Brane States and Group Representation Theory

Nkululeko Nokwara

A thesis submitted to the Faculty of Science, University of the Witwatersrand,
in fulfilment of the requirements for the degree of Doctor of Philosophy.

3rd October 2013

Declaration

I declare that this thesis is my own, unaided work. Chapters 3 and 4 as well as Appendix C are my original work. They are based on the following papers that I published:

1. Robert de Mello Koch, Jeff Murugan and Nkululeko Nokwara, *"Large N Anomalous Dimensions for Large Operators in Leigh-Strassler Deformed SYM",* Phys.Lett., **B721** (2013) 164-170 [arxiv:1212.6624].

2. Robert de Mello Koch, Pablo Diaz, and Nkululeko Nokwara, *"Restricted Schur Polynomials for Fermions and Integrability in the su(2|3) Sector",* JHEP, **1303** (2013) 173 [arXiv:1212.5935].

In the text, these are references [48] and [38], respectively. The thesis is being submitted for the degree of Doctor of Philosophy in the University of Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

NNokwara.
____

3^{rd} day of October, 2013.

Abstract

A complete understanding of quantum gravity remains an open problem. However, the AdS/CFT correspondence which relates quantum field theories that enjoy conformal symmetry to theories of (quantum) gravity is proving to be a useful tool in shedding light on this formidable problem. Recently developed group representation theoretic methods have proved useful in understanding the large N, but non-planar limit of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. In this work, we study operators that are dual to excited giant gravitons, which corresponds to a sector of $\mathcal{N}=4$ super Yang-Mills theory that is described by a large N, but non-planar limit. After a brief review of the work done in the $su(2)$ sector, we compute the spectrum of anomalous dimensions in the $su(2)$ sector of the Leigh-Strassler deformed theory. The result resembles the spectrum of a shifted harmonic oscillator. We then explain how to construct restricted Schur polynomials built using both fermionic and bosonic fields which transform in the adjoint of the gauge group $U(N)$. We show that these operators diagonalise the free field two point function to all orders in $1/N$. As an application of our new operators, we study the action of the one-loop dilatation operator in the $su(2|3)$ sector in a large N, but non-planar limit of $\mathcal{N}=4$ super Yang-Mills theory. As in the $su(2)$ case, the resulting spectrum matches the spectrum of a set of decoupled oscillators. Finally, in an appendix, we study the action of the one-loop dilatation operator in an $sl(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory. Again, the resulting spectrum matches that of a set of harmonic oscillators. In all these cases, we find that the action of the dilatation operator is diagonalised by a double coset ansatz.

Dedication

To my father and the lady who always stood beside him, my mother.

Acknowledgements

The bulk of my gratitude goes to my supervisor, Professor Robert de Mello Koch for tirelessly believing in me while we saw this work through. I also would like to thank Professors D. P. Joubert, J. Carter, J. A. P. Rodrigues and J. E. Lowther to whom I am greatly indebted in various ways. Many thanks go to my colleagues for many fruitful discussions, most notably Garreth James Kemp, Badr Awad Elseid Mohammed, Stephanie Smith, Pablo Diaz, Hesam Soltanpanahi, Pawel Caputa and Dimitrios Giataganas. Special gratitude goes to my friend and colleague Maxwell Vhareta for lending me computing power at a time of need. Last, but definitely not the least, I would like to warmly thank my family and community of friends who bore the greatest social implications of doing this work and yet remained my greatest cheer leaders. This work was made possible by funding from the National Research Foundation, the Centre for Theoretical Physics at the University of the Witwatersrand as well as the J. C. Carstens Trust. Thank you all.

Contents

List of Figures

Chapter 1

Introduction

Quantum gravity, a quantum theory incorporating gravity, remains an open problem today. A leading candidate for such a theory is string theory which was originally developed to describe the strong nuclear force. A string theory description of the strong interaction was abandoned for two reasons: quantum chromodynamics was discovered and immediately received experimental support in deep inelastic scattering, and string theory was found to contain gravitons. Today, string theory is our best hope for unifying general relativity and quantum mechanics.

Unfortunately, because of the limitations of perturbative analyses, we are currently unable to understand the strong coupling limit of string theory directly. The same is true for strongly coupled gauge theories. Fortunately, the conjectured AdS/CFT correspondence [1, 2, 3], described more fully in Section 1.3, is teaching us that the two problems are actually complementary. More concretely, the AdS/CFT correspondence claims an exact equivalence between string theory and gauge theory. The correspondence relates the strongly coupled gauge theory to string theory on a weakly curved background and vice-versa. This way, we can still use tools appropriate for weak coupling on one side to learn non-trivial lessons about the other side.

In string theory, open strings start and end on p-dimensional objects called Dp-branes whose low-energy world volume dynamics is given by supersymmetric versions of Yang-Mills theories. For type IIB string theory, the open strings start and end on $D3$-branes whose low energy world volume theory is $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, a superconformal field theory. It was through studies of the near horizon limit of $D3$ branes that the AdS/CFT duality between $\mathcal{N} = 4$ super Yang-Mills theory and type IIB string theory on asymptotically $AdS_5 \times S^5$ spacetime was discovered [1]. This remains the simplest and most completely understood example of a gauge/gravity duality.

There are other examples of this correspondence, including the one involving ABJM theory [4], a $(2 + 1)$-dimensional supersymmetric Chern-Simons-matter theory. This theory has attracted considerable interest for two reasons. First, it is a candidate for the worldvolume dynamics of M-theory two branes and second, it has the potential to teach us something about the strong coupling limit of certain condensed matter systems. In this work, we will discuss both of these examples of the gauge/gravity duality.

1.1 Conformal symmetry

The quantum field theories that enter the AdS/CFT correspondence enjoy conformal symmetry. Here, we give a short description of this symmetry. For a more detailed study, [5], Chapter 3 of [6] and Chapter 3 of [7] plus references therein are recommended.

The conformal group is bigger than the Poincare group. The most general element of the conformal group can be obtained by composing

i) translations, generated by

$$P_\mu = -i\partial_\mu, \tag{1.1}$$

ii) rotations and boosts, generated by

$$L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right), \tag{1.2}$$

iii) scale transformations (or dilations), generated by

$$D = -ix^\mu\partial_\mu \tag{1.3}$$

and

iv) special conformal transformations generated by

$$K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right). \tag{1.4}$$

On a conformally flat[1] D-dimensional manifold, there are

$$\frac{1}{2}\left(D+2\right)\left(D+1\right) \tag{1.5}$$

linearly independent infinitesimal conformal transformations. The Lie algebra closed by these generators is $so\left(D,2\right)$ in Lorentzian signature, and $so\left(D+1,1\right)$ in Euclidean. Introduce

$$J_{-1,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right), \tag{1.6}$$

$$J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right), \tag{1.7}$$

[1] A D-dimensional manifold is said to be conformally flat if its metric is proportional to the flat metric, i.e. $g_{\mu\nu} = e^{\omega(x)}g^{flat}_{\mu\nu}$.

$$J_{-1,0} = D, \tag{1.8}$$

$$J_{\mu\nu} = L_{\mu\nu}, \tag{1.9}$$

with $\mu,\ \nu = 0, 1 \ldots, D-1$. The commutation relations obeyed by these generators are

$$[J_{ab}, J_{cd}] = -i\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{db}J_{ac}\right), \tag{1.10}$$

where

$$J_{ab} = -J_{ba}$$

and $a,\ b = -1, 0, \ldots D$. We changed from Greek to Roman indices because we are introducing an extra value for the indices. The structure constant η is $diag\left(-1, -1, +1, \ldots\right)$.

Quantum field theories with conformal symmetry have a conserved energy-momentum tensor. In addition, their β-functions are zero, implying that the trace of the energy momentum tensor vanishes at the quantum level. Another important consequence of the conformal symmetry enjoyed by the theory is the fact that the S-matrix is not observable, since the notion of a distant past and distant future is spoiled by scale invariance. There is therefore no notion of asymptotic states which are a key ingredient in defining the S-matrix. Instead of the S-matrix, we compute correlators of gauge-invariant operators. As an example of an observable, the two-point function is given by

$$\langle \mathcal{O}_1\left(x_1\right) \mathcal{O}_2\left(x_2\right)\rangle \sim \frac{1}{\left|x_1 - x_2\right|^{\Delta+\gamma}}, \tag{1.11}$$

where γ, the anomalous dimension, is a quantum correction to the scaling dimension Δ. The combination $\Delta + \gamma$ is known as the conformal dimension of the local operator of the theory. For BPS operators

$$\gamma = 0$$

and we say that the dimension is protected against quantum corrections.

1.2 CFTs and fixed points[2]

Conformal field theories can be used to describe critical phenomena, such as the region near second order phase transitions in condensed matter physics. In such cases, the anomalous dimension of the conformal field theory determines the scaling behaviour of thermodynamic variables.

According to the renormalisation group (RG), the value of the coupling flows along

[2]This section is based on [8] and p84 of [6].

the RG trajectory in response to changes in the scale μ of the effective field theory.[3] The behaviour of the coupling λ is encoded in the β-function

$$\beta(\lambda) \equiv \mu \frac{\partial \lambda}{\partial \mu}. \tag{1.12}$$

At the fixed point,

$$\beta(\lambda) = 0 \tag{1.13}$$

and the coupling takes a specific value - we say the coupling does not run.

The fact that the β-functions vanish implies that we have a scale invariant quantum field theory. Although it has not been proved in more than two dimensions, it is expected that whenever a theory enjoys both Poincare and scale invariance, it enjoys the full conformal symmetry.

1.3 The AdS/CFT correspondence

The AdS/CFT correspondence relates $(D+1)$-dimensional theories of (quantum) gravity on anti-de Sitter (AdS) space[4] to D-dimensional quantum field theories. The extra dimension on the gravity side sets the scale at which we probe the string theory.

As we have explained already, there are various examples of this conjecture, but the most studied one concerns $\mathcal{N}=4$ SYM in $D=4$ dimensions. Another interesting example is ABJM theory [4]. In this section, we will discuss both dualities.

1.3.1 $\mathcal{N}=4$ SYM and type IIB string theory

$\mathcal{N}=4$ SYM theory is a $(3+1)$-dimensional conformal field theory. We will focus on the case of gauge group $U(N)$. This theory has sixteen supercharges and a gauge coupling g_{YM}^2 which is independent of the renormalisation scale μ. The field content of the theory is a vector field A_μ, six scalar fields ϕ_i, $i=1,\ldots 6$ and four two-component Weyl fermions λ_α^a, $a=1\ldots 4$, $\alpha=1,2$. The fields are all in the adjoint of the gauge group. The symmetry of the theory includes the conformal group $SO(4,2)$ and the R-symmetry[5] $SO(6) \simeq SU(4)$. The Yang-Mills coupling constant is related to the 't Hooft coupling constant by

$$\lambda = g_{YM}^2 N \tag{1.14}$$

[3] The effective field theory description is applicable on length scales larger than μ^{-1}.

[4] Anti-de Sitter space is a background with constant negative curvature [9].

[5] R-symmetry is a symmetry that does not commute with the supersymmetries, but does commute with the Poincare group. See page 641 of [7].

and to the string coupling constant by

$$g_{YM}^2 = 4\pi g_s. \tag{1.15}$$

In this particular case, the dual gravitational theory lives on $AdS_5 \times S^5$ with AdS radius

$$R = \lambda^{\frac{1}{4}} l_s, \tag{1.16}$$

where l_s is the string length scale.[6]

One consequence of the AdS/CFT correspondence is [2, 3]

$$\left\langle \exp\left(-\int \phi_0 \mathcal{O}_\phi\right)\right\rangle_{CFT} = Z_{quantum\, gravity}\left[\phi_0\right]. \tag{1.17}$$

where $\mathcal{O}$ is a local operator on the conformal field theory, ϕ is a field on the gravity side and ϕ_0 is a boundary condition for ϕ. Concretely, when we compute the right hand side of equation (1.17) in a path integral approach, we integrate over all fields ϕ that take the value ϕ_0 on the boundary of the spacetime. As we explain later, both the CFT and the gravity side share the same global symmetries. This implies that the two sides have the same conserved quantum numbers, which we use to build a dictionary. As an example, if $\mathcal{O}$ (in equation (1.17)) has scaling dimension Δ and the field ϕ in AdS_{d+1} has mass m, then

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{2} + R^2 m^2}. \tag{1.18}$$

Equation (1.17) relates the generating functional of correlators in the conformal field theory on the left hand side, to the partition function of the quantum gravity theory on the right hand side. We emphasise the fact that the source of the operator in the conformal field theory generating functional is the boundary condition of the corresponding field on the gravity side.

In what follows, we motivate the AdS/CFT correspondence, following the discussion of [10]. Consider N parallel $D3$-branes sitting very close together, i.e. separated by distances less than the string length l_s.[7] These branes extend in a $(3+1)$-dimensional plane within a $(9+1)$-dimensional spacetime. With this set-up, we get two kinds of excitations, namely closed strings and open strings stretching between the branes. The closed strings are the excitations of empty space, while the open strings describe the excitations of the branes.

In the low-energy limit, i.e. energies lower than the string scale $1/l_s$, only massless

[6] The string length scale is a scale equal to the length of fundamental strings. This is in the region of the Planck length, $\sim 10^{-33} cm$. This scale sets the energy of the first stringy excitation.

[7] The fact that the branes are separated by l_s means that there are string states corresponding to strings stretching between branes, with energies low enough that they survive the low energy limit.

string states can be excited and we can write an effective theory for these states. The massless states for the closed strings give a gravity supermultiplet in 10 dimensions whose effective Lagrangian is that of type IIB supergravity. On the other hand, the open string massless states give an $\mathcal{N}=4$ vector multiplet in $(3+1)$-dimensions. In the low-energy limit, the effective Lagrangian is $\mathcal{N}=4$ $U(N)$ SYM theory [11].

The complete low-energy effective action for the massless states takes the form

$$S = S_{bulk} + S_{brane} + S_{int}, \tag{1.19}$$

where S_{bulk} is the action of the ten dimensional supergravity in flat, ten dimensional Minkowski space. S_{brane} is the brane action defined on the $(3+1)$-dimensional world-volume. It consists of $\mathcal{N}=4$ SYM and some higher derivative corrections. S_{int} describes the interactions between the brane and the bulk modes.

The bulk action can be expanded as a free quadratic part that describes the propagation of the free massless modes plus some interactions. Let us focus on the dynamics of the graviton. Expanding about flat space gives

$$S_{bulk} \sim \frac{1}{2\kappa^2}\int \sqrt{g}\mathcal{R} \sim \int (\partial h)^2 + \kappa\,(\partial h)^2\, h + \cdots\,, \tag{1.20}$$

where

$$\kappa = \sqrt{8\pi G_N}, \tag{1.21}$$

G_N is Newton's constant and $\mathcal{R}$ is the Ricci scalar. In equation (1.20) we have written the metric as

$$g = \eta + \kappa h, \tag{1.22}$$

with η the metric of flat Minkowski space. The interaction term, S_{int}, is proportional to positive powers of κ. In the low-energy limit, all the terms proportional to κ drop out. To see this more clearly, we can keep the energy fixed and send $l_s \to 0$, i.e. $\alpha' \to 0$, keeping all the dimensionless parameters fixed (including g_s and N). This way, the string vibrations admit no more than the lowest vibration mode. This follows from dimensional analysis: sending $l_s \to 0$ means that the energy of the first mode will be infinite. We then have

$$\kappa \sim g_s \alpha'^2 \to 0 \tag{1.23}$$

in the low-energy limit, where

$$\alpha' = l_s^2 \tag{1.24}$$

is the Regge slope parameter and has dimensions of length squared. The interaction Lagrangian involving the bulk and the brane vanishes.

Also, all the higher derivative terms in the brane action vanish at low energies. This leaves only the pure $\mathcal{N}=4$ $U(N)$ gauge theory in $3+1$ dimensions. We therefore realise two decoupled systems in the low energy limit: free supergravity theory on one hand and four dimensional gauge theory on the other.

This far, we have argued for the emergence of decoupled systems by studying the dynamics of a D-brane in string theory on ten dimensional Minkowski space. Thus our description includes both open and closed strings. Let us now consider the same system from a different point of view, i.e. by considering a description that uses closed strings only.

The sources for supergravity fields are massive, charged objects known as p-branes. These p-branes are conjectured to be the same as D-branes. We can find a $p3$-brane solution of the form

$$ds^2 = f^{-\frac{1}{2}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + f^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_5^2\right), \tag{1.25}$$

with a five-form flux

$$F_5 = (1+\star)\, dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1}, \tag{1.26}$$

where

$$f = 1 + \left(\frac{R}{r}\right)^4 \tag{1.27}$$

and

$$R^4 \equiv 4\pi g_s \alpha'^2 N. \tag{1.28}$$

Since g_{tt} is not constant, the energy E_p of an object as measured by an observer at a constant position r and the energy E measured by an observer at infinity are related by the redshift factor

$$E = f^{-\frac{1}{4}} E_p. \tag{1.29}$$

This means that the same object brought closer and closer to $r=0$ would appear to have lower and lower energy for the observer at infinity. Let us take the low energy limit in the background described by equation (1.25). From the point of view of an observer at infinity, there are two kinds of low energy excitations, namely

i) massless particles propagating in the bulk with wavelengths that become very large, and

ii) any kind of excitation that we bring closer and closer to $r=0$.

These two excitations decouple from one another in the low-energy limit. The bulk massless particles decouple from the near horizon region (around $r=0$) because the low energy absorption cross-section goes like

$$\sigma = \omega^3 R^8, \tag{1.30}$$

where ω is the energy. In this limit, the wavelength of the low energy supergravity modes becomes much bigger than the typical gravitational size of the brane. The excitations that live very close to $r = 0$ find it hard to climb the gravitational potential and escape to the asymptotic region. Therefore, the low-energy theory consists of two decoupled pieces, namely free bulk supergravity and the other piece in the near-horizon region of the geometry. In the near-horizon region, $r \ll R$ and

$$f \sim \left(\frac{R}{r}\right)^4 \tag{1.31}$$

so that the metric (1.25) becomes

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2 \tag{1.32}$$

which is the geometry of $AdS_5 \times S^5$ spacetime.

We see here that either approach gives us two decoupled systems in the low-energy limit. In both cases, one of the decoupled theories is supergravity in flat space. It is therefore natural to identify the other decoupled systems. As a result, we can conjecture that $(3+1)$-dimensional $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ is dual to type IIB superstring theory on $AdS_5 \times S^5$ [1].

The isometries of AdS space are in one-to-one correspondence with the generators of the conformal group of the field theory. The CFT is defined on $\mathbb{R}^{3,1}$ with metric

$$ds^2 = dt^2 - dx_1^2 - dx_2^2 - dx_3^2. \tag{1.33}$$

By Wick rotating we get the metric of $\mathbb{R}^4$,

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \\ &= dr^2 + r^2 d\Omega_3^2, \end{aligned} \tag{1.34}$$

where we have converted to spherical coordinates in the last line. We can substitute

$$r = e^{\tilde{t}}, \quad dr = e^{\tilde{t}} d\tilde{t} \tag{1.35}$$

to get

$$ds^2 = e^{2\tilde{t}}\left(d\tilde{t}^2 + d\Omega_3^2\right). \tag{1.36}$$

Performing a conformal transformation gives us

$$ds^2 = d\tilde{t}^2 + d\Omega_3^2 \tag{1.37}$$

which is the metric of $\mathbb{R} \times S^3$. In other words, we see here that the CFT can also be written on $\mathbb{R} \times S^3$.

The global symmetry on the CFT side includes scalings

$$r = e^{\tilde{t}} \to e^{\tilde{t}+a} = e^a r \tag{1.38}$$

in terms of which the $\tilde{t}$ coordinate is translated

$$\tilde{t} \to \tilde{t} + a. \tag{1.39}$$

The generator of scale transformations on the CFT is the dilatation operator D.

On the gravity side, the boundary of $AdS_5 \times S^5$, written $\partial \left(AdS_5 \times S^5\right)$, is $\mathbb{R} \times S^3$. We can perform a time translation on $\mathbb{R}$,

$$t \to t + a. \tag{1.40}$$

The generator of this time translation is the Hamiltonian. We can thus identify the dimensions of operators in the CFT with the energy of states in the string theory.

On the string theory side, N is the flux of the five-form Ramond-Ramond field strength on S^5,

$$\int_{S^5} F_5 = N. \tag{1.41}$$

This N arises in the string theory because the branes each carry unit charge. Stacking N of them together leads to equation (1.41). The same N arises on the CFT side as the rank of the gauge group $U(N)$.

The string theory has a parameter α' that does not appear on the gauge theory side. The ratio of the radius of curvature to α' does appear as a parameter in the gauge theory. As a result, α' sets the units for any physical quantity computed. The radius of curvature is usually set to one in gravity calculations by writing the metric as

$$ds = R^2 d\tilde{s}. \tag{1.42}$$

In that case,

$$\alpha' \sim \frac{1}{\sqrt{g_s N}} \tag{1.43}$$

which implies that any quantity computed without taking into account stringy effects will be independent of $g_s N$. It will depend only on N. α' corrections to this gravity result will be proportional to powers of $1/\sqrt{g_s N}$.

Perturbative Yang-Mills theory can be trusted when

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1. \tag{1.44}$$

On the other hand, a classical gravity description is reliable when the radius of curvature R (of AdS_5 as well as of S^5) is very large compared to the string length l_s, i.e.

$$\frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1. \tag{1.45}$$

We see from this that when one theory is strongly coupled, the other is weakly coupled and vice-versa. This is the point that makes the duality both useful and difficult to prove. Here, we have assumed that $g_s < 1$. The radius of curvature, in Planck units, is

$$\frac{R^4}{l_p} \sim N. \tag{1.46}$$

It is therefore necessary, but not sufficient, to have large N in order to have a weakly coupled supergravity description. The strongest version of the AdS/CFT correspondence claims equivalence for all values of N and g_s.

1.3.1.1 Symmetry matching[8]

One piece of evidence for the duality is symmetry matching. On both sides of the duality, the complete symmetry is given by the superalgebra $PSU(2,2|4)$. A superalgebra of the form $SU(m|n)$ has bosonic subalgebra $SU(m) \times SU(n) \times U(1)$. For the special case $m = n$, the $U(1)$ factor decouples from the rest of the algebra. P tells us that this $U(1)$ factor is not there. The bosonic subgroup for this superalgebra is $SU(2,2) \times SU(4)$.

In addition to conformal symmetry, $\mathcal{N} = 4$ SYM theory is also invariant under supersymmetry transformations. There are eight supercharges that, together with their conjugates, generate the supersymmetry transformations. Combining these supercharges with the generators of the conformal group we discussed in Section 1.1, gives the superconformal algebra. Thus in addition to the conformal algebra, we now have the following commutation and anti-commutation relations

$$\left\{ Q_\alpha^a, \tilde{Q}_{\dot{\alpha}}^b \right\} = \gamma_{\alpha\dot{\alpha}}^\mu \delta^{a\bar{b}} P_\mu, \tag{1.47}$$

[8]This discussion is based on [7, 12, 10].

$$\left\{Q^a_\alpha, Q^b_\beta\right\} = \left\{\tilde{Q}^{\bar{a}}_{\dot{\alpha}}, \tilde{Q}^b_{\dot{\beta}}\right\} = 0, \tag{1.48}$$

$$[P_\mu, Q^a_\alpha] = \left[P_\mu, \tilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = 0, \tag{1.49}$$

$$[D, Q^a_\alpha] = -\frac{i}{2} Q^a_\alpha, \tag{1.50}$$

$$\left[D, \tilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = -\frac{i}{2} \tilde{Q}^{\bar{a}}_{\dot{\alpha}}, \tag{1.51}$$

$$[L^{\mu\nu}, Q^a_\alpha] = i\sigma^{\mu\nu}_{\alpha\beta} \epsilon^{\beta\gamma} Q^a_\gamma, \tag{1.52}$$

$$\left[L^{\mu\nu}, \tilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = i\sigma^{\mu\nu}_{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \tilde{Q}^{\bar{a}}_{\dot{\gamma}}, \tag{1.53}$$

$$[K^\mu, Q^a_\alpha] = \gamma^\mu_{\alpha\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{S}^a_{\dot{\beta}}, \tag{1.54}$$

and

$$\left[K^\mu, \tilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = \gamma^\mu_{\alpha\dot{\alpha}} \epsilon^{\alpha\beta} S^{\bar{a}}_\beta, \tag{1.55}$$

where $\alpha, \dot{\alpha}, \beta, \dot{\beta} = 1, 2$ label the fundamental representations of the two independent $SU(2)$ algebras that make up the four-dimensional Lorentz group. The indices $a, \bar{a}, b, \bar{b} = 1, \ldots 4$ label the fundamental and anti-fundamental representations of an internal $SU(4) \simeq SO(6)$ symmetry known as R-symmetry. $S^{\bar{a}}_\alpha$ and $\tilde{S}^a_{\dot{\alpha}}$ which obey

$$\left\{S^{\bar{a}}_\alpha, \tilde{S}^b_{\dot{\alpha}}\right\} = \gamma^\mu_{\alpha\dot{\alpha}} \delta^{\bar{a}b} K_\mu \tag{1.56}$$

and

$$\left\{S^{\bar{a}}_\alpha, S^{\bar{b}}_\alpha\right\} = \left\{\tilde{S}^a_{\dot{\alpha}}, \tilde{S}^b_{\dot{\alpha}}\right\} = 0, \tag{1.57}$$

are special conformal supercharges. Together with the other supercharges, they give a total of 32 supercharges. The two types of supercharges satisfy

$$\left\{Q^a_\alpha, S^{\bar{b}}_\beta\right\} = -i\varepsilon_{\alpha\beta} \sigma^{ij}_{a\bar{b}} R_{ij} + \sigma^{\mu\nu}_{\alpha\beta} \delta^{a\bar{b}} L_{\mu\nu} - \varepsilon_{\alpha\beta} \delta^{a\bar{b}} D \tag{1.58}$$

and

$$\left\{\tilde{Q}^{\bar{a}}_{\dot{\alpha}}, \tilde{S}^b_{\dot{\beta}}\right\} = +i\varepsilon_{\dot{\alpha}\dot{\beta}} \sigma^{ij}_{\bar{a}b} R_{ij} + \sigma^{\mu\nu}_{\dot{\alpha}\dot{\beta}} \delta^{\bar{a}b} L_{\mu\nu} - \varepsilon_{\dot{\alpha}\dot{\beta}} \delta^{\bar{a}b} D \tag{1.59}$$

from which we get a new set of generators R_{ij} with $i, j = 1, \ldots 6$. These generate the $SU(4)$ R-symmetry. The matrices $\sigma^{ij}_{\bar{a}b}$ are the $SO(6)$ generators in the fundamental representation.

On the string theory side, the AdS_5 space has isometry $SO(2,4)$ which follows from the fact that $(p+2)$-dimensional anti-de Sitter space (AdS_{p+2}) can be represented as the hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = R^2 \tag{1.60}$$

in a flat $(p+3)$-dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2. \tag{1.61}$$

By construction, this space has isometry $SO(2, p+1)$ in addition to being homogeneous and isotropic. In our special case, $p = 3$, the $SO(4,2)$ isometry is the same as the conformal group in $3+1$ dimensions.

There is also an $SO(6)$ symmetry that rotates the S^5 sphere. This $SO(6)$ symmetry can be identified with the $SU(4)$ R-symmetry group we have seen in the field theory.

The $SO(2,4)$ isometry of AdS_5 has a supersymmetric extension known as an AdS supergroup. We will explain how this enhencement is realised for $\mathcal{N} = 1$ supergravity with cosmological constant Λ [13]

$$S = \int d^4x \left(-\sqrt{g}\,(\mathcal{R} - 2\Lambda) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma^5\gamma_\nu\tilde{D}_\rho\psi_\sigma \right), \tag{1.62}$$

where

$$\tilde{D}_\mu = D_\mu + \frac{i}{2}\sqrt{\frac{\Lambda}{3}}\gamma_\mu, \tag{1.63}$$

D_μ is the standard covariant derivative, $\mathcal{R}$ is the Ricci scalar, g is the determinant of the metric and γ^5 and γ_μ are gamma matrices. The local supersymmetry transformation for the vierbein $V_{a\mu}$ and the gravitino ψ_μ are

$$\delta V_{a\mu} = -i\bar{\epsilon}\,(x)\,\gamma_a\psi_\mu \tag{1.64}$$

and

$$\delta\psi_\mu = \tilde{D}_\mu\epsilon\,(x)\,, \tag{1.65}$$

where ϵ is a spinor. To realise a global supersymmetry of the supergravity background, the gravitino variation must vanish, i.e.

$$\delta\psi_\mu = 0. \tag{1.66}$$

This is the Killing spinor equation. There are as many solutions to the equation as there are independent components of the spinor. Since the theory has fermions that belong to spinor representations, it is better to refer to groups $SU(2,2)$ and $SU(4)$ instead. This way, the bosonic subgroup of the supergroup is realised geometrically.

On AdS_5, there are $\mathcal{N} = 2, 4, 6, 8$ gauged supergravities[9] with supersymmetry $SU(2, 2|\mathcal{N}/2)$ [14]. The $\mathcal{N} = 8$ case [15, 16] is the one that is conjectured to be dual to $\mathcal{N} = 4$ SYM theory in four dimensions. It is known to have a gauge group $SU(4) \simeq SO(6)$.

As we have explained, on the gauge theory side, the $SU(4)$ symmetry arises as the global $SU(4)$ R-symmetry, while $SO(4, 2)$ is the conformal group. Bringing everything together, we get the $PSU(2, 2|4)$ supergroup.

1.3.1.2 Large N limit

We study operators

$$\mathcal{O} = tr\left(Z^n\right) tr\left(Z^m\right) \cdots tr\left(Y Z^n Y^m Z\right) \tag{1.67}$$

that are made of $N \times N$ matrices, Z, Y. There are various large N limits that one can take. In each of the examples we consider, we take the limit $N \to \infty$ while holding λ fixed. With this condition, we can keep the number of fields inside $\mathcal{O}$ fixed, i.e. $n,\ m \sim O(1)$. We can also consider $n,\ m \sim O\left(\sqrt{N}\right)$, $n,\ m \sim O(N)$ or $n,\ m \sim O\left(N^2\right)$.[10]

When we take $N \to \infty$ while keeping λ fixed, the Feynman diagrams arrange according to their genus. In particular, the leading order terms will consist of planar diagrams and so forth. By planar diagrams we mean diagrams that can be drawn on a plane without their ribbons crossing.[11]

Since the string coupling g_s is related to the 't Hooft coupling λ through

$$g_s \sim \frac{\lambda}{N}, \tag{1.68}$$

the $1/N$ expansion at fixed λ corresponds to the loop expansion of the dual string theory.

1.3.2 ABJM theory

ABJM theory [4], which bears the names of its authors, is a $(2 + 1)$-dimensional super Chern-Simons-matter theory with $\mathcal{N} = 6$ superconformal symmetry and gauge group $U(N)_k \times U(N)_{-k}$. Here, k is the Chern-Simons level which sets the coupling strength of the theory. When k is large, the theory is weakly coupled.

The theory is conjectured to be dual to gravitational theories describing N parallel $M2$-branes stacked together. The R-symmetry of the gauge theory is $SU(4)$. When $k = 1$, the theory is conjectured to describe N $M2$ branes in flat space.

[9] Gauged supergravities are supergravity theories with non-Abelian gauge fields in the supermultiplet of the graviton [10].

[10] What each of these limits corresponds to is mentioned in Section 1.6.

[11] Since the fields are matrices, the Feynman diagrams have double lines for propagators. It is these double lines that we call ribbons.

The field content of the Chern-Simons theory includes two gauge fields A_m and $\bar{A}_m$, complex scalar fields Y^I as well as Majorana spinors Ψ_I, where $I = 1, \ldots 4$ for both fields.

The corresponding 't Hooft coupling for this theory is

$$\lambda = \frac{N}{k}. \tag{1.69}$$

The 't Hooft limit takes $N \to \infty$ and $k \to \infty$, while holding λ fixed.

Unlike the $D3$-brane case discussed above, the $U(N) \times U(N)$ theory describes the low energy limit of N $M2$-branes probing a $\mathbf{C}^4/\mathbf{Z}_k$ singularity. At large N and $k \ll N$, the theory is dual to M-theory on $AdS_4 \times S^7/\mathbf{Z}_k$. In the 't Hooft limit described above, the theory is conjectured to describe N $D2$-branes in flat space. In this limit, the theory is dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ background.

1.4 Giant gravitons

Giant gravitons are given by Dp-branes wrapping some sphere. Their excited states are described by attaching strings. Giant gravitons play a central role in this thesis.

Consider a massless particle (a graviton) moving along a circle in S^5. It has been shown in [17] that as the momentum of the particle is increased, the coupling of the particle to the background flux becomes more important for the dynamics of the particle. As a result of the coupling, the particle expands into a sphere inside the S^5 of $AdS_5 \times S^5$. There is a cut-off on the size of the giant graviton arising because the S^3 is contained inside the S^5. A particle in this state is known as a giant graviton.

We can also have a dual giant graviton [18, 19] if the expansion happens in the AdS_5 space. In this case, there is no upper bound on the size of the giant graviton. Such a particle is called a dual giant graviton.

1.4.1 Sphere giants

Let us start by reviewing the giant gravitons expanding in the sphere part of the geometry. The discovery draws much from non-commutative field theory [20, 21]. The particles described by such theories have a spatial extension which is proportional to their momentum.

Let us consider a pair of unit charges of opposite sign moving on a plane with a constant magnetic field B. The coordinates of these charges are x_1 and x_2. The Lagrangian is

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{B}{2}\epsilon_{ij}\left(\dot{x}_1^i \dot{x}_1^j - \dot{x}_2^i \dot{x}_2^j\right) - \frac{K}{2}\left(x_1 - x_2\right)^2, \tag{1.70}$$

where the first term is the kinetic energy, the second is the interaction of the charges with the magnetic field and the last term is the harmonic potential between the charges. Let us assume that the mass is small enough to be ignored so that the first term in the Lagrangian can be neglected.

Let us now introduce the centre of mass and relative coordinates

$$X = \frac{x_1 + x_2}{2} \quad \& \quad \triangle = \frac{x_1 - x_2}{2} \tag{1.71}$$

in terms of which the Lagrangian becomes

$$\mathcal{L} = B\epsilon_{ij}\dot{X}^i\triangle^j - 2K\triangle^2. \tag{1.72}$$

Here, $\triangle^j$ refers to the component form of the coordinates. The variables X and $\triangle$ do not commute. Instead, they satisfy

$$\left[X^i, \triangle^j\right] = i\frac{\epsilon_{ij}}{B}. \tag{1.73}$$

The momentum conjugate to X is

$$P_i = B\epsilon_{ij}\triangle^j. \tag{1.74}$$

Therefore, when moving with momentum P in a particular direction, the dipole is stretched to a size

$$|\triangle| = \frac{|P|}{B} \tag{1.75}$$

in the perpendicular direction. This is true because $\triangle$ gives the relative coordinates of the two charge system.

Now let us allow the dipole to move on the surface of a sphere of radius R and magnetic flux N. This can be realised by placing a magnetic monopole of strength

$$2\pi N = \Omega_2 BR^2 \tag{1.76}$$

at the centre of the sphere. When the momentum of the dipole reaches $2BR$, the dipole is as big as the sphere. The angular momentum of the dipole at this point is maximum,

$$L = PR \sim BR^2, \tag{1.77}$$

which is order N, the total flux through the surface of the sphere. This rough analysis agrees with the results of a more precise analysis from which we learn that the angular momentum is exactly cut off at N.

In the case of $AdS_5 \times S^5$, the radius of the five sphere is

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s, \tag{1.78}$$

where the symbols used have already been defined. Let us consider the 't Hooft limit $N \to \infty$ with

$$\lambda = g_s N$$

fixed and large. With this set up, let us consider the exact classical analysis of a $D3$ brane wrapping an S^3 that moves inside the S^5. The bosonic Lagrangian, which is the sum of the Dirac-Born-Infeld and Chern-Simons terms, is

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{CS} = -T_{D3}\Omega_3 r^3 \sqrt{1 - (R^2 - r^2)\,\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4}, \tag{1.79}$$

where

$$T_{D3} = \frac{1}{(2\pi)^3\, l_s^4 g_s}$$

is the tension of the $D3$ brane. Using equation (1.78), we write

$$T_{D3}\Omega_3 = \frac{N}{R^4}, \tag{1.80}$$

and get the angular momentum

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{m\dot{\phi}\left(R^2 - r^2\right)}{\sqrt{1 - (R^2 - r^2)\,\dot{\phi}^2}} + N\frac{r^4}{R^4}, \tag{1.81}$$

where

$$m = T_{D3}\Omega_3 r^3 = \frac{N r^3}{R^4}. \tag{1.82}$$

Since

$$0 \leq r \leq R$$

and

$$0 \leq \dot{\phi} R \leq 1,$$

the angular momentum is again bounded by N.

The energy of this configuration is

$$E = \sqrt{m^2 + \frac{\left(L - N r^4/R^4\right)^2}{R^2 - r^2}}. \tag{1.83}$$

Varying this energy with respect to r for fixed L, we find a stable minimum at

$$r^2 = \frac{L}{N} R^2 \tag{1.84}$$

corresponding to

$$E = \frac{L}{R} \tag{1.85}$$

for large L. This matches the BPS bound of the energy.

1.4.2 AdS (Dual) giants

Following the discovery of the sphere giants whose size is bounded by the five-sphere, [18] and [19] independently discovered stable giants expanding in the AdS part of the geometry. These so-called dual giants are not bounded by the space in which they expand since AdS space is not bounded.

Let us consider a spherical $D3$ brane in AdS_5 moving along the equator of S^5. If the angular velocity of the brane is $\dot{\phi}$, the Lagrangian of the configuration is

$$\mathcal{L} = -T\Omega_3 R^4 \left(\tan^3 \rho \sqrt{\sec^2 \rho - \dot{\phi}^2} - \tan^4 \rho \right) \tag{1.86}$$

and the corresponding energy is

$$E = N \left(\sec \rho \sqrt{\frac{L^2}{N^2} + \tan^6 \rho} - \tan^4 \rho \right) \tag{1.87}$$

with

$$T\Omega_3 R^4 = N. \tag{1.88}$$

The Lagrangian (1.86) comes from embedding a $D3$-brane wrapping the Ω_3 of the AdS_5 background. Working in global coordinates, the Dirac-Born-Infeld Lagrangian for this configuration is

$$\mathcal{L} = -\left(T\sqrt{\left(-g_{tt} - \omega^2 g_{\Omega_5\Omega_5}\right) g^3_{\Omega_3\Omega_3}} - C_{t\Omega_3\Omega_3\Omega_3} \right) \tag{1.89}$$

where

$$C_{t\Omega_3\Omega_3\Omega_3} = TR^4 \tan^4 \rho.$$

The metric of AdS_5 in these coordinates is

$$ds^2 = \frac{R^5}{\cos^2 \rho} \left(-d\tau^2 + d\rho^2 + \sin^2 \rho d\Omega_3^2 \right) + R^2 d\Omega_5^2. \tag{1.90}$$

This then leads to equation (1.86).

The energy corresponding to the local minima

$$\tan\rho = 0 \quad \& \quad \tan\rho = \sqrt{\frac{L}{N}} \tag{1.91}$$

is

$$E = \frac{L}{R}. \tag{1.92}$$

We see here that the quantum numbers of the two giant gravitons match and are equal to that of the point-like graviton. In addition to the maximum sizes of the giants, another difference is that the sphere giant couples magnetically to the background field, while the dual (AdS) couples electrically. In other words, the AdS giant can be thought of as a dielectric brane that couples electrically to the background field, while the sphere giant behaves as a diamagnetic brane.

1.5 Planar limit

A lot of work has been done, employing integrability to solve $\mathcal{N} = 4$ SYM theory in the planar limit. As we explain elsewhere, the planar limit consists of Feynman diagrams whose ribbons do not cross when drawn on a plane. In this section, we review the work done in the planar limit using integrability. For a more comprehensive review, the reader is referred to [22].

1.5.1 $\mathcal{N} = 4$ SYM theory and type IIB string theory (AdS_5/CFT_4)

Let us start by elucidating the conjectured relationship between $\mathcal{N} = 4$ SYM theory and type IIB string theory. On the gauge theory side, one can perform a $1/N$ expansion in the limit $N \to \infty$ for fixed λ. The graphs whose ribbons do not cross when drawn on a plane - the so-called planar diagrams - constitute the leading terms. The non-planar diagrams represent quantum corrections. In other words, the planar limit in figure 1.1 (taken from [22]) consists of these planar diagrams.

When λ is small, the gauge theory is weakly coupled and the background of the string theory is highly curved. On the other hand, when λ is very large, the gauge theory is strongly coupled and the background of the dual string theory is weakly curved. This radius of curvature R is related to the effective string tension T through

$$T = \frac{R^2}{2\pi\alpha'}. \tag{1.93}$$

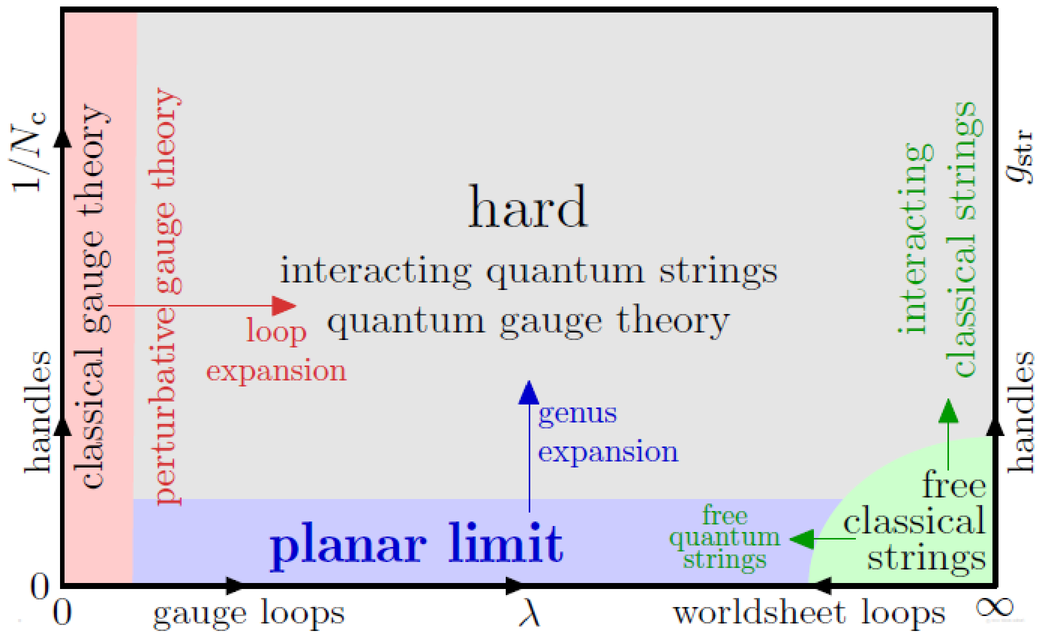

Figure 1.1: The parameter space of $\mathcal{N}=4$ SYM theory and type IIB string theory (from [22]).

In the weak coupling regime,[12] perturbation theory in terms of Feynman diagrams provides a good approximation of the gauge theory. Through the AdS/CFT correspondence, this gives reliable results for the string theory on the highly curved background. In contrast, the same perturbative expansion breaks down in the strong coupling limit of the gauge theory. However, since the string theory background is weakly curved in this region, perturbative string theory is applicable, i.e. we can expand the string theory in terms of the tension. The number of handles can be increased by expanding in terms of the string coupling g_s. The results obtained from this expansion can again be transferred to the strongly coupled gauge theory through the AdS/CFT correspondence.

In the planar limit of $\mathcal{N}=4$ SYM, it has been possible to compute observables at arbitrary gauge coupling λ. The central idea that is employed in this large N limit of the theory is the identification of the dilatation operator D, with the Hamiltonian of an integrable spin system.[13] Integrability predicts the spectrum of planar scaling dimensions for local operators as a function of λ. According to the AdS/CFT correspondence, this spectrum is dual to the energy spectrum of the free string states, i.e. strings that neither break apart nor join together. The results obtained using integrability match on either side of the correspondence, i.e. the energy spectrum obtained on the string theory side

[12] Referring to the gauge coupling λ.
[13] We review this for the $su(2)$ sector of SYM theory in Appendix B.

agrees with the spectrum of anomalous dimensions computed on the gauge theory side. Furthermore, the results obtained using integrability agree with those obtained by the actual computation of the Feynman diagrams in the weak coupling limit of the gauge theory. Similarly, in the strong coupling limit, integrability agrees with perturbative string theory.

1.5.2 ABJM theory and type IIA string theory (AdS_4/CFT_3)

Some work has been done in the weak coupling limit of ABJM theory as well. Specifically, this is in the limit in which ABJM theory is dual to type IIA string theory on $AdS_4 \times CP^3$ spacetime. In this limit, the same approach used to study $\mathcal{N} = 4$ SYM theory has been applied to the case of ABJM theory with appropriate modifications. In particular, the dilatation operator of ABJM theory in this limit is mapped to the Hamiltonian of an integrable spin chain.

In the AdS_5/CFT_4 case we had type IIB string theory on $AdS_5 \times S^5$ with the self dual five-form flux

$$\int F^{(5)} \sim N$$

through AdS_5 and S^5. We now have type IIA string theory on $AdS_4 \times CP^3$ with four form flux

$$\int F^{(4)} \sim N$$

through AdS_4 and two-form flux[14]

$$\int F^{(2)} \sim k$$

through a $CP^1 \subset CP^3$. $D3$ branes are replaced by $M2$ branes.

In the AdS_5/CFT_4 case, the gauge theory is $\mathcal{N} = 4$ SYM theory with coupling g_{YM} and gauge group $U(N)$ on $\mathbb{R}^{1,3}$. In the AdS_4/CFT_3 case this is replaced by ABJM theory which is $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory with gauge group $U(N) \times U(N)$ on $\mathbb{R}^{1,2}$. The Yang-Mills coupling g_{YM} is replaced by the Chern-Simons level k. After rescaling the fields in ABJM theory in particular way, all interactions are suppressed by powers of $1/k$ so that large values of k correspond to the weak coupling regime. One can take a planar limit in which

$$k,\ N \to \infty$$

[14] Where k is the Chern-Simons level.

and

$$\lambda \equiv \frac{N}{k}$$

is held constant. It has been shown that in this limit, weakly coupled ABJM theory can be mapped onto a spin chain problem whose Hamiltonian is known to be integrable [23].

In the planar limit, the string coupling g_s goes to zero and the strings do not split or join. When λ is small, the background is highly curved and the strings get large quantum corrections. On the other hand, when λ is very large, the background is weakly curved and the strings behave classically.

Whereas one has all spins pointing in one direction in the $\mathcal{N} = 4$ SYM case and then considers one excitation moving through the spin chain, in ABJM theory the integrable spin chain has alternating spins. This means that the ABJM spin chain must have an even number of spin sites.

1.6 Non-planar limit

As we have already mentioned, the AdS/CFT dictionary identifies the conformal dimension of the field theory operators with the energy of the corresponding state in the quantum gravity. Operators with dimension ~ 1 are identified with point-like gravitons [3, 2]. If the dimension is $\sqrt{N}$ then they are string states [24]. More interestingly, operators with dimension N are identified with D-brane states [25, 26, 27], while new geometries are associated with operators with dimension N^2 [28, 29]. The large N limit of the last two operators is not captured by summing planar diagrams only [25]. This is because large combinatoric factors that arise from many fields enhance the non-planar contributions [30]. We therefore need to work in a large N, but non-planar limit.

Since summing the large number of Feynman diagrams is such a daunting task, a new approach is needed. By using Schur polynomials,[15] it has been shown how all possible diagrams can be summed in a much easier way [26] in the free field theory, in a half-BPS sector. In this basis, the two-point function of the theory is diagonal and the higher-point correlators take a simple form. These results were then explained in terms of projection operators in [31].

Operators that are dual to excited giant gravitons, restricted Schur polynomials, were first proposed in [32]. Using the technology developed in [33, 34, 35], the two-point function of these restricted Schur polynomials was computed in the free field theory limit [36]. These operators provide a basis for gauge invariant operators built using only scalar fields [37]. These operators are equally good for describing gauge invariant operators with more scalar [38] and fermionic fields [39], as well as gauge fields [39]. In

[15] Explained in Chapter 2.

the basis of these operators, the two-loop dilatation operator has also been studied [40]. As a basis, the restricted Schur polynomials diagonalise the two-point function in the free field theory and they mix weakly at one loop [34, 35], an important property which we explain in Chapter 2.

In [41, 42], numerical studies of the spectrum of anomalous dimensions were carried out, and the results resembled a set of decoupled harmonic oscillators. This inspired analytic work on the action of the dilatation operator on the restricted Schur polynomials. Initially, this work concentrated on two sphere or two AdS giants [43] before considering a more general number of branes [44]. One main difference between these two cases is that in the two giant graviton case, the problem is simplified by the absence of multiplicity indices. We will explain this simplification in Chapter 2.

The analytic study of the action of the dilatation operator makes use of the relationship between symmetric and unitary group. In particular, it employs the so-called Schur-Weyl duality. We will explain this important concept in Sections 2.6 and 3.2.2.

Analytically, the dilatation operator acting on the restricted Schur polynomials has been diagonalised in [43, 44, 38, 39]. In all these cases, restricted Schur polynomials built from a large number of fields, Z, doped slightly with impurities was considered. The diagonalisation problem separated into two problems, one associated with the impurities and the other associated with the Z fields. The diagonalisation problem associated with the Z fields was solved in [45], while the one associated with the impurities is solved through a double coset asantz [46]. The result of the diagonalisation is a spectrum of a set of decoupled oscillators. This signals integrability in this large N, but non-planar limit. In other words, it is found that $\mathcal{N} = 4$ SYM theory is integrable beyond the planar limit.

ABJM theory has also been written in terms of restricted Schur polynomials which diagonalise the free field theory [47]. Again, the spectrum of anomalous dimensions was computed and was found to resemble the spectrum of a set of decoupled oscillators. This was done for systems of two excited giant gravitons represented by two long rows or columns. Interestingly, the technology developed to study the $su(2)$ sector of SYM theory is sufficient to handle this case. Complications immediately arise the moment one considers more than two rows. It is not yet known if the action of the dilatation operator is diagonalised by the double coset ansatz developed for $\mathcal{N} = 4$ SYM theory.

1.6.1 Large N, but non-planar limit

As we explain in Section 1.5 and Appendix B.3, in the case of single trace operators built using $O(1)$ matrix fields, it is sufficient to sum only the planar diagrams in the large N limit. These operators are dual to point gravitons. On the contrary, operators that are dual to giant gravitons are made of $O(N)$ fields. To compute the anomalous dimensions of these large operators, one needs to sum both the planar and the non-planar diagrams. In this thesis, we work in this large N, but non-planar limit. This point will become much clearer in the next chapter.

1.7 Outline of this thesis

In this work, we compute the spectrum of anomalous dimensions for a marginally deformed super Yang-Mills theory as well as the $su(2|3)$ and $sl(2)$ sectors of $\mathcal{N}=4$ SYM theory. The tool we employ is the group representation theory of symmetric and unitary groups reviewed in Chapter 2. The gauge invariant operators we will study are restricted Schur polynomials built from $O(N)$ scalar fields Z, doped with a smaller number, but $O(N)$ impurity fields. In each case, we will notice that the action of the dilatation operator always factorises into a problem associated with the Z fields and a problem associated with the impurities. Each of these two parts will be diagonalised.

We will then take the continuum limit for each of the problems we study. This will help us to see any hints of integrability clearly. At large N, since there are order N boxes in each row of the Young diagram, row lengths effectively become continuous variables and a continuum limit is justified.

Our calculations do not depend on the spacetime coordinates. If needed, the spacetime dependence can easily be incorporated at the end. As a result, we will not include the spacetime dependence in our calculations [26].

1.7.1 Marginally deformed $\mathcal{N}=4$ SYM

Studying $\mathcal{N}=4$ SYM theory may give some insight into quantum chromodynamics and related theories. However, this theory is maximally super-symmetric and conformally invariant, while quantum chromodynamics is not. A natural course to take therefore, is to break some of the supersymmetry - we do not break the conformal symmetry and consequently, the gravitational theories we consider are on AdS backgrounds. There is a deformation that breaks the supersymmetry in $\mathcal{N}=4$ SYM theory down to $\mathcal{N}=1$. This example was first introduced by Leigh and Strassler in [48]. This is the case we study here, i.e. we will consider the action of the deformed dilatation operator on the restricted Schur polynomials in the $su(2)$ sector. This work, reported here in Chapter 3 was published in [49]. It is my original work.

A gravitational dual for this theory was found by Lunin and Maldacena in [50]. The key idea they employed was that marginal deformations of a conformal field theory preserve the conformal symmetry. The conformal group is $SO(2,4)$ and is the isometry of the gravity dual. This isometry gives us AdS_5 spacetime. As a result, Lunin and Maldacena only deformed the S^5 part of the $AdS_5 \times S^5$ spacetime. In particular, they performed a T-duality, followed by a shift and finally by another T-duality (a TsT transformation) on the five-sphere. The result was a theory on the deformed spacetime, $AdS_5 \times \tilde{S}^5$.

Generalisations of this deformation to non-supersymmetric cases exist [51]. In this case, one obtains the Lunin-Maldacena example by equating all the deformation parameters, i.e. by setting $\gamma_i = \gamma$.

1.7.2 $su(2|3)$ sector

The beautiful work done on the $SU(2)$ sector of $\mathcal{N} = 4$ SYM only includes two scalar fields. Three scalar fields do not make up a closed sector because these fields mix with fermions. The next closed sector therefore consists of three scalars and two fermions, the so-called $su(2|3)$ sector [52].

In order to firmly establish the existence of integrability in the large N, but non-planar limit of the theory, we need to include fermions and gauge fields in our study. Chapter 4, published in [39], fills this important gap - this is my original work. In particular, we will explain how to construct restricted Schur polynomials that include both fermions and bosons. These restricted Schur polynomials continue to diagonalise the free field two point function to all orders in $1/N$. We find that these new restricted Schur polynomials continue to diagonalise the free field two point function. In addition, the number of these polynomials matches the expected number of multi-field multi-trace gauge invariant operators. We also show how to transform between the trace basis and the basis provided by the polynomials we construct.

As an application of our results, we study the $su(2|3)$ sector of the theory. It is closed to all orders under the action of the dilatation operator [52, 53]. At the one loop level, the dilatation operator has a simple action in this sector. We explain how to construct the restricted Schur polynomials for the $su(2|3)$ sector and then compute the action of the dilatation operator in this sector. The problem associated with the Z fields in this case is similar to the one solved in [45]. As a result, we only need to diagonalise the impurity problem, which we accomplish by employing a slightly modified version of the double coset ansatz.

1.7.3 The $sl(2)$ sector

There is another closed sector of $\mathcal{N} = 4$ SYM theory that consists of one type of scalar field Z, say, and covariant derivatives - the $sl(2)$ sector [54]. In Appendix C we diagonalise the one-loop dilatation operator acting on restricted Schur polynomials in this sector of the theory. These operators were built in [38]. In this work, we only complete that work by writing the action of the dilatation operator in the Gauss graph basis. We find that this sector is also diagonalised by the double coset ansatz. This too, is my own original work.

Finally, we conclude our work in Chapter 5.

Chapter 2

Group representation theory

In this chapter we review the tools used to study the large N, but non-planar limits of Yang-Mills theories. These tools include the group representation theory of symmetric and unitary groups, as well as the relationships between them. We use the same tools, with appropriate modifications, in the chapters that follow. In this chapter, we will only review the components that are necessary to understand the chapters that follow. The rest will be similar to what we have in Chapter 3.

2.1 Multi-trace operators and gravity

As we discussed in Chapter 1, the gauge invariant operators we wish to study are multi-trace operators. In the $su(2)$ sector of $\mathcal{N} = 4$ SYM theory, we study operators built from $N \times N$ complex scalar fields Z and Y. There are n Z fields and m Y fields. We compute correlators of the form

$$\langle \mathcal{O}_{n,m} \rangle = \left\langle \mathcal{O}_n \mathcal{O}_m \mathcal{O}^{\dagger}_{n+m} \right\rangle , \tag{2.1}$$

where the operator $\mathcal{O}_n$ is

$$\mathcal{O}_n = \frac{1}{\sqrt{nN^n}} tr\left(Z^n\right). \tag{2.2}$$

Our goal here is to find a (simple) formula for (2.1). To proceed, let us study a simpler problem first, i.e. we consider the case $n \neq 0$ and $m = 0$

$$\left\langle tr\left(Z^n\right) tr\left(Z^{\dagger n}\right) \right\rangle .$$

Let us start with $n = 1$ and increase the values of n until we notice a pattern. To compute the correlators in each case, we consider all possible contractions between the Z and $Z^{\dagger}$ fields (by Wick's theorem), draw the corresponding Feynman diagrams and then count the number of closed loops for each case. For each closed loop we write down a factor of N. The case in which $n = 1$ gives one Feynman diagram with one closed loop. We therefore find

$$\left\langle tr\left(Z\right)tr\left(Z^{\dagger}\right)\right\rangle = N. \tag{2.3}$$

Similarly, when $n = 2$, we get two Feynman diagrams from the two possible contractions. Each of these diagrams has two closed loops. We therefore find

$$\left\langle tr\left(Z^2\right)tr\left(Z^{\dagger 2}\right)\right\rangle = 2N^2. \tag{2.4}$$

For $n = 3$ we get three N^3 diagrams as well as three other diagrams with one closed loop. We therefore find

$$\left\langle tr\left(Z^3\right)tr\left(Z^{\dagger 3}\right)\right\rangle = 3N^3 + 3N. \tag{2.5}$$

The $n = 4$ case gives

$$\left\langle tr\left(Z^4\right)tr\left(Z^{\dagger 4}\right)\right\rangle = 4N^4 + 20N^2. \tag{2.6}$$

At this stage we notice pattern for the leading term. To leading order, we can write[1]

$$\left\langle tr\left(Z^n\right)tr\left(Z^{\dagger n}\right)\right\rangle = nN^n. \tag{2.7}$$

This is where the normalisation in equation (2.2) comes from.

We can also derive equation (2.7) in a second way. We use the following identity

$$\int\left[dZdZ^{\dagger}\right]\frac{d}{dZ^i_j}\left\{tr\left(Z^n\right)\left(Z^{\dagger n-1}\right)^i_j e^{-tr\left(ZZ^{\dagger}\right)}\right\} = 0. \tag{2.8}$$

Using

$$\frac{d}{dZ^i_j}e^{-tr\left(ZZ^{\dagger}\right)} = -\left(Z^{\dagger}\right)^j_i e^{-tr\left(ZZ^{\dagger}\right)} \tag{2.9}$$

and

$$\frac{d}{dZ^i_j}tr\left(Z^n\right) = n\left(Z^{n-1}\right)^j_i \tag{2.10}$$

we find

$$\int\left[dZdZ^{\dagger}\right]\left\{ntr\left(Z^{n-1}Z^{\dagger n-1}\right) - tr\left(Z^n\right)tr\left(Z^{\dagger n}\right)\right\}e^{-tr\left(ZZ^{\dagger}\right)} = 0 \tag{2.11}$$

[1] The subleading terms are suppressed by powers of N^2 when compared to the leading term.

which we rewrite as

$$\left\langle tr\left(Z^{n}\right) tr\left(Z^{\dagger n}\right)\right\rangle = n\left\langle tr\left(Z^{n-1}Z^{\dagger n-1}\right)\right\rangle . \tag{2.12}$$

the left hand side is what we originally want to compute. A question we can ask at this point is whether we have cast the problem into a simpler form. It turns out that we have. The trace on the right hand side gives fewer Feynman diagrams than the left hand side. In particular, we get only one diagram that has n loops, i.e. there is only one diagram which gives us N^n, the leading term.[2] We have therefore recast the problem into a simpler one and to leading order, we write (as we found earlier)

$$\left\langle tr\left(Z^{n}\right) tr\left(Z^{\dagger n}\right)\right\rangle = n\left\langle tr\left(Z^{n-1}Z^{\dagger n-1}\right)\right\rangle = nN^{n}. \tag{2.13}$$

Having computed the easier problem,

$$\left\langle tr\left(Z^{n}\right) tr\left(Z^{\dagger n}\right)\right\rangle ,$$

let us now turn our attention to our original problem, equation (2.1). Disregarding the normalisation for the time being, we compute

$$\int \left[dZdZ^{\dagger}\right] \frac{d}{dZ^{\dagger i}_{j}} \left\{tr\left(Z^{n}\right)\left(Z^{m-1}\right)^{i}_{j} tr\left(Z^{\dagger n+m}\right) e^{-tr\left(ZZ^{\dagger}\right)}\right\} = 0 \tag{2.14}$$

which gives

$$\left\langle tr\left(Z^{n}\right) tr\left(Z^{m}\right) tr\left(Z^{\dagger n+m}\right)\right\rangle = \left\langle (n+m)\, tr\left(Z^{n}\right) tr\left(Z^{\dagger n+m-1}Z^{m-1}\right)\right\rangle . \tag{2.15}$$

Now,

$$tr\left(Z^{\dagger n+m-1}Z^{m-1}\right) = m tr\left(Z^{\dagger n}\right). \tag{2.16}$$

To see this, we can study the $m = 3$ and $n = 4$ case. In this case we have

$$tr\left(Z^{\dagger n+m-1}Z^{m-1}\right) = tr\left(Z^{\dagger 6}Z^{2}\right) \tag{2.17}$$

and the only planar contraction that we get has two closed loops resulting from contracting the two Zs with two of the six $Z^{\dagger}$s. Each of the closed loops gives a factor of N, while the remainder of the $Z^{\dagger}$s give us a factor of $tr\left(Z^{\dagger 4}\right)$. We therefore have

$$tr\left(Z^{\dagger 6}Z^{2}\right) = N^{2} tr\left(Z^{\dagger 4}\right). \tag{2.18}$$

[2]This follows immediately upon noting that only one of the diagrams contributing to the right hand side of (2.12) is planar.

The 4 in $tr\left(Z^{\dagger 4}\right)$ is the difference between the $n+m-1$ and $m-1$.

Substituting equation (2.16) into equation (2.15) yields

$$\left\langle tr\left(Z^{n}\right) tr\left(Z^{m}\right) tr\left(Z^{\dagger n+m}\right)\right\rangle = m\left(n+m\right) N^{m-1}\left\langle tr\left(Z^{n}\right) tr\left(Z^{\dagger n}\right)\right\rangle . \tag{2.19}$$

Using equation (2.13) we get

$$\left\langle tr\left(Z^{n}\right) tr\left(Z^{m}\right) tr\left(Z^{\dagger n+m}\right)\right\rangle = mn\left(n+m\right) N^{n+m-1}. \tag{2.20}$$

Finally, including the normalisation we arrive at

$$\left\langle \mathcal{O}_n \mathcal{O}_m \mathcal{O}^{\dagger}_{n+m}\right\rangle = \frac{\sqrt{mn\left(n+m\right)}}{N}. \tag{2.21}$$

To complete our motivation, we also need

$$\left\langle \mathcal{O}_n \mathcal{O}^{\dagger}_{m}\right\rangle = \delta_{mn} + \textit{subleading terms} \tag{2.22}$$

which agrees with

$$\langle \mathbf{p}_2 | \mathbf{p}_1 \rangle = \delta_{\mathbf{p}_1 \mathbf{p}_2}. \tag{2.23}$$

Looking at equation (2.21) we see that

$$\left\langle \mathcal{O}_n \mathcal{O}_m \mathcal{O}^{\dagger}_{n+m}\right\rangle = 0 \tag{2.24}$$

when $m,\ n = O\left(1\right)$ and $N \to \infty$. Similarly,

$$\left\langle \mathcal{O}_{n_1} \mathcal{O}_{n_2} \ldots \mathcal{O}_{n_l} \mathcal{O}^{\dagger}_{m_1} \mathcal{O}^{\dagger}_{m_2} \ldots \mathcal{O}^{\dagger}_{m_k}\right\rangle = 0 \tag{2.25}$$

when

$$l \neq k, \;\; m_i \neq n_i$$

for $i = 1 \ldots l$. Thus the multi-trace operators do not mix.

We now propose a dictionary between the single trace operators and the supergravity Fock space. If we think of n_i as momenta and also consider

$$\langle \mathbf{p}_2, \mathbf{p}_3 | \mathbf{p}_1 \rangle = 0, \tag{2.26}$$

we see that the number of traces should be identified with the number of particles. This reproduces the supergravity Fock space. We know that gravitational interaction increases as energy increases. Therefore a bigger n implies more momentum, more energy and therefore more interaction.

When
$$n \sim m \sim O\left(N^{\frac{2}{3}}\right), \tag{2.27}$$
we have
$$\left\langle \mathcal{O}_n \mathcal{O}_m \mathcal{O}^\dagger_{n+m} \right\rangle \neq 0$$
and the traces begin to mix.

We see from this section that we can learn something about supergravity by studying multi-trace operators.

2.2 Action of $\sigma \in S_n$.

To study the multi-trace operators, we will develop a description motivated by the representation theory of symmetric and unitary groups. With this in mind, we now introduce some notation and study the action of the symmetric group on the trace operator.

Consider a matrix Z that acts on an N-dimensional vector space V as follows
$$Z : V \to V,$$
i.e.
$$|w\rangle = Z\,|v\rangle\,, \tag{2.28}$$
where
$$|w\rangle\,,\ |v\rangle \in V.$$
We can also consider an N^n-dimensional vector space $V^{\otimes n}$, with elements
$$|u\rangle \otimes |v\rangle \otimes |w\rangle$$
for $n = 3$. In this case,
$$Z^{\otimes 3}\,|u\rangle \otimes |v\rangle \otimes |w\rangle = Z\,|u\rangle \otimes Z\,|v\rangle \otimes Z\,|w\rangle\,. \tag{2.29}$$
We can rewrite these relationships in index notation as
$$w^i = Z^i_j v^j \tag{2.30}$$
for (2.28), and
$$Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{i_3}_{j_3} u^{j_1} v^{j_2} w^{j_3} \tag{2.31}$$
for (2.29), where repeated indices are summed.

Now, we can consider the action of $\sigma \in S_n$ on the vector space $V^{\otimes n}$. Considering in particular $\sigma = (12)$, which swaps elements 1 and 2 around, we have

$$(12)\, u^{j_1} v^{j_2} w^{j_3} = u^{j_2} v^{j_1} w^{j_3}. \tag{2.32}$$

We can therefore write

$$(12)^I_J = \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \delta^{i_3}_{j_3}. \tag{2.33}$$

In general, we can write

$$\sigma\left(v_1 \otimes v_2 \otimes \ldots v_n\right) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots v_{\sigma(n)} \tag{2.34}$$

with

$$(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \ldots \delta^{i_n}_{j_{\sigma(n)}}. \tag{2.35}$$

Using this notation, we can write

$$tr\left(\sigma Z^{\otimes n}\right) = \sigma^I_J \left(Z^{\otimes n}\right)^J_I = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \ldots Z^{i_n}_{i_{\sigma(n)}}. \tag{2.36}$$

As an example, we can consider the case $n = 3$ for which the possible values of σ are

$$\sigma = 1,\ (12)\,,\ (13)\,,\ (23)\,,\ (123)\,,\ (132)\,. \tag{2.37}$$

For these values of σ, we find

$$tr\left(1 \cdot Z^{\otimes 3}\right) = tr\,(Z)^3\,, \tag{2.38}$$

$$tr\left((12) \cdot Z^{\otimes 3}\right) = tr\left((13) \cdot Z^{\otimes 3}\right) = tr\left((23) \cdot Z^{\otimes 3}\right) = tr\left(Z^2\right) tr\,(Z) \tag{2.39}$$

and

$$tr\left((123) \cdot Z^{\otimes 3}\right) = tr\left((132) \cdot Z^{\otimes 3}\right) = tr\left(Z^3\right). \tag{2.40}$$

We learn here that the conjugacy classes of the symmetric group correspond to specific multi-trace structures.

2.3 Correlation functions

We would like to write down a general formula for computing the n-point function for the case in which we have one type of field. We know that the two point function of our fields is

$$\left\langle Z^i_j Z^{\dagger k}_l \right\rangle = \delta^i_l \delta^k_j.$$

Similarly,

$$\left\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{\dagger k_1}_{l_1} Z^{\dagger k_2}_{l_2} \right\rangle = \delta^{i_1}_{l_1} \delta^{i_2}_{l_2} \delta^{k_1}_{j_1} \delta^{k_2}_{j_2} + \delta^{i_1}_{l_2} \delta^{i_2}_{l_1} \delta^{k_1}_{j_2} \delta^{k_2}_{j_1}. \tag{2.41}$$

Looking at the right hand side of equation (2.41) carefully, we notice that the lower indices in the first term are numbered exactly as the upper indices, while in the second term, the l labels are permuted relative to the i labels. The same follows for the $k's$ and $j's$. This action is similar to that of $\sigma \in S_n$. However, studying the $n = 3$ case establishes the fact that the $j's$ are actually acted on by σ^{-1}, while the $l's$ are acted on by σ. We can therefore write the 2n-point function as

$$\begin{aligned} \left\langle Z^{i_1}_{j_1} \dots Z^{i_n}_{j_n} Z^{\dagger k_1}_{l_1} \dots Z^{\dagger k_n}_{l_n} \right\rangle &= \sum_{\sigma \in S_n} \delta^{i_1}_{l_{\sigma(1)}} \dots \delta^{i_n}_{l_{\sigma(n)}} \delta^{k_1}_{j_{\sigma^{-1}(1)}} \dots \delta^{k_n}_{j_{\sigma^{-1}(n)}} \\ &\equiv \sum_{\sigma \in S_n} (\sigma)^I_L \left(\sigma^{-1}\right)^K_J . \end{aligned} \tag{2.42}$$

We can also write a general formula for

$$\left\langle \left(Z^{\otimes n} \otimes Y^{\otimes m}\right)^I_J \left(Z^{\dagger \otimes n} \otimes Y^{\dagger \otimes m}\right)^K_L \right\rangle$$

as we did in equation (2.42). By studying the $n = 3,\ m = 2$ case for example, we learn that

$$\begin{aligned} \left\langle \left(Z^{\otimes n} \otimes Y^{\otimes m}\right)^I_J \left(Z^{\dagger \otimes n} \otimes Y^{\dagger \otimes m}\right)^K_L \right\rangle = &\sum_{\sigma \in S_n \times S_m} \delta^{i_1}_{l_{\sigma(1)}} \dots \delta^{i_n}_{l_{\sigma(n)}} \delta^{i_{n+1}}_{l_{\sigma(n+1)}} \dots \delta^{i_{n+m}}_{l_{\sigma(n+m)}} \\ &\times \delta^{k_1}_{j_{\sigma^{-1}(1)}} \dots \delta^{k_n}_{j_{\sigma^{-1}(n)}} \delta^{k_{n+1}}_{j_{\sigma^{-1}(n+1)}} \dots \delta^{k_{n+m}}_{j_{\sigma^{-1}(n+m)}}. \end{aligned} \tag{2.43}$$

In general, we see that the correlation function for the multi-trace operators can be expressed entirely in terms of the symmetric group elements.

2.4 Schur polynomials

If we consider operators constructed using one type of field Z, a class of operators we can build are the Schur polynomials defined as

$$\chi_R(Z) \equiv \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\, tr\left(\sigma Z^{\otimes n}\right), \tag{2.44}$$

where

$$\chi_R(\sigma) = tr\left(\Gamma_R(\sigma)\right) \tag{2.45}$$

is the character of the group element $\sigma \in S_n$ in representation R. $\Gamma_R(\sigma)$ is a matrix representing σ in representation R. In the case $n = 3$, there are six distinct permutations σ, but only three distinct trace structures

$$tr\left(\sigma Z^{\otimes 3}\right) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} Z^{i_3}_{i_{\sigma(3)}} \tag{2.46}$$

and we have three distinct Schur polynomials. Let R_1 be the trivial representation, R_2 be the sign representation and R_3 be the standard representation of S_3. The Schur polynomials we have are

$$\chi_{R_1}(Z) = \frac{1}{3!}\left\{(trZ)^3 + 3tr(Z)\,tr\left(Z^2\right) + 2tr\left(Z^3\right)\right\}, \tag{2.47}$$

$$\chi_{R_2}(Z) = \frac{1}{3!}\left\{(trZ)^3 - 3tr(Z)\,tr\left(Z^2\right) + 2tr\left(Z^3\right)\right\} \tag{2.48}$$

and

$$\chi_{R_3}(Z) = \frac{1}{3}\left\{(trZ)^3 - tr\left(Z^3\right)\right\}. \tag{2.49}$$

Now,

$$\left\langle \chi_{R_1}(Z)\,\chi^{\dagger}_{R_1}(Z)\right\rangle = N^3 + 3N^2 + 2N \tag{2.50}$$

and

$$\left\langle \chi_{R_1}(Z)\,\chi^{\dagger}_{R_3}(Z)\right\rangle = 0. \tag{2.51}$$

The right hand side of equation (2.50) is actually the product of the factors of the irreducible representation (irrep) R, f_R. In order to understand what these factors are, let us consider an example. Let us compute the f_R for the Young diagram of S_3 shown in figure 2.1. We label the top left hand box N, then as we move to the right, we add one and subtract one when we move down. This way, the factors of the Young tableaux shown in figure 2.1 are N, $N+1$ and $N-1$ as indicated. The product of factors for this Young tableaux is therefore

$$f_R = N(N+1)(N-1). \tag{2.52}$$

Both equations (2.50) and (2.51), follow from the more general formula of the two point function

$$\left\langle \chi_R(Z)\,\chi^{\dagger}_S(Z)\right\rangle = \delta_{RS} f_R. \tag{2.53}$$

A more general proof of equation (2.53) will be presented in Section 2.4.2.

As we recounted in Chapter 1, these Schur polynomials were first studied in the context of giant gravitons in [26].

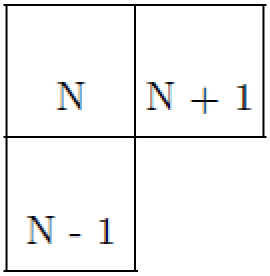

Figure 2.1: The standard Young tableaux for S_3.

2.4.1 Projectors

Let us define the operator

$$P_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\, \sigma. \tag{2.54}$$

If $\gamma \in S_n$ also, then P_R commutes with γ, i.e.

$$P_R \cdot \gamma = \gamma \cdot P_R. \tag{2.55}$$

To verify this, we can compute both sides of equation (2.55) for the group S_3. In particular, if we may consider the standard representation of S_3 with

$$\sigma^I_J = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \delta^{i_3}_{j_{\sigma(3)}} \tag{2.56}$$

and

$$\gamma^J_K = (23)^J_K = \delta^{j_1}_{k_1} \delta^{j_2}_{k_3} \delta^{j_3}_{k_2} \tag{2.57}$$

we find that

$$P_R \cdot \gamma = \frac{1}{3!} \{2\gamma - (13) - (12)\} \tag{2.58}$$

which indeed agrees with $\gamma \cdot P_R$.

For the more general proof we can write

$$P_R \cdot \gamma = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\, \sigma\gamma \tag{2.59}$$

and introduce a change of variable

$$\psi = \gamma^{-1} \sigma \gamma \tag{2.60}$$

to get

$$\begin{aligned}P_R \cdot \gamma &= \frac{1}{n!}\sum_{\psi\in S_n}\chi_R\left(\gamma\psi\gamma^{-1}\right)\gamma\psi\gamma^{-1}\gamma \\ &= \frac{1}{n!}\sum_{\psi\in S_n} tr\left(\Gamma_R\left(\gamma\psi\gamma^{-1}\right)\right)\gamma\psi \\ &= \frac{1}{n!}\sum_{\psi\in S_n} tr\left(\Gamma_R\left(\gamma^{-1}\gamma\psi\right)\right)\gamma\psi \\ &= \gamma\frac{1}{n!}\sum_{\psi\in S_n}\chi_R\left(\psi\right)\psi \\ &= \gamma\cdot P_R \end{aligned} \tag{2.61}$$

which completes the proof.

Using the fundamental orthogonality relation of a group G, we can also show that

$$P_R \cdot P_S = \frac{1}{d_R}\delta_{RS}P_R, \tag{2.62}$$

where d_R is the dimension of the Young diagram R. Thus we see that the operators P_R are projection operators. For an S_n group,

$$d_R = \frac{n!}{hooks_R}, \tag{2.63}$$

where $hooks_R$ stands for the product of hook-lengths of the Young diagram R a representation of S_n. The fundamental orthogonality relation of a group G states that

$$\sum_{g\in G}\Gamma_R\left(g\right)_{ab}\Gamma_S\left(g^{-1}\right)_{cd} = \frac{|G|}{d_R}\delta_{RS}\delta_{ad}\delta_{bc}, \tag{2.64}$$

where $|G|$ is the order of the group. For the symmetric group we have

$$|S_n| = n!. \tag{2.65}$$

Setting $G = S_n$ and $g = \sigma$ in our case, we compute

$$P_R \cdot P_S = \frac{1}{n!n!}\sum_{\sigma\epsilon S_n}\sum_{\gamma\in S_n} tr\left(\Gamma_R\left(\sigma\right)\right) tr\left(\Gamma_S\left(\gamma\right)\right)\sigma\cdot\gamma. \tag{2.66}$$

Further, we set

$$\gamma = \sigma^{-1}\rho \tag{2.67}$$

and use

$$\Gamma_R(\gamma) = \Gamma_R\left(\sigma^{-1}\rho\right) = \Gamma_R\left(\sigma^{-1}\right)\Gamma_R(\rho) \tag{2.68}$$

together with equation (2.64) to get

$$P_R \cdot P_S = \frac{1}{n! d_R}\delta_{RS}\delta_{lj}\sum_{\rho\in S_n}\Gamma_S(\rho)_{jl}\,\rho \tag{2.69}$$

which simplifies to equation (2.62).

We can also use the fact that $\chi_R(U)$ (a Schur polynomial evaluated with $Z = U \in U(N)$) equals the character of the group element U in irrep R to evaluate the trace of the projector P_R. In this case,

$$\chi_R(U) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\, tr\left(\sigma U^{\otimes n}\right). \tag{2.70}$$

One element of the unitary group $U(N)$ that we know is the identity $\mathbf{1}$. Its character is the dimension of the irrep R, i.e.

$$\chi_R(\mathbf{1}) = Dim_N(R). \tag{2.71}$$

If we consider $\sigma = (12)$,

$$(\sigma)^I_J = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1} \tag{2.72}$$

and

$$tr((12)) = N \tag{2.73}$$

while

$$tr(\mathbf{1}) = N^2. \tag{2.74}$$

From this, we can see that the power of N equals the number of cycles of σ, $C(\sigma)$. Thus

$$tr(\sigma) = N^{C(\sigma)}. \tag{2.75}$$

We also know that

$$tr(\sigma\cdot\mathbf{1}) = tr(\sigma). \tag{2.76}$$

Using these results, we can calculate the trace of the projector P_R,

$$\begin{aligned} tr\left(P_R\right) &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R\left(\sigma\right)tr\left(\sigma\right) \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R\left(\sigma\right)tr\left(\sigma\mathbf{1}\right) \\ &= \chi_R\left(\mathbf{1}\right) \\ &= Dim_N\left(R\right), \end{aligned} \tag{2.77}$$

where

$$Dim_N\left(R\right) = \frac{\prod_{i,j}\left(N-i+j\right)}{hooks_R} \tag{2.78}$$

is the dimension of the Young diagram R as a representation of the group $U\left(N\right)$.

2.4.2 The two-point function of Schur polynomials

We now derive the two-point function of Schur polynomials, equation (2.53), by following the original argument given in [26]. We will also need

$$\sum_{\sigma\in S_n}\chi_R\left(\sigma^{-1}\right)\chi_S\left(\sigma\right) = n!\delta_{RS} \tag{2.79}$$

which also follows from the fundamental orthogonality relation, equation (2.64), as well as the delta function of a group,

$$\delta\left(\rho\right) = \frac{1}{n!}\sum_R d_R\chi_R\left(\rho\right). \tag{2.80}$$

To prove equation (2.53), we first convert the sum over contractions to a sum over symmetric groups. Thus

$$\begin{aligned}
\left\langle \chi_R(Z)\,\chi_S^\dagger(Z)\right\rangle &= \left\langle \sum_{\sigma\in S_n}\frac{\chi_R(\sigma)}{n!}tr\,(\sigma Z)\sum_{\tau\in S_n}\frac{\chi_S^\star(\tau)}{n!}tr\left(\tau^{-1}Z^\dagger\right)\right\rangle \\
&= \sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}\left\langle Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_n}_{i_{\sigma(n)}}Z^{\dagger j_1}_{j_{\tau^{-1}(1)}}\dots Z^{\dagger j_n}_{j_{\tau^{-1}(n)}}\right\rangle \\
&= \sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}\delta^{i_1}_{j_{\alpha\tau^{-1}(1)}}\dots\delta^{i_n}_{j_{\alpha\tau^{-1}(n)}}\delta^{j_1}_{i_{\alpha^{-1}\sigma(1)}}\dots\delta^{j_n}_{i_{\alpha^{-1}\sigma(n)}} \\
&= \sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}\delta^{i_{\tau\alpha^{-1}(1)}}_{j_1}\dots\delta^{i_{\tau\alpha^{-1}(n)}}_{j_n}\delta^{j_1}_{i_{\alpha^{-1}\sigma(1)}}\dots\delta^{j_n}_{i_{\alpha^{-1}\sigma(n)}} \\
&= \sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}\delta^{i_{\tau\alpha^{-1}(1)}}_{i_{\alpha^{-1}\sigma(1)}}\dots\delta^{i_{\tau\alpha^{-1}(n)}}_{i_{\alpha^{-1}\sigma(n)}} \\
&= \sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}\delta^{i_{\sigma^{-1}\alpha\tau\alpha^{-1}(1)}}_{i_1}\dots\delta^{i_{\sigma^{-1}\alpha\tau\alpha^{-1}(n)}}_{i_n} \\
&= \sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}tr\left(\sigma^{-1}\alpha\tau\alpha^{-1}\right) \\
&= \sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}N^{C\left(\sigma^{-1}\alpha\tau\alpha^{-1}\right)}, \qquad (2.81)
\end{aligned}$$

where we have used equation (2.42) to write the third equality and equation (2.75) to get the last line. At this stage, let us introduce a new variable

$$\rho = \sigma^{-1}\alpha\tau\alpha^{-1} \qquad (2.82)$$

in terms of which the two-point function (2.81) becomes

$$\begin{aligned}
\left\langle \chi_R(Z)\,\chi_S^\dagger(Z)\right\rangle &= \sum_{\rho\in S_n}\sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\sum_{\tau\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\tau^{-1}\right)}{n!}N^{C(\rho)}\delta\left(\rho^{-1}\sigma^{-1}\alpha\tau\alpha^{-1}\right) \\
&= \sum_{\rho\in S_n}\sum_{\alpha\in S_n}\sum_{\sigma\in S_n}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S\left(\alpha^{-1}\rho^{-1}\sigma^{-1}\alpha\right)}{n!}N^{C(\rho)}. \qquad (2.83)
\end{aligned}$$

Summing over $\alpha \in S_n$ in this case gives a factor of $n!$ so that we have

$$\left\langle \chi_R(Z)\,\chi_S^\dagger(Z)\right\rangle = \frac{1}{n!}\sum_{\rho\in S_n}\sum_{\sigma\in S_n}\chi_R(\sigma)\,\chi_S\left(\rho^{-1}\sigma^{-1}\right)N^{C(\rho)}. \qquad (2.84)$$

Writing this in matrix notation gives

$$\left\langle \chi_R(Z)\,\chi_S^\dagger(Z)\right\rangle = \frac{1}{n!}\sum_{\rho\in S_n}\sum_{\sigma\in S_n}\Gamma_R(\sigma)_{ii}\,\Gamma_S\left(\sigma^{-1}\right)_{jk}\Gamma_S\left(\rho^{-1}\right)_{kj}N^{C(\rho)}. \tag{2.85}$$

Now we can apply the fundamental orthogonality relation (2.64) in order to perform the over $\sigma \in S_n$. This yields

$$\left\langle \chi_R(Z)\,\chi_S^\dagger(Z)\right\rangle = \frac{\delta_{RS}}{d_R}\sum_{\rho\in S_n}\chi_S\left(\rho^{-1}\right)N^{C(\rho)}. \tag{2.86}$$

Using the result (2.77) then yields

$$\left\langle \chi_R(Z)\,\chi_S^\dagger(Z)\right\rangle = \frac{n!\delta_{RS}Dim_N(R)}{d_R}. \tag{2.87}$$

From equations (2.63) and (2.78),

$$\frac{n!Dim_N(R)}{d_R} = \prod_{i,j}(N-i+j) \equiv f_R. \tag{2.88}$$

Putting this into equation (2.86) yields precisely the correlation function (2.53) and this concludes the proof.

2.5 Restricted Schur polynomials

$\mathcal{N}=4$ super Yang-Mills theory has six hermitian Higgs fields ϕ_i, with $i=1,2,\ldots 6$. It is from these fields that we build the complex matrices

$$Z=\phi_1+i\phi_2, \quad X=\phi_3+i\phi_4, \quad Y=\phi_5+i\phi_6. \tag{2.89}$$

The space of 1/2 BPS representations in $\mathcal{N}=4$ SYM theory is in one-to-one correspondence with the Schur polynomials built using Z [26]. Furthermore, these Schur polynomials have diagonal two-point functions as we have seen in Section 2.4. Employing insights from the dual quantum gravity theory, restricted Schur polynomials were identified as the excitations of these 1/2 BPS states [32]. Given a Schur polynomial, a *restricted* Schur polynomial is obtained by attaching (or replacing some of the Z fields with) impurities or open string words W. The letters of these open string words can be fermions, gauge fields or any of the other Higgs fields. If the word W contains $O\left(\sqrt{N}\right)$ letters, it is dual to an open string. With $O(N)$ fields, the restricted Schur polynomial is dual to a membrane with open strings attached, while $O\left(N^2\right)$ fields describe strings moving in a new geometry.

In the $su(2)$ sector of $\mathcal{N}=4$ SYM theory, we can define the following restricted Schur polynomial,

$$\chi_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n},Y^{\otimes m}\right)=\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\mu_1\mu_2}(\sigma)\,Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right), \tag{2.90}$$

in which the irrep (r,s) is one of the irreps that arises when R, an irrep of S_{n+m}, is restricted to the subgroup $S_n\times S_m$. We can remove m boxes from the Young diagram R to remain with the Young diagram $r\vdash n$. Assembling these m boxes gives the Young diagram s. In general, the irrep (r,s) occurs with some multiplicity. The indices μ_1 and μ_2 tell us which copy of (r,s) we are considering. The restricted trace

$$\chi_{R,(r,s)\mu_1\mu_2}(\sigma)=Tr\left(P_{R\to(r,s)\mu_1\mu_2}\Gamma_R(\sigma)\right) \tag{2.91}$$

is taken over the space labelled by the Young diagrams r and s. To ensure this, we have a projector $P_{R\to(r,s)\mu_1\mu_2}$ that takes us from the space labelled by R to that labelled by (r,s). In this case, we are interested in the case in which both n and m are order $O(N)$, with $n\gg m$.

The Young diagrams r and s are subduced from $R\vdash n+m$ with some multiplicities specified by μ_1 and μ_2. Starting with a Young diagram $R\vdash n+m$, we can remove m boxes associated to the impurity labels, to remain with $r\vdash n$. Assembling the m boxes we removed gives us the diagram s in more ways than one. To specify the particular copy of s we are considering, we have the multiplicity indices μ_1 and μ_2. More information about this is provided in Chapter 3.

2.5.1 Projectors for restricted Schurs

Strictly speaking, the projectors that appear in restricted Schur polynomials are intertwinners[3] that bear some of the properties of projectors. The term projector in this case is therefore used loosely. We consider

$$P_{R,(r,s)\alpha\beta}\equiv\frac{1}{(n+m)!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\cdot\sigma \tag{2.92}$$

[3] An interwtinner is a map between isomorphic irreps.

where R is an irrep of S_{n+m} while (r,s) is an irrep of $S_n \times S_m$. The restricted character[4] in equation (2.92) is defined as

$$\chi_{R,(r,s)\alpha\beta}(\sigma) \equiv \sum_i \langle i,(r,s)\,\alpha|\Gamma_R(\sigma)|i,(r,s)\,\beta\rangle \tag{2.93}$$

where α and β specify which copies of (r,s) we take. The index i is a state label.

It is of interest to compute the product of these projectors. To this end, we compute

$$\begin{aligned} P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta} &= \frac{1}{(n+m)!}\cdot\frac{1}{(n+m)!}\sum_{\sigma,\tau\in S_{n+m}}\sum_{i,j}\langle i,(r,s)\,\alpha|\Gamma_R(\sigma)|i,(r,s)\,\beta\rangle \\ &\times \langle j,(t,u)\,\gamma|\Gamma_T(\tau)|j,(t,u)\,\delta\rangle\,\sigma\tau \\ &= \frac{1}{(n+m)!}\cdot\frac{1}{(n+m)!}\sum_{\sigma,\tau\in S_{n+m}}\sum_{i,j}\langle i,(r,s)\,\alpha|\,\Gamma_R(\sigma)\,\delta_{rt}\delta_{su}\delta_{\beta\gamma}\delta_{ij} \\ &\times \Gamma_T(\tau)\,|j,(t,u)\,\delta\rangle\,\sigma\tau. \end{aligned} \tag{2.94}$$

Setting

$$\tau = \sigma^{-1}\mu \tag{2.95}$$

yields

$$\begin{aligned} P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta} &= \frac{1}{(n+m)!}\cdot\frac{1}{(n+m)!}\sum_{\sigma,\mu\in S_{n+m}}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\sum_i\langle i,(r,s)\,\alpha|\,\Gamma_R(\sigma) \\ &\times \Gamma_T\left(\sigma^{-1}\right)\Gamma_T(\mu)\,|i,(t,u)\,\delta\rangle\,\mu. \end{aligned} \tag{2.96}$$

Using the fundamental orthogonality relation (2.64) to sum over $\sigma \in S_{n+m}$ gives

$$P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta} = \frac{1}{d_R}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\frac{1}{(n+m)!}\sum_{\mu\in S_{n+m}}\sum_i\langle i,(r,s)\,\alpha|\,\Gamma_T(\mu)\,|i,(t,u)\,\delta\rangle\,\mu \tag{2.97}$$

in which we recognise the definition of the projector. Writing this out, the product of the projectors then works out to

$$P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta} = \frac{\delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\gamma}}{d_R}P_{T,(t,u)\alpha\delta}. \tag{2.98}$$

To compute the trace of the projector, we use the properties of Jucys-Murphys

[4] Restricted characters were first studied in [33].

elements. This was done in Appendix F of [33] whose result we now use. We obtain

$$\begin{aligned} Tr\left(P_{R,(r,s)\alpha\beta}\right) &= \frac{1}{(n+m)!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)tr\left(\sigma\right) \\ &= \frac{1}{(n+m)!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)N^{C(\sigma)} \\ &= \frac{1}{(n+m)!}d_r d_s \delta_{\alpha\beta} f_R. \end{aligned} \tag{2.99}$$

2.5.2 Two-point function

To compute the two-point function of restricted Schur polynomials, we will proceed as we did with Schur polynomials. In what follows, the sum over Wick contractions is performed by a sum over $S_n \times S_m$. Using the free field result, equation (2.43), we have

$$\begin{aligned} &\left\langle \chi_{R,(r,s)\alpha\beta}\left(Z^{\otimes n}, Y^{\otimes m}\right)\chi^{\dagger}_{T,(t,u)\delta\gamma}\left(Z^{\otimes n}, Y^{\otimes m}\right)\right\rangle = \sum_{\sigma,\tau\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi^{\dagger}_{T,(t,u)\delta\gamma}\left(\tau\right)}{n!m!} \\ &\times\left\langle Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}Z^{\dagger j_1}_{j_{\tau^{-1}(1)}}\cdots Z^{\dagger j_n}_{j_{\tau^{-1}(n)}}Y^{\dagger j_{n+1}}_{j_{\tau^{-1}(n+1)}}\cdots Y^{\dagger j_{n+m}}_{j_{\tau^{-1}(n+m)}}\right\rangle \\ &= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\tau\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\tau^{-1}\right)}{n!m!}\delta^{i_1}_{j_{\xi\tau^{-1}(1)}}\cdots\delta^{i_{n+m}}_{j_{\xi\tau^{-1}(n+m)}}\delta^{j_1}_{i_{\xi^{-1}\sigma(1)}}\cdots\delta^{j_{n+m}}_{i_{\xi^{-1}\sigma(n+m)}} \\ &= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\tau\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\tau^{-1}\right)}{n!m!}\delta^{i_{\tau\xi^{-1}(1)}}_{j_1}\cdots\delta^{i_{\tau\xi^{-1}(n+m)}}_{j_{n+m}}\delta^{j_1}_{i_{\xi^{-1}\sigma(1)}}\cdots\delta^{j_{n+m}}_{i_{\xi^{-1}\sigma(n+m)}} \\ &= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\tau\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\tau^{-1}\right)}{n!m!}\delta^{i_{\tau\xi^{-1}(1)}}_{i_{\xi^{-1}\sigma(1)}}\cdots\delta^{i_{\tau\xi^{-1}(n+m)}}_{i_{\xi^{-1}\sigma(n+m)}} \\ &= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\tau\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\tau^{-1}\right)}{n!m!}\delta^{i_{\sigma^{-1}\xi\tau\xi^{-1}(1)}}_{i_1}\cdots\delta^{i_{\sigma^{-1}\xi\tau\xi^{-1}(n+m)}}_{i_{n+m}} \\ &= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\tau\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\tau^{-1}\right)}{n!m!}tr\left(\sigma^{-1}\xi\tau\xi^{-1}\right) \\ &= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\tau\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\tau^{-1}\right)}{n!m!}N^{C\left(\sigma^{-1}\xi\tau\xi^{-1}\right)}. \end{aligned} \tag{2.100}$$

Introducing a new variable

$$\rho = \sigma^{-1}\xi\tau\xi^{-1}, \tag{2.101}$$

equation (2.100) becomes

$$\begin{aligned}
&\left\langle \chi_{R,(r,s)\alpha\beta}\left(Z^{\otimes n},Y^{\otimes m}\right)\chi^{\dagger}_{T,(t,u)\delta\gamma}\left(Z^{\otimes n},Y^{\otimes m}\right)\right\rangle \\
&= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\tau,\rho\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\tau^{-1}\right)}{n!m!}N^{C(\rho)}\delta\left(\rho^{-1}\sigma^{-1}\xi\tau\xi^{-1}\right) \\
&= \sum_{\xi\in S_n\times S_m}\sum_{\sigma,\rho\in S_{n+m}}\frac{\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)}{n!m!}\frac{\chi_{T,(t,u)\gamma\delta}\left(\xi^{-1}\rho^{-1}\sigma^{-1}\xi\right)}{n!m!}N^{C(\rho)} \\
&= \frac{1}{n!m!}\sum_{\sigma,\rho\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}\left(\sigma\right)\chi_{T,(t,u)\gamma\delta}\left(\rho^{-1}\sigma^{-1}\right)N^{C(\rho)},
\end{aligned} \tag{2.102}$$

where the sum over $\xi \in S_n \times S_m$ gives $n!m!$ again. We now use the fundamental orthogonality relation to sum over $\sigma \in S_{n+m}$. The result is

$$\begin{aligned}
&\left\langle \chi_{R,(r,s)\alpha\beta}\left(Z^{\otimes n},Y^{\otimes m}\right)\chi^{\dagger}_{T,(t,u)\delta\gamma}\left(Z^{\otimes n},Y^{\otimes m}\right)\right\rangle \\
&= \frac{\delta_{RT}}{n!m!d_R}\sum_{\rho\in S_{n+m}}Tr\left(P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta}\Gamma_T\left(\rho^{-1}\right)\right)N^{C(\rho)}.
\end{aligned} \tag{2.103}$$

Using equation (2.98) yields

$$\begin{aligned}
&\left\langle \chi_{R,(r,s)\alpha\beta}\left(Z^{\otimes n},Y^{\otimes m}\right)\chi^{\dagger}_{T,(t,u)\delta\gamma}\left(Z^{\otimes n},Y^{\otimes m}\right)\right\rangle \\
&= \frac{(n+m)!}{n!m!d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\sum_{\rho\in S_{n+m}}Tr\left(P_{T,(t,u)\alpha\delta}\Gamma_T\left(\rho^{-1}\right)\right)N^{C(\rho)} \\
&= \frac{(n+m)!}{n!m!d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\sum_{\rho\in S_{n+m}}\chi_{T,(t,u)\alpha\delta}\left(\rho^{-1}\right)N^{C(\rho)}.
\end{aligned} \tag{2.104}$$

Finally, using Appendix F of [33] and simplifying, we get

$$\left\langle \chi_{R,(r,s)\alpha\beta}\left(Z^{\otimes n},Y^{\otimes m}\right)\chi^{\dagger}_{T,(t,u)\delta\gamma}\left(Z^{\otimes n},Y^{\otimes m}\right)\right\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\delta_{\alpha\delta}\frac{hooks_R}{hooks_r hooks_s}f_R. \tag{2.105}$$

The calculation given here closely follows the original derivation in [36].

2.5.3 Action of the dilatation operator

The dilatation operator acts on the restricted Schur polynomials in the $su(2)$ sector of $\mathcal{N}=4$ SYM theory to give [46]

$$DO_{R,(r,s)\mu_1\mu_2} = -g^2_{YM}\sum_{u,\nu_1,\nu_2}\sum_{i<j}M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}\triangle_{ij}O_{R,(r,u)\nu_1\nu_2}, \tag{2.106}$$

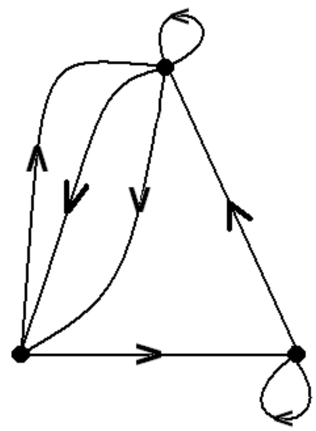

Figure 2.2: An example of a valid open string configuration consisting of $p = 3$ branes and $m = 7$ strings.

where $M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}$ is a matrix that acts only on the impurity labels and $\triangle_{ij}$ acts only the Young diagrams R, r associated with the Z fields. Because the action of the dilatation operator is thus factorised, we can diagonalise the impurity labels separately from the Z fields. The problem of diagonalising the impurity labels was solved by means of a double coset ansatz [46], reviewed in Section 2.6. On the other hand, the problem associated with the Z fields was solved in [45]. Notably, on the R, r labels, the action of the dilatation operator reduces to the Hamiltonian of a set of decoupled oscillators. Because the harmonic oscillator is known to be integrable, we conclude that the $su(2)$ sector of $\mathcal{N} = 4$ SYM theory is integrable in the non-planar limit.

2.6 Double coset ansatz

In this section we review the double coset ansatz [46] for the $su(2)$ sector of $\mathcal{N} = 4$ SYM theory. We first argue that the number of states of an excited system of separated giant gravitons is equal to the number of restricted Schur polynomials that are labelled by Young diagrams in the widely separated corners limit.

2.6.1 Gauss graphs

The giant gravitons we discussed in Section 1.4 have compact world volumes. For compact world volumes, Gauss law implies that the total charge on the giant graviton's world volume must vanish. This gives a constraint on the number of open string configurations that are allowed since the open string ends are charged. In particular, the number of strings emanating from a given giant graviton must equal the number of strings termin-

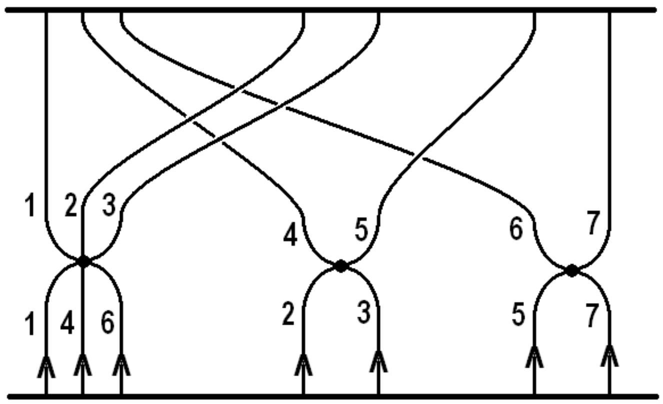

Figure 2.3: A labelled graph for the open string configuration shown in figure 2.2.

ating on it. For each allowed configuration, we can draw a Gauss graph. An example is shown in figure 2.2 in which the dots represent the giant gravitons and the directed lines are the open strings. The arrows on the open strings strings show the string orientation.

A mathematical description of the Gauss graphs can be achieved by labelling the graphs with some numbers. Let us say we have p branes and m strings which we 'cut' into halves. Let us label the half outgoing strings with numbers $1, \dots m$ and the half incoming strings with the same set of numbers. The way in which these two sets of strings are joined is provided by the permutation $\sigma \in S_m$. Let us also label the branes with numbers $1, \dots p$ so that the number of incoming (or outgoing) strings on the p^{th} brane is m_p. It should be clear that

$$m_1 + m_2 + \cdots m_p = m. \tag{2.107}$$

Further, let the labelling on the strings be such that the strings emanating from the p^{th} brane are $m_{p-1}+1$, $m_{p-1}+2$, $\dots m_p$. This way, the configuration shown in figure 2.2 can be mapped into the labelled graph shown in figure 2.3. In this case (figure 2.3), the top bold line must be identified with the bottom bold line.

As we have already mentioned, the structure of the labelled graph is encoded in the permutation $\sigma \in S_m$, but there is a redundancy in the coding since the incoming strings that terminate on the p^{th} giant are indistinguishable. In other words, labelled graphs which differ only by swapping the end points that connect to the same brane give the same configuration. This is immediately clear from figure 2.3. One way to resolve this is to relabel the outgoing half-strings by permutations in their symmetry group $\prod_i S_{m_i}$. This results in the multiplication of σ from the left. Doing the same with the incoming half-strings results in multiplying σ from the right. This way, we see that the open string configurations are in one-to-one correspondence with the elements of the double coset

$$H \setminus S_m / H, \tag{2.108}$$

where

$$H = S_{m_1} \times S_{m_2} \times \cdots S_{m_p}. \tag{2.109}$$

Each element of the double coset gives a distinct Gauss graph.

Through the Burnside Lemma [55], the number of open string configurations (or Gauss graphs) is

$$N_C = \frac{1}{|H|^2} \sum_{\alpha_1 \in H} \sum_{\alpha_2 \in H} \sum_{\sigma_1 \in S_m} \delta\left(\alpha_2 \sigma_1^{-1} \alpha_1^{-1} \sigma_1\right). \tag{2.110}$$

The delta function of the symmetric group

$$\delta\left(\rho\right) = \frac{1}{n!} \sum_R d_R \chi_R\left(\rho\right), \tag{2.111}$$

where R is a Young diagram with n boxes, is defined to be one when $\rho \in S_n$ is identity and zero otherwise. Using this together with the fundamental orthogonality relation, equation (2.110) can be written as

$$N_C = \frac{1}{|H|^2} \sum_{\alpha_1 \in H} \sum_{\alpha_2 \in H} \sum_{s \vdash m} \chi_s\left(\alpha_2\right) \chi_s\left(\alpha_1\right). \tag{2.112}$$

The sums over α_1 and α_2 give projection operators which project onto the trivial representation of H. Equation (2.112) is equivalent to

$$N_C = \sum_{s \vdash m} \left(\mathcal{M}^s_{1_H}\right)^2 \tag{2.113}$$

where $\mathcal{M}^s_{1_H}$ is the multiplicity of the of the one-dimensional representation of H when the irreducible representation s of S_m is decomposed into representations of the subgroup H.

2.6.2 Counting

In this section, we show that the number of restricted Schur polynomials, the operators that describe the giant gravitons with strings attached (as reviewed in Section 1.6), is equal to the number of Gauss graph operators, equation (2.113). To start with, we recall that the Young diagram $R \vdash n+m$ that appears in restricted Schur polynomials, equation (2.90), can be decomposed into smaller Young diagrams $r \vdash n$ and $s \vdash m$. We can restrict any irrep of S_{n+m} to its $S_n \times S_m$ subgroup. Generically, we will get a redu-

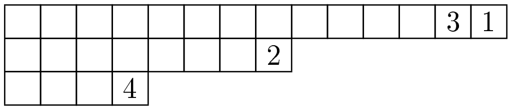

Figure 2.4: A Young diagram with $p = 3$ rows and $m = 4$ impurity boxes.

cible representation. Irreducible representations into which this reducible representation decomposes are labelled by (r, s). This is accomplished by removing m boxes from R to remain with r, and then assembling the m boxes into a diagram s. In general, there are various ways of obtaining the same diagram s. The particular copy of the diagram s is specified by the multiplicity labels $\mu_1\mu_2$. The Young diagrams s label a vector space $V_p^{\otimes m}$ for which there are two ways to decompose. The vector space V_p can be written as a sum of one-dimensional vector spaces V_i, i.e.

$$V_p = \bigoplus_{i=1}^{p} V_i. \tag{2.114}$$

In the restricted Schur polynomial for long rows, a state in V_i corresponds to an impurity box in the i^{th} row. As an example, consider a Young diagram with $p = 3$ rows, in which we label the impurity boxes as shown figure 2.4. The $m = 4$ impurity boxes in the diagram correspond to

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.115}$$

Using Schur-Weyl duality[5], we can write $V_p^{\otimes m}$ in terms of irreps of the groups $U(p)$ and S_m,

$$V_p^{\otimes m} = \bigoplus_{s \vdash m, c_1(s) \le p} V_s^{U(p)} \otimes V_s^{S_m}, \tag{2.116}$$

where $c_1(s)$ is the number of parts of s. If the way in which we remove the m boxes from the Young diagram $R \vdash n + m$ is given by $\vec{m}$ with

$$\sum_{i=1}^{p} m_i = m, \tag{2.117}$$

then we can write equation (2.116) as

$$V_p^{\otimes m} = \bigoplus_{s \vdash m, c_1(s) \le p} \bigoplus_{\vec{m}} \bigotimes_{1}^{m} V_{m_i}^{U_i(1)} \otimes V_{s \to \vec{m}}^{U(p) \to U(1)^p} \otimes V_s^{S_m}. \tag{2.118}$$

[5] We review Schur-Weyl duality in Section 3.2.2.

Here, we decomposed the $U(p)$ irrep into $U(1)^p$ irreps, summing over all the irreps of this group labelled by $\vec{m}$ (which gives the $U(1)$ charges). $V_{m_i}^{U_i(1)}$ is the one-dimensional irrep which transforms with charge i under the i^{th} $U(1)$. In the language of restricted Schur polynomials, these are the numbers of boxes in the i^{th} row. Each set of the $U(1)$ charges $\vec{m}$ comes with a multiplicity label as we have already explained above. the multiplicity labels span a vector space $V_{s\to\vec{m}}^{U(p)\to U(1)^p}$, whose dimension is the number of times the irrep $\vec{m}$ appears when irrep s is decomposed under the subgroup $U(1)^p$. These are the Kostka numbers [56], $\mathcal{K}_{s\vec{m}}$, which are reviewed in Appendix B of [46]. Since the restricted Schur polynomials are labelled by a pair of multiplicity labels, the total number of restricted Schur polynomials is given by the sum of the squares of the Kostka numbers,

$$N_C = \sum_{s\vdash m, c_1(s)\le p} \left(\mathcal{K}_{s\vec{m}}\right)^2 . \tag{2.119}$$

To complete our proof, we now need to show that equation (2.119) is equivalent to equation (2.113). Our proof hinges on the Schur-Weyl duality which we now develop more fully. We will work at the level of a basis for $V_p^{\otimes m}$ in which case the reduction coefficients that arise in the final step are branching coefficients for irrep s of $U(p)$ into the irrep $\vec{m}$ of $\mathcal{H}\equiv U(1)^p$. Let

$$|I\rangle \equiv |i_1, i_2, \ldots i_p\rangle \tag{2.120}$$

be a basis for the tensor product. We know from Schur-Weyl duality that there is a change of basis to

$$|I\rangle = \sum_{s,m_s,M_s} |s, M_s, m_s\rangle \langle s, M_s, m_s|I\rangle , \tag{2.121}$$

where M_s is a state label for the $U(p)$ irrep s, corresponding to semi-standard Young tableaux [44]. In the same vein, m_s is a state label for the irrep s of S_m, which can be described by standard Young tableaux. Decomposing into $U(1)^p$, we get

$$|I\rangle = \sum_{\vec{m},\nu} \sum_{s,m_s,M_s} C_{M_s}^{\vec{m},\nu} |s, M_s, m_s\rangle \langle s, M_s, m_s|I\rangle , \tag{2.122}$$

where the coefficient $C_{M_s}^{\vec{m},\nu}$ gives the decomposition of a $U(p)$ irrep into $U(1)^p$ irreps, and ν is a multiplicity label which labels states in $V_{s\to\vec{m}}^{U(p)\to U(1)^p}$.

We have decomposed $V_p^{\otimes m}$ into irreps of $\mathcal{H}$ in one way which is equivalent to equation (2.118), but there is another way which uses permutations of S_m. Our choice of $\vec{m}$ implies that there are m_1 copies of v_1, m_2 copies of v_2 and so forth, where v_i is a vector belonging to V_i. One state we can have this way is

$$\left|\bar{v}, \vec{m}\right\rangle \equiv \left|v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots v_p^{\otimes m_p}\right\rangle . \tag{2.123}$$

All other states are related to (2.123) by permutations, i.e. we can write the general state as

$$\left|v_\sigma\right\rangle \equiv \sigma \left|v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots v_p^{\otimes m_p}\right\rangle , \tag{2.124}$$

where

$$\sigma \left|v_{i_1} \otimes v_{i_2} \otimes \cdots v_{i_p}\right\rangle = \left|v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \cdots v_{i_{\sigma(p)}}\right\rangle . \tag{2.125}$$

We note that not all $\sigma \in S_m$ give independent vectors since

$$\left|v_\sigma\right\rangle = \left|v_{\sigma\gamma}\right\rangle , \tag{2.126}$$

with $\gamma \in H$. We can also write

$$\left|v_\sigma\right\rangle = \frac{1}{|H|}\sum_{\gamma\in H} \left|v_{\sigma\gamma}\right\rangle , \tag{2.127}$$

which demonstrates that the states correspond to elements of S_m/H.

Using this notation, we can write the representation basis as[6]

$$\begin{aligned}
\left|v_{s,i,j}\right\rangle &= \sum_{\sigma\in S_m} \Gamma^{(s)}_{ij}\left(\sigma\right)\left|v_\sigma\right\rangle \\
&= \frac{1}{|H|}\sum_{\sigma\in S_m}\sum_{\gamma\in H}\Gamma^{(s)}_{ij}\left(\sigma\right)\left|v_{\sigma\gamma}\right\rangle \\
&= \frac{1}{|H|}\sum_{\sigma\in S_m}\sum_{\gamma\in H}\Gamma^{(s)}_{ij}\left(\sigma\gamma\right)\left|v_{\sigma}\right\rangle \\
&= \frac{1}{|H|}\sum_{\sigma\in S_m}\sum_{\gamma\in H}\Gamma^{(s)}_{ik}\left(\sigma\right)\Gamma^{(s)}_{kj}\left(\gamma\right)\left|v_{\sigma}\right\rangle .
\end{aligned} \tag{2.128}$$

We can decompose the matrix of the H projector using

$$\frac{1}{|H|}\sum_{\gamma\in H}\Gamma^{(s)}_{ik}\left(\gamma\right) = \sum_{\mu} B^{s\to 1_H}_{i\mu} B^{s\to 1_H}_{k\mu} , \tag{2.129}$$

where μ is a multiplicity index for the trivial representation of H under the reduction

[6] Repeated indices are summed.

of irrep s from $S_m \to H$ and $B_{i\mu}^{s\to 1_H}$ is a branching coefficient, to get

$$|v_{s,i,j}\rangle = \sum_{\sigma\in S_m}\sum_{\mu}\Gamma_{ik}^{(s)}(\sigma)\, B_{k\mu}^{s\to 1_H} B_{j\mu}^{s\to 1_H}\, |v_\sigma\rangle\,. \tag{2.130}$$

We can now introduce

$$|\vec{m}, s, \mu; i\rangle \equiv \sum_{j} B_{j\mu}^{s\to 1_H}\, |v_{s,i,,j}\rangle = \sum_{j} B_{j\mu}^{s\to 1_H} \sum_{\sigma\in S_m} \Gamma_{ij}^{(s)}(\sigma)\, |v_\sigma\rangle\,, \tag{2.131}$$

which is equivalent to the decomposition

$$V_p^{\otimes m} = \bigoplus_{\vec{m}} \bigoplus_{s} V_s^{S_m} \otimes V_{s\to \mathbf{1}}^{S_m \to H(\vec{m})} \bigotimes_{i=1}^{p} V_{m_i}^{U(1)^p}. \tag{2.132}$$

Comparing equations (2.118) and (2.132), we deduce that

$$\mathcal{M}_{\mathbf{1}_H}^s \equiv \left|V_{s\to\mathbf{1}}^{S_m\to H(\vec{m})}\right| = \left|V_{s\to\vec{m}}^{U(p)\to U(1)^p}\right| \equiv \mathcal{K}_{s\vec{m}}, \tag{2.133}$$

which concludes the proof that the number of Gauss graphs is equivalent to the number of restricted Schur polynomials that label the open string configurations.

2.6.3 Gauss graph operators

Having proved the equivalence between the Gauss graphs and restricted Schur polynomials, we would like to exploit this basis to diagonalise the dilatation operator. In order to do so, we now construct the operators in the Gauss graph basis, the Gauss graph operators, that provide an alternative basis to the restricted Schur polynomials.

Given an object $\mathcal{O}_\tau$ that is determined by a permutation τ, we can form linear combinations $\mathcal{O}_{ij}^s$ that are labelled by an irrep s and state labels i, j, i.e.

$$\mathcal{O}_{ij}^s = \sum_{\sigma\in S_m} \Gamma_{ij}^{(s)}(\sigma)\, \mathcal{O}_\sigma. \tag{2.134}$$

These matrix elements provide a resolution of the delta function on the group since

$$\sum_s \frac{d_s}{m!} \Gamma_{ij}^{(s)}(\sigma)\, \Gamma_{ij}^{(s)}(\tau) = \delta\left(\sigma\tau^{-1}\right),$$

and behave like Fourier coefficients. If $\mathcal{O}_\tau$ is invariant under left and right multiplication of $\tau \in S_m$ by $\gamma_1, \gamma_2 \in H$, where

$$H = H(\vec{m}) = \prod_i S_{m_i}, \tag{2.135}$$

we can write[7]

$$\begin{aligned}
\mathcal{O}_\tau &= \frac{1}{|H|^2}\sum_{\gamma_1,\gamma_2\in H}\mathcal{O}_{\gamma_1\tau\gamma_2} \\
&= \frac{1}{|H|^2}\sum_s\frac{d_s}{m!}\sum_{\gamma_1,\gamma_2\in H}\Gamma^{(s)}_{ij}\left(\gamma_1\tau\gamma_2\right)\mathcal{O}^s_{ij} \\
&= \frac{1}{|H|^2}\sum_s\frac{d_s}{m!}\sum_{\gamma_1,\gamma_2\in H}\Gamma^{(s)}_{ik}\left(\gamma_1\right)\Gamma^{(s)}_{kl}\left(\tau\right)\Gamma^{(s)}_{lj}\left(\gamma_2\right)\mathcal{O}^s_{ij}.
\end{aligned} \tag{2.136}$$

Using equation (2.129), we get

$$\begin{aligned}
\mathcal{O}_\tau &= \sum_s\sum_{\mu_1,\mu_2}\left(\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}\left(\tau\right)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}\right)\left(\sqrt{\frac{d_s}{m!}}B^{s\to 1_H}_{i\mu_1}B^{s\to 1_H}_{j\mu_2}\mathcal{O}^s_{ij}\right) \\
&= \sum_s\sum_{\mu_1,\mu_2}\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}\left(\tau\right)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}\mathcal{O}^s_{\mu_1\mu_2},
\end{aligned} \tag{2.137}$$

where we have defined

$$\mathcal{O}^s_{\mu_1\mu_2} = \sqrt{\frac{d_s}{m!}}B^{s\to 1_H}_{i\mu_1}B^{s\to 1_H}_{j\mu_2}\mathcal{O}^s_{ij}. \tag{2.138}$$

We now show that the group theoretic coefficients

$$C^s_{\mu_1\mu_2}\left(\tau\right) = |H|\sqrt{\frac{d_s}{m!}}\Gamma^{(s)}_{kl}\left(\tau\right)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2} \tag{2.139}$$

provide an orthogonal transformation between double coset elements σ and $\mathcal{O}^s_{\mu_1\mu_2}$. The normalisation $|H|$ is placed here for convenience. We can show that

$$\begin{aligned}
C^s_{\mu_1\mu_2}\left(\tau\right)C^s_{\mu_1\mu_2}\left(\sigma\right) &= |H|^2\sum_s\frac{d_s}{m!}\Gamma^{(s)}_{kl}\left(\tau\right)B^{s\to 1_H}_{k\mu_1}B^{s\to 1_H}_{l\mu_2}\Gamma^{(s)}_{pq}\left(\sigma\right)B^{s\to 1_H}_{p\mu_1}B^{s\to 1_H}_{q\mu_2} \\
&= \sum_s\sum_{\gamma_1,\gamma_2\in H}\frac{d_s}{m!}\Gamma^{(s)}_{kl}\left(\tau\right)\Gamma^{(s)}_{kp}\left(\gamma_1\right)\Gamma^{(s)}_{lq}\left(\gamma_2\right)\Gamma^{(s)}_{pq}\left(\sigma\right) \\
&= \sum_s\sum_{\gamma_1,\gamma_2\in H}\frac{d_s}{m!}\chi\left(\gamma_1\sigma\gamma_2^{-1}\tau\right) \\
&= \sum_{\gamma_1,\gamma_2\in H}\delta\left(\gamma_1\sigma\gamma_2^{-1}\tau\right)
\end{aligned} \tag{2.140}$$

which expresses orthogonality since the right hand side is the delta function on the double coset.

A natural form of the Gauss graph operators (that are dual to the Gauss configuration

[7] All Latin indices are summed.

σ) is[8]

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{jk} \sum_{s \vdash m} \sum_{\mu_1,\mu_2} \sqrt{d_s} \Gamma^{(s)}_{jk}(\sigma) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2} O_{R,(r,s)\mu_1,\mu_2}, \qquad (2.141)$$

where $O_{R,(r,s)\mu_1,\mu_2}$ is a restricted Schur polynomial. Using

$$\left\langle O_{R,(r,s)\mu_1,\mu_2} O^\dagger_{T,(t,u)\nu_1,\nu_2} \right\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2} \qquad (2.142)$$

we compute the two-point function of the Gauss graph operators. We get

$$\begin{aligned}
\left\langle O_{R,r}(\sigma_1) O^\dagger_{T,t}(\sigma_2) \right\rangle &= \frac{|H|^2}{m!} \sum_{s,u \vdash m} \sum_{\mu_1\mu_2\nu_1\nu_2} \sqrt{d_s d_u} \Gamma^{(s)}_{jk}(\sigma_1) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2} \\
&\times \Gamma^{(s)}_{lm}(\sigma_2) B^{s\to 1_H}_{l\nu_1} B^{s\to 1_H}_{m\nu_2} \left\langle O_{R,(r,s)\mu_1,\mu_2} O^\dagger_{T,(t,u)\nu_1,\nu_2} \right\rangle \\
&= \frac{|H|^2}{m!} \sum_{s \vdash m} \sum_{\mu_1\mu_2} d_s \Gamma^{(s)}_{jk}(\sigma_1) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2} \Gamma^{(s)}_{lm}(\sigma_2) B^{s\to 1_H}_{l\mu_1} B^{s\to 1_H}_{m\mu_2} \\
&= \frac{1}{m!} \sum_{s \vdash m} \sum_{\gamma_1\gamma_2 \in H} d_s \Gamma^{(s)}_{jk}(\sigma_1) \Gamma^{(s)}_{jl}(\gamma_1) \Gamma^{(s)}_{lm}(\sigma_2) \Gamma^{(s)}_{mk}(\gamma_2) \\
&= \frac{1}{m!} \sum_{s \vdash m} \sum_{\gamma_1\gamma_2 \in H} d_s \chi\left(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2\right) \\
&= \sum_{\gamma_1,\gamma_2 \in H} \delta\left(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2\right). \qquad (2.143)
\end{aligned}$$

Again, the right hand side is the delta function on the double coset that sets $\sigma_1 = \sigma_2$ so that if σ_1 and σ_2 are the same double coset element, the two-point function is one, otherwise it is zero.

[8]The overall factor is chosen for convenience.

Chapter 3

Leigh-Strassler deformed SYM

In this chapter that is based on [49] and is my original work, we use group representation theory to compute the spectrum of anomalous dimensions of the marginally-deformed super Yang-Mills (SYM) theory. As we mentioned earlier, understanding $\mathcal{N} = 4$ SYM theory may contribute to the direction of understanding better quantum chromodynamics (QCD) and related theories, but one major difference is that it is maximally super-symmetric and conformally invariant, while QCD is not. To move closer to QCD and/or QCD-like theories, we can consider breaking at least some of the supersymmetry in $\mathcal{N} = 4$ SYM theory. The special case we consider here is a marginally-deformed theory, for which a corresponding gravity dual is known.

As we have already discussed in the first chapter, the most studied example of the AdS/CFT correspondence relates type IIB string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ super-Yang-Mills theory. The marginal deformations we consider were first studied in [48] and the gravity dual of the deformed gauge theory was found by Lunin and Maldacena in [50]. The deformed theory studied by Lunin and Maldacena preserves some of the supersymmetry, in particular, the gauge theory has $\mathcal{N} = 1$ superconformal invariance.

The idea employed by Lunin and Maldacena is motivated by the fact that marginal deformations of a conformal field theory preserve conformal symmetry. The conformal group is $SO(2,4)$, which must be the isometry of the gravity dual. This isometry gives us AdS_5 spacetime. As a result, Lunin and Maldacena only introduced the deformation on the S^5 part of the $AdS_5 \times S^5$ spacetime. To do so, they performed a T-duality, followed by a shift and finally by another T-duality (a TsT transformation) on the five-sphere. The result was a theory on the deformed spacetime, $AdS_5 \times \tilde{S}^5$.

Generalisations of this deformation to non-supersymmetric cases exist [51]. In this case, one obtains the Lunin-Maldacena solution by equating all the deformation parameters, i.e. by setting $\gamma_i = \gamma$.

In [17], a giant graviton (D3 brane) sitting at the center of AdS_5, wrapping an S^3

onto the S^5 part of the geometry was found. It had exactly the same quantum numbers as those of a graviton. The energy of this giant graviton was found to be

$$E = \frac{J}{R}, \tag{3.1}$$

where J is its angular momentum and R is the radius of the background.

In [18] and [19], dual giant gravitons were found in the undeformed theory. These dual giants extend into the AdS part of the geometry and their energy is also given by equation (3.2): both giants saturate a BPS bound for their energy.

More recently, Pirrone [57] found stable (dual) giants in a non-supersymmetric deformed background. By setting all the deformation parameters equal to each other, Pirrone's result gives stable (dual) giants in the Lunin-Maldacena background as well.

In what follows, we will review Pirrone's discussion and quote his results before carrying out our calculation on the gauge theory side. In the conclusion, we will compare our results in the case of a single giant to the string theory case reported by Pirrone.

3.1 The string theory case[1]

The metric of the $AdS_5 \times \tilde{S}^5$ can be written as a sum of the metric of the AdS_5 and that of the deformed five-sphere $\tilde{S}^5$. In the string frame, and setting $\alpha' = 1$, we have

$$ds^2 = ds^2_{AdS_5} + ds^2_{\tilde{S}^5}, \tag{3.2}$$

where

$$ds^2_{AdS_5} = -\left(1 + \frac{l^2}{R^2}\right) dt^2 + \frac{dl^2}{1 + \frac{l^2}{R^2}} + l^2 \left(d\alpha_1^2 + \sin^2\alpha_1 \left(d\alpha_2^2 + \sin^2\alpha_2 d\alpha_3^2\right)\right) \tag{3.3}$$

and

$$ds^2_{\tilde{S}^5} = R^2 \left(\frac{dr^2}{R^2 - r^2} + \frac{r^2}{R^2} d\theta^2 + G\sum_{i=1}^{3} \rho_i^2 d\varphi_i^2\right) + R^2 G \rho_1^2 \rho_2^2 \rho_3^2 \left(\sum_{i=1}^{3} \gamma_i d\varphi_i\right)^2. \tag{3.4}$$

In equation (3.4),

$$G^{-1} = 1 + \gamma_1^2 \rho_2^2 \rho_3^2 + \gamma_2^2 \rho_1^2 \rho_3^2 + \gamma_3^2 \rho_1^2 \rho_2^2 \tag{3.5}$$

and ρ_i are the cartesian coordinates of the sphere which we can parametrise as

$$\rho_1^2 = 1 - \frac{r^2}{R^2},$$

[1] This whole section is due to [57].

$$\rho_2^2 = \frac{r^2}{R^2}\cos^2\theta$$

and

$$\rho_3^2 = \frac{r^2}{R^2}\sin^2\theta.$$

These coordinates satisfy

$$\sum_{i=1}^{3}\rho_i^2 = 1.$$

Also,

$$0 \le r \le R.$$

The dilaton of the deformed background, ϕ, is related to that of the undeformed background, ϕ_0, by

$$e^{2\phi} = e^{2\phi_0}G \tag{3.6}$$

and the 't Hooft coupling is, as usual,

$$\lambda = 4\pi e^{\phi_0}N = R^4. \tag{3.7}$$

The non-zero NS-NS[2] two-form is

$$B = R^2G\left(\gamma_3^2\rho_1^2\rho_2^2 d\varphi_1\wedge d\varphi_2 + \gamma_1^2\rho_2^2\rho_3^2 d\varphi_2\wedge d\varphi_3 + \gamma_2^2\rho_1^2\rho_3^2 d\varphi_3\wedge d\varphi_1\right)$$

and the non-zero Ramond-Ramond (R-R) forms are

$$C_2 = -4R^2e^{-\phi_0}\omega_1\wedge\sum_{i=1}^{3}\gamma_i d\varphi_i,$$

$$d\omega_1 = \frac{r^3}{R^4}\sin\theta\cos\theta dr\wedge d\theta$$

and

$$C_4 = e^{-\phi_0}\frac{l^4}{R}\sin^2\alpha_1\sin\alpha_2 dt\wedge d\alpha_1\wedge d\alpha_2\wedge d\alpha_3 + 4R^4e^{-\phi_0}G\omega_1\wedge d\varphi_1\wedge d\varphi_2\wedge d\varphi_3.$$

The five-form field strength of the background is given by

$$F_5 = dC_4 - C_2\wedge dB$$

and it satisfies

$$\star F_5 = F_5.$$

[2]NS stands for Neveu-Schwarz.

This set-up reduces to the Lunin-Maldacena background when $\gamma_i = \gamma$ as we have already mentioned.

Because in the undeformed background $AdS_5 \times S^5$ there are three different particle states characterised by the same quantum numbers, one considers here three similar scenarios, as we discuss below.

3.1.1 Rotating point particle

In this case, one considers a massless point particle rotating on the $\tilde{S}^5$, minimising its energy in this space. It is convenient to consider first a point particle of mass M, which can then be set to zero at the end. In ten spacetime dimensions, the action is

$$S = -M \int dt \sqrt{-g}$$

where

$$g = G_{MN} \dot{X}^M \dot{X}^N$$

and M, $N = 0, \ldots 9$. This particle sits at the centre of the AdS_5 space and spins in the φ_1 direction. This implies that

$$g = G_{tt} + G_{\varphi_1 \varphi_1} \dot{\varphi}_1^2$$

so that the action becomes

$$S = -M \int dt \sqrt{1 - Q^2 \dot{\varphi}_1^2}, \tag{3.8}$$

where we have introduced the positive quantity

$$Q^2 = R^2 G^2 \rho_1^2 \left(1 + \gamma_1^2 \rho_2^2 \rho_3^2\right).$$

The action (3.8) does not have explicit φ_1 dependence, we can replace $\dot{\varphi}_1$ by its conjugate momentum

$$J = \frac{\partial L}{\partial \dot{\varphi}_1} = \frac{Q^2 M \dot{\varphi}_1}{\sqrt{1 - Q^2 \dot{\varphi}_1^2}}, \tag{3.9}$$

where L is the Lagrangian.

The Hamiltonian of the theory is

$$H = \dot{\varphi}_1 J - L = \frac{J}{Q} \tag{3.10}$$

in the limit $M \to 0$.

The minimum of this Hamiltonian is when Q is maximum, i.e. when $r = 0$ and

$Q = R$. The energy (minimum) of the rotating point particle is therefore

$$E = \frac{J}{R}. \tag{3.11}$$

In other words, this energy is equal to the angular momentum of the particle in units of $\frac{1}{R}$. It is the same as that of a graviton in the undeformed theory.

3.1.2 Giant graviton

In this case we consider a $D3$ brane (wrapping an S^3) expanding in the deformed $\tilde{S}^5$ and sitting at the centre of the AdS_5 space.

The dynamics of any $D3$ brane in a given background is given by [58]

$$S = S_{DBI} + S_{WS}, \tag{3.12}$$

where the Dirac-Born-Infeld [59, 60] part is

$$S_{DBI} = -T_3 \int_{\sum_4} d\tau d^3\sigma e^{-\phi} \sqrt{-det\left(g_{ab} + \mathcal{F}_{ab}\right)} \tag{3.13}$$

and the Wess-Zumino term is [61, 62, 63]

$$S_{WS} = T_3 \int_{\sum_4} P\left[\sum_q C_q e^{-B}\right] e^{2\pi F}, \tag{3.14}$$

where $P[\ldots]$ is the pullback[3] and

$$F = dA + A^2.$$

$$T_3 = \frac{1}{(2\pi)^3}$$

is the tension of the brane and $\sum_4$ is its worldvolume.

$$g_{ab} = G_{MN}\partial_a X^M \partial_b X^N$$

and

$$\mathcal{F}_{ab} = 2\pi F_{ab} - b_{ab}$$

[3]If $\varphi : M \to N$ is a smooth map between smooth manifolds M and N, then there is an associated linear map from the space of 1-forms on N to the space of 1-forms on M. This linear map is what is known as the pullback (by φ). In general, any covariant tensor field – any differential form – on N may be pulled back to M using φ.

(with F_{ab} the gauge field strength and $b_{ab} = B_{MN}\partial_a X^M \partial_b X^N$) is the modified field strength.

On the deformed background one chooses a static gauge such that

$$\tau = t, \quad \sigma_1 = \theta \in \left[0, \frac{\pi}{2}\right], \quad \sigma_2 = \varphi_2 \in [0, 2\pi], \quad \sigma_3 = \varphi_3 \in [0, 2\pi].$$

Keeping only scalar field contributions as well as terms quadratic in the wedge product from the Wess-Zumino term, (3.12) becomes

$$S = -T_3 \int_{\Sigma_4} dt d\theta d\varphi_2 d\varphi_3 e^{-\phi} \sqrt{-det\,(g_{ab} - b_{ab})} + T_3 \int_{\Sigma_4} P\left[C_4 - C_2 \wedge B\right]. \qquad (3.15)$$

One way of embedding the brane that gives a minimum energy configuration is to have the graviton of constant radius, r_0, orbit the $\tilde{S}^5$ in the φ_1 direction at a constant angular velocity ω_0. This leads to an ansatz

$$r = r_0, \quad \varphi_1 = \omega_0 t, \quad l = \alpha_1 = \alpha_2 = \alpha_3 = 0$$

which leads to the Lagrangian

$$L = -h\sqrt{1 - a^2 \dot{\varphi}_1^2} + m\dot{\varphi}_1, \qquad (3.16)$$

where

$$h = N\frac{r_0^3}{R^4}, \qquad a^2 = R^2 - r_0^2, \qquad m = N\frac{r_0^4}{R^4}.$$

To get this result we also use

$$A_3 T_3 e^{-\phi_0} = \frac{N}{R^4},$$

where A_3 is the area of a unit 3-sphere.

The size of the brane cannot exceed the radius of the space containing it. This implies that $r_0 \leq R$ and $a^2 \geq 0$. Interestingly, this Lagrangian matches that of the undeformed case [17, 18, 19] and one finds the Hamiltonian

$$H = \dot{\varphi}_1 J - L = \sqrt{h^2 + \frac{(J-m)^2}{a^2}} \qquad (3.17)$$

with

$$J = \frac{\partial L}{\partial \dot{\varphi}_1} = \frac{h a^2 \dot{\varphi}_1}{\sqrt{1 - a^2 \dot{\varphi}_1^2}} + m$$

being the angular momentum of the $D3$ brane.

Differentiating the Hamiltonian (3.17) with respect to r_0, one finds minima at $r_0 = 0$

and $r_0 = R\sqrt{\frac{J}{N}}$. The corresponding energy is the same as that of the single graviton we studied above. The angular velocity is

$$\omega_0 = \dot{\varphi}_1 = \frac{1}{R}. \tag{3.18}$$

3.1.3 Dual giant graviton

In this case, we consider a $D3$ brane expanding in the AdS_5 part of the geometry. In other words, we consider a $D3$ brane wrapping a 3-sphere $(\alpha_1, \alpha_2, \alpha_3)$ contained in AdS_5. One chooses static gauge for the worldvolume coordinates

$$\tau = t, \quad \sigma_1 = \alpha_1 \in [0, \pi], \quad \sigma_2 = \alpha_2 \in [0, \pi], \quad \sigma_3 = \alpha_3 \in [0, 2\pi].$$

This so-called dual giant graviton has constant radius l_0 and orbits the $\tilde{S}^5$ in the same direction φ_1 and constant angular velocity ω_0. This leads to the ansatz

$$l = l_0, \quad \varphi_1 = \omega_0 t, \quad r = \varphi_2 = \varphi_3 = 0, \quad \theta = \frac{\pi}{4}$$

which in turn leads to the effective Lagrangian

$$L = -\tilde{h}\sqrt{\tilde{b}^2 - R^2\dot{\varphi}_1^2} + \tilde{m}, \tag{3.19}$$

where

$$\tilde{h} = N\frac{l_0}{R^4}, \qquad \tilde{b}^2 = 1 + \frac{l_0}{R^2}, \qquad \tilde{m} = N\frac{l_0^4}{R^5}.$$

Again, this Lagrangian agrees with the undeformed case [18, 19].

The conjugate momentum for this case is

$$J = \frac{\partial L}{\partial \dot{\varphi}_1} = \frac{\tilde{h}R^2\dot{\varphi}_1}{\sqrt{\tilde{b}^2 - R^2\dot{\varphi}_1^2}}$$

so that the Hamiltonian is given by

$$H = \dot{\varphi}_1 J - L = \tilde{b}\sqrt{\tilde{h}^2 + \frac{J^2}{R^2}} - \tilde{m}. \tag{3.20}$$

Differentiating (3.20) respect to l_0, we find minima at $l_0 = 0$ and $l_0 = R\sqrt{\frac{J}{N}}$. The minimum energy as well as the angular velocity agree with the previous section. In this case though, the energy of the giant graviton is not bounded because AdS_5 space is not compact.

3.1.4 Stability of the (giant) gravitons

This far, we have reviewed the existence of the graviton, giant graviton and dual giant graviton in the deformed space as reported in [57]. A natural question to ask is whether or not these giant gravitons are stable. The answer is found in the same reference [57]. To answer this question, Pirrone perturbed the giants about their equilibrium positions

$$X = X_0 + \varepsilon\delta X\left(t, \sigma_i\right), \tag{3.21}$$

where X_0 is a classical solution to the equations of motion and ε is a small perturbation parameter. He then expanded the action of the probe brane in powers of ε to get

$$S = \int dt d^3\sigma \left(\mathcal{L}_0 + \varepsilon\mathcal{L}_1 + \varepsilon^2\mathcal{L}_2 + \cdots\right). \tag{3.22}$$

Here, $\mathcal{L}_1$ tells us whether or not the solutions obtained earlier really minimise the action, while $\mathcal{L}_2$ tells us whether the configurations obtained earlier are stable or not, amongst other things. In both cases (for the giant and the dual giant gravitons), $\mathcal{L}_1$ vanishes telling us that we are indeed expanding about a solution to the equations of motion.

As far as $\mathcal{L}_2$ is concerned, Pirrone found real and non-negative frequencies for both cases which imply that the giant gravitons as well as the dual giant gravitons are stable. However, in the giant graviton case, the frequencies depend on the radius of the giants, r_0, in the deformed theory, which is not the case in the undeformed theory. The frequencies of the dual giants are independent of the radius $l_{0,}$ and agree with the undeformed case.

All in all, both the giant gravitons and the dual giant gravitons were found to be stable in [57].

3.2 The gauge theory

After deformation, the superpotential depends on three parameters

$$W = i\kappa\left[Tr\left(XYZ - qXZY\right) + \frac{h}{3}Tr\left(X^3 + Y^3 + Z^3\right)\right]. \tag{3.23}$$

In this work, we consider the simplest case of a β-deformation for which

$$q = e^{-2i\pi\gamma}, \quad h = 0$$

and γ is real. This deformation preserves integrability [64, 65, 66, 67].

With this deformation, the dilatation operator we consider is [64]

$$D_\gamma = -g_{YM}^2 Tr\left(ZY\partial_Y\partial_Z + YZ\partial_Z\partial_Y - ZY\partial_Z\partial_Y e^{2\pi i\gamma} - YZ\partial_Y\partial_Z e^{-2\pi i\gamma}\right). \tag{3.24}$$

We would like to determine the action of D_γ on restricted Schur polynomials

$$\chi_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n},Y^{\otimes m}\right)=\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\mu_1\mu_2}\left(\sigma\right)Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right) \tag{3.25}$$

where the restricted character is

$$\chi_{R,(r,s)\mu_1\mu_2}\left(\sigma\right)=Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left(\sigma\right)\right)=Tr\left(P_{R\to(r,s)\mu_1\mu_2}\Gamma_R\left(\sigma\right)\right) \tag{3.26}$$

and

$$Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)=Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}. \tag{3.27}$$

When the partial derivatives in D_γ act on $\chi_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n},Y^{\otimes m}\right)$ we get the following terms

$$Tr\left(ZY\partial_Y\partial_Z\right)Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)=n{\cdot}m\left(ZY\right)^{i_n}_{i_{\sigma(n+1)}}\delta^{i_{n+1}}_{i_{\sigma(n)}}Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}}Y^{i_{n+2}}_{i_{\sigma(n+2)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}},$$

$$Tr\left(YZ\partial_Z\partial_Y\right)Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)=n{\cdot}m\left(YZ\right)^{i_{n+1}}_{i_{\sigma(n)}}\delta^{i_n}_{i_{\sigma(n+1)}}Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}}Y^{i_{n+2}}_{i_{\sigma(n+2)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}},$$

$$\begin{aligned}Tr\left(ZY\partial_Z\partial_Y e^{2\pi i\gamma}\right)Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)&=n\cdot m\left(ZY\right)^{i_{n+1}}_{i_{\sigma(n)}}\delta^{i_n}_{i_{\sigma(n+1)}}e^{2\pi i\gamma}Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}}\\&\times Y^{i_{n+2}}_{i_{\sigma(n+2)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}\end{aligned}$$

and

$$\begin{aligned}Tr\left(YZ\partial_Y\partial_Z e^{-2\pi i\gamma}\right)Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)&=n\cdot m\left(YZ\right)^{i_n}_{i_{\sigma(n+1)}}\delta^{i_{n+1}}_{i_{\sigma(n)}}e^{-2\pi i\gamma}Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n-1}}_{i_{\sigma(n-1)}}\\&\times Y^{i_{n+2}}_{i_{\sigma(n+2)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}.\end{aligned}$$

By setting

$$\sigma=\rho\left(n,n+1\right)$$

we can write

$$\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R(\sigma)\right) Tr\left(ZY\partial_Y\partial_Z\right) Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)$$
$$= \frac{1}{(n-1)!\,(m-1)!}\sum_{\rho\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left(\rho\left(n,n+1\right)\right)\right) Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n-1}}_{i_{\rho(n-1)}}\left(ZY\right)^{i_n}_{i_{\rho(n)}}$$
$$\times\,\delta^{i_{n+1}}_{i_{\rho(n+1)}} Y^{i_{n+2}}_{i_{\rho(n+2)}}\cdots Y^{i_{n+m}}_{i_{\rho(n+m)}}$$

for the first term.

In the second term we use the following identity

$$\prod_{j=1}^{n} Z^{i_j}_{i_{\beta(j)}} = \prod_{j=1}^{n} Z^{i_{\psi(j)}}_{i_{\beta(\psi(j))}} = \prod_{j=1}^{n} Z^{i_{\beta^{-1}(j)}}_{i_j} \tag{3.28}$$

which teaches us that operating on the lower indices with an element of the symmetric group is equivalent to operating on the upper indices with the inverse of the group element. To get the last equality in equation (3.28) we set $\psi=\beta^{-1}$. The first equality follows from the fact that the permutation ψ (which acts on both the upper and the lower indices) only changes the order in which the product appears, but it does not change the overall value of the product.

With this lesson in mind, we then set

$$\sigma = (n,n+1)\,\rho$$

and change the sum over $\sigma\in S_{n+m}$ to a sum over $\rho\in S_{n+m}$. It now appears that we are acting with $(n,n+1)$ on the lower indices. This is equivalent to operating on the upper indices by the inverse of $(n,n+1)$ which swaps the n^{th} and the $(n+1)^{th}$ indices in the upper indices. This way, the second term can be written as

$$\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R(\sigma)\right) Tr\left(YZ\partial_Z\partial_Y\right) Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)$$
$$= \frac{1}{(n-1)!\,(m-1)!}\sum_{\rho\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left(\left(n,n+1\right)\rho\right)\right) Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n-1}}_{i_{\rho(n-1)}}\left(YZ\right)^{i_n}_{i_{\rho(n)}}$$
$$\times\,\delta^{i_{n+1}}_{i_{\rho(n+1)}} Y^{i_{n+2}}_{i_{\rho(n+2)}}\cdots Y^{i_{n+m}}_{i_{\rho(n+m)}}.$$

The third term can be manipulated in the same way we did the second to get

$$\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left(\sigma\right)\right) Tr\left(ZY\partial_Z\partial_Y e^{2\pi i\gamma}\right) Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)$$
$$= \frac{1}{(n-1)!\,(m-1)!}\sum_{\rho\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left((n,n+1)\,\rho\right)\right) Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n-1}}_{i_{\rho(n-1)}}\left(ZY\right)^{i_n}_{i_{\rho(n)}}$$
$$\times\,\delta^{i_{n+1}}_{i_{\rho(n+1)}} e^{2\pi i\gamma} Y^{i_{n+2}}_{i_{\rho(n+2)}}\cdots Y^{i_{n+m}}_{i_{\rho(n+m)}},$$

while the last term can be manipulated as we did the first to get

$$\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left(\sigma\right)\right) Tr\left(YZ\partial_Y\partial_Z e^{-2\pi i\gamma}\right) Tr\left(\sigma Z^{\otimes n}Y^{\otimes m}\right)$$
$$= \frac{1}{(n-1)!\,(m-1)!}\sum_{\rho\in S_{n+m}} Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left(\rho\,(n,n+1)\right)\right) Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n-1}}_{i_{\rho(n-1)}}\left(YZ\right)^{i_n}_{i_{\rho(n)}}$$
$$\times\,\delta^{i_{n+1}}_{i_{\rho(n+1)}} e^{-2\pi i\gamma} Y^{i_{n+2}}_{i_{\rho(n+2)}}\cdots Y^{i_{n+m}}_{i_{\rho(n+m)}},$$

respectively.

To enable us to do the sum over ρ we first write

$$\sum_{\rho\in S_{n+m}} \to \sum_{\rho\in S_{n+m-1}}$$

in which case the latter sum runs over the elements that leave $n+1$ inert.[4] Following the reduction rule for restricted Schur polynomials [68, 33] we get

$$D_\gamma\chi_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n},Y^{\otimes m}\right) = \frac{-g^2_{YM}}{(n-1)!\,(m-1)!}\sum_{\rho\in S_{n+m-1}}\sum_{R'} c_{RR'}$$
$$\times\Big[Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_{R'}\left(\rho\right)\Gamma_R\left((n,n+1)\right)\right) Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n-1}}_{i_{\rho(n-1)}}\left(\left(ZY\right)^{i_n}_{i_{\rho(n)}} - \left(YZ\right)^{i_n}_{i_{\rho(n)}} e^{-2\pi i\gamma}\right)$$
$$\times\, Y^{i_{n+2}}_{i_{\rho(n+2)}}\cdots Y^{i_{n+m}}_{i_{\rho(n+m)}}$$
$$+\,Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R\left((n,n+1)\right)\Gamma_{R'}\left(\rho\right)\right) Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n-1}}_{i_{\rho(n-1)}}\left(\left(YZ\right)^{i_n}_{i_{\rho(n)}} - \left(ZY\right)^{i_n}_{i_{\rho(n)}} e^{2\pi i\gamma}\right)$$
$$\times\, Y^{i_{n+2}}_{i_{\rho(n+2)}}\cdots Y^{i_{n+m}}_{i_{\rho(n+m)}}\Big] \quad (3.29)$$

where $c_{RR'}$ is the weight of the single box that must be removed from the Young diagram

[4] This choice to leave $n+1$ inert is not unique. We could have chosen to leave any of the other Y fields inert, but not a Z field slot. This is because from our Young diagram R, we can remove any of the Y boxes, but not the Z ones. In other words, this $(n+1)^{th}$ slot may correspond to the first box we remove from the Young diagram R to get r.

R to obtain R'. The sum over R' follows from writing the sum over S_{n+m} as a sum over the subgroup S_{n+m-1} and its cosets. The subgroup S_{n+m-1} keeps only the permutations that leave $n+1$ inert.

Now writing[5]

$$(ZY)^{i_n}_{i_{\rho(n)}} = Z^{i_n}_{i_{\rho(n+1)}} Y^{i_{n+1}}_{i_{\rho(n)}}$$

and

$$(YZ)^{i_n}_{i_{\rho(n)}} = Y^{i_n}_{i_{\rho(n+1)}} Z^{i_{n+1}}_{i_{\rho(n)}}$$

followed by swapping the indices again allows us to write equation (3.29) as

$$\begin{aligned} D_\gamma \chi_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n}, Y^{\otimes m}\right) = & \frac{-g_{YM}^2}{(n-1)!\,(m-1)!} \sum_{\rho\in S_{n+m-1}} \sum_{R'} c_{RR'} \\ & \times \Big[Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_{R'}(\rho)\,\Gamma_R((n,n+1))\right) \\ & \times \left(Tr\left(\rho\,(n,n+1)\,Z^{\otimes n}Y^{\otimes m}\right) - e^{-2\pi i\gamma} Tr\left((n,n+1)\,\rho Z^{\otimes n}Y^{\otimes m}\right)\right) \\ & + Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R((n,n+1))\,\Gamma_{R'}(\rho)\right) \\ & \times \left(Tr\left((n,n+1)\,\rho Z^{\otimes n}Y^{\otimes m}\right) - e^{2\pi i\gamma} Tr\left(\rho\,(n,n+1)\,Z^{\otimes n}Y^{\otimes m}\right)\right)\Big]. \end{aligned} \tag{3.30}$$

We now use the identity [37]

$$Tr\left(\psi Z^{\otimes n}Y^{\otimes m}\right) = \sum_{T,(t,u)\nu_2\nu_1} \frac{d_T n! m!}{d_t d_u\,(n+m)!} \chi_{T,(t,u)\nu_2\nu_1}\left(\psi^{-1}\right) \chi_{T,(t,u)\nu_1\nu_2}(Z,Y) \tag{3.31}$$

in equation (3.30) to get

$$D_\gamma \chi_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n}, Y^{\otimes m}\right) = \sum_{T,(t,u)\nu_2\nu_1} M_{R,(r,s)\mu_1\mu_2;T(t,u)\nu_2\nu_1} \chi_{T,(t,u)\nu_1\nu_2}(Z,Y) \tag{3.32}$$

[5] We can do this because $\rho(n+1) = n+1$.

where

$$M_{R,(r,s)\mu_1\mu_2;T(t,u)\nu_2\nu_1} = \frac{-n\cdot mg_{YM}^2}{(n+m)!}\sum_{\rho\in S_{n+m-1}}\sum_{R'}c_{RR'}\frac{d_T}{d_td_u}$$
$$\times\Big[Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_{R'}(\rho)\,\Gamma_R((n,n+1))\right)$$
$$\times\left(\chi_{T,(t,u)\nu_2\nu_1}\left((n,n+1)\rho^{-1}\right)-e^{-2\pi i\gamma}\chi_{T,(t,u)\nu_2\nu_1}\left(\rho^{-1}(n,n+1)\right)\right)$$
$$+Tr_{(r,s)\mu_1\mu_2}\left(\Gamma_R((n,n+1))\,\Gamma_{R'}(\rho)\right)$$
$$\times\left(\chi_{T,(t,u)\nu_2\nu_1}\left(\rho^{-1}(n,n+1)\right)-e^{2\pi i\gamma}\chi_{T,(t,u)\nu_2\nu_1}\left((n,n+1)\rho^{-1}\right)\right)\Big]. \tag{3.33}$$

We now do the sum over $\rho\epsilon S_{n+m-1}$ using the fundamental orthogonality relation (together with equation (3.26))

$$\sum_{\rho\in S_{n+m-1}}\Gamma_{R'}(\rho)_{ab}\,\Gamma_{T'}\left(\rho^{-1}\right)_{cd} = \frac{(n+m-1)!}{d_{R'}}\delta_{R'T'}\delta_{ad}\delta_{bc}$$
$$= \frac{(n+m-1)!}{d_{R'}}\delta_{R'T'}\left(I_{R'T'}\right)_{ad}\left(I_{T'R'}\right)_{cb} \tag{3.34}$$

where $I_{R'T'}$ is an intertwiner that takes us from R' to T'. The result we get is

$$M_{R,(r,s)\mu_1\mu_2;T(t,u)\nu_2\nu_1} = -g_{YM}^2\sum_{R'}c_{RR'}\frac{d_Tn\cdot m}{d_{R'}d_td_u(n+m)}\delta_{R'T'}$$
$$\times\Big[Tr\left(\left(\Gamma_R((n,n+1))\,P_{R\to(r,s)\mu_1\mu_2}\right)I_{R'T'}\left(P_{T\to(t,u)\nu_2\nu_1}\Gamma_T((n,n+1))\right)I_{T'R'}\right)$$
$$-e^{-2\pi i\gamma}Tr\left(\left(\Gamma_R((n,n+1))\,P_{R\to(r,s)\mu_1\mu_2}\right)I_{R'T'}\left(\Gamma_T((n,n+1))\,P_{T\to(t,u)\nu_2\nu_1}\right)I_{T'R'}\right)$$
$$+Tr\left(\left(P_{R\to(r,s)\mu_1\mu_2}\Gamma_R((n,n+1))\right)I_{R'T'}\left(\Gamma_T((n,n+1))\,P_{T\to(t,u)\nu_2\nu_1}\right)I_{T'R'}\right)$$
$$-e^{2\pi i\gamma}Tr\left(\left(P_{R\to(r,s)\mu_1\mu_2}\Gamma_R((n,n+1))\right)I_{R'T'}\left(P_{T\to(t,u)\nu_2\nu_1}\Gamma_T((n,n+1))\right)I_{T'R'}\right)\Big]. \tag{3.35}$$

Finally, in order to compute the spectrum of anomalous dimensions, we use normalised operators $O_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n},Y^{\otimes m}\right)$ rather than $\chi_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n},Y^{\otimes m}\right)$. The two-point function for restricted Schur polynomials has been found to be

$$\left\langle\chi_{R,(r,s)\mu_1\mu_2}(Z,Y)\,\chi_{T,(t,u)\nu_1\nu_2}(Z,Y)^\dagger\right\rangle = \delta_{RT}\delta_{(r,s)(t,u)}\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1}f_R\frac{hooks_R}{hooks_r hooks_s} \tag{3.36}$$

where f_R is the product of weights of the Young diagram R and $hooks_R$ is the product of hook-lengths of diagram R. In computing this two-point function, the order in which the Greek indices appear is related to our convention

$$\chi_{T,(t,u)\nu_1\nu_2}(Z,Y)^\dagger \equiv \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr\left(P_{T\to(t,u)\nu_1\nu_2}\Gamma_T(\sigma)\right) Tr\left(\sigma Z^{\dagger\otimes n}Y^{\dagger\otimes m}\right). \quad (3.37)$$

From equation (3.36) we deduce that

$$\chi_{R,(r,s)\mu_1\mu_2}(Z,Y) = \sqrt{\frac{f_R hooks_R}{hooks_r hooks_s}} O_{R,(r,s)\mu_1\mu_2}(Z,Y). \quad (3.38)$$

In terms of these normalised operators, the action of the dilatation operator is

$$D_\gamma O_{R,(r,s)\mu_1\mu_2}\left(Z^{\otimes n}, Y^{\otimes m}\right) = \sum_{T,(t,u)\nu_2\nu_1} N_{R,(r,s)\mu_1\mu_2;T(t,u)\nu_2\nu_1} O_{T,(t,u)\nu_1\nu_2}(Z,Y) \quad (3.39)$$

where

$$\begin{aligned} N_{R,(r,s)\mu_1\mu_2;T(t,u)} &= -g_{YM}^2 \sum_{R'} c_{RR'} \frac{d_T n\cdot m}{d_{R'}d_t d_u (n+m)} \delta_{R'T'} \sqrt{\frac{f_T hooks_T hooks_r hooks_s}{f_R hooks_R hooks_t hooks_u}} \\ &\times \Big[Tr\left(\left(\Gamma_R((n,n+1)) P_{R\to(r,s)\mu_1\mu_2}\right) I_{R'T'} \left(P_{T\to(t,u)\nu_2\nu_1}\Gamma_T((n,n+1))\right) I_{T'R'}\right) \\ &- e^{-2\pi i\gamma} Tr\left(\left(\Gamma_R((n,n+1)) P_{R\to(r,s)\mu_1\mu_2}\right) I_{R'T'} \left(\Gamma_T((n,n+1)) P_{T\to(t,u)\nu_2\nu_1}\right) I_{T'R'}\right) \\ &+ Tr\left(\left(P_{R\to(r,s)\mu_1\mu_2}\Gamma_R((n,n+1))\right) I_{R'T'} \left(\Gamma_T((n,n+1)) P_{T\to(t,u)\nu_2\nu_1}\right) I_{T'R'}\right) \\ &- e^{2\pi i\gamma} Tr\left(\left(P_{R\to(r,s)\mu_1\mu_2}\Gamma_R((n,n+1))\right) I_{R'T'} \left(P_{T\to(t,u)\nu_2\nu_1}\Gamma_T((n,n+1))\right) I_{T'R'}\right)\Big]. \end{aligned} \quad (3.40)$$

We will now evaluate equation (3.40) under the following conditions

i) $n \gg m$,

ii) we assume that the Young diagram R has p long rows,

iii) p is held fixed while we take $N \to \infty$ and

iv) R has well separated corners.

3.2.1 The traces

The projectors take us from a space labeled by R, a representation of S_{n+m}, to a space labeled by (r,s), a representation of $S_n \times S_m$. To construct these projectors we first go from S_{n+m} to $S_n \times (S_1)^m$ and then, by employing Schur-Weyl duality, from $S_n \times (S_1)^m$ to $S_n \times S_m$.

S_{n+m} **to** $S_n \times (S_1)^m$

This step is accomplished by pulling off m boxes from the Young diagram R, leaving a representation of S_n, $r \vdash n$. There are different ways of pulling off the same set of m boxes from R that always leave the same diagram r. These different ways of pulling off the m boxes give us a different sub-spaces with the same irreducible representation r. This multiplicity is resolved by specifying the order in which we pull off the m boxes from R. This can be done by numbering the m boxes 1 to m, where box number 1 is to be pulled off first and box number m is pulled off last. The numbering should be such that each time we pull off a box we remain with a legal Young diagram.

Now what remains is assembling the individual boxes we pulled off R to get irreducible representations of the $S_n \times S_m$ sub-group. To do so, we employ the Schur-Weyl duality.

3.2.2 Schur-Weyl duality

The Schur-Weyl duality we discuss here relates the actions of unitary and symmetric groups on a vector space. We follow the discussion given in [44].

Let us consider a Young diagram R with p rows and built from $n+m \sim O(N)$ boxes, where $m \ll n$. We also want each row in the diagram to consist of $O(N)$ boxes. Let us label m of the boxes as we described earlier. Two boxes in two different rows will then have factors c_i and c_j if they carry labels i and j, respectively, such that

$$c_i - c_j \sim O(N). \tag{3.41}$$

If we think of the partially labeled diagram as a Young-Yamonouchi state and let S_m (a sub-group of S_{n+m}) act on these states, (3.41) results in a significant simplification in the representations of S_m. When adjacent permutations $(i, i+1)$ act on the boxes that belong to the same row, the diagram is unchanged, and when the diagrams belong to different rows, the boxes are swapped.

Considering the diagram with p rows, there are p^m different ways of removing m boxes from the same diagram R. This gives us p^m different partially labeled diagrams.

We can associate a p dimensional vector to each box that is labeled, giving a total of m vectors $\overrightarrow{v}(i)$, where $i = 1, 2, \ldots m$. Let us denote the components of these vectors $\overrightarrow{v}(i)_n$, where $n = 1, \ldots p$. This way, if we pull box i from the j^{th} row, we have

$$\overrightarrow{v}(i)_n = \delta_{ni}.$$

For each labeled box we have a vector space V_p. The tensor product of these vector spaces is another vector space $V_p^{\otimes m}$.

Now, $\sigma \in S_m$ has the following action on $V_p^{\otimes m}$

$$\sigma \cdot \left(\overrightarrow{v}(1) \otimes \overrightarrow{v}(2) \otimes \cdots \overrightarrow{v}(m)\right) = \overrightarrow{v}(\sigma(1)) \otimes \overrightarrow{v}(\sigma(2)) \otimes \cdots \overrightarrow{v}(\sigma(m)). \tag{3.42}$$

In other words, the symmetric group element moves vectors between slots, but it does not permute elements of a vector.

The action of the unitary group $U(p)$ on $V_p^{\otimes m}$ is

$$U \cdot \left(\overrightarrow{v}(1) \otimes \overrightarrow{v}(2) \otimes \cdots \overrightarrow{v}(m)\right) = D(U)\overrightarrow{v}(1) \otimes D(U)\overrightarrow{v}(2) \otimes \cdots D(U)\overrightarrow{v}(m), \tag{3.43}$$

where $D(U)$ is the $p \times p$ unitary matrix representing group element $U \in U(p)$ in the fundamental representation. We see here that the unitary group element $U \in U(p)$ changes the value of the vector in the i^{th} slot, but it does not move it to another slot. The action of the unitary group element $U \in U(p)$ therefore commutes with that of the symmetric group element $\sigma \in S_m$, i.e.

$$\begin{aligned} U \cdot \left(\sigma \cdot \left(\overrightarrow{v}(1) \otimes \overrightarrow{v}(2) \otimes \cdots \overrightarrow{v}(m)\right)\right) &= U \cdot \left(\overrightarrow{v}(\sigma(1)) \otimes \overrightarrow{v}(\sigma(2)) \otimes \cdots \overrightarrow{v}(\sigma(m))\right) \\ &= D(U)\overrightarrow{v}(\sigma(1)) \otimes D(U)\overrightarrow{v}(\sigma(2)) \otimes \cdots D(U)\overrightarrow{v}(\sigma(m)) \\ &= \sigma \cdot \left(D(U)\overrightarrow{v}(1) \otimes D(U)\overrightarrow{v}(2) \otimes \cdots D(U)\overrightarrow{v}(m)\right) \\ &= \sigma \cdot \left(U \cdot \left(\overrightarrow{v}(1) \otimes \overrightarrow{v}(2) \otimes \cdots \overrightarrow{v}(m)\right)\right). \end{aligned}$$

As a result, we can write

$$V_p^{\otimes m} = \bigoplus_s V_p^{U(p)} \otimes V_p^{S_m}, \tag{3.44}$$

i.e. the vector space $V_p^{\otimes m}$ built using the m boxes we removed can be written as a tensor product of a vector space $V_p^{U(p)}$ that is labeled by representations of the unitary group and another vector space $V_p^{S_m}$ labeled by representations of the symmetric group S_m. The Young diagrams s are representations of both the unitary and the symmetric groups.

The dimension of the vector space $V_p^{\otimes m}$ can also be written in terms of the unitary group representation, $Dim(s)$, and that of the symmetric group representation, d_s, where s is the Young diagram subduced from R in the way we described earlier. We have

$$p^m = \sum_s Dim(s) \cdot d_s. \tag{3.45}$$

Thus to identify states with good S_m labels we only need to identify states with good $U(p)$ labels, and vice-versa.

$S_n \times (S_1)^m$ to $S_n \times S_m$

Thus far, we have constructed an $S_n \times (S_1)^m$ subgroup of S_{n+m} by removing m boxes from the Young diagram R, with the different ways of removing the boxes giving us different sub-spaces. Assembling the m boxes together gives us a Young diagram s, an irreducible representation of S_m. From the Schur-Weyl duality we discussed above, the diagram s is also a good label for the unitary group $U(p)$, where p is the number of rows in R. The projectors $P_{R\to(r,s)\mu_1\mu_2}$ carry good labels for $U(p)$ and can therefore be constructed solely from $U(p)$ group theory.

To evaluate the action of the dilatation operator we remove $m+1$ boxes, i.e. we remove all the Y boxes and one Z box. Each box is represented by a vector in V_p, allowing us to easily evaluate the action of $\Gamma_R((n,n+1))$ and $\Gamma_T((n,n+1))$ in equation (3.40).

The Young diagrams R and T differ at most by the placement of one box. After removing this single box from R (to get R') and T (to get T'), R' and T' agree. The intertwiner $I_{R'T'}$ can then be written as $E^{(n+1)}_{ij}$ if one box is removed from the i^{th} row of R and the other from the j^{th} row of T. $E^{(n+1)}_{ij}$ can be written as

$$\sum_l E^{(n+1)}_{ij} E^{(n)}_{ll}$$

so that

$$\begin{aligned} E^{(n+1)}_{ij}\Gamma_R((n,n+1)) &= \sum_l E^{(n+1)}_{ij} E^{(n)}_{ll}\Gamma_R((n,n+1)) \\ &= \sum_l E^{(n+1)}_{il} E^{(n)}_{lj} \end{aligned}$$

and

$$\begin{aligned} \Gamma_R((n,n+1)) E^{(n+1)}_{ij} &= \Gamma_R((n,n+1)) \sum_l E^{(n+1)}_{ij} E^{(n)}_{ll} \\ &= \sum_l E^{(n+1)}_{lj} E^{(n)}_{il}. \end{aligned}$$

This is how we will manipulate the traces in equation (3.40) to find the action of $\Gamma((n,n+1))$ on the intertwiners. We will also write the projectors $P_{R\to(r,s)\mu_1\mu_2}$ and $P_{T\to(t,u)\nu_2\nu_1}$ as

$$P_{\vec{m};R,(r,s)\mu_1\mu_2} = \mathbf{1}_r \otimes |\vec{m}, s, \mu_1; a\rangle\langle \vec{m}, s, \mu_2; a| \tag{3.46}$$

and

$$P_{\vec{n};T,(t,u)\nu_2\nu_1} = \mathbf{1}_t \otimes |\vec{n}, u, \nu_2; b\rangle\langle \vec{n}, u, \nu_1; b|, \tag{3.47}$$

respectively. Here, $\overrightarrow{m}$ and $\overrightarrow{n}$ specify how we remove boxes from R and T respectively. In this notation, the Greek indices are multiplicity labels while a and b label the states of s and u. These state labels are summed.

Case 1: $R = T$

Let us consider first, the case when $R = T$. In this case, we obtain R' and T' by removing single boxes from the same row in R as in T and

$$c_{RR'}\sqrt{\frac{f_T}{f_R}} = c_{RR'}. \tag{3.48}$$

For a non-zero result, $r = t$ because the product

$$\mathbf{1}_r\mathbf{1}_t = \delta_{r,t}$$

appears when we multiply the projectors.[6] In other words, since we start with the Young diagrams $R = T$, these diagrams must still agree after we remove the $m = n$ boxes from each.

$$\begin{aligned}
&Tr\left(\left(\Gamma_R\left((n,n+1)\right)P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}\right)I_{R'T'}\left(P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}\Gamma_T\left((n,n+1)\right)\right)I_{T'R'}\right)\\
&= Tr\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{ii}P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}\Gamma_T\left((n,n+1)\right)E^{(n+1)}_{ii}\Gamma_R\left((n,n+1)\right)\right)\\
&= \sum_l Tr\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{ii}P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}\Gamma_T\left((n,n+1)\right)E^{(n+1)}_{ii}E^{(n)}_{ll}\Gamma_R\left((n,n+1)\right)\right)\\
&= \sum_l Tr\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{ii}P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}E^{(n+1)}_{ll}E^{(n)}_{ii}\right)\\
&= \delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}\sum_j Tr_r\left(E^{(n)}_{ii}\right)\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{jj}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&= \delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}\sum_j \delta_{s,u}d_{r'(i)}\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{jj}\,|\overrightarrow{m},s,\mu_1;a\rangle.
\end{aligned}$$

[6] See equations (3.46) and (3.47).

Similarly,

$$\begin{aligned}
&Tr\left(\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}\Gamma_R\left((n,n+1)\right)\right)I_{R'T'}\left(\Gamma_T\left((n,n+1)\right)P_{T\to(t,u)}P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}\right)I_{T'R'}\right)\\
&=\sum_l Tr\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}\Gamma_R\left((n,n+1)\right)E^{(n+1)}_{ii}E^{(n)}_{ll}\Gamma_T\left((n,n+1)\right)P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}E^{(n+1)}_{ii}\right)\\
&=\delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}\sum_j Tr_r\left(E^{(n)}_{ii}\right)\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&=\delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}\sum_j\delta_{s,u}d_{r'(i)}\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle,
\end{aligned}$$

$$\begin{aligned}
&e^{-2\pi i\gamma}Tr\left(\left(\Gamma_R\left((n,n+1)\right)P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}\right)I_{R'T'}\left(\Gamma_T\left((n,n+1)\right)P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}\right)I_{T'R'}\right)\\
&=e^{-2\pi i\gamma}\sum_{l,k}Tr\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{ii}E^{(n)}_{ll}\Gamma_T\left((n,n+1)\right)P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}E^{(n+1)}_{ii}E^{(n)}_{kk}\Gamma_R\left((n,n+1)\right)\right)\\
&=e^{-2\pi i\gamma}\sum_{l,k}Tr\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{il}E^{(n)}_{li}P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}E^{(n+1)}_{ik}E^{(n)}_{ki}\right)\\
&=e^{-2\pi i\gamma}\delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}Tr_r\left(E^{(n)}_{ii}\right)\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&=e^{-2\pi i\gamma}\delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}d_{r'(i)}\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle
\end{aligned}$$

and

$$\begin{aligned}
&e^{2\pi i\gamma}Tr\left(\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}\Gamma_R\left((n,n+1)\right)\right)I_{R'T'}\left(P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}\Gamma_T\left((n,n+1)\right)\right)I_{T'R'}\right)\\
&=e^{2\pi i\gamma}\sum_{l,k}Tr\left(P_{\overrightarrow{m};R,(r,s)\mu_1\mu_2}\Gamma_R\left((n,n+1)\right)E^{(n+1)}_{ii}E^{(n)}_{ll}P_{\overrightarrow{n};T,(t,u)\nu_2\nu_1}\Gamma_T\left((n,n+1)\right)E^{(n+1)}_{ii}E^{(n)}_{kk}\right)\\
&=e^{2\pi i\gamma}\delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}Tr_r\left(E^{(n)}_{ii}\right)\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&=e^{2\pi i\gamma}\delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}d_{r'}\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle.
\end{aligned}$$

Summing everything together, we obtain for $R=T$,

$$\begin{aligned}
&\delta_{\overrightarrow{m},\overrightarrow{n}}\delta_{r,t}\sum_{j\neq i}d_{r'(i)}\delta_{s,u}\Big[\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{jj}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&\qquad+\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&+2(1-\cos(2\pi\gamma))\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\overrightarrow{n},u,\nu_2;b\rangle\langle\overrightarrow{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\Big].
\end{aligned}\tag{3.49}$$

Case 2: $R \neq T$

In this case, we can obtain $R' = T'$ if we remove one box from row i of Young diagram R and one box from row j of Young diagram T, where $i \neq j$. Also, from the coefficient that we compute in Section 3.2.3

$$c_{RR'}\sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'}c_{TT'}}. \tag{3.50}$$

For a non-zero result,[7] $r'(i) = t'(j)$ emanating from the trace over r subspace. Each of the non-zero traces come multiplied by $\delta_{r'(i),t'(j)}$. In other words, since the diagrams R and T disagree by the placement of a single box, they still disagree (in the same way) after we remove the first $m = n$ boxes to get r and t. However, if we remove the one extra box from row i of r and one from row j of t to get r' and t', the diagrams then agree.

We therefore have

$$\begin{aligned}
&e^{-2\pi i\gamma}Tr\left(\left(\Gamma_R\left((n,n+1)\right)P_{\vec{m};R,(r,s)\mu_1\mu_2}\right)I_{R'T'}\left(\Gamma_T\left((n,n+1)\right)P_{\vec{n};T,(t,u)\nu_2\nu_1}\right)I_{T'R'}\right)\\
&= e^{-2\pi i\gamma}\sum_{l,k}Tr\left(P_{\vec{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{ij}E^{(n)}_{ll}\Gamma_T\left((n,n+1)\right)P_{\vec{n};T,(t,u)\nu_2\nu_1}E^{(n+1)}_{ji}E^{(n)}_{kk}\Gamma_R\left((n,n+1)\right)\right)\\
&= e^{-2\pi i\gamma}\sum_{l,k}Tr\left(P_{\vec{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{il}E^{(n)}_{lj}P_{\vec{n};T,(t,u)\nu_2\nu_1}E^{(n+1)}_{jk}E^{(n)}_{ki}\right)\\
&= e^{-2\pi i\gamma}\delta_{\vec{m},\vec{n}}\sum_{l,k}Tr_r\left(\mathbf{1}_rE^{(n)}_{lj}\mathbf{1}_tE^{(n)}_{ki}\right)\langle\vec{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\vec{n},u,\nu_2;b\rangle\langle\vec{n},u,\nu_1;b|\,E^{(n+1)}_{jj}\,|\vec{m},s,\mu_1;a\rangle\\
&= e^{-2\pi i\gamma}\delta_{\vec{m},\vec{n}}\delta_{r'(i),t'(j)}d_{r'(i)}\langle\vec{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\vec{n},u,\nu_2;b\rangle\langle\vec{n},u,\nu_1;b|\,E^{(n+1)}_{jj}\,|\vec{m},s,\mu_1;a\rangle
\end{aligned}$$

and

$$\begin{aligned}
&e^{2\pi i\gamma}Tr\left(\left(P_{\vec{m};R,(r,s)\mu_1\mu_2}\Gamma_R\left((n,n+1)\right)\right)I_{R'T'}\left(P_{\vec{n};T,(t,u)\nu_2\nu_1}\Gamma_T\left((n,n+1)\right)\right)I_{T'R'}\right)\\
&= e^{2\pi i\gamma}\sum_{l,k}Tr\left(P_{\vec{m};R,(r,s)\mu_1\mu_2}\Gamma_R\left((n,n+1)\right)E^{(n+1)}_{ij}E^{(n)}_{ll}P_{\vec{n};T,(t,u)\nu_2\nu_1}\Gamma_T\left((n,n+1)\right)E^{(n+1)}_{ji}E^{(n)}_{kk}\right)\\
&= e^{2\pi i\gamma}\sum_{l,k}Tr\left(P_{\vec{m};R,(r,s)\mu_1\mu_2}E^{(n+1)}_{lj}E^{(n)}_{il}P_{\vec{n};T,(t,u)\nu_2\nu_1}E^{(n+1)}_{ki}E^{(n)}_{jk}\right)\\
&= e^{2\pi i\gamma}\delta_{\vec{m},\vec{n}}\sum_{l,k}Tr_r\left(\mathbf{1}_rE^{(n)}_{il}\mathbf{1}_tE^{(n)}_{jk}\right)\langle\vec{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|\vec{n},u,\nu_2;b\rangle\langle\vec{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\vec{m},s,\mu_1;a\rangle\\
&= e^{2\pi i\gamma}\delta_{\vec{m},\vec{n}}\delta_{r'(i),t'(j)}d_{r'(i)}\langle\vec{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|\vec{n},u,\nu_2;b\rangle\langle\vec{n},u,\nu_1;b|\,E^{(n+1)}_{ii}\,|\vec{m},s,\mu_1;a\rangle.
\end{aligned}$$

[7] A note on notation: $r'(i)$ means that we are removing one box (represented by one $'$) from row i of Young diagram r.

We write $\delta_{r'(i),t'(j)}$ in the last line because we are removing one box from row i of Young diagram r and one box from row j of Young diagram t.

The other two traces evaluate to zero since

$$\mathbf{1}_r \mathbf{1}_t = \delta_{r,t} = 0$$

and

$$\begin{aligned} P_{\vec{n};T,(t,u)\nu_2\nu_1} E_{ii}^{(n+1)} P_{\vec{m};R,(r,s)\mu_1\mu_2} &= \mathbf{1}_r \mathbf{1}_t \otimes |\vec{n}, u, \nu_2; b\rangle\langle \vec{n}, u, \nu_1; b| \, E_{ii}^{(n+1)} \\ &\times |\vec{m}, s, \mu_1; a\rangle \langle \vec{m}, s, \mu_2; a| \\ &= 0. \end{aligned} \tag{3.51}$$

This is because r and t are different subspaces in this case.

Thus, when $R \neq T$, the sum of the traces is

$$\begin{aligned} -d_{r'(i)} \delta_{\vec{m},\vec{n}} \delta_{r'(i),t'(j)} \Big[& e^{-2\pi i\gamma} \langle \vec{m}, s, \mu_2; a| \, E_{ii}^{(n+1)} \, |\vec{n}, u, \nu_2; b\rangle\langle \vec{n}, u, \nu_1; b| \, E_{jj}^{(n+1)} \, |\vec{m}, s, \mu_1; a\rangle \\ & + e^{2\pi i\gamma} \langle \vec{m}, s, \mu_2; a| \, E_{jj}^{(n+1)} \, |\vec{n}, u, \nu_2; b\rangle\langle \vec{n}, u, \nu_1; b| \, E_{ii}^{(n+1)} \, |\vec{m}, s, \mu_1; a\rangle \Big]. \end{aligned} \tag{3.52}$$

3.2.3 The coefficient

Now we calculate the coefficient

$$c_{RR'} \frac{d_T n \cdot m}{d_{R'} d_t d_u \, (n+m)} \sqrt{\frac{f_T hooks_T hooks_r hooks_s}{f_R hooks_R hooks_t hooks_u}}$$

for $N \to \infty$. Let us start by writing the coefficient as

$$c_{RR'} \sqrt{\frac{f_T}{f_R}} \cdot \frac{d_T n \cdot m}{d_{R'} d_t d_u \, (n+m)} \sqrt{\frac{hooks_T hooks_r hooks_s}{hooks_R hooks_t hooks_u}} \tag{3.53}$$

and then write out

$$d_T = \frac{(n+m)!}{hooks_T}, \qquad d_{R'} = \frac{(n+m-1)!}{hooks_{R'}}, \qquad d_t = \frac{n!}{hooks_t}.$$

After some simplification, this gives

$$
\begin{aligned}
c_{RR'}\sqrt{\frac{f_T}{f_R}}\cdot\frac{m}{d_{r'}d_u}\sqrt{\frac{hooks_s}{hooks_u}} &= c_{RR'}\sqrt{\frac{f_T}{f_R}}\cdot\frac{m}{d_{r'}}\sqrt{\frac{1}{d_u^2}\frac{hooks_s}{hooks_u}} \\
&= c_{RR'}\sqrt{\frac{f_T}{f_R}}\cdot\frac{m}{d_{r'}}\sqrt{\frac{1}{d_u}\frac{hooks_u}{m!}\frac{hooks_s}{hooks_u}} \\
&= c_{RR'}\sqrt{\frac{f_T}{f_R}}\cdot\frac{m}{d_{r'}}\sqrt{\frac{1}{d_u}\frac{hooks_s}{m!}} \\
&= c_{RR'}\sqrt{\frac{f_T}{f_R}}\cdot\frac{m}{d_{r'}}\cdot\frac{1}{\sqrt{d_s d_u}}. \qquad (3.54)
\end{aligned}
$$

3.2.4 Action of the dilatation operator

Bringing everything together, the action of the dilatation operator can be written as

$$
\begin{aligned}
D_\gamma O_{R,(r,s)\mu_1\mu_2} = {}& -g_{YM}^2 \sum_{u\nu_1\nu_2}\frac{m}{\sqrt{d_s d_u}}\delta_{\vec{m},\vec{n}}\sum_{i=1}^{p} \\
& \times\Bigg[\sum_{j\neq i}^{p}(\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{jj}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle \\
& +\langle\vec{m},s,\mu_2;a|\,E_{jj}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle)\Delta_{ij}^0\delta_{r,t} \\
& -(e^{-2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{jj}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle \\
& +e^{2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{jj}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle) \\
& \times\Delta_{ij}^-\delta_{r'(i),t'(j)} \\
& -\left(e^{-2\pi i\gamma}+e^{2\pi i\gamma}-2\right)\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)} \\
& \times|\vec{m},s,\mu_1;a\rangle\,\Delta_{ii}^0\delta_{r,t}\Bigg]O_{R,(r,u)\nu_1\nu_2}, \qquad (3.55)
\end{aligned}
$$

where we have introduced

$$\Delta_{ij}^0 \quad \& \quad \Delta_{ij}^\pm.$$

In order to explain this notation, let r_i be the row length of Young diagram r. Further, let r_{ij}^+ be the diagram obtained by moving a box from row j to row i and r_{ij}^- be the diagram obtained by moving a box from row i to row j. We then have

$$\Delta_{ij}^0 O_{R,(r,s)\mu_1\mu_2} = (2N + r_i + r_j)\,O_{R,(r,s)\mu_1\mu_2} \qquad (3.56)$$

for the case in which $R = T$.

When $R \neq T$ we have

$$\Delta_{ij}^{+} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N+r_i)(N+r_j)}\, O_{R_{ij}^{+},(r_{ij}^{+},s)\mu_1\mu_2} \tag{3.57}$$

and

$$\Delta_{ij}^{-} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N+r_i)(N+r_j)}\, O_{R_{ij}^{-},(r_{ij}^{-},s)\mu_1\mu_2}. \tag{3.58}$$

The sum over i follows from the sum over R' which encodes the various ways of removing a single box from Young diagram R.

In equation (3.55), the sum over $j \neq i$ can be written as a sum over $j > i$ as follows.[8]

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j\neq i}^{p}\Big[\langle \overrightarrow{m}, s, \mu_2; a|\, E_{ii}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{jj}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle \\
&\quad + \langle \overrightarrow{m}, s, \mu_2; a|\, E_{jj}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{ii}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle\Big]\Delta_{ij}^{0} \\
&= \sum_{i=1}^{p}\sum_{j>i}^{p}\Big[\langle \overrightarrow{m}, s, \mu_2; a|\, E_{ii}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{jj}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle \\
&\quad + \langle \overrightarrow{m}, s, \mu_2; a|\, E_{jj}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{ii}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle\Big]\Delta_{ij}^{0} \\
&+ \sum_{i=1}^{p}\sum_{j<i}^{p}\Big[\langle \overrightarrow{m}, s, \mu_2; a|\, E_{ii}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{jj}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle \\
&\quad + \langle \overrightarrow{m}, s, \mu_2; a|\, E_{jj}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{ii}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle\Big]\Delta_{ij}^{0}.
\end{aligned}$$

Swapping i and j in the $\sum_{j<i}^{p}$ term then allows us to write the sum over $j \neq i$ as

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j>i}^{p}\Big[\langle \overrightarrow{m}, s, \mu_2; a|\, E_{ii}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{jj}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle \\
&\quad + \langle \overrightarrow{m}, s, \mu_2; a|\, E_{jj}^{(n+1)}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E_{ii}^{(n+1)}\, |\overrightarrow{m}, s, \mu_1; a\rangle\Big]\Delta_{ij}^{0}.
\end{aligned} \tag{3.59}$$

[8]In what follows, we will not explicitly write $\delta_{r,t}$ and $\delta_{r'(i),t'(j)}$.

Similarly,

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j\neq i}^{p}\Big[e^{-2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{jj}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle \\
&\qquad +e^{2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{jj}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle\Big]\Delta_{ij}^{-} \\
&=\sum_{i=1}^{p}\sum_{j>i}^{p}\Big[e^{-2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{jj}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle \\
&\qquad +e^{2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{jj}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle\Big]\Delta_{ij}^{-} \\
&+\sum_{i=1}^{p}\sum_{j<i}^{p}\Big[e^{-2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{jj}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle \\
&\qquad +e^{2\pi i\gamma}\langle\vec{m},s,\mu_2;a|\,E_{jj}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle\Big]\Delta_{ij}^{-}.
\end{aligned}$$

In this case, we are removing a single box from the i^{th} row of Young diagram r and we have Δ_{ij}^{-}. If the box comes from row j of diagram r we have Δ_{ij}^{+}. With this in mind, swapping i and j changes Δ_{ij}^{-} to Δ_{ij}^{+}. Doing this for the $\sum_{j<i}^{p}$ term and simplifying yields

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j>i}^{p}\Big[(e^{-2\pi i\gamma}\Delta_{ij}^{-}+e^{2\pi i\gamma}\Delta_{ij}^{+})\,\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{jj}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle \\
&\qquad (e^{-2\pi i\gamma}\Delta_{ij}^{+}+e^{2\pi i\gamma}\Delta_{ij}^{-})\,\langle\vec{m},s,\mu_2;a|\,E_{jj}^{(n+1)}\,|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle\Big].
\end{aligned} \tag{3.60}$$

Now, substituting expressions (3.59) and (3.60) into equation (3.55) then yields

$$\begin{aligned}
D_{\gamma}O_{R,(r,s)\mu_1\mu_2} &= -g_{YM}^2\sum_{u\nu_1\nu_2}\frac{m}{\sqrt{d_s d_u}}\delta_{\vec{m},\vec{n}}\sum_{i=1}^{p}\Big[\sum_{j>i}^{p} \\
&\times\Big((\Delta_{ij}^{0}-(e^{-2\pi i\gamma}\Delta_{ij}^{+}+e^{2\pi i\gamma}\Delta_{ij}^{-}))\,\langle\vec{m},s,\mu_2;a|\,E_{jj}^{(n+1)} \\
&\times|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle \\
&+(\Delta_{ij}^{0}-(e^{-2\pi i\gamma}\Delta_{ij}^{-}+e^{2\pi i\gamma}\Delta_{ij}^{+}))\,\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)} \\
&\times|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{jj}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle\Big) \\
&-\Big(e^{-2\pi i\gamma}+e^{2\pi i\gamma}-2\Big)\,\langle\vec{m},s,\mu_2;a|\,E_{ii}^{(n+1)} \\
&\times|\vec{n},u,\nu_2;b\,\rangle\langle\,\vec{n},u,\nu_1;b|\,E_{ii}^{(n+1)}\,|\vec{m},s,\mu_1;a\rangle\,\Delta_{ii}^{0}\Big]O_{R,(r,u)\nu_1\nu_2}.
\end{aligned} \tag{3.61}$$

We notice here that in each of these three terms, the action of the dilatation operator factorises into a part that acts on the R, r & T, t labels and a part that acts only on the s, u labels. In the next section we will take a continuum limit for the R, r & T, t part.

3.2.5 The continuum limit

The continuum limit we take here is such that

$$N + b_0 \to \infty$$

while the variables

$$x_i = \frac{l_i}{\sqrt{N + b_0}}$$

are held constant. Here, b_0 is the length of the bottom most row in our Young diagrams so that the lengths of the other rows are $b_0 + l_i$. In spacetime, this corresponds to having giant gravitons that are very close to one another.

For operators of a good scaling dimension we can make the ansatz

$$O = \sum_{b_0, l_1, \ldots l_{p-1}} f\left(b_0, l_1, \ldots l_{p-1}\right) O\left(b_0, l_1, \ldots l_{p-1}\right),$$

where p is the number of rows in each Young diagram. This way, we can write

$$\begin{aligned} \Delta_{ij} O &= \sum_{b_0, l_1, \ldots l_{p-1}} f\left(b_0, l_1, \ldots l_{p-1}\right) \Delta_{ij} O\left(b_0, l_1, \ldots l_{p-1}\right) \\ &= \sum_{b_0, l_1, \ldots l_{p-1}} \tilde{\Delta}_{ij} f\left(b_0, l_1, \ldots l_{p-1}\right) O\left(b_0, l_1, \ldots l_{p-1}\right) \end{aligned}$$

where

$$\tilde{\Delta}_{ij} f\left(b_0, l_1, \ldots l_{p-1}\right) = \left(2N + 2b_0 + l_i + l_j\right) f\left(b_0, l_1, \ldots l_{p-1}\right)$$

for

$$\tilde{\Delta}_{ij} = \Delta^0_{ij}$$

for example.

First term: $\Delta^0_{ij} - (e^{-2\pi i\gamma}\Delta^+_{ij} + e^{2\pi i\gamma}\Delta^-_{ij})$

Here,

$$\Delta^0_{ij} f\left(b_0, l_1, \ldots l_{p-1}\right) = \left(2N + 2b_0 + l_i + l_j\right) f\left(b_0, l_1, \ldots l_{p-1}\right) \tag{3.62}$$

as we have already mentioned, while

$$\Delta_{ij}^{+} f\left(b_0, l_1, \ldots l_{p-1}\right) = \sqrt{\left(N + b_0 + l_i\right)\left(N + b_0 + l_j\right)} f\left(b_0, \ldots, l_i + 1, \ldots, l_j - 1, \ldots\right) \tag{3.63}$$

and

$$\Delta_{ij}^{-} f\left(b_0, l_1, \ldots l_{p-1}\right) = \sqrt{\left(N + b_0 + l_i\right)\left(N + b_0 + l_j\right)} f\left(b_0, \ldots, l_i - 1, \ldots, l_j + 1, \ldots\right). \tag{3.64}$$

Expanding

$$\sqrt{\left(N + b_0 + l_i\right)\left(N + b_0 + l_j\right)} = N + b_0 + \frac{x_i + x_j}{2}\sqrt{N + b_0} - \frac{\left(x_i - x_j\right)^2}{8} + \ldots$$

and

$$\begin{aligned} f\left(b_0, \ldots, l_i + 1, \ldots, l_j - 1, \ldots\right) \rightarrow\; & f\left(b_0, \ldots, x_i + \frac{1}{\sqrt{N + b_0}}, \ldots, x_j - \frac{1}{\sqrt{N + b_0}}, \ldots\right) \\ &= f\left(b_0, \ldots, x_i, \ldots, x_j, \ldots\right) + \frac{1}{\sqrt{N + b_0}}\frac{\partial f}{\partial x_i} - \frac{1}{\sqrt{N + b_0}}\frac{\partial f}{\partial x_j} \\ &+ \frac{1}{2\left(N + b_0\right)}\frac{\partial^2 f}{\partial x_i^2} + \frac{1}{2\left(N + b_0\right)}\frac{\partial^2 f}{\partial x_j^2} - \frac{1}{N + b_0}\frac{\partial^2 f}{\partial x_i \partial x_j} + \ldots \end{aligned}$$

followed by taking the continuum limit gives

$$\begin{aligned} \left(\Delta_{ij}^{0} - \left(e^{-2\pi i\gamma}\Delta_{ij}^{+} + e^{2\pi i\gamma}\Delta_{ij}^{-}\right)\right) f = & \left(2N + 2b_0 + l_i + l_j\right) f \\ & - \cos\left(2\pi\gamma\right)\left(2N + 2b_0 + \left(x_i + x_j\right)\sqrt{N + b_0} - \frac{\left(x_i - x_j\right)^2}{4}\right) f \\ & + i\sin\left(2\pi\gamma\right)\left(2\sqrt{N + b_0} + x_i + x_j\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) f \\ & - \cos\left(2\pi\gamma\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2 f \end{aligned}$$

which we can re-write as

$$\begin{aligned} \left(\Delta_{ij}^{0} - \left(e^{-2\pi i\gamma}\Delta_{ij}^{+} + e^{2\pi i\gamma}\Delta_{ij}^{-}\right)\right) f = & \left(2N + 2b_0 + l_i + l_j\right)\left(1 - \cos\left(2\pi\gamma\right)\right) f \\ & + i\sin\left(2\pi\gamma\right)\left(2\sqrt{N + b_0} + x_i + x_j\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) f \\ & + \cos\left(2\pi\gamma\right)\left(\frac{\left(x_i - x_j\right)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right) f. \end{aligned} \tag{3.65}$$

Second term: $\Delta^0_{ij} - (e^{-2\pi i\gamma}\Delta^-_{ij} + e^{2\pi i\gamma}\Delta^+_{ij})$

In this case we get

$$\begin{aligned}\left(\Delta^0_{ij} - (e^{-2\pi i\gamma}\Delta^-_{ij} + e^{2\pi i\gamma}\Delta^+_{ij})\right) f =& \left(2N + 2b_0 + l_i + l_j\right) f \\ &- \cos\left(2\pi\gamma\right)\left(2N + 2b_0 + \left(x_i + x_j\right)\sqrt{N + b_0} - \frac{\left(x_i - x_j\right)^2}{4}\right) f \\ &- i\sin\left(2\pi\gamma\right)\left(2\sqrt{N + b_0} + x_i + x_j\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) f \\ &- \cos\left(2\pi\gamma\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2 f\end{aligned}$$

in the continuum limit. We can also re-write this equation as

$$\begin{aligned}\left(\Delta^0_{ij} - (e^{-2\pi i\gamma}\Delta^-_{ij} + e^{2\pi i\gamma}\Delta^+_{ij})\right) f =& \left(2N + 2b_0 + l_i + l_j\right)\left(1 - \cos\left(2\pi\gamma\right)\right) f \\ &- i\sin\left(2\pi\gamma\right)\left(2\sqrt{N + b_0} + x_i + x_j\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) f \\ &+ \cos\left(2\pi\gamma\right)\left(\frac{\left(x_i - x_j\right)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right) f.\end{aligned} \tag{3.66}$$

Last term: $(e^{-2\pi i\gamma} + e^{2\pi i\gamma} - 2)\,\Delta^0_{ii}$

This term yields

$$\left(e^{-2\pi i\gamma} + e^{2\pi i\gamma} - 2\right)\Delta^0_{ii} f = 2\left(2N + 2b_0 + 2l_i\right)\left(\cos\left(2\pi\gamma\right) - 1\right) f. \tag{3.67}$$

Thus, putting everything together, the action of the dilatation operator in the continuum limit is

$$
\begin{aligned}
D_\gamma O = & -g_{YM}^2 \sum_{b_0,l_1,\ldots l_{p-1}} \sum_{u\nu_1\nu_2} \frac{m}{\sqrt{d_s d_u}} \delta_{\overrightarrow{m},\overrightarrow{n}} \sum_{i=1}^{p} \Bigg[\sum_{j>i}^{p} [((2N + 2b_0 + l_i + l_j)\,(1-\cos(2\pi\gamma)) \\
& + i\sin(2\pi\gamma)\left(2\sqrt{N+b_0} + x_i + x_j\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \\
& + \cos(2\pi\gamma)\left(\frac{(x_i-x_j)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right)) M^{(ji)}_{s\mu_1\mu_2;u\nu_1\nu_2} \\
& + ((2N + 2b_0 + l_i + l_j)\,(1-\cos(2\pi\gamma)) \\
& - i\sin(2\pi\gamma)\left(2\sqrt{N+b_0} + x_i + x_j\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \\
& + \cos(2\pi\gamma)\left(\frac{(x_i-x_j)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right)) M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}] \\
& - 2\,(2N + 2b_0 + 2l_i)\,(\cos(2\pi\gamma) - 1)\, M^{(ii)}_{s\mu_1\mu_2;u\nu_1\nu_2} \Bigg] \\
& \times f\,(b_0, \ldots, x_i, \ldots, x_j, \ldots)\, O\,(b_0, l_1, \ldots l_{p-1})\,,
\end{aligned}
\tag{3.68}
$$

where

$$M^{(ji)}_{s\mu_1\mu_2;u\nu_1\nu_2} = \langle \overrightarrow{m}, s, \mu_2; a|\, E^{(n+1)}_{jj}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E^{(n+1)}_{ii}\, |\overrightarrow{m}, s, \mu_1; a\rangle\,,$$

$$M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} = \langle \overrightarrow{m}, s, \mu_2; a|\, E^{(n+1)}_{ii}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E^{(n+1)}_{jj}\, |\overrightarrow{m}, s, \mu_1; a\rangle$$

and

$$M^{(ii)}_{s\mu_1\mu_2;u\nu_1\nu_2} = \langle \overrightarrow{m}, s, \mu_2; a|\, E^{(n+1)}_{ii}\, |\overrightarrow{n}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{n}, u, \nu_1; b|\, E^{(n+1)}_{ii}\, |\overrightarrow{m}, s, \mu_1; a\rangle\,.$$

Simplifying yields

$$D_\gamma O = -g_{YM}^2 \sum_{b_0,l_1,\ldots l_{p-1}} \sum_{u\nu_1\nu_2} \frac{m}{\sqrt{d_s d_u}} \delta_{\vec{m},\vec{n}} \sum_{i=1}^{p} \Bigg[\sum_{j>i}^{p} [((2N + 2b_0 + l_i + l_j)(1 - \cos(2\pi\gamma))$$
$$+ \cos(2\pi\gamma) \left(\frac{(x_i - x_j)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)^2 \right)) \left(M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} + M^{(ji)}_{s\mu_1\mu_2;u\nu_1\nu_2} \right)$$
$$- i \sin(2\pi\gamma) \left(2\sqrt{N + b_0} + x_i + x_j \right) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \left(M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} - M^{(ji)}_{s\mu_1\mu_2;u\nu_1\nu_2} \right)]$$
$$- 2(2N + 2b_0 + 2l_i)(\cos(2\pi\gamma) - 1) M^{(ii)}_{s\mu_1\mu_2;u\nu_1\nu_2} \Bigg]$$
$$\times f(b_0, \ldots, x_i, \ldots, x_j, \ldots) O(b_0, l_1, \ldots l_{p-1}). \tag{3.69}$$

3.2.6 Gauss graph operators

In this section, we compute the matrix elements of the deformed dilatation operator, D_γ, in the Gauss graph basis $\langle O_{T,t}(\sigma_2) D_\gamma O_{R,r}(\sigma_1) \rangle$. The Gauss graph operators can be written as

$$O_{R,r}(\sigma_1) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s\vdash m} \sum_{\mu_1,\mu_2} \sqrt{d_s} \Gamma^{(s)}_{jk}(\sigma_1) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2} O_{R,(r,s)\mu_1\mu_2}, \tag{3.70}$$

where

$$H = S_{m_1} \times S_{m_2} \times \ldots S_{m_p}$$

is a sub-group of the symmetric group S_m and $B^{s\to 1_H}_{j\mu_1}$ is a branching coefficient that allows us to project from s, an irrep of S_m, to 1_H, a one-dimensional representation of H. d_s is the dimension of the irrep s. The order of the sub-group H is

$$|H| = \prod_{i=1}^{p} m_i!. \tag{3.71}$$

From the first term in equation (3.61) we have[9]

$$\begin{aligned}\left\langle O^{\dagger}_{T,t}(\sigma_2) D_\gamma O_{R,r}(\sigma_1)\right\rangle_1 &= \frac{|H|^2}{m!}\sum_{s,u\vdash m}\sum_{\mu_1,\mu_2,\nu_1,\nu_2}\sqrt{d_s d_u}\Gamma^{(s)}_{jk}(\sigma_1) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2}\\ &\times \Gamma^{(u)}_{lm}(\sigma_2) B^{u\to 1_H}_{l\nu_1} B^{u\to 1_H}_{m\nu_2}\left\langle O^{\dagger}_{T,(t,u)\nu_1\nu_2} D_\gamma O_{R,(r,s)\mu_1\mu_2}\right\rangle_1\\ &= -g^2_{YM}\frac{|H|^2}{m!}\sum_{i=1}^{p}\sum_{j>i}^{p}\sum_{s,u\vdash m}\sum_{\mu_1,\mu_2,\nu_1,\nu_2}\Gamma^{(s)}_{jk}(\sigma_1) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2}\\ &\times \Gamma^{(u)}_{lm}(\sigma_2) B^{u\to 1_H}_{l\nu_1} B^{u\to 1_H}_{m\nu_2}(\Delta^0_{ij} - (e^{-2\pi i\gamma}\Delta^+_{ij} + e^{2\pi i\gamma}\Delta^-_{ij}))\\ &\times m\,\langle \overrightarrow{m}, s, \mu_2; a|\, E^{(n+1)}_{jj}\, |\overrightarrow{m}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{m}, u, \nu_1; b|\, E^{(n+1)}_{ii}\, |\overrightarrow{m}, s, \mu_1; a\rangle\,.\end{aligned} \tag{3.72}$$

Let us sum over u first. We study

$$\begin{aligned}\sum_{s,u\vdash m}\sum_{\mu_1,\mu_2,\nu_1,\nu_2}\Gamma^{(s)}_{jk}(\sigma_1) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2}\Gamma^{(u)}_{lm}(\sigma_2) B^{u\to 1_H}_{l\nu_1} B^{u\to 1_H}_{m\nu_2}\\ \times m\,\langle \overrightarrow{m}, s, \mu_2; a|\, E^{(n+1)}_{jj}\, |\overrightarrow{m}, u, \nu_2; b\,\rangle\langle\, \overrightarrow{m}, u, \nu_1; b|\, E^{(n+1)}_{ii}\, |\overrightarrow{m}, s, \mu_1; a\rangle\,.\end{aligned} \tag{3.73}$$

and use the following definition for the ket [46]

$$|\overrightarrow{m}, u, \nu_2; b\rangle \equiv \sum_p\sum_{\sigma\in S_m} B^{u\to 1_H}_{p\nu_2}\Gamma^{(u)}_{bp}(\sigma)\,|v_\sigma\rangle\,, \tag{3.74}$$

where

$$|v_\sigma\rangle = \sigma\,|v\rangle = \sigma\left|v_1^{\otimes m_1}\otimes v_2^{\otimes m_2}\otimes\cdots v_p^{\otimes m_p}\right\rangle. \tag{3.75}$$

We also define the bra as [46]

$$\langle\overrightarrow{m}, u, \nu_1; b| \equiv \frac{d_u}{m!\,|H|}\sum_q\sum_{\tau\in S_m}\langle v_\tau|\,\Gamma^{(u)}_{bq}(\tau)\,B^{u\to 1_H}_{q\nu_1}, \tag{3.76}$$

where

$$\langle v_\tau| = \langle v|\,\tau^{-1} = \left\langle v_1^{\otimes m_1}\otimes v_2^{\otimes m_2}\otimes\cdots v_p^{\otimes m_p}\right|\tau^{-1}. \tag{3.77}$$

Using this together with

$$\frac{1}{|H|}\sum_{\gamma\in H}\Gamma^{(u)}_{ik}(\gamma) = \sum_{\nu_1} B^{u\to 1_H}_{i\nu_1} B^{u\to 1_H}_{k\nu_1}, \tag{3.78}$$

[9]In what follows, we will not explicitly write out the sum over the matrix indices, but they must be understood as summed.

expression (3.73) becomes

$$\begin{aligned}
&\frac{md_u}{|H|^3\, m!}\sum_{s,u\vdash m}\sum_{\sigma,\tau\in S_m}\sum_{\gamma_1,\gamma_2\in H}\sum_{\mu_1,\mu_2}\Gamma^{(s)}_{jk}(\sigma_1)\, B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2}\Gamma^{(u)}_{lm}(\sigma_2)\,\Gamma^{(u)}_{ql}(\gamma_1)\\
&\times\Gamma^{(u)}_{pm}(\gamma_2)\,\Gamma^{(u)}_{bp}(\sigma)\,\Gamma^{(u)}_{bq}(\tau)\,\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|v_\sigma\rangle\langle v_\tau|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&=\frac{md_u}{|H|^3\, m!}\sum_{s,u\vdash m}\sum_{\sigma,\tau\in S_m}\sum_{\gamma_1,\gamma_2\in H}\sum_{\mu_1,\mu_2}\Gamma^{(s)}_{jk}(\sigma_1)\, B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2}\Gamma^{(u)}_{bp}(\sigma)\,\Gamma^{(u)}_{pm}(\gamma_2)\\
&\times\Gamma^{(u)}_{ml}\left(\sigma_2^{-1}\right)\Gamma^{(u)}_{lq}\left(\gamma_1^{-1}\right)\Gamma^{(u)}_{qb}\left(\tau^{-1}\right)\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|v_\sigma\rangle\langle v_\tau|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&=\frac{md_u}{|H|^3\, m!}\sum_{s,u\vdash m}\sum_{\sigma,\tau\in S_m}\sum_{\gamma_1,\gamma_2\in H}\sum_{\mu_1,\mu_2}\Gamma^{(s)}_{jk}(\sigma_1)\, B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2}\\
&\times\chi_u\left(\sigma\gamma_2\sigma_2^{-1}\gamma_1^{-1}\tau^{-1}\right)\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|v_\sigma\rangle\langle v_\tau|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\\
&=\frac{m}{|H|^3}\sum_{s\vdash m}\sum_{\sigma,\tau\in S_m}\sum_{\gamma_1,\gamma_2\in H}\sum_{\mu_1,\mu_2}\Gamma^{(s)}_{jk}(\sigma_1)\, B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2}\\
&\times\delta\left(\sigma\gamma_2\sigma_2^{-1}\gamma_1^{-1}\tau^{-1}\right)\langle\overrightarrow{m},s,\mu_2;a|\,E^{(n+1)}_{jj}\,|v_\sigma\rangle\langle v_\tau|\,E^{(n+1)}_{ii}\,|\overrightarrow{m},s,\mu_1;a\rangle\,.
\end{aligned}$$

To get the last line, we used the definition of the delta function for the symmetric group S_m,

$$\sum_u\frac{d_u}{m!}\chi_u(\sigma\tau)=\delta(\sigma\tau)\,. \tag{3.79}$$

The sum over s can be done in exactly the same way. The result is

$$\frac{m}{|H|^6}\sum_{\alpha,\beta,\sigma,\tau\in S_m}\sum_{\gamma_i\in H}\delta\left(\gamma_3\sigma_1\gamma_4^{-1}\beta^{-1}\alpha\right)\delta\left(\sigma\gamma_2\sigma_2^{-1}\gamma_1^{-1}\tau^{-1}\right)\langle v_\beta|\,E^{(n+1)}_{jj}\,|v_\sigma\rangle\langle v_\tau|\,E^{(n+1)}_{ii}\,|v_\alpha\rangle \tag{3.80}$$

which we can re-write as

$$\begin{aligned}
&\frac{m}{|H|^6}\sum_{\alpha,\beta,\sigma,\tau\in S_m}\sum_{\gamma_i\in H}\delta\left(\gamma_3\sigma_1\gamma_4^{-1}\beta^{-1}\alpha\right)\delta\left(\sigma\gamma_2\sigma_2^{-1}\gamma_1^{-1}\tau^{-1}\right)\langle v|\,\beta^{-1}E^{(n+1)}_{jj}\sigma\\
&\qquad\times|v\rangle\langle v|\,\tau^{-1}E^{(n+1)}_{ii}\alpha\,|v\rangle\\
&=\frac{m}{|H|^6}\sum_{\alpha,\beta,\sigma,\tau\in S_m}\sum_{\gamma_i\in H}\delta\left(\gamma_3\sigma_1\gamma_4^{-1}\beta^{-1}\alpha\right)\delta\left(\sigma\gamma_2\sigma_2^{-1}\gamma_1^{-1}\tau^{-1}\right)\langle v|\,E^{\beta^{-1}(n+1)}_{jj}\beta^{-1}\sigma\\
&\qquad\times|v\rangle\langle v|\,E^{\tau^{-1}(n+1)}_{ii}\tau^{-1}\alpha\,|v\rangle\,.
\end{aligned} \tag{3.81}$$

The delta functions are non-zero when

$$\alpha^{-1}=\gamma_3\sigma_1\gamma_4^{-1}\beta^{-1}$$

and

$$\sigma^{-1} = \gamma_2\sigma_2^{-1}\gamma_1^{-1}\tau^{-1}$$

from which we deduce

$$\alpha = \beta\gamma_4\sigma_1^{-1}\gamma_3^{-1}$$

and

$$\sigma = \tau\gamma_1\sigma_2\gamma_2^{-1}.$$

Using this together with the invariance of $|v\rangle$ under H, we get

$$\frac{m}{|H|^4}\sum_{\beta,\tau\in S_m}\sum_{\gamma_1,\gamma_4\in H}\langle v|\, E_{jj}^{\beta^{-1}(n+1)}\beta^{-1}\tau\gamma_1\sigma_2\,|v\rangle\langle v|\, E_{ii}^{\tau^{-1}(n+1)}\tau^{-1}\beta\gamma_4\sigma_1^{-1}\,|v\rangle\,. \tag{3.82}$$

Now, $\langle v|\, E_{jj}^{\beta^{-1}(n+1)}$ gives $\langle v|$ if $\beta^{-1}(n+1)$ belongs to a set of integers S_j in the range

$$(m_1 + m_2 + \cdots m_{j-1} + 1)\,, \ldots, (m_1 + m_2 + \cdots m_j)\,,$$

both inclusive. Also, (3.82) gives zero except when

$$\beta^{-1}(n+1) \in S_j$$

and

$$\tau^{-1}(n+1) \in S_i.$$

Using this in conjunction with

$$\langle v|\,\sigma\,|v\rangle = \sum_{\gamma\in H}\delta(\sigma\gamma) \tag{3.83}$$

we get

$$\frac{m}{|H|^4}\sum_{\beta,\tau\in S_m}\sum_{\gamma_i\in H}\delta\left(\beta^{-1}\tau\gamma_1\sigma_2\gamma_2\right)\delta\left(\tau^{-1}\beta\gamma_4\sigma_1^{-1}\gamma_3\right)\sum_{k\in S_i}\sum_{l\in S_j}\delta\left(\tau^{-1}(n+1)\,,k\right)\delta\left(\beta^{-1}(n+1)\,,l\right). \tag{3.84}$$

At this point, we can introduce $\beta \to \beta^{-1}\alpha$ and $\tau^{-1} \to \tau^{-1}\alpha$, where $\alpha \in Z_m$.[10] This adds up to replacing $n+1$ by $\alpha(n+1)$ in (3.84). The sum over $\alpha \in Z_m$ is normalised by $\frac{1}{m}$.

[10] Here, Z_m is a group of products of cyclic permutations.

Performing this sum gives us

$$\frac{1}{|H|^4}\sum_{\beta,\tau\in S_m}\sum_{\gamma_i\in H}\delta\left(\beta^{-1}\tau\gamma_1\sigma_2\gamma_2\right)\delta\left(\tau^{-1}\beta\gamma_4\sigma_1^{-1}\gamma_3\right)\sum_{k\in S_i}\sum_{l\in S_j}\delta\left(\beta^{-1}\tau\left(k\right),l\right)$$
$$=\frac{m!}{|H|^4}\sum_{\beta\in S_m}\sum_{\gamma_i\in H}\delta\left(\beta^{-1}\gamma_1\sigma_2\gamma_2\right)\delta\left(\beta\gamma_4\sigma_1^{-1}\gamma_3\right)\sum_{k\in S_i}\sum_{l\in S_j}\delta\left(\beta^{-1}\left(k\right),l\right).$$

At this point, we realise the number of strings leaving brane i and terminating on brane j

$$n_{ij}^{+}\left(\beta^{-1}\right)=\sum_{k\in S_i}\sum_{l\in S_j}\delta\left(\beta^{-1}\left(k\right),l\right) \tag{3.85}$$

yielding

$$\frac{m!}{|H|^4}\sum_{\beta\in S_m}\sum_{\gamma_i\in H}\delta\left(\beta^{-1}\gamma_1\sigma_2\gamma_2\right)\delta\left(\beta\gamma_4\sigma_1^{-1}\gamma_3\right)n_{ij}^{+}\left(\beta^{-1}\right)$$
$$=\frac{m!}{|H|^4}\sum_{\gamma_i\in H}\delta\left(\gamma_4\sigma_1^{-1}\gamma_3\gamma_1\sigma_2\gamma_2\right)n_{ij}^{+}\left(\gamma_4\sigma_1^{-1}\gamma_3\right).$$

Finally, $n_{ij}^{+}(\sigma)$ is invariant under left and right multiplication by H so that we get[11]

$$\frac{m!}{|H|^2}\sum_{\gamma_1,\gamma_2\in H}\delta\left(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2\right)n_{ij}^{+}\left(\sigma_1\right). \tag{3.86}$$

With this result, equation (3.72) becomes

$$\left\langle O_{T,t}^{\dagger}\left(\sigma_2\right)D_{\gamma}O_{R,r}\left(\sigma_1\right)\right\rangle_1=-g_{YM}^2\sum_{i=1}^{p}\sum_{j>i}^{p}\sum_{\gamma_1,\gamma_2\in H}\delta\left(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2\right)n_{ij}^{+}\left(\sigma_1\right)$$
$$\times\left(\Delta_{ij}^{0}-\left(e^{-2\pi i\gamma}\Delta_{ij}^{+}+e^{2\pi i\gamma}\Delta_{ij}^{-}\right)\right). \tag{3.87}$$

[11] Since $n_{ij}^{+}\left(\sigma_1^{-1}\right)=n_{ij}^{+}\left(\sigma_1\right)$.

From the second term in equation (3.61) we get

$$\begin{aligned}\left\langle O^{\dagger}_{T,t}(\sigma_2)\,D_\gamma O_{R,r}(\sigma_1)\right\rangle_2 &= \frac{|H|^2}{m!}\sum_{s,u\vdash m}\sum_{\mu_1,\mu_2,\nu_1,\nu_2}\sqrt{d_s d_u}\,\Gamma^{(s)}_{jk}(\sigma_1)\,B^{s\to 1_H}_{j\mu_1}B^{s\to 1_H}_{k\mu_2}\\ &\times \Gamma^{(u)}_{lm}(\sigma_2)\,B^{u\to 1_H}_{l\nu_1}B^{u\to 1_H}_{m\nu_2}\left\langle O^{\dagger}_{T,(t,u)\nu_1\nu_2}D_\gamma O_{R,(r,s)\mu_1\mu_2}\right\rangle_2\\ &= -g^2_{YM}\frac{|H|^2}{m!}\sum_{i=1}^{p}\sum_{j>i}^{p}\sum_{s,u\vdash m}\sum_{\mu_1,\mu_2,\nu_1,\nu_2}\Gamma^{(s)}_{jk}(\sigma_1)\,B^{s\to 1_H}_{j\mu_1}B^{s\to 1_H}_{k\mu_2}\\ &\times \Gamma^{(u)}_{lm}(\sigma_2)\,B^{u\to 1_H}_{l\nu_1}B^{u\to 1_H}_{m\nu_2}(\Delta^0_{ij}-(e^{-2\pi i\gamma}\Delta^-_{ij}+e^{2\pi i\gamma}\Delta^+_{ij}))\\ &\times m\,\langle \vec{m},s,\mu_2;a|\,E^{(n+1)}_{ii}\,|\vec{m},u,\nu_2;b\,\rangle\langle\,\vec{m},u,\nu_1;b|\,E^{(n+1)}_{jj}\,|\vec{m},s,\mu_1;a\rangle\end{aligned} \tag{3.88}$$

which we evaluate in exactly the same way we did (3.72). The analog of equation (3.86) in this case is

$$\frac{m!}{|H|^2}\sum_{\gamma_1,\gamma_2\in H}\delta\left(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2\right)n^-_{ij}(\sigma_1), \tag{3.89}$$

where

$$n^-_{ij}(\sigma_1)\equiv\sum_{k\in S_i}\sum_{l\in S_j}\delta\left(\sigma_1^{-1}(l),k\right) \tag{3.90}$$

is the number of strings emanating from brane j and terminating on brane i. Equation (3.88) then gives

$$\begin{aligned}\left\langle O^{\dagger}_{T,t}(\sigma_2)\,D_\gamma O_{R,r}(\sigma_1)\right\rangle_2 &= -g^2_{YM}\sum_{i=1}^{p}\sum_{j>i}^{p}\sum_{\gamma_1,\gamma_2\in H}\delta\left(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2\right)n^-_{ij}(\sigma_1)\\ &\times(\Delta^0_{ij}-(e^{-2\pi i\gamma}\Delta^-_{ij}+e^{2\pi i\gamma}\Delta^+_{ij})).\end{aligned} \tag{3.91}$$

Similarly, the last term in equation (3.61) gives

$$\begin{aligned}\left\langle O^{\dagger}_{T,t}(\sigma_2)\,D_\gamma O_{R,r}(\sigma_1)\right\rangle_3 &= -g^2_{YM}\sum_{i=1}^{p}\sum_{j>i}^{p}\sum_{\gamma_1,\gamma_2\in H}\delta\left(\sigma_1^{-1}\gamma_1\sigma_2\gamma_2\right)\\ &\times\left(e^{-2\pi i\gamma}+e^{2\pi i\gamma}-2\right)n_{ii}(\sigma_1)\,\Delta^0_{ii},\end{aligned} \tag{3.92}$$

where $n_{ii}(\sigma_1)$ is the number of strings that begin and end on the same brane.

With this, the action of the dilatation operator in the Gauss graph basis can be written as

$$
\begin{aligned}
D_\gamma O_{R,r}(\sigma_1) = -g_{YM}^2 \sum_{i=1}^{p} [\sum_{j>i}^{p} [& n_{ij}^+(\sigma_1)(\Delta_{ij}^0 - (e^{-2\pi i\gamma}\Delta_{ij}^+ + e^{2\pi i\gamma}\Delta_{ij}^-)) \\
& + n_{ij}^-(\sigma_1)(\Delta_{ij}^0 - (e^{-2\pi i\gamma}\Delta_{ij}^- + e^{2\pi i\gamma}\Delta_{ij}^+))] \\
& + n_{ii}(\sigma_1)\left(2 - \left(e^{-2\pi i\gamma} + e^{2\pi i\gamma}\right)\right)\Delta_{ii}^0] O_{R,r}(\sigma_1) .
\end{aligned} \tag{3.93}
$$

3.2.7 Continuum limit in the Gauss graph basis

Taking the continuum limit in the Gauss graph basis gives

$$
\begin{aligned}
D_\gamma O(\sigma_1) = -g_{YM}^2 \sum_{R,r} \sum_{b_0, l_1, \ldots l_{p-1}} \sum_{i=1}^{p} \Bigg[\sum_{j>i}^{p} & [((2N + 2b_0 + l_i + l_j)(1 - \cos(2\pi\gamma)) \\
& + \cos(2\pi\gamma)\left(\frac{(x_i - x_j)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right)) n_{ij}(\sigma_1) \\
& - i\sin(2\pi\gamma)\left(2\sqrt{N + b_0} + x_i + x_j\right)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\left(n_{ij}^-(\sigma_1) - n_{ij}^+(\sigma_1)\right)] \\
& + 2(2N + 2b_0 + 2l_i)(1 - \cos(2\pi\gamma))\, n_{ii}(\sigma_1) \Bigg] \\
& \times O_{R,r}(\sigma_1) ,
\end{aligned} \tag{3.94}
$$

where

$$
n_{ij}(\sigma_1) = n_{ij}^-(\sigma_1) + n_{ij}^+(\sigma_1) \tag{3.95}
$$

is the total number of strings stretching between branes i and j.

The eigenvalue problem

$$
DO = \Gamma O \tag{3.96}
$$

implies that

$$\begin{aligned}
& g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} n_{ij}^{-} \Big[\left((2N + r_i + r_j) \left(1 - \cos\left(2\pi\gamma\right)\right) f\left(r_0, l_1, \ldots, l_{p-1}\right) \right. \\
& - i \sin\left(2\pi\gamma\right) \left(2\sqrt{N + r_0} + x_i + x_j\right) \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \\
& + \cos\left(2\pi\gamma\right) \left(\frac{\left(x_i - x_j\right)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right) f \Big] \\
& + g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} n_{ij}^{+} \Big[\left((2N + r_i + r_j) \left(1 - \cos\left(2\pi\gamma\right)\right) f\left(r_0, l_1, \ldots, l_{p-1}\right) \right. \\
& + i \sin\left(2\pi\gamma\right) \left(2\sqrt{N + r_0} + x_i + x_j\right) \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \\
& + \cos\left(2\pi\gamma\right) \left(\frac{\left(x_i - x_j\right)^2}{4} - \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right) f \Big] \\
& + 4 g_{YM}^2 \sin^2\left(\pi\gamma\right) \sum_{i=1}^{p} \left(N + r_i\right) n_{ii} f\left(r_0, l_1, \ldots, l_{p-1}\right) = \Gamma f\left(r_0, l_1, \ldots, l_{p-1}\right),
\end{aligned} \tag{3.97}$$

where we have written

$$O = \sum_{r_0, l_1, \ldots l_{p-1}} f\left(r_0, l_1, \ldots, l_{p-1}\right) O\left(\sigma, r_0, l_1, \ldots, l_{p-1}\right) \tag{3.98}$$

Using the trig identity

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{3.99}$$

and introducing a set of coordinates

$$y_i = \sqrt{N + r_0} + x_i \tag{3.100}$$

we get

$$\begin{aligned}
& g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} n_{ij}^{-} \Bigg[\left((2N + r_i + r_j)\, 2\sin^2(\pi\gamma)\, f \right. \\
& - \sin(2\pi\gamma)\,(y_i + y_j)\, P_{ij} f \\
& + \cos(2\pi\gamma) \left(\frac{y_{ij}^2}{4} + P_{ij}^2 \right) f \Bigg] \\
& + g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} n_{ij}^{+} \Bigg[\left((2N + r_i + r_j)\, 2\sin^2(\pi\gamma)\, f \right. \\
& + \sin(2\pi\gamma)\,(y_i + y_j)\, P_{ij} f \\
& + \cos(2\pi\gamma) \left(\frac{y_{ij}^2}{4} + P_{ij}^2 \right) f \Bigg] \\
& + 4 g_{YM}^2 \sin^2(\pi\gamma) \sum_{i=1}^{p} (N + r_i)\, n_{ii} f = \Gamma f,
\end{aligned} \tag{3.101}$$

where

$$P_{ij} = i \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j} \right) \tag{3.102}$$

and

$$y_{ij} = y_i - y_j. \tag{3.103}$$

Rearranging gives

$$\begin{aligned}
& g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left[n_{ij} \cos(2\pi\gamma) \left(\frac{y_{ij}^2}{4} + P_{ij}^2 \right) f + \left(n_{ij}^{+} - n_{ij}^{-} \right) \sin(2\pi\gamma)\,(y_i + y_j)\, P_{ij} f \right] \\
& + 2 g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} \sin^2(\pi\gamma)\,(2N + r_i + r_j)\, n_{ij} f = \Gamma f.
\end{aligned} \tag{3.104}$$

In the last line, we have included the term proportional to n_{ii} thereby changing the sum over j. This gives the partial differential equation that must be solved in order to obtain the anomalous dimensions for the deformed theory.

In the undeformed case, configurations with $n_{ij} = 0$ and $n_{ii} \neq 0$ correspond to BPS operators. One of the implications of this is that any excitation of a single giant graviton, i.e. any restricted Schur polynomial built using only Zs and Ys, labelled by Young diagrams that have only a single row or column are BPS. In the deformed case, we see that this is not so, i.e. $n_{ii} \neq 0$ leads to operators that are not BPS. In other words, in the deformed case, the excitations of a single giant graviton are not BPS.

3.2.8 Spectrum

Equation (3.104) can now be written as

$$Hf = \tilde{\Gamma} f \tag{3.105}$$

with

$$H = g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left[n_{ij} \cos(2\pi\gamma) \left(P_{ij}^2 + \frac{1}{4} y_{ij}^2 \right) + \left(n_{ij}^+ - n_{ij}^- \right) \sin(2\pi\gamma) (y_i + y_j) P_{ij} \right] \tag{3.106}$$

and

$$\tilde{\Gamma} = -2 g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} \sin^2(\pi\gamma)(2N + r_i + r_j) n_{ij} + \Gamma. \tag{3.107}$$

We note that $y_i + y_j$ commutes with H and hence is a constant of the motion. It thus makes sense to shift

$$P_{ij} \to P_{ij} + \alpha (y_i + y_j). \tag{3.108}$$

By choosing

$$\alpha = \frac{\left(n_{ij}^- - n_{ij}^+ \right) \tan(2\pi\gamma)(y_i + y_j)}{2 n_{ij}} \tag{3.109}$$

we find

$$H = g_{YM}^2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} \left[n_{ij} \cos(2\pi\gamma) \left(P_{ij}^2 + \frac{1}{4} y_{ij}^2 \right) - \frac{\left(n_{ij}^+ - n_{ij}^- \right)^2 \sin^2(2\pi\gamma)(y_i + y_j)^2}{4 n_{ij} \cos(2\pi\gamma)} \right]. \tag{3.110}$$

The second term inside the square braces commutes with the Hamiltonian and is thus a constant. Noting that

$$\left[P_{ij}, \frac{y_i - y_j}{2} \right] = i, \tag{3.111}$$

it is clear that H is equivalent to a (shifted) harmonic oscillator whose spectrum depends on the parameter γ. We also note that from the result (3.110) there appears to be a singularity in H at

$$\gamma = \frac{1}{4}. \tag{3.112}$$

Looking back at equation (3.106), we see that at this value of γ the term that is quadratic in the momenta vanishes, leaving only a term linear in P_{ij}. At this point, since $y_i + y_j$ is a constant of motion, the Hamiltonian becomes proportional to P_{ij} and the operators of a good scaling dimension are plane waves. We need to require that these wavefunctions vanish whenever $y_i = y_j$. This boundary condition will quantise the P_{ij} momentum eigenvalues so that we obtain an evenly spaced spectrum. Intuitively, since the original Hamiltonian depends smoothly on γ, we expect that there is nothing singular about the point $\gamma = 1/4$.

3.3 Discussion

Our goal has been to compute the spectrum of anomalous dimensions of the Leigh-Strassler deformed $\mathcal{N} = 4$ super-Yang Mills theory. The operators we have studied are AdS/CFT dual to systems of giant gravitons. This implies that although we work at large N, we are not in the planar limit of the theory.

We found that the action of the dilatation operator continues to to factorise into an action on the impurity labels $s\mu_1\mu_2; u\nu_1\nu_2$ (associated with the Y fields) and an action on the $R, r; T, t$ labels associated with the Z fields. The deformed dilatation operator picks up an extra term when compared to the undeformed case. This extra term is diagonal in the Gauss graph basis so that the double coset ansatz continues to diagonalise the impurity labels.

We also studied the diagonalisation problem associated to the Z fields. Though this problem is different from the undeformed case, we find that it can be reduced to a set of decoupled oscillators. However, the deformed dilatation operator picks up an additional term which produces an extra shift in the anomalous dimension. The shift is positive as it should be - a negative shift would produce operators with a dimension less than their $\mathcal{R}$-charge which is not possible in a unitary conformal field theory. This predicts that all excitations of the giant gravitons in the deformed theory are not BPS. In a system of $p = 2$ giant gravitons for example, we have

$$\begin{aligned}\Gamma_k = & 4g_{YM}^2 (N + r_1) n_{11} \sin^2(\pi\gamma) + 4g_{YM}^2 (N + r_2) n_{22} \sin^2(\pi\gamma) \\ & + 2g_{YM}^2 (2N + n) n_{12} \sin^2(\pi\gamma) + 4g_{YM}^2 n_{12} \cos(2\pi\gamma) k,\end{aligned} \tag{3.113}$$

where k is any non-negative integer. When $\gamma = 0$ we recover the anomalous dimensions of the undeformed theory [43].

Since our operators are not BPS, their anomalous dimensions are not protected quantities. Owing the strong/weak coupling duality of the AdS/CFT correspondence, a direct comparison of our results with those of AdS/CFT predictions [50, 57, 69] is almost sure to fail. More precisely, the dual gravitational system is defined in the limit of large 't Hooft coupling λ and small γ ($\gamma^2\lambda$ is fixed) while our field theory computation is valid for small λ and arbitrary γ. However, since the quantum numbers of our operators become parametrically large with N, a comparison may still be possible [24, 70, 71, 72, 73]. We leave this interesting question for future research.

Chapter 4

Including fermions

In this chapter, we study large operators built using both bosonic and fermionic fields. The operators we study are dual to excited giant gravitons. In this case, the large N and planar limits do not coincide, meaning that to compute the large N observables, we need to sum more than just the planar diagrams. This problem can be solved completely by using the group theory of symmetric and unitary groups as well as the relations between them. Using representation theory, the two point functions can be solved exactly in the free field limit [26, 33, 74, 75, 76, 77, 36, 78, 79, 80, 81, 82].

In what follows, we explain how to build the restricted Schur polynomials that incorporate both bosons and fermions. We show that the number of these polynomials matches the number of multi-field, multi-trace gauge invariant operators. We also show how to transform between the trace basis and the basis provided by the restricted Schur polynomials that we construct.

As a concrete application of our results, we study the $su\,(2|3)$ sector of SYM theory. This sector consists of operators built from two fermions and three bosons, hence the name. The $su\,(2|3)$ sector is closed to all orders under the action of the dilatation operator [52, 53]. At the one loop level, the dilatation operator has a simple action in this sector - see formula (2.1) of [53] or the H_2 piece in Table 1 of [52]. After explaining how to build restricted Schur polynomials for this sector, we compute the action of the dilatation operator on these polynomials. Finally, we show that the double coset ansatz [46] diagonalises the dilatation operator in this sector of the theory. This chapter is based on a paper that I published [39] - it is my original work.

4.1 Warm up: single fermion

Consider a single fermion ψ^i_j transforming in the adjoint of the gauge group $U(N)$. The relevant two point function is

$$\left\langle \psi^i_j \left(\psi^\dagger\right)^k_l \right\rangle = \delta^i_l \delta^k_j. \tag{4.1}$$

The fermionic fields are Grassman valued, so that swapping them costs a minus sign. Our conventions for ordering the fields are as follows

$$\left(\psi^{\otimes n}\right)^I_J = \psi^{i_1}_{j_1}\psi^{i_2}_{j_2}\cdots\psi^{i_n}_{j_n} \tag{4.2}$$

$$\left(\psi^{\dagger\otimes n}\right)^K_L = \psi^{\dagger k_n}_{l_n}\cdots\psi^{\dagger k_2}_{l_2}\psi^{\dagger k_1}_{l_1}. \tag{4.3}$$

We then see that

$$\left\langle \left(\psi^{\otimes n}\right)^I_J \left(\psi^{\dagger\otimes n}\right)^K_L \right\rangle = \sum_{\sigma\in S_n} sgn\left(\sigma\right)\sigma^I_L\left(\sigma^{-1}\right)^K_J, \tag{4.4}$$

where $sgn\left(\sigma\right)$ is the sign of the permutation σ which is computed by decomposing the permutation into a product of transpositions. This decomposition is not unique. Then

$$sgn\left(\sigma\right) = (-1)^m, \tag{4.5}$$

where m is the number of transpositions in the product. $sgn\left(\sigma\right)$ is well defined, i.e. it does not depend on the specific decomposition of σ into transpositions. The ordering in equation (4.3) is used to ensure that no extra n dependent phases appear in equation (4.4).

The Grassman nature of ψ implies that the trace of an even number of fields vanishes. As an example, consider

$$Tr\left(\psi^4\right) = \psi^i_j\psi^j_k\psi^k_l\psi^l_i = -\psi^j_k\psi^i_j\psi^k_l\psi^l_i = \psi^j_k\psi^k_l\psi^i_j\psi^l_i = -\psi^j_k\psi^k_l\psi^l_i\psi^i_j = -Tr\left(\psi^4\right). \tag{4.6}$$

Further, the product of two traces with the same number of fields in each trace vanishes, e.g.

$$\begin{aligned} Tr\left(\psi^3\right)Tr\left(\psi^3\right) &= \psi^i_j\psi^j_k\psi^k_i Tr\left(\psi^3\right) = -\psi^i_j\psi^j_k Tr\left(\psi^3\right)\psi^k_i = \psi^i_j Tr\left(\psi^3\right)\psi^j_k\psi^k_i \\ &= -Tr\left(\psi^3\right)\psi^i_j\psi^j_k\psi^k_i = -Tr\left(\psi^3\right)Tr\left(\psi^3\right). \end{aligned} \tag{4.7}$$

Let us now consider polynomials built from the adjoint fermion. Since we want a gauge invariant operator, consider polynomials built as a linear combination of traces[1]

$$\sum_{\sigma\in S_n} C_\sigma Tr_{V^{\otimes n}}\left(\sigma\psi^{\otimes n}\right). \tag{4.8}$$

By changing the summation variables to $\gamma^{-1}\sigma\gamma$ and using the Grassman nature of the fermionic fields we find

$$\begin{aligned}\sum_{\sigma\in S_n} C_\sigma Tr_{V^{\otimes n}}\left(\sigma\psi^{\otimes n}\right) &= \sum_{\sigma,\gamma\in S_n} C_{\gamma^{-1}\sigma\gamma} Tr_{V^{\otimes n}}\left(\sigma\gamma\psi^{\otimes n}\gamma^{-1}\right)\\ &= \sum_{\sigma,\gamma\in S_n} C_{\gamma^{-1}\sigma\gamma}\, sgn\left(\gamma\right) Tr_{V^{\otimes n}}\left(\sigma\psi^{\otimes n}\right). \end{aligned} \tag{4.9}$$

Thus the coefficients used to define our polynomial must obey

$$C_{\gamma^{-1}\sigma\gamma} = sgn\left(\gamma\right) C_\sigma. \tag{4.10}$$

A natural way to achieve this is to consider

$$\chi_R^F\left(\psi\right) = \sum_{\alpha\in S_n} S^{[1^n]RR}{}_{m'm}\Gamma^R_{m'm}\left(\alpha\right) Tr_{V^{\otimes n}}\left(\alpha\psi^{\otimes n}\right), \tag{4.11}$$

where $\Gamma^R_{m'm}(\alpha)$ is the matrix representing $\alpha\in S_n$ in irrep R and $S^{[1^n]RR}{}_{m'm}$ is the Clebsch-Gordan coefficient for $R\times R$ to couple to the antisymmetric irrep $[1^n]$. This formula can be viewed as a "degeneration" of the operators constructed in [75, 76],

$$\sum_{\sigma\in S_n} B_{j\beta} S^{\tau;\Lambda RR}{}_{j\ pq}\Gamma^\Lambda_{pq}\left(\sigma\right) Tr_{V^{\otimes n}}\left(\sigma \mathbf{X}^{\mu\otimes n}\right), \tag{4.12}$$

which provides a basis for M species of complex matrix (different species indexed by μ). The basis thus obtained has good $U(M)$ quantum numbers (see the first formula in Section 1.1 of [75]. Since $[1^n]$ appears only once in $R\otimes R$ the analogue of the multiplicity label τ which appears in (4.12) is not needed in (4.11). Equation (4.11) is the simplest way to turn the "counting formula" (equation (106) of [75]) into a "construction formula".

To simplify the notation, we write the Schur polynomials for fermions as

$$\chi_R^F\left(\psi\right) = \sum_{\sigma\in S_n} C_\sigma Tr_{V^{\otimes n}}\left(\sigma\psi^{\otimes n}\right) = \sum_{\alpha\in S_n} Tr\left(O\Gamma^R\left(\alpha\right)\right) Tr_{V^{\otimes n}}\left(\alpha\psi^{\otimes n}\right), \tag{4.13}$$

where

$$O_{mm'} = S^{[1^n]RR}{}_{m'm}.$$

[1]Each of these single traces in $V^{\otimes n}$ can give rise to any multi-trace structure involving the n fields. Here, V is isomorphic to the carrier space of the fundamental representation of $U(N)$.

The Clebsch-Gordan coefficients of the symmetric group obey (see formula (7-186) of [83])

$$\Gamma^{\mu}_{ij}(\sigma)\,\Gamma^{\nu}_{kl}(\sigma)\,S^{\lambda\tau_\lambda\mu\nu}_{s\;\;\;\;jl} = \Gamma^{\lambda\tau_\lambda}_{s's}(\sigma)\,S^{\lambda\tau_\lambda\mu\nu}_{s'\;\;\;\;ik}. \tag{4.14}$$

To specialise this to our problem, let us first replace μ, ν by R and λ by $[1^n]$. There is no need for the multiplicity label τ_λ. Also, because $[1^n]$ is one dimensional, there is no need for indices s, s' and we replace $\Gamma^{\lambda\tau_\lambda}_{s's}(\sigma) \to sgn(\sigma)$. The equation for the Clebsch-Gordan coefficients becomes

$$\Gamma^{R}_{ij}(\sigma)\,\Gamma^{R}_{kl}(\sigma)\,S^{[1^n]RR}_{\;\;\;\;\;\;jl} = sgn(\sigma)\,S^{[1^n]RR}_{\;\;\;\;\;\;ik}. \tag{4.15}$$

Since we may assume without loss of generality that we have an orthogonal representation, equation (4.15) implies that

$$S^{[1^n]RR}_{\;\;\;\;\;\;ml}\Gamma^{R}_{lk}(\sigma) = sgn(\sigma)\,\Gamma^{R}_{mi}(\sigma)\,S^{[1^n]RR}_{\;\;\;\;\;\;ik}. \tag{4.16}$$

This proves that

$$\Gamma^{S}(\sigma)\,O = sgn(\sigma)\,O\Gamma^{S}(\sigma). \tag{4.17}$$

Clearly then, O^2 commutes with every element of the group and is proportional to the identity matrix (by Schur's Lemma). Thus, (perhaps after a normalisation) we have

$$O^2 = 1. \tag{4.18}$$

This immediately implies that characters for all odd elements (those with sign -1) of the symmetric group vanish since

$$\begin{aligned} Tr\left(\Gamma^{R}(\sigma)\right) &= Tr\left(O^2\Gamma^{R}(\sigma)\right) = sgn(\sigma)\,Tr\left(O\Gamma^{R}(\sigma)\,O\right) \\ &= sgn(\sigma)\,Tr\left(OO\Gamma^{R}(\sigma)\right) = sgn(\sigma)\,Tr\left(\Gamma^{R}(\sigma)\right), \end{aligned} \tag{4.19}$$

where we have used equation (4.17) and the cyclicity of the trace. The representation s^T which is conjugate to s is defined by flipping the Young diagram as shown in figure 4.1. O can only be non-zero for self conjugate irreps because it is only for these that the characters of all odd elements vanish. Indeed, $S^{[1^n]RR}_{\;\;\;\;\;\;mm'}$ is only non-zero for self-conjugate

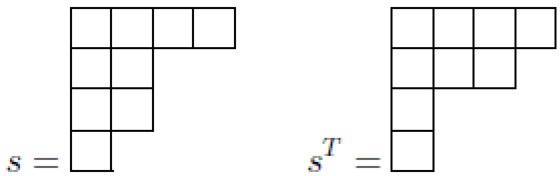

Figure 4.1: Relationship between the Young diagrams s and s^T.

irreps. The above observations all follow from

$$\begin{aligned} C_{\gamma^{-1}\sigma\gamma} &= Tr\left(O\Gamma^R\left(\gamma^{-1}\sigma\gamma\right)\right) \\ &= Tr\left(O\Gamma^R\left(\gamma^{-1}\right)\Gamma^R\left(\sigma\right)\Gamma^R\left(\gamma\right)\right) \\ &= sgn\left(\gamma\right)Tr\left(\Gamma^R\left(\gamma^{-1}\right)O\Gamma^R\left(\sigma\right)\Gamma^R\left(\gamma\right)\right) \\ &= sgn\left(\gamma\right)C_\sigma \end{aligned} \tag{4.20}$$

which proves that the coefficients of our polynomials do indeed obey (4.10).

Spelling out index structures, our conventions are

$$\chi_R\left(\psi\right) = \frac{1}{n!}\sum_{\sigma\in S_n} Tr\left(O\Gamma^R\left(\sigma\right)\right)\psi^{i_1}_{i_{\sigma(1)}}\cdots\psi^{i_n}_{i_{\sigma(n)}} \tag{4.21}$$

and

$$\chi^\dagger_R\left(\psi\right) = \frac{1}{n!}\sum_{\sigma\in S_n} Tr\left(\Gamma^R\left(\sigma\right)O\right)\psi^{\dagger i_n}_{i_{\sigma(n)}}\cdots\psi^{\dagger i_1}_{i_{\sigma(1)}}. \tag{4.22}$$

One difference between fermionic variables and bosonic variables is that fermionic variables anti-commute. As a result, different choices for populating the slots with fermionic fields can differ by a sign. It is for this reason that we must spell things out.

The two-point function of the fermionic Schur polynomials is

$$\begin{aligned}\left\langle \chi_R \chi_S^\dagger \right\rangle &= \frac{1}{(n!)^2} \sum_{\sigma,\rho,\gamma\in S_n} Tr\left(O\Gamma^R(\sigma)\right) Tr\left(\Gamma^S(\rho)\,O\right) sgn(\gamma)\, Tr_{V^{\otimes n}}\left(\gamma\sigma\gamma^{-1}\rho\right) \\ &= \frac{1}{(n!)^2} \sum_{\beta,\rho,\gamma\in S_n} Tr\left(O\Gamma^R\left(\gamma^{-1}\beta\gamma\right)\right) Tr\left(\Gamma^S(\rho)\,O\right) sgn(\gamma)\, Tr_{V^{\otimes n}}(\beta\rho) \\ &= \frac{1}{(n!)^2} \sum_{\beta,\rho\in S_n} Tr\left(O\Gamma^R(\beta)\right) Tr\left(\Gamma^S(\rho)\,O\right) Tr_{V^{\otimes n}}(\beta\rho) \\ &= \frac{1}{n!} \sum_{\psi,\rho\in S_n} Tr\left(O\Gamma^R(\psi)\,\Gamma^R\left(\rho^{-1}\right)\right) Tr\left(\Gamma^S(\rho)\,O\right) Tr_{V^{\otimes n}}(\psi) \\ &= \frac{\delta_{RS}}{d_R} \sum_{\psi\in S_n} Tr\left(\Gamma^R(\psi)\right) Tr_{V^{\otimes n}}(\psi) \\ &= \delta_{RS} f_R. \end{aligned} \tag{4.23}$$

This completes the construction of Schur polynomials for a single fermion. We now want to construct restricted Schur polynomials for an arbitrary number of fermionic and bosonic matrix flavors. We will first consider the counting of these operators. For the counting relevant for a single fermionic variable, see equation (106) of [75]. As we highlighted earlier, our construction is motivated by this counting and the number of operators we have matches this counting.

4.2 Counting

We will start with a quick review of counting for bosons [84]. Thereafter, we will consider the counting of operators built from fermions and bosons.

4.2.1 Warm up: bosons

We will count the number of operators built with k species of bosonic fields. This should equal the number of restricted Schur polynomials $\chi_{R,(r_1,r_2,\ldots r_k)}$. Let us start from the $U(N)$ partition function as quoted in [85], equation (3.7), for the case of k bosonic fields

$$\mathcal{Z}_{U(N)}(t) = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \prod_{j=1}^{k} \prod_{r,s=1}^{N} \frac{1}{1 - t_j z_r z_s^{-1}}. \tag{4.24}$$

Using the Cauchy-Littlewood formula

$$\prod_{i=1}^{L} \prod_{j=1}^{M} \frac{1}{1 - x_i y_j} = \sum_{r,l(r)\leq \min(L,M)} \chi_r(x)\, \chi_r(y) \tag{4.25}$$

we write the partition function as

$$\mathcal{Z}_{U(N)}(t)=\frac{1}{(2\pi i)^N N!}\oint\prod_{i=1}^{N}\frac{dz_i}{z_i}\triangle(z)\triangle\left(z^{-1}\right)\prod_{j=1}^{k}\sum_{r_j,l(r_j)\leq N}\chi_{r_j}(t_jz)\chi_{r_j}\left(z^{-1}\right). \quad (4.26)$$

Since the Schur polynomial $\chi_r(z)$ is a homogeneous polynomial of order $|r|\equiv$ the number of boxes in r, we know that

$$\mathcal{Z}_{U(N)}(t)=\frac{1}{(2\pi i)^N N!}\oint\prod_{i=1}^{N}\frac{dz_i}{z_i}\triangle(z)\triangle\left(z^{-1}\right)\prod_{j=1}^{k}\sum_{r_j,l(r_j)\leq N}(t_j)^{|r_j|}\chi_{r_j}(z)\chi_{r_j}\left(z^{-1}\right). \quad (4.27)$$

Using the Littlewood-Richardson rule to perform the product of the Schur polynomials we find

$$\mathcal{Z}_{U(N)}(t)=\frac{1}{(2\pi i)^N N!}\sum_{r_1,\ldots,r_{k+2},l(r_i)\leq N}(t_1)^{|r_1|}(t_2)^{|r_2|}\cdots(t_k)^{|r_k|}g(r_1,r_2,\ldots r_k,r_{k+1})$$
$$\times g(r_1,r_2,\ldots r_k,r_{k+2})\oint\prod_{i=1}^{N}\frac{dz_i}{z_i}\triangle(z)\triangle\left(z^{-1}\right)\chi_{r_{k+1}}(z)\chi_{r_{k+2}}\left(z^{-1}\right). \quad (4.28)$$

Now,

$$\langle g,h\rangle_N\equiv\frac{1}{(2\pi i)^N N!}\oint\prod_{i=1}^{N}\frac{dz_i}{z_i}\triangle(z)\triangle\left(z^{-1}\right)g(z)h\left(z^{-1}\right) \quad (4.29)$$

and

$$\langle\chi_r,\chi_t\rangle_N=\delta_{rt} \quad (4.30)$$

so that

$$\mathcal{Z}_{U(N)}(t)=\frac{1}{(2\pi i)^N N!}\sum_{r_1,\ldots,r_k,R,l(r_i)\leq N,l(R)\leq N}(t_1)^{|r_1|}(t_2)^{|r_2|}\cdots(t_k)^{|r_k|}\left(g(r_1,r_2,\ldots r_k,R)\right)^2. \quad (4.31)$$

From the coefficient of $(t_1)^{n_1}(t_2)^{n_2}\cdots(t_k)^{n_k}$ we learn how many operators can be built using n_k fields of species k. This is in turn equal to the number of restricted Schur polynomials $\chi_{R,(r_1,r_2,\ldots r_k)}$ with $r_i\vdash n_i$ and $R\vdash n_1+n_2+\cdots n_k$ [84].

4.2.2 One fermion and one boson

In this subsection, we will count the number of operators built with one bosonic species and one fermionic species of field. We will use r for the bosonic and s for the fermionic Young diagrams. Again, we start from the $U(N)$ partition function as quoted in formula

(3.13) of [85] for the case of one bosonic field and one fermionic field

$$\mathcal{Z}_{U(N)}(f,b) = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \prod_{r,s=1}^{N} \frac{1 - f z_r z_s^{-1}}{1 - b z_r z_s^{-1}}. \tag{4.32}$$

Using the Cauchy-Littlewood formula (4.25) and Littlewood's formula

$$\prod_{i=1}^{L}\prod_{j=1}^{M} 1 + x_i y_j = \sum_{s,l(s)\leq L, l(s^T)\leq M} \chi_s(x)\chi_{s^T}(y), \tag{4.33}$$

where s^T is conjugate to s, the partition function (4.32) becomes

$$\begin{aligned} \mathcal{Z}_{U(N)}(f,b) = {} & \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \\ & \times \sum_{r,s,l(r)\leq N, l(s)\leq N, l(s^T)\leq N} \chi_r(bz)\chi_r\left(z^{-1}\right)\chi_s(fz)\chi_{s^T}\left(z^{-1}\right). \end{aligned} \tag{4.34}$$

Since the Schur polynomial $\chi_t(z)$ is a homogeneous polynomial of order $|t| \equiv$ the number of boxes in t, we know that

$$\begin{aligned} \mathcal{Z}_{U(N)}(f,b) = {} & \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \\ & \times \sum_{r,s,l(r)\leq N, l(s)\leq N, l(s^T)\leq N} b^{|r|} f^{|s|} \chi_r(z)\chi_r\left(z^{-1}\right)\chi_s(z)\chi_{s^T}\left(z^{-1}\right). \end{aligned} \tag{4.35}$$

Using the Littlewood-Richardson rule to perform the product of the Schur polynomials, we get

$$\begin{aligned} \mathcal{Z}_{U(N)}(f,b) = {} & \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \sum_{r,s,l(r)\leq N, l(s)\leq N, l(s^T)\leq N} \\ & \times \sum_{R_1,R_2,l(R_i)\leq N} b^{|r|} f^{|s|} g(r,s,R_1)\, g\left(r,s^T,R_2\right) \chi_{R_1}(z)\chi_{R_2}\left(z^{-1}\right). \end{aligned} \tag{4.36}$$

Finally, using (4.30), we obtain

$$\mathcal{Z}_{U(N)}(f,b) = \sum_{r,s,l(r)\leq N, l(s)\leq N, l(s^T)\leq N} \sum_{R,l(R)\leq N} b^{|r|} f^{|s|} g(r,s,R)\, g\left(r,s^T,R\right). \tag{4.37}$$

Equation (4.37) reflects the fermionic statistics. Since the fermionic matrix is a matrix of Grassman variables, any product with more than N^2 factors of the fermionic matrix will vanish. We note that since both $l(s) \leq N$ and $l\left(s^T\right) \leq N$, s can have at most N^2 boxes, i.e. we never get operators with a product of more than N^2 factors of the fermionic matrix. We also note that in general

$$g(r, s, R) \neq g\left(r, s^T, R\right) \tag{4.38}$$

so that this counting is genuinely different from (4.31).

4.2.3 Fermions and bosons

We will now count the number of operators built with n_b species of bosonic fields and n_f species of fermionic fields. Again, we start from the $U(N)$ partition function for n_b bosons and n_f fermions, formula (3.13) in [85],

$$\mathcal{Z}_{U(N)}(f, b) = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \prod_{j=1}^{n_f} \prod_{k=1}^{n_b} \prod_{r,s=1}^{N} \frac{1 - f_j z_r z_s^{-1}}{1 - b_k z_r z_s^{-1}}. \tag{4.39}$$

Using the Cauchy-Littlewood formula (4.25) and Littlewood's formula (4.33), we rewrite this partition function as

$$\begin{aligned} \mathcal{Z}_{U(N)}(f, b) = \frac{1}{(2\pi i)^N N!} \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \prod_{j=1}^{n_f} \prod_{k=1}^{n_b} \sum_{r_k, s_j, l(r_k) \leq N, l(s_j) \leq N, l\left(s_j^T\right) \leq N} \\ \times \chi_{r_k}(b_k z) \chi_{r_k}\left(z^{-1}\right) \chi_{s_j}(f_j z) \chi_{s_j^T}\left(z^{-1}\right). \end{aligned} \tag{4.40}$$

Again, since the Schur polynomial $\chi_t(z)$ is a homogeneous polynomial of order $|t|$ and using the Littlewood-Richardson rule to perform the product of the Schur polynomials, we find

$$\begin{aligned} \mathcal{Z}_{U(N)}(f, b) = \frac{1}{(2\pi i)^N N!} & \sum_{r_1, \ldots r_{n_b}, l(r_a) \leq N, s_1, \ldots s_{n_f}, l(s_b) \leq N, l\left(s_b^T\right) \leq N} \; \sum_{R_1, R_2, l(R_i) \leq N} \\ & \times (f_1)^{|s_1|} \ldots \left(f_{n_f}\right)^{\left|s_{n_f}\right|} (b_1)^{|r_1|} \ldots \left(b_{n_b}\right)^{\left|r_{n_b}\right|} \\ & \times g\left(r_1, \ldots r_{n_b}, s_1, \ldots s_{n_f}, R_1\right) g\left(r_1, \ldots r_{n_b}, s_1^T, \ldots s_{n_f}^T, R_2\right) \\ & \times \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \triangle(z) \triangle\left(z^{-1}\right) \chi_{R_1}(z) \chi_{R_2}\left(z^{-1}\right). \end{aligned} \tag{4.41}$$

Using equation (4.30) again yields

$$\mathcal{Z}_{U(N)}(f,b)=\sum_{r_1,\ldots r_{n_b},l(r_a)\leq N,}\sum_{s_1,\ldots s_{n_f},l(s_b)\leq N,l\left(s_b^T\right)\leq N}\sum_{R,l(R)\leq N}(f_1)^{|s_1|}\cdots\left(f_{n_f}\right)^{\left|s_{n_f}\right|}(b_1)^{|r_1|}\cdots(b_{n_b})^{|r_{n_b}|}$$
$$\times\, g\left(r_1,\ldots r_{n_b},s_1,\ldots s_{n_f},R\right)g\left(r_1,\ldots r_{n_b},s_1^T,\ldots s_{n_f}^T,R\right). \tag{4.42}$$

We note again that in general,

$$g\left(r_1,\ldots r_{n_b},s_1,\ldots s_{n_f},R\right)\neq g\left(r_1,\ldots r_{n_b},s_1^T,\ldots s_{n_f}^T,R\right). \tag{4.43}$$

4.3 Restricted Schurs for $su(2|3)$

Now that we have learnt how to count the operators built using both fermionic and bosonic fields, we now consider their construction.

4.3.1 Preliminary comments

How many times does $[1^n]$ appear in $s\otimes s^T$? In general, we have

$$s\otimes s^T=\oplus_t a_t t. \tag{4.44}$$

To determine the positive integer a_t with $t=[1^n]$, we start from the formula for the character of a direct product representation

$$\chi_s(g)\,\chi_{s^T}(g)=\sum_t a_t\chi_t(g) \tag{4.45}$$

and use the character orthogonality relation

$$\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi_R(g)\,\chi_S\left(g^{-1}\right)=\delta_{RS} \tag{4.46}$$

to obtain

$$\begin{aligned} a_{[1^n]} &= \frac{1}{|\mathcal{G}|}\sum_g \chi_s(g)\,\chi_{s^T}(g)\,\chi_{[1^n]}\left(g^{-1}\right) \\ &= \frac{1}{|\mathcal{G}|}\sum_g \chi_s(g)\,\chi_{s^T}(g)\,sgn(g) \\ &= \frac{1}{|\mathcal{G}|}\sum_g \chi_s(g)\,\chi_s(g) \\ &= \frac{1}{|\mathcal{G}|}\sum_g \chi_s\left(g^{-1}\right)\chi_s(g) \\ &= 1. \end{aligned} \tag{4.47}$$

Thus there is no need for a multiplicity label. To get the third line we used

$$\chi_{s^T}(g)\,sgn(g) = \chi_s(g)\,, \tag{4.48}$$

and to get fourth equality we used

$$\chi_s(g) = \chi_s\left(g^{-1}\right). \tag{4.49}$$

In this case, Hammermesh's formula reads

$$\Gamma^s_{ij}(\sigma)\,\Gamma^{s^T}_{kl}(\sigma)\,S^{[1^n]ss^T}_{\;\;jl} = sgn(\sigma)\,S^{[1^n]ss^T}_{\;\;ik}. \tag{4.50}$$

Using the fact that we have an orthogonal representation we find

$$\Gamma^s_{ij}(\sigma)\,\hat{O}_{jp} = sgn(\sigma)\,\hat{O}_{ik}\Gamma^{s^T}_{kp}(\sigma)\,, \tag{4.51}$$

where

$$\hat{O}_{jl} = S^{[1^n]ss^T}_{\;\;jl}. \tag{4.52}$$

$\hat{O}_{jl}$ is a map from s^T to s. $\hat{O}^T\hat{O}$ maps from s^T to s^T and commutes with all elements of the group. Therefore, it is proportional to the identity. $\hat{O}\hat{O}^T$ maps from s to s and commutes with all elements of the group. Similarly, it is proportional to the identity. By normalising correctly, we can choose

$$\hat{O}^T\hat{O} = \mathbf{1}_{s^T} \quad \& \quad \hat{O}\hat{O}^T = \mathbf{1}_s. \tag{4.53}$$

In what follows we will subduce two irreps from R, namely $(r, s\alpha)$ and $\left(r, s^T\beta\right)$, where α and β are multiplicity labels. To spell out the fact that these multiplicity labels belong to s and s^T rather than $[1^n]$, we will write

$$\hat{O}_{jl}\left(s\alpha; s^T\beta\right) \equiv S^{[1^n] s,\alpha}{}_{j}{}^{s^T,\beta}{}_{l}. \tag{4.54}$$

Making use of the operators (4.54) is the simplest way to turn the counting formula (4.37) into a construction formula.

4.3.2 Construction

In terms of the operators

$$P_{R,(r,s)\alpha\beta} = \mathbf{1}_r \otimes \hat{O}\left(s\alpha; s^T\beta\right) \qquad P^{\dagger}_{R,(r,s)\alpha\beta} = \mathbf{1}_r \otimes \hat{O}\left(s^T\beta; s\alpha\right) \tag{4.55}$$

we can write the restricted Schur polynomials as

$$\chi_{R,(r,s)\alpha\beta}\left(Z,\psi\right) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left(\sigma\right)\right)\psi^{i_1}_{i_{\sigma(1)}}\cdots\psi^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}} \tag{4.56}$$

and

$$\chi^{\dagger}_{R,(r,s)\alpha\beta}\left(Z,\psi\right) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr\left(P^{\dagger}_{R,(r,s)\alpha\beta}\Gamma^R\left(\sigma\right)\right)\psi^{\dagger i_m}_{i_{\sigma(m)}}\cdots\psi^{\dagger i_1}_{i_{\sigma(1)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}. \tag{4.57}$$

The specific choice of which slots we use for Z or ψ is unimportant - they are related by performing an automorphism on S_{n+m}, which is a symmetry of the Schur polynomial. The ordering of the Z fields is completely arbitrary, while the ordering of the ψ fields fixes a sign. We note that

$$P_{R,(r,s)\alpha\beta}\Gamma^r\left(\sigma_1\right)\circ\Gamma^{s^T}\left(\sigma_2\right) = sgn\left(\sigma_2\right)\Gamma^r\left(\sigma_1\right)\circ\Gamma^s\left(\sigma_2\right)P_{R,(r,s)\alpha\beta} \tag{4.58}$$

which implies that $P_{R,(r,s)\alpha\beta}$ is an intertwining map in the carrier space of R from the subspace $\left(r,s^T\right)$ to the subspace $\left(r,s\right)$. Further,

$$P_{R,(r,s)\alpha\beta}P^{\dagger}_{T,(t,u)\delta\gamma} = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\gamma}\bar{P}_{R,(r,s)\alpha\delta} \tag{4.59}$$

where

$$\bar{P}_{R,(r,s)\alpha\delta} = \mathbf{1}_r \otimes \sum_j \left|s,\alpha;j\right\rangle\left\langle s,\delta;j\right|. \tag{4.60}$$

It is now easy to show that

$$\left\langle \chi_{R_1,(r_1,s_1)\alpha\beta}(Z,\psi)\,\chi^\dagger_{R_2,(r_2,s_2)\gamma\delta}(Z,\psi)\right\rangle = \delta_{R_1R_2}\delta_{r_1r_2}\delta_{s_1s_2}\delta_{\beta\delta}\delta_{\alpha\gamma}\frac{f_{R_1}hooks_{R_1}}{hooks_{r_1}hooks_{s_1}}. \tag{4.61}$$

The generalisation to many fermions and bosons is straight forward. For the $su(2|3)$ sector in particular, we have

$$P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}} = \mathbf{1}_{r_1}\otimes\sum_j |r_2,\alpha_1;j\rangle\langle r_2,\beta_1;j|\otimes\sum_k|r_3,\alpha_2;k\rangle\langle r_3,\beta_2;k| \\ \otimes\hat{O}\left(s_1\alpha_3;s_1^T\beta_3\right)\otimes\hat{O}\left(s_2\alpha_4;s_2^T\beta_4\right). \tag{4.62}$$

We have written this with a specific procedure for the construction of $P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}}$ in mind. We imagine that boxes are removed from the Young diagram R until r_1 is remains. The boxes that are removed are then assembled to produce the representations r_2, r_3, s_1, s_2. Following this construction, r_1 has no multiplicities, r_2 has multiplicities α_1 and β_1, r_3 has multiplicities α_2 and β_2, s_1 has multiplicity α_3, s_1^T has multiplicity β_3, s_2 has multiplicity α_4 and s_2^T has multiplicity β_4. Our conventions for the ordering of the fermionic fields are

$$\chi_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}}(Z,X,Y,\psi_1,\psi_2) = \frac{1}{n_1!n_2!n_3!m_1!m_2!}\sum_{\sigma\in S_{n_1+n_2+n_3+m_1+m_2}} Tr\left(P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}}\Gamma^R(\sigma)\right) \\ \times\psi^{i_1}_{1i_{\sigma(1)}}\cdots\psi^{i_{m_1}}_{1i_{\sigma(m_1)}}\psi^{i_{m_1+1}}_{2i_{\sigma(m_1+1)}}\cdots\psi^{i_{m_1+m_2}}_{2i_{\sigma(m_1+m_2)}}X^{i_{m_1+m_2+1}}_{i_{\sigma(m_1+m_2+1)}}\cdots \tag{4.63}$$

and

$$\chi^\dagger_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}}(Z,X,Y,\psi_1,\psi_2) = \frac{1}{n_1!n_2!n_3!m_1!m_2!}\sum_{\sigma\in S_{n_1+n_2+n_3+m_1+m_2}} Tr\left(P^\dagger_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}}\Gamma^R(\sigma)\right) \\ \times\psi^{\dagger i_{m_1+m_2}}_{2i_{\sigma(m_1+m_2)}}\cdots\psi^{\dagger i_{m_1+1}}_{2i_{\sigma(m_1+1)}}\psi^{\dagger i_{m_1}}_{1i_{\sigma(m_1)}}\cdots\psi^{\dagger i_1}_{1i_{\sigma(1)}}X^{\dagger i_{m_1+m_2+1}}_{i_{\sigma(m_1+m_2+1)}}\cdots \tag{4.64}$$

As far as the bosons go, the X fields occupy slots m_1+m_2+1 to $m_1+m_2+n_2$, the Y fields occupy slots $m_1+m_2+n_2+1$ to $m_1+m_2+n_2+n_3$, while the Z fields occupy slots $m_1+m_2+n_2+n_3+1$ to $m_1+m_2+n_2+n_3+n_1$. As is evident in equation (4.64), the boson slots are not reordered by †. The two-point function that follows from (4.63) and (4.64) is

$$\left\langle \chi_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}}\left(Z,X,Y,\psi_1,\psi_2\right)\chi^{\dagger}_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}}\left(Z,X,Y,\psi_1,\psi_2\right)\right\rangle$$
$$= \delta_{RT}\prod_{i=1}^{3}\delta_{r_i t_i}\prod_{j=1}^{2}\delta_{s_j u_j}\prod_{k=1}^{4}\delta_{\alpha_k\gamma_k}\prod_{l=1}^{4}\delta_{\beta_l\delta_l}\frac{f_R hooks_R}{\prod_m hooks_{r_m}\prod_n hooks_{s_n}}. \tag{4.65}$$

4.4 Action of dilatation operator in the $su\,(2|3)$ sector

We now want to compute the action of the dilatation operator on the restricted Schur polynomials of the $su\,(2|3)$ sector of SYM theory. To simplify the formula of the one loop dilatation operator, we set $\phi_1 \equiv Z$, $\phi_2 \equiv X$ and $\phi_3 \equiv Y$. From formula (2.1) of [53] or the H_2 piece of table 1 in [52], we find the following one loop dilatation operator

$$D = -g_{YM}^2\Big(\sum_{i>j=1}^{3} Tr\left(\left[\phi_i,\phi_j\right]\left[\partial_{\phi_i},\partial_{\phi_j}\right]\right) + \sum_{i=1}^{3}\sum_{a=1}^{2} Tr\left(\left[\phi_i,\psi_a\right]\left[\partial_{\phi_i},\partial_{\psi_a}\right]\right)$$
$$+ Tr\left(\left\{\psi_1,\psi_2\right\}\left\{\partial_{\psi_1},\partial_{\psi_2}\right\}\right)\Big). \tag{4.66}$$

We will study the limit in which the number of ϕ_1s (n_1) is much greater than the number of ϕ_2s (n_2), ϕ_3s (n_3), ψ_1s (m_1) and ψ_2s (m_2). In this limit, we can simplify the dilatation operator to

$$D = -g_{YM}^2\Big(\sum_{j=2}^{3} Tr\left(\left[\phi_1,\phi_j\right]\left[\partial_{\phi_1},\partial_{\phi_j}\right]\right) + \sum_{a=1}^{2} Tr\left(\left[\phi_1,\psi_a\right]\left[\partial_{\phi_1},\partial_{\psi_a}\right]\right). \tag{4.67}$$

This simpler expression (4.67) is obtained from (4.66) simply by noting that a derivative with respect to ϕ_1 will generate n_1 terms. Since $n_1 \gg n_2, n_3, m_1, m_2$, this is a lot more terms than is generated by differentiating with respect to any other field.

The simplest example to start with is when the operator is built using only one fermion ψ_1 and one boson $\phi_1 \equiv Z$. One of the terms we need to evaluate is

$$Z^i_j \psi^j_{1k}\frac{d}{dZ^l_k}\frac{d}{d\psi^i_{1l}}\left(\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} Tr_{(r,s)\alpha\beta}\left(\Gamma^R\left(\sigma\right)\right)\psi^{i_1}_{1i_{\sigma(1)}}\cdots\psi^{i_m}_{1i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}\right). \tag{4.68}$$

To take this derivative, we need to use the product rule and hit each of the m factors of ψ_1 and each of the n factors of Z. We know that the contribution from each Z derivative is the same so that we simply get an overall n multiplied by the term obtained when the derivative hits (say) the Z in slot $m+1$. The first thing we want to argue is that the contribution from each ψ_1 derivative is also the same so that we can write these m

terms as m multiplied by the term obtained when the derivative hits (say) the ψ_1 in slot 1. Let us start by thinking of

$$\psi^j_{1k}\frac{d}{d\psi^i_{1l}} \tag{4.69}$$

as our operator. It is Grassman even so it commutes with all other variables. This allows us to move it into any slot without costing any signs. We now consider

$$\begin{aligned}\sum_{\rho\in S_{n+m}} & Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left((1,m+1)\,\rho\right)\right)\delta^{i_1}_{i_{\rho(1)}}\psi^{i_1}_{1i_{\rho(m+1)}}\psi^{i_2}_{1i_{\rho(2)}}\cdots\psi^{i_m}_{1i_{\rho(m)}}Z^{i_{m+1}}_{i_{\rho(1)}}Z^{i_{m+2}}_{i_{\rho(m+2)}}\cdots Z^{i_{m+n}}_{i_{\rho(m+n)}}\\ &=\sum_{\rho\in S_{n+m}}Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left((1,m+1)\,\rho\right)\right)\delta^{i_1}_{i_{\rho(1)}}Tr_{V^{\otimes n+m}}\left(\rho\,(1,m+1)\,\psi_1^{\otimes m}Z^{\otimes n}\right).\end{aligned} \tag{4.70}$$

We can now change variable from ρ to $\gamma=(1,l)\,\rho\,(1,l)$ to obtain

$$\sum_{\gamma\in S_{n+m}}Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left((1,m+1)\,(1,l)\,\gamma\,(1,l)\right)\right)\delta^{i_l}_{i_{\gamma(l)}}Tr_{V^{\otimes n+m}}\left((1,l)\,\gamma\,(1,l)\,(1,m+1)\,\psi_1^{\otimes m}Z^{\otimes n}\right). \tag{4.71}$$

Now,

$$\begin{aligned}Tr_{V^{\otimes n+m}}&\left((1,l)\,\gamma\,(1,l)\,(1,m+1)\,\psi_1^{\otimes m}Z^{\otimes n}\right)=Tr_{V^{\otimes n+m}}\left(\gamma\,(l,m+1)\,(1,l)\,\psi_1^{\otimes m}Z^{\otimes n}\,(1,l)\right)\\ &=\psi^{i_l}_{1i_{\gamma(m+1)}}\psi^{i_2}_{1i_{\gamma(2)}}\cdots\psi^{i_{l-1}}_{1i_{\gamma(l-1)}}\psi^{i_1}_{1i_{\gamma(1)}}\psi^{i_{l+1}}_{1i_{\gamma(l+1)}}\cdots\psi^{i_m}_{1i_{\gamma(m)}}Z^{i_{m+1}}_{i_{\gamma(l)}}Z^{i_{m+2}}_{i_{\gamma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\gamma(m+n)}}\\ &=-\psi^{i_1}_{1i_{\gamma(1)}}\psi^{i_2}_{1i_{\gamma(2)}}\cdots\psi^{i_{l-1}}_{1i_{\gamma(l-1)}}\psi^{i_l}_{1i_{\gamma(m+1)}}\psi^{i_{l+1}}_{1i_{\gamma(l+1)}}\cdots\psi^{i_m}_{1i_{\gamma(m)}}Z^{i_{m+1}}_{i_{\gamma(l)}}Z^{i_{m+2}}_{i_{\gamma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\gamma(m+n)}}.\end{aligned} \tag{4.72}$$

Also,

$$\begin{aligned}Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left((1,m+1)\,(1,l)\,\gamma\,(1,l)\right)\right)&=Tr\left(\Gamma^R\left((1,l)\right)P_{R,(r,s)\alpha\beta}\Gamma^R\left((1,l)\,(l,m+1)\,\gamma\right)\right)\\ &=-Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left((l,m+1)\,\gamma\right)\right).\end{aligned} \tag{4.73}$$

Thus we find

$$\begin{aligned}\sum_{\rho\in S_{n+m}}&Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left((1,m+1)\,\rho\right)\right)\delta^{i_1}_{i_{\rho(1)}}\psi^{i_1}_{1i_{\rho(m+1)}}\psi^{i_2}_{1i_{\rho(2)}}\cdots\psi^{i_m}_{1i_{\rho(m)}}Z^{i_{m+1}}_{i_{\rho(1)}}Z^{i_{m+2}}_{i_{\rho(m+2)}}\cdots Z^{i_{m+n}}_{i_{\rho(m+n)}}\\ =\sum_{\gamma\in S_{n+m}}&Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R\left((l,m+1)\,\gamma\right)\right)\delta^{i_l}_{i_{\gamma(l)}}\psi^{i_1}_{1i_{\gamma(1)}}\psi^{i_2}_{1i_{\gamma(2)}}\cdots\psi^{i_{l-1}}_{1i_{\gamma(l-1)}}\psi^{i_l}_{1i_{\gamma(m+1)}}\psi^{i_{l+1}}_{1i_{\gamma(l+1)}}\cdots\psi^{i_m}_{1i_{\gamma(m)}}\\ &\times Z^{i_{m+1}}_{i_{\gamma(l)}}Z^{i_{m+2}}_{i_{\gamma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\gamma(m+n)}}.\end{aligned} \tag{4.74}$$

The left hand side of this last identity is obtained when we differentiate the ψ_1 in slot 1, while the right hand side is obtained by differentiating ψ_1 in slot l. Thus this last identity proves that the contribution from each ψ_1 derivative is the same. Therefore,

$$Z^i_j \psi^j_{1k} \frac{d}{dZ^l_k} \frac{d}{d\psi^i_{1l}} \left(\frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} Tr_{(r,s)\alpha\beta} \left(\Gamma^R(\sigma)\right) \psi^{i_1}_{1i_{\sigma(1)}} \cdots \psi^{i_m}_{1i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}} \right)$$
$$= \frac{1}{(n-1)!(m-1)!} \sum_{\rho \in S_{n+m}} Tr\left(P_{R,(r,s)\alpha\beta} \Gamma^R((1,m+1)\rho)\right) \delta^{i_1}_{i_{\rho(1)}} \psi^j_{1i_{\rho(m+1)}} \psi^{i_2}_{1i_{\rho(2)}} \cdots \psi^{i_m}_{1i_{\rho(m)}}$$
$$\times Z^{i_1}_j Z^{i_{m+2}}_{i_{\rho(m+2)}} \cdots Z^{i_{m+n}}_{i_{\rho(m+n)}}. \tag{4.75}$$

At this point, it is now easy to find

$$D\chi_{R,(r,s)\alpha\beta}(\psi_1, Z) = -\frac{2g^2_{YM}}{(4\pi)^2} Tr\left([Z,\psi_1][\partial_Z, \partial_{\psi_1}]\right) \chi_{R,(r,s)\alpha\beta}(\psi_1, Z)$$
$$= \frac{2g^2_{YM}}{(4\pi)^2 (n-1)!(m-1)!} \sum_{\rho \in S_{n+m}} \delta^{i_1}_{i_{\rho(1)}} Tr_{(r,s)\alpha\beta}\left(\Gamma^R([(1,m+1),\rho])\right)$$
$$\times Tr_{V^{\otimes n+m}}\left([(1,m+1),\rho]\, \psi_1^{\otimes m} Z^{\otimes n}\right). \tag{4.76}$$

Our next task is to express $Tr_{V^{\otimes n+m}}\left([(1,m+1),\rho]\, \psi_1^{\otimes m} Z^{\otimes n}\right)$ as a sum over restricted Schur polynomials. We will generalise the argument given in [37] (and used in Chapter 3) which provides the identity for restricted Schur polynomials built entirely out of bosonic fields. First, we need an identity. The irrep (r,s) of $S_n \times S_m$ will, in general, be subduced by irrep R of S_{n+m} more than once. We label these different copies with β. It is convenient to switch to a bra-ket notation in which the operators used to define the restricted Schur polynomials we have constructed are

$$\left[P_{R,(r,s)\alpha\beta}\right]_{JI} = \sum_{a,b,i} \langle R, J|s, b; r, i; \alpha\rangle \left\langle s^T, a; r, i; \beta|R, I\right\rangle O_{ba}. \tag{4.77}$$

We will make use of the identity [75, 37]

$$\sum_\beta \langle R, I|r, b; s, i; \beta\rangle \langle r, a; s, j; \beta|R, J\rangle = \frac{d_r d_s}{n!m!} \sum_{\alpha_1 \in S_m} \sum_{\alpha_2 \in S_n} \Gamma^s\left(\alpha_1^{-1}\right)_{ij} \Gamma^r\left(\alpha_2^{-1}\right)_{ab} \Gamma^R(\alpha_1 \circ \alpha_2)_{IJ} \tag{4.78}$$

in what follows. Consider the sum

$$\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s (n+m)!}\chi_{R,(r,s)\alpha\beta}(\tau)\,\chi^\dagger_{R,(r,s)\beta\alpha}(\sigma)$$
$$=\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s (n+m)!}Tr\left(P_{R,(r,s)\alpha\beta}\Gamma^R(\tau)\right)Tr\left(P^\dagger_{R,(r,s)\beta\alpha}\Gamma^R(\sigma)\right)$$
$$=\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s (n+m)!}\left[P_{R,(r,s)\alpha\beta}\right]_{IJ}\left[\Gamma^R(\tau)\right]_{JI}\left[P^\dagger_{R,(r,s)\beta\alpha}\right]_{KL}\left[\Gamma^R(\sigma)\right]_{LK}. \qquad (4.79)$$

Rewriting both projectors using bra-ket notation, we find

$$\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s (n+m)!}\chi_{R,(r,s)\alpha\beta}(\tau)\,\chi^\dagger_{R,(r,s)\beta\alpha}(\sigma)$$
$$=\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s (n+m)!}\sum_{a,b,i,I,J}\langle R,I|s,b;r,i;\alpha\rangle\left\langle s^T,a;r,i;\beta|R,J\right\rangle O_{ba}\left[\Gamma^R(\tau)\right]_{JI}$$
$$\times\sum_{c,d,j,K,L}\left\langle R,L|s^T,d;r,j;\beta\right\rangle\langle s,c;r,j;\alpha|R,K\rangle\left(O^T\right)_{dc}\left[\Gamma^R(\sigma)\right]_{LK}. \qquad (4.80)$$

The sum over the multiplicity labels can now be performed using the identity (4.78). We get

$$\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s (n+m)!}\chi_{R,(r,s)\alpha\beta}(\tau)\,\chi^\dagger_{R,(r,s)\beta\alpha}(\sigma)$$
$$=\sum_{R,(r,s)\alpha\beta}\sum_{\gamma_1,\tau_1\in S_m}\sum_{\gamma_2,\tau_2\in S_n}\frac{d_R d_r d_s}{(n+m)!n!m!}Tr\left(\Gamma^r(\gamma_2\tau_2)\right)Tr\left(\Gamma^{s^T}(\gamma_1)\,O^T\Gamma^s(\tau_1)\,O\right)$$
$$\times Tr\left(\Gamma^R(\tau\cdot\gamma_1\circ\gamma_2\cdot\sigma\cdot\tau_1\circ\tau_2)\right). \qquad (4.81)$$

In this last expression, we recognise the delta function on the group

$$\sum_R\frac{d_R}{|\mathcal{G}|}\chi_R(\sigma)=\delta(\sigma), \qquad (4.82)$$

where R is a complete set of irreps of group $\mathcal{G}$. We therefore now have

$$\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s (n+m)!}\chi_{R,(r,s)\alpha\beta}(\tau)\,\chi^\dagger_{R,(r,s)\beta\alpha}(\sigma)$$
$$=\sum_{R\vdash n+m}\sum_{\tau_1\in S_m}\sum_{\tau_2\in S_n}sgn(\tau_1)\frac{d_R}{(n+m)!}\chi_R\left(\tau\cdot\tau_1^{-1}\circ\tau_2^{-1}\cdot\sigma\cdot\tau_1\circ\tau_2\right)$$
$$=\sum_{\tau_1\in S_m}\sum_{\tau_2\in S_n}sgn(\tau_1)\,\delta\left(\tau\cdot\tau_1^{-1}\circ\tau_2^{-1}\cdot\sigma\cdot\tau_1\circ\tau_2\right). \qquad (4.83)$$

This identity is all that is needed to prove that

$$Tr_{V^{\otimes n+m}}\left(\sigma\psi_1^{\otimes m}Z^{\otimes n}\right)=\sum_{R,(r,s)\alpha\beta}\frac{d_R n!m!}{d_r d_s(n+m)!}\chi^{\dagger}_{R,(r,s)\alpha\beta}(\sigma)\,\chi_{R,(r,s)\beta\alpha}(\psi_1,Z)\,. \quad (4.84)$$

Using this, we find

$$D\chi_{R,(r,s)\alpha\beta}(\psi_1,Z)=\sum_{T,(t,u)\gamma\delta}M_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta}\chi_{T,(t,u)\gamma\delta}(\psi_1,Z)\,, \quad (4.85)$$

where

$$M_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta}=-g_{YM}^2\sum_{R'}\frac{c_{RR'}d_T nm}{d_{R'}d_t d_u(n+m)}Tr\Big(\left[\Gamma^R((1,m+1)),P_{R,(r,s)\alpha\beta}\right]I_{R'T'} \times\left[\Gamma^T(1,m+1),P_{T,(t,u)\delta\gamma}\right]I_{T'R'}\Big). \quad (4.86)$$

As before, to obtain the spectrum of anomalous dimensions, it is convenient to consider the action of the dilatation operator on operators whose two point functions are normalised to unity. In this particular case, we have

$$\chi_{R,(r,s)\alpha\beta}(\psi_1,Z)=\sqrt{\frac{f_R hooks_R}{hooks_r hooks_s}}O_{R,(r,s)\alpha\beta}(\psi_1,Z)\,. \quad (4.87)$$

In terms of these normalised operators, the action of the dilatation operator is

$$DO_{R,(r,s)\alpha\beta}(\psi_1,Z)=\sum_{T,(t,u)\gamma\delta}N_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta}O_{T,(t,u)\gamma\delta}(\psi_1,Z)\,, \quad (4.88)$$

where

$$N_{R,(r,s)\alpha\beta;T,(t,u)\gamma\delta}=-g_{YM}^2\sum_{R'}\frac{c_{RR'}d_T nm}{d_{R'}d_t d_u(n+m)}\sqrt{\frac{f_T hooks_T hooks_r hooks_s}{f_R hooks_R hooks_t hooks_u}} \times Tr_{R\oplus T}\Big(\left[\Gamma^R((1,m+1)),P_{R,(r,s)\alpha\beta}\right]I_{R'T'}\left[\Gamma^T(1,m+1),P_{T,(t,u)\delta\gamma}\right]I_{T'R'}\Big). \quad (4.89)$$

We have explicitly indicated that the last trace is taken over the direct sum of the carrier spaces of R and T. Remarkably, this takes a very similar form to what was obtained in the $SU(2)$ sector [86]. As a result, we know that the operators with a definite scaling dimension can be constructed using the ideas of the double coset ansatz [46] reviewed in Section 2.6. A few of the details are different though, so that it is worth describing some of the steps involved.

As we have already mentioned, we remove boxes from R to produce r. The number of boxes that must be removed from row i of R is m_i. The m_i can be assembled to produce the vector label $\vec{m}$ which is conserved by the one loop dilatation operator. There are two types of branching coefficients

$$\sum_{\mu} B^{s\to 1_H}_{k\mu} B^{s\to 1_H}_{l\mu} = \frac{1}{|H|} \sum_{\gamma\in H} \Gamma^s(\gamma)_{kl} \tag{4.90}$$

and

$$\sum_{\mu} B^{s^T\to 1^m}_{k\mu} B^{s^T\to 1^m}_{l\mu} = \frac{1}{|H|} \sum_{\gamma\in H} sgn(\gamma)\, \Gamma^{s^T}(\gamma)_{kl}\,, \tag{4.91}$$

where

$$H = S_{m_1}\times S_{m_2}\times\cdots S_{m_p}. \tag{4.92}$$

The branching coefficients $B^{s\to 1_H}_{k\mu}$ resolve the multiplicities that arise when we restrict irrep s of S_m to the identity representation 1_H of H for which

$$\Gamma^{1_H}(\gamma) = 1, \quad \forall\gamma. \tag{4.93}$$

On the other hand, the branching coefficients $B^{s^T\to 1^m}_{k\mu}$ resolve the multiplicities that arise when we restrict irrep s of S_m to the representation 1^m of H for which

$$\Gamma^{1^m}(\gamma) = sgn(\gamma)\,, \quad \forall\gamma. \tag{4.94}$$

Notice that

$$\begin{aligned}
\sum_{\mu} B^{s^T\to 1^m}_{k\mu} B^{s^T\to 1^m}_{l\mu} &= \frac{1}{|H|} sgn(\gamma) \sum_{\gamma\in H} \Gamma^{s^T}(\gamma)_{kl} \\
&= \frac{1}{|H|} \sum_{\gamma\in H} \left(O^T \Gamma^s(\gamma)\, O\right)_{kl} \\
&= O^T_{km} \sum_{\mu} B^{s\to 1_H}_{m\mu} B^{s\to 1_H}_{n\mu} O_{nl}
\end{aligned} \tag{4.95}$$

so that we can identify

$$B^{s\to 1_H}_{n\mu} O_{nl} = B^{s^T\to 1^m}_{l\mu}. \tag{4.96}$$

This argument suggests that the multiplicity problem of $s\to 1_H$ can be identified with the multiplicity problem of $s^T\to 1^m$. To prove that this is indeed the case, we denote the multiplicity of 1_H in s by $n^s_{1_H}$ and the multiplicity of 1^m in s^T by $n^{s^T}_{1^m}$. We then have

$$\begin{aligned}
n^s_{1_H} &= \frac{1}{|H|}\sum_\sigma \chi_s(\sigma)\,\chi_{1_H}(\sigma) \\
&= \frac{1}{|H|}\sum_\sigma \chi_s(\sigma) \\
&= \frac{1}{|H|}\sum_\sigma \chi_{s^T}(\sigma)\, sgn(\sigma) \\
&= \frac{1}{|H|}\sum_\sigma \chi_{s^T}(\sigma)\,\chi_{1^m}(\sigma) \\
&= n^{s^T}_{1^m}
\end{aligned} \tag{4.97}$$

which completes the proof.

Now, following what was done in the $SU(2)$ sector [46], we identify

$$|\vec{m}, s, \mu; i\rangle = \sum_j B^{s\to 1_H}_{j\mu} \sum_{\sigma\in S_m} \Gamma^s(\sigma)_{ij}\,|v_\sigma\rangle\,. \tag{4.98}$$

The components m_i of the vector label $\vec{m}$ appearing in the above ket record the number of boxes that must be removed from row i of R to produce r. These are the basis vectors in s that are used to construct the projectors appearing in the restricted Schur polynomials. To construct the projectors, we also need to make use of a basis for s^T. The basis for s^T should be constructed using $\hat{O}^T$ which provides a map from the carrier space of s^T to the carrier space of s. Using $\hat{O}^T$, we find

$$\sum_i \left(\hat{O}^T\right)_{ki} |\vec{m}, s, \mu; i\rangle = \sum_j B^{s^T\to 1^m}_{j\mu} \sum_{\sigma\in S_m} sgn(\sigma)\,\Gamma^{s^T}(\sigma)_{ij}\,|v_\sigma\rangle\,. \tag{4.99}$$

Given these bases, it is now easy to verify that the projectors appearing in the restricted Schur polynomials can be written as

$$O\left(s\alpha, s^T\beta\right) = \frac{d_s}{m!\,|H|}\sum_{\sigma,\tau\in S_m} B^{s\to 1_H}_{c\alpha}\,\Gamma^s_{ac}(\sigma)\,|v_\sigma\rangle\langle v_\tau|\,B^{s^T\to 1^m}_{d\beta}\, sgn(\tau)\,\Gamma^{s^T}_{bd}(\tau)\,O^T_{ba}. \tag{4.100}$$

Using these expressions, one can verify that

$$O\left(s\alpha, s^T\beta\right) O\left(s^T\beta, s\alpha\right) = \mathbf{1}_s \tag{4.101}$$

and

$$O\left(s^T\beta, s\alpha\right) O\left(s\alpha, s^T\beta\right) = \mathbf{1}_{s^T}. \tag{4.102}$$

In terms of the branching coefficients, let us introduce the quantities

$$C^{(s)}_{\mu_1\mu_2}\left(\tau\right)=|H|\sqrt{\frac{d_s}{m!}}\left(\Gamma^s\left(\tau\right)\hat{O}\right)_{km}B^{s\to 1_H}_{k\mu_1}B^{s^T\to 1^m}_{m\mu_2} \tag{4.103}$$

which define an orthogonal transformation

$$\begin{aligned}C^{(s)}_{\mu_1\mu_2}\left(\tau\right)C^{(s)}_{\mu_1\mu_2}\left(\sigma\right)&=\sum_{s\vdash m}\sum_{\gamma_1,\gamma_2\in H}|H|^2\frac{d_s}{m!}sgn\left(\gamma_2\right)Tr\left(\Gamma^s\left(\tau\right)\hat{O}\Gamma^{s^T}\left(\gamma_2\right)\hat{O}^T\Gamma^s\left(\sigma^{-1}\right)\Gamma^s\left(\gamma_1\right)\right)\\&=\sum_{s\vdash m}\sum_{\gamma_1,\gamma_2\in H}\frac{d_s}{m!}\chi_s\left(\tau\gamma_2\sigma^{-1}\gamma_1\right)\\&=\sum_{\gamma_1,\gamma_2\in H}\delta\left(\tau\gamma_2\sigma^{-1}\gamma_1\right).\end{aligned} \tag{4.104}$$

It is then rather natural to build operators that are dual to the Gauss graph configuration σ by

$$O_{R,r}\left(\sigma\right)=\sum_{s\vdash m}\sum_{\mu_1,\mu_2}C^{(s)}_{\mu_1\mu_2}\left(\sigma\right)O_{R,(r,s)\mu_1\mu_2}. \tag{4.105}$$

Using (4.104) we find

$$\left\langle O_{R,r}\left(\sigma_1\right)O^{\dagger}_{T,t}\left(\sigma_2\right)\right\rangle=\sum_{\gamma_1,\gamma_2\in H}\delta\left(\gamma_1\sigma_1\gamma_2\sigma_2^{-1}\right). \tag{4.106}$$

At the end of the day, we want to evaluate the action of the dilatation operator on the Gauss graph operators (4.105). To this end, let us revisit the evaluation of the dilatation operator on the normalised restricted Schur polynomials $O_{R,(r,s)\mu_1\mu_2}$, as we did in Chapter 3. We denote the number of rows in the Young diagram labelling the restricted Schur polynomials by p. The one loop dilatation operator (4.89) is exact to all orders in $1/N$. To capture the large N (but non-planar) limit we use the displaced corners approximation. Recall that to subduce $r\vdash n$ from $R\vdash n+m$ we remove m boxes from R. Each box in row i and column j of the Young diagram R can be assigned a factor which is equal to $N-i+j$. The displaced corners approximation applies when the difference between the factors of any two boxes (of the m boxes removed) is of order N whenever the removed boxes come from different rows. The action of the dilatation operator simplifies in this limit because the action of the symmetric group becomes particularly simple [87]. When the displaced corners approximation holds, we associate each removed box with a vector in a p-dimensional vector space V_p. This way, the m removed boxes associated with the ψ_1s define a vector in $V_p^{\otimes m}$. The trace over $R\oplus T$ factorises into a trace over $r\oplus t$ and $V_p^{\otimes m}$. The bulk of the work is in evaluating the trace over $V_p^{\otimes m}$. This trace is evaluated in exactly the same way we followed in Chapter

3. Doing this, we find

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{YM}^2 \sum_{u\nu_1\nu_2} \sum_{i<j} \delta_{\vec{m},\vec{n}} M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} \triangle_{ij} O_{R,(r,u)\nu_1\nu_2}, \tag{4.107}$$

where

$$M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} = \frac{m}{\sqrt{d_s d_u}} \Big[Tr\left(\hat{O}\left(s\mu_1; s^T\mu_2\right) E^{(1)}_{ii} \hat{O}\left(u^T\nu_2; u\nu_1\right) E^{(1)}_{jj}\right) + Tr\left(\hat{O}\left(s\mu_1; s^T\mu_2\right) E^{(1)}_{jj} \hat{O}\left(u^T\nu_2; u\nu_1\right) E^{(1)}_{ii}\right)\Big] \tag{4.108}$$

acts only on the impurity labels, and

$$\triangle_{ij} = \triangle^+_{ij} + \triangle^0_{ij} + \triangle^-_{ij} \tag{4.109}$$

acts only on the Young diagrams R, r. To describe the action of $\triangle_{ij}$, we introduce a little bit more notation. As in Chapter 3, we denote the row lengths of Young diagram r by r_i and let r^+_{ij} be the Young diagram obtained by removing a single box from row j of r and adding it to row i. In the same vein, r^-_{ij} is a Young diagram obtained by removing one box from row i of r and adding it to row j. We then have

$$\triangle^0_{ij} O_{R,(r,s)\mu_1\mu_2} = -\left(2N + r_i + r_j\right) O_{R,(r,s)\mu_1\mu_2}, \tag{4.110}$$

$$\triangle^+_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{\left(N + r_i\right)\left(N + r_j\right)} O_{R^+_{ij},(r^+_{ij},s)\mu_1\mu_2} \tag{4.111}$$

and

$$\triangle^-_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{\left(N + r_i\right)\left(N + r_j\right)} O_{R^-_{ij},(r^-_{ij},s)\mu_1\mu_2}. \tag{4.112}$$

Since the Young diagrams R and r change in exactly the same way, the vector $\vec{m}$ is preserved by the dilatation operator.

With this, we now proceed with computing the action of the dilatation operator on the Gauss graph operators (4.105). Towards this end, we consider

$$\begin{aligned}\left\langle O^{\dagger}_{T,t}\left(\sigma_2\right)DO_{R,r}\left(\sigma_1\right)\right\rangle &= \frac{|H|^2}{m!}\sum_{s,u\vdash m}\sum_{\mu_1\mu_2\nu_1\nu_2}\sqrt{d_s d_u}\left(\Gamma^s\left(\sigma_1\right)\hat{O}\right)_{k_1m_1}B^{s\to 1_H}_{k_1\mu_1}B^{s^T\to 1^m}_{m_1\mu_2}\\ &\times\left(\Gamma^u\left(\sigma_2\right)\hat{O}\right)_{k_2m_2}B^{u\to 1_H}_{k_2\nu_1}B^{u^T\to 1^m}_{m_2\nu_2}\left\langle O^{\dagger}_{T,(t,u)\nu_1\nu_2}DO_{R,(r,s)\mu_1\mu_2}\right\rangle\\ &= -\frac{|H|^2}{m!}g^2_{YM}\sum_{s,u\vdash m}\sum_{\mu_1\mu_2\nu_1\nu_2}\left(\Gamma^s\left(\sigma_1\right)\hat{O}\right)_{k_1m_1}B^{s\to 1_H}_{k_1\mu_1}B^{s^T\to 1^m}_{m_1\mu_2}\\ &\times\left(\Gamma^u\left(\sigma_2\right)\hat{O}\right)_{k_2m_2}B^{u\to 1_H}_{k_2\nu_1}B^{u^T\to 1^m}_{m_2\nu_2}\sum_{i<j}\triangle^{R,r;T,t}_{ij}m\\ &\times\left(\left\langle\vec{m},s^T,\mu_2;a\left|E^{(1)}_{ii}\right|\vec{m},u^T,\nu_2;b\right\rangle\left\langle\vec{m},u,\nu_1;b\left|E^{(1)}_{jj}\right|\vec{m},s,\mu_1;a\right\rangle\right.\\ &\left.+\left\langle\vec{m},s^T,\mu_2;a\left|E^{(1)}_{jj}\right|\vec{m},u^T,\nu_2;b\right\rangle\left\langle\vec{m},u,\nu_1;b\left|E^{(1)}_{ii}\right|\vec{m},s,\mu_1;a\right\rangle\right)\end{aligned} \tag{4.113}$$

and focus on the evaluation of

$$\begin{aligned}&\sum_u\sum_{\nu_1,\nu_2}\left|\vec{m},u^T,\nu_2;b\right\rangle\left\langle\vec{m},u,\nu_1;b\right|\left(\Gamma^u\left(\sigma_2\right)\hat{O}\right)_{k_2m_2}B^{u\to 1_H}_{k_2\nu_1}B^{u^T\to 1^m}_{m_2\nu_2}\\ &=\sum_u\sum_{\nu_1,\nu_2}\sum_{\sigma,\tau\in S_m}\frac{d_u}{|H|\,m!}sgn\left(\tau\right)B^{u^T\to 1^m}_{d\nu_2}\Gamma^{u^T}_{bd}\left(\tau\right)\left|v_\tau\right\rangle O_{cb}\left\langle v_\sigma\right|\Gamma^u_{ce}\left(\sigma\right)B^{u\to 1_H}_{e\nu_1}\\ &\qquad\times\left(\Gamma^u\left(\sigma_2\right)\hat{O}\right)_{k_2m_2}B^{u\to 1_H}_{k_2\nu_1}B^{u^T\to 1^m}_{m_2\nu_2}\\ &=\sum_u\sum_{\sigma,\tau\in S_m}\sum_{\gamma_1,\gamma_2\in H}\frac{d_u}{|H|^3\,m!}sgn\left(\tau\right)sgn\left(\gamma_2\right)\left|v_\tau\right\rangle\left\langle v_\sigma\right|\\ &\times Tr\left(\Gamma^u\left(\gamma_1\right)\Gamma^u\left(\sigma^{-1}\right)\hat{O}\Gamma^{u^T}\left(\tau\right)\Gamma^{u^T}\left(\gamma_2\right)\hat{O}^T\Gamma^u\left(\sigma_2^{-1}\right)\right)\\ &=\sum_u\sum_{\sigma,\tau\in S_m}\sum_{\gamma_1,\gamma_2\in H}\frac{d_u}{|H|^3\,m!}\left|v_\tau\right\rangle\left\langle v_\sigma\right|\chi_u\left(\gamma_1\sigma^{-1}\tau\gamma_2\sigma_2^{-1}\right)\\ &=\sum_{\sigma,\tau\in S_m}\sum_{\gamma_1,\gamma_2\in H}\frac{1}{|H|^3}\left|v_\tau\right\rangle\left\langle v_\sigma\right|\delta\left(\gamma_1\sigma^{-1}\tau\gamma_2\sigma_2^{-1}\right).\end{aligned} \tag{4.114}$$

From this point on, the evaluation proceeds exactly as in [46] (or as we did in Chapter 3 with $\gamma = 0$). The result is

$$\left\langle O^{\dagger}_{T,t}\left(\sigma_2\right)DO_{R,r}\left(\sigma_1\right)\right\rangle = -g^2_{YM}\sum_{\gamma_1,\gamma_2\in H}\delta\left(\gamma_1\sigma_2\gamma_2\sigma_1^{-1}\right)\sum_{i<j}n_{ij}\left(\sigma_1\right)\triangle^{R,r;T,t}_{ij} \tag{4.115}$$

or

$$DO_{R,r}\left(\sigma_1\right) = -g^2_{YM}\sum_{i<j}n_{ij}\left(\sigma_1\right)\triangle_{ij}, \tag{4.116}$$

where $n_{ij}(\sigma_1)$ is the number of strings stretching between branes i and j. This proves that the Gauss graph operators (4.105) indeed diagonalise the impurity labels. The remaining eigenproblem that must be solved has been studied in detail in [45] from which we know that the spectrum of D reduces to the spectrum of set of decoupled oscillators. This signals integrability.

We now consider the general case with three bosons ϕ_1, ϕ_2, ϕ_3 and two fermions ψ_1, ψ_2. After a rather lengthy calculation that resembles what we have already done, we find that the action of the dilatation operator (4.67) is given by

$$DO_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}} = \sum_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} N_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta};T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} O_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}}, \tag{4.117}$$

where

$$\begin{aligned} N_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta};T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} = &-g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T n_1}{d_{R'} \prod_n d_{t_n} \prod_m d_{u_m} (n_1 + K)} \sqrt{\frac{f_T hooks_T \prod_a hooks_{r_a} \prod_b hooks_{s_b}}{f_R hooks_R \prod_c hooks_{t_c} \prod_d hooks_{s_d}}} \\ &\times \Big[m_1 Tr \left(\left[\Gamma^R(1, K+1), P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}} \right] I_{R'T'} \left[\Gamma^T(1, K+1), P_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} \right] I_{T'R'} \right) \\ &+ m_2 Tr \left(\left[\Gamma^R(m_1+1, K+1), P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}} \right] I_{R'T'} \left[\Gamma^T(m_1+1, K+1), P_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} \right] I_{T'R'} \right) \\ &+ n_2 Tr \left(\left[\Gamma^R(m_1+m_2+1, K+1), P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}} \right] I_{R'T'} \left[\Gamma^T(m_1+m_2+1, K+1), P_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} \right] I_{T'R'} \right) \\ &+ n_3 Tr \left(\left[\Gamma^R(K-n_3, K+1), P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}} \right] I_{R'T'} \left[\Gamma^T(K-n_3, K+1), P_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} \right] I_{T'R'} \right) \Big] \end{aligned} \tag{4.118}$$

and $K = n_2 + n_3 + m_1 + m_2$ is the total number of impurities. The projectors $P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}}$ and $P_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}}$ were defined in equation (4.62). We note that these projectors factorise into a product of factors and that in each term above, the product of all but the Z projector and one other have a trivial action. As an example, in the trace

$$T = Tr \left(\left[\Gamma^R(1, K+1), P_{R,(\vec{r},\vec{s})\vec{\alpha}\vec{\beta}} \right] I_{R'T'} \left[\Gamma^T(1, K+1), P_{T,(\vec{t},\vec{u})\vec{\gamma}\vec{\delta}} \right] I_{T'R'} \right) \tag{4.119}$$

the swap $(1, K+1)$ only has a non-trivial action on slots 1 and $K+1$. Slot 1 is populated by ψ_1 and corresponds to representation s_1. Slot $K+1$ is populated by $\phi_1 = Z$ and corresponds to representation r_1. The traces over r_2, r_3 and s_2 are trivial while the trace over $r_1 \oplus s_1$ is performed exactly as we described earlier. Trace (4.119) then gives

$$\begin{aligned} T = &d_{s_2} d_{r_2} d_{r_3} d_{r_1'} \delta_{s_2 u_2} \delta_{r_2 t_2} \delta_{r_3 t_3} \delta_{r_1' t_1'} (Tr \left(\hat{O}\left(s_1\mu_1; s_1^T\mu_2\right) E_{ii}^{(1)} \hat{O}\left(u_1^T\nu_2; u_1\nu_1\right) E_{jj}^{(1)} \right) \\ &+ Tr \left(\hat{O}\left(s_1\mu_1; s_1^T\mu_2\right) E_{jj}^{(1)} \hat{O}\left(u_1^T\nu_2; u_1\nu_1\right) E_{ii}^{(1)} \right). \end{aligned} \tag{4.120}$$

Defining Gauss graph operators for this general case now involves an element of a double coset for each type of impurity. We denote the total number of $(\phi_2, \phi_3, \psi_1, \psi_2)$ impurities by (n_2, n_3, m_1, m_2) and describe the number of boxes removed row i of R for each impurity type by the vectors $(\vec{n}_2, \vec{n}_3, \vec{m}_1, \vec{m}_2)$.[2] As an example, we now have a subgroup

$$H_{\vec{n}_2} = S_{(n_2)_1} \times S_{(n_2)_2} \times \cdots S_{(n_2)_p}. \tag{4.121}$$

The relevant cosets are

$$\begin{aligned} \phi_2 &\leftrightarrow \sigma_{\phi_2} \in H_{\vec{n}_2} \backslash S_{n_2} / H_{\vec{n}_2} \\ \phi_3 &\leftrightarrow \sigma_{\phi_3} \in H_{\vec{n}_3} \backslash S_{n_3} / H_{\vec{n}_3} \\ \psi_1 &\leftrightarrow \sigma_{\psi_1} \in H_{\vec{m}_1} \backslash S_{m_1} / H_{\vec{m}_1} \\ \psi_2 &\leftrightarrow \sigma_{\psi_2} \in H_{\vec{m}_2} \backslash S_{m_2} / H_{\vec{m}_2}. \end{aligned} \tag{4.122}$$

The orthogonal transformation from the restricted Schur basis to the Gauss graph uses both the group theoretic coefficients of [46]

$$C^{(r_i)}_{\mu_1\mu_2} = |H_{\vec{n}_i}| \sqrt{\frac{d_{r_i}}{n_i!}} \Gamma^{(r_i)}(\tau)_{km} B^{r_i \to 1_{H_{\vec{n}_i}}}_{k\mu_1} B^{r_i \to 1_{H_{\vec{n}_i}}}_{m\mu_2} \tag{4.123}$$

to transform the ϕ_2 and ϕ_3 labels, as well as the group theoretic coefficients we have introduced in this chapter

$$C^{(s_i)}_{\mu_1\mu_2} = |H_{\vec{m}_i}| \sqrt{\frac{d_{s_i}}{m_i!}} \left(\Gamma^{(s_i)}(\tau)\,\hat{O}\right)_{km} B^{s_i \to 1_{H_{\vec{m}_i}}}_{k\mu_1} B^{s_i^T \to 1^{m_i}_{H_{\vec{n}_i}}}_{m\mu_2}. \tag{4.124}$$

In terms of these coefficients, the Gauss graph operators are

$$O_{R,r_1}(\vec{\sigma}) = \sum_{r_2 \vdash n_2} \sum_{r_3 \vdash n_3} \sum_{s_1 \vdash m_1} \sum_{s_2 \vdash m_2} \sum_{\vec{\mu},\vec{\nu}} C^{(r_2)}_{\mu_1\nu_1}(\sigma_{\phi_2})\, C^{(r_3)}_{\mu_2\nu_2}(\sigma_{\phi_3})\, C^{(s_1)}_{\mu_3\nu_3}(\sigma_{\psi_1})\, C^{(s_2)}_{\mu_4\nu_4}(\sigma_{\psi_2})\, O_{R,(\vec{r},\vec{s})\vec{\mu}\vec{\nu}}. \tag{4.125}$$

The action of the dilatation operator in the Gauss graph basis then becomes

$$DO_{R,r_1}(\sigma) = -g^2_{YM} \sum_{i<j} \left(n_{ij}(\sigma_{\phi_2}) + n_{ij}(\sigma_{\phi_3}) + n_{ij}(\sigma_{\psi_1}) + n_{ij}(\sigma_{\psi_2})\right) \triangle_{ij} O_{R,r_1}(\sigma). \tag{4.126}$$

Using the results from [45], we see here that the spectrum of the dilatation operator again reduces to a set of decoupled oscillators. This is a clear indication of integrability in this large N limit of the $su(2|3)$ sector.

[2] This way, $\vec{m}_2$ has components $(m_2)_i$ with $i = 1, 2, \ldots p$ and $\sum_i (m_2)_i = m_2$.

4.5 Discussion

In this chapter, we have studied a large N, but non-planar limit of the correlation functions of a class of operators that are AdS/CFT dual to systems of excited AdS giant gravitons. In particular, we have included adjoint fermions for the first time. We started by explaining how to construct restricted Schur polynomials that include both adjoint bosons and adjoint fermions. These operators diagonalise the free field two point functions to all orders in N and are a complete set of local operators. We then explored the one loop anomalous dimensions of these operators. Our study shows that the action of the one loop dilatation operator acting on a sector that includes fermionic fields is diagonalised by a natural extension of the double coset ansatz [46]. The resulting spectrum is identical to the spectrum of a set of decoupled oscillators, clearly indicating integrability in this large N limit of the $su\,(2|3)$ sector of super Yang-Mills theory.

Chapter 5

Conclusion

Gauge/gravity duality relates gravitational theories on backgrounds with constant negative curvature (AdS space) to conformal field theories living on the boundary of these curved backgrounds. In the case of the most studied example of this duality, when the AdS space is highly curved so that we are unable to perform gravity calculations, the dual gauge theory is weakly coupled. Conversely, when the gauge theory is strongly coupled, the gravity theory reduces to classical supergravity. We have worked on the gauge theory side which has the potential to provide non-trivial lessons about the gravity theory.

The two sides of the AdS/CFT correspondence are related by a dictionary according to which states in the gravity theory are dual to operators in the gauge theory. The energies of these states are dual to the scaling dimensions of the operators. To compute the energy spectrum of the states of the gravity theory, one can therefore compute the spectrum of anomalous dimensions on the gauge theory side.

5.1 En-route to restricted Schur polynomials

An important step towards computing the physical observables in the field theory is constructing gauge invariant operators, since all physical observables are gauge invariant. $\mathcal{N} = 4$ super Yang-Mills theory has gauge group $U(N)$. For this reason, the (scalar) fields are $N \times N$ complex matrices. One can use $O(1)$ such scalar fields to construct single-trace gauge invariant operators that are dual to string states. In the limit $N \to \infty$, it turns out that one needs to sum only planar diagrams in order to compute the one-loop anomalous dimensions. This problem is solved by identifying the dilatation operator here with the Hamiltonian of an integrable spin chain.

In our work, we were more interested in large, multi-trace operators built using $O(N)$ fields, that are dual to giant graviton states. Excited giants correspond to these large operators doped with impurities. To compute the one-loop anomalous dimensions for these operators, it is no longer sufficient to sum only the planar diagrams. This leads to a breakdown of the spin chain approach for this class of operators. It therefore becomes imperative to develop a new set of tools that are capable of handling this problem. Our approach is to exploit the representation theory of symmetric and unitary groups as well as the relations between them and the operators we study are restricted Schur polynomials.

5.2 Our results

It is possible to deform the $AdS_5 \times S^5$ background on which type IIB string theory sits in order to study giant gravitons on a Lunin-Maldacena background, $AdS_5 \times \tilde{S}^5$, [50]. On the gauge theory side, this amounts to introducing a real deformation parameter (first catalogued by Leigh and Strassler [48]) into the theory. The result is an $\mathcal{N} = 1$ SYM theory from $\mathcal{N} = 4$ SYM theory. This is the subject of Chapter 3 published in [49].

In particular, we computed the spectrum of anomalous dimensions of restricted Schur polynomials in the $SU(2)$ sector of the deformed theory. We found that the action of the dilatation operator factorises into a problem that is associated with the Z fields and a problem associated with the impurities. The problem associated with the impurities was diagonalised by the double coset ansatz of [46]. The problem associated with the Z fields generalised the corresponding problem in the undeformed theory. We managed to write this problem as the Hamiltonian of a shifted harmonic oscillator, thereby signalling integrability in this sector of the deformed theory.

In Chapter 4 published in [39], we explained how to build restricted Schur polynomials that include both fermions and bosons. These new restricted Schur polynomials continue to diagonalise the two point function in the free field limit. The number of these polynomials is equal to the number of multi-trace operators. We also explained how to transform between the trace basis and the basis provided by the Schur polynomials that we constructed. As a concrete application of our results, we studied the $su(2|3)$ sector of $\mathcal{N} = 4$ SYM theory. This sector consists of operators built using two fermions and three bosons and it is closed to all orders under the action of the dilatation operator.

After building the restricted Schur polynomials for the $su(2|3)$ sector, we computed the spectrum of anomalous dimensions in this sector. We found that the action of the dilatation operator again factorised into a problem associated with the Z fields and another associated with the impurities. The problem associated with the Z fields was similar to the one studied to in the $SU(2)$ sector [45]. The impurity problem was solved by a slightly modified version of the double coset ansatz. The resulting spectrum is identical to the spectrum of a set of decoupled oscillators which is a clear indication of integrability in the large N, but non-planar limit of the $su(2|3)$ sector.

In Appendix C, we computed the spectrum of anomalous dimensions in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM theory. This action of the dilatation operator in this sector of the theory is again diagonalised by the double coset ansatz.

5.3 Some open problems

While group representation theory has proved to be a very powerful tool in studying the large N, but non-planar limit of $\mathcal{N} = 4$ SYM theory as well as its deformations, there are questions that we did not answer.

First, we did not compare our results from Chapter 3 with the predictions of the gravity theory. A direct comparison seemed almost sure to fail since the AdS/CFT correspondence is a weak/strong duality. More precisely, the dual gravitational system is defined in the large 't Hooft coupling λ and small deformation parameter γ ($\gamma^2\lambda$ is fixed), while our field theory calculation is valid when λ is small and γ is arbitrary. However, since the quantum numbers of our operators become parametrically large with N, a comparison may still be possible. We left this interesting problem for future research.

Second, there are reasonably small corrections that we dropped in the $su(2|3)$ calculation. We did not check whether they are integrable or not.

Lastly, our results suggest that the double coset ansatz of [46] together with the extension we described in Chapter 4, may diagonalise the complete one-loop dilatation operator. Since the double coset ansatz is a direct result of Gauss's law, we also expect that it (maybe in a modified version) should diagonalise the dilatation operator even when integrability is not present. It would be nice to verify these two points.

5.4 Conclusion

Though this may be a small step, we hope that we have contributed something to the ultimate understanding of quantum gravity. In particular, we hope that our results do shed light on the properties of excited giant gravitons in type IIB string theory. Undoubtedly, more work still remains to be done in order to understand the problem in its entirety.

Appendix A

Proof that C commutes with all $\sigma \epsilon S_n$.

In this appendix we prove that

$$C = \sum_{i>j} (ij)$$

commutes with all $\sigma \epsilon S_n$.

We start by noting that in the symmetric group S_n, C is a sum of

$$\frac{n(n-1)}{2}$$

terms and then consider

$$(ij) \neq (kl).$$

From here on, we can write

$$\sigma (ij) \neq \sigma (kl)$$

and

$$\sigma (ij) \sigma^{-1} \neq \sigma (kl) \sigma^{-1}.$$

It therefore follows that

$$\sum_{i>j} (ij) = \sigma \sum_{i>j} (ij) \sigma^{-1}.$$

Multiplying by σ from the right yields

$$\sum_{i>j} (ij) \sigma = \sigma \sum_{i>j} (ij)$$

which proves that C commutes with $\sigma \epsilon S_n$.

Appendix B

The spin chain

In this Appendix, based on [12, 88], we describe how the anomalous dimensions were computed in the planar limit of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. In particular, we will compute the spectrum of anomalous dimensions in the $su\,(2)$ sector of the theory. The gauge invariant operators that we consider, $\mathcal{O}\,(x)$, are single trace operators. In the large N and planar limit, the spectrum of local operators comes from these single trace operators [88].

B.1 One loop anomalous dimensions

As we mentioned in Chapter 1, the two point function of an operator is given by

$$\left\langle \mathcal{O}\,(x)\,\bar{\mathcal{O}}\,(y)\right\rangle \approx \frac{1}{|x-y|^{2\triangle}}, \tag{B.1}$$

where

$$\triangle = \triangle_0 + \gamma. \tag{B.2}$$

is the conformal dimension, $\triangle_0$ is the classical mass dimension and γ is the anomalous dimension. The anomalous dimension is a quantum correction to the scaling dimension $\triangle_0$. When the Yang-Mills coupling is small, we have $\gamma \ll \triangle_0$. In this case, the two point function (B.1) is approximately given by

$$\left\langle \mathcal{O}\,(x)\,\bar{\mathcal{O}}\,(y)\right\rangle \approx \frac{1}{|x-y|^{2\triangle_0}}\left(1-\gamma \ln \Lambda^2\,|x-y|^2\right), \tag{B.3}$$

where Λ is a cut-off scale.

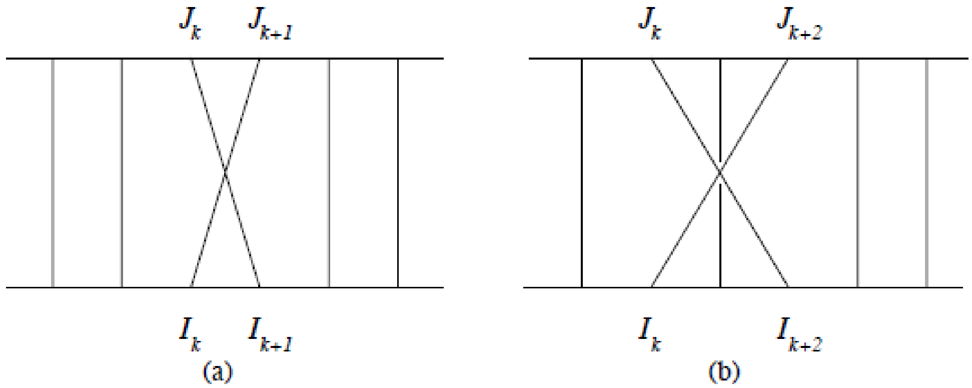

Figure B.1: Planar (a) and non-planar (b) diagrams contributing to the one-loop piece of the two point correlator.

We can write down gauge invariant operators of the form

$$\mathcal{O}_{I_1,I_2,\ldots I_L}(x) = \frac{\left(4\pi^2\right)^{L/2}}{\sqrt{C_{I_1,I_2,\ldots I_L}}N^{L/2}} Tr\left(\phi_{I_1}(x)\,\phi_{I_2}(x)\ldots\phi_{I_L}(x)\right), \tag{B.4}$$

where $C_{I_1,I_2,\ldots I_L}$ is a symmetric factor. For these operators, the leading contribution is

$$\left\langle \mathcal{O}_{I_1,I_2,\ldots I_L}(x)\,\bar{\mathcal{O}}^{J_1,J_2,\ldots J_L}(y)\right\rangle_{tree} = \frac{1}{C_{I_1,I_2,\ldots I_L}}\left(\delta_{I_1}^{J_1}\delta_{I_2}^{J_2}\cdots\delta_{I_L}^{J_L} + cycles\right)\frac{1}{|x-y|^{2L}}. \tag{B.5}$$

To get the one-loop contribution, we need to sum one-loop diagrams whose form is shown in figures B.1 and B.2 (from [88]). In these diagrams, the horizontal lines represent the operators that enter the two point function, while the vertical lines represent the fields that are Wick contracted to get the correlator. Figure B.1 consists of both planar and non-planar diagrams. If $L \ll N$ and $N \to \infty$, we observe two things. First, the number of non-planar diagrams will be much smaller than that of the planar diagrams. Second, the non-planar diagrams will be suppressed by a factor of $1/N^2$ when compared to the planar diagrams. We can therefore drop the non-planar diagrams from our computation. In other words, the one-loop anomalous dimension can be determined by summing the planar diagrams only.

Unlike the diagrams in figure B.1, the gluon diagrams shown in figure B.2 do not mix the index structures. They give the same index structures as the free theory diagrams and are therefore easy to compute.[1] Denoting the contribution from these diagrams by

[1] This is because the R-charge is conserved and gluons do not have an R-charge [88].

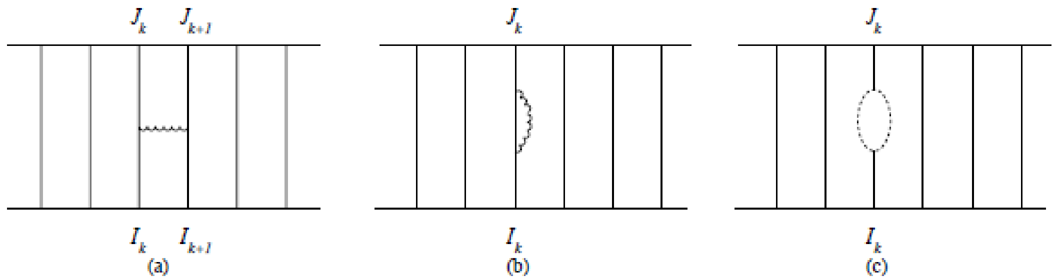

Figure B.2: One loop diagrams that do not change the index structures.

a constant C, the one-loop contribution can be written as [88]

$$\left\langle \mathcal{O}_{I_1,I_2,\ldots I_L}(x)\,\bar{\mathcal{O}}^{J_1,J_2,\ldots J_L}(y)\right\rangle_{one-loop} = \frac{\lambda}{16\pi^2}\frac{\ln\left(\Lambda^2|x-y|^2\right)}{|x-y|^{2L}}\sum_{l=1}^{L}\left(2P_{l,l+1}-K_{l,l+1}-1+C\right) \times \frac{1}{\sqrt{C_{I_1,\ldots I_L}C_{J_1,\ldots J_L}}}\delta_{I_1}^{J_1}\delta_{I_2}^{J_2}\cdots\delta_{I_L}^{J_L} + cycles, \tag{B.6}$$

where $P_{l,l+1}$ is an exchange operator and $K_{l,l+1}$ is a trace operator. Acting on the delta-functions in equation (B.6), $P_{l,l+1}$ exchanges the indices on the l and the $l+1$ sites inside the trace, i.e.

$$P_{l,l+1}\delta_{I_1}^{J_1}\cdots\delta_{I_l}^{J_l}\delta_{I_{l+1}}^{J_{l+1}}\cdots\delta_{I_L}^{J_L} = \delta_{I_1}^{J_1}\cdots\delta_{I_l}^{J_{l+1}}\delta_{I_{l+1}}^{J_l}\cdots\delta_{I_L}^{J_L}, \tag{B.7}$$

while $K_{l,l+1}$ contracts the indices of neighbouring fields, i.e.

$$K_{l,l+1}\delta_{I_1}^{J_1}\cdots\delta_{I_l}^{J_l}\delta_{I_{l+1}}^{J_{l+1}}\cdots\delta_{I_L}^{J_L} = \delta_{I_1}^{J_1}\cdots\delta_{I_lI_{l+1}}\delta^{J_lJ_{l+1}}\cdots\delta_{I_L}^{J_L}. \tag{B.8}$$

These two operators result in operator mixing at the one-loop level.

Adding (B.5) and (B.6) we get

$$\left\langle \mathcal{O}_{I_1,I_2,\ldots I_L}(x)\,\bar{\mathcal{O}}^{J_1,J_2,\ldots J_L}(y)\right\rangle = \frac{1}{|x-y|^{2L}} \times\left(1-\frac{\lambda}{16\pi^2}\ln\left(\Lambda^2|x-y|^2\right)\sum_{l=1}^{L}\left(1-C-2P_{l,l+1}+K_{l,l+1}\right)\right) \times\delta_{I_1}^{J_1}\delta_{I_2}^{J_2}\cdots\delta_{I_L}^{J_L} + cycles. \tag{B.9}$$

Comparing this result with equation (B.3), we see that the anomalous dimension γ has

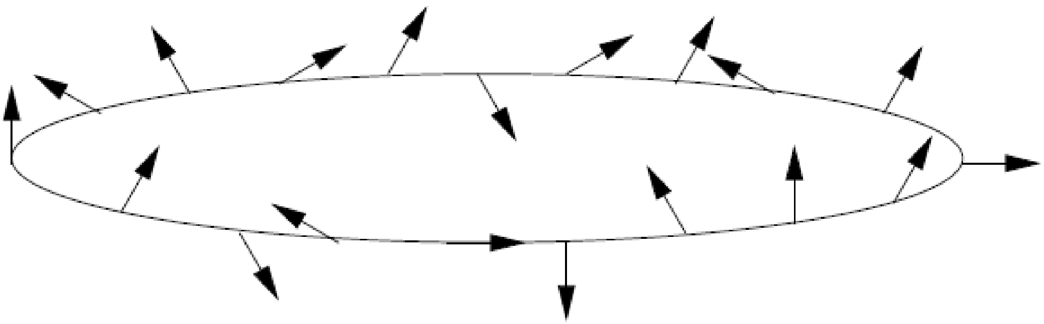

Figure B.3: A spin chain with $SO(6)$ vector sites.

been replaced by the operator

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(1 - C - 2P_{l,l+1} + K_{l,l+1}\right). \tag{B.10}$$

To obtain the one-loop anomalous dimensions, we therefore need to diagonalise Γ.

B.2 Relation to spin chain

The whole class of scalar single trace operators of length L can be mapped to a Hilbert space which is a tensor product of finite dimensional Hilbert spaces

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \mathcal{V}_l \otimes \cdots \mathcal{V}_L, \tag{B.11}$$

where each $\mathcal{V}_l$ is a Hilbert space for an $SO(6)$ vector representation. The Hilbert space (B.11) is the same as the Hilbert space of a one-dimensional spin-chain with L sites shown in figure B.3 (from [88]). At each site in figure B.3, we have an $SO(6)$ vector spin.

Since the trace is cyclic, the Hilbert space (B.11) must be invariant under the shift

$$\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \mathcal{V}_l \otimes \cdots \mathcal{V}_L \to \mathcal{V}_L \otimes \mathcal{V}_1 \otimes \cdots \mathcal{V}_l \otimes \cdots \mathcal{V}_{L-1}. \tag{B.12}$$

Γ acts linearly in this space,

$$\Gamma : \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \mathcal{V}_l \otimes \cdots \mathcal{V}_L \to \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \mathcal{V}_l \otimes \cdots \mathcal{V}_L. \tag{B.13}$$

Also, Γ is Hermitian and commutes with the shift (B.12). Putting this together, we see that Γ can be treated as a Hamiltonian of the spin chain, with the energy eigenstates corresponding to the possible anomalous dimensions for the scalar operators. The Hamiltonian commutes with the shift, so that we can project onto eigenstates that are invariant under the shift. Since the operators $P_{l,l+1}$ and $K_{l,l+1}$ act on neighbouring fields, the Hamiltonian of the spin chain only has nearest neighbour interactions between the spins.

B.2.1 Determining C

We now compute the value of C by using the properties of BPS operators. Consider the chiral primary (BPS) operator[2]

$$\Psi_L = \frac{\left(4\pi^2\right)^{L/2}}{\sqrt{L}N^{L/2}} Tr\left(Z^L\right) \tag{B.14}$$

which is symmetric under the exchange of any fields. The exchange operator acting on (B.14) retains Ψ_L, i.e.

$$P_{l,l+1}\Psi_L = \Psi_L \tag{B.15}$$

for all l. This operator, equation (B.14), contains only Z fields, i.e. there are no $\bar{Z}$ fields. Therefore, the trace operator gives

$$K_{l,l+1}\Psi_L = 0. \tag{B.16}$$

Putting this together, we find

$$\Gamma\Psi_L = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1 - C - 2\right)\Psi_L. \tag{B.17}$$

Now, as we mentioned in Chapter 1, the scaling dimensions of BPS operators are protected by supersymmetry. This implies that

$$1 - C - 2 = 0 \tag{B.18}$$

in equation (B.17). It therefore follows that $C = -1$. Putting this into equation (B.10), we find

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1 - P_{l,l+1} + \frac{1}{2}K_{l,l+1}\right). \tag{B.19}$$

[2] The normalisation is chosen for later convenience.

B.2.2 In terms of projectors

It is convenient to write the Hamiltonian (B.19) in terms of projectors. We use the fact that the tensor product of two $SO(6)$ vector representations is reducible into the traceless symmetric, the antisymmetric and the singlet representations. The operators that project $\mathcal{V}_l \otimes \mathcal{V}_{l+1}$ onto these representations are

$$\prod_{l.l+1}^{sym} = \frac{1}{2}(1 + P_{l,l+1}) - \frac{1}{6}K_{l,l+1}, \tag{B.20}$$

$$\prod_{l,l+1}^{as} = \frac{1}{2}(1 - P_{l,l+1}) \tag{B.21}$$

and

$$\prod_{l,l+1}^{sing} = \frac{1}{6}K_{l,l+1}. \tag{B.22}$$

In terms of these, we can write equation (B.19) as

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(0\prod_{l.l+1}^{sym} + 2\prod_{l,l+1}^{as} + 3\prod_{l,l+1}^{sing}\right). \tag{B.23}$$

We see here that only two of the three projectors contribute to Γ.

B.2.3 Comments

The Hamiltonian that corresponds to Γ for the spin chain is integrable [89], meaning that it can be solved [88], at least in principle. Also, when one goes beyond the first loop, the n-loop contribution to the anomalous dimension can involve up to n neighbouring fields in an effective Hamiltonian [90, 91, 92]. As the coupling λ grows bigger, these longer range interactions become increasingly important. At strong coupling, the spin chain becomes effectively long range and the Hamiltonian is not known above the first two loop orders [90, 91, 93].

B.3 The $su(2)$ sector

Thus far, we have mapped the one-loop dilatation operator to the Hamiltonian of an $SO(6)$ spin chain. We are now in a position to compute the anomalous dimensions of the $O(1)$ single trace operators in the $su(2)$ sector of SYM theory. As mentioned in Chapter 1, this sector, consists of two scalar fields, Z and Y say. These fields transform under a doublet of $SU(2)$ so that we can label the Z field as spin up ($\uparrow$) and the Y

field as spin down $(\downarrow)$. We note that there are no conjugate fields in the operator - we only have Z and Y fields - so that the contribution from $K_{l,l+1}$ is zero. Using this, the Hamiltonian (B.19) becomes

$$\Gamma_{su(2)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1 - P_{l,l+1}\right). \tag{B.24}$$

Equivalently, we can write this in terms of spin operators. We have

$$\Gamma_{su(2)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(\frac{1}{2} - 2\vec{S}_l \cdot \vec{S}_{l+1}\right), \tag{B.25}$$

which is the Hamiltonian of a Heisenberg spin chain with L lattice sites. The total spin

$$\vec{S} = \sum_{l}\vec{S}_l \tag{B.26}$$

commutes with Γ so that the energy eigenstates are also total spin eigenstates [88].

Since the $\vec{S}_l \cdot \vec{S}_{l+1}$ term has a negative sign, the corresponding spin chain is ferromagnetic. The ground state of a ferromagnet has all the spins are aligned, with a total spin of $L/2$ (for L spins). This representation is symmetric and corresponds to the chiral primary operator. In this case, the energy of the Hamiltonian (B.25) is zero. To get non-chiral primary operators, we need to excite the spin chain about its ground state. The total spin of these operators is less than $L/2$ [89]. We now give a partial description of how to get these other states using the S-matrix approach [94, 95].

B.3.1 Single magnon state

To start with, we write the ground state as $|\uparrow\uparrow\uparrow\cdots\uparrow\uparrow\rangle$, corresponding to the chiral primary operator (B.14). The simplest excited state has one spin pointing down. In this case, the Hamiltonian (B.24) acts like a constant plus a hopping term that moves the down spin (magnon) one site to the left or right. If the magnon is at a particular position l, then (B.24) has the action

$$\begin{aligned}\Gamma_{su(2)}\left|\uparrow\cdots\uparrow\overset{l}{\downarrow}\uparrow\cdots\uparrow\uparrow\right\rangle = \frac{\lambda}{8\pi^2}\Bigg(2\left|\uparrow\cdots\uparrow\overset{l}{\downarrow}\uparrow\cdots\uparrow\uparrow\right\rangle - \left|\uparrow\cdots\overset{l-1}{\downarrow}\uparrow\uparrow\cdots\uparrow\uparrow\right\rangle \\ - \left|\uparrow\cdots\uparrow\uparrow\overset{l+1}{\downarrow}\cdots\uparrow\uparrow\right\rangle\Bigg).\end{aligned} \tag{B.27}$$

We can define the eigenstates

$$|p\rangle \equiv \frac{1}{\sqrt{L}}\sum_{l=1}^{L} e^{ipl} \left| \uparrow\uparrow \cdots \overset{l}{\downarrow} \cdots \uparrow\uparrow \right\rangle \tag{B.28}$$

known as single magnon states with momentum p. In terms of these eigenstates, we have

$$\Gamma_{su(2)} |p\rangle = \varepsilon(p) |p\rangle \tag{B.29}$$

where

$$\varepsilon(p) = \frac{\lambda}{2\pi^2} \sin^2\left(\frac{p}{2}\right) \tag{B.30}$$

are the eigenvalues. We can quantise the dispersion $\varepsilon(p)$ as well as the magnon momentum p so that $|p\rangle$ is invariant under the shift $l \to l + L$. Thus

$$p = \frac{2\pi n}{L}. \tag{B.31}$$

The symmetric state has $n = 0$ and total spin $L/2$. The other cases have total spin $L/2 - 1$. Since the trace is cyclic, our states must be invariant under the shift $l \to l + 1$. This means that the only allowed state has $p = 0$. For this state, there are no chiral primary operators with only one Y field [88].

B.3.2 Two-magnon state

Let us now consider a two-magnon state which we construct using an argument that was first presented by Yang and Yang [96]. Given at least two down spins in our operator, it is possible to have excited state that satisfies the trace condition. The argument we follow considers first, an finite spin - instead of a closed one. An unnormalised two-magnon state that we can write down is

$$|p_1, p_2\rangle = \sum_{l_1 < l_2} e^{ip_1 l_1 + ip_2 l_2} \left| \cdots \overset{l_1}{\downarrow} \cdots \overset{l_2}{\downarrow} \cdots \right\rangle + e^{i\phi} \sum_{l_1 > l_2} e^{ip_1 l_1 + ip_2 l_2} \left| \cdots \overset{l_1}{\downarrow} \cdots \overset{l_2}{\downarrow} \cdots \right\rangle, \tag{B.32}$$

where we have assumed that $p_1 > p_2$. Equation (B.32) can be thought of as the scattering state of two magnons, with the first term describing the incoming part, while the second term describes the outgoing part. For this scattering process, the S-matrix - which we denote S_{12} - is given by the phase $e^{i\phi}$. When the two magnons are well separated, i.e. $|l_1 - l_2| \gg 1$, they do not interact with each other. If $|p_1, p_2\rangle$ is an eigenstate of $\Gamma_{su(2)}$, the corresponding eigenvalue is a sum of the two non-interacting magnon states with magnon momenta p_1 and p_2. By considering all the possible ways of placing the two magnons next to each other at sites l and $l + 1$, we find that in order to have an eigenstate, we must satisfy

$$e^{ip_2}\left(2-e^{-ip_1}-e^{ip_2}\right)+e^{ip_1}\left(2-e^{ip_1}-e^{-ip_2}\right)$$
$$=\left(4-e^{-ip_1}-e^{ip_1}-e^{-ip_2}-e^{ip_2}\right)\left(e^{ip_2}+e^{ip_1}e^{i\phi}\right). \tag{B.33}$$

The solution to this equation is

$$e^{i\phi}=S_{12}=-\frac{e^{ip_1+ip_2}-2e^{ip_2}+1}{e^{ip_1+ip_2}-2e^{ip_1}+1}. \tag{B.34}$$

We now consider a closed spin chain of length L. In this case, the cyclicity of the trace means that the total momentum must be

$$p_1+p_2=0. \tag{B.35}$$

Now, transporting one magnon around the circle results in the same state. However, since this process takes the first magnon past the second one, the first magnon picks up a phase $e^{i\phi}$. If the magnon that we transported has momentum p_1, we have

$$e^{ip_1L}e^{i\phi}=1. \tag{B.36}$$

Using equation (B.35) we get

$$e^{i\phi}=e^{-ip_1} \tag{B.37}$$

so that the allowed values of p_1 are

$$p_1=\frac{2\pi n}{L-1}. \tag{B.38}$$

Therefore, the two magnon state has eigenvalues

$$\gamma=\frac{\lambda}{\pi^2}\sin^2\frac{n\pi}{L-1}. \tag{B.39}$$

Again, the case $n=0$ corresponds to the symmetric case with spin $L/2$. All other values of n have spin $L/2-2$.

B.3.3 M magnons and Bethe equations

It is convenient to define the rapidity u such that

$$e^{i\phi}=\frac{u+i/2}{u-i/2}. \tag{B.40}$$

The dispersion relation then becomes

$$\varepsilon(u) = \frac{\lambda}{8\pi^2}\frac{1}{u^2 + 1/4} \tag{B.41}$$

and the S-matrix for magnons with rapidity u_j and u_k is

$$S_{jk} = \frac{u_j - u_k - i}{u_j - u_k + i}. \tag{B.42}$$

For M magnons with momenta $p_1 > p_2 > \cdots p_M$, we have

$$|p_1, p_2, \ldots p_M\rangle = \sum_{l_1 < l_2 \cdots l_M} e^{ip_1 l_1 + ip_2 l_2 + \cdots ip_M l_M} \left| \cdots \overset{l_1}{\downarrow} \cdots \overset{l_2}{\downarrow} \cdots \overset{l_M}{\downarrow} \cdots \right\rangle + \cdots, \tag{B.43}$$

where the last set of dots denotes all the other possible orderings of the magnons with appropriate phase factors. These phase factors are products of the two particle S-matrices [88], implying that the system is integrable. On a circle with lattice sites L, the quantisation condition for the j^{th} magnon is

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{B.44}$$

The state has energy

$$\gamma = \sum_{j=1}^{M} \varepsilon(u_j) \tag{B.45}$$

where $\varepsilon(u_j)$ is given by equation (B.41). The trace condition for the total momentum is

$$\prod_{j=1}^{M} \frac{u_j + i/2}{u_j - i/2} = 1. \tag{B.46}$$

Equations (B.44) are known as Bethe equations for the Heisenberg spin chain [97]. For further solutions to these equations, the reader is referred to [89, 98, 99]. The spin-chain technique in the planar large N limit can be generalised to other sectors of SYM theory including the full $PSU(2,2|4)$. These generalisations are discussed in [100] and reviewed in [101, 102, 103].

Appendix C

The $sl\left(2\right)$ sector[1]

In this Appendix, we diagonalise the action of the dilatation operator in the $sl\left(2\right)$ sector using the double coset ansatz [46]. This sector was first written in the restricted Schur polynomial basis in [38]. The operators for this sector are built using n Z fields and m vector impurities, i.e. we have m covariant derivatives D_+ that act on the n Z fields. These operators do not mix with other operators under the action of the dilatation operator. In other words, they form the closed $sl\left(2\right)$ subsector [104]. The impurities are $Z^{(i)}$ with $i = 0, 1, 2, \dots m$, where

$$Z^{(n)} = \frac{1}{n!} D_+^n Z, \tag{C.1}$$

$$Z^{(n)\dagger} = \frac{1}{n!} D_-^n Z^\dagger \tag{C.2}$$

and $Z^{(0)} \equiv Z$. Denoting the number of $Z^{(i)}$ by n_i, the restricted Schur polynomial is

$$\chi_{R,\{r_i\}\alpha\beta}\left(Z^{(0)}, Z^{(1)}, \dots Z^{(m)}\right) = \prod_{k=0}^{M} \frac{1}{n_k!} \sum_{\sigma\in S_{n_Z}} \chi_{R,\{r_i\}\alpha\beta}\left(\sigma\right) Tr\left(\sigma \prod_{j=0}^{m}\left(Z^{(j)}\right)^{\otimes n_j}\right). \tag{C.3}$$

The label $\{r_i\}\,\alpha\beta$ specifies an irreducible representation of $S_{n_0} \times S_{n_1} \times \cdots S_{n_m}$. It consists of less than m Young diagrams $\{r_i\}$ and a pair of multiplicity labels $\alpha\beta$. As before, a given $S_{n_0} \times S_{n_1} \times \cdots S_{n_m}$ irrep can be subduced more than once: the multiplicity labels therefore tell us which of the degenerate copies are being used by the restricted character $\chi_{R,\{r_i\}\alpha\beta}\left(\sigma\right)$. The free two point function that follows from (C.3) is

$$\left\langle \chi_{R,\{r_i\}\alpha\beta}\left(P\right) \chi^\dagger_{S,\{s_j\}\delta\gamma}\left(Q\right)\right\rangle = \delta_{RS}\delta_{\{r_i\}\{s_j\}}\delta_{\alpha\gamma}\delta_{\beta\delta}\frac{hooks_R}{hooks_{\{r_i\}}} f_R. \tag{C.4}$$

[1]This work was published in [39] - it is my original work.

The delta function $\delta_{\{r_i\}\{s_j\}}$ is 1 if the two $S_{n_0} \times S_{n_1} \times \cdots S_{n_m}$ irreps specified by $\{r_i\}$ and $\{s_j\}$ are identical. The corresponding multiplicity labels must also match [36]. The action of the dilatation operator in this sector then becomes

$$D\chi_{R,(r,s)\alpha\beta}\left(Z, Z^{(q)}\right) = \sum_{S,(t,u)\gamma\delta} M_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma}\chi_{S,(t,u)\delta\gamma}, \tag{C.5}$$

where

$$M_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma} = \frac{1}{q}M^{SU(2)}_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma} + \delta M_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma}. \tag{C.6}$$

Here, $M^{SU(2)}_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma}$ is identical to the usual action of the dilatation operator in the $SU(2)$ sector. However, we notice that we pick up a correction

$$\begin{aligned}\delta M_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma} = {} & g_{YM}^2\left(\frac{1}{q} - \sum_{i=1}^{q}\frac{1}{i}\right)\delta_{RS}\delta_{(r,s)(t,u)}\frac{nm}{d_r d_s} \\ & \times\left(\delta_{\alpha\delta}\chi_{R,(r,s)\beta\gamma}\left((1,m+1)\right) + \delta_{\beta\gamma}\chi_{R,(r,s)\alpha\delta}\left((1,m+1)\right)\right) \\ & - g_{YM}^2\left(\frac{1}{q} - \sum_{i=1}^{q}\frac{1}{i}\right)\sum_{R'}\frac{c_{RR'}d_S nm}{d_t d_u (n+m) d_{R'}} \\ & \times [Tr\left(I_{S'R'}P_{R\to(r,s)\alpha\beta}\left(1,m+1\right)I_{R'S'}\left(1,m+1\right)P_{S\to S,(t,u)\delta\gamma}\right) \\ & + Tr\left(I_{S'R'}\left(1,m+1\right)P_{R\to(r,s)\alpha\beta}I_{R'S'}P_{S\to S,(t,u)\delta\gamma}\left(1,m+1\right)\right)].\end{aligned} \tag{C.7}$$

Since $M^{SU(2)}_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma}$ is the usual action of the dilatation operator in the $SU(2)$ sector, we know that moving to the Gauss graph basis will diagonalise $M^{SU(2)}_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma}$ on its impurity labels, leaving only the problem considered in [45]. Denoting the piece of the dilatation operator that leads to $\delta M_{R,(r,s)\alpha\beta;S,(t,u)\delta\gamma}$ by δD, we find that in the Gauss graph basis we have

$$\delta D O_{R,r}(\sigma) = 2g_{YM}^2 Nm\left(\sum_{i=1}^{q}\frac{1}{i} - \frac{1}{q}\right)O_{R,r}(\sigma). \tag{C.8}$$

Thus the double coset ansatz diagonalises the one loop dilatation operator in the $sl(2)$ sector.

Bibliography

[1] J.M. Maldacena, *"The Large N Limit of Superconformal Field Theories and Supergravity"*, Adv.Theor.Math.Phys **2**, 231 (1998), `hep-th/9711200`.

[2] E. Witten, *"Anti-de Sitter space and holography"*, Adv.Theor.Math.Phys. **2**, 253 (1998), `hep-th/9802150`.

[3] S. Gubser, I.R. Klebanov & A.M. Polyakov, *"Gauge theory correlators from non-critical string theory"*, Phys.Lett. **B428**, 105 (1998), `hep-th/9802109`.

[4] O. Aharony, O. Bergman, D.L. Jafferis & J. Maldacena, *"N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals"*, JHEP **0810**, 091 (2008), `arXiv:0806.1218`.

[5] P. Di Francesco, P. Mathieu & D. Senechal, *"Conformal Field Theory"*, Springer (1997).

[6] C.V. Johnson, *"D-Branes"*, Cambridge University Press (2003).

[7] K. Becker, M. Becker & J.H. Schwarz, *"String Theory and M-Theory: A Modern Introduction"*, Cambridge University Press (2007).

[8] M.E. Peskin & D.V. Schroeder, *"An Introduction to Quantum Field Theory"*, Westview Press (1995).

[9] S. Carroll, *"Spacetime and Geometry: An Introduction to General Relativity"*, Addison Wesley (2004).

[10] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri & Y. Oz, *"Large N field theories, string theory and gravity"*, Phys.Rept. **323**, 183 (2000), `hep-th/9905111`.

[11] E. Witten, *"Bound states of strings and p-branes"*, Nucl.Phys. **B460**, 335 (1996), `hep-th/9510135`.

[12] J.A. Minahan, *"A brief introduction to the Bethe ansatz in N=4 super-Yang-Mills"*, Journal of Physics A: Mathematical and General **39**, 12657 (2006), `http://stacks.iop.org/0305-4470/39/i=41/a=S02`.

[13] P.K. Townsend, *"Cosmological constant in supergravity"*, Phys. Rev. D **15**, 2802 (1977), http://link.aps.org/doi/10.1103/PhysRevD.15.2802.

[14] W. Nahm, *"Supersymmetries and their representations"*, Nuclear Physics B **135**, 149 (1978), http://www.sciencedirect.com/science/article/pii/0550321378902183.

[15] M. Pernici, K. Pilch & P. van Nieuwenhuizen, *"Gauged $N = 8$ $d = 5$ supergravity"*, Nuclear Physics B **259**, 460 (1985), http://www.sciencedirect.com/science/article/pii/0550321385906455.

[16] M. Gunaydin, L. Romans & N. Warner, *"Compact and non-compact gauged supergravity theories in five dimensions"*, Nuclear Physics B **272**, 598 (1986), http://www.sciencedirect.com/science/article/pii/0550321386902373.

[17] J. McGreevy, L. Susskind & N. Toumbas, *"Invasion of the giant gravitons from Anti-de Sitter space"*, JHEP **0006**, 008 (2000), hep-th/0003075.

[18] M.T. Grisaru, R.C. Myers & O. Tafjord, *"SUSY and Goliath"*, JHEP **0008**, 040 (2000), hep-th/0008015.

[19] A. Hashimoto, S. Hirano & N. Itzhaki, *"Large branes in AdS and their field theory dual"*, JHEP **0008**, 051 (2000), hep-th/0008016.

[20] M. Sheikh-Jabbari, *"Open strings in a B field background as electric dipoles"*, Phys.Lett. **B455**, 129 (1999), hep-th/9901080.

[21] D. Bigatti & L. Susskind, *"Magnetic fields, branes and noncommutative geometry"*, Phys.Rev. **D62**, 066004 (2000), hep-th/9908056.

[22] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov, R.A. Janik, V. Kazakov, T. Klose, G.P. Korchemsky, C. Kristjansen, M. Magro, T. McLoughlin, J.A. Minahan, R.I. Nepomechie, A. Rej, R. Roiban, S. Schafer-Nameki, C. Sieg, M. Staudacher, A. Torrielli, A.A. Tseytlin, P. Vieira, D. Volin & K. Zoubos, *"Review of AdS/CFT Integrability: An Overview"*, Lett. Math. Phys. **99,**, 3 (2010), arXiv:1012.3982.

[23] T. Klose, *"Review of AdS/CFT Integrability, Chapter IV.3: N=6 Chern-Simons and Strings on AdS4xCP3"*, Lett.Math.Phys. **99**, 401 (2012), arXiv:1012.3999.

[24] D.E. Berenstein, J.M. Maldacena & H.S. Nastase, *"Strings in flat space and pp waves from N=4 superYang-Mills"*, JHEP **0204**, 013 (2002), hep-th/0202021.

[25] V. Balasubramanian, M. Berkooz, A. Naqvi & M.J. Strassler, *"Giant gravitons in conformal field theory"*, JHEP **0204**, 034 (2002), hep-th/0107119.

[26] S. Corley, A. Jevicki & S. Ramgoolam, *"Exact correlators of giant gravitons from dual N=4 SYM theory"*, Adv.Theor.Math.Phys. **5**, 809 (2002), hep-th/0111222.

[27] D. Berenstein, *"Shape and holography: Studies of dual operators to giant gravitons"*, Nucl.Phys. **B675**, 179 (2003), `hep-th/0306090`.

[28] H. Lin, O. Lunin & J.M. Maldacena, *"Bubbling AdS space and 1/2 BPS geometries"*, JHEP **0410**, 025 (2004), `hep-th/0409174`.

[29] D. Berenstein, *"A Toy model for the AdS / CFT correspondence"*, JHEP **0407**, 018 (2004), `hep-th/0403110`.

[30] C. Kristjansen, *"Review of AdS/CFT Integrability, Chapter IV.1: Aspects of Non-Planarity"*, `arXiv:1012.3997`.

[31] S. Corley & S. Ramgoolam, *"Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory"*, Nucl.Phys. **B641**, 131 (2002), `hep-th/0205221`.

[32] V. Balasubramanian, D. Berenstein, B. Feng & M.x. Huang, *"D-branes in Yang-Mills theory and emergent gauge symmetry"*, JHEP **0503**, 006 (2005), `hep-th/0411205`.

[33] R. de Mello Koch, J. Smolic & M. Smolic, *"Giant Gravitons - with Strings Attached (I)"*, JHEP **0706**, 074 (2007), `hep-th/0701066`.

[34] R. de Mello Koch, J. Smolic & M. Smolic, *"Giant Gravitons - with Strings Attached (II)"*, JHEP **0709:049,2007**, (2007), `hep-th/0701067`.

[35] D. Bekker, R. de Mello Koch & M. Stephanou, *"Giant Gravitons - with Strings Attached (III)"*, JHEP **0802:029,2008**, (2007), `arXiv:0710.5372`.

[36] R. Bhattacharyya, S. Collins & R. de Mello Koch, *"Exact Multi-Matrix Correlators"*, JHEP **0803:044,2008**, (2008), `arXiv:0801.2061`.

[37] R. Bhattacharyya, R. de Mello Koch & M. Stephanou, *"Exact Multi-Restricted Schur Polynomial Correlators"*, JHEP **0806:101,2008**, (2008), `arXiv:0805.3025`.

[38] R. de Mello Koch, P. Diaz & H. Soltanpanahi, *"Non-planar Anomalous Dimensions in the sl(2) Sector"*, Physics Letters B **713**, 509 (2012), `arXiv:1111.6385`.

[39] R. de Mello Koch, P. Diaz & N. Nokwara, *"Restricted Schur Polynomials for Fermions and integrability in the su(2|3) sector"*, JHEP **1303**, 173 (2013), `arXiv:1212.5935`.

[40] R. de Mello Koch, G. Kemp, B.A.E. Mohammed & S. Smith, *"Nonplanar integrability at two loops"*, JHEP **1210**, 144 (2012), `arXiv:1206.0813`.

[41] R. de Mello Koch, G. Mashile & N. Park, *"Emergent Threebrane Lattices"*, Phys.Rev. **D81**, 106009 (2010), `arXiv:1004.1108`.

[42] V. De Comarmond, R. de Mello Koch & K. Jefferies, *"Surprisingly Simple Spectra"*, JHEP **1102**, 006 (2011), `arXiv:1012.3884`.

[43] W. Carlson, R. de Mello Koch & H. Lin, *"Nonplanar Integrability"*, JHEP **1103:105,2011**, (2011), `arXiv:1101.5404`.

[44] R. de Mello Koch, M. Dessein, D. Giataganas & C. Mathwin, *"Giant Graviton Oscillators"*, JHEP **1110**, 009 (2011), `arXiv:1108.2761`.

[45] R. de Mello Koch, G. Kemp & S. Smith, *"From Large N Nonplanar Anomalous Dimensions to Open Spring Theory"*, Phys.Lett. **B711**, 398 (2012), `arXiv:1111.1058`.

[46] R. de Mello Koch & S. Ramgoolam, *"A double coset ansatz for integrability in AdS/CFT"*, JHEP **1206**, 083 (2012), `arXiv:1204.2153`.

[47] R. de Mello Koch, B.A.E. Mohammed, J. Murugan & A. Prinsloo, *"Beyond the Planar Limit in ABJM"*, JHEP **1205**, 037 (2012), `arXiv:1202.4925`.

[48] R.G. Leigh & M.J. Strassler, *"Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory"*, Nucl.Phys. **B447**, 95 (1995), `hep-th/9503121`.

[49] R. de Mello Koch, J. Murugan & N. Nokwara, *"Large N anomalous dimensions for large operators in Leigh-Strassler deformed SYM"*, Phys.Lett. **B721**, 164 (2013), `arXiv:1212.6624`.

[50] O. Lunin & J. Maldacena, *"Deforming field theories with* $U(1) \times U(1)$ *global symmetry and their gravity duals"*, JHEP **0505**, 033 (2005), `hep-th/0502086`.

[51] S. Frolov, *"Lax pair for strings in Lunin-Maldacena background"*, JHEP **0505**, 069 (2005), `hep-th/0503201`.

[52] N. Beisert, *"The su(2|3) dynamic spin chain"*, Nucl.Phys. **B682**, 487 (2004), `hep-th/0310252`.

[53] J. Stefanski, B. & A.A. Tseytlin, *"Super spin chain coherent state actions and AdS(5) x S**5 superstring"*, Nucl.Phys. **B718**, 83 (2005), `hep-th/0503185`.

[54] N. Beisert & M. Staudacher, *"The N=4 SYM integrable super spin chain"*, Nucl.Phys. **B670**, 439 (2003), `hep-th/0307042`.

[55] W. Burnside, *"Theory of Groups of Finite Order"*, Cambridge University Press (1897).

[56] W. Fulton & J. Harris, *"Theory of Group Representations and Applications"*, Springer Verlag (1991).

[57] M. Pirrone, *"Giants On Deformed Backgrounds"*, JHEP **0612**, 064 (2006), `hep-th/0609173`.

[58] R.C. Myers, *"Dielectric branes"*, JHEP **9912**, 022 (1999), `hep-th/9910053`.

[59] M. Born & L. Infeld, *"Foundations of the New Field Theory"*, Proc. R. Soc. Lond. A. **144**, 425 (1934).

[60] P.A.M. Dirac, *"An Extensible Model of the Electron"*, Proc. R. Soc. Lond. A. **268**, 57 (1962).

[61] M. Li, *"Boundary states of D-branes and Dy strings"*, Nucl.Phys. **B460**, 351 (1996), `hep-th/9510161`.

[62] M.R. Douglas, *"Branes within branes"*, `hep-th/9512077`.

[63] M.B. Green, J.A. Harvey & G.W. Moore, *"I-brane inflow and anomalous couplings on d-branes"*, Class.Quant.Grav. **14**, 47 (1997), `hep-th/9605033`.

[64] R. Roiban, *"On spin chains and field theories"*, JHEP **0409**, 023 (2004), `hep-th/0312218`.

[65] D. Berenstein & S.A. Cherkis, *"Deformations of N=4 SYM and integrable spin chain models"*, Nucl.Phys. **B702**, 49 (2004), `hep-th/0405215`.

[66] N. Beisert & R. Roiban, *"Beauty and the twist: The Bethe ansatz for twisted N=4 SYM"*, JHEP **0508**, 039 (2005), `hep-th/0505187`.

[67] S. Frolov, R. Roiban & A.A. Tseytlin, *"Gauge-string duality for superconformal deformations of N=4 super Yang-Mills theory"*, JHEP **0507**, 045 (2005), `hep-th/0503192`.

[68] R. de Mello Koch & R. Gwyn, *"Giant Graviton Correlators from Dual SU(N) super Yang-Mills Theory"*, JHEP **0411**, 081 (2004), `hep-th/0410236`.

[69] A. Hamilton & J. Murugan, *"Giant Gravitons on Deformed pp-waves"*, JHEP **0706**, 036 (2007), `hep-th/0609135`.

[70] S. Gubser, I. Klebanov & A.M. Polyakov, *"A Semiclassical limit of the gauge / string correspondence"*, Nucl.Phys. **B636**, 99 (2002), `hep-th/0204051`.

[71] S. Frolov & A.A. Tseytlin, *"Semiclassical quantization of rotating superstring in AdS(5) x S**5"*, JHEP **0206**, 007 (2002), `hep-th/0204226`.

[72] S. Frolov & A.A. Tseytlin, *"Multispin string solutions in AdS(5) x S**5"*, Nucl.Phys. **B668**, 77 (2003), `hep-th/0304255`.

[73] S. Frolov & A.A. Tseytlin, *"Quantizing three spin string solution in AdS(5) x S**5"*, JHEP **0307**, 016 (2003), `hep-th/0306130`.

[74] Y. Kimura & S. Ramgoolam, *"Branes, anti-branes and brauer algebras in gauge-gravity duality"*, JHEP **0711**, 078 (2007), `arXiv:0709.2158`.

[75] T.W. Brown, P. Heslop & S. Ramgoolam, *"Diagonal multi-matrix correlators and BPS operators in N=4 SYM"*, JHEP **0802**, 030 (2008), `arXiv:0711.0176`.

[76] T.W. Brown, P. Heslop & S. Ramgoolam, *"Diagonal free field matrix correlators, global symmetries and giant gravitons"*, JHEP **0904**, 089 (2009), `arXiv:0806.1911`.

[77] J. Pasukonis & S. Ramgoolam, *"From counting to construction of BPS states in N=4 SYM"*, JHEP **1102**, 078 (2011), `arXiv:1010.1683`.

[78] Y. Kimura & S. Ramgoolam, *"Enhanced symmetries of gauge theory and resolving the spectrum of local operators"*, Phys.Rev. **D78**, 126003 (2008), `arXiv:0807.3696`.

[79] Y. Kimura, *"Non-holomorphic multi-matrix gauge invariant operators based on Brauer algebra"*, JHEP **0912**, 044 (2009), `arXiv:0910.2170`.

[80] Y. Kimura, *"Quarter BPS classified by Brauer algebra"*, JHEP **1005**, 103 (2010), `arXiv:1002.2424`.

[81] J. Pasukonis & S. Ramgoolam, *"Quantum states to brane geometries via fuzzy moduli spaces of giant gravitons"*, JHEP **1204**, 077 (2012), `arXiv:1201.5588`.

[82] Y. Kimura, *"Correlation functions and representation bases in free N=4 Super Yang-Mills"*, Nucl.Phys. **B865**, 568 (2012), `arXiv:1206.4844`.

[83] M. Hamermesh, *"Group Theory and Its Applications to Physical Problems"*, Addison-Wesley (1962).

[84] S. Collins, *"Restricted Schur Polynomials and Finite N Counting"*, Phys.Rev. **D79**, 026002 (2009), `arXiv:0810.4217`.

[85] F. Dolan, *"Counting BPS operators in N=4 SYM"*, Nucl.Phys. **B790**, 432 (2008), `arXiv:0704.1038`.

[86] V.D. Comarmond, R. de Mello Koch & K. Jefferies, *"Surprisingly Simple Spectra"*, JHEP **1102:006,2011**, (2010), `arXiv:1012.3884`.

[87] R. de Mello Koch, N. Ives & M. Stephanou, *"On subgroup adapted bases for representations of the symmetric group"*, Journal of Physics A Mathematical General **45**, 135204 (2012), `arXiv:1112.4316`.

[88] J.A. Minahan, *"Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N=4 Super Yang-Mills"*, Lett.Math.Phys. **99**, 33 (2012), `arXiv:1012.3983`.

[89] J. Minahan & K. Zarembo, *"The Bethe ansatz for N=4 superYang-Mills"*, JHEP **0303**, 013 (2003), `hep-th/0212208`.

[90] N. Beisert, C. Kristjansen & M. Staudacher, *"The Dilatation operator of conformal N=4 superYang-Mills theory"*, Nucl.Phys. **B664**, 131 (2003), `hep-th/0303060`.

[91] N. Beisert, *"The Dilatation operator of N=4 super Yang-Mills theory and integrability"*, Phys.Rept. **405**, 1 (2005), `hep-th/0407277`.

[92] C. Sieg, *"Review of AdS/CFT Integrability, Chapter I.2: The spectrum from perturbative gauge theory"*, Lett.Math.Phys. **99**, 59 (2010), arXiv:1012.3984.

[93] C. Sieg, *"Superspace computation of the three-loop dilatation operator of N=4 SYM theory"*, Phys.Rev. **D84**, 045014 (2011), arXiv:1008.3351.

[94] M. Staudacher, *"Review of AdS/CFT Integrability, Chapter III.1: Bethe Ansátze and the R-Matrix Formalism"*, Lett.Math.Phys. **99**, 191 (2012), arXiv:1012.3990.

[95] C. Ahn & R.I. Nepomechie, *"Review of AdS/CFT Integrability, Chapter III.2: Exact World-Sheet S-Matrix"*, Lett.Math.Phys. **99**, 209 (2012), arXiv:1012.3991.

[96] C.N. Yang & C.P. Yang, *"One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe's Hypothesis for Ground State in a Finite System"*, Phys. Rev. **150**, 321 (1966), http://link.aps.org/doi/10.1103/PhysRev.150.321.

[97] H. Bethe, *"On the Theory of Metals. I. Eigenvalues and Eigenfunctions of the Linear Atom Chain"*, Zeitschrift für Physik **71**, 205 (1931).

[98] N. Beisert, J. Minahan, M. Staudacher & K. Zarembo, *"Stringing spins and spinning strings"*, JHEP **0309**, 010 (2003), hep-th/0306139.

[99] N. Beisert, S. Frolov, M. Staudacher & A.A. Tseytlin, *"Precision spectroscopy of AdS / CFT"*, JHEP **0310**, 037 (2003), hep-th/0308117.

[100] N. Beisert & M. Staudacher, *"Long-range psu(2,2|4) Bethe Ansatze for gauge theory and strings"*, Nucl.Phys. **B727**, 1 (2005), hep-th/0504190.

[101] A. Rej, *"Review of AdS/CFT Integrability, Chapter I.3: Long-range spin chains"*, Lett.Math.Phys. **99**, 85 (2012), arXiv:1012.3985.

[102] S. Schafer-Nameki, *"Review of AdS/CFT Integrability, Chapter II.4: The Spectral Curve"*, Lett.Math.Phys. **99**, 169 (2012), arXiv:1012.3989.

[103] P. Vieira & D. Volin, *"Review of AdS/CFT Integrability, Chapter III.3: The Dressing factor"*, Lett.Math.Phys. **99**, 231 (2012), arXiv:1012.3992.

[104] N. Beisert, *"The complete one loop dilatation operator of N=4 superYang-Mills theory"*, Nucl.Phys. **B676**, 3 (2004), hep-th/0307015.