NEW APPROACHES TO PRIMARY MATHEMATICS IN THE U.S.A.

BY
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The pressures for changes in the teaching of maths in America came downwards from the universities to the high schools, and the changes and advances in mathematics began at the high school level. It was only later that the Americans settled down to look at primary mathematics after having concentrated on high school maths. This was the case with the School Mathematics Study Group, headed by Professor Beagle. There are, however, a couple of American projects which tended to start the other way round, and these are the ones that I shall mention briefly to you. There are several projects that are worth examining; they are carefully described in books which are on display in this hall.

The S.M.S.G. project for the primary grades emanated from Stanford University in California. It started off with three series of texts for what the Americans call Grades 4, 5 and 6, which in age terms are 9, 10 and 11. These texts have been available for several years and 100,000 children are using them. The results of this work will lead to modifications of these texts. Where else in the world will you find a new project throwing up a catchment area of 100,000 children of all kinds? The kind of material dealt with is really the basic arithmetic seen in a new way—the result of the applications of axioms and concepts of all kinds, with a larger amount of intuitive geometry than is normally found at this level. The Americans feel that intuitive geometry is the thing—no predisposition towards Euclid, simply intuitive thinking geometrically. In 1963, the S.M.S.G. decided that even the 4th grade level was a bit late, so they decided to produce material for what they call kindergartens through to grade 3. The kindergarten is the 5 to 6-year-old level in the United States and most children there now go to school at 5, not 6. This material was prepared in 1963 and tried in the autumn, but I have not yet seen the printed text and I do not know anything about it. Side by side with this work, in 1962, the S.M.S.G. project tested 50,000 9-year-olds with a battery of tests, and they propose to apply the battery and additional tests year by year from 1962 onwards for 5-year-olds; in other words to maintain a continuous view of the results of the application of their own material to children. At the end of these five years they are perfectly prepared and can afford to throw away everything that they have done. Of course, they are not expecting to throw *everything* away, but we can expect in 1967 a sound revision of their work.

The S.M.S.G. material is the only material that introduces sets to children as a sort of language of sets and manipulation of sets; they have separate topic booklets and one is on sets; one is on intuitive geometry, another is on number lines and so on. It is really a sort of topic approach, the idea being that what happens in the primary school is laying a good mathematical basis for what will happen in the secondary school.

The interesting thing about the S.M.S.G. project is that it is mathematically orientated and seems to take no account whatsoever of the nature of childhood and the ways in which children learn. The American projects were started by mathematicians, and mathematicians are not necessarily psychologists. Some of the problems arising from the S.M.S.G. material have been simple problems arising from the fact that the mathematicians were saying what should be learned, without their realising that the learning had to be done by children.

Another project, which is a minor one, is the work by Suppes, also a Californian. It is a sound set of books and I commend them to you. The author starts off with sets, numbers and some work in logic. Here then is a number of new mathematical lines being brought down to the younger grades—logic and sets for young children.

The two projects that I want to discuss next are those known as the Illinois Project and the Madison Project.

The Director of the University of Illinois Arithmetic Project is David Page. He is a superb teacher, an amazing manipulator of children, and this is the tendency of the Page project—it is a teacher-centred project with material being taught by the teacher with the child participating through discussion, listening and exercises. In all of the American projects, no account whatsoever is taken of children working with real materials and real situations. The reason for this is that none of the American projects have ever taken any account of the work of Piaget. Until recently they did not know about the developmental stages, the fact that at the stage of concrete operations, you need real experience in order to get hold of clear ideas.

None of these projects uses materials. Last year I taught in the Madison Project with Bob Davies; he taught for three days and I taught for three others. Bob Davies could not understand why the children were not getting hold of certain ideas, so I went away into a corner and got somebody to make some simple materials. When Davies came back three days later, the children could understand, and he was quite fascinated that all his excellent teaching had not done what a few bits of wood and a few simple structured experiences had done for children when they had been allowed to think for themselves.

So the American projects are very much teacher-directed. The S.M.S.G. Project is a sort of work-book, but there is a thick teacher's book which tells you what you say to the child next. The Illinois Project has a thick work-book which tells you what you are to say—it writes it out in detail. The Madison Project is a teaching project, but both David Page and Bob Davies are beginning to wonder whether or not there may not be a real need for parallel experience and perhaps in fact alternative experience.

One thing that David Page has discovered is that the idea of spelling everything out in a linear fashion, taking little tiny steps, is ineffective. He discovered that when you first introduce a topic, the best thing to do is "throw the children in at the deep end." If you want somebody to do something, you do not say, "No, you mustn't do this," and "you mustn't do that" and "you mustn't do the other" and "be careful about this" and "watch out for that", and then give them a little tiny nibble. By that time they will have lost interest completely. If however, you start off with a big bang and you throw them into an interesting and complex situation, they will get on with it. He also discovered that you can go fast in the early stages. The small-step concept may work with a linear programme on a Skinnerian type of teaching machine, but this is not what he was talking about. He was concerned with mathematics, with learning ideas, and he said: "Ideas are complex things; let's face them for what they are. Then let's worry about taking them to pieces later." This could really be explained in my terminology by saying, "Let's have an intuitive look at the whole and then start analysing it."

The Illinois Project is meant for children of about 8 or 9. It does not include much geometry. It starts off with an introduction, for example to equations. An equation in the Illinois view is a mathematical sentence containing an equals sign (or an open sentence with an equals sign). I had better explain some of the terminology of the American mathematics. If I make this statement—

$$3 + 5 = 8 \text{ .... T}$$

that is a mathematical sentence called an equation because it has an equals sign in it and it is said to be true. I welcome the idea of true and false in relation to mathematical statements, because I do not understand what people mean when they say right or wrong. Right or wrong has emotional overtones quite different from true or false and the object of writing statements in mathematics is to write true ones, not false ones. This, on the other hand, is false—

$$3 + 4 = 8 \text{ .... F}$$

but this is neither true nor false—

$$\square + 2 = 13 \text{ .... OPEN}$$

It is in the American terminology an *open statement*. It is open because it has a variable represented in the common American box notation. The idea of a box rather than the literal symbol "x" is to me attractive because young children seeing a box immediately feel that there is something that could go into the box. Nobody could have the same feeling about this at all—

$$x + 2 = 13$$

So the idea of a variable in America is by means of a shape, an enclosed space. This, then, is an open statement. The fact that it is open leads the child and us to suppose that it may be either true or false according to what you put in the box. There will be a set of numbers which may be substituted for the box or put into the box which will make an open statement into a true one: this is called the *truth set*. The truth set of the open statement above is 11. If we put that into the box, then our open statement has become a true one. The notion of true or false statements, open statement with solution sets is basic to both the Illinois and the Madison Projects and is a fundamental one to an understanding of any equation.

You can use more than one shape—

$$\square + \square + \square + 2 = 8$$

Here we are not using more than one shape, but using the same shape three times and of course, there are various other ways of writing this. We could write it—

$$(3 \times \square) + 2 = 8$$

The convention is that if you have one shape in any open statement, then the solution set must fit always into the same shape wherever it occurs. Can you decide what the solution set of that open statement is? The solution set is 2. On the other hand, we could have more than one shape. We could say—

$$\square + 2\triangle + 6 = \square + 20$$

Here we have two shapes—you can have any number of shapes. The fact that they are equilateral triangles or squares is unimportant. You can have clouds, if you like, or moons, in fact the Illinois Project does do this. Accepting the convention that if we put 3 into one box, 3 must go into the other box, it is pretty obvious that this particular open statement has a solution set consisting of ordered pairs because there are two variables in it. If we make a tabulation (and this again is typical of the American projects) we start putting things into some order, so that patterns become obvious.

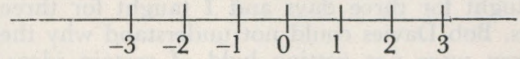
□	△

The next thing that the Illinois Project introduces is the number line, which is the most fascinating way of understanding number systems. For primary grades these will first of all be the natural numbers, and since the natural numbers will not enable you to subtract because the set of natural numbers is not closed in respect to subtraction, (it is closed in respect to addition—you can always find a third number which is the sum of any two natural numbers, but it is not closed in respect of subtraction) you need to extend the set of natural numbers and have some more numbers; so you have the directed numbers, the positive numbers, zero as your origin, your datum line, and the negative numbers, and these form the set of integers. For all practical purposes, you could say that the natural numbers are similar to the positive numbers, and so in mathematics we accept that the natural numbers are isomorphic with the positive numbers. But we still cannot do very much with the integers, so we need a new set of numbers called fractions. The fractions and the integers together form the set of rational numbers, thus:

$$\left.\begin{array}{l}\text{Positive numbers}\\ \text{Negative numbers}\end{array}\right\}\left.\begin{array}{l}\text{Integers}\\ \\ \text{Fractions}\end{array}\right\}\text{Rational numbers}$$

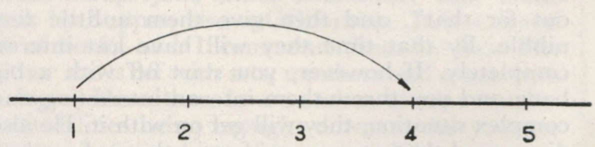
If you want to go on, these do not include numbers like π so you have the irrationals and so on. I do not want to go any further than this because a study of the rational number system is the end of the primary grades according to the American project work.

A study of the systems of numbers is made very clear by use of a number line.



To the right of zero will be positive 1, positive 2 and so on and to the left will be negative 1, negative 2 and so on. We can have a number line with some of the elements of the set of integers upon it. The Illinois project introduces the number line, but not the number line with the negative numbers on it. It simply has a point of origin, usually not marked—in fact it uses a ray rather than a line; it does not mark it with zero, just a starting point and it has congruent segments and numbers them. The idea of a number line goes side by side with the idea of movement on the line. John Flavell was showing you how, conventionally, movement which is approximating to the operation of addition means movement to the right and subtraction is movement to the left.

The Illinois Project introduces a sort of "jumping cricket". The Americans are prone to Disney-like fantasies and they cannot do anything without crickets or rabbits or frogs or strange little men jumping around—you must remember that quite young children do find an appeal in this and some light relief. If we are dealing with the jumping 3 cricket, and we start him off on the 1, one jump will carry him over three spaces and he will arrive at a 4:



So the idea of jumping about on the lines is built up and then instead of having the jumping 3 cricket, David Page uses a new convention that I find very attractive. He says that if you take the sign □, then a jumping 3 cricket statement is this:

$$\square \rightarrow \square + 3$$

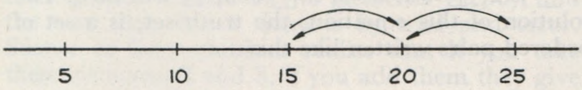
that is, if you take any starting point, you go from that starting point plus 3. And of course, you can do the reverse, which is subtracting. This is a left jumping cricket—

$$\square \rightarrow \square - 3$$

This is a right jumping cricket—

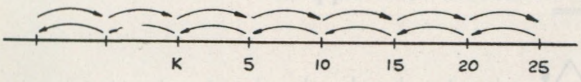
$$\square \rightarrow \square + 3$$

In order to introduce the negative numbers, he would write something like this: "Now, we'll call this jump "A"←— and we'll call this jump "B"—→ Suppose you start at 25 on the line and you take two of these jumps A and two of these jumps B. Where will you land up? Well, let's do it. Let's start at 25 and we'll take two of jumps A: 20, 15, followed by 2 of B, 20, 25."



So it is built up in the minds of the children by very skilful teaching that if you do any number of this kind of jump followed by an equal number of that kind of jump, you land back where you started.

Then we start at, say, 25 and take 7 of these jumps.



Unfortunately, there are not enough segments on our line, but as the line has always been introduced with part of a segment to the left of the starting point, children realise that if you are taking 5 of these jumps, you have reached the point "K". Now take 5 back and you reach 25 again. Now let us take 7 of these jumps. It is not very much of an intellectual leap to realise what it is going to be—must obviously be, because it is axiomatic—that if you take any number of these jumps to the left followed by the same number of these jumps to the right, you are back where you started. You must have gone beyond "K". And so the idea of negative numbers is brought in.

At this point we have to start naming the negative numbers. David Page has a very interesting convention for negative number. Can you tell me, if I start at 25 and take 7 of these jumps to the left, where I get to? Negative 10. But he does not call it negative 10, he calls it "b10"—10 below zero, because he talks to children about the thermometer and the fact that if it gets down to zero and it is still getting colder, obviously it has got to go down a bit further. It does not stop being cold just because there are not enough marks on the scale; this sort of thing makes sense. If you then write down a convention like this: 5 below and you subtract 5, where will you be? You will be 10 below. By means of this simple, skilful teaching, you begin to build up this notion.

I should like to show you how I saw David Page deal with the difficult notion of the multiplication of two negative numbers leading to a positive number. This is always a stumbling block in the minds of most mathematics teachers. Before he goes any further with this kind of thing, Page introduces the notion of laws governing the behaviour of numbers. Notice that if you have invented a law, the numbers have to follow the rules. He then introduces the conventional laws—the algebra of arithmetic if you like—the laws of closure for addition and multiplication, the commutative law for addition and multiplication, the associative law for addition and multiplication, the identity laws for addition and multiplication and the distributive law. The distributive law distributes multiplication over addition. a x (b + c) is the same number as ab + ac. This is the distributive law. It is absolutely essential in the Illinois Project that this idea of laws is built up quite soon. You notice the progression: the idea of an equation, the idea of zero from a number line, the idea of a variable, the idea of more than one variable, the idea of movement on the number line. Then come the axioms. After that the work becomes axiomatic. Suppose you have b2 × 5, obviously by using the commutative law, we can say that that is equal to

5 × b2 = b10. Now consider this—

$$b7\ (5 + b5)$$

Using the distributive law, I can say this—

$$\begin{aligned} b7\ (5 + b5) &= b7.5 + b7.b5 \\ &= b35 + b7.b5 \end{aligned}$$

I have used the distributive law to distribute this. From this we know that b35 is negative 35. I am not sure what b7.b5 is because I have never met any multiplication of two negative numbers before. In order to find out what it is, let us go back. What do we know from our number line about this? We know that 5 + b5 must be zero. What is b7 × zero? Zero. So we can put this side of the equation as being equivalent to zero. I can write this in another way—

$$b35 + \square = 0$$

I can say that b35 plus something, which at the moment is unknown to me, but which corresponds to b7.b5, must equal zero and what is the only solution, the only member of the solution set that will satisfy that? 35. Very elegant. An axiomatic approach to the multiplication of negative numbers with great elegance and power and very easily understood.

That is just an example of the Illinois Project, and it proceeds in this way—skilful teaching, sound mathematics, the use of logic, the use of axioms—and later gets on to some very interesting work. David Page does some very interesting manoeuvres on lattices and things of this kind.

The Madison Project is the only project in the United States that has any regard for the nature of learning, and because it is concerned with children as well as with maths, I must admit that I have a very soft spot for it. I think that the Madison Project is the most promising project for young children at present available in the United States, but I must stress that this is a personal point of view. The Madison Project starts at about the same grade level, about 9, and at the moment goes on to about 12, in other words, it is the upper primary classes here.

The Madison Project work has the same notation—it uses the idea of an open statement, the idea of a truth set, but it also introduces quite early on another kind of open statement which we would have called an inequality. We would, for example, say that ($3 < \square < 7$) is an inequality, but in the language of the Madison Project, this is just another kind of open statement. But notice the difference—the truth set for this open statement has more than one element if you are using the natural numbers only. The truth set for this open statement is (4, 5, 6). So early on we have this introduction of inequalities, and this is fundamental to work in mathematics. *It is no good dealing with equalities unless you deal with inequalities.* The idea of the box is again used, only this time it is used as a place-holder. It is holding a place for a number of other elements—we have not yet decided what they are. The Illinois Project recently has begun to call these things pro-numerals in the way that one thinks of a *pro*noun—something standing in place of a noun—the noun has not yet been named. True, false and open is again there and the idea of tabulation is introduced very early on. What the Madison Project does is to introduce, not only the tabulation of a set of ordered pairs, but also the graphing of them on a set of Cartesian co-ordinates. This I find extremely powerful because the use of Cartesian co-ordinates and the use of graphs to show functions is an important idea. Take for example—

$$\triangle = \square + 3$$

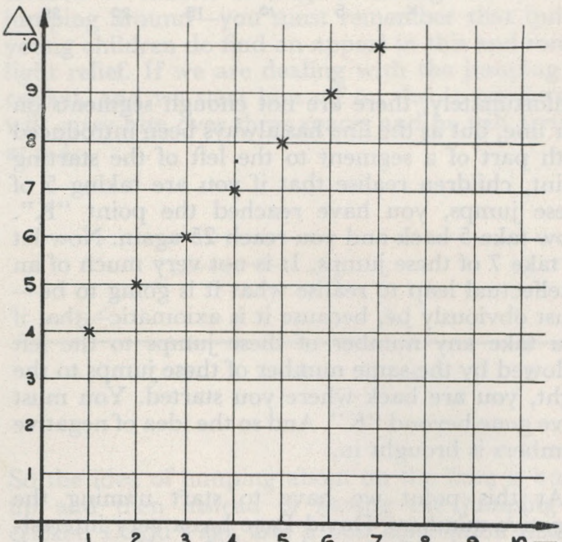
Notice the very simple, very elegant way in which it is written.

$\square$	$\triangle$
1	4
2	5
3	6

Let us consider that $\square$ is 1. What will $\triangle$ be? 4. If $\square$ is 2, $\triangle$ will be 5, if $\square$ is 3, $\triangle$ will be 6. The solution of this equation, the truth set, is a set of ordered pairs written like this—

(1, 4) (2, 5) ....

Do you notice something about it? It is not a finite set. There is an infinite set of ordered pairs. There is an infinite number of elements in this set of ordered pairs. Here you see that elegant, simple ideas (which many people think are complex) can be drawn well if you graph this on Cartesian co-ordinates. Let us graph these ordered pairs and see what happens:



What of course you are really doing is leading to the graphing of a function. The point is you are already building up, laying down a foundation for the fascinating and important work later on in an elegant, interesting and simple way. Why have we not done this before? Why have we not graphed these truth sets from the word "go"? With all good ideas, you kick yourself to death and say "why didn't I think of that?" It takes a first-class mathematician to do so.

Another thing that the Madison Project does is to introduce the idea of trial and error as being eminently respectable. For example, if I write—

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

what is the truth set? It is really a very great surprise to children to find that the truth set for this equation consists of two elements—(2, 3), and the way that this is done is by trial and error. Sooner or later children realise that if you take these elements, 2 and 3, if you add them they give you the numerical co-efficient of this variable, $\square$; if you multiply them, they give you this final term, 6. Trial and error is entirely respectable in the Madison Project approach.

The Madison Project introduces a number o topics quite early. Because the number systems depend upon the use of the positive and directive numbers quite soon, positive and directive numbers are introduced very early to 8-year-olds.

Some of the other work that the Madison Project does is to introduce matrices quite early and the matrix algebra is quite an important algebra: it is a fascinating one because it does not follow the same set of laws that our ordinary algebra does.

I hope that this has given you an idea of the elegance of the American projects, the good ones. I hope it has made you realise that a lot of thought has gone into them, but I hope also that it has made you realise that they are not all child-centred.