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A bilocal description of the conformal algebra at the critical point in 3 dimensions

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Declaration of Authorship

I, Celeste JOHNSON (0303028K), declare that this Thesis titled, 'A bilocal description of the conformal algebra at the critical point in 3 dimensions', is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signed: 

Date: 18 September, 2020

"I am just a child who has never grown up. I still keep asking these 'how' and 'why' questions. Occasionally, I find an answer."

Stephen Hawking

Abstract

Klebanov and Polyakov conjectured in 2002 a duality between $O(N)$ vector models and Vasiliev higher spin theory, providing a useful probe of the AdS/CFT Correspondence[1]. When considering this duality in $d = 2 + 1$ dimensions, the free 3d $O(N)$ vector model is dual to type A minimal Vasiliev higher spin gauge theory with a $\Delta = 1$ scalar at the boundary, and the critical 3d $O(N)$ vector model is dual to type A minimal Vasiliev higher spin theory with scalar having $\Delta = 2$. In particular, in the presence of a relevant interaction, the theory flows from an unstable UV fixed point with a scalar of dimension $\Delta = 1$ to an IR fixed point where $\Delta = 2$. Strong evidence for this conjectured duality was provided by Giombi and Yin around 2010 where they performed explicit tree level and one-loop calculations. For example they showed that at tree level, the three-point functions on both sides of the conjecture match for both $\Delta = 1$ and $\Delta = 2$, and that the one-loop 3-sphere free energy on both sides of the conjecture agree[2, 3, 4]. Further evidence has been the reconstruction of the bulk by de Mello Koch, Jevicki, Jin and Rodrigues[5] (for another construction see Ref. [6]). In 2018 Mulokwe and Rodrigues showed that, making use of a bilocal field approach (taking a $1/N$ expansion) to investigate a three-dimensional $O(N)$ invariant bosonic model with $\frac{\lambda}{N}\left(\phi^a\phi^a\right)^2$ interaction at the critical point / infrared fixed point, there was indeed a state identified to correspond to a $\Delta = 2$ scalar state; the $\Delta = 1$ state was found to vanish from the spectrum in agreement with Polyakov and Klebanov[7].

In this thesis, we make use of Collective field theory, where bilocals are used to encode the invariance of the theory explicitly so that it is the invariant variables which are described by the theory[8]. We build on the constructive approach developed by de Mello Koch, Jevicki, Jin, Rodrigues and Yoon between 2010 and 2015 in both the light-cone and temporal gauge for the free theory, wherein an explicit map between the conformal field theory in $d = 2 + 1$ dimensions and the higher spin theory in $\text{AdS}_4 \times S_1$ was established[5, 9, 10]. In the Hamiltonian approach, the $1 + 2 + 2 = 5$ coordinates of the equal time bilocals map (in phase space) to the coordinates of $\text{AdS}_4 \times S_1$, and in this thesis we investigate the applicability of this map to the interacting theory. Using the Hamiltonian approach in time-like gauge, it is found that the quartic interaction contributes linearly in the bilocal field fluctuation equations so that the spectrum problem is that of a potential scattering

problem. The scattering state solution takes a universal form at the critical point. Using a change of variables from bilocal momenta to bulk momenta (as dictated by the map), and instituting a field redefinition to define bulk higher spin fields, we are able to obtain a bulk description of these boundary scattering states. It is rather remarkably shown directly in the bulk that for the interacting theory, the $s = 0$ state (corresponding to the $\Delta = 0 + 1 = 1$ state) is precisely removed in agreement with Ref. [7]. It is then shown that this approach is equivalent to that taken in Ref. [10], by showing that the bulk algebra agrees with that found in [10] both at the free and interacting critical point.

To My Heavenly Daddy

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Chapter 1

Introduction

1.1 Holography

Holography, the concept that the description of a volume of space, understood as the gravitational bulk, may be encoded on a lower dimensional boundary of that region (and that space-time is an emergent property[18, 19]), has been a tremendous source of inspiring work in theoretical physics over the last few decades, most notably the AdS/CFT correspondence as initiated by Maldacena[20]. A chief instigating factor has been the limitations of the $SU(3) \times SU(2) \times U(1)$ model or Standard model with regard to explaining things like dark energy and dark matter. This model is based on local invariance principles and has been extremely successful in understanding the nuclear Strong, Weak and Electromagnetic forces; in particular the theory predicted the spin-0 Higgs boson required for symmetry breaking (in section 1.2 we review background mathematics useful for physics and in section 1.3 we revise the spontaneous symmetry breaking mechanism in the Standard Model). However, notably, when the Gravitational force is incorporated into the Standard model, the theory becomes non-renormalisable (We review Renormalisation and the Renormalisation Group (RG) flow in section 1.4). For example, the ratio of the one-graviton correction to the zero-graviton amplitude is of the order

$$E^2 G_N = \left(\frac{E}{M_P} \right)^2 \tag{1.1}$$

where $M_p = 1.22 \times 10^{19}$GeV is the Planck mass. For small energies, or large distances, this quantum correction is irrelevant. However at high energies or small distances, this coupling grows stronger, and indeed the divergence gets worse with each additional graviton, and the perturbation theory breaks down. This is the short-distance problem of quantum gravity, the non-trivial UV fixed point, stemming in position space from all the graviton vertices being coincident. [21]

In section 1.5 we review some of the developments in General Relativity. Some key points in its history are as follows. In 1915, Einstein developed his famous Einstein equation, which Schwartzschild famously gave a solution for in 1916 for a static, uncharged black hole. In particular, this black hole has a singularity at $r = 0$ and an horizon at $r = 2GM$. More general black hole solutions were subsequently found. In 1916 and 1918, Reissner and Nordström respectively found the charged, non-rotating solution, while the uncharged, rotating solution was found in 1963 by Kerr and the charged, rotating solution by Kerr and Newman in 1965. A key result obtained by Bekenstein in 1975 and Hawking in 1976 [22, 23] after studying Einstein's equations is that the entropy of a black hole grows as the area of its event horizon and not as its volume as intuition may have suggested. Surface or boundary conditions therefore determine everything within the boundary, a concept known as holography. Quantum mechanically, black holes radiate, and this leads to the realisation that surface gravity and temperature are equivalent concepts in black holes; this may also be used to show that entropy scales as area. (Extremal black holes have zero temperature.) The generalised second law of thermodynamics states that the entropy of the black holes combined with that of the rest of the universe is increasing, implying that the area of the black hole is increasing. The quantum mechanical notion of unitarity conflicts with the Hawking/Bekenstein-derived notion that entropy is ever-increasing (otherwise put, a pure state should evolve to a pure state; for a black hole, the end state is instead a thermal state indicating a paradoxical loss of information). Some interesting developments in this area are also reviewed in section 1.5.

There are various No-Go Theorems, for example those due to Coleman, Mandula, Weinberg and Witten (we review these in section 1.8), which place constraints on the type of theories that may be written. A fruitful avenue of research has been finding ways around these no-go theorems by deviating from the underlying assumptions of these theorems. For instance Supersymmetry (which we review in section 1.9) deviates from the Lie algebraic assumption and instead requires anti-commutators to build non-trivial interactions between internal and external symmetries. Supergravity builds on this and with the introduction of a '$T_{\mu\nu}$' term which one interprets in a gravitational context (we review concepts from General Relativity in subsection 1.5.3, as well as Supergravity in section 1.10). A further way to circumvent the no-go theorems is to assume a non-flat space-time background as we shall see soon; however, we first need to review the role of String theory and D$_p$-branes in holography and how they ultimately gave rise to the AdS/CFT correspondence.

In 1973/1974, Gerard 't Hooft showed that the Lagrangian $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$ for a non-Abelian Yang-Mills theory (in general a matrix theory with trace structure), gives rise to a topological expansion in terms of 't Hooft coupling $\lambda = g_{YM}^2 N$ [24] (we review the argument in 1.6). Over time, these topological structures then sweep out worldsheets in a space with one dimension higher, and this theory is known as String theory. In the 1970s/1980s String theory (some aspects of which we review in section 1.7), where the string tension T is given by $T = \frac{1}{2\pi\alpha'}$ for α' the Regge slope, was found to have highly desirable features as a possible candidate for unification (the string length in this theory is then of order the Planck length ℓ_P). (Initially it had been thought that the dual resonance model was a likely candidate to study hadrons[25], and within the span of 4 years, wherein was the rise of QCD, the dual resonance model became a contender instead for describing non-hadrons [26, 27]). For example, non-renormalisable amplitudes and ultraviolet divergences make it difficult for other quantum field theories to incorporate gravity, whereas the massless spin-2 particle (having a Regge intercept of 2) is understandable as a graviton that is present in any string theory. String theories naturally incorporate gravitons and massless vector particles. The low energy dynamics of the massless sector of closed (super)strings was found to yield the Einstein-Hilbert action.

D-branes or Dirichlet-branes arose from considering boundary conditions of strings. The strings move around within a 'bulk', whereas the string endpoints are confined to moving about certain boundaries known as p-dimensional Dp-branes. For example, the open string / closed string duality states that a closed string moving from one D-brane to another is dual to an open string between two D-branes whose endpoints perform a single circuit in the same direction on their respective D-branes. A further way in which the idea of duality and holography arises was found by Sen, Hull-Townsend and Witten in 1995 [28, 29, 30]. Here five different superstring theories are unified in terms of an eleven dimensional theory called M-theory, all of which are related by dualities; the low energy limit of this theory gives 11 dimensional supergravity. One can retrieve the original 4-dimensional theory by, for example, making use of Kaluza-Klein reduction [31], where the additional dimensions are curled up on a small space like a small sphere or torus. One might, as a simple case, consider a background solution to be

$$g_{\Lambda\Sigma} = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) & 0 \\ 0 & g^{(0)}_{mn}(y) \end{pmatrix}$$

where Λ represents 4-dimensional indices and Σ the remaining indices; $g^{(0)}_{\mu\nu}(x)$ and $g^{(0)}_{mn}(y)$ are the metrics on the 4- and extra-dimensional spaces respectively. One then expands using spherical harmonics,

$$g_{\mu\nu}(x,y) = g^{(0)}_{\mu\nu}(x) + \sum_n g^{(n)}_{\mu\nu}(x) Y_n(y),$$

(with $Y_n(y)$ the spherical harmonic) and then employing dimensional reduction, i.e. only keeping the lowest dimensional mode (the constant one),

$$g_{\Lambda\Sigma} = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) + h_{\mu\nu}(x) & h_{\mu n}(x) \\ h_{m\nu}(x) & g^{(0)}_{mn}(y) + h_{mn}(x) \end{pmatrix}.$$

In 1997, Banks, Fischler, Shenker and Susskind came up with the BFSS matrix model for M-theory [32]. Here, the infinite momentum limit of M-theory is the $N \to \infty$ limit of N coincident D0-branes, given by $U(N)$ symmetry. The D0-branes describe multi-graviton bound states and the eigenvalues of the matrices represent the classical position of the D-branes; these interact via open strings. Some further results include Polchinski describing D-branes as solitons in string theory in 1994 [33, 34], Klebanov contributing to the understanding of stacks of D3 branes in 1997 [35] and Douglas, Polchinski and Strominger making the observation that the near-horizon behaviour can be obtained by taking the large-N limit of a quantum field theory [36]. These set the necessary foundations required for Maldacena to make his breakthrough discovery.

Considering the near-horizon geometry of N parallel Dp-branes of type IIB string theory and string coupling g in the large N limit generally gives a supergravity solution $AdS_{p+2} \times \mathcal{M}$ where AdS describes a space with constant negative curvature, Anti-de-Sitter space. Anti-de Sitter, or AdS, space is the maximally symmetric Lorentzian manifold with a constant negative (scalar) curvature, and $\mathcal{M}$ is typically a compact manifold. For example, the supergravity solution related to D3-branes is given by $AdS_5 \times S^5$. The curvature of the combined space $AdS_{p+2} \times \mathcal{M}$ (in Planck units) is a positive power of $1/N$ [37], so that, in the large N limit it gives the required asymptotically flat universe. For $AdS_5 \times S^5$, the radius of the Anti-de Sitter space is the same as that of the five-sphere and are both proportional to $N^{1/4}$. Moreover, in the large-N limit, this results in a conformal field theory (CFT) on the brane given by $\mathcal{N} = 4$ Super Yang-Mills (SYM). The presence of the graviton together with the holographic principle led to the assertion in the 1990s by Maldacena[20] (and fleshed out by Gubser, Klebanov and Polyakov[38] and Witten[39]) that non-Abelian gauge theories correspond to quantum theories of gravity and that they are related by the value of λ [40]. In particular $\mathcal{N} = 4SYM$ is equivalent to type IIB strings in $AdS_5 \times S^5$, where the coupling constants are related as $g^2_{YM} = g_s$ and the radius of both S_5 and AdS_5 are related to λ by

$$R^4 = (4\pi\alpha')^2\lambda$$

The duality is established by the value of λ. Large λ, corresponding to a non-perturbative regime on strongly coupled $\mathcal{N} = 4$ SYM, corresponds to a large radius of curvature R, or weakly coupled gravity. This duality is referred to as the AdS/CFT correspondence, which is reviewed in section 1.11.

Another avenue for circumventing the No-Go Theorems came in the form of the Fronsdal and Vasiliev programmes which challenged the underlying flat space-time background assumption and builds higher-spin theories on curved space-times where there is no S-matrix. These are reviewed in section 1.12.2.

An example of the AdS/CFT correspondence is the conjectured duality between large N conformal theories in d dimensions which contain N component vector fields and theories of an infinite number of higher-spin massless gauge fields in AdS_{d+1}; in particular Klebanov and Polyakov conjectured this for $d = 2 + 1$ dimensions in 2002 [1]. Later, Giombi and Yin were able to do calculations using the AdS/CFT correspondence to corroborate this conjecture [41, 2, 4]. This conjectured correspondence is reviewed in section 1.13.

It can be shown that the holographic duality that exists for the $\Delta = 2$ critical theory follows from the $\Delta = 1$ free theory using Legendre transformation arguments; this is reviewed in Ref. 1.14.

1.2 Mathematical background

1.2.1 Lie group definitions

The following is based on Ref. [42]. A *Lie Group* is a group G whose underlying set is given a *manifold* structure (a manifold is a set M endowed with bijections to $\mathbb{R}^n$ and various nice properties, that can often be visualised as a surface, for example the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is a manifold). The *dimension* of the Lie group is then given by the dimension of this underlying manifold. Lie groups often come with a *homomorphism*, i.e. a *smooth* map (one where all partial derivatives of all orders exist) $J : G \to H$ such that

$$J(G_1 G_2) = J(G_1)J(G_2)\ \forall G_1, G_2 \in G.$$

Two Lie groups are *isomorphic* if there exists a bijective homomorphism between them, e.g. $J : U(1) \to O(2)$ given by

$$J(e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We tend to like manifolds that are *compact* (closed and bounded, therefore the hyperboloid $\{x, y, z \in \mathbb{R}|\, x^2 - y^2 - z^2 = 1\}$ is a non-compact manifold) and *simply connected* ('connected' means that any 2 points are connected by a path, and 'simply' means that all loops can be contracted to a point, and all paths between the same 2 points can be continuously deformed into each other). For example, while the sphere S^2 is simply connected, the circle S' and the torus are not. Furthermore $SO(3)$ is not simply connected: we can define $\vec{n}$ a rotation axis (corresponding to eigenvalue $\lambda = 1$; the other two eigenvalues will be of the form $e^{\pm i\theta}$ where θ is the rotation angle). Then, since the point $M(\vec{n}, 2\pi - \theta)$ is identified with the point $M(-\vec{n}, \theta)$, we would need to restrict θ to the range $0 \leq \theta < \pi$ to avoid double-counting of points. In particular, anti-podal points $(\vec{n}, \pi)$ and $(-\vec{n}, \pi)$ are identified. As a case in point, the two paths $\theta\vec{n}$ where $\theta \in [0, \pi)$ and $-(2\pi - \theta)\vec{n}$ where $\theta \in [\pi, 2\pi)$ are two paths between the same two points. But since these paths are traversed in opposite directions, it is impossible to continuously deform one path into another, giving the result that $SO(3)$ is not simply connected. Indeed, although SO(3) has the same number of generators ($\frac{3\times 2}{2} = 3$) as $SU(2)$ $(2^2 - 1 = 3)$, $SU(2)$ is actually a double cover of SO(3). For $U \in SU(2)$, $U = a_0\mathbb{1} + i\vec{a} \cdot \sigma$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

necessitating that

$$\begin{aligned} \mathbb{1} = UU^\dagger &= \begin{pmatrix} a_0 + ia_1 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_1 \end{pmatrix} \begin{pmatrix} a_0 - ia_3 & -a_2 - ia_1 \\ a_2 - ia_1 & a_0 + ia_3 \end{pmatrix} \\ &= \left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right)\mathbb{1} \\ \iff & \quad a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1. \end{aligned}$$

There is a global $2:1$ map $f : SU(2) \to SO(3)$ given by

$$f(A)_{ij} = \frac{1}{2}\mathrm{tr}\left(\sigma_i A \sigma_j A^\dagger\right) \in SO(3).$$

Notably $f(A_{ij}) = f(-A_{ij})$. $SO(3)$ and $SU(2)$ satisfy the same algebra $\left[T^a, T^b\right] = \epsilon_{abc}T^c$. Note that the choice we made for $SU(2)$ generators results in real structure constants $f_{abc} = \epsilon_{abc}$ and therefore the choice of generators is called a real basis; later on we will use a new basis called the Cartan-Weyl basis which is not real.

1.2.2 Lie algebra definitions

An *algebra* is a mapping from a group to a vector space, endowed with an operation, and a *Lie algebra* g is a vector space over a field F equipped with a bracket $[\,] : g \times g \to g$ that is anti-symmetric, $[x, y] = -[y, x]$, linear, $[\alpha x + \beta y, z] = \alpha [x, y] + \beta [x, y]$ and obeys the Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$. It has *dimension* $\dim(g) = \dim(\text{vector space}) = \dim\,(\text{basis } T^a)$. Elements in the algebra are given by $x = x_a T^a$

and basis elements satisfy $\left[T^a, T^b\right] = f_c^{ab} T^c$. Two algebras g and g' are *isomorphic* if there exists a bijective linear map $f : g \to g'$ such that

$$[f(x), f(y)] = f\left([x, y]\right)$$

and our aim is to classify Lie algebras up to isomorphism. A *subalgebra* $h \subset g$ is a subset that is also a Lie algebra. An *ideal* h of g is a subalgebra such that $[x, y] \in h \; \forall x \in g, \; \forall y \in h$. Ideals are to Lie algebras what normal subgroups are to groups. Some examples of ideals are the *derived algebra* and the *centre*, respectively

$$\begin{aligned} i(g) &= \{[x, y] : x, y \in g\} \\ J(g) &= \{x \in g : [x, y] = 0 \; \forall \; Y \in g\}. \end{aligned}$$

A Lie algebra is *abelian* if $[x, y] = 0 \; \forall x, y \in g$ in which case $i(g) = \{0\}$ and $J(g) = g$. A Lie algebra is *simple* if it is non-abelian and has no non-trivial ideals, in which case, $i(g) = g$ and $J(g) = \{0\}$.

1.2.3 Obtaining Lie algebras from Lie groups

For any Lie group G (which will come with an underlying manifold/surface), we can define a tangent space (locally flat) around the identity. The mapping from the tangent space to the (matrix representation of the) group is given by

$$\mathcal{T}(G) \ni v^i \frac{\partial}{\partial \theta^i} \mapsto v^i \left. \frac{\partial g(\theta)}{\partial \theta^i} \right|_{\theta=0} \in \mathrm{Mat}_n(\mathbb{F}),$$

where $\mathbb{F}$ is a space such as the real or complex numbers. We can write curves close to the identity parametrised by t in our Lie group $G_i(t)$ in terms of tangent space / Lie algebra elements x, y, w as follows

$$\begin{gathered} g_1(0) = g_2(0)\mathbb{1}; \;\; \dot{g}_1(0) = x_1; \;\; \dot{g}_2(0) = x_2 \\ g_1(t) = \mathbb{1} + x_1 t + w_1 t^2; \;\; g_2(t) = \mathbb{1} + x_2 t + w_2 t^2. \end{gathered}$$

To determine the operation we should define on our tangent space, closure requires that the composition $g_1^{-1}(t) g_2^{-1}(t) g_1(t) g_2(t)$ should also be in our tangent space.

This gives

$$\left(\mathbb{1} - x_1 t - w_1 t^2 + x_1^2 t^2\right)\left(\mathbb{1} - x_2 t - w_2 t^2 + x_2^2 t^2\right)$$
$$\left(\mathbb{1} + x_1 t + w_1 t^2\right)\left(\mathbb{1} + x_2 t + w_2 t^2\right).$$

The terms linear in t clearly cancel, as well as the w_1 and w_2 terms in t^2 and we are left with

$$\begin{aligned} & \mathbb{1} + \left(x_1^2 + x_2^2 + x_1 x_2 - x_1^2 - x_1 x_2 - x_2 x_1 - x_2^2 + x_1 x_2\right) t^2 \\ = \quad & \mathbb{1} + [x_1, x_2]\, t^2. \end{aligned}$$

Here we see that our tangent space is a mapping from a group to a vector space that we can naturally endow with an operation $[x_1, x_2]$ - which is exactly the definition we had for a Lie algebra! That is

$$\mathcal{L}(G) = \{T_e(G), [,]\}.$$

As examples, consider the groups $G = SO(n)$ or $O(n)$. Here

$$\begin{aligned} & R^T(t)R(t) = \mathbb{1} \\ \Rightarrow 0 \quad = \quad & \dot{R}^T(t)R(t) + R^T(t)\dot{R}(t) \\ \Rightarrow 0 \quad = \quad & \dot{R}^T(0)R(0) + R^T(0)\dot{R}(0) = X^T\mathbb{1} + \mathbb{1}X = X + X^T \end{aligned}$$

$\iff$ X is anti-symmetric with dim(G)= $\frac{1}{2}n(n-1)$. For $G = SU(n)$ we similarly obtain $U^\dagger(t)U(t) = 0 \Rightarrow Z^\dagger + Z = 0$, so the dimension of G is given by n^2 constraints, less one since, writing $U = 1 + Zt + \mathcal{O}(t)$, the constraint $\det(U(t)) = 1 \Rightarrow e^{\mathrm{Tr}\ln(1+Zt)} = 1 \Rightarrow \mathrm{Tr}(Z)t = 0$, hence the generators need to be traceless and anti-hermitian of $\dim(G) = n^2 - 1$; for example in $SU(2)$ the 3 generators would be $T^a = -\frac{1}{2}i\sigma_a$. Note that a Lie group is an abstract object and in order to discuss particular types of objects like matrices one needs to talk of that particular representation.

1.2.4 Representations

Recall that an homomorphism was a mapping that came into play in Lie groups, for example two Lie groups were isomorphic if there existed a bijective homomorphism between them. On the other hand, a Lie algebra was associated with a bracket [,]. These properties filter down to their respective representations.

For any group G, not necessarily Lie, a *representation* is a set of non-singular matrices $\{D(G) \in GL(n, \mathbb{F}), G \in G\}$ with a *homomorphism*

$$D(G_1)D(G_2) = D(G_1G_2)\ \forall G_1, G_2 \in G.$$

For any Lie algebra $\mathfrak{g}$, a *representation* is a set of matrices $\{d(g) \in \text{Mat}_n(\mathbb{F}), x \in \mathfrak{g}\}$ such that

$$[d(x_1), d(x_2)] = d\left([(x_1), (x_2)]\right)\ \forall x_1, x_2 \in \mathfrak{g}$$
$$d(\alpha x_1 + \beta x_2) = \alpha d(x_1) + \beta d(x_2)\ \forall x_1, x_2 \in \mathfrak{g};\ \alpha, \beta \in \mathbb{F}.$$

The dimension of the representation is the dimension of the corresponding matrices, and the space $\mathbb{F}^n$ the representation acts on is called the *representation space.*

Given a representation d of the Lie algebra $\mathcal{L}(G)$ we can define a representation $D(G = e^x) = e^{d(x)}$. For example, if $D(G_i) = e^{x_i}$

$$\begin{aligned} D(G_1G_2) &= \exp\left[d\left(x_1 + x_2 + \frac{1}{2}[x_1, x_2] + ...\right)\right] \\ &= \exp\left[d(x_1) + d(x_2) + \frac{1}{2}[d(x_1), d(x_2)] + ...\right] \\ &= \exp\left(d(x_1)\right)\exp\left(d(x_2)\right) = D(G_1)D(G_2), \end{aligned}$$

which follows from the *Baker-Campbell-Haussdorf* formula, which is briefly explained/derived as follows:

$$\begin{aligned} e^z = e^x e^y &= (1 + x + \frac{1}{2}x^2 + ...)(1 + y + \frac{1}{2}y^2 + ...) \\ &= 1 + \left(x + y + \frac{1}{2}[x, y] + ...\right) + \frac{1}{2}\left(x + y + [x, y] + ...\right)^2 + ... \\ \Rightarrow z &= x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] - \frac{1}{12}[y, [x, y]] + ... \end{aligned}$$

This is most useful as we can see a group representation as being generated by an algebra. Some representations of a Lie algebra $\mathfrak{g}$ are the *trivial representation*,

$d_0(x) = 0\ \forall x \in \mathfrak{g}$, the *fundamental representation* $d_f(x) = x\ \forall x \in \mathfrak{g}$ and the *adjoint representation*

$$d_{\text{adj}}(x) = \text{ad}_x\ \forall x \in \mathfrak{g}$$

where $\text{ad}_x(y) = [x, y]\ \ \forall y \in \mathfrak{g}$. We can choose a basis $B = \{T_a\}, a = 1, ..., D$ for $\mathfrak{g}$ so that

$$[\text{ad}_x(y)]_c = [x, y]_c = x_a f_c^{ab} y_b \iff [\text{ad}_x]_c^b = x_a f_c^{ab}$$

which is a $D \times D$ matrix. This is a valid representation since it trivially satisfies the second requirement, and the first requirement is satisfied as follows:

$$\begin{aligned} [d(x), d(y)] = [\text{ad}_x, \text{ad}_y](z) &= [x, [y, z]] - [y, [x, z]] \\ &= [x, [y, z]] + [y, [z, x]] \\ &= -[z, [x, y]] \\ &= [[x, y], z] = \text{ad}_{[x,y]}(z) = d_{[x,y]}. \end{aligned}$$

Two representations R_1 and R_2 are *isomorphic* if $\exists S$ such that $R_2(x) = SR_1(x)S^{-1}$ $\forall x \in \mathfrak{g}$. A representation with representation space V has an *invariant subspace* $U \subset V$ if $R(x)_u \in U\ \forall x \in \mathfrak{g}, u \in U$; the two trivial invariant spaces are $\{0\}$ and V. An *irreducible representation* or *irrep* has no non-trivial invariant subspaces.

If the structure constants in a Lie group representation are real, the basis for the representation is called real. We could complexify the basis (resulting in real generators). For example in $SU(2)$

$$su(2) = \text{span}_{\mathbb{R}}\{-\frac{1}{2}i\sigma_a : a = 1, 2, 3\} \to \text{span}_{\mathbb{C}}\{\sigma_a : a = 1, 2, 3\}.$$

We will use the basis

$$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ E_1, E_2 = \sigma_1 \pm i\sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that $[E_+, E_-] = H$, $[H, E_\pm] = \text{ad}_H(E_\pm) = \pm 2E_\pm$ and $\text{ad}_H(H) = 0$. Therefore ad_H has three eigenvalues $0, \pm 2$; the eigenvalues of the representation of H are known as weights (in general there may be more than one H). It also has three eigenvectors, $H, E_{\pm}$ known as the *roots* of $su(2)$, and $E_{\pm}$ are known as the step operators. We can operate with these step operators multiple times (gaining or losing 2 each time) until we get to states of maximal and mimimal weight Λ and $-\Lambda$. The *finite-dimensional representations* of $su(2)$, R_Λ with weight given by $\{-\Lambda, -\Lambda+2..., \Lambda\}$ has dimension $\dim(R_\Lambda)$=$\Lambda+1$. $SO(3) \cong \frac{SU(2)}{\mathbb{Z}_2}$ results in $SU(2)$ and not $SO(3)$ being able to model spin-$\frac{1}{2}$ particles.

To consider what happens in the above discussion when we add or multiply representation, consider the direct sum of representations which can have different dimensional representation spaces $R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_n$ may be visualised as follows

$$\left(R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_n\right)(x) = \begin{pmatrix} R_1(x) & 0 & & 0 & 0 \\ 0 & R_2(x) & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & R_{n-1}(x) & 0 \\ 0 & 0 & & 0 & R_n(x) \end{pmatrix}$$

which makes evident the reducibility and invariant subspaces; if the representation is reducible then there is a basis in which it will have upper triangular form. When it has block-diagonal form we say that it is completely reducible. The dimension of the above is equal to the sum of the dimensions of the individual representations.

A direct product $(R_1 \bigotimes R_2)(v \bigotimes w) = (R_1(v)) \bigotimes (R_2(w))$ of two representations may be done in one of two ways - multiplying R_1 by each of the components of R_2 or vice versa. The dimension of the resulting representation will then be the product of the dimensions of the individual representations.

With this machinery we can now say that

$$R_{\Lambda_1} \bigotimes R_{\Lambda_2} = R_{|\Lambda_1-\Lambda_2|} \bigoplus R_{|\Lambda_1-\Lambda_2|+2} \bigoplus ... \bigoplus R_{\Lambda_1+\Lambda_2}.$$

For examples, for spin-$\frac{1}{2}$ particles we have that

$$R_1 \bigotimes R_1 = R_2 \bigoplus R_0$$

so we can have a triplet of states or just a singlet.

1.2.5 Cartan subalgebras

In this subsection we will extend the $su(2)$ example above. In order to do this, one needs to be able to normalise things, and for this we define an *inner product* (given a vector space V and field $\mathbb{F}$, this is a symmetric bilinear map $i : V \times V \to \mathbb{F}$) called the *Killing Form* $K : \mathfrak{g} \times \mathfrak{g} \to F$ by

$$K(x, y) = \operatorname{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y).$$

This is indeed an inner product since

$$\begin{aligned}
(\mathrm{ad}_x \circ \mathrm{ad}_y)(z) &= [x, [y, z]] = x_a x_b x_c [t^a, [t^b, t^c]] \\
&= x_a x_b x_c [t^a, f^{bc}_d t^d] = x_a x_b x_c f^{bc}_d f^{ad}_e t^e] \\
&= \left(x_a x_b f^{bc}_d f^{ad}_e\right) z_c t^e = [m(x, y)]^c_e z_c t^e \\
\Rightarrow K(x, y) &= \operatorname{tr}[m(x, y)]^c_e = x_a x_b f^{bc}_d f^{ad}_c.
\end{aligned}$$

These Killing forms have the following property:

$$\begin{aligned}
K([x, y], z) &= \operatorname{tr}\left(\mathrm{ad}_{[x,y]} \circ \mathrm{ad}_z\right) \\
&= \operatorname{tr}([\mathrm{ad}_x, \mathrm{ad}_y] \circ \mathrm{ad}_z) \\
&= \operatorname{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y \circ \mathrm{ad}_z - \mathrm{ad}_y \circ \mathrm{ad}_x \circ \mathrm{ad}_z) \\
&= \operatorname{tr}(\mathrm{ad}_x \circ [\mathrm{ad}_y, \mathrm{ad}_z]) \\
&= K(x, [y, z]) = -K(x, [z, y]) = K(y, [z, x]).
\end{aligned}$$

In particular, an inner product on Lie algebra $\mathfrak{g}$ is called *invariant* if

$$K([z, x], y) + K(x, [z, y]) = K(\mathrm{ad}_z(x), y) + K(x, \mathrm{ad}_z(y)) = 0$$

since elements in $\mathfrak{g}$ transform naturally as $x \to$ Note that the d generators $\{T^a, a = 1, ..., d\}$ satisfy commutation relations

$$\left[T^a, T^b\right] = f^{ab}_c T^c.$$

where f^{ab}_c are the *structure constants*. A Lie algebra is *semi-simple* if it has no abelian ideals, in which case it can be written as the direct sum of simple Lie algebras. Having abelian ideals requires some commutators to $[T^a, T^b] = 0$ which sets certain structure constants to zero; it turns out that the Killing vector (which contains 2 structure constant terms) will be identically zero in this case, and so being semi-simple is equivalent to requiring that the Killing vector is non-degenerate. This is important for us because we will be working with a *Cartan Subalgebra*(CSA) $\mathbb{h}$ ($\mathbb{h}$ is a maximal abelian subalgebra of $\mathbb{g}$ if all its elements are *ad-diagonalisable* (so we can think of them as diagonal generators like H in our previous example; it is, however, true in the more general abstract sense). We can demand $\mathbb{g}$ to be semi-simple and yet contain a CSA since it is just a subalgebra and not an ideal ($H \in \mathbb{h} \Rightarrow [H, H'] = 0$ and if $[E, H] = 0\ \forall h \in \mathbb{h} \Rightarrow E \in \mathbb{h}$).

Extending the $su(n)$ example from before, we can choose $n-1$ generators like the H we chose - $(H^i)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i} - \delta_{\alpha(i+1)}\delta_{\beta(i+1)}$ - that will commute with each other. The rank of the CSA will then be its dimension, $n-1$. The remaining eigenvectors of the CSA $\{E^\alpha : \alpha \in \Phi\}$ are called *step operators* E^α. The *Cartan-Weyl basis* is then given by $\mathcal{B} = \{H^i, i = 1, ...r\}U\{E^\alpha, \alpha \in \Phi\}$. That is

$$
\begin{aligned}
\left[H^i, H^j\right] &= 0 \\
\left[H^i, E^\alpha\right] &= \alpha^i E^\alpha,\ \ i = 1, ..., r,\ \forall \alpha \in \Phi \\
\left[H^i, \left[E^\alpha, E^\beta\right]\right] &= -\left[E^\alpha, \left[E^\beta, H^i\right]\right] - \left[E^\beta, \left[H^i, E^\alpha\right]\right] \\
&= \left(\alpha^i + \beta^i\right)\left[E^\alpha, E^\beta\right] \\
\Rightarrow \left[E^\alpha, E^\beta\right] &= \begin{cases} \propto E^{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ 0 & \text{otherwise}, \alpha \neq -\beta \\ ? & \alpha = -\beta \end{cases} .
\end{aligned}
\tag{1.2}
$$

In an attempt to understand the unknowns in the previous equation, note now that we have the following results (which we prove in reverse):

$$
\begin{aligned}
& K\left(H, E^\alpha\right) = 0 \\
\Longleftrightarrow\quad & \alpha^i K\left(H^i, E^\alpha\right) = 0 \\
\Longleftrightarrow\quad & K\left(H, \alpha^i E^\alpha\right) = 0
\end{aligned}
$$

$$\begin{aligned} \Longleftrightarrow \quad & K\left(H, \left[H^i, E^\alpha\right]\right) = 0 \\ \Longleftrightarrow \quad & K\left(E^\alpha, \left[H, H^i\right]\right) = 0 \end{aligned}$$

and

$$\begin{aligned} & K\left(E^\alpha, E^\beta\right) = 0 \\ \Longleftrightarrow \quad & \left(\alpha^i + \beta^i\right) K\left(E^\alpha, E^\beta\right) = 0 \\ \Longleftrightarrow \quad & K\left(\alpha^i E^\alpha, E^\beta\right) + K\left(E^\alpha, \beta^i E^\beta\right) = 0 \\ \Longleftrightarrow \quad & K\left(\left[H^i, E^\alpha\right], E^\beta\right) + K\left(E^\alpha, \left[H^i, E^\beta\right]\right) \\ \Longleftrightarrow \quad & K\left(\left[H^i, E^\alpha\right], E^\beta\right) + K\left(\left[E^\alpha, H^i\right], E^\beta\right) = 0. \end{aligned}$$

By the previous argument K is non-degenerate, however. Therefore there must exist H' such that $K(H, H') \neq 0$. Therefore, if we define

$$K(H, H') = K(H^i e_i, H^j e'_j) = K^{ij} e_i e'_j \neq 0$$

then K^{ij} is invertible $\Rightarrow \exists K^{-1}_{ij}$ such that $K^{-1}_{ij} K^{jk} = \delta^k_i$, and we can start talking about the 'geometry of the roots'. From the commutators in (1.2) the only option that will lead to a non-degenerate K is if $H'(= H^\alpha) \propto [E^\alpha, E^{-\alpha}]$.

In the $su(2)$ example we had $[E_+, E_-] = H$; consider therefore

$$K\left(\left[E^\alpha, E^{-\alpha}\right], H^i\right) = K\left(\left[H^i, E^\alpha\right], E^{-\alpha}\right) = \alpha^i K\left(E^\alpha, E^{-\alpha}\right). \tag{1.3}$$

Therefore, if we define

$$H^\alpha = \frac{[E^\alpha, E^{-\alpha}]}{K\left(E^\alpha, E^{-\alpha}\right)} \Longleftrightarrow \left[E^\alpha, E^{-\alpha}\right] = K\left(E^\alpha, E^{-\alpha}\right) H^\alpha$$

then equation (1.3) will become

$$K(H^\alpha, H^i) = \alpha^i$$

Our set of commutators has therefore now become

$$\begin{aligned} \left[H^i, H^j\right] &= 0 \\ \left[H^i, E^\alpha\right] &= \alpha^i E^\alpha, \ i = 1, ..., r, \ \forall \alpha \in \Phi \end{aligned}$$

$$\begin{aligned}
\left[H^i, \left[E^\alpha, E^\beta\right]\right] &= -\left[E^\alpha, \left[E^\beta, H^i\right]\right] - \left[E^\beta, \left[H^i, E^\alpha\right]\right] \\
&= \left(\alpha^i + \beta^i\right) \left[E^\alpha, E^\beta\right] \\
\Rightarrow \left[E^\alpha, E^\beta\right] &= \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ 0 & \text{otherwise}, \alpha \neq -\beta \\ K\left(E^\alpha, E^{-\alpha}\right) H^\alpha & \alpha = -\beta \end{cases} \cdot
\end{aligned}$$

Now we can obtain H^α

$$\begin{aligned}
K(H^\alpha, H) &= K\left(H^i e_i^\alpha, H^j e_j\right) = K^{ij} e_i^\alpha e_j = \alpha^i e_i \\
\Rightarrow e_i^\alpha &= (K^{-1})_{ij} \alpha^j \\
\Rightarrow H^\alpha &= (K^{-1})_{ij} \alpha^j H^i.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left[H^\alpha, E^\beta\right] &= (K^{-1})_{ij} \alpha^j \left[H^i, E^\beta\right] \\
&= (K^{-1})_{ij} \alpha^i \beta^j E^\beta \\
&= (\alpha, \beta)\, E^\beta \qquad (1.4)
\end{aligned}$$

so that K^{ij} acts on $\mathbb{h}$ while K_{ij}^{-1} acts on the dual space $\mathbb{h}^*$. If we now define

$$\begin{aligned}
e^\alpha &= \sqrt{\frac{2}{(\alpha,\alpha) K(E^\alpha, E^{-\alpha})}} E^\alpha \\
h^\alpha &= \frac{2}{(\alpha,\alpha)} H^\alpha,
\end{aligned}$$

the commutation relations become

$$\begin{aligned}
\left[h^\alpha, h^\beta\right] &= \left[\frac{2}{(\alpha,\alpha)}\right] \left[\frac{2}{(\beta,\beta)}\right] \left[H^\alpha, H^\beta\right] = 0 \\
\left[h^\alpha, e^\beta\right] &= \frac{2}{(\alpha,\alpha)} \sqrt{\frac{2}{(\beta,\beta) K(E^\beta, E^{-\beta})}} \left[H^\alpha, E^\beta\right] = \frac{2}{(\alpha,\alpha)} \sqrt{\frac{2}{(\beta,\beta) K(E^\beta, E^{-\beta})}} (\alpha,\beta) E^\beta \\
&= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} e^\beta \\
\left[e^\alpha, e^\beta\right] &= \frac{2}{(\alpha,\alpha) K(E^\alpha, E^{-\alpha})} \left[E^\alpha, E^\beta\right]
\end{aligned}$$

$$= \begin{cases} n_{\alpha\beta} e^{\alpha+\beta} & \alpha+\beta \in \Phi \\ \frac{2KH^{\alpha}(E^{\alpha},E^{-\alpha})}{(\alpha,\alpha)K(E^{\alpha},E^{-\alpha})} = h^{\alpha} & \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

where $n_{\alpha\beta} \in \mathbb{C}$ and $(\alpha, \alpha) \neq 0$ is given without proof. Note that $\forall \alpha \in \Phi$ there is a $\mathcal{L}_{\mathbb{C}}(su(2)) \equiv su(2)_{\alpha}$ subalgebra that is generated by $\{h^{\alpha}, e^{\alpha}, e^{-\alpha}\}$:

$$\begin{aligned} \left[h^{\alpha}, e^{\pm\alpha}\right] &= \pm 2e^{\pm\alpha} \\ \left[e^{\alpha}, e^{-\alpha}\right] &= h^{\alpha}. \end{aligned}$$

This will prove important. We will now try to simplify this further by defining *simple roots* which will eliminate the $\alpha + \beta \in \Phi_S$ option since simple roots cannot be written as the sum of 2 roots.

For $\alpha, \beta \in \Phi$ and $\alpha \neq \beta$ the α-string (a root-string) through β is given by

$$S_{\alpha,\beta} = \{\beta + \rho\alpha \in \Phi; \rho \in \mathbb{Z}\}.$$

A given root-string has a corresponding vector space

$$V_{\alpha\beta} = \text{span}_{\mathbb{C}}\{e^{\beta+\rho\alpha}; \beta + \rho\alpha \in \phi\}$$

and, since $\beta + (\rho \pm 1)\,\alpha \neq 0$

$$\begin{aligned} \left[h^{\alpha}, e^{\beta+\rho\alpha}\right] &= \frac{2(\alpha, \beta+\rho\alpha)}{(\alpha,\alpha)} e^{\beta+\rho\alpha} = \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2\rho\right) e^{\beta+\rho\alpha} \\ \left[e^{\pm\alpha}, e^{\beta+\rho\alpha}\right] & \begin{cases} \propto e^{\beta+(\rho\pm1)\alpha} & \beta + (\rho \pm 1)\,\alpha \in \Phi \\ = 0 & \text{otherwise} \end{cases}. \end{aligned}$$

We can make use of the initial $su(2)$ example by noticing that for R a representation of $su(2)_{\alpha}$ (under which $V_{\alpha,\beta}$ is invariant, so that $V_{\alpha\beta}$ may be seen as a representation space for R. Furthermore R is an irrep of $su(2)_{\alpha}$ and therefore must have a highest weight $0 < \Lambda \in \mathbb{Z}$). The weights are thus given by

$$S_R = \{\left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2\rho\right); \beta + \rho\alpha \in \Phi\} = \{-\Lambda, -\Lambda+2, ..., \Lambda\}.$$

(Each weight has multiplicity 1, i.e. corresponds to exactly one state.) Therefore $\rho \in \mathbb{Z}$ and for some integers $n_+ \geq 0$ and $n_- \leq 0$,

$$\begin{aligned} -\Lambda &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + n_- \\ \Lambda &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + n_+. \end{aligned}$$

Adding these two equations together we find that $R_{\alpha,\beta} \equiv \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_- + n_+) \in \mathbb{Z}\ \forall \alpha, \beta \in \Phi$. It is possible to show that the inner product satisfies the necessary requirements to be a Euclidean inner product e.g. $(\alpha,\alpha) \geq 0$ with equality $\iff \alpha = 0$. So we can define the length of a root to be

$$\begin{aligned} |\alpha| &= (\alpha,\alpha)^{1/2} \\ (\alpha,\beta) &= |\alpha||\beta|\cos\phi \end{aligned}$$

where ϕ is the angle between the two roots α and β, as per figure 1.1. Hence

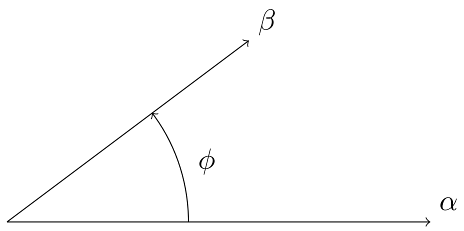


FIGURE 1.1: Diagram showing the angle ϕ between two roots α and β

$$\begin{aligned} & R_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2|\beta|}{|\alpha|}\cos\phi \in \mathbb{Z};\ \ R_{\beta,\alpha} = \frac{2|\alpha|}{|\beta|}\cos\phi \in \mathbb{Z} \\ \Rightarrow\quad & \cos^2\phi = \frac{n}{4},\ \ n \in \mathbb{Z} \Rightarrow \cos\phi = \pm\frac{\sqrt{n}}{2} \end{aligned}$$

so we can divide the root set into *positive roots* Φ_+ and *negative roots* Φ_- (we will draw a hyperplane with negative slope through the origin; roots to the right are called positive and roots to the left are called negative). We further define *simple roots* to be roots that cannot be written as the sum of two positive roots; the set of all simple roots is Φ_S. Some results:

- If $\alpha, \beta \in \Phi_S$ then $\alpha - \beta \notin \Phi_S$ since $\alpha - \beta \in \Phi_+$ implies α is the sum of $\alpha - \beta$ and β (so not simple), and $\alpha - \beta \in \Phi_- \Rightarrow \beta - \alpha \in \Phi_+$ implies β is the sum of

$\beta - \alpha$ and α (so not simple).

- Therefore $S_{\alpha,\beta} = \{\beta + n\alpha; n_- \leq n \leq n_+\} \to S_{\alpha,\beta} = \{\beta + n\alpha; 0 \leq n \leq -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\}$ since $n_- = 0$ for Φ_S and $n_+ + n_- = -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$, and the length of the α-string through β has length $l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{N}$.

- Hence $(\alpha, \beta) \leq 0$.

- Any positive root β may be written as a linear combination of simple roots (since the rank is finite, one can write β as a sum of other roots until all roots are simple). Furthermore any roots $\alpha \in \Phi$ may be written as a linear combination of simple roots, $\alpha = \sum_i d_i \alpha_{(i)}$ where $\alpha_{(i)} \in \Phi_S$. The simple roots therefore *span* $\mathbb{h}^*_{\mathbb{R}}$ (objects that are in $\mathbb{g}$ but not $\mathbb{h}$).

- The simple roots are also *linearly indepependent* since $\lambda \in \mathbb{h}^*_{\mathbb{R}} \Rightarrow \lambda = \sum_i c_i \alpha_{(i)}$ where $c_i \in \mathbb{R}$ and $\alpha_{(i)} \in \Phi_S$. For general non-zero λ, not all $c_i = 0$ and we can define $J_\pm = \{i : c_i \gtrless 0\}$ and $\lambda_\pm = \pm \sum_{i \in J_\pm} c_i \alpha_{(i)}$ so that $\lambda = \lambda_+ - \lambda_-$ and therefore

$$\begin{aligned}(\lambda, \lambda) &= (\lambda_+, \lambda_+) + (\lambda_-, \lambda_-) - 2(\lambda_+, \lambda_-) \\ &> -2(\lambda_+, \lambda_-) \\ &= 2 \sum_{i \in J_+} \sum_{j \in J_-} c_i c_j \alpha_{(i)} \alpha_{(j)} \\ &\geq 0\end{aligned}$$

  the last step is true since $c_i c_j < 0$ and $\alpha_{(i)} \alpha_{(j)} \leq 0$. Hence $(\lambda, \lambda) = 0 \iff \lambda = 0$ completing the proof for linear independence.

- It follows from the previous two points that Φ_S is a basis of $\mathbb{h}^*_{\mathbb{R}}$ implying that $|\Phi_S| = \dim(\mathbb{h}^*_{\mathbb{R}}) = r$, and the simple roots will from now on be labelled $\alpha_{(i)},\ i = 1, ..., r$.

Defining the basis for the Lie algebra to be $\{h^i, e^i = e_+^{\alpha_{(i)}}, e_-^i = e^{-\alpha_{(i)}}\}$ and the (not necessarily symmetric) *Cartan matrix* to be

$$A^{ij} = \frac{2\left(\alpha_{(i)}, \alpha_{(j)}\right)}{\left(\alpha_{(i)}, \alpha_{(i)}\right)} \in \mathbb{Z},$$

the (not necessarily symmetric). The commutation relations (1.5) then become

$$\begin{aligned} \left[h^i, e^i_\pm\right] &= \pm 2 e^i_\pm \\ \left[h^i, e^j_\pm\right] &= \pm 2 A_{ij} e^j_\pm \\ \left[e^i_\pm, e^j_\pm\right] &= h^i. \end{aligned}$$

The Cartan matrix satisfies the following:

- $A^{ii} = 2 \forall\ \mathbb{Z}$.
- $A^{ij} = 0 \iff A^{ji} = 0$.
- $A^{ij} \in \mathbb{Z}^{\leq 0}\ \forall i \neq j$ since $(\alpha, \beta) \leq 0\ \forall \alpha \neq \beta$.
- From equation (1.4) the inner product was defined as $(\alpha, \beta) = \alpha^T k^{-1} \beta > 0 \Rightarrow \det k^{-1} > 0\ A^{ij}$ defines a Euclidean inner product necessitating that $\det(A) > 0$.
- the Lie algebra $\mathfrak{g}$ is simple and hence A^{ij} is irreducible.

These constrain the Cartan matrix, enabling us to classify all Cartan matrices and allows us to uniquely determine all Lie algebras. For rank 2 the constraints above give rise to three Cartan matrices,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

For each Cartan matrix one can draw a *Dynkin diagram* by drawing a node $\circ$ for each simple root $\alpha_{(i)}$, drawing $\max\left(|A^{ij}|, |A^{ji}|\right)$ lines between nodes $\alpha_{(i)}$ and $\alpha_{(j)}$, and drawing an arrowhead from longest to shortest root, as in figure 1.2 respectively for the rank 2 example. All finite dimensional simple complex Lie algebras has one

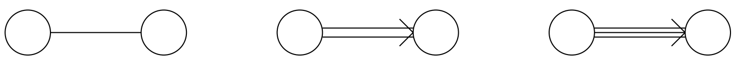
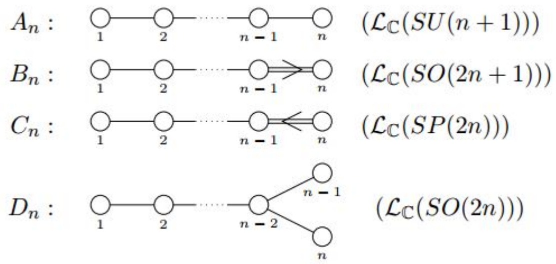


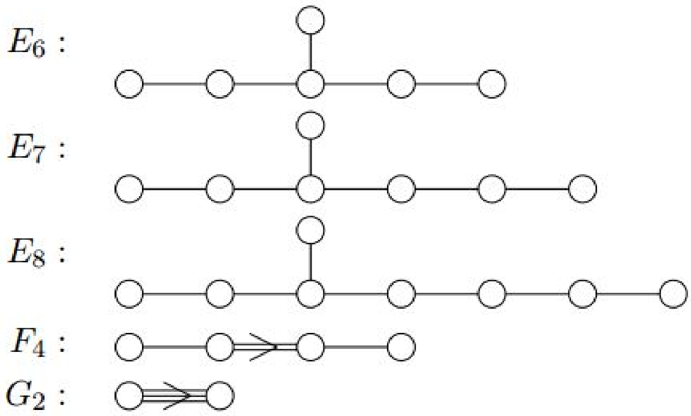
FIGURE 1.2: Dynkin diagram for Rank 2

.

of the following forms:

or be one of the 5 exceptional cases For rank 1, D_1 is not possible but $A_1 = B_1 = C_1$





.

implying that

$$\mathcal{L}_{\mathbb{C}}(SU(2)) \simeq \mathcal{L}_{\mathbb{C}}(SO(3)) \simeq \mathcal{L}_{\mathbb{C}}(SP(2)).$$

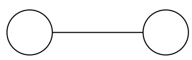
In rank 2 $B_2 = C_2$ so that

$$\mathcal{L}_{\mathbb{C}}(SO(5)) \simeq \mathcal{L}_{\mathbb{C}}(SP(4))$$

and the diagram for D_2 is the same as two A_1 s implying

$$\mathcal{L}_{\mathbb{C}}(SO(4)) \simeq \mathcal{L}_{\mathbb{C}}(SU(2)) \bigoplus \mathcal{L}_{\mathbb{C}}(SU(2)).$$

Now we can use a Dynkin diagram to reconstruct a Lie algebra $\mathfrak{g}$. The Dynkin diagram for A_2 is which corresponds to Cartan matrix



$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The roots, which may be called α and β then satisfy

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\alpha,\beta)}{(\beta,\beta)} = \frac{2|\alpha|}{|\beta|}\cos\phi = \frac{2|\beta|}{|\alpha|}\cos\phi = -1.$$

Hence $|\alpha| = |\beta|$ and $\cos\phi = -\frac{1}{2}$ and therefore $\phi = \frac{2\pi}{3}$. The length of the α-string through β is $\ell_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2$ so that $\alpha+\beta$ is a root. The full set of roots therefore comprises 6 elements, $\Phi = \{\alpha, \beta, -\alpha, -\beta, \alpha+\beta, -\alpha-\beta\}$, which may be visualised in figure 1.3 and the Cartan-Weyl basis of A_2 ($\mathcal{L}_{\mathbb{C}}(SU(3))$) has dimension

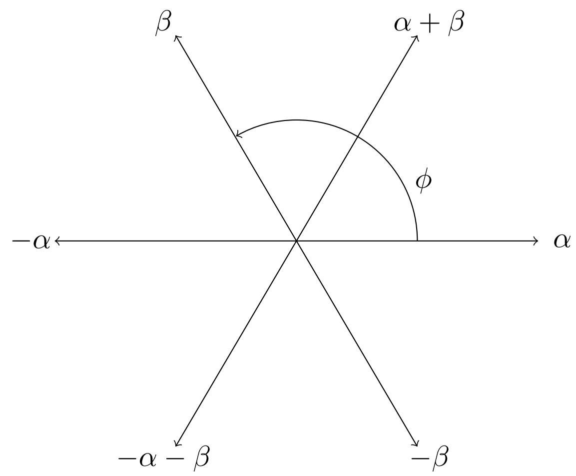


FIGURE 1.3: Diagram showing the full set of roots, comprising 6 elements for rank 2

$2+6=8$ and consists of

$$\mathcal{B}_{CW} = \{H^1, H^2, E^{\pm\alpha}, E^{\pm\beta}, E^{\pm(\alpha+\beta)}\}$$

1.2.6 Root and weight lattices

For R a representation of $\mathfrak{g}$ on V, $R(H^i) \in gl(V)$ and $R(E^\alpha) \in \mathfrak{gl}(V)$ and

$$\left[R(H^i), R(H^j)\right] = R\left([H^i, H^j]\right) = 0.$$

Since the $R(H^i)$ commute, they have a common eigenvector v_λ so $\forall\ H \in \mathfrak{h}$, $R(H)v_\lambda = \lambda(H)v_\lambda$ for some $\lambda(H)$ and $\lambda \in \mathfrak{h}^*$ is called a *weight* of R; the set of all weights S_R

of R is called the *weight set* and V_λ is the subspace of vectors v that satisfy

$$R(H)v = \lambda(H)v.$$

Since V_λ need not be 1-dimensional, the weights can have *multiplicity*

$$m_\lambda = \mathrm{dim} V_\lambda \geq 1;$$

(Note that these weights are integers). For example,

$$[H^i, E^\alpha] = \mathrm{ad}_{H^i} E^\alpha = R_{\mathrm{adj}}(H^i)E_\alpha = \alpha^i E_\alpha,$$

so that the roots α are the weights of the adjoint representation. We have that

$$\begin{aligned} R(H^i)R(E^\alpha)v &= R(E^\alpha)R(H^i)v + \left[R(H^i), R(E^\alpha)\right] v \\ &= \lambda^i R(E^\alpha)v + \alpha^i R(E^\alpha)v = \left(\lambda^i + \alpha^i\right) R(E^\alpha)v. \end{aligned}$$

Therefore, $v \in V_\lambda \Rightarrow R(E^\alpha)v \in V_{\lambda+\alpha}$, but if $V_{\lambda+\alpha} = \{0\}$ then $\lambda + \alpha$ is not a weight. Hence the Cartan elements H^i preserve the weights and the step operators E^α increment the weights by α.

1.3 Standard model - Higgs mechanism

In 1964-1968, Brout, Englert, Guralnik, Hagen, Kibble, and Higgs recognised that spontaneous symmetry breaking results in gauge fields acquiring mass [43, 44, 45, 46]. This led to the building of ATLAS (A Toroidal LHC Apparatus) and a full-blown search for the Higgs particle by the collaboration under team leader Fabiola Gianotti, which culminated in finding it and thus providing the last vindication of the Standard Model in regimes where coupling to gravity is not important. For this work Peter Higgs and François Englert received the Nobel Prize in 2013.

The Standard Model Lagrangian is given by

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{D}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{Higgs}},$$

where $\mathcal{L}_{\text{YM}}$ gives the dynamics of the gauge fields, $\mathcal{L}_{\text{D}}$ gives the dynamics of the matter fields (fermions) and their interaction with the gauge field, $\mathcal{L}_{\text{Yukawa}}$ gives the mass terms for the matter fields and $\mathcal{L}_{\text{Higgs}}$ gives the dynamics and potential of the Higgs. In particular $\mathcal{L}_{\text{YM}}$ describes the $SU(3)_{\text{Colour}} \bigotimes SU(2)_{\text{Left}} \bigotimes U(1)_{\text{Hypercharge}}$ as follows:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4g_1^2} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4g_2^2} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4g_3^2} G_{A\mu\nu} G^{A\mu\nu}.$$

The first term represents the $U(1)$ Hypercharge part with boson $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$; the second term is the $SU(2)$ weak isospin felt by left-handed fermions with $W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - \epsilon^{abc} W^b_\mu W^c_\nu$, where $a = 1, 2, 3$ and before symmetry breaking these are not the $W^\pm$- and Z- bosons, and includes only left-handed objects, i.e. maximally violating symmetry; the third term is the $SU(3)$ strong force colour felt by quarks, $G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f^{ABC} A^B_\mu A^C_\nu$, where $A = 1, ..., 8$ and A^A_μ is the gluon. Now

$$\mathcal{L}_{\text{Higgs}} = \left(D^\mu \phi\right)^\dagger \left(D^\mu \phi\right) - V(\phi)$$

with ϕ a scalar field, and where (with rescaling $B_{\mu\nu} \to g_1 B_{\mu\nu}$ and $W^A_{\mu\nu} \to g_2 W^A_{\mu\nu}$)

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi \mathbb{1} + \frac{i}{2} g_2 W^a_\mu \sigma^a + \frac{i}{2} g_1 B_\mu \mathbb{1} \\ V(\phi) &= -\mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi\right)^2, \end{aligned} \tag{1.6}$$

i.e. just involving the $U(1)$ and $SU(2)$ gauge fields. We require $\lambda > 0$ for vacuum stability. Since this field ϕ is a scalar complex doublet, it has 4 degrees of freedom, while W^a_μ represents 3 complex numbers and hence 6 degrees of freedom, and similarly B_μ has 2 degrees of freedom; there are therefore 12 degrees of freedom before symmetry breaking, in total. Since, prior to symmetry breaking, the field has a zero VEV (vacuum expectation value), all of these fields are massless prior to symmetry breaking (as we would see in nature above a threshold temperature). If $\mu^2 > 0$, then the minimum is at $\phi^\dagger \phi = \frac{\mu^2}{2\lambda} = \frac{\nu^2}{2}$ (so $\mu^2 = \lambda \nu^2$) and the scalar field develops a non-zero VEV $\langle 0 \left| \phi \right| 0 \rangle = \begin{pmatrix} 0 \\ \frac{\nu}{\sqrt{2}} \end{pmatrix}$ which spontaneously breaks the symmetry. Using equation (1.6), the Higgs covariant derivative term becomes:

$$\begin{aligned}
(D_\mu\phi)^\dagger D_\mu\phi &= \left|\left(\partial_\mu\phi\mathbb{1} + \frac{i}{2}g_2 W_\mu^a\sigma^a + \frac{i}{2}g_1 B_\mu\mathbb{1}\right)\begin{pmatrix} 0 \\ \frac{\nu}{\sqrt{2}} \end{pmatrix}\right|^2 \\
&= \frac{\nu^2}{8}\left|\left(g_2 W_\mu^a\sigma^a + g_1 B_\mu\mathbb{1}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 \\
&= \frac{\nu^2}{8}\left|\left[\begin{pmatrix} g_2W_\mu^3 + g_1B_\mu & g_2W_\mu^1 + ig_2W_\mu^2 \\ g_2W_\mu^1 - ig_2W_\mu^2 & -g_2W_\mu^3 + g_1B_\mu \end{pmatrix}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 \\
&= \frac{\nu^2}{8}\left|\begin{pmatrix} g_2W_\mu^1 + ig_2W_\mu^2 \\ -g_2W_\mu^3 + g_1B_\mu \end{pmatrix}\right|^2 \\
&= \frac{\nu^2}{8}\left[\left(g_2W_\mu^1 + ig_2W_\mu^2\right)\left(g_2W_\mu^1 - ig_2W_\mu^2\right) + \left(-g_2W_\mu^3 + g_1B_\mu\right)^2\right],
\end{aligned}$$

and with fluctuations around the minimum potential, this becomes

$$\begin{aligned}
(D_\mu\phi)^\dagger D_\mu\phi &= \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{(\nu+h)^2}{8}\left[\left(g_2W_\mu^1 + ig_2W_\mu^2\right)\left(g_2W_\mu^1 - ig_2W_\mu^2\right)\right. \\
&\quad \left. + \left(-g_2W_\mu^3 + g_1B_\mu\right)^2\right] \\
V(\phi) &= -\mu^2\frac{(\nu+h)^2}{2} + \lambda\frac{(\nu+h)^4}{4} \\
&= = -\lambda\nu^2\frac{\nu^2 + 2\nu h + h^2}{2} + \lambda\frac{\nu^4 + 4\nu^3 h + 6\nu^2h^2 + 4\nu h^3 + h^4}{4} \\
&= -\frac{1}{4}\lambda\nu^4 + \lambda\nu^2h^2 + \lambda\nu h^3 + \frac{1}{4}\lambda h^4,
\end{aligned}$$

Here we see that the scalar field has reduced from having 4 degrees (a scalar complex doublet) of freedom to having 1 (choosing the vacuum out of the infinite options that satisfy $\phi^\dagger\phi = \frac{\nu^2}{2}$ is what spontaneously breaks the symmetry). The terms multiplying ν^2 have developed a mass; in particular the Higgs scalar h has a mass. So too the $W_\mu^\pm$ bosons defined as

$$\begin{aligned}
W_\mu^\pm &= \left[W_\mu^1 \mp iW_\mu^2\right] \\
Z_\mu &= \frac{1}{\sqrt{g_1^2 + g_2^2}}\left(g_2W_\mu^3 - g_1B_\mu\right)
\end{aligned}$$

have masses $m_W = \frac{g_2\nu}{2}$ and $m_Z = \frac{\sqrt{g_1^2+g_2^2}\nu}{2}$. These 3 bosons have now got an additional longitudinal degree of freedom, at the expense of the 'eating' of 3 Goldstone bosons. In addition we have a massless photon

$$A_\mu \quad = \quad \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(g_2 W_\mu^3 - g_1 B_\mu \right) .$$

Furthermore, in the Yukawa sector, the Higgs boson couples to fermions, whose masses may be similarly read off. One can define the Weinberg angle θ_W such that

$$\cos^2 \theta_W = \frac{m_W^2}{m_Z^2} \iff \theta_W = \arctan\left(\frac{g_1}{g_2}\right)$$

and in practice it is found that θ_W runs with energy.

1.4 Renormalisation

The following is based on Ref. [47]. At very high energies (UV) there is a vast mess of interactions, as may be seen at the LHC. However, as one flows down to energies at the scale we live, may of these interactions seem to be 'interacted out' and to become irrelevant. Renormalisation, which was discovered in 1947–49 by Hans Kramers, Hans Bethe, Julian Schwinger, Richard Feynman, and Shin'ichiro Tomonaga, and systematised by Freeman Dyson in 1949 [48, 49, 50, 51, 52], is a procedure which explains how one may understand this. Kenneth Wilson made his important discovery regarding the Renormalisation Group (RG) in 1971 [53, 54]. We will need to work in Euclidean signature $k^\mu k_\mu < \Lambda$ for an effective cutoff.

1.4.1 Relevant, irrelevant and marginal operators

Consider a path integral over momentum modes less than Λ given by

$$\mathcal{Z} = \int [D\phi]_\Lambda \text{exp}\left(- \int d^d x \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 \right) ,$$

that is over fields $\phi(\vec{x}) = \int \frac{d^d k}{2\pi} e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k})$ and one could set $[D\phi]_\Lambda = \prod_{\vec{k}<\Lambda} d\phi(\vec{k})$. The renormalisation procedure follows 3 steps.

In the first step, we break the path integral into two parts $\phi = \phi_< + \phi_>$ using a parameter $b << 1$:

$$\begin{aligned}
\phi_<(\vec{k}) &= \begin{cases} \phi(\vec{k}), & \text{for } 0 \leq |\vec{k}| < \Lambda \cdot (1-b) \\ 0, & \text{for } \Lambda \cdot (1-b) < |\vec{k}| \leq \Lambda \end{cases} \\
\phi_<(\vec{k}) &= \begin{cases} 0, & \text{for } 0 \leq |\vec{k}| < \Lambda \cdot (1-b) \\ \phi(\vec{k}), & \text{for } \Lambda \cdot (1-b) < |\vec{k}| \leq \Lambda \end{cases}
\end{aligned}$$

In the second step we integrate out the higher momentum modes as follows:

$$\begin{aligned}
\mathcal{S}(\phi) &= \mathcal{S}\left(\phi_< + \phi_> + \frac{\lambda}{4!}\phi^4\right) \\
&= \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2\right) \\
&= \int d^d x \left(\frac{1}{2}(\partial_\mu (\phi_< + \phi_>))^2 + \frac{1}{2}m^2 (\phi_< + \phi_>)^2 + \frac{\lambda}{4!}(\phi_< + \phi_>)^4\right) \\
&= \mathcal{S}(\phi_<) + \widetilde{S},
\end{aligned}$$

where $\widetilde{S} = \mathcal{S}_0 + \mathcal{S}_{\text{int}}$ with $\mathcal{S}_0$ a propagator: $\mathcal{S}_0 = \int d^d x \frac{1}{2}(\partial_\mu \phi)^2$. Note that since

$$\frac{\int [D\phi] e^{-\mathcal{S}_0}\phi(\vec{k})\phi(\vec{p})}{\int [D\phi] e^{-\mathcal{S}_0}} = \frac{(2\pi)^d \delta^{(d)}(\vec{k} + \vec{p})}{\vec{k}^2},$$

the term $\int d^d x \frac{1}{2}(\partial_\mu \phi_>)(\partial_\mu \phi_<)$ drops out, and so $\mathcal{S}_0 = \int d^d x \frac{1}{2}(\partial_\mu \phi_>)^2$ and

$$e^{-\mathcal{S}_{\text{eff}}} = e^{-\mathcal{S}(\phi_<)} \int [D\phi_>] e^{-\mathcal{S}_0} \sum_{n=0}^{\infty} \frac{(\mathcal{S}_{\text{int}})^n}{n!}.$$

Obviously it would make sense to define that if $\mathcal{S} = \frac{1}{2}\int (\partial_\mu \phi)^2 d^d x$, then $\mathcal{S} = \mathcal{S}_{\text{eff}}$. We define this as the free field fixed point, and we assume very small changes due to this step so that we remain close to the free field fixed point (which is also the zero coupling fixed point).

The third step involves rescaling of parameters:

$$[D\phi_<] = \prod_{|\vec{k}| < (1-b)\Lambda} d\phi(\vec{k})$$

$$\begin{aligned} k' &= \frac{k}{(1-b)} \\ x' &= x(1-b) \\ \phi' &= (1-b)^{-\frac{d-2}{2}}\phi \\ \Rightarrow [D\phi_<] &= \prod_{|\vec{k}|<\Lambda} d\phi(\vec{k}) \to \prod_{|\vec{k}'|<\Lambda} d\phi'(\vec{k}') = [D\phi'_<] = [D\phi']_\Lambda. \end{aligned}$$

Iterating through steps 1-3 will result in parameters changing continuously, known as a 'Renormalisation Group flow'. Notice that we have (in mass dimensions)

$$\begin{aligned} & \left[\int d^dx'\partial'^M\phi'^N\right] = (1-b)^{\left[d-M-\frac{N(d-2)}{2}\right]}\left[\int d^dx\partial^M\phi^N\right] \\ = \quad & (1-b)^{-\Delta}\left[\int d^dx\partial^M\phi^N\right] \\ \Rightarrow \quad & \lambda\int d^dx'\partial^M\phi^N \to \lambda'\int d^dx\partial'^M\phi'^N = \lambda'\,(1-b)^{-\Delta}\int d^dx\partial^M\phi^N \\ \Rightarrow \quad & \lambda' = (1-b)^{\Delta}\,\lambda \end{aligned}$$

so that these couplings, which were previously dimensionless, now have mass dimension $[\lambda] = M^\Delta$, where

$$\Delta = -\left[d - M - \frac{N(d-2)}{2}\right].$$

For example the term

$$\int d^dx\, m^2\phi^2 \tag{1.7}$$

will have $\Delta = -2$ (independent of d) and therefore coupling $m^2 \to \frac{m^2}{(1-b)^2}$. Since $b < 1$, m^2 grows along the RG-flow and so this operator is called relevant. On the other hand, the term

$$\int d^dx\lambda_6\phi^6$$

will have coupling $\lambda'_6 = (1-b)^{2d-6}\lambda_6$ so that for $d = 4$, $\Delta = 2$ and $\lambda'_6 = (1-b)^2\lambda_6$ and the coupling λ_6 dies along the RG-flow. Such a coupling is called irrelevant. Clearly the term

$$\int d^4x\lambda_4\phi^4 \tag{1.8}$$

has $\Delta = 0$ so that the coupling does not change over the RG-flow. Such an operator is called 'marginal'. We have therefore learned that however big the 'high energy mess' we started with, all the operators with positive Δ will be irrelevant and are called non-renormalisable interactions. This explains why quantum field theories of particle interactions are given by 'renormalisable' quantum field theories.

Interestingly, masses need to be fine-tuned at scale Λ to explain experiments - this is known as the hierarchy problem. Furthermore, while it was previously understood that Λ was simply a regulator needed for mathematical convenience, this procedure shows that Λ is a scale telling us that we may not describe nature with larger energies than that cutoff, or equivalently length scales lower than $\frac{\hbar}{\Lambda}$. Also of interest is that, when one deals with λ_4 carefully, one actually finds that it is irrelevant. One rarely finds marginal operators. In particular, a general theory with

$$\mathcal{S} = \int d^d x \, \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \lambda\phi^n \right)$$

for which

$$[\phi] = M^{d/2-1} \Rightarrow \left[\int d^d x \, \phi^n \right] = M^{-(d-n(d/2-1))} = M^\Delta$$

will have a marginal operator if $\Delta = 0$, or equivalently

$$d = -\frac{n}{1 - n/2}. \tag{1.9}$$

This equation has particular positive integer solutions when $(n, d) = (3, 6), (4, 4)$ or $(6, 3)$. There are often quantum corrections to these solutions which will make these theories 'marginally relevant' or 'marginally irrelevant' as shall be explored further in subsection 1.4.3.

1.4.2 Matching conditions

Before we jump to the Callan-Symanzik equations, we quickly review the initial work done on renormalisation.

Here the aim is to find the Lagrangian in terms of physical couplings

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_p^2\phi^2 - g_p\phi^4$$

What we have access to is a Lagrangian for a bare field ϕ_0 with quantum corrections turned off (i.e. $\hbar = 0$)given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m^2\phi_0^2 - g\phi_0^4,$$

and a Lagrangian in terms of a dressed field with a mess of all sorts of multiparticle interactions shielding a particle, $\phi = z^{-\frac{1}{2}}\phi_0$ in which

$$z = \frac{1}{1 + \int_0^\infty da^2\sigma(a^2)},$$

where the integral over the spectral density, $\int_0^\infty da^2\sigma(a^2)$, quantifies the sum of all these quantum corrections:

$$\mathcal{L} = \frac{1}{2}z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}zm_p^2\phi^2 - zg_p\phi^4.$$

We introduce counter-weights δz, δm^2 and δg and write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$$

where

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_p^2\phi^2 \\ \mathcal{L}_{\text{int}} &= -g_p\phi^4 + \frac{1}{2}\delta z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \delta g\phi^4.\end{aligned}$$

We will find out what the counter-weights are by solving to one-loop and using matching conditions. The free two-point function is given by

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\rangle &= \int [D\phi]e^{i\int d^4x\mathcal{L}_0}\phi(x_1)\phi(x_2) \\ &= \int \frac{d^4k}{(2\pi)^4}e^{ik\cdot(x_1-x_2)}\frac{i}{k^2 - m_p^2 + i\epsilon} = \Delta(x_1 - x_2).\end{aligned}$$

For the complete 2-point function we obtain

$$\begin{aligned}
\langle\phi(x_1)\phi(x_2)\rangle &= \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\int[D\phi]e^{i\int d^4x\mathcal{L}_0}\sum_{p=0}^{\infty}\frac{1}{p!}\left(i\int d^4x\mathcal{L}_{\text{int}}(\phi)\right)^p e^{\int d^4xJ(x)\phi(x)}\Big|_{J=0}\\
&= \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\left(1+i\int d^4x\frac{\delta z}{2}\left(\partial_\mu\frac{\delta}{\delta J(x)}\right)\left(\partial_\mu\frac{\delta}{\delta J(x)}\right)+...\right)\\
&\quad e^{\frac{i}{2}\int d^4\bar{x}\int d^4\bar{y}J(\bar{x})\Delta(\bar{x}-\bar{y})J(\bar{y})}\\
&= \Delta(x_1-x_2)+\int d^4x\Delta(x_1-x_2)i\frac{\delta z}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\Delta(x-\bar{x})|_{x=\bar{x}}\\
&\quad +i\delta z\int d^4x\left(\frac{\partial}{\partial x^\mu}\Delta(x_1-x)\right)\left(\frac{\partial}{\partial x_\mu}\Delta(x_2-x)\right).
\end{aligned}$$

The second term in the above is merely a vacuum contribution. The third term, however, may be simplified as:

$$\begin{aligned}
&i\delta z\int d^4x\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\left(\frac{i(ik_{1\mu})}{k_1^2-m_p^2+i\epsilon}\frac{i(ik_\mu^2)}{k_2^2-m_p^2+i\epsilon}\right)e^{i(k_1\cdot x_1-k_2\cdot x_2+x\cdot(k_1-k_2))}\\
=\ &i\delta z\int\frac{d^4k}{(2\pi)^4}k^2\left(\frac{i}{k^2-m_p^2+i\epsilon}\frac{i}{k^2-m_p^2+i\epsilon}\right)e^{ik\cdot(x_1-x_2)}.
\end{aligned}$$

Now, to extend this to one-loop (i.e. to one factor in $\hbar$), consider that the mass counter-weight will appear in a similar integral to the one above, so combining these (and changing the dummy variable k to p) we get

$$\int\frac{d^4p}{(2\pi)^4}\left(ip^2\delta z-i\delta m^2\right)\left(\frac{i}{p^2-m_p^2+i\epsilon}\frac{i}{p^2-m_p^2+i\epsilon}\right)e^{ip\cdot(x_1-x_2)}.$$

For g_p we have incoming and outgoing momenta p, and momentum in the loop k, coming with a combinatorial factor $\frac{4!}{2}=12$, giving us the following expression:

$$-12ig_p\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\frac{i}{p^2-m_p^2+i\epsilon}\frac{i}{k^2-m_p^2+i\epsilon}\frac{i}{p^2-m_p^2+i\epsilon}e^{ip\cdot(x_1-x_2)}.$$

The correlator at one-loop is thus given by

$$\begin{aligned}
&\int\frac{d^4p}{(2\pi)^4}e^{ik\cdot(x_1-x_2)}\frac{i}{p^2-m_p^2+i\epsilon}\left[1-12ig_p\frac{i}{p^2-m_p^2+i\epsilon}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_p^2+i\epsilon}\right.\\
&\left.+\left(ip^2\delta z-i\delta m^2\right)\left(\frac{i}{p^2-m_p^2+i\epsilon}\right)\right]
\end{aligned}$$

$$= \quad \int \frac{d^4p}{(2\pi)^4} e^{ik\cdot(x_1-x_2)} \frac{i}{p^2 - m_p^2 + 12ig_p \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_p^2+i\epsilon} - (ip^2\delta z - i\delta m^2)}.$$

Since we require the pole to be at $p^2 = m_p^2$, the matching conditions will be obtained from

$$12ig_p \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_p^2 + i\epsilon} = \left(ip^2\delta z - i\delta m^2\right)$$

for which

$$\begin{aligned}
\delta z &= 0 \\
\delta m^2 &= -12ig_p \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_p^2 + i\epsilon} \\
&= -12ig_p \int \frac{1}{(2\pi)^4} \int_{S^3} \Omega_3 \int_0^\infty dk \frac{k^3}{[k^2 - m_p^2]} \\
&= -12ig_p \left[\frac{1}{2} \int \frac{1}{(2\pi)^4} \int_{S^3} \Omega_3 \int_0^\infty dt \frac{t}{[t - m_p^2]}\right].
\end{aligned}$$

Setting $x = \frac{-m_p^2}{t-m_p^2}$ and $1 - x = \frac{t}{t-m_p^2}$ so that $dx = \frac{m_p^2}{(t-m_p^2)^2} dt = -\left(\frac{x^2}{m_p^2}\right) dt$, we obtain that

$$\begin{aligned}
\delta m^2 &= -12ig_p \left[\frac{1}{2} \int \frac{1}{(2\pi)^4} \int_{S^3} \Omega_3 \int_0^\infty dx \frac{1-x}{x^2} m_p^2\right] \\
&= -12ig_p \left[\frac{1}{2} \frac{\Gamma(2)\Gamma(-1)}{\Gamma(1)}\right] \left[\frac{1}{m_p^2 8\pi^2}\right] \\
&= -12ig_p \left[\frac{\Gamma(-1)}{m_p^2(4\pi)^2}\right].
\end{aligned}$$

Of course $\Gamma(-1)$ is not finite, but this gives the general idea behind matching conditions! Let us move onto the Callan Symanzik equation, which was derived independently Curtis Callan and Kurt Symanzik in 1970.

1.4.3 Callan-Symanzik equation

Before matching, one would be unable to read the S-matrix elements off the correlators defined for bare fields (note that throughout the time-ordered product is intended),

$$\int [D\phi] e^{i\mathcal{S}} \phi_0(x_1)...\phi_0(x_n) = \langle 0 \left| \phi_0(x_1)...\phi_0(x_n) \right| 0 \rangle$$

where

$$\mathcal{S} = \int d^4x \mathcal{L}, \qquad \mathcal{L} = \frac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{1}{2}m^2\phi_0^2 - \frac{g}{4!}\phi_0^4.$$

However, after matching the dressed fields ($\phi = z^{-1/2}\phi_0$ where z depends on M and ϕ_0 depends on g and the UV cutoff Λ but not on M),

$$\langle 0 \left| \phi(x_1)...\phi(x_n) \right| 0 \rangle = z^{-n/2} \langle 0 \left| \phi_0(x_1)...\phi_0(x_n) \right| 0 \rangle \tag{1.10}$$

with

$$\langle 0 \left| \phi(x_1)...\phi(x_n) \right| 0 \rangle = F(M, g_p), \tag{1.11}$$

we do obtain correctly normalised wave functions since the matching conditions replace Λ, g and z with the renormalisation scale M (what we previously called m_p) and g_p. Consider the following variations, where we set $\delta\eta = -\frac{1}{2}\delta \text{log} z$:

$$\begin{aligned} M &\rightarrow M + \delta M \\ g_p &\rightarrow g_p + \delta g_p \\ \phi &\rightarrow (1 + \delta\eta)\phi. \end{aligned}$$

For the expression in equation (1.10),

$$G^{(n)} \rightarrow (1 + \delta\eta)^n G^{(n)} \Rightarrow \delta G^{(n)} = n\delta\eta G^{(n)}$$

and from the expression in equation (1.11),

$$\delta G^{(n)} = \left[\delta M \frac{\partial}{\partial M} + \delta g_p \frac{\partial}{\partial g_p} \right] G^{(n)}.$$

Setting them equal to each other leads to the Callan-Symanzik equation which was found around 1970 [55, 56],

$$\begin{aligned} 0 &= \left[\delta M \frac{\partial}{\partial M} + \delta g_p \frac{\partial}{\partial g_p} - n\delta\eta\right] G^{(n)} \\ &= \left[\frac{\partial}{\partial M} + \frac{\delta g_p}{\delta M}\frac{\partial}{\partial g_p} - n\frac{\delta\eta}{\delta M}\right] G^{(n)} \\ &= \left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial g_p} + n\gamma\right] G^{(n)} \end{aligned}$$

where $\beta = M\frac{\delta g_p}{\delta M}$ tells you how the coupling changes as you go to higher energies (although we are generally more interested in how the coupling flows from the UV to the IR). For example in QCD, $\beta < 0$ so that at high energies / small distances (UV) the strength of interactions is small, an effect known as asymptotic freedom. As you flow from the UV to the IR (low energies/large distances) the strength of interaction becomes very strong, which is known as confinement). Also, $\gamma = -\frac{\delta\eta}{\delta M}M$ is called the anomalous dimension, which gives the quantum corrections to the dimension. Consider the marginal operators obtained in equation (1.9). For a coupling term of the form $\frac{g}{4!}\phi^4$ in 4 dimensions one has that the one-loop beta function is given by

$$\beta = \frac{3}{16\pi^2}g^2 > 0$$

so that the operator is marginally irrelevant. To make this theory once again scale independent, one can add a relevant contribution, obtained as follows. Consider working in dimension $d = 4 \pm \epsilon$, where $[\phi] = M^{1\pm\frac{\epsilon}{2}}$ so that $[g] = M^{\mp\epsilon}$. A relevant contribution corresponds to $[g] = M^{\epsilon}$ and thus $d = 4 - \epsilon$ dimensions. One can then write $[g] = [\bar{g}]M^{-\epsilon}$ to obtain a critical coupling

$$\beta^* = -g\epsilon + \frac{3}{16\pi^2}g^2 = 0 \Rightarrow g^* = \frac{16\pi^2}{3}\epsilon + \mathcal{O}(\epsilon^2)$$

which is weakly coupled for epsilon small, so that one can make use of it for perturbation theory. Similarly for a theory with coupling $\frac{g}{3!}\phi^3$ in $d = 6$ dimensions which is marginally relevant since

$$\beta = -\frac{3g^4}{256\pi^3} < 0,$$

we work in $d = 6 + \epsilon$ dimensions for which $[g] = M^{-\epsilon/2}$, which deforms β to give a critical coupling as follow:

$$\beta^* = \frac{\epsilon}{2} g - \frac{3g^4}{256\pi^3} = 0 \Rightarrow g^* = \left(\frac{128\pi^3}{3}\epsilon\right)^{1/3}.$$

Furthermore, a theory with a coupling term $\lambda\phi^6$ in 3 dimensions is marginally irrelevant with $\beta = \frac{3\lambda^2}{2\pi^2} > 0$ can be made scale invariant by working in $d = 3 - \epsilon$ dimensions where $[\lambda] = M^{2\epsilon}$, and the critical exponent is found as

$$\beta^* = -2\epsilon\lambda + \frac{3\lambda^2}{2\pi^2} = 0 \to \lambda^* = \frac{4\pi^2}{3}\epsilon.$$

1.5 Black Holes, holography and the information paradox

This section is based on Refs. [57, 58, 59, 60, 61]. Gravity is non-renormalisable as can be seen for fluctuations in the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P}.$$

where M_P is the Planck mass. For small energy scales the space-time metric is approximately equal to the flat metric and perturbation theory works satisfactorily. However for larger energies tending towards the Planck mass this is clearly not the case any longer. In the quest to find a UV-complete theory, black holes become of great importance as objects with both relativistic and quantum behaviour.

A star that has used up its nuclear fuel will use quantum mechanical sources to support itself against gravitational collapse, known as degeneracy pressure; when this source is a gas of cold fermions a white dwarf is formed, possible for stars less than 1.4 solar masses, which is the Chandrasekhar limit. When the pressure increases further to that of nuclear density, it is the degeneracy pressure of neutrons which becomes important, resulting in neutron stars. Since at the surface of a neutron star the gravitational potential is of order 0.1 and no longer $<< 1$ required for Newtonian gravity, General Relativity considerations need to start being applied. General Relativity predicts a maximum mass of 3 solar masses for such a star, and so hot star masses more massive than this may not end their life as a cold star but rather as a black hole.

Hawking and Bekenstein showed that the the energy spectrum of photos radiated from a black hole is the same as the black body radiation where the temperature is given by (which we shall justify later in this chapter).

$$k_B T = \frac{\hbar}{8\pi GM} \Rightarrow T = \frac{\hbar}{8\pi c^3 k_B GM}.$$

This gives a temperature of $60nK$ for a solar mass black hole and even less for a larger black hole - significantly less than the background temperature of the universes which is $2.73K$. The smaller the black hole, however, the hotter it is, and in order for a black hole to have a temperature of $2.73K$, it would have to have a mass comparable with that of the moon, $4.5 \times 10^2 2kg$. Since the internal energy U of a black hole is just its mass M, we have the statistical mechanics relation

$$\begin{aligned} \frac{\partial S}{\partial U} &= \frac{1}{T} \\ \Rightarrow \frac{\partial S}{\partial M} &= \frac{8\pi c^3 k_B GM}{\hbar} \\ \Rightarrow S &= \frac{4\pi c^3 k_B GM^2}{\hbar} \\ &= \frac{c^3 k_B A}{4\hbar G} \end{aligned}$$

where we have recalled that the radius of the event horizon is $2GM$ and the surface area of a sphere is $4\pi r^2$. Of great interest is the fact that black holes are the classical solutions of Einstein's equations, and yet this result for S contains the quantum mechanical quantity $\hbar$, together with quantities c and G_N which are indicators of relativity and gravity. Defining the Planck length as $\ell_P = \sqrt{\frac{G\hbar}{c^3}}$, the entropy can be given by

$$S = \frac{k_B A}{4\ell_P^2}.$$

The area thus scales as the area rather than as the perhaps more intuitive volume, which is a hint of the holographic principle.

1.5.1 Differential geometry notation

In d dimensions, a p-form $w^{(p)}$ can be constructed as

$$w^{(p)} = \frac{1}{p!} w_{\mu_1} w_{\mu_2} ... w_{\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p} \tag{1.12}$$

where the wedge ($\wedge$) product is antisymmetric. The exterior derivative d maps p-forms onto (p+1)-forms as follows

$$\begin{aligned} dw^{(p)} &= \frac{1}{p!} \partial_\mu w_{\mu_1} w_{\mu_2} ... w_{\mu_p} dx^{\mu} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p} \\ \Rightarrow d^2 w^{(p)} &= \frac{1}{p!} \partial_{\mu_a} \partial_{\mu_b} w_{\mu_1} w_{\mu_2} ... w_{\mu_p} dx^{\mu_a} \wedge dx^{\mu_b} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p} \\ d\left(w^{(p)} \wedge w^{(q)}\right) &= dw^{(p)} \wedge w^{(q)} + (-)^p w^{(p)} \wedge dw^{(q)}. \end{aligned}$$

The latter two properties are referred to as the nilpotent and distributive properties respectively. One is able to go from a p-form to a (d-p)-form using the Hodge star product. Here the wedge product indices in (1.12) are absorbed into an ϵ-term so that a wedge product on the dual directions can be instead incorporated, and a factor of $\frac{1}{(d-p)!}$ compensates for over-counting:

$$*w^{(p)} = \frac{1}{p!(d-p)!} w_{\mu_1} w_{\mu_2} ... w_{\mu_p} \epsilon^{\mu_1 ... \mu_p}{}_{\mu_{p+1} ... \mu_d} dx^{\mu_{p+1}} \wedge dx^{\mu_{p+2}} \wedge ... \wedge dx^{\mu_d}.$$

Notice that doing this twice will then give you $w^{(p)}$ back up to a sign,

$$\begin{aligned} * * w &= \frac{1}{(p!)^2((d-p)!)^2} w_{\mu_1} w_{\mu_2} ... w_{\mu_p} \epsilon^{\mu_{p+1} ... \mu_d}{}_{\nu_1 ... \nu_p} \\ &\quad \epsilon^{\mu_1 ... \mu_p}{}_{\mu_{p+1} ... \mu_d} dx^{\nu_1} \wedge dx^{\nu_2} \wedge ... \wedge dx^{\nu_p} \\ &= \frac{(-1)^{t+p(d-p)}}{(p!)^2((d-p)!)^2} w_{\mu_1} w_{\mu_2} ... w_{\mu_p} \epsilon_{\nu_1 ... \nu_p}{}^{\mu_{p+1} ... \mu_d} \\ &\quad \epsilon^{\mu_1 ... \mu_p}{}_{\mu_{p+1} ... \mu_d} dx^{\nu_1} \wedge dx^{\nu_2} \wedge ... \wedge dx^{\nu_p} \\ &= \frac{(-1)^{t+p(d-p)}}{(p!)^2((d-p)!)^2} w_{\mu_1} w_{\mu_2} ... w_{\mu_p} p!(d-p)!(d-p)! dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p} \\ &= \frac{(-1)^{t+p(d-p)}}{(p!)} w_{\mu_1} w_{\mu_2} ... w_{\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_p} \\ &= (-1)^{t+p(d-p)} w \end{aligned}$$

where t is the number of time-like directions we have (if in Minkowski signature; for Euclidean signature we can set t-0). This comes about due to the definition

$$\epsilon^{\mu_1,\ldots,\mu_p,\lambda_{p+1}\ldots\lambda_d}\epsilon_{\nu_1\ldots\nu_p\lambda_{p+1}\ldots\lambda_d} = (-)^t p!(d-p)!\delta^{\mu_1\ldots\mu_p}_{[\nu_1\ldots\nu_p]}$$

Performing the exterior derivative on the Hodge star product of a p-form gives

$$d*w^{(p)} = \frac{1}{p!(d-p)!(d-p)!}\partial_\mu w_{\mu_1}w_{\mu_2}\ldots w_{\mu_p}\epsilon^{\mu_1\ldots\mu_p}{}_{\mu_{p+1}\ldots d}dx^\mu \wedge dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_d}$$
$$\Rightarrow *d*w^{(p)} = \frac{1}{p!(d-p)!(d-p)!}\partial_\mu w_{\mu_1}w_{\mu_2}\ldots w_{\mu_p}\epsilon^{\mu_{p+1}\ldots\mu_d}{}_{\nu_1\ldots\nu_p}\epsilon^{\mu_1\ldots\mu_p}{}_{\mu_{p+1}\ldots\mu_d}$$
$$dx^\mu\wedge dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_d}.$$

Seeing that $\epsilon_{0,1,\ldots,d} = \sqrt{|g|}$ and hence the partial derivatives give

$$\partial_\mu\sqrt{|g|} = \sqrt{|g|}\Gamma^\rho_{\rho\mu},$$

and therefore

$$*d*w = \frac{1}{(p-1)!}(-)^{t+(p-1)(p-d)}\nabla^\nu w_{\nu_{\mu\ldots\mu_{p-1}}}dx^{\mu_1}\wedge\ldots\wedge dx^{p-1}. \tag{1.13}$$

1.5.2 Stokes' Theorem

Stokes' Theorem gives that

$$\int_M dW = \int_{\partial M} W.$$

From equation (1.13) we can see that $\nabla_\mu j^\mu \iff d*j$, and therefore a conservation law where a manifold M is covered by surfaces Σ_{t_1} and Σ_{t_2} given by

$$Q(t) = \int_{\Sigma_{t_2}} *j - \int_{\Sigma_{t_1}} *j = \int_M d*j = 0 \Rightarrow Q(t_1) = Q(t_2)$$

Also Maxwell's equations, which are given by $\nabla^\nu F_{\mu\nu} = -4\pi j_\mu \iff d*F = 4\pi*j$. The conservation law is therefore given by

$$Q = \int_\Sigma *j = \frac{1}{4\pi}\int_{\partial\Sigma} *F$$

so that the conserved electric charge is $Q = \frac{1}{4\pi}\lim_{R\to\infty}\int_{S_R^2} *F$ and magnetic charge is $P = \frac{1}{4\pi}\lim_{R\to\infty}\int_{S_R^2} F$.

1.5.3 Concepts from General Relativity

First recall the standard methods for obtaining Riemann tensor coefficients. The Euler-Lagrange equations of motion which maximise proper time given by $\mathcal{S} = \int d^4x\mathcal{L}$ for which $\mathcal{L} = \sqrt{-ds^2} = \sqrt{-g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau}}$ are

$$\frac{\partial}{\partial\tau}\left(g_{\mu\alpha}\frac{\partial x^\alpha}{\partial\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau} \tag{1.14}$$

Relativity relations follow simply from defining vierbeins e_α (the prefix comes from the number of dimensions they live in) such that $g_{\alpha\beta} = e_\alpha e_\beta$ and $\partial_\alpha e_\beta = \Gamma^\mu_{\alpha\beta}e_\mu$. The covariant derivative comes from

$$\begin{aligned} & \frac{dA}{dx^\alpha} = \frac{\partial\left(A^\nu e_\nu\right)}{\partial x^\alpha}\left[=\left(\nabla_\alpha A^\mu\right)e_\mu\right] = \frac{\partial A^\nu}{\partial x^\alpha}e_\nu + A^\nu\Gamma^\mu_{\alpha\nu}e_\mu \\ \Rightarrow \quad & \nabla_\alpha A^\mu = \partial_\alpha A^\mu + A^\nu\Gamma^\mu_{\alpha\nu} \quad ; \quad \nabla_\alpha A_\mu = \partial_\alpha A_\mu - A^\nu\Gamma^\mu_{\alpha\nu}. \end{aligned}$$

The second of these follows from requiring that for scalars $\nabla_\alpha\left(A^\mu B_\mu\right) = \partial_\alpha\left(A^\mu B_\mu\right)$. The Christoffel definition comes from adding the 3 cyclic permutations of $\partial_\alpha g_{\mu\nu}$

$$\begin{aligned} & \partial_\alpha g_{\mu\nu} = \left(\partial_\alpha e_\mu\right)e_\nu + e_\mu\left(\partial_\alpha e_\nu\right) = \Gamma^\sigma_{\alpha\mu}g_{\sigma\nu} + \Gamma^\sigma_{\alpha\nu}g_{\mu\sigma} \\ \Rightarrow \quad & \Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}\left(-\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu}\right). \end{aligned}$$

which is clearly symmetric in lower indices. To determine the Christoffel symbols for a given metric, one derives an alternative form of the geodesic equation to compare with (1.14); the starting point is $0 = \frac{\partial}{\partial\tau}\left(\frac{\partial x^\mu}{\partial\tau}e_\mu\right)$ which results in:

$$\frac{\partial^2 x^\mu}{\partial\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau}.$$

The Riemann tensor is obtained by considering $\nabla_c\nabla_b V_a - \nabla_b\nabla_c V_a$:

$$\begin{aligned} \nabla_c\nabla_b V_a &= \partial_c\left(\nabla_b V_a\right) - \Gamma^d_{bc}\nabla_d V_a - \Gamma^d_{ca}\nabla_b V_d \\ &= \partial_c\left(\partial_b V_a - \Gamma^d_{ba}V_d\right) - \Gamma^d_{bc}\left(\partial_d V_a - \Gamma^e_{da}V_e\right) - \Gamma^d_{ca}\left(\partial_b V_d - \Gamma^e_{bd}V_e\right). \end{aligned}$$

Now, the first term and middle 2 terms are symmetric in swapping b and c and therefore

$$\begin{aligned}
& \nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a \\
= \; & \partial_b \left(\Gamma^d_{ca} V_d\right) - \partial_c \left(\Gamma^d_{ba} V_d\right) + \Gamma^d_{ba} \left(\partial_c V_d - \Gamma^e_{cd} V_e\right) - \Gamma^d_{ca} \left(\partial_b V_d - \Gamma^e_{bd} V_e\right) \\
= \; & \left(\partial_b \Gamma^d_{ca} - \partial_c \Gamma^d_{ba}\right) V_d + \left(\Gamma^d_{ca} \Gamma^e_{bd} - \Gamma^d_{ba} \Gamma^e_{cd}\right) V_e \\
\Rightarrow \; & R^d_{abc} = \left(\partial_b \Gamma^d_{ca} - \partial_c \Gamma^d_{ba}\right) + \left(\Gamma^e_{ca} \Gamma^d_{bd} - \Gamma^e_{ba} \Gamma^d_{ce}\right). \qquad (1.15)
\end{aligned}$$

The stress-energy $T^{\mu\nu}$ arises from realising that quantities like $\vec{E} = \left(\frac{kQ}{r^2}\right) \hat{r}$ and $\vec{g} = \left(-\frac{GM}{r^2}\right) \hat{r}$ assume action at a distance which displeased Einstein, so that the better starting point to consider local field equations of the form $\vec{\nabla} \cdot \vec{E} = 4\pi k\rho$ and $-\vec{\nabla} \cdot \vec{g} = -\vec{\nabla} \cdot \left(\vec{\nabla}\phi\right) = \nabla^2 \phi = 4\pi G\rho$, the tensorial extension of which is $\nabla_\nu T^{\mu\nu} = 0$ where T^{00} is the energy density, T^{0i} is the x^i-momentum density and T^{ij} is the i-flux of the j-momentum, in other words the stresses exerted by the fluid. The Einstein tensor and Einstein equation may then be constructed using combinations of $g^{\mu\nu}$ and $R^{\mu\nu}$ terms using tensorial arguments, and are respectively given by

$$\begin{aligned}
G^{\mu\nu} + \Lambda g^{\mu\nu} &= \kappa T_{\mu\nu} \qquad (1.16) \\
\nabla_\mu G^{\mu\nu} &= 0 \\
G^{\mu\nu} &= R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \\
\Rightarrow R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} &= \kappa T_{\mu\nu} \qquad (1.17) \\
\Rightarrow -R + 4\Lambda &= \kappa T \\
\Rightarrow R^{\mu\nu} &= \kappa \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} T\right) + \Lambda g^{\mu\nu}
\end{aligned}$$

where $\kappa = 8\pi G$.

1.5.4 Cartan formalism for obtaining the Riemann tensor

Riemann tensor calculations can be tedious and slow to give intuition. However, these may be done quicker using Cartan formalism and differential forms (for example, see lectures by Professor Ruth Gregory at the Perimeter Institute), which will be expedited below with the help of a simple example. Consider the metric for a sphere,

$$ds^2 = r^2 dr^2 + r^2 \sin^2\theta d\theta^2.$$

We choose the basis e^a where $e^a_\mu dx^\mu$, such that $ds^2 = \eta_{ab} e^a e^b$. In this case

$$e^r = rdr \ ; \ e^\theta = r\sin\theta d\theta.$$

The exterior derivative of a 1-form is $de^{(1)} = \partial_\mu e_{\mu_1} dx^\mu \wedge dx^{\mu_1}$ and therefore

$$\begin{aligned} de^r &= \partial_\mu r dx^\mu \wedge dr = 0 \\ de^\theta &= \partial_\mu r\sin\theta dx^\mu \wedge d\theta = \partial_r r\sin\theta dr \wedge d\theta = \frac{1}{r^2} e^r \wedge e^\theta. \end{aligned}$$

To obtain the connection components w^a_b, we use the first structure component,

$$\begin{aligned} & de^a + w^a_b \wedge e^b = 0 \\ a = r: \quad & 0 = w^r_b \wedge e^b \Rightarrow w^r_\theta = 0 \\ a = \theta: \quad & 0 = \frac{1}{r^2} de^r \wedge de^\theta + w^\theta_b \wedge e^b = \frac{1}{r^2} de^r \wedge de^\theta + w^\theta_r \wedge e^r \Rightarrow w^\theta_r = \frac{1}{r^2} e^\theta. \end{aligned}$$

Thus $w^r_\theta = w^r_r = w^\theta_\theta = 0$ and $w^\theta_r = \frac{\sin\theta}{r}\theta$. The second structure equation gives us a formula for the Ricci tensor, $R^a_b = dw^a_b + w^a_c w^c_b$. Hence

$$\begin{aligned} R^\theta_\theta &= R^r_\theta = R^r_r = 0 \\ R^\theta_r &= dw^\theta_r = d\left(\frac{\sin\theta}{r} d\theta\right) = \partial_\mu \left(\frac{\sin\theta}{r}\right) dx^\mu \wedge d\theta = -\left(\frac{\sin\theta}{r^2}\right) dr \wedge d\theta. \end{aligned}$$

Thus $R^\theta_r = \frac{1}{r^4} d\theta \wedge dr$. From $R^a_b = R^a_{bcd} e^c \wedge e^d$ we see that $R^a_{b\theta r} = \frac{1}{r^4} e^\theta \wedge e^r \wedge e^c \wedge e^d$, and one may use

$$R^\lambda_{\mu\nu\sigma} = (e^{-1})^\lambda_a R^a_{bcd} e^b_\mu e^c_\nu e^d_\sigma$$

to convert back to an orthonormal basis, and find for example that $R^\theta_{r\theta r} = \sin^2\theta$ as expected for a sphere.

1.5.5 Solutions to Einstein equations

1.5.5.1 The Schwartzschild Solution

First we define that a spacetime is stationary if it admits a Killing vector field k^μ that is everywhere timelike (i.e. $g_{\mu\nu}k^\mu k^\nu < 0$ for $g_{00} < 0$). In particular if we define a hypersurface Σ and assign coordinates (t, x^i) to give a coordinate chart, then $k^\mu = (\partial/\partial t)^\mu$ will always point in the time direction and hence is timelike and the spacetime will be static. Similarly the space-like killing vector that generates space-time isometries (i.e. $\phi \to \phi + 2\pi$) is $\tilde{k}^\mu = (\partial/\partial\phi)^\mu$; spacetimes that admit such solutions are called axisymmetric.

The spherically symmetric solution of the vacuum Einstein equation gives a stationary and uncharged blackhole as described by the Schwartzschild metric (or a solution which is isometric to the Schwartzschild solution as proven by Hawking and Ellis and which theorem goes by the name of the Birkhoff theorem):

$$ds^2 = -\left(1 - \frac{2GM}{R}\right)dt^2 + \left(1 - \frac{2GM}{R}\right)^{-1} dR^2 + R^2 d\Omega^2. \tag{1.18}$$

The solution has a further isometry - the hypersurface-orthogonal $\partial/\partial t$ which is timelike for $r > 2M$ implying a static universe for this region. Furthermore $r = 2M$ constitutes an event horizon where nothing passing into the black hole will ever be able to escape, including light. The metric's t- and ϕ- independence tell us that the relativistic energy per unit mass at infinity, as well as the relativistic angular momentum per unit mass are conserved. This has an Innermost Stable Circular Orbit (ISCO) at $R = 6GMR$.

1.5.5.2 The weak field solution

Since $T^{\mu\nu}$ is a symmetric 4×4 tensor and Einstein's equations are given by $G^{\mu\nu} = 8\pi G T^{\mu\nu}$, we see that energy conservation, $\nabla_\mu T^{\mu\nu} = 0 \iff \nabla_\mu G^{\mu\nu} = 0$, gives 10 constraints. However since this equation still holds if we perform an arbitrary coordinate transformation (4 equations), we actually only have 6 constraints. In a Local Inertial Frame (LIF) (locally flat so that if we consider $g'_{\mu\nu} = \frac{\partial x^\alpha \partial x^\beta}{\partial x'^\mu \partial x'^\nu} g_{\alpha\beta} = \eta_{\mu\nu}$ at P and where $\frac{\partial x^\alpha}{\partial x'^\mu} = a^\alpha_{\ \mu} + b^\alpha_{\ \mu\nu}\delta x'^\nu + c^\alpha_{\ \mu\nu\beta}\delta x'^\nu \delta x'^\beta = a^\alpha_{\ \mu}$ at P, and hence to first order

$$\left[g'_{\mu\nu}\right]_P = a^{\alpha}_{\ \mu} a^{\beta}_{\ \nu} \left[g_{\alpha\beta}\right]_P = \eta_{\mu\nu}$$

which involves 10 independent conditions on 16 components $a^{\alpha}_{\ \mu}$ so that the 6 leftover degrees of freedom may be interpreted as 3 rotations and 3 boosts giving a Local Observer Frame (LOF). Using similar counting arguments to second and third order ones finds one has sufficient degrees of freedom to set first derivatives to zero (which is the key result for LIFs) but not second derivatives. Thus in this case the Riemann tensor in equation (1.15) becomes

$$\begin{aligned} R^d_{abc} &\rightarrow \partial_b \Gamma^d_{ca} - \partial_c \Gamma^d_{ba} \\ &= \partial_b \left[\frac{1}{2} g^{de} \left(-\partial_e g_{ca} + \partial_c g_{ea} + \partial_a g_{be}\right)\right] - \partial_c \left[\frac{1}{2} g^{de} \left(-\partial_e g_{ba} + \partial_b g_{ea} + \partial_a g_{be}\right)\right] \end{aligned}$$

$$\begin{aligned} \Rightarrow R_{dabc} &= \frac{1}{2}\left[-\partial_b\partial_d g_{ca} + \partial_b\partial_c g_{da} + \partial_b\partial_a g_{bd} + \partial_c\partial_d g_{ba} - \partial_c\partial_b g_{da} - \partial_c\partial_a g_{bd}\right] \\ &= \frac{1}{2}\left[\partial_b\partial_a g_{bd} + \partial_c\partial_d g_{ba} - \partial_b\partial_d g_{ca} - \partial_c\partial_a g_{bd}\right]. \end{aligned}$$

For the weak field solution we set $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for $|h_{\mu\nu}| << 1$, in which case

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \frac{1}{2}\left(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}\right) \\ \Rightarrow R_{\beta\nu} &= \frac{1}{2}\eta^{\alpha\mu}\left(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\beta\partial_\nu h_{\alpha\mu} - \partial_\alpha\partial_\mu h_{\beta\nu}\right) \\ &= \frac{1}{2}\left(\partial_\beta H_\nu + \partial_\nu H_\beta - \eta^{\alpha\mu}\partial_\alpha\partial_\mu h_{\beta\nu}\right) \end{aligned}$$

where $H_\nu = \eta^{\alpha\mu}\left(\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial_\nu h_{\alpha\mu}\right)$ which we can set to 0 to get back to 10 constraints, and the Ricci tensor reduces to

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right)$$

since both $R_{\mu\nu}$ and $T_{\mu\nu}$ are of order $h_{\mu\nu}$. For a slowly rotating (about the z-axis), time-independent object, we obtain (in polar coordinates $x = R\cos\theta\sin\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$)

$$ds^2 = -\left(1 - \frac{2GM}{R}\right)dt^2 + \left(1 + \frac{2GM}{R}\right)\left(dx^2 + dy^2 + dz^2\right) - \frac{4GS}{R^3}(xdy - ydx)dt$$

$$\begin{aligned} &= -\left(1-\frac{2GM}{R}\right)dt^2+\left(1+\frac{2GM}{R}\right)\left(dR^2+R^2d\theta^2+R^2\sin^2\theta d\phi^2\right) \\ &\quad -\frac{4GS}{R}\sin^2\theta d\phi dt, \end{aligned}$$

where S is the angular momentum which we may write as $S = Ma$. This reduces to the flat solution as $R \to \infty$.

1.5.5.3 The Kerr Solution

The solution for a spinning black hole turns out to be the Kerr solution (for example see figure 1.4), for which

$$\begin{aligned} ds^2 &= -\left(1-\frac{2GMR}{R^2+a^2\cos^2\theta}\right)dt^2+\left(\frac{R^2+a^2\cos^2\theta}{R^2-2GMR+a^2}\right)dR^2 \\ &\quad +\left(R^2+a^2\cos^2\theta\right)d\theta^2+\left(R^2+a^2+\frac{2GMRa^2\sin^2\theta}{R^2+a^2\cos^2\theta}\right)\sin^2\theta d\phi^2 \\ &\quad -\frac{4GMRa\sin^2\theta}{R^2+a^2\cos^2\theta}d\phi dt. \end{aligned}$$

Clearly, for a non-spinning black hole (i.e. $a = 0$), this reduces to the Schwartzschild solution. Furthermore, $R \to \left(1+\frac{2GM}{R}\right)^{1/2} R$ gives the weak field solution to first order in $\frac{GM}{R}$. The event horizon is found at $R = \left(1+\sqrt{1-a^2}\right)GM$ which reduces to the Schwartschild case when $a = 0$ and to $R = GM$ for a maximally spinning black hole where $a = 1$. If $a > 1$ in 'superspinners', there is a naked singularity which the cosmic censorship hypothesis forbids. Avoiding this situation involves defining the 'irreducible mass' of a black hole to be $M_{ir} = \frac{\sqrt{2GMR_+}}{2G}$ where $R_+ = 2GM$, implying that the area of the event horizon is given by $A = 4\pi\left(2GM_{ir}\right)^2$. Thus a statement that mass can't decrease is equivalent to one that area can't decrease, and indeed that entropy can't decrease. This is related to the 'information paradox' (since pure states evolve into thermal or mixed states), an area where Maldacena has recently made quite some progress, which we shall discuss later. The Innermost Stable Circular Orbit (ISCO) for a Kerr black hole is found by solving

$$R^2 - 6GMR - 3a^2 \pm 8a\sqrt{GMR} = 0$$

where the $+$ is for orbits in the same direction as rotation, and the $-$ is for opposite direction. When $a = 0$ this reduces to the Schwartzschild case where $R = 6GM$, and for extremal black holes where $a - 1$ there are two solutions, $R = GM$ and $R = 9GM$. Note that the 'No-hair theorem' states that any isolated black hole is

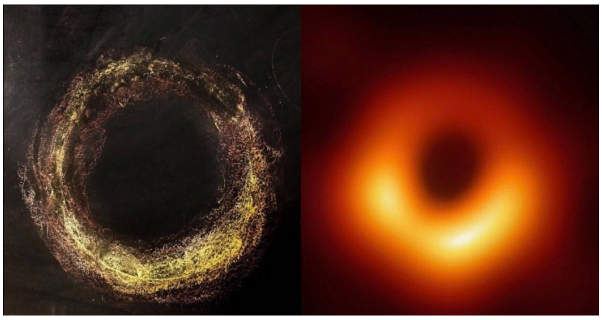


FIGURE 1.4: First photograph taken of a black hole, or rather of the accretion disc surrounding it, on 10 April 2019[11, 12, 13, 14, 15, 16]; the results are consistent with a spinning uncharged Kerr black hole with spinning parameter $a = 0.9 \pm 0.1$[17]. This black hole is located in the elliptical galaxy Messier 87 located 50 million light years away, and the photograph was taken by the Event Horizon Telescope, a collaboration of 8 Earth-based telescopes. (Image: © EHT Collaboration)

described only in terms of its mass, spin and electrical charge.

1.5.5.4 The Kerr-Newman Solution

The Kerr-Newman solution gives a rotating charged solution of the Einstein-Maxwell theory and is given by

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - 2a\sin^2\theta \frac{r^2 + a^2 - \Delta}{\Sigma} dt d\phi \\ &+ \frac{\left(R^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dR^2 + \Sigma d\theta^2 \end{aligned}$$

where $\Sigma = R^2 + a^2\cos^2\theta$ and $\Delta = R^2 - 2MR + a^2 + P^2 + Q^2$. When $a = 0$ this gives a Reissner-Nördstrom black hole, that is, a stationary and charged black hole.

1.5.6 Penrose Diagrams

Consider the flat metric

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) = -dt^2 + dr^2 + r^2 d\Omega^2_{(2)}$$

where $t \in (-\infty, \infty)$, $r \in [0, \infty)$, $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$. Consider the following change of coordinates:

$$\begin{aligned} \bar{t} &= \arctan(t+r) + \arctan(t-r) \in [-\pi, \pi) \\ \bar{r} &= \arctan(t+r) - \arctan(t-r) \in [0, \pi) \\ \Rightarrow \bar{t} + \bar{r} &= 2\arctan(t+r) \in [-\pi, \pi) \\ \bar{t} - \bar{r} &= 2\arctan(t-r) \in [-\pi, \pi) \end{aligned}$$

We have captured these boundary lines $\bar{t} = \pm\bar{r} \pm \pi$ in Figure 1.5, and drawn the

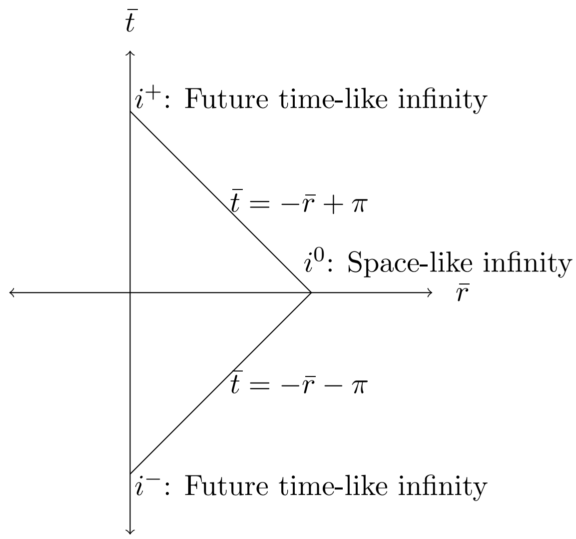


FIGURE 1.5: Penrose diagram for flat space.

Penrose diagram for a Schwartzschild black hole in Figure 1.6. Note that $|r| = |t| \iff |\bar{r}| = |\bar{t}|$ implying that light rays still follow 45° trajectories. Furthermore, from

$$\begin{aligned} t &= \frac{1}{2}\left(\tan(\bar{t}+\bar{r}) + \tan(\bar{t}-\bar{r})\right) \\ r &= \frac{1}{2}\left(\tan(\bar{t}+\bar{r}) - \tan(\bar{t}-\bar{r})\right) \end{aligned}$$

we see the following special points on the diagram:

- i^+, i^-: $(\bar{r} = 0, \bar{t} = \pm\pi) \iff (r = 0, t = \pm\infty) \iff$ time-like future / past infinity

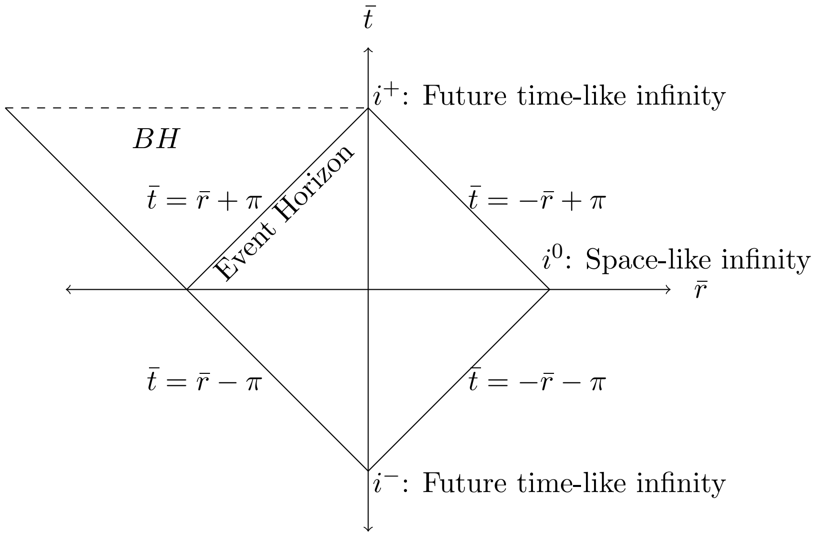


FIGURE 1.6: Penrose diagram for observer falling into a black hole.

- $i^0 : (\bar{r} = \pi, \bar{t} = 0) \iff (r = \pi, t = 0) \iff$ space-like infinity

1.5.7 Hawking radiation - QFT in a curved space time

Two observers may disagree on the number of particles they see; this may be understood from the following argument. The Killing vector satisfies $\left[k, \nabla^2\right] f = 0$. Since ∇^2 (where $\nabla^2 f = g^{\mu\nu}\nabla_\mu \partial_\nu f$) and $k = ik^\mu \partial_\mu$ are both self-adjoint, they admit a complete set of common eigenfunctions,

$$\begin{aligned} \nabla^2 f &= m^2 f \\ ik^\mu \partial_\mu f &= \nu f, \end{aligned}$$

where $ik^\mu \partial_\mu = i\partial_t$ is time-like and $f = e^{ik\dot{x}}$ explaining the frequency eigenvalue $\nu = k^0$; likewise $kf^* = -\nu f^*$. In a spacetime admitting a timelike Killing vector, we can mode expand our field. In particular for two observers with different timelike killing vectors, and the same field will admit two different mode expansions:

$$\varphi(U) = \int_0^\infty \frac{d\nu}{2\pi\sqrt{2\nu}} \left(a_\nu e^{-i\nu U} + a_\nu^\dagger e^{i\nu U}\right) = \int_0^\infty \frac{d\omega}{2\pi\sqrt{2\omega}} \left(b_\omega e^{-i\omega u} + b_\omega^\dagger e^{i\nu u}\right) = \varphi(u).$$

where $\left[a_\nu, a_{\nu'}^\dagger\right] = 2\pi\delta(\nu - \nu')$ and $\left[b_\omega, b_{\omega'}^\dagger\right] = 2\pi\delta(\omega - \omega')$. The first expansion $\varphi(U)$ we assign is for an in-falling observer near the horizon and the second $\varphi(u)$ for an observer very far away. Then

$$b_\omega = \int_0^\infty du\sqrt{2\omega}\varphi(u)e^{i\omega u}$$

We could expand these in terms of the $a, a^\dagger$ models and obtain

$$\begin{aligned} b_\omega &= \int_0^\infty d\omega\sqrt{2\omega}\int_0^\infty \frac{d\nu}{2\pi\sqrt{2\nu}}\left(a_\nu e^{-i\nu U} + a_\nu^\dagger e^{i\nu U}\right)e^{i\omega u} \\ &= \int_0^\infty \frac{d\nu}{2\pi}\left(\alpha_{\omega\nu}a_\nu + \beta_{\omega\nu}a_\nu^\dagger\right) \end{aligned}$$

where

$$\alpha_{\omega\nu} = \sqrt{\frac{2\omega}{2\nu}}e^{i\omega u - i\nu U} \;\; ; \;\; \beta_{\omega\nu} = \sqrt{\frac{2\omega}{2\nu}}e^{i\omega u + i\nu U}$$

are the Bogoliubov coefficients.

The observer near the event horizon will experience massive time dilation so that, since the quantum system changes slowly compared to the spacing between energy levels, by the adiabatic principle the probability of an excitation is exponentially small. That is, the lowest energy state is given by $a_\nu\left|\varphi\right\rangle = 0\;\forall\nu$, and therefore this observer sees no particles,

$$\left\langle\varphi\right| a_\nu^\dagger a_{\nu'}\left|\varphi\right\rangle = 0.$$

The distant observer, however,

$$\left\langle\varphi\right| b_\omega^\dagger b_{\omega'}\left|\varphi\right\rangle = \int_0^\infty \frac{d\nu}{2\pi}\int_0^\infty \frac{d\nu'}{2\pi}\beta_{\omega\nu}^*\beta_{\omega'\nu'}\left[a_\nu^\dagger, a_{\nu'}\right] = \int_0^\infty \frac{d\nu}{2\pi}\beta_{\omega\nu}^*\beta_{\omega'\nu'}.$$

An explicit calculation reveals this to be

$$\left\langle\varphi\right| b_\omega^\dagger b_{\omega'}\left|\varphi\right\rangle = \frac{2\pi\delta(\omega-\omega')}{e^{4\pi R_S\omega}-1}$$

where $R_S = 2GM = 2M$ is the Schwartzschild radius. Comparing this with the expression for a bosonic field blackbody,

$$\frac{2\pi\delta(\omega-\omega')}{e^{\hbar\omega/T_H}-1}$$

we see that this radiation can indeed be seen as blackbody radiation with Hawking temperature given by

$$T_H = \frac{\hbar}{4\pi R_S} = \frac{\hbar\kappa}{2\pi}$$

where $\kappa = \frac{1}{2R_S}[= \frac{c^4}{4GM}]$ is the surface gravity of the black hole. Amazingly, the surface gravity is equivalent to the temperature for a black hole. This observation naturally leads to the relationship between entropy and area as seen in the introduction. The mechanism behind this 'black hole evapouration' is the pair production that happens very near to the event horizon. Particles of positive energy are radiated out while particles of 'negative energy' enter the black hole. These particles are not really negative energy particles inside the event horizon, however, since the Killing vector goes from timelike to spacelike, and these should rather be interpreted as particles with negative momentum.

Another point of interest is that, since $T \propto \frac{1}{M}$ and the mass of a black hole is also its internal energy,

$$C = \frac{dU}{dT} = -\frac{1}{8\pi M^2} < 0$$

meaning that the black hole will be perpetually losing mass and gaining temperature. This, of course, is a quantum mechanical effect; classically the black hole is stable.

The Hawking temperature may also be deduced by considering regularity of Euclidean geometry. Allowing $t = -i\tau$ in the Schwartzschild metric, we obtain

$$ds^2 = \left(1 - \frac{R_S}{R}\right) d\tau^2 + \left(1 - \frac{R_S}{R}\right)^{-1} dr^2 + R^2 d\Omega^2.$$

which has signature

$$(+,+,+,+) \quad \text{for} \quad R > R_S$$
$$(-,-,+,+) \quad \text{for} \quad R < R_S.$$

We exclude the $R = 0$ region by asserting that the region of validity is just $R_S \leq R < \infty$. For $R >> R_S$, the space time is asymptotically flat, i.e.

$$ds^2 \sim d\tau^2 + dR^2 + R^2 d\Omega_2.$$

In particular, since Euclidean time $t = -i\tau$ necessitates periodic conditions to be placed on it, this metric will be $S_1 \bigotimes \mathbb{R}^3$, with 'cigar' topology (considering just one of the spatial dimensions.

When R gets closer to R_S, if we let $R = R_S + \frac{\rho^2}{4R_S} = R_S\left(1 + \frac{\rho^2}{4R_S}\right)$ where $\rho << R_S$, then

$$\begin{aligned} 1 - \frac{R_S}{R} &\approx \frac{\rho^2}{4R_S^2} \\ dR &= \frac{\rho}{2R_S} d\rho \end{aligned}$$

so that the conical singularity where $R = R_S$ is smoothed out; the metric becomes

$$ds^2 = \frac{\rho^2}{4R_S^2} d\tau^2 + d\rho^2 + R_S^2 d\Omega^2,$$

which is $\mathbb{R}^2 \bigotimes S^2_{R_S}$. As mentioned before, τ is necessarily periodic and we need to identify $\frac{\tau}{2R_S}$ with $\frac{\tau+\beta}{2R_S}$. The reason that this makes sense here is that, unlike when we set $c = \hbar = 1$ for which $[L] = [T] = [M]^{-1}$, here we set $G = c = 1$, for which $[L] = [T] = [M]$, and hence $[\beta] = [E] = [L] = [T]$. This gives us a periodic condition

$$\beta = 4\pi R_S = \frac{2\pi}{\kappa} = \frac{1}{T_H}.$$

1.5.8 Information Paradox

Hawking and Bekenstein's results, as shown in a previous section, convey that entropy is monotonically increasing as plotted on a 'Page curve'. This however, would result in a non-unitary process since pure states go into the black hole, but at the end of the life of the black hole there are just thermal / mixed states, representing a loss of information. Quantum effects are usually fairly small, so this result was surprising (discrepancy is large, of order $\mathcal{O}(1/G_N)$), and in Ref. [62] (for AdS_2; AdS is discussed in Section 1.11.1) it was argued that the formula for entropy being used (von Neumann - $S_{vN} = -\text{Tr}\rho_R\text{log}\rho_R$) required refining; instead the entropy for the black hole is given by that on the quantum-corrected Ryu-Takayanagi surface,

$$S_B = \text{ext}_Q \left[\frac{\text{Area}(Q)}{4G_N} + S_{\text{matter}}(B) \right]$$

where Q is the quantum extremal surface. The issue is understood to result from entanglement of quantum fields across the horizon, and the paradox lies in whether the calculation measures the Page curve of the black hole and or of the radiation. Ref. [62] further recommended glueing the right-hand-side of an AdS_2 black hole (that is of the Penrose diagram thereof) to the origin of an auxiliary Minkowski space (which effectively acts as a bath) - this allows the AdS_2 black hole to radiate. In Ref. [63], the idea of 'quantum extremal islands' emerged, regions in the gravitational theory side with matter entangled with the external quantum system. In Ref. [64], further machinery was developed for calculating the entropy of the union of the island with the Hawking radiation, and in Ref. [65], the battle was conquered with the use of saddles and replica wormholes, and a trick which works as follows. In Ref. [66] it was shown that the quantum Rényi entropy defined by

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{Tr} \left(\frac{\rho^\alpha}{\rho} \right), \quad \alpha \in (0,1) \cup (1,\infty)$$

(which indeed is monotonically increasing for increasing α) reduces to the von Neumann entropy in the limit as $\alpha \to 1$. This may be understood using L'Hospital's rule since when $\alpha = 1$, both the numerator and denominator equal zero.

$$\begin{aligned} \lim_{\alpha\to1} S_\alpha(\rho) &= \frac{1}{1-\alpha} \log \text{Tr} \left(\frac{\rho^\alpha}{\rho} \right), \quad \alpha \in (0,1) \cup (1,\infty) \\ &= -\sum \lambda_i \log \lambda_i \end{aligned}$$

$$= -\mathrm{Tr}\rho \log \rho$$
$$= S_{vN}.$$

Ultimately, including these islands was pivotal to the result.

1.6 't Hooft's large N expansion for matrix-valued fields

In 1973/1974, Gerard 't Hooft showed that the Lagrangian $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$, derived in the previous section for a non-Abelian Yang-Mills theory (in general a matrix theory with trace structure), gives rise to a topological expansion; this result will be derived here.

Consider the Feynman rules associated with this Lagrangian. The term $\mathcal{L}_2$ describes a propagator, the term $\mathcal{L}_3$ incorporating a factor g describes a cubic interaction, and the term $\mathcal{L}_4$ incorporating a factor g^2 describes a quartic interaction, where g is the coupling constant of the theory. A graph describing the interactions of this theory would then involve edges, 3-vertices and 4-vertices respectively.

Hence, the Feynman diagrams in this theory will be associated with the factor [24] (assuming there are no quark loops)

$$r = g^{V_3+2V_4} N^F, \tag{1.19}$$

where V_3 is the number of cubic vertices, V_4 is the number of quartic vertices and $\delta^i_i = N'$. A famous formula discovered first by Leonhard Euler describes a topological invariant for polyhedra. This topological invariant is called the Euler characteristic and is given by the formula

$$\chi(H) = F - P + V = 2 - 2H, \tag{1.20}$$

where F is the number of faces, V the number of vertices, and P the number of edges / 'propagators'. H is called the genus of the topological surface and counts the number of holes in the topology, for example a doughnut has one hole and so its topology, together with any other one-hole object, would be described by an $H = 1$ topology. A derivation of this formula may be found in Appendix A.

The total number of vertices is given by

$$V = V_3 + V_4$$

and the number of outgoing propagators from each vertex, given by $3V_3+4V_4$ double-counts the actual number of propagators so that

$$P = \frac{1}{2}\left(3V_3 + 4V_4\right).$$

Hence

$$P - V = \frac{1}{2}V_3 + V_4,$$

so that (1.20) gives

$$\begin{aligned} F &= P - V + 2 - 2H \\ &= \left(\frac{1}{2}V_3 + V_4\right) + (2 - 2H). \end{aligned}$$

Finally, substituting this in (1.19) gives

$$r = \left(g^2 N'\right)^{\frac{1}{2}V_3+V_4} N'^{2-2H}. \tag{1.21}$$

Hence, clearly this factor admits an infinite sum over topologies which one equates to the partition function $\mathcal{Z}$

$$\begin{aligned} log\mathcal{Z} &= \sum_{h=0}^{\infty} N^{2-2H} f_H\left(g_{YM}^2 N\right) \\ &= N^2\left(f_0(\lambda) + \frac{1}{N^2} f_1(\lambda) + \frac{1}{N'^4} f_2(\lambda) + ...\right), \end{aligned} \tag{1.22}$$

for which, in the limit where $N \to \infty$ and $\lambda = g_{YM}^2 N$ is constant (referred to as 't Hooft's coupling constant), all but the planar Feynman diagrams drop out giving rise to an unexpected simplification of the calculation [24]. This is a topological expansion which is the signature of a string theory (since a string moving in spacetime gives rise to a world sheet), with string coupling given by $g_s^2 = \frac{1}{N^2}$, or $g_s = \frac{1}{N'}$. An example of this is the correspondence between type IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4SYM$ referred to earlier. This is a topological expansion which is the signature of a string theory (since a string moving in spacetime gives rise to a world sheet); here it is observed that the string coupling is given by $g_s^2 = \frac{1}{N^2}$, or $g_s = \frac{1}{N}$. This was one of the first hints of a gauge/gravity duality: λ is of the order R^4, so when considering strong coupling where λ is large (and where non-perturbative methods are required), the curvature is small so that the supergravity side of the duality is realised. For small λ, the conformal field theory dominates.

1.7 String Theory

1.7.1 Quantum chromodynamics and strings

String theory was initially conceived in the 1960's as a mechanism to explain the great number of particles being discovered in terms of different oscillatory modes of a smaller subset of fundamental strings [67]; it was thought that these fundamental strings, with length of the order of the size of the nucleus, could explain the strong interaction. Veneziano developed a theory of dual models (which would now be called a 26-dimensional bosonic string theory) where scattering amplitudes were calculated using approximations to the Euler gamma and Euler beta functions and this theory was later shown to have a string interpretation. For example, Veneziano model's predicts that particles with greatest spin for a particular mass follow Regge trajectories given by the formula $\alpha' m^2 = J + constant$ [40], so that α' is the slope of these Regge (and daughter) trajectories (as seen in the Chew-Frautschi plot in Figure 1.7); in string theory, string rotations explain the resonances along the Regge trajectories while vibrational modes give the daughter trajectories; α' emerges as the inverse string tension. The agreement between string S-matrix scattering amplitudes and the results of meson scattering amplitudes of the time gave strong support to the theory.

However, this theory which predicted the high-energy, fixed scattering amplitude falling off exponentially with s (squared sum of incoming momenta) was published at about the same time as the publication of the first experimental evidence [68] of

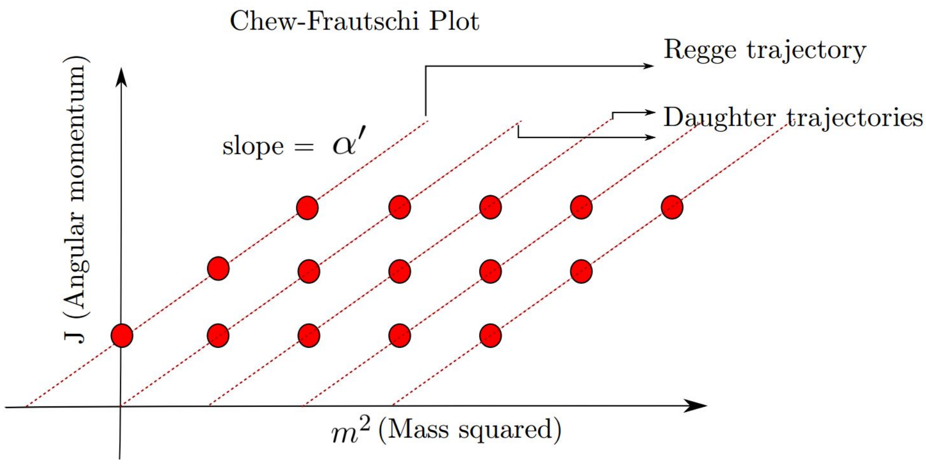


FIGURE 1.7: Chew-Frautschi plot showing Regge trajectories with Regge slope α'

the parton-like behaviour of the strong interaction (the scattering amplitude actually falls off according to a power law in these processes) [69]. There were also concerns over the large number of extra dimensions required for the theory to be consistent and the massless particles apart from spin-1 gluon, that the theory predicted (the open-string spectrum predicted massless vector particles and the closed-string spectrum predicted massless tensor particles [68]). Furthermore, a particle known as the tachyon, with its undesirable property of violating causality, resulted in infrared divergences in the loop diagrams in these 'consistent dual models'. By 1973/1974 it had been accepted that the strong interaction could be explained using an SU(3) colour gauge theory QCD (Quantum Chromodynamics), and this was backed up well by experiment, including the parton-like behaviour resulting from aymptotic freedom [70]. SU(3) colour gauge theory emerged as the successful theory of the strong interaction.

String theory, however, was found to have highly desirable features as a possible candidate for unification (the string length in this theory is then of the order the Planck length ℓ_P). For example, non-renormalisable amplitudes and ultraviolet divergences make it difficult for other quantum field theories to incorporate gravity, whereas the massless spin-2 particle (having a Regge intercept of 2) is understandable as a graviton that is present in any string theory. String theories naturally incorporate gravitons and massless vector particles. The low energy dynamics of the massless sector of closed (super)strings yields the Einstein-Hilbert action, which, when taking into account the variational principle, gives rise to the Einstein equation:

$$\begin{aligned}\mathcal{S}_{\mathrm{EH}} &= \tfrac{c^4}{16\pi G_{\mathrm{N}}}\int \mathrm{d}^4x\,\sqrt{-\det|g|}\,(R-2\Lambda)+\int \mathrm{d}^4x\,\sqrt{-\det|g|}\,\mathcal{L}_{\mathrm{M}}\\ \Rightarrow&\quad R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda\, g_{\mu\nu}=\frac{8\pi G_{\mathrm{N}}}{c^4}\,T_{\mu\nu}.\end{aligned}$$

In the 1980s, it was extablished that SO(32) type I superstrings are anomaly free[71]: 5 superstring theories are consistent, each of which are defined in 10 dimensions. These are Type I, non-chiral IIA and chiral IIB, and heterotic SO(32) and $E_8 \times E_8$; in compactifying from a 9+1 spacetime to a 3+1 spacetime, string dynamics require the resulting 6-dimensional manifolds are Calabi-Yau spaces which display similar features to the Standard Model [72]. The 1990's then gave rise to the discovery that all of the 5 10-dimensional superstring theories are perturbative expansions (consistent at every finite order) of one unique (although not well understood) underlying 11-dimensional theory. This 11-dimensional theory is referred to as M-theory, the low energy limit of which is 11-dimensional supergravity. Ref. [73, 74] gives a good overview of these milestones.

1.7.2 String theory and the Nambu-Goto and Polyakov action

This subsection is based on Refs. [21, 75, 76]. In contrast to how most field theories developed, where the second quantized theory can be completely derived from one action, String theory was initially developed as a first quantized theory, where things such as the vertices, interaction choice and weights of perturbation diagrams need to be put in by hand, and the unitarity explicitly checked. In first quantized relativistic point particle theories, scattering amplitudes are given by a sum over graphs of different topologies, not manifolds. Thus one can place an arbitrary number of different spins at interaction points corresponding to an infinite number of point particle theories. Furthermore, one can 'pinch' internal lines in the graph to zero which results in locally deformed topologies with ultraviolet singularities for each 'pinch'. In String theory, however, the string worldsheet cannot be 'pinched' to form an ultraviolet singularity from a topological perspective (although the pinched diagram could be understood as an infrared divergence involving the emission into the vacuum of massless spin-0 particles - but supersymmetry eliminates these infrared divergences). Furthermore, the worldsheet is a manifold / Riemannian surface and the selection of interactions consistent with the propagator are limited by conformal symmetry, supersymmetry and modular invariance, resulting in only a few allowable string theories. In this subsection we overview some of the fundamental theory of String theory.

Strings in space-time sweep out an area element, and this gives rise to the Nambu-Goto action,

$$S_{NG} = -T \int dA$$

where the string tension T is given by $T = \frac{1}{2\pi\alpha'}$ and α' is the Regge slope. Now this can be written in terms of the induced metric, $h_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$, as

$$S_{NB} = -T \int d\sigma\, d\tau \sqrt{-h}. \tag{1.23}$$

Note that $\mu, \nu = 0, 1, ..., D$ and $a, b = 0, 1$, and $X^\mu(\sigma, \tau)$ describes a one-dimensional object sweeping out a two-dimensional worldsheet in terms of two parameters and; this $X^\mu(\sigma, \tau)$ is then just a vector extending from the origin to a point on the two-dimensional manifold while the induced metric h_{ab} is the contraction of two tangent vectors of this two-dimensional manifold, where $-\infty < \tau < \infty$ is taken as a time parameter and $0 < \sigma < \pi$ as a spatial parameter. Here the Lagrangian simplifies to

$$\mathcal{L} = T\sqrt{\dot{X}_\mu^2 X'^{\mu 2} - \left(\dot{X}_\mu X'^\mu\right)^2} \tag{1.24}$$

where the dot is a τ-differentiation and the prime a σ-differentiation. The square root proves cumbersome, and it is simpler to work with the Polyakov action,

$$S_P = -\frac{T}{2} \int d\sigma\, d\tau \left(-\gamma\right)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu$$

where we have introduced an independent worldsheet metric $\gamma_{ab}(\tau, \sigma)$ with signature $(-, +)$. We identify this action as being that of massless Klein-Gordon scalars X^μ coupled to the metric γ_{ab}, and thus bosonic. The Nambu-Goto and Polyakov are indeed related, as can be shown since for any real symmetric matrix A we have that $4\det A \leq (\mathrm{tr} A)^2$,

$$4\det\left(\gamma^{ac} h_{cb}\right) \leq \left(\gamma^{ac} h_{ac}\right)^2$$

$$
\begin{aligned}
\Rightarrow \qquad & \sqrt{\det\left(-h_{cb}\right)} \leq \frac{1}{2}\sqrt{\det\left(-\gamma_{ac}\right)}\gamma^{ac}h_{ac} \\
\Rightarrow \qquad & S_{NG} \leq S_P
\end{aligned}
$$

and equality holds when $h_{ab} \propto \gamma_{ab}$. Varying the Polyakov action with respect to X^μ we obtain the following equations of motion

$$
\begin{aligned}
0 = \delta S_P &= \delta\left[-\frac{T}{2}\int d\sigma\, d\tau\, (-\gamma)^{1/2}\, \partial_a X^\mu \partial^a X_\mu\right] \\
&= -T\int d\sigma\, d\tau\, (-\gamma)^{1/2}\left[\partial_a\left(\delta X^\mu \partial^a X_\mu\right) - \delta X^\mu \nabla^2 X_\mu\right].
\end{aligned}
$$

The second term gives the equations of motion which is a wave equation, and we require the first term (the surface term) to vanish,

$$
-T\int d\tau\, (-\gamma)^{1/2}\left[\delta X^\mu \partial^\sigma X_\mu\right]_{\sigma=0}^{\sigma=\pi} = 0.
$$

This can happen in two ways. Firstly, it vanishes if $\partial^\sigma X_\mu$ vanishes at the spatial boundaries, implying there is no momentum flow at the ends; here no constraint is placed on X_μ - in other words this is an open string whose ends can move freely in spacetime. These are known as Neumann boundary conditions. The second way the surface term can vanish is if

$$
\begin{aligned}
& X^\mu\left(\tau, 0\right) = X^\mu\left(\tau, \pi\right) \\
& \partial^\sigma X^\mu\left(\tau, 0\right) = \partial^\sigma X^\mu\left(\tau, \pi\right) \\
& \gamma_{ab}\left(\tau, 0\right) = \gamma_{ab}\left(\tau, \pi\right),
\end{aligned}
$$

which can happen if the fields are periodic, and the endpoints are joined to form a loop - in other words closed strings.

In order to learn more about the open string spectrum, one can go to light cone coordinates and make conditions on the metric, namely

$$
\begin{aligned}
& X^+ = X^0 + X^1 = \tau \\
& \partial_\sigma \gamma_{\sigma\sigma} = 0 \\
& \det\gamma_{ab} = -1.
\end{aligned}
$$

Then, taking various boundary conditions into account, the Polyakov lagrangian becomes

$$\mathcal{L} = -\frac{\pi}{T}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{2T}\int_0^\pi d\sigma \left(\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i \partial_\sigma X^i\right),$$

and, after obtaining the momentum conjugate to x^-, the spacetime coordinate, and $X^i(\tau,\sigma)$, the worldsheet coordinate, one finds that the Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &= p_-\partial_\tau x^- - \mathcal{L} + \int_0^\pi d\sigma \Pi_i \partial_\tau X^i \\ &= \frac{\pi}{2Tp^+}\int_0^\pi d\sigma \left(T\Pi^i\Pi^i + \frac{1}{T}\partial_\sigma X^i \partial_\sigma X^i\right).\end{aligned}$$

The equations of motion that arise from this Hamiltonian result in the general solution to a wave equation, given in terms of centre-of-mass variables by

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i\left(\frac{T}{\pi}\right)^{1/2}\sum_{n=-\infty n\neq 0}^{\infty}\frac{1}{n}\alpha_n^i \exp\left(-\frac{\pi i n c \tau}{\ell}\right)\cos\frac{\pi n \sigma}{\ell}.$$

Imposing equal time canonical commutation relations from which a general state is obtained, one arrives at an Hamiltonian in terms of raising and lowering operators,

$$H = \frac{p^i p^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i \alpha_n^i + A\right).$$

The conformal gauge approach, which makes the covariance explicit, gives the value of A to be (correctly regularised)

$$\begin{aligned}A &= \frac{D-2}{2}\sum_{n=1}^{\infty} n \\ &= \frac{2-D}{24}.\end{aligned}$$

[Here we used that $\sum_{n=1}^{\infty} n = -\frac{1}{12}$; one way of seeing this is by introducing a regulator ϵ and writing

$$\sum_{n=1}^{\infty} n \;\rightarrow\; \sum_{n=1}^{\infty} n e^{-n\epsilon} = \sum_{n=1}^{\infty} -\frac{d}{d\epsilon}e^{-n\epsilon} = -\frac{d}{d\epsilon}\frac{1}{1-e^{-\epsilon}} = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2}.$$

and, using Mathematica, this has an expansion $\frac{1}{x^2} - \frac{1}{12} + \frac{x^2}{240} - \frac{x^4}{6048}...$, for which the only regulator-independent term is $-\frac{1}{12}$.] This cutoff-independent term is a Casimir energy due to the string having finite length. The mass can now be expressed in terms of the level (N) as

$$m^2 = \frac{1}{\alpha'}\left(N + \frac{2-D}{24}\right).$$

From this we notice two of the interesting aspects of bosonic string theory. Firstly for $D > 0$ the square of the mass is less than zero, and the state is tachyonic. Recalling that the potential energy for, say, a scalar field, is $\frac{1}{2}m^2\phi^2$, we see that this vacuum is unstable. With greater study, however, one finds that there are tachyon-free string theories. The second thing to notice is that the first excited state is given by

$$m^2 = \frac{1}{\alpha'}\left(\frac{26-D}{24}\right),$$

and this state must be massless, hence D=26 giving rise to the well-known statement about bosonic string theory that it is only consistent, or Lorentz-invariant, in 26 dimensions. A further item of interest is that the allowed eigenstates of the Hamiltonian are products of the Fock spaces of harmonic oscillators,

$$\prod_{n,i} \alpha^{\dagger}_{n,\mu} \left|0\right\rangle,$$

where the vacuum is given by

$$a_{n\mu} \left|0\right\rangle = 0,$$

for $n \geq 0$ and the low lying states are catalogued [77]:

$$\begin{aligned}
\left|0\right\rangle &\equiv \text{Tachyon} \\
a_1^{\dagger\mu} \left|0\right\rangle &\equiv \text{Massless vector} \\
k_\mu a_1^{\dagger\mu} \left|0\right\rangle &\equiv \text{Massless scalar} \\
a_1^{\dagger\mu} a_1^{\dagger\nu} \left|0\right\rangle &\equiv \text{Massive spin} - 2 \\
a_1^{\dagger\mu} \left|0\right\rangle &\equiv \text{Massless.}
\end{aligned}$$

The following involves supersymmetry, which will only be discussed later in section 1.9, but is nevertheless useful to mention here. The tachyons in the spectrum are an instability, but they can be fixed with supersymmetry. Also, string perturbation theory does not converge, and there exist non-perturbative objects such as instantons in Yang-Mills of mass $1/g_{YM}^2$ and D-branes in string theory, also of mass $1/g_{YM}^2 \sim 1/g_{str}$.

The supersymmetrised Polyakov action (which now involves fermions too) is given by

$$S_P \;=\; -\frac{T}{2}\int d^2\sigma\eta^{ab}\left[\partial_a X^\mu \partial_b X^\nu + i\bar{\psi}^\mu\gamma_a\partial_b\psi^\nu\right]g_{\mu\nu}(x),$$

which is explicitly invariant under infinitesimal supersymmetric transformations $\delta_\epsilon X^\mu = \bar{\epsilon}\psi$ and $\delta_\epsilon\psi^\mu = \gamma^a\partial_a X^\mu\epsilon$ (normally we just use $g_{\mu\nu}(x) = \eta_{\mu\nu}$). The resulting equations of motion, $\partial_-\chi_L = \partial_+\chi_R = 0$, give rise to left- and right- moving solutions,

$$\chi_L = \chi_L(\sigma+\tau); \qquad \chi_R = \chi_R(\sigma-\tau).$$

Varying the action, we find,

$$\delta S = \frac{T}{2}\int d\tau\left[\chi_L^\mu\delta\chi_{L\mu} - \chi_R^\mu\delta\chi_{R\mu}\right]_{\sigma=-\ell/2}^{\sigma=\ell/2}$$

for which one requires that $\chi_L^\mu(\tau,-\ell/2) = \pm\chi_R^\mu(\tau,-\ell/2)$ and $\chi_L^\mu(\tau,\ell/2) = \pm\chi_R^\mu(\tau,\ell/2)$, which leads to two types boundary conditions, Ramond for which $\chi_L^\mu(\tau,\ell/2) = \chi_R^\mu(\tau,\ell/2)$, with expansion

$$\chi_L^\mu(\tau,\ell/2) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma},$$

and Neveu-Schwartz for which $\chi_L^\mu(\tau,\ell/2) = -\chi_R^\mu(\tau,\ell/2)$, with expansion

$$\chi_L^\mu(\tau,\ell/2) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma}.$$

These satisfy the relations $\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m,-n}$ and $\{b_m^\mu, b_n^\nu\} = \eta^{\mu\nu}\delta_{m,-n}$ respectively. We observe that $b_r^\mu\left|0\right\rangle_{NS} = 0,\ \forall r > 0$ with $b_r^\mu,\ r < 0$ being creation operators. We similarly have that $d_m^\mu\left|0\right\rangle_R = 0,\ \forall m > 0$ and that d_m^μ for $m < 0$ are creation operators. However, in this Ramond case, $\{d_m^\mu, d_0^\mu\} = 0,\ \forall m > 0$, and thus the ground state is degenerate; d_0^μ in fact takes one ground state to another ground state. For the Neveu-Schwartz sector, the state $b^i_{-1/2}\left|0\right\rangle_{NS} = 0$ has mass-squared of

$$m^2 = \frac{1}{\alpha'}\left(\frac{1}{2} - \frac{D-2}{16}\right),$$

for which there are massless modes in $D = 10$ dimensions, giving the well-known result that supersymmetric string theory is only consistent in $D = 10$ dimensions (synonymous with that of bosonic string theory only being consistent in $D = 26$ dimensions). States here transform as a vector in $SO(8)$. Furthermore, the Neveu-Schwartz ground state is tachyonic with $m^2 = -\frac{1}{2\alpha'}$, but this is solved by introducing the GSO projection (named after Gliozzi, Scherk and Olive),

$$e^{i\pi F} = (-1)^F,$$

which counts the number of fermions and keeps only states with an odd number of creation operators applied to $\left|0\right\rangle_{NS}$, which we won't elaborate on in great detail here. Since there are both left- and right- moving states, written schematically as (L, R), there are 4 possibilities for boundary conditions giving rise to the Ramond-Ramond sector or (R, R) sector, the (R, NS) sector, the (NS, R) sector and the (NS, NS) sector (we will see these later in section 1.10). We further have possibilities $R\pm$ and $NS\pm$ depending on the sign of the GSO projection. In Type IIB string theory, only L and R fermions with the same 'space-time chirality' are allowed, whereas in Type IIA this is not the case. The allowed possibilities are given by

$$\begin{aligned}
\text{Type IIA}: &\quad (NS+, NS+), (R+, NS+), (NS+, R-), (R+, R-)\\
\text{Type IIB}: &\quad (NS+, NS+), (R+, NS+), (NS+, R+), (R+, R+).
\end{aligned}$$

For Type IIA, we have the followed possibilities

$$\begin{aligned}
&\text{Potential}: C_1, C_3, C_5, C_7, C_9\\
&\text{Field strengths}: F_2, F_4, F_6, F_8, F_{10}\\
&\text{Sources}: D0, D2, D4, D6, D8
\end{aligned}$$

and for Type IIB,

$$\text{Potential}: C_0 C_2, C_4, C_6, C_8$$

$$\text{Field strengths}: F_1, F_3, F_5, F_7, F_9$$
$$\text{Sources}: D(-1), D(1), D(3), D(5), D(7).$$

Here Hodge dualities relate, for example, F_2 and F_8 (indeed, wherever the number of dimensions adds up to 10; we saw some details about the Hodge star product in subsection 1.5.1).

1.8 No-Go Theorems

Presented below are no-go theorems that need to be taken into account when constructing higher spin theories. These no-go results concern flat space-time theories and an S-matrix, however, and the concluding purpose here is therefore that one should work in a curved space-time background where there is no S-matrix.

1.8.1 Coleman-Mandula Theorem

In Ref. [78], Sydney Coleman and Jeffrey Mandula proved a theorem describing the impossibility of combining internal symmetries and space-time symmetries in any way but the trivial way. It was general in that it was applicable to infinite-parameter groups, and it assumed an S-matrix. Unitary operators which commute with the S-matrix turn one-particle states into one-particle states, and transform many-particle states as tensor products, where G is the symmetry group of the S matrix that was investigated. Under 5 general assumptions it was shown that this (connected) group G needed to be locally isomorphic to the direct product of the Poincaré group and an internal symmetry, in other words the trivial combination. Otherwise put, due to the fact that theories we observe contain a mass gap, apart from the generators of the Poincaré group, the only conserved quantities, must be Lorentz scalars.

1.8.2 Weinberg

Another theorem (see Ref. [79] for a nice summary) in which general properties of the S-matrix were invoked was due to Steven Weinberg in 1964. Here he considered

N particles with 4-momentum p_i emitting a massless spin-s particle of momentum q and polarisation vector $\epsilon^{m_1 m_2 \ldots m_s}(q)$. In the soft limit, i.e. where the momentum of the emitted particle goes to 0 and the scattering is elastic, the S-matrix could factorise,

$$S(p_1, ..., p_N, q, \epsilon) \approx \sum_{i=1}^{N} g^i \frac{p^i_{m_1} p^i_{m_2} ... p^i_{m_s} \epsilon^{m_1 \ldots m_s}(q)}{2p^i q} S(p_1 ... p_N).$$

The polarisation vector, being transverse and traceless, has only two physical polarisations, so there are redundancies which can be removed by imposing

$$\sum_{i=1}^{N} g_i^{(s)} p^i_{m_1} ... p^i_{m_{s-1}} = 0 \; \forall p_i.$$

Thus

$$\begin{aligned}
s = 1 \quad &\Rightarrow \quad \sum_{i=1}^{N} g_i^{(s)} = 0 \Rightarrow \text{Conservation of charge} \\
s = 2 \quad &\Rightarrow \quad \kappa \sum_{i=1}^{N} p^i_m = 0 \Rightarrow \text{Conservation of energy/momentum} \\
& \qquad g^i = \kappa \Rightarrow \text{Equal interaction with massless spin} - 2 \text{ particle} \\
s > 2 \quad &\Rightarrow \quad \text{No solution for generic momenta.}
\end{aligned}$$

This argument showed that the only particles able to produce long-range interactions are scalars, vectors and one spin-2 particle (otherwise the 2 different spin-2 particles would interact with all matter with different couplings g and g' say).

1.8.3 Weinberg-Witten Theorem

In Ref. [80], Steven Weinberg and Edward Witten proved two theorems, the first one stating that a theory which allows a Lorentz-covariant conserved four-vector current J^μ to be constructed can't contain massless particles of spin $j > \frac{1}{2}$ since in these cases the conserved charge $\int J^0 d^3x$ will vanish. The second theorem states that a theory which allows a Lorentz-covariant conserved energy-momentum tensor $\theta^{\mu\nu}$ to be constructed for which $\int \theta^{0\nu} d^3x$ is the energy-momentum 4-vector can't contain massless particles of spin $j > 1$. To do this they studied matrix elements describing scattering of one-massless-particle states of helicity $\pm j$ and four-momenta p' and p off a soft graviton [81]. They took the limit as $p \to p'$ (i.e. nearly forward scattering) and not $p' - p = 0$ and thus the graviton described was off-shell and spacelike, nearly light-like, whereas $p + p'$ is timelike, and from Lorentz invariance the limit is given by

$$\langle p', \pm j \,| J^{\mu} |\, p, \pm j \rangle \to \frac{g p^{\mu}}{E(2\pi)^3}$$
$$\langle p', \pm j \,| \theta^{\mu\nu} |\, p, \pm j \rangle \to \frac{f p^{\mu} p^{\nu}}{E(2\pi)^3}$$

Because the calculation of matrix elements involves the rotation matrix $R(\theta)$ which has Fourier components $e^{i\theta}$, $e^{-i\theta}$ and 1, the matrix elements vanish unless $2j = 0, 1$, implying that $g = 0$ for $f > \frac{1}{2}$ and $f = 0$ for $j > 1$, thus concluding the proof.

This theorem is applicable to all known renormalisable quantum field theories, for example in quantum chromodynamics, since one may construct Lorentz covariant energy-momentum tensor and Noether current for all symmetries commuting with local symmetries, it may be concluded that quantum chromodynamics has no massless bound states with $j > 1$ and no flavour non-singlet massless bound states with $j > \frac{1}{2}$. Importantly, this theorem does not apply in theories such as Yang-Mills theories and General Relativity where, under Lorentz transformation $x^{\mu} \to \Lambda^{\mu}_{\nu} x^{\nu}$ fields transform as a Lorentz transformation plus a term proportional to a derivative, for example $\partial^{\mu}\Phi$.

1.8.4 Solution around no-go theorems

The following is based on Ref. [82]. Since the notion of massless particles is only unequivocal in theories with Poincaré-invariant vacua, the previously mentioned no-go theorems relate specifically to theories on a flat space-time background. In constantly curved non-flat space-times, one has that the mass operator (∇^2) is related to the eigenvalues of the quadratic Casimir operators of the space-time isometry and Lorentz algebras and, in Minkowski space-time for example, one may interpret massless higher-spin particles as limits of their respective massive particles. These particles are consistent at low energies and observed experimentally in hadronic physics as unstable resonances as opposed to fundamental particles.

The discussions regarding the S-matrix no-go theorems incorporate couplings that involve the same number of derivatives as the number of spins, for example two-derivative couplings between the graviton and other fields. If spin s couplings contain more than s derivatives, this S matrix argument needs re-examination. Indeed a more suitable starting point is using purely Lagrangian arguments. Indeed at the classical level, all of the consistent cubic vertices found in Minkowski and (A)dS space-times have exhibited higher-order derivatives.

The gauge principle states that, "A sensible perturbation theory requires compatibility between the interactions and some deformed version of the abelian gauge symmetries of the free limit". Massless particles are seen as representations of the space-time isometry group - different Lorentz tensors obeying their respective free equations of motion. A subset of these "carriers" are the primary curvature tensors and their derivatives which transform tensorially under isometry, and the construction of interaction Hamiltonians from Lorentz- and hence gauge- invariant non-linear Lagrangians is the guiding principle behind what is known as the "Fronsdal Programme". Ultimately, it was Mikhail Vasiliev and Efim Fradkin who conceived of the principal escape route where the cosmological constant play the role of both the infrared and ultraviolet regulator.

1.9 Supersymmetry

Here we highlight key points regarding Supersymmetry following Ref. [83]. Space-time symmetries may be external, for example continuous symmetries like those of the Poincaré group and the discrete parity P, charge conjugation C and time reversal T, or internal, such as the global isospin and flavour symmetries and the local $U(1)$, $SU(2)$ and $SU(3)$ of electromagnetism, the weak interaction and the strong interaction respectively. Internal symmetry generators R^a obey

$$\left[R^a, R^b\right] = 2if^{abc}R^c$$

and commute with the Casimirs of the Poincaré group, $[R^a, P^2] = [R^a, W^2] = 0$ implying that particles related by the internal symmetry have same mass and spin. In spite of the fact that Sidney Coleman and Jeffrey Mandula showed that one cannot mix internal and external symmetries in their no-go theorem of 1967, the pre-quark era showed for example that spin-$\frac{1}{2}$ baryons are in the **8** and spin-$\frac{3}{2}$ are in the **10** representation of flavour $SU(3)$. In other words, there is a link between the external symmetry spin and the internal symmetry of the $SU(3)$ representation. The Loophole in the Coleman-Mandula 'no-go' theorem was that symmetries were assumed to be Lie algebraic - so one could bypass it by considering spinorial symmetries having half-integer spin which are described by (non-Lie algebraic) fermionic / anticommutating algebras. This led to a new fermionic symmetry call Supersymmetry.

Supersymmetry involves extending the Poincaré group by adding anticommuting spin-$\frac{1}{2}$ operators, meaning that the 4-component Dirac spinor may be written in terms of a Weyl spinor and its conjugate

$$Q_D = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}$$

where the generators Q_D are called supercharges. One wishes, then to add these generators in such a way as to obtain a consistent algebra. (Having more supercharges is called an extended supersymmetry, e.g. $\mathcal{N} = 1 \rightarrow \mathcal{N} = 2, \mathcal{N} = 4$ etc.)

1.9.1 $\mathcal{N} = 1$ SUSY

According to the O'Raifeartaigh's theorem which predates Coleman-Mandula, translations commute with all generators beyond the Lorentz group generators,

$$[P^\mu, Q_\alpha] = \left[P_\mu, \bar{Q}_{\dot{\alpha}}\right] = 0.$$

Further, the Qs transform like spin-$\frac{1}{2}$ states when commuting with $\mathcal{J}_{\mu\nu}$ so that

$$\begin{aligned} [Q_\alpha, \mathcal{J}_{\mu\nu}] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \quad (\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}\beta} - \sigma^\nu_{\alpha\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta}\right) \\ \left[\bar{Q}_{\dot{\alpha}}, \mathcal{J}_{\mu\nu}\right] &= i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} Q_{\dot{\beta}}, \quad (\sigma^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} = \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\gamma}\bar{\sigma}^\nu_{\gamma\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\gamma}\bar{\sigma}^\mu_{\gamma\dot{\beta}}\right) \end{aligned}$$

(recall in the Wey representation these appeared as $\Sigma_{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}$). As per the discussion above the Qs are fermionic and hence relations involve fermionic anticommutators.

For $\mathcal{N} = 1$ (1 supercharge), since the only Lorentz-invariant object that is symmetric in α and β is $\epsilon_{\alpha\beta}$ whereas $\{Q_\alpha, Q_\beta\}$ is symmetric, we have that

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0.$$

For the relation between Q_α, an object in the $(\frac{1}{2}, 0)$ representation and $\bar{Q}_\alpha$ an object in the $(0, \frac{1}{2})$ representation, we recognise that the product should be in the $(\frac{1}{2}, \frac{1}{2})$ representation which is schematically a spacetime vector for which there is only on possibility - P_μ; matching indices gives

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2P_{\alpha\dot{\alpha}}.$$

We further have an R-symmetry which is a $U(1)$ phase rotation,

$$Q_\alpha \to e^{-i\alpha} Q_\alpha \qquad \bar{Q}_{\dot{\alpha}} \to e^{i\alpha} \bar{Q}_{\dot{\alpha}}. \tag{1.25}$$

It is an internal symmetry with respect to the Poincaré group, so that $[R, P_\mu] = [R, \mathcal{J}_{\mu\nu}] = [R, R] = 0$, however it is an external symmetry of the full super-Poincaré group, and charges the Qs in a way that differentiates between left- and right- handed charges,

$$[Q_\alpha, R] = Q_\alpha \qquad [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}}. \tag{1.26}$$

The full super-Poincaré algebra is called a graded Lie algebra since it contains both bosonic and fermionic commutators; the set of generators comprises $\{P_\mu, \mathcal{J}_{\rho\sigma}, Q_\alpha, \bar{Q}_{\dot{\alpha}}, R\}$ which are all bosonic except for Q_α and $\bar{Q}_{\dot{\alpha}}$. This algebra is not arbitrary since the graded Jacobi identities they have to satisfy are very strong consistency relations. They look as follows (where b are bosonic elements and f are fermionic elements):

$$\begin{aligned}
&[b_1, [b_2, b_3]] + [b_2, [b_3, b_1]] + [b_3, [b_1, b_2]] = 0 \\
&[b_1, [b_2, f]] + [b_2, [f, b_1]] + [f, [b_1, b_2]] = 0 \\
&[b, \{f_1, f_2\}] + \{f_1, [f_2, b]\} - \{f_2, [b, f_1]\} = 0 \\
&\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0.
\end{aligned}$$

It is a useful exercise to check that the identities hold for all possible choices of 3 generators. We first consider combinations with 2 Qs and one of P_μ, R and $\mathcal{J}_{\mu\nu}$ (we just consider Q_α and $Q_{\dot\alpha}$ since it is trivial to show this for 2 Q_αs and 2 $Q_{\dot\alpha}$s; it is for this same reason that we don't consider 3 Qs):

$$\begin{aligned}
& [P_\mu, \{\bar{Q}_{\dot\alpha}, Q_\alpha\}] + \{\bar{Q}_{\dot\alpha}, [Q_\alpha, P_\mu]\} - \{Q_\alpha, [P_\mu, \bar{Q}_{\dot\alpha}]\} \\
= & [P_\mu, 2\sigma^\nu_{\alpha\dot\alpha} P_\nu] + \{\bar{Q}_{\dot\alpha}, 0\} - \{Q_\alpha, 0\} = 0 \\
& [R, \{\bar{Q}_{\dot\alpha}, Q_\alpha\}] + \{\bar{Q}_{\dot\alpha}, [Q_\alpha, R]\} - \{Q_\alpha, [R, \bar{Q}_{\dot\alpha}]\} \\
= & [R, 2\sigma^\nu_{\alpha\dot\alpha} P_\nu] + \{\bar{Q}_{\dot\alpha}, Q_\alpha\} - \{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 0 \\
& [\mathcal{J}_{\rho\sigma}, \{\bar{Q}_{\dot\alpha}, Q_\alpha\}] + \{\bar{Q}_{\dot\alpha}, [Q_\alpha, \mathcal{J}_{\rho\sigma}]\} - \{Q_\alpha, [\mathcal{J}_{\rho\sigma}, \bar{Q}_{\dot\alpha}]\} \\
= & [\mathcal{J}_{\rho\sigma}, -2\sigma^\mu_{\alpha\dot\alpha} P_\mu] + \{\bar{Q}_{\dot\alpha}, i(\sigma_{\rho\sigma})_\alpha{}^\beta Q_\beta\} + \{Q_\alpha, i(\bar\sigma_{\rho\sigma})^{\dot\beta}{}_{\dot\alpha} Q_{\dot\beta}\} \\
= & 2i\sigma^\mu_{\alpha\dot\alpha}(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho) - 2i(\sigma_{\rho\sigma})_\alpha{}^\beta \sigma^\mu_{\beta\dot\alpha} P_\mu + 2i(\bar\sigma_{\rho\sigma})^{\dot\beta}{}_{\dot\alpha} \sigma^\mu_{\alpha\dot\beta} P_\mu \\
= & 2i\left(\sigma_{\rho\alpha\dot\alpha}P_\sigma - \sigma_{\sigma\alpha\dot\alpha}P_\rho)\right) - 2i\frac{1}{4}\left(\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\beta} - \sigma_{\sigma\alpha\dot\gamma}\bar\sigma_\rho^{\dot\gamma\beta}\right)\sigma^\mu_{\beta\dot\alpha}P_\mu \\
& +2i\frac{1}{4}\left(\bar\sigma_\rho^{\dot\beta\gamma}\sigma_{\sigma\gamma\dot\alpha} - \bar\sigma_\sigma^{\dot\beta\gamma}\sigma_{\rho\gamma\dot\alpha}\right)\sigma^\mu_{\alpha\dot\beta}P_\mu \\
= & 2i\left(\sigma_{\rho\alpha\dot\alpha}P_\sigma - \sigma_{\sigma\alpha\dot\alpha}P_\rho)\right) - 2i\frac{1}{4}\left(2\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\beta} + 2\eta_{\rho\sigma}\delta^\beta_\alpha\right)\sigma^\mu_{\beta\dot\alpha}P_\mu \\
& +2i\frac{1}{4}\left(2\bar\sigma_\rho^{\dot\beta\gamma}\sigma_{\sigma\gamma\dot\alpha} - \eta_{\rho\sigma}\delta^{\dot\beta}_{\dot\alpha}\right)\sigma^\mu_{\alpha\dot\beta}P_\mu \\
= & 2i\left(\sigma_{\rho\alpha\dot\alpha}P_\sigma - \sigma_{\sigma\alpha\dot\alpha}P_\rho)\right) - 2i\frac{1}{4}\left(2\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\beta}\sigma^\mu_{\beta\dot\alpha}P_\mu - 2\bar\sigma_\rho^{\dot\beta\gamma}\sigma_{\sigma\gamma\dot\alpha}\sigma^\mu_{\alpha\dot\beta}P_\mu\right) \\
= & 2i\left(\sigma_{\rho\alpha\dot\alpha}P_\sigma - \sigma_{\sigma\alpha\dot\alpha}P_\rho)\right) - 2i\frac{1}{4}\left(2\sigma_{\rho\alpha\dot\gamma}\left(2\eta^\mu_\sigma\delta^{\dot\gamma}_{\dot\alpha} - \bar\sigma^{\mu\dot\gamma\beta}\sigma_{\sigma\beta\dot\alpha}\right)P_\mu\right. \\
& \left.+2\sigma_{\sigma\gamma\dot\alpha}\left(2\eta^\mu_\rho\delta^\gamma_\alpha - \bar\sigma^{\mu\dot\beta\gamma}\sigma_{\rho\alpha\dot\beta}\right)P_\mu\right) \\
= & 2i\left(\sigma_{\rho\alpha\dot\alpha}P_\sigma - \sigma_{\sigma\alpha\dot\alpha}P_\rho)\right) - 2i\left(\sigma_{\rho\alpha\dot\alpha}P_\sigma - 2\sigma_{\sigma\alpha\dot\alpha}P_\rho\right) \\
= & 0.
\end{aligned}$$

For combinations with 1 Q and 2 from P_μ, R and $\mathcal{J}_{\mu\nu}$, there are 6 possibilities (not including the identities involving $\bar{Q}_{\dot\alpha}$ which follow in the same way) - 3 involving R:

$$\begin{aligned}
& [R, [R, Q_\alpha]] + [R, [Q_\alpha, R]] + [Q_\alpha, [R, R]] = 0 \\
& [R, [P_\mu, Q_\alpha]] + [P_\mu, [Q_\alpha, R]] + [Q_\alpha, [R, P_\mu]] \\
= & 0 + [P_\mu, Q_\alpha] + [Q_\alpha, 0] = 0 \\
& [R, [J_{\mu\nu}, Q_\alpha]] + [J_{\mu\nu}, [Q_\alpha, R]] + [Q_\alpha, [R, J_{\mu\nu}]] \\
= & [R, -(\sigma_{\mu\nu})^\beta_\alpha Q_\beta] + [J_{\mu\nu}, Q_\alpha] + [Q_\alpha, 0] \\
= & (\sigma_{\mu\nu})^\beta_\alpha Q_\beta - (\sigma_{\mu\nu})^\beta_\alpha Q_\beta = 0,
\end{aligned}$$

and 3 not involving R:

$$\begin{aligned}
& [P_\mu,[P_\nu,Q_\alpha]]+[P_\nu,[Q_\alpha,P_\mu]]+[Q_\alpha,[P_\mu,P_\nu]]=0\\
& [\mathcal{J}_{\rho\sigma},[P_\mu,Q_\alpha]]+[P_\mu,[Q_\alpha,\mathcal{J}_{\rho\sigma}]]+[Q_\alpha,[\mathcal{J}_{\rho\sigma},P_\mu]]\\
= \; & 0+[P_\mu,\left(i(\sigma_{\rho\sigma})_\alpha{}^\beta Q_\beta\right)]+[Q_\alpha,(-i(\eta_{\mu\rho}P_\sigma-\eta_{\mu\sigma}P_\rho)]=0\\
& [\mathcal{J}_{\mu\nu},[\mathcal{J}_{\rho\sigma},Q_\alpha]]+[\mathcal{J}_{\rho\sigma},[Q_\alpha,\mathcal{J}_{\mu\nu}]]+[Q_\alpha,[\mathcal{J}_{\mu\nu},\mathcal{J}_{\rho\sigma}]]\\
= \; & [\mathcal{J}_{\mu\nu},-\left(i(\sigma_{\rho\sigma})_\alpha{}^\beta Q_\beta\right)]+\left[\mathcal{J}_{\rho\sigma},\left(i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta\right)\right]\\
& +[Q_\alpha,i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma}+\eta_{\mu\sigma}\mathcal{J}_{\nu\rho}-\eta_{\mu\rho}\mathcal{J}_{\nu\sigma}-\eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right)]\\
= \; & i(\sigma_{\rho\sigma})_\alpha{}^\beta(\sigma_{\mu\nu})_\beta{}^\gamma Q_\gamma-i(\sigma_{\mu\nu})_\alpha{}^\beta(\sigma_{\rho\sigma})_\beta{}^\gamma Q_\gamma\\
& +i\left[\eta_{\nu\rho}(\sigma_{\mu\sigma})_\alpha{}^\beta Q_\beta+\eta_{\mu\sigma}(\sigma_{\nu\rho})_\alpha{}^\beta Q_\beta-\eta_{\mu\rho}(\sigma_{\nu\sigma})_\alpha{}^\beta Q_\beta-\eta_{\nu\sigma}(\sigma_{\mu\rho})_\alpha{}^\beta Q_\beta\right]
\end{aligned} \tag{1.27}$$

Now

$$\begin{aligned}
(\sigma_{\rho\sigma})_\alpha{}^\beta(\sigma_{\mu\nu})_\beta{}^\gamma &= \frac{1}{4^2}\left(\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\beta}-\sigma_{\sigma\alpha\dot\gamma}\bar\sigma_\rho^{\dot\gamma\beta}\right)\left(\sigma_{\mu\beta\dot\gamma}\bar\sigma_\nu^{\dot\gamma\gamma}-\sigma_{\nu\beta\dot\gamma}\bar\sigma_\mu^{\dot\gamma\gamma}\right)\\
&= \frac{1}{4}\left(\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\beta}-\eta_{\rho\sigma}\delta_\alpha^\beta\right)\left(\sigma_{\mu\beta\dot\gamma}\bar\sigma_\nu^{\dot\gamma\gamma}-\eta_{\mu\nu}\delta_\beta^\gamma\right)\\
(\sigma_{\mu\nu})_\alpha{}^\beta(\sigma_{\rho\sigma})_\beta{}^\gamma &= \frac{1}{4}\left(\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\nu^{\dot\gamma\beta}-\eta_{\mu\nu}\delta_\alpha^\beta\right)\left(\sigma_{\rho\beta\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\gamma}-\eta_{\rho\sigma}\delta_\beta^\gamma\right).
\end{aligned}$$

After cancellations we get that $(\sigma_{\rho\sigma})_\alpha{}^\beta(\sigma_{\mu\nu})_\beta{}^\gamma-(\sigma_{\mu\nu})_\alpha{}^\beta(\sigma_{\rho\sigma})_\beta{}^\gamma$ is equal to

$$\begin{aligned}
= \; & \frac{1}{4}\left[\left(\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\beta}\right)\left(\sigma_{\mu\beta\dot\delta}\bar\sigma_\nu^{\dot\delta\gamma}\right)-\left(\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\nu^{\dot\gamma\beta}\right)\left(\sigma_{\rho\beta\dot\delta}\bar\sigma_\sigma^{\dot\delta\gamma}\right)\right]\\
= \; & \frac{1}{4}\left[\left(\sigma_{\rho\alpha\dot\gamma}\left(2\eta_{\mu\sigma}\delta_{\dot\delta}^{\dot\gamma}-\bar\sigma_\mu^{\dot\gamma\beta}\sigma_{\sigma\beta\dot\delta}\right)\bar\sigma_\nu^{\dot\delta\gamma}\right)\right.\\
& \left.-\left(\sigma_{\mu\alpha\dot\gamma}\left(2\eta_{\nu\rho}\delta_{\dot\delta}^{\dot\gamma}-\bar\sigma_\rho^{\dot\gamma\beta}\sigma_{\nu\beta\dot\delta}\right)\bar\sigma_\sigma^{\dot\delta\gamma}\right)\right]\\
= \; & \frac{1}{2}\left[\eta_{\mu\sigma}\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\nu^{\dot\gamma\gamma}-\eta_{\nu\rho}\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\gamma}\right]\\
& -\frac{1}{4}\left[\left(2\eta_{\mu\rho}\delta_\alpha^\beta-\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\rho^{\dot\gamma\beta}\right)\sigma_{\sigma\beta\dot\delta}\bar\sigma_\nu^{\dot\delta\gamma}-\right.\\
& \left.\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\rho^{\dot\gamma\beta}\left(2\eta_{\nu\sigma}\delta_\beta^\gamma-\sigma_{\sigma\beta\dot\delta}\bar\sigma_\nu^{\dot\delta\gamma}\right)\right]\\
= \; & \frac{1}{2}\left[\eta_{\mu\sigma}\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\nu^{\dot\gamma\gamma}-\eta_{\nu\rho}\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\gamma}-\eta_{\mu\rho}\sigma_{\sigma\alpha\dot\gamma}\bar\sigma_\nu^{\dot\gamma\gamma}+\eta_{\nu\sigma}\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\rho^{\dot\gamma\beta}\right]\\
= \; & \frac{1}{4}\left[\eta_{\mu\sigma}\left(\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\nu^{\dot\gamma\gamma}-\sigma_{\nu\alpha\dot\gamma}\bar\sigma_\rho^{\dot\gamma\gamma}\right)-\eta_{\nu\rho}\left(\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\gamma}-\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\gamma}\right)\right.\\
& \left.-\eta_{\mu\rho}\left(\sigma_{\sigma\alpha\dot\gamma}\bar\sigma_\nu^{\dot\gamma\gamma}-\sigma_{\nu\alpha\dot\gamma}\bar\sigma_\sigma^{\dot\gamma\gamma}\right)+\eta_{\nu\sigma}\left(\sigma_{\mu\alpha\dot\gamma}\bar\sigma_\rho^{\dot\gamma\beta}-\sigma_{\rho\alpha\dot\gamma}\bar\sigma_\mu^{\dot\gamma\beta}\right)\right]
\end{aligned}$$

$$= \quad [-\eta_{\mu\sigma}(\sigma_{\nu\rho})_\alpha{}^\gamma - \eta_{\nu\rho}(\sigma_{\mu\sigma})_\alpha{}^\gamma + \eta_{\mu\rho}(\sigma_{\nu\sigma})_\alpha{}^\gamma + \eta_{\nu\sigma}(\sigma_{\mu\rho})_\alpha{}^\gamma] \,,$$

which cancels with (1.27) to give the desired 0. All the other Jacobi identities do not involve Q and come from the standard Poincaré algebra. The full $\mathcal{N} = 1$ super-Poincaré algebra (excluding special conformal transformations) is therefore given by

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\mu, \mathcal{J}_{\rho\sigma}] &= i\eta_{\mu\rho}P_\sigma - i\eta_{\mu\sigma}P_\rho \\
[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right) \\
[P_\mu, Q_\alpha] &= \left[P_\mu, \bar{Q}_{\dot{\alpha}}\right] = 0 \\
[Q_\alpha \mathcal{J}_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^\beta Q_\beta \\
\left[\bar{Q}_{\dot{\alpha}} \mathcal{J}_{\mu\nu}\right] &= -\bar{Q}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})_\alpha^\beta \\
\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2\sigma_{\alpha\dot{\alpha}}P_\mu = 2P_{\alpha\dot{\alpha}} \\
\{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \\
[Q_\alpha, R] &= Q_\alpha \\
\left[\bar{Q}_{\dot{\alpha}}, R\right] &= -\bar{Q}_{\dot{\alpha}} \\
[R, R] &= [R, P_\mu] = [R, \mathcal{J}_{\mu\nu}] = 0
\end{aligned} \tag{1.28}$$

The supersymmetric generators are fermionic, so acting twice on a state should return the original state; that is

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle \qquad Q\,|\text{fermion}\rangle = |\text{boson}\rangle$$

The superparticles these represent contain both fermionic and bosonic components which belong to the same supermultiplet. The Casimirs which commute with supersymmetric generators Q_α turn out to be the usual $C_1 = P_\mu P^\mu$ (which follows directly from $[P_\mu, Q_\alpha] = 0$ and shows that all fields in a supersymmetry multiplet have the same mass), and the not so usual $C_2' = C_{\mu\nu}C^{\mu\nu}$ where $C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu$ with $B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\beta}Q_\beta$ (compared with the usual $C_2 = W_\mu W^\mu$). Here we recall that the second Casimir operator, known as the Pauli-Lubanski pseudovector, is defined as

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{J}^{\nu\rho}P^\sigma$$

The next thing to consider are the massless multiplets of $\mathcal{N} = 1$ SUSY. Recall that a covariant definition of helicity entails observing that we have tensors (here, vectors) that are both null and orthogonal to each other

$$P_\mu P^\mu |k^\mu\rangle = W_\mu W^\mu |k^\mu\rangle = W_\mu P^\mu |k^\mu\rangle = 0$$

so that W^μ is proportional to P^μ and therefore their eigenvalues satisfy $w^\mu = \lambda p^\mu$. Boosting to the frame $P_\mu = (E, 0, 0, E)$ and specifying a state with helicity λ, we have that

$$\begin{aligned} & W_\mu = \lambda P_\mu \qquad W_0 = \lambda P_0 \\ \Rightarrow \qquad & W_0 |E, \lambda\rangle = \lambda P_0 |E, \lambda\rangle = \lambda E |E, \lambda\rangle \,, \end{aligned}$$

from which we would like to build a supersymmetry multiplet. Recalling that in $\sigma^\mu_{\alpha\dot{\alpha}}$, the top index tells us which Pauli matrix we are talking about

1.10 Supergravity

The following is based on Ref. [84]. Supergravity theories are interesting as they are field theories with local supersymmetry, are the only known field theories with interacting spin-3/2 fields and they enable us to write supersymmetric extensions of General Relativity (including the adding of higher order R-curvature terms where the algebra may not close so on which is a feature quantum mechanics is built on). Of key interest today is the phenomenologically interesting 4d $\mathcal{N} = 1$ where new patterns of SUSY-breaking with a positive semi-definite scalar potential allow for a smaller cosmological constant and provide new dark matter candidates (e.g. gravitino) and places constraints on inflation models. It is further the natural interface between string theory and low energy string phenomenology, which can shed light on non-perturbative aspects of string theory such as black holes and AdS/CFT dualities. While Supersymmetry is a global theory, Supergravity is a local theory:- it is gauged Supersymmetry.

For the (free) Wess-Zumino model, one could use superspace notation,

$$\mathcal{L}_{WZ} = \int d^4\theta \; \Phi^* \Phi$$

where $\Phi(x, \theta)$ is a chiral superfield ($\mathcal{D}\Phi = 0$), $\Phi^*(x, \theta)$ its hermitean conjugate, $K(\Phi, \Phi^*) = \Phi^*\Phi$ is the canonical Kähler potential and $W(\Phi) = 0$ is the vanishing superpotential. However, here we will rather use 4-component spinor notation.

1.10.1 Notation

The following notation is useful conceptually as well as for doing calculations. We have the following relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}_4 \qquad \Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$$

and Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\sigma_a\sigma_b = \delta_{ab}\mathbf{1}_2 + i\epsilon_{abc}\sigma_c \qquad \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \mathbf{1}_2$$

For any even dimension one may, as de Witt and Friedman in the 1980s did, work in a "friendly representation" [1], defined as (the convention used in this section is $\eta_{\mu\nu} = (-1, 1, 1, 1)$)

$$\gamma_0^\dagger = -\gamma_0 \qquad \gamma_i^\dagger = -\gamma_i \qquad \gamma^{\mu T} = \pm\gamma^\mu.$$

An example of a 'friendly representation' is the Weyl representation

$$\gamma_\mu = \begin{pmatrix} 0 & i\bar{\sigma}_\mu \\ i\sigma_\mu & 0 \end{pmatrix}$$

where $\bar{\sigma}_0 = \sigma_0 = \mathbf{1}_2$ and $\bar{\sigma}_i = -\sigma_i$. Thus

$$\begin{aligned} \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3 \\ &= i \cdot i^4 \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \end{aligned}$$

[1] There are various representations (which won't be used here), for example in the Dirac representation, $\gamma_0 = i\sigma_3 \bigotimes \mathbf{1}_2 = \begin{pmatrix} i\mathbf{1}_2 & 0 \\ 0 & -i\mathbf{1}_2 \end{pmatrix}$ and $\gamma_i = \sigma_2 \bigotimes \sigma_i$ and in the Majorana representation $\gamma_0 = i\sigma_2 \bigotimes \sigma_3$, $\gamma_1 = -\sigma_1 \bigotimes \mathbf{1}_2$, $\gamma_2 = \sigma_2 \bigotimes \sigma_2$ and $\gamma_3 = \sigma_3 \bigotimes \mathbf{1}_2$.

$$= \begin{pmatrix} -i\sigma_0\sigma_1\sigma_2\sigma_3 & 0 \\ 0 & 0-(-i\sigma_0\sigma_1\sigma_2\sigma_3) \end{pmatrix} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}$$

so that

$$\begin{aligned} \{\gamma_\mu, \gamma_5\} &= 0 \\ \Sigma_{\mu\nu} &= \frac{1}{4}[\gamma_\mu, \gamma_\nu] = -\frac{1}{4}\begin{pmatrix} \bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu & 0 \\ 0 & \sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu \end{pmatrix} = \begin{pmatrix} er^*_{\mu\nu}e^{-1} & 0 \\ 0 & r_{\mu\nu} \end{pmatrix} \end{aligned}$$

where $r_{\mu\nu} = -\frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$ and $r^*_{\mu\nu} = e^{-1}\left[-\frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)\right]e$ where e is just a convenient tool to write $\bar{\sigma}$ in term of σ:

$$e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \bar{\sigma}_\mu = e^{-1}\sigma_\mu e$$

Since $\Sigma_{\mu\nu}$ is reducible, the 4-component Dirac spinor decomposes naturally into two 2-component spinors

$$(\psi_\alpha) = \begin{pmatrix} e^{\dot{A}\dot{B}}w_{\dot{B}} \\ \lambda_A \end{pmatrix}$$

where the number of degrees of freedom may be cut down by setting $w = 0$ or $\lambda = 0$ for a Weyl spinor or setting $w^* = \lambda$ for a Majorana spinor in Weyl representation. Furthermore, $\Sigma_{\mu\nu}$ may be understood as a generator of rotations, for example

$$\begin{aligned} \Sigma_{12} &= -\frac{1}{4}\begin{pmatrix} \bar{\sigma}_1\sigma_2 - \bar{\sigma}_2\sigma_1 & 0 \\ 0 & \sigma_1\bar{\sigma}_2 - \sigma_2\bar{\sigma}_1 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \frac{i}{2}\mathrm{diag}(1,-1,1,-1) \\ \Rightarrow R(\theta) &= e^{\frac{\theta}{2}\Sigma_{12}} = \cos\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\Sigma_{12}. \end{aligned}$$

Noticeably, $R(2\pi) = -1 \to$ a full rotation leads to a negative sign in fermions. Since $\{\gamma_\mu, \gamma_5\} = 0$ and $\gamma_0^2 = -\mathbf{1}_4$, a friendly representation dictates that $\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu$. It is now simple to define projection operators:

$$P_{L,R} = \frac{1}{2}(\mathbf{1}_4 \pm \gamma_5) \qquad P^*_{L,R} = P_{L,R}$$

$$P_L\psi = \psi_L \qquad P_R\psi = \psi_R \qquad \gamma_5\psi_L = \psi_L \qquad \gamma_5\psi_R = -\psi_R$$

$$P_L\gamma_0 = \gamma_0 P_R \qquad P_R\gamma_0 = \gamma_0 P_L$$

$$\gamma_0\Sigma^\dagger_{\mu\nu}\gamma_0 = -\frac{1}{4}\left(\gamma_0\gamma^\dagger_\nu\gamma_0\gamma_0\gamma^\dagger_\mu\gamma_0 - \gamma_0\gamma^\dagger_\mu\gamma_0\gamma_0\gamma^\dagger_\nu\gamma_0\right) = -\frac{1}{4}\left(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu\right) = \Sigma_{\mu\nu}.$$

Note further that the 16 matrices defined from γ_5, that is $\gamma_{\mu\nu\rho\sigma} = i\hat{\epsilon}_{\mu\nu\rho\sigma}\gamma_5$, $\gamma_{\mu\nu\rho\sigma} = i\hat{\epsilon}_{\mu\nu\rho}\gamma_5\gamma^\sigma$, $\gamma_{\mu\nu} = -\frac{i}{2}\hat{\epsilon}_{\mu\nu\rho\sigma}\gamma_5\gamma^{\rho\sigma}$ and $\gamma_\mu = -\frac{i}{6}\hat{\epsilon}_{\mu\nu\rho\sigma}\gamma_5\gamma^{\nu\rho\sigma}$ is a basis for all (complex) 4×4 matrices. Now, defining the Dirac conjugate to be $\bar{\psi} = i\psi^\dagger\gamma^0 = -i\psi^\dagger\gamma_0$, one has

$$\overline{\psi_R} \;=\; \overline{(P_R\psi)} = (P_R\psi)^\dagger i\gamma^0 = i\psi^\dagger\gamma^0 P_L = \bar{\psi}P_L = \bar{\psi}_L; \quad \overline{\psi_L} = \bar{\psi}_R.$$

Recall that Majorana spinors are of the form $\psi = \begin{pmatrix} e\cdot\chi \\ \chi \end{pmatrix}$. The Majorana condition is a reality condition ($\psi^C = \psi$) on a four-component Dirac spinor) requires self-consistency ($\psi^{**} = \psi$) and Lorentz covariance. One defines a charge-conjugation matrix in a friendly representation so that,

$$C^T = C^{-1} = -C \qquad \gamma^T_\mu = -C\gamma_\mu C^{-1} \qquad \psi^T C = \bar{\psi}.$$

[C may be written explicitly in Weyl representation as $C = i\gamma_0\gamma_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$].

Using $\gamma_{\mu\nu...} = \frac{1}{n!}\left(\gamma_\mu\gamma_\nu... + ...\right)$ (where combinations that are odd permutations come with a negative sign and if any of $\mu,\nu,...$ are the same, then $\gamma_{\mu\nu...} = 0$), we obtain the following identities (where we note that swapping a χ and a ψ costs a minus sign for anti-commuting spinors),

$$\begin{aligned}
\bar{\chi}\gamma_\mu\psi &= \left(\bar{\chi}\gamma_\mu\psi\right)^T = -\psi^T\gamma^T_\mu\bar{\chi}^T = -\psi^T\left(-C\gamma_\mu C^{-1}\right)\bar{\chi}^T = -\bar{\psi}\gamma_\mu C^{-1}C^T\chi = -\bar{\psi}\gamma_\mu\chi \qquad (1.29)\\
\bar{\chi}\gamma_{\mu\nu}\psi & \quad \left[\sim -\psi^T\left(-C\gamma_\nu C^{-1}\right)\left(-C\gamma_\mu C^{-1}\right)(\chi^T C)^T\right] = -\bar{\psi}^T\gamma_{\mu\nu}\bar{\chi}\\
\bar{\chi}\gamma_{\mu\nu\rho}\psi & \quad \left[\sim -\psi^T\left(-C\gamma_\rho C^{-1}\right)\left(-C\gamma_\nu C^{-1}\right)\left(-C\gamma_\mu C^{-1}\right)(\chi^T C)^T = -\bar{\psi}\gamma_\rho\gamma_\nu\gamma_\mu\chi\right]\\
&= \bar{\psi}^T\gamma_{\mu\nu\rho}\bar{\chi}\\
\bar{\chi}\gamma_{\mu\nu\rho\sigma}\psi & \quad \left[\sim -\psi^T\left(-C\gamma_\sigma C^{-1}\right)\left(-C\gamma_\rho C^{-1}\right)\left(-C\gamma_\nu C^{-1}\right)\left(-C\gamma_\mu C^{-1}\right)(\chi^T C)^T\right.\\
&= \left.-\bar{\psi}\gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\mu\chi\right]\bar{\psi}^T\gamma_{\mu\nu\rho\sigma}\bar{\chi}
\end{aligned}$$

1.10.2 Simplest globally supersymmetric model in 4d

The kinetic term of the Wess-Zumino model is given by

$$\mathcal{L}_{WZ} = \int d^4\theta\ \Phi^*\Phi = -(\partial_\mu\phi)(\partial^\mu\phi) - \bar{\chi}\partial\!\!\!/\chi + |F|^2$$

where setting $F = 0$ just enforces the equations of motion, so that the on-shell Lagrangian is just

$$\mathcal{L}_{WZ} = -(\partial_\mu\phi)(\partial^\mu\phi) - \bar{\chi}\partial\!\!\!/\chi.$$

where

$$\begin{aligned}
\mathcal{L}_{\text{ferm}} &= -\bar{\chi}\partial\!\!\!/\chi = -\bar{\chi}\partial\!\!\!/\left(P_L + P_R\right)\chi = -\chi^T C\gamma^\mu\partial_\mu\left(P_L^2 + P_R^2\right)\chi \\
&= -\chi^T C P_R\gamma^\mu\partial_\mu P_L\chi - \chi^T C P_L\gamma^\mu\partial_\mu P_R\chi = -\bar{\chi_R}\partial\!\!\!/\chi_L - \bar{\chi_L}\partial\!\!\!/\chi_R \\
&= -2\bar{\chi_L}\partial\!\!\!/\chi_R
\end{aligned}$$

The supersymmetric transformations take bosons to fermions and vice versa and the Wess-Zumino action is invariant under the following changes of fields and has the following currents:

$$\begin{aligned}
\delta\phi &= \bar{\chi_L}\epsilon_L \\
\delta\phi^* &= \bar{\chi_R}\epsilon_R \\
\delta\chi_L &= \frac{1}{2}\left(\partial\!\!\!/\phi\right)\epsilon_R \\
\delta\chi_R &= \frac{1}{2}\left(\partial\!\!\!/\phi^*\right)\epsilon_L \\
J_L^\mu &= -\bar{\chi_L}\gamma^\mu(\partial\!\!\!/\phi^*) \\
J_R^\mu &= -\bar{\chi_R}\gamma^\mu(\partial\!\!\!/\phi)
\end{aligned}$$

Here, up to total derivatives (where we use equation (1.29) and $\partial\!\!\!/\partial\!\!\!/ = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \Box\mathbf{1}_4$),

$$\begin{aligned}
\delta\mathcal{L}_{\text{bos}} &= \delta\left(-(\partial_\mu\phi)(\partial^\mu\phi)\right) = \delta\phi\Box\phi^* + \delta\phi^*\Box\phi \\
\delta\mathcal{L}_{\text{ferm}} &= -2\delta(\bar{\chi_L}\partial\!\!\!/\chi_R) = -2\delta(\bar{\chi_L})\partial\!\!\!/\chi_R) - 2\bar{\chi_L}\partial\!\!\!/\delta(\chi_R) \\
&= -2(\bar{\chi}_R\partial\!\!\!/\delta(\chi_L)) - 2\bar{\chi_L}\partial\!\!\!/\delta(\chi_R) = -2(\bar{\chi}_R\partial\!\!\!/\delta(\chi_L)) + CC
\end{aligned}$$

$$
\begin{aligned}
&= -2\bar{\chi_R}\partial\!\!\!/(\frac{1}{2}\left(\partial\!\!\!/\phi\right)\epsilon_R) + CC = -\bar{\chi_R}\partial\!\!\!/\left(\partial\!\!\!/\phi\right)\epsilon_R + CC \\
&= -\bar{\chi_R}\left(\Box\phi\right)\epsilon_R - \bar{\chi_R}\gamma^\mu\left(\partial\!\!\!/\phi\right)\partial_\mu\epsilon_R + CC \\
&= -\delta\phi^*\left(\Box\phi\right) - J^\mu_R\partial_\mu\epsilon_R + CC \\
\Rightarrow \delta\mathcal{L}_{\text{WZ}} &= \delta\mathcal{L}_{\text{bos}} + \delta\mathcal{L}_{\text{ferm}} = J^\mu_L\partial_\mu\epsilon_L + J^\mu_R\partial_\mu\epsilon_R.
\end{aligned}
$$

The Wess-Zumino Lagrangian is thus locally but not globally invariant. This is similar to QED, where $\mathcal{L}_{\text{kin}} = -\bar{\lambda}\partial\!\!\!/\lambda$:

$$
\begin{aligned}
\lambda \to e^{\imath\alpha}\lambda \;\; &\Rightarrow \;\; -\bar{\lambda}\gamma^\mu\partial_\mu\lambda \to -\bar{\lambda}\gamma^\mu\lambda(i\partial_\mu\alpha) - \bar{\lambda}\gamma^\mu\partial_\mu\lambda \\
&\Rightarrow \;\; \delta_{u(1)}\mathcal{L}_{\text{kin}} = J^\mu_{\text{EM}}\partial_\mu\alpha = \left(-i\bar{\lambda}\gamma^\mu\lambda\right)\partial_\mu\alpha.
\end{aligned}
$$

where $J^\mu_{\text{EM}} = -i\bar{\lambda}\gamma^\mu\lambda$. In QED the solution would be to make $U(1)$ a local symmetry by introducing a field $A_\mu(x)$, requiring that $\delta_{u(1)}A_\mu(x) = \partial_\mu\alpha(x)$, and then adding a term $\mathcal{L}'$ to the Lagrangian

$$
\begin{aligned}
\mathcal{L}' &= J^\mu_{\text{EM}}A_\mu \Rightarrow \delta_{u(1)}\mathcal{L}' = -\delta_{u(1)}(J_{\text{EM}^\mu})A_\mu - J_{\text{EM}}\partial_\mu\alpha \\
\Rightarrow \delta_{u(1)}(\mathcal{L}_{\text{kin}} + \mathcal{L}') &= -\delta_{u(1)}(J^\mu_{\text{EM}})A_\mu
\end{aligned}
$$

Since the fermion is a Majorana fermion, the λ and $\bar{\lambda}$ terms cancel out, so that this goes to zero. The Lagrangian that is locally invariant in QED is therefore

$$
\mathcal{L}_{\text{kin}} + \mathcal{L}' = -i\bar{\lambda}D_\mu\lambda.
$$

We follow a similar procedure for our Wess-Zumino action: introduce a field $\psi_\mu(x)$ (if we expanded indices, this would really read $\psi_{\mu\alpha}(x)$ since the expanded form of the action is given by $\delta\mathcal{L}_{WZ} = J^\mu_{L\alpha}\epsilon_{L\alpha} + J^\mu_{R\alpha}\epsilon_{R\alpha}$). Next we require that

$$
\delta_\epsilon\psi_\mu = \frac{1}{\kappa}\partial_\mu\epsilon \tag{1.30}
$$

κ is required due to the for dimensional analyis to be consistent: in $c = 1$ ($[L] = [T]$) and $\hbar = 1$ ($[[p, x]] = 1 \Rightarrow [M] = [L]^{-1}$) coordinates, ϕ has mass dimension 1 and ψ has mass dimension $\frac{3}{2}$ (since $\int d^4x\partial_\mu\phi\partial^\mu\phi$ and $\int d^4x\psi\partial_\mu\psi$ are dimensionless) and therefore ϵ has mass dimension $-\frac{1}{2}$ (since $\delta\phi = \bar{\chi}_L\epsilon_L$). Hence (1.30) has dimension

$\frac{3}{2}$ and therefore κ has dimension -1. Lastly we add the term

$$\begin{aligned} \mathcal{L}' &= -\kappa J_L^\mu \psi_{\mu L} - \kappa J_R^\mu \psi_{\mu R} \\ \Rightarrow \delta_\epsilon \mathcal{L}' &= -\kappa \delta_\epsilon (J_L^\mu) \psi_{\mu L} - \kappa \delta_\epsilon (J_R^\mu) \psi_{\mu R} - \kappa J_L^\mu \partial_\mu \epsilon_L - \kappa J_R^\mu \partial_\mu \epsilon_R \\ \Rightarrow \delta_\epsilon \left(\mathcal{L}_{\text{WZ}} + \mathcal{L}' \right) &= -\kappa \delta_\epsilon (J_L^\mu) \psi_{\mu L} - \kappa \delta_\epsilon (J_R^\mu) \psi_{\mu R}. \end{aligned}$$

In QED $\delta_\epsilon \left(\mathcal{L}_{\text{WZ}} + \mathcal{L}' \right) = 0$ because of the Majorana condition. Let us consider what happens in SUSY:

$$\begin{aligned} \delta_\epsilon (J_L^\mu) &= -\delta_\epsilon \left(\bar{\chi_L} \gamma^\mu \left(\partial\!\!\!/ \phi^* \right) \right) = - \left(\delta_\epsilon \bar{\chi_L} \right) \gamma^\mu \left(\partial\!\!\!/ \phi^* \right) - \bar{\chi_L} \gamma^\mu \partial\!\!\!/ \left(\delta_\epsilon (\phi^*) \right) \\ &\propto -\bar{\epsilon}_R \partial\!\!\!/ \phi \gamma^\mu \left(\partial\!\!\!/ \phi^* \right) - \bar{\chi_L} \gamma^\mu \partial\!\!\!/ \bar{\epsilon_R} \chi_R \\ &= -\bar{\epsilon_R} \gamma^\mu \left[\text{''} \partial_\phi \partial \phi - \bar{\chi_L} \partial\!\!\!/ \chi_L \text{''} \right] \end{aligned}$$

This expression is similar to $T^{\mu\nu} \sim \mathcal{L} g^{\mu\nu}$ (foreshadowing the link to gravity), and we therefore write

$$\delta_\epsilon \left(\mathcal{L}_{\text{WZ}} + \mathcal{L}' \right) = -\bar{\epsilon} \gamma_\mu \gamma_\nu T^{\mu\nu}(\phi, \chi).$$

Unlike QED, now, we need follow the Noether procedure and introduce a new field $g_{\mu\nu}(x)$ with

$$\delta a g_{\mu\nu}(x) = \kappa \bar{\epsilon} \gamma_\mu \psi_\nu \tag{1.31}$$

and add $\mathcal{L}'' = g_{\mu\nu} T^{\mu\nu}$. Now ψ_μ, with helicity $\pm\frac{3}{2}$ (gravitino) is the superpartner of $g_{\mu\nu}$ with helicity ± 2 (graviton) and we identify $\kappa = \frac{1}{m_P} \approx (10^{18} GeV)^{-1}$. In QED, we would add an interacting Lagrangian $-J_{\text{EM}}^\mu A_\mu$

$$\begin{aligned} \mathcal{L}_{\text{tot}}(\lambda, A_\mu) &= \mathcal{L}_{\text{kin}}(A_\mu) + \mathcal{L}_{\text{kin}}(\lambda) + \mathcal{L}'(\lambda, A_\mu) \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \lambda \partial\!\!\!/ \lambda - J_{\text{EM}}^\mu A_\mu. \end{aligned}$$

Similarly now we have a Lagrangian for SUSY,

$$\begin{aligned} \mathcal{L}_{\text{tot}} &= \mathcal{L}_{\text{WZ}}(\phi, \chi) + \mathcal{L}'(\phi, \chi, \psi_\mu) + \mathcal{L}''(\phi, \chi, g_{\mu\nu}) \\ &= \left[-(\partial_\mu \phi)(\partial_\mu \phi^*) - \bar{\chi} \partial\!\!\!/ \chi \right] + \left[-J_L^\mu \psi_{\mu L} - J_R^\mu \psi_{\mu R} \right] + g_{\mu\nu} T^{\mu\nu} \end{aligned}$$

and one for supergravity, the first term involving gravitons with 2 vector indices and

Usual coordinate space	Orthonormal tangent space
$V^\mu \to \left(\frac{\partial x^{\mu'}}{\partial x^\nu}\right) V^\nu$	$\chi_\alpha \to R(\Lambda)_{\alpha\beta}\chi_\beta$
Inner product $g(\partial_\mu, \partial_\nu) = g_{\mu\nu}$	Inner product $\eta(e_a, e_b) = \eta_{ab}$
$e_a \to e'_a = \Lambda_a{}^b(x) e_b$ has no effect on ∂_μ	$\partial_\mu \to \left(\frac{\partial x^\nu}{\partial x^\mu}\right) \partial_\nu$ has no effect on e_a
Covariant derivative: $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$ $\nabla_\rho g_{\mu\nu} = 0$ (metric compatible) $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ (torsion free)	Covariant derivative: $D_\mu V^a = \partial_\mu V^a + w_\mu{}^a{}_b V^b$ Metric compatible: impose $D_\mu \eta^{ab} = 0$ $\Rightarrow w_\mu{}^a{}_b \eta^{bc} + \eta^{ab} w_\mu{}^a{}_b = 0$ $\Rightarrow w_\mu{}^{ab} = -w_\mu{}^{ba}$ Torsion-free: Impose D_μ equivalent to ∇_μ with respect to different bases $\Rightarrow D_\mu V^a = e^a{}_\nu \nabla_\mu V^\nu$ (else $D_{[\mu} e_{\nu]} = \frac{1}{2} S^a_{\mu\nu} \neq 0$ where $S^a_{\mu\nu}$ is the torsion tensor.)

TABLE 1.1: Table comparing the usual coordinate space and orthogonal tangent space

vierbeins with 1 spinor index, and the second term involving gravitinos - fermions with vector and spinor indices:

$$
\begin{aligned}
\mathcal{L}(g_{\mu\nu}, \psi_\mu) &= \mathcal{L}_{\text{kin}}(g_{\mu\nu}) + \mathcal{L}_{\text{kin}}(\psi_\mu) + \mathcal{L}_{\text{int}}[\psi_\mu, g_{\mu\nu}] \\
&= \left[-\frac{1}{2\kappa^2}\sqrt{g}R(g)\right] + \left[-\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \psi_\rho\right] + ?. \qquad (1.32)
\end{aligned}
$$

Here, $\delta g_{\mu\nu} = \kappa \bar{\epsilon} \gamma_{(\mu} \psi_{\nu)} + ?$, and $\delta \psi_\mu = \left[\frac{1}{\kappa}\partial_\mu \epsilon\right]_{\text{cov}} + ?$. Vector fields transform according to $V^\mu \to \Lambda^\mu{}_\nu V^\nu$, that is $GL(4, \boldsymbol{R})$ and spinor fields according to $\chi_\alpha \to R(\Lambda)_{\alpha\beta}\chi_\beta$, that is $SO(1,3)$. Now $SO(1,3) \subset GL(4, \boldsymbol{R})$; we therefore start to utilise spinors in curved spacetime, so we now have two types of inner products on an usual coordinate space and an orthonormal tangent / local Lorentz space respectively.

Notice that the w term has the same number of indices as a Γ term, and performs the same type of function. Further it is just a function of e. The inner product satisfies

$$
\begin{aligned}
\eta_{ab} &= e^\mu_a g_{\mu\nu} e^\nu{}_b \\
g_{\mu\nu} &= e^a{}_\mu \eta_{ab} e^b{}_\nu. \qquad (1.33)
\end{aligned}
$$

We now define $\tilde{\nabla}$ to include both covariant derivatives:

$$\tilde{\nabla}_\mu {e_\nu}^a = \partial_\mu e_\nu^a + {w_\mu}^a{}_b {e^b}_\nu - \Gamma^\rho_{\mu\nu} {e_\rho}^a = 0$$

The Vierbein postulate is then given by

$$\tilde{\nabla}_\mu {e^a}_\nu = 0.$$

We then have

$$\tilde{\nabla}_{[\mu} e^a_{\nu]} = \partial_{[\mu} e^a_{\nu]} + w_{[\mu}{}^a{}_b e^b_{\nu]} = 0 \Rightarrow D_{[\mu} e^a_{\nu]} = 0$$

For fermions we have (recalling that $\Sigma_{ab} = \frac{1}{4}\{\gamma_a, \gamma_b\}$),

$$\begin{aligned}
\tilde{\nabla}_\mu \epsilon &= D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} w_\mu^{ab}(\epsilon)\Sigma_{ab}\epsilon = \partial_\mu \epsilon + \frac{1}{4} w_\mu^{ab}(e)\gamma_{ab}\epsilon \\
\Rightarrow \tilde{\nabla}_\mu (R(\Lambda(x))\epsilon) &= R(\Lambda(x))\tilde{\nabla}_\mu \epsilon \\
\tilde{\nabla}_\mu \psi_\nu &= D_\mu \psi_\nu - \Gamma^\rho_{\mu\nu}(g)\psi_\rho \\
\Rightarrow \tilde{\nabla}_\mu \left[(R(\Lambda(x))\epsilon) \frac{\partial x^\rho}{\partial x^\nu} \psi_\rho \right] &= R(\Lambda(x)) \left(\frac{\partial x^\rho}{\partial x^\nu} \right) \tilde{\nabla}_\mu \psi_p
\end{aligned}$$

[Note that, while spinors live according to the symmetry of flat spacetime, Supergravity lives in a curved spacetime. At every point in curved spacetime we therefore define a flat spacetime where our objects can transform. Hence the natural objects related to spinors are γ_a - if we see objects γ_μ we are really talking about $e^a_\mu \gamma_a$. For example,

$$\tilde{\nabla}_\mu \gamma^\nu = \tilde{\nabla}_\mu {e^\nu}_a \gamma^a = \partial_\mu ({e_a}^a \gamma^a) + ... = \gamma^a \tilde{\nabla}_\mu {e_a}^a.]$$

The following gamma matrix identities are of use:

$$\begin{aligned}
\gamma^a &= \text{const.} \\
\{\gamma^a, \gamma^b\} &= 2\eta^{ab} \\
\gamma^\mu &= e^\mu_a(x)\gamma^a = \gamma^\mu(x) \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}(x) \\
\tilde{\nabla}_\mu \gamma^\nu &= 0 \\
\gamma^{\mu\nu\rho} D_\mu \psi_\nu &= \gamma^{\mu\nu\rho} \tilde{\nabla}_\mu \psi_\nu.
\end{aligned}$$

General relativity may be written in Vierbein formalism as

$$\mathcal{L}_{\rm EH} = \frac{\sqrt{-g}}{2\kappa^2} e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab}(w(e)),$$

where $e \equiv \det(e_\mu{}^a) = \sqrt{-g}$ and

$$R_{\mu\nu}{}^{ab}(w) = 2\partial_{[\mu} w^{ab}_{\nu]} + 2w^{ac}_{[\mu} w^{b}_{\nu]c}.$$

Now, defining a 'flat ϵ-tensor' $\hat{\epsilon}^{abcd}$ ($\hat{\epsilon}^{0123} = 1 \iff \hat{\epsilon}_{0123} = -1$) with

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} &= e e^\mu_a e^\nu_b e^\rho_c e^\sigma_d \hat{\epsilon}^{abcd} \\ \Rightarrow \epsilon^{0123} &= e\det(e^\nu_a) = e.e^{-1} = 1 \\ \Rightarrow \epsilon^{\mu\nu\rho\sigma} &= \text{const.} \\ \Rightarrow \hat{\epsilon}^{efcd}\hat{\epsilon}_{abcd} &= -2(\delta^e_a\delta^f_b - \delta^f_a\delta^e_b) \\ \Rightarrow \epsilon^{\mu\nu\rho\sigma}\hat{\epsilon}_{abcd}e^c_\rho e^d_\sigma &= -4e(e^\mu_{[a}e^\nu_{b]}), \end{aligned}$$

so that the Einstein-Hilbert action becomes,

$$\mathcal{L}_{\rm EH} = \frac{1}{8\kappa^2}\epsilon^{\mu\nu\rho\sigma}\hat{\epsilon}_{abcd}e^c_\rho e^d_\sigma R_{\mu\nu}{}^{ab}.$$

For the variation with respect to w, we rewrite the R part of the action to obtain (performing an integration by parts to get to the second line),

$$\begin{aligned} \delta_w\mathcal{L}_{\rm EH} &= \frac{1}{8\kappa^2}\epsilon^{\mu\nu\rho\sigma}\hat{\epsilon}_{abcd}e^c_\rho e^d_\sigma D_{[\mu}\delta w_{\nu]}{}^{ab} = 0 \\ D_{[\mu}\epsilon_{\sigma]}{}^d &= 0 \iff w_\mu{}^a{}_b = w_\mu{}^a{}_b(e). \end{aligned} \tag{1.34}$$

This was just the torsionless condition that we imposed. For the variation with respect to e, consider

$$\begin{aligned} & M = e^{\ln M} \Rightarrow \det M = e^{\rm tr\ln M} \Rightarrow \delta\det M = \det M.{\rm tr}(M^{-1}\delta M) \\ \Rightarrow\quad & \delta e = e e^\rho_c \delta e^c_\rho \\ & e^\mu_a e^c_\mu = \delta^c_a \Rightarrow \delta e^\mu_a = -e^\rho_a(\delta e^c_\rho)e^\mu_c \\ \Rightarrow\quad & \delta_e\mathcal{L}_{\rm EH} = 0 \Rightarrow \frac{e}{2\kappa^2}\left[e^\rho_c e^\mu_a e^\nu_b - 2e^\rho_a e^\mu_c e^\nu_b\right] R^{ab}_{\mu\nu}\delta e^c_\rho \Rightarrow -e^\rho_c R + 2R^\rho_c = 0 \\ \iff\quad & R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \end{aligned}$$

which is just Einstein's equation!

1.10.3 Pure 4d $\mathcal{N} = 1$ SUGRA

We can now write down an expression of a supersymmetric theory for $g_{\mu\nu}$ and ψ_μ alone as per the mission in equation (1.32):

$$\begin{aligned}
\mathcal{L}(g_{\mu\nu}, \psi_\mu) &= \mathcal{L}_{\text{kin}}(g_{\mu\nu}) + \mathcal{L}_{\text{kin}}(\psi_\mu) + \mathcal{L}_{\text{int}}[\psi_\mu, g_{\mu\nu}] \\
&= \left[-\frac{1}{2\kappa^2}\sqrt{g}R(g)\right] + \left[-\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho\right] + ? \\
\Rightarrow \mathcal{L}_{\text{Pure}} &= \mathcal{L}_{\text{kin}}(g_{\mu\nu}) + \mathcal{L}_{\text{kin}}(\psi_\mu) \\
&\to \frac{1}{8\kappa^2}\epsilon^{\mu\nu\rho\sigma}\hat{\epsilon}_{abcd} e^c_\rho e^d_\sigma R_{\mu\nu}{}^{ab}(w(e)) - \frac{e}{2}\bar{\psi}\gamma^{\mu\nu\rho}\tilde{\nabla}_\nu \psi_\rho. \qquad (1.35)
\end{aligned}$$

where the last step was possible since $\Gamma^\sigma_{[\nu\rho]} = 0$. Here we recall concisely that having torsion-free spin-connection $w_\mu{}^a{}_b(e)$ implies that

$$\begin{aligned}
\tilde{\nabla}_\mu e_\nu{}^a &= \partial_\mu e^a_\nu + w_\mu{}^a{}_b e^b{}_\nu - \Gamma^\rho_{\mu\nu} e_\rho{}^a = 0 \\
D_{[\mu} e_{\nu]} &= 0 \iff \frac{\delta \mathcal{L}_{\text{EH}}}{\delta w_\mu{}^a{}_b} = 0 \\
\tilde{\nabla}_\rho \gamma^\mu &= 0
\end{aligned}$$

and that

$$\tilde{\nabla}_\mu \psi_\nu = \partial_\nu \psi_\rho + \frac{1}{4} w^{ab}_\mu(e)\gamma_{ab}\psi_\nu - \Gamma^\rho_{\mu\nu}(g)\psi_\rho = D_\mu \psi_\nu - \Gamma^\rho_{\mu\nu}(g)\psi_\rho.$$

We will use that

$$\begin{aligned}
\delta e^a_\mu &= \frac{\kappa}{2}\bar{\epsilon}\gamma^a \psi_\mu \Rightarrow \delta g_{\mu\nu} = \kappa \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} \\
\delta \psi_\mu &= \frac{1}{\kappa} D_\mu(w(e))\epsilon + \mathcal{O}(k\psi^2 \epsilon).
\end{aligned}$$

Making use of (some of which we have already proven, others which one could prove),

$$\begin{aligned}
\bar{\lambda}\gamma^{\mu\nu\rho}\chi &= \bar{\chi}\gamma^{\mu\nu\rho}\lambda \\
\bar{\lambda}\gamma^\mu \chi &= -\bar{\chi}\gamma^\mu \lambda \\
\overline{(\gamma_{ab}\epsilon)} &= -\bar{\epsilon}\gamma_{ab} \\
D_{[\mu} D_{\nu]}\chi &= \frac{1}{2}[D_\mu, D_\nu]\chi = \frac{1}{8} R^{ab}_{\mu\nu}\gamma_{ab}\chi
\end{aligned}$$

$$
\begin{aligned}
\{\gamma^{\mu\nu\rho}, \gamma_{ab}\} &= -4\gamma^{\rho} e^{\mu}_{[a} e^{\nu}_{b]} + 8 e^{\rho}_{[a} \gamma^{[\mu} e^{\nu]}_{b]} \\
\delta_w \mathcal{L}_{\mathrm{RS}} &= \frac{1}{2\kappa^2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} e^{c}_{\rho} \left[-\frac{\kappa}{4} \bar{\psi}_{\mu} \gamma^{d} \psi_{\sigma} \right] \delta^{ab}_{\nu} \\
e\gamma^{\mu\nu\rho} &= -i\epsilon^{\mu\nu\rho\sigma} \gamma_{\nu}\gamma_5 \\
\left(\bar{\psi}_{[\mu}\gamma_a \chi\right)\left(\bar{\epsilon}\gamma^{a}\psi_{\nu]}\right) &= \frac{1}{2}\left(\bar{\psi}_{[\mu}\gamma_b\psi_{\nu]}\right)\left(\bar{\epsilon}\gamma^{b}\chi\right)
\end{aligned}
$$

we will show that

$$\delta_{\mathrm{SUSY}} \mathcal{L}_{\mathrm{pure}} = 0 + \mathcal{O}(\kappa\psi^2\epsilon)$$

where $\delta_{\mathrm{SUSY}}\mathcal{L}(e^{a}_{\mu}, \psi_{\mu}, e_{\mu}{}^{ab}(e)) = \delta_e\mathcal{L} + \delta_{\psi}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta w^{ab}_{\nu}}\Big|_{w=w(e)} \cdot \frac{\delta w_{\nu}{}^{ab}(\epsilon)}{\delta e^{c}_{\mu}} \delta e^{c}_{\mu}$ and

$$\mathcal{L} \;=\; \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{RS}} + \mathcal{L}_{4F}.$$

Let us then first consider the variations with respect to e: Since the last term involves 4 fermion terms we do not consider it ($\delta_e\mathcal{L}_{4F} = \mathcal{O}(\kappa\epsilon\psi^5)$, $\delta_w\mathcal{L}_{4F} = \mathcal{O}(\kappa\epsilon\psi^3)$). Further

$$
\begin{aligned}
\delta_e \mathcal{L}_{\mathrm{EH}} &= \frac{e}{2\kappa^2}\left[e^{\rho}_{c} e^{\mu}_{a} e^{\nu}_{b} - 2 e^{\rho}_{a} e^{\mu}_{c} e^{\nu}_{b}\right] R^{ab}_{\mu\nu} \delta e^{c}_{\rho} \\
&= \frac{e}{2\kappa^2}\left[e^{\rho}_{c} e^{\mu}_{a} e^{\nu}_{b} - 2 e^{\rho}_{a} e^{\mu}_{c} e^{\nu}_{b}\right] R^{ab}_{\mu\nu} \left(\frac{\kappa}{2}\bar{\epsilon}\gamma^{c}\psi_{\rho}\right) \\
\delta_e \mathcal{L}_{\mathrm{RS}} &= \delta_e\left[-\frac{e}{2}\bar{\psi}_{\mu}\gamma^{\mu\nu\rho} D_{\nu}\psi_{\rho}\right] = \mathcal{O}(\kappa\psi^3\epsilon).
\end{aligned}
$$

For $w = w(e)$

$$
\begin{aligned}
\delta_w \mathcal{L}_{\mathrm{EH}} &= 0 \\
\delta_w \mathcal{L}_{\mathrm{RS}} &= \delta_w\left[-\frac{e}{2}\bar{\psi}_{\mu}\gamma^{\mu\nu\rho} D_{\nu}\psi_{\rho}\right] \\
&= -\left[\frac{e}{2}\delta_w(\bar{\psi}_{\mu})\gamma^{\mu\nu\rho} D_{\nu}\psi_{\rho}\right] - \left[\frac{e}{2}\bar{\psi}_{\mu}\gamma^{\mu\nu\rho} D_{\nu}\delta_w(\psi_{\rho})\right]\delta_w + \mathcal{O}(\kappa\psi^3\epsilon) \\
&= -\left[\frac{e}{2}\left(\frac{1}{\kappa}\overline{D_{\mu}\epsilon}\right)\gamma^{\mu\nu\rho} D_{\nu}\psi_{\rho}\right] - \left[\frac{e}{2}\bar{\psi}_{\mu}\gamma^{\mu\nu\rho} D_{\nu} D_{\rho}\epsilon)\right]\delta_w + \mathcal{O}(\kappa\psi^3\epsilon).
\end{aligned}
$$

Using the identities, it becomes precisely

$$-\frac{e}{2\kappa^2}\left[e^{\rho}_{c} e^{\mu}_{a} e^{\nu}_{b} - 2 e^{\rho}_{a} e^{\mu}_{c} e^{\nu}_{b}\right] R^{ab}_{\mu\nu}\left(\frac{\kappa}{2}\bar{\epsilon}\gamma^{c}\psi_{\rho}\right) + \mathcal{O}(\kappa\psi^3\epsilon)$$

which cancels with $\delta_e \mathcal{L}_{\text{EH}}$, so that indeed

$$\delta_{\text{SUSY}} \mathcal{L}_{\text{pure}} = 0 + \mathcal{O}(\kappa \psi^2 \epsilon).$$

In fact it is possible to show this result for all orders by absorbing the $\mathcal{L}_{4F}$ and $\mathcal{O}(\kappa \psi^3 \epsilon)$ into a modified spin connection with torsion,

$$\hat{w}_\mu{}^{ab}(e, \psi) = w_\mu{}^{ab} + \kappa_\mu{}^{ab}(e, \psi)$$

which is a fascinating result!

1.10.4 Pure 4d $\mathcal{N} = 1$ SUGRA with a cosmological constant

If one adds a cosmological constant to the Einstein-Hilbert action,

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{EH}} - \frac{1}{\kappa^2} \sqrt{-g} \Lambda \\ G_{\mu\nu} &= -\Lambda g_{\mu\nu}, \end{aligned}$$

it turns out that SUSY is broken. However in the case where $\Lambda < 0$, (AdS), SUSY can be restored by adding an additional term to $\mathcal{L}$. A way to understand why this is only possible for AdS and not dS is as follows: Just as the Poincaré algebra has an extension to the Super-Poincaré algebra, so too does the AdS isometry algebra $SO(2,3)$ have an extension to the $\mathcal{N} = 1$ superalgebra. However, for the dS isometry algebra $SO(1,4)$, such an extension cannot be made which still satisfies the super-Jacobi identities. One could get deSitter solutions in 4d, $\mathcal{N} = 1$ SUGRA theories, but only by adding matter multiplets - and any de Sitter solution of such a theory will have that SUSY is spontaneously broken - similarly for all d and $\mathcal{N}$!

1.10.5 Matter coupled to 4d, $\mathcal{N} = 1$ SUGRA

Using κ as $\frac{1}{M_P}$, so $\mathcal{L}_{\text{EH}} = -\frac{\kappa}{2} M_P^2 R$ and so on, one can consider some background material on non-linear σ-models which is outlined below: Given n_s scalar fields, $\psi^i(x)$, $(i = 1, ..., n_s)$ in terms of a kinetic term and a scalar potential $V(\phi, \phi^*)$ term as follows:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\left(b_{ij}+b_{lij}\varphi^k+b_{klij}\varphi^k\varphi^l+...\right)\partial_\mu\varphi^i\partial^\mu\varphi^j \\ &\quad -\left(a+a_i\varphi^i+a_{ij}\varphi^i\varphi^j+...\right).\end{aligned}$$

Here we identify the term $\left(b_{ij}+b_{lij}\varphi^k+b_{klij}\varphi^k\varphi^l+...\right)\to g^{ij}(\varphi)$ as a metric on a scalar manifold which has the same dimension as the number of scalars; this makes sense since

$$\begin{aligned} & \varphi^i \to \tilde{\varphi}^i(\varphi) \\ \Rightarrow \quad & g_{ij}(\varphi)\partial_\mu\varphi^i\partial^\mu\varphi^j \to \tilde{g}_{ij}(\tilde{\varphi})\partial_\mu\tilde{\varphi}^i\partial^\mu\tilde{\varphi}^j \\ \text{where} \quad & \tilde{g}_{ij}(\tilde{\varphi}) = \frac{\partial\varphi^k}{\partial\tilde{\varphi}^i}g_{kl}(\varphi\tilde{\varphi})\frac{\partial\varphi^l}{\partial\tilde{\varphi}^j}.\end{aligned}$$

One would require the following constraints:

- $g_{ij}=g_{ji}$
- g_{ij} is invertible (so we don't get non-singular kinetic terms).
- Must be Riemannian (i.e. we require g_{ij} to be positive definite to ensure unitarity and that we won't have to include ghosts for example).

We will then be able to form all the usual geometric quantities, like $\Gamma^k_{ij}(g)$, $R^k_{ij}(\Gamma(g))$ etc., and if ${R_{ij}}^k=0$, we can just define $g_{ij}(\varphi)=\delta_{ij}$ so that the theory is renormalisable. Of course, since gravity is non-renormalisable, we can consider curved scalar manifolds which is the case for many extended SUGRA theories. For global SUSY, we have chiral multiplets involving a complex scalar field and a Majorana spinor, ϕ^m,χ^m, where $m=1,...,n_c$ and we define $\phi^{\bar{m}}=(\phi^m)^*$. It turns out that the most general action with at most 2 soacetime derivatives in each term and global $\mathcal{N}=1$ SUSY is completely determined by two functions, a real function called a Kähler potential $K(\phi^m,\phi^{\bar{m}})$, and a holomorphic function called the Superpotential $W(\phi^m)$, where $\frac{\partial W}{\partial\phi^{\bar{m}}}=\partial_{\bar{m}}W=0$. Since SUSY distinguishes between left- and right- handed ϕ^m and $\phi^{\bar{m}}$ via $\phi^m\to\chi^m_L$ and $\phi^{\bar{m}}\to\chi^{\bar{m}}_R$, one requires a restricted linear-sigma model and so one requires that $\mathcal{M}_{\text{scalar}}$ be a complex manifold with n_c complex coordinates and holomorphic transition functions. Further, since only $\phi^m\to\tilde{\phi}^m(\phi^m)$ may be allowed to preserve left-right transitions, the metric is constrained,

$$g_{mn} = g_{\bar{m}\bar{n}} = 0$$
$$g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K.$$

Such manifolds are called Kähler manifolds. From a superspace approach,

$$\mathcal{L} \;=\; \int d^4\theta \mathcal{K}(\Phi^m, \Phi^{m*}) + \left[\int d^2\theta W(\Phi) + \text{h.c.}\right].$$

Using holonomy arguments, an equivalent definition of Kähler manifolds is that $\text{Hol}(\mathcal{M}_{\text{scalar}}) \subset U(n_C)$ which comes from the form of the SUSY transformations (the holonomy group is the set of different paths you sweep out). The Superpotential $W(\phi^m)$ determines $V(\phi, \phi^*)$ as well as Yukawa interactions and fermionic mass terms. The scalar potential is given by

$$V = V_F = g^{m\bar{n}}(\partial_m W)\partial_{\bar{n}} W^* \geq 0$$

where V_F is called the F-potential term, and $g^{m\bar{n}} g_{\bar{n}l} = \delta^m_l$. The Yukawa mass term is given by

$$\mathcal{L}_{\text{Yuk/mass}} \;=\; -\left[(\mathcal{D}\partial_n W)\bar{\chi}^m_L \chi^n_L + \text{h.c.}\right]$$

where $\mathcal{D}\partial_n W = \partial_m \partial_n W - \Gamma^l_{mn}\partial_l W$, defined on the scalar manifold. The theory is then invariant under Kähler manifolds

$$K \to K' + r(\phi^m) + r^*(\phi^{\bar{m}})$$

since for holomorphic functions $g_{m\bar{n}} K = \partial_m \partial_{\bar{n}} K'$.

1.10.6 Chiral and vector multiplets in the global SUSY

Here one adds to the previous theory n_v vector multiplets (A^I_μ, λ^I) where $I = 1, ..., n_v$ which requires adding a kinetic Lagrangian involving the sum of terms that are products of 2 non-Abelian $F^I_{\mu\nu} = 2\partial_{[\mu} A^I_{\nu]} + f^I_{JK} A^J_\mu A^k_\nu$ with real or imaginary part of the holomorphic 'gauge kinetic function' $f_{IJ}(\phi^m)$. In order to do this, one also needs to add certain terms bilinear in fermions χ^m and λ^I, and one may either have the case with no gauged isometries where nothing further happens to $\mathcal{L}$ and $V = V_F = g^{m\bar{n}}(\partial_m W)(\partial_{\bar{n}} W^*)$, or one could have the case where there are gauged isometries of $\mathcal{M}_{scalar}$. In this case, the field content is given by $(\phi^m, \chi^m)\bigoplus(A^I_\mu, \lambda^I)$ and

$$\hat{\partial}_\mu \phi^m = \partial_\mu \phi^m - A^I_\mu \xi^m_I(\phi)$$

where $\xi^m_I(\phi)$ are the holomorphic killing vectors. One then needs to add to V_F a new term called the D-term potential involving two terms $P_I = i[\xi^m_I \partial_m K - \eta_I]$, where η_I are called the Fayet-Iliopoulos terms (only for Abelian gauge group factors). If $K(\phi, \phi^*)$ changes under the isometry generated by $\xi^m_I(\phi)$, then a further term needs to be added to P_I. From holomorphicity of $\xi^m_I(\phi)$, one can show that

$$\xi^m_I(\phi) - i g^{m\bar{n}}(\partial_{\bar{m}} P_I)$$

and one obtains for global SUSY

$$\begin{aligned} &V = V_F + V_D \geq 0 \\ &\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\text{Re}f)_{IJ} F^I_{\mu\nu} F^{\mu\nu I} - \frac{1}{2} g_{ij} \partial_\mu \varphi^i \partial^\mu \phi^j \end{aligned}$$

1.10.7 Coupling n_c chiral and n_V vector multiplets to SUGRA

This process requires covariantisation and adding various M_P-suppressed Noether couplings to φ_μ to $\mathcal{L}_{\text{pure}}$. Thereafter W and $\partial_m W$ are replaced and further M_P-suppressed Noether terms involving W and not the derivative are added. The result is that $V = V_F + V_D$ becomes more complicated. However the upside of doing all this is that for $V < 0$ one is able to have both AdS and dS solutions because the new K and W are no longer independent in SUGRA due to their redefinitions, and once can rewrite $\mathcal{L}$ in terms of $G = K + M_P^2 \ln\frac{W}{M_P^2} + \text{CC}$.

1.10.8 Extended SUGRA

[31]One may extend SUGRA in 4d by increasing the number of supercharges to $\mathcal{N} = 2$ or higher. See the table below for $\mathcal{N} \leq 8$ which is the maximal supergravity with spin$\leq$ 2; chiral multiplets of spins are given by (1/2,0), gauge multiplets by

	Fields	Multiplets
$\mathcal{N}=1$	(e^a_μ, ψ_μ)	(2,3/2)
$\mathcal{N}=2$	$(e^a_\mu, \psi^i_\mu, A_\mu)$, $i=1,2$	(2,3/2) + (3/2,1)
$\mathcal{N}=3$	$(e^a_\mu, \psi^i_\mu, A^i_\mu, \lambda)$, $i=1,2,3$	(2,3/2) + 2(3/2,1)+(1,1/2)
$\mathcal{N}=4$	$(e^a_\mu, \psi^i_\mu, A^k_\mu, B^k_\mu, \lambda^i, B)$	(2,3/2) + 3(3/2,1)+3(1,1/2)+(1/2,0)
	$i=1,2,3,4$, $k=1,2,3$	
$\mathcal{N}=8$	$(e^a_\mu, 8\psi^i_\mu, 28A^{IJ}_\mu, 56\chi_{ijk}, 70\nu)$	(2,3/2) + 7(3/2,1)+21(1,1/2)+35(1/2,0)

TABLE 1.2: Table comparing fields and multiplets for different supercharge number

(1,1/2), gravitino multiplets by (3/2,1)), and supergravity multiplets by (2,3/2); the fields are given by the graviton e^a_μ, the gravitino ψ^i_μ, the vector/photon A^i_μ, axial vector B^i_μ, the fermion χ and scalars ϕ, ν, B. Each results in a different type of scalar manifold. Note that the numbers of fields and multiplets starts to look like numbers in Pascal's triangle.

For $\mathcal{N} \geq 2$ (unlike $\mathcal{N}=1$) it is not possible to have a gauged SUGRA (with gauged interactions) such that $V=0$. Also, while $SO(8)$ does not contain the standard model, $SO(10)$ does.

Another way to extend SUGRA is to take it to arbitrary dimension. Here one needs to understand spinor representations of $SO(1, D-2)$ which has starting point the irreps of the Clifford algebra Cliff$(1, D-1)$, $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1}$ for $\mu, \nu = 0, 1, ..., d-1)$. For even d one finds that there exists only on irrep of the algebra with complex dimension $2^{D/2}$, so that for $d=2$ the Γ_μ matrices are 2×2-dimensional, i.e. $\boldsymbol{C}^2$, $d=4$ is $\boldsymbol{C}^4$ and $d=6$ is $\boldsymbol{C}^8$. When d is odd, one chooses an irrep from Cliff$(1, d-2)$. There are 2 definitions for choosing the set of Γs in this case, and one may have the case where there are two inequivalent irreps of Cliff$(1, d-1)$. One may impose chirality and/or reality conditions (possible or not depending on the value of d). One finds that different spinors exist in different dimensions. For example Weyl spinors, where $\gamma^2 = \mathbf{1}$, the spinors may only exist in even dimension d. Majorana spinors on the other hand may exist for all d except when $d = 5, 6, 7\text{mod}8$. Majorana-Weyl spinors exist only for $d = 2\text{mod}8$. Symplectic Majorana are a better choice in 5 or 7 dimensions. This leads to the following table (notice that 'M' and 'W' cuts dimensions in half, while 'MW' cuts it to a quarter of original dimension.

Minimal number of real components of a spinor rep			
Original # components	d	# real components	Spinor type
4	2	1	MW
4	3	2	M
8	4	4	M or W
8	5	8	SM
16	6	8	SMW
16	7	16	SM
32	8	16	M or W
32	9	16	M
64	10	16	MW
64	11	32	M
128	12	64	M or W

Table 1.3: Table showing spinor types allowed in various dimensions, and reduction in the number of components as a result of these allowed spinor types

1.10.9 $\mathcal{N} = 1$ SUGRA in 11d

In 11d SUGRA, only $\mathcal{N} = 1$ is possible, and Q has 32 real components. The field content of the multiplet will include a Majorana spinor ψ_μ together with the Vierbein e^a_μ and a 3-form $C_{\mu\nu\rho}$, $\mu, \nu, \rho, a, b, c = 0, 1, ..., 10)$. The field strength may then be derived from the 3-form as

$$dC_3 = \frac{1}{4!} G_{\nu\nu\rho\sigma} dx^\mu \wedge ... \wedge dx^\sigma \iff G_{\mu\nu\rho\sigma} = 4\partial_{[\mu} C_{\nu\rho\sigma]}.$$

The 11-dimensional action is then invariant under $C_3 \to C_3 + d\Lambda_2$ for Λ_2 and arbitrary 2-form; this invariance is known as p-form gauge invariance. 11 dimensional SUGRA is known to be the low energy limit of M-theory, and it is the highest dimensional theory that has supergravity.

1.10.10 $\mathcal{N} = 1$ SUGRA in 10d

In type I SUGRA there are 2 types of multiplets: there is $(e^a_\mu, \psi^a_\mu, B_{\mu\nu}, \chi^-, \phi)$ which includes the real scalar dilaton and the negative helicity dilatino. These represent the bosons and form the Neveu Schwartz-Neveu-Schwartz (NSNS) sector. Then there are the n_v vector multiplets (A_μ, λ^+) which includes the gaugino. These are the fermionic elements and for the Ramond-Ramond (RR) sector. The chiral spinor fields have gravitational and gauge anomalies except for

$$G = E_8 \times E_8, SO(32), \left[E_8 \times U(1)^{248}, U(1)^{496}\right]$$

where the bracketed groups have no string theoretic applications. The rather complicated action includes a term $-\frac{1}{2}|H_3|^2 = \frac{1}{3!}H_{\mu\nu\rho}H^{\mu\nu\rho}$ where $H_3 = dB_2 + \frac{\alpha'}{4}(\Omega_{\rm YM} - \Omega_L)$ requires a Chern-Simons form to allow for anomally cancellation:

$$\Omega_{\rm YM} = {\rm tr}\left(A \wedge A \wedge A - \frac{2}{3} i A \wedge A \wedge A\right).$$

Type IIA SUGRA is the low energy limit of type IIA string theory; here some SUSY generators have + chirality while others have - chirality. There is also type IIB SUGRA which is the low energy limit of type IIB string theory.

1.11 AdS/CFT Correspondence

1.11.1 Anti-de Sitter space (AdS)

1.11.1.1 Vacuum solutions to Einstein's equations in AdS

From equations (1.16) and (1.17),

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} \quad = \quad \kappa T_{\mu\nu}$$

where $\kappa = 8\pi G_N$. From dimensional analysis where $[M]^{-1} = [T] = [L]$ we have that for $D = 4$, $[F] = [MLT^{-2}] = [M]^2 \Rightarrow [G_{\mu\nu}] = [\kappa] = [M]^{-2}, [\rho] = [p] = [M]^4$. For general spacetime dimension D, these become $[\kappa] = [L]^{D-2} = [M]^{2-D}, [T_{\mu\nu}] = [\rho] = [p] = [L]^{-D} = [M]^D \Rightarrow [\kappa T_{\mu\nu}] = [M]^2 = [L]^{-2} \Rightarrow [\Lambda] = [L]^{-2}$. The vacuum solution to equation (1.36) (i.e. where $T_{\mu\nu} = 0$) are found by finding the trace to obtain

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R &= -\Lambda g^{\mu\nu} \qquad (1.36) \\ R\left(1 - \frac{D}{2}\right) &= -\Lambda D \\ \Rightarrow R &= \frac{2D\Lambda}{D-2} \end{aligned}$$

$$R^{\mu\nu} = \left(\frac{D\Lambda}{D-2} - \Lambda\right) g^{\mu\nu} = \left(\frac{2\Lambda}{D-2}\right) g^{\mu\nu} = \frac{R}{D} g^{\mu\nu}.$$

We can thus see that we have three cases

$\Lambda = 0$	no curvature	Minkowski (flat)
$\Lambda > 0$	positive curvature	de Sitter (dS)
$\Lambda < 0$	negative curvature	Anti-de Sitter (AdS).

Noting that equation (1.36) remains true if we rescale Λ and $R^{\mu\nu}$ in the same way, we can define Λ in AdS (thus negative) in the following way to avoid having factors involving D in the denominator of either Λ or R

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2} \iff R = -\frac{D(D-1)}{L^2},$$

where L is the AdS length.

1.11.1.2 AdS metric in various coordinates

Anti-de Sitter, or AdS, space is the maximally symmetric Lorentzian manifold with a constant negative (scalar) curvature; just as one would use a Minkowski metric when working in Euclidean space, we use an AdS space when working in hyperbolic space. Consider the hyperboloid on $\mathbb{R}^{d,2}$ and its related manifold, given by

$$(X^0)^2 + (X^{d+1})^2 - \sum_{i=1}^{d} (X^i)^2 = L^2$$

$$\Rightarrow \qquad ds^2 = -dX^0 dX^0 - dX^{d+1} dX^{d+1} + \sum_{i=1}^{d} dX^i dX^i. \qquad (1.37)$$

This space clearly has $SO(2, d)$ isometry (and therefore has maximal component subgroup $SO(2) \times SO(d)$. There are multiple parametrisations / diffeomorphisms (an invertible function that maps one differential manifold to another such that both the function and its inverse are smooth, that is, an isomorphism of smooth manifolds) possible, for instance

$$X^0 = R\text{cosh}\rho \cos\tau$$
$$X^i = R\text{sinh}\rho w_i, \qquad i = 1, ..., d$$
$$X^{d+1} = R\text{cosh}\rho \sin\tau$$

where $\sum_{i=1}^{d} w_i^2 = 1$ with

$$\begin{aligned} w_1 &= \cos\theta_1 \\ w_2 &= \sin\theta_1\cos\theta_2 \\ &\vdots \\ w_{d-1} &= \sin\theta_1\sin\theta_2\ldots\sin\theta_{d-2}\cos\theta_{d-1} \\ w_d &= \sin\theta_1\sin\theta_2\ldots\sin\theta_{d-2}\sin\theta_{d-1}, \qquad \theta_i\in[0,\pi),\ \theta_{d-1}\in[0,2\pi), \end{aligned}$$

induces the metric

$$ds^2 = R^2\left(-\cosh^2\!\rho d\tau^2 + d\rho^2 + \sinh^2\!\rho d\Omega_{d-1}^2\right) \tag{1.38}$$

where Ω_{d-1} is the metric of the $d-1$-sphere S^{d-1}. $\rho > 0$ and $\tau \in [0, 2\pi)$ covers the whole hyperboloid. This situation is illustrated in figure 1.8. Near $\rho = 0$, this metric

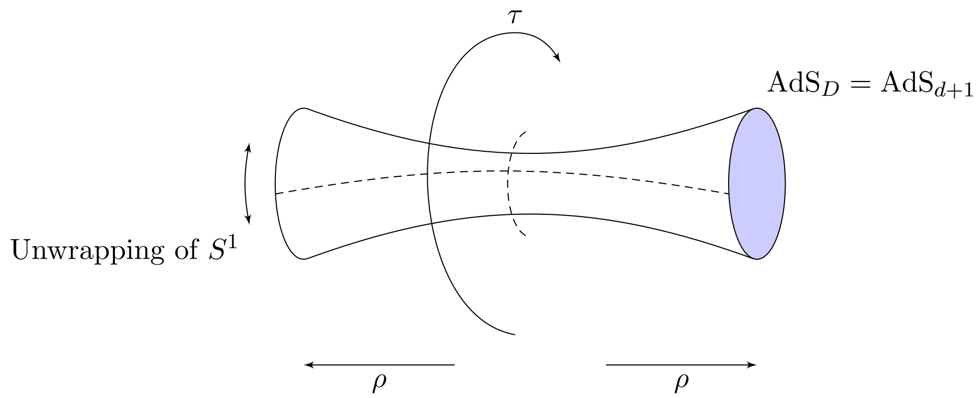


FIGURE 1.8: Diagram illustrating the AdS metric given in equation (1.38).

has topology $S^1 \times \mathbb{R}^d$ where the S^1 is indicative of having closed timelike curves in the τ direction.

$$ds^2 \approx L^2\left(d\tau^2 + d\rho^2 + \rho^2 d\Omega_{d-1}^2\right).$$

Now, the metric for the Einstein static universe on $\mathbb{R}^{1,d}$ is given as follows, where, in the second step we make use of the transformation $u_\pm = t \pm r \Rightarrow -du_+du_- = -dt^2 + dr^2$, in the third line we set $u_\pm = \tan v_\pm$ (so that $(u_+ - u_-)^2 = (\tan v_+ - \tan v_-)^2 =$

$\frac{(\sin v_+ \cos v_- - \sin v_- \cos v_+)^2}{\cos^2 v_- \cos^2 v_+} = \frac{\sin^2(v_+ - v_-)}{\cos^2 v_- \cos^2 v_+}$) and in the fourth line we set $v_\pm = \frac{\tau \pm \theta}{2}$:

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2 \\ &= -du_+ du_- + \frac{1}{4}(u_+ - u_-)^2 d\Omega_{d-1}^2 \\ &= \frac{1}{\cos^2 v_- \cos^2 v_+}\left(-dv_+ dv_- + \frac{1}{4}\sin^2(v_+ - v_-) d\Omega_{d-1}^2\right) \\ &= \frac{1}{\cos^2 v_-(\tau,\theta)\cos^2 v_+(\tau,\theta)}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2\right). \end{aligned}$$

Upon analytic continuation we obtain

$$ds'^2 = -dt^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2.$$

The Penrose diagram for this is illustrated in Figure 1.9, and has isometry $\mathbb{R} \times S^d$. where $\theta \in [0, \pi]$, $t \in (-\infty, \infty)$. We could further define $\tan\theta = \sinh\rho$ for $\theta \in$

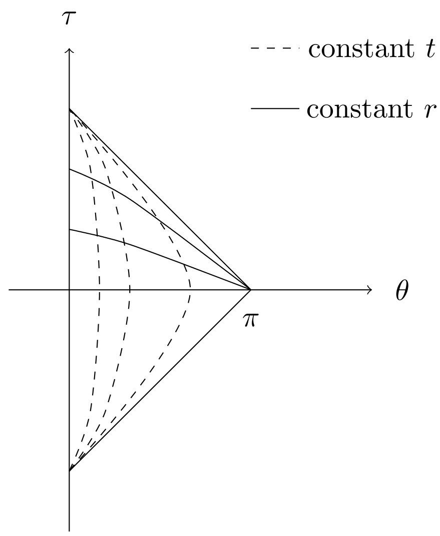


FIGURE 1.9: Diagram showing the Penrose diagram for Einstein static universe, with lines of constant t and r displayed.

$[0, \pi/2)$ in equation (1.38) (which is equivalent to decompactifying the circle so that $\tau \in (-\infty, \infty)$), and obtain (using $\cosh^2\rho - \sinh^2\rho = 1 \Rightarrow \sinh\rho = \sec\theta$),

$$\begin{aligned} ds^2 &= \frac{R^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2\right) \\ \rightarrow ds'^2 &= -d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2, \end{aligned} \tag{1.39}$$

which is the metric of one half of the Einstein static universe. This space-time is found illustrated in figure 1.10. As in the comment below equation (1.18), the time-like Killing vector ∂_τ is everywhere non-zero (since multiplied by a square number); further in conformally rescaled coordinates it is of constant norm, and so τ is just a global time coordinate. Alternatively, defining

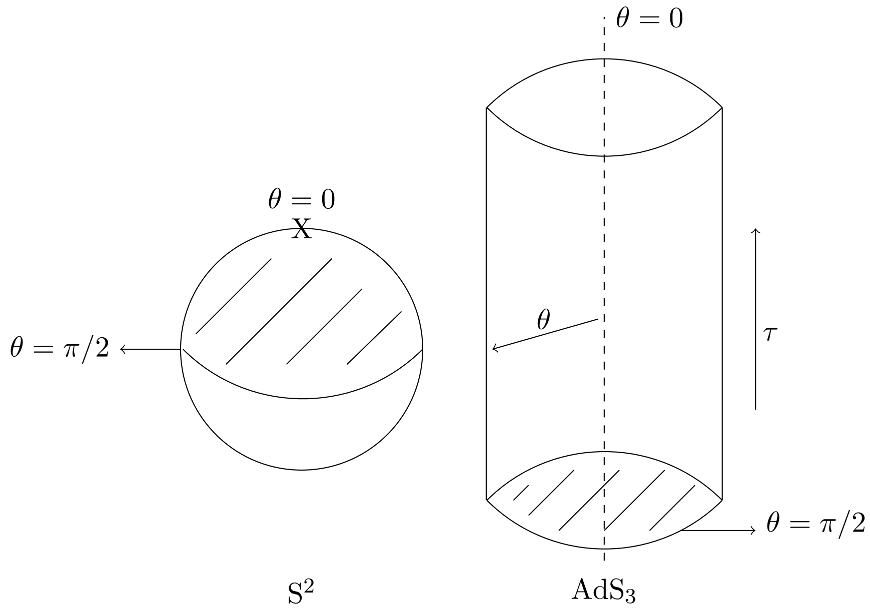


FIGURE 1.10: Diagram illustrating the AdS metric given in equation (1.39).

$$
\begin{aligned}
X^0 &= \sqrt{L^2 - r^2} \sin \frac{t}{L} \\
X^i &= r w_i \quad i = 1, ..., d \\
X^{d+1} &= \sqrt{L^2 - r^2} \cos \frac{t}{L}.
\end{aligned}
$$

in equation (1.37), will result in the metric

$$
ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2,
$$

which, with $L = 2(\times G)M$ and $d = 3$ is the metric of the Schwartzschild black hole in AdS_5; this may be compared to the the Schwartzschild black hole in Minkowski given in equation (1.18).

The last coordinate system we will consider is the Poincaré coordinate system. Here

$$\begin{aligned}
X^0 &= \frac{1}{2u}\Big(1+u^2(L^2+\vec{X}^2-t^2)\Big) \\
\Rightarrow dX^0 &= \frac{1}{2}\left(-\frac{1}{u^2}+(L^2+\vec{X}^2-t^2)\right)du+\frac{u}{2}\sum X^j dX^j-\frac{u}{2}tdt \quad (1.40)\\
X^i &= LuX^i, \quad i=1,...,d-1 \\
\Rightarrow dX^i &= LX^i du+LudX^i \quad (1.41)\\
X^d &= \frac{1}{2u}\Big(1-u^2(L^2-\vec{X}^2+t^2)\Big) \\
\Rightarrow dX^d &= \frac{1}{2}\left(-\frac{1}{u^2}-(L^2-\vec{X}^2+t^2)\right)du+\frac{u}{2}\sum X^j dX^j-\frac{u}{2}tdt \quad (1.42)\\
X^{d+1} &= Lut \\
\Rightarrow dX^{d+1} &= Ltdu+Ludt. \quad (1.43)
\end{aligned}$$

Plugging this into equation (1.37) we obtain for the du^2 term

$$\begin{aligned}
&-\left[\frac{1}{2}\left(-\frac{1}{u^2}+(L^2+\vec{X}^2-t^2)\right)\right]^2+L^2\vec{X}^2 \\
&+\left[\frac{1}{2}\left(-\frac{1}{u^2}-(L^2-\vec{X}^2+t^2)\right)\right]^2-L^2t^2,
\end{aligned}$$

which simplifies beautifully to

$$L^2\left(\frac{du^2}{u^2}\right).$$

Terms arising from the last 2 terms of equation (1.40) cancel identically with equation (1.42) and the only terms arising from cross-terms with these are

$$-L^2\Big(u\sum X^j dudX^j-utdudt\Big),$$

which cancels with the cross terms arising from equations (1.41) and (1.43), and we finally obtain (with coordinate transformation $z=\frac{1}{u}$ in the second line) the AdS metrics given by

$$\begin{aligned}
ds^2 &= L^2\left(\frac{du^2}{u^2}+u^2(-dt^2+d\vec{x}^2)\right) \\
\Rightarrow ds^2 &= \frac{L^2}{z^2}\left(dz^2-dt^2+d\vec{x}^2\right),
\end{aligned}$$

in terms of Poincaré coordinates $(u, t, \vec{x})$ and $(z, t, \vec{x})$ respectively. We see that the conformal boundary at $z = 0$ has topology $\mathbb{R}^{2,d-1}$. The isometries $SO(1, d-1)$ and $SO(1, 1)$ are clearly manifest. However, these Poincaré coordinates cover only half of space-time, in particular, only when $z > 0$ or equivalently $u > 0$.

1.11.2 Conformal Field Theory

1.11.2.1 Conformal Transformations

This subsection is based on Ref. [85]. The conformal group in d dimensions will be shown to be described as a non-compact group SO(d+1,1). A conformal transformation is an invertible mapping $x \to x'$ such that $g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x)$, i.e. locally equivalent to a rotation and dilation, and the Poincare group ($\Lambda = 1$) is a subgroup.

Such transformations conserve angle but not distance as can be seen as follows:

$$\begin{aligned}
\cos\theta' &= \frac{A' \cdot B'}{(A' \cdot A')^{1/2}\,(B' \cdot B')^{1/2}} \\
&= \frac{g'_{\mu\nu} A'^{\mu} B'^{\nu}}{\left(g'_{\mu\nu} A'^{\mu} A'^{\nu}\right)^{1/2} \left(g'_{\rho\sigma} A'^{\rho} B'^{\sigma}\right)^{1/2}} \\
&= \frac{\Lambda g_{\mu\nu} A'^{\mu} B'^{\nu}}{\left(\Lambda g_{\mu\nu} A'^{\mu} A'^{\nu}\right)^{1/2} \left(\Lambda g_{\rho\sigma} A'^{\rho} B'^{\sigma}\right)^{1/2}} \\
&= \frac{g_{\mu\nu} A'^{\mu} B'^{\nu}}{\left(g_{\mu\nu} A'^{\mu} A'^{\nu}\right)^{1/2} \left(g_{\rho\sigma} A'^{\rho} B'^{\sigma}\right)^{1/2}} \\
&= \cos\theta
\end{aligned}$$

For a transformation $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$,

$$g'_{\rho\sigma} \frac{dx'^{\rho}}{dx^{\mu}} \frac{dx'^{\sigma}}{dx^{\nu}} = g_{\mu\nu} \Rightarrow g'_{\mu\nu} = g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right)$$

which, to be consistent with the conformal condition requires that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}. \tag{1.44}$$

This is called the conformal Killing vector equation. (For Poincaré transformations, the right had side would be 0 identically, and for metrics with space-time dependence

$g_{\mu\nu} = g_{\mu\nu}(x)$, the left hand-side would become

$$\begin{aligned} & \epsilon^{\alpha} g_{\mu\nu,\alpha} + (\partial_{\mu}\epsilon^{\alpha})\, g_{\alpha\nu} + (\partial_{\nu}\epsilon^{\alpha})\, g_{\alpha\mu} \\ = \;& \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} + \epsilon^{\alpha}\left(g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu} - g_{\alpha\mu,\nu}\right) \\ = \;& \nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu}). \end{aligned}$$

We will now prove that $\partial^2 f = 0$. Contracting equation (1.44) with $g^{\mu\nu}$ gives

$$2\partial_{\rho}\epsilon^{\rho} = df(x). \tag{1.45}$$

A second thing that can be done is acting with ∂_{ρ} on equation (1.44) to obtain

$$\partial_{\rho}\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\partial_{\rho}\epsilon_{\mu} = \partial_{\rho} f(x) g_{\mu\nu}.$$

Writing the two other cyclic permutations and using arithmetic to isolate the $\partial_{\rho}\partial_{\mu}\epsilon_{\nu}$ term we get

$$2\partial_{\rho}\partial_{\mu}\epsilon_{\nu} = \left[g_{\mu\nu}\partial_{\rho} + g_{\nu\rho}\partial_{\mu} - g_{\rho\mu}\partial_{\nu}\right] f(x) \tag{1.46}$$

and finally contracting with $g^{\mu\rho}$ we obtain

$$\partial^2 \epsilon_{\nu} = \frac{2-d}{2} \partial_{\nu} f\left(x\right). \tag{1.47}$$

Now, acting with ∂^2 on equation (1.44) and using our expression in equation (1.47) we get

$$\left(2-d\right)\partial_{\mu}\partial_{\nu} f = \partial^2 f g_{\mu\nu}$$

and, contracting with $g^{\mu\nu}$ gives

$$\left(2-d\right)\partial^2 f = d\partial^2 f \Rightarrow \left(2-2d\right)\partial^2 f = 0 \Rightarrow \partial^2 f = 0 \tag{1.48}$$

Just a brief comment on this, a solution to $\partial^2 f = 0$ is called harmonic as can be seen as follows. Writing $f = A(x)B(y)...$, the solution is given by $\frac{1}{A}\frac{d^2A}{dx^2} + \frac{1}{B}\frac{d^2B}{dy^2} + ... = 0$ and hence each of the individual terms equals a constant and the constants sum to zero. This implies that $\frac{d^2A}{dx^2} = \text{const} \times A$ which admits periodic/harmonic solutions. Now, for $d = 1$, equation (1.48) does not imply that f is harmonic since $2 - 2d = 0$,

but we see that in this case $\frac{d^2A}{dx^2} = 0$ which does not admit sinusoidal solutions but just a linear solution.
Now, since $f(x) = \frac{2\partial_\rho \epsilon^\rho}{d}$, our transformation parameter takes the form

$$\begin{aligned} \epsilon_\mu &= a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\sigma}x^\nu x^\sigma \\ \Rightarrow f(x) &= \frac{2}{d}b^\lambda{}_\lambda + \left(c^\sigma{}_{\lambda\sigma}x^\lambda + c^\lambda{}_{\lambda\sigma}x^\sigma\right) \end{aligned}$$

where $c_{\mu\nu\sigma} = c_{\mu\sigma\nu}$. From equations (1.44), (1.45) and (1.47), $\epsilon_\mu = a_\mu$ holds automatically and corresponds to translations

$$x'^\mu = x^\mu + A^\mu.$$

Substituting $\epsilon_\mu = b_{\mu\rho}x^\rho$ into equation (1.44) gives

$$\begin{aligned} f(x)g_{\mu\nu} &= \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu \\ \Rightarrow \frac{2}{d}b^\lambda{}_\lambda g_{\mu\nu} &= b_{\mu\nu} + b_{\nu\mu}. \end{aligned}$$

If we now limit ourselves to diagonal $g_{\mu\nu} = \eta_{\mu\nu}$, it is clear that $b_{\mu\nu}$ consists of a diagonal part and an antisymmetric part ($m_{\mu\nu} = -m_{\nu\mu}$),

$$b_{\mu\nu} = p\eta_{\mu\nu} + m_{\mu\nu}$$

which corresponds to dilations and rigid rotations,

$$\begin{aligned} x'^\mu &= Px^\mu \\ x'^\mu &= M^\mu{}_\nu x^\nu. \end{aligned}$$

Substituting $\epsilon_\nu = c_{\nu\lambda\sigma}x^\lambda x^\sigma$ and $f(x) = c^\lambda{}_{\lambda\sigma}x^\sigma = b_\sigma x^\sigma$ into equation (1.47), we obtain,

$$\begin{aligned} 2\partial_\rho\partial_\mu\epsilon_\nu &= \left[\eta_{\mu\nu}\partial_\rho + \eta_{\nu\rho}\partial_\mu - \eta_{\rho\mu}\partial_\nu\right] f(x) \\ c_{\nu\rho\mu} &\equiv \eta_{\mu\nu}b_\rho + \eta_{\nu\rho}b_\mu - \eta_{\rho\mu}b_\nu \\ \Rightarrow \epsilon_\nu = c_{\nu\rho\mu}x^\rho x^\mu &\equiv 2x \cdot b\, x_\nu - x^2 b_\nu. \end{aligned}$$

Notice that

$$\begin{aligned}\epsilon_\nu &= 2x\cdot b\, x_\nu - x^2 b_\nu \\ &= (x\cdot b)\, x_\nu + (b^\nu x_\mu - x^\mu b_\nu)\, x_\mu\end{aligned}$$

and so the special conformal transformation corresponds to a local scaling (first term) and a local rotation (since the bracketed term in the second bracket is antisymmetric). To find the finite transformation associated with this, one needs to solve

$$\dot{x_\mu} = 2x\cdot b\, x_\mu - x^2 b_\mu.$$

Now, if we let $y_\mu = \frac{x_\mu}{x^2}$ then

$$\begin{aligned}\dot{y}_\mu &= \frac{\dot{x}_\mu}{x^2} - \frac{2\,(x\cdot\dot{x})\,x_\mu}{x^4} \\ &= \frac{2x\cdot b\,x_\mu - x^2 b_\mu}{x^2} - \frac{2\left(2\,(x\cdot b)\,x^2 - x^2\,(x\cdot b)\right)x_\mu}{x^4} \\ &= -b_\mu \\ \Rightarrow y_\mu &= \frac{x_\mu}{x^2} = \frac{(x_0)_\mu}{x_0^2} - tb_\mu.\end{aligned}$$

Squaring both sides of this equation to obtain a consistency equation, we obtain

$$\frac{1}{x^2} = \frac{1 - 2tx_0\cdot b + tb^2 x_0^2}{x_0^2}$$

and therefore

$$\begin{aligned}x'_\mu &= x^2\left(\frac{(x_0)_\mu}{x_0^2} - tb_\mu\right) \\ &= \frac{x_0^2}{1 - 2tx_0\cdot b + t^2 b^2 x_0^2}\left(\frac{(x_0)_\mu - tb_\mu x_0^2}{x_0^2}\right) \\ &= \frac{(x_0)_\mu - B_\mu x_0^2}{1 - 2x_0\cdot B + B^2 x_0^2}.\end{aligned}$$

Notice that this finite form of the special conformal transformation could be obtained by a series of inversion, translation, inversion as follows:

$$x_\mu \to \frac{x_\mu}{x^2}$$

$$\begin{aligned}
&\rightarrow \quad \frac{x_\mu}{x^2} - B_\mu = \frac{x_\mu - x^2 B_\mu}{x^2} \\
&\rightarrow \quad \frac{x^2\left(x_\mu - x^2 B_\mu\right)}{\left(x_\mu - x^2 B_\mu\right)\left(x^\mu - x^2 B^\mu\right)} = \frac{x_\mu - x^2 B_\mu}{1 - 2x\cdot B + x^2 B^2}.
\end{aligned}$$

Furthermore, the infinitesimal version of this can easily be re-obtained by expanding $\frac{x_\mu - x^2 B_\mu}{1-2x\cdot B + x^2 B^2}$ to first order in B:

$$x'_\mu = \left(x_\mu - x^2 B_\mu\right)\left(1 + 2x\cdot B\right) = x_\mu + 2x\cdot B x_\mu - x^2 B_\mu.$$

In summary, then, the finite transformations resulting from the conformal transformation are given by

$$\begin{aligned}
\text{Translation} \quad & x'^\mu = x^\mu + \epsilon^\mu \\
\text{Dilation} \quad & x'^\mu = \epsilon x^\mu \\
\text{Rigid rotation} \quad & x'^\mu = \epsilon^\mu{}_\nu x^\nu \\
\text{Special Conformal Transformation} \quad & x'^\mu = \frac{x^\mu - x^2\epsilon^\mu}{1 - 2x\cdot\epsilon + \epsilon^2 x^2}.
\end{aligned}$$

The scale factor $\Lambda(x)$ is roughly the square of the coefficient of x_μ, namely

$$\Lambda(x) = \frac{1}{\left(1 - 2x\cdot B + B^2 x^2\right)^2}$$

1.11.2.2 The Witt and Virasoro algebras

The following is based on Refs. [86, 87]. Any holomorphic map in 2 dimensions is conformal. First let's review the definition of an holomorphic map. For $f = u(z) + iv(z)$ to be analytic where $z = x + iy$, $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial f}{i\partial y}$ leading to the Cauchy-Riemann equations, $v_y = u_x$, $v_x = -u_y$. An holomorphic map is one for which $\partial_z f \neq 0$, i.e. the Jacobian,

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ -u_y & u_x \end{vmatrix} = u_x^2 + u_y^2 = |\partial_z f|^2 \neq 0$$

A change of coordinates to $r = |\partial_z f|$ and $\tan\theta = \frac{u_y}{u_x}$ results in the following Jacobian matrix,

$$r\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{1.49}$$

which is a composition of a scaling and rotation and hence is a conformal map that preserves angles. In particular, the infinitesimal conformal generators,

$$L_n = -z^{n+1}\frac{\partial}{\partial z}$$

(for $n \in \mathbb{Z}$) are holomorphic maps. There are an infinite number of these generators. The commutation relations for these generators are given by

$$[L_n, L_m] = (n-m)\, L_{m+n}$$

with the Lie algebra relating to this being called the Witt algebra. In 1+1 dimesions, we can define all these generators except for rigid rotations (so 3 generators in total), and these obey the Witt algebra with $m = -1, 0, 1$ which coincides with the bosonic case. These generators could either be written in terms of derivatives,

$$\begin{array}{ccc} \text{Time translations}: & L_{-1} = -\partial_t & \rightarrow \text{Conserved charge is momentum} \\ \text{Scaling}: & L_0 = -t\partial_t & \rightarrow \text{Conserved charge is dilatation operator} \\ \text{Sp. Conformal}: & L_1 = -t^2\partial_t & , \end{array}$$

where L_0 is identified as the energy operator and must hence be bounded from below. These generators may also be written in terms of the 3 generators of $S\ell(2,\mathbb{Z})$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ ad - bc = 1,$$

which can also be written as

$$z' = \frac{az+b}{cz+d}.$$

These generators are given by the exponents below:

$$\text{Time translations} : z' = \frac{z+\epsilon}{0+1} \quad \rightarrow \quad \begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix} = e^{\begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix}} \Rightarrow L_{-1} = \begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix}$$

$$\text{Dilation} : z' = \frac{(1+\epsilon)\, z}{1-\epsilon} \quad \rightarrow \quad \begin{bmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{bmatrix} = e^{\begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix}} \Rightarrow L_{0} = \begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix}$$

$$\text{Sp. Conformal} : z' = \frac{z}{1+\epsilon z} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} = e^{\begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix}} \Rightarrow L_{1} = \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix}$$

where the last expression follows from

$$z \rightarrow \frac{1}{z} \rightarrow \frac{1}{z} + \epsilon \rightarrow \frac{z}{1+\epsilon z}$$

Now, in quantum theory, symmetry transformations act projectively on states. We dissect this comment here. Groups have composition laws, so if g_1 and g_2 are in the group, there must be a third element such that

$$g_1 g_2 = g_3$$

which is a representation of the group. QM and QFT are probabilistic and hence we have what is called a projective representation,

$$g_1 g_2 = e^{i\phi_{12}} g_3.$$

It has been shown that one can obtain a projective representation if either the group has a non-trivial topology, or if the algebra develops a central charge.

We will consider the option where it develops a central charge, denoted by c. Physically, c measures the number of degrees of freedom in the theory (after complicated analysis). It is called 'central' because it commutes with everything in the algebra, and 'charge' because it commutes with the Hamiltonian and is hence conserved. Now, we want the algebra to have the same results as we had classically for L_{-1}, L_0, L_1, and so the commutation relations become,

$$[L_n, L_m] = (n-m)\, L_{m+n} + \frac{c}{12}(n-1)n(n+1)\delta_{n+m,0}\mathbb{1}.$$

This is the quantum mechanical version of the Witt algebra and is known as the Virasoro algebra.

1.11.2.3 Scaling dimension Δ

Scale transformations are defined as

$$\begin{aligned} x' &= \lambda x \\ \phi'(\lambda x) &= \lambda^{-\Delta}\phi(x). \end{aligned}$$

Hence

$$\begin{aligned} S &= \int d^d x \mathcal{L}\left(\phi, \partial_\mu \phi\right) \\ &\to \lambda^d \int d^d x \mathcal{L}\left(\lambda^{-\Delta}\phi, \lambda^{-1-\Delta}\partial_\mu\phi\right). \end{aligned}$$

Therefore, for an action $S = \int d^d x \partial_\mu \phi \partial^\mu \phi$, it is scale invariant if

$$d - 2\left(1+\Delta\right) = 0 \Rightarrow \Delta = \frac{d}{2} - 1,$$

and for an action $S = \int d^d x\, \partial_\mu\phi\partial^\mu\phi + \phi^n$,

$$\begin{aligned} d - n\Delta = d - n\left(\frac{d}{2}-1\right) &= 0 \\ \Rightarrow n &= \frac{d}{\Delta} = \frac{2d}{d-2} \end{aligned}$$

which is an integer for $d = 3, 4, 6$ and even if $d = 3, 4$, which ties back to what we saw in equation (1.9).

1.11.2.4 Noether's Theorem - generators

For a transformation (actively) $x'^{\mu} = x^{\mu} + \delta x^{\mu} = x^{\mu} + \epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a}$,

$$\begin{aligned} \phi'(x') &= \phi(x) + \epsilon_a \frac{\delta F(x)}{\delta \epsilon_a} \\ &= \phi(x') - \epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a} \partial_{\mu} \phi(x') + \epsilon_a \frac{\delta F(x')}{\delta \epsilon_a}. \end{aligned} \tag{1.50}$$

Here, the F is defined to satisfy whatever symmetry the transformation corresponds to, so under translations, $\frac{dF}{d\epsilon} = 0$, and under the Lorentz transformation the additional term $S^{\rho\nu}$ obeys the Lorentzian algebra, and so forth. In the above equation, the first line was written in terms of x and the second line in terms of x' (which we can do since, to first order in ϵ, $\epsilon_a \frac{\delta F(x')}{\delta \epsilon_a} = \epsilon_a \frac{\delta F(x)}{\delta \epsilon_a}$). Now, we define the following at point x:

$$\delta\phi(x) = \phi'(x) - \phi(x) = -i\epsilon_a G_a \phi(x)$$

Therefore, noting that equation (1.50) is defined at a single point x', it is clear that

$$\begin{aligned} \delta\phi(x) &= -i\epsilon_a G_a \phi(x) = -\epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a} \partial_{\mu} \phi + \epsilon_a \frac{\delta F}{\delta \epsilon_a} \\ \Rightarrow iG_a \phi &= \frac{\delta x^{\mu}}{\delta \epsilon_a} \partial_{\mu} \phi - \frac{\delta F}{\delta \epsilon_a}. \end{aligned} \tag{1.51}$$

We therefore obtain the following generators:

Translation $\quad \phi(x^{\mu}) = \phi(x'^{\mu} - \epsilon^{\mu})$

$\Rightarrow P_{\mu} = -i\partial_{\mu}$

Rotation $\quad \phi(x^{\mu}) = \phi(x'^{\mu} - \epsilon^{\mu}{}_{\nu} x^{\nu})$

$\Rightarrow \mathcal{J}_{\mu}{}^{\nu} = -ix^{\nu}\partial_{\mu} \Rightarrow \mathcal{J}_{\mu\nu} \equiv i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) = -\left[x_{\mu}P_{\nu} - x_{\nu}P_{\mu}\right]$

Dilation $\quad \phi(x^{\mu}) = \phi(x'^{\mu} - \epsilon x^{\mu})$

$\Rightarrow D = -ix^{\mu}\partial_{\mu} = x^{\mu}P_{\mu}$

SCT $\quad \phi(x^{\mu}) = \phi(x'^{\mu} - 2\left(x \cdot \epsilon\right) x^{\mu} + x^2 \epsilon^{\mu})$

$\Rightarrow K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}\right) = 2x_{\mu}x^{\nu}P_{\nu} - x^2 P_{\mu} = 2x_{\mu}D - x^2 P_{\mu}.$

The commutators are given as follows (for detailed calculations, see Appendix C.1):

$$\begin{aligned}
[P_\mu, \mathcal{J}_{\rho\sigma}] &= i\eta_{\mu\rho}P_\sigma - i\eta_{\mu\sigma}P_\rho \\
[D, P_\nu] &= iP_\nu \\
[D, K_\nu] &= -iK_\nu \\
[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right) \\
[K_\mu, P_\nu] &= 2i\left[\eta_{\mu\nu}D - \mathcal{J}_{\mu\nu}\right] \\
[K_\rho, \mathcal{J}_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right).
\end{aligned}$$

Note that one could alter the statement for Noether's theorem in equation (1.51) to

$$T_a\phi = i\partial_\mu\left[\frac{\delta x^\mu}{\delta\epsilon_a}\phi\right],$$

in which case the above generators become the following

$$\begin{aligned}
&\text{Translation} \quad P_\mu \to -i\partial_\mu \\
&\text{Rotation} \quad \mathcal{J}_{\mu\nu} \to i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right), \text{ since } \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \\
&\text{Dilation} \quad D \to -ix^\mu\partial_\mu - id_\phi, \text{ where } d_\phi = \text{number of dimensions} \\
&\text{SCT} \quad K_\mu \to -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2d_\phi x_\mu\right).
\end{aligned}$$

Counting the number of elements described by these generators we obtain d from P_μ, d from K_μ, $\frac{d(d-1)}{2}$ from $\mathcal{J}_{\mu\nu}$ and 1 from D, i.e.

$$\frac{d(d-1)}{2} + d + d + 1 = \frac{d^2+3d+2}{2} = \frac{(d+1)(d+2)}{2}$$

hence we can describe this using a (d+2)-dimensional antisymmetric matrix.

1.11.2.5 Redefining generators - the SO(d+1,1) group

We define a new antisymmetric object J_{ab} (where $\mu = 1, ..., d$ and $a = -1, 0, ..., d$)($\eta_{\mu\nu}$ defined here as being Euclidean, all +1 on the diagonal),

$$\begin{aligned}
J_{\mu\nu} &= \mathcal{J}_{\mu\nu} \\
J_{0,\mu} &= \frac{1}{2}\left(P_\mu + K_\mu\right)
\end{aligned}$$

$$\begin{aligned} J_{-1,\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right) \\ J_{-1,0} &= D \end{aligned}$$

This new group then satisfies the commutation relations

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right)$$

where $\eta_{ab} = \mathrm{diag}\left(-1, 1, 1, ..., 1\right)$. Let's show this.

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= [\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho,\sigma}] \\
&= i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right) \\
&= i\left(\eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}\right) \\
[J_{0\mu}, J_{0\nu}] &= \frac{1}{4}\left([P_\mu, K_\nu] + [K_\mu, P_\nu]\right) \\
&= \frac{1}{4}\left(-2i\left[\eta_{\nu\mu}D - \mathcal{J}_{\nu\mu}\right] + \left(2i\left[\eta_{\mu\nu}D - \mathcal{J}_{\mu\nu}\right)\right]\right) \\
&= -i\mathcal{J}_{\mu\nu} = -i\eta_{00}\mathcal{J}_{\mu\nu} \\
[J_{-1\mu}, J_{-1\nu}] &= -\frac{1}{4}\left([P_\mu, K_\nu] + [K_\mu, P_\nu]\right) \\
&= -\frac{1}{4}\left(-2i\left[\eta_{\nu\mu}D - \mathcal{J}_{\nu\mu}\right] + \left(2i\left[\eta_{\mu\nu}D - \mathcal{J}_{\mu\nu}\right)\right]\right) \\
&= i\mathcal{J}_{\mu\nu} = -i\eta_{-1\,-1}\mathcal{J}_{\mu\nu} \\
[J_{-1\,0}, J_{-1\,0}] &= [D, D] = 0 \\
[J_{\mu\nu}, J_{0\rho}] &= \left[\mathcal{J}_{\mu\nu}, \frac{1}{2}\left(P_\rho + K_\rho\right)\right]\frac{1}{2}\left([\mathcal{J}_{\mu\nu}, P_\rho] + [\mathcal{J}_{\mu\nu}, K_\rho]\right) \\
&= \frac{1}{2}\left(i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right) - i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right)\right) \\
&= i\left(\eta_{\rho\mu}J_{-1\,\nu} - \eta_{\rho\nu}J_{-1\,\mu}\right) \\
[J_{\mu\nu}, J_{-1\rho}] &= \left[\mathcal{J}_{\mu\nu}, \frac{1}{2}\left(P_\rho - K_\rho\right)\right] = \frac{1}{2}\left([L_{\mu\nu}, P_\rho] - [L_{\mu\nu}, K_\rho]\right) \\
&= \frac{1}{2}\left(i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right) + i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right)\right) \\
&= i\left(\eta_{\rho\mu}J_{0\,\nu} - \eta_{\rho\nu}J_{0\,\mu}\right) \\
[J_{\mu\nu}, J_{-10}] &= [L_{\mu\nu}, D] = 0 \\
[J_{-1\mu}, J_{0\nu}] &= \left[\frac{1}{2}\left(P_\mu - K_\mu\right), \frac{1}{2}\left(P_\nu + K_\nu\right)\right] = \frac{1}{4}\left([P_\mu, K_\nu] - [K_\mu, P_\nu]\right) \\
&= \frac{1}{4}\left(-2i\left(\eta_{\nu\mu}D - L_{\nu\mu}\right) - 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right)\right) \\
&= -i\eta_{\mu\nu}J_{-1,0}
\end{aligned}$$

$$\begin{aligned}
[J_{-1\,0}, J_{0\mu}] &= \left[D, \frac{1}{2}(P_\mu + K_\mu)\right] = \frac{1}{2}([D, P_\mu] + [D, K_\mu]) \\
&= \frac{1}{2}(iP_\mu - iK_\mu) \\
&= i\eta_{00} J_{-1\,\mu} \\
[J_{-1\,0}, J_{-1\mu}] &= \left[D, \frac{1}{2}(P_\mu - K_\mu)\right] = \frac{1}{2}([D, P_\mu] - [D, K_\mu]) \\
&= \frac{1}{2}(iP_\mu + iK_\mu) \\
&= -i\eta_{-1,-1} J_{0\,\mu}
\end{aligned}$$

These 10 commutators represent all possibilities, and hence we have shown that the algebra holds. Let us now construct the invariants $\Gamma(x)$ that are left invariant by all conformal transformations. Translation and rotation invariance requires that Γ depends only on the distance between points, $|x_i - x_j|$. Furthermore, scale invariance implies that Γ depends only on ratios of these distances, $\frac{|x_i - x_j|}{|x_k - x_l|}$.
For special conformal transformations, we have

$$\begin{aligned}
\left(\vec{x}'_i - \vec{x}'_j\right) &= \frac{\vec{x}_i - \vec{b}x_i^2}{1 - 2b\cdot x_i + b^2 x_i^2} - \frac{\vec{x}_j - \vec{b}x_j^2}{1 - 2b\cdot x_j + b^2 x_j^2} \\
x_i'^2 &= \frac{x_i^2}{1 - 2b\cdot x_i + b^2 x_i^2} \\
\Rightarrow \left|\vec{x}'_i - \vec{x}'_j\right|^2 &= \frac{1}{\left(1 - 2b\cdot x_i + b^2 x_i^2\right)\left(1 - 2b\cdot x_j + b^2 x_j^2\right)} \\
&\quad \left[x_i^2\left(1 - 2b\cdot x_j + b^2 x_j^2\right) + x_j^2\left(1 - 2b\cdot x_i + b^2 x_i^2\right)\right. \\
&\quad \left. -2\left(x_i\cdot x_j - x_i^2 x\cdot b - x_j^2 x\cdot b + 2b x_i^2 x_j^2\right)\right] \\
&= \frac{x_i^2 + x_j^2 - 2x_i\cdot x_j}{\left(1 - 2b\cdot x_i + b^2 x_i^2\right)\left(1 - 2b\cdot x_j + b^2 x_j^2\right)} \\
\Rightarrow \left|\vec{x}'_i - \vec{x}'_j\right| &= \frac{|\vec{x}_i - \vec{x}_j|}{\left(1 - 2b\cdot x_i + b^2 x_i^2\right)^{1/2}\left(1 - 2b\cdot x_j + b^2 x_j^2\right)^{1/2}}
\end{aligned}$$

hence it is clear that the simplest invariants are cross-ratios involving 4 points, e.g.

$$\Gamma = \frac{|\vec{x}_1 - \vec{x}_2|\,|\vec{x}_3 - \vec{x}_4|}{|\vec{x}_1 - \vec{x}_3|\,|\vec{x}_2 - \vec{x}_4|}, \frac{|\vec{x}_1 - \vec{x}_2|\,|\vec{x}_3 - \vec{x}_4|}{|\vec{x}_1 - \vec{x}_4|\,|\vec{x}_2 - \vec{x}_3|},$$

of which $\frac{N(N-3)}{2}$ may be constructed.

As an example, notice this invariance in the (1,1)-dimensional case

$$
\begin{aligned}
\Gamma &= \frac{\left(\frac{az_1-b}{cz_1-d}-\frac{az_2-b}{cz_2-d}\right)\left(\frac{az_3-b}{cz_3-d}-\frac{az_4-b}{cz_4-d}\right)}{\left(\frac{az_1-b}{cz_1-d}-\frac{az_3-b}{cz_3-d}\right)\left(\frac{az_2-b}{cz_2-d}-\frac{az_4-b}{cz_4-d}\right)} \\
&= \frac{((az_1-b)(cz_2-d)-(az_2-b)(cz_1-d))}{((az_1-b)(cz_3-d)-(az_3-b)(cz_1-d))}\times \\
&\quad \frac{((az_3-b)(cz_4-d)-(az_4-b)(cz_3-d))}{((az_2-b)(cz_4-d)-(az_4-b)(cz_2-d))}\times \\
&\quad \frac{(cz_1-d)(cz_2-d)(cz_3-d)(cz_4-d)}{(cz_1-d)(cz_2-d)(cz_3-d)(cz_4-d)} \\
&= \frac{(z_1(-ad+bc)+z_2(-bc+ad))\,(z_3(-ad+bc)+z_4(-bc+ad))}{(z_1(-ad+bc)+z_3(-bc+ad))\,(z_2(-ad+bc)+z_4(-bc+ad))} \\
&= \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}.
\end{aligned}
$$

1.11.2.6 Noether's Theorem continued - conserved currents and charges

In subsubsection 1.11.2.4 we considered the transformation $x'^{\mu} = x^{\mu} + \epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a}$. Writing

$$\phi'(x') = \mathcal{F}(\phi(x)) = \phi(x) + \epsilon_a \frac{\delta x^{\mu}}{\delta \epsilon_a},$$

we can write the new action as follows:

$$
\begin{aligned}
S' &= \int d^d x \mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right) \\
&= \int d^d x' \mathcal{L}\left(\phi'(x'), \partial_\mu \phi'(x')\right) \\
&= \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \left(\partial x^{\nu}/\partial x'^{\mu}\right)\partial_\nu \mathcal{F}(\phi(x))\right)
\end{aligned}
$$

If we consider ϵ_a infinitesimal (so $\det(1+\epsilon) \approx \text{Tr}\epsilon$), and local, i.e. allow ϵ_a to vary with x, we can write

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \delta^{\nu}_{\mu} + \partial_\mu\left(\epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right), \quad \frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu} - \partial_\mu\left(\epsilon_a \frac{\delta x^{\nu}}{\delta \epsilon_a}\right)$$

$$\left|\frac{\partial x'}{\partial x}\right| \approx 1 + \partial_\mu \left(\epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\right).$$

Therefore, to calculate δS to first order in ϵ_a,

$$\begin{aligned}
S' &= \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \left(\partial x^\nu / \partial x'^\mu\right) \partial_\nu \mathcal{F}(\phi(x))\right) \\
&= \int d^d x \left[1 + \partial_\mu \left(\epsilon_a \frac{\delta x^\nu}{\delta \epsilon_a}\right)\right] \\
&\quad \mathcal{L}\left(\phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}, \left(\delta^\nu_\mu - \partial_\mu \left(\epsilon_a \frac{\delta x^\nu}{\delta \epsilon_a}\right)\right) \partial_\nu \left(\phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}\right)\right) \\
&= \int d^d x \left[1 + \partial_\mu \left(\epsilon_a \frac{\delta x^\nu}{\delta \epsilon_a}\right)\right] \\
&\quad \mathcal{L}\left(\phi(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}, \partial_\mu \phi(x) + \left(\partial_\mu \left(\epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}\right) - \partial_\mu \left(\epsilon_a \frac{\delta x^\nu}{\delta \epsilon_a}\right) \partial_\nu \phi(x)\right)\right) \\
\Rightarrow \delta S &= \int d^d x \left[\frac{\partial \mathcal{L}}{\partial \phi} \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a} + \partial_\mu \left(\epsilon_a \frac{\delta x^\nu}{\delta \epsilon_a}\right) \mathcal{L}\right. \\
&\quad \left. + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \left(\partial_\mu \left(\epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a}\right) - \partial_\mu \left(\epsilon_a \frac{\delta x^\nu}{\delta \epsilon_a}\right) \partial_\nu \phi(x)\right)\right].
\end{aligned}$$

Then, since this is a symmetry, the action is stationary against variation in the fields and so δS vanishes for any position-dependent parameters $\epsilon(x)$. Therefore,

$$\begin{aligned}
\delta S &= \int d^d x \left[\left(\partial_\mu \epsilon_a\right) \frac{\delta x^\nu}{\delta \epsilon_a} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \left(\left(\partial_\mu \epsilon_a\right) \frac{\delta \mathcal{F}}{\delta \epsilon_a} - \left(\partial_\mu \epsilon_a\right) \frac{\delta x^\nu}{\delta \epsilon_a} \partial_\nu \phi(x)\right)\right] \\
&= -\int d^d x j^\mu_a \partial_\mu \epsilon_a \qquad (1.52) \\
&= 0
\end{aligned}$$

where

$$j^\mu_a = \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right] \frac{\partial x^\nu}{\partial \epsilon_a} - \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \frac{\delta \mathcal{F}}{\delta \epsilon_a} \qquad (1.53)$$

Integrating equation (1.52) by parts, we get the conservation law.

$$\delta S = \int d^d x \epsilon_a \partial_\mu j^\mu_a = 0 \Rightarrow \partial_\mu j^\mu_a = 0,$$

and if we define the charge

$$Q_a = \int d^{d-1} x \, j^0_a$$

then

$$\begin{aligned} \dot{Q}_a &= \int d^{d-1}x\, \partial_0 j_a^0 \\ &= -\int d^{d-1}x\, \partial_i j_a^i \\ &= 0 \end{aligned}$$

since it is the integral of a total derivative and is thus a surface term which must vanish at infinity. So the charge is indeed conserved.

We take a quick digression to the Ward Identities. Defining $X = \phi(x_1)\phi(x_2)...\phi(x_n)$, we have

$$\begin{aligned} \langle X \rangle &= \frac{1}{Z}\int [d\phi]\, X \exp(-S(\phi)) \\ &= \frac{1}{Z}\int \left[d\phi'\right] (X + \delta X) \exp(-S(\phi) - \int dx \partial_\mu j_a^\mu \epsilon_a(x)) \\ \Rightarrow \langle \delta X \rangle &= \int dx \, \langle \partial_\mu j_a^\mu X \rangle \, \epsilon_a(x) \end{aligned}$$

the integrand of which can be written explicitly (bar the ϵ term) using $\delta\phi(x) = \phi'(x) - \phi(x) = -i\epsilon_a G_a(x)$ to give the *Ward identity*,

$$\langle \partial_\mu \left(j_a^\mu \phi(x_1)\phi(x_2)...\phi(x_n) \right) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \phi(x_1)...G_a\phi(x_i)...\phi(x_n) \rangle$$

which vanishes because the LHS is a surface term. Defining now $Y = \phi(x_2)\phi(x_3)...\phi(x_n)$ and defining $t = x_i^0$ different to all the times in Y and bounded in a thin pill-box $t_- < t < t_+$ and by spatial infinity which excludes all other points $x_2, x_3, ...x_n$:

$$\int ds_\mu \langle (j_a^\mu \phi(x_1) Y) \rangle = -i \langle \phi(x_1) G_a Y \rangle \,.$$

Only the time components don't vanish, and this is true for arbitrary x, giving (for the second line, in the limit $t_- \to t_+ \to t$, that charge Q_a generates infinitesimal symmetry transformations; for Minkowski, replace Q_a with $-iQ_a$):

$$\begin{aligned} Q_a(t_+)\phi(x) - \phi(x)Q_a(t_-) &= -i\phi(x)G_a \\ [Q_a, \phi] &= -iG_a\phi \end{aligned}$$

1.11.2.7 Energy-Momentum Tensor

For a translation $x^{'\mu} = x^\mu + \epsilon^\mu$, $\frac{\partial x^\mu}{\partial \epsilon^\rho} = \eta^\mu_\rho$ and $\frac{\delta F}{\delta \epsilon^\rho} = 0$, and equation (1.53) becomes

$$\begin{aligned} j^\mu &= \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial_\rho \phi - \eta^\mu_\rho \mathcal{L}\right]\frac{\partial x^\rho}{\partial \epsilon^\nu} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\frac{\delta \mathcal{F}}{\delta \epsilon^\nu} \\ &= \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial_\rho \phi - \eta^\mu_\rho \mathcal{L}\right] x^\rho - 0 \\ &\sim T^{\mu\rho} x_\rho. \end{aligned}$$

We thus obtain the conservation law $\partial_\mu T^{\mu\nu} = 0$ where

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial^\nu \phi,$$

where (recapping) $\eta^{\mu\nu}$ is defined as $(1, 1, 1..., 1)$; the conserved charge is the momentum

$$P^\nu = \int d^{d-1}x T^{0\nu}$$

and energy is the 0-component of this,

$$P^0 = \int d^{d-1}x T^{00} = \int d^{d-1}x \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L}$$

and this generates time translations

$$[P_0, \phi] = -i\partial_0 \phi$$

The conserved charge of the energy-momentum tensor gives

$$[P_\mu, \phi] = -i\partial_\mu \phi$$

Under a transformation $x^{'\mu} = x^\mu + \delta x^\mu$, we can see that, by acting on it with the stress-energy tensor, the conserved current would then be given by

$$j_\mu = T_{\mu\nu}\delta x^\nu.$$

For translations, this would be given by

$$j_\mu = T_{\mu\nu}\epsilon^\nu$$

so pretty much $T_{\mu\nu}$. For scalings it would be given by

$$j_\mu = T_{\mu\nu}x^\nu\epsilon$$

Here the conservation law gives

$$\begin{aligned} \partial^\mu j_\mu &= 0 \\ \Rightarrow T_{\mu\nu}\delta^\nu_\mu &= 0 \end{aligned}$$

which implies that the stress-energy tensor is traceless. Furthermore the conserved charge for scaling is given by

$$Q = \int d^{d-1}xj_0 = \int d^{d-1}xT_{0\nu}x^\nu$$

and this conserved charge is called the dilatation operator. For the special conformal transformation, $\delta x_\nu = 2x \cdot bx_\nu - x^2 b_\nu$ so the conservation current is

$$j^\mu = T^{\mu\nu}\left(2x \cdot bx_\nu - x^2 b_\nu\right)$$

for which the conserved charge is

$$Q = \int d^{d-1}xj_0 = \int d^{d-1}xT_{0\nu}\left(2x \cdot bx^\nu - x^2 b^\nu\right)$$

which corresponds to the conserved charge of a local scaling and a local rotation (as was seen before). A result which will be used expansively later on is that, for an x-dependent transformation $x^{'\mu} = x^\mu + \epsilon^\mu(x)$, the induced change in the action can be written as (for $T_{\mu\nu}$ assumed to be symmetric)

$$\begin{aligned} \delta S &= \int d^d x T^{\mu\nu}\partial_\mu\epsilon_\nu \\ &= \frac{1}{2}\int d^d x T^{\mu\nu}\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right) \\ &= -\frac{1}{2}\int d^d x T^{\mu\nu}\delta g_{\mu\nu} \end{aligned}$$

whose correctness is not dependent on the equations of motion being obeyed. In particular, we have shown that $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} = \frac{2}{d}\partial_\rho \epsilon^\rho g_{\mu\nu}$ and therefore

$$\delta S \quad = \quad -\frac{1}{d}\int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho$$

which is zero due to the tracelessness of the stress-energy tensor.

1.11.2.8 Classical Field Theory

We now consider the subgroup of the Poincaré group which leaves the point $x = 0$ invariant by introducing spin $S_{\mu\nu}$ and thereafter, in a similar way, Δ:

$$L_{\mu\nu}\phi(0) = S_{\mu\nu}\phi(0).$$

Now, the Hausdorff formula is given by

$$e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}\left[[B, A], A\right] ....$$

Hence, to translate $L_{\mu\nu}$ to a non-zero value of x, we consider,

$$e^{ix^\sigma P_\sigma} L_{\mu\nu} e^{-ix^\rho P_\rho} \quad = \quad S_{\mu\nu} + \left[L_{\mu\nu}, -ix^\rho P_\rho\right].$$

However, we are assuming that $[L_{\mu\nu}(0), x^\sigma] = 0$ for x a constant, therefore,

$$\begin{aligned} e^{-D} L_{\mu\nu} e^{D} = e^{ix^\sigma P_\sigma} L_{\mu\nu} e^{-ix^\rho P_\rho} &\rightarrow L_{\mu\nu}(0) - ix^\rho \left[L_{\mu\nu}, P_\rho\right] \\ &= L_{\mu\nu}(0) - ix^\rho \left[-i\left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\right)\right] \\ &= L_{\mu\nu}(0) - \left(x_\mu P_\nu - x_\nu P_\mu\right). \end{aligned}$$

Similarly,

$$\begin{aligned} e^{ix^\sigma P_\sigma} D e^{-ix^\rho P_\rho} &\rightarrow D(0) - ix^\rho \left[D, P_\rho\right] \\ &= D(0) - ix^\rho \left(iP_\rho\right) \\ &= D(0) - ix^\rho \partial_\rho \\ e^{ix^\sigma P_\sigma} K_\mu e^{-ix^\rho P_\rho} &\rightarrow k_\mu - ix^\rho \left[K_\mu, P_\rho\right] \\ &= K_\mu(0) - ix^\rho \left[2i\left(\eta_{\mu\rho} D - L_{\mu\rho}\right)\right] \end{aligned}$$

$$= K_\mu(0) + 2x^\rho \left(\eta_{\mu\rho} D - L_{\mu\rho}\right).$$

Hence

$$\begin{aligned}
L_{\mu\nu}\phi(x) &= \left(L_{\mu\nu}(0) + i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\right)\phi(x) \\
D\phi(x) &= \left(D(0) - ix^\nu\partial_\nu\right)\phi(x) \\
K_\mu\phi(x) &= \left[K_\mu(0) + 2\left(x_\mu D - x^\nu L_{\mu\nu}\right)\right]\phi(x) \\
&= \left[K_\mu(0) + 2x_\mu D(0) - 2x^\nu L_{\mu\nu}(0) - 2ix_\mu x^\nu\partial_\nu - 2ix^\nu\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\right]\phi(x) \\
&= \left[K_\mu(0) + 2x_\mu D(0) - x^\nu L_{\mu\nu}(0) - 4ix_\mu x^\nu\partial_\nu + 2ix^2\partial_\mu\right]\phi(x).
\end{aligned}$$

We can require that $\phi(x)$ belong to an irreducible representation of the Lorentz group, and since $\tilde{\Delta}$, which we have called $D(0)$, commutes with all generators $k_{\mu\nu}$, it must be a multiple of the identity which then forces k_μ to vanish.

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d}\phi(x)$$

and the Jacobian is related to the scale factor in the conformal metric transformation ($g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x)$) as follows

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-d/2}.$$

Fields transforming as such are called "quasi-primary".

1.11.3 Overview of the AdS/CFT correspondence

This subsection reviews Ref. [20], where Maldacena sketches the key ideas and building blocks of the AdS/CFT correspondence. In section 1.9 we found the commutator relations obeyed by the super-Poincaré group, and in section 1.11.2 we found the commutator relations obeyed by the Conformal group. Ref. [88] went further and found the commutator relations obeyed by the superconformal group; indeed the superconformal group has twice the number of supersymmetries as the corresponding super-Poincaré group has. They further showed that the supersymmetry group of $AdS_5 \times S^5$ is the same as the superconformal group in 3+1 dimensions. This was one of the key insights that gave rise to Maldacena's paper Ref. [20].

The metric for the supergravity solution carrying D3-brane charge is given by

$$ds^2 = f^{-1/2} dx_{||}^2 + f^{1/2}(dr^2 + r^2 d\Omega_5^2), \qquad f = 1 + \frac{4\pi g N \alpha'^2}{r^4}$$

where $x_{||}$ are the four coordinates along the world-volume of the 3-brane, N is the number of branes in the stack, and r is the distance between the D3 branes. One may then define the decoupling limit, in which the theories on the branes decouple from the gravitational bulk (where the branes move together but the Higgs expectation corresponding to the separation remains fixed) as follows

$$\begin{aligned} &\text{Energies fixed} \\ &\alpha' \to 0 \\ &U \equiv \frac{r}{\alpha'} \iff \text{Mass of stretched strings fixed.} \end{aligned} \tag{1.54}$$

In this limit $f \to \frac{4\pi g N \alpha'^2}{r^4}$, and

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g N}} dx_{||}^2 + \sqrt{4\pi g N} \frac{dU^2}{U^2} + \sqrt{4\pi g N} d\Omega_5^2 \right],$$

which one recognises as the metric for Anti-de Sitter space-time,

$$ds^2 = \frac{U^2}{R^2} dx^2 + R^2 \frac{dU^2}{U^2},$$

so that it becomes evident that both the radius of the five-sphere, as well as the AdS_5 space is proportional to $N^{1/4}$, and that this supergravity solution may be trusted when $gN >> 1$ (in particular $N >> 1$). Even in the $\alpha' \to 0$ limit, proper energies (E_{proper}) and proper wavelengths (ω) remain fixed, with

$$E_{\text{proper}} = \frac{1}{\sqrt{\alpha'}} \frac{\omega (gN4\pi)^{1/4}}{U},$$

and the space-time action becomes

$$S \sim \frac{1}{\alpha'^4} \int d^{10}x \sqrt{G} R + \ldots.$$

Since, in this limit, this theory had a finite Planck length, one can find fields propagating on the AdS background; furthermore, the Hawking temperature is finite implying a finite flux of energy from the black hole to the AdS space-time. One expects that $\mathcal{N} = 4$, d=4 U(N) SYM includes in its Hilbert space states of type IIB supergravity on $(AdS_5 \times S^5)_N$ due to the fact that it itself is a unitary theory. In particular there is a duality conjectured between the two, supported by the fact that the supersymmetry group of $AdS_5 \times S^5$ is the same as the superconformal group in $3 + 1$ dimensions; the theory on the brane is 4-dimensional $\mathcal{N} = 4$, d=4 U(N) SYM at the superconformal point where $r = 0$. The superconformal $\mathcal{N} = 4$ SYM has symmetry given by the Lie supergroup $PSU(2, 2|4)$, which has the bosonic subgroup $SU(2, 2) \times SU(4) = SO(4, 2) \times SO(6)$. The correspondence is a strong/weak correspondence since at large gN we cannot trust perturbative calculations in Yang-Mills theory but we can trust calculations in supergravity on $(AdS_5 \times S^5)_N$, while at small gN quantum effects such as Hawking radiation are $1/N$ effects and are challenging in $(AdS_5 \times S^5)_N$ since the radius is small and hence the curvature very large. The SYM coupling given by

$$\frac{1}{g_{YM}^2} + i\frac{\theta}{8\pi^2} = \frac{1}{2\pi}\left(\frac{1}{g} + i\frac{\chi}{2\pi}\right),$$

where χ is the value of the RR scalar. A point to consider is that compactifying the theory breaks conformal invariance, leaving a Poincaré supersymmetry. However, we can trust the supergravity solutions if the physical lengths of the compact circles are big in α' units and we stay far from the horizon at $U = 0$, i.e.

$$U >> \frac{(gN)^{1/4}}{R_i}.$$

1.11.4 AdS/CFT dictionary

Here we review Ref. [39], where Witten was able to concretise the concepts from Ref. [20], proposing that correlation functions in conformal field theory are given by the dependence of the supergravity action on the asymptotic behaviour at infinity. An interesting claim in Ref.[20] is noted - that in order to describe $\mathcal{N} = 4$ $d = 4$ SYM, one should use, not just the low energy supergravity on AdS_5 but the whole ∞ tower of massive Kaluza-Klein states on $AdS_5 \times S^5$, and the chiral fields in $\mathcal{N} = 4$ $d = 4$ SYM correspond to Kaluza-Klein harmonics on $AdS_5 \times S^5$. Indeed it was shown that the Kaluza-Klein excitations of $AdS_5 \times S^5$ match precisely with certain operators of the $\mathcal{N} = 4$ theory.

$SO(2,d)$ acts on AdS_{d+1} as a group of ordinary symmetries, and on M_d, the boundary of AdS_{d+1}, as a group of conformal symmetries; so too, there are two ways of getting physics with $SO(2,d)$ symmetry: in a relativistic field theory, with or without gravity on AdS_{d+1} or in a CFT on M_d. Minkowski space arises as the boundary of AdS space, which may be understood as follows: $SO(2,d)$ acts on

$$uv - \sum_{i,j} \eta_{ij} x^i x^j = 0, \tag{1.55}$$

(with scale invariance $u, v, x^i \to su, sv, sx^i$ for $0 \neq s \in \mathbb{R}$), which is a compactification (since scale invariance allows us to set $v = 1$) of Minkowski space-time (for generic $v \neq 0$), known as the 'quadric', which preserves the metric form

$$-dudv + \sum_{i,j} \eta_{i,j} dx^i dx^j.$$

(Note that the quadric has an additional point at infinity where $v = 0$ when compared with the Minkowski metric; in the next paragraph's example this infinity gives exactly what is needed to make a sphere.) Topologically, this is just $S^1 \times S^{d-1}/\mathbb{Z}_2$ where we mod by $\mathbb{Z}_2$ because we have scaled out $s = -1$. Meanwhile, AdS_{d+1} coordinates obey

$$uv - \sum_{i,j} \eta_{ij} x^i x^j = 1,$$

which clearly reduces to (1.55) for u, v, x^i large (i.e. at the boundary of AdS_{d+1}).

Euclidean language makes understanding of the boundary behaviour easier. Here, AdS_{d+1} is identified as the open ball B_{d+1} with $\sum_{i=0}^{d} y_i^2 < 1$ in Euclidean $\mathbb{R}^{d+1}$, and with metric

$$ds^2 = \frac{4\sum_{i=1}^{d} {dy_i}^2}{(1-|y|^2)^2}, \tag{1.56}$$

while compactifying gives the closed unit ball $\bar{B}_{d+1}$ defined by $\sum_{i=0}^{d} y_i^2 \leq 1$ with boundary S^d, identified as the boundary of AdS_{d+1}, i.e. Minkowski space. Of course B_{d+1} does not extend over $\bar{B}_{d+1}$ due to the singularity at $|y| = 1$, so we could define an f such as $f = 1 - |y|^2$ so that

$$d\tilde{s}^2 = f^2 ds^2$$

where f is well defined up to conformal transformations $f = fe^w$ so that

$$d\tilde{s}^2 \to e^{2w} d\tilde{s}^2, \qquad w \in \mathbb{R}$$

which is precisely the metric of S^d. Putting this all together, this is saying that AdS_{d+1} has a metric invariant under $SO(1, d+1)$ and the boundary S^d has only a conformal structure that is preserved by the action of $SO(1, d+1)$.

Equation (1.56) could be written differently, using $r = \tanh(y/2)$ and $\sum_{i=1}^d dy_i^2 = dr^2 + r^2 d\Omega_{d-1}^2$:

$$\begin{aligned} dr &= \frac{1}{2}\text{sech}^2(y/2)dy \\ &= \frac{1}{2}\left(1-r^2\right)dy \\ \Rightarrow dy^2 &= \frac{4}{(1-r^2)^2}dr^2 \\ \Rightarrow ds^2 &= \frac{4dr^2}{1-r^2} + \frac{4r^2 d\Omega_{d-1}}{1-r^2} \\ ds^2 &= dy^2 + \sinh^2 d\Omega^2, \end{aligned} \tag{1.57}$$

where $0 \leq y < \infty$ and the boundary is at $y = \infty$. Another representation is where $x_0 > 0$ and

$$ds^2 = \frac{1}{x_0^2}\left(\sum_{i=0}^{d}(dx^i)^2\right).$$

Here the boundary is at $x_0 = 0$ (where the boundary is $\mathbb{R}^d$), as well as where $x^0 = \infty$ so that the other directions vanish so that this corresponds to a single point. These boundaries together give a sphere S^d.

The next thing to understand is the massless field equations (Einstein's equations with a negative cosmological constant). In AdS_{d+1}, $\forall \phi \in S^d, \exists$ a unique function on $\bar{B}_{d+1}$ with given boundary conditions that obeys the field equation $D_i D^i \phi = 0$. The

Laplace equation in representation (1.57) reads

$$\left(-\frac{1}{(\text{sinh}y)^d}\frac{d}{dy}(\text{sinh}y)^d\frac{d}{dy}+\frac{L^2}{\text{sinh}^2 y}\right)\phi=0,$$

so that for large y and ϕ written as $\phi=\sum_\alpha \phi_\alpha(y)f_\alpha(\Omega)$,

$$\frac{d}{dy}e^{dy}\frac{d}{dy}\phi_\alpha=0.$$

We thus have the following: any metric on B_{d+1} with a double pole on the boundary induces a conformal structure on the boundary S^d and conversely, a theorem by Graham and Lee [89] states that any conformal structure on S^d sufficiently close to the standard one can be shown to arise from a unique metric on B_{d+1} which obeys the massless field equations and has a double pole at the boundary.

This formalism can then be extended to Yang-Mills field A with curvature F with minimal Yang-Mills equations $D^iF_{ij}=0$. Any A on S^d that is sufficiently close to $A=0$ is the boundary value of a solution of Yang-Mills equations on B_{d+1}, unique up to gauge transformation.

As an ansatz for the effective action, suppose that there is a field ϕ in AdS_{d+1} obeying $D_iD^i\phi=0$ on AdS_{d+1} (where there is no mass/curvature coupling and we have uniqueness of the solution on $\bar{B}_{d+1}$ for given boundary values. Let ϕ_0 be the restriction of ϕ to the boundary of AdS_{d+1} and assume that ϕ_0 couples to the conformal field $\mathcal{O}$ via

$$\int_{S^d}\phi_0\mathcal{O}.$$

Consider, now, the supergravity / string partition function $Z_S(\phi_0)$ which may be written in terms of the classical supergravity action I_S and also in terms of the conformal field theory as follows (with several caveats)

$$Z_S(\phi_0)=\exp(-I_S(\phi))=\left\langle \exp\int_{S^d}\phi_0\mathcal{O}\right\rangle_{CFT}.$$

One could likewise have an AdS theory with gauge group G (and gauge fields A^a being the global symmetry group of the conformal field theory boundary (with currents

J_a on the boundary). Writing $A \to_\infty A_0$, the propoal becomes

$$Z_S(A_0) = \exp(-I_S(A)) = \left\langle \exp \int_{S^d} J_a A_0^a \right\rangle_{CFT}.$$

As an example, one can write $ds^2 = \frac{1}{x_0^2}\sum_{i=0}^d (dx_i)^2$ (which we recall has boundaries at $x_0 = 0, \infty$), with K (thought of as a Green's function) satisfying the Laplace equation

$$\frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} K(x_0) = 0,$$

with vanishing solution at $x_0 = 0$ given by $K(x_0) = cx_0^d$, which tends to ∞ at ∞. Consider the following mapping

$$x_i \to \frac{x_i}{x_0^2 + \sum_{j=1}^d x_j^2}, \qquad i = 0, 1, ..., d$$

which will clearly map boundary points to the origin $x_i = 0, i = 0, ..., d$. Thus K will be given by $c\frac{x_0^d}{(x_0^2+\sum_{j=1}^d x_j^2)^d}$, which has the correct properties to be a delta function (for example it vanishes as $x \to 0$ except where $x_1 = x_2 = ... = x_d = 0$ and is positive since it is assumed $x_0 > 0$). Hence the field $\phi(x_0, x_i)$ is a 'multiple' of K, and one can ultimately write (where $x_0 \to 0$)

$$I(\phi) = \lim_{\epsilon\to 0} \int_{T_\epsilon} d\mathbf{x}\sqrt{h}\phi(\vec{n}\cdot\vec{\nabla})\phi = \frac{cd}{2}\int d\mathbf{x}d\mathbf{x}' \frac{\phi_0(\mathbf{x})\phi_0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{2d}}.$$

Adding a mass term to the Laplace equation,

$$\left(-\frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} + m^2\right) K(x_0) = 0,$$

and repeating the procedure, one obtains $\lambda(\lambda + d) = m^2$ (from $\phi \sim e^{yd}$) and

$$I(\phi) = \frac{c'(d+\lambda_+)}{2}\int d\mathbf{x}d\mathbf{x}' \frac{\phi_0(\mathbf{x})\phi_0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{2(\lambda_++d)}}.$$

In other words, a field of mass m in AdS space is related to the two-point function of a conformal field $\mathcal{O}$ with dimension $\Delta = \lambda_+ + d$, so that

$$\Delta(\Delta - d) = m^2 \quad \Rightarrow \quad \Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right)$$

$$\Rightarrow \quad \lambda_+ = \frac{1}{2}\left(-d + \sqrt{d^2 + 4m^2}\right)$$

Note that

$$\begin{aligned} m^2 < 0 \quad &\Longleftrightarrow \quad \lambda_+ < 0 \\ m^2 \geq -d^2/4 \quad &\Longleftrightarrow \quad \lambda_+ \geq -d/2 \Longleftrightarrow \Delta \geq d/2 \quad \text{lower bound.} \end{aligned}$$

1.12 Higher spins - Fronsdal and Vasiliev

Working in a curved background is one way to circumvent the various no-go theorems which prohibit higher-spin theories. Fronsdal and Vasiliev did pioneering work in this field.

1.12.1 Fronsdal Equations

We look now into the results obtained in Ref. [90, 91]. Christian Fronsdal wrote down consistent actions describing arbitrary particles for spin greater than two for free theories. We will perform the construction, motivated by considering the elementary linearised General Relativity (i.e. by approximating the metric as small fluctuations about the flat metric) as follows (ignoring terms quadratic in h): The generalised Christoffel symbol for the spin-s gauge field is written as

$$\Gamma^{(1)}_{\nu;\mu_1\ldots\mu_s} = \partial_\nu \phi_{\mu_1\ldots\mu_s} - s\partial_{(\mu_1}\phi_{\mu_2\ldots\mu_s)\nu};$$

for $s = 2$ these reduce to the usual Christoffel symbols up to a multiplicative factor

$$g_{\mu_1\mu_2} = \eta_{\mu_1\mu_2} + h_{\mu_1\mu_2} \Rightarrow \Gamma_{\nu;\mu_1\mu_2} = \frac{1}{2}\left(-\partial_\nu h_{\mu_1\mu_2} + \partial_{\mu_1} h_{\mu_2\nu} + \partial_{\mu_2} h_{\nu\mu_1}\right).$$

Using Ref. [91], one may iteratively define higher-rank Christoffel symbols,

$$\Gamma^{(m)}_{\nu_1\ldots\nu_m,\mu_1\ldots\mu_s} = \partial_{\nu_1}\Gamma^{(m-1)}_{\nu_1\ldots\nu_m;\mu_1\ldots\mu_s} - \frac{s}{m}\partial_{(\mu_1}\Gamma^{(m-1)}_{|\nu_2\ldots\nu_m,\nu_1|\mu_2\ldots\mu_s)}.$$

(When $m = s$ the Christoffel symbol is gauge invariant, giving us a generalised curvature tensor.) The Fronsdal tensor is now defined as $\mathcal{F}_{\mu_1\ldots\mu_s} = \Gamma^{(2)\rho}_{\rho \mu_1\ldots\mu_s}$ where

$$\begin{aligned}\Gamma^{(2)}_{\nu_1\nu_2,\mu_1\ldots\mu_s} &= \partial_{\nu_1}\Gamma^{(1)}_{\nu_2,\mu_1\ldots\mu_s} - \frac{s}{2}\partial_{(\mu_1}\Gamma^{(1)}_{|\nu_2,\nu_1|\mu_2\ldots\mu_s)} \\ &= \partial_{\nu_1}\partial_{\nu_2}\phi_{\mu_1\ldots\mu_s} - s\partial_{\nu_1}\partial_{(\mu_1}\phi_{\mu_2\ldots\mu_s)\nu_2} + \frac{1}{2}s(s-1)\partial_{(\mu_1}\partial_{\mu_2}\phi_{\mu_3\ldots\mu_s)\nu_1\nu_2},\end{aligned}$$

and therefore, setting $\mathcal{F}_{\mu_1\ldots\mu_s} = 0$, we obtain the gauge invariant[2] equation obtained by Fronsdal,

$$\mathcal{F}_{\mu_1\ldots\mu_s} = \Box\phi_{\mu_1\ldots\mu_s} - s\partial_{(\mu_1}\partial^{\rho}\phi_{\mu_2\ldots\mu_s)\rho} + \frac{1}{2}s(s-1)\partial_{(\mu_1}\partial_{\mu_2}\phi^{\rho}_{\mu_3\ldots\mu_s)\rho} = 0, \quad (1.58)$$

or simply

$$\mathcal{F} = 0.$$

Now, in the General Relativity case, fixing the gauge reduced the linearised equations (i.e. no quadratic terms in h) to the wave equation as follows:

$$\begin{aligned}R_{\mu\nu\rho\sigma} &= \frac{1}{2}\left(\partial_\rho\Gamma_{\mu;\sigma\nu} - \partial_\sigma\Gamma_{\mu;\rho\nu} + \Gamma_{\mu;\rho\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma_{\mu;\sigma\lambda}\Gamma^{\lambda}_{\rho\nu}\right) \\ &= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\rho\nu} - \partial_\rho\partial_\mu h_{\sigma\nu} - \partial_\sigma\partial_\nu h_{\mu\rho}\right) \\ \Rightarrow R_{\mu\nu} &= R_{\mu\rho\nu\sigma}\,(\eta + h)^{\rho\sigma} \\ &= \frac{1}{2}\left(\partial_\rho\partial_\nu h_\mu^{\ \rho} + \partial_\rho\partial_\mu h_\nu^{\ \rho} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right). \quad (1.59)\end{aligned}$$

In de Donder gauge where $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0$, the vacuum Einstein equations which reduce to the Ricci flat condition $R_{\mu\nu} = 0$ since

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \iff R - 2R = 0 \iff R = 0 \iff R_{\mu\nu} = 0$$

causes equation (1.59) to reduce to the wave equation

$$\Box h_{\mu\nu} = 0.$$

[2] It can be shown that $\delta\mathcal{F} = \frac{1}{2}s(s-1)(s-2)\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\zeta^{\rho}_{\mu_4\ldots\mu_s)\rho}$; thus, for $s = 0, 1, 2$ the gauge invariance is trivial; for spin>2 it turns out also to be gauge-invariant but is non-trivial.

Following the same philosophy in this higher spin case, we find the analogue of the Ricci tensor to be related to the antisymmetrisation of the Fronsdal tensor:

$$R_{\mu_1\nu_1\ldots\mu_s\nu_s}\eta^{\nu_1\nu_2} = -\frac{1}{2}\mathcal{F}_{\mu_1\mu_2[\mu_3[\ldots[\mu_s,\nu_s]\ldots]\nu_3]},$$

for example for spin 2

$$R_{\mu\rho\nu\sigma}\eta^{\rho\sigma} = R_{\mu\nu} = -\frac{1}{2}\mathcal{F}_{\mu\nu},$$

so that $\mathcal{F} = 0$ gives the Ricci flat condition. The analogue of Einstein's tensor is the generalised Einstein tensor,

$$\mathcal{G}_{\mu_1\ldots\mu_s} = \mathcal{F}_{\mu_1\ldots\mu_s} - \frac{1}{4}s(s-1)\eta_{(\mu_1\mu_2}\mathcal{F}_{\mu_3\ldots\mu_s)\rho}{}^{\rho}.$$

The analogue of the vacuum Einstein equations is

$$\mathcal{G}_{\mu_1\ldots\mu_s} = 0$$

and contracting with $\eta^{\mu_1\mu_2}$ requires

$$\left(1 - s(s-1)\right)\mathcal{F}_{\mu_3\ldots\mu_s)\rho}{}^{\rho} = 0.$$

Plugging this back into the expression for the generalised Einstein tensor gives

$$\mathcal{G}_{\mu_1\ldots\mu_s} = \mathcal{F}_{\mu_1\ldots\mu_s} = 0.$$

The extension of the de Donder gauge condition is given by

$$D_{\mu_2\ldots\mu_s} = \partial^\rho\phi_{\rho\mu_2\ldots\mu_s} - \frac{1}{2}s(s-1)\partial_{(\mu_2}\phi^{\rho}{}_{\mu_3\ldots\mu_s)\rho} = 0.$$

This results in the second and third term in our expression for $\mathcal{F}$ in equation (1.58) to cancel, leaving us with the wave equation

$$\Box\phi_{\mu_1\ldots\mu_s} = 0.$$

Continuing on the GR side, we insert $\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ into the de Donder gauge condition to obtain

$$\partial^\mu \partial_\mu \epsilon_\nu + \partial^\mu \partial_\nu \epsilon_\mu - \frac{1}{2} \cdot 2\partial_\nu \partial^\mu \epsilon_\mu = \Box \epsilon_\nu = 0,$$

giving gauge freedom to gauge away the trace, as well as showing that we have two wave equations giving plane wave solutions

$$\begin{aligned} h &= -\eta^{\mu\nu} \delta h_{\mu\nu} = -2\partial^\rho \epsilon_\rho \\ h_{\mu\nu} &= H_{\mu\nu} e^{ikx} \\ \epsilon_\mu &= C_\mu e^{ikx} \end{aligned}$$

where $k^2 = 0$, thus describing propagating massless particles. Therefore

$$He^{ikx} = h = -2\partial^\rho \epsilon_\rho = -2\partial^\rho C_\rho e^{ikx} = -2\left(k \cdot C\right) e^{ikx} = -2k^0 C_0 e^{ikx},$$

so that the trace may be gauged away with appropriate choice of C_0 and the de Donder condition becomes the transversality condition $\partial^\mu h_{\mu\nu} = 0$, we are free to set $-2\partial^\rho \epsilon_\rho = 0$. This gives us that the fully gauge-fixed field contains exactly $(10 - 4 - 1 - 3) = 2$ degrees of freedom so that the linearised Einstein equations may correctly be interpreted as the classical field theory of massless spin-2 particles propagating in a flat background.

The way this is played out in the higher-spin sphere is that we first construct a canonical Lagrangian which give the correct equations of motion $\frac{\partial \mathcal{L}}{\partial \phi} = 0$,

$$\mathcal{L} = \phi_{\mu_1 \ldots \mu_s} \mathcal{G}^{\mu_1 \ldots \mu_s}.$$

Requiring $\delta S = 0$ where we have the spin-s gauge transformation $\delta \phi_{\mu_1 \ldots \mu_s} = s\partial_{(\mu} \zeta_{\mu_2 \ldots \mu_s)}$, results in a vanishing total derivative and a term $\partial \cdot \mathcal{G}$. Then, as in the GR case we observe that we have plane wave solutions for both the field as well as the parameter. The trace can be gauged away and we are left with a transversality condition and we can set the analogue of the divergence of the parameter to zero, and counting once again results in two physical polarisation states.

It is worth noting that in AdS space where $G_{\mu\nu} = -\Lambda g_{\mu\nu}$, the cosmological term generates a mass tern for the higher-spin fields. This mass may take on negative values and is bounded from below by the Breitenlohner-Freedman (BF) bound. It may be derived by considering the action for a massive scalar field ϕ in AdS_d (where R is the radius of AdS_{d+1}, η is a normalisation constant,

$$S(\phi) \quad = \quad -\frac{\eta}{2}\int \frac{dzd^dx}{z^{d+1}} z^2\partial_z\phi\partial_z\phi + z^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2R^2\phi^2$$

and rescaling the field $\phi = z^{\frac{d}{2}}\psi$ and changing coordinates $y = -\ln z$ so $dz = -zdy$. This action incorporates surface terms which can arbitrarily be made to be positive, as well as a term

$$S(\phi) = -\frac{\eta}{2}\int dyd^dx\left[\frac{d^2}{4} + m^2R^2\right]\psi^2,$$

requiring that

$$m^2R^2 \geq -\frac{d^2}{4}.$$

Fronsdal's equations in AdS_4 are given by

$$\left(\nabla^2 - m^2\right)\phi_{\mu_1\ldots\mu_s} - s\nabla_{(\mu_1}\nabla^\rho\phi_{\mu_2\ldots\mu_s)} + \frac{s(s-1)}{4s}g_{(\mu_1\mu_2}\nabla^{\nu_1}\nabla^{\nu_2}\phi_{\mu_3\ldots\mu_s)\nu_1\nu_2} = 0$$

where $m^2 = s^2 - 2s - 2$.

1.12.2 Vasiliev's Higher Spin Theories

We now take a look the way Mikhail Vasiliev and collaborators overcame the no-go theorems, starting in around 1980 and working relentlessly through several decades (see Refs.[92, 93, 94, 95, 96, 97]). Explicitly written, the problem he set out to solve was to find a non-linear higher spin gauge theory such that it has the correct free limit, i.e. is free of ghosts and is equivalent to the Fronsdal theory for the case of totally symmetric fields. Secondly, it needed to contain unbroken higher spin gauge symmetries since massive higher spin particles are not seen in nature. And, thirdly, it was required to have non-Abelian global higher spin symmetry of a vacuum solution, necessary to ensure non-trivial interactions such as Yang-Mills interactions [97]. It was inspired by geometric reformulations of Einstein gravity in which the vierbein and Lorentz spin-connections are used to characterise local-Lorentz frames on the manifold rather than the metric as proposed by Macdowell, Mansouri, Stelle and West[98, 99, 100, 101, 102]. These, together with higher-spin fields that propagate on the background are all contained in a single dynamical one-form called a master field. The construction could then be generalised to higher-spins using the generalised curvatures due to Ref. [91] as reviewed in the previous subsection. One needed to include a space-time zero-form to ensure consistency of the interacting higher-spin equations, endowing Vasiliev's theory with a massive scalar field. Fundamentally, one obtains a higher-spin algebra by introducing a twistor space with non-commutative star product. This twistor space is spanned by a pair of commuting spinors

$$(Y, Z) = \left(y^{\alpha}, \bar{y}^{\dot{\alpha}}, z^{\alpha}, \bar{z}^{\dot{\alpha}}\right)$$

or 'ghosts', where barred variables transform as left-handed spinors and unbarred variables as right-handed spinors of $SU(2)$. These satisfy commutator relations

$$[y^{\alpha}, y^{\beta}] = [z^{\alpha}, z^{\beta}] = [\bar{y}^{\dot{\alpha}}, \bar{y}^{\dot{\beta}}] = [\bar{z}^{\dot{\alpha}}, \bar{z}^{\dot{\beta}}] = 0$$

and may be transformed using vierbeins so that spinors contracted with themselves vanish,

$$y^{\alpha} y_{\alpha} = \epsilon^{\alpha\beta} y_{\beta} y_{\alpha} = \epsilon^{\alpha\beta} y_{\alpha} y_{\beta} = -\epsilon^{\beta\alpha} y_{\alpha} y_{\beta} = -y^{\beta} y_{\beta} \iff y^{\alpha} y_{\alpha} = 0$$

The inner product on the twistor space is called the star product defined as

$$\begin{aligned} f(Y,Z) * g(Y,Z) \quad &= \quad f(Y,Z)\mathrm{exp}\left[\epsilon^{\alpha\beta}\left(\overleftarrow{\partial}_{y^{\alpha}} + \overleftarrow{\partial}_{z^{\alpha}}\right)\left(\overrightarrow{\partial}_{y^{\beta}} - \overrightarrow{\partial}_{z^{\beta}}\right)\right. \\ &\qquad \left. + \, \epsilon^{\dot{\alpha}\dot{\beta}}\left(\overleftarrow{\partial}_{\bar{y}^{\dot{\alpha}}} + \overleftarrow{\partial}_{\bar{z}^{\dot{\alpha}}}\right)\left(\overrightarrow{\partial}_{\bar{y}^{\dot{\beta}}} - \overrightarrow{\partial}_{\bar{z}^{\dot{\beta}}}\right)\right] g(Y,Z). \end{aligned}$$

The star-commutation relations are then calculated as follows (we give one example just to elucidate the method),

$$\left[y^{\alpha}, y^{\beta}\right]_{*} \quad = \quad y^{\alpha}\left(1 + \epsilon^{ab}\overleftarrow{\partial}_{y^{a}}\overrightarrow{\partial}_{y^{b}}\right) y^{\beta} - y^{\beta}\left(1 + \epsilon^{ab}\overleftarrow{\partial}_{y^{a}}\overrightarrow{\partial}_{y^{b}}\right) y^{\alpha} = 2\epsilon^{\alpha\beta}.$$

Similarly $\left[z^\alpha, z^\beta\right]_* = -2\epsilon^{\alpha\beta}$. The twistor fields are auxiliary so physical fields do not depend on them. Notice that the star product combines barred terms or unbarred terms but not a combination of barred and unbarred terms. The following functions are defined

$$K(t) = e^{ty_\alpha z^\alpha} \qquad \bar{K}(t) = e^{t\bar{y}_{\dot{\alpha}}\bar{z}^{\dot{\alpha}}} \tag{1.60}$$

which are called Kleinians K and $\bar{K}$ when $t = 1$. The functions (1.60) satisfy

$$f(y,z) * K = f(-z,-y)K, \qquad K * f(y,z) = Kf(z,y).$$

This is best understood by considering the alternative definition of the star product given by the convolution formula[2],

$$f(y,z) * g(y,z) \;\;=\;\; \int d^2u d^2v e^{u^\alpha v_\alpha} f(y+u, z+u) g(y+v, z-v),$$

normalised with $f * 1 = f$ etc. Thus

$$\begin{aligned} f(y,z) * K(t) &= \int d^2u d^2v e^{u^\alpha v_\alpha} f(y+u, z+u) e^{t(y_\alpha + v_\alpha)(z^\alpha - v^\alpha)} \\ &= \int d^2u d^2v e^{u^\alpha v_\alpha} f(y+u, z+u) e^{t(y_\alpha z^\alpha - y_\alpha v^\alpha + v_\alpha z^\alpha - v_\alpha v^\alpha)}. \end{aligned}$$

However, $v_\alpha v^\alpha = \epsilon^{\alpha\beta} v_\alpha v_\beta = 0$ from symmetry-antisymmetry arguments; also $y_\alpha v^\alpha = \epsilon^{\alpha\beta} v_\alpha y_\beta = -\epsilon^{\beta\alpha} v_\alpha y_\beta = -y^\alpha v_\alpha$, so that

$$\begin{aligned} f(y,z) * K(t) &= \int d^2u d^2v e^{u^\alpha v_\alpha} f(y+u, z+u) e^{t(y^\alpha z_\alpha + (y^\alpha + z^\alpha) v_\alpha)} \\ &= \int d^2u d^2v e^{[u^\alpha + t(y^\alpha + z^\alpha)] v_\alpha} f(y+u, z+u) e^{ty^\alpha z_\alpha} \\ &= \int d^2\tilde{u} d^2v e^{\tilde{u}^\alpha v_\alpha} f(y + \tilde{u} - t\,(y+z)\,, z + \tilde{u} - t\,(y+z)) e^{ty^\alpha z_\alpha} \\ &= \int d^2\tilde{u} f(y(1-t) - tz + \tilde{u}, z(1-t) - ty + \tilde{u}) \int d^2v e^{\tilde{u}^\alpha v_\alpha} e^{ty^\alpha z_\alpha} \\ &= \int d^2\tilde{u} f(y(1-t) - tz + \tilde{u}, z(1-t) - ty + \tilde{u}) \delta(\tilde{u}) K(t) \\ &= f(y(1-t) - tz, z(1-t) - ty) K(t). \end{aligned}$$

Similarly,

$$K(t) * F(y,z) = F((1-t)y + tz, (1-t)z + ty)K(t),$$

from which one may now easily obtain

$$f(y,z) * K = f(-z,-y)K, \qquad K * f(y,z) = Kf(z,y).$$

The master fields which depend on twistor coordinates (lower case) and space-time coordinates (upper case) are given by the physical one-form and scalar field W and B, and the non-physical auxilliary one-form S, that is by

$$W(x|Y,Z) = W_\mu(x|Y,Z)dx^\mu;\ B(x|Y,Z);\ S(x|Y,Z) = S^\alpha(x|Y,Z)dz_\alpha + \bar{S}^{\dot{\alpha}}(x|Y,Z)d\bar{z}_{\dot{\alpha}}.$$

We group the one-form fields into the field $\hat{\mathcal{A}}$,

$$\hat{\mathcal{A}} = W + S + \frac{1}{2}z^\alpha dz_\alpha + \frac{1}{2}\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}}.$$

Vasiliev then gave gauge invariant and non-linear master equations of motion in terms of this field (where f is an arbitrary function and f_* is obtained from f by replacing products with star products; the operation π swaps the sign of y, z and dz):

$$\begin{aligned} d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} * \hat{\mathcal{A}} &= f_*(B * K)dz^2 + \bar{f}_*(B * \bar{K})d\bar{z}^2 \\ d_x B + \hat{\mathcal{A}} * B - B * \pi(\hat{\mathcal{A}}) &= 0 \end{aligned} \tag{1.61}$$

which are invariant under

$$\delta\mathcal{A} = d\epsilon + [\mathcal{A}, \epsilon]_*; \qquad \delta B = -\epsilon * B + B * \pi(\epsilon).$$

Requiring that $[K\bar{K}, W]_* = \{K\bar{K}, S\}_* = [K\bar{K}, B]_* = 0$ necessitates that W and B are even functions and that S is an odd function, i.e. $W(x|y,\bar{y},z,\bar{z}) = W(x|-y,-\bar{y},-z,-\bar{z})$, $B(x|y,\bar{y},z,\bar{z}) = B(x|-y,-\bar{y},-z,-\bar{z})$ and $S(x|y,\bar{y},z,\bar{z}) = -S(x|-y,-\bar{y},-z,-\bar{z})$. The function f allows one to encode the infinitely many higher spins

$$f(x) \;=\; \frac{1}{4} + xe^{i\theta(x)},$$

where $\theta(x) = \sum_{n=0}^{\infty} \theta_{2n} x^{2n}$. From parity invariance of the master equations of motion, one has that θ may take on values 0 and $\frac{\pi}{2}$ only, giving rise to type-A and type-B Vasiliev theories; other values of θ would describe Vasiliev theories that have broken parity symmetry. We now give an a priori condition that physical degrees of freedom in the master fields are given where $Z = 0$, and therefore define

$$C(x|Y) = B(x|Y,Z)|_{Z=0} \qquad \Omega_\mu(x|Y) = \hat{W}_\mu(x|Y,Z)|_{Z=0}$$

which are, respectively, a massive scalar field and an infinite tower of (bosonic) higher-spin fields. This may be seen by expanding f into (m,n) components as follows (where we define the vector Y^μ spinorially as $Y^\mu = \sigma^\mu_{\alpha\dot{\alpha}} y^\alpha \bar{y}^{\dot{\alpha}}$, called the 'Winding Dictionary')

$$\begin{aligned} f(x|Y) &= \sum_{m,n=0}^{\infty} f^{(m,n)}(x)_{\alpha_1 \ldots \alpha_n \dot{\alpha}_1 \ldots \dot{\alpha}_m} y^{\alpha_1} \ldots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1} \ldots \bar{y}^{\dot{\alpha}_m} \\ \Rightarrow \frac{\delta}{\delta Y^{\mu_1}} \ldots \frac{\delta}{\delta Y^{\mu_n}} f(x|Y)|_{Y=0} &= f^{(n,n)}(x)_{\mu_1 \ldots \mu_n} \end{aligned}$$

for $m+n$ even. $\Omega^{(s-1+k,s-1-k)}_{\mu,\alpha_1\ldots\alpha_{s-2}\dot{\beta}_1\ldots\dot{\beta}_{s-2}}(x)$ is related to the k^{th} Christoffel symbol; for example, (applying treatment similar to that above to the Ωs) one has schematically that

$$\Omega^{(s-1+1,s-1-1)}_{\mu,\alpha_1\ldots\alpha_{s-2}\dot{\beta}_1\ldots\dot{\beta}_{s-2}}(x) \sim \sigma^{\mu_1}_{\alpha_1\dot{\beta}_1} \ldots \sigma^{\mu_1}_{\alpha_{s-2}\dot{\beta}_{s-2}} . \sigma^{\rho\sigma}_{\alpha\beta} \Gamma^{(1)}_{\rho,\sigma\nu\mu_1\ldots\mu_{s-2}}$$

where we have used the usual $(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{1}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)_\alpha^{\ \beta}$. Hence, these objects describe the higher-spin curvatures of de Wit and Friedman that we observed in the previous subsection!

Furthermore, the maximally symmetric solution for $\mathcal{A}$ is given by

$$\mathcal{A} = W_0(x|Y) = (e_0)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} + (w_0)_{\alpha\beta} y^\alpha y^\beta + (w_0)_{\dot{\alpha}\dot{\beta}} y^{\dot{\alpha}} y^{\dot{\beta}},$$

and e_0 and w_0 are now interpreted as vierbein and spin-connection. When we put $B = S = 0$ in the master equation of motion (1.61), gives

$$(d_x W_0)^{\mu\nu} + W_0^\mu * W_0^\nu = 0.$$

After some detailed manipulations, one obtains

$$
\begin{aligned}
&d_x(e_0)_{\alpha\dot{\beta}} + 4(e_0)_{\gamma\dot{\beta}} \wedge (w_0)_{\alpha\dot{\delta}}\epsilon^{\gamma\delta} - 4(e_0)_{\alpha\dot{\gamma}} \wedge (w_0)_{\dot{\delta}\dot{\beta}}\epsilon^{\dot{\gamma}\dot{\delta}} = 0 \\
&d_x(w_0)_{\alpha\beta} + (e_0)_{\alpha\dot{\gamma}} \wedge (e_0)_{\beta\dot{\delta}}\epsilon^{\dot{\gamma}\dot{\delta}} + 4(w_0)_{\alpha\gamma} \wedge (w_0)_{\beta\delta}\epsilon^{\gamma\delta} = 0 \\
&d_x(w_0)_{\dot{\alpha}\dot{\beta}} + (e_0)_{\gamma\dot{\alpha}} \wedge (e_0)_{\delta\dot{\beta}}\epsilon^{\gamma\delta} + 4(w_0)_{\dot{\alpha}\dot{\gamma}} \wedge (w_0)_{\dot{\beta}\dot{\delta}}\epsilon^{\dot{\gamma}\dot{\delta}} = 0,
\end{aligned}
$$

and using the definitions

$$
(e_0)_{\alpha\dot{\beta}} = \frac{1}{4}e^a\sigma^a_{\alpha\dot{\beta}} \qquad (w_0)_{\alpha\dot{\beta}} = \frac{1}{16}w^{ab}\sigma^{ab}_{\alpha\beta} \qquad (w_0)_{\dot{\alpha}\dot{\beta}} = -\frac{1}{16}w^{ab}\bar{\sigma}^{ab}_{\dot{\alpha}\dot{\beta}}
$$

results in

$$
\begin{aligned}
&d_x e_a + w_{ab} \wedge e_b = 0 \\
&d_x w_{ab} + w_{ac} \wedge w_{cb} = 6 e_b \wedge e_a
\end{aligned}
$$

which, respectively, are the torsion-free equation and the Ricci tensor in terms of the vierbein in the geometry of AdS_4!

If, now, instead of considering the maximally symmetric vacuum solution where $W = W_0$ and $S = B = 0$ we consider small perturbations of order λ around this solution, i.e.

$$
W = W_0 + \lambda\hat{W}; \qquad S = \lambda\hat{S}; \qquad B = \lambda\hat{B},
$$

and consider terms linear in λ in the equations of motion, we may equation coefficients in λ to obtain

$$
\begin{aligned}
&D_0\hat{W} = 0 \\
&d_Z\hat{W} + D_0\hat{S} = 0 \\
&d_Z\hat{S} = e^{i\theta_0}(\hat{B} * K) + e^{-i\theta_0}(\hat{B} * \bar{K}) \\
&\widetilde{D}_0\hat{B} = 0 \\
&d_Z\hat{B} = 0
\end{aligned}
$$

where $D_0 = d_x + [W_0, \cdot]_*$ and $\widetilde{D}_0 = d_x + W_0 * \cdot - \cdot * \pi(W_0)$. One can indeed show that these equations do describe a massive scalar together with a tower of higher-spin fields in AdS_4.

For the scalar field, one can expand $\widetilde{D}_0\hat{B} = 0$ and set $Z = 0$ to derive the equation

$$-\frac{1}{2}\left(\nabla^{\mu}\partial_{\mu} + 2\right) C^{(0,0)}(x) = 0$$

which is just the Klein-Gordon equation for a scalar field with mass-squared given by $m^2 = -2$ in AdS_4.

1.13 Higher spins - Klebanov, Polyakov, Giombi,Yin

This follows key concepts from Refs. [1, 41, 2, 4, 103]. While much progress has been made in understanding the AdS/CFT correspondence when the duality is between a strongly coupled CFT and a weakly coupled gravity/string theory (taking the radius of AdS large)[67], somewhat less progress has been made in understanding the correspondence in the opposite limit, where the radius of AdS is small. Historically, steps that led to noting that such a duality should exist came in by considering that, in the tensionless limit, String theory's massive excitations become massless. In this limit, Type IIB String theory was therefore thought to be dual to a free $\mathcal{N} = 4$ SYM theory [104, 105, 106, 107], although this idea has never been made concrete. Klebanov and Polyakov provided the first plausible duality pair in which a weakly coupled, and possibly free gauge theory at large N is dual to a known bulk theory. Around a decade later, Giombi and Yin were able to do calculations using the AdS/CFT correspondence to corroborate this conjecture [41, 2, 4], and the conjecture gained momentum as a result.

The theory was worked out in $d = 2 + 1$ dimensions in Ref. [1]. One reason that $d = 2 + 1$ dimensional theories are of interest is that they act as a portal to problems which are challenging in $d = 3 + 1$ dimensions. For example in Ref. [108, 109], Polyakov tackled 'the infrared catastrophe' and was able to demonstrate confinement by showing that the presence of instantons / monopoles enables apparently broken gauge symmetries to be restored. Peskin further introduced the particle / vortex duality [110] to try and explain confinement in $d = 3 + 1$ dimensions.

In this example of the AdS/CFT correspondence, the starting point is the simplest CFT, which is given by the $O(N)$ vector model with action

$$\mathcal{S} = \int d^3x \left(\frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \frac{1}{2} m^2 (\phi^a \phi^a) + \frac{\lambda}{4} (\phi^a \phi^a)^2 \right),$$

for $a = 1, ..., N$. This is a 3-dimensional free theory of N massless scalar fields in the singlet sector. Being a conformal field theory, this singlet condition requires that the set of physical operators must be $O(N)$ invariant (an attribute that makes it amenable to the introduction of Collective field theory as we shall see in Chapter 2. This action has a UV fixed point at $\lambda = 0$ and an IR fixed point for $2 < d < 4$ (relevant when $d < 4$)which gives rise an RG flow from a UV fixed point in the free theory to an IR fixed point in the interacting theory. In $d = 4 - \epsilon$, this IR fixed point is found by considering the one-loop beta function, $\beta_\lambda = -\epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + ...$, for which the (weakly coupled) IR fixed point occurs when

$$\lambda = \frac{8\pi^2}{N+1}.$$

(since $\epsilon = 1$ in 3d). The $O(N)$ vector model has an infinite number of conserved currents, for example

$$\begin{aligned} J_\mu &= \phi \partial_\mu \phi \\ J_{\mu\nu} &= T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{4(d-1)} \left[(d-2)\partial_\mu \partial_\nu + g_{\mu\nu}\partial^2 \right] \phi^2. \end{aligned}$$

These conserved currents all have the form

$$J_{\mu_1 \ldots \mu_s} = \phi^a \partial_{(\mu_1} \ldots \partial_{\mu_s)} \phi^a + \ldots$$

and could be repackaged in the manner of Giombi and Yin in Ref. [41] as follows

$$\phi_i(x) f\left(\epsilon_\mu, \overleftarrow{\partial}_\mu, \overrightarrow{\partial}_\mu\right) \phi_i(x), \tag{1.62}$$

where conservation and traceless conditions on the currents require that

$$(\vec{u} + \vec{v}) \cdot \vec{\partial}_\epsilon f(\vec{\epsilon}, \vec{u}, \vec{v}) = \vec{\partial}_\epsilon^2 f(\vec{\epsilon}, \vec{u}, \vec{v}) = 0.$$

This has solution

$$e^{\vec{\alpha}_\pm \cdot \vec{\epsilon}}, \qquad \vec{\alpha}_\pm = \vec{u} - \vec{v} \pm \sqrt{\frac{-2}{u \cdot v}} \vec{u} \times \vec{v}.$$

and the massless equations of motion require $u^2 = v^2 = 0$, and so

$$f\left(\overrightarrow{\epsilon}, \overrightarrow{\partial}_\mu, \vec{u}, \vec{v}\right) = \frac{e^{\vec{\alpha}_+ \vec{\epsilon}} + e^{\vec{\alpha}_- \vec{\epsilon}}}{2} = e^{(u-v)\cdot\epsilon} \cos\left[\sqrt{2(u \cdot v)\epsilon^2 - 4(u \cdot \epsilon)(v \cdot \epsilon)}\right].$$

In the duality, these $O(N)$ vector models are conjectured to be dual to higher spin theories[103]. In section (1.12.2), it was seen that Vasiliev was able to construct exact non-linear equations of motion (without the use of an action), and that they have a vacuum solution corresponding to AdS. The master fields with $Z = 0$ gave scalar field of mass-square $m^2 = -2/L^2$ (with L, the radius of AdS, making this relation dimensionally correct) together with an infinite tower of bosonic higher spin fields. (Initially Vasiliev's equations were written in AdS_4, but these have been generalised to AdS in all dimensions.) The $s = 2$ graviton is present in all higher spin theories, and generalises Einstein theory to include an infinite tower of (massless) higher spins (one can minimally truncate this to involve only a spectrum of even spins). Furthermore, we saw in section 1.11 that in the AdS/CFT dictionary we have the identification

$$J_\mu \leftrightarrow A_\mu, \quad T_{\mu\nu} \leftrightarrow g_{\mu\nu}, \quad J_{\mu_1 \ldots \mu_s} \leftrightarrow \phi_{\mu_1 \ldots \mu_s}.$$

It is fully conceivable, therefore that a theory with conserved currents as in equation (1.62) as the $O(N)$ vector model has, would be dual to a gravitational theory having an infinite tower of massless higher spins such as Vasiliev higher spin theory. Indeed, while much successful work has been done to understand the correspondence between matrix-valued fields (which would transform in the adjoint representation) and higher-spin massless gauge theories, $O(N)$ vectors models have the desirable property that there exists a conserved current for every even spin, and thus a 1-1 correspondence between the spectrum of currents and the spectrum of massless higher-spin fields in the minimal bosonic theory in AdS_4[1]. Vector models are indeed seen to be one of the simplest ways of probing the AdS/CFT correspondence (see Table 1.4, courtesy of Ref. [111]) and are 'solvable' in the large-N limit, allowing a more detailed study of the AdS/CFT correspondence.

Comparison of models probing AdS/CFT			
	Matrix models	SYK	Vector models
Large N Feynman diagrams	Planar	Melons	Bubbles
Difficulty	Hard	Easy/medium	Easy
A CFT example	$\mathcal{N}=4$ Yang-Mills (4 dimensions)	SYK, at strong coupling (1 dimension)	Free/critical O(N) vector model (3 dimensions)
AdS dual theory	String theory	?	Vasiliev higher spin
Gravitational sector	Stress-tensor $\leftrightarrow$ Einstein gravity	h=2 mode (Schwartzian) $\leftrightarrow$ Jackiw-Teitelboim gravity	Inapplicable
Gauge invariant operators	$\text{tr}\left(X\partial^k XYY...\right)$	$\chi_i\partial_\tau^{1+2n}\chi_i$	$\phi^a\partial_{(\mu_1}...\partial_{\mu_s)}\phi^a$
Anomalous dimensions	Large (at large 't Hooft coupling)	Order 1	order 1/N
Dual of these operators	Stringy modes	Tower of scalars	Tower of massless higher spin fields

TABLE 1.4: Table showing comparison between Matrix models, SYK and Vector models for probing the AdS/CFT correspondence

Furthermore, according to Klebanov and Polyakov, the conjecture is that the free 3d $O(N)$ vector model is dual to type A minimal Vasiliev higher spin gauge theory with $\Delta = 1$, while the critical 3d $O(N)$ vector model is dual to type A minimal Vasiliev Higher spin theory with scalar having $\Delta = 2$ [1, 4] (here Δ is the dimension of the scalar field that is in the infinite tower of higher-spin fields). Near the AdS_4 boundary, the bulk scalar field $\varphi(\vec{x}, z)$ has falloff behaviour[41]

$$\varphi \sim az + bz^2 + \mathcal{O}(z^3).$$

Here there are two consistent conformally invariant boundary conditions

Dirichlet with $a = 0$ $\longrightarrow$ Dual operator with dimension $\Delta = 2$
Neumann with $b = 0$ $\longrightarrow$ Dual operator with dimension $\Delta = 1$

The large N limit of the 3-dimensional $(\phi^a\phi^a)^2$ theory (where ϕ^a is an N-component scalar field transforming according to the fundamental representation of $O(N)$) is well-known to describe critical points of $O(N)$ magnets and to be conformal. According to the AdS/CFT dictionary, conserved currents in the boundary correspond to massless gauge fields in the bulk of a gravitational theory or string theory. There is further a conjectured connection between theories of massless higher-spins in AdS_{d+1} and free fields in d dimensions.

The AdS/CFT prescription suggests that we can write

$$\left\langle \exp \int d^3x h_0^{(\mu_1 \dots \mu_s)} J_{(\mu_1 \dots \mu_s)} \right\rangle = e^{S[h_0]},$$

so that boundary values of the fields are identified with sources $h_0^{(\mu_1 \dots \mu_s)}$ in the dual field theory. Position space calculations for propagators are then fairly simple. Defining $x_{12} = |x_1 - x_2|$ and spin-0 'current' as $J = \phi^a \phi^a$ etc. we obtain expressions such as

$$\begin{aligned} \left\langle \phi^a(x_1)\phi^b(x_2) \right\rangle &= \frac{\delta^{ab}}{x_{12}} \\ \langle J(x_1)J(x_2) \rangle &\sim \frac{N}{x_{12}^2} \\ \langle J(x_1)J(x_2)J(x_3) \rangle &\sim \frac{N}{x_{12}x_{13}x_{23}}. \end{aligned}$$

Note that the dimension of J is 1 which is less that $d/2 = 3/2$, and thus (where L is the radius of AdS_4),

$$\Delta_- = 1 = \frac{d}{2} - \sqrt{\frac{d^2}{4} + (mL)^2} \Rightarrow m^2 = -2/L^2.$$

We observe that $\Delta_- = 1$ is related to the free $O(N)$ theory. Now, since the radius of AdS_4 is given by $R = -\frac{12}{L^2}$ we can use the mass found above to make an effective Lagrangian for the scalar field h of AdS_4,

$$S = \frac{N}{2} \int d^4x \sqrt{g} \left[(\partial_\mu h)^2 + \frac{1}{6} R h^2 + \alpha h^3 + \dots \right].$$

It turns out that the CFT corresponding to the case where the operator J has dimension $\Delta_+ = 2$ is in fact the interacting $d = 3$ $O(N)$ vector model,

$$S = \int d^3x \left[\frac{1}{2} (\partial_\mu \phi^a)^2 + \frac{\lambda}{2N} (\phi^a \phi^a)^2 \right].$$

The standard approach for obtaining a $1/N$ expansion[112, 113] is by introducing an auxiliary field $\sigma(x)$, where $\sigma = \phi^a\phi^a$, and integrating the (now quadratic in ϕ^a) action over the ϕ^a fields. Here $\mathcal{L}$ is made to be of the form

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + a\sigma^2 + b\sigma + \frac{1}{2}(\phi^a\phi^a)\sigma = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \frac{\lambda}{2N}(\phi^a\phi^a)^2 \\ &= \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + a\left(\sigma + \frac{1}{2a}\left(b + \frac{1}{2}(\phi^a\phi^a)\right)\right)^2 - \frac{1}{4a}\left(b^2 + b(\phi^a\phi^a) + \frac{1}{4}(\phi^a\phi^a)^2\right);\end{aligned}$$

matching coefficients one obtains an action that can easily be expanded in terms of N,

$$S = \int d^3x \left[\frac{1}{2}(\partial_\mu\phi^a)^2 + \sigma\phi^a\phi^a - \frac{N}{2\lambda}\sigma^2\right],$$

with interaction term which may be written as $\lambda J^2/(2N)$. When a relevant interaction (that is the tachyonic, negative mass squared terms, but nevertheless doesn't lead to instability) is included in the interaction, the theory flows from an unstable UV fixed point with dimension $\Delta_- = 1$ to an IR fixed point where J has dimension $\Delta_+ = 2$[114].

Note that in this thesis, use will be made of the Collective field theory approach rather than this standard auxiliary field approach.

There have been numerous validations of the $O(N)$ Vector model - Higher spin theory conjecture in the form of calculations that were done on both sides of the duality and found to match with one another. These include the reconstruction of the bulk [5], the one-loop test of the duality involving the calculation of the $\mathcal{O}(N)$ correction to the Weyl anomaly for even d and to free energy for odd d [4, 4] and the matching of the three-point function for both $\Delta = 1$ and $\Delta = 2$ (which we review below). These have played no minor role in attracting attention to the duality.

In Ref. [2], it was shown that the tree level three-point functions match. This was done as follows. Making use of Vasiliev higher spin theory as reviewed in section 1.12.2, it is noted that no explicit Lagrangian is known for the theory, and correlation functions of boundary operators $J(\vec{x})$ are instead computed using dual bulk fields near the boundary $z \to 0$. In particular,

$$\langle\varphi_s(\vec{x}_0, z \to 0)\rangle_\phi \to z^{s+1}\mathcal{C}_s\left\langle J_s(\vec{x}_0 e^{\sum a_i \int d^3\vec{x} J_i(\vec{x})\phi_i(\vec{x})}\right\rangle_{CFT}$$

for some normalisation constant $\mathcal{C}_s$, so that

$$\frac{\langle\varphi_s(\vec{x}, z \to 0)\rangle_{(s_i;\vec{x}_i), i=1,\ldots,n-1}}{\langle\varphi_s(\vec{x}, z \to 0)\rangle_{(s;\vec{x}')}} \to \frac{\Pi_{i=1}^{n-1} a_{s_i}}{a_s} \frac{\langle J_s(\vec{x}) J_{s_1}(\vec{x}_1) \ldots J_{s_{n-1}}(\vec{x}_{n-1})\rangle_{CFT}}{\langle J_s(\vec{x}) J_s(\vec{x}'_1)\rangle_{CFT}};$$

the left hand side of this can be calculated using Vasiliev's equations of motion (up to normalisation factors). Stripping off polarisation and $\vec{x}$ dependence ($J_s(\vec{x}) \equiv J_s(\vec{x}, \epsilon) \to J_s$ and $\vec{x}_{12} \to 0$ at the boundary), this will be denoted $C(s_1, s_2; s_3)$ so that

$$C(s_1, s_2; s_3) \sim \frac{\langle J_{s_1} J_{s_2} J_{s_3}\rangle}{\langle J_{s_1} J_{s_2}\rangle}$$

where the two currents J_{s_1} and J_{s_2} were chosen to be sources on the boundary. The perturbative form of Vasiliev's equations of motion are given by (writing $W = W_0 + \hat{W}$),

$$\begin{aligned}
&D_0 \hat{W} = -\hat{W} * \hat{W} \\
&d_Z \hat{W} + D_0 S = -\{\hat{W}, S\}_* \\
&d_Z S - B * \Theta = -S * S \\
&\tilde{D}_0 B = -\hat{W} * B + B * \pi(\hat{W}) \\
&d_Z B = -S * B + B * \pi(S).
\end{aligned}$$

We recall that all of the physical degrees of freedom may be found in $\hat{W}$ and B restricted to $z_\alpha = \bar{z}_{\dot{\alpha}} = 0$. In particular, making use of a Taylor expansion

$$\begin{aligned}
\Omega \;&=\; \hat{W}\Big|_{z=\bar{z}=0} = \sum_{m,n=0}^{\infty} \Omega^{(n,m)}_{\alpha_1\ldots\alpha_n\dot{\beta}_1\ldots\dot{\beta}_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\beta}_1}\ldots\bar{y}^{\dot{\beta}_m} \\
B|_{z=\bar{z}=0} \;&=\; \sum_{n,m=0}^{\infty} B^{(n,m)}_{\alpha_1\ldots\alpha_n\dot{\beta}_1\ldots\dot{\beta}_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\beta}_1}\ldots\bar{y}^{\dot{\beta}_m},
\end{aligned}$$

one finds that all the spin-s degrees of freedom are found in $\Omega^{(s-1+n,s-1-n)}$ for $|n| \leq s-1$ (where $\Omega^{(s-1.s-1)}$ is a symmetric s-tensor field) as well as in $B^{(2s+m,m)}$ and $B^{(m,2s+m)}$ for $m \geq 0$ (where $B^{(2s,0)}$ incorporates up to s space-time derivatives of $\Omega^{(s-1,s-1)}$). Using the linearised equations of motion for master fields B and W, the boundary-to-bulk propagators can then be derived. For B, the linearised equations of motion are given by

$$d_Z B = 0$$
$$dB + [w_0^L, B]_* + \{e_0, B\}_* = 0.$$

The first says that $B(x|y, \bar{y})$ is independent of z_α and $\bar{z}_{\dot{\alpha}}$, and the solution of the second is given by the scalar boundary-to-bulk propagator

$$B^{(0,0)} = K(x,z)^\Delta, \qquad \text{for } \Delta = 1 \text{ or } \Delta = 2$$
$$\Rightarrow B_{\text{scalar}} = Ke^{-y\Sigma\bar{y}}, \qquad \text{for } \Delta = 1$$
$$\Rightarrow B_{(s)} = \textstyle\sum_{m=0}^{\infty}(-)^m \frac{(2s)!}{m!(2s+m)!} z^{-s-m} \left(z^2 y \partial\!\!\!/ \bar{y}\right)^m z^s C(x|y) + c.c., \text{for } \Delta = 1,$$

where $K(x,z) \equiv \frac{z}{x^2+z^2}$ (which is the correct boundary-to-bulk propagator for the scalar field $B^{(0,0)}$), $\Sigma = \sigma^z - \frac{2z}{x^2}\boldsymbol{x}$ with $\boldsymbol{x} \equiv x^\mu \sigma_\mu = x^i \sigma_i + z\sigma^z$, and $C(x|y) = B|_{\bar{y}=0}$. For $\Delta = 2$, the scalar field then has the form $C(\vec{x}, z) = K^2 = \frac{z^2}{(\vec{x}^2+z^2)^2}$, and the scalar component of the master field B is given by

$$B^{\Delta=2}_{\text{scalar}}(x|y,\bar{y}) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \frac{1}{z^n} \left(-z^2 y \partial\!\!\!/ \bar{y}\right)^n C(x).$$

The master field $\hat{W}$ is now written as a fluctuation away from the vacuum configuration as follows,

$$\hat{W}(x|y,\bar{y},z,\bar{z}) = \Omega(x|y,\bar{y}) + W'(x|y,\bar{y},z,\bar{z}) \tag{1.63}$$

where $\Omega = \hat{W}\Big|_{z=\bar{z}=0}$. This obeys the linearised equation

$$D_0\hat{W} = 0 \Rightarrow D_0\Omega = -D_0 W' = -\left. D_0 W'\right|_{x=\bar{z}=0}.$$

When this is solved (in light cone coordinates, it is found that the entire boundary-to-bulk propagator for Ω comes from Ω^n_{++}, given by

$$\Omega^n_{++} = \frac{2^{-n-2}}{(2s-1)!} z^s (x^-)^{s+n} \left(y\boldsymbol{x}\sigma^{-z}\boldsymbol{x}y\right)^n \partial_+^{2s} \frac{(y\boldsymbol{x}\bar{y})^{s-n}}{x^2}$$

Solving for B at second order near the boundary then requires defining a homogeneous function in y'_α of degree $2s$, $\mathcal{K}(\vec{x}, z|y, \partial_{y'})$ which is the same as the boundary-to-bulk propagator for the scalar field when $s = 0$, but for $s > 0$ obeys different boundary conditions to the boundary-to-bulk propagator of $B^{(2s,0)}(\vec{x},z|y)$. In particular, the helicity$=s$ part of $\mathcal{K}$, denoted $\mathcal{K}_{(s)}$, is given by

$$\begin{aligned}\mathcal{K}_{(s)}(\vec{x},z|y,\lambda) &= 2^{-2s}z^{2-s}\int_0^\infty dt(1+t)^{-2s} \\ &\quad\times\left[\frac{(y\sigma^a\partial\!\!\!/\lambda)^{2s}}{(2s)!}\left[-\frac{\gamma(2-2s)(x^2)^{s-1}}{2\pi^2|\vec{x}|}\sin\left(2(s-1)\text{arctan}\frac{|\vec{x}|}{z}\right)\right]\right].\end{aligned}$$

Here,

$$\begin{aligned}\mathcal{K}_{(0)}^{\Delta=1}(\vec{x},z) &= \frac{1}{2\pi^2}\frac{z}{\vec{x}^2+z^2} \\ \mathcal{K}_{(0)}^{\Delta=2}(\vec{x},z) &= \frac{1}{\pi^2}\frac{z^2}{(\vec{x}^2+z^2)^2},\end{aligned}$$

and the 3-point function coefficient $C(s_1,s_2;s)$ can be computed from

$$\begin{aligned}&\lim_{z\to0} z^{-s-1}B_{h=s}^{(2s,0)} \\ &= \int\frac{d^3\vec{x}_0dz_0}{z_0^4}\mathcal{K}_{(s)}\left(\vec{x}-\vec{x}_0,z_0|y,\partial_{y_0}\right)\left[-\frac{2z_0}{s+1}J(\vec{x}_0,z_0;\vec{x}_1,\vec{x}_2|y_0)\right].\end{aligned}$$

In particular, considering the limit where the two sources collide, so that $\vec{\delta}=\vec{x}_2-\vec{x}_1$ is close to zero, and using translational invariance to set $\vec{x}_1=0$, the scalar becomes

$$\begin{aligned}\lim_{z\to0} z^{-1}B_{\Delta=1}^{(0,0)}(\vec{x},z) &\to -\frac{\delta^{-s_1-s_2-1}}{\pi^2|\vec{x}|^2}\int dz'd^3\vec{x}'(z')^{-2}J(\vec{x}'.z';0.\hat{\delta}) \\ \lim_{z\to0} z^{-1}B_{\Delta=2}^{(0,0)}(\vec{x},z) &\to -\frac{2\delta^{-s_1-s_2}}{\pi^2|\vec{x}|^4}\int dz'd^3\vec{x}'(z')^{-1}J(\vec{x}'.z';0.\hat{\delta})\end{aligned}$$

Using their normalisation convention, it is found that

$$\begin{aligned}C(s_1.s_2;0) &= -\frac{\sqrt{\pi}}{2}\Gamma\left(s_1+s_2+\frac{1}{2}\right) \\ C(0,s_1;s_2) &= -\frac{\sqrt{\pi}}{2}2^{-s_2}\frac{\Gamma(s_1+\frac{1}{2})}{s_2!}, \qquad s_1>s_2.\end{aligned}$$

For example, for the $\Delta=1$ 3-point function involving two scalars and one spin-s current (i.e. $C(0,s;0)$), one can deduce that

$$\left\langle J_0(0)J_{\tilde{s}}(\vec{\delta};\vec{\epsilon})J_0(\vec{x})\right\rangle \to \frac{g}{a_s}C(0,s;0)\frac{(\vec{\delta}\cdot\vec{\epsilon})^s}{|\vec{x}|^2\delta^{2s+1}}, \quad \frac{\delta}{|\vec{x}|}\to0, \tag{1.64}$$

where a_s is a normalisation factor and g is the coupling constant of Vasiliev theory, and where

$$C(0,s;0) = -\frac{\pi^{1/2}}{2}\Gamma\left(s+\frac{1}{2}\right)$$

up to normalisation factor a_s which depends only on boundary-to-bulk propagator (which is at his point undetermined since the normalisation of the 2-point function is not known). Similarly, for the 3-point function comprising one scalar and two higher spin currents, one obtains

$$\left\langle J_s(0;\vec{\epsilon})J_{\tilde{s}}(\vec{\delta};\vec{\epsilon})J_0(\vec{x})\right\rangle \to g\frac{a_0}{a_a a_{\tilde{s}}}C(s,\tilde{s};0)\frac{(\vec{\delta}\cdot\vec{\epsilon})^{s+\tilde{s}}}{|\vec{x}|^2\delta^{2s+2\tilde{s}+1}}, \quad \frac{\delta}{|\vec{x}|}\to 0. \tag{1.65}$$

For the free $O(N)$ vector theory (recalling equation (1.62)), the conserved currents have got the form

$$\mathcal{O}_f(\vec{x};\epsilon) = \phi_i(\vec{x})f\left(\epsilon_\mu, \overleftarrow{\partial}_\mu, \overrightarrow{\partial}_\mu\right)\phi_i(\vec{x}) = \sum_{s=0}^{\infty} J_{\mu_1\ldots\mu_s}(\vec{x})\epsilon^{\mu_1}\ldots\epsilon^{\mu_s}$$

where

$$f\left(\overrightarrow{\epsilon}, \overrightarrow{\partial}_\mu, \overrightarrow{\partial}_\mu\right) \equiv e^{(u-v)\cdot\epsilon}\cos\left[\sqrt{4(u\cdot\epsilon)(v\cdot\epsilon)-2(u\cdot v)\epsilon^2}\right].$$

From this one can compute 3-point functions; firstly

$$\left\langle J_0(\vec{x})J_0(\vec{x}-\vec{\delta})J_s(0;\vec{\epsilon})\right\rangle \to 8N\frac{2^s\pi^{-\frac{1}{2}}\Gamma\left(s+\frac{1}{2}\right)}{s!|\delta|(x^2)^{s+1}}\left[\vec{\epsilon}\cdot\left(\vec{\delta}-\frac{2\vec{\delta}\cdot\vec{x}}{x^2}\vec{x}\right)\right]^s,$$

whose coefficient may be compared with $C(0,s;0)$ from equation (1.64), to identify that the normalised 3-point function coefficients for the free $O(N)$ theory are given by

$$C^{\text{free}}_{00s} = N^{-\frac{1}{2}}2^{3-s}\pi^{-\frac{1}{4}}\sqrt{\frac{\Gamma\left(s+\frac{1}{2}\right)}{s!}}.$$

Secondly

$$\left\langle J_{s_1}(\vec{x},\vec{\epsilon})J_{s_2}(\vec{\delta},\vec{\epsilon})J_0(0)\right\rangle_{\vec{\epsilon}\cdot\vec{x}=0} \to 8N2^{s_1+s_2}\pi^{-1}\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)}{s_1!s_2!}\frac{(\vec{\epsilon}\cdot\vec{\delta})^{s_1+s_2}}{|\vec{x}|^{2s_1+2}|\vec{\delta}|^{2s_2+1}}$$

whose coefficient should be similarly be compared with equation (1.65). Furthermore,

$$\lim_{z\to 0} z^{-s'-1} B^{(2s',0)}_{h=s}\left(\vec{x}, z|y=\lambda\right) \to -\frac{2^{s'-2}\pi^{\frac{1}{2}}}{s'!}\Gamma\left(s+\frac{1}{2}\right)\frac{(\vec{\delta}\cdot\vec{\epsilon})^{s+s'}}{(\vec{x}^2)^{s'+1}\delta^{2s+1}},$$

from which

$$C(0,s;s') = -\frac{2^{-2s'-1}\pi^{\frac{1}{2}}}{s'!}\Gamma\left(s+\frac{1}{2}\right).$$

Comparing this with

$$\frac{\left\langle \mathcal{O}_s(0,\vec{\epsilon})\mathcal{O}_{s'}(\vec{\delta},\vec{\epsilon})\mathcal{O}_0(\vec{x})\right\rangle}{\langle \mathcal{O}_0\mathcal{O}_0\rangle} \to gC(s,s';0)\frac{(\vec{\delta}\cdot\vec{\epsilon})^{s+s'}}{(\vec{x}^2)\delta^{2s+2s'+1}},$$

and making the association $\mathcal{O}_s = a_s J_s$, one finds that the 3-point functions once again agree given that

$$\frac{a_s}{a_0} = 2^{-s}s!.$$

By comparing this with the free field theory, one obtains explicitly that

$$a_0 = \frac{1}{\sqrt{N}}, \qquad g = -\frac{16}{\pi\sqrt{N}}.$$

Performing a similar procedure for $\Delta = 2$, we obtain

$$C^{\Delta=2}(0,s;0) = -\frac{\pi^{\frac{1}{2}}}{2}s\Gamma\left(s+\frac{1}{2}\right),$$

with corresponding normalised 3-point function in Vasiliev theory given by

$$C^{\Delta=2}_{00s} = g\frac{a_0' a_s}{a_0'}C^{\Delta=2}(0,s;0) = ga_sC^{\Delta=2}(0.s;0).$$

where the prime indicates the normalisation constant for the $\Delta = 2$ case; no prime indicates values are the same as for the $\Delta = 1$ case. For $C(s,\tilde{s};0)$,

$$\int d^3\vec{x}\lim_{z\to 0} z^{-2}B^{(0,0)}_{\Delta=2}(\vec{x},z) = -\pi^{\frac{5}{2}}\left(\Gamma(s+\tilde{s}+\frac{1}{2}\right)\frac{(\vec{\delta}\cdot\vec{\epsilon})^{s+\tilde{s}}}{\delta^{2s+2\tilde{s}+1}},$$

which, together with the normalisation factor $g\frac{a_s a_{\tilde{s}}}{a_0'}$, gives the desired 3-point function in Vasiliev theory,

$$\frac{\int d^3\vec{x}\left\langle J_s(\vec{\delta},\vec{\epsilon})J_{\tilde{s}}(0,\vec{\epsilon})J_0'(\vec{x})\right\rangle}{\langle J_0' J_0'\rangle} = g\frac{a_0'}{a_s a_{\tilde{s}}}\int d^3\vec{x}\lim_{z\to 0} z^{-2}B^{(0,0)}_{\Delta=2}(\vec{x},z).$$

We see that Giombi and Yin provided some remarkable evidence for the duality conjectured by Klebanov and Polyakov, that the free 3d $O(N)$ vector model is dual to type A minimal Vasiliev higher spin gauge theory with $\Delta = 1$, while the critical 3d $O(N)$ vector model is dual to type A minimal Vasiliev Higher spin theory with scalar having $\Delta = 2$. This will be one of the main themes in this thesis.

1.14 Holographic duality for the IR fixed point follows from that of the UV fixed point

On the surface it is surprising that there exists an holographic duality between critical 3d $O(N)$ vector models and type A minimal Vasiliev Higher spin theory with scalar having $\Delta = 2$ (that is, for the interacting IR fixed point case) since the higher symmetry in the bulk is broken through loop effects. The simplest example to show that this happens at leading order in $1/N$ (noting that the $O(N)$ vector model has only even spin currents) is for the three point function involving spins 4,2 and 0[115]. Computing the three-point function with null polarisation vectors ϵ_1 and ϵ_2 in momentum space gives the following result for the free theory (calculated using the $U(N)$ version for simplicity):

$$\left\langle J^{(0)}\left(-p_1-p_2\right)J^{(3)}\left(p_1,\epsilon_1\right)J^{(1)}\left(p_2,\epsilon_2\right)\right\rangle = \int d^3q\frac{\epsilon_2\cdot\left(2q+p_2\right)f_3\left(\epsilon_1\cdot q,\epsilon_1\cdot\left(p_1-q\right)\right)}{q^2\left(q-p_1\right)^2\left(q+p_2\right)^2}.$$

The equivalent value in the critical theory is obtained by multiplying the free theory value by $-|p_1+p_2|$. Now, while the divergence of the spin 3 current $J^{(3)}(p_1)$ in the free case is composed of two analytic terms, when we obtain the critical theory values (by once again multiplying by $-|p_1+p_2|$, one of the terms turns out to be non-analytic.

The duality at criticality, however, actually follows from the $\Delta = 1$ case (for the free UV fixed point case). In particular, Refs. [114] and [116] showed that at large N, the generating functional of the correlators at the UV and IR fixed point are related by a Legendre transformation. In Ref. [114], it was conveyed that in moving between a given bulk and boundary condition, there is a switching of the dimension of the operator from λ to $d - \lambda$, similar to Liouville theory and results from the old matrix model where a suitable double trace interaction reversed the 'gravitational dressing' of the interaction. This was done by finding the expected value of

$$\exp\left(-N^2 W\right) = \exp\left(-N^2 \int d^n x f(x) g(\mathcal{O}_i)\right).$$

where $g(\mathcal{O}_i)$ denotes that this procedure works for general functions of a series of operators (we denote by β_i the expectation of $\mathcal{O}_i$ with dimension λ in the boundary CFT). The expected value (i.e. the boundary functional) is found by replacing $\mathcal{O}_i$ throughout by β_i to obtain the so-called 'multitrace boundary interaction' $W \equiv E(x, \beta, d\beta, ...)$. The boundary condition may be stated as $\alpha_i = \frac{\delta W}{\delta \beta_i}$, so that α_i are related to the source of the operators. Now note that the near-boundary behaviour of two scalar fields ϕ_1 and ϕ_2 (with equal mass squared) in $D = d + 1$-dimensional AdS space is given by

$$\phi_i \sim \alpha_i(x) r^{d-\lambda} + \beta_i(x) r^{\lambda}..$$

For $d/2 - 1 < \lambda < d/2$, two methods of quantisation is possible, where the scalar fields are related to primary operators of dimension either λ or $d - \lambda$. Considering a marginal operator $f\mathcal{O}_1\mathcal{O}_2$, one may one the first method to the first scalar field, and the second method to the second scalar field, so that

$$\begin{aligned} \phi_1 &\sim \alpha_1(x) r^{d-\lambda} + \beta_1(x) r^{\lambda} \\ \phi_2 &\sim \beta_2'(x) r^{d-\lambda} + \alpha_2'(x) r^{\lambda}, \end{aligned}$$

with boundary conditions $\alpha_1 = f\beta_2'$ and $\alpha_2' = f\beta_1$. There is a $f \to 1/f$ symmetry which gives rise to a $\phi_1 \leftrightarrow \phi_2$ symmetry, so that the value of f, while not changing the physics, defines which scalar field is related to the operator of dimension λ and which to the operator of dimension $d - \lambda$. Importantly, note that for $d/2 - 1 < \lambda < d/2$ and $\phi \sim \alpha r^{d-\lambda} + \beta r^{\lambda}$, we could perturb a theory with boundary condition $\alpha = 0$

with relevant perturbation $W = \frac{g}{2}\beta^2$ so that the boundary condition is $\alpha = g\beta$. Then as $g \to \infty$, the boundary condition condition $\beta = 0$ is approached, suitable for the quantisation of a field of dimension $d - \lambda$. (Note that in the reverse direction, the perturbation $W = \frac{g}{2}\alpha^2$ would be irrelevant.) Thus, the renormalisation group flow leads from one quantisation method to the other. This was expanded upon in Ref. [117], resulting in 'an improved correspondence formula for AdS/CFT with multi-trace'. Furthermore, the generating functionals of the correlation functions are related to each other by a Legendre transform[118].

Ref. [119] applied Witten's prescription as set out in the previous paragraph to fields ϕ_a in AdS with a mostly positive metric. The classical action that was considered was

$$S = \frac{1}{2\kappa^2}\int d^{d+1}z\sqrt{g}\,(R - \Lambda_0) + \int d^{d+1}z\sqrt{g}\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right).$$

A key point investigated here was the concept that an RG flow should 'interpolate' between the UV Δ_- theory and the IR Δ_+ theory - the idea of a line of RG fixed points. The focus, however, was not on the full interpolating solution but on properties of the AdS endpoints and the central charges of the CFTs dual to these endpoints. The one-loop scalar correction to the gravitational Lagrangian is

$$-\sqrt{g}^{-1}\delta\mathcal{L} = V(z; m^2, f) = -\frac{i}{2}\mathrm{tr}\log(-\Box + m^2)$$

Making use of the Breitenlohner-Freedman bound (the smallest mass with normalisable modes in AdS) one can then find the difference in V when $f = 0$ and when $f = \infty$ (corresponding to the change in central charge as one flows from the UV to the IR. To do this, making it possible for the change in one-loop self-energy between the UV and IR endpoints of an holographic RG flow to be evaluated.

Meanwhile, Ref. [116] tackled the question from the field theory side. In the undeformed (UV) CFT,

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{x^{2\Delta}},$$

and we assume a suppression of higher point functions of $\mathcal{O}$. To trigger an RG flow we include a relevant deformation $\frac{f}{2}\mathcal{O}^2$; using a Hubbard-Stratonovich transformation (involving the introduction of an auxiliary field σ)we have

$$Z_f[J] \quad = \quad \left\langle e^{-\int \frac{f}{2}\mathcal{O}^2+\int J\mathcal{O}} \right\rangle_0 = \sqrt{\det\left(-\frac{1}{f}\mathbb{1}\right)} \int \mathcal{D}\sigma \left\langle e^{\int\left(\frac{1}{2f}\sigma^2+\sigma\mathcal{O}+J\mathcal{O}\right)} \right\rangle_0 .$$

Indeed, the correlator in the IR is found to be $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_f \propto \frac{1}{x^{2(d-\Delta)}}$ for $x >> f^{-\frac{1}{d-2\Delta}}$. Defining $\mathcal{W}$ to be the generating function of correlators in the UV CFT and with some rescaling and discarding of surface terms, one can write

$$Z_f \sim \int \mathcal{D}\tilde{\sigma} e^{\mathcal{W}+\int \tilde{\sigma}\tilde{J}}.$$

In the planar limit (using a saddle point approximation), one is able to realise the Legendre transformation between the IR generating functional and UV one, $\tilde{J} = -\frac{\delta \mathcal{W}}{\delta \tilde{\sigma}}$, and conversely. The difference in central charges then emerges using the following calculation,

$$c_{IR} - c_{UV} = \frac{1}{d} R \frac{\partial}{\partial R} \left(W_f[R] - W_0[R]\right),$$

where $W_f[R] \equiv \log Z_f \left[J = 0, S_R^d\right]$, with $Z_f[J = 0, S_R^d]$ being the partition function of the CFT on the S_R^d sphere deformed by $\int \frac{f}{2}\mathcal{O}^2$. Without going too much into the mathematical detail which involves zeta-functions and various polynomials, one can write

$$W_f[R] = -\frac{1}{2}(V_1 + V_2).$$

For d even there is good agreement with the results in Ref. [119], for example the gravity (AdS_5) result was that

$$\frac{c_+ - c_-}{c_-} = \frac{\kappa_5^2}{120\pi^2 L^3} \left[\frac{(\Delta_- - 2)^3}{3} - \frac{(\Delta_- - 2)^5}{5}\right],$$

and for the CFT result it was found that the total anomalous part of V_2 to order $(\Delta - 2)^5$ was

$$\frac{1}{6}\log\left(R\Lambda\right)\left(\frac{(\Delta - 2)^3}{3} - \frac{(\Delta - 2)^5}{5}\right).$$

The above discussion shows indeed that the $\Delta = 2$ duality is a direct result of the $\Delta = 1$ duality to leading order.

Ref. [115] then extended the above results to all orders in $1/N$. In particular, if it holds for the free $O(N)$ theory that bulk tree level diagrams reproduce the correct n-point functions and all loop corrections cancel with the boundary condition, then the theory with the $\Delta = 2$ boundary condition has a UV finite perturbative expansion that matches order by order with the $1/N$ expansion of the critical $O(N)$ vector model. This was done by finding the difference between critical and free n-point correlators, where a scalar operator of dimension 2 (α was included to allow for the building of loops. The relation

$$\begin{aligned} G_{\Delta=2}\left(q;z,z'\right) - G_{\Delta=2}\left(q;z,z'\right) &= -|q| K_{\Delta=1}\left(q;z\right) K_{\Delta=1}\left(-q;z'\right) \\ &= -\frac{1}{|q|} K_{\Delta=2}(q;z) K_{\Delta=2}(-q;z') \end{aligned}$$

is used, where $K_\Delta(q;z)$ is the Fourier transform of the boundary to bulk propagator which obeys the correct relation between correlators of the free and critical $O(N)$ vector models,

$$K_{\Delta=2}(q;z) = -|q| K_{\Delta=1}(q;z),$$

and $G_\Delta(x,x')$ is the Fourier transform of the bulk scalar propagator. The relation can be shown quite beautifully using Witten diagrams, as a cutting of bulk n-point functions. The ultimate result here is that the perturbative expansion of the theory with $\Delta = 2$ boundary condition matches the $1/N$ expansion of the critical $O(N)$ vector model order by order. Furthermore, the higher spin symmetry breaking caused by the $\Delta = 2$ boundary condition is controlled by the bulk coupling constant $1/N$, with anomalous dimensions of boundary higher spin currents being suppressed by $1/N$.

1.15 Aim of the Thesis

One of the 'easiest' ways of probing the AdS/CFT correspondence is the duality conjectured by Klebanov and Polyakov[1] between free/critical O(N) vector models in $d = 2 + 1$ dimensions and higher spin theories, where gauge invariant operators $\phi^a \partial_{(\mu_1} ... \partial_{\mu_s)} \phi^a$ are dual to a tower of massless higher spin fields[111] (for example, see table 1.4). In this duality, the free 3d $O(N)$ vector model is dual to type A minimal Vasiliev higher spin gauge theory with $\Delta = 1$ and the critical 3d $O(N)$ vector model is dual to type A minimal Vasiliev Higher spin theory with scalar having $\Delta = 2$[1, 4]. In particular, it was claimed that when a relevant interaction is present, the theory flows from an unstable UV fixed point with a dimension $\Delta = 1$ scalar at the boundary to an IR fixed point where $\Delta = 2$.

Remarkable evidence for Klebanov and Polyakov's $O(N)$ vector model / higher spin theory conjecture has come in the form of the reconstruction of the bulk [5], the one-loop test of the duality involving the calculation of the $\mathcal{O}(N)$ correction to the Weyl anomaly for even d and to free energy for odd d and the matching of the three-point function for both $\Delta = 1$ and $\Delta = 2$[4, 2]. Furthermore, Refs. [115, 114, 116] show how using a Legendre transformation the $\Delta = 2$ duality follows from the $\Delta = 1$ duality. In Ref. [1], the standard auxilliary field approach is used to obtain the 1/N expansion (and take N large). In this thesis we will instead make use of the Collective field theory approach, where bilocals are used to encode the invariance of the theory explicitly. It is then the invariant variables which are described by the Collective field theory[8].

In Ref. [7], the three-dimensional $O(N)$ invariant bosonic model with $\frac{\lambda}{N}(\phi^a\phi^a)^2$ interaction was studied at its infrared fixed point. This was done making use of a bilocal field approach, in a $1/N$ expansion. At the critical point / infrared fixed point, a state was indeed identified to correspond to a $\Delta = 2$ scalar state and the $\Delta = 1$ state was found to vanish from the spectrum in agreement with Polyakov and Klebanov.

In this thesis, we build on the constructive approach which has been developed in Refs. [5, 9, 10] in both the light-cone gauge and the temporal gauge for the free theory, in which an explicit map between the conformal field theory in $d = 2 + 1$ dimensions and the higher spin theory in $AdS_4 \times S^1$ was established. In the Hamiltonian approach, the $1 + 2 + 2 = 5$ coordinates of the equal time bilocals map (in phase space) to the coordinates of $\mathrm{AdS}_4 \times S^1$. In this thesis, the chief aim is to investigate the applicability of this map to the interacting theory [120] and to make contact with the work in Ref. [7].

We will use the Hamiltonian approach in a time like gauge[121]. The quartic interaction contributes linearly in the bilocal field fluctuation equations, and the spectrum problem is then that of a potential scattering problem. The scattering state solutions take a universal form at the critical point.

In order to obtain the bulk description of these boundary scattering states, we develop a first principles approach consisting of a simple change of variables from bilocal momenta to bulk momenta, as dictated by the map, but requiring a field redefinition in defining the bulk higher spin field. Remarkably, it will then be shown that the $s = 0$ state (equivalent to $\Delta = s + 1 = 1$ state) is precisely removed from the bulk higher spin field, which is equivalent to the observation found in Ref. [7], by studying correlators and the spectrum on the boundary.

The equivalence of this approach to that developed in Ref. [10] then has to be established. This is indeed done by showing that the conformal algebra, in bulk coordinates, both at the free and interacting critical point, agrees with that obtained previously in Ref. [10].

I have endeavoured to make this PhD thesis as self-contained as possible, ensuring adequate explanations are provided where necessary, without sacrificing elegance and conciseness of the arguments, in the spirit of Einstein's comment,"Everything should be made as simple as possible, but no simpler."

Chapter 2

Collective Field Theory

In this thesis, the Collective field theory approach[8] is the approach taken in describing the large N limit of $O(N)$ invariant vector theories. The Collective field theory approach re-expresses these theories directly in terms of bilocal fields which encode the invariance explicitly.

Collective field theory is based on a non-trivial change of variables from the original variables of a theory to the set of invariant variables appropriate to the description of the theory's large N limit, which thereby reduces the number of degrees of freedom for the theory. Below is an overview of this method, which uses the Hamiltonian of the system to obtain an expression for the Jacobian.

2.1 General Method

A change of variables is performed in a Hilbert space, from original variables x^a to invariant variables ϕ_α (where Greek letters α, β etc. are used to label these invariant variables). The kinetic energy portion of the Hamiltonian is proportional to ∇^2 which can be written as follows using the chain rule:

$$\nabla^2 = \omega_\alpha \partial_\alpha + \Omega_{\alpha\beta}\partial_\alpha \partial_\beta \tag{2.1}$$

$\partial_\alpha \equiv \frac{\partial}{\partial \phi_\alpha}$
$\omega_\alpha = \sum_{a=1}^{N} \frac{\partial^2 \phi_\alpha}{\partial (x^a)^2}$

$\Omega_{\alpha\beta} = \sum_{a=1}^{N} \frac{\partial\phi_\alpha}{\partial x^a}\frac{\partial\phi_\beta}{\partial x^a}$. The inner product is invariant in Hilbert space and so

$$\int \psi^*(x)\,\psi(x) = \int J\Phi^*(\phi)\,\Phi(\phi)$$
$$\int \psi^*(x)\,\partial_\alpha\psi(x) = \int J\Phi^*(\phi)\,\partial_\alpha\Phi(\phi)\,,$$

where J is the Jacobian of the transformation. Equivalently, we could absorb the transformation into the individual wavefunctions and so obtain

$$\begin{aligned}\int \psi^*(x)\,\psi(x) &= \int J^{\frac{1}{2}}\Phi^*(\phi)\,J^{\frac{1}{2}}\Phi(\phi)\\ \int \psi^*(x)\,\partial_\alpha\psi(x) &= \int J^{\frac{1}{2}}\Phi^*(\phi)\left(J^{\frac{1}{2}}\partial_\alpha J^{-\frac{1}{2}}\right)J^{\frac{1}{2}}\Phi(\phi)\,.\end{aligned}$$

Hence

$$\begin{aligned}\partial_\alpha f &\rightarrow J^{\frac{1}{2}}\partial_\alpha J^{-\frac{1}{2}}f = J^{\frac{1}{2}}\left(J^{-\frac{1}{2}}\partial_\alpha - \frac{1}{2}J^{-\frac{3}{2}}\partial_\alpha J\right)f = \left(\partial_\alpha - \frac{1}{2}\partial_\alpha \ln J\right)f\\ &\Rightarrow \partial_\alpha \rightarrow \partial_\alpha - \frac{1}{2}\partial_\alpha \ln J. \qquad (2.2)\end{aligned}$$

Using (2.2) in (2.1) results in,

$$\begin{aligned}\nabla^2 &= \Omega_{\alpha\beta}\left(\partial_\alpha - \frac{1}{2}\partial_\alpha\ln J\right)\left(\partial_\beta - \frac{1}{2}\partial_\beta\ln J\right) + \omega_\alpha\left(\partial_\alpha - \frac{1}{2}\partial_\alpha\ln J\right)\\ &= \Omega_{\alpha\beta}\partial_\alpha\partial_\beta - \frac{1}{2}\Omega_{\alpha\beta}\partial_\alpha\left(\partial_\beta\ln J\right) - \frac{1}{2}\Omega_{\alpha\beta}\left(\partial_\alpha\ln J\right)\partial_\beta\\ &\quad + \frac{1}{4}\Omega_{\alpha\beta}\left(\partial_\alpha\ln J\right)\left(\partial_\beta\ln J\right) + \omega_\alpha\partial_\alpha - \frac{1}{2}\omega_\alpha\left(\partial_\alpha\ln J\right)\\ &= \partial_\alpha\Omega_{\alpha\beta}\partial_\beta - \left(\partial_\alpha\Omega_{\alpha\beta}\right)\partial_\beta - \frac{1}{2}\Omega_{\alpha\beta}\left(\partial_\alpha\partial_\beta\ln J\right) - \frac{1}{2}\Omega_{\alpha\beta}\left(\partial_\beta\ln J\right)\partial_\alpha\\ &\quad - \frac{1}{2}\Omega_{\alpha\beta}\left(\partial_\alpha\ln J\right)\partial_\beta + \frac{1}{4}\left(\partial_\alpha\ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta\ln J\right) + \omega_\alpha\partial_\alpha\\ &\quad - \frac{1}{2}\omega_\alpha\left(\partial_\alpha\ln J\right)\\ &= \partial_\alpha\Omega_{\alpha\beta}\partial_\beta - \left(\partial_\alpha\Omega_{\alpha\beta}\right)\partial_\beta - \Omega_{\alpha\beta}\left(\partial_\alpha\ln J\right)\partial_\beta - \frac{1}{2}\Omega_{\alpha\beta}\left(\partial_\alpha\partial_\beta\ln J\right)\\ &\quad + \frac{1}{4}\left(\partial_\alpha\ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta\ln J\right) + \omega_\beta\partial_\beta - \frac{1}{2}\omega_\alpha\left(\partial_\alpha\ln J\right), \qquad (2.3)\end{aligned}$$

where the last step was possible because $\Omega_{\alpha\beta}$ is symmetric. Now since kinetic energy is an observable, its operator must be explicitly hermitian and hence (2.3) must be hermitian. Operators linear in a derivative must be accompanied by the imaginery i to be hermitian (for example the momentum operator) unlike scalar terms and terms quadratic in the derivative. Hence the real terms linear in ∂_β in (2.3) must be set to zero, forming an important constraint which can be written in two useful forms due to the symmetry of $\Omega_{\alpha\beta}$,

$$\partial_\alpha \Omega_{\alpha\beta} = \omega_\beta - \Omega_{\alpha\beta}\left(\partial_\alpha \ln J\right) \tag{2.4}$$

$$\Omega_{\alpha\beta}\left(\partial_\beta \ln J\right) = \omega_\alpha - \partial_\beta \Omega_{\alpha\beta}. \tag{2.5}$$

Hence (2.1) becomes

$$\begin{aligned}\nabla^2 \;&=\; -\frac{1}{2}\omega_\alpha\left(\partial_\alpha \ln J\right) - \frac{1}{2}\Omega_{\alpha\beta}\left(\partial_\alpha\partial_\beta \ln J\right)\\ &\quad + \left[\partial_\alpha \Omega_{\alpha\beta}\partial_\beta + \frac{1}{4}\left(\partial_\alpha \ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta \ln J\right)\right].\end{aligned} \tag{2.6}$$

Now, using equations (2.4) and (2.5) the first line in equation (2.6) simplifies to

$$\begin{aligned}\nabla^2_{(1)} \;&=\; -\frac{1}{2}\omega_\beta\left(\partial_\beta \ln J\right) - \frac{1}{2}\partial_\alpha\left(\Omega_{\alpha\beta}\partial_\beta \ln J\right) + \frac{1}{2}\left(\partial_\alpha \Omega_{\alpha\beta}\right)\left(\partial_\beta \ln J\right)\\ &=\; -\frac{1}{2}\left[\omega_\beta - \left(\partial_\alpha \Omega_{\alpha\beta}\right)\right]\left(\partial_\beta \ln J\right) - \frac{1}{2}\partial_\alpha\left(\Omega_{\alpha\beta}\partial_\beta \ln J\right)\\ &=\; -\frac{1}{2}\left[\Omega_{\alpha\beta}\left(\partial_\alpha \ln J\right)\right]\left(\partial_\beta \ln J\right) - \frac{1}{2}\partial_\alpha\left(\omega_\alpha - \partial_\beta \Omega_{\alpha\beta}\right).\end{aligned}$$

Hence, combining with the second line in equation (2.6) results in

$$\begin{aligned}\nabla^2 &= \partial_\alpha \Omega_{\alpha\beta}\partial_\beta - \frac{1}{4}\left(\partial_\alpha \ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta \ln J\right) - \frac{1}{2}\partial_\alpha\omega_\alpha + \frac{1}{2}\partial_\alpha\partial_\beta\Omega_{\alpha\beta}\\ \Rightarrow \frac{1}{2}\nabla^2 &= \frac{1}{2}\partial_\alpha \Omega_{\alpha\beta}\partial_\beta - \frac{1}{8}\left(\partial_\alpha \ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta \ln J\right) - \frac{1}{4}\partial_\alpha\omega_\alpha + \frac{1}{4}\partial_\alpha\partial_\beta\Omega_{\alpha\beta}.\end{aligned} \tag{2.7}$$

Note that the standard way for writing the term $\frac{1}{8}\left(\partial_\alpha \ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta \ln J\right)$ in the literature (for example see [122]) is $\frac{1}{8}\omega_\alpha\Omega^{-1}_{\alpha\beta}\omega_\beta$; to leading order these two terms are equivalent by making use of equation 2.4. (For this thesis we only need the first two terms of equation (2.7), so we are further at liberty to write it as above, which avoids the need to invert $\Omega_{\alpha\beta}$. The first of these is a kinetic energy term, and the second contributes to the effective potential.) Except for the derivative terms, it is evident that this expression should be incorporated with the potential to give rise to an effective potential, necessary in determining the ground state (for an example of this, see Ref. [123]).

2.2 $O(N)$ Hamiltonian

We obtain an expression for the Jacobian of an O(N) uncharged bosonic scalar field using the Collective field theory method, and that of the Collective field Hamiltonian. We follow Refs. [124, 125] with the addition of flavour indices α, β for a change of variables

$$\phi^a_\alpha(x) \to \sigma_{\alpha\beta}(x,y) = \phi^a_\alpha(x)\phi^a_\beta(y),$$

where x is a d-dimensional parameter and a=1,2,...,N. It is useful to define an upper triangular matrix $\Phi_{\alpha\beta}(x,y)$ where

$$\begin{aligned} \sigma_{\alpha\beta}(x,y) = \quad & \Phi_{\alpha\beta}(x,y) \text{ for } \alpha \geq \beta \\ & \Phi_{\beta\alpha}(y,x) \text{ for } \alpha < \beta. \end{aligned} \tag{2.8}$$

and similarly for x, y indices. We will be making use of equation (2.4)

$$\partial_{\tilde{\alpha}}\Omega_{\tilde{\alpha}\tilde{\beta}} = \omega_{\tilde{\beta}} - \Omega_{\tilde{\alpha}\tilde{\beta}}\left(\partial_{\tilde{\alpha}} \ln J\right)$$

where (with $\delta^{aa} = N$ for colour indices and $\delta_{\alpha\alpha} = m$ for flavour indices)[1],

$$\begin{aligned} \omega_{\tilde{\alpha}} &\equiv \int \mathrm{d}^d z \frac{\delta^2}{\delta\phi^a_\delta(z)\,\delta\phi^a_\delta(z)}\sigma_{\alpha\beta}(x,y) \\ &= \int \mathrm{d}^d z \frac{\delta}{\delta\phi^a_\delta(z)}\Big(\delta_{\alpha\delta}\phi^a_\beta(y)\delta^d(x-z) + \delta_{\beta\delta}\phi^a_\alpha(x)\delta^d(z-y)\Big) \\ &= \int \mathrm{d}^d z \Big(N\delta_{\alpha\delta}\delta_{\beta\delta}\delta^d(x-z)\delta^d(z-y) + N\delta_{\alpha\delta}\delta_{\beta\delta}\delta^d(y-z)\delta^d(z-x)\Big) \\ &= 2N\delta_{\alpha\beta}\delta^d(x-y) \end{aligned} \tag{2.9}$$

$$\begin{aligned} \Omega_{\tilde{\alpha}\tilde{\beta}} &\equiv \int \mathrm{d}^d z \frac{\delta}{\delta\phi^a_\delta(z)}\sigma_{\alpha\beta}(x,y)\,\frac{\delta}{\delta\phi^a_\delta(z)}\sigma_{\alpha'\beta'}(x',y') \\ &= \int d^d z\,\Big[\Big(\delta_{\alpha\delta}\phi^a_\beta(y)\delta^d(x-z) + \delta_{\beta\delta}\phi^a_\alpha(x)\delta^d(y-z)\Big) \\ &\quad \Big(\delta_{\alpha'\delta}\phi^a_{\beta'}(y')\delta^d(x'-z) + \delta_{\beta'\delta}\phi^a_{\alpha'}(x')\delta^d(z-y')\Big)\Big] \\ &= \Big[\sigma_{\alpha\alpha'}(x,x')\delta_{\beta\beta'}\delta^d(y-y') + \sigma_{\alpha\beta'}(x,y')\delta_{\beta\alpha'}\delta^d(y-x') \\ &\quad + \sigma_{\beta\alpha'}(y,x')\delta_{\alpha\beta'}\delta^d(x-y') + \sigma_{\beta\beta'}(y,y')\delta_{\alpha\alpha'}\delta^d(x-x')\Big]. \end{aligned} \tag{2.10}$$

[1]Here we change labels α to $\tilde{\alpha}$ to convey that these labels now represent both space-time and flavour degrees of freedom.

Finding derivatives with respect to $\Phi_{\alpha\beta}(x,y)$ will then single out only the upper triangular copy of variables, as follows:

$$\begin{aligned}
\partial_{\tilde{\beta}}\Omega_{\tilde{\alpha}\tilde{\beta}} &= \int_{x'>y'} \mathrm{d}^d x' \mathrm{d}^d y' \frac{\delta}{\delta\Phi_{\alpha'\beta'}(x',y')} \\
&\Big[\sigma_{\alpha\alpha'}(x,x')\delta_{\beta\beta'}\delta^d(y-y') + \sigma_{\alpha\beta'}(x,y')\delta_{\beta\alpha'}\delta^d(y-x') \\
&+\sigma_{\beta\alpha'}(y,x')\delta_{\alpha\beta'}\delta^d(x-y') + \sigma_{\beta\beta'}(y,y')\delta_{\alpha\alpha'}\delta^d(x-x')\Big] \\
&= \int \mathrm{d}^d x' \int \mathrm{d}^d y' \Big[\delta_{\alpha\alpha'}\delta_{\beta'\alpha'}\delta_{\beta\beta'}\delta^d(x-x')\delta^d(x'-y')\delta^d(y-y') \\
&+\delta_{\alpha\alpha'}\delta_{\beta'\beta'}\delta_{\beta\alpha'}\delta^d(x-x')\delta^d(y'-y')\delta^d(y-x') \\
&+\delta_{\beta\alpha'}\delta_{\beta'\alpha'}\delta_{\alpha\beta'}\delta^d(y-x')\delta^d(x'-y')\delta^d(x-y') \\
&+\delta_{\beta\alpha'}\delta_{\beta'\beta'}\delta_{\alpha\alpha'}\delta^d(y-x')\delta^d(y'-y')\delta^d(x-x') \\
&= 2(m\delta^d(0)L^d+1)\delta_{\alpha\beta}\delta^d(x-y)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\Omega_{\tilde{\alpha}\tilde{\beta}}\left(\partial_{\tilde{\beta}}\ln J\right) \\
&= \int_{x'>y'} \mathrm{d}^d x' \mathrm{d}^d y' \Big[\sigma_{\alpha\alpha'}(x,x')\delta_{\beta\beta'}\delta^d(y-y') + \sigma_{\alpha\beta'}(x,y')\delta_{\beta\alpha'}\delta^d(y-x') \\
&\quad +\sigma_{\beta\alpha'}(y,x')\delta_{\alpha\beta'}\delta^d(x-y') + \sigma_{\beta\beta'}(y,y')\delta_{\alpha\alpha'}\delta^d(x-x')\Big] \frac{\delta}{\delta\Phi_{\alpha'\beta'}(x',y')}\ln J \\
&= \left(\int d^d x'\sigma_{\alpha\alpha'}(x,x')\frac{\delta}{\delta\Phi_{\alpha'\beta}(x',y)} + \int d^d y'\sigma_{\alpha\beta'}(x,y')\frac{\delta}{\delta\Phi_{\beta\beta'}(y,y')}\right. \\
&\quad \left. + \int d^d x'\sigma_{\beta\alpha'}(y,x')\frac{\delta}{\delta\Phi(x',x)} + \sigma_{\beta\beta'}(y,y')\frac{\delta}{\delta\Phi_{\alpha\beta'}(x,y')}\right)\ln J \\
&= 4\int d^d z\sigma_{\alpha\gamma}(x,z)\frac{\delta\ln J}{\delta\Phi_{\beta\gamma}(y,z)}.
\end{aligned}$$

And therefore equation (2.4) becomes

$$N\delta_{\alpha\beta}\delta^d(x-y) - (m\delta^d(0)L^d+1)\delta_{\alpha\beta}\delta^d(x-y) = 2\int d^d z\sigma_{\alpha\gamma}(x,z)\frac{\delta\ln J}{\delta\Phi_{\beta\gamma}(y,z)} \quad . \tag{2.11}$$

Now note that

$$\Omega_{\tilde{\alpha}\tilde{\beta}}\partial_{\tilde{\beta}}\ln\det(\sigma) = \int_{x'>y'}\Omega\frac{\delta}{\delta\Phi_{\alpha'\beta'}(x',y')}\ln\det(\sigma)$$

$$\begin{aligned}
&= \left(\int d^dz\sigma_{\alpha\delta}(x,z)\frac{\delta}{\delta\Phi_{\delta\beta}\left(z,y\right)} + \int d^dz\sigma_{\alpha\delta}(x,z)\frac{\delta}{\delta\Phi_{\beta\delta}\left(y,z\right)}\right. \\
&\quad \left. + \int d^dz\sigma_{\beta\delta}(y,z)\frac{\delta}{\delta\Phi_{\delta\alpha}\left(z,x\right)} + \sigma_{\beta\delta}(y,z)\frac{\delta}{\delta\Phi_{\alpha\delta}\left(x,z\right)}\right)\ln\det(\sigma) \\
&= \left(\int d^dz\sigma_{\alpha\delta}(x,z)\sigma^{-1}_{\beta\delta}(y,z) + \int d^dz\sigma_{\alpha\delta}(x,z)\sigma^{-1}_{\delta\beta}(z,y)\right. \\
&\quad \left. + \int d^dz\sigma_{\beta\delta}(y,z)\sigma^{-1}_{\delta\alpha}(z,x) + \sigma_{\beta\delta}(y,z)\sigma^{-1}_{\delta\alpha}(z,x)\right) \\
&= 4\delta^d(x-y)\delta_{\alpha\beta} \qquad (2.12)
\end{aligned}$$

where we have used the symmetry of σ. Hence

$$lnJ = \tfrac{1}{2}\left[N - \left(mL^d\delta^d(0)+1\right)\right]\ln\det(\sigma) = \tfrac{1}{2}\left[N - \left(mL^d\delta^d(0)+1\right)\right]\mathrm{tr}\ln(\sigma). \qquad (2.13)$$

This result is consistent with both Ref. [124] as well as the author's MSc dissertation. We can now obtain the Hamiltonian (with flavour indices now omitted) as follows

$$\begin{aligned}
\partial_\alpha\Omega_{\alpha\beta}\partial_\beta &= \int dx\int_{x<y} dy\frac{\partial}{\partial\Phi(x,y)}\left(\int d^dz\sigma(x,z)\frac{\delta}{\delta\Phi\left(z,y\right)} + \int d^dz\sigma(x,z)\frac{\delta}{\delta\Phi\left(y,z\right)}\right. \\
&\quad \left. + \int d^dz\sigma(y,z)\frac{\delta}{\delta\Phi\left(z,x\right)} + \sigma(y,z)\frac{\delta}{\delta\Phi\left(x,z\right)}\right) \\
&= 4\int dx\int_{z<y<x} dydz\frac{\partial}{\partial\Phi(y,x)}\sigma(x,z)\frac{\delta}{\delta\Phi\left(z,y\right)} \\
&= 4\mathrm{Tr}\frac{\partial}{\partial\Phi}\sigma\frac{\partial}{\partial\Phi}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\left(\partial_\alpha\ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta\ln J\right) &= \int dx\int dy\left(\frac{\partial}{\partial\Phi(x,y)}\frac{1}{2}\left(N-(\delta^d(0)L^d+1)\right)\ln\det\sigma\right) \\
&\quad \left(2N-2(\delta^d(0)L^d+1)\right)\delta^d(x-y) \\
&= \int dx\int dy\left(N-(\delta^d(0)L^d+1)\right)^2\sigma^{-1}(y,x)\delta^d(x-y) \\
&\to N^2\mathrm{Tr}(\sigma^{-1})
\end{aligned}$$

and the kinetic energy from equation (2.7), where $\omega \sim N$, $\Omega \sim 1$ and $\ln J \sim N$, becomes

$$\begin{aligned} K &= -\frac{1}{2}\partial_\alpha \Omega_{\alpha\beta}\partial_\beta + \frac{1}{8}\Omega_{\alpha\beta}\left(\partial_\alpha \ln J\right)\left(\partial_\beta \ln J\right) \\ &= -2\mathrm{Tr}\frac{\partial}{\partial\Phi}\sigma\frac{\partial}{\partial\Phi} + \frac{N^2}{8}\mathrm{Tr}(\sigma^{-1}) \\ &\to \frac{2}{N}\mathrm{Tr}\Pi\sigma\Pi + \frac{N}{8}\mathrm{Tr}(\sigma^{-1}), \end{aligned} \tag{2.14}$$

where $\Pi_{xy} = -i\frac{\delta}{\delta\Phi(xy)}$ and $\sigma \to N\sigma$. For the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a - \frac{g}{4!}\left(\phi^a\phi^a\right)^2$$

the Hamiltonian density may be written as

$$\mathcal{H} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \frac{1}{2}m^2\phi^a\phi^a + \frac{g}{4!}\left(\phi^a\phi^a\right)^2,$$

so this contribution to the Hamiltonian may be written in terms of bilocals as follows

$$\begin{aligned} H &= -\frac{1}{2}\int dx \int dy\delta(x-y)\partial_y^2\sigma_{xy} + \frac{1}{2}\int dx m^2\sigma_{xx} + \frac{g}{4!}\int dx\sigma_{xx}^2 \\ &\to N\left(-\frac{1}{2}\int dx \int dy\delta(x-y)\partial_y^2\sigma_{xy} + \frac{1}{2}\int dx m^2\sigma_{xx} + \frac{\lambda}{4!}\int dx\sigma_{xx}^2\right) \end{aligned}$$

where $\lambda = gN$ is the t'Hooft coupling. The complete Hamiltonian is then given by

$$\begin{aligned} H &= \frac{2}{N}\mathrm{Tr}\Pi\psi\Pi + \frac{N}{8}\mathrm{Tr}(\psi^{-1}) \\ &\quad + N\int d^{d-1}\vec{x}\left(-\frac{1}{2}\lim_{\vec{x}\to\vec{y}}\partial_{\vec{y}}^2\psi_{\vec{x}\vec{y}} + \frac{1}{2}m^2\psi_{\vec{x}\vec{x}} + \frac{\lambda}{4!}\psi_{\vec{x}\vec{x}}^2\right). \end{aligned} \tag{2.15}$$

Here we note that as $N \to \infty$, fluctuations about the saddle point of the effective potential are suppressed. In this formalism, the large N configuration is the semiclassical approximation determined by the effective potential.

2.3 $O(N)$ Path Integral

Here we follows what was done in Ref. [125]. The Schwinger-Dyson equations for an uncharged bosonic scalar field are defined as follows:

$$\begin{aligned}
&\int \mathrm{D}\phi \, \frac{\delta}{\delta\phi^a_\beta(y)} \left[\phi^a_\alpha(x) F[\sigma] e^{i\mathcal{S}}\right] = 0 \\
= &\int \mathrm{D}\sigma \int \mathrm{d}^d z \int_{x'<y'} d^d x' d^d y' \\
&\left[\frac{\delta}{\delta\Phi_{\alpha'\beta'}(x',y')} \left(\sigma_{\alpha\alpha'}(x,x')\delta^d(y-y') + \sigma_{\alpha\beta'}(x,y')\delta^d(y-x')\right)\right] JF[\sigma]e^{i\mathcal{S}}. \\
&+\int \mathrm{D}\sigma \int \mathrm{d}^d z \int d^d z \left(\sigma_{\alpha\gamma}(x,z)\frac{\delta}{\delta\Phi_{\gamma\beta}(z,y)} + \sigma_{\alpha\gamma}(x,z)\frac{\delta}{\delta\Phi_{\beta\gamma}(y,z)}\right)\left[JF[\sigma]e^{i\mathcal{S}}\right].
\end{aligned} \tag{2.16}$$

Using the chain rule we have

$$\begin{aligned}
\frac{\delta}{\delta\phi^a_\beta(y)} &\equiv \int \mathrm{d}^d z \int \mathrm{d}^d x \frac{\delta\sigma_{\gamma\alpha}(z,x)}{\delta\phi^a_\beta(y)} \frac{\delta}{\delta\Phi_{\gamma\alpha}(z,x)} \\
&= \int \mathrm{d}^d z \int \mathrm{d}^d x \left(\delta_{\beta\gamma}\delta^d(y-z)\phi^a_\alpha(x) + \delta_{\beta\alpha}\delta^d(x-y)\phi^a_\gamma(z)\right) \frac{\delta}{\delta\Phi_{\gamma\alpha}(z,x)} \\
&= \int d^d x \phi^a_\alpha(x) \frac{\delta}{\delta\Phi_{\beta\alpha}(y,x)} + \int d^d z \phi^a_\gamma(z) \frac{\delta}{\delta\Phi_{\gamma\beta}(z,y)} \\
&= 2\int d^d z \phi^a_\gamma(z) \frac{\delta}{\delta\Phi_{\beta\gamma}(y,z)}.
\end{aligned} \tag{2.17}$$

Substituting equation (2.17) into equation (2.16), finding expectation values for the Schwinger-Dyson equations, and making the necessary cancellations, we obtain

$$N\delta_{\alpha\beta}\delta^d(x-y) = \int d^d z \left(mL^d\delta^d(0)+1\right)\delta_{\alpha\beta}\delta^d(x-y) + 2\int d^d z \sigma_{\alpha\gamma}(y,z)\frac{\delta lnJ}{\delta\Phi_{\beta\gamma}(y,z)}$$

in other words, the same solution as was found in equation (2.11) using the Collective Field Theory, for which the Jacobian once again is given by (see equation (2.13))

$$lnJ = \tfrac{1}{2}\left[N - \left(mL^d\delta^d(0)+1\right)\right]\ln\det(\sigma) = \tfrac{1}{2}\left[N - \left(mL^d\delta^d(0)+1\right)\right]\mathrm{tr}\ln(\sigma),$$

the leading term of which is

$$\ln J = \frac{1}{2}N\ln\det\sigma = \frac{N}{2}\mathrm{tr}\ln\sigma. \tag{2.18}$$

Note that this has the same functional form as for the Hamiltonian formalism, but with the time coordinate included.

Chapter 3

Large N infrared fixed point of $3-d$ $O(N)$ invariant vector theory - a bilocal boundary study

Here we review some of the results in Refs. [7, 126], where the system is studied as a $d = 2 + 1$ dimensional conformal field theory, using bilocal fields. The emphasis in this thesis will be on the Hamiltonian (single time) formalism; for $d = 2 + 1$ dimensions in this formalism, bilocal fields have $2 + 2 + 1 = 5$ degrees of freedom, which is the same as the number of degrees of freedom as $AdS_4 \times S^1$[8]. In Refs. [5, 9], explicit maps were constructed, which will be discussed in detail at a later stage. Considering two-dimensional vectors $\vec{x_1}$ and $\vec{x_2}$, $O(N)$ singlet sector equal-time (Hamiltonian formalism) bilocals are given by

$$\psi_{\vec{x_1}\vec{x_2}} = \sum_{a=1}^{N} \phi^a \left(t, \vec{x_1}\right) \phi^a \left(t, \vec{x_2}\right). \tag{3.1}$$

We will compare the Collective field Hamiltonian approach which we focus on in this thesis with that of the Lagrangian (two-time) covariant formalism in which most of the literature is given. Bilocals in the Lagrangian covariant formalism have $3+3 = 6$

degrees of freedom. Bilocals here are given by

$$\psi_{x_1 x_2} = \sum_{a=1}^{N} \phi^a (x_1) \phi^a (x_2). \tag{3.2}$$

Note that we differentiate notationally between $\vec{x}$ which has only spatial coordinates, and $x \equiv x^\mu$, which is a contravariant vector and thus has both spatial and temporal coordinates.

3.1 Non-linear sigma model

As highlighted before, the approach taken in this thesis is the Hamiltonian/Collective field theory approach. In this section, however, we will obtain expressions for the $\Delta = 1$ and $\Delta = 2$ propagators using the standard auxiliary field theory approach, which we will compare later with those found, first using covariant bilocals, and then using the Hamiltonian/Collective field theory approach; these will be shown to be identical.

At the IR fixed point, the standard way for treating the critical $O(N)$ vector model is by describing it using a non-linear sigma model. This non-linear sigma model has a bilocal description which can be written in Euclidean signature as

$$S = N \int d^d x \left(\frac{1}{2} \partial_\mu \vec{S} \partial_\mu \vec{S} + \frac{\alpha(x)}{2} \left(\vec{S}(x) \cdot \vec{S}(x) - \frac{1}{\lambda} \right) \right).$$

Covariant bilocals are given by

$$\psi_{xy} = \vec{S}(x) \cdot \vec{S}(y).$$

The leading term of log of the Jacobian for non-charged bosonic fields was found in equation (2.13) to be

$$logJ = \frac{N}{2} Trlog\psi$$

Hence, in the large-N limit, the $O(N)$ vector-model at its infra-red critical point is described by a non-linear sigma model with an effective action

$$S_{eff} = N\left[-\frac{1}{2}Trln\psi + \int d^dx\left(-\frac{1}{2}\lim_{y\to x}\partial^2\psi_{xy} + \frac{1}{2}\alpha_x\psi_{xx} - \frac{1}{2\lambda}\alpha_x\right)\right].$$

where $\alpha_x = \alpha$ is constant-valued at the saddle point / leading large N configuration and $\int d^dx\alpha_x$ is a multiplicative volume which factors out. Using the large N translational invariant ansatz

$$\psi_{xy} = \int \frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\psi_k, ,$$

a saddle point analysis leads to

$$\begin{aligned} 0 &= -\frac{1}{2}\psi_k^{-1} + \frac{k^2}{2} + \frac{\alpha}{2} \\ \Rightarrow \psi_k^0 &= \frac{1}{k^2+\alpha} \\ \Rightarrow \psi_{xy} &= \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{k^2+\alpha}. \end{aligned}$$

Varying the effective action with respect to α, yields the gap equation

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+\alpha} = \frac{1}{\lambda}; \quad \sqrt{\alpha} = -\frac{4\pi}{\lambda}.$$

The bilocals at the conformal fixed point ($\alpha \to 0$, equivalently at the infrared fixed point, where $\lambda \to \infty$) are given by

$$\psi_k^0 = \frac{1}{k^2}.$$

The $1/N$ corrections about this large-N background are

$$\alpha_x = 0 + \frac{1}{\sqrt{N}}\tilde{\alpha}(x); \quad \psi_{xy} = \psi_{xy}^0 + \frac{1}{\sqrt{N}}\eta_{xy}.$$

Substituting these into the effective action

$$\begin{aligned} S_{eff} &= N\left[-\frac{1}{2}Trln\psi + \int d^dx\left(-\tfrac{1}{2}\lim_{y\to x}\partial^2\psi_{xy} + \frac{1}{2}\alpha_x\psi_{xx} - \frac{1}{2\lambda}\alpha_x\right)\right] \\ &\to N\left[-\frac{1}{2}Trln\left(\psi^0 + \frac{1}{\sqrt{N}}\eta\right) + \int d^dx\left(-\tfrac{1}{2}\lim_{y\to x}\partial^2\left(\psi_{xy}^0 + \frac{1}{\sqrt{N}}\eta_{xy}\right)\right.\right. \end{aligned}$$

$$+\frac{1}{2}\frac{1}{\sqrt{N}}\tilde{\alpha}(x)\left(\psi^0_{xy}+\frac{1}{\sqrt{N}}\eta_{xy}\right)_{x=y}-\frac{1}{2\lambda}\frac{1}{\sqrt{N}}\tilde{\alpha}(x)\right)\Bigg].$$

It is clear that the quadratic part of S_{eff}, for which the factors of N cancel come from the first and third terms:

$$\begin{aligned} S_{eff} &= N\left[-\frac{1}{2}Trln\psi^0\left(1+\frac{1}{\sqrt{N}}\left(\psi^0\right)^{-1}\eta\right)\right. \\ &\left.+\frac{1}{2}\int d^dx\frac{1}{\sqrt{N}}\tilde{\alpha}(x)\left(\psi^0_{xy}+\frac{1}{\sqrt{N}}\eta_{xy}\right)_{x=y}\right], \end{aligned}$$

so that the quadratic part of the action is given by

$$S^2_{eff}=\frac{1}{4}Tr\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right)+\frac{1}{2}Tr\left(\tilde{\alpha}\eta\right)$$

Applying a shift $\tilde{\eta}=(\psi_0\tilde{\alpha}\psi_0)+\eta$ to decouple the η and $\tilde{\alpha}$ terms, this becomes

$$\begin{aligned} S^2_{eff} &= \frac{1}{4}Tr\left(\psi_0^{-1}\left(-\left(\psi_0\tilde{\alpha}\psi_0\right)+\tilde{\eta}\right)\psi_0^{-1}\left(-\left(\psi_0\tilde{\alpha}\psi_0\right)+\tilde{\eta}\right)\right)+\frac{1}{2}Tr\left(\tilde{\alpha}\left(\left(-\psi_0\tilde{\alpha}\psi_0\right)+\tilde{\eta}\right)\right) \\ &= \frac{1}{4}Tr\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0-\psi_0^{-1}\tilde{\eta}\tilde{\alpha}\psi_0-\tilde{\alpha}\tilde{\eta}+\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right)+\frac{1}{2}Tr\left(-\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0+\tilde{\alpha}\tilde{\eta}\right) \\ &= \frac{1}{4}Tr\left(\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right)-\frac{1}{4}Tr\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\right) \\ &\equiv \frac{1}{4}\int d^dxd^dyd^dzd^dw\left(\psi_0\right)^{-1}_{xy}\tilde{\eta}_{yz}\left(\psi_0\right)^{-1}_{zw}\tilde{\eta}_{wx}-\frac{1}{4}\int d^dxd^dy\tilde{\alpha}_x\psi_{0xy}\tilde{\alpha}_y\psi_{0yx} \end{aligned} \quad (3.3)$$

we that we obtain the quadratic effective action which may be schematically written as,

$$S^{(2)}_{eff} = \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\tilde{\eta}\psi_0^{-1}\tilde{\eta}\right)-\frac{1}{4}\mathrm{Tr}\left(\tilde{\alpha}\psi_0\tilde{\alpha}\psi_0\right).$$

In momentum space,

$$\begin{aligned} \tilde{\eta}_{xy} &= \int\frac{d^dk_1}{(2\pi)^{d/2}}\int\frac{d^dk_2}{(2\pi)^{d/2}}e^{ik_1x}e^{ik_2y}\tilde{\eta}_{k_1k_2} \\ \tilde{\alpha}_x &= \int\frac{d^dk}{(2\pi)^{d/2}}e^{ikx}\tilde{\alpha}_k. \end{aligned}$$

Hence, the first term in equation (3.3) becomes

$$S^{2_1}_{eff} = \frac{1}{4}\int\frac{d^dk_1}{(2\pi)^{d/2}}\int\frac{d^dk_2}{(2\pi)^{d/2}}\int\frac{d^dk_3}{(2\pi)^{d/2}}\int\frac{d^dk_4}{(2\pi)^{d/2}}\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}$$

$$
\begin{aligned}
& (\psi_0)^{-1}_{p_1}\,\tilde{\eta}_{k_1k_2}\,(\psi_0)^{-1}_{p_2}\,\tilde{\eta}_{k_3k_4}\int d^dx d^dy d^dz d^dw \\
& e^{ip_1(x-y)}e^{ik_1y}e^{ik_2z}e^{ip_2(z-w)}e^{ik_3w}e^{ik_4x} \\
= & \int d^dk_1\int d^dk_2\,(\psi_0)^{-1}_{k_1}\,\tilde{\eta}_{k_1k_2}\,(\psi_0)^{-1}_{-k_2}\,\tilde{\eta}_{-k_2-k_1}
\end{aligned}
$$

and similarly for the second term, and therefore the quadratic effective action for $d=3$ is given by,

$$
\begin{aligned}
S^{(2)}_{eff} &= \frac{1}{4}\int d^3k_1\int d^3k_2\tilde{\eta}_{k_1k_2}k_1^2k_2^2\tilde{\eta}_{-k_2,-k_1} \\
&\quad -\frac{1}{4}\int d^3k_1\tilde{\alpha}_{k_1}\left(\frac{1}{8\left|\vec{k}_1\right|}\right)\tilde{\alpha}_{-k_1}.
\end{aligned}
$$

From this action, the propagators are read off to be the following,

$$
\langle\tilde{\eta}_{k_1k_2}\tilde{\eta}_{k_3k_4}\rangle = \frac{2}{k_1^2k_2^2}\delta(k_1+p_1)\delta(k_2+p_2) \tag{3.4}
$$

$$
\langle\tilde{\alpha}_{k_1}\tilde{\alpha}_{k_2}\rangle = -16|k_1|\delta(k_1+k_2). \tag{3.5}
$$

In coordinate space these are given by

$$
\begin{aligned}
\langle\eta_{x_1x_2}\eta_{x_3x_4}\rangle &= 2\int\frac{d^dk_1}{(2\pi)^d}\frac{e^{ik_1(x_1-x_4)}}{k_1^2}\int\frac{d^dk_2}{(2\pi)^d}\frac{e^{ik_2(x_2-x_3)}}{k_2^2} \\
&= \int\frac{d^dk_1}{(2\pi)^d}\frac{e^{ik_1(x_1-x_4)}}{k_1^2}\int\frac{d^dk_2}{(2\pi)^d}\frac{e^{ik_2(x_2-x_3)}}{k_2^2}+(x_3\leftrightarrow x_4)\,.
\end{aligned}
$$

In equation (B.6) we show that

$$
\int\frac{d^3p}{(2\pi)^3}\frac{1}{(p^2)^1}e^{ipx} = \frac{\pi^{3/2}}{(2\pi)^3\Gamma(1)}\frac{\Gamma(1/2)}{(x^2/4)^{1/2}}=\frac{1}{4\pi|x|}
$$

so that

$$
\langle\eta_{x_1x_2}\eta_{x_3x_4}\rangle = \left(\frac{1}{4\pi}\right)^2\left(\frac{1}{\left(x_{13}^2\right)^{1/2}}\frac{1}{\left(x_{24}^2\right)^{1/2}}+\frac{1}{\left(x_{14}^2\right)^{1/2}}\frac{1}{\left(x_{23}^2\right)^{1/2}}\right).
$$

Similarly, with $\alpha=-1/2$ and $\Gamma(-1/2)=-2\Gamma(1/2)=-2\sqrt{\pi}$,

$$
\langle\tilde{\alpha}_{x_1}\tilde{\alpha}_{x_2}\rangle = -16\int\frac{d^dk}{(2\pi)^d}e^{ik(x_1-x_2)}|k| = \frac{\pi^{d/2}}{(2\pi)^d\Gamma(\alpha)}\frac{\Gamma(d/2-\alpha)}{(x^2/4)^{d/2-\alpha}}.
$$

$$= \frac{16}{\pi^2}\frac{1}{\left(x_{12}^2\right)^2}.$$

By setting $x_1 = x_2$ and $x_3 = x_4$, it becomes clear that the field η_{xx} has scaling dimension $\Delta = 1$ and that the Lagrange multiplier field $\tilde{\alpha}$ has scaling dimension $\Delta = 2$. This model thus gives rise to both a $\Delta = 1$ state (identical to the free theory) and $\Delta = 2$ state at the IR fixed point. However, from Section 1.13 we know that at the IR fixed point there should only be a $\Delta = 2$ state. This we shall verify in the next section by using directly the quartic theory.

3.2 Path Integral at the critical point with a quartic interaction

The effective action may be written in Euclidean signature as

$$\begin{aligned}\mathcal{S}_{eff} &= N\int d^dx\left[\frac{1}{2}\left(-\lim_{y\to x}\partial_y^2\psi_{xy}\right) - \frac{1}{2}m^2\psi_{xx} - \frac{\lambda}{4!}\left(\psi_{xx}\right)^2\right] \\ &\quad -\frac{N}{2}\text{Trln}\psi.\end{aligned}$$

We make use of the translationally invariant ansatz

$$\psi_{xy} = \int\frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\psi_k,$$

to perform a saddle point analysis

$$\begin{aligned}S_{eff} &= N\int d^dx\left[\frac{1}{2}\left(-\lim_{y\to x}\int\frac{d^dk'}{(2\pi)^d}e^{ik'(x-y)}\psi_{k'}\left(-k'^2\right)\right) - \frac{1}{2}m^2\int\frac{d^dk'}{(2\pi)^d}\psi_{k'}\right. \\ &\quad\left.-\frac{\lambda}{4!}\left(\int\frac{d^dk'}{(2\pi)^d}\psi_{k'}\right)^2\right] - \frac{N}{2}\text{Trln}\psi_{k'},\end{aligned}$$

which leads to the mass gap equation

$$\begin{aligned}\frac{\delta S_{eff}}{\delta\psi_k} = 0 &= N\int d^dx\left[\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\psi_k k^2 - \frac{1}{2}m^2 - \frac{2\lambda}{4!}\left(\int\frac{d^dk'}{(2\pi)^d}\psi_{k'}\right)\right] - \frac{N}{2}\psi_k^{-1} \\ \Rightarrow 0 &= \frac{1}{2}k^2 - \frac{1}{2}m^2 - \frac{\lambda}{12}\int\frac{d^dk'}{(2\pi)^d}\psi_{k'} - \frac{1}{2}\psi_k^{-1}\end{aligned}$$

$$\Rightarrow \psi^0_k \;=\; \frac{1}{k^2 - m^2 - \frac{\lambda}{6}\int \frac{d^dk'}{(2\pi)^d}\psi^0_{k'}}.$$

The large-N conformal background is then given by (for a more detailed explanation of this, see the discussion after equation (3.13))

$$\psi^0_k = \frac{1}{k^2}. \tag{3.6}$$

To obtain the $1/N$ corrections we consider the fluctuations when expanding the effective action about the large-N background,

$$\begin{aligned}
\psi_{xy} &= \psi^0_{xy} + \frac{1}{\sqrt{N}}\eta_{xy} \\
\Rightarrow S^{(2)}_{eff} &= \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) - \frac{\lambda}{4!}\int d^dx'\eta^2_{x'x'} \\
&= \frac{1}{4}① - \frac{\lambda}{4!}②.
\end{aligned} \tag{3.7}$$

We desire to write this quadratic effective action as

$$S^{(2)}_{eff} = \frac{1}{2}\int d^dk_1\int d^dk_2\int d^dk_3\int d^dk_4\eta_{k_1k_2}\mathcal{O}_{k_1k_2k_3k_4}\eta_{k_3k_4},$$

Expanding the circled terms we obtain

$$\begin{aligned}
① &= \mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) \\
&= \int d^dx\int d^dy\int d^dz\int d^dw \\
&\quad \int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^{d/2}}\int\frac{d^dk_3}{(2\pi)^{d/2}}\int\frac{d^dk_4}{(2\pi)^d}\int\frac{d^dk_5}{(2\pi)^{d/2}}\int\frac{d^dk_6}{(2\pi)^{d/2}} \\
&\quad e^{ik_1(x-y)}\psi^{-1}_{0k_1}e^{-ik_2y}e^{-ik_3z}\eta_{k_2k_3}e^{ik_4(w-z)}\psi^{-1}_{0k_4}e^{-ik_5y}e^{-ik_6z}\eta_{k_5k_6}.
\end{aligned}$$

The 4 spatial integrals result in 4 Dirac delta functions multiplied by a factor of $(2\pi)^{4d}$; thereafter, integrating over k_2, k_4, k_5 and k_6 sets $k_6 = k_1$, $k_2 = -k_1$, $k_4 = -k_3$ and $k_5 = k_4 = -k_3$ to give

$$\begin{aligned}
① &= \int d^dk_1\int d^dk_3\psi^{-1}_{0k_1}\eta_{-k_1k_3}\psi^{-1}_{0-k_3}\eta_{-k_3k_1} \\
&= \int d^dk_1\int d^dk_2\eta_{k_1k_2}\psi^{-1}_{0-k_1}\psi^{-1}_{0-k_2}\eta_{-k_2-k_1} \\
&= \int d^dk_1\int d^dk_2\eta_{k_1k_2}\left[\delta^d(k_1+k_4)\delta^d(k_2+k_3)\psi^{-1}_{0k_3}\psi^{-1}_{0k_4}\right]\eta_{k_3k_4}
\end{aligned}$$

. Similarly

$$\begin{aligned} ② &= \int d^dx\eta_{xx}^2 \\ &= \int d^dx \int \frac{d^dk_1}{(2\pi)^{d/2}} \int \frac{d^dk_2}{(2\pi)^{d/2}} \int \frac{d^dk_3}{(2\pi)^{d/2}} \int \frac{d^dk_4}{(2\pi)^{d/2}} e^{-ik_1x}e^{-ik_2x}e^{-ik_3x}e^{-ik_4x}\eta_{k_1k_2}\eta_{k_3k_4} \\ &= \int d^dk_1 \int d^dk_2 \int d^dk_3 \int d^dk_4\eta_{k_1k_2}\left[\frac{1}{(2\pi)^d}\delta^d(k_1+k_2+k_3+k_4)\right]\eta_{k_3k_4}, \end{aligned}$$

and hence

$$\begin{aligned} S_{eff}^{(2)} &= \frac{1}{2}\int d^dk_1 \int d^dk_2 \int d^dk_3 \int d^dk_4\eta_{k_1k_2} \\ &\quad \left[\frac{1}{2}\delta^d(k_1+k_4)\delta^d(k_2+k_3)\psi_{0k_3}^{-1}\psi_{0k_4}^{-1}\right. \\ &\quad \left.-\frac{2\lambda}{4!}\frac{1}{(2\pi)^d}\delta^d(k_1+k_2+k_3+k_4)\right]\eta_{k_3k_4}. \end{aligned}$$

Therefore

$$S_{eff}^{(2)} = \frac{1}{2}\int d^dk_1 \int d^dk_2 \int d^dk_3 \int d^dk_4\eta_{k_1k_2}\mathcal{O}_{k_1k_2;k_3k_4}\eta_{k_3k_4}$$

where[125]

$$\begin{aligned} \mathcal{O}_{k_1k_2;k_3k_4} &= \frac{1}{2}\delta^d(k_1+k_4)\delta^d(k_2+k_3)\psi_{0k_3}^{-1}\psi_{0k_4}^{-1} \\ &\quad -\frac{2\lambda}{4!}\frac{1}{(2\pi)^d}\delta^d(k_1+k_2+k_3+k_4). \end{aligned}$$

In Minkowski signature, this may be written as [7]

$$iS_{eff}^{(2)} = -\frac{1}{2}\int d^dk_1 \int d^dk_2 \int d^dk_3 \int d^dk_4\eta_{k_1k_2}\mathcal{O}_{k_1k_2;k_3k_4}\eta_{k_3k_4}$$

where

$$\begin{aligned} \mathcal{O}_{k_1k_2;k_3k_4} &\to \frac{1}{2}\delta^d(k_1+k_4)\delta^d(k_2+k_3)(\psi_0)_{k_3}^{-1}(\psi_0)_{k_4}^{-1} \\ &\quad +\frac{2i\lambda}{4!}\frac{1}{(2\pi)^d}\delta^d(k_1+k_2+k_3+k_4), \end{aligned} \tag{3.8}$$

and

$$\psi_k^0 = \frac{i}{k^2}.$$

The effective propagator, which is given by the inverse of this operator, is thus[125]

$$\begin{aligned}\mathcal{O}^{-1}_{k_1k_2;p_1p_2} &= 2\psi^0_{p_1}\psi^0_{p_2}\delta^3\left(k_1+p_1\right)\delta^3\left(k_2+p_2\right)\\ &+\frac{-\frac{i\lambda}{3}\frac{1}{(2\pi)^3}\psi^0_{k_1}\psi^0_{k_2}\psi^0_{p_1}\psi^0_{p_2}}{1+\frac{\lambda}{48}\frac{1}{|p_1+p_2|_E}}\delta^3\left(k_1+k_2+p_1+p_2\right). \qquad (3.9)\end{aligned}$$

For finite λ there is a pole condition given by

$$\begin{aligned} & 1+\frac{\lambda}{48}\frac{1}{|p_1+p_2|_E}=0\\ \Longleftrightarrow\quad & |p_1+p_2|_E=\sqrt{-(E_1+E_2)^2+(\vec{p}_1+\vec{p}_2)^2}=-\frac{\lambda}{48}\\ \Longleftrightarrow\quad & (E_1+E_2)^2-(\vec{p}_1+\vec{p}_2)^2=-\frac{\lambda^2}{48^2}; \qquad (3.10)\end{aligned}$$

as $\lambda\to\infty$, we note the infinite "pole" in the two-point function, which is consistent with conformal invariance and dimensional analysis. At criticality ($\lambda\to\infty$), equation (3.9) takes a finite form, independent of λ:

$$\begin{aligned}\mathcal{O}^{-1}_{k_1k_2;p_1p_2} &= 2\frac{i}{p_1^2}\frac{i}{p_2^2}\delta^3\left(k_1+p_1\right)\delta^3\left(k_2+p_2\right)\\ &-\frac{i}{k_1^2}\frac{i}{k_2^2}\frac{16i|p_1+p_2|_E}{(2\pi)^3}\frac{i}{p_1^2}\frac{i}{p_2^2}\delta^3\left(k_1+k_2+p_1+p_2\right). \qquad (3.11)\end{aligned}$$

This is the sum of a disconnected piece (associated with the UV critical point) and a connected piece which is a $\Delta=2$ state that emerges at criticality. Up to leg factors, these results are in direct agreement with the results found using the non-linear sigma model in section 3.1. It is highlighted that this result is finite and that no regularisation scheme was required. We now examine the $\Delta=1$ propagator $\langle\eta_{xx}\eta_{yy}\rangle$ in the path integral approach,

$$\begin{aligned}&\langle\eta_{xx}\eta_{yy}\rangle\\ =&\int\frac{d^3k_1}{(2\pi)^{3/2}}\int\frac{d^3k_2}{(2\pi)^{3/2}}\int\frac{d^3p_1}{(2\pi)^{3/2}}\int\frac{d^3p_2}{(2\pi)^{3/2}}e^{-ix(k_1+k_2)}e^{-iy(p_1+p_2)}\mathcal{O}^{-1}_{k_1k_2;p_1p_2}\\ =&\int\frac{d^3k_1}{(2\pi)^{3/2}}\int\frac{d^3k_2}{(2\pi)^{3/2}}\int\frac{d^3p_1}{(2\pi)^{3/2}}\int\frac{d^3p_2}{(2\pi)^{3/2}}e^{-ix(k_1+k_2)}e^{-iy(p_1+p_2)}\times\\ &\left[2\frac{i}{p_1^2}\frac{i}{p_2^2}\delta^3\left(k_1+p_1\right)\delta^3\left(k_2+p_2\right)\right.\\ &\left.-\frac{i}{k_1^2}\frac{i}{k_2^2}\frac{16i|p_1+p_2|_E}{(2\pi)^3}\frac{i}{p_1^2}\frac{i}{p_2^2}\delta^3\left(k_1+k_2+p_1+p_2\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \left[\frac{-2e^{-ix(p_1+p_2)}e^{-iy(p_1+p_2)}}{p_1^2 p_2^2} \right. \\
&\quad \left. - i^2 \frac{i}{8|p_1+p_2|_E} \frac{16i|p_1+p_2|_E}{1} \frac{i}{p_1^2} \frac{i}{p_2^2} e^{-ix(p_1+p_2)} e^{-iy(p_1+p_2)} \right] \\
&= 0,
\end{aligned}
$$

so that indeed the $\Delta = 1$ state is removed from the spectrum at the IR critical fixed point. Note that this results from a combination of the connected and disconnected pieces of the propagator.

The identification of the $\Delta = 2$ field as suggested by equation (3.11) may be written in coordinate space as

$$
\begin{aligned}
\alpha(x)\delta(x-y) &= \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{xy} \\
\Rightarrow \left\langle \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{x_1y_1} \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{x_2y_2} \right\rangle &= \delta(x_1-y_1)\delta(x_2-y_2)\left\langle \alpha(x_1)\alpha(x_2)\right\rangle
\end{aligned}
$$

Making use of equations (3.4) and (3.5) -

$$
\begin{aligned}
\langle \tilde{\eta}_{k_1k_2}\tilde{\eta}_{k_3k_4}\rangle &= \frac{2}{k_1^2k_2^2}\delta(k_1+p_1)\delta(k_2+p_2) \\
\langle \tilde{\alpha}_{k_1}\tilde{\alpha}_{k_2}\rangle &= -16|k_1|\delta(k_1+k_2),
\end{aligned}
$$

we have that for the connected piece,

$$
\begin{aligned}
\left\langle \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{x_1y_1} \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{x_2y_2} \right\rangle &= \delta(x_1-y_1)\delta(x_2-y_2)\left\langle \alpha(x_1)\alpha(x_2)\right\rangle \\
&= \delta(x_1-y_1)\delta(x_2-y_2)\int \frac{d^3p}{(2\pi)^3} e^{ip(y_2-y_1)}\left(-16|p|\right)
\end{aligned}
$$

and that the disconnected piece satisfies

$$
\begin{aligned}
\left\langle \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{x_1y_1} \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{x_2y_2} \right\rangle &= \int \frac{d^3k_1}{(2\pi)^{3/2}} \int \frac{d^3k_2}{(2\pi)^{3/2}} \int \frac{d^3p_1}{(2\pi)^{3/2}} \int \frac{d^3p_2}{(2\pi)^{3/2}} \\
&\quad e^{ik_1x_1}e^{ik_2x_2}e^{ip_1y_1}e^{ip_2y_2}k_1^2k_2^2p_1^2p_2^2\left\langle \eta_{k_1k_2}\eta_{p_1p_2}\right\rangle \\
&= \int \frac{d^3k_1}{(2\pi)^{3/2}} \int \frac{d^3k_2}{(2\pi)^{3/2}} \int \frac{d^3p_1}{(2\pi)^{3/2}} \int \frac{d^3p_2}{(2\pi)^{3/2}} \\
&\quad e^{ik_1x_1}e^{ik_2x_2}e^{ip_1y_1}e^{ip_2y_2}2p_1^2p_2^2\delta(k_1+p_1)\delta(k_1+p_2) \\
&= \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{ik_1(x_1-y_1)}e^{ik_2(x_2-y_2)}2k_1^2k_2^2.
\end{aligned}
$$

From equation (B.5):

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{(p^2)^\alpha}e^{ipx} \quad = \quad \frac{\pi^{d/2}}{\Gamma(\alpha)}\frac{\Gamma(d/2-\alpha)}{(x^2/4)^{d/2-\alpha}},$$

we see that $\alpha = -1$ and so this disconnected piece goes to zero since $\frac{1}{\Gamma(-1)} \to 0$.

3.3 Hamiltonian Approach

The Hamiltonian formalism builds upon the concepts from chapter 2. The collective Hamiltonian was found in equation (2.15) to be

$$\begin{aligned} H \quad &= \quad \frac{2}{N}\mathrm{Tr}\Pi\psi\Pi + \frac{N}{8}\mathrm{Tr}(\psi^{-1}) \\ &\quad + N\int d^{d-1}\vec{x}\left(-\frac{1}{2}\lim_{\vec{x}\to\vec{y}}\partial_{\vec{y}}^2\psi_{\vec{x}\vec{y}} + \frac{1}{2}m^2\psi_{\vec{x}\vec{x}} + \frac{\lambda}{4!}\psi_{\vec{x}\vec{x}}^2\right) \end{aligned}$$

Using the translationally invariant ansatz,

$$\psi_{\vec{x}\vec{y}} = \int \frac{d^{d-1}k}{(2\pi)^d}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\psi_{\vec{k}}.$$

For this Hamiltonian, performing the saddle point analysis with respect to $\psi_{\vec{k}}$, we find that

$$\begin{aligned} -\frac{1}{8}(\psi_{\vec{k}}^0)^{-2} + \frac{1}{2}\vec{k}^2 + \frac{1}{2}m^2 + \frac{\lambda}{12}\psi_{\vec{k}}^0 = 0 \\ \Rightarrow \psi_{\vec{k}}^0 \quad &= \quad \left[8\left(\frac{1}{2}\vec{k}^2 + \frac{1}{2}m^2 + \frac{\lambda}{12}\psi_{\vec{k}}^0\right)\right]^{-1/2} \\ &= \quad \frac{1}{2}\left(\vec{k}^2 + m^2 + \frac{\lambda}{6}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}\right)^{-1/2}. \end{aligned} \tag{3.12}$$

Setting $s = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}$ and integrating, we may write the above as

$$s = \frac{1}{2}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{\sqrt{\vec{k}^2 + m^2 + \frac{\lambda}{6}s}}$$

Writing $\alpha = m^2 + \frac{\lambda}{6}s$ the above becomes

$$\frac{6}{\lambda}\left(\alpha - m^2\right) = \frac{1}{2}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{\sqrt{\vec{k}^2+\alpha}} \tag{3.13}$$

Now notice that (where the signature used is (+,-,-), and using equation (B.2) in 3 dimensions),

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\alpha}[= i\frac{\sqrt{\alpha}}{4\pi}] \quad = \quad \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\int \frac{dE}{2\pi}\frac{1}{E^2-(\vec{k}^2+\alpha)}.$$

The energy integral is given by (closing in the UHP)

$$\begin{aligned}\int \frac{dE}{2\pi}\frac{1}{E^2-(\vec{k}^2+\alpha)} &= \int \frac{dE}{2\pi}\frac{1}{(E-\sqrt{\vec{k}^2+\alpha}+i\epsilon)(E+\sqrt{\vec{k}^2+\alpha}-i\epsilon)} \\ &= \frac{2\pi i}{2\pi}\frac{-1}{2\sqrt{\vec{k}^2+\alpha}}\end{aligned}$$

implying that

$$\begin{aligned} i\frac{\sqrt{\alpha}}{4\pi} &= -\frac{i}{2}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{\sqrt{\vec{k}^2+\alpha}} \\ \Rightarrow \frac{1}{2}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{\sqrt{\vec{k}^2+\alpha}} &= -\frac{\sqrt{\alpha}}{4\pi}. \end{aligned} \tag{3.14}$$

Therefore, equation (3.13) becomes

$$\begin{aligned}\frac{6}{\lambda}\left(\alpha - m^2\right) &= -\frac{\sqrt{\alpha}}{4\pi} \\ \Rightarrow \sqrt{\alpha} &= \frac{24\pi}{\lambda}\left(m^2-\alpha\right) \\ &= \frac{24\pi m^2}{\lambda} - \left(\frac{24\pi}{\lambda}\right)^3 m^4 \ldots\end{aligned}$$

The IR fixed point is approached by keeping m^2 finite and letting $\lambda \to \infty$, in which situation equation (3.12) reduces to the background operator in conformal form - the $O(N)$ invariant two-point function of the scalar fields:

$$\psi^0_{\vec{k}} \quad = \quad \frac{1}{2}\left(\vec{k}^2 + m^2 + \frac{\lambda}{6}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\psi_{\vec{k}}\right)^{-1/2} \to \frac{1}{2\sqrt{\vec{k}^2}}$$

so the large-N saddle point analysis reveals that at the IR fixed point, the conformal background has the form

$$\psi^0_{\vec{k}} = \frac{1}{2\left|\vec{k}\right|}. \tag{3.15}$$

We now obtain $1/N$ corrections (and study the spectrum of fluctuations about the large N conformal background) using

$$\psi = \psi_0 + \frac{1}{\sqrt{N}}\eta; \qquad \Pi = \sqrt{N}\pi \tag{3.16}$$

$$\begin{aligned} H &= \frac{2}{N}\mathrm{Tr}\Pi\psi\Pi + \frac{N}{8}\mathrm{Tr}(\psi^{-1}) \\ &+N\int d^{d-1}\vec{x}\left(-\frac{1}{2}\lim_{\vec{x}\to\vec{y}}\partial^2_{\vec{y}}\psi_{\vec{x}\vec{y}} + \frac{1}{2}m^2\psi_{\vec{x}\vec{x}} + \frac{\lambda}{4!}\psi^2_{\vec{x}\vec{x}}\right). \end{aligned}$$

The quadratic Hamiltonian is then given by

$$H^{(2)} = 2\mathrm{Tr}\left(\pi\psi_0\pi\right) + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) + \frac{\lambda}{4!}\int d^{d-1}\vec{x}\eta^2_{\vec{x}\vec{x}}. \tag{3.17}$$

3.3.1 Quadratic fluctuations and the Eigenvalue equation

The equations of motion for the quadratic Hamiltonian given in equation (3.17) are as follows:

$$\begin{aligned} \dot{\eta}_{\vec{x}\vec{y}} &= \frac{\delta H^{(2)}}{\delta\pi_{\vec{x}\vec{y}}} = 2\left[\psi_0\pi + \pi\psi_0\right]_{\vec{y}\vec{x}} \\ \dot{\pi}_{\vec{x}\vec{y}} &= -\frac{\delta H^{(2)}}{\delta\eta_{\vec{x}\vec{y}}} \\ &= -\frac{1}{8}\left[\left(\psi_0^{-1}\right)^2\eta\psi_0^{-1} + \psi_0^{-1}\eta\left(\psi_0^{-1}\right)^2\right]_{\vec{y}\vec{x}} - \frac{\lambda}{12}\eta_{\vec{x}\vec{x}}\delta(\vec{x}-\vec{y}). \end{aligned}$$

To decouple this we see that

$$\begin{aligned} \ddot{\eta}_{\vec{x}\vec{y}} &= 2\left[\psi_0\dot{\pi} + \dot{\pi}\psi_0\right]_{\vec{y}\vec{x}} \\ &= \Bigg[-\frac{1}{4}\psi_0\left[\left(\psi_0^{-1}\right)^2\eta\psi_0^{-1} + \psi_0^{-1}\eta\left(\psi_0^{-1}\right)^2\right] - \frac{1}{4}\left[\left(\psi_0^{-1}\right)^2\eta\psi_0^{-1} + \psi_0^{-1}\eta\left(\psi_0^{-1}\right)^2\right]\psi_0 \\ &\quad -\frac{\lambda}{6}\eta_{\vec{x}\vec{x}}\delta(\vec{x}-\vec{y})\psi_0 - \frac{\lambda}{6}\psi_0\eta_{\vec{x}\vec{x}}\delta(\vec{x}-\vec{y})\Bigg]_{\vec{y}\vec{x}} \end{aligned}$$

$$= \left[-\frac{1}{4}\left[\psi_0^{-1}\eta\psi_0^{-1} + \eta\left(\psi_0^{-1}\right)^2 + \left(\psi_0^{-1}\right)^2\eta + \psi_0^{-1}\eta\psi_0^{-1}\right]\right.$$
$$\left. -\frac{\lambda}{6}\delta(\vec{x}-\vec{y})\left(\eta_{\vec{x}\vec{x}}\psi_0 + \psi_0\eta_{\vec{x}\vec{x}}\right)\right]_{\vec{y}\vec{x}}.$$

In momentum space,

$$\eta_{\vec{x}\vec{y}} = \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{i\left(\vec{k}_1\cdot\vec{x}+\vec{k}_2\cdot\vec{y}\right)}\eta_{\vec{k}_1\vec{k}_2}$$
$$(\psi_0)_{\vec{x}\vec{y}} = \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{d-1}}\left(\psi_0\right)_{k_1} e^{i\vec{k}_1\cdot(\vec{x}-\vec{y})}$$

and substituting this into equation (3.18) term by term (noting that, from equation (3.15) $(\psi_0)_{\vec{k}} \equiv (\psi_0)_{\vec{k}}$ and that $\eta_{\vec{k}_1\vec{k}_2} = \eta_{\vec{k}_2\vec{k}_1}$), we find that

$$\ddot{\eta}_{\vec{x}\vec{y}} = \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{i\left(\vec{k}_1\cdot\vec{x}+\vec{k}_2\cdot\vec{y}\right)}\eta_{\vec{k}_1\vec{k}_2}\left[-E^2\right]$$

$$\left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{\vec{y}\vec{x}} = \int d^{d-1}\vec{x}_1 \int d^{d-1}\vec{x}_2 \left(\psi_0^{-1}\right)_{\vec{y}\vec{x}_1}\eta_{\vec{x}_1\vec{x}_2}\left(\psi_0^{-1}\right)_{\vec{x}_2\vec{x}}$$
$$= \int d^{d-1}\vec{x}_1 d^{d-1}\vec{x}_2 \frac{d^{d-1}k_1 d^{d-1}k_2 d^{d-1}k_3 d^{d-1}k_4}{(2\pi)^{3(d-1)}}$$
$$e^{-iEt}e^{i\vec{k}_1\cdot(\vec{y}-\vec{x}_1)}\left(\psi_0^{-1}\right)_{\vec{k}_1} e^{i\left(\vec{k}_2\cdot\vec{x}_1+\vec{k}_3\cdot\vec{x}_2\right)}\eta_{\vec{k}_2\vec{k}_3} e^{i\vec{k}_4\cdot(\vec{x}_2-\vec{x})}\left(\psi_0^{-1}\right)_{\vec{k}_4}$$
$$= \int \frac{d^{d-1}k_1 d^{d-1}k_3}{(2\pi)^{(d-1)}} e^{-iEt}e^{i\vec{k}_1\cdot(\vec{y})}\left(\psi_0^{-1}\right)_{\vec{k}_1}\eta_{\vec{k}_1\vec{k}_3} e^{i\vec{k}_3\cdot\vec{x}}\left(\psi_0^{-1}\right)_{-\vec{k}_3}$$
$$= \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{i\left(\vec{k}_1\cdot\vec{x}+\vec{k}_2\cdot\vec{y}\right)}\eta_{\vec{k}_1\vec{k}_2}\left[\left(\psi_0^{-1}\right)_{\vec{k}_1}\left(\psi_0^{-1}\right)_{\vec{k}_2}\right]$$

$$\left(\eta\left(\psi_0^{-1}\right)^2\right)_{\vec{y}\vec{x}} = \int d^{d-1}\vec{x}_1 \int d^{d-1}\vec{x}_2 \eta_{\vec{y}\vec{x}_1}\left(\psi_0^{-1}\right)_{\vec{x}_1\vec{x}_2}\left(\psi_0^{-1}\right)_{\vec{x}_2\vec{x}}$$
$$= \int d^{d-1}\vec{x}_1 d^{d-1}\vec{x}_2 \frac{d^{d-1}k_1 d^{d-1}k_2 d^{d-1}k_3 d^{d-1}k_4}{(2\pi)^{3(d-1)}}$$
$$e^{i\left(\vec{k}_1\cdot\vec{y}+\vec{k}_2\cdot\vec{x}_1\right)}\eta_{\vec{k}_1\vec{k}_2} e^{-iEt} e^{i\vec{k}_3\cdot(\vec{x}_1-\vec{x}_2)}\left(\psi_0^{-1}\right)_{\vec{k}_3} e^{i\vec{k}_4\cdot(\vec{x}_2-\vec{x})}\left(\psi_0^{-1}\right)_{\vec{k}_4}$$
$$= \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{i\left(\vec{k}_1\cdot\vec{y}+\vec{k}_2\cdot\vec{x}\right)}\eta_{\vec{k}_1\vec{k}_2}\left[\left(\psi_0^{-1}\right)_{-\vec{k}_2}\left(\psi_0^{-1}\right)_{-\vec{k}_2}\right]$$
$$= \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{i\left(\vec{k}_1\cdot\vec{x}+\vec{k}_2\cdot\vec{y}\right)}\eta_{\vec{k}_1\vec{k}_2}\left[\left(\psi_0^{-1}\right)_{\vec{k}_2}\left(\psi_0^{-1}\right)_{\vec{k}_2}\right]$$

$$\left(\left(\psi_0^{-1}\eta\right)^2\right)_{\vec{y}\vec{x}} = \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}}\int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{-iEt} e^{i\left(\vec{k}_1\cdot\vec{x}+\vec{k}_2\cdot\vec{y}\right)}\eta_{\vec{k}_1\vec{k}_2}\left[\left(\psi_0^{-1}\right)_{\vec{k}_1}\left(\psi_0^{-1}\right)_{\vec{k}_1}\right].$$

The last term in equation (3.18) is given by

$$\begin{aligned}\eta_{\vec{y}\vec{y}}\left(\psi_0\right)_{\vec{y}\vec{x}} &= \int\frac{d^{d-1}\vec{k_1}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_2}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_3}}{(2\pi)^{d-1}}e^{i\left(\vec{k_1}\cdot\vec{y}+\vec{k_2}\cdot\vec{y}\right)}\eta_{\vec{k_1}\vec{k_2}}e^{i\vec{k_3}(\vec{y}-\vec{x})}\left(\psi_0\right)_{k_3}\\ &= \int\frac{d^{d-1}\vec{k_1}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_2}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_3}}{(2\pi)^{d-1}}e^{i\left(\vec{k_1}+\vec{k_2}+\vec{k_3}\right)\cdot\vec{y}}e^{-i\vec{k_3}\cdot\vec{x}}\eta_{\vec{k_1}\vec{k_2}}\left(\psi_0\right)_{k_3},\end{aligned}$$

and we will use a change of variables $\vec{p_1}=\vec{k_1}+k_2+\vec{k_3}$, $\vec{p_2}=-\vec{k_3}$ and $\vec{p_3}=\vec{k_2}$ (for which the Jacobian $\left|\frac{\partial p_i}{\partial k_j}\right|=1$); thus $\vec{k_1}=\vec{p_1}+\vec{p_2}-\vec{p_3}$, and we obtain:

$$\begin{aligned}\eta_{\vec{y}\vec{y}}\left(\psi_0\right)_{\vec{y}\vec{x}} &= \int\frac{d^{d-1}\vec{p_1}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{p_2}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{p_3}}{(2\pi)^{d-1}}e^{i(\vec{p_1}\cdot\vec{y}+\vec{p_2}\cdot\vec{x})}\eta_{\vec{p_1}+\vec{p_2}-\vec{p_3},\vec{p_3}}\left(\psi_0\right)_{\vec{p_2}}\\ &= \int\frac{d^{d-1}\vec{p_1}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{p_2}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{p_3}}{(2\pi)^{d-1}}e^{i(\vec{p_1}\cdot\vec{x}+\vec{p_2}\cdot\vec{y})}\eta_{\vec{p_1}+\vec{p_2}-\vec{p_3},\vec{p_3}}\left(\psi_0\right)_{\vec{p_1}}.\end{aligned}$$

Similarly, changing variables to $\vec{p_1}=\vec{k_2}+\vec{k_3}-\vec{k_1}$, $\vec{p_2}=\vec{k_1}$ and $\vec{p_3}=\vec{k_3}$

$$\begin{aligned}\left(\psi_0\right)_{\vec{y}\vec{x}}\eta_{\vec{x}\vec{x}} &= \int\frac{d^{d-1}\vec{k_1}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_2}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_3}}{(2\pi)^{d-1}}e^{i\vec{k_1}(\vec{y}-\vec{x})}\left(\psi_0\right)_{\vec{k_1}}e^{i\left(\vec{k_2}\cdot\vec{x}+\vec{k_3}\cdot\vec{x}\right)}\eta_{\vec{k_2}\vec{k_3}}\\ &= \int\frac{d^{d-1}\vec{k_1}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_2}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{k_3}}{(2\pi)^{d-1}}e^{i\left(\vec{k_2}+\vec{k_3}-\vec{k_1}\right)\cdot\vec{x}}e^{i\vec{k_1}\cdot\vec{y}}\left(\psi_0\right)_{\vec{k_1}}\eta_{\vec{k_2}\vec{k_3}}\\ &= \int\frac{d^{d-1}\vec{p_1}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{p_2}}{(2\pi)^{\frac{d-1}{2}}}\int\frac{d^{d-1}\vec{p_3}}{(2\pi)^{d-1}}e^{i(\vec{p_1}\cdot\vec{x}+\vec{p_2}\cdot\vec{y})}\left(\psi_0\right)_{\vec{p_2}}\eta_{\vec{p_1}+\vec{p_2}-\vec{p_3},\vec{p_3}}.\end{aligned}$$

Relabelling indices, equation (3.18) simplifies to

$$\begin{aligned}-E^2\eta_{\vec{k_1}\vec{k_2}} &= -\frac{1}{4}\left[\left(\psi_0^{-1}\right)^2_{\vec{k_1}}+2\left(\psi_0^{-1}\right)_{\vec{k_1}}\left(\psi_0^{-1}\right)_{\vec{k_2}}+\left(\psi_0^{-1}\right)^2_{\vec{k_2}}\right]\eta_{\vec{k_1}\vec{k_2}}\\ &\quad -\frac{\lambda}{6}\left[\left(\psi_0\right)_{\vec{k_1}}+\left(\psi_0\right)_{\vec{k_2}}\right]\int\frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k_1}+\vec{k_2}-\vec{l},\vec{l}}\\ \Rightarrow E^2\eta_{\vec{k_1}\vec{k_2}} &= \frac{1}{4}\left(\left(\psi_0^{-1}\right)_{\vec{k_1}}+\left(\psi_0^{-1}\right)_{\vec{k_2}}\right)^2\eta_{\vec{k_1}\vec{k_2}}+\frac{\lambda}{6}\left[\left(\psi_0\right)_{\vec{k_1}}+\left(\psi_0\right)_{\vec{k_2}}\right]\\ &\quad\int\frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k_1}+\vec{k_2}-\vec{l},\vec{l}},\end{aligned}\tag{3.18}$$

resulting in the eigenvalue equation,

$$\left[E^2-\frac{1}{4}\left(\left(\psi_0^{-1}\right)_{\vec{k_1}}+\left(\psi_0^{-1}\right)_{\vec{k_2}}\right)^2\right]\eta_{\vec{k_1}\vec{k_2}} = \frac{\lambda}{6}\left[\left(\psi_0\right)_{\vec{k_1}}+\left(\psi_0\right)_{\vec{k_2}}\right]\int\frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k_1}+\vec{k_2}-\vec{l},\vec{l}}.\tag{3.19}$$

3.3.2 Quadratic Hamiltonian

The quadratic Hamiltonian was found in equation (3.17) to be

$$H^{(2)} = 2\mathrm{Tr}\left(\pi\psi_0\pi\right) + \frac{1}{8}\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1} + \frac{\lambda}{4!}\int d^{d-1}\vec{x}\eta^2_{\vec{x}\vec{x}} \tag{3.20}$$

so, for the free theory

$$H^{(2)} = 2\mathrm{Tr}\left(\pi\psi_0\pi\right) + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right).$$

We obtain as an equation of motion that

$$\begin{aligned} \dot{\eta}_{xy} &= \frac{\delta H^{(2)}}{\delta\pi_{xy}} \\ &= 2\left(\pi\psi_0 + \psi_0\pi\right)_{yx}. \end{aligned} \tag{3.21}$$

Writing

$$\begin{aligned} \eta\left(t;\vec{x_1},\vec{x_2}\right) &= \int \frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\left(e^{-iE_{\vec{k}_1\vec{k}_2}t}e^{i\vec{k}_1\cdot\vec{x}}e^{i\vec{k}_2\cdot\vec{y}}\alpha_{\vec{k}_1\vec{k}_2}a_{\vec{k}_1\vec{k}_2} + \text{h.c.}\right) \\ \dot{\eta}\left(t;\vec{x_1},\vec{x_2}\right) &= \int \frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\left(-iE_{\vec{k}_1\vec{k}_2}\right)\left(e^{-iE_{\vec{k}_1\vec{k}_2}t}e^{i\vec{k}_1\cdot\vec{x}}e^{i\vec{k}_2\cdot\vec{y}}\alpha_{\vec{k}_1\vec{k}_2}a_{\vec{k}_1\vec{k}_2} - \text{h.c.}\right) \\ \pi\left(t;\vec{x_1},\vec{x_2}\right) &= \int \frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\left(e^{-iE_{\vec{k}_1\vec{k}_2}t}e^{i\vec{k}_1\cdot\vec{x}}e^{i\vec{k}_2\cdot\vec{y}}\beta_{\vec{k}_1\vec{k}_2}a_{\vec{k}_1\vec{k}_2} - \text{h.c.}\right) \\ \left(\psi_0\right)_{xy} &= \int \frac{d^{d-1}p}{(2\pi)^{d-1}}e^{ip(x-y)}\psi^0_p \end{aligned}$$

where

$$\begin{aligned} \left[a_{\vec{k}_1\vec{k}_2}, a^\dagger_{\vec{k}'_1\vec{k}'_2}\right] &= \delta_{\vec{k}_1,\vec{k}'_1}\delta_{\vec{k}_2,\vec{k}'_2} \\ \left[\eta\left(t,\vec{x}_1,\vec{x}_2\right),\pi\left(t,\vec{y}_1,\vec{y}_2\right)\right] &= i\delta\left(\vec{x}_1-\vec{y}_1\right)\delta\left(\vec{x}_2-\vec{y}_2\right), \end{aligned}$$

we see that

$$\beta_{\vec{k}_1\vec{k}_2} = \frac{-iE_{\vec{k}_1\vec{k}_2}}{2\left(\psi^0_{\vec{k}_1}+\psi^0_{\vec{k}_2}\right)}\alpha_{\vec{k}_1\vec{k}_2}.$$

and

$$\alpha_{\vec{k}_1\vec{k}_2} = \sqrt{\frac{\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)}{E_{\vec{k}_1\vec{k}_2}}}.$$

From equation (3.15)

$$\psi^0_{\vec{k}} = \frac{1}{2\left|\vec{k}\right|}.$$

From the eigenvalue equation (equation (3.19)) we see that for the free theory,

$$E_{\vec{k}_1,\vec{k}_2} = \frac{1}{2\psi^0_{\vec{k}_1}} + \frac{1}{2\psi^0_{\vec{k}_2}} = \omega_{\vec{k}_1} + \omega_{\vec{k}_2}.$$

Hence

$$\begin{aligned} \alpha_{\vec{k}_1\vec{k}_2} &= \sqrt{2\psi_{\vec{k}_1}\psi_{\vec{k}_2}} = \frac{1}{\sqrt{2\omega_{\vec{k}_1}\omega_{\vec{k}_2}}} = \alpha^*_{\vec{k}_1\vec{k}_2} \\ \left(\frac{\left(\omega_{\vec{k}_1} + \omega_{\vec{k}_2}\right)}{2(\psi_{\vec{k}_2} + \psi_{\vec{k}_2})}\right)\alpha_{\vec{k}_1\vec{k}_2} &= \sqrt{\frac{\omega_{\vec{k}_1}\omega_{\vec{k}_2}}{2}}. \end{aligned}$$

The mode expansions are therefore given by

$$\boxed{\begin{aligned} \eta_{\vec{x}\vec{y}} = &\int \frac{d^{d-1}k_1}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}k_2}{(2\pi)^{(d-1)/2}}\frac{1}{\sqrt{2\omega_{\vec{k}_1}\omega_{\vec{k}_2}}}\left(e^{-i\left(\omega_{\vec{k}_1}+\omega_{\vec{k}_2}\right)t}e^{i\vec{k}_1\cdot\vec{x}}e^{i\vec{k}_2\cdot\vec{y}}a_{\vec{k}_1\vec{k}_2}+\right. \\ &\left. e^{i\left(\omega_{\vec{k}_1}+\omega_{\vec{k}_2}\right)t}e^{-i\vec{k}_1\cdot\vec{x}}e^{-i\vec{k}_2\cdot\vec{y}}a^\dagger_{\vec{k}_1\vec{k}_2}\right) \\ \pi_{\vec{x}\vec{y}} = &\int \frac{d^{d-1}k_1}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}k_2}{(2\pi)^{(d-1)/2}}\left(-i\sqrt{\frac{\omega_{\vec{k}_1}\omega_{\vec{k}_2}}{2}}\right) \\ &\left(e^{-i\left(\omega_{\vec{k}_1}+\omega_{\vec{k}_2}\right)t}e^{i\vec{k}_1\cdot\vec{x}}e^{i\vec{k}_2\cdot\vec{y}}a_{\vec{k}_1\vec{k}_2} - e^{i\left(\omega_{\vec{k}_1}+\omega_{\vec{k}_2}\right)t}e^{-i\vec{k}_1\cdot\vec{x}}e^{-i\vec{k}_2\cdot\vec{y}}a^\dagger_{\vec{k}_1\vec{k}_2}\right). \end{aligned}} \tag{3.22}$$

Substituting these mode expansions into the quadratic Hamiltonian (see Appendix C.2 for the calculation), we obtain

$$\begin{aligned} H^{(2)} &= \frac{1}{2}\int d^2k_1 \int d^2k_2 \left(\omega_{\vec{k}_1} + \omega_{\vec{k}_2}\right)\left(a_{\vec{k}_1\vec{k}_2}a^\dagger_{\vec{k}_2\vec{k}_1} + a^\dagger_{\vec{k}_1\vec{k}_2}a_{\vec{k}_2\vec{k}_1}\right) \\ &= \frac{1}{2}\int d^2k_1 \int d^2k_2 E_{\vec{k}_1\vec{k}_2}\left(a_{\vec{k}_1\vec{k}_2}a^\dagger_{\vec{k}_2\vec{k}_1} + a^\dagger_{\vec{k}_1\vec{k}_2}a_{\vec{k}_2\vec{k}_1}\right). \end{aligned}$$

3.3.3 Scattering States

We wish to express the scattering solution in terms of the free solutions. The full scattering solution of equation (3.19) is given by

$$\eta^{\vec{p}_1,\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} = \psi^{\vec{p}_1\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} + \frac{\lambda}{12}\frac{1}{E^2_{\vec{p}_1\vec{p}_2} - \left(|\vec{k}| + |\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|} + \frac{1}{|\vec{p}-\vec{k}|}\right)\int \frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta^{\vec{p}_1\vec{p}_2}_{\vec{p}-\vec{l},\vec{l}},$$

where $E_{\vec{p}_1\vec{p}_2} = |\vec{p_1}| + |\vec{p_2}|$ and $\psi^{\vec{p}_1\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} = \delta^{d-1}(\vec{p}_1 - (\vec{p}-\vec{k}))\delta^{d-1}(\vec{p}_2 - \vec{k})$ is the solution to the free equation. Let us write (with $\vec{k}_1 + \vec{k}_2 = \vec{p}$),

$$\begin{aligned}
\psi^{\vec{p}_1,\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} &= \delta^{d-1}(\vec{p}_2 - (\vec{p}-\vec{k}))\delta^{d-1}(\vec{p}_1 - \vec{k}) \\
&= \delta^{d-1}(\vec{p}_2 - \vec{k}_1 - \vec{k}_2 + \vec{k})\delta^{d-1}(\vec{p}_1 - \vec{k}) \\
\Rightarrow \int d^{d-1}k\, \psi^{\vec{p}_1,\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} &= \delta^{d-1}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)
\end{aligned}$$

and hence,

$$\begin{aligned}
&\int \frac{d^{d-1}k}{(2\pi)^{d-1}}\eta^{\vec{p}_1,\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} = \int \frac{d^{d-1}k}{(2\pi)^{d-1}}\psi^{\vec{p}_1\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} \\
&+ \int \frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{\lambda}{12}\frac{1}{E^2_{\vec{p}_1\vec{p}_2} - \left(|\vec{k}| + |\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|} + \frac{1}{|\vec{p}-\vec{k}|}\right)\int \frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta^{\vec{p}_1\vec{p}_2}_{\vec{p}-\vec{l},\vec{l}},
\end{aligned}$$

resulting in

$$\begin{aligned}
\int \frac{d^{d-1}k}{(2\pi)^{d-1}}\eta^{\vec{p}_1,\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} &= \frac{1}{(2\pi)^{d-1}}\frac{\delta^{d-1}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)}{1 - \frac{\lambda}{12}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{E^2_{\vec{p}_1\vec{p}_2} - \left(|\vec{k}|+|\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|} + \frac{1}{|\vec{p}-\vec{k}|}\right)} \\
&= \frac{\delta^{d-1}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)}{(2\pi)^{d-1}\left(1 + \frac{2\lambda}{12i}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-p)^2}\right)}, \qquad (3.23)
\end{aligned}$$

where we have used equation (B.7),

$$\int \frac{dE}{(2\pi)}\frac{1}{k^2(k-p)^2} = -\frac{i}{2}\frac{1}{E^2_{\vec{p}} - \left(|\vec{k}| + |\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|} + \frac{1}{|\vec{p}-\vec{k}|}\right).$$

From equation (B.4),

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2(q-k)^2} \quad = \quad \frac{i}{8\,|p|_E},$$

where Minkowski subscripts were dropped in the last line. Equation (3.23) becomes

$$\begin{aligned}\int \frac{d^2k}{(2\pi)^{d-1}}\eta^{\vec{p}_1,\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} &= \frac{\delta^2(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)}{1-\frac{2\lambda}{12i}\int\frac{d^3k}{(2\pi)^d}\frac{1}{k^2(k-p)^2}} \\ &= \frac{\delta^2(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)}{1+\frac{\lambda}{48|p|_E}},\end{aligned}$$

where we may write $|\vec{p}_E| = \sqrt{-\left(|p_1|+|p_2|\right)^2+(\vec{p}_1+\vec{p}_2)^2}$. The scattering solution is therefore given by

$$\begin{aligned}\eta^{\vec{p}_1,\vec{p}_2}_{\vec{p}-\vec{k},\vec{k}} &= \delta^2(\vec{p}_1-(\vec{p}-\vec{k}))\delta^{d-1}(\vec{p}_2-\vec{k}) \\ &+\frac{\lambda}{12(2\pi)^2}\frac{1}{E^2_{\vec{p}_1\vec{p}_2}-\left(|\vec{k}|+|\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|}+\frac{1}{|\vec{p}-\vec{k}|}\right)\frac{\delta^2(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)}{1+\frac{\lambda}{48|p|_E}},\end{aligned}$$

which may be rewritten as

$$\begin{aligned}\eta^{\vec{p}_1,\vec{p}_2}_{\vec{k}_1,\vec{k}_2} &= \delta^2(\vec{p}_1-\vec{k}_1)\delta^2(\vec{p}_2-\vec{k}_2) \\ &+\frac{\lambda}{12(2\pi)^2}\frac{1}{E^2_{\vec{p}_1\vec{p}_2}-\left(|\vec{k}_1|+|\vec{k}_2|\right)^2}\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\frac{\delta^2(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)}{1+\frac{\lambda}{48|p|_E}}.\end{aligned}$$

At the critical point ($\lambda\to\infty$), it takes a finite form

$$\boxed{\eta^{\vec{p}_1,\vec{p}_2}_{\vec{k}_1,\vec{k}_2} = \delta^2(\vec{p}_1-\vec{k}_1)\delta^2(\vec{p}_2-\vec{k}_2)+\frac{|p|_E}{\pi^2}\frac{\delta^2(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)}{E^2_{\vec{p}_1\vec{p}_2}-\left(|\vec{k}_1|+|\vec{k}_2|\right)^2}\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)} \tag{3.24}$$

where $E^2_{\vec{p}_1\vec{p}_2} = (|\vec{p}_1|+|\vec{p}_2|)^2$. Thus, at the infra-red conformal fixed point ($\lambda\to\infty$ as per this equation), we have that (once again using the equations from Appendix B),

$$\eta_{\vec{x}\vec{x}} \quad\sim\quad \int d^2\vec{k}_1 d^2\vec{k}_2\ \eta^{\vec{p}_1,\vec{p}_2}_{\vec{k}_1,\vec{k}_2}$$

$$\begin{aligned}
&= \int d^2\vec{k}_1 d^2\vec{k}_2 \Bigg[\delta^2(\vec{p}_1-\vec{k}_1)\delta^2(\vec{p}_2-\vec{k}_2) \\
&\quad + \frac{|p|_E}{\pi^2}\frac{\delta^2(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)}{E^2_{\vec{p}_1\vec{p}_2}-\left(|\vec{k}_1|+|\vec{k}_2|\right)^2}\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\Bigg] \\
&= 1+\frac{|p|_E}{\pi^2}\int d^2\vec{k}\frac{1}{E_{\vec{p}}-\left(|\vec{k}|+|\vec{k}-\vec{p}|\right)^2}\left(\frac{1}{|\vec{k}|}+\frac{1}{|\vec{k}-\vec{p}|}\right) \\
&= 1+|p|_E\times i\int\frac{d^3k}{\pi^3}\frac{1}{k^2(k-p)^2} \\
&= 1+|p|_E\times i\left(\frac{i}{|p|_E}\right) \\
&\to 0
\end{aligned}$$

so this $\Delta = 1$ state is removed from the spectrum [7].

3.3.4 Bound state

The eigenvalue equation (3.19) leads to a bound state, given by

$$\eta_{\vec{k}_1\vec{k}_2} = \frac{\lambda}{6}\frac{(\psi_0)_{\vec{k}_1}+(\psi_0)_{\vec{k}_2}}{E^2-\frac{1}{4}\left(\left(\psi_0^{-1}\right)_{\vec{k}_1}+\left(\psi_0^{-1}\right)_{\vec{k}_2}\right)^2}\int\frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k}_1+\vec{k}_2-\vec{l},\vec{l}}. \tag{3.25}$$

We now act on both sides of equation (3.25) with $\delta(\vec{k}_1+\vec{k}_2-\vec{q}_1-\vec{q}_2)$ and integrate over $\vec{k}_1$ and $\vec{k}_2$ so that the left hand side becomes

$$\begin{aligned}
\text{LHS} &= \int\frac{d^{d-1}\vec{k}_1}{(2\pi)^{(d-1)/2}}\int\frac{d^{d-1}\vec{k}_2}{(2\pi)^{(d-1)/2}}\eta_{\vec{k}_1\vec{k}_2}\delta(\vec{k}_1+\vec{k}_2-\vec{q}_1-\vec{q}_2) \\
&= \int\frac{d^{d-1}\vec{k}_2}{(2\pi)^{d-1}}\eta_{\vec{q}_1+\vec{q}_2-\vec{k}_2,\vec{k}_2}
\end{aligned} \tag{3.26}$$

and the right hand side becomes

$$\begin{aligned}
\text{RHS} = \frac{\lambda}{6}\int\frac{d^{d-1}\vec{k}_1}{(2\pi)^{(d-1)/2}}\int\frac{d^{d-1}\vec{k}_2}{(2\pi)^{(d-1)/2}}\frac{(\psi_0)_{\vec{k}_1}+(\psi_0)_{\vec{k}_2}}{E^2-\frac{1}{4}\left(\left(\psi_0^{-1}\right)_{\vec{k}_1}+\left(\psi_0^{-1}\right)_{\vec{k}_2}\right)^2} \\
\int\frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k}_1+\vec{k}_2-\vec{l},\vec{l}}\delta(\vec{k}_1+\vec{k}_2-\vec{q}_1-\vec{q}_2)
\end{aligned}$$

$$= \frac{\lambda}{6}\int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{(d-1)/2}} \frac{(\psi_0)_{\vec{q}_1+\vec{q}_2-\vec{k}_2}+(\psi_0)_{\vec{k}_2}}{E^2-\frac{1}{4}\left(\left(\psi_0^{-1}\right)_{\vec{q}_1+\vec{q}_2-\vec{k}_2}+\left(\psi_0^{-1}\right)_{\vec{k}_2}\right)^2}\int \frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{q}_1+\vec{q}_2-\vec{l},\vec{l}}$$

and consistency of these equations requires that (relabelling $\vec{k}_2 \to \vec{k}$ and $\vec{q}_1+\vec{q}_2 \to \vec{p}$, and recalling that $\psi_0 = \frac{1}{2|k|}$):

$$\boxed{1 = \frac{\lambda}{12}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{E_p^2-\left(|\vec{k}|+|\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|}+\frac{1}{|\vec{p}-\vec{k}|}\right)}, \tag{3.27}$$

in agreement with Ref. [7]; this is also in agreement with the pole condition, as we will now show.

The pole condition from equation (3.10) can be written as

$$1 = \frac{\lambda i}{6}\frac{1}{(2\pi)^d}\int d^d k \frac{1}{k^2(p_1+p_2-k)^2}$$

From equation (B.7),

$$\int \frac{dE}{(2\pi)}\frac{1}{k^2(k-p)^2} = -\frac{i}{2}\frac{1}{E_p^2-\left(|\vec{k}|+|\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|}+\frac{1}{|\vec{p}-\vec{k}|}\right).$$

Finally,

$$1 = \frac{\lambda}{12}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{E_p^2-\left(|\vec{k}|+|\vec{p}-\vec{k}|\right)^2}\left(\frac{1}{|\vec{k}|}+\frac{1}{|\vec{p}-\vec{k}|}\right),$$

which agrees with the result for the Hamiltonian formalism given by equation (3.27).

Chapter 4

A constructive higher spin AdS/CFT correspondence

In section 1.11 the concept of a duality between a conformal field theory in d dimensions, CFT_d, and a gravitational theory in $d+1$ dimensions, AdS_{d+1} was developed (indicating the rise of an additional AdS space coordinate, z). A useful example of this to consider is the duality between CFT_3 and AdS_4. In particular, in the Hamiltonian approach in $d = 2+1$ dimensions, the $1+2+2 = 5$ coordinates of the equal time bilocals map (in phase space) to the 5 coordinates of $\text{AdS}_4 \times S_1$. S_1, which was added to ensure that the degrees of freedom on both sides of the map match, is then parametrised by an angle θ. It is then conceivable that an explicit map exists between the Hamiltonian CFT_3 coordinates of the form $(t, x_1^1, x_1^2, x_2^1, x_2^2)$ and the $\text{AdS}_4 \times S_1$ coordinates of the form $(t, x^1, x^2, z, p^\theta)$ (the notation arose that θ be treated as a momentum variable and p^θ as a spatial variable). The map was originally understood in terms of light-cone gauge and thereafter moved on to the more general temporal gauge; highlights of this development are explicated below.

The aim of Ref. [5] was to construct higher-spin theory in AdS_4 from the CFT_3 obtained using the $O(N)$ vector model written in terms of canonical collective fields. Light-cone gauge, where the physical degrees of freedom of the gauge theory are most clear, was a natural starting point[127, 128, 129]. Indeed, in the light-cone gauge phase space, an exact map was achieved between these two spaces in null-plane quantisation and the collective coordinates of the bi-local field were successfully used to generate AdS_4 space-time. An exact one-to-one reconstruction of higher-spin fields and bulk AdS_4 was thus obtained in light-cone gauge. This was done by comparing the AdS_4 generators in light-cone gauge found by Metsaev in Ref. [130] and comparing them with the second quantised conformal generators obtained for the collective fields, written as follows

$$G = \int dx_1^- dx_2^- d\vec{x}_1 d\vec{x}_2 A^\dagger \hat{g} A = \int dx_1^- dx_2^- d\vec{x}_1 d\vec{x}_2 A^\dagger \left(\hat{g}_1 + \hat{g}_2\right) A.$$

In this thesis, we wish to obtain the conformal generators for the collective fields in the temporal gauge (and at the IR critical point), necessary to construct the bulk using a similar approach to that used in Ref. [5]. These will then be compared with Ref. [10], where a kernel was used to obtain an exact one-to-one reconstruction of higher-spin fields and bulk AdS_4 in both light-cone gauge and temporal gauge.

To this end, we begin by summarising the conformal algebra and its generators in section 4.1, which we will then write in light-cone gauge in section 4.2 and in temporal gauge in section 4.3 in $d = 2+1$ dimensions. A summary of the light-cone results of Refs. [5, 10] are included in section 4.2, and a summary of the temporal results of Ref. [10] are included in section 4.3.

4.1 Conformal Algebra

In section 1.11.2 it was shown that the 1st quantised conformal algebra consists of the following generators,

$$\begin{aligned}
\text{Translation} \quad & \phi(x^\mu) = \phi(x^{'\mu} - \epsilon^\mu) \\
& \Rightarrow P_\mu = -i\partial_\mu \\
\text{Rotation} \quad & \phi(x^\mu) = \phi(x^{'\mu} - \epsilon^\mu{}_\nu x^\nu) \\
& \Rightarrow \mathcal{J}_\mu{}^\nu = -ix^\nu \partial_\mu \Rightarrow \mathcal{J}_{\mu\nu} \equiv i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) = -\left[x_\mu P_\nu - x_\nu P_\mu\right] \\
\text{Dilation} \quad & \phi(x^\mu) = \phi(x^{'\mu} - \epsilon x^\mu) \\
& \Rightarrow D = -ix^\mu \partial_\mu = x^\mu P_\mu \\
\text{SCT} \quad & \phi(x^\mu) = \phi(x^{'\mu} - 2\left(x\cdot\epsilon\right) x^\mu + x^2 \epsilon^\mu) \\
& \Rightarrow K_\mu = -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) = 2x_\mu x^\nu P_\nu - x^2 P_\mu = 2x_\mu D - x^2 P_\mu,
\end{aligned}$$

obeying the commutation relations

$$\begin{aligned}
[P_\mu, \mathcal{J}_{\rho\sigma}] &= i\eta_{\mu\rho}P_\sigma - i\eta_{\mu\sigma}P_\rho \\
[D, P_\nu] &= iP_\nu \\
[D, K_\nu] &= -iK_\nu \\
[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right) \\
[K_\mu, P_\nu] &= 2i\left[\eta_{\mu\nu}D - \mathcal{J}_{\mu\nu}\right] \\
[K_\rho, \mathcal{J}_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right).
\end{aligned}$$

4.2 Light Cone

4.2.1 Light Cone coordinates

This subsection follows the notation of Refs. [131, 130]. In CFT$_3$, the light cone coordinates are given by

$$\begin{aligned}
x^\pm &= \frac{1}{\sqrt{2}}(x \pm t) \\
x^\perp &= y
\end{aligned}$$

and hence

$$\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 \\
\rightarrow ds^2 &= 2dx^+dx^- + dx^{\perp 2} \\
&= 2g_{ab}dx^a dx^b + dx^{\perp 2}
\end{aligned}$$

where a, b run over $+, -$ and

$$g_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly the metric has convention $\eta = (-, +, +)$. Note that since $x_a = g_{ab}x^b$, $x_+ = x^-$ and $x_- = x^+$. Furthermore,

$$\partial_t = \frac{\partial x^+}{\partial t}\partial_+ + \frac{\partial x^-}{\partial t}\partial_- = \frac{1}{\sqrt{2}}(\partial_+ - \partial_-)$$

$$\begin{aligned}
\partial_x &= \frac{\partial x^+}{\partial x}\partial_+ + \frac{\partial x^-}{\partial x}\partial_- = \frac{1}{\sqrt{2}}\left(\partial_+ + \partial_-\right) \\
\Rightarrow \partial_+ &= \frac{1}{\sqrt{2}}\left(\partial_x + \partial_t\right) \\
\partial_- &= \frac{1}{\sqrt{2}}\left(\partial_x - \partial_t\right)
\end{aligned}$$

for which $\partial_\pm$ has the property that $\partial^\pm = \frac{\partial}{\partial x_\pm} = \frac{\partial}{\partial x^\mp} = \partial_\mp$. The Lagrangian (which is negative due to the metric convention) is given by

$$\begin{aligned}
\mathcal{L} = -\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i \quad \rightarrow \quad & -\frac{1}{2}\partial_+\phi^i\partial^+\phi^i - \frac{1}{2}\partial_-\phi^i\partial^-\phi^i - \frac{1}{2}\partial_\perp\phi^i\partial^\perp\phi^i \\
&= -\partial_-\phi^i\partial_+\phi^i - \frac{1}{2}\partial_\perp\phi^i\partial_\perp\phi^i
\end{aligned} \tag{4.1}$$

which we recognise is no longer quadratic in 'velocity' but rather linear. To obtain the Hamiltonian, we notice that if we make the identification of x^+ with time, so that the conjugate momentum to this is

$$\Pi^{+i} = \frac{\delta\mathcal{L}}{\delta\left(\partial_+\phi^i\right)} = -\partial_-\phi^i$$

the Hamiltonian, which is in general of the form $H = \Pi\dot{\phi} - \mathcal{L}$, is now given by (noting that ∂_+ now takes the role of ∂_t),

$$\begin{aligned}
H &= -\partial_-\phi^i\partial_+\phi^i - \left(-\partial_-\phi^i\partial_+\phi^i - \frac{1}{2}\partial_\perp\phi^i\partial_\perp\phi^i\right) \\
&= \frac{1}{2}\partial_\perp\phi^i\partial_\perp\phi^i.
\end{aligned}$$

The equations of motion are in general given by $\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right) = \frac{\delta\mathcal{L}}{\delta\phi} = 0$ and for the particular lagrangian in equation (4.1) are given by

$$\begin{aligned}
0 &= \partial_+\left(\frac{\delta\mathcal{L}}{\delta\left(\partial_+\phi^i\right)}\right) + \partial_-\left(\frac{\delta\mathcal{L}}{\delta\left(\partial_-\phi^i\right)}\right) + \partial_\perp\left(\frac{\delta\mathcal{L}}{\delta\left(\partial_\perp\phi^i\right)}\right) \\
&= -\partial_+\partial_-\phi^i - \partial_-\partial_+\phi^i - \partial_\perp\partial_\perp\phi^i \\
\Rightarrow & \quad \left(2\partial_+\partial_- + \partial_\perp\partial_\perp\right)\phi^i = 0
\end{aligned}$$

which has solutions of the form

$$\phi^i \sim e^{ix^\mu p_\mu} = e^{i\left(x^+p_+ + x^-p_- + x^\perp p_\perp\right)} = e^{i\left(x^+p^- + x^-p^+ + x^\perp k\right)}.$$

We see that p^- is the energy conjugate to the time x^+ and p^+ is the momentum conjugate to the spatial coordinate x^-. We wish to find out the scalar field in standard time-like quantisation, where the generic scalar field has the form

$$\phi\left(x,t\right) = \int \frac{d^dk}{(\sqrt{2\pi})^d}\frac{1}{\sqrt{2\omega_k}}\left(e^{-i\omega_k t+i\vec{k}\cdot\vec{x}}a_{\vec{k}} + e^{i\omega_k t-i\vec{k}\cdot\vec{x}}a^\dagger_{\vec{k}}\right).$$

Since the time component of the metric is negative, the Klein-Gordon equation takes the form,

$$\begin{aligned}
& p_\mu p^\mu = -m^2 \\
\Rightarrow \quad & p_+p^+ + p_-p^- + p_\perp p^\perp = -m^2 \\
\Rightarrow \quad & 2p^-p^+ = -m^2 - p_\perp p^\perp \\
\Rightarrow \quad & p^- = \frac{-m^2 - p_\perp p^\perp}{2p^+} \\
\Rightarrow \quad & p^- \underset{m^2=0}{\longrightarrow} \frac{-p_\perp^2}{2p^+}.
\end{aligned}$$

Light cone variables are associated with the infinite momentum frame, in particular partons which travel parallel and anti-parallel to the direction of the boost (notice that the sign of p^+ is opposite that of p^-; this sign relates to left- and right- moving massless bosons). The infinitely massive ones drop off from the spectrum. The scalar field may thus be written as follows:

$$\begin{aligned}
\phi^i\left(x^+;x^-,x^\perp\right) &= \int_0^\infty \frac{dp^+}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2p^+}} \\
&\quad \left(e^{i\left(p^-x^++p^+x^-+kx_\perp\right)}A^i_{p^+k} + e^{-i\left(p^-x^++p^+x^-+kx_\perp\right)}A^{\dagger i}_{p^+k}\right).
\end{aligned}$$

4.2.2 Commutation relations in light cone gauge

Making use of the relation for the raising and annihilation operators,

$$\left[A_{p_1^+k_1}, A^\dagger_{p_2^+,k_2}\right] = \delta\left(p_1^+ - p_2^+\right)\delta\left(k_1 - k_2\right),$$

the equal time commutator between the scalar field and the conjugate momentum in light cone coordinates is given by

$$\left[\Pi^{+i}\left(x^+,x_1^-,x_2^\perp\right),\phi^j\left(x^+,x_2^-,x_2^\perp\right)\right]_{x^+} = -\frac{i}{2}\delta(x_1^- - x_2^-)\delta(x_{1\perp} - x_{2\perp})\delta_{ij}$$

(the details of this, together with the commutator that follows, are found in Appendix C.3.1). This comes with a factor of $\frac{1}{2}$ since half of the degrees of freedom are missing. The equal time commutator between fields is given by

$$\left[\phi^i\left(x_1^-, x_{1\perp}\right), \phi^j\left(x_2^-, x_{2\perp}\right)\right] = \frac{1}{2}\delta_{ij}\delta(x_{1\perp} - x_{2\perp})i\epsilon\left(x_1^- - x_2^-\right)$$

where

$$\epsilon\left(x_1^- - x_2^-\right) = \int_0^\infty \frac{dp_1^+}{2\pi}\frac{1}{ip_1^+}\left(-e^{i\left(p_1^+(x_2^- - x_1^-)\right)} + e^{i\left(p_1^+(x_1^- - x_2^-)\right)}\right)$$

satisfies

$$\frac{d}{dx_1^-}\epsilon(x_1^-) = \delta(x_1^-),$$

which does not vanish.

4.2.3 First and Second quantised operators

. In light-cone coordinates, the metric has the property that $g^{+-} = g^{-+} = g^{ii} = g_{+-} = g_{-+} = g_{ii} = 1$ and $g^{\mu\nu} = 0$ elsewhere, and the conserved charges are given by

$$\begin{aligned} j^+ &= -\left[\left(\mathcal{L}g_\nu^+ - \frac{\delta\mathcal{L}}{\delta\partial_+\phi}\partial_\nu\phi\right)\delta x^\nu + \frac{\delta\mathcal{L}}{\delta\partial_+\phi}\delta\phi\right] \\ &= \pi\partial_\nu\phi\delta x^\nu - \mathcal{L}g_\nu^+\delta x^\nu - \pi\delta\phi \\ &= \pi\partial_\nu\phi\delta x^\nu - \left(\pi\partial_+\phi - \mathcal{H}\right)g_\nu^+\delta x^\nu - \pi\delta\phi, \end{aligned}$$

from which we obtain first quantised generators in light cone gauge (details of which may be found in Appendix C.3.2):

$$\begin{aligned} P^+ &= -\int dx^- dx^i \pi^2 \\ P^- &= H = \int dx^- dx^i \frac{1}{2}\left(\partial_i\phi\right)^2 \\ P^i &= \int dx^- dx^i \left(\pi\partial_i\phi\right) \\ M^{+-} &= x^+H - \int dx^- dx^i \left(\pi^2 x^-\right) \\ M^{+i} &= \int dx^- dx^i \pi x^+ \partial_i\phi + x^i\pi^2 \end{aligned}$$

$$
\begin{aligned}
M^{-i} &= \int dx^- dx^i \left(x^- \pi \partial_i \phi - x^i \mathcal{H}\right) \\
M^{ij} &= \int dx^- dx^\perp \left(\pi x^i \partial_j \phi - \pi x^j \partial_i \phi\right) \\
D &= x^+ H + \int dx^- dx^\perp \ \left(\pi d_\phi \phi + \pi x^i \partial_i \phi - x^- \pi^2\right) \\
K^i &= \int dx^- dx^i \left(x^i \mathcal{D} - \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\pi \partial_i \phi\right) \\
K^+ &= \int dx^- dx^\perp \left(x^+ \mathcal{D} + \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\pi^2\right) \\
K^- &= \int dx^- dx^\perp \left(x^- \left[x^+ \mathcal{H} - \pi^2 x^- + x^j \partial_j \phi + \pi d_\phi \phi\right] - \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\mathcal{H}\right).
\end{aligned}
$$

We now describe in more detail the approach of Ref. [5]. Here we note that second quantised conformal generators satisfy $G = \int dx_1^- dx_2^- d\vec{x}_1 d\vec{x}_2 A^\dagger \hat{g} A)$ (with the momenta conjugate given by $(p_1^+, p_2^+, p_1^i, p_2^i)$). The second quantised generators are summarised as follows (details of these calculations may be found in Appendix C.3.3):

$$
\begin{aligned}
P^+ &= -\int dx^- dx^i \pi^2 \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger(p^+, x^i)\left(-p^+ a(p^+, x^i)\right) \\
\delta a(p^+, x^i)_{P^+} &= p^+ a(p^+, x^i) \\
\Rightarrow \hat{p}^+ &= p_1^+ + p_2^+ \qquad (4.2) \\
P^i &= \int dx^- dx^i \left(\pi \partial_i \phi\right) \\
&= \int dx^i \int_0^\infty dp_1^+ a^\dagger(p^+, x^i)\left(i\partial_i a(p^+, x^i)\right) \\
\delta a(p^+, x^i)_{P^i} &= i\partial_i a(p^+, x^i) \\
\Rightarrow \hat{p}^i &= i\partial_i = p_1^i + p_2^i \qquad (4.3) \\
P^- &= \int dx^- dx^i \left(\frac{1}{2}\left(\partial_i \phi\right)^2\right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger\left(p^+, x^i\right)\left(-\frac{1}{2p^+}\partial_i^2 a\left(p^+, x^i\right)\right) \\
\delta a(p^+, x^i)_{P^-} &= -\frac{\partial_i^2}{2p^+} a\left(p^+, x^i\right) \\
\Rightarrow \hat{p}^- &= -\left(\frac{p_1^i p_1^i}{2p_1^+} + \frac{p_2^i p_2^i}{2p_2^+}\right) \qquad (4.4) \\
M^{+-} &= x^+ H + \int dx^- dx^i \left(x^- \pi^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \int dx^i \int_0^\infty dp^+ a^\dagger\left(p^+, x^i\right)\left(\frac{x^+}{2p^+}\partial_i^2 + i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right) \\
&\quad \left(a\left(p^+, x^i\right)\right) \\
\delta a(p^+, x^i)_{M^{+-}} &= \left(\frac{x^+}{2p^+}\partial_i^2 - i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)\left(a\left(p^+, x^i\right)\right) \\
\Rightarrow \hat{m}^{+-} &= t\hat{p}^- - x_1^- p_1^+ - x_2^- p_2^+ \qquad (4.5) \\
M^{+i} &= \int dx^i dx^- \left(x^+ \pi \partial_i \phi + x^i \pi^2\right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger(p^+, x^i)\left(ix^+\partial_i + x^i p^+\right) a(p^+, x^i) \\
\delta a(p^+, x^i)_{M^{+i}} &= \left(ix^+\partial_i - x^i p^+\right) a(p^+, x^i) \\
\Rightarrow \hat{m}^{+i} &= t\hat{p}^i - x_1^i p_1^+ - x_2^- p_2^+ \qquad (4.6) \\
M^{-i} &= \int dx^- dx^i \left(x^- \pi \partial_i \phi - x^i \mathcal{H}\right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger\left(p^+, x^i\right)\left(-\partial_i \frac{\partial}{\partial p^+} - \frac{\partial_j x^i \partial_j}{2p^+}\right) \\
&\quad a(p^+, x^i) \\
\delta a(p^+, x^i)_{M^{-i}} &= \left(-\partial_i \frac{\partial}{\partial p^+} - \frac{\partial_j x^i \partial_j}{2p^+}\right) a(p^+, x^i) \\
\Rightarrow \hat{m}^{-i} &= p_1^i x_1^- + p_2^i x_2^- + x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+} \qquad (4.7) \\
M^{ij} &= \int dx^- dx^\perp \left(x^i \pi \partial_j \phi - x^j \pi \partial_i \phi\right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger(p_1^+, x^i)\left(ix^i\partial_j - ix^j\partial_i\right) a(p^+, x^i) \\
\delta a(p^+, x^i)_{M^{ij}} &= \left(ix^i\partial_j - ix^j\partial_i\right) a(p^+, x^i) \\
\Rightarrow \hat{m}^{ij} &= x_1^i p_1^j + x_2^i p_2^j - x_1^j p_1^i - x_2^j p_2^i \qquad (4.8) \\
D &= x^+ H + \int dx^- dx^\perp \ \left(\pi d_\phi \phi + \pi x^i \partial_i \phi - x^- \pi^2\right) \\
&= \int dx^\perp \int_0^\infty dp^+ a^\dagger(p^+, x^i) \\
&\quad \left(\frac{x^+}{2p^+}\partial_i^2 + id_\phi + ix^i\partial_i - i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right) a(p^+, x^i) \\
\delta a(p^+, x^i)_D &= \left(\frac{x^+}{2p^+}\partial_i^2 + id_\phi + ix^i\partial_i + i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right) a(p^+, x^i) \\
\Rightarrow \hat{d} &= t\hat{p}^- + 2d_\phi + x_1^i p_1^i + x_2^i p_2^i + x_1^- p_1^+ + x_2^- p_2^+ \qquad (4.9) \\
K^+ &= \int dx^- dx^i \left(x^+ \mathcal{D} + \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\pi^2\right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger(p^+, x^i)
\end{aligned}
$$

$$
\begin{aligned}
& \left[x^+\left(\frac{x^+}{2p^+}\partial_i^2+id_\phi+ix^i\partial_i\right)+\frac{1}{2}x_ix^ip^+\right]a(p^+,x^i) \\
\delta a(p^+,x^i)_{K^+} &= \left[x^+\left(\frac{x^+}{2p^+}\partial_i^2+id_\phi+ix^i\partial_i\right)-\frac{1}{2}x_ix^ip^+\right]a(p^+,x^i) \\
\Rightarrow \hat{k}^+ &= t^2\hat{p}^-+t(2d_\phi+x_1^ip_1^i+x_2^ip_2^i)-\frac{1}{2}\left(x_1^ix_1^ip_1^++x_2^ix_2^ip_2^+\right) \qquad (4.10) \\
K^i &= \int dx^-dx^i\left(x^i\mathcal{D}-\frac{1}{2}\left(2x^+x^-+x^jx_j\right)\pi\partial_i\phi\right) \\
\delta a(p^+,x^i)_{K^i} &= \left[x^+\frac{\partial_jx^i\partial_j}{2p^+}+x^+\partial_i\frac{\partial}{\partial p^+}-\frac{i}{2}x^jx^j\partial_i\right. \\
& \left.+ix^i\left[d_\phi+x^j\partial_j+\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right]\right]a(p^+,x^i) \\
\Rightarrow \hat{k}^i &= -t\left(x_1^i\frac{p_1^jp_1^j}{2p_1^+}+x_2^i\frac{p_2^jp_2^j}{2p_2^+}+p_1^ix_1^-+p_2^ix_2^i\right) \\
& -\frac{1}{2}\left(x_1^jx_1^jp_1^i+x_2^jx_2^jp_2^i\right)+x_1^i\left(d_\phi+x_1^jp_1^j+p_1^+x_1^-\right) \\
& +x_2^i\left(d_\phi+x_2^jp_2^j+p_2^+x_2^-\right) \qquad (4.11) \\
K^- &= \int dx^-dx^i\left(x^-\mathcal{D}-\frac{1}{2}\left(2x^+x^-+x^jx_j\right)\mathcal{H}\right) \\
\delta a(p^+,x^i)_{K^-} &= \left[-\frac{\partial_jx^ix^i\partial_j}{4p^+}-\sqrt{p^+}\frac{\partial}{\partial p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}-x^i\partial_i\frac{\partial}{\partial p^+}\right. \\
& \left.-d_\phi\frac{1}{\sqrt{p^+}}\frac{\partial}{\partial p^+}\sqrt{p^+}\right]a(p^+,x^i). \\
\Rightarrow \hat{k}^- &= x_1^ix_1^i\frac{p_1^jp_1^j}{4p_1^+}+x_2^ix_2^i\frac{p_2^jp_2^j}{4p_2^+}+x_1^-\left(x_1^-p_1^++x_1^ip_1^i+d_\phi\right) \\
& +x_2^-\left(x_2^-p_2^++x_2^ip_2^i+d_\phi\right) \qquad (4.12)
\end{aligned}
$$

4.2.4 Map between AdS_4 generators and CFT_3 generators

The generators in AdS_4 higher spin theory were first worked out in Ref. [130] and then summarised in Ref. [5], and were used to establish a map between bilocal and $AdS_4 \times S_1$ coordinates. Here $G = \int dx^-dxdzd\theta\bar{\Phi}\hat{g}\Phi$, and the conjugate momenta were given by $(p^+, p^x, p^z, p^\theta)$):

$$
\begin{aligned}
\hat{p}^- &= -\frac{p^xp^x+p^zp^z}{2p^+} \\
\hat{p}^+ &= p^+ \\
\hat{p}^x &= p^x
\end{aligned}
$$

$$
\begin{aligned}
&\hat{m}^{+-} = t\hat{p}^{-} - x^{-}p^{+} \\
&\hat{m}^{+x} = t\hat{p}^{x} - xp^{+} \\
&\hat{m}^{-x} = x^{-}p^{x} - x\hat{p}^{-} + \frac{p^{\theta}p^{z}}{p^{+}} \\
&\hat{d} = t\hat{p}^{-} + x^{-}p^{+} + xp^{x} + zp^{z} + d_a \\
&\hat{k}^{-} = -\frac{1}{2}(x^2+z^2)\hat{p}^{-} + x^{-}(x^{-}p^{+} + xp^{x} + zp^{z} + d_a) \\
&+\frac{1}{p^{+}}\left((xp^{z} - zp^{x})p^{\theta} + (p^{\theta})^2\right) \\
&\hat{k}^{+} = x^{+2}\hat{p}^{-} + x^{+}(xp^{x} + zp^{z} + d_a) - \frac{1}{2}(x^2+z^2)p^{+} \\
&\hat{k}^{x} = x^{+}(x\hat{p}^{-} - x^{-}p^{x} - \frac{p^{\theta}p^{z}}{p^{+}}) + \frac{1}{2}(x^2-z^2)p^{x} \\
&+x(x^{-}p^{+} + zp^{z} + d_a) + zp^{\theta}
\end{aligned}
$$

where $d_a = 1$ for AdS_4. Comparing these two sets and generators results in the following map between CFT and $AdS_4 \times S_1$ (worked out explicitly in Appendix C.3.4):

$$
\begin{aligned}
\hat{p}^{+} &= p_1^{+} + p_2^{+} \\
\hat{p}^{i} &= p_1^{i} + p_2^{i} \\
\hat{p}^{-} &= -\left(\frac{p_1^i p_1^i}{2p_1^{+}} + \frac{p_2^i p_2^i}{2p_2^{+}}\right) = -\frac{pp + p^z p^z}{2p^{+}} \\
x &= \frac{x_1 p_1^{+} + x_2 p_2^{+}}{p_1^{+} + p_2^{+}} \\
x^{-} &= \frac{x_1^{-} p_1^{+} + x_2^{-} p_2^{+}}{p_1^{+} + p_2^{+}} \\
z &= \frac{(x_1 - x_2)\sqrt{p_1^{+} p_2^{+}}}{p_1^{+} + p_2^{+}} \\
p^{z} &= \frac{p_1 p_2^{+} - p_1^{+} p_2}{\sqrt{p_1^{+} p_2^{+}}} \\
p^{\theta} &= \left(x_1^{-} - x_2^{-}\right)\sqrt{p_1^{+} p_2^{+}} + \frac{\left(x_1^{-} - x_2^{-}\right)}{2}\left[\sqrt{\frac{p_2^{+}}{p_1^{+}}}p_1 - \sqrt{\frac{p_1^{+}}{p_2^{+}}}p_2\right]
\end{aligned}
\tag{4.13}
$$

$$
\theta = 2\arctan\sqrt{\frac{p_2^{+}}{p_1^{+}}}. \tag{4.14}
$$

4.2.5 Map established through Kernel

Another approach, generalisable to the temporal gauge, was developed in Ref. [10]. The collective fields are canonical, i.e. $[\Psi_x, \Pi_{c'}] = \delta_{c,c'}$ where the equal time bilocals are given by

$$\Psi(t; \vec{x}_1 . \vec{x}_2) = \sum_{i=1}^{N} \varphi^i(t, \vec{x}_1)\varphi^i(t, \vec{x}_2).$$

The large N expansion of the collective Hamiltonian is given by

$$H_{\rm col} = NH_0 + H_2 + \frac{1}{\sqrt{N}} H_3 + \frac{1}{N} H_4 + \ldots.$$

In subsection 1.12.2 and section 1.13, it was observed that theories could be written in terms of an infinite tower of higher spins. In the present case there is a similar situation as a result of the S_1 parameter θ which was added to ensure that CFT degrees of freedom match with the AdS degrees of freedom. A Fourier transformation of a sequence of higher spin fields yields

$$\mathcal{H}\left(x^+; x^-, x, z, \theta\right) = \sum_{s=0:\text{even}}^{\infty} \cos\left(s\theta\right) \mathcal{H}_s\left(x^+; x^-, x, z\right), \tag{4.15}$$

representing a field on $AdS_4 \times S^1$. The pointwise one-to-one momenta map in equations (4.14) between bi-local ($SO(2,3)$) and bulk ($AdS_4 \times S^1$) fields is summarised below:

$$\begin{aligned} p^+ &= p_1^+ + p_2^+ && (4.16)\\ p &= p_1 + p_2 && (4.17)\\ p^z &= p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} && (4.18)\\ \theta &= 2\arctan\sqrt{\frac{p_2^+}{p_1^+}}. && (4.19) \end{aligned}$$

This may be inverted as follows: from (4.19) we may obtain expressions for p_1^+ and p_2^+,

$$\tan^2\frac{\theta}{2} = \frac{p_2^+}{p_1^+}$$

$$
\begin{aligned}
\Rightarrow \tan^2\frac{\theta}{2}+1 &= \sec^2\frac{\theta}{2}=\frac{1}{\cos^2\frac{\theta}{2}}=\frac{p_1^+ + p_2^+}{p_1^+}=\frac{p^+}{p_1^+}\\
\Rightarrow p_1^+ &= p^+\cos^2\frac{\theta}{2}\\
\Rightarrow p_2^+ &= p^+ - p_1^+\\
&= p^+\sin^2\frac{\theta}{2}.
\end{aligned}
$$

Plugging (4.17) and (4.19) into (4.18) we obtain

$$
\begin{aligned}
p^z &= p_1\tan\frac{\theta}{2}-(p-p_1)\frac{1}{\tan\frac{\theta}{2}}\\
\Rightarrow p_1\left(\tan\frac{\theta}{2}+\frac{1}{\tan\frac{\theta}{2}}\right) &= p_1\frac{\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}}=p^z+p\frac{1}{\tan\frac{\theta}{2}}\\
\Rightarrow p_1 &= p^z\cos^2\frac{\theta}{2}\tan\frac{\theta}{2}+p\cos^2\frac{\theta}{2}\\
&= p^z\cos\frac{\theta}{2}\sin\frac{\theta}{2}+p\cos^2\frac{\theta}{2}\\
&= \frac{1}{2}\left(p^z\sin\theta+p\left(\cos\theta+1\right)\right)\\
\Rightarrow p_2 &= p-p_1\\
&= \frac{1}{2}\left(p\left(1-\cos\theta\right)-p^z\sin\theta\right).
\end{aligned}
$$

We may therefore write the map between fields as

$$
\tilde{\mathcal{H}}\left(p^+,p,p^z,\theta\right)=\int dp_1^+dp_1dp_2^+dp_2\mathcal{K}\left(p^+,p,p^z,\theta;p_1^+,p^1,p_2^+,p_2\right)\tilde{\Psi}\left(p_1^+,p_1,p_2^+,p_2\right)
$$

where the Kernel $\mathcal{K}$ is given by

$$
\begin{aligned}
&\mathcal{K}\left(p^+,p,p^z,\theta;p_1^+,p^1,p_2^+,p_2\right)\\
=\;&\delta\left[p_1-\frac{1}{2}\left(p^z\sin\theta+p\left(\cos\theta+1\right)\right)\right]\delta\left[p_2-\frac{1}{2}\left(p\left(1-\cos\theta\right)-p^z\sin\theta\right)\right]\\
&\delta\left[p_1^+-p^+\cos^2\frac{\theta}{2}\right]\delta\left[p_2^+-p^+\sin^2\frac{\theta}{2}\right].
\end{aligned}
$$

This map may be used to obtain the identifications as found in the previous section. For example, the identification between the extra AdS_4 coordinate and the relative bi-local space found in equation (4.13),

$$z = \frac{(x_1 - x_2)\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}.$$

The map is established using the kernel as follows[10]

$$L^{AdS}\tilde{\mathcal{H}}\left(p^+, p, p^z, \theta\right) = \int dp_1^+ dp_1 dp_2^+ dp_2\, \mathcal{K} L^{\text{bi-local}} \tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right).$$

Some examples of obtaining generators using the kernel may be found in AppendixC.3.5. We can rewrite the Kernel in terms of the AdS coordinates,

$$\begin{aligned} \mathcal{K} &= \delta\left(p_1^+ + p_2^+ - p^+\right)\delta\left(p_1 + p_2 - p\right)\delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z\right) \\ &\quad \delta\left(\arctan\sqrt{\frac{p_2^+}{p_1^+}} - \frac{\theta}{2}\right)\mathrm{J} \end{aligned}$$

where the Jacobian J$=\frac{\partial\left(p^+, p, p^z, \theta\right)}{\partial\left(p_1^+, p_1, p_2^+, p_2\right)}$ of the transformation is given by

$$\det\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{p_2^+ p_1\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)}{2p_1^{+2}\sqrt{\frac{p_2^+}{p_1^+}}} - \frac{p_2\left(\sqrt{\frac{p_2^+}{p_a^+}}\right)}{2p_2^+\sqrt{\frac{p_2^+}{p_1^+}}} & \sqrt{\frac{p_2^+}{p_1^+}} & \frac{p_2^+ p_2\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)}{2p_1^{+2}\sqrt{\frac{p_2^+}{p_1^+}}} + \frac{p_2\left(\sqrt{\frac{p_1^+}{p_2^+}}\right)}{2p_2^+\sqrt{\frac{p_1^+}{p_2^+}}} & -\sqrt{\frac{p_1^+}{p_2^+}} \\ -\frac{p_2^+}{p_1^{+2}\left(\frac{p_2^+}{p_1^+}+1\right)\sqrt{\frac{p_2^+}{p_1^+}}} & 0 & \frac{1}{p_1^+\left(\frac{p_2^+}{p_1^+}+1\right)\sqrt{\frac{p_2^+}{p_1^+}}} & 0 \end{vmatrix}$$

which simplifies to

$$\begin{aligned} \mathrm{J} &= \sqrt{\frac{p_2^+}{p_1^+}}\frac{1}{p_1^+\left(\frac{p_2^+}{p_1^+}+1\right)\sqrt{\frac{p_2^+}{p_1^+}}} - \left(-\sqrt{\frac{p_1^+}{p_2^+}}\frac{1}{p_1^+\left(\frac{p_2^+}{p_1^+}+1\right)\sqrt{\frac{p_2^+}{p_1^+}}}\right)\left(-\sqrt{\frac{p_1^+}{p_2^+}}\right) \\ &\quad \left(-\frac{p_2^+}{p_1^{+2}\left(\frac{p_2^+}{p_1^+}+1\right)\sqrt{\frac{p_2^+}{p_1^+}}}\right) + \sqrt{\frac{p_2^+}{p_1^+}}\left(\frac{p_2^+}{p_1^{+2}\left(\frac{p_2^+}{p_1^+}+1\right)\sqrt{\frac{p_2^+}{p_1^+}}}\right) \end{aligned} \quad (4.20)$$

$$\begin{aligned} &= \frac{1}{p_1^+ + p_2^+} + \frac{p_1^+}{p_2^+}\frac{1}{p_1^+ + p_2^+} + \frac{1}{p_1^+ + p_2^+} + \frac{p_2^+}{p_1^+}\frac{1}{p_1^+ + p_2^+} \\ &= \frac{1}{p_1^+} + \frac{1}{p_2^+} \end{aligned}$$

and we may therefore rewrite

$$\begin{aligned} &\tilde{\mathcal{H}}\left(p^+, p, p^z, \theta\right) \\ =& \int dp_1^+ dp_1 dp_2^+ dp_2 \, \mathcal{K}\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right) \\ =& \int dp_1^+ dp_1 dp_2^+ dp_2 \, J\delta\left(p_1^+ + p_2^+ - p^+\right)\delta\left(p_1 + p_2 - p\right) \\ & \delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z\right)\delta\left(\arctan\sqrt{\frac{p_2^+}{p_1^+}} - \frac{\theta}{2}\right)\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right). \end{aligned}$$

Now recalling what was initially written and performing a Fourier transform,

$$\begin{aligned} & \mathcal{H}\left(x^+; x^-, x, z, \theta\right) = \sum_{s=0:\text{even}}^{\infty} \cos\left(s\theta\right)\mathcal{H}_s\left(x^+; x^-, x, z\right) \\ \Rightarrow\quad & \mathcal{H}_s\left(x^+; x^-, x, z\right) \\ =& \int d\theta \mathcal{H}\left(x^+; x^-, x, z, \theta\right)\cos(s\theta) \\ =& \int d\theta \int d^4p \, \delta\left(2p^+p^- + p^2 - (p^z)^2\right) e^{ix^\mu p_\mu}\tilde{\mathcal{H}}\left(p^+, p, p^z, \theta\right)\cos\left(s\theta\right) \\ =& \int d\theta \int d^4p \, \delta\left(2p^+p^- + p^2 - (p^z)^2\right) e^{ix^\mu p_\mu}\int dp_1^+ dp_1 dp_2^+ dp_2 \\ & \mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right)\delta\left(p_1 + p_2 - p\right)\delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z\right) \\ & \delta\left(\arctan\sqrt{\frac{p_2^+}{p_1^+}} - \frac{\theta}{2}\right)\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right)\cos\left(s\theta\right) \end{aligned}$$

where in the third line, this Fourier transform would normally be over 3 variables but since we are now integrating over 4 variables it introduces an additional delta function which enforces the on-shell condition.

Now, since

$$\cos\frac{\theta}{2} = \sqrt{\frac{p_1^+}{p_1^+ + p_2^+}} \Rightarrow \theta = 2\arccos\sqrt{\frac{p_1^+}{p_1^+ + p_2^+}},$$

$\mathcal{H}_s\left(x^+; x^-, x, z\right)$ reduces to

$$\begin{aligned}
&\int d\theta \int d^4p\, \delta\left(2p^+p^- + p^2 - (p^z)^2\right) e^{ix^\mu p_\mu} \int dp_1^+ dp_1 dp_2^+ dp_2 \\
&\mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right) \delta\left(p_1 + p_2 - p\right) \delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z\right) \\
&\delta\left(\arccos\sqrt{\frac{p_1^+}{p_1^+ + p_2^+}} - \frac{\theta}{2}\right) \cos\left(s\theta\right) \tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right) \\
= &\int dp^+ dp dp^z d\theta \int d^4p\, \delta\left(2p^+p^- + p^2 - (p^z)^2\right) e^{ix^\mu p_\mu} \int dp_1^+ dp_1 dp_2^+ dp_2 \\
&\mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right) \delta\left(p_1 + p_2 - p\right) \delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z\right) \\
&\cos\left(2\arccos\sqrt{\frac{p_1^+}{p_1^+ + p_2^+}} s\right) \tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right),
\end{aligned}$$

Orthogonal polynomials are a useful tool to understand this further. By way of introduction, these polynomials are solutions to second order differential equations, and some insight may be gained by viewing them with regard to their generating functions, below of which are a few examples (defined when $|z| < 1$; for Jacobi polynomials, R is defined as $R = 1 - 2xz + z^2$):

$$\begin{array}{cc}
\text{Legendre}: & \frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n \\
\text{Chebyshev}: & \frac{1-xz}{1-2xz+z^2} = \sum_{n=0}^{\infty} T_n(x) z^n \\
\text{Ultraspherical}\,(\propto \text{Gegenbauer}): & \frac{1}{(1-2xz+z^2)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) z^n \\
\text{Jacobi}: & \frac{2^{\alpha+\beta}}{R(1+R-z)^{\alpha}(1+R+z)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n \\
\text{Hypergeometric}: & F(a,b,c;z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s s!} z^s
\end{array}$$

Here we see that Gegenbauer polynomials are generalisations of Legendre polynomials. Gradschteyn and Ryzhik's table of integrals , series and products[132], Szego's Orthogonal Polynomials [133] and Slater's Generalized Hypergeometric Function [134] give us some ideas about how to proceed. Firstly note that Chebyshev's polynomials of the first kind are given by (8.940 Ref. [132])

$$T_n(x) \;=\; \cos\left(n \arccos x\right)$$

and thus

$$\begin{aligned}\mathcal{H}_s\left(x^+;x^-,x,z\right) &= \int dp^+dpdp^zd\theta\; e^{ix^\mu p_\mu}\int dp_1^+dp_1dp_2^+dp_2\, J\delta\left(p_1^++p_2^+-p^+\right)\\ &\quad \delta\left(p_1+p_2-p\right)\delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}}-p_2\sqrt{\frac{p_1^+}{p_2^+}}-p^z\right)\tilde{\Psi}\left(p_1^+,p_1,p_2^+,p_2\right)\\ &\quad T_{2s}\left(\sqrt{\frac{p_1^+}{p_1^++p_2^+}}\right).\end{aligned}$$

In order to calculate $T_{2s}\left(\sqrt{\frac{p_1^+}{p_1^++p_2^+}}\right)$, we take note of the following formulae:

$$T_n(x) = \frac{2^{2n}(n!)^2}{(2n)!}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \qquad 8.962,\ [132]$$

$$P_{2\nu}^{\alpha,\alpha}(x) = (-1)^\nu\frac{\Gamma(2\nu+\alpha+1)\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)\Gamma(2\nu+1)}P_\nu^{(-\frac{1}{2},\alpha)}(1-2x^2), \qquad \text{Theorem 4.1},\ [133]$$

$$P_n^{(\alpha,\beta)}(1) = \begin{pmatrix} n+\alpha \\ n\end{pmatrix} \qquad 4.1.1,\ [133]$$

The calculation requires us to treat $T_{2s}\left(\sqrt{\frac{p_1^+}{p_1^++p_2^+}}\right)$ as a normalised quantity. Hence we obtain,

$$\begin{aligned}T_{2s}\left(\sqrt{\frac{p_1^+}{p_1^++p_2^+}}\right) &= \frac{\frac{2^{2s}(s!)^2}{(2s)!}P_{2s}^{(-\frac{1}{2},-\frac{1}{2})}\left(\sqrt{\frac{p_1^+}{p_1^++p_2^+}}\right)}{\frac{2^{2s}(s!)^2}{(2s)!}P_{2s}^{(-\frac{1}{2},-\frac{1}{2})}(0)}\\ &= \frac{P_{2s}^{(-\frac{1}{2},-\frac{1}{2})}\left(\sqrt{\frac{p_1^+}{p_1^++p_2^+}}\right)}{P_{2s}^{(-\frac{1}{2},-\frac{1}{2})}(0)}\\ &= \frac{(-1)^s\frac{\Gamma(2s-\frac{1}{2}+1)\Gamma(s+1)}{\Gamma(s-\frac{1}{2}+1)\Gamma(2s+1)}P_s^{(-\frac{1}{2},-\frac{1}{2})}(1-2\left(\sqrt{\frac{p_1^+}{p_1^++p_2^+}}\right)^2)}{(-1)^s\frac{\Gamma(2s-\frac{1}{2}+1)\Gamma(s+1)}{\Gamma(s-\frac{1}{2}+1)\Gamma(2s+1)}P_s^{(-\frac{1}{2},-\frac{1}{2})}(1)}\\ &= \frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+-p_2^+}{p_1^++p_2^+}\right)}{P_s^{(-\frac{1}{2},-\frac{1}{2})}(1)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+-p_2^+}{p_1^++p_2^+}\right)}{\begin{pmatrix} s-\frac{1}{2} \\ s \end{pmatrix}} \\
&= \frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+-p_2^+}{p_1^++p_2^+}\right)}{\frac{\Gamma(s+\frac{1}{2})}{s!\Gamma(-\frac{1}{2})}} \\
&= \Gamma(-\frac{1}{2})\frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+-p_2^+}{p_1^++p_2^+}\right)s!}{\Gamma(s+\frac{1}{2})},
\end{aligned}
\tag{4.21}
$$

and therefore

$$
\begin{aligned}
&\mathcal{H}_s\left(x^+;x^-,x,z\right) \\
= \; &\Gamma(-\frac{1}{2})\int dp^+dpdp^zdp^-\ \delta\left(2p^+p^-+p^2+(p^z)^2\right)e^{ix^\mu p_\mu}\int dp_1^+dp_1dp_2^+dp_2 \\
&\mathrm{J}\delta\left(p_1^++p_2^+-p^+\right)\delta\left(p_1+p_2-p\right)\delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}}-p_2\sqrt{\frac{p_1^+}{p_2^+}}-p^z\right) \\
&\frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+-p_2^+}{p_1^++p_2^+}\right)s!}{\Gamma(s+\frac{1}{2})}\tilde{\Psi}\left(p_1^+,p_1,p_2^+,p_2\right).
\end{aligned}
\tag{4.22}
$$

in agreement (up to a factor of $-2\sqrt{\pi}$) with equation 30 of Ref. [10]. Equation 33 of Ref. [10] (which has a detailed but yet insightful derivation and hence is included in Appendix C.3.6) tells us that in the limit $z \to 0$,

$$
\begin{aligned}
\mathcal{H}_s\left(x^+;x^-,x,z\right) \underset{z=0}{\longrightarrow} &\frac{s!}{\Gamma\left(s+\frac{1}{2}\right)}\int_{-2p^+p^--p^2>0}dp^+dpdp^- \\
&\left(\frac{1}{(-i\partial_-)^s}e^{i\left(x^+p^-+x^-p^++xp\right)}\right)\tilde{\mathcal{O}}_s\left(p^-,p^+,p\right),
\end{aligned}
$$

which shows that the Collective field map to AdS fields at $z=0$ (which is the boundary of AdS) leads to the conserved primary operators of the Conformal Field Theory; this acts as a good consistency check.

4.3 Temporal Gauge

The map found in section 4.2 as enforced by a kernel was generalised to temporal gauge in Ref. [10]. We review the construction of this map below.

4.3.1 Operator derivations in temporal gauge

In temporal gauge, where

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\
&= -\frac{1}{2}\left(\partial_t\phi\partial^t\phi + \partial_x\phi\partial^x\phi + \partial_y\phi\partial^y\phi\right) \\
&= \frac{1}{2}\left(\partial_t\phi\partial_t\phi - \partial_x\phi\partial_x\phi - \partial_y\phi\partial_y\phi\right) \\
\Rightarrow \pi &= \frac{\partial\mathcal{L}}{\partial_t\phi} \\
&= \partial_t\phi \\
\Rightarrow \mathcal{H} &= \pi^2 - \mathcal{L} = \pi\partial_t\phi - \mathcal{L} \\
&= \frac{1}{2}\left(\pi^2 + \partial_x\phi\partial_x\phi + \partial_y\phi\partial_y\phi\right)
\end{aligned}$$

we have

$$\begin{aligned}
j^\mu &= -\left[\left(\mathcal{L}g^\mu_\nu - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\nu\phi\right)\delta x^\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi\right] \\
\Rightarrow j^t &= -\left[\left(\mathcal{L}g^t_\nu - \frac{\delta\mathcal{L}}{\delta\partial_t\phi}\partial_\nu\phi\right)\delta x^\nu + \frac{\delta\mathcal{L}}{\delta\partial_t\phi}\delta\phi\right] \\
&= -\left[\left(\mathcal{L}g^t_\nu - \pi\partial_\nu\phi\right)\delta x^\nu + \pi\delta\phi\right] \\
&= \pi\partial_\nu\phi\delta x^\nu - \mathcal{L}g^t_\nu\delta x^\nu - \pi\delta\phi \\
&= \pi\partial_\nu\phi\delta x^\nu - \left(\pi\partial_t\phi - \mathcal{H}\right)g^t_\nu\delta x^\nu - \pi\delta\phi
\end{aligned}$$

For translations (where caligraphic writing indicates operator densities), noting that $g^t_\nu = g^{t\mu} g_{\nu\mu} \Rightarrow g^t_t = 1$,

$$\begin{aligned}
\delta x^\mu &= \epsilon^\mu \\
\delta\phi &= \partial_\mu \left[\epsilon^\mu\right] \phi = 0 \\
\Rightarrow j^t &= \pi\partial_\nu\phi\delta x^\nu - \left(\pi\partial_t\phi - \mathcal{H}\right) g^t_\nu \delta x^\nu - \pi\delta\phi \\
&= \pi\partial_\nu\phi\epsilon^\nu - \left(\pi\partial_t\phi - \mathcal{H}\right) g^t_\nu \epsilon^\nu \\
&= \left(\pi\partial_\nu\phi - \left(\pi\partial_t\phi - \mathcal{H}\right) g^t_\nu\right) \epsilon^\nu \\
\Rightarrow \mathcal{P}^\nu = T^{t\nu} = j^{t\nu} &= \pi\partial_\nu\phi - \left(\pi\partial_t\phi - \mathcal{H}\right) g^t_\nu \\
\mathcal{P}^x &= \pi\partial_x\phi \\
\Rightarrow P^x &= \int dx\, dy\, \pi\partial_x\phi \\
\mathcal{P}^y &= \pi\partial_y\phi \\
\Rightarrow P^y &= \int dx\, dy\, \pi\partial_y\phi \\
\mathcal{P}^t &= \mathcal{H} \\
\Rightarrow P^t &= \int dx\, dy\, \mathcal{H}.
\end{aligned}$$

For Lorentz transformations,

$$\begin{aligned}
\delta x_\mu &= \left(\epsilon_{\rho\mu} - \epsilon_{\mu\rho}\right) x^\rho \\
\delta\phi &= -\partial^\mu \left[\left(\epsilon_{\rho\mu} - \epsilon_{\mu\rho}\right) x^\rho\right] \phi = 0 \text{ (antisymmetric parameter)} \\
\Rightarrow j^\mu &= \left(\mathcal{L} g^\mu_\nu - \pi\partial_\nu\phi\right) \delta x^\nu \\
&= \left(\mathcal{L} g^{\mu\nu} - \pi\partial^\nu\phi\right) \left(\epsilon_{\rho\nu} - \epsilon_{\nu\rho}\right) x^\rho \\
&= \epsilon_{\rho\nu} \left[\left(\mathcal{L} g^{\mu\nu} - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\nu\phi\right) x^\rho - \left(\mathcal{L} g^{\mu\rho} - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\rho\phi\right) x^\nu\right] \\
\Rightarrow j^{\mu\nu\rho} &= \left[\left(\mathcal{L} g^{\mu\nu} - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\nu\phi\right) x^\rho - \left(\mathcal{L} g^{\mu\rho} - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\rho\phi\right) x^\nu\right] \\
\Rightarrow \mathcal{M}^{\nu\rho} = j^{t\nu\rho} &= \left[\left(\mathcal{L} g^{t\nu} - \frac{\delta\mathcal{L}}{\delta\partial_t\phi}\partial^\nu\phi\right) x^\rho - \left(\mathcal{L} g^{t\rho} - \frac{\delta\mathcal{L}}{\delta\partial_t\phi}\partial^\rho\phi\right) x^\nu\right] \\
&= \left[\left(\mathcal{L} g^{t\nu} - \pi\partial^\nu\phi\right) x^\rho - \left(\mathcal{L} g^{t\rho} - \pi\partial^\rho\phi\right) x^\nu\right] \\
\Rightarrow \mathcal{M}^{tt} &= \left[\left(-(\pi^2 - \mathcal{H}) + \pi^2\right) t - \left(-(\pi^2 - \mathcal{H}) + \pi^2\right) t\right] \\
&= 0 \\
\Rightarrow M^{tt} &= 0
\end{aligned}$$

as expected for an antisymmetric operator. Furthermore,

$$\begin{aligned}
\mathcal{M}^{tx} &= \left[\left(\mathcal{L}g^{tt} - \pi\partial^t\phi\right)x - \left(\mathcal{L}g^{tx} - \pi\partial^x\phi\right)t\right] \\
&= \left[\left(-\left(\pi^2 - \mathcal{H}\right) + \pi^2\right)x + \left(\pi\partial_x\phi\right)t\right] \\
&= \mathcal{H}x + \left(\pi\partial_x\phi\right)t \\
\Rightarrow M^{tx} &= \int dxdy\ \mathcal{H}x + \left(\pi\partial_x\phi\right)t \\
\Rightarrow M^{ty} &= \int dxdy\ \mathcal{H}y + \left(\pi\partial_y\phi\right)t \\
\Rightarrow M^{xy} &= \int dxdy\ -\left(\pi\partial_x\phi\right)y + \left(\pi\partial_y\phi\right)x.
\end{aligned}$$

For Dilations,

$$\begin{aligned}
\delta x^\mu &= \epsilon x^\mu \\
\delta\phi &= -\partial_\mu\left[\epsilon x^\mu\right]\phi = -\epsilon d_\phi\phi \\
j^t &= \pi\partial_\nu\phi\delta x^\nu - \left(\pi\partial_t\phi - \mathcal{H}\right)g^t_\nu\delta x^\nu - \pi\delta\phi \\
&= \pi\epsilon\left(\pi t + \partial_x\phi x + \partial_y\phi y\right) - \epsilon\left(\pi^2 - \mathcal{H}\right)t + \epsilon\pi d_\phi\phi \\
\Rightarrow \mathcal{D} &= \left(\pi\partial_x\phi\right)x + \left(\pi\partial_y\phi\right)y + \mathcal{H}t + \pi d_\phi\phi \\
\Rightarrow D &= \int dxdy\left[\left(\pi\partial_x\phi\right)x + \left(\pi\partial_y\phi\right)y + \mathcal{H}t + \pi d_\phi\phi\right] \\
&= Ht + \int dxdy\left[\left(\pi\partial_x\phi\right)x + \left(\pi\partial_y\phi\right)y + \pi d_\phi\phi\right].
\end{aligned}$$

For Special Conformal Transformations,

$$\begin{aligned}
\delta x^\mu &= 2\left(x\cdot\epsilon\right)x^\mu - x^2\epsilon^\mu = \left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right]\epsilon_\alpha \\
\delta\phi &= -\partial_\mu\left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right]\epsilon_\alpha\phi = -2d_\phi\epsilon^\mu x_\mu\phi \\
j^t &= \pi\partial_\nu\phi\delta x^\nu - \left(\pi\partial_t\phi - \mathcal{H}\right)g^t_\nu\delta x^\nu - \pi\delta\phi \\
&= \pi\partial_\nu\phi\left[2x^\alpha x^\nu - x^2 g^{\alpha\nu}\right]\epsilon_\alpha - \left(\pi^2 - \mathcal{H}\right)g^t_\nu\left[2x^\alpha x^\nu - x^2 g^{\alpha\nu}\right]\epsilon_\alpha + 2d_\phi\pi\epsilon_\alpha x^\alpha\phi \\
\Rightarrow \mathcal{K}^\alpha &= \pi\partial_\nu\phi\left[2x^\alpha x^\nu - x^2 g^{\alpha\nu}\right] - \left(\pi^2 - \mathcal{H}\right)g^t_\nu\left[2x^\alpha x^\nu - x^2 g^{\alpha\nu}\right] + 2d_\phi\pi x^\alpha\phi \\
&= \pi\partial_\nu\phi\left[2x^\alpha x^\nu - x^2 g^{\alpha\nu}\right] - \left(\pi^2 - \mathcal{H}\right)\left[2x^\alpha t - x^2 g^{\alpha t}\right] + 2d_\phi\pi x^\alpha\phi \\
\Rightarrow \mathcal{K}^t &= \pi\partial_\nu\phi\left[2tx^\nu - x^2 g^{t\nu}\right] - \left(\pi^2 - \mathcal{H}\right)\left[2t^2 + x^2\right] + 2d_\phi\pi t\phi \\
&= 2\pi t\left(\pi t + \partial_x\phi x + \partial_y\phi y\right) + \pi^2 x^2 - 2\pi^2 t^2 - \pi^2 x^2 + 2\mathcal{H}t^2 + \mathcal{H}x^2 + 2d_\phi\pi t\phi \\
&= 2\pi t\left(\pi t + \partial_x\phi x + \partial_y\phi y\right) - 2\pi^2 t^2 + 2\mathcal{H}t^2 + \mathcal{H}x^2 + 2d_\phi\pi t\phi \\
&= 2t\mathcal{D} - \pi^2 t^2 + \mathcal{H}x^2
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow K^t &= \int dxdy\left[2t\mathcal{D}-\pi^2t^2+\mathcal{H}x^2\right]\\
\mathcal{K}^x &= \pi\partial_\nu\phi\left[2xx^\nu-x^2g^{x\nu}\right]-\left(\pi^2-\mathcal{H}\right)\left[2xt-x^2g^{xt}\right]+2d_\phi\pi x\phi\\
&= 2\pi x\left(\pi t+\partial_x\phi x+\partial_y\phi y\right)-2xt\left(\pi^2-\mathcal{H}\right)+2d_\phi\pi x\phi\\
&= 2x\mathcal{D}+2\pi^2xt-2\pi^2xt\\
&= 2x\mathcal{D}\\
\Rightarrow K^x &= \int dxdy\left[2x\mathcal{D}\right]\\
K^y &= \int dxdy\left[2y\mathcal{D}\right].
\end{aligned}
$$

4.3.2 Temporal Gauge Map

The temporal gauge map was achieved in Ref. [10], and we review the construction of this map here. The map requires that old co-ordinates and new co-ordinates must close the same Lie algebra - $so(d,2)$, and this change of co-ordinates must be a canonical transformation. These have been mapped, in the UV, to $AdS_4\times S^1$ [5] and generalised to temporal gauge [9]. In momentum space, this temporal gauge map, is a point-like transformation that reads:

$$
\begin{aligned}
E &= E_1+E_2=|\vec{k}_1|+\vec{k}_2|\\
\vec{k} &= \vec{k}_1+\vec{k}_2\\
k^z &= \pm2\sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sin\left(\frac{\varphi_2-\varphi_1}{2}\right)\\
\theta &= \arctan\left(\frac{2\vec{k}_2\times\vec{k}_1}{\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)k^z}\right)
\end{aligned}
\tag{4.23}
$$

Let us understand this map a little better. Working in phase space, the centre of momentum of the bilocal space is identified with the AdS_4 momentum,

$$
\begin{aligned}
\vec{k} &= \vec{k}_1+\vec{k}_2\\
k^0 &= \left|\vec{k}_1\right|+\left|\vec{k}_2\right|,
\end{aligned}
$$

and, writing

$$
\vec{k}_1 = \left(\left|\vec{k}_1\right|\cos\varphi_1,\left|\vec{k}_1\right|\sin\varphi_1\right)
$$

$$\vec{k}_2 = \left(\left|\vec{k}_2\right|\cos\varphi_2, \left|\vec{k}_2\right|\sin\varphi_2\right),$$

k^z is defined from the mass-shell condition as

$$\begin{aligned}
k^z &= \sqrt{(k^0)^2 - \vec{k}^2} && (4.24)\\
&= \sqrt{2\left|\vec{k}_1\right|\left|\vec{k}_2\right| - 2\vec{k}_1\cdot\vec{k}_2}\\
&= \sqrt{2\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sqrt{1-(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)}\\
&= \sqrt{2\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sqrt{1-\cos(\varphi_1-\varphi_2)}\\
&= \sqrt{2\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sqrt{1-\left(\cos^2\left(\frac{\varphi_1-\varphi_2}{2}\right)-\sin^2\left(\frac{\varphi_1-\varphi_2}{2}\right)\right)}\\
&= 2\sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right). && (4.25)
\end{aligned}$$

So k^z is defined as

$$k^z = \pm 2\sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right)$$

with the sign changing under exchange of $\vec{k}_1$ and $\vec{k}_2$. θ and k^θ are a bit more tricky. The derivation of these is sketched here, but the full calculation may be found in appendix D. k^θ was found by relating it to the quadratic Casimir using Ref. [135], so that

$$\mathcal{C}_{\mathrm{SO}(2,3))} \sim 2s^2 \sim 2(k^\theta)^2.$$

Working in collective gauge (in the language of [9]) which is different to that given in [135]), where

$$\begin{aligned}
(\vec{x}_1)^2 &= 0\\
(\vec{x}_2)^2 &= 0\\
\vec{x}_1\cdot\vec{k}_1 &= 0\\
\vec{x}_2\cdot\vec{k}_2 &= 0\\
\left(\vec{k}_2\right)^2 &= 0\\
\left(\vec{k}_1\right)^2 &= 0.
\end{aligned}$$

The full expression for the Casimir then simplified to[9]

$$
\begin{aligned}
(k^\theta) &= \sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1+\varphi_2}{2}\left(x_2^1-x_1^1\right)+\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1+\varphi_2}{2}\left(x_2^2-x_1^2\right). \\
&= F_+(\vec{k}_1,\vec{k}_2)\left(x_2^1-x_1^1\right)+F_-(\vec{k}_1,\vec{k}_2)\left(x_2^2-x_1^2\right).
\end{aligned}
$$

Imposing Poisson brackets, in particular,

$$
\begin{aligned}
\{x^1,\theta\}_{\rm PB} &= 0 \\
\{x^2,\theta\}_{\rm PB} &= 0 \\
\{z,\theta\}_{\rm PB} &= 0 \\
\{k^\theta,\theta\}_{\rm PB} &= 1,
\end{aligned}
$$

with the assumption that all AdS coordinates are functions only of bilocal coordinates $\vec{k}_1$ and $\vec{k}_2$ and not of $\vec{x}_i$, we obtain that

$$
\theta = \arctan\left(\frac{2\vec{k}_2\times\vec{k}_1}{\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)k^z}\right).
$$

The map in (4.23) can be inverted to obtain[10] (writing $w\equiv\sqrt{(k^z)^2+\vec{k}^2}$):

$$
\begin{aligned}
k_1^1 &= \frac{w-|\vec{k}|\cos\theta}{2|\vec{k}|\left((k^z)^2+\vec{k}^2\sin^2\theta\right)}\Big[\left(w|\vec{k}|\sin^2\theta-(k^z)^2\cos\theta\right)k^1 \\
&\quad-\left(k^z|\vec{k}|\sin\theta\cos\theta+k^zw\sin\theta\right)k^2\Big] \\
k_1^2 &= \frac{w-|\vec{k}|\cos\theta}{2|\vec{k}|\left((k^z)^2+\vec{k}^2\sin^2\theta\right)}\Big[\left(k^z|\vec{k}|\sin\theta\cos\theta+k^zw\sin\theta\right)k^1 \\
&\quad+\left(w|\vec{k}|\sin^2\theta-(k^z)^2\cos\theta\right)k^2\Big] \\
k_2^1 &= \frac{w+|\vec{k}|\cos\theta}{2|\vec{k}|\left((k^z)^2+\vec{k}^2\sin^2\theta\right)}\Big[\left(w|\vec{k}|\sin^2\theta+(k^z)^2\cos\theta\right)k^1 \\
&\quad-\left(k^z|\vec{k}|\sin\theta\cos\theta-k^zw\sin\theta\right)k^2\Big] \\
k_2^2 &= \frac{w+|\vec{k}|\cos\theta}{2|\vec{k}|\left((k^z)^2+\vec{k}^2\sin^2\theta\right)}\Big[\left(k^z|\vec{k}|\sin\theta\cos\theta-k^zw\sin\theta\right)k^1 \\
&\quad+\left(w|\vec{k}|\sin^2\theta+(k^z)^2\cos\theta\right)k^2\Big].
\end{aligned}
$$

4.3.3 Map established through Kernel

As in the light cone case, the map (4.23) may be established using a kernel, as follows:

$$\mathcal{H}\left(\vec{k},k^z,\theta\right) = \int d\vec{k}_1 d\vec{k}_2 \mathcal{K}\left(\vec{k},k^z,\theta;\vec{k}_1,\vec{k}_2\right)\Psi\left(\vec{k}_1,\vec{k}_2\right). \tag{4.26}$$

The kernel $\mathcal{K}\left(\vec{k},k^z,\theta;\vec{k}_1,\vec{k}_2\right)$ is given by

$$\begin{aligned}\mathcal{K}\left(\vec{k},k^z,\theta;\vec{k}_1,\vec{k}_2\right) &= \mathcal{J}\delta^2\left(\vec{k}_1+\vec{k}_2-\vec{k}\right)\delta\left(2\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right)-k^z\right)\\ &\quad\times\delta\left(\arctan\left(\frac{2\vec{k}_2\times\vec{k}_1}{(|\vec{k}_2|-|\vec{k}_1|)k^z}\right)-\theta\right),\end{aligned}$$

where

$$\mathcal{J} = \left|\frac{\partial\left(\vec{k},k^z,\theta\right)}{\partial\left(k_1^1,k_1^2,k_2^1,k_2^2\right)}\right| = \frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}.$$

Using the partial derivatives[136] (noting that $\sqrt{(k^z)^2+\vec{k}^2} = |\vec{k}_1|+|\vec{k}_2| = k^0$),

$$\begin{aligned}
\frac{\partial k_1^1}{\partial k^1} &= \frac{|\vec{k}_1|}{|\vec{k}_1|+|\vec{k}_2|}+\frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|+k_1^1k_2^1-k_1^2k_2^2\right)}k^2}{\vec{k}^2k^0}\\
\frac{\partial k_1^2}{\partial k^1} &= \frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|-k_1^1k_2^1+k_1^2k_2^2\right)}k^2}{\vec{k}^2k^0}\\
\frac{\partial k_2^1}{\partial k^1} &= \frac{|\vec{k}_2|}{|\vec{k}_1|+|\vec{k}_2|}-\frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|+k_1^1k_2^1-k_1^2k_2^2\right)}k^2}{\vec{k}^2k^0}\\
\frac{\partial k_2^2}{\partial k^1} &= -\frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|-k_1^1k_2^1+k_1^2k_2^2\right)}k^2}{\vec{k}^2k^0}\\
\frac{\partial k_1^1}{\partial k^2} &= -\frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|+k_1^1k_2^1-k_1^2k_2^2\right)}k^1}{\vec{k}^2k^0}\\
\frac{\partial k_1^2}{\partial k^2} &= \frac{|\vec{k}_1|}{|\vec{k}_1|+|\vec{k}_2|}-\frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|-k_1^1k_2^1+k_1^2k_2^2\right)}k^1}{\vec{k}^2k^0}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial k_2^1}{\partial k^2} &= \frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| + k_1^1k_2^1 - k_1^2k_2^2\right)}k^1}{\vec{k}^2k^0} \\
\frac{\partial k_2^2}{\partial k^2} &= \frac{|\vec{k}_2|}{|\vec{k}_1| + |\vec{k}_2|} + \frac{k^z\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| - k_1^1k_2^1 + k_1^2k_2^2\right)}k^1}{\vec{k}^2k^0} \\
\frac{\partial k_1^1}{\partial k^z} &= -\frac{1}{|\vec{k}_1| + |\vec{k}_2|}\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| - k_1^1k_2^1 + k_1^2k_2^2\right)} \\
\frac{\partial k_1^2}{\partial k^z} &= \frac{1}{|\vec{k}_1| + |\vec{k}_2|}\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| + k_1^1k_2^1 - k_1^2k_2^2\right)} \\
\frac{\partial k_2^1}{\partial k^z} &= \frac{1}{|\vec{k}_1| + |\vec{k}_2|}\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| - k_1^1k_2^1 + k_1^2k_2^2\right)} \\
\frac{\partial k_2^2}{\partial k^z} &= -\frac{1}{|\vec{k}_1| + |\vec{k}_2|}\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| + k_1^1k_2^1 - k_1^2k_2^2\right)} \\
\frac{\partial k_1^1}{\partial \theta} &= \sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| + k_1^1k_2^1 - k_1^2k_2^2\right)} \\
\frac{\partial k_1^2}{\partial \theta} &= \sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| - k_1^1k_2^1 + k_1^2k_2^2\right)} \\
\frac{\partial k_2^1}{\partial \theta} &= -\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| + k_1^1k_2^1 - k_1^2k_2^2\right)} \\
\frac{\partial k_2^2}{\partial \theta} &= -\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| - k_1^1k_2^1 + k_1^2k_2^2\right)},
\end{aligned}
$$

the map in (4.26) may be used to find the $\mathrm{AdS}_4 \times S^1$ coordinates in terms of bi-local variables, for example

$$
\begin{aligned}
k^\theta\mathcal{H}\left(\vec{k}, k^z, \theta\right) &= -\frac{\partial}{\partial\theta}\mathcal{H}\left(\vec{k}, k^z, \theta\right) \\
&= \int d\vec{k}_1 d\vec{k}_2 \mathcal{K}\left(\vec{k}, k^z, \theta; \vec{k}_1, \vec{k}_2\right)\left(\frac{\partial\vec{k}_1}{\partial\theta}\cdot\vec{x}_1 + \frac{\partial\vec{k}_2}{\partial\theta}\cdot\vec{x}_2\right)\Psi\left(\vec{k}_1, \vec{k}_2\right).
\end{aligned}
$$

Now, these partial derivatives may be written in terms of trigonometric functions as follows,

$$
\begin{aligned}
\frac{\partial k_1^1}{\partial\theta} &= \sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2| + k_1^1k_2^1 - k_1^2k_2^2\right)} \\
&= \sqrt{\frac{1}{2}|\vec{k}_1||\vec{k}_2|}\sqrt{(1 + \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2)} \\
&= \sqrt{\frac{1}{2}|\vec{k}_1||\vec{k}_2|}\sqrt{\left(1 + \cos\left(\frac{\varphi_1 + \varphi_2}{2}\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{\frac{1}{2}|\vec{k}_1||\vec{k}_2|}\sqrt{\left(1+\cos^2\left(\frac{\varphi_1+\varphi_2}{2}\right)-\sin^2\left(\frac{\varphi_1+\varphi_2}{2}\right)\right)} \\
&= \sqrt{|\vec{k}_1||\vec{k}_2|}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right),
\end{aligned}
$$

and thus

$$
k^\theta = \sqrt{|\vec{k}_1||\vec{k}_2|}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\left(x_2^1-x_1^1\right)+\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\left(x_2^2-x_1^2\right).
$$

Similarly,

$$
\begin{aligned}
x^1 &= \frac{|\vec{k}_1|x_1^1+|\vec{k}_2|x_2^1}{k^0}-\frac{k^2k^zk^\theta}{\vec{k}^2k^0} \\
x^2 &= \frac{|\vec{k}_1|x_1^2+|\vec{k}_2|x_2^2}{k^0}+\frac{k^1k^zk^\theta}{\vec{k}^2k^0} \\
z &= \frac{(\vec{x}_1-\vec{x}_2)\cdot\vec{k}_1|\vec{k}_2|-(\vec{x}_1-\vec{x}_2)\cdot\vec{k}_2|\vec{k}_1|}{k^zk^0}.
\end{aligned}
$$

One can do the same for the generators, to obtain

$$
\begin{aligned}
P^t_{\rm AdS} &= \sqrt{\vec{k}^2+(k^z)^2} \\
P^x_{\rm AdS} &= k^1 \\
P^y_{\rm AdS} &= k^2 \\
J^{xy}_{\rm AdS} &= x^1k^2-x^2k^1 \\
J^{tx}_{\rm AdS} &= -x^1k^0-\frac{k^2k^zk^\theta}{\vec{k}^2} \\
J^{yt}_{\rm AdS} &= x^2k^0-\frac{k^1k^zk^\theta}{\vec{k}^2} \\
D_{\rm AdS} &= x^1k^1+x^2k^2+zp^z \\
K^t_{\rm AdS} &= -\frac{1}{2}\left(\vec{x}^2+z^2\right)k^0\frac{k^zk^\theta}{\vec{k}^2}J^{xy}-\frac{k^0(k^\theta)^2}{2\vec{k}^2} \\
K^x_{\rm AdS} &= -\frac{1}{2}\left(\vec{x}^2+z^2\right)k^1+x^1D+\frac{zk^2k^0k^\theta}{\vec{k}^2}+\frac{k^1(k^\theta)^2}{2\vec{k}^2} \\
K^y_{\rm AdS} &= -\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2+x^2D+\frac{zk^1k^0k^\theta}{\vec{k}^2}+\frac{k^2(k^\theta)^2}{2\vec{k}^2}.
\end{aligned}
\tag{4.27}
$$

Chapter 5

A First Principles Construction of the Bulk

In this thesis, we investigate if, in any way, the map developed in Ref. [10] needs adjustment or if it applies to the higher spin/CFT correspondence at the IR critical point.

In order to answer this question, we arrive at what can be called a first principles construction ('derivation') of the bulk, in momentum space, which simply (conceptually, but not necessarily technically) requires:

- a change of variables from the original variables of the theory to bilocal fields using the Collective Field method;
- a change of variables from bilocal to AdS and spin momentum variables accompanied by a field redefinition;
- and the use of the chain rule.

This approach is different, but as we will show, equivalent to the approach developed in Ref. [10] for the full conformal algebra, at the quadratic level.

Moreover, the critical scattering states, which are the exact eigenstates of the interacting theory and which, on the boundary, have an extremely non-trivial form, are shown, when expressed in bulk variables, to correspond precisely to the removal of the $s = 0$ ($\Delta = 1$) state from the spectrum. This highly non-trivial result confirms the applicability of the method described above.

5.1 Change of variables, field redefinition and Hamiltonian in the bulk

The conformal generator P^t is given by

$$\begin{aligned} P^t = H &= \int d\vec{z}\,\mathcal{H} \\ &= \int d^2z\,\frac{1}{2}\pi^2(\vec{z}) + V \\ &= \int d^2z\,\frac{1}{2}\frac{\partial^2}{\partial\phi_C(\vec{z})\partial\phi_C(\vec{z})} + V, \end{aligned}$$

which we recognise from equation (2.14) (ignoring V) to correspond to

$$P^t = -\frac{1}{2}\partial_\alpha\Omega_{\alpha\beta}\partial_\beta + \frac{1}{8}\left(\partial_\alpha \ln J\right)\Omega_{\alpha\beta}\left(\partial_\beta \ln J\right).$$

Although the key elements of this calculation have been done in chapters 2 and 3, we review it schematically here as this will be useful in understanding other conformal generators. We have from equation (2.18) that

$$\begin{aligned} \ln J &= \frac{N}{2}\mathrm{tr}\ln\psi \\ \Rightarrow \partial_\alpha \ln J &= \frac{\partial \ln J}{\partial\psi(\vec{x},\vec{y})} = \frac{N}{2}\psi^{-1}_{\vec{y}\vec{x}}, \end{aligned}$$

and further $\partial_\alpha \equiv \frac{\delta}{\delta\Phi(\vec{x},\vec{y})} = i\pi_{\vec{x}\vec{y}}$; therefore

$$P^t = \int d^2x \int d^2y \int d^2x' \int d^2y'\,\frac{1}{2}\pi_{\vec{x}\vec{y}}\Omega_{\vec{x}\vec{y};\vec{x}'\vec{y}'}\pi_{\vec{x}'\vec{y}'} + \frac{N^2}{32}\psi^{-1}_{\vec{y}\vec{x}}\Omega_{\vec{x}\vec{y};\vec{x}'\vec{y}'}\psi^{-1}_{\vec{y}'\vec{x}'}. \quad (5.1)$$

Now, from equation (2.10) we have that

$$\begin{aligned} \Omega_{\vec{x}\vec{y};\vec{x}'\vec{y}'} = &\big[\psi(\vec{x},\vec{x}')\delta^2(\vec{y}-\vec{y}') + \psi(\vec{x},\vec{y}')\delta^2(\vec{y}-\vec{x}') \\ &+\psi(\vec{y},\vec{x}')\delta^2(\vec{x}-\vec{y}') + \psi(\vec{y},\vec{y}')\delta^d(\vec{x}-\vec{x}')\big] \end{aligned}$$

and therefore

$$P^t = \mathrm{Tr}\pi\psi\pi + \mathrm{Tr}\pi\overbrace{\pi\psi} + \tfrac{N^2}{8}\mathrm{Tr}\psi^{-1}.$$

Making use of the fluctuations as per equation (3.16) and the comments leading up to it (where to leading order in momentum space, ψ is diagonal and hence commutative), we obtain

$$P^t = 2\mathrm{Tr}\pi\psi_0\pi + \frac{1}{8}\mathrm{Tr}(\psi_0^{-2})\eta(\psi_0^{-1})\eta(\psi_0^{-1}), \tag{5.2}$$

equivalent to what was found in equation (3.17); here, however, the trace is taken schematically as an integral over coordinates $\vec{x}, \vec{y}, \vec{x}', \vec{y}'$ and not momenta. We rewrite $\eta_{\vec{x}\vec{y}}$ and $\pi_{\vec{x}\vec{y}}$ using the Fourier transform,

$$\begin{aligned}
\eta_{\vec{x}\vec{y}} &= \int \frac{d^2k_1}{2\pi}\int\frac{d^2k_2}{2\pi}\eta_{\vec{k}_1\vec{k}_2}e^{i\vec{k}_1\cdot\vec{x}}e^{i\vec{k}_2\cdot\vec{y}}\\
\pi_{\vec{x}\vec{y}} &= \int \frac{d^2k_1}{2\pi}\int\frac{d^2k_2}{2\pi}\pi_{\vec{k}_1\vec{k}_2}e^{i\vec{k}_1\cdot\vec{x}}e^{i\vec{k}_2\cdot\vec{y}}\\
(\psi_0)_{\vec{x}\vec{y}} &= \int \frac{d^2p}{(2\pi)^2}\psi^0_{\vec{k}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}.
\end{aligned} \tag{5.3}$$

The first term in equation (5.2), written schematically as $\pi_{xy}(\psi_0)_{yy'}\pi_{y'x}$, is thus given as follows:

$$\int\frac{d^2k_1}{2\pi}\int\frac{d^2k_2}{2\pi}\int\frac{d^2k_3}{2\pi}\int\frac{d^2k_4}{2\pi}\int\frac{d^2p}{(2\pi)^2}\pi_{\vec{k}_1\vec{k}_2}\psi^0_{\vec{p}}\pi_{\vec{k}_3\vec{k}_4}$$
$$\int d^2x\int d^2y\int d^2y' e^{i\vec{k}_1\cdot\vec{k}}e^{i\vec{k}_2\cdot\vec{y}}e^{i\vec{p}\cdot(\vec{y}-\vec{y}')}e^{i\vec{k}_3\cdot\vec{y}'}e^{i\vec{k}_4\cdot\vec{x}}.$$

Integrating over the $\vec{x}, \vec{y}, \vec{y}'$ followed by the $\vec{k}_3, \vec{k}_4, \vec{p}$ cancels all denominator factors of (2π) and sets $\vec{k}_4 \to -\vec{k}_1$, $\vec{p} \to -\vec{k}_2$ and $\vec{k}_3 \to \vec{p} \to -\vec{k}_2$, resulting in

$$2\mathrm{Tr}\pi\psi^0\pi = 2\int d^2k_1\int d^2k_2\pi_{\vec{k}_1\vec{k}_2}\psi^0_{-\vec{k}_2}\pi_{-\vec{k}_2-\vec{k}_1}.$$

Similarly, the second term in equation (5.2), which can be written schematically as $(\psi_0)^{-2}_{\vec{x}_1\vec{x}_2}\eta_{\vec{x}_2\vec{x}_3}(\psi_0)^{-1}_{\vec{x}_3\vec{x}_4}\eta_{\vec{x}_4\vec{x}_1}$ and is given by

$$\int\frac{d^2k_1}{2\pi}\int\frac{d^2k_2}{2\pi}\int\frac{d^2k_3}{2\pi}\int\frac{d^2k_4}{2\pi}\int\frac{d^2p_1}{(2\pi)^2}\int\frac{d^2p_2}{(2\pi)^2}(\psi^0)^{-2}_{\vec{p}_1}\eta_{\vec{k}_1\vec{k}_2}(\psi^0_{\vec{p}_2})^{-1}\eta_{\vec{k}_3\vec{k}_4}$$
$$\int d^2x_1\int d^2x_2\int d^2x_3\int d^2x_4 e^{i\vec{p}_1\cdot(\vec{x}_1-\vec{x}_2)}e^{i\vec{k}_1\cdot\vec{x}_2}e^{i\vec{k}_2\cdot\vec{x}_3}e^{i\vec{p}_2\cdot(\vec{x}_3-\vec{x}_4)}e^{i\vec{k}_3\cdot\vec{x}_4}e^{i\vec{k}_4\cdot\vec{x}_1}.$$

Here $\vec{p}_1 \to \vec{k}_1$, $\vec{k}_4 \to -\vec{p}_1 \to -\vec{k}_1$, $\vec{p}_2 \to -\vec{k}_2$ and $\vec{k}_3 \to \vec{p}_2 \to -\vec{k}_2$, and therefore, symmetrising, we obtain

$$\begin{aligned} &\frac{1}{16}\mathrm{Tr}(\psi_0^{-2})\eta(\psi_0^{-1})\eta(\psi_0^{-1}) \\ = &\int \frac{d^2k_1}{2\pi}\int \frac{d^2k_2}{2\pi}\eta_{\vec{k}_1\vec{k}_2}\left[(\psi^0_{\vec{k}_1})^{-2}(\psi^0_{\vec{k}_2})^{-1} + (\psi^0_{\vec{k}_2})^{-2}(\psi^0_{\vec{k}_1})^{-1}\right]\eta_{-\vec{k}_2-\vec{k}_1}\end{aligned}$$

and the symmetrised quadratic collective Hamiltonian is therefore given by (with vector notation),

$$\begin{aligned} H_2 \;=\; &\int d^2k_1 \int d^2k_2 \left(\pi_{\vec{k}_1\vec{k}_2}\left(\psi^0_{\vec{k}_1} + \psi^0_{\vec{k}_2}\right)\pi_{-\vec{k}_2-\vec{k}_1}\right) \\ &+\frac{1}{16}\int d^2k_1 \int d^2k_2\, \eta_{\vec{k}_1\vec{k}_2}\left((\psi^0_{\vec{k}_1})^{-2}(\psi^0_{\vec{k}_2})^{-1} + (\psi^0_{\vec{k}_2})^{-2}(\psi^0_{\vec{k}_1})^{-1}\right)\eta_{-\vec{k}_2-\vec{k}_1}.\end{aligned}$$

At the conformal fixed point where masses drop away, we found that

$$\psi_{0\vec{k}} = \frac{1}{2|\vec{k}|},$$

for which the Hamiltonian becomes

$$\begin{aligned} H_2 \;=\; &\int d^2k_1 \int d^2k_2\, \frac{1}{2}\left(\pi_{\vec{k}_1\vec{k}_2}\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\pi_{-\vec{k}_2-\vec{k}_1}\right) \\ &+\frac{1}{2}\eta_{\vec{k}_1\vec{k}_2}\left(|\vec{k}_1|^2|\vec{k}_2| + |\vec{k}_2|^2|\vec{k}_1|\right)\eta_{-\vec{k}_2-\vec{k}_1},\end{aligned}$$

and we notice that

$$\begin{aligned} \dot{\eta}_{\vec{k}_1\vec{k}_2} &= \frac{\delta H_2}{\delta \pi_{\vec{k}_1\vec{k}_2}} = \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\pi_{-\vec{k}_2-\vec{k}_1} \\ \dot{\pi}_{-\vec{k}_2-\vec{k}_1} &= -\frac{\delta H_2}{\delta \eta_{-\vec{k}_2-\vec{k}_1}} = -\left(|\vec{k}_1|^2|\vec{k}_2| + |\vec{k}_2|^2|\vec{k}_1|\right)\eta_{\vec{k}_1\vec{k}_2} \\ \Rightarrow \ddot{\eta}_{\vec{k}_1\vec{k}_2} &= -\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\left(|\vec{k}_1|^2|\vec{k}_2| + |\vec{k}_2|^2|\vec{k}_1|\right)\eta_{\vec{k}_1\vec{k}_2} \\ &= -\left(|\vec{k}_1| + |\vec{k}_2|\right)^2\eta_{\vec{k}_1\vec{k}_2},\end{aligned} \tag{5.4}$$

from which we can immediately read off that the energy in bilocal coordinates is given by

$$P^t = P^0 = \left(|\vec{k}_1| + |\vec{k}_2|\right) = k^0$$

and in AdS is defined from the on-shell condition (equation (4.24)),

$$P^0_{AdS} = \sqrt{\vec{k}^2 + (k^z)^2}$$

Furthermore, $\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)$ is the same expression as was obtained for the Jacobian for the change of variables from bilocals to AdS coordinates in Ref. [10],

$$\left|\frac{\partial k_{AdS\times S^1}}{\partial k_{\rm bilocal}}\right| = \frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}. \tag{5.5}$$

Now in this canonical formulation, since we have the general property written schematically as

$$\begin{aligned} & \int \delta(x - x_0)dx = \int \delta(f(x) - f(x_0))df(x) = \int \delta(f(x) - f(x_0))f'(x)dx \\ \Rightarrow \quad & \delta(x - x_0) = \delta(f(x) - f(x_0))\det\left|f'(x)\right| \end{aligned}$$

we require that

$$\begin{aligned} \left[\pi_{\vec{k}_1\vec{k}_2}, \eta_{\vec{k}'_1\vec{k}'_2}\right] &= -i\delta^2(\vec{k}_1 - \vec{k}'_1)\delta^2(\vec{k}_2 - \vec{k}'_2) \\ &= -i\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\delta(\vec{k}_{AdS\times S^1} - \vec{k}'_{AdS\times S^1}) \\ \Rightarrow \frac{\left[\pi_{\vec{k}_1\vec{k}_2}, \eta_{k'_1k'_2}\right]}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}} &= -i\delta(\vec{k}_{AdS\times S^1} - \vec{k}'_{AdS\times S^1}) \end{aligned}$$

This suggests that we redefine the fields as follows (from equation (5.4) we see that it is η that should be rescaled and not π):

$$\begin{aligned} \mathcal{H}\left(\vec{k}.k^z, \theta\right) &\equiv \left.\frac{\eta_{\vec{k}_1\vec{k}_2}}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right|_{\vec{k}_1,\vec{k}_2\left(\vec{k},k^z,\theta\right)} \\ \Pi_{\mathcal{H}}\left(\vec{k}.k^z, \theta\right) &= \left.\pi_{\vec{k}_1\vec{k}_2}\right|_{\vec{k}_1,\vec{k}_2\left(\vec{k},k^z,\theta\right)}. \end{aligned}$$

The potential term may be massaged as follows:

$$\eta_{\vec{k}_1\vec{k}_2}\left(|\vec{k}_1|^2|\vec{k}_2| + |\vec{k}_2|^2|\vec{k}_1|\right)\eta_{-\vec{k}_2-\vec{k}_1}$$

$$\begin{aligned}
&= \left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\frac{\eta_{\vec{k}_1\vec{k}_2}\left(|\vec{k}_1|^2|\vec{k}_2|+|\vec{k}_2|^2|\vec{k}_1|\right)\eta_{-\vec{k}_2-\vec{k}_1}}{\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)} \\
&= \frac{\eta_{\vec{k}_1\vec{k}_2}\left(|\vec{k}_1|+|\vec{k}_2|\right)^2\eta_{-\vec{k}_2-\vec{k}_1}}{\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)} \\
&= \left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\frac{\eta_{\vec{k}_1\vec{k}_2}\left(|\vec{k}_1|+|\vec{k}_2|\right)^2\eta_{-\vec{k}_2-\vec{k}_1}}{\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)^2}
\end{aligned}$$

implying that we can write the collective quadratic Hamiltonian as an integral over the AdS coordinates by first factorising out this Jacobian from both expressions,

$$\begin{aligned}
H_2 &= \int d^2k_1\int d^2k_2\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\frac{1}{2}\left[\left(\pi_{\vec{k}_1\vec{k}_2}\pi_{-\vec{k}_2-\vec{k}_1}\right)\right. \\
&\left.+\frac{\eta_{\vec{k}_1\vec{k}_2}\left(|\vec{k}_1|+|\vec{k}_2|\right)^2\eta_{-\vec{k}_2-\vec{k}_1}}{\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)}\right]. \qquad (5.6)
\end{aligned}$$

From [10], $|\vec{k}_1|+|\vec{k}_2|=(P^0)_{AdS}=\sqrt{\vec{k}^2+(k^z)^2}=k^0$, and therefore, using equation (5.5), this Hamiltonian may be written as an integral over $AdS\times S^1$ coordinates as follows,

$$\frac{1}{2}\int d\vec{k}_{AdS\times S^1}\ \left[\left(\Pi_{\mathcal{H}}(\vec{k},k^z,\theta)\Pi_{\mathcal{H}}(-\vec{k},-k^z,-\theta)\right)+\left(P^0\right)^2\mathcal{H}(\vec{k},k^z,\theta)\mathcal{H}(-\vec{k},-k^z,-\theta)\right].$$

We define $\vec{\kappa}\equiv\vec{k}_{AdS\times S^1}$, and may therefore write

$$H_2=\frac{1}{2}\int d\vec{\kappa}\left[\Pi(\vec{\kappa})\Pi(-\vec{\kappa})+(P^0)^2\mathcal{H}(\vec{\kappa})\mathcal{H}(-\vec{\kappa})\right].$$

This may be mode expanded using

$$\begin{aligned}
\mathcal{H}_{\vec{z}} &= \int\frac{d\vec{\kappa}}{(2\pi)^2}\frac{1}{\sqrt{2P^0}}\left(e^{-iP^0t}e^{i\vec{\kappa}\cdot\vec{z}}a_{\vec{\kappa}}+e^{iP^0t}e^{-i\vec{\kappa}\cdot\vec{z}}a^\dagger_{\vec{\kappa}}\right) \\
\Rightarrow\mathcal{H}(\vec{\kappa}) &= \frac{1}{\sqrt{2P^0}}\left(a_{\vec{\kappa}}(t)+a^\dagger_{-\vec{\kappa}}(t)\right) \\
&= \frac{1}{\sqrt{2P^0}}\left(a_{\vec{\kappa}}+a^\dagger_{-\vec{\kappa}}\right) \\
\Pi_{\vec{z}} &= \int\frac{d\vec{\kappa}}{(2\pi)^2}\left(-i\sqrt{\frac{P^0}{2}}\right)\left(e^{-iP^0t}e^{i\vec{\kappa}\cdot\vec{z}}a_{\vec{\kappa}}-e^{iP^0t}e^{-i\vec{\kappa}\cdot\vec{z}}a^\dagger_{\vec{\kappa}}\right)
\end{aligned}$$

$$\begin{aligned}\Rightarrow \Pi(\vec{\kappa}) &= -\sqrt{\frac{P^0}{2}}i\left(a_{\vec{\kappa}}(t) - a^\dagger_{-\vec{\kappa}}(t)\right) \\ &= -\sqrt{\frac{P^0}{2}}i\left(a_{\vec{\kappa}} - a^\dagger_{-\vec{\kappa}}\right)\end{aligned}$$

to give

$$\begin{aligned}H_2 &= \frac{1}{2}\int d\vec{\kappa}\left[\Pi(\vec{\kappa})\Pi(-\vec{\kappa}) + (P^0)^2\mathcal{H}(\vec{\kappa})\mathcal{H}(-\vec{\kappa})\right] \\ &= \frac{1}{2}\int d\vec{\kappa}\left[\left[-\sqrt{\frac{P^0}{2}}i\left(a_{\vec{\kappa}} - a^\dagger_{-\vec{\kappa}}\right)\right]\left[-\sqrt{\frac{P^0}{2}}i\left(a_{-\vec{\kappa}} - a^\dagger_{\vec{\kappa}}\right)\right]\right. \\ &\quad \left. +(P^0)^2\left[\frac{1}{\sqrt{2P^0}}\left(a_{\vec{\kappa}} + a^\dagger_{-\vec{\kappa}}\right)\right]\left[\frac{1}{\sqrt{2P^0}}\left(a_{-\vec{\kappa}} + a^\dagger_{\vec{\kappa}}\right)\right]\right] \\ &= \int d\vec{\kappa}\; P^0\left[a_{\vec{\kappa}}a^\dagger_{\vec{\kappa}} + a^\dagger_{\vec{\kappa}}a_{\vec{\kappa}}\right]. \qquad (5.7)\end{aligned}$$

5.2 Scattering States in the Bulk

In equation (3.24), the universal/critical scattering states, independent of λ as $\lambda \to \infty$ were found to be

$$\eta^{\vec{p}_1,\vec{p}_2}_{\vec{k}_1,\vec{k}_2} = \delta^{d-1}(\vec{p}_1 - \vec{k}_1)\delta^{d-1}(\vec{p}_2 - \vec{k}_2) + \frac{|p|_E}{\pi^2}\frac{\delta^{d-1}(\vec{p}_1 + \vec{p}_2 - \vec{k}_1 - \vec{k}_2)}{E^2_{\vec{p}_1\vec{p}_2} - \left(|\vec{k}_1| + |\vec{k}_2|\right)^2}\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right).$$

Here we note that $\vec{k}_1$ and $\vec{k}_2$ are momentum 'coordinates' and that $\vec{p}_1$ and $\vec{p}_2$ label the energy eigenfunctions with energies

$$E^2_{\vec{p}_1\vec{p}_2} = (|\vec{p}_1| + |\vec{p}_2|)^2 .$$

We further recall notationally that in Minkowski coordinates, $p^\mu \equiv (|\vec{p}_1| + |\vec{p}_2|, \vec{p}_1 + \vec{p}_2)$, so that $|p|_E$ may be defined as

$$|p|_E = \sqrt{E_E^2 + \vec{p}^2} = \sqrt{-E^2 + \vec{p}^2} = i\sqrt{E^2 - \vec{p}^2}.$$

Moreover, we see that

$$|p|_E = = i\sqrt{(|p_1| + |p_2|)^2 - (\vec{p}_1 + \vec{p}_2)\cdot(\vec{p}_1 + \vec{p}_2)} \equiv i|p^z|.$$

We may therefore expand the universal / critical scattering states in their energy eigenstates as follows:

$$\begin{aligned}\eta_{\vec{k}_1\vec{k}_2} &= \int d\vec{p}_1 \int d\vec{p}_2 \eta^{\vec{p}_1,\vec{p}_2}_{\vec{k}_1,\vec{k}_2} \psi_{\vec{p}_1\vec{p}_2} \\ &= \psi_{\vec{k}_1\vec{k}_2} + \frac{i}{\pi^2}\int d\vec{p}_1 \int d\vec{p}_2 \, |p^z| \, \frac{\delta^{d-1}(\vec{p}_1+\vec{p}_2-\vec{k}_1-\vec{k}_2)\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)}{E^2_{\vec{p}_1\vec{p}_2} - \left(|\vec{k}_1|+|\vec{k}_2|\right)^2}\psi_{\vec{p}_1\vec{p}_2}.\end{aligned}$$

To change to higher spin variables, we make the identification

$$\begin{aligned}H(\vec{\kappa}) &= \frac{\eta_{\vec{k}_1\vec{k}_2}}{\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)} \\ h(\vec{\kappa}) &= \frac{\psi_{\vec{k}_1\vec{k}_2}}{\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)},\end{aligned}$$

resulting in all the terms being multiplied by a factor of $\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)$ when writing in terms of AdS coordinates; the additional factor of $\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)$ in the last term allows for the bilocal integral to be converted into an AdS integral. Hence we obtain

$$\begin{aligned}H(\vec{\kappa}) &= h(\vec{\kappa}) + \frac{i}{\pi^2}\int d\vec{p}\int dp^z \int d\theta \, |p^z| \, \frac{\delta^2(\vec{p}-\vec{k})}{(p^z)^2+(\vec{p})^2-(k^z)^2-(\vec{k})^2} h(p^z,\vec{p},\theta) \\ &= h(k^z,\vec{k},\theta_{\vec{k}}) + \frac{i}{\pi^2}\int dp^z \int d\theta \frac{|p^z|}{(p^z)^2-(k^z)^2} h(p^z,\vec{k},\theta).\end{aligned}$$

Under mild assumptions of the behaviour of $h(p^z,\vec{k},\theta)$ as $|p^z| \to \infty$ where we also require that $h(-k^z) = h(k^z)$,

$$\int dp^z \frac{|p^z| \, h(p^z,\vec{k},\theta)}{(p^z)^2-(k^z)^2-i\epsilon} = i\pi h(k^z,\vec{k},\theta),$$

so that

$$H(k^z,\vec{k},\theta_{\vec{k}}) = h(k^z,\vec{k},\theta_{\vec{k}}) - \frac{1}{\pi}\int_0^\pi d\theta \; h(k^z,\vec{k},\theta_{\vec{k}}). \tag{5.8}$$

Using

$$h(k^z,\vec{k},\theta) = \sum_{s=0}^{\infty} \cos(s\theta) h_s(k^z,\vec{k})$$

for $0 < \theta < \pi$. The latter term in equation (5.8) removes the $s = 0$ term. Since $\Delta = E = s + 1$, $s = 0$ being removed is the same as removing $\Delta = 1$ state, as seen in the previous observations. This gives great credibility to the identification. The interacting field may thus be written as

$$H(k^z, \vec{k}, \theta) = \sum_{s \neq 0} h_s(k^z, \vec{k}) \cos(s\theta).$$

It is then important that this approach is shown to be equivalent to Ref. [10]. The following procedure yields the desired AdS generators (dropping vector signs for notational simplification):

1. The generators are calculated according to the formulae which were found in subsection 4.3.2 to be (where expressions are first written in terms of the original $O(N)$ invariant vector fields):

$$\begin{aligned}
P^t &= \int dx\, dy\, \mathcal{H} \\
P^x &= \int dx\, dy\, \pi \partial_x \phi \\
P^y &= \int dx\, dy\, \pi \partial_y \phi \\
M^{xy} &= \int dxdy\ - (\pi \partial_x \phi)\, y + (\pi \partial_y \phi)\, x \\
M^{tx} &= \int dxdy\ \mathcal{H}x + (\pi \partial_x \phi)\, t \\
M^{ty} &= \int dxdy\ \mathcal{H}y + (\pi \partial_y \phi)\, t \\
D &= Ht + \int dxdy\, [(\pi \partial_x \phi)\, x + (\pi \partial_y \phi)\, y + \pi d_\phi \phi] \\
K^t &= \int dxdy\, \left[2t\mathcal{D} - \pi^2 t^2 + \mathcal{H}x^2\right] \\
K^x &= \int dxdy\, [2x\mathcal{D}] \\
K^y &= \int dxdy\, [2y\mathcal{D}]\, ,
\end{aligned}$$

   where we set $t = 0$.

2. The expression is then Fourier transformed to give generators in terms of momentum space vectors k_1 and k_2; this step may involve first obtaining a

Collective field description, or expanding using the chain rule.

3. A change of variables $k_{\text{orig}} \to k_{\text{new}}$ is performed given by $k_{\text{bilocal}} \to k_{\text{AdS}\times\text{CFT}}$ with associated Jacobian given by

$$\begin{aligned} \left|\frac{\partial k_{\text{new}}}{\partial k_{\text{orig}}}\right| &= \left|\frac{\partial k_{\text{AdS}\times S^1}}{\partial k_{\text{bilocal}}}\right| \\ \Rightarrow \qquad \int dk_{\text{bilocal}} \left|\frac{\partial k_{\text{AdS}\times S^1}}{\partial k_{\text{bilocal}}}\right| &= \int dk_{\text{AdS}\times S^1} = \int d\vec{\kappa}. \end{aligned}$$

Additional terms that arise due to this change of variables need to be dealt with appropriately.

4. Second quantisation is performed if necessary, that is, the expression is expanded in terms of $a_{\vec{\kappa}}$ and $a^\dagger_{\vec{\kappa}}$ and compare the expression of the AdS generator with those in equations (4.27)[10]:

$$\begin{aligned}
P^t_{\text{AdS}} &= \sqrt{\vec{k}^2 + (k^z)^2} \\
P^x_{\text{AdS}} &= k^1 \\
P^y_{\text{AdS}} &= k^2 \\
J^{xy}_{\text{AdS}} &= x^1 k^2 - x^2 k^1 \\
J^{tx}_{\text{AdS}} &= -x^1 k^0 - \frac{k^2 k^z k^\theta}{\vec{k}^2} \\
J^{yt}_{\text{AdS}} &= x^2 k^0 - \frac{k^1 k^z k^\theta}{\vec{k}^2} \\
D_{\text{AdS}} &= x^1 k^1 + x^2 k^2 + z k^z \\
K^t_{\text{AdS}} &= -\frac{1}{2}\left(\vec{x}^2 + z^2\right) k^0 - \frac{k^z k^\theta}{\vec{k}^2} J^{xy} - \frac{k^0 (k^\theta)^2}{2\vec{k}^2} \\
K^x_{\text{AdS}} &= -\frac{1}{2}\left(\vec{x}^2 + z^2\right) k^1 + x^1 D + \frac{z k^2 k^0 k^\theta}{\vec{k}^2} + \frac{k^1 (k^\theta)^2}{2\vec{k}^2} \\
K^y_{\text{AdS}} &= -\frac{1}{2}\left(\vec{x}^2 + z^2\right) k^2 + x^2 D + \frac{z k^1 k^0 k^\theta}{\vec{k}^2} + \frac{k^2 (k^\theta)^2}{2\vec{k}^2}.
\end{aligned}$$

5.3 Bulk Conformal Algebra of the Free Theory

In this subsection we wish to derive the conformal generators. In order to calculate the conformal generators, we shall require partial derivatives the opposite to those which we used in Ref. [10]. According to the map, the AdS variables are given in terms of the bilocal

$$
\begin{aligned}
k^1 &= k_1^1 + k_2^1 \\
k^2 &= k_1^2 + k_2^2 \\
k^z &= \sqrt{2|\vec{k}_1||\vec{k}_2| - 2\vec{k}_1\vec{k}_2} \\
&= \sqrt{2\sqrt{(k_1^1)^2 + (k_1^2)^2}\sqrt{(k_2^1)^2 + (k_2^2)^2} - 2\left(k_1^1 k_2^1 + k_1^2 k_2^2\right)} \\
\theta &= \arctan\left(\frac{2\vec{p}_2 \times \vec{p}_1}{\left(|\vec{p}_2| - |\vec{p}_1|\right) p^z}\right).
\end{aligned}
$$

The k^z partial derivatives are given by

$$
\begin{aligned}
(k^z)^2 &= 2\sqrt{(k_1^1)^2 + (k_1^2)^2}\sqrt{(k_2^1)^2 + (k_2^2)^2} - 2\left(k_1^1 k_2^1 + k_1^2 k_2^2\right) \\
\Rightarrow 2k^z \frac{\partial k^z}{\partial k_1^1} &= 2\left(k_1^1 \frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right) \\
\Rightarrow \frac{\partial k^z}{\partial k_1^1} &= \frac{1}{k^z}\left(k_1^1 \frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right)
\end{aligned}
$$

and so forth. We can simplify the expression for θ as follows:

$$
\begin{aligned}
\tan\theta &= \frac{2|\vec{k}_1||\vec{k}_2|\sin(\varphi_2 - \varphi_1)}{\left(|\vec{k}_1| - |\vec{k}_2|\right)\sqrt{2\left(|\vec{k}_1||\vec{k}_2| - |\vec{k}_1||\vec{k}_2|\cos(\varphi_2 - \varphi_1)\right)}} \\
&= \frac{2|\vec{k}_1||\vec{k}_2| 2\sin\left(\frac{\varphi_2-\varphi_1}{2}\right)\cos\left(\frac{\varphi_2-\varphi_1}{2}\right)}{\left(|\vec{k}_1| - |\vec{k}_2|\right) 2\sin\left(\frac{\varphi_2-\varphi_1}{2}\right)} \\
&= \frac{2\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\left(\frac{\varphi_2-\varphi_1}{2}\right)}{\left(|\vec{k}_1| - |\vec{k}_2|\right)} \qquad (5.9)\\
&= \frac{2\sqrt{|\vec{k}_1||\vec{k}_2|}\sqrt{\frac{\cos(\varphi_2-\varphi_1)+1}{2}}}{\sqrt{|\vec{k}_1| - |\vec{k}_2|}} \\
&= \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2}}{\left(|\vec{k}_1| - |\vec{k}_2|\right)} \\
\Rightarrow \theta &= \arctan\left[\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2}}{\left(|\vec{k}_1| - |\vec{k}_2|\right)}\right].
\end{aligned}
$$

Thus,

$$\begin{aligned}
\tan\theta &= \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{|\vec{k}_1| - |\vec{k}_2|} \qquad (5.10)\\
\Rightarrow \sec^2\theta &= \frac{2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)}{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2} + 1\\
&= \frac{|\vec{k}_1|^2 + |\vec{k}_2|^2 + 2\vec{k}_1\cdot\vec{k}_2}{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2}\\
&= \frac{\vec{k}^2}{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2}.
\end{aligned}$$

We will make use of $\sec^2\theta\frac{\partial\theta}{\partial k_i^j} = \frac{\partial A}{\partial k_i^j} \Rightarrow \frac{\partial\theta}{\partial k_i^j} = \frac{1}{\sec^2\theta}\frac{\partial A}{\partial k_i^j}$.

$$\begin{aligned}
\frac{\partial A}{\partial k_1^1} &= \frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^1\right]}{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}\left(|\vec{k}_1| - |\vec{k}_2|\right)} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2}\frac{k_1^1}{|\vec{k}_1|}\\
\Rightarrow \frac{\partial\theta}{\partial k_1^1} &= \frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^1\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}
\end{aligned}$$

and so forth. The partial derivatives we require are therefore given by

$$\begin{aligned}
\frac{\partial k^1}{\partial k_1^1} &= \frac{\partial k^1}{\partial k_2^1} = \frac{\partial k^2}{\partial k_1^2} = \frac{\partial k^2}{\partial k_2^2} = 1\\
\frac{\partial k^1}{\partial k_1^2} &= \frac{\partial k^1}{\partial k_2^2} = \frac{\partial k^2}{\partial k_1^1} = \frac{\partial k^2}{\partial k_2^1} = 0\\
\frac{\partial k^z}{\partial k_1^1} &= \frac{1}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right)\\
\frac{\partial k^z}{\partial k_1^2} &= \frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right)\\
\frac{\partial k^z}{\partial k_2^1} &= \frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right)\\
\frac{\partial k^z}{\partial k_2^2} &= \frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial\theta}{\partial k_1^1} &= \frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|} \\
\frac{\partial\theta}{\partial k_1^2} &= \frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|} \\
\frac{\partial\theta}{\partial k_2^1} &= \frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|} \\
\frac{\partial\theta}{\partial k_2^2} &= \frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}.
\end{aligned}
\tag{5.11}
$$

5.3.1 Calculation of P^x and P^y

The first conformal generator we shall calculate is

$$\boxed{P^x = \int dx\, dy\, \pi\partial_x\phi.}$$

Writing $z = (x', y')$ (omitting explicit vector notation),

$$P^x \;=\; \int dx'\, dy'\, \pi\partial_{x'}\phi = \int d^2z\, \partial_{x'}\phi^a(z)\frac{\partial}{\partial\phi^a(z)}$$

The chain rule associated with $\phi^a(x) \to \psi(x,y) = \sum_c \phi^c(x)\phi^c(y)$ is given by

$$
\begin{aligned}
\frac{\partial}{\partial\phi^a(z)} &= \int d^2x \int d^2y \frac{\partial\psi(x,y)}{\partial\phi^a(z)}\frac{\partial}{\partial\psi(x,y)} \\
&= \int d^2x \int d^2y \left(\delta^2(x-z)\phi^a(y) + \delta^2(y-z)\phi^a(x)\right)\frac{\partial}{\partial\psi(x,y)} \\
&= \int d^2x \int d^2y \left(\delta(x_1-x')\delta(x_2-y')\phi^a(y) + \delta(y_1-x')\delta(y_2-y')\phi^a(x)\right) \\
&\quad \frac{\partial}{\partial\psi(x,y)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P^x &= \int d^2x \int d^2y \int d^2z\, \partial_{x'}\phi^a(x) \\
&\quad \left(\delta(x_1-x')\delta(x_2-y')\phi^a(y) + \delta(y_1-x')\delta(y_2-y')\phi^a(x)\right)\frac{\partial}{\partial\psi(x,y)}
\end{aligned}
$$

$$= \int d^2x \int d^2y \int d^2\vec{z}\, [\partial_{x_1}\phi^a(x)\phi^a(y) + \partial_{y_1}\phi^a(y)\phi^a(x)] \frac{\partial}{\partial\psi(x,y)}$$
$$= \int d^2x \int d^2y\, [(\partial_{x_1} + \partial_{y_1})\, \eta(y,x)]\, \pi(x,y)$$

[1]. Using equation (5.3):

$$\eta_{xy} = \int \frac{d^2k_1}{2\pi} \int \frac{d^2k_2}{2\pi} \eta_{k_1k_2} e^{ik_1x} e^{ik_2y}$$
$$\pi_{xy} = \int \frac{d^2k_1}{2\pi} \int \frac{d^2k_2}{2\pi} \pi_{k_1k_2} e^{ik_1x} e^{ik_2y},$$

we have

$$P^x = \int \frac{d^2k_1}{(2\pi)} \int \frac{d^2k_2}{(2\pi)} \int \frac{d^2k_3}{(2\pi)} \int \frac{d^2k_4}{(2\pi)} \left(k_1^1 + k_2^1\right) \eta(k_1,k_2) p(k_3,k_4)$$
$$\int d^2x \int d^2y e^{ik_1y} e^{ik_2x} e^{ik_3x} e^{ik_4y}.$$

Integrating out over x, y, k_3 and k_4 sets $k_3 \to -k_2$ and $k_4 \to -k_1$ and cancels factors of 2π, and therefore (re-introducing the vector notation)

$$\begin{aligned}
P^x &= \int d^2k_1 \int d^2k_2 \left(k_1^1 + k_2^1\right) \eta(\vec{k}_1, \vec{k}_2) \pi(-\vec{k}_2, -\vec{k}_1) \\
&= \int d^2k_1 \int d^2k_2 \left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right] \left(k_1^1 + k_2^1\right) \frac{\eta(k_1,k_2)}{\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_1|}\right]} \pi(-\vec{k}_2, -\vec{k}_1) \\
&= \int d\vec{\kappa}\; k^1 H(\vec{\kappa}) \Pi(-\vec{\kappa}) \qquad (5.12) \\
&= \int d\vec{\kappa}\; k^1 \left[\frac{1}{\sqrt{2P^0}} \left(a_{\vec{\kappa}} + a^\dagger_{-\vec{\kappa}}\right)\right] \left[-\sqrt{\frac{P^0}{2}} i \left(a_{-\vec{\kappa}} - a^\dagger_{\vec{\kappa}}\right)\right] \\
&= -\frac{i}{2} \int d\vec{\kappa}\;\; k^1 \left[a_{\vec{\kappa}} a_{-\vec{\kappa}} - a_{\vec{\kappa}} a^\dagger_{\vec{\kappa}} + a^\dagger_{-\vec{\kappa}} a_{-\vec{\kappa}} - a^\dagger_{-\vec{\kappa}} a^\dagger_{\vec{\kappa}}\right].
\end{aligned}$$

where aa and $a^\dagger a^\dagger$ integrals go to zero by symmetry under $\vec{\kappa} \to -\vec{\kappa}$. Hence (using a dummy substitution in the second step),

$$\begin{aligned}
P^x &= -\frac{i}{2} \int d\vec{\kappa}\;\; k^1 \left[-a_{\vec{\kappa}} a^\dagger_{\vec{\kappa}} + a^\dagger_{-\vec{\kappa}} a_{-\vec{\kappa}}\right] \\
&= \frac{i}{2} \int d\vec{\kappa}\;\; k^1 \left[a_{\vec{\kappa}} a^\dagger_{\vec{\kappa}} + a^\dagger_{\vec{\kappa}} a_{\vec{\kappa}}\right].
\end{aligned}$$

[1]The second term in the similarity transformation $\partial_\alpha \to \partial_\alpha - \frac{1}{2}\partial_\alpha \ln J$ does not contribute.

so that the AdS generator is

$$P^x_{AdS} = k^1.$$

Similarly

$$\boxed{\begin{aligned} P^y &= \int dx\, dy\, \pi \partial_y \phi \\ &= \frac{i}{2} \int d\vec{\kappa}\ \ k^2 \ \left[a_{\vec{\kappa}} a^{\dagger}_{\vec{\kappa}} + a^{\dagger}_{\vec{\kappa}} a_{\vec{\kappa}} \right], \end{aligned}}$$

with associated AdS generator

$$P^y_{AdS} = k^2.$$

as expected.

5.3.2 Calculation of M^{xy}

We now calculate M^{xy},

$$\boxed{M^{xy} = \int dxdy\ \ -(\pi \partial_x \phi)\, y + (\pi \partial_y \phi)\, x,}$$

Using again $z = (x', y')$, we obtain

$$\begin{aligned} M^{xy} &= \int dx'dy'\ \ -(\pi \partial_{x'} \phi)\, y' + \big(\pi \partial_{y'} \phi\big)\, x' \\ &= \int d^2z\ \left[-\left(\partial_{x'} \phi^a(z)\right) y' + \big(\partial_{y'} \phi^a(z)\big)\, x' \right] \frac{\partial}{\partial \phi^a(z)} \\ &= \int d^2x \int d^2y \int d^2z \left[-\left(\partial_{x'} \phi^a(z)\right) y' + \big(\partial_{y'} \phi^a(z)\big)\, x' \right] \\ &\quad \big(\delta^2(x-z)\phi^a(y) + \delta^2(y-z)\phi^a(x)\big) \frac{\partial}{\partial \psi(x,y)} \\ &= \int d^2x \int d^2y \int d^2z \left[-\left(\partial_{x'} \phi^a(z)\right) y' + \big(\partial_{y'} \phi^a(z)\big)\, x' \right] \\ &\quad \big(\delta(x_1 - x')\delta(x_2 - y')\phi^a(y) + \delta(y_1 - x')\delta(y_2 - y')\phi^a(x)\big) \frac{\partial}{\partial \psi(x,y)} \\ &= \int d^2x \int d^2y \left[\left(-\left(\partial_{x_1} \phi^a(x)\right) x_2 + (\partial_{x_2} \phi^a(x)) x_1 \right) \phi^a(y) \right. \end{aligned}$$

$$
\begin{aligned}
& - \left(\left(\partial_{y_1}\phi^a(y)\right) y_2 + \left(\partial_{y_2}\phi^a(y)\right) y_1 \right) \phi^a(x)\right] \frac{\partial}{\partial \psi(x,y)} \\
= & \int d^2x \int d^2y \left[\left(-x_2\partial_{x_1} + x_1\partial_{x_2} - y_2\partial_{y_1} + y_1\partial_{y_2}\right) \psi(x,y)\right] \frac{\partial}{\partial \psi(y,x)} \\
\to & \int d^2x \int d^2y \left[\left(-x_2\partial_{x_1} + x_1\partial_{x_2} - y_2\partial_{y_1} + y_1\partial_{y_2}\right) \eta(x,y)\right] \pi(y,x).
\end{aligned}
$$

Using equation (5.3) once again,

$$
\begin{aligned}
\eta_{xy} &= \int \frac{d^2k_1}{2\pi} \int \frac{d^2k_2}{2\pi} \eta_{k_1k_2} e^{ik_1x} e^{ik_2y} \\
\pi_{xy} &= \int \frac{d^2k_1}{2\pi} \int \frac{d^2k_2}{2\pi} \pi_{k_1k_2} e^{ik_1x} e^{ik_2y},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
M^{xy} = & \int \frac{d^2k_1}{(2\pi)} \int \frac{d^2k_2}{(2\pi)} \int \frac{d^2k_3}{(2\pi)} \int \frac{d^2k_4}{(2\pi)} \left[\left(-\partial_{k_1^2}k_1^1 + \partial_{k_1^1}k_1^2 - \partial_{k_2^2}k_2^1 + \partial_{k_2^1}k_2^2 \right) \eta(k_1,k_2) \right] \\
& p(k_3,k_4) \int d^2x \int d^2y e^{ik_1x} e^{ik_2y} e^{ik_3y} e^{ik_4x} \\
= & \int d^2k_1 \int d^2k_2 \left[\left(-\partial_{k_1^2}k_1^1 + \partial_{k_1^1}k_1^2 - \partial_{k_2^2}k_2^1 + \partial_{k_2^1}k_2^2 \right) \eta(k_1,k_2) \right] \pi(-k_2,-k_1).
\end{aligned}
$$

Now,

$$
\partial_{k_1^2} = \frac{\partial k^1}{\partial k_1^2}\partial_{k^1} + \frac{\partial k^2}{\partial k_1^2}\partial_{k^2} + \frac{\partial k^z}{\partial k_1^2}\partial_{k^z} + \frac{\partial k^\theta}{\partial k_1^2}\partial_{k^\theta},
$$

and so forth. Using the partial derivatives given in equations (5.11), we find that

$$
\begin{aligned}
& -k_1^1\partial_{k_1^2} + k_1^2\partial_{k_1^1} - k_2^1\partial_{k_2^2} + k_2^2\partial_{k_2^1} \\
= & -k_1^1 \left[\partial_{k^2} + \left[\frac{1}{k^z} \left(k_1^2 \frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2 \right) \right] \partial_{k^z} \right] + k_1^2 \left[\partial_{k^1} + \left[\frac{1}{k^z} \left(k_1^1 \frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1 \right) \right] \partial_{k^z} \right] \\
& -k_2^1 \left[\partial_{k^2} + \left[\frac{1}{k^z} \left(k_2^2 \frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2 \right) \right] \partial_{k^z} \right] + k_2^2 \left[\partial_{k^1} + \left[\frac{1}{k^z} \left(k_2^1 \frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1 \right) \right] \partial_{k^z} \right] + \\
& \left[-k_1^1 \left[\frac{\left[k_1^2 \frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^2 \right] \left(|\vec{k}_1| - |\vec{k}_2| \right)}{\sqrt{2}\vec{k}^2 \sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2} \frac{k_1^2}{|\vec{k}_1|} \right] \right. \\
& +k_1^2 \left[\frac{\left[k_1^1 \frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^1 \right] \left(|\vec{k}_1| - |\vec{k}_2| \right)}{\sqrt{2}\vec{k}^2 \sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2} \frac{k_1^1}{|\vec{k}_1|} \right]
\end{aligned}
$$

$$\begin{aligned}
& -k_2^1\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]\\
& +k_2^2\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]\Bigg]\partial_\theta\\
= & -\left(k_1^1+k_2^1\right)\partial_{k^2}+\left(k_1^2+k_2^2\right)\partial_{k^1}\\
= & \; k^2\partial_{k^1}-k^1\partial_{k^2},
\end{aligned}$$

and therefore M^{xy} becomes

$$\begin{aligned}
M^{xy} &= \int d^2k_1\int d^2k_2\left[\left(-\partial_{k_1^2}k_1^1+\partial_{k_1^1}k_1^2-\partial_{k_2^2}k_2^1+\partial_{k_2^1}k_2^2\right)\eta(k_1,k_2)\right]\pi(-k_2,-k_1)\\
&= \int d^2k_1\int d^2k_2\left[\left(k^2\partial_{k^1}-k^1\partial_{k^2}\right)\eta(k_1,k_2)\right]\pi(-k_2,-k_1).
\end{aligned}$$

Consider now

$$\begin{aligned}
& \left[-\partial_{k_1^2}k_1^1+\partial_{k_1^1}k_1^2-\partial_{k_2^2}k_2^1+\partial_{k_2^1}k_2^2\right]\left[\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right]\\
= & \left[-\partial_{k_1^2}k_1^1+\partial_{k_1^1}k_1^2-\partial_{k_2^2}k_2^1+\partial_{k_2^1}k_2^2\right]\left[\frac{1}{\sqrt{(k_1^1)^2+(k_1^2)^2}}+\frac{1}{\sqrt{(k_2^1)^2+(k_2^2)^2}}\right]\\
= & \left[\frac{k_1^1k_1^2-k_1^1k_1^2}{\sqrt{(k_1^1)^2+(k_1^2)^2}^3}+\frac{k_2^1k_2^2-k_2^1k_2^2}{\sqrt{(k_2^1)^2+(k_2^2)^2}^3}\right]\\
= & \; 0.
\end{aligned}$$

Hence we can write

$$\begin{aligned}
M^{xy} &= \int d^2k_1\int d^2k_2\left[\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right]\\
&\quad \left[\left(-\partial_{k_1^2}k_1^1+\partial_{k_1^1}k_1^2-\partial_{k_2^2}k_2^1+\partial_{k_2^1}k_2^2\right)\frac{\eta(\vec{k}_1,\vec{k}_2)}{\left[\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right]}\right]\pi(-\vec{k}_2,-\vec{k}_1)\\
&= \int d\vec{k}\int dk^z\int d\theta\left[\left(k^2\partial_{k^1}-k^1\partial_{k^2}\right)\mathcal{H}(\vec{k},k^z,\theta)\right]\Pi(-\vec{k},-k^z,-\theta)\\
&= \int d\vec{\kappa}\left[\left(k^2\partial_{k^1}-k^1\partial_{k^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \qquad (5.13)\\
&= \int d\vec{\kappa}\left[\left(k^2\partial_{k^1}-k^1\partial_{k^2}\right)\frac{1}{\sqrt{2P^0}}\left(a_{\vec{\kappa}}+a^\dagger_{-\vec{\kappa}}\right)\right]\left[-\sqrt{\frac{P^0}{2}}i\left(a_{-\vec{\kappa}}-a^\dagger_{\vec{\kappa}}\right)\right]
\end{aligned}$$

$$= \frac{i}{2}\int d\vec{\kappa}\ \left[-a_{\vec{\kappa}}\left[k^2\partial_{k^1}-k^1\partial_{k^2}\right]a_{-\vec{\kappa}}+a_{\vec{\kappa}}\left[k^2\partial_{k^1}-k^1\partial_{k^2}\right]a^\dagger_{\vec{\kappa}}\right.$$
$$\left.-a^\dagger_{-\vec{\kappa}}\left[k^2\partial_{k^1}-k^1\partial_{k^2}\right]a_{-\vec{\kappa}}+a^\dagger_{-\vec{\kappa}}\left[k^2\partial_{k^1}-k^1\partial_{k^2}\right]a^\dagger_{\vec{\kappa}}\right].$$

Since k^1 and k^2 are dummy variables which, under exchange, don't change the $a_{\vec{\kappa}}a_{-\vec{\kappa}}$ and $a^\dagger_{-\vec{\kappa}}a^\dagger_{\vec{\kappa}}$ terms, these cancel and we obtain

$$M^{xy} = \frac{i}{2}\int d\vec{\kappa}\ \left[a_{\vec{\kappa}}\left[k^2\partial_{k^1}-k^1\partial_{k^2}\right]a^\dagger_{\vec{\kappa}}-ia^\dagger_{\vec{\kappa}}\left[k^2\partial_{k^1}-k^1\partial_{k^2}\right]a_{\vec{\kappa}}\right].$$

Thus,

$$M^{xy}_{AdS}=x^1k^2-x^2k^1.$$

5.3.3 Calculation of D

Let us now calculate the dilatation operator,

$$\boxed{D=Ht+\int dxdy\left[\left(\pi\partial_x\phi\right)x+\left(\pi\partial_y\phi\right)y+\pi d_\phi\phi\right].}$$

Setting $t=0$ and $z=(x',y')$ and noting that $d_\phi=\frac{2-2}{2}=0$, we obtain

$$\begin{aligned}
D &= \int dx'dy'\left[\left(\pi\partial_{x'}\phi\right)x'+\left(\pi\partial_{y'}\phi\right)y'\right]\\
&= \int d^2z\ \left[\left(\partial_{x'}\phi^a(z)\right)x'+\left(\partial_{y'}\phi^a(z)\right)y'\right]\frac{\partial}{\partial\phi^a(z)}\\
&= \int d^2x\int d^2y\int d^2z\ \left[\left(\partial_{x'}\phi^a(z)\right)x'+\left(\partial_{y'}\phi^a(z)\right)y'\right]\\
&\quad \left(\delta^2(x-z)\phi^a(y)+\delta^2(y-z)\phi^a(x)\right)\frac{\partial}{\partial\psi(x,y)}\\
&= \int d^2x\int d^2y\int d^2z\ \left[\left(\partial_{x'}\phi^a(z)\right)x'+\left(\partial_{y'}\phi^a(z)\right)y'\right]\\
&\quad \left(\delta(x_1-x')\delta(x_2-y')\phi^a(y)+\delta(y_1-x')\delta(y_2-y')\phi^a(x)\right)\frac{\partial}{\partial\psi(x,y)}\\
&= \int d^2x\int d^2y\ \left[\left(\partial_{x_1}\phi^a(x)\right)x_1+\left(\partial_{x_2}\phi^a(x)\right)x_2\right]\phi^a(y)\frac{\partial}{\partial\psi(x,y)}\\
&\quad +\int d^2x\int d^2y\ \left[\left(\partial_{y_1}\phi^a(y)\right)y_1+\left(\partial_{y_2}\phi^a(y)\right)y_2\right]\phi^a(x)\frac{\partial}{\partial\psi(x,y)}\\
&= \int d^2x\int d^2y\ \left[\left(x_1\partial_{x_1}+x_2\partial_{x_2}\right)\psi(y,x)\right]\frac{\partial}{\partial\psi(x,y)}
\end{aligned}$$

$$
\begin{aligned}
&+\int d^2x\int d^2y\ \left[\left(y_1\partial_{y_1}+y_2\partial_{y_2}\right)\psi(y,x)\right]\frac{\partial}{\partial\psi(x,y)}\\
\to\ &\int d^2k_1\int d^2k_2\int d^2k_3\int d^2k_4\int d^2x\int d^2y\\
&\left[\left(k_1^1\partial_{k_1^1}+k_1^2\partial_{k_1^2}+k_2^1\partial_{k_2^1}+k_2^2\partial_{k_2^2}\right)\eta(k_1,k_2)\right]\pi(k_3,k_4)\\
&e^{i\left(k_1^1y_1+k_1^2y_2\right)}e^{i\left(k_2^1x_1+k_2^2x_2\right)}e^{i\left(k_3^1x_1+k_3^2x_2\right)}e^{i\left(k_4^1y_1+k_4^2y_2\right)}\\
=\ &\int d^2k_1\int d^2k_2\\
&\left[\left(k_1^1\partial_{k_1^1}+k_1^2\partial_{k_1^2}+k_2^1\partial_{k_2^1}+k_2^2\partial_{k_2^2}\right)\eta(k_1,k_2)\right]\pi(-k_2,-k_1).
\end{aligned}
$$

Now

$$
\begin{aligned}
&k_1^1\partial_{k_1^1}+k_1^2\partial_{k_1^2}+k_2^1\partial_{k_2^1}+k_2^2\partial_{k_2^2}\\
=\ &k_1^1\left[\partial_{k^1}+\left[\frac{1}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k_2^1\right)\right]\partial_{k^z}\right]+k_1^2\left[\partial_{k^2}+\left[\frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}-k_2^2\right)\right]\partial_{k^z}\right]\\
&k_2^1\left[\partial_{k^1}+\left[\frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^1\right)\right]\partial_{k^z}\right]+k_2^2\left[\partial_{k^2}+\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^2\right)\right]\partial_{k^z}\right]\\
&+k_1^1\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right]\partial_\theta\\
&+k_1^2\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]\partial_\theta\\
&k_2^1\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]\partial_\theta\\
&+k_2^2\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]\partial_\theta\\
=\ &k_1^1\left[\partial_{k^1}+\left[\frac{1}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k_2^1\right)\right]\partial_{k^z}\right]+k_1^2\left[\partial_{k^2}+\left[\frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}-k_2^2\right)\right]\partial_{k^z}\right]\\
&+k_2^1\left[\partial_{k^1}+\left[\frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^1\right)\right]\partial_{k^z}\right]+k_2^2\left[\partial_{k^2}+\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^2\right)\right]\partial_{k^z}\right]\\
&+\left[\frac{2\left(|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2\right)\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\left[|\vec{k}_1|-|\vec{k}_2|\right]\right]\partial_\theta\\
=\ &\left(k_1^1+k_2^1\right)\partial_{k^1}+\left(k_1^2+k_2^2\right)\partial_{k^2}+\frac{1}{k^z}\frac{|\vec{k}_2|}{|\vec{k}_1|}\left((k_1^1)^2+(k_1^2)^2\right)\\
&+\frac{1}{k^z}\frac{|\vec{k}_1|}{|\vec{k}_2|}\left((k_2^1)^2+(k_2^2)^2\right)\partial_{k^z}-\frac{1}{k^z}\left(2k_1^1k_2^1+2k_1^2k_2^2\right)\partial_{k^z}
\end{aligned}
$$

$$
\begin{aligned}
&= k^1\partial_{k^1} + k^2\partial_{k^2} + \frac{1}{k^z}|\vec{k}_1||\vec{k}_2|\partial_{k^z} + \frac{1}{k^z}|\vec{k}_1||\vec{k}_2|\partial_{k^z} - \frac{1}{k^z}\left(2\vec{k}_1\cdot\vec{k}_2\right)\partial_{k^z} \\
&= k^1\partial_{k^1} + k^2\partial_{k^2} + \frac{1}{k^z}\left((k^z)^2\right)\partial_{k^z} \\
&= k^1\partial_{k^1} + k^2\partial_{k^2} + k^z\partial_{k^z}.
\end{aligned} \tag{5.14}
$$

The Dilatation operator therefore becomes

$$
D = \int d^2k_1 \int d^2k_2 \left[\left(k^1\partial_{k^1} + k^2\partial_{k^2} + k^z\partial_{k^z} + 2d_\phi\right)\eta(\vec{k}_1,\vec{k}_2)\right]\pi(-\vec{k}_2,-\vec{k}_1).
$$

Since

$$
\begin{aligned}
&\left[k_1^1\partial_{k_1^1} + k_1^2\partial_{k_1^2} + k_2^1\partial_{k_2^1} + k_2^2\partial_{k_2^2}\right]\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right] \\
&= \left[k_1^1\partial_{k_1^1} + k_1^2\partial_{k_1^2} + k_2^1\partial_{k_2^1} + k_2^2\partial_{k_2^2}\right]\left[\frac{1}{\sqrt{(k_1^1)^2+(k_1^2)^2}} + \frac{1}{\sqrt{(k_2^1)^2+(k_2^2)^2}}\right] \\
&= -\left[\frac{(k_1^1)^2+(k_1^2)^2}{\sqrt{(k_1^1)^2+(k_1^2)^2}^3} + \frac{(k_2^1)^2+(k_2^2)^2}{\sqrt{(k_2^1)^2+(k_2^2)^2}^3}\right] \\
&= -\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right],
\end{aligned}
$$

the integrand of the dilatation operator D is given by

$$
\begin{aligned}
&\left[\left(k^1\partial_{k^1} + k^2\partial_{k^2} + k^z\partial_{k^z}\right)\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]\left[\frac{1}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right]\eta(\vec{k}_1,\vec{k}_2)\right]\pi(-\vec{k}_2,-\vec{k}_1) \\
&= \left[\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]\left(k^1\partial_{k^1} + k^2\partial_{k^2} + k^z\partial_{k^z}\right)\left[\frac{1}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right]\eta(\vec{k}_1,\vec{k}_2)\right]\pi(-\vec{k}_2,-\vec{k}_1) \\
&\quad + \left[\left(k_1^1\partial_{k_1^1} + k_1^2\partial_{k_1^2} + k_1^2\partial_{k_1^2} + k_2^2\partial_{k_2^2}\right)\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]\right]\left[\frac{1}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right]\eta(\vec{k}_1,\vec{k}_2)\pi(-\vec{k}_2,-\vec{k}_1) \\
&= \left[\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]\left(k^1\partial_{k^1} + k^2\partial_{k^2} + k^z\partial_{k^z}\right)\left[\frac{1}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right]\eta(\vec{k}_1,\vec{k}_2)\right]\pi(-\vec{k}_2,-\vec{k}_1) \\
&\quad - \left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]\left[\frac{1}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right]\eta(\vec{k}_1,\vec{k}_2)\pi(-\vec{k}_2,-\vec{k}_1) \\
&= \left[\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]\left(k^1\partial_{k^1} + k^2\partial_{k^2} + k^z\partial_{k^z} - 1\right)\left[\frac{1}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right]\eta(\vec{k}_1,\vec{k}_2)\right]\pi(-\vec{k}_2,-\vec{k}_1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
D &= \int d\vec{\kappa}\ \left[\left(k^1\partial_{k^1} + k^2\partial_{k^2} + k^z\partial_{k^z} - 1\right) H(\kappa)\right] \Pi(-\vec{\kappa}) \qquad (5.15)\\
&= \int d\vec{\kappa}\ \Pi(\vec{\kappa}) \left[k^j\partial_{k^j} - 1\right] H(-\vec{\kappa})\\
&= \int d\vec{\kappa}\ \left(-i\sqrt{\frac{P^0}{2}}a_{\vec{\kappa}} + i\sqrt{\frac{P^0}{2}}a^\dagger_{-\vec{\kappa}}\right)\left[k^j\partial_{k^j} - 1\right]\left(\frac{1}{\sqrt{2P^0}}a_{-\vec{\kappa}} + \frac{1}{\sqrt{2P^0}}a^\dagger_{\vec{\kappa}}\right)\\
&= \frac{1}{2}\int d\vec{\kappa}\left[-ia_{\vec{\kappa}}\left[k^j\partial_{k^j} - 1\right]a_{-\vec{\kappa}} - ia_{\vec{\kappa}}\left[k^j\partial_{k^j} - 1\right]a^\dagger_{\vec{\kappa}}\right.\\
&\quad \left. + ia^\dagger_{-\vec{\kappa}}\left[k^j\partial_{k^j} - 1\right]a_{-\vec{\kappa}} + a^\dagger_{-\vec{\kappa}}\left[k^j\partial_{k^j} - 1\right]a^\dagger_{\vec{\kappa}}\right]. \qquad (5.16)
\end{aligned}
$$

Here the aa and $a^\dagger a^\dagger$ terms don't cancel as described in the comment on page 4 of Ref. [137], which referenced Ref. [138].

These problems can be resolved by using the new improved stress energy momentum tensor defined as [137]:

$$\Theta_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} - \frac{1}{4}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)\phi^2.$$

Using the new improved stress energy momentum tensor, we have

$$
\begin{aligned}
\mathcal{D} &= x^\mu\Theta_{\mu 0}\\
&= x^0\left(\partial_0\phi\pi - \mathcal{L} - \frac{1}{4}\partial_0\left(\partial_0\phi^2\right)\right) + x^i\left(\partial_i\phi\pi - g_{0i}\mathcal{L}\right) - \frac{x^i}{4}\partial_i\left(\left(\partial_0\phi^2\right)\right)\\
&= 0 + x^i\partial_i\phi\pi - \frac{x^i}{2}\partial_i\phi\pi - \frac{x^i}{2}\phi\partial_i\pi\\
&= \frac{x^i}{2}\partial_i\phi\pi - \frac{x^i}{2}\phi\partial_i\pi.
\end{aligned}
$$

Moving to momentum space, we obtain (written schematically)

$$
\begin{aligned}
D &= \int dz\left(\frac{x^i}{2}\pi\partial_i\phi - \frac{x^i}{2}\phi\partial_i\pi\right)\\
&= \frac{1}{2}\int dz\frac{dk}{(2\pi)}\frac{dk'}{(2\pi)}\left[\left(-i\sqrt{\frac{P^0}{2}}\right)\left(a_k e^{ikz} - a^\dagger_{-k}e^{-ikz}\right)\left(k^i\partial_{k^i} - 1\right)\right.\\
&\quad \left(a_{-k'}e^{ik'z} + a^\dagger_{k'}e^{-ik'z}\right)\sqrt{\frac{1}{2P^0}} - \left(-i\sqrt{\frac{P^0}{2}}\right)\left(a_k e^{ikz} + a^\dagger_{-k}e^{-ikz}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \left(k^i\partial_{k^i}-1\right)\left(a_{-k'}e^{ik'z}-a^\dagger_{k'}e^{-ik'z}\right)\sqrt{\frac{1}{2P^0}}\Bigg] \\
= & -\frac{i}{2}\int dk\frac{1}{2}\Big[\left(\left(k^i\partial_{k^i}-1\right)a_{-k}\right)a_k-\left(\left(k^i\partial_{k^i}-1\right)a_k\right)a^\dagger_k+\left(k^i\partial_{k^i}a^\dagger_k\right)a_k \\
& -\left(\left(k^i\partial_{k^i}-1\right)a^\dagger_k\right)a^\dagger_{-k}\Big]-\frac{1}{2}\Big[\left(\left(k^i\partial_{k^i}-1\right)a_k\right)a_{-k}+\left(\left(k^i\partial_{k^i}-1\right)a_k\right)a^\dagger_k \\
& -\left(\left(k^i\partial_{k^i}-1\right)a^\dagger_k\right)a_k-\left(\left(k^i\partial_{k^i}-1\right)a^\dagger_k\right)a^\dagger_{-k}\Big] \\
= & -\frac{i}{2}\int dk\left[-\left(\left(k^i\partial_{k^i}-1\right)a_k\right)a^\dagger_k+\left((k^i\partial_{k^i}-1)a^\dagger_k\right)a_k\right] \\
= & -\frac{i}{2}\int dk\left[\left(\left(k^i\partial_{k^i}-1\right)a^\dagger_k\right)a_k-a^\dagger_k\left(\left(k^i\partial_{k^i}-1\right)a_k\right)\right],
\end{aligned}
$$

where the last expression has been normal ordered. In AdS, therefore

$$
D = -\frac{i}{2}\int d\vec{\kappa}\left[a_{\vec{\kappa}}\left[k^j\partial_{k^j}-1\right]a^\dagger_{\vec{\kappa}}-a^\dagger_{\vec{\kappa}}\left[k^j\partial_{k^j}-1\right]a_{\vec{\kappa}}\right].
$$

Removing the zero point energy we thus obtain

$$
\begin{aligned}
D &= -\frac{i}{2}\int d\vec{\kappa}\left[a_{\vec{\kappa}}\left[k^j\partial_{k^j}\right]a^\dagger_{\vec{\kappa}}-a^\dagger_{\vec{\kappa}}\left[k^j\partial_{k^j}\right]a_{\vec{\kappa}}\right] \\
&= -\frac{i}{2}\int d\vec{\kappa}\Big[a_{\vec{\kappa}}\left[x^1k^1+x^2k^2+zk^z\right]a^\dagger_{\vec{\kappa}} \\
&\quad -a^\dagger_{\vec{\kappa}}\left[x^1k^1+x^2k^2+zk^z\right]a_{\vec{\kappa}}\Big];
\end{aligned}
$$

the dilatation operator in AdS is therefore

$$
D_{AdS}=x^1k^1+x^2k^2+zk^z.
$$

5.3.4 Calculation of M^{tx} and M^{ty}

We now turn to the generators that involve the Hamiltonian density and a coordinate

$$
\boxed{\begin{aligned}
M^{tx} &= \int dxdy\ \mathcal{H}x \\
M^{ty} &= \int dxdy\ \mathcal{H}y.
\end{aligned}}
$$

Recall from equation (5.1),

$$H_2 = \int d^2x \int d^2y \int d^2x' \int d^2y' \; \frac{1}{2}\pi_{xy}\Omega_{xy;x'y'}\pi_{x'y'} + \frac{N^2}{32}\psi^{-1}_{yx}\Omega_{xy;x'y'}\psi^{-1}_{y'x'}.$$

with (where x,y,x',y' are all 2-vectors)

$$\begin{aligned}\Omega_{xy;x'y'} &= \int d^2z \frac{\partial\psi_{xy}}{\partial\phi_C(z)}\frac{\partial\psi_{x'y'}}{\partial\phi_C(z)} \\ &= \Big[\psi(x,x')\delta^d(y-y') + \psi(x,y')\delta^d(y-x') \\ &\quad +\psi(y,x')\delta^d(x-y') + \psi(y,y')\delta^d(x-x')\Big].\end{aligned}$$

We can obtain M^{tx} by adjusting this as follows:

$$\begin{aligned}\tilde{\Omega}_{xy;x'y'} &= \int d^2z\; z^x \frac{\partial\psi_{xy}}{\partial\phi_C(z)}\frac{\partial\psi_{x'y'}}{\partial\phi_C(z)} \\ &= \int d^2z\; z^x \left(\delta(y-z)\phi(x) + \delta(x-z)\phi(y)\right)\big(\delta(y'-z)\phi(x') + \delta(x'-z)\phi(y')\big) \\ &= \Big[y^x\psi(x,x')\delta^d(y-y') + y^x\psi(x,y')\delta^d(y-x') \\ &\quad +x^x\psi(y,x')\delta^d(x-y') + x^x\psi(y,y')\delta^d(x-x')\Big].\end{aligned}$$

Thus

$$\begin{aligned}M^{tx} &= \int d^2x \int d^2y \int d^2x' \int d^2y' \; \frac{1}{2}\pi_{xy}\tilde{\Omega}_{xy;x'y'}\pi_{x'y'} + \frac{N^2}{32}\psi^{-1}_{yx}\tilde{\Omega}_{xy;x'y'}\psi^{-1}_{y'x'} \\ &\equiv M^{tx}_1 + M^{tx}_2.\end{aligned}$$

Therefore, integrating either over y' or x' but then keeping dummy index y',

$$\begin{aligned}M^{tx}_1 &= \frac{1}{2}\int d^2x \int d^2y \int d^2x' \int d^2y' \; \Big[y^x\pi_{xy}\psi(x,x')\delta^d(y-y')\pi_{x'y'} \\ &\quad +y^x\pi_{xy}\psi(x,y')\delta^d(y-x')\pi_{x'y'} + x^x\pi_{xy}\psi(y,x')\delta^d(x-y')\pi_{x'y'} \\ &\quad +x^x\pi_{xy}\psi(y,y')\delta^d(x-x')\pi_{x'y'}\Big] \\ &= \frac{1}{2}\int d^2x \int d^2y \int d^2y' \; y^x\pi_{xy}\psi(x,y')\pi_{y'y} + y^x\pi_{xy}\psi(x,y')\pi_{yy'} \\ &\quad +x^x\pi_{xy}\psi(y,y')\pi_{y'x} + x^x\pi_{xy}\psi(y,y')\pi_{xy'} \\ &= \mathrm{Tr}\left(x^x\pi\psi\pi\right) + \mathrm{Tr}\left(y^x\pi\psi\pi\right), \qquad (5.17)\end{aligned}$$

and (where we integrate by parts in line (5.18)),

$$
\begin{aligned}
M_2^{tx} &= \frac{N^2}{32}\int d^2x\int d^2y\int d^2x'\int d^2y'\ \Big[y^x\psi^{-1}_{yx}\psi_{xx'}\delta^d(y-y')\psi^{-1}_{y'x'} \\
&\quad +y^x\psi^{-1}_{yx}\psi_{xy'}\delta^d(y-x')\psi^{-1}_{y'x'}+x^x\psi^{-1}_{yx}\psi_{yx'}\delta^d(x-y')\psi^{-1}_{y'x'} \\
&\quad +x^x\psi^{-1}_{yx}\psi_{yy'}\delta^d(x-x')\psi^{-1}_{y'x'}\Big] \\
&= \frac{N^2}{32}\int d^2x\int d^2y\int d^2y'\ \Big[y^x\psi^{-1}_{yx}\psi_{xy'}\psi^{-1}_{yy'}+y^x\psi^{-1}_{yx}\psi_{xy'}\psi^{-1}_{y'y} \\
&\quad +x^x\psi^{-1}_{yx}\psi_{yy'}\psi^{-1}_{xy'}+x^x\psi^{-1}_{yx}\psi_{yy'}\psi^{-1}_{y'x}\Big] \\
&= \frac{N^2}{8}\int d^2x\ \left[x^x\psi^{-1}_{xx}\right] \\
&= \frac{N^2}{8}\mathrm{Tr}\left(x^x\psi^{-1}\right) \\
&\to \frac{1}{8}\int d^2x\int d^2y\int d^2x_1\int d^2x_2\int d^2x_3\left[x^x\psi^{-1}_{0xy}\eta_{yx_1}\psi^{-1}_{0x_1x_2}\eta_{x_2x_3}\psi^{-1}_{0x_3x}\right] \\
&= -\frac{1}{8}\int\frac{d^2k_1}{2\pi}\int\frac{d^2k_2}{2\pi}\int\frac{d^2k_3}{2\pi}\int\frac{d^2k_4}{2\pi}\int\frac{d^2p_1}{(2\pi)^2}\int\frac{d^2p_2}{(2\pi)^2}\int\frac{d^2p_3}{(2\pi)^2} \\
&\quad \left[\frac{\partial_{p_1^x}}{i}\psi^{-1}_{0p_1}\eta_{k_1k_2}\psi^{-1}_{0p_2}\eta_{k_3k_4}\psi^{-1}_{0p_3}\right]\int d^2x\int d^2y\int d^2x_1\int d^2x_2\int d^2x_3 \\
&\quad \left[e^{ip_1(x-y)}e^{i(k_1y+k_2x_1)}e^{ip_2(x_1-x_2)}e^{i(k_3x_2+k_4x_3)}e^{ip_3(x_3-x)}\right] \qquad (5.18) \\
&= \frac{i}{8}\int d^2k_1\int d^2k_2\left[\left(\partial_{k_1^x}\psi^{-1}_{0k_1}\right)\eta_{k_1k_2}\psi^{-1}_{0-k_2}\eta_{-k_2-k_1}\psi^{-1}_{0k_1}\right] \\
&= \frac{i}{8}\int d^2k_1\int d^2k_2\left[\frac{2k_1^x}{\sqrt{\vec{k}_1^2}}\eta_{k_1k_2}\psi^{-1}_{0-k_2}\eta_{-k_2-k_1}\psi^{-1}_{0k_1}\right] \\
&= i\int d^2k_1\int d^2k_2\left[k_1^x\sqrt{\vec{k}_2^2}\eta_{k_1k_2}\eta_{-k_2-k_1}\right] \qquad (5.19) \\
&= 0
\end{aligned}
$$

since it is odd in k_1. Therefore,

$$
\boxed{M^{tx}=\mathrm{Tr}\left(x^x\pi\psi\pi\right)+\mathrm{Tr}\left(y^x\pi\psi\pi\right).}
$$

The first term in (5.17) becomes

$$
\int\frac{d^2k_1}{2\pi}\int\frac{d^2k_2}{2\pi}\int\frac{d^2k_3}{2\pi}\int\frac{d^2k_4}{2\pi}\int\frac{d^2p}{(2\pi)^2}\left[\pi_{k_1k_2}\psi^0_p\pi_{k_3k_4}\left(-i\partial_{k_1^x}\right)\right]
$$
$$
\int d^2x\int d^2y\int d^2y'\ e^{ik_1x}e^{ik_2y}e^{p(y-y')}e^{ik_3x}e^{ik_4y'}
$$

$$
\begin{aligned}
&= i\int \frac{d^2k_1}{2\pi}\int \frac{d^2k_2}{2\pi}\int \frac{d^2k_3}{2\pi}\int \frac{d^2k_4}{2\pi}\int \frac{d^2p}{(2\pi)^2}\left[\frac{\partial \pi_{k_1k_2}}{\partial k_1^x}\pi_{k_3k_4}\psi_p^0\right] \\
&\quad \int d^2x \int d^2y \int d^2y'\; e^{ik_1x}e^{ik_2y}e^{p(y-y')}e^{ik_3x}e^{ik_4y'} \\
&= i\int d^2k_1 \int d^2k_2 \left[\frac{\partial \pi_{k_1k_2}}{\partial k_1^x}\psi^0_{-k_2}\pi_{-k_2-k_1}\right] \\
&= i\int d^2k_1 \int d^2k_2 \left[\frac{\partial \pi_{k_1k_2}}{\partial k_1^x}\frac{1}{2|\vec{k}_2|}\pi_{-k_2-k_1}\right] \qquad (5.20)
\end{aligned}
$$

and similarly for the second one. Equation (5.17) is therefore given by

$$
\begin{aligned}
M^{tx} &= i\int d^2k_1 \int d^2k_2 \left[\left(\frac{1}{2|\vec{k}_2|}\frac{\partial}{\partial k_1^x} + \frac{1}{2|\vec{k}_1|}\frac{\partial}{\partial k_2^x}\right)\pi_{k_1k_2}\pi_{-k_2-k_1}\right] \\
&= \frac{i}{2}\int d^2k_1 \int d^2k_2 \left[\left(\frac{1}{|\vec{k}_2|}\partial_{k_1^x} + \frac{1}{|\vec{k}_1|}\partial_{k_2^x}\right)\pi_{k_1k_2}\right]\pi_{-k_2-k_1} \\
&= \frac{i}{2}\int d^2k_1 \int d^2k_2 \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\frac{\left[\left(\frac{1}{|\vec{k}_2|}\partial_{k_1^x} + \frac{1}{|\vec{k}_1|}\partial_{k_2^x}\right)\pi_{k_1k_2}\right]}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)}\pi_{-k_2-k_1} \\
&\quad + \frac{i}{2}\int d^2k_1 \int d^2k_2 \frac{\left[\left(\frac{1}{|\vec{k}_2|}\partial_{k_1^x} + \frac{1}{|\vec{k}_1|}\partial_{k_2^x}\right)\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\right]}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)}\pi_{k_1k_2}\pi_{-k_2-k_1}.
\end{aligned}
$$

The second term has a bracketed expression given by

$$
\left[\left(\frac{1}{|\vec{k}_2|}\partial_{k_1^x} + \frac{1}{|\vec{k}_1|}\partial_{k_2^x}\right)\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\right] = -\left[\frac{k_1^x}{|\vec{k}_2||\vec{k}_1|^3} + \frac{k_2^x}{|\vec{k}_1||\vec{k}_2|^3}\right];
$$

this integral is odd in k_1^x and k_2^x and hence goes to zero; therefore

$$
M^{tx} = \frac{i}{2}\int d^2k_1 \int d^2k_2 \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)\frac{\left[\left(\frac{1}{|\vec{k}_2|}\partial_{k_1^x} + \frac{1}{|\vec{k}_1|}\partial_{k_2^x}\right)\pi_{k_1k_2}\right]}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)}\pi_{-k_2-k_1}.
$$

Changing to $AdS \times S^1$ coordinates,

$$
\begin{aligned}
M^{tx} &= \frac{i}{2}\int d\vec{\kappa}\frac{\left[\left(|\vec{k}_1|\partial_{k_1^x} + |\vec{k}_2|\partial_{k_2^x}\right)\pi_{k_1k_2}\right]}{\left(|\vec{k}_1| + |\vec{k}_2|\right)}\pi_{-k_2-k_1} \\
&= \frac{i}{2}\int \frac{d\vec{\kappa}}{P^0}\left[\left(|\vec{k}_1|\partial_{k_1^x} + |\vec{k}_2|\partial_{k_2^x}\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa})
\end{aligned}
$$

By applying the chain rule, we have (writing $\partial_{k_1^x} \equiv \partial_{k_1^1}$ and $\partial_{k_2^x} \equiv \partial_{k_2^1}$)

$$\partial_{k_1^1} = \frac{\partial k^1}{\partial k_1^1}\frac{\partial}{\partial k^1} + \frac{\partial k^z}{\partial k_1^1}\frac{\partial}{\partial k^z} + \frac{\partial \theta}{\partial k_1^1}\frac{\partial}{\partial \theta} \tag{5.21}$$

$$\partial_{k_2^1} = \frac{\partial k^1}{\partial k_2^1}\frac{\partial}{\partial k^1} + \frac{\partial k^z}{\partial k_2^1}\frac{\partial}{\partial k^z} + \frac{\partial \theta}{\partial k_2^1}\frac{\partial}{\partial \theta}. \tag{5.22}$$

From equations (D.14)

$$\begin{aligned}
0 &= |\vec{k}_1|\frac{\partial \theta}{\partial k_1^1} + |\vec{k}_2|\frac{\partial \theta}{\partial k_2^1} - \frac{k^2 k^z}{\vec{k}^2} \\
\frac{\partial \theta}{\partial k_1^1} &= \frac{1}{|\vec{k}_1| + |\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \frac{k^2 k^z}{\vec{k}^2}\right] \\
\Rightarrow \frac{\partial \theta}{\partial k_2^1} &= \frac{1}{|\vec{k}_2|}\left[\frac{k^2 k^z}{\vec{k}^2} - \frac{|\vec{k}_1|}{|\vec{k}_1| + |\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \frac{k^2 k^z}{\vec{k}^2}\right]\right] \\
&= \frac{1}{|\vec{k}_1| + |\vec{k}_2|}\left[\frac{k^2 k^z}{\vec{k}^2} - \sqrt{\frac{|\vec{k}_1|}{|\vec{k}_2|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\right] \\
0 &= |\vec{k}_1|\frac{\partial \theta}{\partial k_1^2} + |\vec{k}_2|\frac{\partial \theta}{\partial k_2^2} + \frac{k^1 k^z}{\vec{k}^2} \\
\frac{\partial \theta}{\partial k_1^2} &= \frac{1}{|\vec{k}_1| + |\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) - \frac{k^1 k^z}{\vec{k}^2}\right]. \\
\Rightarrow \frac{\partial \theta}{\partial k_2^2} &= \frac{1}{|\vec{k}_2|}\left[-\frac{k^1 k^z}{\vec{k}^2} - \frac{|\vec{k}_1|}{|\vec{k}_1| + |\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) - \frac{k^1 k^z}{\vec{k}^2}\right]\right] \\
&= \frac{1}{|\vec{k}_1| + |\vec{k}_2|}\left[-\sqrt{\frac{|\vec{k}_1|}{|\vec{k}_2|}}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) - \frac{k^1 k^z}{\vec{k}^2}\right]
\end{aligned}$$

and the other partial derivatives from equations (5.11), we obtain

$$\left|\vec{k_1}\right|\frac{\partial k_1}{\partial k_1^1}\partial_{k^1} + \left|\vec{k_2}\right|\frac{\partial k_1}{\partial k_2^1}\partial_{k^1} = \left|\vec{k_1}\right|\partial_{k^1} + \left|\vec{k_2}\right|\partial_{k^1} = P^0\partial_{k^1}$$

and

$$\begin{aligned}
\left|\vec{k_1}\right|\frac{\partial k^z}{\partial k_1^1} + \left|\vec{k_2}\right|\frac{\partial k^z}{\partial k_2^1} &= \frac{1}{k^z}\left[\left(k_1^1|\vec{k}_2| - k_2^1|\vec{k}_1|\right) + \left(k_2^1|\vec{k}_1| - k_1^1|\vec{k}_2|\right)\right] = 0 \\
\left|\vec{k_1}\right|\frac{\partial \theta}{\partial k_1^1} + \left|\vec{k_2}\right|\frac{\partial \theta}{\partial k_2^1} &= \frac{\left|\vec{k_1}\right|}{|\vec{k}_1| + |\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \frac{k^2 k^z}{\vec{k}^2}\right]
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\left|\vec{k}_2\right|}{|\vec{k}_1|+|\vec{k}_2|}\left[\frac{k^2k^z}{\vec{k}^2}-\sqrt{\frac{|\vec{k}_1|}{|\vec{k}_2|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\right] \\
= & \frac{k^2k^z}{\vec{k}^2}
\end{aligned}
\tag{5.23}
$$

and therefore

$$
\begin{aligned}
\left|\vec{k}_1\right|\partial_{k_1^1}+\left|\vec{k}_2\right|\partial_{k_2^1} &= P^0\partial_{k_1}+\left|\vec{k}_1\right|\frac{\partial\theta}{\partial k_1^1}\frac{\partial}{\partial\theta}+\left|\vec{k}_2\right|\frac{\partial\theta}{\partial k_2^1}\frac{\partial}{\partial\theta} \\
&= P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta .
\end{aligned}
$$

The generator is then given by (performing the mode expansion)

$$
\begin{aligned}
M^{tx} &= \frac{i}{2}\int\frac{d\vec{\kappa}}{P^0}\left[\left(|\vec{k}_2|\partial_{k_1^x}+|\vec{k}_1|\partial_{k_2^x}\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= \frac{i}{2}\int\frac{d\vec{\kappa}}{P^0}\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= \frac{i}{2}\int\frac{d\vec{\kappa}}{P^0}\left[\left(P^0\partial_{k_1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)\left(-\sqrt{\frac{P^0}{2}}i\left(a_{\vec{\kappa}}-a^\dagger_{-\vec{\kappa}}\right)\right)\right] \\
&\quad \left(-\sqrt{\frac{P^0}{2}}i\left(a_{-\vec{\kappa}}-a^\dagger_{\vec{\kappa}}\right)\right) \\
&= -\frac{i}{4}\int d\vec{\kappa}\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)\left(a_{\vec{\kappa}}-a^\dagger_{-\vec{\kappa}}\right)\right]\left(a_{-\vec{\kappa}}-a^\dagger_{\vec{\kappa}}\right) \\
&= \frac{i}{4}\int d\vec{\kappa}-\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a_{\vec{\kappa}}\right]a_{-\vec{\kappa}}-\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\
&\quad -\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a^\dagger_{-\vec{\kappa}}\right]a_{-\vec{\kappa}}+\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a^\dagger_{-\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\
&= \frac{i}{4}\int d\vec{\kappa}\left[-\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}}+\left[-\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a^\dagger_{-\vec{\kappa}}\right]a_{-\vec{\kappa}} \\
&= \frac{i}{4}\int d\vec{\kappa}\left[-\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}}-\left[-\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a^\dagger_{\vec{\kappa}}\right]a_{\vec{\kappa}},
\end{aligned}
$$

so we have

$$
M^{tx}_{AdS}=-x^1P^0-\frac{k^2k^zk^\theta}{\vec{k}^2}.
$$

Similarly,

$$
\boxed{M^{ty}=\mathrm{Tr}\left(y^x\pi\psi\pi\right)+\mathrm{Tr}\left(y^x\pi\psi\pi\right),}
$$

and therefore

$$\begin{aligned} M^{ty} &= \frac{i}{2}\int d^2k_1 \int d^2k_2 \left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right) \frac{\left[\left(\frac{1}{|\vec{k}_2|}\partial_{k_1^y}+\frac{1}{|\vec{k}_1|}\partial_{k_2^y}\right)\pi_{k_1k_2}\right]}{\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)}\pi_{-k_2-k_1} \\ &= \frac{i}{2}\int \frac{d\vec{\kappa}}{P^0}\left[\left(|\vec{k}_1|\partial_{k_1^y}+|\vec{k}_2|\partial_{k_2^y}\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}). \end{aligned}$$

Here

$$\begin{aligned} \left(|\vec{k}_1|\partial_{k_1^y}+|\vec{k}_2|\partial_{k_2^y}\right) &= |\vec{k}_1|\left(\frac{\partial k^2}{\partial k_1^2}\frac{\partial}{\partial k^2}+\frac{\partial k^z}{\partial k_1^2}\frac{\partial}{\partial k^z}+\frac{\partial \theta}{\partial k_1^2}\frac{\partial}{\partial \theta}\right) \\ &\quad +|\vec{k}_2|\left(\frac{\partial k^2}{\partial k_2^2}\frac{\partial}{\partial k^2}+\frac{\partial k^z}{\partial k_2^2}\frac{\partial}{\partial k^z}+\frac{\partial \theta}{\partial k_2^2}\frac{\partial}{\partial \theta}\right) \\ &= P^0\partial_{k^2}+0\partial_{k^z}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta, \end{aligned}$$

and a mode expansion gives

$$M^{ty} = -\frac{i}{4}\int d\vec{\kappa}\left[\left(P^0\partial_{k^2}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}}-\left[\left(P^0\partial_{k^2}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta\right)a^\dagger_{\vec{\kappa}}\right]a_{\vec{\kappa}},$$

so that the AdS generator is given by

$$M^{ty}_{AdS} = x^2P^0-\frac{k^1k^zk^\theta}{\vec{k}^2}.$$

5.3.5 Calculation of K^t

Let us now understand the special conformal generators, the first of which is

$$\boxed{K^t = \int dxdy\left[\mathcal{H}x^2\right].}$$

Comparing with the expression for M^{tx} in equation (5.17), we obtain that

$$K^t_1 = \mathrm{Tr}\left((x^x)^2\pi\psi\pi\right)+\mathrm{Tr}\left((y^x)^2\pi\psi\pi\right).$$

Going to momentum space, we then compare with the equivalent expression for M^{tx} in equation (5.20) to obtain

$$\begin{aligned} K_1^t &= i\int d^2k_1 \int d^2k_2 \left[\left(\frac{1}{2|\vec{k}_2|}\left(\frac{\partial}{\partial k_1^x}\right)^2 \pi_{k_1k_2}\right)\pi_{-k_2-k_1} \right. \\ &\left. + \left(\frac{1}{2|\vec{k}_1|}\left(\frac{\partial}{\partial k_2^x}\right)^2 \pi_{k_1k_2}\right)\pi_{-k_2-k_1}\right]. \end{aligned}$$

This we can symmetrise to give

$$\begin{aligned} K_1^t &= \frac{i}{2}\int d^2k_1 \int d^2k_2 \left[\left(\frac{1}{2|\vec{k}_2|}\left(\frac{\partial}{\partial k_1^x}\right)^2 + \frac{1}{2|\vec{k}_2|}\left(\frac{\partial}{\partial k_1^y}\right)^2 \right.\right. \\ &\left.\left. + \frac{1}{2|\vec{k}_1|}\left(\frac{\partial}{\partial k_2^x}\right)^2 + \frac{1}{2|\vec{k}_1|}\left(\frac{\partial}{\partial k_2^y}\right)^2\right)\pi_{k_1k_2}\right]\pi_{-k_2-k_1}. \end{aligned}$$

In AdS, this becomes

$$\begin{aligned} K_1^t &= \frac{i}{2}\int d^2k_1 \int d^2k_2 \left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]\frac{1}{\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}}\left[\left(\frac{1}{2|\vec{k}_2|}\left(\frac{\partial}{\partial k_1^x}\right)^2 + \frac{1}{2|\vec{k}_2|}\left(\frac{\partial}{\partial k_1^y}\right)^2 \right.\right. \\ &\left.\left. + \frac{1}{2|\vec{k}_1|}\left(\frac{\partial}{\partial k_2^x}\right)^2 + \frac{1}{2|\vec{k}_1|}\left(\frac{\partial}{\partial k_2^y}\right)^2\right)\pi_{k_1k_2}\right]\pi_{-k_2-k_1} \\ &= \frac{i}{4}\int d\vec{\kappa}\frac{1}{P^0}\left[\left(|\vec{k}_1|\left(\frac{\partial}{\partial k_1^x}\right)^2 + |\vec{k}_1|\left(\frac{\partial}{\partial k_1^y}\right)^2\right.\right. \\ &\left.\left. + |\vec{k}_2|\left(\frac{\partial}{\partial k_2^x}\right)^2 + |\vec{k}_2|\left(\frac{\partial}{\partial k_2^y}\right)^2\right)\pi_{k_1k_2}\right]\pi_{-k_2-k_1}. \end{aligned}$$

The square partial derivatives are of the form

$$\begin{aligned} (\partial_{k_i^j})^2 &= \partial_{k_i^j}\left[\partial_{k^j} + \left(\partial_{k_i^j}k^z\right)\partial_{k^z} + \left(\partial_{k_i^j}\theta\right)\partial_\theta\right] \\ &= +(\partial_{k^j})^2 + \left(\partial_{k_i^j}k^z\right)^2(\partial_{k^z})^2 + \left(\partial_{k_i^j}\theta\right)^2(\partial_\theta)^2 \\ &\quad +2\left(\partial_{k_i^j}k^z\right)\partial_{k^j}\partial_{k^z} + 2\left(\partial_{k_i^j}\theta\right)\partial_{k^j}\partial_\theta + 2\left(\partial_{k_i^j}k^z\right)\left(\partial_{k_i^j}\theta\right)\partial_{k^z}\partial_\theta \\ &\quad +\partial_{k_i^j}\left(\partial_{k_i^j}k^z\right)\partial_{k^z} + \partial_{k_i^j}\left(\partial_{k_i^j}\theta\right)\partial_\theta. \end{aligned} \tag{5.24}$$

Thus, using the partial derivatives in equations (5.11) in calculating

$$\frac{1}{P^0}\left[|\vec{k}_1|\left(\frac{\partial}{\partial k_1^1}\right)^2+|\vec{k}_1|\left(\frac{\partial}{\partial k_1^2}\right)^2+|\vec{k}_2|\left(\frac{\partial}{\partial k_2^1}\right)^2+|\vec{k}_2|\left(\frac{\partial}{\partial k_2^2}\right)^2\right],$$

we find terms involving $(\partial_{k^1})^2$ and $(\partial_{k^2})^2$ give

$$\frac{1}{P^0}\left[|\vec{k}_1|\,(\partial_{k^1})^2+|\vec{k}_1|\,(\partial_{k^2})^2+|\vec{k}_2|\,(\partial_{k^1})^2+|\vec{k}_2|\,(\partial_{k^2})^2\right]=\frac{P^0}{P^0}(x^2+y^2)=\vec{x}^2_{\text{AdS}}\tag{5.25}$$

Terms involving $\left(\partial_{k_i^j}k^z\right)^2(\partial_{k^z})^2$ sum to

$$\begin{aligned}
&\frac{1}{P^0}\left[|\vec{k}_1|\left[\frac{1}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k_2^1\right)\right]^2+|\vec{k}_1|\left[\frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}-k_2^2\right)\right]^2\right.\\
&\left.+|\vec{k}_2|\left[\frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^1\right)\right]^2+|\vec{k}_2|\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^2\right)\right]^2\right](\partial_{k^z})^2\\
=\;&\frac{1}{P^0(k^z)^2}\left[\frac{1}{|\vec{k}_1|}\left[|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right]^2+\frac{1}{|\vec{k}_1|}\left[|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right]^2\right.\\
&\left.+\frac{1}{|\vec{k}_2|}\left[|\vec{k}_1|k_2^1-|\vec{k}_2|k_1^1\right]^2+\frac{1}{|\vec{k}_2|}\left[|\vec{k}_1|k_2^2-|\vec{k}_2|k_1^2\right]^2\right](\partial_{k^z})^2\\
=\;&\frac{1}{P^0(k^z)^2}\left[\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right]\left[\left[|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right]^2+\left[|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right]^2\right](\partial_{k^z})^2\\
=\;&\frac{1}{|\vec{k}_1||\vec{k}_2|(k^z)^2}\left[\left[|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right]^2+\left[|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right]^2\right](\partial_{k^z})^2\,.
\end{aligned}\tag{5.26}$$

Now

$$\begin{aligned}
|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1 &= |\vec{k}_1||\vec{k}_2|\,(\cos\varphi_1-\cos\varphi_2)\\
&= -2|\vec{k}_1||\vec{k}_2|\left(\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\\
&= -\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1+\varphi_2}{2}k^z\\
|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2 &= |\vec{k}_1||\vec{k}_2|\,(\sin\varphi_1-\sin\varphi_2)\\
&= 2|\vec{k}_1||\vec{k}_2|\left(\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\\
&= \sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1+\varphi_2}{2}k^z
\end{aligned}$$

so that the expression in (5.26) becomes

$$\frac{|\vec{k}_1||\vec{k}_2|(k^z)^2}{|\vec{k}_1||\vec{k}_2|(k^z)^2}\left(\partial_{k^z}\right)^2 = \left(\partial_{k^z}\right)^2 = z_{\text{AdS}}^2. \tag{5.27}$$

Continuing, terms of the form $\left(\partial_{k_i^j}\theta\right)^2\left(\partial_\theta\right)^2$ in equation (5.24) sum to

$$\begin{aligned}
= \quad & \frac{1}{P^0}\left[|\vec{k}_1|\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right]^2\right.\\
& +|\vec{k}_1|\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]^2\\
& +|\vec{k}_2|\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]^2\\
& \left.+|\vec{k}_2|\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]^2\right]\left(\partial_\theta\right)^2\\
. &
\end{aligned} \tag{5.28}$$

The sum of all the first terms squared gives

$$\begin{aligned}
& \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{2P^0(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2\right)}\left[|\vec{k}_1|\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]^2+|\vec{k}_1|\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]^2\right.\\
& \left.+|\vec{k}_2|\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]^2+|\vec{k}_2|\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]^2\right]\\
= \quad & \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{2P^0(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2\right)}\left[\frac{1}{|\vec{k}_1|}\left[|\vec{k}_2|k_1^1+|\vec{k}_1|k_2^1\right]^2+\frac{1}{|\vec{k}_1|}\left[|\vec{k}_2|k_1^2+|\vec{k}_1|k_2^2\right]^2\right.\\
& \left.+\frac{1}{|\vec{k}_2|}\left[|\vec{k}_1|k_2^1+|\vec{k}_2|k_1^1\right]^2+|\vec{k}_2|\left[|\vec{k}_1|k_2^2+|\vec{k}_2|k_1^2\right]^2\right]\\
= \quad & \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{2P^0(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2\right)}\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)
\end{aligned}$$

$$
\begin{aligned}
& \left[\left[|\vec{k}_2|k_1^1 + |\vec{k}_1|k_2^1\right]^2 + \left[|\vec{k}_2|k_1^2 + |\vec{k}_1|k_2^2\right]^2\right] \\
= \;& \frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2}{2|\vec{k}_1||\vec{k}_2|(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)} \left[\left[|\vec{k}_1||\vec{k}_2|\left(\cos\varphi_1 + \cos\varphi_2\right)\right]^2 + \left[|\vec{k}_1||\vec{k}_2|\left(\sin\varphi_1 + \sin\varphi_2\right)\right]^2\right] \\
= \;& \frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2 |\vec{k}_1||\vec{k}_2|}{2(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)} \left[2 + 2\cos\varphi_1\cos\varphi_2 + 2\sin\varphi_1\sin\varphi_2\right] \\
= \;& \frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2 |\vec{k}_1||\vec{k}_2| \left[1 + \cos(\varphi_2 - \varphi_1)\right]}{(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)} \\
= \;& \frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2 |\vec{k}_1||\vec{k}_2| \left[2\cos^2\frac{\varphi_2-\varphi_1}{2}\right]}{(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)}.
\end{aligned}
\tag{5.29}
$$

From equations (5.9) and (5.10)

$$
\begin{aligned}
\tan\theta &= \frac{2\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\left(\frac{\varphi_2-\varphi_1}{2}\right)}{\left(|\vec{k}_1| - |\vec{k}_2|\right)} = \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{|\vec{k}_1| - |\vec{k}_2|} \\
\Rightarrow \cos^2\left(\frac{\varphi_2-\varphi_1}{2}\right) &= \frac{\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)}{2|\vec{k}_1||\vec{k}_2|}
\end{aligned}
$$

so that the expression in (5.29) becomes

$$
\frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2}{(\vec{k}^2)^2}. \tag{5.30}
$$

The sum of the middle terms in equation (5.28) becomes

$$
\begin{aligned}
& \frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)}{P^0\vec{k}^4}\left[-\frac{k_1^1}{|\vec{k}_1|}\left[|\vec{k}_2|k_1^1 + |\vec{k}_1|k_2^1\right] - \frac{k_1^2}{|\vec{k}_1|}\left[|\vec{k}_2|k_1^2 + |\vec{k}_1|k_2^2\right]\right. \\
& \left. + \frac{k_2^1}{|\vec{k}_2|}\left[|\vec{k}_1|k_2^1 + |\vec{k}_2|k_1^1\right] + \frac{k_2^2}{|\vec{k}_2|}\left[|\vec{k}_1|k_2^2 + |\vec{k}_2|k_1^2\right]\right] \\
= \;& \frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)}{P^0\vec{k}^4}\left[-\frac{|\vec{k}_2|}{|\vec{k}_1|}\left((k_1^1)^2 + (k_1^2)^2\right) + \frac{|\vec{k}_1|}{|\vec{k}_2|}\left((k_2^1)^2 + (k_2^2)^2\right)\right] \\
= \;& \frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)}{P^0\vec{k}^4}\left[-|\vec{k}_1||\vec{k}_2| + |\vec{k}_1||\vec{k}_2|\right]
\end{aligned}
$$

$$= \quad 0 \tag{5.31}$$

and the sum of the last terms in equation (5.28) becomes

$$\begin{aligned}
& \frac{2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2\right)}{P^0 \vec{k}^4} \left[\frac{(k_1^1)^2}{|\vec{k}_1|} + \frac{(k_1^2)^2}{|\vec{k}_1|} + \frac{(k_2^1)^2}{|\vec{k}_2|} + \frac{(k_2^2)^2}{|\vec{k}_2|}\right] \\
= \quad & \frac{2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2\right)}{P^0 \vec{k}^4} \left[|\vec{k}_1| + |\vec{k}_2|\right] \\
= \quad & \frac{2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2\right)}{\vec{k}^4}.
\end{aligned} \tag{5.32}$$

Adding equations (5.30), (5.31) and (5.32) gives

$$\frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)^2}{\vec{k}^4} + \frac{2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2\right)}{\vec{k}^4} = \frac{\vec{k}^2}{\vec{k}^4} = \frac{1}{\vec{k}^2}.$$

Therefore, equation (5.28) becomes

$$\frac{1}{\vec{k}^2} \left(\partial_\theta\right)^2 = \frac{1}{\vec{k}^2} \left(k^\theta\right)^2. \tag{5.33}$$

Next, consider terms involving $\left(\partial_{k_i^j} k^z\right) \partial_{k^j} \partial_{k^z}$, which sum to

$$\begin{aligned}
& \frac{2}{P^0} \left[|\vec{k}_1| \left[\frac{1}{k^z} \left(k_1^1 \frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right)\right] \partial_{k^1} + |\vec{k}_1| \left[\frac{1}{k^z} \left(k_1^2 \frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right)\right] \partial_{k^2} \right. \\
& \left. + |\vec{k}_2| \left[\frac{1}{k^z} \left(k_2^1 \frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right)\right] \partial_{k^1} + |\vec{k}_2| \left[\frac{1}{k^z} \left(k_2^2 \frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right)\right] \partial_{k^2} \right] \partial_{k^z} \\
= \quad & 0.
\end{aligned} \tag{5.34}$$

For terms involving $\left(\partial_{k_i^j} \theta\right) \partial_{k^j} \partial_\theta$, recall that from equation (5.23),

$$\begin{aligned}
\left|\vec{k}_1\right| \frac{\partial \theta}{\partial k_1^1} + \left|\vec{k}_2\right| \frac{\partial \theta}{\partial k_2^1} &= \frac{k^2 k^z}{\vec{k}^2} \\
\left|\vec{k}_1\right| \frac{\partial \theta}{\partial k_1^2} + \left|\vec{k}_2\right| \frac{\partial \theta}{\partial k_2^2} &= -\frac{k^1 k^z}{\vec{k}^2}
\end{aligned}$$

so that the $\left(\partial_{k_i^j}\theta\right)\partial_{k^j}\partial_\theta$ contribution is given by

$$\begin{aligned}
2\left(\frac{k^2k^z}{P^0\vec{k}^2}\partial_{k^1}\partial_\theta - \frac{k^1k^z}{P^0\vec{k}^2}\partial_{k^2}\partial_\theta\right) &= 2\left(\frac{k^2k^zx^1k^\theta}{P^0\vec{k}^2} - \frac{k^1k^zx^2k^\theta}{P^0\vec{k}^2}\right) \\
&= 2\left(\frac{k^zk^\theta}{P^0\vec{k}^2}\left[x^1k^2 - x^2k^1\right]\right) \\
&= \frac{2k^zk^\theta}{P^0\vec{k}^2}M^{xy} \qquad (5.35)
\end{aligned}$$

For terms involving $2\left(\partial_{k_i^j}k^z\right)\left(\partial_{k_i^j}\theta\right)\partial_{k^z}\partial_\theta$, we have (ignoring the factor of $2\partial_{k^z}\partial_\theta$ for now)

$$\begin{aligned}
&|\vec{k}_1|\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right]\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^1\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right] \\
&+|\vec{k}_1|\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right]\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^2\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right] \\
&+|\vec{k}_2|\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right]\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} + k_1^1\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right] \\
&+|\vec{k}_2|\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right]\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} + k_1^2\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right] \\
=&\left[\frac{\left[|\vec{k}_2|^2(k_1^1)^2 - |\vec{k}_1|^2(k_2^1)^2\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}|\vec{k}_1|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right]k_1^1}{\vec{k}^2}\right] \\
&+\left[\frac{\left[|\vec{k}_2|^2(k_1^2)^2 - |\vec{k}_1|^2(k_2^2)^2\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}|\vec{k}_1|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} - \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right]k_1^2}{\vec{k}^2}\right] \\
&+\left[\frac{\left[|\vec{k}_1|^2(k_2^1)^2 - |\vec{k}_2|^2(k_1^1)^2\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}|\vec{k}_2|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right]k_2^1}{\vec{k}^2}\right] \\
&+\left[\frac{\left[|\vec{k}_1|^2(k_2^2)^2 - |\vec{k}_2|^2(k_1^2)^2\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}|\vec{k}_2|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} + \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right]k_2^2}{\vec{k}^2}\right] \\
&. \qquad (5.36)
\end{aligned}$$

The first terms in (5.36) give

$$
\begin{aligned}
& \frac{\left[|\vec{k}_2|^2(k_1^1)^2 - |\vec{k}_1|^2(k_2^1)^2 + |\vec{k}_2|^2(k_1^2)^2 - |\vec{k}_1|^2(k_2^2)^2\right]\left(|\vec{k}_1||\vec{k}_2| - |\vec{k}_2|^2\right)}{\sqrt{2}|\vec{k}_1||\vec{k}_2|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} \\
& + \frac{\left[|\vec{k}_1|^2(k_2^1)^2 - |\vec{k}_2|^2(k_1^1)^2 + |\vec{k}_1|^2(k_2^2)^2 - |\vec{k}_2|^2(k_1^2)^2\right]\left(|\vec{k}_1|^2 - |\vec{k}_1||\vec{k}_2|\right)}{\sqrt{2}|\vec{k}_1||\vec{k}_2|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} \\
= \quad & \frac{\left[(k_1^1)^2 + (k_1^2)^2\right]\left[-|\vec{k}_2|^4 + 2|\vec{k}_1||\vec{k}_2|^3 - |\vec{k}_1|^2|\vec{k}_2|^2\right]}{\sqrt{2}|\vec{k}_1||\vec{k}_2|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} \\
& + \frac{\left[(k_2^1)^2 + (k_2^2)^2\right]\left[|\vec{k}_1|^4 - 2|\vec{k}_1|^3|\vec{k}_2| + |\vec{k}_1|^2|\vec{k}_2|^2\right]}{\sqrt{2}|\vec{k}_1||\vec{k}_2|\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}} \\
= \quad & 0.
\end{aligned}
$$

and the second terms in (5.36) give

$$
\begin{aligned}
& \frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\left[-\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right]k_1^1 - \left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right]k_1^2\right. \\
& \left. + \left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right]k_2^1 + \left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right]k_2^2\right] \\
= \quad & 0
\end{aligned} \tag{5.37}
$$

so that the $2\left(\partial_{k_i^j}k^z\right)\left(\partial_{k_i^j}\theta\right)\partial_{k^z}\partial_\theta$ terms contribute zero. Penultimately, for the $\partial_{k_i^j}\left(\partial_{k_i^j}k^z\right)\partial_{k^z}$ terms we have

$$
\begin{aligned}
& \left[-\frac{1}{(k^z)^2}\left[\frac{|\vec{k}_1|}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right)^2\right] + \frac{|\vec{k}_1|}{k^z}\left[\frac{|\vec{k}_2|}{|\vec{k}_1|} - \frac{(k_1^1)^2|\vec{k}_2|}{|\vec{k}_1|^3}\right]\right. \\
& -\frac{1}{(k^z)^2}\left[\frac{|\vec{k}_1|}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right)^2\right] + \frac{|\vec{k}_1|}{k^z}\left[\frac{|\vec{k}_2|}{|\vec{k}_1|} - \frac{(k_1^2)^2|\vec{k}_2|}{|\vec{k}_1|^3}\right] \\
& -\frac{1}{(k^z)^2}\left[\frac{|\vec{k}_2|}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right)^2\right] + \frac{|\vec{k}_2|}{k^z}\left[\frac{|\vec{k}_1|}{|\vec{k}_2|} - \frac{(k_2^1)^2|\vec{k}_1|}{|\vec{k}_2|^3}\right] \\
& \left.-\frac{1}{(k^z)^2}\left[\frac{|\vec{k}_2|}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right)^2\right] + \frac{|\vec{k}_2|}{k^z}\left[\frac{|\vec{k}_1|}{|\vec{k}_2|} - \frac{(k_2^2)^2|\vec{k}_1|}{|\vec{k}_2|^3}\right]\right]\partial_{k^z}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{(k^z)^3}\left[\frac{1}{|\vec{k}_1|}\left(|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right)^2-\left[|\vec{k}_2|-\frac{(k_1^1)^2|\vec{k}_2|}{|\vec{k}_1|^2}\right](k^z)^2\right.\\
&\quad+\frac{1}{|\vec{k}_1|}\left(|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right)^2-\left[|\vec{k}_2|-\frac{(k_1^2)^2|\vec{k}_2|}{|\vec{k}_1|^2}\right](k^z)^2\\
&\quad+\frac{1}{|\vec{k}_2|}\left(|\vec{k}_1|k_2^1-|\vec{k}_2|k_1^1\right)^2-\left[|\vec{k}_1|-\frac{(k_2^1)^2|\vec{k}_1|}{|\vec{k}_2|^2}\right](k^z)^2\\
&\quad\left.+\frac{1}{|\vec{k}_2|}\left(|\vec{k}_1|k_2^2-|\vec{k}_2|k_1^2\right)^2-\left[|\vec{k}_1|-\frac{(k_2^2)^2|\vec{k}_1|}{|\vec{k}_2|^2}\right](k^z)^2\right]\partial_{k^z}\\
&= -\frac{1}{(k^z)^3}\left[\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\left[\left(|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right)^2+\left(|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right)^2\right]\right.\\
&\quad\left.-\left(2|\vec{k}_1|+2|\vec{k}_2|-|\vec{k}_1|-|\vec{k}_2|\right)(k^z)^2\right]\partial_{k^z}\\
&= -\frac{1}{(k^z)^3}\left[\left(\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right)\left[2|\vec{k}_2|^2|\vec{k}_1|^2-2|\vec{k}_1||\vec{k}_2|\left(k_1^1k_2^1+k_1^2k_2^2\right)\right]-P^0(k^z)^2\right]\partial_{k^z}\\
&= -\frac{1}{(k^z)^3}\left[P^0\left(2|\vec{k}_1||\vec{k}_2|-2\vec{k}_1\cdot\vec{k}_2\right)-P^0(k^z)^2\right]\partial_{k^z}\\
&= 0.
\end{aligned}
\tag{5.38}
$$

Finally, for terms involving $\partial_{k_i^j}\left(\partial_{k_i^j}\theta\right)\partial_\theta$, we need to calculate

$$
\begin{aligned}
&|\vec{k}_1|\partial_{k_1^1}\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right]\\
&+|\vec{k}_1|\partial_{k_1^2}\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]\\
&+|\vec{k}_2|\partial_{k_2^1}\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]\\
&+|\vec{k}_2|\partial_{k_2^2}\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right],
\end{aligned}
\tag{5.39}
$$

which involves a sum of 4 'first terms' and a sum of 4 'second terms'. Since

$$
\partial_{k_1^1}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2} = \frac{\frac{|\vec{k}_2|}{|\vec{k}_1|}k_1^1+k_2^1}{2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}},
$$

derivatives coming from the $\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}$ in the numerator of the second terms of (5.39) are given by

$$\begin{aligned}
&\left[\frac{\sqrt{2}}{2\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\right]\left[-k_1^1\left(\frac{|\vec{k}_2|}{|\vec{k}_1|}k_1^1+k_2^1\right)-k_1^2\left(\frac{|\vec{k}_2|}{|\vec{k}_1|}k_1^2+k_2^2\right)\right.\\
&\left.+k_2^1\left(\frac{|\vec{k}_1|}{|\vec{k}_2|}k_2^1+k_1^1\right)+k_2^2\left(\frac{|\vec{k}_1|}{|\vec{k}_2|}k_2^2+k_1^2\right)\right]\\
=&\left[\frac{\sqrt{2}}{2\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\right]\left[-|\vec{k}_1||\vec{k}_2|-\vec{k}_1\cdot\vec{k}_2+|\vec{k}_1||\vec{k}_2|+2\vec{k}_1\cdot\vec{k}_2\right]\\
=&\ 0.
\end{aligned}$$

Derivatives resulting from the $\left[|\vec{k}_1|-|\vec{k}_2|\right]$ in the first terms give

$$\begin{aligned}
&\frac{|\vec{k}_1|\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(\frac{k_1^1}{|\vec{k}_1|}\right)+|\vec{k}_1|\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(\frac{k_1^2}{|\vec{k}_1|}\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\\
&-\frac{|\vec{k}_2|\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(\frac{k_2^1}{|\vec{k}_2|}\right)+|\vec{k}_2|\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(\frac{k_2^2}{|\vec{k}_2|}\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\\
=&\ \frac{\frac{1}{|\vec{k}_1|}\left[|\vec{k}_2|\left((k_1^1)^2+(k_1^2)^2\right)+|\vec{k}_1|\left(k_1^1k_2^1+k_1^2k_2^2\right)\right]}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\\
&-\frac{\frac{1}{|\vec{k}_2|}\left[|\vec{k}_1|\left((k_2^1)^2+(k_2^2)^2\right)+|\vec{k}_2|\left(k_1^1k_2^1+k_1^2k_2^2\right)\right]}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\\
=&\ 0.
\end{aligned}$$

Contributions from derivatives of k_i^j in the second terms of equation (5.39) sum to

$$-\frac{2\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}+\frac{2\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}=0.$$

and from the $\frac{1}{|\vec{k}_i|}$ in the second terms,

$$\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\left[|\vec{k}_1|\frac{(k_1^1)^2}{|\vec{k}_1|^3}+|\vec{k}_1|\frac{(k_1^2)^2}{|\vec{k}_1|^3}-|\vec{k}_2|\frac{(k_2^1)^2}{|\vec{k}_2|^3}-|\vec{k}_2|\frac{(k_2^2)^2}{|\vec{k}_2|^3}\right]$$

$$= \quad 0.$$

Derivatives resulting from the k_i^j in the numerator of the first terms of equation (5.39) become

$$\frac{\left[2|\vec{k}_2| + 2|\vec{k}_1|\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}; \tag{5.40}$$

derivatives resulting from the $\frac{|\vec{k}_i|}{|\vec{k}_j|}$ from the first terms gives

$$\frac{\left[-\frac{|\vec{k}_1||\vec{k}_2|\left[(k_1^1)^2+(k_1^2)^2\right]}{|\vec{k}_1|^3} - \frac{|\vec{k}_2||\vec{k}_1|\left[(k_2^1)^2+(k_2^2)^2\right]}{|\vec{k}_2|^3}\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}$$

$$= \quad -\frac{\left[|\vec{k}_1| + |\vec{k}_2|\right]\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}. \tag{5.41}$$

and derivatives coming from the $\frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}$ in the first term in (5.39) are given by

$$-\frac{1}{2}\frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)^{3/2}}\left[|\vec{k}_1|\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^1\right]^2 + |\vec{k}_1|\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^2\right]^2\right.$$
$$\left.+|\vec{k}_2|\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} + k_1^1\right]^2 + |\vec{k}_2|\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} + k_1^2\right]^2\right]$$
$$= \quad -\frac{1}{2}\frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)^{3/2}}\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]$$
$$\left[\left[|\vec{k}_2|k_1^1 + |\vec{k}_1|k_2^1\right]^2 + \left[|\vec{k}_2|k_1^2 + |\vec{k}_1|k_2^2\right]^2\right]$$
$$= \quad -\frac{1}{2}\frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\left(|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2\right)^{3/2}}\left[\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right]$$
$$\left[2|\vec{k}_1|^2|\vec{k}_2|^2 + 2|\vec{k}_1||\vec{k}_2|\vec{k}_1\cdot\vec{k}_1\right]$$
$$= \quad -\frac{\left(|\vec{k}_1| - |\vec{k}_2|\right)\left(|\vec{k}_1| + |\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}}, \tag{5.42}$$

so that (5.40), (5.41) and (5.42) cancel to give zero.

Now

$$\begin{aligned}
\partial_{k_1^1}\frac{1}{\vec{k}^2} &= \partial_{k_1^1}\frac{1}{|\vec{k}_1|^2+|\vec{k}_2|^2+2\vec{k}_1\cdot\vec{k}_2} \\
&= -\frac{2k_1^1+2k_2^1}{\vec{k}^4}.
\end{aligned}$$

Thus, derivative terms coming from $\frac{1}{\vec{k}^2}$ of the first term of (5.39) become

$$\begin{aligned}
& -\left[\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\right]\frac{1}{\vec{k}^4}\left[|\vec{k}_1|\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right)\left(2k_1^1+2k_2^1\right)\right. \\
& +|\vec{k}_1|\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(2k_1^2+2k_2^2\right)+|\vec{k}_2|\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(2k_2^1+2k_1^1\right) \\
& \left.+|\vec{k}_2|\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(2k_2^2+2k_1^2\right)\right] \\
= & -\left[\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\right]\frac{1}{\vec{k}^4} \\
& \left[\left(|\vec{k}_2|k_1^1+|\vec{k}_1|k_2^1\right)\left(2k_1^1+2k_2^1\right)+\left[|\vec{k}_2|k_1^2+|\vec{k}_1|k_2^2\right]\left(2k_1^2+2k_2^2\right)\right] \\
= & -\left[\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\right]\frac{1}{\vec{k}^4} \\
& \left[4|\vec{k}_2||\vec{k}_1|^2+4|\vec{k}_1||\vec{k}_2|^2+4\left(|\vec{k}_1|+|\vec{k}_2|\right)\vec{k}_1\cdot\vec{k}_2\right] \\
= & -\left[\frac{4\left(|\vec{k}_1|-|\vec{k}_2|\right)\left(|\vec{k}_1|+|\vec{k}_2|\right)}{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\right]\frac{1}{\vec{k}^4}\left[|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2\right] \\
= & -\frac{2\sqrt{2}}{\vec{k}^4}\left(|\vec{k}_1|-|\vec{k}_2|\right)\left(|\vec{k}_1|+|\vec{k}_2|\right)\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2} \qquad (5.43)
\end{aligned}$$

and derivative terms coming from $\frac{1}{\vec{k}^2}$ of the second term of (5.39) become

$$\begin{aligned}
& \frac{1}{\vec{k}^4}\left[\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}k_1^1\left(2k_1^1+2k_2^1\right)\right. \\
& +\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}k_1^2\left(2k_1^2+2k_2^2\right) \\
& -\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}k_2^1\left(2k_2^1+2k_1^1\right)
\end{aligned}$$

$$
\begin{aligned}
& -\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}\,k_2^2\left(2k_2^2 + 2k_1^2\right) \\
= \;& \frac{1}{\vec{k}^4}\left[\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1\cdot\vec{k}_2}\left(2|\vec{k}_1|^2 - 2|\vec{k}_2|^2\right)\right]. \qquad (5.44)
\end{aligned}
$$

Terms (5.43) and (5.44) therefore sum to give zero, so that all the terms involving $\partial_{k_i^j}\left(\partial_{k_i^j}\theta\right)\partial_\theta$ sum to zero. Thus, adding all the contributions from equations (5.25), (5.27), (5.33), (5.34), (5.35), (5.37), (5.38) and we find that equation (5.24) becomes

$$
K_1^t = \vec{x}_{\text{AdS}}^2 + z_{\text{AdS}}^2 + \frac{1}{\vec{k}^2}\left(k^\theta\right)^2 + \frac{2k^z k^\theta}{P^0\vec{k}^2}M^{xy}.
$$

Performing the mode expansion we obtain

$$
\begin{aligned}
K_1^t &= \frac{i}{4}\int d\vec{\kappa}\left[\left(\vec{x}^2 + z^2 + \frac{\left(k^\theta\right)^2}{\vec{k}^2} + \frac{2k^z k^\theta}{P^0\vec{k}^2}M^{xy}\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= \frac{i}{4}\int d\vec{\kappa}\left[\left(\vec{x}^2 + z^2 + \frac{\left(k^\theta\right)^2}{\vec{k}^2} + \frac{2k^z k^\theta}{P^0\vec{k}^2}M^{xy}\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= \frac{i}{4}\int d\vec{\kappa}\left[\left(\vec{x}^2 + z^2 + \frac{\left(k^\theta\right)^2}{\vec{k}^2} + \frac{2k^z k^\theta}{P^0\vec{k}^2}M^{xy}\right)\left(-\sqrt{\frac{P^0}{2}}i\left(a_{\vec{\kappa}} - a_{-\vec{\kappa}}^\dagger\right)\right)\right] \\
&\quad \left(-\sqrt{\frac{P^0}{2}}i\left(a_{-\vec{\kappa}} - a_{\vec{\kappa}}^\dagger\right)\right) \\
&= -\frac{i}{8}\int d\vec{\kappa}\left[\left(\left(\vec{x}^2 + z^2\right)P^0 + \frac{\left(k^\theta\right)^2 P^0}{\vec{k}^2} + \frac{2k^z k^\theta}{\vec{k}^2}M^{xy}\right)\left(a_{\vec{\kappa}} - a_{-\vec{\kappa}}^\dagger\right)\right]\left(a_{-\vec{\kappa}} - a_{\vec{\kappa}}^\dagger\right) \\
&= \frac{i}{8}\int d\vec{\kappa} - \left[\left(\left(\vec{x}^2 + z^2\right)P^0 + \frac{\left(k^\theta\right)^2 P^0}{\vec{k}^2} + \frac{2k^z k^\theta}{\vec{k}^2}M^{xy}\right)a_{\vec{\kappa}}\right]a_{-\vec{\kappa}} \\
&\quad + \left[\left(\left(\vec{x}^2 + z^2\right)P^0 + \frac{\left(k^\theta\right)^2 P^0}{\vec{k}^2} + \frac{2k^z k^\theta}{\vec{k}^2}M^{xy}\right)a_{\vec{\kappa}}\right]a_{\vec{\kappa}}^\dagger \\
&\quad + \left[\left(\left(\vec{x}^2 + z^2\right)P^0 + \frac{\left(k^\theta\right)^2 P^0}{\vec{k}^2} + \frac{2k^z k^\theta}{\vec{k}^2}M^{xy}\right)a_{-\vec{\kappa}}^\dagger\right]a_{-\vec{\kappa}} \\
&\quad - \left[\left(\left(\vec{x}^2 + z^2\right)P^0 + \frac{\left(k^\theta\right)^2 P^0}{\vec{k}^2} + \frac{2k^z k^\theta}{\vec{k}^2}M^{xy}\right)a_{-\vec{\kappa}}^\dagger\right]a_{\vec{\kappa}}^\dagger \\
&= \frac{i}{8}\int d\vec{\kappa}\left[-\left(\left(\vec{x}^2 + z^2\right)P^0 + \frac{\left(k^\theta\right)^2 P^0}{\vec{k}^2} + \frac{2k^z k^\theta}{\vec{k}^2}M^{xy}\right)a_{\vec{\kappa}}\right]a_{\vec{\kappa}}^\dagger \\
&\quad + \left[-\left(\left(\vec{x}^2 + z^2\right)P^0 + \frac{\left(k^\theta\right)^2 P^0}{\vec{k}^2} + \frac{2k^z k^\theta}{\vec{k}^2}M^{xy}\right)a_{-\vec{\kappa}}^\dagger\right]a_{-\vec{\kappa}}
\end{aligned}
$$

$$= \frac{i}{4}\int d\vec{\kappa}\left[\left(-\frac{\left(\vec{x}^2+z^2\right)P^0}{2}-\frac{\left(k^\theta\right)^2P^0}{2\vec{k}^2}-\frac{k^zk^\theta}{\vec{k}^2}M^{xy}\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}}$$
$$+\left[\left(-\frac{\left(\vec{x}^2+z^2\right)P^0}{2}-\frac{\left(k^\theta\right)^2P^0}{2\vec{k}^2}-\frac{k^zk^\theta}{\vec{k}^2}M^{xy}\right)a^\dagger_{\vec{\kappa}}\right]a_{\vec{\kappa}},$$

so we have

$$K^t_{AdS} = -\frac{\left(\vec{x}^2+z^2\right)P^0}{2}-\frac{\left(k^\theta\right)^2P^0}{2\vec{k}^2}-\frac{k^zk^\theta}{\vec{k}^2}M^{xy}.$$

5.3.6 Calculation of K^x

We now calculate the special conformal generator K^x,

$$\boxed{K^x = \int dxdy\,[2x\mathcal{D}]\,,}$$

which is done as follows:

$$\begin{aligned}
K^x &= \int dxdy\,[2x\mathcal{D}] \\
&= 2\int dx'dy'\,\left[(\pi\partial_{x'}\phi)\,(x')^2+\left(\pi\partial_{y'}\phi\right)x'y'\right] \\
&= 2\int d^2z\,\left[(\partial_{x'}\phi^a(z))\,(x')^2+\left(\partial_{y'}\phi^a(z)\right)x'y'\right]\frac{\partial}{\partial\phi^a(z)} \\
&= 2\int d^2x\int d^2y\int d^2z\,\left[(\partial_{x'}\phi^a(z))\,(x')^2+\left(\partial_{y'}\phi^a(z)\right)x'y'\right] \\
&\quad \left(\delta^2(x-z)\phi^a(y)+\delta^2(y-z)\phi^a(x)\right)\frac{\partial}{\partial\psi(x,y)} \\
&= 2\int d^2x\int d^2y\int d^2z\,\left[(\partial_{x'}\phi^a(z))\,(x')^2+\left(\partial_{y'}\phi^a(z)\right)x'y'\right] \\
&\quad \left(\delta(x_1-x')\delta(x_2-y')\phi^a(y)+\delta(y_1-x')\delta(y_2-y')\phi^a(x)\right)\frac{\partial}{\partial\psi(x,y)} \\
&= 2\int d^2x\int d^2y\,\left[(\partial_{x_1}\phi^a(x))\,(x_1)^2+(\partial_{x_2}\phi^a(x))\,x_1x_2\right]\phi^a(y)\frac{\partial}{\partial\psi(x,y)} \\
&\quad +\int d^2x\int d^2y\,\left[(\partial_{y_1}\phi^a(y))\,(y_1)^2+(\partial_{y_2}\phi^a(y))\,y_1y_2\right]\phi^a(x)\frac{\partial}{\partial\psi(x,y)} \\
&= 2\int d^2x\int d^2y\,\left[\left((x_1)^2\partial_{x_1}+x_1x_2\partial_{x_2}\right)\psi(y,x)\right]\frac{\partial}{\partial\psi(x,y)} \\
&\quad +\int d^2x\int d^2y\,\left[\left((y_1)^2\partial_{y_1}+y_1y_2\partial_{y_2}\right)\psi(y,x)\right]\frac{\partial}{\partial\psi(x,y)}
\end{aligned}$$

$$
\begin{aligned}
\to \quad & 2\int d^2k_1 \int d^2k_2 \int d^2k_3 \int d^2k_4 \int d^2x \int d^2y \\
& \left[\left(k_1^1(\partial_{k_1^1})^2 + k_1^2\partial_{k_1^2}\partial_{k_1^1} + k_2^1(\partial_{k_2^1})^2 + k_2^2\partial_{k_2^2}\partial_{k_2^1}\right)\eta(k_1,k_2)\right] \\
& \pi(k_3,k_4)e^{i\left(k_1^1y_1+k_1^2y_2\right)}e^{i\left(k_2^1x_1+k_2^2x_2\right)}e^{i\left(k_3^1x_1+k_3^2x_2\right)}e^{i\left(k_4^1y_1+k_4^2y_2\right)} \\
= \quad & 2\int d^2k_1 \int d^2k_2 \\
& \left[\left(k_1^1(\partial_{k_1^1})^2 + k_1^2\partial_{k_1^2}\partial_{k_1^1} + k_2^1(\partial_{k_2^1})^2 + k_2^2\partial_{k_2^2}\partial_{k_2^1}\right)\eta(k_1,k_2)\right]\pi(-k_2,-k_1).
\end{aligned}
$$

Symmetrising this, we obtain

$$
\begin{aligned}
K^x \quad = \quad & 2\int d^2k_1 \int d^2k_2 \\
& \left[\left(k_1^1(\partial_{k_1^1})^2 + k_2^1(\partial_{k_2^1})^2 + k_1^2\partial_{k_1^2}\partial_{k_1^1} + k_2^2\partial_{k_2^2}\partial_{k_2^1}\right)\eta(k_1,k_2)\right]\pi(-k_2,-k_1) \\
= \quad & 2\int d^2k_1 \int d^2k_2 \ \left[\left(k_1^1(\partial_{k_1^1})^2 + k_2^1(\partial_{k_2^1})^2\right.\right. \\
& \left.\left.+2k_1^2\partial_{k_1^2}\partial_{k_1^1} + 2k_2^2\partial_{k_2^2}\partial_{k_2^1} - k_1^1\partial_{k_1^2}\partial_{k_1^2} - k_2^1\partial_{k_2^2}\partial_{k_2^2}\right)\eta(k_1,k_2)\right]\pi(-k_2,-k_1),
\end{aligned}
$$

and moving to AdS this becomes

$$
\begin{aligned}
K^x \quad = \quad & 2\int d^2k_1 \int d^2k_2 \ \left[\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right]\left[\frac{1}{\left[\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}\right]}\left[\left(k_1^1(\partial_{k_1^1})^2 + k_2^1(\partial_{k_2^1})^2\right.\right.\right. \\
& \left.\left.\left.+2k_1^2\partial_{k_1^2}\partial_{k_1^1} + 2k_2^2\partial_{k_2^2}\partial_{k_2^1} - k_1^1\partial_{k_1^2}\partial_{k_1^2} - k_2^1\partial_{k_2^2}\partial_{k_2^2}\right)\eta(k_1,k_2)\right]\pi(-k_2,-k_1)\right] \\
= \quad & 2\int d\vec{\kappa} \left[\left(k_1^1(\partial_{k_1^1})^2 + k_2^1(\partial_{k_2^1})^2\right.\right. \\
& \left.\left.+2k_1^2\partial_{k_1^2}\partial_{k_1^1} + 2k_2^2\partial_{k_2^2}\partial_{k_2^1} - k_1^1\partial_{k_1^2}\partial_{k_1^2} - k_2^1\partial_{k_2^2}\partial_{k_2^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}).
\end{aligned}
$$

We now calculate

$$
\left(k_1^1(\partial_{k_1^1})^2 + k_2^1(\partial_{k_2^1})^2 + 2k_1^2\partial_{k_1^2}\partial_{k_1^1} + 2k_2^2\partial_{k_2^2}\partial_{k_2^1} - k_1^1\partial_{k_1^2}\partial_{k_1^2} - k_2^1\partial_{k_2^2}\partial_{k_2^2}\right).
$$

We find terms involving $(\partial_{k^1})^2$, $(\partial_{k^2})^2$ and $(\partial_{k^1})(\partial_{k^2})$ give

$$
\begin{aligned}
& \left(k_1^1+k_2^1\right)(\partial_{k^1})^2 + 2\left(k_1^2+k_2^2\right)(\partial_{k^1})(\partial_{k^2}) - k^1(\partial_{k^2})^2 \\
, \quad = \quad & 2k^1(x^1)^2 - k^1(x^1)^2 + 2k^2x^1x^2 - k^1(x^2)^2 \\
= \quad & -k^1\left((x^1)^2+(x^2)^2\right) + 2x^1(k^1x^1+k^2x^2) \\
= \quad & -k^1\vec{x}_{AdS}^2 + 2x^1(k^1x^1+k^2x^2)
\end{aligned} \tag{5.45}
$$

Terms involving $\left(\partial_{k^n_m} k^z\right)\left(\partial_{k^j_i} k^z\right)$ which will be multiplied by $(\partial_{k^z})^2$ sum to

$$
\begin{aligned}
& k^1_1\left[\frac{1}{k^z}\left(k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k^1_2\right)\right]^2+k^1_2\left[\frac{1}{k^z}\left(k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k^1_1\right)\right]^2 \\
& +2k^2_1\left[\frac{1}{k^z}\left(k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k^1_2\right)\right]\left[\frac{1}{k^z}\left(k^2_1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k^2_2\right)\right] \\
& +2k^2_2\left[\frac{1}{k^z}\left(k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k^1_1\right)\right]\left[\frac{1}{k^z}\left(k^2_2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k^2_1\right)\right] \\
& -k^1_1\left[\frac{1}{k^z}\left(k^2_1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k^2_2\right)\right]^2-k^1_2\left[\frac{1}{k^z}\left(k^2_2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k^2_1\right)\right]^2 \\
= \; & \frac{k^1_1}{(k^z)^2|\vec{k}_1|^2}\left[-\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1+\varphi_2}{2}k^z\right]^2 \\
& +\frac{k^1_2}{(k^z)^2|\vec{k}_2|^2}\left[\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1+\varphi_2}{2}k^z\right]^2 \\
& +2\frac{k^2_1}{(k^z)^2|\vec{k}_1|^2}\left[-\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1+\varphi_2}{2}k^z\right]\left[\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1+\varphi_2}{2}k^z\right] \\
& +2\frac{k^2_2}{(k^z)^2|\vec{k}_1|^2}\left[\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1+\varphi_2}{2}k^z\right]\left[-\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1+\varphi_2}{2}k^z\right] \\
& -\frac{k^1_1}{(k^z)^2|\vec{k}_1|^2}\left[\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1+\varphi_2}{2}k^z\right]^2 \\
& -\frac{k^1_2}{(k^z)^2|\vec{k}_2|^2}\left[-\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1+\varphi_2}{2}k^z\right]^2 \\
= \; & \sin^2\frac{\varphi_1+\varphi_2}{2}\left[k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right] \\
& -2\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1+\varphi_2}{2}\left[k^2_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^2_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right] \\
& -\cos^2\frac{\varphi_1+\varphi_2}{2}\left[k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right] \\
= \; & \frac{1}{2}\left[1-\cos\left(\varphi_1+\varphi_2\right)\right]\left[k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right]-\sin\left(\varphi_1+\varphi_2\right)\left[k^2_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^2_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right] \\
& -\frac{1}{2}\left[1+\cos\left(\varphi_1+\varphi_2\right)\right]\left[k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right] \\
= \; & -\cos\left(\varphi_1+\varphi_2\right)\left[k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right]-\sin\left(\varphi_1+\varphi_2\right)\left[k^2_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^2_2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\left[\frac{k_1^1 k_2^1 - k_1^2 k_2^2}{|\vec{k}_1||\vec{k}_2|}\right]\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}\right] - \left[\frac{k_1^2 k_2^1 + k_1^1 k_2^2}{|\vec{k}_1||\vec{k}_2|}\right]\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} + k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}\right] \\
&= -\left[\frac{1}{|\vec{k}_1|^2}\left((k_1^1)^2 k_2^1 - k_1^2 k_2^2 k_1^1 + (k_1^2)^2 k_2^1 + k_1^1 k_2^2 k_1^2\right)\right] \\
&\quad -\left[\frac{1}{|\vec{k}_2|^2}\left((k_2^1)^2 k_1^1 - k_1^2 k_2^2 k_2^1 + k_1^2 k_2^1 k_2^2 + k_1^1 (k_2^2)^2\right)\right] \\
&= -\left[k_2^1 + k_1^1\right] \\
&= -k^1,
\end{aligned}
$$

so the contribution from this expressions is given by

$$-k^1(\partial_{k^z})^2 = -k^1 z^2. \tag{5.46}$$

Terms involving $\partial_{k^1}\partial_{k^z}$ and $\partial_{k^2}\partial_{k^z}$ are calculated as follows.

$$
\begin{aligned}
&2k_1^1\left[\frac{1}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right)\right]\partial_{k^1}\partial_{k^z} + 2k_2^1\left[\frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right)\right]\partial_{k^1}\partial_{k^z} \\
&+2k_1^2\left[\frac{1}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right)\right]\partial_{k^2}\partial_{k^z} + 2k_1^2\left[\frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right)\right]\partial_{k^1}\partial_{k^z} \\
&+2k_2^2\left[\frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right)\right]\partial_{k^2}\partial_{k^z} + 2k_2^2\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right)\right]\partial_{k^1}\partial_{k^z} \\
&-2k_1^1\left[\frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right)\right]\partial_{k^2}\partial_{k^z} - 2k_2^1\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right)\right]\partial_{k^2}\partial_{k^z} \\
= \;&2k_1^1\left[\frac{1}{k^z}\left(k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^1\right)\right]\partial_{k^1}\partial_{k^z} + 2k_2^1\left[\frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^1\right)\right]\partial_{k^1}\partial_{k^z} \\
&+2k_1^2\left[\frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|} - k_2^2\right)\right]\partial_{k^1}\partial_{k^z} + 2k_2^2\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|} - k_1^2\right)\right]\partial_{k^1}\partial_{k^z} \\
= \;&\frac{2}{k^z}\left[\frac{|\vec{k}_2|}{|\vec{k}_1|}\left((k_1^1)^2 + (k_1^2)^2\right) + \frac{|\vec{k}_1|}{|\vec{k}_2|}\left((k_2^1)^2 + (k_2^2)^2\right)\right. \\
&\left.-2\left(k_1^1 k_2^1 + k_2^1 k_2^2\right)\right]\partial_{k^1}\partial_{k^z} \\
= \;&\frac{2}{k^z}\left[2|\vec{k}_1||\vec{k}_2| - 2\vec{k}_1\cdot\vec{k}_2\right]\partial_{k^1}\partial_{k^z} \\
= \;&2k^z\partial_{k^1}\partial_{k^z} \\
= \;&2x^1 k^z z.
\end{aligned} \tag{5.47}
$$

Continuing, terms of the form $\left(\partial_{k_i^j}\theta\right)\left(\partial_{k_m^n}\theta\right)\left(\partial_\theta\right)^2$ sum to

$$\begin{aligned}
&\left[k_1^1\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right]^2\right.\\
&+k_2^1\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]^2\\
&+2k_1^2\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]\\
&\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right]\\
&+2k_2^2\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]\\
&\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]\\
&-k_1^1\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]^2\\
&\left.-\,k_2^1\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]^2\right]\left(\partial_\theta\right)^2.
\end{aligned}$$

The sum of all the first terms squared (omitting $(\partial_\theta)^2$ for now) gives

$$\begin{aligned}
&\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{2(\vec{k}^2)^2\left(|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2\right)}\left[k_1^1\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]^2+k_2^1\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]^2\right.\\
&+2k_1^2\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]+2k_2^2\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\\
&\left.-k_1^1\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]^2-k_2^1\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]^2\right] \qquad (5.48)
\end{aligned}$$

Using

$$
\begin{aligned}
\sqrt{|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2} &= \sqrt{|\vec{k}_1||\vec{k}_2|}\sqrt{1+\cos(\varphi_1-\varphi_2)} \\
&= \sqrt{2|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1-\varphi_2}{2} \\
|\vec{k}_2|k_1^1 + |\vec{k}_1|k_2^1 &= |\vec{k}_1||\vec{k}_2|\left(\cos\varphi_1+\cos\varphi_2\right) \\
&= 2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2} \\
|\vec{k}_2|k_1^1 - |\vec{k}_1|k_2^1 &= -2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2} \\
|\vec{k}_2|k_1^2 + |\vec{k}_1|k_2^2 &= |\vec{k}_1||\vec{k}_2|\left(\sin\varphi_1+\sin\varphi_2\right) \\
&= 2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2} \\
|\vec{k}_2|k_1^2 - |\vec{k}_1|k_2^2 &= 2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}
\end{aligned}
\tag{5.49}
$$

expression (5.48) becomes

$$
\begin{aligned}
&\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{2(\vec{k}^2)^2\left(2|\vec{k}_1||\vec{k}_2|\cos^2\frac{\varphi_1-\varphi_2}{2}\right)}\left[\frac{k_1^1}{|\vec{k}_1|^2}\left[\left[2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right]^2\right.\right. \\
&-\left.\left[2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right]^2\right] \\
&+2\frac{k_1^2}{|\vec{k}_1|^2}\left[2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right] \\
&\left[2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right] \\
&+2\frac{k_2^2}{|\vec{k}_2|^2}\left[2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right] \\
&\left[2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right] \\
&+\frac{k_2^1}{|\vec{k}_2|^2}\left[\left[2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right]^2\right. \\
&-\left.\left[2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right]^2\right] \\
= \quad &\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2|\vec{k}_1||\vec{k}_2|}{(\vec{k}^2)^2}\left[\frac{k_1^1}{|\vec{k}_1|^2}\left[\cos^2\frac{\varphi_1+\varphi_2}{2}-\sin^2\frac{\varphi_1+\varphi_2}{2}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
& +2\frac{k_1^2}{|\vec{k}_1|^2}\left[\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2}\right]+2\frac{k_2^2}{|\vec{k}_2|^2}\left[\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2}\right] \\
& +\frac{k_2^1}{|\vec{k}_2|^2}\left[\cos^2\frac{\varphi_1+\varphi_2}{2}-\sin^2\frac{\varphi_1+\varphi_2}{2}\right]\Bigg] \\
= & \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2|\vec{k}_1||\vec{k}_2|}{(\vec{k}^2)^2}\Bigg[\frac{k_1^1}{|\vec{k}_1|^2}\cos(\varphi_1+\varphi_2)+\frac{k_1^2}{|\vec{k}_1|^2}\sin(\varphi_1+\varphi_2) \\
& +\frac{k_2^2}{|\vec{k}_2|^2}\sin(\varphi_1+\varphi_2)+\frac{k_2^1}{|\vec{k}_2|^2}\cos(\varphi_1+\varphi_2) \\
= & \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{(\vec{k}^2)^2|\vec{k}_1||\vec{k}_2|}\left[\left(|\vec{k}_2|^2k_1^1+|\vec{k}_1|^2k_2^1\right)\cos(\varphi_1+\varphi_2)\right. \\
& +\left(|\vec{k}_2|^2k_1^2+|\vec{k}_1|^2k_2^2\right)\sin(\varphi_1+\varphi_2)\,.
\end{aligned}
$$

Now

$$
\begin{aligned}
& k_1^1\cos(\varphi_1+\varphi_2)+k_1^2\sin(\varphi_1+\varphi_2) \\
= & |\vec{k}_1|\left[\cos^2\varphi_1\cos\varphi_2-\cos\varphi_1\sin\varphi_1\sin\varphi_2+\sin^2\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2\cos\varphi_1\right] \\
= & |\vec{k}_1|\cos\varphi_2 \\
& k_2^1\cos(\varphi_1+\varphi_2)+k_2^2\sin(\varphi_1+\varphi_2) \\
= & |\vec{k}_2|\left[\cos\varphi_1\cos^2\varphi_2-\cos\varphi_2\sin\varphi_1\sin\varphi_2+\sin\varphi_1\sin\varphi_2\cos\varphi_2+\sin^2\varphi_2\cos\varphi_1\right] \\
= & |\vec{k}_2|\cos\varphi_1,
\end{aligned}
$$

so that expression (5.48) further becomes

$$
\begin{aligned}
& \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{(\vec{k}^2)^2|\vec{k}_1||\vec{k}_2|}\left[|\vec{k}_1||\vec{k}_2|^2\cos\varphi_2+|\vec{k}_2||\vec{k}_1|^2\cos\varphi_1\right] \\
= & \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{(\vec{k}^2)^2}\left[k_2^1+k_1^1\right] \\
= & \frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)^2}{(\vec{k}^2)^2}k^1
\end{aligned}
\tag{5.50}
$$

The sum of the middle terms is given by

$$
\frac{(|\vec{k}_1|-|\vec{k}_2|)}{\vec{k}^4}\left[-2\frac{(k_1^1)^2}{|\vec{k}_1|^2}\left(k_1^1|\vec{k}_2|+k_2^1|\vec{k}_1|\right)+2\frac{(k_2^1)^2}{|\vec{k}_2|^2}\left(k_2^1|\vec{k}_1|+k_1^1|\vec{k}_2|\right)\right.
$$

$$
\begin{aligned}
& -2\frac{k_1^2 k_1^1}{|\vec{k}_1|^2}\left(k_1^2|\vec{k}_2| + k_2^2|\vec{k}_1|\right) - 2\frac{(k_1^2)^2}{|\vec{k}_1|^2}\left(k_1^1|\vec{k}_2| + k_2^1|\vec{k}_1|\right) \\
& +2\frac{k_2^2 k_2^1}{|\vec{k}_2|^2}\left(k_2^2|\vec{k}_1| + k_1^2|\vec{k}_2|\right) + 2\frac{(k_2^2)^2}{|\vec{k}_2|^2}\left(k_2^1|\vec{k}_1| + k_1^1|\vec{k}_2|\right) \\
& +2\frac{k_1^1 k_1^2}{|\vec{k}_1|^2}\left(k_1^2|\vec{k}_2| + k_2^2|\vec{k}_1|\right) - 2\frac{k_2^1 k_2^2}{|\vec{k}_2|^2}\left(k_2^2|\vec{k}_1| + k_1^2|\vec{k}_2|\right)\Bigg] \\
= \; & \frac{(|\vec{k}_1| - |\vec{k}_2|)}{\vec{k}^4}\Bigg[-2\frac{(k_1^1)^2 + (k_1^2)^2}{|\vec{k}_1|^2}\left(k_1^1|\vec{k}_2| + k_2^1|\vec{k}_1|\right) \\
& +2\frac{(k_2^1)^2 + (k_2^2)^2}{|\vec{k}_2|^2}\left(k_2^1|\vec{k}_1| + k_1^1|\vec{k}_2|\right) \\
= \; & \frac{(|\vec{k}_1| - |\vec{k}_2|)}{\vec{k}^4}\left[-2\left(k_1^1|\vec{k}_2| + k_2^1|\vec{k}_1|\right) + 2\left(k_2^1|\vec{k}_1| + k_1^1|\vec{k}_2|\right)\right. \\
= \; & 0.
\end{aligned} \tag{5.51}
$$

The sum of the last terms gives

$$
\begin{aligned}
& \frac{2(|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2)}{\vec{k}^4}\Bigg[k_1^1\frac{(k_1^1)^2}{|\vec{k}_1|^2} + k_2^1\frac{(k_2^1)^2}{|\vec{k}_2|^2} + 2\frac{(k_1^2)^2 k_1^1}{|\vec{k}_1|^2} + 2\frac{(k_2^2)^2 k_2^1}{|\vec{k}_2|^2} \\
& -k_1^1\frac{(k_1^2)^2}{|\vec{k}_1|^2} - k_2^1\frac{(k_2^2)^2}{|\vec{k}_2|^2}\Bigg] \\
= \; & \frac{2(|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2)}{\vec{k}^4}\left[k_1^1 + k_2^1\right] \\
= \; & \frac{2(|\vec{k}_1||\vec{k}_2| + \vec{k}_1 \cdot \vec{k}_2)}{\vec{k}^4}k^1.
\end{aligned} \tag{5.52}
$$

Summing the 3 expressions (5.50), (5.51) and (5.52) we obtain that the $(\partial_\theta)^2$ terms is given by

$$
\begin{aligned}
\frac{k^1}{\vec{k}^4}\left[|\vec{k}_1|^2 - 2|\vec{k}_1||\vec{k}_2| + |\vec{k}_2|^2 + 2|\vec{k}_1||\vec{k}_2| + 2\vec{k}_1 \cdot \vec{k}_2\right](\partial_\theta)^2 &= \frac{k^1}{\vec{k}^2}(\partial_\theta)^2 \\
&= \frac{k^1(k^\theta)^2}{\vec{k}^2}.
\end{aligned} \tag{5.53}
$$

Terms involving $\partial_{k^1}\partial_{k^\theta}$ and $\partial_{k^2}\partial_{k^\theta}$ are given as follows.

$$\begin{aligned}
&2k_1^1\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right]\partial_{k^1}\partial_{k^\theta}\\
&+2k_2^1\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]\partial_{k^1}\partial_{k^\theta}\\
&+2k_1^2\left[\frac{\left[k_1^1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^1}{|\vec{k}_1|}\right]\partial_{k^2}\partial_{k^\theta}\\
&+2k_1^2\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]\partial_{k^1}\partial_{k^\theta}\\
&+2k_2^2\left[\frac{\left[k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^1}{|\vec{k}_2|}\right]\partial_{k^2}\partial_{k^\theta}\\
&+2k_2^2\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]\partial_{k^1}\partial_{k^\theta}\\
&-2k_1^1\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]\partial_{k^2}\partial_{k^\theta}\\
&-2k_2^1\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]\partial_{k^2}\partial_{k^\theta}\\
=&\left[\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\left[2\frac{|\vec{k}_2|}{|\vec{k}_1|}\left((k_2^1)^2+k_2^2k_2^1+(k_2^2)^2-k_2^1k_2^2\right)\right]\right.\\
&+2\frac{|\vec{k}_2|}{|\vec{k}_1|}\left((k_1^1)^2+k_1^1k_1^2+(k_1^2)^2-k_1^1k_1^2\right)\\
&+2\left(2k_1^1k_2^1+k_1^2k_2^1+k_1^1k_2^2+k_2^2k_1^1+k_2^2k_1^2-k_1^1k_2^2-k_1^2k_2^1\right)\\
&+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\left[\frac{1}{|\vec{k}_1|}\left(-2(k_1^1)^2-2k_1^2k_1^1-2(k_1^2)^2+2k_1^1k_1^2\right)\right.\\
&\left.\left.+\frac{1}{|\vec{k}_2|}2\left((k_2^1)^2+2k_2^1k_2^2+2(k_2^2)^2-2k_2^1k_2^2\right)\right]\right]\partial_{k^2}\partial_{k^\theta}
\end{aligned}$$

$$
\begin{aligned}
= & \left[\frac{\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\left[4|\vec{k}_1||\vec{k}_2|+4\vec{k}_1\cdot\vec{k}_2\right]\right. \\
& \left.+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\left[-2|\vec{k}_1|+2|\vec{k}_1|\right]\right] \\
= & \frac{2\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\left(|\vec{k}_1|-|\vec{k}_2|-|\vec{k}_1|+|\vec{k}_2|\right) \\
= & 0. \qquad (5.54)
\end{aligned}
$$

Terms involving $\left[\left[\partial_{k^n_m}k^z\right]\left[\partial_{k^j_i}\theta\right]+\left[\partial_{k^n_m}\theta\right]\left[\partial_{k^j_i}k^z\right]\right]\partial_{k^z}\partial_\theta$ are now calculated (where we omit $\partial_{k^z}\partial_\theta$).

$$
\begin{aligned}
& 2k^1_1\left[\frac{\left[k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^1_2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k^1_1}{|\vec{k}_1|}\right] \\
& \left[\frac{1}{k^z}\left(k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k^1_2\right)\right] \\
& +2k^1_2\left[\frac{\left[k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k^1_1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k^1_2}{|\vec{k}_2|}\right] \\
& \left[\frac{1}{k^z}\left(k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k^1_1\right)\right] \\
& +2k^2_1\left[\frac{\left[k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^1_2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k^1_1}{|\vec{k}_1|}\right] \\
& \left[\frac{1}{k^z}\left(k^2_1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k^2_2\right)\right] \\
& +2k^2_1\left[\frac{\left[k^2_1\frac{|\vec{k}_2|}{|\vec{k}_1|}+k^2_2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k^2_1}{|\vec{k}_1|}\right] \\
& \left[\frac{1}{k^z}\left(k^1_1\frac{|\vec{k}_2|}{|\vec{k}_1|}-k^1_2\right)\right] \\
& +2k^2_2\left[\frac{\left[k^1_2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k^1_1\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k^1_2}{|\vec{k}_2|}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^2\right)\right]\\
&+2k_2^2\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]\\
&\left[\frac{1}{k^z}\left(k_2^1\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^1\right)\right]\\
&-2k_1^1\left[\frac{\left[k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}+k_2^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}-\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_1^2}{|\vec{k}_1|}\right]\\
&\left[\frac{1}{k^z}\left(k_1^2\frac{|\vec{k}_2|}{|\vec{k}_1|}-k_2^2\right)\right]\\
&-2k_2^1\left[\frac{\left[k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}+k_1^2\right]\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}+\frac{\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{\vec{k}^2}\frac{k_2^2}{|\vec{k}_2|}\right]\\
&\left[\frac{1}{k^z}\left(k_2^2\frac{|\vec{k}_1|}{|\vec{k}_2|}-k_1^2\right)\right]. \qquad (5.55)
\end{aligned}
$$

Summing the first terms in expression (5.55) we obtain

$$
\begin{aligned}
&\frac{2\left(|\vec{k}_1|-|\vec{k}_2|\right)}{k^z\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}\left[\frac{1}{|\vec{k}_1|^2}\left[k_1^1\left(|\vec{k}_2|k_1^1+|\vec{k}_1|k_2^1\right)\left(|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right)\right.\right.\\
&+k_1^2\left(|\vec{k}_2|k_1^1+|\vec{k}_1|k_2^1\right)\left(|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right)\\
&+k_1^2\left(|\vec{k}_2|k_1^2+|\vec{k}_1|k_2^2\right)\left(|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right)\\
&\left.-k_1^1\left(|\vec{k}_2|k_1^2+|\vec{k}_1|k_2^2\right)\left(|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right)\right]\\
&+\frac{1}{|\vec{k}_2|^2}\left[k_2^1\left(|\vec{k}_1|k_2^1+|\vec{k}_2|k_1^1\right)\left(|\vec{k}_1|k_2^1-|\vec{k}_2|k_1^1\right)\right.\\
&+k_2^2\left(|\vec{k}_1|k_2^1+|\vec{k}_2|k_1^1\right)\left(|\vec{k}_1|k_2^2-|\vec{k}_2|k_1^2\right)\\
&+k_2^2\left(|\vec{k}_1|k_2^2+|\vec{k}_2|k_1^2\right)\left(|\vec{k}_1|k_2^1-|\vec{k}_2|k_1^1\right)\\
&\left.\left.-k_2^1\left(|\vec{k}_1|k_2^2+|\vec{k}_2|k_1^2\right)\left(|\vec{k}_1|k_2^2-|\vec{k}_2|k_1^2\right)\right]\right]\\
=\quad&\frac{2\left(|\vec{k}_1|-|\vec{k}_2|\right)}{k^z\sqrt{2}\vec{k}^2\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1-\varphi_2}{2}}\left[\frac{1}{|\vec{k}_1|^2}\right[
\end{aligned}
$$

$$k_1^1 \left(2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right) \left(-2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)$$
$$k_1^2 \left(2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right) \left(2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)$$
$$k_1^2 \left(2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right) \left(-2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)$$
$$-k_1^1 \left(2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right) \left(2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)\Bigg]$$
$$+\frac{1}{|\vec{k}_2|^2} \Bigg[k_2^1 \left(2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right)$$
$$\left(2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)$$
$$+k_2^2 \left(2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right)$$
$$\left(-2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)$$
$$+k_2^2 \left(2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right) \left(2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)$$
$$-k_2^1 \left(2|\vec{k}_1||\vec{k}_2| \sin\frac{\varphi_1+\varphi_2}{2} \cos\frac{\varphi_1-\varphi_2}{2}\right)$$
$$\left(-2|\vec{k}_1||\vec{k}_2| \cos\frac{\varphi_1+\varphi_2}{2} \sin\frac{\varphi_1-\varphi_2}{2}\right)\Bigg]\Bigg]$$
$$= \frac{4\sqrt{2}\left(|\vec{k}_1|-|\vec{k}_2|\right)|\vec{k}_1||\vec{k}_2|\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1-\varphi_2}{2}}{k^z\vec{k}^2}$$
$$\Bigg[\frac{1}{|\vec{k}_1|^2}\Bigg[-k_1^1\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2} + k_1^2\cos^2\frac{\varphi_1+\varphi_2}{2}$$
$$-k_1^2\sin^2\frac{\varphi_1+\varphi_2}{2} - k_1^1\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2}\Bigg]$$
$$+\Bigg[\frac{1}{|\vec{k}_2|^2}\Bigg[k_2^1\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2} - k_2^2\cos^2\frac{\varphi_1+\varphi_2}{2}$$
$$+k_2^2\sin^2\frac{\varphi_1+\varphi_2}{2} + k_2^1\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2}\Bigg]$$
$$= \frac{2\left(|\vec{k}_1|-|\vec{k}_2|\right)|\vec{k}_1||\vec{k}_2|k^z}{k^z\vec{k}^2}\Bigg[\frac{1}{|\vec{k}_1|^2}\left[-k_1^1\sin(\varphi_1+\varphi_2)+k_1^2\cos(\varphi_1+\varphi_2)\right.$$
$$\left.+\left[\frac{1}{|\vec{k}_2|^2}\left[k_2^1\sin(\varphi_1+\varphi_2)-k_2^2\cos(\varphi_1+\varphi_2)\right]\right]\right.$$

$$
\begin{aligned}
= \quad & \frac{2\left(|\vec{k}_1|-|\vec{k}_2|\right)|\vec{k}_1||\vec{k}_2|}{\vec{k}^2}\left[\left(-\frac{k_1^1}{|\vec{k}_1|^2}+\frac{k_2^1}{|\vec{k}_2|^2}\right)\sin\left(\varphi_1+\varphi_2\right)\right.\\
& \left.+\left(\frac{k_1^2}{|\vec{k}_1|^2}-\frac{k_2^2}{|\vec{k}_2|^2}\right)\cos\left(\varphi_1+\varphi_2\right)\right]\\
= \quad & \frac{2\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\vec{k}^2|\vec{k}_1||\vec{k}_2|}\left[|\vec{k}_1|^2\left[k_2^1\sin\left(\varphi_1+\varphi_2\right)-k_2^2\cos\left(\varphi_1+\varphi_2\right)\right]\right.\\
& \left.-|\vec{k}_2|^2\left[k_1^1\sin\left(\varphi_1+\varphi_2\right)-k_1^2\cos\left(\varphi_1+\varphi_2\right)\right]\right]\\
= \quad & \frac{2\left(|\vec{k}_1|-|\vec{k}_2|\right)}{\vec{k}^2|\vec{k}_1||\vec{k}_2|}\left[|\vec{k}_1|^2\left[\frac{|\vec{k}_2|}{|\vec{k}_1|}k_1^2\right]-|\vec{k}_2|^2\left[\frac{|\vec{k}_1|}{|\vec{k}_2|}k_2^2\right]\right]\\
= \quad & \frac{2\left(|\vec{k}_1|-|\vec{k}_2|\right)\left(k_1^2-k_2^2\right)}{\vec{k}^2}
\end{aligned} \tag{5.56}
$$

Summing the second terms in (5.55) gives

$$
\begin{aligned}
& \frac{2\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{k^z\vec{k}^2}\left[-\frac{(k_1^1)^2}{|\vec{k}_1|^2}\left(|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right)+\frac{(k_2^1)^2}{|\vec{k}_2|^2}\left(|\vec{k}_1|k_2^1-|\vec{k}_2|k_1^1\right)\right.\\
& -\frac{k_1^2k_1^1}{|\vec{k}_1|^2}\left(|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right)-\frac{(k_1^2)^2}{|\vec{k}_1|^2}\left(|\vec{k}_2|k_1^1-|\vec{k}_1|k_2^1\right)\Big]\\
& +\frac{k_2^2k_2^1}{|\vec{k}_2|^2}\left(|\vec{k}_1|k_2^2-|\vec{k}_2|k_1^2\right)+\frac{(k_2^2)^2}{|\vec{k}_2|^2}\left(|\vec{k}_1|k_2^1-|\vec{k}_2|k_1^1\right)\Big]\\
& \left.+\frac{k_1^1k_1^2}{|\vec{k}_1|^2}\left(|\vec{k}_2|k_1^2-|\vec{k}_1|k_2^2\right)-\frac{k_2^1k_2^2}{|\vec{k}_2|^2}\left(|\vec{k}_1|k_2^2-|\vec{k}_2|k_1^2\right)\right]\\
= \quad & \frac{2\sqrt{2}\sqrt{|\vec{k}_1||\vec{k}_2|+\vec{k}_1\cdot\vec{k}_2}}{k^z\vec{k}^2}\left[-\frac{(k_1^1)^2}{|\vec{k}_1|^2}\left(-2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\right.\\
& +\frac{(k_2^1)^2}{|\vec{k}_2|^2}\left(2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\\
& -\frac{k_1^2k_1^1}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\\
& -\frac{(k_1^2)^2}{|\vec{k}_1|^2}\left(-2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\Big]\\
& +\frac{k_2^2k_2^1}{|\vec{k}_2|^2}\left(-2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{(k_2^2)^2}{|\vec{k}_2|^2}\left(2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\Bigg] \\
& +\frac{k_1^1k_1^2}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right) \\
& -\frac{k_2^1k_2^2}{|\vec{k}_2|^2}\left(-2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\Bigg] \\
= & \frac{2\sqrt{2|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1-\varphi_2}{2}}{\sqrt{2|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1-\varphi_2}{2}\vec{k}^2}\left[\frac{(k_1^1)^2}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\right. \\
& +\frac{(k_2^1)^2}{|\vec{k}_2|^2}\left(2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right) \\
& \frac{(k_1^2)^2}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\Bigg] \\
& +\frac{(k_2^2)^2}{|\vec{k}_2|^2}\left(2|\vec{k}_1||\vec{k}_2|\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right)\Bigg] \\
& -\frac{k_1^2k_1^1}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right) \\
& -\frac{k_2^2k_2^1}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right) \\
& +\frac{k_1^2k_1^1}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right) \\
& +\frac{k_2^2k_2^1}{|\vec{k}_1|^2}\left(2|\vec{k}_1||\vec{k}_2|\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}\right) \\
= & \frac{8\cos\frac{\varphi_1-\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2}|\vec{k}_1||\vec{k}_2|}{\vec{k}^2} \\
= & \frac{4|\vec{k}_1||\vec{k}_2|}{\vec{k}^2}(\sin\varphi_1+\sin\varphi_2) \\
= & \frac{4}{\vec{k}^2}\left(|\vec{k}_2|k_1^2+|\vec{k}_1|k_2^2\right).
\end{aligned}
\tag{5.57}
$$

Adding expression (5.56) and (5.57) together gives a contribution to K^t of

$$
\frac{2\left(|\vec{k}_1|+|\vec{k}_2|\right)\left(k_1^2+k_2^2\right)}{\vec{k}^2}\partial_{k^z}\partial_\theta=\frac{2P^0k^2}{\vec{k}^2}zk^\theta \tag{5.58}
$$

Hence, the special conformal generator K^x is given by the sum of contributions from equations (5.45), (5.46), (5.47), (5.53), (5.54), (5.58)

$$\begin{aligned} & -k^1\vec{x}^2_{AdS} + 2x^1\left(k^1x^1 + k^2x^2\right) - k^1z^2 + 2x^1k^zz + \frac{2P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{\vec{k}^2} \\ = & -k^1\left(\vec{x}^2_{AdS} + z^2\right) + 2x^1D + \frac{2P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{\vec{k}^2}. \end{aligned}$$

The mode expansion is then given as follows

$$\begin{aligned} K^x &= 4\int d\vec{\kappa}\left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\ &= 4\int d\vec{\kappa}\left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)\right. \\ &\quad \left.\left(\sqrt{\frac{1}{\sqrt{2P^0}}}\left(a_{\vec{\kappa}} + a^\dagger_{-\vec{\kappa}}\right)\right)\right]\left(-\sqrt{\frac{P^0}{2}}i\left(a_{-\vec{\kappa}} - a^\dagger_{\vec{\kappa}}\right)\right) \\ &= -4i\int d\vec{\kappa}\left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)\right. \\ &\quad \left.\left(a_{\vec{\kappa}} - a^\dagger_{-\vec{\kappa}}\right)\right]\left(a_{-\vec{\kappa}} - a^\dagger_{\vec{\kappa}}\right) \\ &= 4i\int d\vec{\kappa}\left[-\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a_{\vec{\kappa}}\right]a_{-\vec{\kappa}} \\ &\quad + \left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\ &\quad + \left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a^\dagger_{-\vec{\kappa}}\right]a_{-\vec{\kappa}} \\ &\quad - \left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a^\dagger_{-\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\ &= -4i\int d\vec{\kappa}\left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\ &\quad + \left[\left(-\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a^\dagger_{-\vec{\kappa}}\right]a_{-\vec{\kappa}}, \end{aligned}$$

so we have

$$K^x_{AdS} = -\frac{1}{2}k^1\left(\vec{x}^2+z^2\right) + x^1D + \frac{P^0k^2}{\vec{k}^2}zk^\theta + \frac{k^1(k^\theta)^2}{2\vec{k}^2}.$$

5.3.7 Calculation of K^y

In a manner similar to that used for K^x we have that

$$\boxed{K^y = \int dxdy\,[2y\mathcal{D}]\,,}$$

which leads to

$$K^y = 4\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2 + x^2 D + \frac{zk^1 P^0 k^\theta}{\vec{k}^2} + \frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}).$$

Mode expanding this gives

$$\begin{aligned} K^y &= -4i\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2 + x^2 D + \frac{zk^1 P^0 k^\theta}{\vec{k}^2} + \frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\ &\quad + \left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2 + x^2 D + \frac{zk^1 P^0 k^\theta}{\vec{k}^2} + \frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)a^\dagger_{-\vec{\kappa}}\right]a_{-\vec{\kappa}} \end{aligned}$$

and therefore

$$K^y_{\text{AdS}} = -\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2 + x^2 D + \frac{zk^1 P^0 k^\theta}{\vec{k}^2} + \frac{k^2(k^\theta)^2}{2\vec{k}^2}.$$

In summary we have

$$\begin{aligned} P^t &= \frac{1}{2}\int d\vec{\kappa}\left[\Pi(\vec{\kappa})\Pi(-\vec{\kappa}) + (P^0)^2\mathcal{H}(\vec{\kappa})\mathcal{H}(-\vec{\kappa})\right] \\ &= \int d\vec{\kappa}\ P^0\left[a_{\vec{\kappa}}a^\dagger_{\vec{\kappa}} + a^\dagger_{\vec{\kappa}}a_{\vec{\kappa}}\right] \\ P^x &= \int d\vec{\kappa}k^1\mathcal{H}(\vec{\kappa})\Pi(-\vec{\kappa}) \\ &= \frac{i}{2}\int d\vec{\kappa}\ \ k^1\ \left[a_{\vec{\kappa}}a^\dagger_{\vec{\kappa}} + a^\dagger_{\vec{\kappa}}a_{\vec{\kappa}}\right] \\ P^y &= \int d\vec{\kappa}k^2\mathcal{H}(\vec{\kappa})\Pi(-\vec{\kappa}) \\ &= \int dx\,dy\,\pi\partial_y\phi \\ &= \frac{i}{2}\int d\vec{\kappa}\ \ k^2\ \left[a_{\vec{\kappa}}a^\dagger_{\vec{\kappa}} + a^\dagger_{\vec{\kappa}}a_{\vec{\kappa}}\right] \\ M^{xy} &= \int d\vec{\kappa}\left(k^2\partial_{k^1} - k^1\partial_{k^2}\right)\mathcal{H}(\vec{\kappa})\Pi(-\vec{\kappa}) \\ &= \frac{i}{2}\int d\vec{\kappa}\ \ \left[a_{\vec{\kappa}}\left[k^2\partial_{k^1} - k^1\partial_{k^2}\right]a^\dagger_{\vec{\kappa}} - ia^\dagger_{\vec{\kappa}}\left[k^2\partial_{k^1} - k^1\partial_{k^2}\right]a_{\vec{\kappa}}\right] \end{aligned}$$

$$
\begin{aligned}
D &= \int d\vec{\kappa}\left[\left(k^j\partial_{k^j}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= -\frac{i}{2}\int d\vec{\kappa}\Big[a_{\vec{\kappa}}\left[x^1k^1+x^2k^2+zk^z\right]a^\dagger_{\vec{\kappa}} \\
&\quad -a^\dagger_{\vec{\kappa}}\left[x^1k^1+x^2k^2+zk^z\right]a_{\vec{\kappa}}\Big] \\
M^{tx} &= \frac{i}{2}\int d\vec{\kappa}\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= \frac{i}{4}\int d\vec{\kappa}\left[-\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}}-\left[-\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)a^\dagger_{\vec{\kappa}}\right]a_{\vec{\kappa}} \\
M^{ty} &= \frac{i}{2}\int d\vec{\kappa}\left[\left(P^0\partial_{k^2}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= -\frac{i}{4}\int d\vec{\kappa}\left[\left(P^0\partial_{k^2}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}}-\left[\left(P^0\partial_{k^2}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta\right)a^\dagger_{\vec{\kappa}}\right]a_{\vec{\kappa}} \\
K^t &= \frac{i}{4}\int d\vec{\kappa}\left[\left(\vec{x}^2+z^2+\frac{\left(k^\theta\right)^2}{\vec{k}^2}+\frac{2k^zk^\theta}{P^0\vec{k}^2}M^{xy}\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= \frac{i}{4}\int d\vec{\kappa}\left[\left(-\frac{\left(\vec{x}^2+z^2\right)P^0}{2}-\frac{\left(k^\theta\right)^2P^0}{2\vec{k}^2}-\frac{k^zk^\theta}{\vec{k}^2}M^{xy}\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\
&\quad +\left[\left(-\frac{\left(\vec{x}^2+z^2\right)P^0}{2}-\frac{\left(k^\theta\right)^2P^0}{2\vec{k}^2}-\frac{k^zk^\theta}{\vec{k}^2}M^{xy}\right)a^\dagger_{\vec{\kappa}}\right]a_{\vec{\kappa}} \\
K^x &= 4\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^1+x^1D+\frac{P^0k^2}{\vec{k}^2}zk^\theta+\frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= -4i\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^1+x^1D+\frac{P^0k^2}{\vec{k}^2}zk^\theta+\frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\
&\quad +\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^1+x^1D+\frac{P^0k^2}{\vec{k}^2}zk^\theta+\frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)a^\dagger_{-\vec{\kappa}}\right]a_{-\vec{\kappa}} \\
K^y &= 4\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2+x^2D+\frac{zk^1P^0k^\theta}{\vec{k}^2}+\frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= -4i\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2+x^2D+\frac{zk^1P^0k^\theta}{\vec{k}^2}+\frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)a_{\vec{\kappa}}\right]a^\dagger_{\vec{\kappa}} \\
&\quad +\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2+x^2D+\frac{zk^1P^0k^\theta}{\vec{k}^2}+\frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)a^\dagger_{-\vec{\kappa}}\right]a_{-\vec{\kappa}}
\end{aligned}
$$

5.4 Bulk Conformal Algebra at the Infrared Fixed Point

The bulk conformal algebra at the infrared fixed point is now simply obtained as a result of the following observations.

The first is that the interaction term $\frac{\lambda}{4!}\int d^{d-2}\vec{x}\,(\eta_{\vec{x}\vec{x}})^2$ does not contribute to the Hamiltonian (or to the potential) at the critical point, since

$$\eta_{\vec{x}\vec{x}} \sim 1/\lambda + \mathcal{O}(1/\lambda)^2,$$

and, as such, the expressions for the conformal generators of the previous section remain valid.

The second is, of course, that as shown previously, the (scattering) states of the system correspond to the removal of the $s = 0$ spin in the bulk and we may write each of the generators in equations (5.59) as a sum over an infinite tower of spins as follows

$$\begin{aligned}
P^t &= \frac{1}{2}\int d\vec{\kappa}\left[\Pi(\vec{\kappa})\Pi(-\vec{\kappa}) + (P^0)^2\mathcal{H}(\vec{\kappa})\mathcal{H}(-\vec{\kappa})\right] \\
&= \frac{1}{2}\sum_{s=0}^{\infty}\int d\vec{\kappa}\left[\Pi_s(\vec{\kappa})\Pi_s(-\vec{\kappa}) + (P^0)^2\mathcal{H}_s(\vec{\kappa})\mathcal{H}_s(-\vec{\kappa})\right] \\
P^x &= \int d\vec{\kappa} k^1\mathcal{H}(\vec{\kappa})\Pi(-\vec{\kappa}) \\
&= \sum_{s=0}^{\infty}\int d\vec{\kappa} k^1\mathcal{H}_s(\vec{\kappa})\Pi_s(-\vec{\kappa}) \\
P^y &= \int d\vec{\kappa} k^2\mathcal{H}(\vec{\kappa})\Pi(-\vec{\kappa}) \\
&= \sum_{s=0}^{\infty}\int d\vec{\kappa} k^2\mathcal{H}_s(\vec{\kappa})\Pi_s(-\vec{\kappa}) \\
M^{xy} &= \int d\vec{\kappa}\left(k^2\partial_{k^1} - k^1\partial_{k^2}\right)\mathcal{H}(\vec{\kappa})\Pi(-\vec{\kappa}) \\
&= \sum_{s=0}^{\infty}\int d\vec{\kappa}\left(k^2\partial_{k^1} - k^1\partial_{k^2}\right)\mathcal{H}_s(\vec{\kappa})\Pi_s(-\vec{\kappa}) \\
D &= \int d\vec{\kappa}\left[\left(k^j\partial_{k^j}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa}) \\
&= \sum_{s=0}^{\infty}\int d\vec{\kappa}\left[\left(k^j\partial_{k^j}\right)\mathcal{H}_s(\vec{\kappa})\right]\Pi_s(-\vec{\kappa})
\end{aligned}$$

$$
\begin{aligned}
M^{tx} &= \frac{i}{2}\int d\vec{\kappa}\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa})\\
&= \sum_{s=0}^{\infty}\frac{i}{2}\int d\vec{\kappa}\left[\left(P^0\partial_{k^1}+\frac{k^2k^z}{\vec{k}^2}\partial_\theta\right)\Pi_s(\vec{\kappa})\right]\Pi_s(-\vec{\kappa})\\
M^{ty} &= \frac{i}{2}\int d\vec{\kappa}\left[\left(P^0\partial_{k^2}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa})\\
&= \sum_{s=0}^{\infty}\frac{i}{2}\int d\vec{\kappa}\left[\left(P^0\partial_{k^2}-\frac{k^1k^z}{\vec{k}^2}\partial_\theta\right)\Pi_s(\vec{\kappa})\right]\Pi_s(-\vec{\kappa})\\
K^{t} &= \frac{i}{4}\int d\vec{\kappa}\left[\left(\vec{x}^2+z^2+\frac{\left(k^\theta\right)^2}{\vec{k}^2}+\frac{2k^zk^\theta}{P^0\vec{k}^2}M^{xy}\right)\Pi(\vec{\kappa})\right]\Pi(-\vec{\kappa})\\
&= \sum_{s=0}^{\infty}\frac{i}{4}\int d\vec{\kappa}\left[\left(\vec{x}^2+z^2+\frac{\left(k^\theta\right)^2}{\vec{k}^2}+\frac{2k^zk^\theta}{P^0\vec{k}^2}M^{xy}\right)\Pi_s(\vec{\kappa})\right]\Pi_s(-\vec{\kappa})\\
K^{x} &= 4\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^1+x^1D+\frac{P^0k^2}{\vec{k}^2}zk^\theta+\frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa})\\
&= \sum_{s=0}^{\infty}4\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^1+x^1D+\frac{P^0k^2}{\vec{k}^2}zk^\theta+\frac{k^1(k^\theta)^2}{2\vec{k}^2}\right)\right.\\
&\quad \left.\mathcal{H}_s(\vec{\kappa})\right]\Pi_s(-\vec{\kappa})\\
K^{y} &= 4\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2+x^2D+\frac{k^1P^0}{\vec{k}^2}zk^\theta+\frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)\mathcal{H}(\vec{\kappa})\right]\Pi(-\vec{\kappa})\\
&= \sum_{s=0}^{\infty}4\int d\vec{\kappa}\left[\left(-\frac{1}{2}\left(\vec{x}^2+z^2\right)k^2+x^2D+\frac{k^1P^0}{\vec{k}^2}zk^\theta+\frac{k^2(k^\theta)^2}{2\vec{k}^2}\right)\right.\\
&\quad \left.\mathcal{H}_s(\vec{\kappa})\right]\Pi_s(-\vec{\kappa}).
\end{aligned}
$$

At the critical point, the $s=0$ contribution will vanish, and the summation will become $\sum_{s\neq 0}$.

Chapter 6

Outlook / Conclusions

6.1 Conclusions

The result in 1975-1976 by Bekenstein and Hawking, that the entropy of a black hole grows as the area of its event horizon[22, 23], has been a springboard for new ideas in theoretical physics related to the concept of holography - that boundary conditions determine everything within the boundary. Various no-go theorems exist which relate specifically to theories on a flat space-time background, making it necessary to construct theories on a curved space-time [78, 80, 81]. In the 1990s, Maldacena, Witten, Gubser, Polyakov and Klebanov developed an understanding of how gravitational theories, in the form of type IIB strings in curved (but asymptotically flat) $AdS_5 \times S^5$ could be seen to be dual to conformal field theories in the form of $\mathcal{N} = 4$ SYM[20, 38, 39]). This duality relates a strongly coupled theory on one side to a weakly coupled theory on the other side, allowing for the simplification of complex calculations.

One of the simplest non-trivial ways of probing the AdS/CFT correspondence is the duality conjectured by Klebanov and Polyakov[1]. This duality relates free/critical O(N) vector models in $d = 2 + 1$ dimensions with higher spin theories, and the gauge invariant operators $\phi^a \partial_{(\mu_1} ... \partial_{\mu_s)} \phi^a$ are dual to a tower of massless higher spin fields[111]. Here, the free 3d $O(N)$ vector model (restricted to the $O(N)$ singlet sector) is dual to type A minimal Vasiliev higher spin gauge theory with $\Delta = 1$ and the critical 3d $O(N)$ vector model is dual to type A minimal Vasiliev Higher spin theory with scalar having $\Delta = 2$[1, 4]. (Note that the 'minimal' bosonic Vasiliev's theory with spins $s = 2, 4, 6, ...$ is simply a consistent truncation of the more general 'non-minimal' theory which also includes the odd spins. The dual of this non-minimal theory would be expected to be a $U(N)$ vector model but restricted to the $U(N)$ singlet sector.) While it is perhaps not surprising that this duality exists in the free case, considering it is the only known example of a Conformal field theory in a dimension larger than 2 that has higher spin currents that are exactly conserved, it is somewhat more unexpected that it work for $\Delta = 2$ boundary condition case, where the higher spin symmetry in the bulk is broken through loop effects[115]. A fundamental claim that this duality makes is that when a relevant interaction is present, the theory flows from an unstable UV fixed point with a dimension $\Delta = 1$ scalar at the boundary to an IR fixed point where $\Delta = 2$. Klebanov and Polyakov's $O(N)$ vector model / higher spin theory conjecture received a remarkable boost due to evidence in the form of tree-level and one-loop tests of the duality involving the calculation of the $\mathcal{O}(N)$ correction to the Weyl anomaly for even d and to free energy for odd d, the matching of the three-point function for both $\Delta = 1$ and $\Delta = 2$[2, 3, 4] and the reconstruction of the bulk[5] (done in an alternative manner in Ref. [6]).

In Ref. [1], the standard auxiliary field approach was used to obtain the 1/N expansion (taking N large) and in Refs. [115, 114, 116] it is shown how the $\Delta = 2$ duality follows from the $\Delta = 1$ duality using a Legendre transformation; in this thesis we instead made use of the Collective field theory approach. Here bilocals are used to encode the invariance of the theory explicitly and the invariant variables themselves are described by the Collective field theory[8]. Within the Collective field theory framework, the three-dimensional $O(N)$ invariant bosonic model with $\frac{\lambda}{N}\left(\phi^a\phi^a\right)^2$ interaction was considered in Ref. [7]. Here, a bilocal field approach was taken and the a $1/N$ expansion studied at its infrared fixed point. At the critical point / infrared fixed point, a state was in fact identified to correspond to a $\Delta = 2$ scalar state and, in agreement with Polyakov and Klebanov above, the $\Delta = 1$ state was found to vanish from the spectrum.

In recent years, remarkable progress has been made towards achieving a map between CFT_3 and $\mathrm{AdS}_4 \times S_1$ in the free case. In Refs. [5, 9, 10, 121] this was done explicitly by construction in both the light-cone gauge and the temporal gauge, where the $1 + 2 + 2 = 5$ coordinates of the bilocals in the conformal field theory in $d = 2 + 1$ dimensions (in the equal time Hamiltonian approach) were mapped to the 5 coordinates of higher spin theory in $AdS_4 \times S_1$ (in phase space). In this thesis, the aim was to make progress in developing an explicit map between $AdS_4 \times S_1$ and CFT_3 in the interacting case and indeed, this was achieved. We made use of the Hamiltonian approach in a time like gauge[121]. The quartic interaction contributes linearly in the bilocal field fluctuation equations, and the spectrum problem is then that of a potential scattering problem. The scattering state solutions take a universal form at the critical point. The bulk description of these boundary scattering states was obtained by developing a first principles approach, consisting of a simple change of variables from bilocal momenta to bulk momenta, as dictated by the map, but requiring a field redefinition in defining the bulk higher spin field,

$$\begin{aligned} \mathcal{H}\left(\vec{k}.k^z,\theta\right) &\equiv \left.\frac{\eta_{\vec{k}_1\vec{k}_2}}{\frac{1}{|\vec{k}_1|}+\frac{1}{|\vec{k}_2|}}\right|_{\vec{k}_1,\vec{k}_2\left(\vec{k},k^z,\theta\right)} \\ \Pi_{\mathcal{H}}\left(\vec{k}.k^z,\theta\right) &= \left.\pi_{\vec{k}_1\vec{k}_2}\right|_{\vec{k}_1,\vec{k}_2\left(\vec{k},k^z,\theta\right)}. \end{aligned}$$

Note that this differs from the pre-existent matrix-model case which actually involved no redefinition of the scalar field, $\eta(q) = \eta(x)$, and a redefinition of the conjugate momentum field $\Pi(q) = \pi\phi_0 P(x)$, as outlined in Appendix F. Remarkably, it was shown directly in the bulk that the $s = 0$ state (equivalent to $\Delta = s + 1 = 1$ state) is precisely removed from the bulk higher spin field, which is equivalent to the observation found in Ref. [7], by studying correlators and the spectrum on the boundary.

The conformal algebra, in bulk coordinates, was then shown to agree with that obtained previously in Ref. [10], both at the free and interacting critical point, thus establishing the equivalence of our approach with that developed in Ref. [10].

6.2 Outlook

The results of this research are quite exciting and below are highlighted some aspects that are of particular interest for further investigation.

1. This thesis is based on the canonical transformation for the momentum coordinates, and the map has been taken from the point of view of a momentum

change of coordinates, which is point-wise. We have not examined the map for space coordinate changes. In this case, the previously established map yielded, for example

$$\begin{aligned} x^1 &= \frac{|\vec{k}_1|x_1^1 + |\vec{k}_2|x_2^1}{k^0} - \frac{k^2 k^z k^\theta}{\vec{k}^2 k^0} \\ x^2 &= \frac{|\vec{k}_1|x_1^2 + |\vec{k}_2|x_2^2}{k^0} + \frac{k^1 k^z k^\theta}{\vec{k}^2 k^0} \\ z &= \frac{(\vec{x}_1 - \vec{x}_2)\cdot \vec{k}_1 |\vec{k}_2| - (\vec{x}_1 - \vec{x}_2)\cdot \vec{k}_2 |\vec{k}_1|}{k^z k^0}. \end{aligned}$$

It is of interest to consider whether the field redefinition introduced in this thesis changes these identifications, by investigating the full phase space map. If in this case the momentum dependence in the space coordinates is still present, the question would be whether it takes a new form consistent with it still being a canonical space transformation.

2. In this thesis we considered only the quadratic Hamiltonian. One could now consider making use of the redefinition and change of variables for the cubic Hamiltonian - this is still an open problem. Indeed Vasiliev's higher spin theory has a very large symmetry and the issue of a gauge fixed three-point function derived from this theory has not been fully understood. There has been some progress using light cone gauge (see Refs. [139, 140, 141]). One could then also look at applying the principles developed in this thesis to higher point vertices.

3. The final point, perhaps the most exciting, was that in the earlier stages of this thesis, we set out to establish or convince ourselves of the completeness and/or orthogonality of energy eigenfunctions of the Hamiltonian in the boundary. This was done using the Collective field theory with bilocals specified in x-space. Despite the many attempts, this could not be achieved in a consistent manner. However, by making use of the map, we were indeed able to find the exact eigen-states in the bulk! If we were to start with the states in the bulk and perform the change back to the boundary, we should find that the completeness and orthonormality was missing some weight functions. This links with work done recently in Ref. [121], and could explain the leg factors that

were introduced in the discussion about conformal partial waves in Collective field theory, in the context of higher spins.

Appendix A

A derivation of Euler's formula

Euler's formula may be proven in two steps [142]:

1. Distorting a polyhedron in any way leaves the Euler characteristic invariant.
2. Introducing a hole reduces the Euler characteristic by 2.

To prove the first point, note that a cube, for example, has Euler characteristic of $\chi = F - P + V = 6 - 12 + 8 = 2$; similarly for tetrahedrons, octahedrons, pyramids etc (polyhedra with no holes). Now consider any polyhedron: there are three fundamental ways to distort the polygonal surface:

- Any deformation (e.g. rescaling of edges) to the surface that leaves P, V and F unchanged will leave the Euler characteristic invariant.
- Shrinking an edge to a point simultaneously reduces P and V by 1, thus leaving χ unchanged and
- Shrinking an N-sided face to a point reduces F by 1, P by N and V by N-1, once again leaving χ unchanged.

Any deformation to the surface is given by some combination of these fundamental deformations.

To prove the second point, consider that introducing a hole requires matching two non-adjacent N-sided faces with each other and eliminating them, resulting in F decreasing by 2, P by N and V by N; hence χ reduces by two for each hole introduced.

Appendix B

Integral Calculations

B.1 Solid Angle

The formula for the solid angle is given by

$$\int d\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

This formula may be recalled using the trick that

$$I = (\pi)^{d/2} = \int_{-\infty}^{\infty} d^d x e^{-\vec{x}\cdot\vec{x}} = \int d\Omega_{d-1} \int_0^{\infty} dr r^{d-1} e^{-r^2}$$

which becomes (with $t = r^2 \Rightarrow dt = 2rdr$),

$$\begin{aligned} I &= \int d\Omega_{d-1} \int_0^{\infty} \frac{1}{2} t^{d/2-1} e^{-t} dt = \frac{1}{2} \int d\Omega_{d-1} \Gamma(d/2) \\ \Rightarrow \int d\Omega_{d-1} &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \end{aligned}$$

B.2 Schwinger Parametrisation

We derive a formula Schwinger derived for parametrisation:

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 ... \int_0^{\infty} dt_n e^{-(t_1 A_1 + t_2 A_2 + ... + t_n A_n)}$$

We change variable as follows:

$$\begin{aligned}\alpha &= t_1 + t_2 + ... + t_n \in [0, \infty) \\ \alpha_i &= \frac{t_i}{t_1 + t_2 + ... + t_n} \in [0, 1] \,.\end{aligned}$$

Here, $i = 1, ..., n-1$ and $\sum_{i=1}^{n-1} \alpha_i \leq 1$. This is a change of variables from n variables to n variables, and we shall calculate the Jacobian as follows: we see that $t_i = \alpha\alpha_i$ for $i = 1...n-1$, so the $n-1$ dimensional change of variables is given by

$$\det \left| \frac{\partial t_i}{\partial \alpha_i} \right| = \alpha^{n-1}$$

We need to then multiply this by the 1 dimensional

$$\frac{\partial t_n}{\partial \alpha} = \left(\frac{\partial \alpha}{\partial t_n} \right)^{-1} = 1$$

Our expression therefore becomes

$$\begin{aligned}\frac{1}{A_1 A_2 ... A_n} &= \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 ... \int_0^1 d\alpha_{n-1} \int_0^\infty d\alpha \, \alpha^{n-1} \\ &\quad e^{-\alpha(\alpha_1 A_1 + \alpha_2 A_2 + ... + \alpha_{n-1} A_{n-1} + (1 - \alpha_1 - ... - \alpha_{n-1}) A_n)} \\ &= \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 ... \int_0^1 d\alpha_{n-1} \\ &\quad \frac{\Gamma(n)}{(\alpha_1 A_1 + \alpha_2 A_2 + ... + \alpha_{n-1} A_{n-1} + (1 - \alpha_1 - ... - \alpha_{n-1}) A_n)^n} .\end{aligned}$$

The more general expression is calculated as follows:

$$\begin{aligned}\frac{1}{A_1^{\lambda_1} A_2^{\lambda_2} ... A_n^{\lambda_n}} &= \frac{1}{\Gamma(\lambda_1)\Gamma(\lambda_2)...\Gamma(\lambda_n)} \int_0^\infty dt_1 \, t_1^{\lambda_1 - 1} \int_0^\infty dt_2 \, t_2^{\lambda_2 - 1} ... \int_0^\infty dt_n \\ &\quad t_n^{\lambda_n - 1} e^{-(t_1 A_1 + t_2 A_2 + ... + t_n A_n)} \\ &= \frac{1}{\Gamma(\lambda_1)\Gamma(\lambda_2)...\Gamma(\lambda_n)} \int_0^\infty d\alpha \int_0^1 d\alpha_1 \, (\alpha\alpha_1)^{\lambda_1 - 1} ... \int_0^1 d\alpha_{n-1} \\ &\quad (\alpha\alpha_{n-1})^{\lambda_{n-1} - 1} \alpha^{n-1} \left[\alpha (1 - \alpha_1 - ... - \alpha\alpha_{n-1})\right]^{\lambda_n - 1} \\ &\quad e^{-\alpha(\alpha_1 A_1 + ... + \alpha_{n-1} A_{n-1} + (1 - \alpha_1 - ... - \alpha_{n-1}) A_n)} \\ &= \frac{1}{\Gamma(\lambda_1)\Gamma(\lambda_2)...\Gamma(\lambda_n)} \int_0^\infty d\alpha \int_0^1 d\alpha_1 \, \alpha_1^{\lambda_1 - 1} ... \int_0^1 d\alpha_{n-1} \, \alpha_{n-1}^{\lambda_{n-1} - 1} \\ &\quad \alpha^{\lambda_1 + \lambda_2 + ... + \lambda_n - 1} (1 - \alpha_1 - ... - \alpha\alpha_{n-1})^{\lambda_n - 1}\end{aligned}$$

$$
\begin{aligned}
& e^{-\alpha(\alpha_1 A_1+...+\alpha_{n-1}A_{n-1}+(1-\alpha_1-...-\alpha_{n-1})A_n)} \\
= & \frac{\Gamma(\lambda_1+\lambda_2+...+\lambda_n)}{\Gamma(\lambda_1)\Gamma(\lambda_2)...\Gamma(\lambda_n)} \int_0^1 d\alpha_1 ... \int_0^1 d\alpha_{n-1} \\
& \frac{\alpha_1^{\lambda_1-1}...\alpha_{n-1}^{\lambda_{n-1}-1}(1-\alpha_1-...-\alpha_{n-1})^{\lambda_n-1}}{(\alpha_1 A_1+...+\alpha_{n-1}A_{n-1}+(1-\alpha_1-...-\alpha_{n-1})A_n)^{\lambda_1+\lambda_2+...+\lambda_n}} \\
= & \frac{\Gamma(\lambda_1+\lambda_2+...+\lambda_n)}{\Gamma(\lambda_1)\Gamma(\lambda_2)...\Gamma(\lambda_n)} \int_0^1 dt_1 ... \int_0^1 dt_{n-1} \\
& \frac{t_1^{\lambda_1-1}...t_{n-1}^{\lambda_{n-1}-1}(1-t_1-...-t_{n-1})^{\lambda_n-1}}{(t_1 A_1+...+t_{n-1}A_{n-1}+(1-t_1-...-t_{n-1})A_n)^{\lambda_1+\lambda_2+...+\lambda_n}}
\end{aligned}
$$

B.3 D-Dimensional Integral Calculations

From the Euler Gamma function,

$$\frac{1}{(k^2)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty d\alpha\, \alpha^{\lambda-1} e^{-k^2\alpha}. \tag{B.1}$$

Therefore

$$
\begin{aligned}
\frac{1}{(k^2-m^2)^\lambda} &= \frac{1}{\Gamma(\lambda)} \int_0^\infty d\alpha\, \alpha^{\lambda-1} e^{-(k^2-m^2)\alpha} \\
\Rightarrow \int_{-\infty}^\infty \frac{d^dk}{(k^2-m^2)^\lambda} &= \frac{1}{\Gamma(\lambda)} \int_0^\infty d\alpha\, \alpha^{\lambda-1} \int d^dk e^{-(k^2-m^2)\alpha} \\
&= \frac{1}{\Gamma(\lambda)} \int_0^\infty d\alpha\, \alpha^{\lambda-1} e^{-(-m^2)\alpha} \frac{\pi^{d/2}}{\alpha^{d/2}} \\
&= \frac{\pi^{d/2}}{\Gamma(\lambda)} \int_0^\infty d\alpha\, \alpha^{\lambda-\frac{d}{2}-1} e^{-(-m^2)\alpha} \\
&= \frac{\pi^{d/2}}{\Gamma(\lambda)} \frac{\Gamma(\lambda-\frac{d}{2})}{(-m^2)^{\lambda-\frac{d}{2}}} \\
\Rightarrow \int_{-\infty}^\infty \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2-\alpha)} &= \frac{\pi^{3/2}}{\Gamma(1)(2\pi)^3} \frac{\Gamma(-\frac{1}{2})}{(-\alpha)^{-\frac{1}{2}}} = \frac{\pi^{3/2}(-2\sqrt{\pi})\sqrt{\alpha}}{8\pi^3 i} \\
&= i\frac{\sqrt{\alpha}}{4\pi}
\end{aligned} \tag{B.2}
$$

The following is a simple example of an integral that requires Schwinger parametrisation: Let us now calculate

$$I = \int \frac{d^dk}{(k^2)^{\lambda_1}\left[(q-k)^2\right]^{\lambda_2}}$$

$$= \int d^d k \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 dt \, \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{\left(t\left(k^2\right)+(1-t)\left[(q-k)^2\right]\right)^{\lambda_1+\lambda_2}}$$

The bracketed term in the denominator of the integrand is expanded and simplified as follows:

$$\begin{aligned} t\left(k^2\right)+(1-t)\left[(q-k)^2\right] &= k^2-2(1-t)\,q\cdot k+(1-t)\,q^2 \\ &= (k-(1-t)\,q)^2+\left[(1-t)\,q^2-(1-t)^2\,q^2\right] \\ &= (k-(1-t)\,q)^2+t\,(1-t)\,q^2 \end{aligned}$$

Now there is a general property of d dimensional integrals that

$$\int d^d p F[p] = \int d^d p F[p+q]$$

and hence we may therefore write

$$\begin{aligned} I &= \int d^d k \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 dt \, \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{\left(k^2+t(1-t)\,q^2\right)^{\lambda_1+\lambda_2}} \\ &= \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 dt \int d^d k \frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{\left(k^2+t(1-t)\,q^2\right)^{\lambda_1+\lambda_2}} \\ &= \frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^1 dt \; t^{\lambda_1-1}(1-t)^{\lambda_2-1}\pi^{d/2} \\ &\quad \frac{\Gamma(\lambda_1+\lambda_2-d/2)}{\Gamma(\lambda_1+\lambda_2)} \frac{1}{\left(t(1-t)\,q^2\right)^{\lambda_1+\lambda_2-d/2}} \\ &= \pi^{d/2} \frac{\Gamma(\lambda_1+\lambda_2-d/2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\left(q^2\right)^{\lambda_1+\lambda_2-d/2}} \int_0^1 dt\, t^{\lambda_1-1}(1-t)^{\lambda_2-1} \\ &\quad \frac{1}{\left(t(1-t)\right)^{\lambda_1+\lambda_2-d/2}} \\ &= \pi^{d/2} \frac{\Gamma(\lambda_1+\lambda_2-d/2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\left(q^2\right)^{\lambda_1+\lambda_2-d/2}} \int_0^1 dt\, t^{-\lambda_2+\frac{d}{2}-1}(1-t)^{-\lambda_1+\frac{d}{2}-1} \\ &= \pi^{d/2} \frac{\Gamma(\lambda_1+\lambda_2-d/2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\left(q^2\right)^{\lambda_1+\lambda_2-d/2}} \frac{\Gamma\left(-\lambda_2+\frac{d}{2}\right)\Gamma\left(-\lambda_1+\frac{d}{2}\right)}{\Gamma(-\lambda_1-\lambda_2+d)}. \end{aligned}$$

For example, if $\lambda_1=\lambda_2=1$ and $d=3$,

$$\int \frac{d^3 k}{k^2(q-k)^2} = \frac{\pi^3}{|q|}. \tag{B.3}$$

In Euclidean signature, this becomes

$$\begin{aligned}\int \frac{d^3k}{k^2(k-q)_M^2} &= \int \frac{idE_E d^2k_E}{(-k_E^2)\left(-(q-k)_E^2\right)} \\ &= \frac{i\pi^3}{|q|_E}. \end{aligned} \tag{B.4}$$

We now integrate

$$\begin{aligned}\int d^dp \frac{1}{(p^2)^\alpha} e^{ipx} &= \int_0^\infty dt \int d^dp \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-p^2t+ipx} \\ &= \int_0^\infty dt \int d^dp \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t\left(p-i\frac{x}{2t}\right)^2} e^{-x^2/4t} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty dt \frac{\pi^{d/2}}{t^{d/2}} t^{\alpha-1} e^{-x^2/4t} \\ &= \frac{\pi^{d/2}}{\Gamma(\alpha)} \int_0^\infty dt\, t^{\alpha-d/2-1} e^{-x^2/4t} \\ &= \frac{\pi^{d/2}}{\Gamma(\alpha)} \int_0^\infty du\, u^{-\alpha+d/2+1-2} e^{-x^2u/4} \\ &= \frac{\pi^{d/2}}{\Gamma(\alpha)} \frac{\Gamma(d/2-\alpha)}{(x^2/4)^{d/2-\alpha}} \end{aligned} \tag{B.5}$$

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2)^1} e^{ipx} = \frac{\pi^{3/2}}{(2\pi)^3\Gamma(1)} \frac{\Gamma(1/2)}{(x^2/4)^{1/2}} = \frac{1}{4\pi|x|} \tag{B.6}$$

We now solve (as per Ref. [7])

$$\begin{aligned} I &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k - p_1 - p_2)^2} \\ &= \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int \frac{dE}{2\pi} \frac{1}{\left(E^2 - \vec{k}^2 + i\epsilon\right)\left((E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon\right)} \\ &= \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \int \frac{dE}{2\pi} \frac{1}{\left(E - \left(|\vec{k}| - i\epsilon\right)\right)\left(E + \left(|\vec{k}| - i\epsilon\right)\right)} \\ &\quad \frac{1}{\left[E - \left(E_p - \left|\vec{k} - \vec{p}\right| - i\epsilon\right)\right]\left[E - \left(E_p + \left|\vec{k} - \vec{p}\right| - i\epsilon\right)\right]} \end{aligned}$$

The integral over energy (closing in LHP) is given by

$$\begin{aligned} I_E &= -i \left[\frac{1}{2\left|\vec{k}\right|\left(\left|\vec{k}\right| - E_p - \left|\vec{k} - \vec{p}\right|\right)\left(\left|\vec{k}\right| - E_p + \left|\vec{k} - \vec{p}\right|\right)} \right. \\ &\quad \left. + \frac{1}{\left(E_p + \left|\vec{k} - \vec{p}\right| + \left|\vec{k}\right|\right)\left(E_p + \left|\vec{k} - \vec{p}\right| - \left|\vec{k}\right|\right)\left(2\left|\vec{k} - \vec{p}\right|\right)} \right] \end{aligned}$$

Returning to $\vec{k} = \vec{k}_1$ and $\vec{p} - \vec{k} = \vec{k}_2$,

$$\begin{aligned} I_E &= -i \frac{1}{2\left|\vec{k}_1\right|\left(E_p - \left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)\left(E_p - \left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)} \\ &\quad + \frac{1}{\left(E_p + \left|\vec{k}_2\right| + \left|\vec{k}_1\right|\right)\left(E_p + \left|\vec{k}_2\right| - \left|\vec{k}_1\right|\right)\left(2\left|\vec{k}_2\right|\right)} \\ &= -\frac{i}{2} \frac{\left(E_p + \left|\vec{k}_2\right| + \left|\vec{k}_1\right|\right)\left(\left|\vec{k}_2\right|\right) + \left|\vec{k}_1\right|\left(E_p - \left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|\left(E_p - \left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)\left(E_p - \left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)\left(E_p + \left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)} \\ &= -\frac{i}{2} \frac{\left(E_p + \left|\vec{k}_2\right| - \left|\vec{k}_1\right|\right)\left(\left|\vec{k}_2\right|\right) + \left|\vec{k}_1\right|\left(E_p - \left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|\left(E_p - \left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)\left(E_p - \left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)\left(E_p + \left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)} \\ &= -\frac{i}{2} \frac{\left|\vec{k}_2\right| + \left|\vec{k}_1\right|}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|\left(E_p - \left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)\left(E_p + \left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)} \\ &= -\frac{i}{2} \frac{1}{E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2} \left(\frac{1}{\left|\vec{k}_1\right|} + \frac{1}{\left|\vec{k}_2\right|}\right) \end{aligned} \tag{B.7}$$

Appendix C

Detailed Calculations

C.1 Conformal Generators

The exhaustive calculations of the commutation relations of the conformal generators written down in Subsection 1.11.2 is as follows:

$$
\begin{aligned}
[x_\mu, P_\nu] &= i\eta_{\mu\nu} \\
[P_\mu, P_\nu] &= 0 = [x_\mu, x_\nu] \\
[D, P_\nu] &= [x^\mu P_\mu, P_\nu] = [x^\mu, P_\nu] P_\mu = iP_\nu \\
[D, x_\nu] &= [x^\mu P_\mu, x_\nu] = x^\mu [P_\mu, x_\nu] = -ix_\nu \\
\left[D, x^2\right] &= [x^\mu P_\mu, x_\nu x^\nu] = x^\mu (x_\nu [P_\mu, x^\nu] + [P_\mu, x_\nu] x^\nu) = -2ix^2 \\
[D, D] &= [x^\mu P_\mu, x^\nu P_\nu] = x^\mu [P_\mu, x^\nu] P_\nu + x^\nu [x^\mu, P_\nu] P_\mu = 0 \\
[D, x_\mu P_\nu] &= [x^\sigma P_\sigma, x_\mu P_\nu] = x^\sigma [P_\sigma, x_\mu] P_\nu + x_\mu [x^\sigma, P_\nu] P_\sigma = 0 \\
[D, \mathcal{J}_{\mu\nu}] &= -[D, x_\mu P_\nu] + [D, x_\nu P_\mu] = 0 \\
[D, K_\mu] &= 2[D, x_\mu] D + 2x_\mu [D, D] - \left[D, x^2\right] P_\mu - x^2 [D, P_\mu] \\
&= -2ix_\mu D + 2ix^2 P_\mu - ix^2 P_\mu = -iK_\mu \\
[P_\rho, x_\mu P_\nu] &= -i\eta_{\rho\mu} P_\nu \\
[P_\rho, \mathcal{J}_{\mu\nu}] &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
[x_\rho, x_\mu P_\nu] &= i\eta_{\rho\mu} x_\nu \\
[x_\rho, \mathcal{J}_{\mu\nu}] &= i(\eta_{\rho\mu} x_\nu - \eta_{\rho\nu} x_\mu)
\end{aligned}
$$

$$
\begin{aligned}
\left[x^2, \mathcal{J}_{\mu\nu}\right] &= 2ix^\rho \left(\eta_{\rho\mu} x_\nu - \eta_{\rho\nu} x_\mu\right) = 0 \\
\left[K_\rho, \mathcal{J}_{\mu\nu}\right] &= \left[2x_\rho D - x^2 P_\rho, \mathcal{J}_{\mu\nu}\right] \\
&= 2i \left(\eta_{\rho\mu} x_\nu - \eta_{\rho\nu} x_\mu\right) D + 0 + x^2 i \left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\right) - 0 \\
&= i \left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\right) \\
\left[x_\mu P_\nu, x_\rho P_\sigma\right] &= x_\mu \left[P_\nu, x_\rho\right] P_\sigma + x_\rho \left[x_\mu, P_\sigma\right] P_\nu = -i\eta_{\nu\rho} x_\mu P_\sigma + i\eta_{\mu\sigma} x_\rho P_\nu \\
\left[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}\right] &= \left[x_\mu P_\nu, x_\rho P_\sigma\right] - \left[x_\mu P_\nu, x_\sigma P_\rho\right] - \left[x_\nu P_\mu, x_\rho P_\sigma\right] + \left[x_\nu P_\mu, x_\sigma P_\rho\right] \\
&= -i\eta_{\nu\rho} x_\mu P_\sigma + i\eta_{\mu\sigma} x_\rho P_\nu + i\eta_{\nu\sigma} x_\mu P_\rho - i\eta_{\mu\rho} x_\sigma P_\nu \\
&\quad +i\eta_{\mu\rho} x_\nu P_\sigma - i\eta_{\nu\sigma} x_\rho P_\mu - i\eta_{\mu\sigma} x_\nu P_\rho + i\eta_{\nu\rho} x_\sigma P_\mu \\
&= i \left(\eta_{\nu\rho} \mathcal{J}_{\mu\sigma} + \eta_{\mu\sigma} \mathcal{J}_{\nu\rho} - \eta_{\mu\rho} \mathcal{J}_{\sigma\nu} - \eta_{\nu\sigma} \mathcal{J}_{\mu\rho}\right) \\
\left[x^2, P_\nu\right] &= 2ix_\nu \\
\left[K_\mu, P_\nu\right] &= \left[2x_\mu D - x^2 P_\mu, P_\nu\right] \\
&= 2\left[x_\mu, P_\nu\right] D + 2x_\mu \left[D, P_\nu\right] - \left[x^2, P_\nu\right] P_\mu \\
&= 2i\eta_{\mu\nu} D + 2ix_\mu P_\nu - 2ix_\nu P_\mu \\
&= 2i \left[\eta_{\mu\nu} D - \mathcal{J}_{\mu\nu}\right] \\
\left[x_\mu D, x_\nu D\right] &= x_\nu \left[x_\mu, D\right] D + x_\mu \left[D, x_\nu\right] D \\
&= ix_\nu x_\mu D - ix_\mu x_\nu D = 0 \\
\left[x_\mu D, x^2 P_\nu\right] &= x_\mu \left[D, x^2 P_\nu\right] + \left[x_\mu, x^2 P_\nu\right] D \\
&= x_\mu \left[D, x^2\right] P_\nu + x_\mu x^2 \left[D, P_\nu\right] + x^2 \left[x_\mu, P_\nu\right] D \\
&= -2ix_\mu x^2 P_\nu + ix_\mu x^2 P_\nu + ix^2 \eta_{\mu\nu} D \\
&= -ix_\mu x^2 P_\nu + ix^2 \eta_{\mu\nu} D \\
\left[x^2, P_\mu\right] &= 2ix_\mu \\
\left[x^2 P_\mu, x^2 P_\nu\right] &= x^2 \left[P_\mu, x^2\right] P_\nu + x^2 \left[x^2, P_\nu\right] P_\mu \\
&= -2ix^2 x_\mu P_\nu + 2ix^2 x_\nu P_\mu \\
&= 2ix^2 \mathcal{J}_{\mu\nu} \\
\left[K_\mu, K_\nu\right] &= \left[2x_\mu D - x^2 P_\mu, 2x_\nu D - x^2 P_\nu\right] \\
&= -2\left(-ix_\mu x^2 P_\nu + ix^2 \eta_{\mu\nu} D\right) + 2\left(-ix_\nu x^2 P_\mu + ix^2 \eta_{\nu\mu} D\right) + 2ix^2 \mathcal{J}_{\mu\nu} \\
&= 0,
\end{aligned}
$$

which may be summarised as

$$
\left[P_\mu, \mathcal{J}_{\rho\sigma}\right] = i\eta_{\mu\rho} P_\sigma - i\eta_{\mu\sigma} P_\rho
$$

$$\begin{aligned}
[D, P_\nu] &= iP_\nu \\
[D, K_\nu] &= -iK_\nu \\
[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right) \\
[K_\mu, P_\nu] &= 2i\left[\eta_{\mu\nu}D - \mathcal{J}_{\mu\nu}\right] \\
[K_\rho, \mathcal{J}_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right).
\end{aligned}$$

C.2 Quadratic Hamiltonian Mode Expansions

We now substitute these mode expansions into our free quadratic hamiltonian

$$H^{(2)} = 2\mathrm{Tr}\left(p\psi_0 p\right) + \frac{1}{8}\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}.$$

The first term is given by

$$\begin{aligned}
2\mathrm{Tr}\left(p\psi_0 p\right) &= 2\int \frac{d^{d-1}k_1}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}k_2}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{d^{d-1}k_1'}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}k_2'}{(2\pi)^{(d-1)/2}} \\
&\quad \int d^dx\, d^dy\, d^dz \left(-i\sqrt{\frac{\omega_{k_1}\omega_{k_2}}{2}}\right)\left(-i\sqrt{\frac{\omega_{k_1'}\omega_{k_2'}}{2}}\right) e^{ik(y-z)}\psi_k \\
&\quad \left(e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik_1x}e^{ik_2y}a_{k_1k_2} - e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-ik_1x}e^{-ik_2y}a^\dagger_{k_1k_2}\right) \\
&\quad \left(e^{-i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}e^{ik_1'z}e^{ik_2'x}a_{k_1'k_2'} - e^{i\left(\omega_{k_1'}+\omega_{k_2'}\right)t}e^{-ik_1'z}e^{-ik_2'x}a^\dagger_{k_1'k_2'}\right).
\end{aligned}$$

For the $a_{k_1k_2}a_{k_1'k_2'}$ term, the x, y and z integrals result in 3 delta functions: $(2\pi)^{3(d-1)}$ and $k_2' \to -k_1$, $k \to -k_2$ and $k_1' \to k \to -k_2$:

$$2\mathrm{Tr}\left(p\psi_0 p\right)_{aa} = -\int d^{d-1}k_1 d^{d-1}k_2\left(\omega_{k_1}\omega_{k_2}\right)e^{-2i\left(\omega_{k_1}+\omega_{k_2}\right)t}\psi_{k_2}a_{k_1k_2}a_{-k_2-k_1}.$$

For the $a_{k_1k_2}a^\dagger_{k_1'k_2'}$ term, these integrals set $k_2' \to k_1$, $k \to -k_2$ and $k_1' \to -k \to k_2$:

$$2\mathrm{Tr}\left(p\psi_0 p\right)_{aa^\dagger} = \int d^{d-1}k_1 d^{d-1}k_2\left(\omega_{k_1}\omega_{k_2}\right)\psi_{k_2}a_{k_1k_2}a^\dagger_{k_2k_1}.$$

For the $a^\dagger_{k_1k_2}a_{k_1'k_2'}$ term, these integrals set $k_2' \to k_1$, $k \to k_2$ and $k_1' \to k \to k_2$:

$$2\mathrm{Tr}\left(p\psi_0 p\right)_{a^\dagger a} = \int d^{d-1}k_1 d^{d-1}k_2\left(\omega_{k_1}\omega_{k_2}\right)\psi_{k_2}a^\dagger_{k_1k_2}a_{k_2k_1}$$

and for the $a^\dagger_{k_1k_2}a^\dagger_{k'_1k'_2}$ term, they set $k'_2 \to -k_1$, $k \to k_2$ and $k'_1 \to -k \to -k_2$:

$$2\mathrm{Tr}\left(p\psi_0 p\right)_{a^\dagger a^\dagger} = -\int d^{d-1}k_1 d^{d-1}k_2 \left(\omega_{k_1}\omega_{k_2}\right) e^{2i(\omega_{k_1}+\omega_{k_2})t}\psi_{k_2} a^\dagger_{k_1k_2} a^\dagger_{-k_2-k_1}.$$

The second term $\frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) = \frac{1}{8}\mathrm{Tr}\left(\eta\psi_0^{-1}\eta\psi_0^{-2}\right)$ is given by

$$\int \frac{d^{d-1}k_1}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}k_2}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}l_1}{(2\pi)^{d-1}}\frac{d^{d-1}k'_1}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}k'_2}{(2\pi)^{(d-1)/2}}\frac{d^{d-1}l_2}{(2\pi)^{d-1}}$$
$$\int d^dx_1\, d^dx_2\, d^dx_3\, d^dx_4 \frac{1}{\sqrt{2\omega_{k_1}\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k'_1}\omega_{k'_2}}} e^{il_1(x_2-x_3)}\psi^{-1}_{l_1} e^{il_2(x_4-x_1)}\psi^{-2}_{l_2}$$
$$\left(e^{-i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{ik_1x_1}e^{ik_2x_2}a_{k_1k_2} + e^{i\left(\omega_{k_1}+\omega_{k_2}\right)t}e^{-ik_1x_1}e^{-ik_2x_2}a^\dagger_{k_1k_2}\right)$$
$$\left(e^{-i\left(\omega_{k'_1}+\omega_{k'_2}\right)t}e^{ik'_1x_3}e^{ik'_2x_4}a_{k'_1k'_2} + e^{i\left(\omega_{k'_1}+\omega_{k'_2}\right)t}e^{-ik'_1x_3}e^{-ik'_2x_4}a^\dagger_{k'_1k'_2}\right).$$

For the $a_{k_1k_2}a_{k'_1k'_2}$ term, the x_1, x_2, x_3 and x_4 integrals result in 4 delta functions: $(2\pi)^{4(d-1)}$ and $l_2 \to k_1$, $l_1 \to -k_2$, $k'_1 \to l_1 \to -k_2$ and $k'_2 \to -l_2 \to -k_1$:

$$\frac{1}{8}\mathrm{Tr}\left(\eta\psi_0^{-1}\eta\psi_0^{-2}\right)_{aa} = \frac{1}{16}\int d^{d-1}k_1 d^{d-1}k_2 \left(\frac{1}{\omega_{k_1}\omega_{k_2}}\right) e^{-2i(\omega_{k_1}+\omega_{k_2})t}\psi^{-1}_{k_2}\psi^{-2}_{k_1} a_{k_1k_2}a_{-k_2-k_1}.$$

For the $a_{k_1k_2}a^\dagger_{k'_1k'_2}$ term, these integrals set $l_2 \to k_1$, $l_1 \to -k_2$, $k'_1 \to -l_1 \to k_2$ and $k'_2 \to l_2 \to k_1$:

$$\frac{1}{8}\mathrm{Tr}\left(\eta\psi_0^{-1}\eta\psi_0^{-2}\right)_{aa^\dagger} = \frac{1}{16}\int d^{d-1}k_1 d^{d-1}k_2 \left(\frac{1}{\omega_{k_1}\omega_{k_2}}\right)\psi^{-1}_{k_2}\psi^{-2}_{k_1} a_{k_1k_2}a^\dagger_{k_2k_1}.$$

For the $a^\dagger_{k_1k_2}a_{k'_1k'_2}$ term, these integrals set $l_2 \to -k_1$, $l_1 \to k_2$, $k'_1 \to l_1 \to k_2$ and $k'_2 \to -l_2 \to k_1$:

$$\frac{1}{8}\mathrm{Tr}\left(\eta\psi_0^{-1}\eta\psi_0^{-2}\right)_{a^\dagger a} = \frac{1}{16}\int d^{d-1}k_1 d^{d-1}k_2 \left(\frac{1}{\omega_{k_1}\omega_{k_2}}\right)\psi^{-1}_{k_2}\psi^{-2}_{k_1} a^\dagger_{k_1k_2}a_{k_2k_1}.$$

For the $a^\dagger_{k_1k_2}a^\dagger_{k'_1k'_2}$ term, these integrals set $l_2 \to -k_1$, $l_1 \to k_2$, $k'_1 \to -l_1 \to -k_2$ and $k'_2 \to l_2 \to -k_1$:

$$\frac{1}{8}\mathrm{Tr}\left(\eta\psi_0^{-1}\eta\psi_0^{-2}\right)_{a^\dagger a^\dagger} = \frac{1}{16}\int d^{d-1}k_1 d^{d-1}k_2 \left(\frac{1}{\omega_{k_1}\omega_{k_2}}\right) e^{2i(\omega_{k_1}+\omega_{k_2})t}\psi^{-1}_{k_2}\psi^{-2}_{k_1} a^\dagger_{k_1k_2}a^\dagger_{-k_2-k_1}.$$

Grouping all of these results together leads to

$$\begin{aligned}
H^{(2)} &= \int dk_1 \int dk_2 \left(\omega_{k_1}\omega_{k_2}\psi_{k_2} + \frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_2}^{-2}\right) a_{k_1k_2}a^\dagger_{k_2k_1} \\
&= +\int dk_1 \int dk_2 \left(\omega_{k_1}\omega_{k_2}\psi_{k_2} + \frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_2}^{-2}\right) a^\dagger_{k_1k_2}a_{k_2k_1} \\
&= +\int dk_1 \int dk_2 \left(-\omega_{k_1}\omega_{k_2}\psi_{k_2} + \frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_2}^{-2}\right) e^{-2i(\omega_{k_1}+\omega_{k_2})t}a_{k_1k_2}a_{-k_2-k_1} \\
&= +\int dk_1 \int dk_2 \left(-\omega_{k_1}\omega_{k_2}\psi_{k_2} + \frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_2}^{-2}\right) e^{2i(\omega_{k_1}+\omega_{k_2})t}a^\dagger_{k_1k_2}a^\dagger_{-k_2-k_1}.
\end{aligned}$$

From symmetry and using $\psi_k = \frac{1}{2\omega_k}$, we may write

$$\begin{aligned}
\omega_{k_1}\omega_{k_2}\psi_{k_2} &= \frac{1}{2}\omega_{k_1}\omega_{k_2}\left(\psi_{k_1}+\psi_{k_2}\right) = \frac{1}{4}\left(\omega_{k_1}+\omega_{k_2}\right) \\
\frac{1}{16\omega_{k_1}\omega_{k_2}}\psi_{k_2}^{-1}\psi_{k_2}^{-2} &= \frac{1}{4}\frac{1}{2}\left(\psi_{k_1}^{-1}+\psi_{k_2}^{-1}\right) = \frac{1}{4}\left(\omega_{k_1}+\omega_{k_2}\right)
\end{aligned}$$

and hence the quadratic Hamiltonian becomes

$$\begin{aligned}
H^{(2)} &= \frac{1}{2}\int dk_1 \int dk_2 \left(\omega_{k_1}+\omega_{k_2}\right)\left(a_{k_1k_2}a^\dagger_{k_2k_1} + a^\dagger_{k_1k_2}a_{k_2k_1}\right) \\
&= \frac{1}{2}\int dk_1 \int dk_2 E_{\vec{k}_1\vec{k}_2}\left(a_{k_1k_2}a^\dagger_{k_2k_1} + a^\dagger_{k_1k_2}a_{k_2k_1}\right).
\end{aligned}$$

C.3 Light Cone

C.3.1 Equal Time Commutators

The calculation regarding the equal time commutator between the field and conjugate momentum in light cone coordinates from Subsection 4.2.2 is as follows

$$\begin{aligned}
&\left[\Pi^{+i}\left(x^+, x_1^-, x_2^\perp\right), \phi^j\left(x^+, x_2^-, x_2^\perp\right)\right]_{x^+} = \left[-\frac{\partial}{\partial x^-}\phi^i, \phi^j\right]_{x^+} \\
&= -\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{dk_1}{\sqrt{2\pi}}\sqrt{\frac{p_1^+}{2}}\int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{dk_2}{\sqrt{2\pi}}\frac{1}{\sqrt{2p_2^+}} \\
&\quad i\left[e^{i\left(p_1^-x^+ + p_1^+x_1^- + k_1x_{1\perp}\right)}A^i_{p_1^+k_1} - e^{-i\left(p_1^-x^+ + p_1^+x_1^- + k_1x_{1\perp}\right)}A^{\dagger i}_{p_1^+k_1},\right. \\
&\quad \left. e^{i\left(p_2^-x^+ + p_2^+x_2^- + k_2x_{2\perp}\right)}A^j_{p_2^+k_2} + e^{-i\left(p_2^-x^+ + p_2^+x_2^- + k_2x_{2\perp}\right)}A^{\dagger j}_{p_2^+k_2}\right]_{x^+}
\end{aligned}$$

$$\begin{aligned}
&= -\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_1}{\sqrt{2\pi}} \sqrt{\frac{p_1^+}{2}} \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_2}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \\
&\quad i\Big[e^{i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^i_{p_1^+ k_1}, A^k_{p_2^+ k_2}\right] \\
&\quad - e^{-i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^{\dagger i}_{p_1^+ k_1}, A^j_{p_2^+ k_2}\right] \\
&\quad + e^{i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{-i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^i_{p_1^+ k_1} A^{\dagger j}_{p_2^+ k_2}\right] \\
&\quad - e^{-i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{-i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^{\dagger i}_{p_1^+ k_1} A^{\dagger j}_{p_2^+ k_2}\right]\Big] \\
&= i\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_1}{\sqrt{2\pi}} \sqrt{\frac{p_1^+}{2}} \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_2}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \\
&\quad + e^{-i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^{\dagger i}_{p_1^+ k_1}, A^j_{p_2^+ k_2}\right] \\
&\quad - e^{i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{-i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^i_{p_1^+ k_1} A^{\dagger j}_{p_2^+ k_2}\right] \\
&= i\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_1}{\sqrt{2\pi}} \sqrt{\frac{p_1^+}{2}} \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_2}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \\
&\quad + e^{-i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left(-\delta(p_1^+ - p_2^+)\delta(k_1 - k_2)\delta_{ij}\right) \\
&\quad - e^{i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{-i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \delta(p_1^+ - p_2^+)\delta(k_1 - k_2)\delta_{ij} \\
&= -i\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_1}{\sqrt{2\pi}} \frac{1}{2}\frac{1}{2\pi} \\
&\quad \left[e^{i\left(p_1^+(x_2^- - x_1^-) + k_1(x_{2\perp} - x_{1\perp})\right)} + e^{i\left(p_1^+(x_1^- - x_2^-) + k_1(x_{1\perp} - x_{2\perp})\right)}\right]\delta_{ij} \\
&= -\frac{i}{2}\left[\frac{1}{2} \times 2\delta(x_1^- - x_2^-)\delta(x_{1\perp} - x_{2\perp})\right]\delta_{ij} \\
&= -\frac{i}{2}\delta(x_1^- - x_2^-)\delta(x_{1\perp} - x_{2\perp})\delta_{ij}.
\end{aligned}$$

Similarly, the equal time commutator between fields is given by

$$\begin{aligned}
&\left[\phi^i\left(x_1^-, x_{1\perp}\right), \phi^j\left(x_2^-, x_{2\perp}\right)\right] \\
&= \int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_1}{\sqrt{2\pi}} \sqrt{\frac{1}{2p_1^+}} \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_2}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \\
&\quad e^{-i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^{\dagger i}_{p_1^+ k_1}, A^j_{p_2^+ k_2}\right] \\
&\quad + e^{i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{-i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \left[A^i_{p_1^+ k_1} A^{\dagger j}_{p_2^+ k_2}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_1}{\sqrt{2\pi}} \sqrt{\frac{1}{2p_1^+}} \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{dk_2}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \\
&\quad \delta(p_1^+ - p_2^+)\delta(k_1 - k_2)\delta_{ij} \\
&\quad \Big(-e^{-i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)} \\
&\quad + e^{i\left(p_1^- x^+ + p_1^+ x_1^- + k_1 x_{1\perp}\right)} e^{-i\left(p_2^- x^+ + p_2^+ x_2^- + k_2 x_{2\perp}\right)}\Big) \\
&= \int_0^\infty \frac{dp_1^+}{2\pi} \int_{-\infty}^\infty \frac{dk_1}{2\pi} \frac{1}{2p_1^+}\delta_{ij} \\
&\quad \Big(-e^{i\left(p_1^+(x_2^- - x_1^-) + k_1(x_{2\perp} - x_{1\perp})\right)} + e^{i\left(p_1^+(x_1^- - x_2^-) + k_1(x_{1\perp} - x_{2\perp})\right)}\Big) \\
&= \int_0^\infty \frac{dp_1^+}{2\pi} \frac{1}{2p_1^+}\delta_{ij}\delta(x_{1\perp} - x_{2\perp}) \left(-e^{i\left(p_1^+(x_2^- - x_1^-)\right)} + e^{i\left(p_1^+(x_1^- - x_2^-)\right)}\right) \\
&= \frac{1}{2}\delta_{ij}\delta(x_{1\perp} - x_{2\perp}) i\epsilon\left(x_1^- - x_2^-\right)
\end{aligned}
$$

C.3.2 First Quantised Generators

For translations, where derivatives of the field inside the $\delta\phi$ term are set to zero (where caligraphic writing indicates operator densities),

$$
\begin{aligned}
\delta x^\mu &= \epsilon^\mu \\
\delta\phi &= \partial_\mu \left[\epsilon^\mu\right]\phi = 0 \\
\Rightarrow j^+ &= \pi\partial_\nu\phi\delta x^\nu - \left(\pi\partial_+\phi - \mathcal{H}\right) g_\nu^+ \delta x^\nu - \pi\delta\phi \\
&= \pi\partial_\nu\phi\epsilon^\nu - \left(\pi\partial_+\phi - \mathcal{H}\right) g_\nu^+ \epsilon^\nu \\
&= \left(\pi\partial_\nu\phi - \left(\pi\partial_+\phi - \mathcal{H}\right) g_\nu^+\right)\epsilon^\nu \\
\Rightarrow \mathcal{P}^\nu = T^{+\nu} = j^{+\nu} &= \pi\partial^\nu\phi - \left(\pi\partial_+\phi - \mathcal{H}\right) g^{\nu+} \\
\mathcal{P}^+ &= \pi\partial_-\phi \\
\Rightarrow P^+ &= -\int dx^- dx^\perp \pi^2 \\
\mathcal{P}^- &= \pi\partial_+\phi - \left(\pi\partial_+\phi - \mathcal{H}\right) \\
\Rightarrow P^- &= H = \int dx^- dx^\perp \frac{1}{2}\left(\partial_\perp\phi\right)^2 \\
\mathcal{P}^\perp &= \pi\partial_\perp\phi \\
\Rightarrow P^\perp &= \int dx^- dx^\perp\, \pi\partial_\perp\phi.
\end{aligned}
$$

For Lorentz transformations,

$$
\begin{aligned}
\delta x_\mu &= \left(\epsilon_{\rho\mu} - \epsilon_{\mu\rho}\right) x^\rho \\
\delta\phi &= -\partial^\mu \left[\left(\epsilon_{\rho\mu} - \epsilon_{\mu\rho}\right) x^\rho\right]\phi = 0 \text{ (antisymmetric parameter)} \\
\Rightarrow j^\mu &= \left(\mathcal{L}g^\mu_\nu - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\nu\phi\right)\delta x^\nu \\
&= \left(\mathcal{L}g^\mu_\nu - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\nu\phi\right)\left(\epsilon_{\rho\nu} - \epsilon_{\nu\rho}\right) x^\rho \\
&= \epsilon_{\rho\nu}\left[\left(\mathcal{L}g^\mu_\nu - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\nu\phi\right) x^\rho - \left(\mathcal{L}g^\mu_\rho - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\rho\phi\right) x^\nu\right] \\
\Rightarrow j^{\mu\nu\rho} &= \left[\left(\mathcal{L}g^{\mu\nu} - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\nu\phi\right) x^\rho - \left(\mathcal{L}g^{\mu\rho} - \frac{\delta\mathcal{L}}{\delta\partial^\mu\phi}\partial^\rho\phi\right) x^\nu\right] \\
\Rightarrow M^{\nu\rho} = j^{+\nu\rho} &= \left[\left(\mathcal{L}g^{+\nu} - \pi\partial^\nu\phi\right) x^\rho - \left(\mathcal{L}g^{+\rho} - \pi\partial^\rho\phi\right) x^\nu\right] \\
\Rightarrow \mathcal{M}^{+-} &= \left[\left(\mathcal{L}g^{++} - \pi\partial^+\phi\right) x^- - \left(\mathcal{L}g^{+-} - \pi\partial^-\phi\right) x^+\right] \\
&= \left[-\pi^2 x^- - \left(\pi\partial_+\phi - \mathcal{H}\right) x^+ + \pi x^+\partial_+\phi\right] \\
&= x^+\mathcal{H} - x^-\pi^2 \\
\Rightarrow M^{+-} &= x^+ H - \int dx^- dx^\perp\ x^-\pi^2 \\
\mathcal{M}^{+\perp} &= \left[\left(\mathcal{L}g^{++} - \pi\partial^+\phi\right) x^\perp - \left(\mathcal{L}g^{+\perp} - \pi\partial^\perp\phi\right) x^+\right] \\
\Rightarrow M^{+\perp} &= \int dx^- dx^\perp\ \pi x^+\partial_\perp\phi + x^\perp\pi^2 \\
\mathcal{M}^{-\perp} &= \left[\left(\mathcal{L}g^{+-} - \pi\partial^-\phi\right) x^\perp - \left(\mathcal{L}g^{+\perp} - \pi\partial^\perp\phi\right) x^-\right] \\
&= \left(\pi\partial_+\phi - \mathcal{H} - \pi\partial_+\phi\right) x^\perp + \pi\partial_\perp\phi x^- \\
\Rightarrow M^{-\perp} &= \int dx^- dx^\perp\ \left(x^-\pi\partial_\perp\phi - x^\perp\mathcal{H}\right) \\
\mathcal{M}^{\perp_j\perp_k} &= \left[\left(\mathcal{L}g^{+\perp_j} - \pi\partial^{\perp_j}\phi\right) x^{\perp_k} - \left(\mathcal{L}g^{+\perp_k} - \pi\partial^{\perp_k}\phi\right) x^{\perp_j}\right] \\
\Rightarrow M^{\perp_j\perp_k} &= \int dx^- dx^\perp\ \left(\pi x^{\perp_j}\partial_{\perp_k}\phi - \pi x^{\perp_k}\partial_{\perp_j}\phi\right).
\end{aligned}
$$

For Dilatations,

$$
\begin{aligned}
\delta x^\mu &= \epsilon x^\mu \\
\delta\phi &= -\partial_\mu\left[\epsilon x^\mu\right]\phi = -\epsilon d_\phi\phi \\
\Rightarrow j &= \pi\partial_\nu\phi\delta x^\nu - \left(\pi\partial_+\phi - \mathcal{H}\right)\delta^+_\nu\delta x^\nu - \pi\delta\phi \\
&= \pi\partial_\nu\phi\epsilon x^\nu - \left(\pi\partial_+\phi - \mathcal{H}\right)\delta^+_\nu\epsilon x^\nu + \epsilon\pi d_\phi\phi \\
\Rightarrow \mathcal{D} &= \pi\partial_\nu\phi x^\nu - \left(\pi\partial_+\phi - \mathcal{H}\right)\delta^+_\nu x^\nu + \pi d_\phi\phi
\end{aligned}
$$

$$
\begin{aligned}
&= \pi\left(x^+\partial_+\phi + x^-\partial_-\phi + x^\perp\partial_\perp\phi\right) - \left(\pi\partial_+\phi - \mathcal{H}\right)\delta^+_\nu x^\nu + \pi d_\phi\phi \\
&= x^+\mathcal{H} + \pi\left(x^-\partial_-\phi + x^\perp\partial_\perp\phi\right) + \pi d_\phi\phi \\
\Rightarrow D &= x^+H + \int dx^- dx^\perp\ \left(\pi\left(x^-\partial_- + x^\perp\partial_\perp\right)\phi + \pi d_\phi\phi\right) \\
&= x^+H + \int dx^- dx^\perp\ \left(-x^-\pi^2 + x^\perp\partial_\perp\phi + \pi d_\phi\phi\right).
\end{aligned}
$$

For Special Conformal Transformations, writing $i \equiv \perp_i$ etc.,

$$
\begin{aligned}
\delta x^\mu &= 2\left(x\cdot\epsilon\right)x^\mu - x^2\epsilon^\mu = \left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right]\epsilon_\alpha \\
\delta\phi &= -\partial_\mu\left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right]\epsilon_\alpha\phi = -2d_\phi\epsilon^\mu x_\mu\phi \\
\Rightarrow j &= \pi\partial_\nu\phi g x^\nu - \left(\pi\partial_+\phi - \mathcal{H}\right)g^+_\nu\delta x^\nu - \pi\delta\phi \\
&= \pi\partial_\mu\phi\left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right]\epsilon_\alpha - \left(\pi\partial_+\phi - \mathcal{H}\right)g^+_\mu\left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right]\epsilon_\alpha \\
&\quad +2\pi d_\phi\epsilon^\mu x_\mu\phi \\
\Rightarrow \mathcal{K}^\alpha &= \pi\partial_\mu\phi\left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right] - \left(\pi\partial_+\phi - \mathcal{H}\right)g^+_\mu\left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right] + 2\pi d_\phi x^\alpha\phi \\
&= \pi\partial_\mu\phi\left[2x^\alpha x^\mu - x^2 g^{\alpha\mu}\right] - \left(\pi\partial_+\phi - \mathcal{H}\right)\left[2x^\alpha x^+ - x^2 g^{\alpha+}\right] + 2\pi d_\phi x^\alpha\phi \\
\Rightarrow \mathcal{K}^i &= 2\pi\left(x^i x^j\partial_j\phi + x^i x^+\partial_+\phi + x^i x^-\partial_-\phi\right) - x^2\pi\partial_i\phi \\
&\quad -2\pi x^i x^+\partial_+\phi + 2x^i x^+\mathcal{H} + 2\pi d_\phi x^i\phi \\
&= 2\pi\left(x^i x^j\partial_j\phi + x^i x^-\partial_-\phi\right) - x^2\pi\partial_i\phi + 2x^i x^+\mathcal{H} + 2\pi d_\phi x^i\phi \\
&= 2x^i\left[x^+\mathcal{H} + \pi x^-\partial_-\phi + \pi x^j\partial_j\phi + \pi d_\phi\phi\right] - x^2\pi\partial_i\phi \\
&= 2x^i\left[x^+\mathcal{H} - \pi^2 x^- + \pi x^j\partial_j\phi + \pi d_\phi\phi\right] - x^2\pi\partial_i\phi \\
&= 2x^i\mathcal{D} - \left(2x^+x^- + x^j x_j\right)\pi\partial_i\phi \\
\Rightarrow K^i &= \int dx^- dx^\perp\ \left(x^i\mathcal{D} - \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\pi\partial_i\phi\right) \\
\mathcal{K}^+ &= 2\pi\left(x^+x^j\partial_j\phi + x^+x^+\partial_+\phi + x^+x^-\partial_-\phi\right) - x^2\pi\partial_-\phi \\
&\quad -2\pi x^+x^+\partial_+\phi + 2x^+x^+\mathcal{H} + 2\pi d_\phi x^+\phi \\
&= 2\pi\left(x^+x^j\partial_j\phi + x^+x^-\partial_-\phi\right) - x^2\pi\partial_-\phi + 2x^+x^+\mathcal{H} + 2\pi d_\phi x^+\phi \\
&= 2x^+\left[x^+\mathcal{H} - \pi^2 x^- + \pi x^j\partial_j\phi + \pi d_\phi\phi\right] - x^2\pi\partial_-\phi \\
&= 2x^+\mathcal{D} + \left(2x^-x^+ + x^j x_j\right)\pi^2 \\
\Rightarrow K^+ &= \int dx^- dx^\perp\ \left(x^+\mathcal{D} + \frac{1}{2}\left(2x^-x^+ + x^j x_j\right)\pi^2\right) \\
\mathcal{K}^- &= 2\pi\left(x^-x^j\partial_j\phi + x^-x^+\partial_+\phi + x^-x^-\partial_-\phi\right) - x^2\pi\partial_+\phi \\
&\quad -2\pi x^-x^+\partial_+\phi + 2x^-x^+\mathcal{H} + x^2\pi\partial_+\phi - x^2\mathcal{H} + 2\pi d_\phi x^-\phi
\end{aligned}
$$

$$\begin{aligned}
&= 2\pi\left(x^- x^j \partial_j\phi + x^- x^- \partial_-\phi\right) - x^2\pi\partial_+\phi + 2x^- x^+\mathcal{H} + x^2\pi\partial_+\phi \\
&\quad -x^2\mathcal{H} + 2\pi d_\phi x^-\phi \\
&= 2x^-\left[x^+\mathcal{H} - \pi^2 x^- + x^j\partial_j\phi + \pi d_\phi\phi\right] + x^2\pi\partial_+\phi - x^2\mathcal{H} - x^2\pi\partial_+\phi \\
&= 2x^-\left[x^+\mathcal{H} - \pi^2 x^- + x^j\partial_j\phi + \pi d_\phi\phi\right] - \left(2x^- x^+ + x^j x_j\right)\mathcal{H} \\
\Rightarrow K^- &= \int dx^- dx^\perp\,\left(x^-\left[x^+\mathcal{H} - \pi^2 x^- + x^j\partial_j\phi + \pi d_\phi\phi\right] - \frac{1}{2}\left(2x^- x^+ + x^j x_j\right)\mathcal{H}\right).
\end{aligned}$$

In summary,

$$\begin{aligned}
P^+ &= -\int dx^- dx^i \pi^2 \\
P^- &= H = \int dx^- dx^i \frac{1}{2}\left(\partial_i\phi\right)^2 \\
P^i &= \int dx^- dx^i \left(\pi\partial_i\phi\right) \\
M^{+-} &= x^+ H - \int dx^- dx^i \left(\pi^2 x^-\right) \\
M^{+i} &= \int dx^- dx^i \pi x^+ \partial_i\phi + x^i\pi^2 \\
M^{-i} &= \int dx^- dx^i \left(x^-\pi\partial_i\phi - x^i\mathcal{H}\right) \\
M^{ij} &= \int dx^- dx^\perp \left(\pi x^i\partial_j\phi - \pi x^j\partial_i\phi\right) \\
D &= x^+ H + \int dx^- dx^\perp\,\left(\pi d_\phi\phi + \pi x^i\partial_i\phi - x^-\pi^2\right) \\
K^i &= \int dx^- dx^i \left(x^i\mathcal{D} - \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\pi\partial_i\phi\right) \\
K^+ &= \int dx^- dx^\perp \left(x^+\mathcal{D} + \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\pi^2\right) \\
K^- &= \int dx^- dx^\perp \left(x^-\left[x^+\mathcal{H} - \pi^2 x^- + x^j\partial_j\phi + \pi d_\phi\phi\right] - \frac{1}{2}\left(2x^+x^- + x^j x_j\right)\mathcal{H}\right).
\end{aligned}$$

C.3.3 Second Quantised Generators

Using the conventions in Ref. [5], noting that second quantised conformal generators satisfy $G = \int dx_1^- dx_2^- d\vec{x}_1 d\vec{x}_2 A^\dagger \hat{g} A)$ (with the momenta conjugate given by $(p_1^+, p_2^+, p_1^i, p_2^i)$), these second quantised generators are derived here. Writing the mode expansion as the Fourier transform of the fields $\phi(x^-, x^i)$ and $\pi(x^-, x^i)$ in the

x^- direction as

$$\begin{aligned}\phi(x^-,x^i) &= \int_0^\infty \frac{dp^+}{\sqrt{2\pi}}\frac{1}{\sqrt{2p^+}}\Big(a(p^+,x^i)e^{ip^+x^-}+a^\dagger(p^+,x^i)e^{-ip^+x^-}\Big)\\ \pi(x^-,x^i) &= -i\int_0^\infty \frac{dp^+}{\sqrt{2\pi}}\sqrt{\frac{p^+}{2}}\Big(a(p^+,x^i)e^{ip^+x^-}-a^\dagger(p^+,x^i)e^{-ip^+x^-}\Big),\end{aligned}$$

we obtain

$$\begin{aligned}P^- &= \int dx^- dx^i\left(\frac{1}{2}\left(\partial_i\phi\right)^2\right)\\ &= \frac{1}{2}\int dx^- dx^i\left(\partial_i\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}}\frac{1}{\sqrt{2p_1^+}}\Big(a(p_1^+,x^i)e^{ip_1^+x^-}+a^\dagger(p_1^+,x^i)e^{-ip_1^+x^-}\Big)\right)\\ &\quad\left(\partial_i\int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}}\frac{1}{\sqrt{2p_2^+}}\Big(a(p_2^+,x^i)e^{ip_2^+x^-}+a^\dagger(p_2^+,x^i)e^{-ip_2^+x^-}\Big)\right)\end{aligned}$$

As before, the aa and $a^\dagger a^\dagger$ terms vanish since $p^+>0$,

$$\begin{aligned}P^- &= \frac{1}{2}\int dx^i\int_0^\infty dp_1^+\frac{1}{\sqrt{2p_1^+}}\int_0^\infty dp_2^+\frac{1}{\sqrt{2p_2^+}}\delta(p_1^+-p_2^+)\\ &\quad\Big[\left(\partial_i a(p_1^+,x^i)\right)\Big(\partial_i a^\dagger(p_2^+,x^i)\Big)+\Big(\partial_i a^\dagger(p_1^+,x^i)\Big)\left(\partial_i a(p_2^+,x^i)\right)\Big]\\ &= \frac{1}{2}\int dx^i\int_0^\infty dp_1^+\frac{1}{2p_1^+}\\ &\quad\Big[\left(\partial_i a(p_1^+,x^i)\right)\Big(\partial_i a^\dagger(p_1^+,x^i)\Big)+\Big(\partial_i a^\dagger(p_1^+,x^i)\Big)\left(\partial_i a(p_1^+,x^i)\right)\Big].\end{aligned} \quad (C.1)$$

Integrating by parts over x^i and dropping surface terms, this becomes

$$P^- = \int dx^i\int_0^\infty dp^+ a^\dagger\left(p^+,x^i\right)\left(-\frac{1}{2p^+}\partial_i^2 a\left(p^+,x^i\right)\right),$$

and therefore, for P^- we have

$$\delta a\left(p^+,x^i\right) = -\frac{1}{2p^+}\partial_i^2 a\left(p^+,x^i\right).$$

Similarly for P^+ we obtain

$$P^+ = -\int dx^- dx^i \pi^2$$

$$
\begin{aligned}
&= \int dx^- dx^i \left(\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \sqrt{\frac{p_1^+}{2}} \left(a(p_1^+, x^i) e^{ip_1^+ x^-} - a^\dagger(p_1^+, x^i) e^{-ip_1^+ x^-} \right) \right) \\
&\quad \left(\int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \sqrt{\frac{p_2^+}{2}} \left(a(p_2^+, x^i) e^{ip_2^+ x^-} - a^\dagger(p_2^+, x^i) e^{-ip_2^+ x^-} \right) \right) \\
&= \int dx^i \int_0^\infty dp_1^+ \sqrt{\frac{p_1^+}{2}} \int_0^\infty dp_2^+ \sqrt{\frac{p_2^+}{2}} \delta(p_1^+ - p_2^+) \\
&\quad \left(-a(p_1^+, x^i) a^\dagger(p_2^+, x^i) - a^\dagger(p_1^+, x^i) a(p_2^+, x^i) \right) \\
&= -\int dx^i \int_0^\infty dp_1^+ \frac{p_1^+}{2} \left(a(p_1^+, x^i) a^\dagger(p_1^+, x^i) + a^\dagger(p_1^+, x^i) a(p_1^+, x^i) \right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger(p^+, x^i) \left(-p^+ a(p^+, x^i) \right).
\end{aligned}
$$

and therefore, for P^+ we have

$$
\delta a\left(p^+, x^i\right) = -p^+ a(p^+, x^i).
$$

For P^i we obtain

$$
\begin{aligned}
P^i &= \int dx^- dx^i \left(\pi \partial_i \phi \right) \\
&= -i \int dx^- dx^i \left(\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \sqrt{\frac{p_1^+}{2}} \left(a(p_1^+, x^i) e^{ip_1^+ x^-} - a^\dagger(p_1^+, x^i) e^{-ip_1^+ x^-} \right) \right) \\
&\quad \left(\partial_i \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \left(a(p_2^+, x^i) e^{ip_2^+ x^-} + a^\dagger(p_2^+, x^i) e^{-ip_2^+ x^-} \right) \right) \\
&= -i \int dx^i \int_0^\infty dp_1^+ \sqrt{\frac{p_1^+}{2}} \int_0^\infty dp_2^+ \frac{1}{\sqrt{2p_2^+}} \delta(p_1^+ - p_2^+) \\
&\quad \left(a(p_1^+, x^i) \partial_i a^\dagger(p_2^+, x^i) - a^\dagger(p_1^+, x^i) \partial_i a(p_2^+, x^i) \right) \\
&= -\frac{i}{2} \int dx^i \int_0^\infty dp_1^+ - 2a^\dagger(p_1^+, x^i) \partial_i a(p_1^+, x^i) \\
&= \int dx^i \int_0^\infty dp_1^+ a^\dagger(p_1^+, x^i) \left(i \partial_i a(p_1^+, x^i) \right).
\end{aligned}
$$

Therefore, for P^i we have

$$
\delta a\left(p^+, x^i\right) = i \partial_i a(p_1^+, x^i)
$$

Now for the Lorentz generators. For M^{+-} we obtain

$$M^{+-} = x^+H + \int dx^-dx^i\left(x^-\pi^2\right) = x^+P^- + \int dx^-dx^i\left(x^-\pi^2\right).$$

The second integral is given by

$$\begin{aligned}
&\int dx^-dx^i\left(x^-\pi^2\right)\\
=\;& -\int dx^-dx^i\left(x^-\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}}\sqrt{\frac{p_1^+}{2}}\left(a(p_1^+,x^i)e^{ip_1^+x^-} - a^\dagger(p_1^+,x^i)e^{-ip_1^+x^-}\right)\right)\\
&\left(\int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}}\sqrt{\frac{p_2^+}{2}}\left(a(p_2^+,x^i)e^{ip_2^+x^-} - a^\dagger(p_2^+,x^i)e^{-ip_2^+x^-}\right)\right)\\
=\;& -\int dx^-dx^ix^-\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}}\sqrt{\frac{p_1^+}{2}}\int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}}\sqrt{\frac{p_2^+}{2}}\\
&\left(a(p_1^+,x^i)e^{ip_1^+x^-} - a^\dagger(p_1^+,x^i)e^{-ip_1^+x^-}\right)\\
&\left(a(p_2^+,x^i)e^{ip_2^+x^-} - a^\dagger(p_2^+,x^i)e^{-ip_2^+x^-}\right)\\
=\;& -\int dx^-dx^ix^-\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}}\sqrt{\frac{p_1^+}{2}}\int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}}\sqrt{\frac{p_2^+}{2}}\\
&\left(-a(p_1^+,x^i)a^\dagger(p_2^+,x^i)e^{i(p_1^+-p_2^+)x^-} - a^\dagger(p_1^+,x^i)a(p_2^+,x^i)e^{-i(p_1^+-p_2^+)x^-}\right)\\
=\;& \int dx^i\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}}\sqrt{\frac{p_1^+}{2}}\int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}}\sqrt{\frac{p_2^+}{2}}\\
&\left[a(p_1^+,x^i)a^\dagger(p_2^+,x^i)(-i)\frac{\partial}{\partial p_1^+}\int dx^-e^{i(p_1^+-p_2^+)x^-}\right.\\
&\left.+a^\dagger(p_1^+,x^i)a(p_2^+,x^i)(-i)\frac{\partial}{\partial p_2^+}\int dx^-e^{i(p_2^+-p_1^+)x^-}\right]\\
=\;& i\int dx^i\int_0^\infty dp_1^+\int_0^\infty dp_2^+\delta(p_1^+-p_2^+)\\
&\left[\frac{\partial}{\partial p_1^+}\left(\sqrt{\frac{p_1^+}{2}}\sqrt{\frac{p_2^+}{2}}a(p_1^+,x^i)a^\dagger(p_2^+,x^i)\right)\right.\\
&\left.+\frac{\partial}{\partial p_2^+}\left(\sqrt{\frac{p_1^+}{2}}\sqrt{\frac{p_2^+}{2}}a^\dagger(p_1^+,x^i)a(p_2^+,x^i)\right)\right]\\
=\;& i\int dx^i\int_0^\infty dp_1^+\int_0^\infty dp_2^+\delta(p_1^+-p_2^+)
\end{aligned}$$

$$\left[\left(\left(\frac{\partial}{\partial p_1^+}\sqrt{\frac{p_1^+}{2}}\right)a(p_1^+,x^i)+\sqrt{\frac{p_1^+}{2}}\frac{\partial}{\partial p_1^+}a(p_1^+,x^i)\right)\sqrt{\frac{p_2^+}{2}}a^\dagger(p_2^+,x^i)\right.$$
$$\left.+\left(\left(\frac{\partial}{\partial p_2^+}\sqrt{\frac{p_2^+}{2}}\right)a^\dagger(p_2^+,x^i)+\sqrt{\frac{p_2^+}{2}}\frac{\partial}{\partial p_2^+}a^\dagger(p_2^+,x^i)\right)\sqrt{\frac{p_1^+}{2}}a(p_1^+,x^i)\right]$$
$$= i\int dx^i\int_0^\infty dp^+\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\left(a^\dagger\left(p^+,x^i\right)a\left(p^+,x^i\right)\right)$$

where the other two terms cancel after integration by parts. Thus

$$\begin{aligned} M^{+-} &= \int dx^i\int_0^\infty dp^+ a^\dagger\left(p^+,x^i\right)\frac{x^+}{2p^+}\partial_i^2 a\left(p^+,x^i\right) \\ &\quad +i\int dx^i\int_0^\infty dp^+\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\left(a^\dagger\left(p^+,x^i\right)a\left(p^+,x^i\right)\right) \\ &= \int dx^i\int_0^\infty dp^+ a^\dagger\left(p^+,x^i\right)\left(\frac{x^+}{2p^+}\partial_i^2+i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)\left(a\left(p^+,x^i\right)\right)\end{aligned}$$

so that

$$\delta a\left(p^+,x^i\right)=\left(\frac{x^+}{2p^+}\partial_i^2+i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)a\left(p^+,x^i\right).$$

For M^{+i} we obtain the following:

$$M^{+i} = \int dx^i dx^-\left(x^+\pi\partial_i\phi+x^i\pi^2\right)$$

Performing the first integral,

$$\begin{aligned}\int dx^i dx^- x^+\pi\partial_i\phi &= x^+P^i \\ &= \int dx^i\int_0^\infty dp^+ a^\dagger(p^+,x^i)\left(ix^+\partial_i a(p^+,x^i)\right).\end{aligned}$$

The second integral is given by

$$\int dx^i dx^- x^i\pi^2$$
$$= -\int dx^- dx^i\, x^i\left(\int_0^\infty\frac{dp_1^+}{\sqrt{2\pi}}\sqrt{\frac{p_1^+}{2}}\left(a(p_1^+,x^i)e^{ip_1^+x^-}-a^\dagger(p_1^+,x^i)e^{-ip_1^+x^-}\right)\right)$$
$$\left(\int_0^\infty\frac{dp_2^+}{\sqrt{2\pi}}\sqrt{\frac{p_2^+}{2}}\left(a(p_2^+,x^i)e^{ip_2^+x^-}-a^\dagger(p_2^+,x^i)e^{-ip_2^+x^-}\right)\right)$$

$$\begin{aligned}
&= -\int dx^i\, x^i \int_0^\infty dp_1^+ \sqrt{\frac{p_1^+}{2}} \int_0^\infty dp_2^+ \sqrt{\frac{p_2^+}{2}} \delta(p_1^+ - p_2^+) \\
&\quad \left(-a(p_1^+, x^i) a^\dagger(p_2^+, x^i) - a^\dagger(p_1^+, x^i) a(p_2^+, x^i)\right) \\
&= \int dx^i \int_0^\infty dp_1^+\, x^i \frac{p_1^+}{2} \left(a(p_1^+, x^i) a^\dagger(p_1^+, x^i) + a^\dagger(p_1^+, x^i) a(p_1^+, x^i)\right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger(p^+, x^i) \left(x^i p^+ a(p^+, x^i)\right).
\end{aligned}$$

and therefore,

$$M^{+i} = \int dx^i \int_0^\infty dp^+ a^\dagger(p^+, x^i) \left(ix^+\partial_i + x^i p^+\right) a(p^+, x^i),$$

and for M^{+i} we have

$$\delta a\left(p^+, x^i\right) = \left(ix^+\partial_i + x^i p^+\right) a(p^+, x^i)$$

For M^{-i} we have

$$M^{-i} \quad = \quad \int dx^- dx^i \left(x^- \pi \partial_i \phi - x^i \mathcal{H}\right).$$

The first integral is given by

$$\begin{aligned}
&\int dx^- dx^i x^- \pi \partial_i \phi \\
&= -i \int dx^- dx^i\, x^- \left(\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \sqrt{\frac{p_1^+}{2}} \left(a(p_1^+, x^i) e^{ip_1^+ x^-} - a^\dagger(p_1^+, x^i) e^{-ip_1^+ x^-}\right)\right) \\
&\quad \left(\partial_i \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \left(a(p_2^+, x^i) e^{ip_2^+ x^-} + a^\dagger(p_2^+, x^i) e^{-ip_2^+ x^-}\right)\right) \\
&= -\frac{i}{2\pi} \int dx^- dx^i\, x^- \int_0^\infty dp_1^+ \sqrt{\frac{p_1^+}{2}} \int_0^\infty dp_2^+ \frac{1}{\sqrt{2p_2^+}} \\
&\quad \left(a(p_1^+, x^i) \partial_i a^\dagger(p_2^+, x^i) e^{i\left(p_1^+ - p_2^+\right)x^-} - a^\dagger(p_1^+, x^i) \partial_i a(p_2^+, x^i) e^{i\left(p_2^+ - p_1^+\right)x^-}\right) \\
&= -\frac{i}{2\pi} \int dx^i \int_0^\infty dp_1^+ \sqrt{\frac{p_1^+}{2}} \int_0^\infty dp_2^+ \frac{1}{\sqrt{2p_2^+}} (-i) \left(a(p_1^+, x^i) \partial_i a^\dagger(p_2^+, x^i)\right. \\
&\quad \left.\frac{\partial}{\partial p_1^+} \int dx^- e^{i\left(p_1^+ - p_2^+\right)x^-} - a^\dagger(p_1^+, x^i) \partial_i a(p_2^+, x^i) \frac{\partial}{\partial p_2^+} \int dx^- e^{i\left(p_2^+ - p_1^+\right)x^-}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\int dx^i \int_0^\infty dp_1^+ \sqrt{\frac{p_1^+}{2}} \int_0^\infty dp_2^+ \frac{1}{\sqrt{2p_2^+}} \Big(a(p_1^+, x^i)\partial_i a^\dagger(p_2^+, x^i) \\
&\quad \frac{\partial}{\partial p_1^+}\delta(p_1^+ - p_2^+) - a^\dagger(p_1^+, x^i)\partial_i a(p_2^+, x^i)\frac{\partial}{\partial p_1^+}\delta(p_1^+ - p_2^+)\Big) \\
&= \int dx^i \int_0^\infty dp_1^+ \int_0^\infty dp_2^+ \delta(p_1^+ - p_2^+) \left[\frac{\partial}{\partial p_1^+} \left(\sqrt{\frac{p_1^+}{2}} a(p_1^+, x^i) \right) \right. \\
&\quad \left. \frac{1}{\sqrt{2p_2^+}}\partial_i a^\dagger(p_2^+, x^i) - \frac{\partial}{\partial p_2^+} \left(\frac{1}{\sqrt{2p_2^+}}\partial_i a(p_2^+, x^i) \right) \sqrt{\frac{p_1^+}{2}} a^\dagger(p_1^+, x^i) \right] \\
&= \int dx^i \int_0^\infty dp_1^+ \int_0^\infty dp_2^+ \delta(p_1^+ - p_2^+) \\
&\quad \left[\left(\frac{1}{2}\sqrt{\frac{1}{2p_1^+}} a(p_1^+, x^i) + \sqrt{\frac{p_1^+}{2}} \frac{\partial}{\partial p_1^+} a(p_1^+, x^i) \right) \frac{1}{\sqrt{2p_2^+}} \partial_i a^\dagger(p_2^+, x^i) \right. \\
&\quad \left. - \left(-\frac{1}{2} \frac{1}{\sqrt{2}\left(p_2^+\right)^{3/2}} \partial_i a(p_2^+, x^i) + \sqrt{\frac{1}{2p_2^+}} \frac{\partial}{\partial p_2^+} \partial_i a(p_2^+, x^i) \right) \sqrt{\frac{p_1^+}{2}} a^\dagger(p_1^+, x^i) \right] \\
&= \int dx^i \int_0^\infty dp^+ \left[\partial_i a^\dagger(p^+, x^i) \left(\frac{1}{4p^+} + \frac{1}{2}\frac{\partial}{\partial p^+} \right) a(p^+, x^i) \right. \\
&\quad \left. - a^\dagger(p^+, x^i) \left(-\frac{1}{4p^+} + \frac{1}{2}\frac{\partial}{\partial p^+} \right) \partial_i a(p^+, x^i) \right] \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger(p^+, x^i) \left(-\frac{\partial}{\partial p^+} \partial_i a(p^+, x^i) \right).
\end{aligned}
$$

The second integral is similar to the calculation for P^- and hence

$$
\begin{aligned}
&\int dx^- dx^i \, x^i \mathcal{H} \\
&= -\frac{1}{2} \int dx^i \, x^i \int_0^\infty dp_1^+ \frac{1}{\sqrt{2p_1^+}} \int_0^\infty dp_2^+ \frac{1}{\sqrt{2p_2^+}} \delta(p_1^+ - p_2^+) \\
&\quad \left[\left(\partial_j a(p_1^+, x^i)\right) \left(\partial_j a^\dagger(p_2^+, x^i)\right) + \left(\partial_j a^\dagger(p_1^+, x^i)\right) \left(\partial_j a(p_2^+, x^i)\right) \right] \\
&= \int dx^i \int_0^\infty dp^+ \frac{1}{2p_1^+} a^\dagger\left(p^+, x^i\right) \partial_j \left(x^i \partial_j a(p^+, x^i) \right) \\
&= \int dx^i \int_0^\infty dp^+ a^\dagger\left(p^+, x^i\right) \frac{\partial_j x^i \partial_j}{2p^+} a(p^+, x^i).
\end{aligned}
$$

Therefore M^{-i} is given by

$$M^{-i} = \int dx^i \int_0^\infty dp^+ a^\dagger \left(p^+, x^i\right) \left(-\partial_i \frac{\partial}{\partial p^+} - \frac{\partial_j x^i \partial_j}{2p^+}\right) a(p^+, x^i)$$

and therefore, for M^{-i} we have

$$\delta a \left(p^+, x^i\right) = -\partial_i \frac{\partial}{\partial p^+} - \frac{\partial_j x^i \partial_j}{2p^+}.$$

For M^{ij} we have

$$M^{ij} = \int dx^- dx^\perp \left(x^i \pi \partial_j \phi - x^j \pi \partial_i \phi\right)$$

The first integral is given by

$$\begin{aligned}
&\int dx^- dx^\perp \, x^i \pi \partial_j \phi \\
= &-i \int dx^- dx^\perp \, x^i \left(\int_0^\infty \frac{dp_1^+}{\sqrt{2\pi}} \sqrt{\frac{p_1^+}{2}} \left(a(p_1^+, x^i) e^{ip_1^+ x^-} - a^\dagger(p_1^+, x^i) e^{-ip_1^+ x^-}\right)\right) \\
&\left(\partial_j \int_0^\infty \frac{dp_2^+}{\sqrt{2\pi}} \frac{1}{\sqrt{2p_2^+}} \left(a(p_2^+, x^i) e^{ip_2^+ x^-} + a^\dagger(p_2^+, x^i) e^{-ip_2^+ x^-}\right)\right) \\
= &-i \int dx^i \, x^i \int_0^\infty dp_1^+ \sqrt{\frac{p_1^+}{2}} \int_0^\infty dp_2^+ \frac{1}{\sqrt{2p_2^+}} \delta(p_1^+ - p_2^+) \\
&\left(a(p_1^+, x^i) \partial_j a^\dagger(p_2^+, x^i) - a^\dagger(p_1^+, x^i) \partial_j a(p_2^+, x^i)\right) \\
= &-\frac{i}{2} \int dx^i \, x^i \int_0^\infty dp_1^+ a^\dagger(p_1^+, x^i) \left[-2x^i \partial_j + \partial_j x^i\right] a(p_1^+, x^i) \\
= &\int dx^i \int_0^\infty dp_1^+ a^\dagger(p_1^+, x^i) \left[i x^i \partial_j - \frac{i}{2} \delta_j^i\right] a(p_1^+, x^i).
\end{aligned}$$

Similarly the second integral is given by

$$\int dx^- dx^i x^j \pi \partial_i \phi = \int dx^i \int_0^\infty dp_1^+ a^\dagger(p_1^+, x^i) \left[i x^j \partial_i - \frac{i}{2} \delta_j^i\right] a(p_1^+, x^i).$$

Therefore,

$$M^{ij} = \int dx^i \int_0^\infty dp_1^+ a^\dagger(p_1^+, x^i) \left(i x^i \partial_j - i x^j \partial_i\right) a(p_1^+, x^i),$$

and so for M^{ij} we have

$$\delta a\left(p^+,x^i\right)=\left(ix^i\partial_j-ix^j\partial_i\right)a(p_1^+,x^i).$$

The Dilitation operator is given by 4 terms

$$D \;=\; x^+H+\int dx^-dx^i\;\left(\pi d_\phi\phi+\pi x^i\partial_i\phi-x^-\pi^2\right).$$

The first and fourth terms closely resemble M^{+-} and we for these that

$$\begin{aligned}&x^+H-\int dx^-dx^i\;x^-\pi^2\\ =\;&\int dx^i\int_0^\infty dp^+a^\dagger\left(p^+,x^i\right)\left(\frac{x^+}{2p^+}\partial_i^2-i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)\left(a\left(p^+,x^i\right)\right).\end{aligned}$$

The second integral looks like P^i without the ∂_i, hence

$$\int dx^-dx^i\;\pi d_\phi\phi \;=\; \int dx^i\int_0^\infty dp^+a^\dagger(p^+,x^i)\left(id_\phi a(p^+,x^i)\right)$$

and the third integral is the same as the first term in M^{ij}, with $j\to i$,

$$\int dx^-dx^i\;\pi x^i\partial_i\phi \;=\; \int dx^i\int_0^\infty dp^+a^\dagger(p^+,x^i)\left(ix^i\partial_i a(p^+,x^i)\right).$$

Putting these all together we obtain,

$$\begin{aligned}D=&\int dx^i\int_0^\infty dp^+a^\dagger(p^+,x^i)\\ &\left(\frac{x^+}{2p^+}\partial_i^2+id_\phi+ix^i\partial_i-i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)a(p^+,x^i)\end{aligned}$$

so that for D we have

$$\delta a\left(p^+,x^i\right)=\left(\frac{x^+}{2p^+}\partial_i^2+id_\phi+ix^i\partial_i-i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)a(p^+,x^i).$$

Finally, we repeat the above for the special conformal generators,

$$K^+ \;=\; \int dx^-dx^i\left(x^+\mathcal{D}+\frac{1}{2}\left(2x^+x^-+x^jx_j\right)\pi^2\right).$$

The first term is given by

$$\int dx^i \int_0^\infty dp^+ a^\dagger(p^+,x^i)\left[x^+\left(\frac{x^+}{2p^+}\partial_i^2 + id_\phi + ix^i\partial_i - i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)a(p^+,x^i)\right]$$

The second term is given by x^+ multiplied by a term in M^{+-} and is therefore

$$\int dx^i \int_0^\infty dp^+ a^\dagger\left(p^+,x^i\right)\left[x^+\left(i\sqrt{p^+}\frac{\partial}{\partial p^+}\sqrt{p^+}\right)\right]a\left(p^+,x^i\right)$$

and the third term is x^i multiplied by a term in M^{+i}, and is given by

$$\int dx^i \int_0^\infty dp^+ a^\dagger(p^+,x^i)\left(\frac{1}{2}x_i x^i p^+\right)a(p^+,x^i).$$

Combining these we get

$$K^+ = \int dx^i \int_0^\infty dp^+ a^\dagger(p^+,x^i)\left[x^+\left(\frac{x^+}{2p^+}\partial_i^2 + id_\phi + ix^i\partial_i\right) + \frac{1}{2}x_i x^i p^+\right]a(p^+,x^i)$$

so that for K^+ we have

$$\delta a\left(p^+,x^i\right) = \left[x^+\left(\frac{x^+}{2p^+}\partial_i^2 + id_\phi + ix^i\partial_i\right) + \frac{1}{2}x_i x^i p^+\right]a(p^+,x^i).$$

C.3.4 Map between CFT and $AdS_4 \times S_1$

In order to establish the map between CFT and $AdS_4 \times S_1$, we compare CFT generators found in Appendix C.3.3) (summarised below)

$$\hat{p}^+ = p_1^+ + p_2^+ \tag{C.2}$$

$$\hat{p}^- = -\left(\frac{p_1^i p_1^i}{2p_1^+} + \frac{p_2^i p_2^i}{2p_2^+}\right) \tag{C.3}$$

$$\hat{m}^{+-} = t\hat{p}^- - x_1^- p_1^+ - x_2^- p_2^+ \tag{C.4}$$

$$\hat{m}^{+i} = t\hat{p}^i - x_1^i p_1^+ - x_2^- p_2^+ \tag{C.5}$$

$$\hat{m}^{-i} = p_1^i x_1^- + p_2^i x_2^- + x_1^i\frac{p_1^j p_1^j}{2p_1^+} + x_2^i\frac{p_2^j p_2^j}{2p_2^+} \tag{C.6}$$

$$\hat{m}^{ij} = x_1^i p_1^j + x_2^i p_2^j - x_1^j p_1^i - x_2^j p_2^i \tag{C.7}$$

$$\hat{d} = t\hat{p}^- + 2d_\phi + x_1^i p_1^i + x_2^i p_2^i + x_1^- p_1^+ + x_2^- p_2^+ \tag{C.8}$$

$$\hat{k}^+ = t^2\hat{p}^- + t(2d_\phi + x_1^i p_1^i + x_2^i p_2^i) - \frac{1}{2}\left(x_1^i x_1^i p_1^+ + x_2^i x_2^i p_2^+\right) \tag{C.9}$$

$$\begin{aligned}
\hat{k}^i &= -t\left(x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+} + p_1^i x_1^- + p_2^i x_2^i\right) \\
\hat{k}^- &= x_1^i x_1^i \frac{p_1^j p_1^j}{4p_1^+} + x_2^i x_2^i \frac{p_2^j p_2^j}{4p_2^+} + x_1^-\left(x_1^- p_1^+ + x_1^i p_1^i + d_\phi\right) \\
&\quad + x_2^-\left(x_2^- p_2^+ + x_2^i p_2^i + d_\phi\right), \qquad \text{(C.10)}
\end{aligned}$$

with higher spin generators in $AdS_4 \times S_1$ (first worked out in Ref. [130] and then summarised in Ref. [5]) given by

$$\begin{aligned}
&\hat{p}^- = -\frac{p^x p^x + p^z p^z}{2p^+} \\
&\hat{p}^+ = p^+ \\
&\hat{p}^x = p^x \\
&\hat{m}^{+-} = t\hat{p}^- - x^- p^+ \\
&\hat{m}^{+x} = t\hat{p}^x - xp^+ \\
&\hat{m}^{-x} = x^- p^x - x\hat{p}^- + \frac{p^\theta p^z}{p^+} \\
&\hat{d} = t\hat{p}^- + x^- p^+ + xp^x + zp^z + d_a \\
&\hat{k}^- = -\frac{1}{2}(x^2 + z^2)\hat{p}^- + x^-(x^- p^+ + xp^x + zp^z + d_a) \\
&+\frac{1}{p^+}\left((xp^z - zp^x)p^\theta + (p^\theta)^2\right) \\
&\hat{k}^+ = x^{+2}\hat{p}^- + x^+(xp^x + zp^z + d_a) - \frac{1}{2}(x^2 + z^2)p^+ \\
&\hat{k}^x = x^+(x\hat{p}^- - x^- p^x - \frac{p^\theta p^z}{p^+}) + \frac{1}{2}(x^2 - z^2)p^x \\
&+x(x^- p^+ + zp^z + d_a) + zp^\theta
\end{aligned}$$

where $d_a = 1$ for AdS_4. Equating the first 5 of these generators we thus obtain

$$\hat{p}^+ = p^+ = p_1^+ + p_2^+ \qquad \text{(C.11)}$$

$$\hat{p}^x = p^x = p = p_1^x + p_2^x = p_1 + p_2 \qquad \text{(C.12)}$$

$$\hat{p}^- = -\left(\frac{p_1^i p_1^i}{2p_1^+} + \frac{p_2^i p_2^i}{2p_2^+}\right) = -\frac{pp + p^z p^z}{2p^+} \qquad \text{(C.13)}$$

$$\hat{m}^{+-} = t\hat{p}^- - x^- p^+ = t\hat{p}^- - x_1^- p_1^+ - x_2^- p_2^+ \qquad \text{(C.14)}$$

$$\hat{m}^{+i} = tp - xp^+ = tp - x_1^i p_1^+ - x_2^- p_2^+. \qquad \text{(C.15)}$$

Equations (C.11), (C.12), (C.14) and (C.15) give respectively

$$\begin{aligned} p^+ &= p_1^+ + p_2^+ \\ p &= p_1 + p_2 \\ x &= \frac{x_1 p_1^+ + x_2 p_2^+}{p_1^+ + p_2^+} \\ x^- &= \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+}. \end{aligned}$$

Equating the rest of the generators gives

$$\begin{aligned} \hat{m}^{-i} &= x^- p - x\hat{p}^- + \frac{p^\theta p^z}{p^+} = p_1 x_1^- + p_2 x_2^- + x_1 \frac{p_1^j p_1^j}{2p_1^+} + x_2 \frac{p_2^j p_2^j}{2p_2^+} \qquad \text{(C.16)} \\ \hat{d} &= t\hat{p}^- + x^- p^+ + xp + zp^z + d_a \\ &= t\hat{p}^- + 2d_\phi + x_1 p_1 + x_2 p_2 + x_1^- p_1^+ + x_2^- p_2^+ \qquad \text{(C.17)} \\ \hat{k}^+ &= t^2 \hat{p}^- + t(xp + zp^z + d_a) - \frac{1}{2}(x^2 + z^2)p^+ \\ &= t^2 \hat{p}^- + t(2d_\phi + x_1 p_1 + x_2 p_2) - \frac{1}{2}\left(x_1 x_1 p_1^+ + x_2 x_2 p_2^+\right) \qquad \text{(C.18)} \\ \hat{k}^x &= t(x\hat{p}^- - x^- p^x - \frac{p^\theta p^z}{p^+}) + \frac{1}{2}(x^2 - z^2)p^x \\ &\quad + x(x^- p^+ + zp^z + d_a) + zp^\theta \\ &= -t\left(x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+} + p_1^i x_1^- + p_2^i x_2^i\right) \\ &\quad - \frac{1}{2}\left(x_1^j x_1^j p_1^i + x_2^j x_2^j p_2^i\right) + x_1^i\left(d_\phi + x_1^j p_1^j + p_1^+ x_1^-\right) \\ &\quad + x_2^i\left(d_\phi + x_2^j p_2^j + p_2^+ x_2^-\right) \qquad \text{(C.19)} \\ \hat{k}^- &= -\frac{1}{2}(x^2 + z^2)\hat{p}^- + x^-(x^- p^+ + xp^x + zp^z + d_a) \\ &\quad + \frac{1}{p^+}\left((xp^z - zp^x)p^\theta + (p^\theta)^2\right) \\ &= x_1^i x_1^i \frac{p_1^j p_1^j}{4p_1^+} + x_2^i x_2^i \frac{p_2^j p_2^j}{4p_2^+} + x_1^-\left(x_1^- p_1^+ + x_1^i p_1^i + d_\phi\right) \\ &\quad + x_2^-\left(x_2^- p_2^+ + x_2^i p_2^i + d_\phi\right). \qquad \text{(C.20)} \end{aligned}$$

From equation (C.17),

$$x^- p^+ + xp + zp^z = x_1 p_1 + x_2 p_2 + x_1^- p_1^+ + x_2^- p_2^+$$

$$
\begin{aligned}
\Rightarrow zp^z &= x_1p_1 + x_2p_2 + x_1^-p_1^+ + x_2^-p_2^+ - \frac{x_1^-p_1^+ + x_2^-p_2^+}{p^+}p^+ - \frac{x_1p_1^+ + x_2p_2^+}{p^+}p \\
&= \frac{1}{p^+}\left[x_1p_1(p_1^+ + p_2^+) + x_2p_2(p_1^+ + p_2^+) + x_1^-p_1^+(p_1^+ + p_2^+)\right. \\
&\quad \left. + x_2^-p_2^+(p_1^+ + p_2^+) - (x_1^-p_1^+ + x_2^-p_2^+)(p_1^+ + p_2^+) - (x_1p_1^+ + x_2p_2^+)(p_1 + p_2)\right] \\
&= \frac{1}{p^+}\left[x_1p_1p_2^+ + x_2p_2p_1^+ + x_1^-p_1^+p_2^+ + x_2^-p_2^+p_1^+\right. \\
&\quad \left. -(x_1^-p_1^+p_2^+ + x_2^-p_2^+p_1^+) - (x_1p_1^+p_2 + x_2p_2^+p_1)\right] \\
&= \frac{1}{p^+}\left[x_1p_1p_2^+ + x_2p_2p_1^+ - (x_1p_1^+p_2 + x_2p_2^+p_1)\right] \\
&= \frac{1}{p_1^+ + p_2^+}\left[(x_1 - x_2)(p_1p_2^+ - p_1^+p_2)\right]. \qquad \text{(C.21)}
\end{aligned}
$$

Hence equation (C.18) becomes

$$
\begin{aligned}
& t(xp + zp^z) - \frac{1}{2}(x^2 + z^2)p^+ = t(x_1p_1 + x_2p_2) - \frac{1}{2}\left(x_1x_1p_1^+ + x_2x_2p_2^+\right) \\
\Rightarrow \quad & t(\left[\frac{x_1p_1^+ + x_2p_2^+}{p^+}\right]p + \frac{(x_1 - x_2)(p_1p_2^+ - p_1^+p_2)}{p^+}) - \frac{1}{2}(\left[\frac{x_1p_1^+ + x_2p_2^+}{p^+}\right]^2 + z^2)p^+ \\
& = t(x_1p_1 + x_2p_2) - \frac{1}{2}\left(x_1x_1p_1^+ + x_2x_2p_2^+\right) \\
\Rightarrow \quad & z^2 = \frac{1}{p^{+2}}\left[t(x_1p_1 + x_2p_2)p^+ - \frac{1}{2}\left(x_1x_1p_1^+ + x_2x_2p_2^+\right)p^+\right. \\
& \left. -t\left(x_1p_1^+ + x_2p_2^+\right)p - t(x_1 - x_2)(p_1p_2^+ - p_1^+p_2) + \frac{1}{2}\left(x_1p_1^+ + x_2p_2^+\right)^2\right] \\
\Rightarrow \quad & z^2 = \frac{1}{p^{+2}}\left[t(x_1p_1p_2^+ + x_2p_2p_1^+) - \frac{1}{2}\left(x_1x_1p_1^+p_2^+ + x_2x_2p_2^+p_1^+\right)\right. \\
& \left. -t\left(x_1p_1^+p_2 + x_2p_2^+p_1\right) - t(x_1 - x_2)(p_1p_2^+ - p_1^+p_2) + \left(x_1p_1^+x_2p_2^+\right)\right] \\
\Rightarrow \quad & z^2 = \frac{1}{p^{+2}}\left[-\frac{1}{2}\left(x_1x_1p_1^+p_2^+ + x_2x_2p_2^+p_1^+\right) + \left(x_1p_1^+x_2p_2^+\right)\right. \\
& \left. -t\left(x_1p_1^+p_2 + x_2p_2^+p_1\right) + tx_1p_1^+p_2 + tx_2p_1p_2^+\right] \\
= \quad & \frac{p_1^+p_2^+(x_1 - x_2)^2}{p^{+2}} \\
\Rightarrow \quad & z = \frac{(x_1 - x_2)\sqrt{p_1^+p_2^+}}{p_1^+ + p_2^+}. \qquad \text{(C.22)}
\end{aligned}
$$

Equation (C.16) becomes

$$
x^-p - x\hat{p}^- + \frac{p^\theta p^z}{p^+} = p_1x_1^- + p_2x_2^- + x_1\frac{p_1^jp_1^j}{2p_1^+} + x_2\frac{p_2^jp_2^j}{2p_2^+}
$$

$$\Rightarrow \quad \left(\frac{x_1^- p_1^+ + x_2^- p_2^+}{p^+}\right) p + \left(\frac{x_1 p_1^+ + x_2 p_2^+}{p^+}\right)\left(\frac{p_1^2}{2p_1^+} + \frac{p_2^2}{2p_2^+}\right) + \frac{p^\theta p^z}{p^+}$$
$$= p_1 x_1^- + p_2 x_2^- + x_1 \frac{p_1^2}{2p_1^+} + x_2 \frac{p_2^2}{2p_2^+}$$
$$\Rightarrow \quad p^\theta p^z = p^+ \left[p_1 x_1^- + p_2 x_2^- + x_1 \frac{p_1^2}{2p_1^+} + x_2 \frac{p_2^2}{2p_2^+}\right]$$
$$- \left(x_1^- p_1^+ + x_2^- p_2^+\right) p - \left(x_1 p_1^+ + x_2 p_2^+\right)\left(\frac{p_1^2}{2p_1^+} + \frac{p_2^2}{2p_2^+}\right)$$
$$= \left[p_1 x_1^- p_2^+ + p_2 x_2^- p_1^+ + x_1 \frac{p_1^2}{2p_1^+} p_2^+ + x_2 \frac{p_2^2}{2p_2^+} p_1^+\right]$$
$$- \left(x_1^- p_1^+ p_2 + x_2^- p_2^+ p_1\right) - \left(x_1 p_1^+ \frac{p_2^2}{2p_2^+} + x_2 p_2^+ \frac{p_1^2}{2p_1^+}\right)$$
$$= \left(x_1^- - x_2^-\right)\left(p_1 p_2^+ - p_1^+ p_2\right) + \left(x_1^- - x_2^-\right)\left[\frac{p_1^2}{2p_1^+} p_2^+ - \frac{p_2^2}{2p_2^+} p_1^+\right]. \quad \text{(C.23)}$$

We obtain p^z by dividing equation (C.21) by (C.22):

$$p^z = \frac{p_1 p_2^+ - p_1^+ p_2}{\sqrt{p_1^+ p_2^+}} \qquad \text{(C.24)}$$

and p^θ by dividing (C.23) by (C.24):

$$p^\theta = \left(x_1^- - x_2^-\right)\sqrt{p_1^+ p_2^+}\left[1 + \frac{\left[\frac{p_1^2 p_2^+ - p_1 p_1^+ p_2}{2p_1^+} - \frac{p_2^2 p_1^+ - p_2 p_2^+ p_1}{2p_2^+}\right]}{p_1 p_2^+ - p_1^+ p_2}\right]$$
$$= \left(x_1^- - x_2^-\right)\sqrt{p_1^+ p_2^+}\left[1 + \left[\frac{p_1}{2p_1^+} - \frac{p_2}{2p_2^+}\right]\right]$$
$$= \left(x_1^- - x_2^-\right)\sqrt{p_1^+ p_2^+} + \frac{\left(x_1^- - x_2^-\right)}{2}\left[\sqrt{\frac{p_2^+}{p_1^+}} p_1 - \sqrt{\frac{p_1^+}{p_2^+}} p_2\right].$$

C.3.5 Example of obtaining generators using the kernel

The map is established using the kernel as follows[10]

$$L^{AdS}\tilde{\mathcal{H}}\left(p^+, p, p^z, \theta\right) = \int dp_1^+ dp_1 dp_2^+ dp_2\, \mathcal{K} L^{\text{bi-local}} \tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right).$$

where on the left-hand side we will have an AdS/Bulk operator in terms of p^+, p, p^z, θ and on the right-hand side is the bilocal operator in terms of p_1^+, p_1, p_2^+, p_2. Let's first obtain the above result:

$$z\tilde{\mathcal{H}}\left(p^+,p,p^z,\theta\right) = -\frac{\partial}{\partial p^z}\int dp_1^+dp_1dp_2^+dp_2\,\mathcal{K}\tilde{\Psi}\left(p_1^+,p_1,p_2^+,p_2\right).$$

But

$$\begin{aligned}
\frac{\partial}{\partial p^z} &= \frac{\partial p_1}{\partial p^z}\frac{\partial}{\partial p_1}+\frac{\partial p_2}{\partial p^z}\frac{\partial}{\partial p_2}+\frac{\partial p_1^+}{\partial p^z}\frac{\partial}{\partial p_1^+}+\frac{\partial p_2^+}{\partial p^z}\frac{\partial}{\partial p_2^+}\\
&= \frac{\sin\theta}{2}\frac{\partial}{\partial p_1}-\frac{\sin\theta}{2}\frac{\partial}{\partial p_2}\\
&= -\frac{1}{2}\sin\theta\left(x_1-x_2\right)
\end{aligned}$$

and

$$\frac{1}{2}\sin\theta=\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{\sqrt{p_1}}{\sqrt{p_1^++p_2^+}}\frac{\sqrt{p_2^+}}{\sqrt{p_1^++p_2^+}}=\frac{\sqrt{p_1^+p_2^+}}{p_1^++p_2^+}$$

and therefore

$$\begin{aligned}
z\tilde{\mathcal{H}}\left(p^+,p,p^z,\theta\right) &= \int dp_1^+dp_1dp_2^+dp_2\,\mathcal{K}\frac{\sqrt{p_1^+p_2^+}}{p_1^++p_2^+}(x_1-x_2)\tilde{\Psi}\left(p_1^+,p_1,p_2^+,p_2\right)\\
\Rightarrow z_{\mathrm{AdS}} &= \left[\frac{\sqrt{p_1^+p_2^+}}{p_1^++p_2^+}(x_1-x_2)\right]_{\mathrm{Bilocal}}.
\end{aligned}$$

As another example of implementing this equation, we obtained for M^{+i} that

$$\begin{aligned}
\delta a(p^+,x^i) &= \left(ix^+\partial_i+x^ip^+\right)a(p^+,x^i)\\
\Rightarrow m^{+i} &= x^+\hat{p}^i+xp^+
\end{aligned}$$

Inserting this we obtain

$$\begin{aligned}
m^{+i}_{\mathrm{AdS}}\tilde{\mathcal{H}} &= \left(x^+\hat{p}^i+xp^+\right)\tilde{\mathcal{H}}\left(p^+,p,p^z,\theta\right)\\
&= \int dp_1^+dp_1dp_2^+dp_2\,\mathcal{K}\left(\frac{\partial}{\partial p^-}\hat{p}^i+\frac{\partial}{\partial p}p^+\right)\tilde{\Psi}\left(p_1^+,p_1,p_2^+,p_2\right)\\
&= \int dp_1^+dp_1dp_2^+dp_2\,\mathcal{K}\left(\hat{p}^i\frac{\partial}{\partial p^-}+p^+\frac{\partial}{\partial p}\right)\tilde{\Psi}\left(p_1^+,p_1,p_2^+,p_2\right)
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\partial}{\partial p} &= \frac{\partial p_1}{\partial p}\frac{\partial}{\partial p_1} + \frac{\partial p_2}{\partial p}\frac{\partial}{\partial p_2} = \frac{1}{2}(1+\cos\theta)\frac{\partial}{\partial p_1} + \frac{1}{2}(1-\cos\theta)\frac{\partial}{\partial p_2} \\
&= \cos^2\frac{\theta}{2}\frac{\partial}{\partial p_1} + \sin^2\frac{\theta}{2}\frac{\partial}{\partial p_2} \\
&= \frac{p_2^+ x_2 + p_1^+ x_2}{p^+} \\
&= x.
\end{aligned}$$

Hence

$$\begin{aligned}
m^{+i}_{\mathrm{AdS}}\tilde{\mathcal{H}} &= -\int dp_1^+ dp_1 dp_2^+ dp_2\, \mathcal{K}\left(\hat{p}^i x^+ + p^+ x\right)\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right) \\
\Rightarrow m_B^{+i} &= \hat{p}^i x^+ + p^+ x.
\end{aligned}$$

C.3.6 Derivation of equation 33 of Ref. [10]

In equation (4.22), x_μ and p^μ are AdS_4 quantities with first two coordinates related to light-cone variables and second two coordinates to the i and z direction. Hence $e^{ix^\mu p_\mu} = e^{i\left(x^+p^- + x^-p^+ + xp + zp^z\right)}$. The first delta function is of the form $\delta\left(x^2 - a^2\right)$, and since

$$\delta\left(f(x) - f(x')\right) = \frac{\delta(x - x')}{f'(x')}$$

for all real roots of f, we have that $\delta\left(x^2 - a^2\right) = \frac{1}{2a}\left(\delta\left(x-a\right) + \delta\left(x+a\right)\right)$, and therefore

$$\begin{aligned}
&\delta\left((p^z)^2 - \left(2p^+p^- + p^2\right)\right) \\
= &\frac{1}{2\sqrt{-2p^+p^- - p^2}}\left(\delta\left(p^z - \sqrt{-2p^+p^- - p^2}\right) + \delta\left(p^z + \sqrt{-2p^+p^- - p^2}\right)\right).
\end{aligned}$$

Integrating over p^z (with $(p^z)^2$ positive) in equation (4.22) then gives

$$\begin{aligned}
&\mathcal{H}_s\left(x^+; x^-, x, z\right) \\
= &\Gamma(-\frac{1}{2})\int_{-2p^+p^- - p^2 > 0} dp^+ dp dp^- \frac{1}{2\sqrt{-2p^+p^- - p^2}} e^{i\left(x^+p^- + x^-p^+ + xp\right)}
\end{aligned}$$

$$\int dp_1^+ dp_1 dp_2^+ dp_2 \, \mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right)\delta\left(p_1 + p_2 - p\right)\frac{P_s^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}\right)s!}{\Gamma(s+\frac{1}{2})}$$
$$\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right)\left[\delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - \sqrt{-2p^+p^- - p^2}\right)e^{i\sqrt{-2p^+p^- - p^2}z}\right.$$
$$\left.+\delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} + \sqrt{-2p^+p^- - p^2}\right)e^{-i\sqrt{-2p^+p^- - p^2}z}\right].$$

Consider now

$$\delta\left(g(x)\right) = \delta\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} \pm \sqrt{-2p^+p^- - p^2}\right).$$

Here,

$$\begin{aligned}
g(x) = 0 \quad &\Rightarrow \quad p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} = \pm\sqrt{-2p^+p^- - p^2} \\
&\Rightarrow \quad \left(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}}\right)^2 = -2p^+p^- - (p_1 + p_2)^2 \\
&\Rightarrow \quad p_1^2\frac{p_2^+}{p_1^+} - 2p_1p_2 + p_2^2\frac{p_1^+}{p_2^+} = -2p^+p^- - p_1^2 - 2p_1p_2 - p_2^2 \\
&\Rightarrow \quad -2p^+p^- = p_1^2\left(\frac{p_2^+ + p_1^+}{p_1^+}\right) + p_2^2\left(\frac{p_1^+ + p_2^+}{p_2^+}\right) = p_1^2\left(\frac{p^+}{p_1^+}\right) + p_2^2\left(\frac{p^+}{p_2^+}\right) \\
&\Rightarrow \quad -p^- = \frac{p_1^2}{2p_1^+} + \frac{p_2^2}{p_2^+}.
\end{aligned}$$

Furthermore,

$$\left|\frac{\partial g(x)}{\partial p^-}\right| = \frac{2p^+}{2\sqrt{-2p^+p^- - p^2}}.$$

Therefore,

$$\delta(g(x)) = \delta\left(p^- + \frac{p_1^2}{2p_1^+} + \frac{p_2^2}{p_2^+}\right)\frac{2\sqrt{-2p^+p^- - p^2}}{2p^+}$$

and hence

$$\mathcal{H}_s\left(x^+; x^-, x, z\right)$$

$$
\begin{aligned}
= \;& \Gamma(-\frac{1}{2}) \int_{-2p^+p^- - p^2 > 0} dp^+ dp dp^- \; \frac{1}{2\sqrt{-2p^+p^- - p^2}} e^{i\left(x^+p^- + x^-p^+ + xp\right)} \\
& \int dp_1^+ dp_1 dp_2^+ dp_2 \, \mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right) \delta\left(p_1 + p_2 - p\right) \frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}\right) s!}{\Gamma(s+\frac{1}{2})} \\
& \tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right) \delta\left(p^- + \frac{p_1^2}{2p_1^+} + \frac{p_2^2}{p_2^+}\right) \frac{2\sqrt{-2p^+p^- - p^2}}{2p^+} \\
& \left(e^{i\sqrt{-2p^+p^- - p^2} z} + e^{-i\sqrt{-2p^+p^- - p^2} z}\right) \\
= \;& \Gamma(-\frac{1}{2}) \int_{-2p^+p^- - p^2 > 0} dp^+ dp dp^- \; e^{i\left(x^+p^- + x^-p^+ + xp\right)} \int dp_1^+ dp_1 dp_2^+ dp_2 \\
& \mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right) \delta\left(p_1 + p_2 - p\right) \frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}\right) s!}{\Gamma(s+\frac{1}{2})} \tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right) \\
& \delta\left(p^- + \frac{p_1^2}{2p_1^+} + \frac{p_2^2}{p_2^+}\right) \frac{1}{2p^+} \left(2\cos\sqrt{-2p^+p^- - p^2} z\right).
\end{aligned}
$$

From Mathematica, the Bessel function of the first kind satisfies

$$
\begin{aligned}
J_{-\frac{1}{2}}(x) &= \frac{\sqrt{\frac{2}{\pi}}\cos(x)}{\sqrt{x}} \\
\Rightarrow \cos(x) &= \sqrt{\frac{\pi x}{2}} J_{-\frac{1}{2}}(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \mathcal{H}_s\left(x^+; x^-, x, z\right) \\
= \;& \Gamma(-\frac{1}{2}) \int_{-2p^+p^- - p^2 > 0} dp^+ dp dp^- \; e^{i\left(x^+p^- + x^-p^+ + xp\right)} \int dp_1^+ dp_1 dp_2^+ dp_2 \\
& \mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right) \delta\left(p_1 + p_2 - p\right) \frac{P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}\right) s!}{\Gamma(s+\frac{1}{2})} \tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right) \\
& \delta\left(p^- + \frac{p_1^2}{2p_1^+} + \frac{p_2^2}{p_2^+}\right) \frac{1}{2p^+} 2\sqrt{\frac{\pi z\sqrt{-2p^+p^- - p^2}}{2}} J_{-\frac{1}{2}}\left(z\sqrt{-2p^+p^- - p^2}\right) \\
= \;& \Gamma(-\frac{1}{2}) \int_{-2p^+p^- - p^2 > 0} dp^+ dp dp^- \; e^{i\left(x^+p^- + x^-p^+ + xp\right)} \frac{s!}{\Gamma\left(s+\frac{1}{2}\right) p^+ \left(p^+\right)^s} \\
& \sqrt{\frac{\pi z\sqrt{-2p^+p^- - p^2}}{2}} J_{-\frac{1}{2}}\left(z\sqrt{-2p^+p^- - p^2}\right) \\
& \int dp_1^+ dp_1 dp_2^+ dp_2 \, \mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right) \delta\left(p_1 + p_2 - p\right) \left(p_1^+ + p_2^+\right)^s
\end{aligned}
$$

$$\begin{aligned} & P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}\right)\delta\left(p^- + \frac{p_1^2}{2p_1^+} + \frac{p_2^2}{p_2^+}\right)\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right) \\ = \ & \Gamma(-\frac{1}{2})\int_{-2p^+p^- - p^2>0} dp^+ dp dp^- \ e^{i\left(x^+p^- + x^-p^+ + xp\right)}\frac{s!}{\Gamma\left(s+\frac{1}{2}\right)p^+\left(p^+\right)^s} \\ & \sqrt{\frac{\pi z\sqrt{-2p^+p^- - p^2}}{2}}J_{-\frac{1}{2}}\left(z\sqrt{-2p^+p^- - p^2}\right)\tilde{\mathcal{O}}_s\left(p^-, p^+, p\right), \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathcal{O}}_s\left(p^-, p^+, p\right) &= \int dp_1^+ dp_1 dp_2^+ dp_2 \, \mathrm{J}\delta\left(p_1^+ + p_2^+ - p^+\right)\delta\left(p_1 + p_2 - p\right)\left(p_1^+ + p_2^+\right)^s \\ & \quad P_s^{(-\frac{1}{2},-\frac{1}{2})}\left(\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}\right)\delta\left(-\frac{p_1^2}{2p_1^+} - \frac{p_2^2 - p^-}{p_2^+}\right)\tilde{\Psi}\left(p_1^+, p_1, p_2^+, p_2\right). \end{aligned}$$

Reverting now back to the equation above as written in the paper,

$$\begin{aligned} \mathcal{H}_s\left(x^+; x^-, x, z\right) &= \int_{-2p^+p^- - p^2>0} dp^+ dp dp^- \ e^{i\left(x^+p^- + x^-p^+ + xp\right)}\frac{s!}{\Gamma\left(s+\frac{1}{2}\right)\left(p^+\right)^s} \\ & \quad \sqrt{\frac{\pi z\sqrt{-2p^+p^- - p^2}}{2}}J_{-\frac{1}{2}}\left(z\sqrt{-2p^+p^- - p^2}\right)\tilde{\mathcal{O}}_s\left(p^-, p^+, p\right), \end{aligned}$$

where $\sqrt{\frac{\pi z\sqrt{-2p^+p^- - p^2}}{2}}J_{-\frac{1}{2}}\left(z\sqrt{-2p^+p^- - p^2}\right) = \cos\left(z\sqrt{-2p^+p^- - p^2}\right) \rightarrow 1$ as $z \rightarrow 0$, so that in this limit the equation may be written as

$$\begin{aligned} \mathcal{H}_s\left(x^+; x^-, x, z\right) \ \underset{z=0}{\longrightarrow} \ & \frac{s!}{\Gamma\left(s+\frac{1}{2}\right)}\int_{-2p^+p^- - p^2>0} dp^+ dp dp^- \\ & \left(\frac{1}{\left(-i\partial_-\right)^s}e^{i\left(x^+p^- + x^-p^+ + xp\right)}\right)\tilde{\mathcal{O}}_s\left(p^-, p^+, p\right), \end{aligned}$$

as in equation (33) of Ref. [10].

Appendix D

θ and k^{θ} in temporal gauge

The quadratic Casimir is related to it as follows,

$$\mathcal{C}_{\mathrm{SO}(2,3)} = 2\left(k^{\theta}\right)^{2}.$$

We take a quick digression to the commutator relations for the conformal operators, which will help in calculating this Casimir. Counting the number of elements described by these generators we obtain d from P_{μ}, d from K_{μ}, $\frac{d(d-1)}{2}$ from $L_{\mu\nu}$ and 1 from D, i.e.

$$\frac{d(d-1)}{2} + d + d + 1 = \frac{d^2+3d+2}{2} = \frac{(d+1)(d+2)}{2}$$

hence we can describe this using a (d+2)-dimensional antisymmetric matrix. A new antisymmetric object J_{AB} (where $a = 1, ..., d$ replaces μ above, $A = -1, 0, ..., d$, and where $\eta_{AB} = \mathrm{diag}\left(-1, 1, 1, ..., 1\right)$),

$$\begin{aligned} J_{ab} &= L_{ab} \\ J_{0,a} &= \frac{1}{2}\left(P_a + K_a\right) \\ J_{-1,a} &= \frac{1}{2}\left(P_a - K_a\right) \\ J_{-1,0} &= D. \end{aligned}$$

This new group then satisfies the commutation relations

$$[J_{AB}, J_{CD}] = i\left(\eta_{AD}J_{BC} + \eta_{BC}J_{AD} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC}\right).$$

The Casimir is then given by

$$\begin{aligned}
\mathcal{C}_2 &= \frac{1}{2}J_{AB}J^{AB} \\
&= \frac{1}{2}J_{ab}J^{ab} + J_{0,a}J^{0,a} + J_{-1,a}J^{-1,a} + J_{-1,0}J^{-1,0} \\
&= \frac{1}{2}J_{ab}J^{ab} + \frac{(P_a + K_a)(P_a + K_a)}{4} - \frac{(P_a - K_a)(P_a - K_a)}{4} - D^2 \\
&= \frac{1}{2}J_{ab}J^{ab} + \frac{P_aK_a}{2} + \frac{K_aP_a}{2} - D^2 \\
&= \vec{L}^2 + P_aK_a + \frac{1}{2}[K_a, P_a] - D^2 \\
&= \vec{L}^2 + P_aK_a + (i\eta_{aa}D - iL_{aa}) - D^2 \\
&= \vec{L}^2 + P_aK_a - D(D - id). \qquad \text{(D.1)}
\end{aligned}$$

The eigenvalues of the above generators acting on the vacuum are given by

$$\begin{aligned}
\vec{L}^2 |\Omega\rangle &= s(s+1)|\Omega\rangle \\
K_a |\Omega\rangle &= 0 |\Omega\rangle \\
D |\Omega\rangle &= -i\Delta |\Omega\rangle \equiv iE_0 |\Omega\rangle,
\end{aligned}$$

where E_0 is the lowest energy and s is the spin; therefore, with $d = 3$, we have that the Casimir is given by

$$\mathcal{C}_{\mathrm{SO}(2,3))} = s(s+1) + E_0(E_0 - 3).$$

From Ref. [135], $E_0 \geq s + \frac{1}{2}$ and therefore

$$\mathcal{C}_{\mathrm{SO}(2,3))} \sim 2s^2 \sim 2(k^\theta)^2.$$

where this k^θ which labels internal spin degrees of freedom will be the subject of the upcoming calculations. The $SO(3,2)$ bilocal generators for rotations are given by

$$J^{AB} = x_1^A k_1^B - x_1^B k_1^A + x_2^A k_2^B - x_2^B k_2^A$$

and therefore the quadratic Casimir becomes

$$\begin{aligned}
\mathcal{C}_2 &= \frac{1}{2} J_{AB} J^{AB} \\
&= \frac{1}{2} \left(x_{1A} k_{1B} - x_{1B} k_{1A} + x_{2A} k_{2B} - x_{2B} k_{2A} \right) \\
&= \frac{1}{2} \left(x_{1A} k_{1B} - x_{1B} k_{1A} + x_{2A} k_{2B} - x_{2B} k_{2A} \right) \left(x_1^A k_1^B - x_1^B k_1^A + x_2^A k_2^B - x_2^B k_2^A \right) \\
&= \frac{1}{2} \big(x_{1A} k_{1B} x_1^A k_1^B - x_{1A} k_{1B} x_1^B k_1^A + x_{1A} k_{1B} x_2^A k_2^B - x_{1A} k_{1B} x_2^B k_2^A \\
&\quad - x_{1B} k_{1A} x_1^A k_1^B + x_{1B} k_{1A} x_1^B k_1^A - x_{1B} k_{1A} x_2^A k_2^B + x_{1B} k_{1A} x_2^B k_2^A \\
&\quad + x_{2A} k_{2B} x_1^A k_1^B - x_{2A} k_{2B} x_1^B k_1^A + x_{2A} k_{2B} x_2^A k_2^B - x_{2A} k_{2B} x_2^B k_2^A \\
&\quad - x_{2B} k_{2A} x_1^A k_1^B + x_{2B} k_{2A} x_1^B k_1^A - x_{2B} k_{2A} x_2^A k_2^B + x_{2B} k_{2A} x_2^B k_2^A \big) \\
&= \frac{1}{2} \left((\vec{x}_1)^2 \left(\vec{k}_1\right)^2 - \left(\vec{x}_1 \cdot \vec{k}_1\right)^2 + (\vec{x}_1 \cdot \vec{x}_2) \left(\vec{k}_1 \cdot \vec{k}_2\right) - \left(\vec{x}_1 \cdot \vec{k}_2\right) \left(\vec{x}_2 \cdot \vec{k}_1\right) \right. \\
&\quad - \left(\vec{x}_1 \cdot \vec{k}_1\right)^2 + (\vec{x}_1)^2 \left(\vec{k}_1\right)^2 - \left(\vec{x}_1 \cdot \vec{k}_2\right) \left(\vec{x}_2 \cdot \vec{k}_1\right) + (\vec{x}_1 \cdot \vec{x}_2) \left(\vec{k}_1 \cdot \vec{k}_2\right) \\
&\quad + (\vec{x}_1 \cdot \vec{x}_2) \left(\vec{k}_1 \cdot \vec{k}_2\right) - \left(\vec{x}_1 \cdot \vec{k}_2\right) \left(\vec{x}_2 \cdot \vec{k}_1\right) + (\vec{x}_2)^2 \left(\vec{k}_2\right)^2 - \left(\vec{x}_2 \cdot \vec{k}_2\right)^2 \\
&\quad \left. - \left(\vec{x}_1 \cdot \vec{k}_2\right) \left(\vec{x}_2 \cdot \vec{k}_1\right) + (\vec{x}_1 \cdot \vec{x}_2) \left(\vec{k}_1 \cdot \vec{k}_2\right) - \left(\vec{x}_1 \cdot \vec{k}_1\right) \left(\vec{x}_2 \cdot \vec{k}_2\right) + (\vec{x}_2)^2 \left(\vec{k}_2\right)^2 \right) \\
&= (\vec{x}_1)^2 (\vec{k}_1)^2 + (\vec{x}_2)^2 (\vec{k}_2)^2 - (\vec{x}_1 \cdot \vec{k}_1)^2 - (\vec{x}_2 \cdot \vec{k}_2)^2 + 2 (\vec{x}_1 \cdot \vec{x}_2)(\vec{k}_1 \cdot \vec{k}_2) \\
&\quad - 2 (\vec{x}_1 \cdot \vec{k}_2)(\vec{x}_2 \cdot \vec{k}_1).
\end{aligned}$$

We can satisfy the above relation by choosing the following constraints:[1]

$$(\vec{x}_1)^2 = 0 \tag{D.2}$$

$$(\vec{x}_2)^2 = 0 \tag{D.3}$$

$$\vec{x}_1 \cdot \vec{k}_1 = 0 \tag{D.4}$$

$$\vec{x}_2 \cdot \vec{k}_2 = 0 \tag{D.5}$$

$$\left(\vec{k}_2\right)^2 = 0 \tag{D.6}$$

$$\left(\vec{k}_1\right)^2 = 0. \tag{D.7}$$

[1]These constraints (also called collective gauge in the language of [9]) are different to the ones given in [135].

Using these constraints we may write (noting that Yoon's expression for k^θ is in terms of $x_{21} = (x_2 - x_1)$)

$$\begin{aligned}
\left(k^\theta\right)^2 &= s^2 = \frac{1}{2}\left[2\left(\vec{x}_1\cdot\vec{x}_2\right)\left(\vec{k}_1\cdot\vec{k}_2\right) - 2\left(\vec{x}_1\cdot\vec{k}_2\right)\left(\vec{x}_2\cdot\vec{k}_1\right)\right] \\
&= \left[-\left(\vec{x}_2-\vec{x}_1\right)^2\left(\vec{k}_1\cdot\vec{k}_2\right) + \left(\left(\vec{x}_2-\vec{x}_1\right)\cdot\vec{k}_2\right)\left(\left(\vec{x}_2-\vec{x}_1\right)\cdot\vec{k}_1\right)\right] \\
&= \left[-\vec{x}_{21}^2\left(\vec{k}_1\cdot\vec{k}_2\right) + \left(\left(\vec{x}_{21}\right)\cdot\vec{k}_2\right)\left(\left(\vec{x}_{21}\right)\cdot\vec{k}_1\right)\right] \\
&= \left[\left(\vec{k}_1\cdot\vec{x}_{21}\right)\left(\vec{x}_{21}\cdot\vec{k}_2\right) - \left(\vec{k}_1\cdot\vec{k}_2\right)\vec{x}_{21}^2\right]
\end{aligned}$$

Here we notice that the cross product of 2 vectors,

$$\begin{aligned}
(A\times B)\cdot(C\times D) &= \epsilon^{abc}A^bB^c\epsilon_{ade}C^dD^e \\
&= \left(\delta^b_d\delta^c_e - \delta^b_e\delta^c_d\right)A^bB^cC^dD^e \\
&= (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)
\end{aligned}$$

and therefore, recalling that $k_1^1 = |\vec{k}_1|\cos\varphi_1$ and so forth,

$$\begin{aligned}
(k^\theta)^2 &= \left[\left(\vec{k}_1\cdot\vec{x}_{21}\right)\left(\vec{x}_{21}\cdot\vec{k}_2\right) - \left(\vec{k}_1\cdot\vec{k}_2\right)\vec{x}_{21}^2\right] \\
&= \left(\vec{k}_1\times\vec{x}_{21}\right)\cdot\left(\vec{x}_{21}\times\vec{k}_2\right) \\
&= -\left(k_1^1x_{21}^2 - k_1^2x_{21}^1\right)\left(k_2^1x_{21}^2 - k_2^2x_{21}^1\right) \\
&= -\left(\left|\vec{k}_1\right|\cos\varphi_1x_{21}^2 - \left|\vec{k}_1\right|\sin\varphi_1x_{21}^1\right)\left(\left|\vec{k}_2\right|\cos\varphi_2x_{21}^2 - \left|\vec{k}_2\right|\sin\varphi_2x_{21}^1\right) \\
&= -\left|\vec{k}_1\right|\left|\vec{k}_2\right|\left(\cos\varphi_1x_{21}^2 - \sin\varphi_1x_{21}^1\right)\left(\cos\varphi_2x_{21}^2 - \sin\varphi_2x_{21}^1\right) \\
&= \left|\vec{k}_1\right|\left|\vec{k}_2\right|\left[-(x_{21}^2)^2\cos\varphi_1\cos\varphi_2 + (x_{21}^2)(x_{21}^1)\left(\cos\varphi_1\sin\varphi_2 + \sin\varphi_1\cos\varphi_2\right)\right. \\
&\quad \left. -(x_{21}^1)^2\sin\varphi_1\sin\varphi_2\right].
\end{aligned}$$

From the identities

$$\begin{aligned}
\sin(\varphi_1)\cos(\varphi_2) &= \frac{1}{2}\left(\sin(\varphi_1+\varphi_2) + \sin(\varphi_1-\varphi_2)\right) \\
&= \left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \sin\left(\frac{\varphi_1-\varphi_2}{2}\right)\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)\right] \\
\sin(\varphi_2)\cos(\varphi_1) &= \left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) - \sin\left(\frac{\varphi_1-\varphi_2}{2}\right)\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)\right]
\end{aligned}$$

we have that

$$\sin(\varphi_1)\cos(\varphi_2) + \sin(\varphi_2)\cos(\varphi_1) = 2\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\cos\left(\frac{\varphi_1+\varphi_2}{2}\right).$$

Furthermore,

$$\begin{aligned}(x_{21}^1)^2 + (x_{21}^2)^2 &= (x_2^1 - x_1^1)^2 + (x_2^2 - x_1^2)^2 \\ &= \left[(x_2^1)^2 + (x_2^2)^2\right] - 2\left[(x_2^1)(x_1^1) + (x_2^2)(x_1^2)\right] + \left[(x_1^1)^2 + (x_1^2)^2\right]\end{aligned}$$

and since

$$\begin{aligned}(x_1)^2 = (x_2)^2 = 0 \quad &\Rightarrow \quad \left[(x_2^1)^2 + (x_2^2)^2\right] = \left[(x_1^1)^2 + (x_1^2)^2\right] = 0 \\ \text{and} \quad &\Rightarrow \quad (x_1^2) = i(x_1^1); \;\; (x_2^2) = i(x_2^1)\end{aligned}$$

we obtain that

$$\begin{aligned}&(x_{21}^1)^2 + (x_{21}^2)^2 = 0 \\ \Rightarrow \quad &(x_{21}^1)^2 = -(x_{21}^2)^2.\end{aligned}$$

$$\begin{aligned}\cos(\varphi_1)\cos(\varphi_2) &= \frac{1}{2}\left(\cos(\varphi_1+\varphi_2) + \cos(\varphi_1-\varphi_2)\right) \\ &= \frac{1}{2}\left[2\cos^2\left(\frac{\varphi_1+\varphi_2}{2}\right) - 1 + 2\cos^2\left(\frac{\varphi_1-\varphi_2}{2}\right) - 1\right] \\ &= \left[\cos^2\left(\frac{\varphi_1+\varphi_2}{2}\right) + \cos^2\left(\frac{\varphi_1-\varphi_2}{2}\right) - 1\right] \\ &= \left[-\sin^2\left(\frac{\varphi_1+\varphi_2}{2}\right) + \cos^2\left(\frac{\varphi_1-\varphi_2}{2}\right)\right] \\ \sin(\varphi_1)\sin(\varphi_2) &= \frac{1}{2}\left(\cos(\varphi_1-\varphi_2) - \cos(\varphi_1+\varphi_2)\right) \\ &= \left[\cos^2\left(\frac{\varphi_1-\varphi_2}{2}\right) - \cos^2\left(\frac{\varphi_1+\varphi_2}{2}\right)\right]\end{aligned}$$

and therefore

$$\begin{aligned}&\left|\vec{k}_1\right|\left|\vec{k}_2\right|\left[-(x_{21}^2)^2\cos\varphi_1\cos\varphi_2 - (x_{21}^1)^2\sin\varphi_1\sin\varphi_2\right] \\ = \;&\left|\vec{k}_1\right|\left|\vec{k}_2\right|\left[(x_{21}^2)^2\sin^2\left(\frac{\varphi_1+\varphi_2}{2}\right) + (x_{21}^1)^2\cos^2\left(\frac{\varphi_1+\varphi_2}{2}\right)\right].\end{aligned}$$

(k^θ) is thus given as follows:

$$
\begin{aligned}
(k^\theta)^2 &= \left|\vec{k}_1\right|\left|\vec{k}_2\right|\left[(x_{21}^1)^2\cos^2\left(\frac{\varphi_1+\varphi_2}{2}\right)+2(x_{21}^2)(x_{21}^1)\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\right. \\
&\quad \left.+(x_{21}^2)^2\sin^2\left(\frac{\varphi_1+\varphi_2}{2}\right)\right] \\
&= \left|\vec{k}_1\right|\left|\vec{k}_2\right|\left[(x_{21}^1)\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)+(x_{21}^2)\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\right]^2 \\
\Rightarrow (k^\theta) &= \sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\left[(x_{21}^1)\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)+(x_{21}^2)\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\right]
\end{aligned}
$$

which is identical to the expression in Ref. [10].

$$
\begin{aligned}
(k^\theta) &= \sqrt{|\vec{k}_1||\vec{k}_2|}\cos\frac{\varphi_1+\varphi_2}{2}\left(x_2^1-x_1^1\right)+\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1+\varphi_2}{2}\left(x_2^2-x_1^2\right). \\
&= F_+(\vec{k}_1,\vec{k}_2)\left(x_2^1-x_1^1\right)+F_-(\vec{k}_1,\vec{k}_2)\left(x_2^2-x_1^2\right)
\end{aligned}
$$

Note that we may also write $F_+(\vec{k}_1,\vec{k}_2)=\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|+k_1^1k_2^1-k_1^2k_2^2\right)}$ and $F_-(\vec{k}_1,\vec{k}_2)=\sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|-k_1^1k_2^1+k_1^2k_2^2\right)}$ since

$$
\begin{aligned}
F_+(\vec{k}_1,\vec{k}_2) &= \sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|+k_1^1k_2^1-k_1^2k_2^2\right)} \qquad \text{(D.8)} \\
&= \sqrt{\frac{|\vec{k}_1||\vec{k}_2|}{2}\left(1+\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2\right)} \\
&= \sqrt{\frac{|\vec{k}_1||\vec{k}_2|}{2}\left(1+\cos\left(\varphi_1+\varphi_2\right)\right)} \\
&= \sqrt{\frac{|\vec{k}_1||\vec{k}_2|}{2}\left(1+\cos^2\left(\frac{\alpha_1+\varphi_2}{2}\right)-\sin^2\left(\frac{\varphi_1+\varphi_2}{2}\right)\right)} \\
&= \sqrt{|\vec{k}_1||\vec{k}_2|}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) \\
F_-(\vec{k}_1,\vec{k}_2) &= \sqrt{\frac{1}{2}\left(|\vec{k}_1||\vec{k}_2|-k_1^1k_2^1+k_1^2k_2^2\right)} \\
&= \sqrt{\frac{|\vec{k}_1||\vec{k}_2|}{2}\left(1-\cos\left(\varphi_1+\varphi_2\right)\right)} \\
&= \sqrt{|\vec{k}_1||\vec{k}_2|}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)
\end{aligned}
$$

Let us now find the θ conjugate to k^θ From [136], the result from equating two sets of generators is given by

$$
\begin{aligned}
x^1 &= \frac{|\vec{k}_1|x_1^1 + |\vec{k}_2|x_2^1}{\sqrt{(k^z)^2+\vec{k}^2}} - \frac{k^2k^zk^\theta}{\vec{k}^2\sqrt{(k^z)^2+\vec{k}^2}} \\
x^2 &= \frac{|\vec{k}_1|x_1^2 + |\vec{k}_2|x_2^2}{\sqrt{(k^z)^2+\vec{k}^2}} + \frac{k^1k^zk^\theta}{\vec{k}^2\sqrt{(k^z)^2+\vec{k}^2}} \\
z &= \frac{(x_2^1-x_1^1)F_-(\vec{k}_1,\vec{k}_2) - (x_2^2-x_1^2)F_+(\vec{k}_1,\vec{k}_2)}{\sqrt{(k^z)^2+\vec{k}^2}} \qquad \text{(D.9)}
\end{aligned}
$$

θ is found by considering Poisson brackets,

$$
\{f,g\}_{\mathrm{PB}} = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}\right)
$$

and following Ref. [143], the best one to start with is (note that θ is considered a momentum and not a coordinate); it is further assumed that all AdS coordinates, including θ, are functions only of bilocal coordinates $\vec{k}_1$ and $\vec{k}_2$ and not of of $\vec{x}_i$-coordinates.

$$
\begin{aligned}
0 &= \{x^1,\theta\}_{\mathrm{PB}} = \sum_i \left(\frac{\partial x^1}{\partial x_i^1}\frac{\partial \theta}{\partial k_i^1} - \frac{\partial x^1}{\partial k_i^1}\frac{\partial \theta}{\partial x_i^1}\right) = \frac{\partial x^1}{\partial x_1^1}\frac{\partial \theta}{\partial k_1^1} + \frac{\partial x^1}{\partial x_2^1}\frac{\partial \theta}{\partial k_2^1} + \frac{\partial x^1}{\partial k^\theta}\frac{\partial \theta}{\partial \theta} \\
&= \left(\frac{|\vec{k}_1|}{\sqrt{(k^z)^2+\vec{k}^2}}\right)\frac{\partial \theta}{\partial k_1^1} + \left(\frac{|\vec{k}_2|}{\sqrt{(k^z)^2+\vec{k}^2}}\right)\frac{\partial \theta}{\partial k_2^1} - \frac{k^2k^z}{\vec{k}^2\sqrt{(k^z)^2+\vec{k}^2}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
0 &= \{x^2,\theta\}_{\mathrm{PB}} \\
\Rightarrow 0 &= \left(\frac{|\vec{k}_1|}{\sqrt{(k^z)^2+\vec{k}^2}}\right)\frac{\partial \theta}{\partial k_1^2} + \left(\frac{|\vec{k}_2|}{\sqrt{(k^z)^2+\vec{k}^2}}\right)\frac{\partial \theta}{\partial k_2^2} + \frac{k^1k^z}{\vec{k}^2\sqrt{(k^z)^2+\vec{k}^2}},
\end{aligned}
$$

and therefore we obtain,

$$
|\vec{k}_1|\frac{\partial \theta}{\partial k_1^1} + |\vec{k}_2|\frac{\partial \theta}{\partial k_2^1} - \frac{k^2k^z}{\vec{k}^2} = 0 \qquad \text{(D.10)}
$$

$$
|\vec{k}_1|\frac{\partial \theta}{\partial k_1^2} + |\vec{k}_2|\frac{\partial \theta}{\partial k_2^2} + \frac{k^1k^z}{\vec{k}^2} = 0. \qquad \text{(D.11)}
$$

where from equation (4.25), $k^z = 2\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\frac{\varphi_1-\varphi_2}{2}$. Furthermore,

$$\begin{aligned}
0 &= \{z,\theta\}_{\rm PB} = \frac{\partial z}{\partial x_1^1}\frac{\partial\theta}{\partial k_1^1} + \frac{\partial z}{\partial x_2^1}\frac{\partial\theta}{\partial k_2^1} + \frac{\partial z}{\partial x_1^2}\frac{\partial\theta}{\partial k_1^2} + \frac{\partial z}{\partial x_2^2}\frac{\partial\theta}{\partial k_2^2} + \frac{\partial z}{\partial k^\theta}\frac{\partial\theta}{\partial\theta} \\
\Rightarrow 0 &= F_-(\vec{k}_1,\vec{k}_2)\left(\frac{\partial\theta}{\partial k_2^1} - \frac{\partial\theta}{\partial k_1^1}\right) - F_+(\vec{k}_1,\vec{k}_2)\left(\frac{\partial\theta}{\partial k_2^2} - \frac{\partial\theta}{\partial k_1^2}\right) \qquad \text{(D.12)}
\end{aligned}$$

and

$$\begin{aligned}
1 &= \{k^\theta,\theta\}_{\rm PB} = \frac{\partial k^\theta}{\partial x_1^1}\frac{\partial\theta}{\partial k_1^1} + \frac{\partial k^\theta}{\partial x_2^1}\frac{\partial\theta}{\partial k_2^1} + \frac{\partial k^\theta}{\partial x_1^2}\frac{\partial\theta}{\partial k_1^2} + \frac{\partial k^\theta}{\partial x_2^2}\frac{\partial\theta}{\partial k_2^2} + \frac{\partial k^\theta}{\partial k^\theta}\frac{\partial\theta}{\partial\theta} \\
\Rightarrow 1 &= F_+(\vec{k}_1,\vec{k}_2)\left(\frac{\partial\theta}{\partial k_2^1} - \frac{\partial\theta}{\partial k_1^1}\right) + F_-(\vec{k}_1,\vec{k}_2)\left(\frac{\partial\theta}{\partial k_2^2} - \frac{\partial\theta}{\partial k_1^2}\right). \qquad \text{(D.13)}
\end{aligned}$$

Now equations (D.12) and (D.13) are of the form

$$\begin{aligned}
& F_- \times A - F_+ \times B = 0 \\
& -F_+ \times A - F_- \times B = 1 \\
\Rightarrow \quad & \begin{pmatrix} F_- & F_+ \\ F_+ & F_- \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial k_2^1} - \frac{\partial\theta}{\partial k_1^1} \\ \frac{\partial\theta}{\partial k_2^2} - \frac{\partial\theta}{\partial k_1^2} \end{pmatrix} \\
= \quad & \frac{1}{F_-^2+F_+^2}\begin{pmatrix} F_- & F_+ \\ -F_+ & F_- \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{F_+}{F_+^2+F_-^2} \\ -\frac{F_-}{F_+^2+F_-^2} \end{pmatrix}.
\end{aligned}$$

Note that

$$\begin{aligned}
F_+(\vec{k}_1,\vec{k}_2) &= \sqrt{|\vec{k}_1||\vec{k}_2|}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) \\
F_-(\vec{k}_1,\vec{k}_2) &= \sqrt{|\vec{k}_1||\vec{k}_2|}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) \\
\Rightarrow F_+^2 + F_-^2 &= |\vec{k}_1||\vec{k}_2|.
\end{aligned}$$

Therefore

$$\begin{pmatrix} \frac{\partial\theta}{\partial k_2^1} - \frac{\partial\theta}{\partial k_1^1} \\ \frac{\partial\theta}{\partial k_2^2} - \frac{\partial\theta}{\partial k_1^2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) \\ -\frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) \end{pmatrix},$$

and equations (D.10) and (D.11) become

$$
\begin{aligned}
0 &= |\vec{k}_1|\frac{\partial\theta}{\partial k_1^1} + |\vec{k}_2|\frac{\partial\theta}{\partial k_2^1} - \frac{k^2k^z}{\vec{k}^2} \\
&= |\vec{k}_1|\frac{\partial\theta}{\partial k_1^1} + |\vec{k}_2|\left(\frac{\partial\theta}{\partial k_1^1} - \frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\right) - \frac{k^2k^z}{\vec{k}^2} \\
\Rightarrow \frac{\partial\theta}{\partial k_1^1} &= \frac{1}{|\vec{k}_1|+|\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \frac{k^2k^z}{\vec{k}^2}\right] \\
0 &= |\vec{k}_1|\frac{\partial\theta}{\partial k_1^2} + |\vec{k}_2|\frac{\partial\theta}{\partial k_2^2} + \frac{k^1k^z}{\vec{k}^2} \\
&= |\vec{k}_1|\frac{\partial\theta}{\partial k_1^2} + |\vec{k}_2|\left(\frac{\partial\theta}{\partial k_1^2} - \frac{1}{\sqrt{|\vec{k}_1||\vec{k}_2|}}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\right) + \frac{k^1k^z}{\vec{k}^2} \\
\Rightarrow \frac{\partial\theta}{\partial k_1^2} &= \frac{1}{|\vec{k}_1|+|\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) - \frac{k^1k^z}{\vec{k}^2}\right]. \qquad \text{(D.14)}
\end{aligned}
$$

Now, since $k_1^2 = |\vec{k}_1|\sin\varphi_1$ and $k_1^1 = |\vec{k}_1|\cos\varphi_1$, we have

$$
\begin{aligned}
\frac{\partial\theta}{\partial k_1} \quad [\equiv &\ \frac{\partial\theta}{\partial|\vec{k}_1|}] = \cos\varphi_1\frac{\partial\theta}{\partial k_1^1} + \sin\varphi_1\frac{\partial\theta}{\partial k_1^2} \\
= &\ \cos\varphi_1\frac{1}{|\vec{k}_1|+|\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \frac{k^2k^z}{\vec{k}^2}\right] \\
&+\sin\varphi_1\frac{1}{|\vec{k}_1|+|\vec{k}_2|}\left[\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) - \frac{k^1k^z}{\vec{k}^2}\right] \\
= &\ \frac{1}{|\vec{k}_1|+|\vec{k}_2|}\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\left[\cos\varphi_1\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \sin\varphi_1\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\right] \\
&+\frac{k^z}{\vec{k}^2\left(|\vec{k}_1|+|\vec{k}_2|\right)}\left[k^2\cos\varphi_1 - k^1\sin\varphi_1\right].
\end{aligned}
$$

The bracketed term in the first line is given by

$$
\begin{aligned}
&\cos\varphi_1\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \sin\varphi_1\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) \\
= &\left(\cos^2\left(\frac{\varphi_1}{2}\right) - \sin^2\left(\frac{\varphi_1}{2}\right)\right)\left(\cos\left(\frac{\varphi_1}{2}\right)\cos\left(\frac{\varphi_2}{2}\right) - \sin\left(\frac{\varphi_1}{2}\right)\sin\left(\frac{\varphi_2}{2}\right)\right) \\
&+2\sin\left(\frac{\varphi_1}{2}\right)\cos\left(\frac{\varphi_1}{2}\right)\left(\sin\left(\frac{\varphi_1}{2}\right)\cos\left(\frac{\varphi_2}{2}\right) + \sin\left(\frac{\varphi_2}{2}\right)\cos\left(\frac{\varphi_1}{2}\right)\right) \\
= &\left(\cos^2\left(\frac{\varphi_1}{2}\right) + \sin^2\left(\frac{\varphi_1}{2}\right)\right)\left(\cos\left(\frac{\varphi_1}{2}\right)\cos\left(\frac{\varphi_2}{2}\right) + \sin\left(\frac{\varphi_1}{2}\right)\sin\left(\frac{\varphi_2}{2}\right)\right)
\end{aligned}
$$

$$= \quad 1\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)$$

The bracketed term in the second line is given by

$$\begin{aligned}
k^2\cos\varphi_1 - k^1\sin\varphi_1 &= \left(k_1^2+k_2^2\right)\cos\varphi_1 - \left(k_1^1+k_2^1\right)\sin\varphi_1 \\
&= \left(|\vec{k}_1|\sin\varphi_1 + |\vec{k}_2|\sin\varphi_2\right)\cos\varphi_1 - \left(|\vec{k}_1|\cos\varphi_1 + |\vec{k}_2|\cos\varphi_2\right)\sin\varphi_1 \\
&= |\vec{k}_2|\sin\left(\varphi_2-\varphi_1\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{\partial\theta}{\partial k_1} &= \frac{1}{|\vec{k}_1|+|\vec{k}_2|}\sqrt{\frac{|\vec{k}_2|}{|\vec{k}_1|}}\left[\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)\right] + \frac{k^z}{\vec{k}^2(|\vec{k}_1|+|\vec{k}_2|)}|\vec{k}_2|\sin\left(\varphi_2-\varphi_1\right) \\
&= \frac{\left|\vec{k}_2\right|\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)}{\left(|\vec{k}_1|+|\vec{k}_2|\right)\sqrt{|\vec{k}_1||\vec{k}_2|}\vec{k}^2}\left[\vec{k}^2 + 2\sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}k^z\sin\left(\frac{\varphi_1-\varphi_2}{2}\right)\right].
\end{aligned}$$

Using the expression for p^z in equation (4.25)

$$k^z \quad = \quad 2\sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right)$$

the bracketed term becomes

$$\begin{aligned}
& (\vec{k}_1)^2 + (\vec{k}_2)^2 + 2|\vec{k}_1||\vec{k}_2|\cos(\varphi_1-\varphi_2) \\
& +2\sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\left(2\sqrt{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right)\right)\sin\left(\frac{\varphi_1-\varphi_2}{2}\right) \\
= \quad & (\vec{k}_1)^2 + (\vec{k}_2)^2 + 2|\vec{k}_1||\vec{k}_2|\left[\cos^2\left(\frac{\varphi_1-\varphi_2}{2}\right) - \sin^2\left(\frac{\varphi_1-\varphi_2}{2}\right)\right] \\
& +4\left|\vec{k}_1\right|\left|\vec{k}_2\right|\sin^2\left(\frac{\varphi_1-\varphi_2}{2}\right) \\
= \quad & \left(|\vec{k}_1|+|\vec{k}_2|\right)^2
\end{aligned}$$

so that

$$\frac{\partial\theta}{\partial k_1} \quad = \quad \frac{\left|\vec{k}_2\right|\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)\left(|\vec{k}_1|+|\vec{k}_2|\right)}{\sqrt{|\vec{k}_1||\vec{k}_2|}\vec{k}^2},$$

which integrates to

$$
\begin{aligned}
\theta &= \arctan\left(\frac{2\sqrt{|\vec{k}_1||\vec{k}_2|}\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)}{\left(|\vec{k}_2|-|\vec{k}_1|\right)}\right) \\
&= \arctan\left(\frac{2|\vec{k}_1||\vec{k}_2|\sin\left(\varphi_1-\varphi_2\right)}{\left(|\vec{k}_2|-|\vec{k}_1|\right)\left(2\sqrt{|\vec{k}_1||\vec{k}_2|}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right)\right)}\right) \\
&= \arctan\left(2\frac{\vec{k}_2\times\vec{k}_1}{\left(|\vec{k}_2|-|\vec{k}_1|\right)k^z}\right) \\
&= \arctan\left(2\frac{k_2^1k_1^2-k_1^1k_2^2}{\left(|\vec{k}_2|-|\vec{k}_1|\right)k^z}\right)
\end{aligned}
$$

where $\frac{\partial\theta}{\partial\varphi_1}=\frac{\partial\theta}{\partial\varphi_2}=0$, i.e. the constant of integration is 0.

Appendix E

Toy Model

E.1 Non-Relativistic Quantum Mechanical Problem

In equation (3.7), the second order effective action was found to be

$$S^{(2)}_{eff} = \frac{1}{4}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\right) - \frac{\lambda}{4!}\int d^d x' \eta^2_{x'x'}. \tag{E.1}$$

The equations of motion for this effective action are given by

$$\begin{aligned} \frac{1}{2}\left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{xy} - \frac{\lambda}{12}\delta(x-y)\eta_{xx} &= 0 \\ \Rightarrow \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{xy} &= \frac{\lambda}{6}\delta(x-y)\eta_{xx}. \end{aligned} \tag{E.2}$$

Now $\hat{p} = -i\partial_x$, thus

$$\begin{aligned} \langle x|\partial|y\rangle &= i\langle x|\hat{p}|y\rangle \\ &= i\int dp_1 \int dp_2 \langle x|p_1\rangle \langle p_1|\hat{p}|p_2\rangle \langle p_2|y\rangle \\ &= i\int dp_1 \int dp_2 \frac{e^{ixp_1}}{\sqrt{2\pi}}\delta(p_1-p_2)p_2\frac{e^{-iyp_2}}{\sqrt{2\pi}} \\ &= \int \frac{dp_1}{2\pi} e^{ip_1(x-y)}(ip_1) \\ &= \frac{\partial}{\partial x}\delta(x-y). \end{aligned}$$

Similarly,

$$\begin{aligned} \left\langle x|\partial^2|y\right\rangle &= i^2 \left\langle x|\hat{p}|y\right\rangle \\ &= \frac{\partial^2}{\partial x^2}\delta(x-y). \end{aligned}$$

The large-N background for equation (3.6) was found to be

$$\psi_k^0 = \frac{1}{k^2},$$

and therefore

$$\left\langle x \left| \left(\psi_k^0\right)^{-1} \right| y \right\rangle = -\frac{\partial^2}{\partial x^2}\delta\left(x-y\right).$$

The equations of motion in equation (E.2) therefore become

$$\begin{aligned} \frac{\lambda}{6}\delta(x-y)\eta_{xx} &= \left(\psi_0^{-1}\eta\psi_0^{-1}\right)_{xy} \\ &= \int d^d z \int d^d w \left(\psi_0\right)^{-1}_{xz} \eta_{zw} \left(\psi_0\right)^{-1}_{wy} \\ &= \int d^d z \int d^d w \left(-\frac{\partial^2}{\partial z^2}\delta\left(x-z\right)\right)\eta_{zw}\left(-\frac{\partial^2}{\partial w^2}\delta\left(w-y\right)\right) \\ &= -\int d^d z \int d^d w \delta\left(x-z\right)\frac{\partial^2}{\partial z^2}\eta_{zw}\left(\frac{\partial^2}{\partial w^2}\delta\left(w-y\right)\right) \\ &= -\int d^d w \frac{\partial^2}{\partial x^2}\eta_{xw}\left(\frac{\partial^2}{\partial w^2}\delta\left(w-y\right)\right) \\ &= \int d^d w \delta\left(w-y\right)\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial w^2}\eta_{xw} \\ &= \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\eta_{xy}. \end{aligned}$$

Therefore the equations of motion may be written as

$$\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\eta_{xy} = \frac{\lambda}{6}\delta(x-y)\eta_{xx}.$$

This justifies the use of a one-dimensional Dirac delta function potential toy model.

E.2 Dirac delta function potential

For the Dirac delta potential,

$$V(x) = V_0\delta(x),$$

Schrödinger's equation is given by (with $\hbar = 1$),

$$i\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{1}{2m}\frac{d^2}{dx^2} + V_0\delta(x)\right)\psi(x,t).$$

Since the time dependent wavefunction is given by $\psi(x,t) = e^{-iEt}\psi_E(x)$ where E is the time independent part, this equation becomes

$$\left(E + \frac{1}{2m}\frac{d^2}{dx^2}\right)\psi_E(x) = V_0\delta(x)\psi_E(0). \tag{E.3}$$

We will now drop the E subscript. This may be solved as follows: suppose we had a Green's function

$$G\left(x,x'\right) = \int \frac{dk}{2\pi}e^{ik(x-x')}G_k$$

which satisfied

$$\begin{aligned} &\int \frac{dk}{2\pi}\left(E + \frac{1}{2m}\frac{d^2}{dx^2}\right)e^{ikx}G_k = \delta(x-x') \\ \Rightarrow \qquad &\psi(x) = \int dx' G(x,x')V_0\delta(x')\psi(0). \end{aligned}$$

Substituting this Green's function into (E.3) would then result in (treating $\psi(0)$ as a constant and recalling that $\delta(x) = \int \frac{dk}{2\pi}e^{ikx}$,

$$\begin{aligned} G_k &= \frac{1}{E - \frac{k^2}{2m}} \\ \Rightarrow G(x,x') &= \int \frac{dk}{2\pi}e^{ik(x-x')}\frac{1}{E-\frac{k^2}{2m}} \\ &= \int \frac{dk}{2\pi}(-2m)e^{ik(x-x')}\frac{1}{(k+\sqrt{2mE}+i\epsilon)(k-\sqrt{2mE}-i\epsilon)} \\ &= \frac{m}{i\sqrt{2mE}}e^{\sqrt{2mE}|x-x'|}. \end{aligned} \tag{E.4}$$

Hence,

$$\begin{aligned}\psi(x) &= \int dx' G(x,x')V_0\delta(x')\psi(0)\\ &= G(x,0)V_0\psi(0)\\ &= \frac{m}{i\sqrt{2mE}}e^{\sqrt{2mE}|x|}V_0\psi(0).\end{aligned}$$

The general solution thus becomes (noting that $p=\sqrt{2mE}$)

$$\psi(x) = e^{ipx} + \frac{m}{ip}e^{ip|x|}V_0\psi(0). \tag{E.5}$$

Setting $x=0$ we see that $\psi_0 = \frac{1}{1-mV_0/ip}$, so the general solution becomes

$$\psi(x) = e^{ipx} + \frac{mV_0}{ip-mV_0}e^{ip|x|}.$$

It is notable that in the limit $V_0\to\infty$, the $x>0$ wavefunction vanishes.

E.2.1 Second quantised commutators for the free theory

Since Hamilton's equation is given by $i\partial_t\hat{\psi} = H\hat{\psi}$, the Lagrangian is given by

$$\mathcal{L} = \int dx \int dt \hat{\psi}^\dagger(x,t)\left(i\partial_t - h\right)\hat{\psi}(x,t)$$

where

$$h = -\frac{1}{2m}\frac{d^2}{dx^2} + V_0\delta(x).$$

The conjugate momentum is readily obtained to be

$$\pi\left(x,t\right) = \frac{\partial\mathcal{L}}{\partial\left(\partial_t\hat{\psi}\left(x,t\right)\right)} = i\hat{\psi}^\dagger\left(x,t\right).$$

It then follows that the second quantized canonical commutation relations are

$$\left[\hat{\psi}\left(x,t\right),\hat{\psi}^\dagger\left(x',t\right)\right] = \delta\left(x-x'\right).$$

For the scattering states, we have

$$\hat{\psi}(x) \;=\; \int \frac{dk}{2\pi} e^{ikx} \hat{\eta}_k$$

where

$$\hat{\eta}_k = \int dk_0 \eta_k^{k_0} \hat{b}_{k_0}. \tag{E.6}$$

From Schrödinger's equation,

$$\begin{aligned}
& -\frac{1}{2m}\frac{d^2}{dx^2}\psi(x) + V_0\delta(x)\psi(0) = E\psi(x) \\
\Rightarrow\quad & -\frac{1}{2m}\frac{d^2}{dx^2}\int \frac{dk}{2\pi}e^{ikx}\hat{\eta}_k + V_0\psi(0)\int \frac{dk}{2\pi}e^{ikx} = E\int \frac{dk}{2\pi}e^{ikx}\hat{\eta}_k \\
\Rightarrow\quad & \hat{\eta}_k = \frac{V_0\psi(0)}{E-\frac{k^2}{2m}} = \frac{V_0}{E-\frac{k^2}{2m}}\int \frac{dk}{2\pi}e^{ikx}\hat{\eta}_k \\
\Rightarrow\quad & \psi(x) = \int \frac{dk}{2\pi}e^{ikx}\frac{V_0\psi(0)}{E-\frac{k^2}{2m}} \\
\Rightarrow\quad & \psi(0) = \int \frac{dk}{2\pi}\frac{V_0\psi(0)}{E-\frac{k^2}{2m}} \\
\Rightarrow\quad & \hat{\eta}_k = \delta(k-k_0) + \frac{V_0\int \frac{dk}{2\pi}\frac{V_0\psi(0)}{E-\frac{k^2}{2m}}}{E-\frac{k^2}{2m}}.
\end{aligned}$$

Here,

$$\begin{aligned}
\eta_k^{k_0} &= \delta(k-k_0) + \frac{v_0}{2\pi}\frac{1}{\frac{k_0^2}{2m}-\frac{k^2}{2m}}\frac{1}{1-v_0\int \frac{dk}{2\pi}\frac{1}{\frac{k_0^2}{2m}-\frac{k^2}{2m}}} \\
&= \delta(k-k_0) - \frac{1}{\pi}\frac{ik_0}{k_0^2-k^2},
\end{aligned}$$

and $k_0 = \sqrt{2mE}$ is fixed. In momentum space, the critical scattering states[1]

$$\begin{aligned}
\eta_k &= \int dk_0 \eta_k^{k_0} b_{k_0} \\
&= \int dk_0 \delta(k-k_0)\, b_{k_0} - \frac{i}{\pi}\int dk_0 \frac{k_0 b_{k_0}}{k_0^2-k^2}
\end{aligned}$$

[1] Eq. (E.7) are the analogue of the full relativistic bilocal equation:

$$\eta_{k_1k_2} \;=\; \alpha_{k_1k_2}A_{k_1k_2} + \alpha^*_{k_1k_2}A^\dagger_{k_1k_2} + \int dp \frac{4i\,|p_E|\left(\frac{1}{|k_1|}+\frac{1}{|k_2|}\right)}{E^2_{p,p-k_1-k_2} - (|k_1|+|k_2|)^2}\left(\alpha_{p,p-k_1-k_2}A_{p,p-k_1-k_2} + \alpha^*_{k_1k_2}A^\dagger_{k_1k_2}\right).$$

A huge difference between the two cases is that we are able to perform the integral over the momentum label in the non-relativistic case.

$$= \quad b_k - \frac{i}{\pi}\int dk_0 \frac{k_0 b_{k_0}}{k_0^2 - k^2}. \tag{E.7}$$

It then follows that the canonical commutation relations are since $[b_k, b_{k'}] = \delta_{k,k'}$, where in the last line we close above,

$$\begin{aligned}
\left[\eta_k, \eta_{k'}^\dagger\right] &= \left[b_k - \frac{i}{\pi}\int dk_0 \frac{k_0 b_{k_0}}{k_0^2 - k^2}, b_{k'}^\dagger + \frac{i}{\pi}\int dk_0' \frac{k_0' b_{k_0'}^\dagger}{k_0'^2 - k^2}\right] \\
&= \left[b_k, b_{k'}^\dagger\right] + \left[b_k, \frac{i}{\pi}\int dk_0' \frac{k_0' b_{k_0'}^\dagger}{k_0'^2 - k'^2}\right] + \left[-\frac{i}{\pi}\int dk_0 \frac{k_0 b_{k_0}}{k_0^2 - k^2}, b_{k'}^\dagger\right] \\
&\quad + \left[-\frac{i}{\pi}\int dk_0 \frac{k_0 b_{k_0}}{k_0^2 - k^2}, \frac{i}{\pi}\int dk_0' \frac{k_0' b_{k_0'}^\dagger}{k_0'^2 - k'^2}\right] \\
&= \delta_{k,k'} + \frac{i}{\pi}\frac{k}{k^2 - k'^2} - \frac{i}{\pi}\frac{k}{k^2 - k'^2} + \frac{1}{\pi^2}\int d_{k_0} \frac{k_0^2}{\left(k_0^2 - k^2\right)\left(k_0^2 - k'^2\right)} \\
&= \delta_{k,k'} + \frac{i}{\pi}\frac{k}{k^2 - k'^2} - \frac{i}{\pi}\frac{k'}{k^2 - k'^2} + \frac{1}{\pi^2}\left[\frac{2\pi i k^2}{2k\left(k^2 - k'^2\right)} + \frac{2\pi i k'^2}{2k'\left(k'^2 - k^2\right)}\right] \\
&= \delta_{k,k'}.
\end{aligned}$$

E.3 Normalisation Constant

We now set out to normalise this Dirac delta function potential. Defining

$$f(p) = \frac{mV_0}{ip - mV_0}, \quad f^*(p) = \frac{-mV_0}{ip + mV_0},$$

the wave-functions are given by

$$\begin{aligned}
\psi_p(x) &= \begin{cases} e^{ipx} + f(p)e^{ipx} & x > 0 \\ e^{ipx} + f(p)e^{-ipx} & x < 0 \end{cases} \\
\psi_k^\dagger(x) &= \begin{cases} e^{-ikx} + f^*(k)e^{-ikx} & x > 0 \\ e^{-ikx} + f^*(k)e^{ikx} & x < 0 \end{cases}
\end{aligned}$$

We now find that

$$\delta^d(k - p) = \int_{-\infty}^{\infty} dx A^\dagger \psi^\dagger(x) A \psi_p(x)$$

$$
\begin{aligned}
&= |A|^2 \left[\int_{-\infty}^{0} dx \left(e^{-ikx} + f^*(k) e^{ikx} \right) \left(e^{ipx} + f(p) e^{-ipx} \right) \right. \\
&\quad \left. + \int_0^{\infty} \left(e^{-ikx} + f^*(k) e^{-ikx} \right) \left(e^{ipx} + f(p) e^{ipx} \right) \right] \\
&= |A|^2 \left[2\pi\delta^d(p-k) + f(p) \int_0^{\infty} dx \left(e^{ipx} \left(e^{-ikx} + e^{ikx} \right) \right. \right. \\
&\quad \left. \left. + f^*(k) e^{-ikx} \left(e^{-ipx} + e^{ipx} \right) \right) + 2 f(p) f^*(k) e^{ipx} e^{-ikx} \right] \\
&= |A|^2 \left[2\pi\delta^d(p-k) + f(p) \int_0^{\infty} dx \left((\cos px + i \sin px) (2 \cos kx) \right. \right. \\
&\quad + f^*(k) (\cos kx - i \sin kx) (2 \cos px)) \\
&\quad + 2 f(p) f^*(k) (\cos px + i \sin px) (\cos kx - i \sin kx)] . \qquad \text{(E.8)}
\end{aligned}
$$

Consider now

$$
\begin{aligned}
\int_0^{L/2} dx \sin \frac{2\pi n}{L} x \cos \frac{2\pi m}{L} x &= \frac{1}{2} \int_0^{L/2} dx \left[\sin \frac{2\pi}{L} (m+n) x - \sin \frac{2\pi}{L} (m-n) x \right] \\
&= -\frac{1}{2} \left[\frac{\cos \frac{2\pi}{L} (m+n)}{\frac{2\pi}{L} (m+n)} - \frac{\cos \frac{2\pi}{L} (m-n)}{\frac{2\pi}{L} (m-n)} \right]_0^{L/2} \\
&= -\frac{1}{2} \left[\frac{(-1)^{m+n} - 1}{\frac{2\pi}{L} (m+n)} - \frac{(-1)^{m-n} - 1}{\frac{2\pi}{L} (m-n)} \right] .
\end{aligned}
$$

If the denominators were not different, this would reduce to zero since $(-1)^n = (-1)^{-n}$. The terms containing both cos and sin in equation (E.8) are as follows

$$
\begin{aligned}
I &= \int_0^{\infty} dx f(p) 2i \sin px \cos kx - f^*(k) 2i \sin kx \cos px \\
&\quad + 2 f(p) f^*(k) (i \sin px \cos kx - i \cos px \sin kx) \qquad \text{(E.9)}
\end{aligned}
$$

To calculate this, we work with the discrete case:

$$
\begin{aligned}
I_{dis} &= \int_0^{L/2} dx f(\frac{2\pi n}{L}) 2i \sin \frac{2\pi n}{L} x \cos \frac{2\pi m}{L} x - f^*(\frac{2\pi m}{L}) 2i \sin \frac{2\pi m}{L} x \cos \frac{2\pi n}{L} x \\
&\quad + 2 f(\frac{2\pi n}{L}) f^*(\frac{2\pi m}{L}) \left(i \sin \frac{2\pi n}{L} x \cos \frac{2\pi m}{L} x - i \cos \frac{2\pi n}{L} x \sin \frac{2\pi m}{L} x \right) .
\end{aligned}
$$

The first line in equation (E.9) becomes

$$
I_{dis_1} = \int_0^{L/2} dx f(\frac{2\pi n}{L}) 2i \sin \frac{2\pi n}{L} x \cos \frac{2\pi m}{L} x - f^*(\frac{2\pi m}{L}) 2i \sin \frac{2\pi m}{L} x \cos \frac{2\pi n}{L} x
$$

$$\begin{aligned}
&= 2i\left(-\frac{1}{2}\right)\frac{mV_0}{i\frac{2\pi n}{L}-mV_0}\left\{\left[\frac{(-1)^{m+n}-1}{\frac{2\pi}{L}(m+n)}-\frac{(-1)^{m-n}-1}{\frac{2\pi}{L}(m-n)}\right]\right. \\
&\quad \left.+\frac{mV_0}{i\frac{2\pi m}{L}+mV_0}\left[\frac{(-1)^{m+n}-1}{\frac{2\pi}{L}(m+n)}-\frac{(-1)^{n-m}-1}{\frac{2\pi}{L}(n-m)}\right]\right\} \\
&= -i\frac{mV_0}{(i\frac{2\pi n}{L}-mV_0)(i\frac{2\pi m}{L}+mV_0)}\left[\left(\frac{(-1)^{m+n}-1}{\frac{2\pi}{L}(m+n)}\right)\left(i\frac{2\pi m}{L}+mV_0\right.\right. \\
&\quad \left.\left.+i\frac{2\pi n}{L}-mV_0\right)-\left(\frac{(-1)^{m-n}-1}{\frac{2\pi}{L}(m-n)}\right)\left(i\frac{2\pi m}{L}+mV_0-\left(i\frac{2\pi n}{L}-mV_0\right)\right)\right] \\
&= -i\frac{mV_0}{(i\frac{2\pi n}{L}-mV_0)(i\frac{2\pi m}{L}+mV_0)}\left[i\left((-1)^{m+n}-1\right)-i\left((-1)^{m-n}-1\right)\right. \\
&\quad \left.-2mV_0\left(\frac{(-1)^{m-n}-1}{\frac{2\pi}{L}(m-n)}\right)\right] \\
&= \frac{2i(mV_0)^2}{(i\frac{2\pi n}{L}-mV_0)(i\frac{2\pi m}{L}+mV_0)}\left[\left(\frac{(-1)^{m-n}-1}{\frac{2\pi}{L}(m-n)}\right)\right] \qquad \text{(E.10)}
\end{aligned}$$

The bracketed term in the second line of equation (E.9) evaluates to

$$\begin{aligned}
&-\frac{1}{2}i\left[\frac{(-1)^{m+n}-1}{\frac{2\pi}{L}(m+n)}-\frac{(-1)^{m-n}-1}{\frac{2\pi}{L}(m-n)}-\left(\frac{(-1)^{m+n}-1}{\frac{2\pi}{L}(m+n)}-\frac{(-1)^{n-m}-1}{\frac{2\pi}{L}(n-m)}\right)\right] \\
&= i\left(\frac{(-1)^{m-n}-1}{\frac{2\pi}{L}(m-n)}\right).
\end{aligned}$$

Hence

$$I_{dis_2} = 2i\left(\frac{mV_0}{i\frac{2\pi n}{L}-mV_0}\right)\left(\frac{-mV_0}{i\frac{2\pi m}{L}+mV_0}\right)\left(\frac{(-1)^{m-n}-1}{\frac{2\pi}{L}(m-n)}\right). \qquad \text{(E.11)}$$

Adding the terms (E.10) and (E.11) together, we finally obtain that these ”extra terms” equate to

$$\begin{aligned}
& I_{dis} = 0 \\
\Rightarrow\quad & I = 0.
\end{aligned}$$

Now, for $m \neq n$

$$\int_0^{L/2} dx \cos\frac{2\pi n}{L}x\cos\frac{2\pi m}{L}x = \frac{1}{2}\int_0^{L/2} dx \cos\frac{2\pi}{L}(m+n)x+\cos\frac{2\pi}{L}(m-n)x$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{\sin\frac{2\pi}{L}(m+n)x}{\frac{2\pi}{L}(m+n)} + \frac{\sin\frac{2\pi}{L}(m-n)x}{\frac{2\pi}{L}(m-n)}\right]_0^{L/2} \\
&= 0 \\
&= \int_0^{L/2} dx \sin\frac{2\pi n}{L}x \sin\frac{2\pi m}{L}x,
\end{aligned}
$$

whereas for $m = n$,

$$
\begin{aligned}
\int_0^{L/2} dx \cos^2\frac{2\pi n}{L}x &= \frac{1}{2}\int_0^{L/2} dx \left(1+\cos\frac{4\pi n}{L}x\right) = \frac{L}{4} \\
\int_0^{L/2} dx \sin^2\frac{2\pi n}{L}x &= \frac{1}{2}\int_0^{L/2} dx \left(1-\cos\frac{4\pi n}{L}x\right) = \frac{L}{4},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_0^{L/2} dx \cos\frac{2\pi n}{L}x \cos\frac{2\pi m}{L}x = \int_0^{L/2} dx \sin\frac{2\pi n}{L}x \sin\frac{2\pi m}{L}x = \frac{L}{4}\delta_{m,n} \\
\Rightarrow\quad &\int_0^{\infty} dx \cos px \cos kx = \int_0^{\infty} dx \sin px \sin kx = \frac{2\pi}{L}\frac{L}{4}\delta^d(k-p) = \frac{\pi}{2}\delta^d(k-p).
\end{aligned}
$$

We therefore find that (E.8) reduces to

$$
\begin{aligned}
\delta^d(k-p) &= |A|^2\Big[2\pi\delta^d(p-k) + f(p)\int_0^{\infty} dx\,((\cos px + i\sin px)\,(2\cos kx) \\
&\quad +f^*(k)\,(\cos kx - i\sin kx)\,(2\cos px)) \\
&\quad +2f(p)f^*(k)\,(\cos px + i\sin px)\,(\cos kx - i\sin kx)\Big] \\
&\to |A|^2\Big[2\pi\delta^d(p-k) + 2f(p)\int_0^{\infty} dx\cos px\cos kx + 2f^*(k)\cos kx\cos px \\
&\quad +2f(p)f^*(k)\cos px\cos kx + \sin px\sin kx\Big] \\
&= |A|^2\Big[2\pi\delta^d(p-k) + 2f(p)\frac{\pi}{2}\delta^d(k-p) + 2f^*(k)\frac{\pi}{2}\delta^d(k-p) \\
&\quad +2f(p)f^*(k)\left(\frac{\pi}{2}\delta^d(k-p) + \frac{\pi}{2}\delta^d(k-p)\right)\Big] \\
&= |A|^2\Big[2\pi\delta^d(p-k) + 2f(p)\frac{\pi}{2}\delta^d(k-p) + 2f^*(k)\frac{\pi}{2}\delta^d(k-p) \\
&\quad +2f(p)f^*(k)\frac{\pi}{2}\delta^d(k-p) + \frac{\pi}{2}\delta^d(k-p)\Big] \\
&= |A|^2\pi\delta^d(k-p)\left[2 + f(p) + f^*(k) + 2f(p)f^*(k)\right] \\
&= |A|^2\pi\delta^d(k-p)\left[2 + \frac{mV_0}{ip-mV_0} - \frac{mV_0}{ik+mV_0} - 2\frac{mV_0}{ip-mV_0}\frac{mV_0}{ik+mV_0}\right] \\
&= \frac{|A|^2\pi\delta^d(k-p)}{(ip-mV_0)(ik+mV_0)}\big[2(ip-mV_0)(ik+mV_0) + mV_0(ik+mV_0) \\
&\quad -mV_0(ip-mV_0) - 2(mV_0)^2\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{|A|^2\pi\delta^d(k-p)}{(ip-mV_0)(ik+mV_0)}\left[2(ip-mV_0)(ik+mV_0)+mV_0(ik-ip)\right] \\
&= |A|^2\pi\delta^d(k-p)\left[2+\frac{mV_0(ik-ip)}{(ip-mV_0)(ik+mV_0)}\right] \\
\Rightarrow |A|^2 &= \frac{1}{\pi\left[2+\frac{mV_0(ik-ip)}{(ip-mV_0)(ik+mV_0)}\right]}.
\end{aligned}
$$

Considering the presence of the $\delta^d(k-p)$, we may set $k=p$ and so

$$
|A| = \frac{1}{\sqrt{2\pi}}.
$$

E.4 Normalisation in momentum space

In terms of the Green's function (E.4), the wavefunction may be written

$$
\psi(x) = e^{i\sqrt{2mE}x} + \int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{e^{ipx}}{E-p^2/2m}V_0\psi(0),
$$

which we may write as

$$
\begin{aligned}
\psi(x) &= e^{ik_0x} + \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{k_0^2/2m-k^2/2m}V_0\psi(0) \qquad \text{(E.12)} \\
&\equiv \psi_1(x)+\psi_2(x).
\end{aligned}
$$

We make use of the convention

$$
\begin{aligned}
\psi(x) &= \int\frac{dk}{2\pi}e^{ikx}\psi_k \\
\psi_k &= \int dx\psi(x)e^{-ikx}.
\end{aligned}
$$

Hence

$$
\psi(0) = \int\frac{dk}{2\pi}\psi_k
$$

and

$$
\begin{aligned}
\psi_{1k} &= \int dx e^{ik_0x}e^{-ikx} \\
&= 2\pi\delta(k-k_0).
\end{aligned}
$$

The normalisation would be $\int \psi_k = 2\pi = A$ if it were free; in order to normalize the whole of $\psi(x)$ we write

$$\begin{aligned} \psi_{1k} &= A\delta(k-k_0) \\ \Rightarrow \psi_1(x) &= \int \frac{dk}{2\pi} A\delta(k-k_0)e^{ikx} \end{aligned}$$

and we will need to solve for A. Thus equation (E.12) becomes

$$\begin{aligned} \int \frac{dk}{2\pi} e^{ikx}\psi_k &= \psi(x) = \int \frac{dk}{2\pi} A\delta(k-k_0)e^{ikx} + \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{e^{ikx}}{E-k^2/2m}V_0\psi(0) \\ \Rightarrow \psi_k &= A\delta(k-k_0) + \frac{V_0}{k_0^2/2m - k^2/2m}\int \frac{dk'}{2\pi}\psi_{k'}. \end{aligned} \tag{E.13}$$

Integrating now on both sides with respect to k we get

$$\begin{aligned} \int \frac{dk}{2\pi}\psi_k &= A/2\pi + \int \frac{dk}{2\pi}\frac{V_0}{k_0^2/2m - k^2/2m}\int \frac{dk'}{2\pi}\psi_{k'} \\ \Rightarrow \int \frac{dk'}{2\pi}\psi_{k'} &= \frac{A/2\pi}{1 - \frac{1}{2\pi}\int dk \frac{V_0}{k_0^2/2m - k^2/2m}} \\ &= \frac{A/2\pi}{1 - \frac{mV_0}{ik_0}} \end{aligned}$$

and equation (E.13) becomes

$$\psi_k = A\delta(k-k_0) + \frac{V_0}{k_0^2/2m - k^2/2m}\frac{A/2\pi}{1-\frac{mV_0}{ik_0}}. \tag{E.14}$$

In the limit as $V_0 \to \infty$ we obtain a finite form which is the critical form of scattering states, given by

$$\begin{aligned} \psi_k &= A\delta(k-k_0) + \frac{V_0}{k_0^2/2m - k^2/2m}\frac{A/2\pi}{1-\frac{mV_0}{ik_0}} \\ &= A\delta(k-k_0) - \frac{A}{2\pi}\frac{2ik_0}{k_0^2 - k^2} \end{aligned} \tag{E.15}$$

For expression (E.15) we therefore obtain

$$\begin{aligned} \delta(k_0 - k_0') &= \int dk \psi_{k,k_0}\psi_{k,k_0'} \\ &= \int dk \left(A\delta(k-k_0) - \frac{A}{2\pi}\frac{2ik_0}{k_0^2-k^2}\right)\left(A\delta(k-k_0') - \frac{A}{2\pi}\frac{2ik_0'}{k_0'^2-k^2}\right) \end{aligned}$$

whose solution has 4 parts:

$$\begin{aligned}
a &= A^2\delta(k_0 - k_0') \\
b &= -\frac{A^2}{2\pi}\frac{2ik_0}{k_0^2 - k_0'^2} \\
c &= -\frac{A^2}{2\pi}\frac{2ik_0'}{k_0'^2 - k_0^2} \\
d &= \left(\frac{A}{2\pi}\right)^2 \int dk \left(\frac{2ik_0}{k_0^2 - k^2}\right)\left(\frac{2ik_0'}{k_0'^2 - k^2}\right).
\end{aligned}$$

Adding the expression in b and c gives

$$\begin{aligned}
b + c &= -\frac{A^2}{2\pi}\frac{2ik_0}{k_0^2 - k_0'^2} - \frac{A^2}{2\pi}\frac{2ik_0'}{k_0'^2 - k_0^2} \\
&= \frac{2iA^2}{2\pi}\frac{k_0 - k_0'}{(k_0' - k_0)(k_0' + k_0)} \\
&= -\frac{2iA^2}{2\pi(k_0' + k_0)}.
\end{aligned}$$

and d is given by

$$\begin{aligned}
d &= \left(\frac{A}{2\pi}\right)^2 (2ik_0)(2ik_0') \int dk \frac{1}{(k^2 - k_0'^2)(k^2 - k_0^2)} \\
&= \left(\frac{A}{2\pi}\right)^2 (2ik_0)(2ik_0') \int dk \frac{1}{(k - k_0')(k + k_0')(k - k_0)(k + k_0) + i\epsilon} \\
&= \left(\frac{A}{2\pi}\right)^2 (2ik_0)(2ik_0') \int dk \frac{1}{(k - k_0' - i\epsilon)(k + k_0' + i\epsilon)(k - k_0 - i\epsilon)(k + k_0 + i\epsilon)}.
\end{aligned}$$

We include poles on the positive x-axis and close above and thus obtain

$$\begin{aligned}
I &= \left(\frac{A}{2\pi}\right)^2 (2ik_0)(2ik_0')(2\pi i)\left(\frac{1}{(k_0 - k_0')(k_0 + k_0')(2k_0)} + \frac{1}{(2k_0')(k_0' - k_0)(k_0' + k_0)}\right) \\
&= -\frac{A^2 2k_0k_0' i}{2\pi}\left(\frac{k_0' - k_0}{(k_0 - k_0')(k_0 + k_0')(k_0k_0')}\right) \\
&= \frac{A^2 2k_0k_0' i}{2\pi}\left(\frac{1}{(k_0 + k_0')(k_0k_0')}\right) \\
&= \frac{2A^2 i}{2\pi}\frac{1}{(k_0 + k_0')}.
\end{aligned}$$

This clearly cancels with our expression for $b + c$ and we obtain finally that

$$\delta(k_0 - k_0') = \int dk \psi_{k,k_0} \psi_{k,k_0'} = A^2 \delta(k_0 - k_0')$$

and therefore $|A| = 1$ so our expression for ψ_k is already correctly normalized. Doing a similar exercise for expression (E.14) we find

$$\begin{aligned}
\delta(k_0 - k_0') &= \int dk \psi_{k,k_0} \psi_{k,k_0'} \\
&= \int dk \left(A\delta(k - k_0) + \frac{V_0}{k_0^2/2m - k^2/2m} \frac{A/2\pi}{1 - \frac{mV_0}{ik_0}} \right) \\
&\quad \left(A\delta(k - k_0') + \frac{V_0}{k_0'^2/2m - k^2/2m} \frac{A/2\pi}{1 - \frac{mV_0}{ik_0'}} \right),
\end{aligned}$$

whose solution has 4 parts:

$$\begin{aligned}
a &= A^2 \delta(k_0 - k_0') \\
b &= \frac{A^2}{2\pi} \frac{V_0}{k_0'^2/2m - k_0^2/2m} \frac{1}{1 - \frac{mV_0}{ik_0'}} = \frac{A^2}{2\pi} \frac{2mV_0}{k_0'^2 - k_0^2} \frac{1}{1 - \frac{mV_0}{ik_0'}} \\
c &= \frac{A^2}{2\pi} \frac{V_0}{k_0^2/2m - k_0'^2/2m} \frac{1}{1 - \frac{mV_0}{ik_0}} = \frac{A^2}{2\pi} \frac{2mV_0}{k_0^2 - k_0'^2} \frac{1}{1 - \frac{mV_0}{ik_0}} \\
d &= \left(\frac{A}{2\pi}\right)^2 \frac{V_0^2}{(1 - \frac{mV_0}{ik_0})(1 - \frac{mV_0}{ik_0'})} \int dk \frac{1}{k_0'^2/2m - k^2/2m} \frac{1}{k_0^2/2m - k^2/2m}
\end{aligned}$$

Adding the expression in b and c gives

$$\begin{aligned}
b + c &= \frac{A^2}{2\pi} \frac{2mV_0}{k_0^2 - k_0'^2} \left[\frac{1}{1 - \frac{mV_0}{ik_0}} - \frac{1}{1 - \frac{mV_0}{ik_0'}} \right] \\
&= \frac{A^2}{2\pi} \frac{2mV_0}{k_0^2 - k_0'^2} \left[\frac{\frac{mV_0}{ik_0} - \frac{mV_0}{ik_0'}}{\left(1 - \frac{mV_0}{ik_0}\right)\left(1 - \frac{mV_0}{ik_0'}\right)} \right] \\
&= -\frac{A^2 i}{2\pi} \frac{2(mV_0)^2}{k_0^2 - k_0'^2} \left[\frac{\frac{k_0' - k_0}{k_0' k_0}}{\left(1 - \frac{mV_0}{ik_0}\right)\left(1 - \frac{mV_0}{ik_0'}\right)} \right] \\
&= \frac{A^2 i}{2\pi} \frac{2(mV_0)^2}{k_0 k_0'(k_0 + k_0')\left(1 - \frac{mV_0}{ik_0}\right)\left(1 - \frac{mV_0}{ik_0'}\right)}
\end{aligned}$$

The integral in d is given by

$$\begin{aligned} I &= \int dk \frac{(2m)^2}{(k^2 - k_0'^2)(k^2 - k_0^2)} \\ &= \int dk \frac{(2m)^2}{(k - k_0')(k + k_0')(k - k_0)(k + k_0) + i\epsilon} \\ &= \int dk \frac{(2m)^2}{(k - k_0' - i\epsilon)(k + k_0' + i\epsilon)(k - k_0 - i\epsilon)(k + k_0 + i\epsilon)}. \end{aligned}$$

We include poles on the positive x-axis and close above and thus obtain

$$\begin{aligned} I &= (2m)^2 \frac{2\pi i}{2} \left(\frac{1}{(k_0 - k_0')(k_0 + k_0')(2k_0)} + \frac{1}{(2k_0')(k_0' - k_0)(k_0' + k_0)} \right) \\ &= (2m)^2 \frac{2\pi i}{2} \left(\frac{k_0' - k_0}{(k_0 - k_0')(k_0 + k_0')(k_0 k_0')} \right) \\ &= -(2m)^2 \frac{2\pi i}{2} \left(\frac{1}{(k_0 + k_0')(k_0 k_0')} \right). \end{aligned}$$

Therefore

$$\begin{aligned} d &= \left(\frac{A}{2\pi} \right)^2 \frac{V_0^2}{(1 - \frac{mV_0}{ik_0})(1 - \frac{mV_0}{ik_0'})} I \\ &= \left(\frac{A}{2\pi} \right)^2 \frac{V_0^2}{(1 - \frac{mV_0}{ik_0})(1 - \frac{mV_0}{ik_0'})} \left[-(2m)^2 \frac{2\pi i}{2} \left(\frac{1}{(k_0 + k_0')(k_0 k_0')} \right) \right] \\ &= -\frac{A^2 i}{2\pi} \frac{2\,(mV_0)^2}{(1 - \frac{mV_0}{ik_0})(1 - \frac{mV_0}{ik_0'})(k_0 + k_0')(k_0 k_0')}. \end{aligned} \tag{E.16}$$

This clearly cancels with our expression for $b + c$ and we obtain finally that

$$\delta(k_0 - k_0') = \int dk \psi_{k,k_0} \psi_{k,k_0'} = A^2 \delta(k_0 - k_0'),$$

and therefore, once again, $|A| = 1$ so our expression for ψ_k is already correctly normalised.

Appendix F

Previous example of the need for a redefinition of fields in matrix models.

The construction of an explicit map between O(N) Vector models and higher spin theories was a major theme of this thesis. This involved a field redefinition involving the Jacobian for a change of variables from bilocals to AdS coordinates, which is given by

$$\left|\frac{\partial k_{AdS\times S^1}}{\partial k_{bilocal}}\right| = \frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}.$$

The quadratic Hamiltonian at the conformal fixed point was found to be

$$\begin{aligned} H_2 \quad &= \quad \int d^2k_1 \int d^2k_2 \left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right) \frac{1}{2} \left[\left(\pi_{\vec{k}_1\vec{k}_2}\pi_{-\vec{k}_2-\vec{k}_1}\right)\right. \\ &\quad + \left.\frac{\eta_{\vec{k}_1\vec{k}_2}}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)} \left(|\vec{k}_1|^2 + |\vec{k}_2|^2\right) \frac{\eta_{-\vec{k}_2-\vec{k}_1}}{\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}\right)}\right], \end{aligned}$$

so that

$$\dot{\eta}_{\vec{k}_1\vec{k}_2} \quad = \quad \left(\frac{1}{|\vec{k}|} + \frac{1}{|\vec{k}_2|}\right)\pi_{-\vec{k}_2-\vec{k}_2} = \left|\frac{\partial k_{AdS\times S^1}}{\partial k_{bilocal}}\right| \pi_{-\vec{k}_2-\vec{k}_2}$$

$$\ddot{\eta}_{\vec{k}_1\vec{k}_2} = -\left(|\vec{k}_1| + |\vec{k}_2|\right)^2 \eta_{\vec{k}_1\vec{k}_2}.$$

The canonical commutation relations required are then given by

$$\frac{\left[\pi_{\vec{k}_1\vec{k}_2}, \eta_{\vec{k}'_1\vec{k}'_2}\right]}{\left|\frac{\partial k_{AdS\times S^1}}{\partial k_{bilocal}}\right|} = -i\delta(\vec{k}_{AdS\times S^1} - \vec{k}'_{AdS\times S^1}).$$

The above is then consistent with the redefinition

$$\begin{aligned} \mathcal{H}\left(\vec{k}.k^z, \theta\right) &\equiv \left.\frac{\eta_{\vec{k}_1\vec{k}_2}}{\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{k}_2|}}\right|_{\vec{k}_1,\vec{k}_2\left(\vec{k},k^z,\theta\right)} \\ \Pi_{\mathcal{H}}\left(\vec{k}.k^z, \theta\right) &= \left.\pi_{\vec{k}_1\vec{k}_2}\right|_{\vec{k}_1,\vec{k}_2\left(\vec{k},k^z,\theta\right)}, \end{aligned}$$

with Hamiltonian now rewritten in AdS variables as

$$\frac{1}{2}\int d\vec{k}_{AdS\times S^1}\ \left[\left(\Pi_{\mathcal{H}}(\vec{k},k^z,\theta)\Pi_{\mathcal{H}}(-\vec{k},-k^z,-\theta)\right) + \left(P^0\right)^2 \mathcal{H}(\vec{k},k^z,\theta)\mathcal{H}(-\vec{k},-k^z,-\theta)\right].$$

It is possible for a similar procedure to be applied to matrix models, whose AdS dual is String theory. The Collective Hamiltonian is given by

$$H = \frac{1}{2N^2}\int dx\, \partial_x \Pi \psi \partial_x \Pi + N^2 \int dx \left(\frac{\pi^2}{6}\psi^3 + \psi\left(v - \psi\right)\right).$$

We now introduce fluctuations for the large N configuration, given by

$$\begin{aligned} \psi(x) &= \phi_0 + \frac{1}{N\sqrt{\pi}}\partial_x \eta \\ \partial_x \Pi &= -N\sqrt{\pi}P(x), \end{aligned}$$

and we therefore obtain the quadratic Hamiltonian

$$H_2 = \frac{\pi}{2}\int dx\, P(x)\phi_0(x)P(x) + \frac{\pi}{2}\int dx\left(\partial_x\eta\right)^2\phi_0(x).$$

Here we have that

$$\dot{\eta} = \frac{\delta H_2}{\delta P(x)} = \phi_0(x)P(x).$$

Now consider a change of variables from x (old) to q (new) with associated Jacobian $\left|\frac{\partial q}{\partial x}\right|$. The canonical commutation relations are given by

$$\frac{[P(x), \eta(x')]}{\left|\frac{\partial q}{\partial x}\right|} = -i\delta(q - q').$$

Defining the variable q to be the 'time of flight' variable (which is the time taken for a particle to go from the origin to a point x)[144]:

$$q = \frac{1}{\pi}\int_x \frac{dx}{\phi_0(x)}; \qquad \frac{dx}{dq} = \pi\phi_0.$$

The Hamiltonian may then be written as

$$\begin{aligned} H_2 &= \frac{\pi}{2}\int dx\, P(x)\phi_0(x)P(x) + \frac{\pi}{2}\int dx\, (\partial_x\eta)^2\,\phi_0(x) \\ &= \frac{\pi}{2}\int dx \left|\frac{\partial q}{\partial x}\right|^2 \frac{P(x)}{\frac{\partial q}{\partial x}}\phi_0(x)\frac{P(x)}{\frac{\partial q}{\partial x}} + \frac{\pi}{2}\int dx \left|\frac{\partial q}{\partial x}\right| \left(\frac{\partial q}{\partial x}\partial_q\eta\right)^2 \phi_0(x)\left(\frac{\partial x}{\partial q}\right) \\ &= \frac{1}{2}\int dq\, \frac{P(x)}{\frac{\partial q}{\partial x}}\frac{P(x)}{\frac{\partial q}{\partial x}} + \frac{1}{2}\int dq\, (\partial_q\eta)^2. \end{aligned}$$

These are satisfied by setting

$$\Pi(q) = \pi\phi_0 P(x); \qquad \eta(q) = \eta(x).$$

This differs from the vector case worked on in this thesis where the field $\eta_{\vec{k}_1\vec{k}_2}$ had the redefinition, but the canonical momentum $\pi_{\vec{k}_1\vec{k}_2}$ remained the same.

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