Ain Shams Engineering Journal 15 (2024) 102381

Contents lists available at ScienceDirect

Ain Shams Engineering Journal

journal homepage: www.sciencedirect.com

Optical solitons and conservation laws for the concatenation model: Power–law nonlinearity☆

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ARTICLE INFO

Keywords:
solitons
Kudryashov
Power–law
Concatenation

ABSTRACT

By investigating the concatenation model that incorporates a power-law nonlinearity, this paper provides an in-depth analysis, and the soliton solutions are derived. By utilizing the extended tanh-function technique and the enhanced Kudryashov's scheme, bright and singular 1-solitons are yielded, as demonstrated in this paper. The conservation laws can be obtained through the multipliers approach, which subsequently exposes the conserved quantities.

1. Introduction

Optical soliton is one of the most captivating areas of Telecommunication Sciences that has caught the attention of a wide range of scientists and Engineers across the globe. The most prevalent model used to describe the dynamics of soliton propagation across intercontinental distances is the nonlinear Schrödinger's equation (NLSE), although there are other models as well. There are several additional models that have subsequently appeared which also describe the soliton propagation under special situation and circumstances. The concatenation model is an innovative approach that integrates various nonlinear Schrödinger equation (NLSE) variants, namely the Sasa-Satsuma equation (SSE) and the Lakshmanan-Porsezian-Daniel (LPD) model. This model has gained considerable attention in recent years, with its initial introduction in 2014 and a subsequent resurgence since 2022. Numerous research studies have been conducted, resulting in a wide range of findings and advancements [1–7]. One significant aspect of the concatenation model is the application of the Painlevé analysis. This mathematical technique allows researchers to investigate the integrability properties of the model, identify its singularities, and gain insights into the underlying dynamics. By employing the Painlevé analysis, researchers have been able to understand the behavior of the concatenation model in various scenarios [1]. Conservation laws have also been extensively studied within the context of the concatenation model. These laws provide fundamental insights into the model's symmetries and the conservation of physical quantities. By analyzing conservation laws, researchers can better comprehend the system's overall behavior and uncover hidden relationships [2]. Another notable area of exploration in the concatenation model is the application of the method of undetermined coefficients. This approach allows researchers to determine the specific forms of the model's solutions by solving a system of equations. By utilizing this method, researchers can obtain explicit expressions for

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https://doi.org/10.1016/j.asej.2023.102381
Received 3 May 2023; Received in revised form 26 June 2023; Accepted 3 July 2023
Available online 18 July 2023

soliton solutions and explore their properties [3]. Rogue waves, which are extreme and rare wave events, have been a subject of interest in the context of the concatenation model. Researchers have investigated the occurrence and characteristics of rogue waves within this framework, aiming to understand their formation mechanisms and predict their behavior. This research contributes to the broader field of nonlinear wave phenomena [4]. The trial equation approach has also been employed to study the concatenation model. This technique involves proposing a trial function that satisfies the model's equations and subsequently analyzing its properties. By utilizing the trial equation approach, researchers can gain insights into the existence and properties of different types of solutions, thereby expanding their understanding of the model's dynamics [5]. Furthermore, the concatenation model has been used to explore quiescent solitons with nonlinear chromatic dispersion (CD). Quiescent solitons refer to stable, localized structures that remain unchanged over time. By considering the effects of nonlinear chromatic dispersion, which describes the dependence of wave propagation on the frequency components, researchers can study the behavior of quiescent solitons under realistic conditions [6,7]. As a result, the concatenation model has attracted significant attention in recent years, leading to numerous investigations into its properties and applications. Through the study of Painlevé analysis, conservation laws, the method of undetermined coefficients, rogue waves, trial equation approach, and quiescent solitons with nonlinear chromatic dispersion, researchers have made notable progress in understanding and expanding the knowledge of this intriguing model [1–7].

Turning a new page. In the past, the concatenation model has been extensively studied in the context of the Kerr law of nonlinearity [1–7]. However, in this current work, a different approach is taken by investigating the model with a power-law of nonlinearity. This paper introduces a generalized version of the self-phase modulation (SPM) effect, expanding its application beyond the limitations of the Kerr law to encompass the power-law nonlinearity. The main objective of this study is to recover soliton solutions for the concatenation model incorporating the power-law of SPM. To achieve this, two integration approaches are employed: the improved extended tanh-function technique and the enhanced Kudryashov's approach. These integration techniques are utilized to solve the mathematical equations governing the concatenation model, allowing for the identification and analysis of soliton solutions. Through the application of both integration approaches, the model under consideration exhibits singular and bright solitons. These solitons represent localized waveforms that retain their shape and velocity during propagation, making them highly significant in various fields such as optics, fluid dynamics, and nonlinear physics. The existence and characterization of solitons in the power-law concatenated model contribute to a deeper understanding of its dynamics and potential applications. Furthermore, this paper derives the conservation laws associated with the model, offering a comprehensive overview of its underlying principles. Conservation laws play a crucial role in understanding the fundamental properties and symmetries of physical systems. By providing a detailed derivation of these laws, this study enhances the overall understanding of the power-law concatenation model. To present a thorough analysis, the subsequent sections of this paper delve into a detailed description of the procedures employed and the results obtained. These sections provide a comprehensive account of the research methodology, including the mathematical formulations, numerical techniques, and analytical findings. Through this comprehensive presentation, readers can gain a complete understanding of the insights and implications derived from the study of the power-law concatenated model with the SPM effect.

1.1. Governing equations

The concatenation model with power–law of SPM reads [8]:

$$\begin{aligned} & iq_t + aq_{xx} + b|q|^{2n}q \\ + \quad & c_1\begin{bmatrix} \sigma_1 q_{xxxx} + \sigma_2 (q_x)^2 q^* + \sigma_3 |q_x|^2 q \\ +\sigma_4 |q|^{2n} q_{xx} + \sigma_5 q^2 q_{xx}^* + \sigma_6 |q|^{2n+2} q \end{bmatrix} \\ + \quad & ic_2\left[\sigma_7 q_{xxx} + \sigma_8 |q|^{2n} q_x + \sigma_9 q^2 q_x^*\right] = 0. \end{aligned} \tag{1}$$

Here a complex-valued function $q(x,t)$ is employed in this study to describe the wave profile. t and x correspond to temporal and spatial coordinates, respectively. The parameter n comes from the power-law of SPM, and $i=\sqrt{-1}$. Here the coefficient of SPM is represented by b, and the coefficient of CD is represented by a. Then, the coefficient of c_1 is from the LPD model while the coefficient of c_2 is from SSE. Thus (1) is indeed a concatenation of NLSE, LPD equation and SSE that have been individually and extensively studied in the past. The remaining coefficients σ_j for $1\leqslant j\leqslant 9$ are constant parameters.

The concatenation model incorporates a power-law nonlinearity and is relevant to several physical phenomena described by nonlinear partial differential equations. Specifically, three important equations associated with this model are the NLSE, the SSE, and the LPD model. The power-law nonlinearity is a mathematical term that represents a nonlinear relationship between variables in a system. It is characterized by an exponent or power term in the equation that governs the system's behavior. The power-law nonlinearity can be found in various physical phenomena, such as the propagation of light in nonlinear optical media, the dynamics of Bose–Einstein condensates, and other systems exhibiting nonlinear responses. The NLSE is a significant equation that appears in the study of wave propagation in nonlinear media. It describes the evolution of complex-valued wave functions, often associated with the behavior of solitons or wave packets that maintain their shape during propagation. The NLSE incorporates a power-law nonlinearity to account for the nonlinear interactions between waves. It has wide applications in various fields, including nonlinear optics, plasma physics, and condensed matter physics. The SSE is another important equation that arises in the context of optical solitons. It was introduced to model the propagation of ultrashort optical pulses in a fiber optic communication system. The equation combines the effects of dispersion, self-phase modulation, and nonlinear gain or loss, and it exhibits soliton solutions. The power-law nonlinearity in the SSE enables the description of the nonlinear interactions and stability properties of solitons. The LPD model is a higher-order nonlinear equation that provides a mathematical description of nonlinear phenomena in physical systems. It includes a power-law nonlinearity, as well as dispersion and nonlinearity of higher orders. This equation has been used to study various physical systems, such as wave propagation in nonlinear lattices, Bose–Einstein condensates, and plasma physics. Its power-law nonlinearity term plays a crucial role in capturing the nonlinear dynamics and phenomena exhibited by these systems.

The concatenation model incorporates a power-law nonlinearity, which is a key feature in understanding and modeling the behavior of various physical phenomena. The NLSE, SSE, and LPD model are specific examples of equations that utilize this power-law nonlinearity to describe and study nonlinear effects in different physical systems.

2. Mathematical methodologies

This paper investigates a governing model, which is described below

$$G(u, u_x, u_t, u_{xt}, u_{xx}, \ldots) = 0. \tag{2}$$

The time and space variables are denoted by t and x, respectively, in the function $u = u(x,t)$ that represents a wave profile.

The wave profile

$$u(x,t) = U(\xi), \ \xi = \mu(x - \upsilon t), \tag{3}$$

condenses (2) to

$$P(U, -\mu\upsilon U', \mu U', \mu^2 U', \ldots) = 0. \tag{4}$$

The wave velocity, wave variable, and wave width are represented by υ, ξ, and μ, respectively.

2.1. Enhanced Kudryashov's procedure

The fundamental idea of the method can be summarized as [9,10]: Step–1: One can obtain the solution to the reduced model (4) as:

$$U(\xi) = \lambda_0 + \sum_{l=1}^{N} \sum_{i+j=l} \lambda_{ij} Q^i(\xi) R^j(\xi), \tag{5}$$

where

$$R'(\xi)^2 = R(\xi)^2(1 - \chi R(\xi)^2), \tag{6}$$

and

$$Q'(\xi) = Q(\xi)(\eta Q(\xi) - 1). \tag{7}$$

The constants λij $(i,j = 0,1,\ldots,N), \lambda_0, \chi$, and η are constants in (4), where N is determined using the balancing procedure.Step–2: Soliton waves can be expressed using Eqs. (6) and (7), as shown below

$$R(\xi) = \frac{4c}{4c^2 e^{\xi} + \chi e^{-\xi}}, \tag{8}$$

and

$$Q(\xi) = \frac{1}{\eta + de^{\xi}}, \tag{9}$$

where the values of c and d are considered as constants.Step–3: The constants needed for Eqs. (3)–(7) can be obtained by plugging Eqs. (5)–(7) into Eq. (4). Next, by substituting the obtained parametric restrictions, together with either Eq. (8) or Eq. (9), into Eq. (5), one can obtain bright, singular and dark solitons.

2.2. Improved extended tanh–function algorithm

According to references [2,3], the algorithm relies on the following central procedure:Step–1: Assuming that the solution to Eq. (4) can be expressed as follows, we proceed:

$$U(\xi) = a_0 + \sum_{i=1}^{N} \left(a_i \Phi^i + b_i \Phi^{-i}\right), \tag{10}$$

where Φ admits

$$\Phi'^2 = \alpha_0 + \alpha_1\Phi + \alpha_2\Phi^2 + \alpha_3\Phi^3 + \alpha_4\Phi^4. \tag{11}$$

A range of fundamental solutions can be obtained from this equation [2].Step–2: Eq. (4) can be utilized to balance the equation and calculate the positive integer N necessary for Eq. (10).Step–3: We can obtain exact solutions to (2) by substituting (10) and (11) into (4) and setting the resulting polynomial of Φ equal to zero. The system of algebraic equations becomes overdetermined, and its solution involves finding the values of a_0, v, k, b_i, and a_i $(i = 1, 2, \ldots)$.

3. Application to the concatenation model

The solution approach for addressing Eq. (1) involves selecting the following solution form:

$$\psi(x,t) = U(\xi)e^{i\phi(x,t)}, \tag{12}$$

where, we express the wave variable ξ as

$$\xi = k(x - vt). \tag{13}$$

Here, the soliton consists of a phase part $\phi(x,t)$ and an amplitude part $U(\xi)$. v represents the solitons' velocity, and the phase part $\phi(x,t)$ reads as

$$\phi(x,t) = -\kappa x + \omega t + \theta_0, \tag{14}$$

where the wave number is denoted by ω, while κ represents the frequency of the solitons, and θ_0 is the phase constant. Substitution of (12) into (1) and decomposition into imaginary and real components results in:

$$\begin{gathered} k^2\left(a - 6c_1\kappa^2\sigma_1 + 3c_2\kappa\sigma_7\right)U'' + \left(-a\kappa^2 + c_1\kappa^4\sigma_1 - c_2\kappa^3\sigma_7 - \omega\right)U \\ + \left(b - c_1\kappa^2\sigma_4 + c_2\kappa\sigma_8\right)U^{2n+1} + c_1k^4\sigma_1 U'''' + c_1k^2\sigma_4 U^{2n}U'' + c_1k^2\sigma_5 U^2 U'' \\ + c_1k^2(\sigma_2 + \sigma_3)UU'^2 + c_1\sigma_6 U^{2n+3} - \kappa(c_1\kappa(\sigma_2 - \sigma_3 + \sigma_5) + c_2\sigma_9)U^3 = 0, \end{gathered} \tag{15}$$

and

$$\begin{gathered} -k\left(2a\kappa - 4c_1\kappa^3\sigma_1 + 3c_2\kappa^2\sigma_7 + v\right)U' + k^3(c_2\sigma_7 - 4c_1\kappa\sigma_1)U''' \\ + (c_2k\sigma_8 - 2c_1\kappa k\sigma_4)U^{2n}U' + k(2c_1\kappa(\sigma_5 - \sigma_2) + c_2\sigma_9)U^2U' = 0. \end{gathered} \tag{16}$$

The soliton speed can be calculated by considering the imaginary part of the equation, which is given by

$$v = -2a\kappa + 4c_1\kappa^3\sigma_1 - 3c_2\kappa^2\sigma_7, \tag{17}$$

with the following parametric restrictions

$$\begin{aligned} &2c_1\kappa(\sigma_5 - \sigma_2) + c_2\sigma_9 = 0, \\ &c_2\sigma_8 - 2c_1\kappa\sigma_4 = 0, \\ &c_2\sigma_7 - 4c_1\kappa\sigma_1 = 0. \end{aligned} \tag{18}$$

Eq. (15) can be indicated as

$$\begin{gathered} k^2U'''' + L_6U^{2n}U'' + L_7U^{2n+1} + L_8U^{2n+3} + L_4U^2U'' + L_3U'' + L_2UU'^2 + L_5U^3 \\ + L_1U = 0, \end{gathered} \tag{19}$$

with

$$\begin{gathered} L_1 = \frac{-a\kappa^2 + c_1\kappa^4\sigma_1 - c_2\kappa^3\sigma_7 - \omega}{c_1k^2\sigma_1}, L_2 = \frac{\sigma_2 + \sigma_3}{\sigma_1}, L_3 = \frac{a - 6c_1\kappa^2\sigma_1 + 3c_2\kappa\sigma_7}{c_1\sigma_1}, \\ L_4 = \frac{\sigma_5}{\sigma_1}, L_5 = -\frac{\kappa(c_1\kappa(\sigma_2 - \sigma_3 + \sigma_5) + c_2\sigma_9)}{c_1k^2\sigma_1}, L_6 = \frac{\sigma_4}{\sigma_1}, \\ L_7 = \frac{b - c_1\kappa^2\sigma_4 + c_2\kappa\sigma_8}{c_1k^2\sigma_1}, L_8 = \frac{\sigma_6}{k^2\sigma_1}. \end{gathered} \tag{20}$$

Using the transformation

$$U = V^{\frac{2}{n}},$$

Eq. (19) collapses to

$$\begin{aligned}&2k^2n^3V''''V^3-6k^2(n-2)n^2V^2V''^2-8k^2(n-2)n^2V'''V^2V'\\&-4k^2(n-2)(n-1)(3n-2)V'^4+24k^2(n-2)(n-1)nVV'^2V''+L_5n^4V^{\frac{4}{n}+4}\\&+L_8n^4V^{\frac{4}{n}+8}+L_7n^4V^8+L_1n^4V^4+2L_4n^3V^{\frac{4}{n}+3}V''+2L_6n^3V^7V''+2L_3n^3V^3V''\\&+2n^2(2L_2-L_4(n-2))V^{\frac{4}{n}+2}V'^2-2L_6(n-2)n^2V^6V'^2-2L_3(n-2)n^2V^2V'^2=0.\end{aligned}\tag{21}$$

For integrability, we consider

$$L_2=L_4=L_5=L_8=0.\tag{22}$$

This leads to

$$\sigma_6=\sigma_5=0,\sigma_2=-\sigma_3,c_2\sigma_9-2c_1\kappa\sigma_3=0.\tag{23}$$

Then Eq. (1) reaches

$$\begin{aligned}&iq_t+aq_{xx}+b|q|^{2n}q+c_1\left[\begin{array}{c}\sigma_1q_{xxxx}+\sigma_3\left(|q_x|^2q-(q_x)^2q^*\right)\\+\sigma_4|q|^{2n}q_{xx}\end{array}\right]\\+&\qquad ic_2\left[\sigma_7q_{xxx}+\sigma_8|q|^{2n}q_x+\sigma_9q^2q_x^*\right]=0,\end{aligned}\tag{24}$$

with the restriction paraments (18)

$$\begin{aligned}&2c_1\kappa\sigma_3+c_2\sigma_9=0,\\&c_2\sigma_8-2c_1\kappa\sigma_4=0,\\&c_2\sigma_7-4c_1\kappa\sigma_1=0.\end{aligned}\tag{25}$$

The last equation in (23) and the first equation in (25) lead to $\sigma_3=\sigma_9=0$. Then Eq. (24) collapses to

$$iq_t+aq_{xx}+b|q|^{2n}q+c_1\left[\sigma_1q_{xxxx}+\sigma_4|q|^{2n}q_{xx}\right]+ic_2\left[\sigma_7q_{xxx}+\sigma_8|q|^{2n}q_x\right]=0.\tag{26}$$

In this case, Eq. (19) becomes

$$k^2U''''+L_6U^{2n}U''+L_7U^{2n+1}+L_3U''+L_1U=0,\tag{27}$$

with

$$\begin{aligned}&L_1=\frac{-a\kappa^2+c_1\kappa^4\sigma_1-c_2\kappa^3\sigma_7-\omega}{c_1k^2\sigma_1},L_3=\frac{a-6c_1\kappa^2\sigma_1+3c_2\kappa\sigma_7}{c_1\sigma_1},\\&L_6=\frac{\sigma_4}{\sigma_1},\ L_7=\frac{b-c_1\kappa^2\sigma_4+c_2\kappa\sigma_8}{c_1k^2\sigma_1},\end{aligned}\tag{28}$$

and Eq. (21) reads

$$\begin{aligned}&2k^2n^3V''''V^3-6k^2(n-2)n^2V^2V''^2-8k^2(n-2)n^2V'''V^2V'\\&-4k^2(n-2)(n-1)(3n-2)V'^4+24k^2(n-2)(n-1)nVV'^2V''+L_7n^4V^8+L_1n^4V^4\\&+2L_6n^3V^7V''+2L_3n^3V^3V''-2L_6(n-2)n^2V^6V'^2-2L_3(n-2)n^2V^2V'^2=0.\end{aligned}\tag{29}$$

3.1. Enhanced Kudryashov's scheme

By applying the balancing principle between V^3V'''' and V^8 in Eq. (29), we obtain $N=1$, which can be used to express the solution to the equation as:

$$V(\xi)=\lambda_0+\lambda_{01}R(\xi)+\lambda_{10}Q(\xi).\tag{30}$$

Eq. (30) is combined with (6) and (7), and then the resulting expressions are plugged into Eq. (29), leading to the following results:

$$\begin{aligned}&\lambda_0=\lambda_{10}=0,\lambda_{01}=\pm\sqrt[4]{\frac{L_3(3n^3+11n^2+12n+4)\chi^2}{L_7n^2(n^2+2n+2)}},k=\frac{1}{2}\sqrt{-\frac{L_3n^2}{n^2+2n+2}},\\&L_1=-\frac{4L_3(n+1)^2}{n^2(n^2+2n+2)},L_6=0.\end{aligned}\tag{31}$$

The solutions of Eq. (26) can be determined by plugging the derived parameters into the corresponding expressions, resulting in

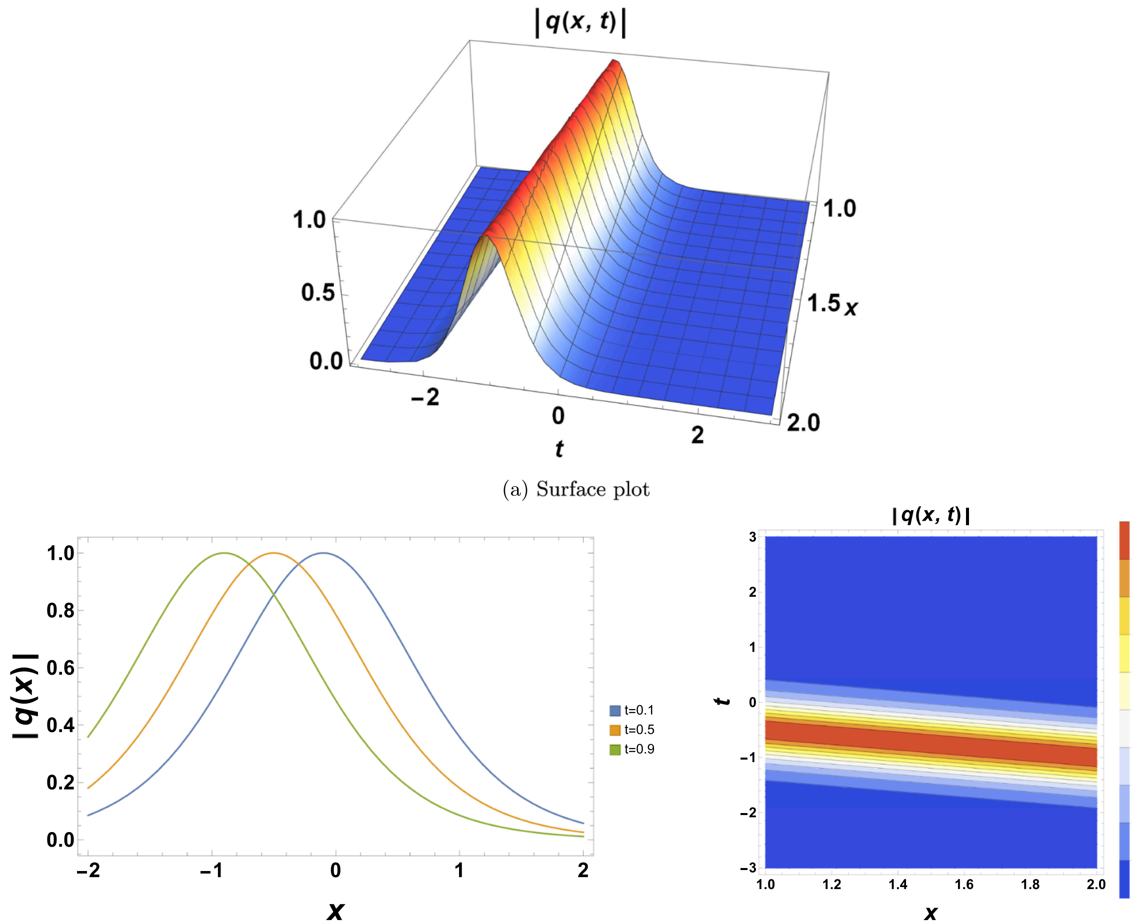
$$q(x,t)=\left\{\frac{\pm4c\sqrt[4]{\frac{L_3(3n^3+11n^2+12n+4)\chi^2}{L_7n^2(n^2+2n+2)}}}{\left(\begin{array}{c}4c^2exp\left[\frac{1}{2}\sqrt{-\frac{L_3n^2}{n^2+2n+2}}(x-vt)\right]\\+\chi exp\left[-\frac{1}{2}\sqrt{-\frac{L_3n^2}{n^2+2n+2}}(x-vt)\right]\end{array}\right)}\right\}^{\frac{2}{n}}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{32}$$

Solution (32) gives rise to a bright soliton if $\chi=\pm4c^2,L_3<0$, and $L_7<0$, and a singular soliton if $\chi=\pm4c^2,L_3<0$ and $L_7>0$, as described below

$$q(x,t)=\left\{\begin{array}{c}\pm\sqrt[4]{\frac{L_3(3n^3+11n^2+12n+4)}{L_7n^2(n^2+2n+2)}}\\\times\text{sech}\left[\frac{1}{2}\sqrt{-\frac{L_3n^2}{n^2+2n+2}}(x-vt)\right]\end{array}\right\}^{\frac{2}{n}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{33}$$

$$q(x,t)=\left\{\begin{array}{c}\pm\sqrt[4]{-\frac{L_3(3n^3+11n^2+12n+4)}{L_7n^2(n^2+2n+2)}}\\\times\text{csch}\left[\frac{1}{2}\sqrt{-\frac{L_3n^2}{n^2+2n+2}}(x-vt)\right]\end{array}\right\}^{\frac{2}{n}}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{34}$$

The method employed is ineffective in deriving dark soliton solutions for the model. Also, in this instance, the condition $\sigma_4=0$ causes σ_8 to be zero.



(a) Surface plot

(b) 2D plots moving in time

(c) Contour plot

Fig. 1. Profile of a bright soliton solution (33).

In Fig. 1, we observe a series of plots representing the bright soliton solution with the governing model (26). These graphical depictions provide a visual understanding of the waveform characteristics and valuable insights into the behavior of the pulses under investigation. The parameter values chosen for these plots are as follows: $n = 0.5, a = 1, \kappa = 1, c_1 = 1, c_2 = 1, \sigma_1 = 1, \sigma_7 = 1, k = 1, b = 1, \sigma_8 = 1$ and $\sigma_4 = 3$.

3.2. Improved extended tanh–function technique

By balancing V^3V''' and V^8 in Eq. (29), we arrive at $N = 1$, which can be used to express the solution to the equation as:

$$V(\xi) = a_0 + a_1\Phi(\xi) + \frac{b_1}{\Phi(\xi)}. \tag{35}$$

By plugging Eq. (35) and Eq. (11) into Eq. (29), we can derive a polynomial that contains $\Phi(\xi)$ and $\frac{1}{\Phi(\xi)}$. To obtain the solutions, we need to address the system of algebraic equations that comes from setting all coefficients of the polynomial to zero. This can be done using Mathematica, as described below

Case 1:. $\alpha_0 = \alpha_1 = \alpha_3 = 0$.

$$\alpha_2 = -\frac{L_1 n^2\left(n^2+2n+2\right)}{4L_3(n+1)^2},\ \alpha_4 = \frac{a_1^2\sqrt{L_1L_7}n^2\left(n^2+2n+2\right)}{2\sqrt{-(n+1)^3L_3^2\left(3n^2+8n+4\right)}},$$
$$a_0 = b_1 = L_6 = 0,\ k = \pm\frac{L_3(n+1)}{\sqrt{L_1\left(n^2+2n+2\right)^2}}. \tag{36}$$

Eq. (26) has a soliton solution that takes the following form under this condition:

$$q\left(x,t\right) = \left\{ \pm\sqrt[4]{\frac{L_1(n+2)(3n+2)}{4L_7(n+1)}} \times\operatorname{sech}\left[\frac{1}{2}\sqrt{-\frac{L_3n^2}{n^2+2n+2}}\left(x-vt\right)\right]\right\}^{\frac{2}{n}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{37}$$

$$q\left(x,t\right) = \left\{ \pm\sqrt[4]{-\frac{L_1(n+2)(3n+2)}{4L_7(n+1)}} \times\operatorname{csch}\left[\frac{1}{2}\sqrt{-\frac{L_3n^2}{n^2+2n+2}}\left(x-vt\right)\right]\right\}^{\frac{2}{n}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{38}$$

When $L_1L_7 > 0$ and $L_3 < 0$, the solitons are bright, whereas they become singular when $L_1L_7 < 0$ and $L_3 < 0$.

The plots presented in Fig. 2 offer a visual depiction of the singular soliton solution with the governing model (26). These visualizations provide valuable insights into the behavior of the investigated pulses by illustrating their waveform characteristics. The parameter values used for these plots are as follows: $n = 0.5, a = 1, \kappa = 1, c_1 = 1, c_2 = 1, \sigma_1 = 1, \sigma_7 = 1, k = 1, b = 1, \sigma_8 = 1, \sigma_4 = 3, \omega = 1$ and $\sigma_7 = -2$.

Case 2:. $\alpha_1 = \alpha_3 = 0, \alpha_0 = \frac{\alpha_2^2}{4\alpha_4}$.

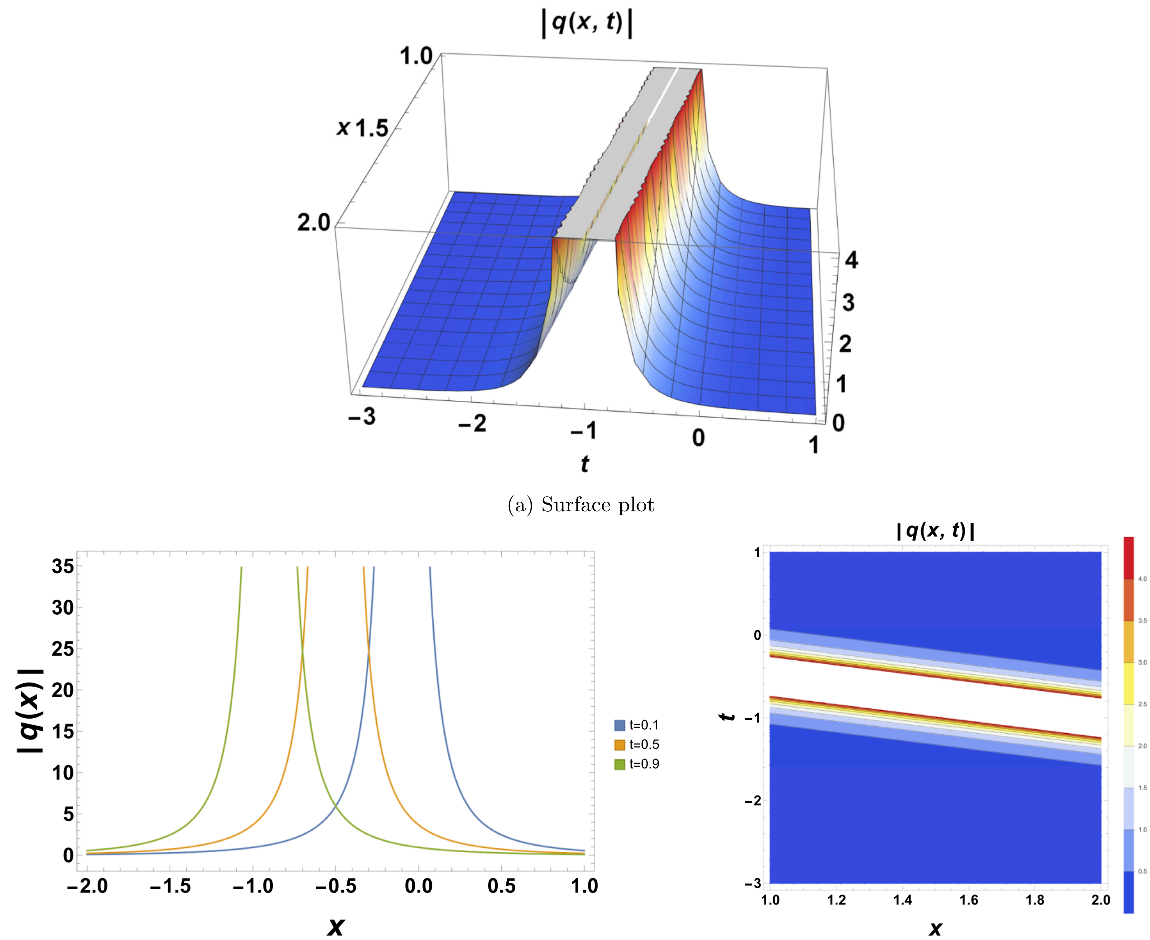


Fig. 2. Profile of a singular soliton solution (38).

$$\alpha_2 = \frac{L_1 n^2\left(n^2+2n+2\right)}{8L_3(n+1)^2},\ \alpha_4 = \frac{a_1 L_1 n^2(n(n+2)+2)}{16 b_1 L_3 (n+1)^2},\ a_0 = L_6 = 0,$$
$$a_1 = \frac{1}{8b_1}\sqrt{-\frac{L_1\left(3n^2+8n+4\right)}{L_7(n+1)}},\ k = \pm\sqrt{\frac{L_3^2(n+1)^2}{L_1(n(n+2)+2)^2}}. \tag{39}$$

This case of Eq. (26) can be solved by expressing its solution as:

$$q\left(x,t\right) = \left\{ \begin{array}{c} \pm\sqrt[4]{-\frac{L_1(n+2)(3n+2)}{4L_7(n+1)}} \\ \times\coth\left[\frac{1}{2}\sqrt{-\frac{L_3 n^2}{n(n+2)+2}}\left(x-vt\right)\right] \end{array} \right\}^{\frac{2}{n}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{40}$$

$$q\left(x,t\right) = \left\{ \begin{array}{c} \mp\sqrt[4]{-\frac{L_1(n+2)(3n+2)}{4L_7(n+1)}} \\ \times\operatorname{csch}\left[\frac{1}{2}\sqrt{-\frac{L_3 n^2}{n(n+2)+2}}\left(x-vt\right)\right] \end{array} \right\}^{\frac{2}{n}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{41}$$

Here the solitons become singular when $L_3 < 0$ and $L_1L_7 < 0$.

Case 3:. $\alpha_1 = \alpha_3 = \alpha_4 = 0$.

$$\alpha_0 = \frac{b_1^2\sqrt{L_1L_7}\,n^2\left(n^2+2n+2\right)}{2\sqrt{L_3^2\left(-(n+1)^3\right)}\left(3n^2+8n+4\right)},\ \alpha_2 = -\frac{L_1 n^2\left(n^2+2n+2\right)}{4L_3(n+1)^2},$$
$$a_0 = L_6 = a_1 = 0,\ k = \pm\frac{L_3(n+1)}{\sqrt{L_1\left(n^2+2n+2\right)^2}}. \tag{42}$$

The solution to Eq. (26) for this particular case is indicated below

$$q\left(x,t\right) = \left\{ \begin{array}{c} \mp\sqrt[4]{-\frac{L_1(n+2)(3n+2)}{4L_7(n+1)}} \\ \times\operatorname{csch}\left[\frac{1}{2}\sqrt{-\frac{L_3 n^2}{n(n+2)+2}}\left(x-vt\right)\right] \end{array} \right\}^{\frac{2}{n}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{43}$$

The solitons exhibit singular when $L_3 < 0$ and $L_1L_7 < 0$, as observed here.

4. Conservation laws

For Eq. (26), we present the conserved densities using the ‘multiplier’ method. First, however, we write the equation as a system obtained by letting $q = u + iv$ and splitting the expression into its imaginary and real components and study the multipliers of the system.

It turns out that conservation laws only exist if

$$\sigma_4 = 0. \tag{44}$$

Thus, the reduced concatenation model (26) further condenses to

$$iq_t + aq_{xx} + b|q|^{2n}q + c_1\sigma_1 q_{xxxx} + ic_2\left[\sigma_7 q_{xxx} + \sigma_8 |q|^{2n} q_x\right] = 0, \tag{45}$$

for its conservation laws.

1. The **power** (P) conserved density can be obtained based on the multiplier $Q = (-u, v)$, as indicated below

$$T^t_P = \frac{1}{2}(u^2 + v^2), \tag{46}$$

and a conserved density for Eq. (45) can be derived:

$$\Phi^t_P = |q|^2. \tag{47}$$

2. From the multiplier $Q = (v_x, u_x)$, the **linear momentum** (M) density is

$$T^t_M = -\frac{1}{2}u_x v + \frac{1}{2}v_x u, \tag{48}$$

and Eq. (45) gives us the **linear momentum density**, which can be expressed as

$$\Phi^t_M = \mathscr{I}(q^* q_x). \tag{49}$$

3. For the multiplier $Q = (v_t, u_t)$, the **Hamiltonian** (H) density on (45) is

$$\begin{aligned}\Phi^t_H = &\frac{1}{2(n+1)}\Big\{(n+1)a\Re\left(qq^*_{xx}\right) + b|q|^{2(n+1)} + \left(n+1\right)c_1\sigma_1 \\ &\times\Re\left(qq^*_{xxxx}\right) + c_2\Big[\left(n+1\right)\sigma_7\Im\left(q^* q_{xxx}\right) - \sigma_8\left|q\right|^{2n}\Im\left(q^* q_x\right)\Big]\Big\}.\end{aligned} \tag{50}$$

The bright 1-soliton solution can be derived from Eq. (33):

$$q(x,t) = A\,\text{sech}^{\frac{2}{n}}[B(x - vt)]e^{i(-\kappa x + \omega t + \theta_0)}, \tag{51}$$

where we can refer to the soliton's inverse width as B, while its amplitude is represented by A. The conserved quantities from the conserved densities evolve as:

$$P = \int_{-\infty}^{\infty}|q|^2 dx = \frac{A^2}{B}\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n} + \frac{1}{2}\right)}, \tag{52}$$

$$M = ia\int_{-\infty}^{\infty}\left(q^* q_x - qq^*_x\right)dx = a\frac{\kappa A^2}{B}\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n} + \frac{1}{2}\right)}, \tag{53}$$

and

$$\begin{aligned}H = &\int_{-\infty}^{\infty}\Phi^t_H dx = \Bigg[-\frac{aA^2}{2n(n+4)B}\left\{4B^2 + n\left(n+4\right)\kappa^2\right\} + \frac{4\left(n+2\right)bA^{2n+2}}{(n+1)(n+4)(3n+4)B} \\ &+ \frac{c_1\sigma_1 A^2}{2n^2\left(n+4\right)\left(3n+4\right)B}\left\{\begin{array}{c}16\left(2n+3\right)B^4 + 24n\left(3n+4\right)\kappa^2 B^2 \\ +n^2\left(n+4\right)\left(3n+4\right)\kappa^4\end{array}\right\} \\ &+ \frac{c_2\sigma_7\kappa A^2}{2n(n+4)B}\left\{12B^2 + n\left(n+4\right)\kappa^2\right\} + \frac{4\left(n+2\right)c_2\sigma_8\kappa A^{2n+2}}{(n+1)(n+4)(3n+4)B}\Bigg]\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n}+\frac{1}{2}\right)}.\end{aligned} \tag{54}$$

5. Conclusions

The concatenation model, expressed using power-law nonlinearity, has been shown to possess optical soliton solutions in this paper. Two integration approaches collectively recovered singular and bright 1–soliton solutions. They are the improved version of the extended tanh–function technique and the enhanced Kudryashov's scheme. Finally, conservation laws for the model are obtained through the use of the multipliers mechanism. The concatenation model for the power–law nonlinearity was drastically reduced to permit the integrability of the scheme. Subsequently, the retrieval of the conservation laws became possible after further dropping a term from the reduced model.

In previous research [1–7], the concatenation model, which involves combining different nonlinear effects in a transmission system, was extensively investigated in conjunction with the Kerr law of nonlinearity. However, our present work adopts a novel approach by examining the model with a power-law of nonlinearity. This paper introduces an expanded and generalized version of the SPM effect, extending its application beyond the traditional Kerr law to encompass the power-law regime. The primary objective of our study is to explore the utilization of the power-law of nonlinearity within the concatenation model. By incorporating this generalized form of SPM, we aim to enhance our understanding of the behavior of optical soliton solutions. Solitons are self-sustained waves that can propagate over long distances without distortion, making them crucial for reliable communication in optical fiber systems. To accomplish our goals, we employ two distinct integration approaches. These techniques enable us to recover soliton solutions for the concatenation model, which now includes the power-law of SPM. Through these integrations, we obtain valuable insights into the dynamics of the system and explore the effects of different parameters on the soliton propagation. By expanding the scope of the concatenation model to incorporate the power-law of nonlinearity, we contribute to the understanding of nonlinear optical phenomena and provide a more comprehensive framework for analyzing soliton-based communication systems. Our findings shed light on the potential advantages and challenges associated with power-law nonlinearity in optical fiber transmission, offering new opportunities for designing and optimizing future optical communication networks.

The recovery of bright soliton solutions gives way to additional avenues to study the model further along. The possibility of quiescent soliton formation could be addressed in future research by exploring the effects of fractional temporal evolution on the model and incorporating nonlinear chromatic dispersion. The quasimionochromatic dynamics of the perturbed solitons needs to be established. This is possible with the usage of soliton perturbation theory. The variational principle needs to be implemented to address the dynamical system of the soliton parameters in addition to the application of collective variables approach and moment method. All of these promissory activities will be published after being aligned with the existing literature [11–27]. Thus a lot lies ahead to consider.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Ankiewicz A, Akhmediev N. Higher–order integrable evolution equation and its soliton solutions. Phys Lett A 2014;378:358–61.
[2] Ankiewicz A, Wang Y, Wabnitz S, Akhmediev N. Extended nonlinear Schrödinger equation with higher–order odd and even terms and its rogue wave solutions. Phys Rev E 2014;89:012907.
[3] Biswas A, Vega-Guzman J, Kara AH, Khan S, Triki H, Gonzalez-Gaxiola O, Moraru L, Georgescu PL. Optical solitons and conservation laws for the concatenation model: undetermined coefficients and multipliers approach. Universe 2023;9(1). Article 15.
[4] Triki H, Sun Y, Zhou Q, Biswas A, Yildirim Y, Alshehri HM. Dark solitary pulses and moving fronts in an optical medium with the higher–order dispersive and nonlinear effects. Chaos, Solitons & Fractals. 2022;164:112622.
[5] Wang M-Y, Biswas A, Yıldırım Y, Moraru L, Moldovanu S, Alshehri HM. Optical solitons for a concatenation model by trial equation approach. Electronics 2023;12 (1). Article 19.
[6] Yildirim Y, Biswas A, Moraru L, Alghamdi AA. Quiescent optical solitons for the concatenation model with nonlinear chromatc dispersion. Mathematics 2023;11 (7). Article 1709.
[7] Kudryashov NA, Biswas A, Borodina AG, Yildirim Y, Alshehri HM. Painleve analysis and optical solitons for a concatenated model. Optik. 2023;272:170255.
[8] Biswas A, Vega-Guzman JM, Yildirim Y, Moshokoa SP, Aphane M, Alghamdi AA. Optical solitons for the concatenation model with power-law nonlinearity: undetermined coefficients. Ukrain J Phys Opt 2023;24(3):185–92.
[9] Arnous AH, Zhou Q, Biswas A, Guggilla P, Khan S, Yıldırım Y, Alshomrani AS, Alshehri HM. Optical solitons in fiber Bragg gratings with cubic–quartic dispersive reflectivity by enhanced Kudryashov's approach. Phys Lett A 2022;422:127797.
[10] Arnous AH, Biswas A, Yıldırım Y, Zhou Q, Liu W, Alshomrani AS, Alshehri HM. Cubic–quartic optical soliton perturbation with complex Ginzburg-Landau equation by the enhanced Kudryashov's method. Chaos, Solitons & Fractals 2022; 155:111748.
[11] Secer A. Stochastic optical solitons with multiplicative white noise via Itô calculus. Optik. 2022;268:169831.
[12] Wang H, Zhou Q, Liu W. Exact analysis and elastic interaction of multi–soliton for a two–dimensional Gross-Pitaevskii equation in the Bose-Einstein condensation. J Adv Res 2022;38:179–90.
[13] Wazwaz A-M. Bright and dark optical solitons for (3+1)–dimensional Schrödinger equation with cubic–quintic–septic nonlinearities. Optik. 2021;225:165752.
[14] Wazwaz A-M. Bright and dark optical solitons of the (2+1)–dimensional perturbed nonlinear Schrödinger equation in nonlinear optical fibers. Optik. 2022;251: 168334.
[15] Yang Z, Hon BYC. An improved modified extended tanh–function method. Zeitschrift für Naturforschung A. 2006;61:103–15.
[16] E.M.E. Zayed, M. El–Horbaty, M.E.M. Alngar & M. El–Shater. "Dispersive optical solitons for stochastic Fokas–Lenells equation with multiplicative white noise. Eng. Volume 3, Issue 4, 523–540. (2022).
[17] Zhou Q. Influence of parameters of optical fibers on optical soliton interactions. Chin Phys Lett 2022;39(1):010501.
[18] Zhou Q, Sun Y, Triki H, Zhong Y, Zeng Z, Mirzazadeh M. Study on propagation properties of one–soliton in a multimode fiber with higher–order effects. Results Phys 2022;41:105898.
[19] Eslami M, Rezazadeh H. The first integral method for Wu–Zhang system with conformable time-fractional derivative. Calcolo 2016;53:475–85.
[20] Neirameh A, Eslami M. New optical soliton of stochastic chiral nonlinear Schrödinger equation. Opt Quant Electron 2023;55(5):444.
[21] Asghari Y, Eslami M, Rezazadeh H. Exact solutions to the conformable time-fractional discretized mKdv lattice system using the fractional transformation method. Opt Quant Electron 2023;55(4):318.
[22] Asghari Y, Eslami M, Rezazadeh H. Soliton solutions for the time-fractional nonlinear differential-difference equation with conformable derivatives in the ferroelectric materials. Opt Quant Electron 2023;55(4):289.
[23] Biswas A, Mirzazadeh M, Eslami M. Dispersive dark optical soliton with Schödinger-Hirota equation by G'/G-expansion approach in power law medium. Optik 2014;125(16):4215–8.
[24] Yaşar E. New travelling wave solutions to the Ostrovsky equation. Appl Math Comput 2010;216(11):3191–4.
[25] Çankal PD, Yaşar E. Optical soliton solutions to a (2+ 1) dimensional Schrödinger equation using a couple of integration architectures. Appl Math Nonlinear Sci 2021;6(1):381–96.
[26] Yaşar E, Yıldırım Y, Yaşar E. New optical solitons of space-time conformable fractional perturbed Gerdjikov-Ivanov equation by sine-Gordon equation method. Results Phys 2018;9:1666–72.
[27] Yaşar E, Yıldırım Y, Adem AR. Perturbed optical solitons with spatio-temporal dispersion in (2+ 1)-dimensions by extended Kudryashov method. Optik 2018;158: 1–14.

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