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DOCTOR OF PHILOSOPHY THESIS

# Symmetry reductions and approximate solutions for heat transfer in slabs and extended surfaces

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# Abstract

In this study we analyse heat transfer models prescribed by reaction-diffusion equations. The focus and interest throughout the work is on models for heat transfer in solid slabs (hot bodies) and extended surface. Different phenomena of interest are heat transfer in slabs and through fins of different shapes and profiles. Furthermore, thermal conductivity and heat transfer coefficients are temperature dependent. As a result, the energy balance equations that are produced are nonlinear. Using the theory of Lie symmetry analysis of differential equations, we endeavor to construct exact solutions for these nonlinear models. We will employ a number of symmetry techniques such as the classical Lie point symmetry methods, the nonclassical symmetry, nonlocal and nonclassical potential symmetry approach to construct the group-invariant solutions. In order to identify the forms of the heat source term that appear in the considered equation for which the principal Lie algebra (PLA) is extended by one element, we first perform preliminary group classification of the transient state problem. Also, we consider the direct group classification method. Invariant solutions are constructed after some reductions have been performed. One-dimensional Differential Transform Method (1D DTM) will be used when it is impossible to determine an exact solution. The 1D DTM has been benchmarked using some exact solutions. To solve the transient/unsteady problem, we use the two-dimensional Differential Transform Method (2D DTM). Effects of parameters appearing in the equations on the temperature distribution will be studied.

# Publication associated with the thesis

1. M. Nkwanazana and R.J Moitsheki, Approximate analytical solutions for a heat transfer in a slab with internal heat generation, Defect and Diffusion Forum, Computational Fluid Dynamics and Mathematical Modelling. Accepted 21/10/2022

2. M. Nkwanazana and R.J Moitsheki, Preliminary group classification of nonlinear reaction-diffusion equation. In preparation.

# Declaration

I declare that the work done in this project is my own, unaided work. It is being submitted in fulfilment of the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

Signed:

Date: $15^{th}$ June 2023

# Dedication

To my family, friends and everyone who showed support throughout my studies.

# Acknowledgment

I really would like to express my admiration to my supervisor, Prof. R. J. Moitsheki, for his guidance and patience throughout the completion of this degree.

I want to express my gratitude to my friends, family, acquaintances, and coworkers for their encouragement and moral support.

For supporting my academic endeavors throughout, I sincerely thank the University of Johannesburg, South African Nuclear Human Asset and Research Programme (SAN-HARP), National Research Foundation (NRF), University of the Witwatersrand, and North West Premiers Office. Finally, I would want to express my sincere gratitude to University Capacity Development Programme (UCDP) for providing financing for staff replacement and to Sefako Makgatho Health Science University for granting me time off work to complete this research.

# Chapter 1

# Introduction

The study of heat conduction and diffusion processes leads to interesting mathematical models which can be formulated in terms of partial differential equations (PDEs). Diffusion with absorption arises in various scientific and engineering fields such as in biological populations, models for the transmission of nerve impulses and heating by microwave radiation. Such physical phenomena are modelled by a reaction-diffusion equation, (see [1, 2, 3, 4, 5]).

The problems being considered are an investigation to determine the analysis for reaction-diffusion models and also to learn probable physical representation where such solutions will be suitable. There are various techniques to apply in finding the solution for the reaction-diffusion equation. A substantial piece of research has been done in the process of analysing the reaction-diffusion equations, (e.g. [6, 7, 9, 8, 10]). In our work we will use Lie symmetry techniques to construct exact solutions. Differential Transform Method (DTM) will also be utilized to obtain approximate analytic solutions. Additionally, we intend to use the nonlocal (potential) symmetry approaches previously introduced by Bluman et al [11]. Also, nonclassical and nonclassical potential symmetry techniques will be employed. Authors in [12, 13, 14] studied reaction-diffusion equations using symmetry analysis.

## 1.1 Literature review

Most scientific problems that arise in mathematical physics are modelled by nonlinear PDEs. These models describe physical phenomena relating space and time derivatives. It is important to study exact and analytical or analytic solutions to gain knowledge or insight of physical phenomena. However, it is not always possible to find solutions explicitly. Nonetheless, in recent years significant progress has been made and many effective techniques for deriving exact solutions of nonlinear PDEs have been developed by various scientists/researchers. Some of these techniques used in the literature are the homogeneous balance method [15], Jacobi elliptic function expansion method [16, 17], Hirota's bilinear method [18], ansatz method [19, 20] and variable separation approach [21].

Heat transfer is a physical process of impulsive, irreversible heat transport from a region of higher temperature to a region of lower temperature. This process occurs when a hot object is placed in cold surroundings. The object loses the internal energy, while the surroundings gain internal energy. Such physical processes basically happen following three different mechanisms known as heat conduction, heat convection and thermal radiation. These phenomena often occur simultaneously though they have different characteristics. Heat conduction, also known as diffusion, is the microscopic exchange of kinetic energy via the border of two systems between particles like molecules or quasiparticles like lattice waves. When a body or its surroundings are at a temperature that is different from an object, heat flows between them until they both reach the same temperature, at which point they are in thermal equilibrium. According to the second rule of thermodynamics, such spontaneous heat transfer always takes place from one area with a high temperature to another with a lower temperature. Heat transfer by conduction simply means heat transfer through direct contact. Heat conduction is the transfer of thermal energy within energetic particles. The thermal conductivity of the material influences the whole process. When a fluid (gas or liquid) flows in its bulk, its heat is carried through the fluid. This is known as heat convection. Heat is also moved in part through diffusion in all convective processes. The movement of a fluid may be influenced by external forces or, less frequently in gravitational fields, buoyant forces brought on by the expansion of the fluid due to heat energy such as in a fire plume. This process is frequently referred to as natural convection. Forced convection occurs when an external force, such as a pump or a fan, is used to drive the fluid flow. The mechanical device applies pressure or suction, forcing the fluid to move in a specific direction. These method is commonly employed in various systems, including pumps in water circulation, fans in air conditioning, and blowers in industrial processes. The transfer of energy by thermal radiation, or electromagnetic waves is known as radiative heat transfer [22]; it can happen in a vacuum or any clear media, including a solid, liquid, or gas. At temperatures above absolute zero, all objects generate thermal radiation as a result of the random motions of matter's atoms and molecules. Due to the protons and electrons that make up these atoms and molecules, when they move, electromagnetic radiation is released, which is a kind of energy transfer. In engineering applications, radiation is often only significant for extremely hot items or objects with a significant temperature differential. The rate of radiant energy transmission is best represented when the objects and spaces between them are big in comparison to the wavelength of thermal radiation. For detailed understanding of the above mentioned heat transfer mechanism, (see e.g. [23]). Heat transfer analysis continues to be a field of interest to engineering applications such as, power systems, the auto-mobile industry, electronic chip cooling, heating and air conditioning and chemical engineering.

In this study we consider heat transfer through a hot body and extended surface (fin). These models are differentiated by the sign of the diffusivity term. Research regarding heat transfer through hot body is important due to its application in various field of study such as heat transfer through the human head [24] and in solids of various geometries [25]. The investigation of heat transfer in human body has been done with regard to the development of new medical treatments [26]. The reader may also refer to the following, see [27, 28, 29].

A fin is a surface device that extends from an object to improve the rate of heat transfer to or from the environment by increasing convection. The analysis of fin heat transfer is of a great impact due to the wide range applications. In particular, fins are used in various industrial applications such as air-cooled craft engines, air conditioning, refrigeration, cooling of computer processor, cooling of oil carrying pipe line and many other devices in which heat is generated. Fins are much longer, which makes it accurate to assume that the temperature varies only in the end-to-end direction. Basically at any point along the length of the fin, temperature is constant across the cross sectional area of the fin, which results in a one-dimensional heat transfer problem. Researchers in [30, 31] documented considerable amount of problems describing heat transfer in fins. Some attention has been paid to fin equations with linear thermal conductivity [32]. Techniques such as symmetry analysis, homotopy perturbation method and homotopy analysis method have been used to analyse nonlinear heat transfer systems, (see e.g. [33, 34, 35, 36]).

## 1.2 Aims and objectives

The aims and objectives of this proposed research are:

- To help advance our understanding of heat transfer processes.
- To employ direct and preliminary group classification. This exercise will give rise to the determination of functions of arbitrary cases for which the equations admit extra Lie point symmetries, beyond the principal Lie algebra.
- To perform symmetry reductions by the elements of the optimal systems.
- To apply the Lie point symmetry analysis; determine the nonclassical symmetries, the nonlocal (potential) symmetries and nonclassical potential symmetries.
- To construct the group-invariant solutions and interpret the solutions.
- To solve the nonlinear steady state problem and transient state problem using 1D DTM and 2D DTM respectively.

## 1.3 Outline

The thesis is outlined as follows:

Chapter 2 presents mathematical formulation for the reaction-diffusion equation. This will show how the governing equation for this study was derived.

Chapter 3 introduces the basic concepts and definitions of the mathematical tools used in this research. We discuss the Lie symmetry methods which include, point symmetries, nonlocal symmetries and nonclassical symmetries. Furthermore, we present the DTM technique.

Chapter 4 seeks to find the exact solutions of the reaction-diffusion equation by calculating the classical Lie point symmetries.

Chapter 5 provides approximate analytical solutions by using the 1D DTM and 2D DTM. Graphical representations of our solutions are presented.

In Chapter 6 Lie group classification of nonlinear reaction-diffusion equation is carried out. The equation studied in this chapter is a special case of the governing equation modelled in chapter 2.

In Chapter 7 we deploy nonclassical potential approach to one-dimensional transient heat conduction equation.

Finally Chapter 8 provides the summary of our research.

# Chapter 2

# Mathematical formulation

This chapter will focus on discussing models representing heat transfer through hot bodies (walls and slaps) and extended surfaces (fins). Many physical phenomena are described by deterministic models given in terms of differential equations. These equations may be linear or nonlinear and contain arbitrary functions which depend on dependent and/or independent variables. The exact solutions of these equations are useful on two folds; first they provide insight into physical phenomena and secondly they may be used as benchmarks for the numerical schemes. Heat is a form of energy while temperature is a property that determines the rate at which heat is transferred. Phase shifts involving work and energy often include heat as a key component. The process of moving heat from one object at a higher temperature to another object at a lower temperature is known as heat transfer. As a result, heat is a measure of the kinetic energy that the particles in a system have. The kinetic energy of the particles in a system will also increase as its temperature rises. Thus, an object's heat measurement can alter over time for a variety of causes. Environmental Conditions: The temperature and humidity of the surrounding environment can impact the heat measurement of an object. For example, if the ambient temperature is higher than the object's temperature, it can absorb heat from the environment and increase its overall heat measurement.

## 2.1 Heat transfer in hot body

The study of heat transfer in hot body is essential in various engineering applications. To achieve heat transmission, engineers also take into account the advection of material containing various chemical species, whether hot or cold. Despite the fact that each of these processes has unique properties, they frequently coexist in the same system. The different forms of geometry of hot body like rectangular, radial or spherical form has been studied in [25]. Heat generation is the process of converting an energy source into heat energy in a medium. The medium's temperature rises as a result of heat production. Consider a body $A$ that exchanges heat with another body $B$ of infinite media. This is a common heat transfer problem.

The heat flow rate is commonly proportional to the temperature difference, with the exceptions of radiation heat transfer and free convection caused by density differences. Resistance heating in wires, exothermic chemical reactions in materials, and nuclear reaction are a few examples of how heat is produced. In the majority of applications, the maximum temperature $T_{max}$ and surface temperature $T_s$ of solids involved in heat generation are of interest. When the outer surface is kept at a constant temperature,

$T_s$, the highest temperature $T_{max}$ that involves homogeneous heat generation will occur at a position most distant from the outer surface, (see figure 2.1). Alternative interpretation: figure 2.1 represents the highest temperature in a symmetrical solid with constant heat generation at the centre while figure 2.2 represents heat transport through a wall.

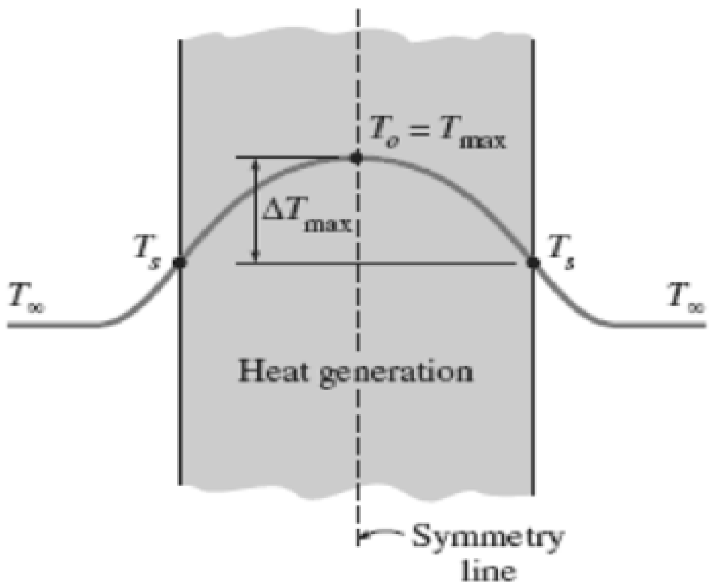



Figure 2.1: Heat transfer in a solid with internal heat generation.

We consider a one-dimensional heat equation given by

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{Q}{k}, \ \alpha = \frac{k}{\rho c}, \tag{2.1}$$

where $k$ and $Q$ are taken as constants. The parameters for functions in equation (2.1) are described in the Table 2.1. If we consider $k$ and $Q$ to be functions of $T$, equation

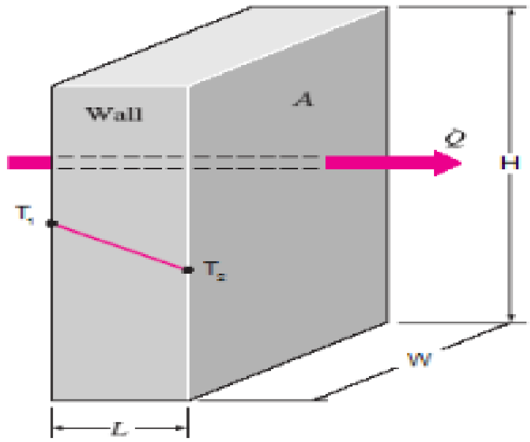



Figure 2.2: Graphical representation of heat transfer through a wall.

(2.1) may be written as

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) + Q(T), \tag{2.2}$$

where $k(T)$ and $Q(T)$ are called the thermal conductivity and heat source term respectively. $T$ is the dimensionless temperature, $t$ and $x$ are dimensionless time and space variables. Heat transfer equations are governed by boundary conditions, hence the imposed boundary conditions are given as

$$\frac{\partial T}{\partial x}(t,0) = 0, \quad T(t,-L) = T_s = T(t,L), \tag{2.3}$$

where $T_s$ is a constant, and

$$T(t,-L) = T_1, \quad T(t,L) = T_2, \tag{2.4}$$

Table 2.1: Nomenclature

| Variable | Description |
|---|---|
| $k$ | Thermal conductivity |
| $\alpha$ | Thermal diffusivity |
| $\rho$ | Density |
| $c$ | Specific heat capacity |
| $Q$ | Internal heat generation |
| $T$ | Temperature |
| $x$ | spatial variable |
| $t$ | Time |
| $L$ | Length |
| $T_1, T_2$ | Temperature at the boundaries |
| $f$ | initial temperature |
| $T_a$ | ambient temperature |
| $k_a$ | thermal conductivity |
| $q_a$ | internal heat generation |
| $\theta$ | dimensionless temperature |

where $T_1 \neq T_2$. Further studies may be done with respect to equation (2.3) for $x \in [0, L]$. Equation (2.4) presents the heat flow from higher to lower temperature. Initial condition is given as

$$T(0, x) = f(x), \tag{2.5}$$

where $f(x)$ represent initial heat profile. The boundary conditions (2.3) and (2.4) are symmetric and asymmetric respectively.

### Dimensional analysis of models describing heat transfer in a slab

Dimensional analysis is a technique employed to reduce a problem to a minimal set of non-dimensional variables. We now present dimensionless ratios of the variables as follows:

$$\bar{x} = \frac{x}{l}, \qquad \bar{t} = \frac{tk_a}{\rho c l^2}, \qquad \theta = \frac{T}{T_a}, \qquad q = \frac{Q}{q_a}, \qquad \bar{k} = \frac{k}{k_a}, \tag{2.6}$$

where $\bar{x}$ is the dimensionless length, $\bar{t}$ is the dimensionless time, $\theta$ is the dimensionless temperature, $q_a$ is a characteristic internal heat generation and $k_a$ is the characteristic thermal conductivity. Utilising the chain rule and change of variable from equation (2.6), we express equation (2.2) in terms of dimensionless variables $\theta$ and $\bar{x}$ and obtain

$$\frac{\partial \theta}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left(\bar{k}(\theta)\frac{\partial \theta}{\partial \bar{x}}\right) + NgQ(\theta), \tag{2.7}$$

where $Ng = \frac{l^2}{k_a T_a}$ represents the internal heat generation term and is considered to be positive. We will discuss the effect of $Ng$ when assumed to be negative in the next section. For simplicity we neglect the bars, hence equation (2.7) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) + NgQ(\theta), \tag{2.8}$$

subject to initial condition

$$\theta(0,x) = 0, \quad 0 \le x \le 1. \tag{2.9}$$

The prescribed boundary conditions are provided as

$$\frac{\partial \theta}{\partial x} = 0, \qquad \theta(t,1) = 1 \tag{2.10}$$

for a wall with identical temperature at the boundaries and

$$\theta(t,-1) = 0.1 \qquad \text{and} \qquad \theta(t,1) = 1 \tag{2.11}$$

for walls with varying temperature at the boundaries along with initial condition

$$\theta(0,x) = f(x). \tag{2.12}$$

The reaction-diffusion equation (2.8) is considered to be a general form of heat transfer model. The rate at which the heat transfer travels from a hot body to the surrounding may be increased by surfaces which extend into that surrounding. Researchers [37, 38, 39, 40] have studied (2.8) for different forms of $k$, $Q$ and the sign of $Ng$.

## 2.2 Heat transfer in fins

Kraus et al. [30] have presented well-documented analysis of heat transmission problems with an emphasis on extended surfaces (fins). Fins are surfaces that extend from a hot object to increase the rate of heat transfer to the surrounding fluid. The steady-state differential equation is a good representation of models that best describe heat transport in fins. This section gives a brief theoretical background on heat transfer in longitudinal one-dimensional fin of cross sectional area $A_c$ as shown by figure 2.3. The

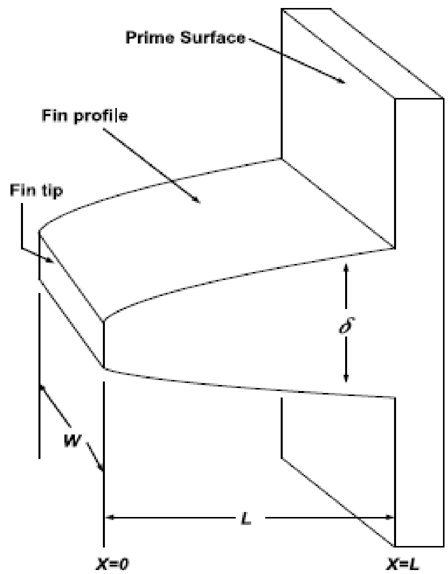



Figure 2.3: Longitudinal fin of an arbitrary profile.

perimeter of the fin is denoted by $P$ and the length of fin by $L$. The fin is attached to a fixed base surface of temperature $T_b$ and extends into a fluid of temperature $T_a$. The fin profile is given by the function $F(X)$, fin thickness $\delta$ depends on the fin profile and the fin thickness at the base is $\delta_b$. The model in question is given by

$$\rho c \frac{\partial T}{\partial t} = \frac{\delta_a}{2} \frac{\partial}{\partial X}\left( F(X)K(T)\frac{\partial T}{\partial X}\right) - \frac{P}{A_c}H(T)(T - T_a), \quad 0 \leq X \leq L. \qquad (2.13)$$

The functions $K$ and $H$ are the non-uniform thermal conductivity and heat transfer coefficients depending on the temperature, $\rho$ is the density, $c$ is the specific heat capacity, $T$ is the temperature distribution, $t$ is the time and $X$ is the space variable. The model given by (2.13) is known as the energy balance for a longitudinal fin of an arbitrary profile. The heat transfer coefficient is the amount of heat which passes through a unit area of a medium or system in a unit time when the temperature difference between the boundaries of the system is 1 degree. The fin length is measured from the tip to the base. Assuming that the fin tip is adiabatic and the base temperature is kept constant, then the boundary conditions are give by

$$T(t, L) = T_b \quad \text{and} \quad \left.\frac{\partial T}{\partial X}\right|_{X=0} = 0, \tag{2.14}$$

and initially the fin is kept at the ambient temperature

$$T(0, X) = T_a. \tag{2.15}$$

**Dimensional analysis for a fin model**

Introducing dimensionless variables (see [30])

$$\begin{aligned} &\bar{x} = \frac{X}{L}, \qquad \bar{t} = \frac{tk_a}{\rho c_v L^2}, \qquad \theta = \frac{T - T_a}{T_b - T_a}, \qquad h = \frac{H}{h_a}, \\ &k = \frac{K}{k_a}, \qquad M^2 = \frac{Ph_b L^2}{A_c k_a}, \qquad f(x) = \frac{\delta_b}{2}F(X). \end{aligned} \tag{2.16}$$

with $k_a$ defined as the thermal conductivity at the ambient temperature and $h_b$ as the heat transfer at the fin base. Using (2.16) reduces equation (2.13) to

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(f(x)k(\theta)\frac{\partial \theta}{\partial x}\right) - M^2 h(\theta)\theta, \quad 0 \leq x \leq 1. \tag{2.17}$$

Given the initial condition

$$\theta(0, x) = 0, \tag{2.18}$$

the sudden alteration in temperature at the base of the fin

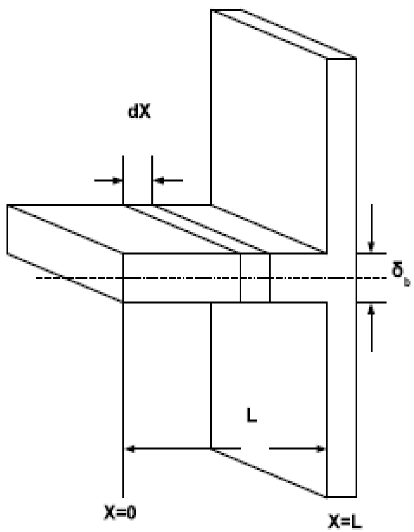
$$\theta(t, 1) = 1, \tag{2.19}$$

and the boundary condition at the tip of the fin

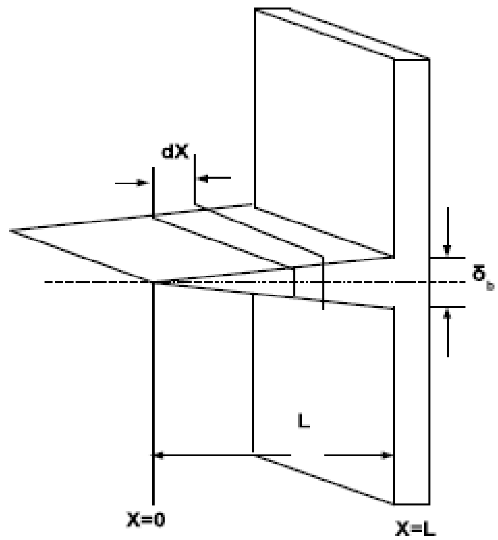
$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0, \tag{2.20}$$

subject to these conditions.

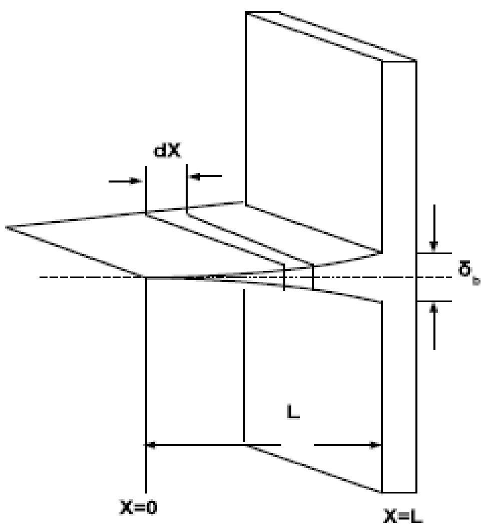
The thermo-geometric fin parameter is represented by the dimensionless variable $M$, $\bar{x}$ is the dimensionless space variable, $k$ is the dimensionless thermal conductivity, $\theta$ is the dimensionless temperature. Figure 2.4 $(a) - (d)$ shows schematic representation of different forms of fin profiles.



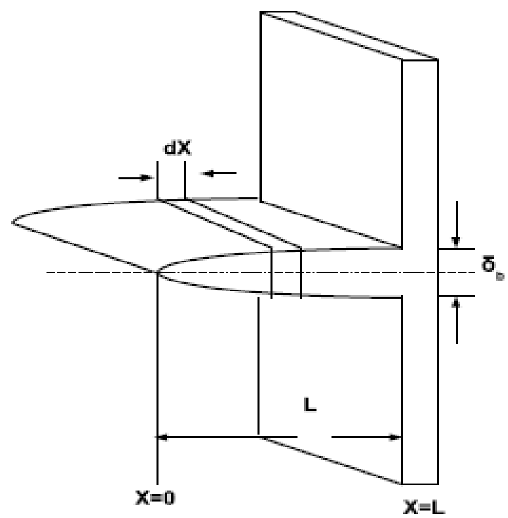

(a) Rectangular profile




(b) Triangular profile




(c) Concave parabolic profile




(d) Convex parabolic profile

Figure 2.4: Schemes of different fin profiles

## 2.3 Concluding remarks

We have covered reaction-diffusion equations based heat transfer models in this chapter. We have formulated our governing equations, with equation (2.1) serving as the basic structure of the heat transfer model. We were able to analyse equation (2.1) in terms of heat transmission through the hot body, subject to the conductivity term and the sign of the heat source term. The overall form of the extended surface is also provided by equation (2.13). In terms of non-dimesional variables, the initial and boundary value problems were presented.

# Chapter 3

# Methods of solution

## 3.1 Historical background

The Lie group analysis also known as the Lie symmetry or classical symmetry method, was developed by the Norwegian mathematician Sophus Lie (1842-1899) and it is one of the useful techniques in finding solutions of (linear or nonlinear) differential equations. He discovered that majority of the methods for solving differential equations could be explained and deduced simply by means of invariance of the differential equations under a continuous group of symmetries. Lie symmetry method has a profound impact on all areas of mathematics, as well as physics, engineering and other mathematically based sciences. The traditional Lie group technique has undergone numerous generalisations for symmetry reductions. Nonclassical symmetry method is one such generalisation of classical symmetry method and it is used for finding additional invariant solutions of PDEs. Bluman and Cole proposed the concept of nonclassical symmetries in 1969, resulting in a new way for discovering symmetries.

Bluman et al. presented an algorithmic technique in [11] that produces new kinds of symmetries of a given PDE that are neither Lie point nor Lie-Bäcklund symmetries. These new kind of symmetries are known as nonlocal (potential) symmetries. In other words, by including additional dependent variables in an auxiliary covering system that surrounds a particular system of PDEs, valuable nonlocal symmetries can be discovered. When a Lie point symmetry of the auxiliary system operates on the space made up of the independent and dependent variables of the given system, as well as the auxiliary variables, the given system acquires a nonlocal symmetry. The equivalent nonlocal symmetries are referred to as potential symmetries since the auxiliary system is created by substituting the given PDE with an appropriate conservation law. When determining nonlocal symmetries, especially when analysing the auxiliary system, caution is required. It is possible that some of the nonlocal symmetry-bearing auxiliary systems are hidden.

Symmetries of differential equations (DEs) may be used to reduce the number of independent variables of the PDEs or the order of the ordinary differential equations (ODEs). These reductions often lead to construction of group-invariant (exact) solutions. It is also possible to determine the cases of arbitrary functions appearing in the given equation for which extra symmetries are obtained. This exercise is called the group classification. Group classification may be performed by direct methods or via equivalence transformations. All these methods will be exploited in pursuit of construction of group-invariant solutions.

## 3.2 Lie symmetry methods

### 3.2.1 Classical Lie point symmetries

Briefly we present basic Lie group theory and the algorithm for calculation of Lie point symmetries of PDEs. The Lie point symmetry method is based on symmetry and invariance principles and it is a systematic method for solving DEs analytically. The mathematical ideas of Lie's theory are presented in; e.g., Olver [41], Bluman and Kumei [42] and Ibragimov [49] and many more. The books mentioned above may be consulted for more information on the definitions and results presented in this work. Symbolic computer packages may also be used to perform calculations involving Lie's theory.

#### Local continuous one-parameter Lie group

Let us take $x = (x^1, ..., x^n)$ to be the independent variable with coordinates $x^i$ and $u = (u^1, .., u^m)$ to be the dependent variable with coordinates $u^\alpha$ ($n$ and $m$ finite).

**Definition 3.1** A set $G$ of transformations

$$T_a : \bar{x}^i = f^i(x, u, a),\ \bar{u}^\alpha = \phi^\alpha(x, u, a),\ i = 1, ..., n,\ \alpha = 1, ..., m, \tag{3.1}$$

where $a$ is a real parameter which continuously takes values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and $f^i$, $\phi^\alpha$ are differentiable functions, is called a *local continuous one-parameter Lie group of transformations* in the space of variables $x$ and $u$ if

(i) For $T_a$, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b\,T_a = T_c \in G$, $c = \phi(a,b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0\,T_a = T_a\,T_0 = T_a$ (Identity), and

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a\,T_{a^{-1}} = T_{a^{-1}}\,T_a = T_0$ (Inverse).

The associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a,b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a,b)) \end{aligned} \tag{3.2}$$

and the function $\phi$ is called the *group composition law*. A group parameter $a$ is called *canonical* if $\phi(a,b) = a + b$.

**Theorem 3.1** For any composition law $\phi(a,b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}$$

where

$$w(s) = \left. \frac{\partial\, \phi(s,b)}{\partial b} \right|_{b=0}.$$

**Prolongation formulas and Group generator**

The derivatives of $u$ with respect to $x$ are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{3.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n, \quad \alpha = 1, ..., m \tag{3.4}$$

is the operator of total differentiation. The collection of all first derivatives $u_i^\alpha$ is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only $u_{ij}^\alpha$ for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$ ,$u_{(5)}$ $\cdots$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (3.3). Thus the $u_s^\alpha$ are called differential variables and a $p$th-order PDE is given as

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{3.5}$$

**Prolonged or extended groups**

If $z = (x, u)$, one-parameter group of transformations $G$ is

$$\begin{aligned} \bar{x}^i &= f^i(x, u, a), \quad f^i|_{a=0} = x^i, \\ \bar{u}^\alpha &= \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \end{aligned} \tag{3.6}$$

According to Lie's theory, the construction of the symmetry group $G$ is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{x}^i \approx x^i + a\,\xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{3.7}$$

obtained from (3.1) by expanding the functions $f^i$ and $\phi^\alpha$ into Taylor series in $a$ about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x, u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{3.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (3.7) as

$$\bar{x}^i \approx (1 + a\,X)x, \quad \bar{u}^\alpha \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\,. \tag{3.9}$$

The differential operator (3.9) is called the infinitesimal operator or generator of the group $G$. Here we see how the derivatives are transformed.

The $D_i$ transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{3.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. Therefore

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots,$$

and

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha. \end{aligned} \tag{3.11}$$

Hence

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{3.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, for small $a$, ie., (3.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{3.13}$$

The transformations in $x, u, u_{(1)}$ space given by (4.35) and (3.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group $G$ and denoted by $G^{[1]}$.

Let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{3.14}$$

to be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (3.7) and (3.14). Higher-order prolongations of $G$, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (3.11).

**Prolonged generators**

Using (3.11) together with (3.7) and (3.14) we get

$$\begin{aligned} D_i(f^j)(\bar{u}^\alpha_j) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u^\alpha_j + a\zeta^\alpha_j) &= D_i(u^\alpha + a\eta^\alpha) \\ (\delta^j_i + aD_i\xi^j)(h^\alpha_j + a\zeta^\alpha_j) &= h^\alpha_i + aD_i\eta^\alpha \\ u^\alpha_i + a\zeta^\alpha_i + au^\alpha_j D_i\xi^j &= u^\alpha_i + aD_i\eta^\alpha \\ \zeta^\alpha_i &= D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j), \quad \text{(sum on } j). \end{aligned} \tag{3.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta^\alpha_{ij} = D_j(\eta^\alpha_i) - u^\alpha_{ik} D_j(\xi^k), \quad \text{(sum on } k). \tag{3.16}$$

By induction (recursively)

$$\zeta^\alpha_{i_1,i_2,\ldots,i_p} = D_{i_p}(\zeta^\alpha_{i_1,i_2,\ldots,i_{p-1}}) - u^\alpha_{i_1,i_2,\ldots,i_{p-1}\,j} D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{3.17}$$

The first and higher prolongations of the group $G$ form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$
\begin{aligned}
X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial h_i^\alpha} \quad \text{(sum on } i, \alpha), \\
&\vdots \\
X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial h_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1,
\end{aligned}
$$

where

$$
X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{3.18}
$$

**Group admitted by a PDE**

**Definition 3.2** The vector field (3.18) is a *point symmetry* of the $p$th-order PDE (3.5), if

$$
X^{[p]}(E) = 0 \tag{3.19}
$$

whenever $E = 0$. This can also be written as

$$
X^{[p]}\, E\big|_{E=0} = 0, \tag{3.20}
$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

**Definition 3.3** Equation (3.19) is called the *determining equation* of (3.5) because it determines all the infinitesimal symmetries of (3.5).

**Definition 3.4 (Symmetry group)** A one-parameter group $G$ of transformations (3.1) is called a symmetry group of equation (3.5) if (3.5) form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{3.21}$$

where the function $E$ is the same as in equation (3.5).

**Group invariants**

**Definition 3.5** A function $F(x, u)$ is called an *invariant of the group of transformation* (3.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a),\, \phi^\alpha(x, u, a)) = F(x, u), \tag{3.22}$$

identically in $x$, $u$ and $a$.

**Theorem 3.2 (Infinitesimal criterion of invariance)** A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{3.23}$$

It follows from the above theorem that every one-parameter group of point transformations (3.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integral:

$$J_1(x,u) = c_1, \cdots, J_n(x,u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^n}{\eta^n(x,u)}.$$

**Theorem 3.3** If the infinitesimal transformation (3.7) or its symbol $X$ is given, then the corresponding one-parameter group $G$ is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x},\bar{u}) \tag{3.24}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

**Lie algebra**

**Definition 3.6** A Lie algebra is formed by a vector space $L$ (over a field of real numbers), together with a binary operator [ , ] called a Lie bracket known as commutator, defined on $L$ such that the following properties hold:

1. Bilinear: for any $X_1, X_2, X_3 \in L$ and $a, b \in \mathbb{R}$,

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3],$$
$$[X_1, aX_2 + bX_3] = a[X_1, X_2] + b[X_1, X_3];$$

2. Skew-symmetric: If $X_1, X_2 \in L$, then

$$[X_1, X_2] = -[X_2, X_1];$$

3. Jacobi identity: If $X_1, X_2, X_3 \in L$, then

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0.$$

**Definition 3.7** Consider a Lie algebra $L$. If the vector space $L$ is finite-dimensional, its dimension is the dimension of the Lie algebra, that is, the finite-dimensional Lie algebra of dimension $r$ is denoted by $L_r$. Lie bracket [ , ] is defined on the set of vector fields $\nu$ as

$$[X_1, X_2] = X_1X_2 - X_2X_1, \quad \text{for any} \quad X_1, X_2 \in \nu, \tag{3.25}$$

where

$$X_1 = \xi_1^i \frac{\partial}{\partial x^i} + \eta_1^\alpha \frac{\partial}{\partial q^\alpha}$$

and

$$X_2 = \xi_2^i \frac{\partial}{\partial x^i} + \eta_2^\alpha \frac{\partial}{\partial q^\alpha}.$$

The binary operation (3.25) makes the space of vector field $\nu$ a Lie algebra. It is possible to determine reduction by any linear combination of the admitted symmetries. However, to determine as minimal set of reductions that cannot be mapped by any point transformation, we construct an optimal system.

### 3.2.2 Nonclassical symmetry techniques

The concept of nonclassical symmetry method was first discussed by Bluman and Cole in 1969 [50] and this methods results in a new way for discovering symmetries. The infinitesimal criterion for invariance is subject to an extra condition; namely, the invariance is sought not only on the solutions of the equation but also on the invariance surface condition (ISC). As a result, equation (3.20) contains an additional condition that can be expressed as

$$X^{[p]}E|_{E=0,ISC} = 0 \tag{3.26}$$

where ISC is given by

$$\xi^i(x,q)\frac{\partial q^\alpha}{\partial x^i} = \eta^\alpha(x,q) \tag{3.27}$$

in a general form. Equation (3.26) results in a system of nonlinear determining equations, in contrast to the situation of conventional Lie point symmetries. Here is an example of how to determine nonclassical symmetries.

We consider the model describing heat transfer in slab given as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] + Ng\theta^3 \tag{3.28}$$

where $Ng$ is a constant. Here $Ng$ is positive and is representing heat transfer through the slab. In order to make the governing equation (3.28) invariant, one should look for one parameter group of transformations given by the vector field

$$\Gamma = \tau(t,x,\theta)\frac{\partial}{\partial t} + \xi(t,x,\theta)\frac{\partial}{\partial x} + \eta(t,x,\theta)\frac{\partial}{\partial \theta}. \tag{3.29}$$

Under the constraint of the invariant surface condition

$$\tau\theta_t + \xi\theta_x = \eta \tag{3.30}$$

we seek invariance of the governing equation, which occasionally results in additional reductions that are not possible with the traditional approach. Without loss of generality, we assume $\tau = 1$. The invariant criterion for determining nonclassical symmetries is given by

$$\Gamma^{[2]}\left(\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] + Ng\theta^3\right)\Bigg|_{(3.28),ISC} = 0 \tag{3.31}$$

where

$$\begin{aligned}\Gamma^{[2]} &= \tau(t,x,\theta)\frac{\partial}{\partial t}+\xi(t,x,\theta)\frac{\partial}{\partial x}+\eta(t,x,\theta)\frac{\partial}{\partial \theta}+\zeta_t\frac{\partial}{\partial \theta_t}+\zeta_x\frac{\partial}{\partial \theta_x} \\ &\quad +\zeta_{xx}\frac{\partial}{\partial \theta_{xx}}+\cdots. \end{aligned} \tag{3.32}$$

The partial derivatives of $\zeta$'s are

$$\begin{aligned}\zeta_x(t,x,\theta) &= D_x[\eta(t,x,\theta)]-\theta_t D_x[\tau(t,x,\theta)]-\theta_x D_x[\xi(t,x,u)],\\ \zeta_t(t,x,\theta) &= D_t[\eta(t,x,\theta)]-\theta_t D_t[\tau(t,x,\theta)]-\theta_x D_t[\xi(t,x,\theta)],\\ \zeta_{xx}(t,x,\theta) &= D_x[\zeta_x(t,x,\theta)]-\theta_{xt} D_x[\tau(t,x,\theta)]-\theta_{xx} D_x[\xi(t,x,\theta)],\\ \zeta_{xt}(t,x,\theta) &= D_t[\zeta_x(t,x,\theta)]-\theta_{xt} D_t[\tau(t,x,\theta)]-\theta_{xx} D_t[\xi(t,x,\theta)],\end{aligned}$$

where $D_x$ and $D_t$ represent the total derivative and are given by

$$D_x = \frac{\partial}{\partial x}+\theta_x\frac{\partial}{\partial \theta}+\theta_{xx}\frac{\partial}{\partial \theta_x}+\theta_{xt}\frac{\partial}{\partial \theta_t}+\cdots, \tag{3.33}$$

$$D_t = \frac{\partial}{\partial t}+\theta_t\frac{\partial}{\partial \theta}+\theta_{tt}\frac{\partial}{\partial \theta_t}+\theta_{xt}\frac{\partial}{\partial \theta_x}+\cdots. \tag{3.34}$$

We employ the interactive software MAPLE to facilitate the calculations. Equation (3.31) leads to a system of nonlinear determining equations for the unknown functions

$\xi(t,x,\theta)$ and $\eta(t,x,\theta)$

$$\xi_{\theta\theta} = 0, \tag{3.35}$$

$$2n\,x^{n+1}\xi_\theta + 2x^2\,\xi\,\xi_\theta - 2\,x^{n+2}\xi_{x\theta} + x^{n+2}\eta_{\theta\theta} = 0, \tag{3.36}$$

$$3\,Ngx^2\theta^3\,\xi_\theta + nx^{n+1}\xi_x + x^2\xi_t - 2x^2\,\eta\,\xi_\theta + 2x^2\,\xi\xi_x + 2\,x^{n+2}\eta_{x_\theta} - x^{n+2}\xi_{xx}$$

$$-nx\,\xi^2 - nx^n\xi = 0, \tag{3.37}$$

$$-Ngnx\theta^3\xi + 3\,Ngx^2\theta^2\eta - Ngx^2\theta^3\eta_\theta + 2\,Ngx^2\theta^3\xi_x + nx\eta\xi + nx^{n+1}\eta_x$$

$$-2x\eta\xi_x + x^{n+2}\eta_{xx} - x^2\eta_t = 0. \tag{3.38}$$

Integrating equation (3.35) with respect to $\theta$ gives

$$\xi \;=\; a(t,x)\theta + b(t,x), \tag{3.39}$$

where $a$ and $b$ are arbitrary functions. Substituting (3.39) into (3.36) and integrate with respect to $\theta$ we find

$$\eta \;=\; -\frac{1}{3}x^{-n}\theta^3a^2 - nx^{-1}\theta^2a - x^{-n}a\,b\,\theta^2 + \theta^2a_x + \alpha(t,x)\theta + \beta(t,x), \tag{3.40}$$

where $\alpha$ and $\beta$ are arbitrary functions. We then substitute (3.39) and (3.40) in equation (3.37) and split with respect to the powers of $\theta$ and find

$$3\,Ng\,x\,a(t,x) + \frac{2}{3}\,x^{1-n}\,a(t,x)^3 = 0, \tag{3.41}$$

$$\Big(n + 2n\,x^2 + 2x^{1-n}b(t,x)\Big)a(t,x)^2 - 4x\,a(t,x)a_x = 0, \tag{3.42}$$

$$3\,x^{n+1}\;a_{xx} - 3nx^n\,a_x - 2\,xb(t,x)a_x - 2\,x\,a(t,x)b_x - 2\,x\,\alpha(t,x)\,a(t,x)$$
$$+x\,a_t + 3n\,x^{n-1}\,a(t,x) + 2\,n\,a(t,x)\,b(t,x) = 0, \tag{3.43}$$
$$n\,x^n b_x - n\,x^{n-1}b(t,x) - 2x\,b(t,x)\beta(t,x) + 2\,x\,b(t,x)b_x + xb_t - nb(t,x)^2$$
$$+2\,x^{n+1}\alpha_x - x^{n+1}b_{xx} = 0. \tag{3.44}$$

From equation (3.41) we find $a(t,x)$ to be equivalent to the following:

$$0,\;\frac{3}{2}\,x^{-n}\sqrt{-2\,x^{-n}Ng},\quad -\frac{3}{2}\,x^{-n}\sqrt{-2\,x^{-n}Ng}.$$

Taking

$$a(t,x) = \frac{3}{2}\,x^{-n}\sqrt{-2\,x^{-n}Ng}$$

and substituting into (3.42) we find

$$b(t,x) = -\frac{1}{2}\,nx^{n-1}.$$

Eliminating $a(t,x)$ and $b(t,x)$ from (3.43) we find

$$\alpha(t,x) = -\frac{1}{8}nx^{n-2}(n-2).$$

Eliminating $a(t,x)$, $b(t,x)$ and $\alpha(t,x)$ from (3.44) we find

$$\beta(t,x) = 0.$$

Finally we can simplify $\tau$, $\xi$ and $\eta$

$$\begin{aligned}\tau &= 1,\\ \xi &= \frac{3}{2}\,x^n\sqrt{-2Ngx^{-n}}\,\theta - \frac{1}{2}\,x^{n-1}n,\\ \eta &= \frac{1}{8}\theta(12\,Ng\,\theta^2 - x^{n-2}n^2 + 2\,x^{n-2}n).\end{aligned}$$

Substituting $\tau$, $\xi$ and $\eta$ into (3.38) we find

$$n(5n^3 - 26n^2 + 44n - 24) = 0.$$

Solving the above equation for $n$ we find three cases of arbitrary $n$ as follows:

$$n = 0, \quad n = 2 \ \text{ and } \ n = \frac{6}{5}.$$

Now for different values of $n$, equation (3.28) admits the following nonclassical symmetry operators together with associated invariant surface conditions:

**Case for arbitrary $n$**

Equation (3.28) admits the following nonclassical symmetry operator

$$\Gamma = \frac{\partial}{\partial t} + \frac{1}{2}\left(3\sqrt{-2Ngx^{-n}}\theta - \frac{n}{x}\right)x^n\frac{\partial}{\partial x} + \frac{1}{8}\left(12Ng\theta^2 - (n-2)\frac{nx^n}{x^2}\right)\theta\frac{\partial}{\partial\theta}, \quad (3.45)$$

and the associated invariant surface condition is given by

$$\theta_t + \frac{1}{2}\left(3\sqrt{-2Ngx^n}\theta - \frac{nx^{-n}}{x}\right)\theta_x = \frac{1}{8}\theta\left(12Ng\theta^2 - (n-2)\frac{nx^n}{x^2}\right). \quad (3.46)$$

**Case for $n = \frac{6}{5}$**

Equation (3.28) admits the following nonclassical symmetry operator

$$\Gamma = \frac{\partial}{\partial t} + 3\left(\frac{1}{2}\sqrt{-2Ng}\,x^{\frac{3}{5}}\,\theta - \frac{1}{5}x^{\frac{1}{5}}\right)\frac{\partial}{\partial x} + 3\left(\frac{Ng}{2}\theta + \frac{1}{25}x^{-\frac{4}{5}}\right)\theta\frac{\partial}{\partial \theta}, \tag{3.47}$$

and the associated ISC is given by

$$\theta_t + 3\left(\frac{1}{2}\sqrt{-2Ng}\,x^{\frac{3}{5}}\,\theta - \frac{1}{5}x^{\frac{1}{5}}\right)\theta_x = 3\left(\frac{Ng}{2}\theta + \frac{1}{25}x^{-\frac{4}{5}}\right)\theta. \tag{3.48}$$

**Case for** $n = 2$

Equation (3.28) admits the nonclassical symmetry operator

$$\Gamma = \frac{\partial}{\partial t} + \left(\frac{3}{2}\sqrt{-2Ng}\theta - 1\right)x\frac{\partial}{\partial x} + \frac{3}{2}Ng\theta^3\frac{\partial}{\partial \theta}, \tag{3.49}$$

and the associated ISC is given by

$$\theta_t + \left(\frac{3}{2}\sqrt{-2Ng}\theta - 1\right)x\theta_x = \frac{3}{2}Ng\theta^3. \tag{3.50}$$

### 3.2.3 Nonlocal symmetry techniques.

A symmetry that has at least one of the coefficient functions (the infinitesimal) depending on the integrals of the dependent variables of DEs is defined as nonlocal symmetry. Nonlocal symmetries can be used to achieve accurate solutions for DEs that cannot be obtained using Lie point symmetries. Bluman et al introduced a model for discovering a new class of symmetries for a PDE in [11, 42]. The prerequisite for a particular

equation to admit the nonlocal symmetries was also given by Pucci and Saccomandi in [43]. The complete list of possible symmetries for various classes of diffusion-convection equations was successfully derived using potential symmetry technique in [44].
Finding potential symmetries of a PDE typically involves writing the PDE in a conserved form with respect to certain choices of its variables. This process helps identify transformations that leave the form of the equation unchanged, providing insight into the underlying symmetries of the system.

To explain the method, we consider a diffusion equation (see [48])

$$\theta_t = [\theta^{-2}\theta_x]_x - [M^2 x]_x. \tag{3.51}$$

Equation (3.51) can be written in conserved form as

$$D_t(\theta) - D_x(\theta^{-2}\theta_x - M^2 x) = 0, \tag{3.52}$$

where $D_t$ and $D_x$ are defined as

$$\begin{aligned} D_t &= \partial_t + \theta_t\partial_\theta + w_t\partial_w + \cdots, \\ D_x &= \partial_x + \theta_x\partial_\theta + w_x\partial_w + \cdots. \end{aligned} \tag{3.53}$$

One can introduce a new variable $w$ (potential variable) to obtain the auxiliary system

$$\begin{aligned} w_x &= \theta, \\ w_t &= \theta^{-2}\theta_x - M^2 x. \end{aligned} \tag{3.54}$$

Given an equation such as (3.54), we seek transformations of the form

$$\begin{aligned}\bar{t} &= t + \epsilon\xi^1(t,x,\theta,w) + \mathcal{O}(\epsilon^2)\\ \overline{x} &= x + \epsilon\xi^2(t,x,\theta,w) + \mathcal{O}(\epsilon^2)\\ \overline{\theta} &= \theta + \epsilon\eta(t,x,\theta,w) + \mathcal{O}(\epsilon^2),\\ \overline{w} &= w + \epsilon\varphi(t,x,\theta,w) + \mathcal{O}(\epsilon^2),\end{aligned}$$

which leave the equation in question invariant. Here $\epsilon$ is the group parameter. These transformations are generated by the vector fields of the form

$$X = \tau(t,x,\theta,w)\frac{\partial}{\partial t} + \xi(t,x,\theta,w)\frac{\partial}{\partial x} + \eta(t,x,\theta,w)\frac{\partial}{\partial \theta} + \varphi(t,x,\theta,w)\frac{\partial}{\partial w} \tag{3.55}$$

if and only if

$$\begin{aligned}&X^{[1]}\left(w_x - \theta\right)\big|_{(3.54)} = 0,\\ &X^{[1]}\left(w_t - \theta^{-2}\theta_x - M^2 x\right)\big|_{(3.54)} = 0\end{aligned} \tag{3.56}$$

where $X^{[1]}$ is the first prolongation

$$\begin{aligned}X^{[1]} &= \tau(t,x,\theta,w)\frac{\partial}{\partial t} + \xi(t,x,\theta,w)\frac{\partial}{\partial x} + \eta(t,x,\theta,w)\frac{\partial}{\partial \theta}\\ &+\varphi(t,x,\theta,w)\frac{\partial}{\partial w} + \zeta_t\frac{\partial}{\partial \theta_t} + \zeta_x\frac{\partial}{\partial \theta_x} + \psi_t\frac{\partial}{\partial w_t} + \psi_x\frac{\partial}{\partial w_x}.\end{aligned} \tag{3.57}$$

Here the extended infinitesimals are given by

$$
\begin{aligned}
\zeta_t &= D_t(\eta) - \theta_t D_t(\tau) - \theta_x D_t(\xi),\\
\zeta_x &= D_x(\eta) - \theta_t D_x(\tau) - \theta_x D_x(\xi),\\
\psi_t &= D_t(\varphi) - w_t D_t(\tau) - w_x D_t(\xi),\\
\psi_x &= D_x(\varphi) - w_t D_x(\tau) - w_x D_x(\xi).
\end{aligned} \tag{3.58}
$$

A genuine nonlocal (potential) symmetry is obtained provided

$$
\tau_w^2 + \xi_w^2 + \eta_w^2 \neq 0. \tag{3.59}
$$

In contrast, a nonlocal symmetry generator will have coefficients depending not only on local variables, but also on the potential variable. We may split (3.57) with respect to the derivatives of $\theta$ since the coefficients of $X$ do not include the derivatives. The resultant overdetermined system of linear homogeneous PDEs, often known as the determining equations, can then be solved. At this point, we omit calculation because they were made easier by the free DIMSYM [45] package, which is part of [46]. Then, the nonlocal symmetries for (3.56) are

$$\begin{cases} X_1 = \partial_t, \\ X_2 = 2t\partial_t - x\partial_x + 2u\partial_u + w\partial_w, \\ X_3 = -\frac{1}{M^2}\left(M^2t^2\partial_t - (w + M^2tx)\partial_x + (u + 2M^2t)\theta\partial_u + M^2tw\partial_w\right), \\ X_4 = \partial_w, \\ X_5 = \frac{1}{M^2}\left(\partial_x - M^2t\partial_w\right). \end{cases} \tag{3.60}$$

Only one symmetry operator from (3.60) represents the nonlocal symmetry of (3.56) and is given by $X_3$.

### Nonlocal symmetry reduction

It is important to explain how nonlocal symmetries can be used to achieve precise solutions through reduction techniques. The invariant surface conditions are

$$\begin{aligned} \tau(t,x,\theta,w)\theta_t + \xi(t,x,\theta,w)\theta_x = \eta(t,x,\theta,w), \\ \tau(t,x,\theta,w)w_t + \xi(t,x,\theta,w)w_x = \varphi(t,x,\theta,w) \end{aligned} \tag{3.61}$$

given a point symmetry for the system.

Three separate integrals

$$\begin{aligned} J_1 &= H_1(t, x, \theta, w), \\ J_2 &= H_2(t, x, \theta, w), \\ J_3 &= H_3(t, x, \theta, w) \end{aligned} \tag{3.62}$$

provide the solutions of the related characteristic system (3.61). One-parameter families of characteristic curves are used to define the solutions of (3.61) and (3.62). Assuming that $J_1 = z$ is a parameter and that $J_2 = h_1(z)$ and $J_3 = h_2(z)$ from (3.62), we arrive at

$$\begin{aligned} &\theta = \Theta(t, x, z, h_1(z), h_2(z)) \\ &w = W(t, x, z, h_1(z), h_2(z)) \\ &F(t, x, z, h_1(z), h_2(z)) = 0. \end{aligned} \tag{3.63}$$

The similarity variable $z$ is defined implicitly as a function of $(x, t)$ in the final equation of (3.63). The $h_i(z)$ solutions of the ordinary system, which are obtained by substitution in (3.54), are the $h_i(z)$ invariant solutions of (3.54), which are supplied by (3.63a) and (3.63b).

### 3.2.4 Nonclassical potential technique

Nonclassical symmetry methods are used to study the linear diffusion equation with a nonlinear source term which includes explicit spatial dependence. In this work classes of symmetries for PDEs which can be written in a conserved form are found. These nonclassical potential symmetries are realized as nonclassical symmetries of the associated potential system and are neither classical potential symmetries realized as Lie symmetries of a related auxiliary system nor nonclassical symmetries of the considered equation.

Suppose a given scalar second order PDE

$$F(t, x, u, u_{(1)}, u_{(2)}) = 0, \tag{3.64}$$

where the subscripts denote the partial derivatives of $u$, respectively. Assume that (3.64) can be represented in a conserved form

$$F = D_x(f(t, x, u, u_{(1)})) - D_t(g(t, x, u, u_{(1)})) = 0, \tag{3.65}$$

for some functions $f$ and $g$. Through the conservation law (3.65) one can introduce an auxiliary potential variable $w$ and form an auxiliary potential system

$$\begin{aligned} w_x &= f(t, x, u, u_{(1)}), \\ w_t &= g(t, x, u, u_{(1)}). \end{aligned} \tag{3.66}$$

For many physical equations one can eliminate $u$ from the potential system (3.66) and form an auxiliary integrated or potential equation (integrated equation approach)

$$H(t, x, w, w_{(1)}, w_{(2)}) = 0, \tag{3.67}$$

for some function $G$ of the indicated arguments.

In [47] two algorithms were proposed which extend the nonclassical method to a potential system (3.66) or a potential equation (3.67):

**Algorithm I:** Nonclassical potential system approach.

The nonclassical method is applied to the associated potential system (3.66). Any Lie group of point transformations

$$X_S = \tau(t,x,u,w)\frac{\partial}{\partial t} + \xi(t,x,u,w)\frac{\partial}{\partial x} + \eta(t,x,u,w)\frac{\partial}{\partial u} + \phi(t,x,u,w)\frac{\partial}{\partial w} \tag{3.68}$$

admitted by (3.66) yields a nonlocal potential symmetry of the given PDE (3.52) if the following condition is satisfied

$$\tau_w^2 + \xi_w^2 + \eta_w^2 \neq 0. \tag{3.69}$$

Considering the invariant surface condition

$$\begin{aligned} \tau(t,x,u,w)u_t + \xi(t,x,u,w)u_x &= \eta(t,x,u,w), \\ \tau(t,x,u,w)w_t + \xi(t,x,u,w)w_x &= \phi(t,x,u,w) \end{aligned} \tag{3.70}$$

arising from the potential system (3.66) without loss of generality, two cases arise $\tau = 1$; $\tau = 0$, $\xi = 1$. The ISC and its differential consequences give additional relation between the derivatives.

**Algorithm II:** Nonclassical potential equation approach.

The nonclassical method is applied to the associated potential (3.67). Any Lie group of point transformations

$$X = \tau(t,x,w)\frac{\partial}{\partial t} + \xi(t,x,w)\frac{\partial}{\partial x} + \phi(t,x,w)\frac{\partial}{\partial w} \tag{3.71}$$

admitted by (3.67) yields a nonlocal potential symmetry of the given PDE (3.52) if the following condition is satisfied

$$\tau_w^2 + \xi_w^2 \neq 0. \tag{3.72}$$

## 3.3 Differential Transform Method

The method outlined in this subsection was first introduced in 1986 by Zhou, [51]. Zhou utilised this method in solving linear and nonlinear boundary value problems. The effectiveness of this method is it's applicability to solve linear and nonlinear DEs without linearization, discretization or perturbation. The following two subsection: 3.3.1 and 3.3.2 describes how to apply DTM.

### 3.3.1 One-Dimensional Differential Transform Method (1D DTM)

Let $\phi(t)$ be an analytic function in a domain $\mathcal{D}$. The Taylor series expansion function of $\phi(t)$ with the center located at $t = t_j$ is given by [51]

$$\phi(t) = \sum_{\kappa=0}^{\infty} \frac{(t-t_j)^\kappa}{\kappa!} \left[ \frac{d^\kappa \phi(t)}{dt^\kappa} \right]_{t=t_j}, \ \forall \ t \in \mathcal{D}. \tag{3.73}$$

The particular case of Equation (3.73) when $t_j = 0$ is referred to as the Maclaurin series expansion of $\phi(t)$ and is expressed as,

$$\phi(t) = \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} \left[ \frac{d^\kappa \phi(t)}{dt^\kappa} \right]_{t=0}, \ \forall\ t \in \mathcal{D}. \tag{3.74}$$

The differential transform of $\phi(t)$ is defined as follows;

$$\Phi(t) = \sum_{\kappa=0}^{\infty} \frac{\mathcal{H}^\kappa}{\kappa!} \left[ \frac{d^\kappa \phi(t)}{dt^\kappa} \right]_{t=0}, \tag{3.75}$$

where $\phi(t)$ is the original analytic function and $\Phi(t)$ is the transformed function. The differential spectrum of $\Phi(t)$ is confined within the interval $t \in [0, \mathcal{H}]$, where $\mathcal{H}$ is a constant. From equations (3.74) and (3.75), the differential inverse transform of $\Phi(t)$ is defined as follows,

$$\phi(t) = \sum_{\kappa=0}^{\infty} \left( \frac{t}{\mathcal{H}} \right)^\kappa \Phi(\kappa), \tag{3.76}$$

and if $\phi(t)$ is expressed by a finite series, then

$$\phi(t) = \sum_{\kappa=0}^{r} \left( \frac{t}{\mathcal{H}} \right)^\kappa \Phi(\kappa). \tag{3.77}$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The values of the function $\Phi(\kappa)$ are referred to as discrete, i.e., $\Phi(0)$ is known as the zero discrete, $\Phi(1)$ as the first discrete, etc. With more discrete available, it is possible to restore the unknown function more precisely. The function $\phi(t)$ consists of the $T$-function $\Phi(k)$, and its value is given by the sum of the $T$-function with $(t/\mathcal{H})^\kappa$ as its coefficient. In real applications, at the right choice of the constant $\mathcal{H}$, the discrete values of the spectrum reduce rapidly with larger values of argument $\kappa$ [52].

Some of the useful mathematical operations performed by the differential transform method are listed in Table 2.1. The delta function $\delta(k-m)$ is given by

$$\delta(k-m) = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$$

Table 3.1: Fundamental operations of the 1D DTM

| Original function | Transformed function |
|---|---|
| $\phi(t) = u(t) \pm v(t)$ | $\Phi(k) = U(k) \pm V(k)$ |
| $\phi(t) = cu(t)$ | $\Phi(k) = cU(k)$ |
| $\phi(t) = \frac{\partial}{\partial t}u(t)$ | $\Phi(k) = (k+1)U(k+1)$ |
| $\phi(t) = \frac{\partial^2}{\partial t^2}u(t)$ | $\Phi(k) = (k+1)(k+2)U(k+2)$ |
| $\phi(t) = \frac{\partial^r}{\partial t^r}u(t)$ | $\Phi(k) = \frac{(k+r)!}{k!}U(k+r)$ |
| $\phi(t) = u(t)v(t)$ | $\Phi(k) = \sum_{\imath=0}^{k} U(i)V(k-i)$ |
| $\phi(t) = 1$ | $\Phi(k) = \delta(k)$ |
| $\phi(t) = t$ | $\Phi(k) = \delta(k-1)$ |
| $\phi(t) = t^m$ | $\Phi(k) = \delta(k-m)$ |
| $\phi(t) = e^{at}$ | $\Phi(k) = \frac{a^h}{h!}$ |
| $\phi(t) = (1+t)^m$ | $\Phi(k) = \frac{m(m-1)...(m-k-1)}{k!}$ |
| $\phi(t) = \sin(at+b)$ | $\Phi(k) = \frac{a^k}{k!}\sin(\frac{\pi k}{2!}+a)$ |
| $\phi(t) = \cos(at+b)$ | $\Phi(k) = \frac{a^k}{k!}\cos(\frac{\pi k}{2!}+a)$ |

**Two-Dimensional Differential Transform Method (2D-DTM)**

Based on the 1D-DTM, the basic fundamental operations of the 2D-DTM are defined as follows

$$\Phi(k,s) = \frac{1}{\kappa!\, s!}\left[\frac{\partial^{\kappa+s}\phi(t,x)}{\partial t^{\kappa}\partial x^{s}}\right]_{(0,0)}. \tag{3.78}$$

The differential inverse transform of $\Phi(k,s)$ is defined as

$$\phi(t,x) = \sum_{\kappa=0}^{\infty}\sum_{s=0}^{\infty}\Phi(k,s)t^{\kappa}x^{s}, \tag{3.79}$$

and from equation (3.78) and (3.79) it can be concluded that

$$\phi(t,x) = \sum_{\kappa=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{\kappa!\, s!}\left[\frac{\partial^{\kappa+s}\phi(t,x)}{\partial t^{\kappa}\partial x^{s}}\right]_{(0,0)} t^{\kappa}x^{s}. \tag{3.80}$$

In real applications, the function $\phi(t,x)$ is expressed by a finite series, and equation (3.79) can be written as:

$$\phi(t,x) = \sum_{\kappa=0}^{m}\sum_{s=0}^{n}\Phi(k,s)t^{\kappa}x^{s}, \tag{3.81}$$

Equation (3.81) implies that

$$\phi(t,x) = \sum_{\kappa=m+1}^{\infty}\sum_{s=n+1}^{\infty}\Phi(k,s)t^{\kappa}x^{s}, \tag{3.82}$$

is negligibly small. Some of the useful mathematical operations performed by the differential transform method are given by the following theorems. See **Table 3.2**

Table 3.2: Fundamental operations of the 2D DTM

| Original function | Transformed function |
|---|---|
| $\phi(t,x) = u(t,x) \pm v(t,x)$ | $\Phi(k,h) = U(k,h) \pm V(k,h)$ |
| $\phi(t,x) = cu(t,x)$ | $\Phi(k,h) = cU(k,h)$ |
| $\phi(t,x) = \frac{\partial}{\partial x} u(t,x)$ | $\Phi(k,h) = (h+1)U(k,h+1)$ |
| $\phi(t,x) = \frac{\partial^{r+s}}{\partial t^r \partial x^s} u(t,x)$ | $\Phi(k,h) = \frac{(k+r)!(h+s)!}{k!h!} U(k+r,h+s)$ |
| $\phi(t,x) = u(t,x)v(t,x)$ | $\Phi(k,h) = \sum_{\imath=0}^{k} \sum_{j=0}^{h} U(k,h)V(k,h)$ |
| $\phi(t,x) = \frac{\partial}{\partial t} u(t,x) \frac{\partial}{\partial x} v(t,x)$ | $\Phi(k,h) = \sum_{\imath=0}^{k} \sum_{j=0}^{h} (k-i+1)(h-j+1)$ |
| | $U(k-i+1,h)V(k,h-j+1)$ |
| $\phi(t,x) = t^m x^n$ | $\Phi(k,h) = \delta(k-m)\delta(h-n)$ |
| $\phi(t,x) = x^m e^{at}$ | $\Phi(k,h) = \frac{a^h}{h!}\delta(k-m)$ |
| $\phi(t,x) = x^m \sin(at+b)$ | $\Phi(k,h) = \frac{a^h}{h!} \sin(\frac{a^h}{h!} + a)$ |

## 3.4 Concluding remarks

In this chapter we provided brief accounts of the methods used in this thesis. The fundamentals of the symmetry analysis were discussed. The relationship between one parameter group and equivalent infinitesimal transformation are outlined. Moreover we also deliberated on the generation of nonclasical, potential and nonclassical potential symmetries. Furthermore the basics of DTM were also outlined.

# Chapter 4

# Classical Lie point symmetry analysis of heat transfer through a hot body

## 4.1 Introduction

In this chapter we analyse (2.8) in terms of heat transfer through the hot body whereby the temperature dependent thermal conductivity $k(\theta)$ and heat source term $Q(\theta)$ may be classified. Direct group classification method will be employed to classify heat source term $Q(\theta)$. In our study we will assume the thermal conductivity to be unity and the profile be a function of space variable denoted by $x^n$, where $n$ is an arbitrary constant. Equation (2.8) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] + NgQ(\theta). \tag{4.1}$$

Nonclassical and classical symmetry techniques have been employed to study equation (4.1) when $n = 2$ and the different forms of the coefficient of the internal heat generation term. (See [6]).

## 4.2 Direct group classification

The Lie point symmetry technique is utilized to analyse equation (4.1). We assume that the vector field of the form

$$\Gamma \;=\; \tau(t,x,\theta)\frac{\partial}{\partial t} + \xi(t,x,\theta)\frac{\partial}{\partial x} + \eta(t,x,\theta)\frac{\partial}{\partial \theta}, \tag{4.2}$$

will generate the symmetries of (4.1) if

$$\Gamma^{[2]}\left(\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] - NgQ(\theta)\right)\bigg|_{(4.1)} = 0, \tag{4.3}$$

where $\Gamma^{[2]}$ is a second prolongation. Equation (4.1) yields the following overdetermined system of linear PDEs:

$$\tau_x = 0, \tag{4.4}$$

$$\tau_\theta = 0, \tag{4.5}$$

$$\xi_\theta = 0, \tag{4.6}$$

$$\eta_{\theta\theta} = 0, \tag{4.7}$$

$$x(\tau_t - 2\xi_x) + n\xi = 0, \tag{4.8}$$

$$(n-1)nx^n\xi + x^2\xi_t + nx^{n+1}(\tau_t - \xi_x) + x^{n+2}(2\eta_{x\theta} - \xi_{xx}) = 0, \tag{4.9}$$

$$nx^n\eta_x - x\eta_t + x^{n+1}\eta_{xx} + Ngx(\xi_t - \eta_\theta)Q + Ngx\eta Q' = 0. \tag{4.10}$$

Solving the above system for arbitrary function $Q(\theta)$, we find that the principal Lie algebra (PLA) consists of one operator, namely;

$$\Gamma_1 = \frac{\partial}{\partial t}.$$

We note that freely software package Dimsym [45] under REDUCE [46] was used to generate PLA. In order for (4.1) to admit additional symmetries, it turns out that $Q(\theta)$ satisfy the first order linear ODE, given by

$$(a\theta + \beta)Q'(\theta) + c\theta + dQ(\theta) + \psi = 0, \tag{4.11}$$

where $a,\ \beta,\ c,\ d$ and $\psi$ are constants. Solving equation (4.11) leads to number of cases listed in Table 4.1, but for our study we only focus on the following two cases, the power law and exponential law:

$$Q(\theta) = \theta^m \quad \text{and} \quad Q(\theta) = e^{m\theta},$$

where $m$ is an arbitrary constant. Please note that, without loss of generality, constant of integration $c$ in the cases we consider is taken as 1.

Table 4.1: Cases obtained from equation (4.11)

| Parameters | $Q(\theta)$ |
|---|---|
| $a = c = d = 0$ | $c_1 - k_1\theta,\ k_1 = \frac{\psi}{a}$ |
| $a = -1,\ c = \psi = 0$ | $c_1(\theta - \beta)^m, \quad m = d$ |
| $a = -1,\ \beta = c = \psi = 0$ | $\theta^m, \quad m = d$ |
| $\beta = -1,\ a = c = \psi = 0$ | $e^{m\theta}, \quad m = d$ |
| $\beta = d = \psi = 0$ | $c_1 - m_2\theta,\ m_2 = \frac{c}{a}$ |

**Case 1** $Q = \theta^m$

Equation (4.1) now becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] + Ng\theta^m, \quad n \neq 0, \quad m \neq 0. \tag{4.12}$$

This case yield two sub-cases base on the parameters $n$ and $m$. The two sub-cases in question are as follows:

**Sub-case 1.1** $n \neq 2$ and $m \neq 1$.

Equation (4.12) extend PLA by one operator

$$\Gamma_2 = t\frac{\partial}{\partial t} - \frac{1}{n-2}x\frac{\partial}{\partial x} - \frac{1}{m-1}\theta\frac{\partial}{\partial \theta} \tag{4.13}$$

**Sub-case 1.2** $n = 2$ **and** $m \neq 1$**.**

Equation (4.12) extend PLA by two operators

$$\begin{cases} \Gamma_2 = t\frac{\partial}{\partial t} - \frac{1}{2}(t - \ln x)x\frac{\partial}{\partial x} - \frac{1}{m-1}\theta\frac{\partial}{\partial \theta}, \\ \\ \Gamma_3 = x\frac{\partial}{\partial x}. \end{cases} \tag{4.14}$$

We now compute the commutation relations for the above Lie point symmetries. By definition of the Lie bracket, we have

$$[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i, \tag{4.15}$$

where the subscripts $i$ and $j$ take values from 1 to 3. Using (4.15), we calculate the commutator $[\Gamma_1, \Gamma_2]$, viz.,

$$\begin{aligned} [\Gamma_1, \Gamma_2] &= \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1 \\ &= \frac{\partial}{\partial t}\left(t\frac{\partial}{\partial t} - \frac{1}{2}(t - \ln x)x\frac{\partial}{\partial x} - \frac{1}{m-1}\theta\frac{\partial}{\partial \theta}\right) - \left(t\frac{\partial}{\partial t} - \frac{1}{2}(t - \ln x)x\frac{\partial}{\partial x} - \frac{1}{m-1}\theta\frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial t} \\ &= 2\Gamma_1 - \Gamma_3. \end{aligned}$$

Likewise, one can obtain the commutation relations between other vector fields. The table below (Table 4.2) present the commutator of the Lie algebra of sub-case 1.2.

Using Table 4.2 of commutators and the formula

$$Ad(\exp(\varepsilon\Gamma_i))\Gamma_j = \Gamma_j - \varepsilon[\Gamma_i, \Gamma_j] + \frac{\varepsilon^2}{2!}[\Gamma_i, [\Gamma_i, \Gamma_j]] + \cdots,$$

Table 4.2: Commutator table for case 1.2

| $[\Gamma_i, \Gamma_j]$ | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ |
|---|---|---|---|
| $\Gamma_1$ | $0$ | $2\Gamma_1 - \Gamma_3$ | $0$ |
| $\Gamma_2$ | $-2\Gamma_1 + \Gamma_3$ | $0$ | $-\Gamma_3$ |
| $\Gamma_3$ | $0$ | $\Gamma_3$ | $0$ |

where $i$ and $j$ take values from 1 to 3, we find the adjoint representation of $\Gamma_i$. Here we will show the detailed calculations for the following adjoint representation:

$$\begin{aligned} Ad(\exp(\varepsilon\Gamma_1))\Gamma_2 &= \Gamma_2 - \varepsilon[\Gamma_1, \Gamma_2] + \frac{\varepsilon^2}{2}[\Gamma_1, [\Gamma_1, \Gamma_2]] + \cdots \\ &= \Gamma_2 - \varepsilon(2\Gamma_1 - \Gamma_3) + \frac{\varepsilon^2}{2}[\Gamma_1, 2\Gamma_1 - \Gamma_3] + \cdots \\ &= -2\varepsilon\Gamma_1 + \Gamma_2 + \varepsilon\Gamma_3 \end{aligned}$$

and

$$\begin{aligned} Ad(\exp(\varepsilon\Gamma_2))\Gamma_3 &= \Gamma_3 - \varepsilon[\Gamma_2, \Gamma_3] + \frac{\varepsilon^2}{2}[\Gamma_2, [\Gamma_2, \Gamma_3]] + \cdots \\ &= \Gamma_3 + \varepsilon\Gamma_3 - \frac{\varepsilon^2}{2}[\Gamma_2, \Gamma_3] + \cdots \\ &= \Gamma_3 + \varepsilon\Gamma_3 + \frac{\varepsilon^2}{2}\Gamma_3 + \cdots \\ &= e^{\varepsilon}\Gamma_3. \end{aligned}$$

All the adjoint representations are summarized in Table 4.3.

Table 4.3: Adjoint table for case 1.2

| $Adj$ | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ |
|---|---|---|---|
| $\Gamma_1$ | $\Gamma_1$ | $-2\varepsilon\Gamma_1 + \Gamma_2 + \varepsilon\Gamma_3$ | $\Gamma_3$ |
| $\Gamma_2$ | $e^{2\varepsilon}\Gamma_1 - e^{\varepsilon}(e^{\varepsilon} - 1)\Gamma_3$ | $\Gamma_2$ | $e^{\varepsilon}\Gamma_3$ |
| $\Gamma_3$ | $\Gamma_1$ | $\Gamma_2 - \varepsilon\Gamma_3$ | $\Gamma_3$ |

## 4.3 Group-invariant solutions given the power law heat source

In this section we use symmetries produced by sub-case 1.2 to reduce PDE (4.12) into ordinary differential equation. We will consider all three (3) operators.

### 4.3.1 Symmetry Reduction using $\Gamma_1$

In order to find symmetry reductions and exact solutions, one has to solve the characteristic equation

$$\frac{dt}{\xi^1(t,x,u)} = \frac{dx}{\xi^2(t,x,u)} = \frac{d\theta}{\eta(t,x,u)}$$

and then substitute the resulting expression into (4.12). Considering symmetry generator $\Gamma_1$ gives the following characteristic equation

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\theta}{0}.$$

Solving the above equation gives rise to the group-invariant solution

$$\theta(t,x) = F(x).$$

The above group-invariant leads to

$$x^2 F''(x) + 2xF'(x) + NgF(x)^m = 0. \tag{4.16}$$

The above equation (4.16) become Euler when $m = 1$, with the solution

$$\theta(t,x) = c_1\, x^{\frac{1}{2}(-1+\sqrt{1-4\,Ng})} + c_2\, x^{\frac{1}{2}(-1-\sqrt{1-4\,Ng})}. \tag{4.17}$$

### 4.3.2 Symmetry Reduction using $\Gamma_2$

Firstly we consider symmetry generator $\Gamma_2$. The characteristic equations for the invariants of the symmetry generator $\Gamma_2$ are

$$\frac{dt}{t} = 2\frac{dx}{(t-\ln x)x} = (m-1)\frac{d\theta}{\theta}.$$

Solving the above characteristic equation yields the following invariants

$$\gamma = -\frac{4(t+\ln x)}{\sqrt{t}} \quad \text{and} \quad \theta(t,x) = (t-mt)^{\frac{1}{1-m}}\varPhi(\gamma).$$

where $\varPhi$ satisfy the ODE

$$32\varPhi''(\gamma) + \gamma\varPhi'(\gamma) + \frac{2}{m-1}\varPhi(\gamma) - \frac{Ng}{2(m-1)}\varPhi^m(\gamma) = 0, \quad m \neq 1. \tag{4.18}$$

We could not generate exact analytic solution for this ODE.

### 4.3.3 Symmetry Reduction using $\Gamma_1$ and $\Gamma_3$

We now take linear combination of symmetry generators $\Gamma_1$ and $\Gamma_3$. The characteristic equation for the above mentioned combination of symmetry generators are

$$\frac{dt}{a_1} = \frac{dx}{a_2 x} = \frac{d\theta}{0}, \quad a_1, a_2 \in I\!R,$$

and gives rise to the group-invariant solutions

$$\alpha = xe^{-\frac{a_2}{a_1}t} \quad \text{and} \quad \theta(t,x) = \Re(\alpha),$$

where $a_1$ and $a_2$ are arbitrary constants. The above group-invariant leads to

$$\alpha^2\Re''(\alpha) + \left(\frac{a_2}{a_1} + 2\right)\alpha\Re'(\alpha) + Ng\Re(\alpha)^m = 0. \tag{4.19}$$

Equation (4.19) admits symmetry

$$X = \alpha\frac{\partial}{\partial\alpha}.$$

We seek to reduce (4.19) to first order ODE by utilizing the differential invariant method. The first prolongation of X above is given by

$$X^{[1]} = \alpha\frac{\partial}{\partial\alpha} - \Re'\frac{\partial}{\partial F'}$$

hence the invariants

$$u = \Re \quad \text{and} \quad v = \alpha \Re'.$$

By writing $v = v(u)$ and from the definitions of $u$ and $v$ and the chain rule, we find the first order ODE

$$vv' + \left(\frac{a_2}{a_1} + 1\right) v + Ngu^m = 0, \quad a_1 \neq 0. \tag{4.20}$$

Setting $a_2 = -a_1$, the solutions for (4.20) are

$$v = \pm\sqrt{\frac{-2Ngu^{m+1} + C(m+1)}{m+1}}, \ m \neq -1. \tag{4.21}$$

where $C$ is a constant of integration. Substituting back for the values of $v$ and $u$ into equation (4.21) yields

$$\alpha\Re' = \pm\sqrt{\frac{-2Ng\Re^{m+1} + C(m+1)}{m+1}}, \quad m \neq -1. \tag{4.22}$$

Integrating (4.22) we get the hypergeometric function solution

$$\Re(\alpha)\sqrt{\frac{C(m+1) - Ng\Re(\alpha)^{m+1}}{2m+2}} \tag{4.23}$$

$${}_2F_1\left(1, \frac{1}{2} + \frac{1}{m+1}; 1 + \frac{1}{m+1}; \frac{Ng\Re(\alpha)^{m+1}}{C(m+1)}\right) = C(c_1 + \ln(\alpha)). \tag{4.24}$$

Otherwise, if constant of integration $C$ in (4.22) vanishes, we find the following solution

$$\Re(\alpha) = \exp\left(\frac{1}{m-1}\left[\ln 2 + \ln\left(-\frac{(m+1)}{Ng\ (m-1)^2\,(c_1 + \ln\gamma)^2}\right)\right]\right), \quad m \neq 1,$$

where $c_1$ is a constant.

Therefore

$$\theta(t,x) = \exp\left(\frac{1}{m-1}\ln\left(-\frac{2(m+1)}{Ng\ (m-1)^2\left(c_1+\ln x+t\right)^2}\right)\right), \quad m \neq 1. \tag{4.25}$$

**Case 2** $Q = e^{m\theta}$.

In this case equation (4.1) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] + Nge^{m\theta}. \tag{4.26}$$

This case also yields two sub-cases based on the parameter $n$. The sub-case $m = 0$ implies the heat generation is constant. We omit this case. The sub-case for when $n \neq 2$, $m \neq 0$ gives

$$\Gamma_2 = t\frac{\partial}{\partial t} - \frac{1}{n-2}x\frac{\partial}{\partial x} - \frac{1}{m}\frac{\partial}{\partial \theta}, \tag{4.27}$$

and when $n = 2$ yields

$$\begin{cases} \Gamma_2 = t\frac{\partial}{\partial t} - \frac{1}{2}(t - lnx)x\frac{\partial}{\partial x} - \frac{1}{m}\frac{\partial}{\partial \theta}, \\ \Gamma_3 = x\frac{\partial}{\partial x}. \end{cases} \tag{4.28}$$

**Note:** Case 2 gives the same commutator and adjoint tables as in case 1.

## 4.4 Group-invariant solutions given exponential heat source

In this section we follow the same procedure as in case 1 to reduce PDE (4.26) into ODEs. The Lie Bracket for the above admitted symmetry algebra given by $e^{m\theta}$ when $n = 2$ yields the same results as in case 1.

### 4.4.1 Symmetry Reduction using $\Gamma_2$

Following the same procedure as above (in case 1) we obtain group-invariants solutions as follows

$$\Upsilon = -\frac{4(t + \ln x)}{\sqrt{t}} \quad \text{and} \quad \theta(t, x) = F(\Upsilon) - \frac{1}{m} \ln t.$$

The above invariant solutions leads to

$$32F''(\Upsilon) + \Upsilon F'(\Upsilon) + 2Nge^{mF(\Upsilon)} + \frac{2}{m} = 0, \ m \neq 0. \tag{4.29}$$

Equation (4.29) does not give an analytic solution. Symmetry reductions using $\Gamma_1$ and linear combination of $\Gamma_1$ and $\Gamma_3$ for case 2 gives the same invariants as obtained in case 1. Here we show the ODEs obtained for case 2 when $n = 2$:

$$\gamma^2 F''(\gamma) + 2\gamma F'(\gamma) + Nge^{mF(\gamma)} = 0 \tag{4.30}$$

and

$$\gamma^2 F''(\gamma) + \Big(\frac{a_2}{a_1} + 2\Big)\gamma F'(\gamma) + Nge^{mF(\gamma)} = 0, \quad a_1 \neq 0. \tag{4.31}$$

Also, equation (4.30) does not give an analytic solution.

Here we construct group invariant solution of (4.31) by employing the same procedure as in section 4.2. Equation (4.31) admits the same group-invariants as (4.19). The following equation leads to finding solutions of (4.31)

$$vv' + \Big(\frac{a_2}{a_1} + 1\Big)v + Nge^{mu} = 0. \tag{4.32}$$

Taking $a_2 = -a_1$ and integrating (4.32) and substituting for values of $v$ and $u$ lead to

$$(\gamma F')^2 = \pm\Big(2k - \frac{2Ng}{m}e^{mF}\Big). \tag{4.33}$$

where $k$ is a constant of integration. Applying the painleve property, $z = e^{mF}$ on (4.33) yields

$$(\gamma z'(\gamma))^2 - 2km^2 z^2(\gamma) + 2Ngmz^3(\gamma) = 0. \tag{4.34}$$

Solving (4.34) we obtain

$$z(\gamma) = \frac{km}{Ng}\Bigg(1 - \tanh^2\Bigg(\frac{\sqrt{k}m}{2}(\pm\sqrt{2}m\ln\gamma - c_1)\Bigg)\Bigg). \tag{4.35}$$

Substituting $z$ back we get

$$F(\gamma) = \frac{1}{m}\ln\Bigg[\frac{km}{Ng}\Bigg(\operatorname{sech}^2\Bigg(\frac{1}{2}\sqrt{k}m(k_1 \pm \sqrt{2}\ln\gamma)\Bigg)\Bigg)\Bigg] \tag{4.36}$$

where $k_1 = c_1\sqrt{m}$ and $c_1$ is a constant of integration. Writing (4.36) in terms of original variables we obtain

$$\theta(t,x) = \frac{1}{m}\ln\left[\frac{km}{Ng}\left(\operatorname{sech}^2\left(\frac{1}{2}\sqrt{k}m(k_1 \pm \sqrt{2}(\ln x - \frac{a_2}{a_1}t)\right)\right)\right], \quad a_1 \neq 0. \qquad (4.37)$$

## 4.5 Concluding remarks

In this chapter direct group classification was carried out and we one PLA was obtained. Two forms of the internal heat generator $Q(\theta)$ were obtained. We then considered two cases; namely the power law and exponential law for which the PLA extends. With this extended PLA, symmetry reductions where carried out, and were possible, we constructed the associated group invariant solutions. While seeking for exact solutions, the power law case gave us hypergeometric function solution and the exponential law case gave us the hyperbolic function solution.

# Chapter 5

# Approximate analytical solutions for a heat transfer in a slab with internal heat generation

## 5.1 Introduction

In this chapter, the differential transform method (DTM) is applied to determine analytic solutions to the boundary value problem describing heat transfer in a hot body with internal heat generation. Internal heat generation is taken as a temperature dependent. We demonstrated in chapter 4, that group invariant (exact) solutions may be difficult to construct. Hence we resort to the DTM in this chapter.

## 5.2 Differential transform method

In this section we consider equation (2.8) with thermal conductivity given as an exponential function and the internal heat generator given as the power law function. The equation in question is given by (4.12). We will first employ 1D DTM to study the steady state model describing the temperature profile in a hot body such as across a wall. The 1D DTM encounters difficulties when the thermal conductivity given by the power law has a fractional value on its exponent. Secondly, we will utilize 2D DTM to compute analytical solutions of transient heat transfer model. The 2D DTM is an extension of 1D DTM which is suitable to compute analytical solutions of PDEs.

### 5.2.1 Application of 1D DTM

We now consider the steady state model describing the temperature profile in a hot body with internal heat generation being a function of temperature given by

$$\frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] + Ng\theta^m = 0, \tag{5.1}$$

subject to the following boundary condition

$$\theta(1) = 1 \quad \text{and} \quad \theta'(0) = 0. \tag{5.2}$$

From Eq (5.1), we assume $y = x^n$ and set $n = \frac{1}{2}$, hence (5.1) becomes

$$\frac{d}{dy}\left[\frac{d\theta}{dy}\right] + 4Ngy\theta^m = 0. \tag{5.3}$$

For more detailed information with regard to the assumption made to find (5.3) see Moradi and Ahmadikia [53]. We now apply 1D DTM subject to (5.2) to compute analytical approximate solutions for (5.3). Let us recall that from chapter 4, $m$ is an arbitrary constant of integration, hence we vary $m$ from 1 to 5. Case one where $m = 1$ gives a detailed calculations on finding analytical solutions for Eq. (5.1).

**Case 1** $m = 1$

Applying 1D DTM on Eq (5.3) subject to (5.2) we obtain the following recurrence relation

$$\Theta(k+2) = -\frac{4Ng}{(k+1)(k+2)}\sum_{i=0}^{k}\delta(i-1)\Theta(k-i) \tag{5.4}$$

and

$$\Theta(0) = a \quad \text{and} \quad \Theta(1) = 0. \tag{5.5}$$

where $a$ is a constant.

The system below is generated by (5.4)

$$\Theta(2) = 0, \tag{5.6}$$

$$\Theta(3) = -\frac{2aNg}{3}, \tag{5.7}$$

$$\Theta(4) = 0, \tag{5.8}$$

$$\Theta(5) = 0, \tag{5.9}$$

$$\Theta(6) = \frac{4aNg^2}{45}, \tag{5.10}$$

$$\Theta(7) = 0, \tag{5.11}$$

$$\Theta(8) = 0, \tag{5.12}$$

$$\Theta(9) = -\frac{2}{405}aNg^3, \tag{5.13}$$

$$\Theta(10) = 0, \tag{5.14}$$

$$\Theta(11) = 0, \tag{5.15}$$

$$\Theta(12) = \frac{2aNg^4}{13365}, \tag{5.16}$$

$$\Theta(13) = 0, \tag{5.17}$$

$$\Theta(14) = 0, \tag{5.18}$$

$$\Theta(15) = -\frac{4aNg^5}{1403325}, \tag{5.19}$$

$$\Theta(16) = 0, \tag{5.20}$$

$$\Theta(17) = 0, \tag{5.21}$$

$$\Theta(18) = \frac{8aNg^6}{214708725}, \tag{5.22}$$

$$\Theta(19) = 0, \tag{5.23}$$

$$\Theta(20) = 0, \tag{5.24}$$

$$\Theta(21) = -\frac{8aNg^7}{22544416125}, \tag{5.25}$$

$$\Theta(22) = 0, \tag{5.26}$$

$$\Theta(23) = 0, \tag{5.27}$$

$$\vdots$$

From the above continuing process, we substitute (5.5), (5.6) - (5.27) into (3.76) and set $H = 1$ to obtain the following solution:

$$\theta(x) = a - \frac{2aNg}{3}x^{\frac{3}{2}} + \frac{4aNg^2}{45}x^3 - \frac{2aNg^3}{405}x^{\frac{9}{2}} + \frac{2aNg^4}{13365}x^6 - \frac{4aNg^5}{1403325}x^{\frac{15}{2}} + \frac{8aNg^6}{214708725}x^9 - \frac{8aNg^7}{22544416125}x^{\frac{21}{2}} + \dots \tag{5.28}$$

In order to obtain the value of $a$ from (5.28), we used (5.2) and obtained the equation below

$$\theta(1) = a - \frac{2aNg}{3} + \frac{4aNg^2}{45} - \frac{2aNg^3}{405} + \frac{2aNg^4}{13365} - \frac{4aNg^5}{1403325} + \frac{8aNg^6}{214708725} - \frac{8aNg^7}{22544416125} + \dots = 1. \tag{5.29}$$

Solving (5.29) by Mathematica software gives the value of $a$. However, one may obtain the expression for $\theta(x)$ upon substituting the obtained value of $a$ into (5.28). Following a similar approach as in Case 1 we obtained the analytical solution for four cases as follows:

**Case 2** $m = 2$

$$\theta(x) = a - \frac{2a^2Ng}{3}x^{3/2} + \frac{8a^3Ng^2}{45}x^3 - \frac{2a^4Ng^3}{45}x^{9/2} + \frac{4a^5Ng^4}{405}x^6 - \dots \tag{5.30}$$

**Case 3** $m = 3$

$$\theta(x) = a - \frac{2a^3Ng}{3}x^{3/2} + \frac{4a^5Ng^2}{15}x^3 - \frac{16a^7Ng^3}{135}x^{9/2} + \frac{232a^9Ng^4}{4455}x^6 - \dots \tag{5.31}$$

**Case 4** $m = 4$

$$\theta(x) = a - \frac{2a^4Ng}{3}x^{3/2} + \frac{16a^7Ng^2}{45}x^3 - \frac{92a^{10}Ng^3}{405}x^{9/2} + \frac{400a^{13}Ng^4}{2673}x^6 - \frac{704096a^{16}Ng^5}{7016625}x^{15/2} + \frac{4302464a^{19}Ng^6}{63149625}x^9 - \frac{61885792a^{22}Ng^7}{1326142125}x^{21/2} + \dots \tag{5.32}$$

**Case 5** $m = 5$

$$\theta(x) = a - \frac{2a^5 Ng}{3}x^{3/2} + \frac{4a^9 Ng^2}{9}x^3 - \frac{10a^{13}Ng^3}{27}x^{9/2} + \frac{290a^{17}Ng^4}{891}x^6 - \frac{68a^{21}Ng^5}{231}x^{15/2} + \frac{86312a^{25}Ng^6}{318087}x^9 - \frac{5079728a^{29}Ng^7}{20039481}x^{21/2} + \dots \quad (5.33)$$

We note that, when we take internal heat generation coefficient $Ng$ varying from 2 to 4 the value of $a$ becomes negative and the graph starts from below the horizontal axis. The graph for solutions (5.30)-(5.33) are depicted in figure 5.1 for various values of $m$ and fixed value of $Ng$.

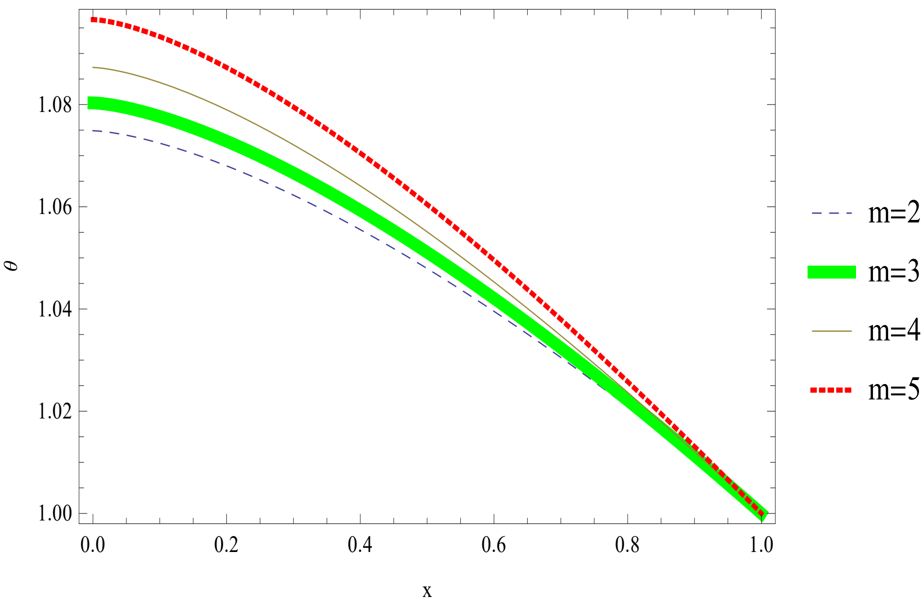



Figure 5.1: Temperature profile with varying values of $m$ and fixed internal heat generation, $Ng = 0.1$

### 5.2.2 Application of 2D DTM

In this subsection we extend 1D DTM to 2D DTM in order to solve PDE (4.12) subject to the initial condition

$$\theta(0,x) = 0, \quad 0 \le x \le 1, \tag{5.34}$$

and the boundary conditions

$$\theta(t,1) = 1, \quad \text{and} \quad \frac{\partial \theta}{\partial x}|_{x=0} = 0. \tag{5.35}$$

We consider equation (4.12) given as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[x^n \frac{\partial \theta}{\partial x}\right] + Ng\theta^m.$$

Now, if we set $n = \frac{1}{2}$ and substitute $y = \frac{1}{2}$ to eliminate the fractional exponent, the equation in question becomes

$$4y\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} + 4Ngy\theta^m. \tag{5.36}$$

From equation (5.36), we consider two cases and follow similar procedure as in subsection 5.2.1 to find analytical solution for (5.36). In this subsection we only consider two values of $m$; namely $m = 1$ and $m = 2$.

**Case 1: m=1**

Taking the two-dimensional differential transform of Eq. (5.36) with $m = 1$ and initial and boundary conditions (5.34) and (5.35 ) we obtain

$$4\sum_{i=0}^{k}\sum_{j=0}^{h}\delta(j-1,k-i)(i+1)\Theta(i+1,h-j) = (h+1)(h+2)\Theta(k,h+2) + 4Ng\sum_{i=0}^{k}\sum_{j=0}^{h}\delta(j-1,k-i)\Theta(i,h-j) \tag{5.37}$$

and

$$\Theta(0,h) = 0, \quad h = 0,1,2,3,..., \tag{5.38}$$

$$\Theta(k,1) = 0, \quad k = 0,1,2,3,..., \tag{5.39}$$

where $\Theta(k,h)$ is the differential transform of $\theta(t,x)$. The other boundary condition is given by

$$\Theta(k,0) = a, \quad a \in R \quad k = 1,2,3,..., \tag{5.40}$$

where is a $a$ constant. Substituting equations (5.38)-(5.40) into (5.37) we obtain the following:

$$\Theta(1,3) = \frac{2}{3}a(2-Ng), \tag{5.41}$$

$$\Theta(2,3) = \frac{2}{3}a(3-Ng), \tag{5.42}$$

$$\Theta(3,3) = \frac{2}{3}a(4-Ng), \tag{5.43}$$

$$\Theta(4,3) = \frac{2}{3}a(5-Ng), \tag{5.44}$$

$$\Theta(5,3) = \frac{2}{3}a(6-Ng), \tag{5.45}$$

$$\vdots$$

Substituting equation (5.38)-(5.45) into (3.76) we obtain the following series solution,

$$\theta(t,x) = at + at^2 + at^3 - \frac{2}{3}a(Ng-2)tx^{3/2} - \frac{2}{3}a(Ng-3)t^2x^{3/2} - \frac{2}{3}a(Ng-4)t^3x^{3/2} + \cdots \tag{5.46}$$

To obtain the value of $a$, we substitute the boundary condition (5.34) into (5.46) at the point $x = 1$. Thus, we have

$$\theta(t,1) = at + at^2 + at^3 - \frac{2}{3}a(Ng-2)t - \frac{2}{3}a(Ng-3)t^2 - \frac{2}{3}a(Ng-4)t^3 + \cdots = 1. \tag{5.47}$$

We then substitute the obtained value of $a$ into equation (5.46) to obtain the expression for $\theta(t,x)$. The graph representing (5.46)

**Case 2: m=2**

$$\theta(t,x) = at + at^2 + at^3 + \frac{4}{3}atx^{3/2} - \frac{2}{3}a(aNg-3)t^2x^{3/2} - \frac{4}{3}a(aNg-2)t^3x^{3/2} + \cdots \tag{5.48}$$

The graphs for solution (5.48) are depicted in figure 5.2 and figure 5.3 for various parameters. figure 5.3 is a plot for the three dimensional distribution of temperature.

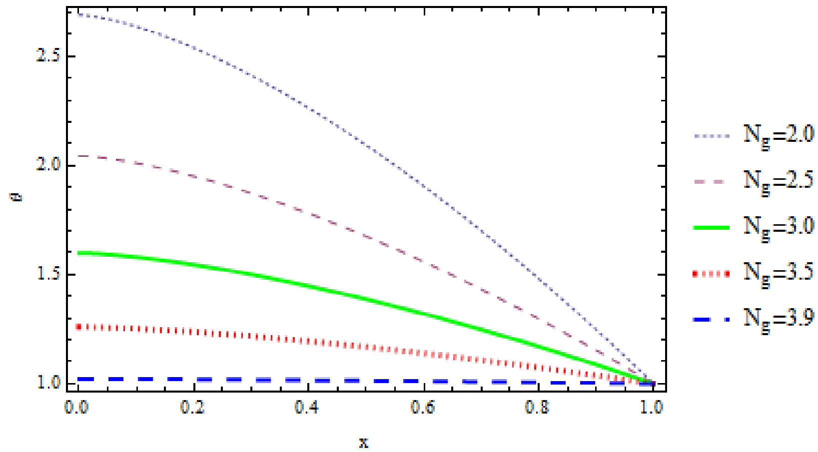



Figure 5.2: Transient temperature distribution with different values of internal heat generation $Ng$ and $m = 2$ and $t = 0.5$.

## 5.3 Concluding remarks

The main focus of this chapter was to generate solutions for the reaction-diffusion model describing heat transfer through the hot body or slab. The 1D DTM and 2D DTM were utilized to construct approximate analytical series solutions for the steady state and for the transient state problems, respectively. When using the 1D DTM, a problem arises if the power law diffusivity term in the equation is given by the fractional exponent. However, some transformation method may be introduced to combat the problem. From the plots we observed the effects of the parameters and constants appearing in the dimensionless models. In figure 5.1, we observe that the temperature profile is proportional exponent $m$. Figure 5.2 depicts the increased values of the parameter $Ng$ decreases the temperature in the slab. The transient temperature profile is presented in figure 5.3 and figure 5.4.

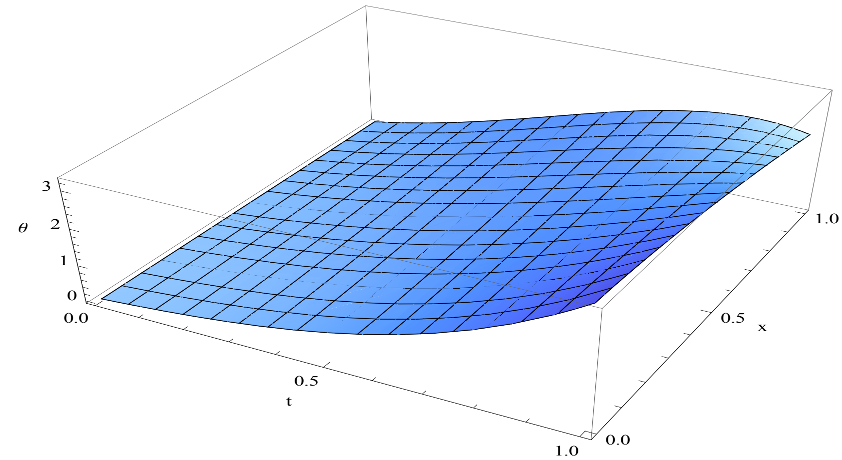



Figure 5.3: Transient temperature distribution with $Ng = 6$, $m = 2$, $x = 1$ and $t = 1.2$.

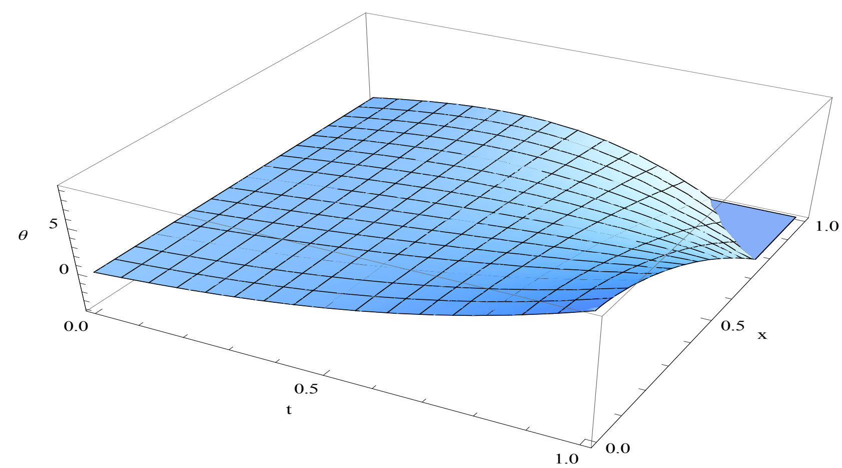



Figure 5.4: Transient temperature distribution with $Ng = 2$, $m = 2$ and $t = 0.5$.

# Chapter 6

# Preliminary group classification of nonlinear reaction-diffusion equation

## 6.1 Introduction

In this chapter we perform Lie group classification of nonlinear reaction-diffusion equation where two arbitrary functions are of interest, namely; $f(x)$ and $k(v_x)$. Preliminary group classification and direct group classification methods will be employed, cases for which the PLA is increased will be generated. Cases which will be of interest will be selected and explored. The chapter is outlined as follows: In section 6.2 we give the mathematical model of our problem. Principal Lie algebra will be determined in section 6.3 and section 6.4 presents equivalence transformations together with the list of cases for which a PLA is extended will be determined in section 6.4. After the process of preliminary group classification in Section 6.5, implementation of direct group classification will be carried out for two special cases, the power law and exponential law in section 6.6. Section 6.7 will generate symmetry reduction and invariant solutions. The conclusion will be in Section 6.8.

## 6.2 Mathematical models

The problem being considered in this chapter is a special case of equation (2.17) that represents the transient heat conduction problem for heat transfer in a straight fin. The model is given as

$$\theta_t = \left[ f(x)k(\theta)\theta_x \right]_x - M^2 h_0 \theta \tag{6.1}$$

where $h_0$ is a constant and $M$ is defined as the thermo-geometric fin parameter. We start by setting $v_x = \theta$ and integrate with respect to $x$, equation (6.1) becomes

$$v_t = f(x)k(v_x)v_{xx} - M^2 h_0 v. \tag{6.2}$$

As mentioned in the section above, equation (6.2) will be analysed in two ways, namely: preliminary and direct group classification.

## 6.3 Principal Lie algebra

In this section we determine the Lie point symmetries admitted by equation (6.2) with arbitrary functions $f$ and $k$, i.e. we seek the PLA. The symmetry group of equation (6.2) will be generated by the vector field of the form

$$X = \tau(t,x,v)\partial_t + \xi(t,x,v)\partial_x + \eta(t,x,v)\partial_v. \tag{6.3}$$

The Lie point symmetry techniques is algorithmic but tedious. The invariance criterion is given by

$$X^{[2]}\left(v_t - f(x)k(v_x)v_{xx} + M^2h_0 v\right)\Big|_{(6.2)} = 0, \tag{6.4}$$

where $X^{[2]}$ is the second prolongation described in Chapter 2. In this context, we employ the Dimsym [45] package within the framework of REDUCE [46]. Consequently, for any given functions $f$ and $k$, equation (6.2) allows for a two-dimensional Lie algebra denoted as $L_2$, which is generated by the following basis vectors:

$$X_1 = \partial_t \quad \text{and} \quad X_2 = e^{-M^2h_0 t}\partial_v. \tag{6.5}$$

We call $L_2$ the PLA for equation (6.2). The classification of coefficients $f(x)$ and $k(v_x)$ such that (6.2) admits an extension of PLA will be carried out by employing preliminary group classification method.

## 6.4 Equivalence transformations

An equivalence transformation (see for example [41]) of (6.2) is an invertible transformation involving the independent variables $t$, $x$ and dependent variable $v$ that maps (6.2) into itself. The operator

$$\begin{aligned} Y &= \tau(t,x,v)\partial_t + \xi(t,x,v)\partial_x + \eta(t,x,v)\partial_v + \mu^1(t,x,v,v_t,v_x,f)\partial_f \\ &\quad + \mu^2(t,x,v,v_t,v_x,k)\partial_k \end{aligned} \tag{6.6}$$

is the generator of the equivalence group for equation (6.2) provided it is admitted by the extended system

$$\begin{cases} v_t - f(x)k(v_x)v_{xx} + M^2 h_0 v = 0, \\ \\ f_t = f_v = f_{v_t} = f_{v_x} = k_t = k_x = k_v = k_{v_t} = 0. \end{cases} \tag{6.7}$$

The prolonged operator for the extended system (6.7) has the form

$$\begin{aligned} \tilde{Y} &= Y^{[2]} + \omega_t^1 \partial_{f_t} + \omega_v^1 \partial_{f_v} + \omega_{v_t}^1 \partial_{f_{v_t}} + \omega_{v_x}^1 \partial_{f_{v_x}} + \omega_t^2 \partial_{k_t} + \omega_x^2 \partial_{k_x} + \omega_v^2 \partial_{k_v} \\ &\quad + \omega_{v_t}^2 \partial_{k_{v_t}}, \end{aligned} \tag{6.8}$$

where $Y^{[2]}$ is the second-prolongation of (6.6) given by

$$\begin{aligned} Y^{[2]} &= \tau(t,x,v)\partial_t + \xi(t,x,v)\partial_x + \eta(t,x,v)\partial_v + \mu^1(t,x,v,f)\partial_f + \mu^2(t,x,v,k)\partial_k \\ &\quad + \zeta_t \partial_{v_t} + \zeta_x \partial_{v_x} + \zeta_{xx} \partial_{v_{xx}}. \end{aligned}$$

The coefficients $\zeta$'s and $\omega$'s are defined by the prolongation formulae

$$\begin{aligned}
\zeta_t &= D_t(\eta) - v_t D_t(\tau) - v_x D_t(\xi), \\
\zeta_x &= D_x(\eta) - v_t D_x(\tau) - v_x D_x(\xi), \\
\zeta_{xx} &= D_x(\zeta_x) - v_{tx} D_x(\tau) - v_{xx} D_x(\xi)
\end{aligned}$$

and

$$\begin{aligned}
&\omega_t^1 = \tilde{D}_t(\mu^1) - f_t\tilde{D}_t(\tau) - f_x\tilde{D}_t(\xi) - f_v\tilde{D}_t(\eta) - f_{v_t}\tilde{D}_t(\zeta_t) - f_{v_x}\tilde{D}_t(\zeta_x), \\
&\omega_v^1 = \tilde{D}_v(\mu^1) - f_t\tilde{D}_v(\tau) - f_x\tilde{D}_v(\xi) - f_v\tilde{D}_v(\eta) - f_{v_t}\tilde{D}_v(\zeta_t) - f_{v_x}\tilde{D}_v(\zeta_x), \\
&\omega_{v_t}^1 = \tilde{D}_{v_t}(\mu^1) - f_t\tilde{D}_{v_t}(\tau) - f_x\tilde{D}_{v_t}(\xi) - f_v\tilde{D}_{v_t}(\eta) - f_{v_t}\tilde{D}_{v_t}(\zeta_t) - f_{v_x}\tilde{D}_{v_t}(\zeta_x), \\
&\omega_{v_x}^1 = \tilde{D}_{v_x}(\mu^1) - f_t\tilde{D}_{v_x}(\tau) - f_x\tilde{D}_{v_x}(\xi) - f_v\tilde{D}_{v_x}(\eta) - f_{v_t}\tilde{D}_{v_x}(\zeta_t) - f_{v_x}\tilde{D}_{v_x}(\zeta_x), \\
&\omega_t^2 = \tilde{D}_t(\mu^2) - k_t\tilde{D}_t(\tau) - k_x\tilde{D}_t(\xi) - k_v\tilde{D}_t(\eta) - k_{v_t}\tilde{D}_t(\zeta_t) - k_{v_x}\tilde{D}_t(\zeta_x), \\
&\omega_x^2 = \tilde{D}_x(\mu^2) - k_t\tilde{D}_x(\tau) - k_x\tilde{D}_x(\xi) - k_v\tilde{D}_x(\eta) - k_{v_t}\tilde{D}_x(\zeta_t) - k_{v_x}\tilde{D}_x(\zeta_x), \\
&\omega_v^2 = \tilde{D}_v(\mu^2) - k_t\tilde{D}_v(\tau) - k_x\tilde{D}_v(\xi) - k_v\tilde{D}_v(\eta) - k_{v_t}\tilde{D}_v(\zeta_t) - k_{v_x}\tilde{D}_v(\zeta_x), \\
&\omega_{v_t}^2 = \tilde{D}_{v_t}(\mu^2) - k_t\tilde{D}_{v_t}(\tau) - k_x\tilde{D}_{v_t}(\xi) - k_v\tilde{D}_{v_t}(\eta) - k_{v_t}\tilde{D}_{v_t}(\zeta_t) - k_{v_x}\tilde{D}_{v_t}(\zeta_x)
\end{aligned}$$

respectively, where

$$D_t = \partial_t + v_t\partial_v + \cdots, \quad D_x = \partial_x + v_x\partial_v + \cdots$$

are the total derivative operators and

$$\tilde{D}_t = \partial_t + f_t\partial_f + k_t\partial_k + \cdots,$$
$$\tilde{D}_x = \partial_x + f_x\partial_f + k_x\partial_k + \cdots,$$
$$\tilde{D}_v = \partial_v + f_v\partial_f + k_v\partial_k + \cdots,$$
$$\tilde{D}_{v_t} = \partial_{v_t} + f_{v_t}\partial_f + k_{v_t}\partial_k + \cdots,$$
$$\tilde{D}_{v_x} = \partial_{v_x} + f_{v_x}\partial_f + k_{v_x}\partial_k + \cdots$$

are the total derivative operators for the extended system. The application of the prolongation (6.8) and the invariance conditions of system (6.7) leads to

$$\begin{aligned} &\tilde{Y}(v_t - f(x)k(v_x)v_{xx} + M^2h_0v) = 0,\\ &\tilde{Y}(f_t) = \tilde{Y}(f_v) = \tilde{Y}(f_{v_t}) = \tilde{Y}(f_{v_x}) = 0,\\ &\tilde{Y}(k_t) = \tilde{Y}(k_x) = \tilde{Y}(k_v) = \tilde{Y}(k_{v_t}) = 0. \end{aligned} \tag{6.9}$$

The above system (6.9) yields the following equivalence generators:

$$\begin{aligned} &Y_1 = \partial_t,\\ &Y_2 = e^{-M^2h_0t}\partial_v,\\ &Y_3 = \partial_x,\\ &Y_4 = x\partial_x + v\partial_v + 2k\partial_k,\\ &Y_H = H(x)(f\partial_f - k\partial_k),\\ &Y_p = p(x)(\partial_f - \frac{k}{f}\partial_k) \end{aligned} \tag{6.10}$$

where $H$ and $p$ are arbitrary functions of $x$.

## 6.5 Preliminary group classification

In this section we follow the technique known as preliminary group classification, see [55]. With the notion of equivalence transformation in section 6.4, we note that (6.2) admits infinite equivalence algebra given by (6.10). Taking the finite-dimensional subalgebra of infinite-dimensional algebra (6.10), we select the subalgebra $L_5$ spanned by the following operators:

$$\begin{aligned} Y_1 &= \partial_t, \\ Y_2 &= e^{-M^2 h_0 t}\partial_v, \\ Y_3 &= \partial_x, \\ Y_4 &= x\partial_x + v\partial_v + 2k\partial_k, \\ Y_5 &= f\partial_f - k\partial_k. \end{aligned} \tag{6.11}$$

Therefore the one-parameter group of equivalence transformation corresponding to each operator is given by

$$
\begin{aligned}
Y_1 &: \ \bar{t} = t + \epsilon_1, \ \bar{x} = x, \ \bar{v} = v, \ \bar{f} = f, \ \bar{k} = k, \\
Y_2 &: \ \bar{t} = t, \ \bar{x} = x, \ \bar{v} = \epsilon_3 e^{-M^2 h_0 t} + v, \ \bar{f} = f, \ \bar{k} = k, \\
Y_3 &: \ \bar{t} = t, \ \bar{x} = x + \epsilon_2, \ \bar{v} = v, \ \bar{f} = f, \ \bar{k} = k, \\
Y_4 &: \ \bar{t} = t, \ \bar{x} = x e^{\epsilon_4}, \bar{v} = v e^{\epsilon_4}, \ \bar{f} = f, \ \bar{k} = k e^{2\epsilon_4}, \\
Y_5 &: \ \bar{t} = t, \ \bar{x} = x, \ \bar{v} = v, \ \bar{f} = f e^{\epsilon_5}, \ \bar{k} = k e^{-\epsilon_5},
\end{aligned}
$$

and their composition gives

$$
\begin{aligned}
\bar{t} &= t + \epsilon_1, \\
\bar{x} &= (x + \epsilon_2) e^{\epsilon_4}, \\
\bar{v} &= (v + \epsilon_3 e^{-M^2 h_0 t}) e^{\epsilon_4}, \\
\bar{f} &= f e^{\epsilon_5}, \\
\bar{k} &= k e^{2\epsilon_4 - \epsilon_5}.
\end{aligned}
$$

Since $f$ and $k$ are dependent variables of $x$ and $v_x$ respectively, we construct the prolongation of (6.11) to the variable $v_x$ and take their projections on the space of $(x, v_x, f, k)$, hence the prolongation stays the same as (6.11). The non-zero projections of the prolongation of (6.11) are as follows:

$$
\begin{aligned}
Z_1 &= pr(\tilde{Y}_2) = \partial_x, \\
Z_2 &= pr(\tilde{Y}_4) = x\partial_x + 2k\partial_k, \\
Z_3 &= pr(\tilde{Y}_5) = f\partial_f - k\partial_k.
\end{aligned}
\tag{6.12}
$$

**Proposition 1** (see, [55]). Let $L_r$ be an $r$-dimensional subalgebra of the algebra $L_3$. Denote by $Z_i, i = 1, ..., r$ a basis of $L_r$ and by $Y^{(i)}$ the elements of the algebra $L_3$ such that $Z_i =$ projections of $Y^{(i)}$, on $(x, v_x, f, k)$. If equations

$$f = \Phi(x), \quad k = \Gamma(v_x) \tag{6.13}$$

are invariant with respect to the algebra $L_r$ then the equation

$$v_t = \Phi(x)\Gamma(v_x)v_{xx} - M^2 h_0 v \tag{6.14}$$

admits the operator $Z_i =$ projection of $Y_i$ on $(t, x, v)$.

**Proposition 2** (see, [55]). Let (6.14) and the equation

$$v_t = \Phi'(x)\Gamma'(v_x)v_{xx} - M^2 h_0 v, \tag{6.15}$$

be constructed according to Proposition 1 via subalgebras $L_r$ and $L'_r$, respectively. If $L_r$ and $L'_r$, are similar subalgebras in $L_3$ then (6.14) and (6.15) are equivalent with respect to the equivalence group $G_5$ generated by $L_r$. These propositions imply that the problem of preliminary group classification of (6.2) is reduced to the algebraic problem of constructing non-similar subalgebras of $L_5$ or optimal system of subalgebras [55]. We will explore the method in [41] to construct the one-dimensional optimal systems.

**Adjoint group of algebra $L_3$**

To compute the Lie brackets known as commutators, defined on $L_3$ and the adjoint representations we follow the same procedure as in section 4.2, alternatively the reader may see see [41]. The results are represented below by table 6.1 and 6.2.

Table 6.1: Commutator table

| $[Z_i, Z_j]$ | $Z_1$ | $Z_2$ | $Z_3$ |
|---|---|---|---|
| $Z_1$ | $0$ | $Z_1$ | $0$ |
| $Z_2$ | $-Z_1$ | $0$ | $0$ |
| $Z_3$ | $0$ | $0$ | $0$ |

Table 6.2: Adjoint table

| $Adj$ | $Z_1$ | $Z_2$ | $Z_3$ |
|---|---|---|---|
| $Z_1$ | $Z_1$ | $Z_2 - \varepsilon Z_1$ | $Z_3$ |
| $Z_2$ | $e^{\varepsilon} Z_1$ | $Z_2$ | $Z_3$ |
| $Z_3$ | $Z_1$ | $Z_2$ | $Z_3$ |

We now construct the one-dimensional optimal system by considering the general operator

$$Z = \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3 \tag{6.16}$$

where $\alpha_1, \alpha_2$ and $\alpha_3$ are arbitrary constants. Referring to table 6.2, we act on (6.16) by $Ad(e^{\varepsilon Z_1})$ to eliminate $\alpha_2 Z_2$ and obtain

$$Z' = Ad(e^{\varepsilon Z_1})Z = [\alpha_1 - \alpha_2 \varepsilon] Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3.$$

Setting $\varepsilon = \frac{\alpha_1}{\alpha_2}$, $\alpha_2 \neq 0$ yields

$$Z' = \alpha_2 Z_2 + \alpha_3 Z_3. \tag{6.17}$$

Since $\varepsilon = \frac{\alpha_1}{\alpha_2}$, $\alpha_2 \neq 0$ hence for simplicity we let $\alpha_2 = 1$. Equation (6.17) becomes

$$Z' = Z_2 + \alpha_3 Z_3. \tag{6.18}$$

Acting $Z'$ by $Ad(e^{\varepsilon Z_2})$ or $Ad(e^{\varepsilon Z_3})$ makes no difference, therefore we move to another assumption.

We now assume $\alpha_2 = 0$ in (6.16), hence (6.16) yields

$$Z'' = \alpha_1 Z_1 + \alpha_3 Z_3. \tag{6.19}$$

Taking $\alpha_3 = 1$, equation (6.19) becomes

$$Z'' = \alpha_1 Z_1 + Z_3. \tag{6.20}$$

Acting on $Z''$ by $Ad(e^{\varepsilon} Z_2)$ yields

$$Y = Z_3 + \alpha_1 e^{\varepsilon} Z_1. \tag{6.21}$$

Depending on the sign of $\alpha_1$ from (6.21), we can make the coefficient of $Z_1$ to be $\pm 1$. Thus for $\alpha_1 > 0$: $Y = Z_3 + Z_1$, $\alpha_1 = 0$: $Z_3$ and $\alpha_1 < 0$: $Y = Z_3 - Z_1$. Hence the one-dimensional optimal system of sub-algebras are:

$$Z_3, \quad Z_3 \pm X_1, \quad Z_2 + \alpha_3 Z_3.$$

The optimal system of one-dimensional sub-algebra is given as follows:

$$
\begin{aligned}
Z^1 &= Z_3, \\
Z^2 &= Z_3 - Z_1, \\
Z^3 &= Z_3 + Z_1, \\
Z^4 &= Z_2 + \gamma Z_3, \quad \gamma \in \mathbb{R}
\end{aligned}
\tag{6.22}
$$

where $\gamma$ is arbitrary constant. Application of Proposition I and II to the set of optimal system (6.22) does not lead to admitting an extension of the PLA $L_p$ because the necessary condition of form (6.13) for existence of invariant solution is not satisfied.

## 6.6 Direct group classification

In this section we employ direct group classification to analyse (6.2) with respect to the two cases where we assume the fin profile to be equivalent to the power law and exponential law. For both cases we will classify (6.2) for different forms of thermal conductivity $k(v_x)$, where sub-cases will arise.

**Case 1: $f(x) = x^n$.**

Assuming that $f(x) = x^n$, equation (6.2) becomes

$$
v_t = x^n k(v_x) v_{xx} - M^2 h_0 v \tag{6.23}
$$

where $n$ is the arbitrary constant. We seek for Lie point symmetries admitted by (6.23). Following the same procedure as in section 6.3, equation (6.23) admits the same PLA

$L_2$ given by (6.5) In the process of computing the PLA $L_2$ for equation (6.23), we find that $k$ must satisfy the 1st order ODE

$$(av_x^2 + bv_x + c)k'(v_x) + (\alpha v_x + \beta)k(v_x) = 0, \tag{6.24}$$

where $a$, $b$, $c$, $\alpha$ and $\beta$ are arbitrary constants. The 1st order ODE (6.24) was flagged by computational software Dimsym. Equation (6.24) leads to the following three cases for the function $k(v_x)$, for which the PLA is extended. Such expressions of $k$ are of the forms:

$$k = (v_x + \lambda)^s, \quad k = v_x^r e^{pv_x} \quad \text{and} \quad k = e^{pv_x}$$

where $\lambda$, $s$, $r$ and $p$ are arbitrary constants. We note that in our current study we will only consider the form where $k = (v_x + \lambda)^s$. The other forms are of no importance to us in our current work hence they will be considered at a later stage. Due to the arbitrary constants in $k$, sub-cases will arise where PLA extends. Substituting $k = (v_x + \lambda)^s$ into (6.23) gives

$$v_t = x^n (v_x + \lambda)^s v_{xx} - M^2 h_0 v \tag{6.25}$$

and it does not extend PLA given by (6.5).

Considering the following constraints, several cases arise where $L_2$ extend:

**Case 1.1:** $n = 2$, $s = -2$ and $\lambda \neq 0$.

In this case the resulting Lie point symmetries (6.25) are

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= e^{-M^2 h_0 t}\partial_v, \\ X_3 &= x\partial_x + v\partial_v. \end{aligned} \tag{6.26}$$

**Case 1.2:** $\lambda = 0$, $n = 2$ and $s = 2$.

The resulting Lie point symmetries are

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= e^{-M^2 h_0 t}\partial_v, \\ X_3 &= e^{2M^2 h_0 t}(M^2 h_0 v\partial_v - \partial_t) \\ X_4 &= v\partial_v + x\partial_x. \end{aligned} \tag{6.27}$$

**Case 1.3:** $\lambda = 0$, $s = -2$ and $n = 2$.

The resulting Lie point symmetries are for equation (6.25) are

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= e^{-M^2 h_0 t}\partial_v, \\ X_3 &= e^{-2M^2 h_0 t}(M^2 h_0 v\partial_v - \partial_t) \\ X_4 &= M^2 h_0(v\partial_v + x\partial_x) - \partial_t. \end{aligned} \tag{6.28}$$

**Case 1.4:** $\lambda = 0$, $s = 2$ and $n = 4$.

The resulting Lie point symmetries for equation (6.25) are

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= e^{-M^2 h_0 t}\partial_v, \\ X_3 &= x\partial_x, \\ X_4 &= e^{2M^2 h_0 t}(M^2 h_0 v\partial_v - \partial_t). \end{aligned} \tag{6.29}$$

**Case 1.5:** $\lambda = 0$, $s = -2$ and $n = 4$.

The resulting Lie point symmetries for equation (6.25) are

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= e^{-M^2 h_0 t}\partial_v, \\ X_3 &= x\partial_x + 2v\partial_v, \\ X_4 &= e^{-2M^2 h_0 t}(M^2 h_0 v\partial_v - \partial_t), \\ X_5 &= e^{M^2 h_0 t}(xv\partial_x + v^2\partial_v). \end{aligned} \tag{6.30}$$

In Case 1.5 we note that symmetry $X_5$ is a nonlocal symmetry. The reduction by the nonlocal symmetry $X_5$ leads to

$$v(t, x) = xF(t)$$

where $F(t)$ satisfies the ODE

$$F' + M^2 h_0 F = 0.$$

The solutions to the above ODE is

$$F(t) = c_1 e^{-M^2 h_0 t}$$

where $c_1$ is a constant of integration. Thus

$$v(t,x) = c_1 \; x \; e^{-M^2 h_0 t}.$$

Reverting back to the governing equation (6.1), we find

$$\theta(t,x) = c_1 e^{-M^2 h_0 t}. \tag{6.31}$$

**One-dimensional optimal system**

In this section we generate the one-dimensional optimal system of sub-algebras for the Cases 1.1, 1.2, 1.3 and 1.4. We follow the same procedure as in chapter 4 to compute commutator and adjoint tables for the cases in question. For detailed computation of the Lie bracket see [41]. Below we present the commutator and adjoint tables as well as the one-dimensional optimal system of sub-algebras for Cases 1.1, 1.2, 1.3, and 1.4 respectively:

Table 6.3: Commutator table: Case 1.1

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | 0 | $-M^2 h_0 X_2$ | 0 |
| $X_2$ | $M^2 h_0 X_2$ | 0 | $X_2$ |
| $X_3$ | 0 | $-X_2$ | 0 |

Table 6.4: Adjoint table: Case 1.1

| $Adj$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | $X_1$ | $e^{M^2h_0\varepsilon}X_2$ | $X_3$ |
| $X_2$ | $X_1 - M^2h_0\varepsilon X_2$ | $X_2$ | $-\varepsilon X_2 + X_3$ |
| $X_3$ | $X_1$ | $e^{\varepsilon}X_2$ | $X_3$ |

One-dimensional optimal system of sub-algebras for Case 1.1 are

$$\{X_1 + X_3, \quad X_3 \pm X_2, \quad X_3, \quad X_2.\}$$

Table 6.5: Commutator table: Case 1.2

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | 0 | $-M^2h_0X_2$ | $2M^2h_0X_4$ | 0 |
| $X_2$ | $M^2h_0X_2$ | 0 | 0 | $X_2$ |
| $X_3$ | $-2M^2h_0X_4$ | 0 | 0 | 0 |
| $X_4$ | 0 | $-X_2$ | 0 | 0 |

We note that the sub-algebras of Case 1.2 also hold for when $s = \pm 2$ and $\lambda = 0$.

**Case 2:** $f(x) = e^{nx}$.

In this case equation (6.2) is given as

$$v_t = e^{nx}k(v_x)v_{xx} - M^2h_0v \tag{6.32}$$

where $n$ is the arbitrary constant. Equation (6.32) gives the same PLA and classification of the function $k(v_x)$ as in Case 1. Taking parameters $n$ and $s$ as arbitrary constants and $k = (v_x + \lambda)^s$ with $\lambda = 0$ and follow the same procedure as in Case 1,

Table 6.6: Adjoint table: Case 1.2

| $Ad$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | $X_1$ | $e^{M^2\varepsilon h_0}X_2$ | $e^{-2M^2h_0\varepsilon}X_3$ | $X_4$ |
| $X_2$ | $X_1 - M^2\varepsilon h_0 X_2$ | $X_2$ | $X_3$ | $X_4 - \varepsilon X_2$ |
| $X_3$ | $X_1 + 2M^2h_0\varepsilon X_4$ | $X_2$ | $X_3$ | $X_4$ |
| $X_4$ | $X_1$ | $e^{\varepsilon}X_2$ | $X_3$ | $X_4$ |

One-dimensional optimal system of sub-algebras for Case 1.2 are

$$\{X_1 \pm X_4, \quad X_1, \quad X_3 \pm X_4, \quad X_3 \pm X_2, \quad X_3\}.$$

Table 6.7: Commutator table: Case 1.3

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | $0$ | $-M^2h_0X_2$ | $-2M^2h_0X_4$ | $0$ |
| $X_2$ | $M^2h_0X_2$ | $0$ | $0$ | $0$ |
| $X_4$ | $2M^2h_0X_4$ | $0$ | $0$ | $-2X_2$ |
| $X_3$ | $0$ | $0$ | $-2X_2$ | $0$ |

we extend (6.5) with two symmetries

$$\begin{aligned} X_3 &= nv\partial_v - s\partial_x, \\ X_4 &= e^{M^2h_0st}(M^2h_0v\partial_v - \partial_t). \end{aligned} \tag{6.33}$$

When we take $\lambda = 0$, $s = \pm 2$ and $n = 1$, we find the following extensions:

**Case 2.1:** $\lambda = 0$, $s = 2$ and $n = 1$.

The resulting Lie point symmetries are

$$\begin{aligned} X_3 &= e^{2M^2h_0t}(\partial_t - M^2h_0v\partial_v), \\ X_4 &= 2\partial_x - v\partial_v. \end{aligned} \tag{6.34}$$

Table 6.8: Adjoint table: Case 1.3

| $Ad$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | $X_1$ | $e^{M^2\varepsilon h_0}X_2$ | $e^{2M^2h_0\varepsilon}X_3$ | $X_4$ |
| $X_2$ | $X_1 - M^2\varepsilon h_0 X_2$ | $X_2$ | $X_3$ | $X_4$ |
| $X_3$ | $X_1 - 2M^2h_0\varepsilon X_4$ | $X_2$ | $X_3$ | $X_4 + 2\varepsilon X_3$ |
| $X_4$ | $X_1$ | $X_2$ | $e^{-2\varepsilon}X_3$ | $X_4$ |

One-dimensional optimal system of sub-algebras for Case 1.3 are

$$\{X_1 \pm X_4, \quad X_1, \quad X_4 \pm X_2, \quad X_4, \quad X_4 \pm X_2, \quad X_3 \pm X_2, \quad X_3\}.$$

Table 6.9: Commutator table: Case 1.4

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | 0 | $-M^2h_0X_2$ | $2M^2h_0X_3$ | 0 |
| $X_2$ | $M^2h_0X_2$ | 0 | 0 | 0 |
| $X_3$ | 0 | 0 | 0 | 0 |
| $X_4$ | $-2M^2h_0X_3$ | 0 | 0 | 0 |

Table 6.10: Adjoint table: Case 1.4

| $Ad$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | $X_1$ | $e^{M^2\varepsilon h_0}X_2$ | $X_3$ | $e^{-2M^2h_0\varepsilon}X_4$ |
| $X_2$ | $X_1 - M^2\varepsilon h_0 X_2$ | $X_2$ | $X_3$ | $X_4$ |
| $X_3$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
| $X_4$ | $X_1 + 2M^2h_0\varepsilon X_3$ | $X_2$ | $X_3$ | $X_4$ |

One-dimensional optimal system of sub-algebras for Case 1.4 are

$$\{X_4, \quad X_3 + \alpha X_4, \quad X_4 \pm X_2, \quad X_1 + \beta X_4, \quad X_4 + X_3 \pm X_2, \quad X_4 - X_3 \pm X_2\}$$

**Case 2.2:** $\lambda = 0$, $s = -2$ and $n = 1$.

The resulting Lie point symmetries are

$$\begin{aligned} X_3 &= e^{-2M^2 h_0 t}(M^2 h_0 v\partial_v - \partial_t), \\ X_4 &= M^2 h_0(v\partial_v + 2\partial_x) - \partial_t. \end{aligned} \tag{6.35}$$

We note that the optimal systems for these two cases are the same as the ones computed in Case 1.2 and Case 1.3 hence we are not displaying them here, only the reductions will be shown in table in the next section.

## 6.7 Symmetry reductions and invariant solutions

In this section we use an optimal system of one-dimensional subalgebras calculated above to find symmetry reductions and invariant solutions. Only some elements of the optimal system which are of interest to our study will be considered, the rest will be done in future or in other studies.

**Reduction and invariant solutions from an optimal system of Case 1.1**

**Consider** $X_3 + X_2$ :

The corresponding characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{x} = \frac{dv}{v - e^{-M^2 h_0 t}}. \tag{6.36}$$

Solving (6.36) we find the general exact solutions for the equation (6.25) using simple computations as

$$v(t, x) = (c_1\ x - 1)e^{-M^2 h_0 t} \tag{6.37}$$

where $c_1$ is a constant of integration.

**Invariance under** $X_3$ :

The characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{x} = \frac{dv}{v}, \tag{6.38}$$

and then yield the following solutions

$$v(t, x) = c_1 \ x \ e^{-M^2 h_0 t} \tag{6.39}$$

where $c_1$ is a constant of integration.

**Invariance under** $X_1 + X_3$ :

The characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{x} = \frac{dv}{v}. \tag{6.40}$$

Equations (6.40) yield

$$v(t, x) = e^t g(z) \tag{6.41}$$

where $g$ satisfies the 2nd-order ODE

$$(1 + M^2 h_0)g(z) - zg'(z) - \frac{z^2 g''(z)}{(g'(z) + \lambda)^2} = 0. \tag{6.42}$$

The above ODE is difficult to solve exactly.

The reductions and invariant solutions for the remaining cases are listed in the tables below:

Table 6.11: Reductions by elements of optimal systems for Case 1.2

| Symmetry | Invariants and reductions | Solutions in $v$ |
|---|---|---|
| $X_3$ | $v(t,x) = e^{-M^2 h_0 t} g(x)$ where $g(x)$ satisfies $g'g'' = 0$ | $v(t,x) = c_1 e^{-M^2 h_0 t}$<br><br>$v(t,x) = (c_2 x + c_3) e^{-M^2 h_0 t}$ where $c_2$ and $c_3$ are constants of integration. |
| $X_3 + X_2$ | $v(t,x) = \frac{\exp(-3M^2 h_0 t)}{2M^2 h_0} + e^{-M^2 h_0 t} h(x)$ where $h(x)$ satisfies $x^2 h'^2 h'' + 1 = 0$ | $v(t,x) = -\frac{1}{2}\sqrt[3]{-3}x^{2/3}\,{}_2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{5}{3}; xc_1\right)$<br><br>$-\sqrt[3]{-3}x^{2/3}\sqrt[3]{c_1 x + 1} + c_2 + \frac{e^{-2h_0 M^2 t}}{2h_0 M^2}$<br>$v(t,x) = -\frac{1}{2}\sqrt[3]{3}x^{2/3}\,{}_2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{5}{3}; xc_1\right)$<br>$+\sqrt[3]{3}x^{2/3}\sqrt[3]{c_1 x + 1} + c_2 + \frac{e^{-2h_0 M^2 t}}{2h_0 M^2}$<br>$v(t,x) = -\frac{1}{2}\sqrt[3]{3}x^{2/3}\,{}_2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{5}{3}; xd_1\right)$<br>$-\sqrt[3]{3}x^{2/3}\sqrt[3]{d_1 x + 1} + d_2 + \frac{e^{-2h_0 M^2 t}}{2h_0 M^2}$ where ${}_2F_1$ represent hypergeometric function, $d_1$ and $d_2$ are constants of integration |

Table 6.12: Reductions by elements of optimal systems for Case 1.3

| Symmetry | Invariants and reductions | Solutions in $v$ |
|---|---|---|
| $X_3$ | $v(t,x) = e^{-M^2 h_0 t} I(x)$ where $I(x)$ satisfies $I'' = 0$ | $v(t,x) = (\kappa_1 x + \kappa_2) e^{-M^2 h_0 t}$ where $\kappa_1$ and $\kappa_2$ are constants of integration |
| $X_3 + X_2$ | $v(t,x) = e^{-M^2 h_0 t} H(x) \mp \frac{1}{2} e^{M^2 h_0 t}$ where $H(x)$ satisfies $x^2 H'' \pm M^2 h_0 H'^2 = 0$ | $v(t,x) = \exp(-M^2 h_0 t) \left( \frac{M^2 h_0 \ln\left(\mu_1 x + M^2 h_0\right)}{\mu_1^2} \right)$ $+ \exp(-M^2 h_0 t) \left( -\frac{x}{\mu_1} + \mu_2 \right) - \frac{1}{2} \exp(M^2 h_0 t)$, $\mu_1 \neq 0$. $\mu_1$ and $\mu_2$ are constants of integration |
| $X_4$ | $\gamma = e^{M^2 h_0 t} x,\ v(t,x) = e^{-M^2 h_0 t} K(\gamma)$ where $K(\gamma)$ satisfies $\gamma K'' - M^2 h_0 K'^3 = 0$ | $v(t,x) = \sigma_2 e^{-M^2 h_0 t} \mp \frac{\exp(\frac{-M^4 {h_0}^2 - \sigma_1}{M^2 h_0})}{\sqrt{M^2 h_0}}$ $\left( \frac{\sqrt{\frac{\pi}{2}}\ \mathrm{erf}\left( \sqrt{\frac{M^4 {h_0}^2 - M^2 h_0 \ln x - \sigma_1}{M^2 h_0}} \right)}{\sqrt{M^2 h_0}} \right)$ where $\sigma_1$ and $\sigma_2$ are constants of integration |

## 6.8 Concluding remarks

In this chapter we carried out preliminary and direct group classification. The principal Lie algebra was found to be two dimensional for both analyses. Equivalence transformation was expressed in order to perform preliminary group classification. We also derived optimal systems for both methods and the existence of invariant solutions was not satisfied for preliminary group classification. Direct group classification led to two cases and we generated group invariant solutions by employing symmetry reductions. Some reductions gave general solution which will be studied further in future. We note that some invariant solutions generated special function solutions such as error function and hypergeometric function.

Table 6.13: Reductions by elements of optimal systems for Case 1.4 ($\lambda = 0$,$s = 2, n = 4$)

| Symmetry | Invariants and reductions | Solutions in $v$ |
|---|---|---|
| $X_3$ | $v(t,x) = Z(t)$, where $Z(t)$ satisfies $M^2 h_0 Z(t) + Z'(t) = 0$ | $v(t,x) = \epsilon_1 e^{-M^2 h_0 t}$ where $\epsilon_1$ is a constants of integration |
| $X_4$ | $v(t,x) = e^{M^2 h_0 t} B(x)$, where $B$ satisfies $B'(x)B''(x) = 0$ | $v(t,x) = \varsigma_1 e^{-M^2 h_0 t}$<br><br>$v(t,x) = (\varsigma_1 x + \varsigma_2) e^{-M^2 h_0 t}$where $\varsigma_1$ and $\varsigma_2$ are constants of integration |
| $X_3 + X_2$ | $v(t,x) = e^{-M^2 h_0 t} \ln x + \Lambda(t)$, where $\Lambda$ satisfies $1 + M^2 h_0 e^{3M^2 h_0 t} \Lambda(t) + e^{3M^2 h_0 t} \Lambda'(t) = 0$ | $v(t,x) = c_1 e^{-M^2 h_0 t} + e^{-M^2 h_0 t} \ln x + \frac{e^{-3M^2 h_0 t}}{2 - M^2 h_0}$ where $c_1$ is a constants of integration |

Table 6.14: Solutions generated by $X_4+X_3$ and $X_4+X_2$

| Symmetry | Solutions in $v$ |
|---|---|
| $X_4+X_3$ | $v(t,x)=c_1e^{-M^2h_0t}$ <br> $v(t,x)=e^{-M^2h_0t}\left(-\sqrt{2a_1x^2\exp(-\frac{e^{-2M^2h_0t}}{M^2h_0})-1}\right)-e^{-M^2h_0t}\arctan\left(\frac{1}{\sqrt{2a_1x^2\exp(-\frac{e^{-2h_0M^2t}}{M^2h_0})-1}}\right)$ <br> $+a_2e^{-M^2h_0t}$ <br> $v(t,x)=e^{-M^2h_0t}\arctan\left(\frac{1}{\sqrt{2z_1x^2\exp(-\frac{e^{-2h_0M^2t}}{M^2h_0})-1}}\right)+e^{-M^2h_0t}\left(\sqrt{2z_1x^2\exp(-\frac{e^{-2M^2h_0t}}{M^2h_0})-1}+z_2\right)$ <br> where $z_1$ and $z_2$ are constants of integration |
| $X_4+X_2$ | $v(t,x)=\frac{e^{-2M^2h_0t}}{2M^2h_0}+e^{-M^2h_0t}\Big(1+1/18\,\pi\sqrt{3}-1/2\,\ln 3+\ln x+1/3\,\ln(b_1)\Big)$ <br> $+e^{-M^2h_0t}\Big(1/9\,x^3b_1\,{}_3\mathrm{F}_2(2/3,1,1;\,2,2;\,-b_1\,x^3)+b_2\Big)$ <br> $v(t,x)=\frac{e^{-2M^2h_0t}}{2M^2h_0}+e^{-M^2h_0t}\Big(-i/2\sqrt{3}-i/12\pi+i/4\sqrt{3}\ln 3-i/2\sqrt{3}\ln x\Big)$ <br> $+e^{-M^2h_0t}\Big(-i/6\sqrt{3}\ln(q_1)-i/18\sqrt{3}x^3q_1\,{}_3\mathrm{F}_2(2/3,1,1;\,2,2;\,-q_1\,x^3)-1/2-1/36\,\pi\sqrt{3}+$ <br> $1/4\,\ln 3-1/2\,\ln x-1/6\,\ln(q_1)-1/18\,x^3q_1\,{}_3\mathrm{F}_2(2/3,1,1;\,2,2;\,-q_1\,x^3)+q_2\Big)$ <br> $v(t,x)=\frac{e^{-2M^2h_0t}}{2M^2h_0}+e^{-M^2h_0t}\Big(i/2\sqrt{3}+i/12\pi-i/4\sqrt{3}\ln 3+i/2\sqrt{3}\ln x+i/6\sqrt{3}\ln(r_1)\Big)$ <br> $e^{-M^2h_0t}\Big(+i/18\sqrt{3}x^3r_1\,{}_3\mathrm{F}_2(2/3,1,1;\,2,2;\,-r_1\,x^3)-1/2-1/36\,\pi\sqrt{3}+1/4\,\ln 3-1/2\,\ln x-$ <br> $1/6\,\ln(r_1)-1/18\,x^3r_1\,{}_3\mathrm{F}_2(2/3,1,1;\,2,2;\,-r_1\,x^3)+r_2\Big)$ where ${}_3F_2$ is hypergeometric function, $r_1$ and $r_2$ are constants of integration |

Table 6.15: Reductions by elements of optimal systems for Case 2.1

| Symmetry | Reductions | Solutions in $v$ |
|---|---|---|
| $X_3$ | $v(t,x) = e^{-M^2 h_0 t}\chi(x)$ where $\chi(x)$ satisfies $\chi'\chi'' = 0$ | $v(t,x) = c_1 e^{-M^2 h_0 t}$<br>$v(t,x) = (c_1 x + c_2)e^{-M^2 h_0 t}$ |
| $X_3 + X_2$ | $v(t,x) = \frac{1}{2}exp^{-3M^2 h_0 t} + e^{-M^2 h_0 t} g(x)$ where $g(x)$ satisfies $M^2 h_0 - e^x g'^2 g'' = 0$ | See solutions in the table below |

Table 6.16: Solutions in $v$ generated by $X_3 + X_2$

| Solutions |
|---|
| $v(t,x) = e^{h_0(-M^2)t}\left(c_2 - \frac{3\sqrt[3]{-3}\sqrt[3]{c_1 - h_0M^2e^{-x}}\left(c_1e^x\left(\left(1-\frac{c_1e^x}{h_0M^2}\right)^{2/3}{}_2F_1\left(\frac{2}{3},\frac{2}{3};\frac{5}{3};\frac{e^xc_1}{M^2h_0}\right)-2\right)+2h_0M^2\right)}{2c_1e^x-2h_0M^2}\right) - \frac{1}{2}e^{-3h_0M^2t}$ |
| $v(t,x) = e^{h_0(-M^2)t}\left(\frac{3\sqrt[3]{3}\sqrt[3]{c_1 - h_0M^2e^{-x}}\left(c_1e^x\left(\left(1-\frac{c_1e^x}{h_0M^2}\right)^{2/3}{}_2F_1\left(\frac{2}{3},\frac{2}{3};\frac{5}{3};\frac{e^xc_1}{M^2h_0}\right)-2\right)+2h_0M^2\right)}{2c_1e^x-2h_0M^2} + c_2\right) - \frac{1}{2}e^{-3h_0M^2t}$ |
| $v(t,x) = e^{h_0(-M^2)t}\left(\frac{3(-1)^{2/3}\sqrt[3]{3}\sqrt[3]{c_1 - h_0M^2e^{-x}}\left(c_1e^x\left(\left(1-\frac{c_1e^x}{h_0M^2}\right)^{2/3}{}_2F_1\left(\frac{2}{3},\frac{2}{3};\frac{5}{3};\frac{e^xc_1}{M^2h_0}\right)-2\right)+2h_0M^2\right)}{2c_1e^x-2h_0M^2} + c_2\right) - \frac{1}{2}e^{-3h_0M^2t}$ |

Table 6.17: Case 2.2 reductions and invariants solutions

| Symmetry | Reductions | Solutions in $v$ |
|---|---|---|
| $X_3$ | $v(t,x) = e^{-M^2h_0t}g(x)$ where $g(x)$ satisfies $g'' = 0$ | $v(t,x) = (c_1x + c_2)e^{-M^2h_0t}$ |
| $X_3 + X_2$ | $v(t,x) = e^{-M^2h_0t}g(x) - \frac{1}{2}e^{M^2h_0t}$ where $g(x)$ satisfies $e^xg'' + M^2h_0g'^2 = 0$ | $v(t,x) = e^{-M^2h_0t}\left(c_2 - \frac{\ln\left(c_1e^x+M^2h_0\right)}{c_1}\right)$ $-\frac{1}{2}e^{M^2h_0t}$ |
| $X_4$ | $\gamma = x+2M^2h_0t$, $v(t,x) = e^{-M^2h_0t}g(\gamma)$ where $g(\gamma)$ satisfies $M^2h_0g'^3 - e^{\gamma}g'' = 0$ | $v(t,x) = -\frac{1}{2}e^{M^2h_0t} + c_2e^{M^2h_0t}$ $\mp M^2h_0t\left(\frac{i\sqrt{2}\ln\left(\sqrt{c_1}\sqrt{c_1M^2h_0e^{x+2M^2h_0t}-2}+c_1e^{x/2+M^2h_0t}\right)}{\sqrt{c_1}}\right)$ |

# Chapter 7

# Nonclassical Potential Symmetries

## 7.1 Introduction

A helpful approach for creating groups of point transformations that DEs admit is provided by Lie group theory. The classical Lie symmetry method, commonly referred to as the Lie point symmetry approach, has proven to be an effective technique for identifying symmetry reductions of PDEs. The classical symmetries are the solutions of a linear PDE system with sufficient determination. The traditional approach for symmetry reductions of PDEs has undergone a number of improvements. The classical method for finding invariant solutions from point symmetries admitted by a certain PDE is generalized by the nonclassical method for obtaining PDE solutions. The nonclassical approach was introduced by Bluman and Cole [50] to investigate the symmetry reductions of the heat equation. Finding all potential solutions to the overdetermined system is challenging since the nonclassical method yields fewer determining equations than the classical method. A technique to discover a new class of symmetries for a PDE was introduced by Bluman et al. [11, 42]. Any PDE that is expressed as a system of two PDEs can be written in a conserved form by adding a new variable known as the potential or auxiliary variable. Nonlocal symmetries can be used to find exact solutions to DEs that cannot be found, for instance, via Lie point symmetries. Pucci and Saccomandi [43] demonstrated more than one way of expressing some PDEs as a system of PDEs known as an auxiliary system. Moitsheki et al [57] has shown that some of the auxiliary systems may be hidden. In comparison to the traditional Lie group technique, these symmetry techniques typically have fewer determining equations. Consequently, it is quite challenging to identify all possible solutions for the overdetermined system. A significantly larger class of symmetry groups is available with this new symmetry approach. Research by Bluman and Yan in [47] has shown that, the nonclassical method, when applied to a potential equation, can produce new solutions to a given PDE that are not possible as invariant solutions from the admitted point symmetries of the given PDE. Such solutions are known as nonclassical potential solutions. For more research on nonclassical potential symmetry technique, the reader is referred to [58, 59, 60, 61, 62].

In this chapter we consider the special cases of the more general model for the transient heat transfer equation discussed in section 2.2 of chapter 2 given by (2.17). We note that in Chapter 6, the Lie group classification was employed for the special case of (2.17). Here are the assumptions we will consider for our work

$$f(x) = 1, \quad k(\theta) = \theta^m \ \text{ and } \ h(\theta) = \theta^n.$$

Now (2.17) becomes

$$\theta_t = [\theta^m \theta_x]_x - M^2 \theta^{n+1} \tag{7.1}$$

where $m$ and $n$ are arbitrary constants. Mhlongo et al obtained the Lie point symmetries and nonlocal symmetries of (7.1) for $m \neq n$ in [48]. In our work we will utilize nonclassical potential symmetry techniques to study the integrated form of (7.1). When we set $n = -1$, equation (7.1) becomes

$$\theta_t = (\theta^m \theta_x)_x - M^2. \tag{7.2}$$

Equation (7.2) is equivalent to

$$D_t(\theta) = D_x(\theta^m \theta_x - M^2 x) \tag{7.3}$$

in a conserved form. Introducing auxiliary term $w = w(t, x)$ equation (7.2) becomes

$$\begin{cases} w_x = \theta, \\ w_t = \theta^m \theta_x - M^2 x. \end{cases} \tag{7.4}$$

Taking $w_x = \theta$ and substituting into (7.2) we get the integrated form of (7.2) as

$$w_t = w_x^m w_{xx} - M^2 x. \tag{7.5}$$

In the following, Section 7.2 we will employ a nonlocal method on equation (7.5) followed by nonlocal symmetry reduction of (7.5) in Section 7.3. Section 7.4 we look into the nonclassical potential symmetries of equation (7.1) by a applying the nonclassical method to auxiliary system (7.4) and the integrated equation (7.5). Finally we give concluding remarks in section 7.5.

## 7.2 Potential Symmetries for equation (7.5)

In this section we compute nonlocal symmetries for (7.5). We note that equation (7.2) does not give any nonlocal symmetries when $m$ is arbitrary and for $m = -3$. In computing the nonlocal symmetries for the integrated form (7.5), we follow the procedure outlined in chapter 2. We consider three cases, namely: $m$ is arbitrary, $m = -2$ and $m = -3$.

**Case 1: m is arbitrary**

In this case (7.5) admits four symmetries and they are not potential symmetries

$$\begin{cases} \Upsilon_1 = \partial_t, \\ \Upsilon_2 = -\frac{1}{m+3}\Big(t\partial_t - (m+2)x\partial_x + (m+3)w\partial_w\Big), \\ \Upsilon_3 = \partial_w, \\ \Upsilon_4 = \partial_x - M^2 t\partial_w. \end{cases} \tag{7.6}$$

**Case 2: m=-2**

In this case (7.5) admits five symmetries as given below

$$
\begin{cases}
\Upsilon_1 = \partial_t, \\
\Upsilon_2 = -2t\partial_t + x\partial_x - w\partial_w, \\
\Upsilon_3 = \partial_w, \\
\Upsilon_4 = \frac{1}{2M^2}\Big(\partial_x - M^2 t\partial_w\Big), \\
\Upsilon_5 = \frac{1}{2M^2}\Big(-M^2t^2\partial_t - (w + M^2 tx)\partial_x + M^2 tw\partial_w\Big).
\end{cases} \tag{7.7}
$$

and $\Upsilon_5$ is a potential symmetry.

**Case 3: m=-3**

This case gives four generators and none of the operators represent nonlocal symmetry

$$
\begin{cases}
\Upsilon_1 = t\partial_t - x\partial_x, \\
\Upsilon_2 = \partial_t, \\
\Upsilon_3 = \partial_w, \\
\Upsilon_4 = \partial_x - M^2 t\partial_w.
\end{cases} \tag{7.8}
$$

## 7.3 Potential symmetry reduction

It is necessary to solve the characteristic equation in order to discover symmetry reductions and accurate solutions. Reduction by $\Upsilon_5$ from (7.7) gives the following invariants

$$
\begin{cases}
J_1 = \frac{w}{t}, \\
\\
J_2 = \frac{w}{M^2} + tx.
\end{cases}
\tag{7.9}
$$

If we set $z = J_1$ and $J_2 = F$ where $F$ is an arbitrary function leading to

$$
F(z) = \frac{w}{M^2} + tx \tag{7.10}
$$

where $F$ satisfies

$$
F''(z) + M^2 F(z) F'(z) = 0. \tag{7.11}
$$

Equation (7.11) can be solved to

$$
F(z) = \frac{\sqrt{2\ c_1}}{M} \tanh\left(\frac{c_1 M^2\ (z + c_2)}{\sqrt{2\ c_1 M^2}}\right). \tag{7.12}
$$

Substituting (7.12) into (7.10) yields

$$
w = \sqrt{2\ c_1 M^2}\ \tanh\left(\frac{c_1 M^2 (w + c_2 t)}{\sqrt{2\ c_1 M^2}\ t}\right) - M^2 tx. \tag{7.13}
$$

Representing equation (7.13) in terms of $\theta$ we find

$$
\theta(t,x) = \frac{M^2 t^2}{c_1 M^2 \mathrm{sech}^2\left(\frac{c_1 M^2 (w + c_2 t)}{\sqrt{2\ c_1 M^2}\ t}\right) - t}. \tag{7.14}
$$

The generalized exact solutions (7.13) and (7.14) are implicit.

## 7.4 Nonclassical potential symmetry for a transient heat conduction equation

In this section we study nonclassical potential symmetries of (7.2) with $m = -2$. The theory behind this technique has been discussed and outlined in section 3.2.4. We will first do the analysis of (7.2) by utilizing the potential system approach, followed by the potential equation approach. It is safe to note that $\theta(x)$ and $w(x)$ solve (7.2) and (7.5), respectively, if $(\theta(x), w(x))$ satisfies (7.4). We require both (7.2) and the invariant surface constraint to be invariant under the infinitesimal generator

$$\Upsilon \;=\; \tau(t,x,\theta,w)\frac{\partial}{\partial t} + \xi(t,x,\theta,w)\frac{\partial}{\partial x} + \eta(t,x,\theta,w)\frac{\partial}{\partial \theta} + \phi(t,x,\theta,w)\frac{\partial}{\partial w}. \quad (7.15)$$

**Algorithm I: Potential system approach**

Here we consider the analysis of the auxiliary system (7.4) by utilizing the nonclassical symmetry method. We need (7.4) and invariant surface condition

$$\begin{aligned} &\tau(t,x,\theta,w)\theta_t + \xi(t,x,\theta,w)\theta_x = \eta(t,x,\theta,w),\\ &\tau(t,x,\theta,w)\theta_t + \xi(t,x,\theta,w)\theta_x = \phi(t,x,\theta,w) \end{aligned} \quad (7.16)$$

for the vector field (7.15). The infinitesimal criterion for the invariance is given by

$$\begin{aligned} &\Upsilon^{[1]}\,(w_x - \theta)\big|_{(7.4),(7.16)} = 0,\\ &\Upsilon^{[1]}\,(w_t - \theta^{-2}\theta_x - M^2 x)\big|_{(7.4),(7.16)} = 0 \end{aligned} \quad (7.17)$$

where $\Upsilon^{[1]}$ is the first prolongation

$$\Upsilon^{[1]} = \Upsilon + \zeta_t \frac{\partial}{\partial \theta_t} + \zeta_x \frac{\partial}{\partial \theta_x} + \psi_t \frac{\partial}{\partial w_t} + \psi_x \frac{\partial}{\partial w_x}. \tag{7.18}$$

The extended infinitesimals are given by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - \theta_t D_t(\tau) - \theta_x D_t(\xi), \\
\zeta_x &= D_x(\eta) - \theta_t D_x(\tau) - \theta_x D_x(\xi), \\
\psi_t &= D_t(\phi) - v_t D_t(\tau) - v_x D_t(\xi), \\
\psi_x &= D_x(\phi) - v_t D_x(\tau) - v_x D_x(\xi).
\end{aligned} \tag{7.19}$$

The nonclassical method applied to the auxiliary system (7.4) gives rise to two determining equations for the infinitesimals $\xi(t, x, \theta, w)$, $\eta(t, x, \theta, w)$ and $\phi(t, x, \theta, w)$. The symmetry determination equation for system (7.4) in the traditional Lie symmetry group theory is

$$\begin{aligned}
&\phi_\theta \theta_x + \phi_w w_x - \Big(\xi_x + \xi_\theta \theta_x + \xi_w w_x\Big) w_x - \Big(\tau_x + \tau_\theta \theta_x + \tau_w w_x\Big) w_t \\
&+\phi_x - \eta = 0,
\end{aligned} \tag{7.20}$$

$$\begin{aligned}
&M^2\theta^3\xi + 2\theta_x\eta - \theta\Big[\eta_x + \eta_\theta\theta_x + \eta_w w_x - \Big(\xi_x + \xi_\theta\theta_x + \xi_w w_x\Big)\theta_x\Big] \\
&+\Big(\phi_\theta\theta_t + \phi_w w_t\Big)\theta^3 - \Big(\tau_x + \tau_\theta\theta_x + \tau_w w_x\Big)\theta\theta_t - \Big(\xi_t + \xi_\theta\theta_t + \xi_w w_t\Big)\theta^3 w_x \\
&-\Big(\tau_t + \tau_\theta\theta_t + \tau_w w_t\Big)w_t + \theta^3\phi_t = 0.
\end{aligned} \tag{7.21}$$

We now combine (7.4) and (7.16) to obtain

$$\begin{aligned}
\theta_t &= \frac{\eta}{\tau} - \frac{\xi}{\tau}\left(M^2 x + \frac{\phi}{\tau} - \frac{\xi}{\tau}\theta\right)\theta^2, \\
\theta_x &= \left(M^2 x + \frac{\phi}{\tau} - \frac{\xi}{\tau}\theta\right)\theta^2.
\end{aligned} \tag{7.22}$$

Without loss of generality, two cases are considered: $\tau = 1$ and $\tau = 0$, $\xi = 1$.

**Case** $\tau = 1$**.**

Substituting (7.22) into (7.20) and (7.21) yields

$$\begin{aligned}
&\xi\xi_\theta\theta^4 - \Big(M^2x\xi_\theta + \phi\xi_\theta + \xi\phi_\theta\Big)\theta^3 + \Big(M^2x\phi_\theta + \phi\phi_\theta - \xi_w\Big)\theta^2 \\
&+\Big(\phi_w - \xi_x\Big)\theta + \phi_x - \eta = 0,
\end{aligned} \tag{7.23}$$

$$\begin{aligned}
&\Big(\xi\phi_\theta - M^2x\xi_w - \phi\xi_\theta + \xi\phi_\theta\Big)\xi\theta^5 \\
&+\Big(M^4x^2\xi_\theta - \xi\phi\phi_\theta - M^2x\xi\phi_\theta + 2M^2x\phi\xi_\theta + \phi^2\xi_\theta\Big)\theta^4 \\
&+\Big(M^2x\xi_w - \xi_t + \xi\phi_\theta - \phi\xi_\theta - \xi\phi_w - \xi\xi_x\Big)\theta^3 \\
&+\Big(\phi\phi_w + \eta\phi_\theta + \phi_t + (\phi + M^2x)\xi_x - (\phi + M^2x)\eta_\theta - (2\eta - M^2)\xi\Big)\theta^2 \\
&+\Big((\phi_\theta - 2\xi)\eta + \phi_t + M^2\xi\Big)\theta^2 + \Big(2\eta\phi - \eta_w + 2M^2x\eta\Big)\theta \\
&-\eta_x = 0
\end{aligned} \tag{7.24}$$

Without first establishing certain assumptions about the dependence of the infinitesimals to streamline the computations, this system cannot be solved since it is underdetermined. We do not use those assertions here.

**Case** $\tau = 0$, $\xi = 1$.

When using the nonclassical technique in the aforementioned situation, we made the assumption that $\tau \neq 0$, which allowed us to adjust $\tau = 1$ without losing generality. We now look at the scenario where $\tau = 0$. Without loss of generality we set $\xi = 1$ and obtain two determining equations

$$\phi_\theta\theta^3 + \Big(M^2x\phi_\theta + \phi\phi_\theta\Big)\theta^2 + \phi_w\theta + \phi_x - \eta = 0, \tag{7.25}$$

$$\Big(\phi_\theta + \phi_\theta\Big)\theta^5 + \Big(-\phi\phi_\theta - M^2x\phi_\theta\Big)\theta^4 + \Big(\phi_\theta - \phi_w\Big)\theta^3$$
$$+\Big((\phi_w - \eta_\theta)\phi - M^2x\eta_\theta + \eta\phi_\theta - 2\eta + \phi_t + M^2\Big)\theta^2$$
$$+\Big(2\eta\phi - \eta_w + 2M^2x\eta\Big)\theta - \eta_x = 0. \tag{7.26}$$

Once more, we observe that the system is underdetermined and that we are unable to find the general solution.

**Algorithm II: Potential equation approach**

Here we analyse the integrated equation (7.5). We seek $\tau(t,x,w)$, $\xi(t,x,w)$ and $\eta(t,x,w)$ in order for the vector field

$$\Upsilon = \tau(t,x,w)\frac{\partial}{\partial t} + \xi(t,x,w)\frac{\partial}{\partial x} + \phi(t,x,w)\frac{\partial}{\partial w} \tag{7.27}$$

to be invariant provided the ISC

$$\tau(t,x,w)w_t + \xi(t,x,w)w_x = \eta(t,x,w), \tag{7.28}$$

is satisfied.

**Case $\tau = 1$.**

Employing nonclassical method to the potential equation (7.5) gives the determining equation

$$M^2x\xi_w\, {w_x}^3 - 2\,\xi\phi_w\, {w_x}^3 - \xi_t\, {w_x}^3 + \xi_{ww}\, {w_x}^3 + 2\,\xi_{xw}\, {w_x}^2 - \phi_{ww}\, {w_x}^2$$

$$+M^2x\phi_w\, {w_x}^2 + M^2\xi\, {w_x}^2 - 2\,\xi\phi_x\, {w_x}^2 + 2\,\phi\phi_w\, {w_x}^2 + \phi_t\, {w_x}^2 + \xi_{xx}\, w_x$$

$$+2\,\phi\phi_x\, w_x + 2\,M^2x\phi_x\, w_x - 2\,\phi_{xw}\, w_x - \phi_{xx} = 0 \tag{7.29}$$

We now separate the determining equation (7.29) with respect to the powers of the derivatives of $w$ and find the following system:

$$\phi_{xx} = 0, \tag{7.30}$$

$$2\,M^2x\phi_x + 2\,\phi\phi_x - 2\,\phi_{xw} + \xi_{xx} = 0, \tag{7.31}$$

$$M^2x\phi_w + M^2\xi - 2\,\xi\phi_x + 2\,\phi\phi_w + \phi_t + 2\,\xi_{xw} - \phi_{ww} = 0, \tag{7.32}$$

$$M^2x\xi_w - 2\,\xi\phi_w - \xi_t + \xi_{ww} = 0 \tag{7.33}$$

From equation (7.30) we find

$$\phi = f_1(t,w)x + f_2(t,w). \tag{7.34}$$

Eliminating $\phi$ in equation (7.31) gives

$$\xi = -\frac{1}{3}x^3\, f_1\left(M^2 + f_1\right) - x^2\, \left(f_1f_2 - \, f_{1w}\right) + x\, f_3 + f_4, \tag{7.35}$$

where $f_3$ and $f_4$ are also arbitrary functions of $t$ and $w$. Substituting (7.34) and (7.35) into equation (7.32) and separating with respect to the powers of $x$ we find

$$M^2 f_4 - 2\, f_1 f_4 + 2\, f_2 \frac{\partial}{\partial w} f_2 + \frac{\partial}{\partial t} f_2 + 2\, \frac{\partial}{\partial w} f_3 - \frac{\partial^2}{\partial w^2} f_2 = 0, \tag{7.36}$$

$$M^2 \frac{\partial}{\partial w} f_2 + M^2 f_3 - 2\, f_3 f_1 - 2\, f_2 \frac{\partial}{\partial w} f_1 - 2\, f_1 \frac{\partial}{\partial w} f_2 + \frac{\partial}{\partial t} f_1$$

$$+3\, \frac{\partial^2}{\partial w^2} f_1 = 0, \tag{7.37}$$

$$-f_1 \left( M^2 f_2 - 2\, f_1 f_2 + 4\, \frac{\partial}{\partial w} f_1 \right) = 0, \tag{7.38}$$

$$-\frac{1}{3}\, M^2 f_1 \left( M^2 + f_1 \right) + \frac{2}{3}\, (f_1)^2 \left( M^2 + f_1 \right) = 0. \tag{7.39}$$

Solving equation (7.39) for $f_1(t, w)$ we find three solutions of $f_1$, namely

$$0, \;\; - M^2, \;\; \frac{1}{2} M^2.$$

Substituting $f_1(t, w) = \frac{1}{2} M^2$ into (7.33), (7.36)−(7.38) and separating with the respect to the powers of $x$ yields

$$\frac{\partial^2}{\partial w^2} f_4 - \frac{\partial}{\partial t} f_4 - 2\, f_4 \frac{\partial}{\partial w} f_2 = 0, \tag{7.40}$$

$$M^2 \frac{\partial}{\partial w} f_4 - 2\, f_3 \frac{\partial}{\partial w} f_2 - \frac{\partial}{\partial t} f_3 + \frac{\partial^2}{\partial w^2} f_3 = 0, \tag{7.41}$$

$$2 \frac{\partial}{\partial w} f_3 + 2 f_2 \frac{\partial}{\partial w} f_2 + \frac{\partial}{\partial t} f_2 - \frac{\partial^2}{\partial w^2} f_2 = 0. \tag{7.42}$$

Solving the system above we obtain

$$\tau(t, x, w) = 1,$$

$$\xi(t, x, w) = -\frac{1}{4}\, M^4 x^3 - \frac{1}{2}\, M^2 f_2(t, w) x^2 + f_3(t, w) x + f_4(t, w),$$

$$\phi(t, x, w) = \frac{1}{2}\, M^2 x + f_2(t, w). \tag{7.43}$$

We note that we may find the arbitrary function if we follow a similar approach as in [58, 59, 60, 61, 62].

When we take $f_1(t,w) = -M^2$, we find that

$$f_2(t,w) = f_3(t,w) = f_4(t,w) = 0.$$

Hence

$$\begin{aligned}
\tau(t,x,w) &= 1,\\
\xi(t,x,w) &= 0,\\
\phi(t,x,w) &= -M^2 x. \qquad (7.44)
\end{aligned}$$

Taking $f_1(t,w) = 0$, we remain with this system

$$M^2 f_4 + 2 f_2 \frac{\partial}{\partial w} f_2 + \frac{\partial}{\partial t} f_2 + 2 \frac{\partial}{\partial w} f_3 - \frac{\partial^2}{\partial w^2} f_2 = 0, \qquad (7.45)$$

$$\frac{\partial}{\partial w} f_2 + f_3 = 0. \qquad (7.46)$$

The above system cannot be solved without undertaking some assumptions since it is underdetermined.

**Case** $\tau = 0$, $\xi = 1$.

Following the same procedure as in the previous case we find

$$\begin{aligned}
&M^2 x \phi_w \, {w_x}^2 + M^2 \, {w_x}^2 + 2\, M^2 x \phi_x \, w_x - 2\, \phi_w \, {w_x}^3 - 2\, \phi_x \, {w_x}^2 + 2\, \phi \phi_w \, {w_x}^2\\
&+2\, \phi \phi_x \, w_x + \phi_t \, {w_x}^2 - \phi_{ww} \, {w_x}^2 - 2\, \phi_{xw} \, w_x + \xi_{xx} \, w_x - \phi_{xx} = 0, \qquad (7.47)
\end{aligned}$$

Likewise we separate the determining equation (7.47) with respect to the powers of the derivatives of $w$ and find the following system:

$$\phi_{xx} = 0, \tag{7.48}$$

$$2\,M^2 x \phi_x + 2\,\phi\phi_x - 2\,\phi_{xw} = 0, \tag{7.49}$$

$$M^2 x \phi_w + M^2 - 2\phi_x + 2\,\phi\phi_w + \phi_t - \phi_{ww} = 0, \tag{7.50}$$

$$\phi_w = 0. \tag{7.51}$$

Solving the above system yields

$$\Upsilon = \frac{\partial}{\partial x} - M^2 x \frac{\partial}{\partial w}. \tag{7.52}$$

It is noted that the symmetry (7.52) is a potential symmetry and not nonclassical.

## 7.5 Concluding remarks

In this chapter we considered a one dimensional transient heat conduction equation. We first presented the governing equation (7.1) in a conserved form as a system of first order DEs, known as the auxiliary system and used the potential variable to obtain the integrated form equation. The main focus of this chapter was to apply nonclassical potential symmetry approach on both the auxiliary system and the integrated form equation. We note that the potential symmetry approach could not yield explicit solutions. Also, the nonclassical potential symmetry approach for the potential system could not give any symmetry except the potential equation. These approaches require one to take some assertions in order to fully examine the potential system. We did find one nonlocal symmetry from the potential system and could not find exact solutions.

For the potential equation we found one point symmetry.

# Chapter 8

# Conclusion

Our work was to analyse heat transfer models prescribed by reaction-diffusion equations. Symmetry analysis was employed to study these models. The main focus was to utilize group classification methods to determine cases of arbitrary functions which diffusion equations admit some Lie point symmetries. Furthermore, to explore whether exact solutions may be constructed using other techniques such as potential (nonlocal) symmetries, nonclassical symmetries and nonclassical potential symmetries. Different phenomena were of our interest, namely heat transfer in slabs or hot bodies and through extended surface of different shapes and profiles.

In chapter two we gave a prescription of a heat transfer model in line with our research. We presented the formulation of our governing equations and analysed the equations in terms of heat transfer through the hot body and extended surfaces.

A brief account of the methodology employed in our work was discussed in chapter three. For more details regarding the analysis of our methodology the reader is referred to the provided references.

Direct group classification was carried out in chapter four and Lie algebra was employed to perform symmetry reductions. Two cases were of our interest, namely the power law and exponential law. We obtained hypergeometric and hyperbolic function solutions respectively.

In the absence of obtaining exact solutions in Chapter 4, we utilized DTM in chapter five and presented the solutions obtained graphically. The 1D DTM was utilized to solve the steady state ordinary differential equation and the transient state problem was solved using 2D DTM.

Preliminary and direct group classification was carried out in Chapter 6. We obtained a two dimensional principal Lie algebra through the equivalence group approach. Additionally, we obtained an optimal system for both approaches, but the presence of invariant solutions did not meet the requirements for preliminary group classification. To address this, we introduced an equivalence transformation to facilitate the preliminary group classification process. By employing direct group classification, we encountered multiple scenarios and produced group invariant solutions through the application of symmetry reductions.

We looked at the one-dimensional transient heat conduction equation in chapter seven. As a system of first order DEs known as the auxiliary system, we first presented the governing equation in a conserved form. We then used the potential variable to get the integrated equation. The primary objective of chapter 7 was to utilize nonclassical approaches on potential systems and potential equations, aiming to obtain nonclassical potential solutions. However, despite our efforts, we were unable to achieve this goal. Moving forward, we plan to revisit the work conducted in chapter 7 and introduce certain assumptions, in the hopes of uncovering potential nonclassical solutions.

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