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PAPER

First integrals, conserved vectors of nonlinear partial difference equations

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Abstract

We perform a symmetry analysis of some nonlinear partial difference equations (nP △ Es), where the discrete version is obtained using some discretization approach. The discrete versions of the wave, diffusion, Fisher and Huxley equations are the subject of this research. At first, the initial invariance approach is the Lie symmetry approach. The first integrals technique that Hydon introduced to be used with discrete ordinary difference equations (O △ Es) serves as our inspiration in this situation. We develop a similar technique for generating the first integral vectors of the nP △ Es without recourse to symmetry generators.

1. Introduction

The technique and theory for analyzing differential equations using invariance under some (local) Lie group is well-known and the standard texts cited include, among others, Ovsyannikov [1], Olver [2], Bluman [3, 4] and Ibragimov [5]. For the case of discrete equations, this extension is relatively new; Hydon [6] being the first to introduce this approach. Similar studies for discrete equations are infrequent. The idea of conservation laws for O∆Es was extended further by Hydon [7] and Rasin [8]. In the investigation of the application of Lie groups and Lie algebras to difference equations (DEs), divergent methodologies are discernible. One such approach involves treating both the DEs and associated lattices as predetermined entities for examination. Subsequent efforts are directed towards furnishing analytical tools for the resolution of these equations, with an emphasis on simplification, as well as the classification of solutions based on integrability and linearity [6–8]. An alternative perspective involves both DEs and their corresponding lattices as auxiliary components. The technique of employing transforming lattices, notably explored by Dorodnitsyn and collaborators [9–14], characterizes this method. Notably, this approach maintains the symmetry structure inherent in differential equations, distinguishing it from alterations typically associated with differential equations in the presence of transforming lattices. There are some applications of these techniques in Hussain *et al* [15], Folly-Gbetoula *et al* [16, 17], and [18, 19]; in the latter, the conservation laws for the discrete sine-Gordon equation and discrete Liouville equation were thoroughly addressed. Also, some third order difference equations were studied using that technique. Several investigations on differential equations can be pursued in accordance with the studies mentioned [20–22].

For obtaining exact solutions to DEs, symmetries and first integrals are helpful tools. For DEs, the relationship between symmetries, conservation laws and integrability was established in [23]. It has been demonstrated that the equation can be reduced twice when the invariance condition connects the symmetry generator and the first integral. So, when discrete analogs of such equations are created, these qualities may very well be preserved. This association was discussed in [16] which supports the continuity of the first integral's invariance under a symmetry generator when dealing with O △ Es. The first integral results of the nonlinear partial differential equations are now employed to be applied to the nP △ Es. Here, we use the notion of first integral vectors followed by a shift operator. Hydon introduced this shift operator [24] and is a key operator in the theory of difference equations. For a *forward shift* the shift operator is defined as $S_n: n \mapsto n + 1$. For a function $f(n)$ via the symmetry approach it leads to

$$S_n^r\{f(n)\} = f(n+r), \quad S_n^r v_i = v_{i+r},$$

whereas for a *backward shift*, the shift operator is S_n: $n \mapsto n-1$, or for a function $f(n)$ it follows as

$$S_n^r\{f(n)\} = f(n-r), \quad S_n^r v_i = v_{i-r}.$$

Here, we study the symmetry generators and first integral vectors for discrete equations like wave, diffusion, Fisher, and Huxley equations. The layout of this article is as follows: section 1 introduces Lie symmetry work in theory of difference equations. The general approach for the symmetry generators and first integral vectors of the nP $\triangle$ Es will be introduced in section 2. In section 3 we apply this method to the discrete forms of the above equations. At the end, a conclusion is provided.

2. Symmetries and first integral vectors

We consider the partial difference equation

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k+1}^l), \tag{1}$$

where k and l are integers, v_k^l is a function that depends on the independent variables k, l and ω is a function of the dependent and independent variables.

Here we consider the transformation [6]

$$\Gamma_1: (k, l, v_k^l, v_k^{l+1}, v_{k+1}^l, v_{k+1}^{l+1}) \mapsto (k, l, \hat{v}_k^l, \hat{v}_k^{l+1}, \hat{v}_{k+1}^l, \hat{v}_{k+1}^{l+1}). \tag{2}$$

If we assume that Γ_1 is a symmetry for equation (1), then we have

$$\hat{v}_{k+1}^{l+1} = \omega(k, l, \hat{v}_k^l, \hat{v}_k^{l+1}, \hat{v}_{k+1}^l), \tag{3}$$

whenever equation (1) holds. Lie symmetries are obtained by linearizing the symmetry condition (3) about the identity. For this purpose, we first seek one-parameter (local) Lie groups of symmetries of the form

$$\hat{v}_k^l = v_k^l + \epsilon Q(k, l, v_k^l) + O(\epsilon^2). \tag{4}$$

The function Q is called the characteristic of the one-parameter group. Also, for the remaining variables, we have

$$\begin{aligned}
\hat{v}_{k+1}^l &= v_{k+1}^l + \epsilon Q(k+1, l, v_{k+1}^l) + O(\epsilon^2),\\
\hat{v}_k^{l+1} &= v_k^{l+1} + \epsilon Q(k, l+1, v_k^{l+1}) + O(\epsilon^2),\\
\hat{v}_{k+1}^{l+1} &= v_{k+1}^{l+1} + \epsilon Q(k+1, l+1, v_{k+1}^{l+1}) + O(\epsilon^2).
\end{aligned} \tag{5}$$

Expanding equations (4)–(5) to first order in ϵ yields the linearized symmetry condition [18, 24]

$$S_k S_l Q - X\omega = 0, \tag{6}$$

where

$$X = Q\frac{\partial}{\partial v_k^l} + (S_k Q)\frac{\partial}{\partial v_{k+1}^l} + (S_l Q)\frac{\partial}{\partial v_k^{l+1}}. \tag{7}$$

In the same way, when we have the partial difference equation of the type

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_{k-1}^{l+1}, v_k^{l+1}), \tag{8}$$

where k and l are integers, v_k^l is a function that depends on the independent variables k, l and ω is a function of the dependent and independent variables.

Here we consider the transformation [6]

$$\Gamma_2: (k, l, v_k^l, v_{k-1}^{l+1}, v_k^{l+1}, v_{k+1}^{l+1}) \mapsto (k, l, \hat{v}_k^l, \hat{v}_{k-1}^{l+1}, \hat{v}_k^{l+1}, \hat{v}_{k+1}^{l+1}). \tag{9}$$

If we assume that Γ_2 is a symmetry for equation (8), then, we have

$$\hat{v}_{k+1}^{l+1} = \omega(k, l, \hat{v}_k^l, \hat{v}_{k-1}^{l+1}, \hat{v}_k^{l+1}), \tag{10}$$

whenever equation (8) holds. Lie symmetries are obtained by linearizing the symmetry condition (10) about the identity. For this purpose, we first seek one-parameter (local) Lie groups of symmetries of the form

$$\hat{v}_k^l = v_k^l + \epsilon Q(k, l, v_k^l) + O(\epsilon^2). \tag{11}$$

The function Q is called the characteristic of the one-parameter group. For the remaining variables we have the transformations

$$\hat{v}_{k-1}^{l+1} = v_{k-1}^{l+1} + \epsilon Q(k-1, l+1, v_{k-1}^{l+1}) + O(\epsilon^2),$$
$$\hat{v}_{k}^{l+1} = v_{k}^{l+1} + \epsilon Q(k, l+1, v_{k}^{l+1}) + O(\epsilon^2),$$
$$\hat{v}_{k+1}^{l+1} = v_{k+1}^{l+1} + \epsilon Q(k+1, l+1, v_{k+1}^{l+1}) + O(\epsilon^2). \tag{12}$$

Expanding equations (11)–(12) to first order in ϵ yields the linearized symmetry condition [18, 24] for this type given by

$$S_k S_l Q - X\omega = 0, \tag{13}$$

where

$$X = Q\frac{\partial}{\partial v_k^l} + (S_{-k}S_l Q)\frac{\partial}{\partial v_{k-1}^{l+1}} + (S_l Q)\frac{\partial}{\partial v_k^{l+1}}. \tag{14}$$

2.1. First integral vectors

We consider the partial difference equation of the form

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k+1}^l), \tag{15}$$

where k and l are integers, v_k^l is a function that depends on the independent variables k, l and ω is a function of the dependent and independent variables. Let

$$\phi = \phi(k, l, v_k^l, v_{k+1}^l), \qquad \psi = \psi(k, l, v_k^l, v_k^{l+1}). \tag{16}$$

Then the first integral condition for this case is as follows

$$S_l\phi + S_k\psi = \phi + \psi, \qquad \frac{\partial\phi}{\partial v_k^l} \neq 0, \qquad \frac{\partial\psi}{\partial v_k^l} \neq 0. \tag{17}$$

In the form of arguments, equation (17) becomes

$$S_l\phi(k, l, v_k^l, v_{k+1}^l) + S_k\psi(k, l, v_k^l, v_k^{l+1}) = \phi(k, l, v_k^l, v_{k+1}^l) + \psi(k, l, v_k^l, v_k^{l+1}), \tag{18}$$

or

$$\phi(k, l+1, v_k^{l+1}, \omega) + \psi(k+1, l, v_{k+1}^l, \omega) = \phi(k, l, v_k^l, v_{k+1}^l) + \psi(k, l, v_k^l, v_k^{l+1}). \tag{19}$$

Now we introduce some notations

$$P_1(k, l, v_k^l) = \frac{\partial\phi}{\partial v_k^l}, \qquad R_1(k, l, v_k^l) = \frac{\partial\psi}{\partial v_k^l},$$
$$P_2 = \frac{\partial\phi}{\partial v_{k+1}^l}, \qquad R_2 = \frac{\partial\psi}{\partial v_k^{l+1}}. \tag{20}$$

Differentiating equation (19) w.r.t. v_k^l, we have

$$\frac{\partial\phi}{\partial v_k^l}(k, l+1, v_k^{l+1}, \omega) + \frac{\partial\psi}{\partial v_k^l}(k+1, l, v_{k+1}^l, \omega) = \frac{\partial\phi}{\partial v_k^l} + \frac{\partial\psi}{\partial v_k^l}. \tag{21}$$

By using the chain rule and writing in the form of shifts (forward), we obtain

$$(S_l P_2 + S_k R_2)\frac{\partial\omega}{\partial v_k^l} = P_1 + R_1. \tag{22}$$

As both the differential and shift operators are linear, we can write them as

$$P_1 = (S_l P_2)\frac{\partial\omega}{\partial v_k^l}, \qquad R_1 = (S_k R_2)\frac{\partial\omega}{\partial v_k^l}. \tag{23}$$

Now, we differentiate equation (19) w.r.t. v_{k+1}^l

$$\frac{\partial\phi}{\partial v_{k+1}^l}(k, l+1, v_k^{l+1}, \omega) + \frac{\partial\psi}{\partial v_{k+1}^l}(k+1, l, v_{k+1}^l, \omega) = \frac{\partial\phi}{\partial v_{k+1}^l} + \frac{\partial\psi}{\partial v_{k+1}^l}. \tag{24}$$

In the form of a shift operator, we can simplify as

$$P_2 = S_l P_2\frac{\partial\omega}{\partial v_{k+1}^l} + S_k R_2\frac{\partial\omega}{\partial v_{k+1}^l}. \tag{25}$$

Next, differentiating equation (19) w.r.t. v_k^{l+1}

$$\frac{\partial\phi}{\partial v_k^{l+1}}(k, l+1, v_k^{l+1}, \omega) + \frac{\partial\psi}{\partial v_k^{l+1}}(k+1, l, v_{k+1}^l, \omega) = \frac{\partial\phi}{\partial v_k^{l+1}} + \frac{\partial\psi}{\partial v_k^{l+1}}. \tag{26}$$

After simplifying, we get

$$R_2 = S_l P_2 \frac{\partial \omega}{\partial v_k^{l+1}} + S_k R_2 \frac{\partial \omega}{\partial v_k^{l+1}}. \tag{27}$$

Solving equations (25) and (27), we have

$$\begin{aligned} P_2 &= \frac{\partial \omega}{\partial v_{k+1}^l}[S_l P_2 + S_l P_2 \frac{\partial \omega}{\partial v_k^{l+1}} + S_k^2 R_2 \frac{\partial \omega}{\partial v_k^{l+1}}],\\ R_2 &= \frac{\partial \omega}{\partial v_k^{l+1}}[S_k R_2 + S_k R_2 \frac{\partial \omega}{\partial v_{k+1}^l} + S_l^2 P_2 \frac{\partial \omega}{\partial v_{k+1}^l}]. \end{aligned} \tag{28}$$

From equation (28), it follows that

$$\begin{aligned} P_2 &= \frac{\partial \omega}{\partial v_{k+1}^l}[S_l P_2 + S_l P_2 \frac{\partial \omega}{\partial v_k^{l+1}} + S_k^2 P_2 \frac{\partial \omega}{\partial v_k^{l+1}}],\\ R_2 &= \frac{\partial \omega}{\partial v_k^{l+1}}[S_k R_2 + S_k R_2 \frac{\partial \omega}{\partial v_{k+1}^l} + S_l^2 R_2 \frac{\partial \omega}{\partial v_{k+1}^l}], \end{aligned} \tag{29}$$

such that $S_l^2 P_2 = S_k^2 R_2$ holds. From equation (20), we get the integrability condition as

$$\frac{\partial P_1}{\partial v_{k+1}^l} = \frac{\partial P_2}{\partial v_k^l}, \qquad \frac{\partial R_1}{\partial v_k^{l+1}} = \frac{\partial R_2}{\partial v_k^l}. \tag{30}$$

Then the first integral components (ϕ, ψ) are given by

$$\begin{aligned} \phi &= \int P_1 dv_k^l + P_2 v_{k+1}^l + G(k, l),\\ \psi &= \int R_1 dv_k^l + R_2 v_k^{l+1} + H(k, l). \end{aligned} \tag{31}$$

Next, we consider the partial difference equation of the form

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1}), \tag{32}$$

where k and l are integers, v_k^l is a function that depends on the independent variables $k,\ l$ and ω is a function of the dependent and independent variables.

Let

$$\phi = \phi(k, l, v_k^l, v_{k-1}^{l+1}), \qquad \psi = \psi(k, l, v_k^l, v_k^{l+1}). \tag{33}$$

Then, the first integral condition for this case follows as

$$S_k^2 \phi + S_k \psi = \phi + \psi, \qquad \frac{\partial \phi}{\partial v_k^l} \neq 0, \qquad \frac{\partial \psi}{\partial v_k^l} \neq 0. \tag{34}$$

In the form of arguments, equation (34) becomes

$$S_k^2 \phi(k, l, v_k^l, v_{k-1}^{l+1}) + S_k \psi(k, l, v_k^l, v_k^{l+1}) = \phi(k, l, v_k^l, v_{k-1}^{l+1}) + \psi(k, l, v_k^l, v_k^{l+1}), \tag{35}$$

or

$$\phi(k+2, l, v_{k+2}^l, \omega) + \psi(k+1, l, v_{k+1}^l, \omega) = \phi(k, l, v_k^l, v_{k-1}^{l+1}) + \psi(k, l, v_k^l, v_k^{l+1}). \tag{36}$$

As before we define some notations

$$\begin{aligned} P_1(k, l, v_k^l) &= \frac{\partial \phi}{\partial v_k^l}, \qquad R_1(k, l, v_k^l) = \frac{\partial \psi}{\partial v_k^l},\\ P_2 &= \frac{\partial \phi}{\partial v_{k-1}^{l+1}}, \qquad R_2 = \frac{\partial \psi}{\partial v_k^{l+1}}. \end{aligned} \tag{37}$$

Differentiating equation (36) w.r.t. v_k^l, we have

$$\frac{\partial \phi}{\partial v_k^l}(k+2, l, v_{k+2}^l, \omega) + \frac{\partial \psi}{\partial v_k^l}(k+1, l, v_{k+1}^l, \omega) = \frac{\partial \phi}{\partial v_k^l} + \frac{\partial \psi}{\partial v_k^l}. \tag{38}$$

By using chain rule and writing in the form of shifts, we get

$$(S_k^2 P_2 + S_k R_2)\frac{\partial \omega}{\partial v_k^l} = P_1 + R_1. \tag{39}$$

As both the differential and shift operators are linear, so we can write

$$P_1 = S_k^2(P_2)\frac{\partial \omega}{\partial v_k^l}, \qquad R_1 = S_k(R_2)\frac{\partial \omega}{\partial v_k^l}. \tag{40}$$

Now we differentiate equation (36) w.r.t. v_{k+1}^l

$$\frac{\partial \phi}{\partial v_{k+1}^l}(k+2, l, v_{k+2}^l, \omega) + \frac{\partial \psi}{\partial v_{k+1}^l}(k+1, l, v_{k+1}^l, \omega) = \frac{\partial \phi}{\partial v_{k+1}^l} + \frac{\partial \psi}{\partial v_{k+1}^l}. \tag{41}$$

In the form of the shift operator, we can simplify it as

$$P_2 = [S_l(P_2) + S_k(R_2)]\frac{\partial \omega}{\partial v_{k+1}^l}. \tag{42}$$

Next, differentiating equation (36) w.r.t. v_{k+2}^l gives

$$\frac{\partial \phi}{\partial v_{k+2}^l}(k+2, l, v_{k+2}^l, \omega) + \frac{\partial \psi}{\partial v_{k+2}^l}(k+1, l, v_{k+1}^l, \omega) = \frac{\partial \phi}{\partial v_{k+2}^l} + \frac{\partial \psi}{\partial v_{k+2}^l}. \tag{43}$$

After simplifying, we get

$$R_2 = S_k^2(P_2) + S_{-k}S_l(R_2)\frac{\partial \omega}{\partial v_{k+2}^l}. \tag{44}$$

Solving equations (42) and (44), we have

$$\begin{aligned} P_2 &= \frac{\partial \omega}{\partial v_{k+1}^l}\left[S_l(P_2) + \frac{\partial \omega}{\partial v_{k+2}^l}\{S_k^2(P_2) + S_l(R_2)\}\right], \\ R_2 &= \frac{\partial \omega}{\partial v_{k+2}^l}\left[S_{-k}S_l(R_2) + \frac{\partial \omega}{\partial v_{k+1}^l}\{S_k^2 S_l(P_2) + S_k(R_2)\}\right]. \end{aligned} \tag{45}$$

From equation (45), it follows that

$$\begin{aligned} P_2 &= S_k\left(\frac{\partial \omega}{\partial v_k^l}\right)\left[S_l(P_2) + S_k^2\left(\frac{\partial \omega}{\partial v_k^l}\right)S_k^2(P_2) + S_k^2\left(\frac{\partial \omega}{\partial v_k^l}\right)S_l(P_2)\right\}\right], \\ R_2 &= S_k^2\left(\frac{\partial \omega}{\partial v_k^l}\right)\left[S_{-k}S_l(R_2) + S_k\left(\frac{\partial \omega}{\partial v_k^l}\right)S_k(R_2) + S_k\left(\frac{\partial \omega}{\partial v_k^l}\right)S_k^2 S_l(R_2)\right\}\right], \end{aligned} \tag{46}$$

such that $S_l(R_2) = S_k^2 S_l(P_2)$ holds. From equation (37), we obtain the integrability condition as

$$\frac{\partial P_1}{\partial v_{k-1}^{l+1}} = \frac{\partial P_2}{\partial v_k^l}, \qquad \frac{\partial R_1}{\partial v_k^{l+1}} = \frac{\partial R_2}{\partial v_k^l}. \tag{47}$$

Then the first integral components (ϕ, ψ) are given by

$$\begin{aligned} \phi &= \int P_1 dv_k^l + P_2 v_{k-1}^{l+1} + G(k, l), \\ \psi &= \int R_1 dv_k^l + R_2 v_k^{l+1} + H(k, l). \end{aligned} \tag{48}$$

3. Derivation of P△ Es

3.1. Discrete wave equation

Here to apply the technique discussed above, we consider the discrete wave equation [24]

$$v_{k+1}^{l+1} = v_{k+1}^l + v_k^{l+1} - v_k^l.$$

We first compute the symmetries of the discrete wave equation.

3.1.1. Symmetries

Consider the discrete wave equation

$$v_{k+1}^{l+1} = v_{k+1}^l + v_k^{l+1} - v_k^l. \tag{49}$$

This equation is of the form

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k+1}^l), \tag{50}$$

where the term on the right hand side $\omega(k, l, v_k^l, v_k^{l+1}, v_{k+1}^l)$ is a smooth function. Let the characteristic of the equation (49) be $Q = Q(k, l, v_k^l)$. Then the symmetry condition equation (6) becomes

$$S_kS_lQ - X\omega = 0, \qquad Q(k+1, l+1, \omega) - X\omega = 0, \tag{51}$$

where equation (7) is

$$X = Q\frac{\partial}{\partial v_k^l} + (S_kQ)\frac{\partial}{\partial v_{k+1}^l} + (S_lQ)\frac{\partial}{\partial v_k^{l+1}}.$$

The symmetry condition equation (51) takes the form

$$Q(k+1, l+1, \omega) - Q(k+1, l, v_{k+1}^l) - Q(k, l+1, v_k^{l+1}) + Q(k, l, v_k^l) = 0. \tag{52}$$

By using the differential operator given by

$$L_0 = \frac{\partial}{\partial v_k^l} + \frac{\partial v_{k+1}^l}{\partial v_k^l}\frac{\partial}{\partial v_{k+1}^l}, \tag{53}$$

we get

$$Q'(k+1, l, v_{k+1}^l) - Q'(k, l, v_k^l) = 0. \tag{54}$$

Solving this equation, we have

$$Q(k, l, v_k^l) = c_1(k, l)v_k^l + c_2(k, l). \tag{55}$$

Using the value of Q in equation (52), and comparing the coefficients

$$\begin{aligned} &v_{k+1}^l\text{: } c_1(k+1, l+1) - c_1(k+1, l) = 0,\\ &v_k^{l+1}\text{: } c_1(k+1, l+1) - c_1(k, l+1) = 0,\\ &\text{Constants: } c_2(k+1, l+1) + c_2(k, l) - c_2(k+1, l) - c_2(k, l+1) = 0. \end{aligned} \tag{56}$$

Solving the system, we obtain

$$Q(k, l, v_k^l) = (-1)^k v_k^l + A. \tag{57}$$

The following symmetry generators are obtained

$$X_1 = \frac{\partial}{\partial v_k^l} - \frac{\partial}{\partial v_{k+1}^l} + \frac{\partial}{\partial v_k^{l+1}}, \qquad X_2 = v_k^l\frac{\partial}{\partial v_k^l} + v_{k+1}^l\frac{\partial}{\partial v_{k+1}^l} + v_k^{l+1}\frac{\partial}{\partial v_k^{l+1}}. \tag{58}$$

3.1.2. First integral vectors

In this section, we implement the main results derived in the aforementioned section (2.1). We derive the first integral vector for the discrete wave equation given by

$$v_{k+1}^{l+1} = v_{k+1}^l + v_k^{l+1} - v_k^l.$$

This equation is of the form that is of the type discussed in the equation (15)

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k+1}^l),$$

where the right hand side $\omega(k, l, v_k^l, v_k^{l+1}, v_{k+1}^l)$ is a smooth function. Then we define $P_2 = P_2(k, l, v_k^l)$ and the first integral condition for P_2 follows from equation (29) as

$$P_2(k+2, l, v_{k+2}^l) + 2P_2(k, l+1, v_k^{l+1}) - P_2(k, l, v_k^l) = 0. \tag{59}$$

By using the differential operator equation (53), we obtain

$$2P_2'(k, l+1, v_k^{l+1}) - P_2'(k, l, v_k^l) = 0. \tag{60}$$

Differentiating this equation w.r.t. v_k^l, and simplifying, we get

$$P_2(k, l, v_k^l) = c_1(k, l)v_k^l + c_2(k, l). \tag{61}$$

Further using equations (59)–(60), we have

$$P_2(k, l, v_k^l) = 2^{1-l}v_k^l + (-1+\sqrt{2})^l + (-1-\sqrt{2})^l. \tag{62}$$

On the other hand the relation for P_1 from equation (23) follows as

$$P_1(k+1, l, v_{k+1}^l) = -2^{1-l}v_{k+1}^l - (-1+\sqrt{2})^l - (-1-\sqrt{2})^l. \tag{63}$$

Using the integrability condition equation (29), we get

$$\frac{\partial P_1}{\partial v_{k+1}^l} = -2^{1-l}, \qquad \frac{\partial P_2}{\partial v_k^l} = 2^{1-l}. \tag{64}$$

Then, from equation (17) the first component ϕ of the first integral vector follows as

$$\psi = \{(-1-\sqrt{2})^l + (-1+\sqrt{2})^l\}(v_{k+1}^l + v_k^l) + G(k, l). \tag{65}$$

Now we proceed to obtain the second component ψ of the first integral vector. For this we define $R_2 = R_2(k, l, v_k^l)$ and the first integral condition for R_2 follows from equation (29) as

$$2R_2(k+1, l, v_{k+1}^l) + R_2(k, l+2, v_k^{l+2}) - R_2(k, l, v_k^l) = 0. \tag{66}$$

By using the differential operator (53), we obtain

$$2R_2'(k+1, l, v_{k+1}^l) - R_2'(k, l, v_k^l) = 0. \tag{67}$$

Differentiating this equation w.r.t. v_k^l, and simplifying,

$$R_2(k, l, v_k^l) = c_1(k, l)v_k^l + c_2(k, l). \tag{68}$$

Further using equations (66)–(67), we have

$$R_2(k, l, v_k^l) = 2^{1-k}v_k^l + (-1+\sqrt{2})^k + (-1-\sqrt{2})^k. \tag{69}$$

On the other hand the relation for R_1 from equation (23) leads to

$$R_1(k, l+1, v_k^{l+1}) = -2^{1-k}v_k^{l+1} - (-1+\sqrt{2})^k - (-1-\sqrt{2})^k. \tag{70}$$

For the integrability condition (30), we get

$$\frac{\partial R_1}{\partial v_k^{l+1}} = -2^{1-k}, \qquad \frac{\partial R_2}{\partial v_k^l} = 2^{1-k}. \tag{71}$$

Then, from equation (31), second component ψ of the first integral vector follows as

$$\psi = \{(-1-\sqrt{2})^k + (-1+\sqrt{2})^k\}(v_k^{l+1} + v_k^l) + H(k, l). \tag{72}$$

We thus have

Theorem 3.1. *The first integral vector for the discrete wave equation (49) has components*

$$\begin{aligned}\phi &= \{(-1-\sqrt{2})^l + (-1+\sqrt{2})^l\}(v_{k+1}^l + v_k^l) + (-1)^k,\\ \psi &= \{(-1-\sqrt{2})^k + (-1+\sqrt{2})^k\}(v_k^{l+1} + v_k^l) + (-1)^l.\end{aligned} \tag{73}$$

3.2. Discrete diffusion equation

Consider the one-dimensional diffusion equation

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2}, \tag{74}$$

where x and t are spatial and temporal variables respectively, D is the diffusion constant and v is dependent variable here. The rate of diffusion increases with D value and has a unit length squared over time. For the discretization of a single domain diffusion equation, we start with an implicit finite difference scheme, which is a backward difference in time and a central second order difference in space. The approximations for the derivatives are given in discrete form as

$$\frac{\partial v}{\partial t} \approx \frac{v_k^{l+1} - v_k^l}{\Delta t} + O(\Delta t)^2, \tag{75}$$

$$\frac{\partial^2 v}{\partial x^2} \approx \frac{v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}}{2(\Delta x)^2} + O(\Delta x)^3. \tag{76}$$

From equation (74), we have

$$\frac{v_k^{l+1} - v_k^l}{\Delta t} + O(\Delta t)^2 = D\frac{v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}}{2(\Delta x)^2} + O(\Delta x)^3. \tag{77}$$

Ignoring the truncated terms and writing in simpler form

$$\lambda(v_k^{l+1} - v_k^l) = v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}, \tag{78}$$

where $\lambda = \frac{2(\Delta x)^2}{D(\Delta t)}$. Rearranging the terms, we get

$$v_{k+1}^{l+1} = Av_k^{l+1} - \lambda v_k^l - v_{k-1}^{l+1},$$

where $A = \lambda + 1$.

3.2.1. *Symmetries*

Consider the discrete diffusion equation derived above

$$v_{k+1}^{l+1} = Av_k^{l+1} - \lambda v_k^l - v_{k-1}^{l+1}. \tag{79}$$

This equation is of the form that is of type discussed in the equation (32)

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1}),$$

where the term on the right hand side $\omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1})$ is a smooth function. Let the characteristic of the equation (79) be $Q = Q(k, l, v_k^l)$. Then the symmetry condition (13) becomes

$$S_k S_l Q - X\omega = 0, \qquad Q(k+1, l+1, \omega) - X\omega = 0, \tag{80}$$

where equation (14) is

$$X = Q\frac{\partial}{\partial v_k^l} + (S_{-k}S_l Q)\frac{\partial}{\partial v_{k-1}^{l+1}} + (S_l Q)\frac{\partial}{\partial v_k^{l+1}}.$$

The symmetry condition (80) takes the form

$$Q(k+1, l+1, \omega) + Q(k-1, l+1, v_{k-1}^{l+1}) - AQ(k, l+1, v_k^{l+1}) + \lambda Q(k, l, v_k^l) = 0. \tag{81}$$

By using the differential operator (53), we get

$$\lambda Q'(k, l, v_k^l) - \lambda Q'(k, l+1, v_k^{l+1}) = 0. \tag{82}$$

Solving this equation, we have

$$Q(k, l, v_k^l) = \frac{1}{\lambda}c_1(k, l)v_k^l + c_2(k, l). \tag{83}$$

Using the value of Q in equation (80) and equation (82), we obtain

$$Q(k, l, v_k^l) = \frac{1}{\lambda}(-1)^k v_k^l + B. \tag{84}$$

The following symmetry generators are obtained

$$X_1 = \frac{\partial}{\partial v_k^l} - \frac{\partial}{\partial v_k^{l+1}} + \frac{\partial}{\partial v_{k-1}^{l+1}}, \qquad X_2 = v_k^l\frac{\partial}{\partial v_k^l} - v_k^{l+1}\frac{\partial}{\partial v_k^{l+1}} + v_{k-1}^{l+1}\frac{\partial}{\partial v_{k-1}^{l+1}}. \tag{85}$$

3.2.2. First integral vectors

In this section, we derive the first integral vector for the discrete diffusion equation (79) given by

$$v_{k+1}^{l+1} = Av_k^{l+1} - \lambda v_k^l - v_{k-1}^{l+1}.$$

This equation is of the form that is of type discussed in the equation (32)

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1}),$$

where the right hand side $\omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1})$ is a smooth function. Then we define $P_2 = P_2(k, l, v_k^l)$ and the first integral condition for P_2 follows from equation (46) as

$$\lambda P_2(k+2, l, v_{k+2}^l) + (\lambda - 1)P_2(k, l+1, v_k^{l+1}) - \frac{1}{\lambda}P_2(k, l, v_k^l) = 0. \tag{86}$$

By using the differential operator (53), we obtain

$$\lambda(1-\lambda)P_2'(k, l+1, v_k^{l+1}) + \frac{A}{\lambda}P_2'(k, l, v_k^l) = 0. \tag{87}$$

Differentiating this equation w.r.t. v_k^l, and simplifying, we get

$$P_2(k, l, v_k^l) = c_1(k, l)v_k^l + c_2(k, l). \tag{88}$$

Further using equation (86) and equation (87), we have

$$P_2(k, l, v_k^l) = c_1\lambda\left(\frac{A}{\lambda^2(\lambda-1)}\right)^{l-1} v_k^l + c_2\left(\frac{\lambda+1}{\lambda}\right)^{l-1}. \tag{89}$$

On the other hand the relation for P_1 follows from equation (40) as

$$P_1(k-1, l+1, v_{k-1}^{l+1}) = -c_1\lambda^2\left(\frac{A}{\lambda^2(\lambda-1)}\right)^l v_{k-1}^{l+1} - c_2\lambda\left(\frac{\lambda+1}{\lambda}\right)^l. \tag{90}$$

Using the integrability condition (47), we get

$$\frac{\partial P_1}{\partial v_{k-1}^{l+1}} = -c_1\lambda^2\left(\frac{A}{\lambda^2(\lambda-1)}\right)^l, \qquad \frac{\partial P_2}{\partial v_k^l} = c_1\lambda\left(\frac{A}{\lambda^2(\lambda-1)}\right)^{l-1}. \tag{91}$$

Then, from equation (48) the first component ϕ of the first integral vector follows as

$$\phi = \left\{1 - \left(\frac{A}{\lambda^2(\lambda - 1)}\right)\right\} c_1 \lambda \left(\frac{A}{\lambda^2(\lambda - 1)}\right)^{l-1} v_k^l v_{k-1}^{l+1} + c_2 \lambda \left(\frac{\lambda + 1}{\lambda}\right)^{l-1} (v_{k-1}^{l+1} - v_k^l) + G(k, l). \quad (92)$$

Now we proceed to obtain the second component ψ of the first integral vector. For this we define $R_2 = R_2(k, l, v_k^l)$ and the first integral condition for R_2 follows from equation (46)

$$\lambda R_2(k + 2, l + 1, v_{k+2}^{l+1}) + \lambda R_2(k + 1, l, v_{k+1}^l) - R_2(k - 1, l + 1, v_{k-1}^{l+1}) - \frac{1}{\lambda} R_2(k, l, v_k^l) = 0. \quad (93)$$

We define the differential operator as

$$L_0 = \frac{\partial}{\partial v_k^l} + \frac{\partial v_{k-1}^{l+1}}{\partial v_k^l} \frac{\partial}{\partial v_{k-1}^{l+1}}. \quad (94)$$

By applying the operator, we get

$$\lambda R_2'(k - 1, l + 1, v_{k-1}^{l+1}) - \frac{1}{\lambda} R_2'(k, l, v_k^l) = 0. \quad (95)$$

Differentiating equation (95) w.r.t. v_k^l, and simplifying

$$R_2(k, l, v_k^l) = \lambda c_1(k, l) v_k^l + c_2(k, l). \quad (96)$$

Further using equations (93)–(67), we have

$$R_2(k, l, v_k^l) = c_1 \lambda^{2k-1} v_k^l + c_2 \left(-\frac{\lambda - 1}{\lambda}\right)^{l-1}. \quad (97)$$

On the other hand the relation for R_1 follows as

$$R_1(k, l + 1, v_k^{l+1}) = -c_1 \lambda^{2k+1} v_k^{l+1} - c_2 \lambda \left(-\frac{\lambda - 1}{\lambda}\right)^l. \quad (98)$$

Using the integrability condition (47) we get

$$\frac{\partial R_1}{\partial v_k^{l+1}} = -c_1 \lambda^{2k+1}, \qquad \frac{\partial R_2}{\partial v_k^l} = c_1 \lambda^{2k-1}. \quad (99)$$

Then, from equation (48) second component ψ of the first integral vector follows

$$\psi = c_2 \lambda \left(-\frac{\lambda - 1}{\lambda}\right)^{l-1} (v_k^{l+1} + v_k^l) + H(k, l). \quad (100)$$

We thus have

Theorem 3.2. *The first integral vector for the discrete wave equation (79) has components*

$$\phi = \left\{1 - \left(\frac{A}{\lambda^2(\lambda - 1)}\right)\right\} c_1 \lambda \left(\frac{A}{\lambda^2(\lambda - 1)}\right)^{l-1} v_k^l v_{k-1}^{l+1} + c_2 \lambda \left(\frac{\lambda + 1}{\lambda}\right)^{l-1} (v_{k-1}^{l+1} - v_k^l) + (-1)^k,$$

$$\psi = c_2 \lambda \left(-\frac{\lambda - 1}{\lambda}\right)^{l-1} (v_k^{l+1} + v_k^l) + (-1)^l. \quad (101)$$

3.3. Discrete Fisher equation

Consider the one-dimensional Fisher equation

$$v_t = v_{xx} + v(1 - v), \quad (102)$$

where x and t are spatial and temporal variables respectively and v is the dependent variable. Based on the type of model, $v(x, t)$ represents the concentration of a fluid, bacteria, or a specific biological cell and $v(1 - v)$ is the reaction or growth term. This equation has been derived by Fisher [25] in its most basic version, who initially analysed it. For the discretization of a single domain diffusion equation, we are starting with an implicit finite difference scheme, which is a backward difference in time and a central second order difference in space. The approximations for the derivatives in case of implicit discretization are defined as

$$v_t \approx \frac{v_k^{l+1} - v_k^l}{\Delta t} + O(\Delta t)^2, \quad (103)$$

$$v_{xx} \approx \frac{v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}}{2(\Delta x)^2} + O(\Delta x)^3. \quad (104)$$

From equation (102), we have

$$\frac{v_k^{l+1} - v_k^l}{\Delta t} + O(\Delta t)^2 = \frac{v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}}{2(\Delta x)^2} + O(\Delta x)^3 + v_k^l(1 - v_k^l). \quad (105)$$

Ignoring the truncated terms and writing in simpler form

$$v_k^{l+1} - v_k^l = \nu(v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}) + (\triangle t)v_k^l(1 - v_k^l), \tag{106}$$

where $\nu = \frac{(\triangle t)}{2(\triangle x)^2}$. Rearranging the terms, we get

$$v_{k+1}^{l+1} = \lambda_1 v_k^{l+1} - \lambda_2 v_k^l - v_{k-1}^{l+1} + \lambda_3 (v_k^l)^2,$$

where $\lambda_1 = \frac{1}{\nu} + 2,\ \ \lambda_2 = \frac{1+\triangle t}{\nu},\ \ \lambda_3 = \frac{\triangle t}{\nu}$. Now we compute here the symmetries of the above equation known as the discrete Fisher equation.

3.3.1. Symmetries

Consider the discrete Fisher equation derived above

$$v_{k+1}^{l+1} = \lambda_1 v_k^{l+1} - \lambda_2 v_k^l - v_{k-1}^{l+1} + \lambda_3 (v_k^l)^2. \tag{107}$$

This equation is of the form that is of type discussed in the equation (32)

$$v_{k+1}^{l+1} = \omega(k,\, l,\, v_k^l,\, v_k^{l+1},\, v_{k-1}^{l+1}),$$

where the term on the right hand side $\omega(k,\, l,\, v_k^l,\, v_k^{l+1},\, v_{k-1}^{l+1})$ is a smooth function. Let the characteristic of the equation (107) be $Q = Q(k,\, l,\, v_k^l)$. Then the symmetry condition (13) becomes

$$S_k S_l Q - X\omega = 0, \qquad Q(k+1,\, l+1,\, \omega) - X\omega = 0, \tag{108}$$

where equation (14) is

$$X = Q\frac{\partial}{\partial v_k^l} + (S_{-k}S_l Q)\frac{\partial}{\partial v_{k-1}^{l+1}} + (S_l Q)\frac{\partial}{\partial v_k^{l+1}}.$$

The symmetry condition (108) takes the form

$$Q(k+1,\, l+1,\, \omega) + Q(k-1,\, l+1,\, v_{k-1}^{l+1}) - \lambda_1 Q(k,\, l+1,\, v_k^{l+1}) + (\lambda_2 - 2\lambda_3 v_k^l)Q(k,\, l,\, v_k^l) = 0. \tag{109}$$

By using the differential operator (53), we get

$$-(\lambda_2 - 2\lambda_3 v_k^l)Q'(k,\, l+1,\, v_k^{l+1}) + (\lambda_2 - 2\lambda_3 v_k^l)Q'(k,\, l,\, v_k^l) - 2\lambda_3 Q(k,\, l,\, v_k^l) = 0. \tag{110}$$

Solving this equation, we have

$$Q(k,\, l,\, v_k^l) = -\frac{1}{4\lambda_3}c_1(k,\, l)(\lambda_2 - 2\lambda_3 v_k^l) + \frac{\lambda_2^2}{4\lambda_3(\lambda_2 - 2\lambda_3 v_k^l)}c_1(k,\, l) + \frac{c_2(k,\, l)}{\lambda_2 - 2\lambda_3 v_k^l}. \tag{111}$$

Using the value of Q in equations (109)–(110), we get

$$Q(k,\, l,\, v_k^l) = \frac{(-1)^{l+1}}{4\lambda_3}\lambda_2 + \frac{(-1)^{l+2}}{2}v_k^l + \frac{(-1)^l\lambda_2^2 + 4\lambda_3 c_1\left(-\frac{1}{\lambda_1}\right)^{k-1}}{(\lambda_2 - 2\lambda_3 v_k^l)}. \tag{112}$$

The following symmetry generators are obtained

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial v_k^l} - \frac{\partial}{\partial v_k^{l+1}} + \frac{\partial}{\partial v_{k-1}^{l+1}},\\
X_2 &= v_k^l\frac{\partial}{\partial v_k^l} - v_k^{l+1}\frac{\partial}{\partial v_k^{l+1}} + v_{k-1}^{l+1}\frac{\partial}{\partial v_{k-1}^{l+1}},\\
X_3 &= \frac{1}{(\lambda_2 - 2\lambda_3 v_k^l)}\frac{\partial}{\partial v_k^l} - \frac{1}{(\lambda_2 - 2\lambda_3 v_k^{l+1})}\frac{\partial}{\partial v_k^{l+1}} + \frac{1}{(\lambda_2 - 2\lambda_3 v_{k-1}^{l+1})}\frac{\partial}{\partial v_{k-1}^{l+1}}.
\end{aligned} \tag{113}$$

3.3.2. First integral vectors

In this section, we derive the first integral vector for the discrete Fisher equation (107) given by

$$v_{k+1}^{l+1} = \lambda_1 v_k^{l+1} - \lambda_2 v_k^l - v_{k-1}^{l+1} + \lambda_3 (v_k^l)^2.$$

This equation is of the form that is of the type discussed in the equation (32)

$$v_{k+1}^{l+1} = \omega(k,\, l,\, v_k^l,\, v_k^{l+1},\, v_{k-1}^{l+1}),$$

where the right hand side $\omega(k,\, l,\, v_k^l,\, v_k^{l+1},\, v_{k-1}^{l+1})$ is a smooth function. Then we define $P_2 = P_2(k,\, l,\, v_k^l)$ and the first integral condition for P_2 follows from equation (46) as

$$P_2(k, l, v_k^l) = -(\lambda_2 + 2\lambda_3 v_{k+1}^l)[P_2(k, l+1, v_k^{l+1}) \\ -(\lambda_2 + 2\lambda_3 v_{k+2}^l)P_2(k+2, l, v_{k+2}^l) - (\lambda_2 + 2\lambda_3 v_{k+2}^l)P_2(k, l+1, v_k^{l+1})]. \tag{114}$$

By using the differential operator (53), we obtain

$$P_2'(k, l, v_k^l) + \frac{1}{\lambda_1}(\lambda_2 + 2\lambda_3 v_k^l)(\lambda_2 + 2\lambda_3 v_{k+1}^l)(1 - \lambda_2 - 2\lambda_3 v_{k+2}^l)P_2'(k, l+1, v_k^{l+1}) = 0. \tag{115}$$

Differentiating this equation w.r.t. v_k^l, and simplifying, we get

$$P_2(k, l, v_k^l) = \lambda_2 c_1(k, l)v_k^l + \lambda_3 c_1(k, l)(v_k^l)^2 + c_2(k, l). \tag{116}$$

Further using equations (114)–(115), we have

$$P_2(k, l, v_k^l) = c_1\lambda_2\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l-1} v_k^l + c_1\lambda_3\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l-1}(v_k^l)^2 + c_2\left(\frac{\lambda_2^2 - 1}{\lambda_2^2(\lambda_2 + 1)}\right)^{l-1}. \tag{117}$$

On the other hand the relation for P_1 follows from equation (40)

$$P_1(k-1, l+1, v_{k-1}^{l+1}) = (-\lambda_2 - 2\lambda_3 v_{k-3}^{l+1})\left\{c_1\lambda_2\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l} v_{k-1}^{l+1} \right. \\ \left. + c_1\lambda_3\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l}(v_{k-1}^{l+1})^2 + c_2\left(\frac{\lambda_2^2 - 1}{\lambda_2^2(\lambda_2 + 1)}\right)^{l}\right\}. \tag{118}$$

For the integrability condition (47), we get

$$\frac{\partial P_1}{\partial v_{k-1}^{l+1}} = (-\lambda_2 - 2\lambda_3 v_{k-3}^{l+1})\left\{c_1\lambda_2\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l} + 2c_1\lambda_3\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l} v_{k-1}^{l+1}\right\}, \\ \times\frac{\partial P_2}{\partial v_k^l} = c_1\lambda_2\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l-1} + c_1\lambda_3\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l-1} v_k^l. \tag{119}$$

Then, from equation (48), the first component ϕ of the first integral vector follows as

$$\phi = c_1\lambda_2\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l-1} v_k^l v_{k-1}^{l+1}\left\{1 - (\lambda_2 + 2\lambda_3 v_{k-3}^{l+1})\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)\right\} \\ + c_1\lambda_3\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right)^{l-1} v_{k-1}^{l+1} v_k^l\left\{v_k^l - (\lambda_2 + 2\lambda_3 v_{k-3}^{l+1})\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2 - 1)}\right) v_{k-1}^{l+1}\right\} \\ + c_2\left(\frac{\lambda_2^2 - 1}{\lambda_2^2(\lambda_2 + 1)}\right)^{l-1}\left\{v_{k-1}^{l+1} - (\lambda_2 + 2\lambda_3 v_{k-3}^{l+1})\left(\frac{\lambda_2^2 - 1}{\lambda_2^2(\lambda_2 - 1)}\right) v_k^l\right\}. \tag{120}$$

Now we proceed to obtain the second component ψ of the first integral vector. For this we define $R_2 = R_2(k, l, v_k^l)$ and the first integral condition for R_2 follows from equation (46) as

$$R_2(k, l, v_k^l) + (\lambda_2 + 2\lambda_3 v_{k+2}^l)[R_2(k-1, l+1, v_{k-1}^{l+1}) \\ -(\lambda_2 + 2\lambda_3 v_{k+1}^l)R_2(k+1, l, v_{k+1}^l) - (\lambda_2 + 2\lambda_3 v_{k+1}^l)R_2(k+2, l+1, v_{k+2}^{l+1})] = 0. \tag{121}$$

By using the differential operator (94), we obtain

$$R_2'(k, l, v_k^l) - (\lambda_2 + 2\lambda_3 v_k^l)(\lambda_2 + 2\lambda_3 v_{k+2}^l)R_2'(k-1, l+1, v_{k-1}^{l+1}) = 0. \tag{122}$$

Differentiating equation (122) w.r.t. v_k^l, and simplifying we get

$$R_2(k, l, v_k^l) = \lambda_2 c_1(k, l)v_k^l + \lambda_3 c_1(k, l)(v_k^l)^2 + c_2(k, l). \tag{123}$$

Further using equations (121)–(122), we get following result

$$R_2(k, l, v_k^l) = c_1(\lambda_2)^{2l-1}v_k^l + c_1\lambda_3(\lambda_2)^{2l-2}(v_k^l)^2 + c_2\left(\frac{\lambda_2 + 1}{\lambda_2}\right)^{l-1}. \tag{124}$$

On the other hand the relation for R_1 follows as

$$R_1(k, l+1, v_k^{l+1}) = -(\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})\left\{c_1(\lambda_2)^{2l+1}v_k^{l+1} + c_1\lambda_3(\lambda_2)^{2l}(v_k^{l+1})^2 + c_2\left(\frac{\lambda_2 + 1}{\lambda_2}\right)^{l}\right\}. \tag{125}$$

For the integrability condition (47), we get

$$\frac{\partial R_1}{\partial v_k^{l+1}} = -(\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})\{c_1(\lambda_2)^{2l+1} + 2c_1\lambda_3(\lambda_2)^{2l}v_k^{l+1}\},$$
$$\frac{\partial R_2}{\partial v_k^l} = c_1(\lambda_2)^{2l-1} + 2c_1\lambda_3(\lambda_2)^{2l-2}v_k^l. \tag{126}$$

Then, from equation (48), second component ψ of the first integral vector follows as

$$\psi = c_1(\lambda_2)^{2l-1}v_k^l v_k^{l+1}\{1 - (\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})\lambda_2^2\} + c_1\lambda_3(\lambda_2)^{2l-2}v_k^l v_k^{l+1}\{v_k^l - \lambda_2^2(\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})v_k^{l+1}\}$$
$$+ c_2\left(\frac{\lambda_2+1}{\lambda_2}\right)^{l-1}\{v_k^{l+1} - (\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})v_k^l\} + H(k, l). \tag{127}$$

We thus have

Theorem 3.3. *The first integral vector for the discrete Fisher equation (107) has components*

$$\phi = c_1\lambda_2\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2-1)}\right)^{l-1}v_k^l v_{k-1}^{l+1}\left\{1 - (\lambda_2 + 2\lambda_3 v_{k-3}^{l+1})\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2-1)}\right)\right\}$$
$$+ c_1\lambda_3\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2-1)}\right)^{l-1}v_{k-1}^{l+1}v_k^l\left\{v_k^l - (\lambda_2 + 2\lambda_3 v_{k-3}^{l+1})\left(\frac{\lambda_1}{\lambda_2^2(\lambda_2-1)}\right)v_{k-1}^{l+1}\right\}$$
$$+ c_2\left(\frac{\lambda_2^2-1}{\lambda_2^2(\lambda_2+1)}\right)^{l-1}\left\{v_{k-1}^{l+1} - (\lambda_2 + 2\lambda_3 v_{k-3}^{l+1})\left(\frac{\lambda_2^2-1}{\lambda_2^2(\lambda_2-1)}\right)v_k^l\right\} + (-1)^k,$$
$$\psi = c_1(\lambda_2)^{2l-1}v_k^l v_k^{l+1}\{1 - (\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})\lambda_2^2\} + c_1\lambda_3(\lambda_2)^{2l-2}v_k^l v_k^{l+1}\{v_k^l - \lambda_2^2(\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})v_k^{l+1}\}$$
$$+ c_2\left(\frac{\lambda_2+1}{\lambda_2}\right)^{l-1}\{v_k^{l+1} - (\lambda_2 + 2\lambda_3 v_{k-2}^{l+2})v_k^l\} + (-1)^l. \tag{128}$$

3.4. Discrete Huxley equation

Consider the one-dimensional Huxley equation [26]

$$v_t = v_{xx} + v^2(1 - v), \tag{129}$$

where x and t are spatial and temporal variables respectively, v is dependent variable here. The approximations for the derivatives in case of implicit discretization are defined as

$$v_t \approx \frac{v_k^{l+1} - v_k^l}{\Delta t} + O(\Delta t)^2, \tag{130}$$

$$v_{xx} \approx \frac{v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}}{2(\Delta x)^2} + O(\Delta x)^3. \tag{131}$$

From equation (129), we have

$$\frac{v_k^{l+1} - v_k^l}{\Delta t} + O(\Delta t)^2 = \frac{v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}}{2(\Delta x)^2} + O(\Delta x)^3 + (v_k^l)^2(1 - v_k^l). \tag{132}$$

Ignoring the truncated terms and writing in simpler form

$$v_k^{l+1} - v_k^l = \nu(v_{k+1}^{l+1} - 2v_k^{l+1} + v_{k-1}^{l+1}) + (\Delta t)(v_k^l)^2(1 - v_k^l), \tag{133}$$

where $\nu = \frac{D(\Delta t)}{2(\Delta x)^2}$. Rearranging the terms, we get

$$v_{k+1}^{l+1} = Av_k^{l+1} - v_{k-1}^{l+1} - Bv_k^l - \lambda(v_k^l)^2 + \lambda(v_k^l)^3,$$

where $A = \frac{1}{\nu} + 2,\ \lambda = \frac{\Delta t}{\nu},\ B = \frac{1}{\nu}$. Now we compute here the symmetries of the equation known as the discrete Huxley equation.

3.4.1. Symmetries

Consider the discrete Huxley equation derived above

$$v_{k+1}^{l+1} = Av_k^{l+1} - v_{k-1}^{l+1} - Bv_k^l - \lambda(v_k^l)^2 + \lambda(v_k^l)^3. \tag{134}$$

This equation is of the form that is of type discussed in the equation (32)

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1}),$$

where the term on the right hand side $\omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1})$ is a smooth function. Let the characteristic of the equation (134) be $Q = Q(k, l, v_k^l)$. Then the symmetry condition (13) becomes

$$S_k S_l Q - X\omega = 0, \qquad Q(k+1, l+1, \omega) - X\omega = 0, \tag{135}$$

where equation (14) is

$$X = Q\frac{\partial}{\partial v_k^l} + (S_{-k}S_l Q)\frac{\partial}{\partial v_{k-1}^{l+1}} + (S_l Q)\frac{\partial}{\partial v_k^{l+1}}.$$

The symmetry condition (108) takes the form

$$Q(k+1, l+1, \omega) + Q(k-1, l+1, v_{k-1}^{l+1}) - AQ(k, l+1, v_k^{l+1}) \\ + (B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)Q(k, l, v_k^l) = 0. \tag{136}$$

By using the differential operator (53), we get

$$-(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)Q'(k, l+1, v_k^{l+1}) + (B + 2\lambda v_k^l \\ -3\lambda(v_k^l)^2)Q'(k, l, v_k^l) + (2\lambda - 6\lambda v_k^l)Q(k, l, v_k^l) = 0. \tag{137}$$

Solving this equation, we have

$$Q(k, l, v_k^l) = c_1(k, l)\left(\frac{1}{3}v_k^l - \frac{1}{9}\right) + \frac{(2\lambda + 6)v_k^l + B}{9(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)}c_1(k, l) + \frac{c_2(k, l)}{(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)}. \tag{138}$$

For simplicity, we take $c_1(k, l) = c_2(k, l) = 1$, so we get

$$Q(k, l, v_k^l) = \frac{1}{3}v_k^l - \frac{1}{9} + \frac{(2\lambda + 6)v_k^l + B}{9(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)} + \frac{1}{(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)}. \tag{139}$$

The following symmetry generators are obtained

$$\begin{aligned} X_1 &= \frac{\partial}{\partial v_k^l} + \frac{\partial}{\partial v_k^{l+1}} + \frac{\partial}{\partial v_{k-1}^{l+1}}, \\ X_2 &= v_k^l\frac{\partial}{\partial v_k^l} + v_k^{l+1}\frac{\partial}{\partial v_k^{l+1}} + v_{k-1}^{l+1}\frac{\partial}{\partial v_{k-1}^{l+1}}, \\ X_3 &= \frac{(2\lambda + 6)v_k^l + B}{(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)}\frac{\partial}{\partial v_k^l} + \frac{(2\lambda + 6)v_k^{l+1} + B}{(B + 2\lambda v_k^{l+1} - 3\lambda(v_k^{l+1})^2)}\frac{\partial}{\partial v_k^{l+1}} \\ &\quad + \frac{(2\lambda + 6)v_{k-1}^{l+1} + B}{(B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)}\frac{\partial}{\partial v_{k-1}^{l+1}}, \\ X_4 &= \frac{1}{(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)}\frac{\partial}{\partial v_k^l} + \frac{1}{(B + 2\lambda v_k^{l+1} - 3\lambda(v_k^{l+1})^2)}\frac{\partial}{\partial v_k^{l+1}} \\ &\quad + \frac{1}{(B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)}\frac{\partial}{\partial v_{k-1}^{l+1}}. \end{aligned} \tag{140}$$

3.4.2. First integral vectors

In this section, we derive the first integral vector for the discrete Huxley equation (134) given by

$$v_{k+1}^{l+1} = Av_k^{l+1} - v_{k-1}^{l+1} - Bv_k^l - \lambda(v_k^l)^2 + \lambda(v_k^l)^3.$$

This equation is of the form discussed in the equation (32)

$$v_{k+1}^{l+1} = \omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1}),$$

where the right hand side $\omega(k, l, v_k^l, v_k^{l+1}, v_{k-1}^{l+1})$ is a smooth function. Then we define $P_2 = P_2(k, l, v_k^l)$ and the first integral condition for P_2 follows from equation (46) as

$$P_2(k, l, v_k^l) + (B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)[P_2(k, l+1, v_k^{l+1}) - (B + 2\lambda v_k^l - 3\lambda(v_k^l)^2) \\ +P_2(k+2, l, v_{k+2}^l) - (B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)P_2(k, l+1, v_k^{l+1})] = 0. \tag{141}$$

By using the differential operator (53), we obtain

$$P_2'(k, l, v_k^l) + \frac{1}{A}(B + 2\lambda v_{k+1}^l - 3\lambda(v_{k+1}^l)^2)(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2) \\ +(1 - B - 2\lambda v_{k+2}^l + 3\lambda(v_{k+2}^l)^2))P_2'(k, l+1, v_k^{l+1}) = 0. \tag{142}$$

Differentiating this equation w.r.t. v_k^l, and simplifying, we get

$$P_2(k, l, v_k^l) = Bc_1(k, l)v_k^l + \lambda c_1(k, l)(v_k^l)^2 - \lambda c_1(k, l)(v_k^l)^3 + c_2(k, l). \tag{143}$$

Further using equations (141)–(142), we have

$$P_2(k, l, v_k^l) = c_3 B\left(\frac{A}{B(B-1)}\right)^{l-1} v_k^l + c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1}(v_k^l)^2$$
$$-c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1}(v_k^l)^3 + c_4(-1)^{l-1}. \tag{144}$$

From equation (40), the relation for P_1 follows as

$$P_1(k-1, l+1, v_{k-1}^{l+1}) = -(B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left\{c_3 B\left(\frac{A}{B(B-1)}\right)^{l} v_{k-1}^{l+1}\right.$$
$$\left.+c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l}(v_{k-1}^{l+1})^2 - c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l}(v_{k-1}^{l+1})^3 + c_4(-1)^l\right\}. \tag{145}$$

Using the integrability condition (47), we get

$$\frac{\partial P_1}{\partial v_{k-1}^{l+1}} = -(B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left\{c_3 B\left(\frac{A}{B(B-1)}\right)^{l}\right.$$
$$\left.+2c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l} v_{k-1}^{l+1} - 3c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l}(v_{k-1}^{l+1})^2\right\},$$
$$\times\frac{\partial P_2}{\partial v_k^l} = c_3 B\left(\frac{A}{B(B-1)}\right)^{l-1} + 2c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1} v_k^l$$
$$-3c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1}(v_k^l)^2. \tag{146}$$

Then, from equation (48) the component ϕ of the first integral vector follows as

$$\phi = c_3 B\left(\frac{A}{B(B-1)}\right)^{l-1} v_k^l v_{k-1}^{l+1}\left\{1 - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left(\frac{A}{B(B-1)}\right)\right\}$$
$$+ c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1} v_{k-1}^{l+1} v_k^l\left\{v_k^l - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left(\frac{A}{B(B-1)}\right)v_{k-1}^{l+1}\right\}$$
$$- c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1} v_{k-1}^{l+1} v_k^l\left\{(v_k^l)^2 - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left(\frac{A}{B(B-1)}\right)(v_{k-1}^{l+1})^2\right\}$$
$$+ c_4(-1)^l\{v_{k-1}^{l+1} - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)v_k^l\} + G(k, l). \tag{147}$$

Now we proceed to obtain the second component ψ of the first integral vector. For this we define $R_2 = R_2(k, l, v_k^l)$ and the first integral condition for R_2 follows from equation (46) as

$$R_2(k, l, v_k^l) + (B + 2\lambda v_{k+2}^l - 3\lambda(v_{k+2}^l)^2)[R_2(k-1, l+1, v_{k-1}^{l+1}) - (B + 2\lambda v_{k+1}^l - 3\lambda(v_{k+1}^l)^2)$$
$$R_2(k+1, l, v_{k+1}^l) - (B + 2\lambda v_{k+1}^l - 3\lambda(v_{k+1}^l)^2)R_2(k+2, l+1, v_{k+2}^{l+1})] = 0. \tag{148}$$

By using the differential operator (94), we obtain

$$R_2'(k, l, v_k^l) - (B + 2\lambda v_{k+2}^l - 3\lambda(v_{k+2}^l)^2)(B + 2\lambda v_k^l - 3\lambda(v_k^l)^2)+$$
$$R_2'(k-1, l+1, v_{k-1}^{l+1}) = 0. \tag{149}$$

Differentiating this equation w.r.t. v_k^l, and simplifying we get

$$R_2(k, l, v_k^l) = c_3(B)^{2k-1}v_k^l + c_3\lambda(B)^{2k-2}(v_k^l)^2 - c_3\lambda B^{2k-2}(v_k^l)^3. \tag{150}$$

On the other hand, the relation for R_1 becomes

$$R_1(k, l+1, v_k^{l+1}) = -(B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)\{c_3(B)^{2k-1}v_k^{l+1}$$
$$+c_3\lambda(B)^{2k-2}(v_k^{l+1})^2 - c_3\lambda B^{2k-2}(v_k^{l+1})^3\}. \tag{151}$$

Using the integrability condition (47), we get

$$\frac{\partial R_1}{\partial v_k^{l+1}} = -(B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)\{c_3(B)^{2k-1} + 2c_3\lambda(B)^{2k-2}v_k^{l+1} - 3c_3\lambda B^{2k-2}(v_k^{l+1})^2\},$$
$$\times\frac{\partial R_2}{\partial v_k^l} = c_3(B)^{2k-1} + 2c_3\lambda(B)^{2k-2}v_k^l - 3c_3\lambda B^{2k-2}(v_k^l)^2. \quad (152)$$

Then, from equation (48), second component ψ of the first integral vector follows as

$$\psi = c_3(B)^{2k-1}v_k^l v_k^{l+1}\{1 - B - 2\lambda v_{k-1}^{l+1} + 3\lambda(v_{k-1}^{l+1})^2\} + c_3\lambda(B)^{2k-2}v_k^l v_k^{l+1}\{v_k^l - (B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)v_k^{l+1}\} - c_3\lambda(B)^{2k-2}v_k^l v_k^{l+1}\{(v_k^l)^2 - (B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)(v_k^{l+1})^2\} + H(k, l). \quad (153)$$

We thus have

Theorem 3.4. *The first integral vector for the discrete Huxley equation (134) has components*

$$\phi = c_3B\left(\frac{A}{B(B-1)}\right)^{l-1}v_k^l v_{k-1}^{l+1}\left\{1 - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left(\frac{A}{B(B-1)}\right)\right\} + c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1}v_{k-1}^{l+1}v_k^l\left\{v_k^l - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left(\frac{A}{B(B-1)}\right)v_{k-1}^{l+1}\right\} - c_3\lambda\left(\frac{A}{B(B-1)}\right)^{l-1}v_{k-1}^{l+1}v_k^l\left\{(v_k^l)^2 - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)\left(\frac{A}{B(B-1)}\right)(v_{k-1}^{l+1})^2\right\} + c_4(-1)^l\{v_{k-1}^{l+1} - (B + 2\lambda v_{k-3}^{l+1} - 3\lambda(v_{k-3}^{l+1})^2)v_k^l\} + (-1)^k,$$
$$\psi = c_3(B)^{2k-1}v_k^l v_k^{l+1}\{1 - B - 2\lambda v_{k-1}^{l+1} + 3\lambda(v_{k-1}^{l+1})^2\} + c_3\lambda(B)^{2k-2}v_k^l v_k^{l+1}\{v_k^l - (B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)v_k^{l+1}\} - c_3\lambda(B)^{2k-2}v_k^l v_k^{l+1}\{(v_k^l)^2 - (B + 2\lambda v_{k-1}^{l+1} - 3\lambda(v_{k-1}^{l+1})^2)(v_k^{l+1})^2\} + (-1)^l. \quad (154)$$

4. Conclusions

The Lie group method stands as a cornerstone in the domain of mathematical physics and engineering, offering a profound means to systematically explore the symmetries embedded within differential equations. This study extends the application of the Lie group method to DEs, broadening its scope. Specifically, the research entails the symmetry analysis of certain nP $\triangle$ Es. The discrete counterparts of well-known equations, such as the wave, diffusion, Fisher, and Huxley equations, were derived through a chosen discretization approach. The symmetry algebra for these equations was determined utilizing Hydon's method, providing valuable insights into the symmetries inherent in the discrete formulations of these fundamental mathematical models. We derived some 'first integral vector' conditions for P$\triangle$ Es independent of the underlying symmetry properties.

Data availability statement

No data was used while preparing this article. The data that support the findings of this study are available upon reasonable request from the authors.

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Authors' contributions

AH Kara and FD Zaman formulated the problems and presented the initial work and supervised the research. Akhtar Hussain performed the detailed calculations and preparations for the manuscript.

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