; Luchko, Y. Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl.* **1998**, *227*, 81–97. [CrossRef]
27. Gazizov, R.K.; Kasatkin, A.A.; Lukashchuk, S.Y. Symmetry properties of fractional diffusion equations. *Phys. Scr.* **2009**, *T136*, 014016. [CrossRef]
28. Gazizov, R.K.; Kasatkin, A.A.; Lukashchuk, S.Y. Continuous transformation groups of fractional differential equations. *Vestn. USATU* **2007**, *9*, 125–135.
29. Bakkyaraj, T. Lie symmetry analysis of system of nonlinear fractional partial differential equations with Caputo fractional derivative. *Eur. Phys. J. Plus* **2020**, *135*, 126. [CrossRef]
30. Leo, R.A.; Sicuro, G.; Tempesta, P. A theorem on the existence of symmetries of fractional PDEs. *C. R. Acad. Sci. Paris Ser. I* **2014**, *352*, 219–222. [CrossRef]
31. Kubayi, J.T.; Jamal, S. Lie Symmetries and Third- and Fifth-Order Time-Fractional Polynomial Evolution Equations. *Fractal Fract.* **2023**, *7*, 125. [CrossRef]
32. Lie, S. On integration of a class of linear partial differential equations by means of definite integrals. *Arch. Math. Og Naturvidenskab* **1881**, *VI*, 328–368. (In Germany)
33. Naz, R.; Johnpillai, A.G. Exact solutions via invariant approach for Black-Scholes model with time-dependent parameters. *Math. Methods Appl. Sci.* **2018**, *41*, 4417–4427. [CrossRef]
34. Naz, R.; Naeem, I. Exact Solutions of a Black-Scholes Model with Time-Dependent Parameters by Utilizing Potential Symmetries. *Discret. Contin. Dyn. Syst. Ser. S* **2020**, *1310*, 2841–2851. [CrossRef]
35. Li, Z.B.; He, J.H. Application of the fractional complex transform to fractional differential equations. *Nonlinear Sci. Lett. A* **2011**, *2*, 121–126.
36. Jamal, S. Imaging Noise Suppression: Fourth-Order Partial Differential Equations and Travelling Wave Solutions. *Mathematics* **2020**, *8*, 2019. [CrossRef]
37. Obaidullah, U.; Jamal, S. A computational procedure for exact solutions of Burgers' hierarchy of non-linear partial differential equations. *J. Appl. Math. Comput.* **2020**, *65*, 541. [CrossRef]
38. Olver, P.J. *Application of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1993.
39. Longstaff, F.A. Are negative option prices possible? The callable U.S. Treasury-Bond puzzle. *J. Bus.* **1992**, *65*, 571–592. [CrossRef]
40. Janowicz, M.; Zembrzuski, A. Symmetry properties of modified Black-Scholes equation. *Quant. Methods Econ.* **2021**, *XXII*, 77–86. [CrossRef]