Eigenvalue Dynamics in the Large N Limit of Multi Matrix Systems

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

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Declaration

I hereby declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Abstract

The Collective Field method is used to investigate various matrix valued problems. First the application to the single hermitian matrix is reviewed after which the method is generalised to multiple hermitian matrices. Two different approaches are considered, both of which return the expected Jacobian. The second approach is then used to explore a system of complex matrices to determine if a Jacobian including both radial and angular dynamics can be revealed. This turns out not to be the case but the analysis is informative, and some avenues for further research are suggested.

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Chapter 1

Introduction

The twentieth century was an eventful and tumultuous time for theoretical physics. The discoveries of General Relativity and Quantum Mechanics led to a multitude of new avenues of research, but also to significant hurdles that remain to this day. One such hurdle is finding a consistent theory that includes all four fundamental forces of nature. Presently String Theory is the most popular candidate for such a theory [19], but it was not always this way. String Theory first emerged as an attempt to describe strong interactions, but it was unsuccessful due to the fact that it predicted a 2-spin particle and a tachyon.

With the discovery of Quantum Chromodynamics as a more viable tool to describe strong interactions, String Theory was put aside until the 1980s when supersymmetry was proposed. One of the features that prevented String Theory from being a theory of strong interactions, the spin 2 particle, instead made it a candidate for a theory of Quantum Gravity, as that is exactly the spin that a graviton is predicted to have.

Although String Theory and Quantum Chromodynamics both hold significant promise, one for developing a consistent unified theory and the other as a theory of strong interactions, they both pose great difficulty for calculations in non-perturbative regimes. It is only in the last two decades that a way around this problem has begun to reveal itself in the Anti-de Sitter/Conformal Field Theory (AdS/CFT) Correspondence, also known as the Gauge/Gravity Duality. The first clue to the existence of the correspondence came in 1974 when Gerard 't Hooft published a paper investigating a Gauge Theory of colour group $U(N)$ [1]. In it he discovered a systematic $\frac{1}{N}$ expansion indexed by the Euler characteristic of the surface on which the Feynman diagrams could be drawn. Additionally he found that when one takes the limit as N tends toward infinity, only Feynman diagrams that can be drawn on a plane or a sphere contribute to the final calculation. This was the first hint that the large N limit of gauge theories might contain strings.

't Hooft's realisation eventually lead to the formulation of the AdS/CFT correspondence by Juan Maldacena in 1997 [4]. Maldacena showed that the large N limit of certain conformal field theories included a sector equivalent to supergravity on various compact manifolds, such as spheres and Anti-de Sitter spacetimes. In particular he showed that the large N limit of $\mathcal{N} = 4$ Super Yang-Mills had a description which contained strings. This was expanded upon by Gubser, Klebanov and Polyakov in 1998 [5] from the other direction, demonstrating that some Green's functions of $\mathcal{N} = 4$ SYM could be found through the boundary of String Theory in an AdS space.

Since the large N limit plays such a significant part in the AdS/CFT correspondence it is important to be able to find this limit, which is not always a straightforward enterprise. One method that shows promise in this regard is the Collective Field method, which was developed in 1980 by A. Jevicki and B. Sakita [6]. The premise of the method is that one reformulates a given gauge theory problem in terms of a (usually overcomplete) set of commuting operators invariant under the original symmetries of the system. Doing so brings the terms of leading order in N to the forefront, and for this reason it is an attractive proposition for investigating problems pertaining to the AdS/CFT correspondence, as this large N background becomes the semi-classical approximation of the effective invariant theory.

The aim of this research is to investigate matrix valued quantum field systems and their eigenvalue dynamics. Matrix models are of interest because many important theories are defined in terms of matrix models. This includes QCD, $\mathcal{N} = 4$ SYM, String Theory in 2D [7] (one of the only non-trivial string theories with an exact solution), and even a possible definition of M-Theory [20]. This paper investigates matrix models in physical contexts which depend only on the eigenvalues of the system. The eigenvalue dynamics are of significance because in the context of the AdS/CFT correspondence it is expected that the AdS geometry emerges from the eigenvalues [9].

This dissertation is structured as follows: in chapter 2 we review the AdS/CFT correspondence, which constitutes the majority of the background information to which this research pertains. In chapter 3 we first recap the dynamics of a system based on a single hermitian matrix, and then give the derivation of the collective field method. Next we review the application of the collective field method to the single hermitian matrix problem, which is the one that Jevicki and Sakita solved in their original paper [6]. In chapter 4 we apply the collective field method to a system based on several hermitian matrices using two different approaches. Finally in chapter 5 we investigate a set of complex matrices, reviewing the efforts in [13] and [14], and then attempt to find a Jacobian that encapsulates both the radial and angular dynamics of the system.

Chapter 2

AdS/CFT Correspondence

2.1 String Theory

String Theory is a broad and complex topic with multiple incarnations, but the essence of it revolves around the idea that the fundamental components of nature are “strings”, one dimensional objects that vibrate and propagate through space-time [19]. These strings can either be “open”, meaning that their ends propagate independently of each other, and the string resembles a line segment, or “closed”, meaning that the ends of the string are attached, and the string resembles a loop. Strings are quantum mechanical in nature, each bearing properties such as mass and charge, and particles are manifest through the vibrational modes of these strings.

String Theory was originally postulated as a theory of particle physics but it was later realised that it also worked effectively as a theory of quantum gravity, since the vibrational modes of closed strings align with the properties predicted for gravitons. This was the first hint that something connecting gauge theories and gravity theories might exist.

Although String Theory is currently one of the foremost candidates for the unification of Quantum Mechanics and General Relativity, it is often subject to several criticisms. Firstly, String Theory is not well defined in the non-perturbative regime, which means that strong-coupling problems are usually difficult or impossible to solve. Secondly, String Theories generally must be formulated in higher dimensions in order to be consistent, for example Bosonic String Theory which requires twenty six dimensions or Superstring Theory which requires ten. This clearly poses a problem given that the universe appears to be four dimensional. Thirdly, physical predictions made by String Theory occur at very high energies or very small scales and thus it is not currently possible to investigate them experimentally.

Not much progress has been made on getting around the third problem, and most likely won't be for some time, but there are ways in which the other problems with String Theory can be dealt with. The issue of higher dimensions is explained either by the idea of "compactification", which suggests that the extra dimensions are curved around themselves at such small scales as to appear invisible, or by the idea of "brane cosmology", which suggests that the perceived universe is the edge of some higher dimensional space, and that observable particles come from open strings attached to that edge.

The non-perturbative problem has also seen substantial progress in recent decades through the AdS/CFT correspondence. The strongly coupled String Theory problems that were once so intractable now might be solved by their weakly coupled gauge counterparts. This is precisely the reason that AdS/CFT is such a significant and exciting field of research in modern theoretical physics.

2.2 Large N Limit

An important part of the AdS/CFT correspondence is the large N limit. To demonstrate this we follow the argument used by Maldacena in [2] based in $U(N)$ using a hermitian Matrix M as the field. For a more general explanation involving non-hermitian matrices see [15] and for 't Hooft's original paper that gave rise to this discovery see [1]. 't Hooft's original aim was to attempt to construct a field theory of strong interactions in which the quarks form inseparable bound states and thus the formulation is more complicated than necessary to demonstrate the topological argument.

The initial Lagrangian will look something like

$$L = \frac{1}{g^2} Tr[(\partial M)^2 + M^2 + M^3 + ...] = \frac{1}{g^2} Tr[(\partial M)^2 + V(M)].$$

and the resulting Feynman diagrams will consist of double line propagators, so as to keep track of the matrix indices, each contributing a factor of g^2 and vertices each contributing a factor of $\frac{1}{g^2}$. All closed lines in the diagrams will contain sums over the gauge indices and thus will contribute a factor of N. The total contribution for a diagram will then be proportional to $(g^2)^{\#Propagators - \#Vertices} N^{\#ClosedLines}$

If we were to draw these diagrams on a two dimensional surface and then imagine each closed line as the face of a geometric object, then the contribution is

$$(g^2)^{\#Edges - \#Vertices} N^{\#Faces}$$

$$= (g^2)^{\#Edges - \#Vertices} N^{\#Faces - \#Edges + \#Vertices + (\#Edges - \#Vertices)}$$

$$= (g^2 N)^{\#Edges - \#Vertices} N^{\#Faces - \#Edges + \#Vertices}$$

and by Euler's relation this is simply equal to

$$= (g^2 N)^{\#Edges - \#Vertices} N^{2-2h}$$

where h is the genus of the two dimensional surface upon which the diagram can be drawn (0 for a plane or sphere, 1 for a torus etcetera). The total contribution will be $\sum_{h=0}^{\infty} N^{2-2h} f_h(g^2 N)$, where f_h encompasses the detailed evaluation of each specific diagram.

The 't Hooft limit is then

$$\lim_{N\to\infty}[\frac{1}{N^2}\sum_{h=0}^{\infty} N^{2-2h} f_h(g^2 N)]$$

with $\lambda \equiv (g^2 N)$, referred to as the "'t Hooft Coupling", fixed. It is clear that only planar diagrams ($h = 0$) contribute in this limit. Additionally, as λ increases the diagrams become dense on the sphere and resemble the worldsheet of a string theory. Further clues to the existence of a correspondence between this and string theory come from the fact that if one includes matter in the calculation one finds diagrams with boundaries, which can be viewed as open strings with a quark and anti-quark at their ends. Finally, this formulation includes Regge trajectories, i.e. that the particle with the highest spin for a particular mass obeys the relation $\alpha' m^2 = J + const$. This is something that is also characteristic of string behaviour.

2.3 AdS

Having observed all of these clues, Maldacena [4] decided to define the worldsheet theory as the result of summing the planar diagrams. The question then is what string theory is applicable? This is not trivial, as summing the diagrams is not necessarily an easy matter. For a four dimensional gauge theory one would expect the corresponding string theory to be four dimensional as well, but this turns out to be incorrect. Weyl symmetry ($g_{ab} \to \Omega g_{ab}$) generates a non-trivial action quantum mechanically. The way this is dealt with is effectively the same as adding another dimension [2].

We are interested in a space of one higher dimension than the one in which the gauge theory resides. This space should have four dimensional Poincaré symmetry, so the metric should resemble

$$ds^2 = \omega(z)^2(dx_{1+3}^2 + dz^2)$$

where the coefficients of dz and dx are set equal through reparameterisation symmetry. In the case of a scale invariant gauge theory the following symmetry should hold:

$$x \to \lambda x.$$

But string theory is not scale invariant; its scale is determined by the string tension. Therefore the only way that this symmetry can hold is if it is an isometry of the metric. In order for this to be true it must be the case that $z \to \lambda z$ and $\omega(z) = \frac{R}{z}$. This yields

$$ds^2 = R^2 \frac{dx^2 + dz^2}{z^2} \tag{2.1}$$

which is the metric of Anti-de-Sitter space. AdS is the most symmetric spacetime with constant negative curvature and is analogous to a hyperbolic plane, which is the most symmetric negatively curved space in Euclidean geometry.

AdS space can also be formulated as a surface in the higher dimensional space $R^{2,4}$:

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2.$$

This space has two time dimensions but one of them is orthogonal to the surface, which is still Lorentzian. The original metric (2.1) is obtained by substituting $X_{-1} + X_4 = \frac{R}{z}, X_\mu = \frac{Rx_\mu}{z}$ for $\mu = 0, ..., 3$. This representation is referred to as

"Poincaré coordinates" and only covers a portion of the AdS space. The surface can also be parameterised in terms of "global coordinates", which are defined by setting $X_{-1} = R\cosh\rho\cos\tau, X_0 = R\cosh\rho\sin\tau, X_\mu = R\sinh\rho\,\Omega_\mu$ for $\mu = 1, ..., 4$, and $\sum\Omega_\mu^2 = 1$. This parameterisation covers the whole space and results in a metric of the form:

$$ds^2 = R^2\big(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2\big). \tag{2.2}$$

The Penrose diagram will be a solid cylinder whose boundary is $S^3 \times R$ with R corresponding to the time direction. The field theory is defined on the boundary of the cylinder and the string theory is defined inside the cylinder. On the boundary the isometries of AdS resemble the conformal group in four dimensions.

2.4 $\mathcal{N} = 4$ SYM and Type IIB String Theory

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory is a theory that contains four supersymmetries in four dimensions for a total of sixteen supercharges, which is doubled by conformal symmetry for a grand total of thirty two supersymmetries. The theory contains one vector field or gauge boson, six scalars and four fermions. The scalars are related by the group $SO(6) = SU(4)$, which is not commutative with the supercharges and is thus referred to as an "R symmetry". With this included, the supergroup for $\mathcal{N} = 4$ SYM is $PSU(2, 2|4)$ [2].

Additionally this theory has an S-duality in which

$$\tau_{YM} = \frac{\theta}{2\pi} + i\frac{2\pi}{g_{YM}^2}$$

is related to $\frac{-1}{\tau}$. Here g_{YM} is the coupling constant and θ is a parameter of the Lagrangian. From this we have all of the indicators necessary to suggest the correspondnce, but for a more detailed description of $\mathcal{N} = 4$ SYM consult [23].

Now consider type IIB string theory. It is the case that the product of five dimensional Anti-de-Sitter space and a five-sphere constitute a solution to the supergravity equations of motion of this theory [2] in such a way that

$$R = (4\pi g_s N)^{1/4} l_s$$

where R is the radius of the sphere and the AdS space (the R in equation (2.2)), g_s is the string coupling and $2\pi l_s^2$ is the inverse of the string tension. String theory on this geometry has thirty two supersymmetries, the same number as $\mathcal{N} = 4$ SYM, and has the overall symmetry group $SO(2,4) \times SO(6)$ (the product of the symmetry groups of AdS_5 and S^5), which is the bosonic subgroup of $PSU(2,2|4)$.

The couplings of the string theory and the Yang-Mills theory are related by

$$g_{YM}^2 = g_s.$$

String theory is a good approximation to gravity in the regime in which R is much larger than l_s, so we can see from the above equation that this will be case when the parameter $g_{YM}^2 N$ is large. When this is not the case, it means that the coupling of the gauge theory is small and the theory is easier to solve. So it is clear from this that when $\mathcal{N} = 4$ SYM is difficult to solve, its dual IIB string theory on $AdS_5 \times S^5$ is easy to solve, and vice versa. This apparent strong/weak duality is what makes the AdS/CFT correspondence a hopeful candidate for further insight.

2.5 AdS/CFT Dictionary

In the preceding sections we have seen the associations between the symmetries of the two sides of the correspondence as well as their couplings, but we still do not have a clear means by which interesting quantities in one might be obtained from their equivalents in the other. This is certainly not a simple matter, but there are some important starting points.

Generally in field theories the quantities in which one is interested are the correlation functions of a given operator, denoted by

$$< \hat{O}(x_1)\hat{O}(x_2)...\hat{O}(x_d) >$$

which are found by taking the derivatives of the so called "generating functional" with respect to a particular source and then setting the source equal to zero. This is represented symbolically as

$$\begin{aligned} &\frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)}...\frac{\partial}{\partial J(x_d)}Z[J]\Bigg|_{J=0} \\ =\ &\frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)}...\frac{\partial}{\partial J(x_d)}\left\langle e^{\int J\hat{O}}\right\rangle\Bigg|_{J=0}. \end{aligned}$$

On the string side there is also a partition function, which is dependent on the boundary conditions of the AdS space in which the theory exists:

$$Z_{bulk}[\phi(x,z)|_{z=0} = \phi_0(x)]$$

where $\phi_0(x)$ is an arbitrary function specifying the boundary value of ϕ, the field in the bulk. The AdS/CFT dictionary dictates that this is equal to the generating function of the field theory with the boundary condition ϕ_0 as its source [5], i.e.

$$Z_{bulk}[\phi(x,z)|_{z=0} = \phi_0(x)] = \left\langle e^{\int [dx]\phi_0(x)\hat{O}(x)}\right\rangle_{field\,theory}.$$

This suggests that changes in the boundary conditions of AdS relate to changes in the Lagrangian of the field theory, and infinitesimal changes relate to the insertion of an operator [2].

More specific calculations involve a boundary condition of the form

$$\phi(x,z)|_{z=\epsilon} = \epsilon^{\alpha_-}\phi_0^r(x)$$

with the limit taken as $\epsilon \to 0$. α_- is dependent on the specific calculation, and the conformal dimension of the operator on the field theory side is

$$4 - \alpha_- \equiv \alpha_+.$$

This $\phi_0^r(x)$ is called the "renormalised" boundary condition and is associated with the fact that the operator in the field theory needs to be renormalised.

Explicitly the AdS/CFT dictionary is encapsulated in the following relations

$$\begin{aligned} g_s &= g_{YM}^2, \\ \frac{R^4}{l_s} &= 4\pi g_{YM}^2 N = 4\pi\lambda, \\ Z_s[\phi_0] &= \left\langle e^{\int \phi_0 \hat{O}} \right\rangle_{CFT}. \end{aligned}$$

We have highlighted some of the foundations for performing calculations using the AdS/CFT correspondence, but actually connecting all quantities in one side with quantities in the other is a substantial problem and the subject of much ongoing research. Nonetheless there have been succesful attempts to connect certain subsets of the quantities in certain limits, one of which we will examine in the next section.

2.6 BMN Limit

Berenstein, Maldacena and Nastase published a paper in 2002 [24] demonstrating that a particular limit of the string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is equivalent to that on a plane wave background, and that the string spectrum of this resulting theory could be arrived at from the large N limit of $\mathcal{N} = 4$ SYM with some specific restrictions. This has since become known as the BMN limit and is of interest particulary because free string theory on the plane wave background is exactly solvable. We will review the account of the BMN limit provided in [25].

Recall the AdS_5 metric (2.2). Combining it with the metric for S^5 yields

$$\frac{ds^2}{R^2} = -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta d\Omega_3'^2.$$

The limit in question then involves focusing on a particle in the "centre" of AdS_5 ($\rho = 0$) traveling along the equator of S^5 (measured by ψ for a specific θ). We introduce the coordinates $\tilde{x}^{\pm} = \frac{1}{2}(\tau \pm \psi)$ and then perform the following rescaling:

$$x^+ = \tilde{x}^+,\ x^- = R^2\tilde{x}^-,\ r = R\rho,\ y = R\theta.$$

The limit of interest is then found by taking $R \to \infty$. This can be thought of as the analogue to the $N \to \infty$ limit in the dual gauge theory.

Since the act of choosing an equator broke the $SO(6)$ symmetry of S^5, something similar must happen on the gauge theory side. This can be done by taking some $U(1)$ subgroup of the R-symmetry group. There is a charge associated with this subgroup referred to as the R-charge. The light cone energy and momentum, denoted respectively by p^- and p^+, are found to be associated with quantities on the gauge theory side in the following way:

$$2p^- = \Delta - J,\ 2p^+ = \frac{\Delta + J}{R^2}$$

where Δ is the conformal dimension of an operator in the dual CFT, which corresponds to the energy in global coordinates in AdS_5, and J is the angular momentum around the equator of S^5, which corresponds to the R-charge of an operator in the CFT. The string theory is limited to states with finite p^+ and p^-, which corresponds to limiting the gauge theory to operators with $\Delta \approx J \approx \sqrt{N}$ and finite $\Delta - J$.

In the light cone gauge the action of IIB string theory describes eight massive bosons and eight massive fermions and is trivially solvable [26]. The bosonic states are formulated in the following terms:

$$a^{I_1}_{n_1}...a^{I_m}_{n_m}|0;p^+>$$

with the Hamiltonian:

$$2p^- = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{8} (a^I_n)^\dagger a^I_n \sqrt{\frac{n^2}{(\alpha' p^+)^2}}.$$

The association between $2p^-$ and $\Delta - J$ suggests then that the ground state $|0;p^+>$ should correspond to an operator for which $\Delta - J = 0$. There is a unique operator that obeys this condition: $Tr(Z^J)$ which depends on the $U(1)$ subgroup associated with the breaking of $SO(6)$ symmetry we mentioned earlier. First excited states correspond with adding into the trace operators for which $\Delta - J = 1$, and higher excited states are found by adding more of these. States for which $n_1, ..., n_m = 0$ correspond to BPS operators, i.e. those whose conformal dimension does not dependent on the coupling. There are also higher states, which have the requirement that

$$n_1 + ... + n_m = 0.$$

This is the condition for states to be physical, and is also a condition for the trace on the operator side not to be zero. More details on the exact association between states and operators can be found in [24], but the important thing to note is that this limit constitutes an example of a context in which states on the one side of the AdS/CFT correspondence relate directly to operators on the other side.

2.7 Emergent Geometry

In the previous sections we have outlined the arguments for the existence of the AdS/CFT correspondence, examined some means by which it might be practically used and investigated a limit in which it is fairly well understood. In this section we will explain why it is that the AdS/CFT is of relevance to the subject of this dissertation, i.e. the eigenvalue dynamics of matrix valued systems. The reason for this is that the eigenvalues are associated with an emergent geometry, which corresponds to the geometry of a dual gravity theory. This was the insight primarily of David Berenstein who has published multiple papers on the matter [8], [9], [10], [11], the essence of which we will now review. An elaboration on these ideas can be found in [12].

In [8] it was determined that the half BPS states of an $\mathcal{N} = 4$ SYM theory could be modelled by a collection of fermions. Such a system can be viewed as a density of particles in a phase space and understood in terms of the statistics of thermodynamics. There is a probability associated with any given configuration of these particles, and quantum mechanically each configuration should be valid. The aim is then to determine which configuration is most statistically likely to occur. This can be done by finding the square of the holomorphic multi-particle ground state wave function, which for a two dimensional phase space has been argued to be [8]

$$\psi_F = \prod_{i<j}(z_i - z_j)e^{-\sum \frac{z\bar{z}}{2}}.$$

The probability density is then

$$p(\vec{x}_i) = \prod_{i<j}|z_i - z_j|^2 e^{-\sum z\bar{z}} = e^{-\sum z\bar{z} + \sum_{i<j} 2\ln|z_i - z_j|}.$$

This can be interpreted as an ensemble of N particles in two dimensions at positions x_i, subject to a potential with pairwise repulsion. This is essentially the same as the classical canonical ensemble, so we will refer to the quantity inside the exponent as βH.

If the configuration of the fermions has a continuous density profile $\rho(\vec{x})$ then the sums can be approximated by integrals that depend on ρ:

$$\beta H = \int d^2\vec{x}\rho(\vec{x})\vec{x}^2 + \int d^2\vec{x} \int d^2\vec{y}\rho(\vec{x})\rho(\vec{y}) \ln |\vec{x} - \vec{y}|.$$

There is also a constraint on the number of particles:

$$N = \int d^2\vec{x}\rho(\vec{x})$$

where N is assumed to be large enough that the statistical approach makes sense. The configuration with the lowest energy will be represented by a peak in the probability density, so it can be found by varying βH with respect to ρ. Introducing a Lagrange multiplier l to enforce the constraint on the number of particles, and constraining the variation to regions in which $\rho(x)$ is positive, yields

$$\vec{x}^2 + l = 2 \int d^2\vec{y}\rho(\vec{y}) \ln |\vec{x} - \vec{y}|.$$

This can be viewed as a Coulomb gas in an electric field produced by a constant density charge. The particles arrange themselves in such a way as to cancel the electric field locally, and rotational symmetry suggests that the configuration they acquire is a circle. This does indeed turn out to be the minimum energy configuration. Thus we have seen that a statistical model of a fermionic system in a two dimensional phase space results in the appearance of a circle.

This illustrates the concept of emergent geometry, but the more interesting case is that of 1/8 BPS states. In [9] Berenstein demonstrated that these states could be modelled by a series of commuting $N \times N$ hermitian matrices. It is important that the matrices commute because there is an association between commuting matrices and the strong coupling limit of the Yang Mills quartic potential. This can be seen by considering a potential of the form

$$g^2\, Tr[X_\mu, X_\nu]^2.$$

In the limit in which $g \to \infty$ it will be necessary for the commutator to tend towards zero in order for the path integral to make sense. The $SU(N)$ symmetry of the system means that each matrix can be diagonalised, reducing its respective degrees of freedom from N^2 to N, and the fact that the matrices commute means that they can all be diagonalised simultaneously. The eigenvalues of the matrices can be viewed as bosons and in a similar fashion to the example discussed above, a repulsive potential emerges and the bosons arrange themselves in a particular geometry. The geometry that emerges is none other than S_5, the very same that is of such interest in the AdS/CFT correspondence.

The purpose of this dissertation is to determine if there is a consistent large N description of systems of matrices in potentials depending only on the eigenvalues of the matrices, and in particular to investigate if methods of the collective field theory can be adapted to the description of this sector. The previous paragraph provides an example of a physical context that depends only on the eigenvalues, and hence evidence that restricting our investigation in this way is physically reasonable. In chapter 4 we will see that the collective field theory methods can in fact be applied to these systems, and in chapter 5 we will investigate if the methods developed in chapter 4 can also be used in the study of the angular degrees of freedom of multi matrix systems described by matrix valued curvilinear coordinates.

Chapter 3

Hermitian Matrices

3.1 Single Matrix Dynamics

Here we review the results obtained from investigating the single hermitian matrix Hamiltonian [7][16] without using the collective field method. In section 3.3 we will review the collective field approach to this problem and see how the results correlate.

We begin with a system in which the degrees of freedom are represented by a single $N \times N$ hermitian matrix M, with a potential depending only on the eigenvalues of the matrix. The Hamiltonian of such a system will be

$$H = -\frac{1}{2}\frac{\partial}{\partial M_{ab}}\frac{\partial}{\partial M_{ba}} + TrV(\lambda).$$

Due to the $SU(N)$ symmetry the matrix can be diagonalised into $U^\dagger DU$ where U is a unitary matrix and $D = diag(\lambda_1, ..., \lambda_N)$ is a diagonal matrix of the eigenvalues of M, denoted by λ_i.

We would like to know the Jacobian for this diagonalisation as well as the

Hamiltonian in terms of the diagonalised variables, which requires the line element dM. This can be written as

$$dM = dU^\dagger DU + U^\dagger dDU + U^\dagger DdU$$

$$= U^\dagger(UdU^\dagger D + dD + DdUU^\dagger)U$$

$$= U^\dagger(-dUU^\dagger D + dD + DdUU^\dagger)U$$

$$= U^\dagger(dD + DdUU^\dagger - dUU^\dagger D)U$$

$$= U^\dagger(dD + [D, dS])U$$

where we have introduced $dS = dUU^\dagger$ with the property

$$dS = dUU^\dagger = -UdU^\dagger = -dS^\dagger.$$

This is true because

$$UU^\dagger = I$$

and thus

$$dUU^\dagger + UdU^\dagger = 0.$$

We also note that

$$dM^\dagger = U^\dagger(dD + [dS^\dagger, D])U$$

$$= U^\dagger(dD - [dS, D])U$$

$$= U^\dagger(dD + [D, dS])U = dM$$

In order to find the metric we need $Tr(dM^2)$ which is

$$Tr(dMdM^\dagger) = Tr(U^\dagger(dD + [D, dS])UU^\dagger(dD + [D, dS])U)$$

$$= Tr(dD^2 + 2dD[D, dS] + [D, dS]^2)$$

$$= Tr(dD^2 + [D, dS]^2).$$

The middle term disappears because of the following identity for traces involving commutators of matrices:

$$Tr([A,B]C) = Tr([B,C]A) = Tr([C,A]B)$$

and the fact that

$$[D, dD] = 0.$$

In component form we have

$$Tr(dM^2) = \sum_i (dM^2)_{ii}$$

$$= Tr(dD^2) + Tr([D, dS]^2)$$

$$= \sum_i (d\lambda_i^2) - \sum_{i,j} (dS)_{ij}(dS)_{ji}(\lambda_i - \lambda_j)^2$$

$$= \sum_i (d\lambda_i^2) + \sum_{i \neq j} (dS)_{ij}(dS^*)_{ij}(\lambda_i - \lambda_j)^2$$

$$= \sum_i (d\lambda_i^2) + \frac{1}{2}\sum_{i \neq j} (\lambda_i - \lambda_j)^2 (dS^*)_{ij}(dS)_{ij} + \frac{1}{2}\sum_{i \neq j} (\lambda_i - \lambda_j)^2 (dS)_{ij}(dS^*)_{ij}$$

$$= \sum_i (d\lambda_i^2) + \sum_{i<j} (\lambda_i - \lambda_j)^2 (dS^*)_{ij}(dS)_{ij} + \sum_{i<j} (\lambda_i - \lambda_j)^2 (dS)_{ij}(dS^*)_{ij}.$$

Next we use

$$Tr(dM^2) = g_{\mu\nu} dX^\mu dX^\nu,$$

with $\mu, \nu = 0, 1, 2$ and $dX = (d\lambda, dS_{ij}, dS^*_{ij})$, to find the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\lambda_i - \lambda_j)^2 & 0 \\ 0 & 0 & (\lambda_i - \lambda_j)^2 \end{pmatrix}$$

from which we can find the Jacobian:

$$J = \sqrt{\det g_{\mu\nu}} = \prod_{i<j} (\lambda_i - \lambda_j)^2 = \Delta^2$$

where Δ is the Vandermonde determinant. Note that there is an association between a Jacobian of this form and the emergent geometry associated with inter-eigenvalue repulsion [7].

The laplacian is obtained from

$$\begin{aligned}\nabla^2 &= \frac{1}{\sqrt{\det g_{\mu\nu}}}\frac{\partial}{\partial X^\mu}\sqrt{\det g_{\mu\nu}}g^{\mu\nu}\frac{\partial}{\partial X^\nu} \\ &= \frac{1}{\Delta^2}\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i} \\ &+\frac{1}{\Delta^2}\frac{\partial}{\partial S_{ij}}\Delta^2\sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2}\frac{\partial}{\partial S^*_{ij}} \\ &+\frac{1}{\Delta^2}\frac{\partial}{\partial S^*_{ij}}\Delta^2\sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2}\frac{\partial}{\partial S_{ij}} \\ &= \frac{1}{\Delta^2}\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i} - \sum_{i\neq j}\frac{1}{(\lambda_i-\lambda_j)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S_{ji}}\end{aligned}$$

In a physical context in which the dynamics depend only on the eigenvalues, the derivatives with respect to S will disappear and thus the eigenvalue problem will look like

$$\Big(-\frac{1}{2}\frac{1}{\Delta^2}\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i}+Tr(V(\lambda))\Big)\Phi = E\Phi.$$

Let us examine the first term on the left hand side:

$$\begin{aligned}&-\frac{1}{2}\frac{1}{\Delta^2}\sum_i\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i} \\ &= -\frac{1}{2}\frac{1}{\Delta^2}\sum_i\Big(\Big(\frac{\partial}{\partial\lambda_i}\Delta^2\Big)\frac{\partial}{\partial\lambda_i}+\Delta^2\frac{\partial^2}{\partial\lambda_i^2}\Big) \\ &= -\frac{1}{2}\frac{1}{\Delta^2}\sum_i\Big(2\Delta\Big(\frac{\partial}{\partial\lambda_i}\Delta\Big)\frac{\partial}{\partial\lambda_i}+\Delta^2\frac{\partial^2}{\partial\lambda_i^2}\Big) \\ &= -\frac{1}{2}\frac{1}{\Delta}\sum_i\Big(\Big(\frac{\partial^2}{\partial\lambda_i^2}\Delta\Big)+\Big(\frac{\partial}{\partial\lambda_i}\Delta\Big)\frac{\partial}{\partial\lambda_i}+\Big(\frac{\partial}{\partial\lambda_i}\Delta\Big)\frac{\partial}{\partial\lambda_i}+\Delta\frac{\partial^2}{\partial\lambda_i^2}\Big)\end{aligned}$$

$$= -\frac{1}{2}\frac{1}{\Delta}\sum_i \frac{\partial}{\partial\lambda_i}\left(\left(\frac{\partial}{\partial\lambda_i}\Delta\right) + \Delta\frac{\partial}{\partial\lambda_i}\right)$$

$$= -\frac{1}{2}\frac{1}{\Delta}\sum_i \frac{\partial^2}{\partial\lambda_i^2}\Delta$$

where in the fourth line we can insert the second derivative of Δ with respect to λ_i since it is equal to zero when summed over. A proof of this can be found in [21]. The eigenvalue problem is now

$$-\frac{1}{2}\frac{1}{\Delta}\sum_i \frac{\partial^2}{\partial\lambda_i^2}\Delta\Phi + Tr(V(M))\Phi = E\Phi$$

Rescaling the wave function to $\Psi = \Delta\Phi$ this becomes

$$-\frac{1}{2}\frac{1}{\Delta}\sum_i \frac{\partial^2}{\partial\lambda_i^2}\Psi + Tr(V(M))\frac{\Psi}{\Delta} = E\frac{\Psi}{\Delta}$$

or

$$\sum_i h_i\Psi(\lambda_i) = E\Psi(\lambda_i)$$

where

$$h_i = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial\lambda_i^2} + Tr(V(\lambda_i)).$$

The Vandermonde determinant is antisymmetric under an exchange of the eigenvalues, and thus the rescaled wave function Ψ is also antisymmetric. This means that the problem is reduced to that of a set of non-relativistic non-interacting fermions.

3.2 Collective Field Method

The aim of the Collective Field Method [6] is to reformulate a given problem from its original coordinates, denoted by $q_1, ..., q_d$, to a new set of coordinates generally chosen to be invariant under the original symmetries of the system, denoted by ϕ_k. This new set is almost always overcomplete.

Beginning with a standard Hamiltonian:

$$H = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial q_i^2} + V(q_1, ..., q_d)$$

the kinetic term can be written in terms of the collective field coordinates with the use of the chain rule as follows:

$$K = -\frac{1}{2}\sum_i \frac{\partial}{\partial q_i}\frac{\partial}{\partial q_i}$$

$$= -\frac{1}{2}\sum_{i,a} \frac{\partial}{\partial q_i}\left(\frac{\partial \phi_a}{\partial q_i}\frac{\partial}{\partial \phi_a}\right)$$

$$= -\frac{1}{2}\sum_{i,a} \frac{\partial^2 \phi_a}{\partial q_i^2}\frac{\partial}{\partial \phi_a} - \frac{1}{2}\sum_{i,a,b} \frac{\partial \phi_a}{\partial q_i}\frac{\partial \phi_b}{\partial q_i}\frac{\partial}{\partial \phi_b}\frac{\partial}{\partial \phi_a}$$

$$= -\frac{1}{2}\sum_a \omega_a \frac{\partial}{\partial \phi_a} - \frac{1}{2}\sum_{a,b} \Omega_{a,b}\frac{\partial}{\partial \phi_b}\frac{\partial}{\partial \phi_a}$$

where

$$\omega_a = \sum_i \frac{\partial^2 \phi_a}{\partial q_i^2}$$

and

$$\Omega_{a,b} = \sum_i \frac{\partial \phi_a}{\partial q_i}\frac{\partial \phi_b}{\partial q_i}$$

are referred to as the “splitting operator” and the “joining operator” respectively.

This change of coordinates introduces a non-trivial Jacobian:

$$\int [dq]\Phi_1^*(q)\Phi_2(q) = \int [d\phi]J(\phi)\Phi_1^*(\phi)\Phi_2(\phi).$$

In order to have a simple scalar product we perform the similarity transformation:

$$\Psi(\phi) = (J(\phi))^{\frac{1}{2}}\Phi(\phi).$$

$$\frac{\partial}{\phi_k} \to (J(\phi))^{\frac{1}{2}}\frac{\partial}{\phi_k}(J(\phi))^{-\frac{1}{2}} = \left(\frac{\partial}{\partial \phi_k} - \frac{1}{2}\frac{\partial \ln J(\phi)}{\partial \phi_k}\right).$$

The kinetic term in the Hamiltonian is then

$$-\frac{1}{2}\omega_a\left(\frac{\partial}{\partial\phi_a}-\frac{1}{2}\frac{\partial\ln J(\phi)}{\partial\phi_a}\right)-\frac{1}{2}\Omega_{a,b}\left(\frac{\partial}{\partial\phi_a}-\frac{1}{2}\frac{\partial\ln J(\phi)}{\partial\phi_a}\right)\left(\frac{\partial}{\partial\phi_b}-\frac{1}{2}\frac{\partial\ln J(\phi)}{\partial\phi_b}\right)$$

$$\begin{aligned}&=-\frac{1}{2}\omega_a\frac{\partial}{\partial\phi_a}+\frac{1}{4}\omega_a\frac{\partial\ln J(\phi)}{\partial\phi_a}-\frac{1}{2}\Omega_{a,b}\frac{\partial}{\partial\phi_a}\frac{\partial}{\partial\phi_b}+\frac{1}{4}\Omega_{a,b}\frac{\partial}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\\&\quad+\frac{1}{4}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_a}\frac{\partial}{\partial\phi_b}-\frac{1}{8}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\end{aligned}$$

$$\begin{aligned}&=-\frac{1}{2}\omega_a\frac{\partial}{\partial\phi_a}+\frac{1}{4}\omega_a\frac{\partial\ln J(\phi)}{\partial\phi_a}-\frac{1}{2}\Omega_{a,b}\frac{\partial}{\partial\phi_a}\frac{\partial}{\partial\phi_b}+\frac{1}{4}\Omega_{a,b}\left(\frac{\partial}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\right)\\&\quad+\frac{1}{4}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_b}\frac{\partial}{\partial\phi_a}+\frac{1}{4}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_a}\frac{\partial}{\partial\phi_b}-\frac{1}{8}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\end{aligned}$$

$$\begin{aligned}&=-\frac{1}{2}\omega_a\frac{\partial}{\partial\phi_a}+\frac{1}{4}\omega_a\frac{\partial\ln J(\phi)}{\partial\phi_a}-\frac{1}{2}\frac{\partial}{\partial\phi_b}\Omega_{a,b}\frac{\partial}{\partial\phi_a}+\frac{1}{2}\left(\frac{\partial}{\partial\phi_b}\Omega_{a,b}\right)\frac{\partial}{\partial\phi_a}\\&\quad+\frac{1}{4}\Omega_{a,b}\left(\frac{\partial}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\right)+\frac{1}{2}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_b}\frac{\partial}{\partial\phi_a}-\frac{1}{8}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\end{aligned}$$

$$\begin{aligned}&=\left(-\frac{1}{2}\omega_a+\frac{1}{2}\left(\frac{\partial}{\partial\phi_b}\Omega_{a,b}\right)+\frac{1}{2}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_b}\right)\frac{\partial}{\partial\phi_a}+\frac{1}{4}\omega_a\frac{\partial\ln J(\phi)}{\partial\phi_a}\\&\quad-\frac{1}{2}\frac{\partial}{\partial\phi_a}\Omega_{a,b}\frac{\partial}{\partial\phi_b}+\frac{1}{2}\Omega_{a,b}\left(\frac{\partial}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\right)-\frac{1}{8}\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_a}\frac{\partial\ln J(\phi)}{\partial\phi_b}\end{aligned}$$

In order for the Hamiltonian to remain hermitian the coefficient of the derivative with respect to ϕ_a must be equal to zero:

$$-\omega_a+\left(\frac{\partial}{\partial\phi_b}\Omega_{a,b}\right)+\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_b}=0.$$

Rearranging this gives us an equation that makes the Jacobian considerably easier to find than usual:

$$\Omega_{a,b}\frac{\partial\ln J(\phi)}{\partial\phi_b}=\omega_a-\left(\frac{\partial}{\partial\phi_c}\Omega_{a,c}\right). \tag{3.1}$$

Combining the last three equations together gives us

$$\frac{1}{4}\omega_a\Omega^{-1}_{a,b}\omega_b - \frac{1}{4}\omega_a\Omega^{-1}_{a,b}\left(\frac{\partial}{\partial\phi_c}\Omega_{b,c}\right)$$
$$-\frac{1}{2}\frac{\partial}{\partial\phi_a}\Omega_{a,b}\frac{\partial}{\partial\phi_b} + \frac{1}{2}\Omega_{a,b}\left(\frac{\partial}{\partial\phi_a}\left(\Omega^{-1}_{b,d}\omega_d - \Omega^{-1}_{b,d}\left(\frac{\partial}{\partial\phi_c}\Omega_{d,c}\right)\right)\right)$$
$$-\frac{1}{8}\Omega_{a,b}\left(\Omega^{-1}_{a,d}\omega_d - \Omega^{-1}_{a,d}\left(\frac{\partial}{\partial\phi_c}\Omega_{d,c}\right)\right)\left(\Omega^{-1}_{b,d}\omega_d - \Omega^{-1}_{b,d}\left(\frac{\partial}{\partial\phi_c}\Omega_{d,c}\right)\right).$$

In practice most of these terms represent corrections to the large N expansion, and what we are really interested in is

$$K = -\frac{1}{2}\frac{\partial}{\partial\phi_a}\Omega_{a,b}\frac{\partial}{\partial\phi_b} + \frac{1}{8}\omega_a\Omega^{-1}_{a,b}\omega_b. \tag{3.2}$$

The ground state energy in the large N limit of the system can then be found by minimising the effective potential, which is the right hand term above combined with the original potential.

3.3 Collective Field for a Single Matrix

Before applying the collective field method to a system of multiple hermitian matrices it is informative to see how it proceeds in the context of one hermitian matrix. This application was the one used by Jevicki and Sakita to illustrate the applicability of the collective field method to problems in quantum field thoery in general [6].

We begin from the same point as in section 3.1 with a single $N \times N$ hermitian matrix that can be diagonalised in terms of its eigenvalues λ_i. The Hamiltonian of this system will be

$$H = -\frac{1}{2}Tr\left(\frac{\partial^2}{\partial M^2}\right) + Tr\left(\frac{1}{2}M^2 + \frac{g}{N}M^4\right)$$

where we are now using a specific potential, the same as [15]. Next is to change variables from M to a set of invariants defined as

$$\phi_k = Tre^{ikM} = \sum_i e^{ik\lambda_i}$$

as well as its fourier transform

$$\phi(x) = \int dk e^{-ikx}\phi_k = \sum_i \delta(x - \lambda_i).$$

Using the identity

$$\frac{\partial}{\partial M_{ij}}\big(e^{ikM}\big)_{ab} = ik\int_0^1 d\alpha\big(e^{i\alpha kM}\big)_{ai}\big(e^{i(1-\alpha)kM}\big)_{jb} \tag{3.3}$$

we find that

$$\begin{aligned}\frac{\partial\phi_k}{\partial M_{ij}} &= ik\int_0^1 d\alpha\big(e^{i\alpha kM}\big)_{ai}\big(e^{i(1-\alpha)kM}\big)_{ja}\\ &= ik\int_0^1 d\alpha\big(e^{i(1-\alpha)kM}\big)_{ja}\big(e^{i\alpha kM}\big)_{ai}\\ &= ik\int_0^1 d\alpha\big(e^{i(1-\alpha)kM+i\alpha kM}\big)_{ji}\\ &= ik\big(e^{ikM}\big)_{ji}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\phi_k}{\partial M_{ij}\partial M_{ji}} &= \frac{\partial}{\partial M_{ij}} ik\big(e^{ikM}\big)_{ij}\\ &= (ik)^2\int_0^1 d\alpha\big(e^{i\alpha kM}\big)_{ii}\big(e^{i(1-\alpha)kM}\big)_{jj}\\ &= -k^2\int_0^1 d\alpha\big(Tre^{i\alpha kM}\big)\big(Tre^{i(1-\alpha)kM}\big)\\ &= -k^2\int_0^k \frac{dk'}{k}\big(Tre^{ik'M}\big)\big(Tre^{i(k-k')M}\big)\\ &= -k\int_0^k dk'\phi_{k'}\phi_{k-k'}.\end{aligned}$$

Therefore the joining and splitting operators are

$$\Omega_{kk'} = \frac{\partial \phi_k}{\partial M_{ij}} \frac{\partial \phi'_k}{\partial M_{ji}}$$

$$= -kk' Tr\left(e^{ikM} e^{ik'M}\right)$$

$$= -kk' \sum_i e^{i(k+k')\lambda_i}$$

and

$$\omega_k = \frac{\partial^2 \phi_k}{\partial M_{ij} \partial M_{ji}} = -k \int_0^k dk' \phi_{k'} \phi_{k-k'}$$

$$= -k \sum_{ij} e^{ik\lambda_j} \int_0^k dk' e^{ik'(\lambda_i - \lambda_j)}$$

$$= -k \sum_{i \neq j} e^{ik\lambda_j} \int_0^k dk' e^{ik'(\lambda_i - \lambda_j)} - k^2 \sum_i e^{ik\lambda_i}$$

$$= ik \sum_{i \neq j} e^{ik\lambda_j} \left[\frac{e^{ik'(\lambda_i - \lambda_j)}}{\lambda_i - \lambda_j} \right]_0^k - k^2 \sum_i e^{ik\lambda_i}$$

$$= ik \sum_{i \neq j} \frac{e^{ik\lambda_i} - e^{ik\lambda_j}}{\lambda_i - \lambda_j} - k^2 \sum_i e^{ik\lambda_i}$$

$$= -2ik \sum_{i \neq j} \frac{e^{ik\lambda_i}}{\lambda_i - \lambda_j} - k^2 \sum_i e^{ik\lambda_i}.$$

In the coordinate representation these become

$$\Omega(x, x') = \int dk \int dk' e^{-ikx} e^{-ik'x'} \Omega_{kk'}$$

$$= \int dk \int dk(-kk') \sum_i e^{ik(\lambda_i - x)} e^{ik'(\lambda_i - x')}$$

$$= \partial_x \partial_{x'} \sum_i \delta(x - \lambda_i) \delta(x' - \lambda_i)$$

$$= \partial_x \partial_{x'} \delta(x - x') \sum_i \delta(x - \lambda_i)$$

$$= \partial_x \partial_{x'} \delta(x - x') \phi(x)$$

and

$$\begin{aligned}\omega(x) &= \int dk e^{-ikx}\omega_k \\ &= 2\partial_x \sum_{i\neq j} \frac{\delta(\lambda_i - x)}{\lambda_i - \lambda_j} + \partial_x^2\phi(x) \\ &= 2\partial_x ⨍ dy \int dz \frac{\delta(z-x)}{z-y}\phi(z)\phi(y) + \partial_x^2\phi(x) \\ &= 2\partial_x ⨍ dy \frac{\phi(x)\phi(y)}{x-y} + \partial_x^2\phi(x).\end{aligned}$$

We can ignore the term on the right since it corresponds to a specific regularisation of the $i = j$ contribution, and is not of leading order in N.

Now recall the restriction on the Jacobian from the derivation of the collective field method:

$$-\omega_a + \left(\frac{\partial}{\partial\phi_b}\Omega_{a,b}\right) + \Omega_{a,b}\frac{\partial \ln J(\phi)}{\partial\phi_b} = 0.$$

which in the coordinate representation becomes

$$-\omega(x) + \int dx' \frac{\partial\Omega(x,x')}{\partial\phi(x')} + \int dx'\Omega(x,x')\frac{\partial \ln J(\phi(x'))}{\partial\phi(x')} = 0. \tag{3.4}$$

Substituting in the expressions for ω and Ω we find that the second term assumes the following form:

$$\begin{aligned}&\int dx' \frac{\partial}{\partial\phi(x')}\partial_x\partial_{x'}\delta(x-x')\phi(x) \\ &= \partial_x \int dx'[\partial_{x'}\delta(x-x')]\delta(x-x') \\ &= \partial_x\delta'(0).\end{aligned}$$

If one considers the delta function as the limit of a Gaussian function as its width tends towards zero and its height tends towards infinity, with its area held constant at 1, then it is easy to see that the derivative of such a function will be zero at the origin, therefore $\delta'(0) = 0$. Equation (3.4) will then be

$$\int dx'\partial_x\partial_{x'}\delta(x-x')\phi(x)\frac{\partial \ln J}{\partial\phi(x')} = 2\partial_x ⨍ dy \frac{\phi(x)\phi(y)}{x-y}.$$

Integration by parts on the left yields

$$\partial_x \frac{\partial \ln J}{\partial \phi(x)} = 2 ⨍ dy \frac{\phi(y)}{x-y}$$

the solution to which is

$$\ln J = \int dx ⨍ dy \phi(x)\phi(y) \ln|x-y|$$

$$= \sum_{i \neq j} \int dx ⨍ dy \delta(x-\lambda_i)\delta(y-\lambda_j) \ln|x-y|$$

$$= \sum_{i \neq j} \ln|\lambda_i - \lambda_j|$$

$$= \ln \prod_{i<j} (\lambda_i - \lambda_j)^2$$

and thus

$$J = \prod_{i<j} (\lambda_i - \lambda_j)^2$$

which is in precise agreement with what was found in section 3.1.

Now in order to find the large N behaviour of the system we need to minimise the effective potential, which is the sum of the initial potential in the Hamiltonian and the term obtained in section 3.2, which in the coordinate representation reads

$$\frac{1}{8} \int dx \int dx' \omega(x) \Omega^{-1}(x,x') \omega(x') = \frac{1}{8} \int dx \omega(x) \frac{\partial \ln J}{\partial \phi(x)} \tag{3.5}$$

$$= \frac{1}{2} \int dx \left(\partial_x ⨍ dy \frac{\phi(x)\phi(y)}{x-y} \right) \left(⨍ dz \phi(z) \ln|x-z| \right)$$

$$= \frac{1}{2} \int dx \phi(x) \left(⨍ dy \frac{\phi(y)}{x-y} \right)^2 = \frac{\pi^2}{6} \int dx \phi^3(x)$$

where in the last line we have used a known identity that can be found in [17]. Incorporating the original potential we arrive at

$$V_{eff} = \int dx \left(\frac{\pi^2}{6} \phi^2(x) + \frac{1}{2} x^2 + \frac{g}{N} x^4 \right) \phi(x).$$

Introducing a Lagrange multiplier l in order to enforce the constraint:

$$\int dx \phi(x) = N \tag{3.6}$$

and then minimising with respect to $\phi(x)$ yields

$$\frac{\pi^2}{2}\phi^2(x) + \frac{1}{2}x^2 + \frac{g}{N}x^4 - l = 0,$$

the solution to which is then clearly

$$\phi_0(x) = \frac{1}{\pi}\sqrt{2l - x^2 - \frac{2g}{N}x^4}$$

where l is determined by (3.6).

Equation (3.2) in this context will be

$$\frac{1}{2}\int dx \int dx' \pi(x)\Omega(x,x')\pi(x') + \frac{1}{8}\int dx \int dx' \omega(x)\Omega^{-1}(x,x')\omega(x')$$

where

$$\pi(x) = -i\frac{\partial}{\partial\phi(x)}.$$

The effective Hamiltonian is therefore

$$H_{eff} = \frac{1}{2}\int dx \partial_x \pi(x)\phi(x)\partial_x\pi(x) + \int dx \left(\frac{\pi^2}{6}\phi^2(x) + \frac{1}{2}x^2 + \frac{g}{N}x^4\right)\phi(x).$$

In order to make the N dependence explcit we will rescale as follows:

$$x \to \sqrt{N}x; \phi \to \sqrt{N}\phi$$

$$\pi \to \frac{1}{N}\pi; \partial_x \to \frac{1}{\sqrt{N}}\partial_x; l \to Nl.$$

The resulting Hamiltonian will be

$$H_{eff} = \frac{1}{2}\int dx\sqrt{N}\frac{1}{\sqrt{N}}\partial_x\pi\frac{1}{N}\sqrt{N}\phi\frac{1}{\sqrt{N}}\partial_x\pi\frac{1}{N}$$

$$+\int dx\sqrt{N}\left(\frac{\pi^2}{6}N\phi^2 + \frac{1}{2}Nx^2 + \frac{g}{N}N^2x^4\right)\sqrt{N}\phi.$$

$$= \frac{1}{N^2}\frac{1}{2}\int dx \partial_x \pi(x)\phi(x)\partial_x \pi(x) + N^2 \int dx \left(\frac{\pi^2}{6}\phi^2(x) + \frac{1}{2}x^2 + \frac{g}{N}x^4 \right)\phi(x)$$

and thus we can see that the large N background is determined by the effective potential.

Chapter 4

Multiple Hermitian Matrices

4.1 Jacobian

In this chapter we will investigate the extent to which the collective field method can be used to describe a system of multiple matrices which depend only on their eigenvalues. The system will consist of several $N \times N$ hermitian matrices denoted by $M^A, A = 1, ..., d$, each of which can be simultaneously diagonalised:

$$M^A = (U^A)^\dagger D^A U^A$$

where U is a unitary matrix and $D^A = diag(\lambda_1^A, ..., \lambda_N^A)$ is a diagonal matrix of the eigenvalues of M^A denoted by λ_i^A.

The advantage of the collective field theory is that it generally begins with invariants that contain only a single trace of the coordinates, but there are no such invariants that can be used in this case. It is however quite easy to find the Jacobian for this multiple matrix system, which hopefully will provide insight as to what invariants will be practical to use. We know from the previous section that an integral in the single matrix case will take the form:

$$\int dMf(M) = \int d\lambda Jf(\lambda) = \int d\lambda \prod_{i<j}(\lambda_i - \lambda_j)^2 f(\lambda).$$

In the multiple matrix case we will have

$$\int ... \int dM^1 dM^2 ... dM^d e^{-\frac{1}{2}g^2\left(Tr(M^1)^2+Tr(M^2)^2+...+Tr(M^d)^2\right)}$$

$$= \left(\int dM^1 e^{-\frac{1}{2}g^2\left(Tr(M^1)^2\right)}\right)\left(\int dM^2 e^{-\frac{1}{2}g^2\left(Tr(M^2)^2\right)}\right)...\left(\int dM^d e^{-\frac{1}{2}g^2\left(Tr(M^d)^2\right)}\right)$$

$$= \left(\int d\lambda^1 \prod_{i<j}(\lambda_i^1 - \lambda_j^1)^2 e^{-\frac{1}{2}g^2\left(\sum_i(\lambda_i^1)^2\right)}\right)\left(\int d\lambda^2 \prod_{i<j}(\lambda_i^2 - \lambda_j^2)^2 e^{-\frac{1}{2}g^2\left(\sum_i(\lambda_i^2)^2\right)}\right)...$$

and thus the Jacobian is

$$J = \prod_A \prod_{i<j}(\lambda_i^A - \lambda_j^A)^2. \tag{4.1}$$

$$= \prod_A \Delta_A^2.$$

Where Δ_A is the Vandermonde determinant for the matrix M^A. Clearly this is a straightforward generalisation of the single matrix case, so hopefully the invariants can be generalised in a similar way.

4.2 Collective Field for Multiple Matrices

Since the Jacobian is just a product of single matrix Jacobians, one clear possibility is to define the collective field invariants as a product of single matrix traces:

$$\phi_{k_1...k_D} = (Tre^{ik_1M^1})(Tre^{ik_2M^2})...(Tre^{ik_DM^D})$$

$$= \prod_{A=1}^{D} Tre^{ik_AM^A} = \prod_{A=1}^{D}\sum_{i=1}^{N} e^{ik_A\lambda_i^A}$$

Taking the fourier transform yields

$$\phi(x_1, ..., x_D) = \int dk_1 ... \int dk_D e^{-ik_1x_1} ... e^{-ik_Dx_D} \phi_{k_1...k_D}$$

$$= \prod_{A=1}^{D} \sum_{i=1}^{N} \delta(\lambda_i^A - x_A) = \prod_{A=1}^{D} \phi^A(x_A)$$

Where each ϕ^A represents an eigenvalue density of the matrix M^A and is subject to the restriction:

$$\int dx_A \phi^A(x_A) = N. \tag{4.2}$$

Using the identity (3.3) to find the derivatives of the collective field coordinate we find that

$$\frac{\partial}{\partial M_{ij}^A} \phi_{k_1 \ldots k_D} = i k_A (e^{i k_A M^A})_{ji} \prod_{B \neq A} (Tr e^{i k_B M^B}).$$

This is similar to the previous section, but now because of the product there are terms that don't depend on M^A which remain as "observers".

The joining operator is then

$$\Omega_{k_1 \ldots k_D, k'_1 \ldots k'_D} = \sum_A \frac{\partial \phi_{k_1 \ldots k_D}}{\partial M_{ij}^A} \frac{\partial \phi_{k'_1 \ldots k'_D}}{\partial M_{ji}^A}$$

$$= \sum_A (-k_A k'_A) Tr(e^{i k_A M^A} e^{i k'_A M^A}) \Big(\prod_{B \neq A} (Tr e^{i k_B M^B}) \Big) \Big(\prod_{C \neq A} (Tr e^{i k'_C M^C}) \Big)$$

$$= -\sum_A (k_A k'_A) \sum_p e^{i(k_A + k'_A)\lambda_p^A} \Big(\prod_{B \neq A} \Big(\sum_i e^{i k_B \lambda_i^B} \Big) \Big) \Big(\prod_{C \neq A} \Big(\sum_j e^{i k'_C \lambda_j^C} \Big) \Big).$$

This is not really a "joining" operation at all. Generally this operator joins two loops into one, but here we still have two loops. This is therefore a fairly non-standard application of the collective field method, but it works nonetheless. Next is to take the fourier transform:

$$\Omega(x_1, \ldots, x_D, x'_1, \ldots, x'_D) =$$

$$\int dk_1 \ldots \int dk_D \int dk'_1 \ldots \int dk'_D e^{-i k_1 x_1} \ldots e^{-i k_D x_D} e^{-i k'_1 x'_1} \ldots e^{-i k'_D x'_D} \Omega_{k_1 \ldots k_D, k'_1 \ldots k'_D}$$

$$= -\sum_A \int dk_A \int dk'_A e^{-ik_A x_A} e^{-ik'_A x'_A} (k_A k'_A) \sum_p e^{i(k_A + k'_A)\lambda_p^A}$$

$$\times \prod_{B \neq A} \Big(\sum_i \delta(\lambda_i^B - x_B) \Big) \prod_{C \neq A} \Big(\sum_j \delta(\lambda_j^C - x'_C) \Big)$$

$$= \sum_A \partial_{x_A} \partial_{x'_A} \sum_p \delta(\lambda_p^A - x_A) \delta(\lambda_p^A - x'_A) \prod_{B \neq A} \Big(\sum_i \delta(\lambda_i^B - x_B) \Big) \prod_{C \neq A} \Big(\sum_j \delta(\lambda_j^C - x'_C) \Big)$$

$$= \sum_A \partial_{x_A} \partial_{x'_A} \sum_p \delta(\lambda_p^A - x_A) \delta(x_A - x'_A) \prod_{B \neq A} \Big(\sum_i \delta(\lambda_i^B - x_B) \Big) \prod_{C \neq A} \Big(\sum_j \delta(\lambda_j^C - x'_C) \Big)$$

$$= \sum_A \partial_{x_A} \partial_{x'_A} \big(\delta(x_A - x'_A) \phi^A(x_A) \big) \prod_{B \neq A} \phi^B(x_B) \prod_{C \neq A} \phi^C(x'_C)$$

The splitting operator follows the same procedure as in the single matrix case, and also includes an observer:

$$\omega_{k_1 \ldots k_D} = -\sum_A \frac{\partial^2 \phi_{k_1 \ldots k_D}}{\partial M_{ij}^A \partial M_{ji}^A}$$

$$= -\sum_A \frac{\partial}{\partial M_{ij}^A} \Big(i k_A (e^{ik_A M_A})_{ij} \prod_{B \neq A} (Tr e^{ik_B M_B}) \Big)$$

$$= \sum_A k_A \int_0^{k_A} dk'_A Tr e^{ik'_A M_A} Tr e^{i(k_A - k'_A) M_A} \Big(\prod_{B \neq A} (Tr e^{ik_B M_B}) \Big)$$

$$= \sum_A k_A \int_0^{k_A} dk'_A \sum_{ij} e^{ik'_A \lambda_i^A} e^{i(k_A - k'_A)\lambda_j^A} \Big(\prod_{B \neq A} \Big(\sum_p e^{ik_B \lambda_p^B} \Big) \Big)$$

$$= \sum_A \Big(-2ik_A \sum_{i \neq j} \frac{e^{ik_A \lambda_i^A}}{\lambda_i^A - \lambda_j^A} \Big) \Big(\prod_{B \neq A} \Big(\sum_p e^{ik_B \lambda_p^B} \Big) \Big)$$

Once again taking the fourier transform yields

$$\omega(x_1, \ldots, x_D) = \int dk_1 \ldots \int dk_D e^{-ik_1 x_1} \ldots e^{ik_D x_D} \omega_{k_1 \ldots k_D}$$

$$= 2 \sum_A \partial_{x_A} \sum_{i \neq j} \frac{\delta(\lambda_i^A - x_A)}{\lambda_i^A - \lambda_j^A} \Big(\prod_{B \neq A} \Big(\sum_p \delta(\lambda_p^B - x_B) \Big) \Big)$$

$$= 2\sum_A \partial_{x_A} \sum_{i,j} \int dy \fint dz \frac{\delta(y-x_A)\delta(\lambda_i^A - y)\delta(\lambda_j^A - z)}{y-z} \Big(\prod_{B\neq A} \Big(\sum_p \delta(\lambda_p^B - x_B) \Big) \Big)$$

$$= 2\sum_A \partial_{x_A} \sum_{i,j} \fint dz \frac{\delta(\lambda_i^A - x_A)\delta(\lambda_j^A - z)}{x_A - z} \Big(\prod_{B\neq A} \Big(\sum_p \delta(\lambda_p^B - x_B) \Big) \Big)$$

$$= 2\sum_A \partial_{x_A} \phi^A(x_A) \fint dz \frac{\phi^A(z)}{x_A - z} \prod_{B\neq A} \phi^B(x_B)$$

The equation for the Jacobian (3.1) in coordinate representation is now

$$\int dx_1' ... dx_D' \Omega(x_1, ..., x_D, x_1', ..., x_D') \frac{\partial \ln J}{\partial \phi(x_1', ..., x_D')} = \omega(x_1, ..., x_D).$$

Substituting in the operators gives

$$\int dx_1' ... dx_D' \sum_A \partial_{x_A} \partial_{x_A'} \big(\delta(x_A - x_A')\phi^A(x_A)\big) \prod_{B\neq A} \phi^B(x_B) \prod_{C\neq A} \phi^C(x_C') \frac{\partial \ln J}{\partial \phi(x_1', ..., x_D')}$$

$$= 2\sum_A \partial_{x_A} \phi^A(x_A) \fint dz \frac{\phi^A(z)}{x_A - z} \prod_{B\neq A} \phi^B(x_B).$$

The sum can be dropped and one of the observers cancels. Integrating by parts with respect to x_A' on the left in order to deal with the delta function, and then cancelling the $\phi^A(x_A)$ results in

$$\int \prod_{B\neq A} [dx_B'] \prod_{C\neq A} \phi^C(x_C') \partial_{x_A} \frac{\partial \ln J}{\partial \phi(x_1, ..., x_D)} = 2 \fint dz \frac{\phi^A(z)}{x_A - z}.$$

Recalling the restriction (4.2) to deal with the integrals and observers on the left hand side, we end up with

$$N^{D-1} \partial_{x_A} \frac{\partial \ln J}{\partial \phi(x_1, ..., x_D)} = 2 \fint dz \frac{\phi^A(z)}{x_A - z}.$$

Since the ϕ^As are taken to be independent, the derivative on the left hand side can be written as

$$\frac{\partial}{\partial \phi^1(x_1)} \cdots \frac{\partial}{\partial \phi^D(x_D)} \ln J.$$

Substituting in the Jacobian yields

$$\frac{\partial}{\partial\phi^1(x_1)}\cdots\frac{\partial}{\partial\phi^D(x_D)}\ln\prod_A\Delta_A^2$$

$$=\frac{\partial}{\partial\phi^1(x_1)}\cdots\frac{\partial}{\partial\phi^D(x_D)}\sum_A\ln\Delta_A^2$$

$$=\sum_A\frac{\partial}{\partial\phi^1(x_1)}\cdots\frac{\partial}{\partial\phi^D(x_D)}\ln\Delta_A^2.$$

Looking once again at restriction (4.2) we see that

$$1=\frac{1}{N}\int dx_B\phi^B(x_B)$$

so we can insert one of these for each derivative except for the one with index A:

$$\sum_A\frac{\partial}{\partial\phi^1(x_1)}\cdots\frac{\partial}{\partial\phi^D(x_D)}\ln\Delta_A^2$$

$$=\sum_A\frac{\partial}{\partial\phi^1(x_1)}\cdots\frac{\partial}{\partial\phi^D(x_D)}\prod_{B\neq A}\left(\frac{1}{N}\int dx'_B\phi^B(x'_B)\right)\ln\Delta_A^2.$$

For a given functional derivative we have

$$\frac{\partial}{\partial\phi^B(x_B)}\int dx'_B\phi^B(x'_B)=\int dx'_B\delta(x'_B-x_B)=1.$$

Therefore

$$\sum_A\frac{\partial}{\partial\phi^1(x_1)}\cdots\frac{\partial}{\partial\phi^D(x_D)}\prod_{B\neq A}\left(\frac{1}{N}\int dx'_B\phi^B(x'_B)\right)\ln\Delta_A^2$$

$$=\frac{1}{N^{D-1}}\sum_A\frac{\partial}{\partial\phi^A(x_A)}\ln\Delta_A^2.$$

Going back to our original equation we now have

$$\partial_{x_A}\sum_B\frac{\partial}{\partial\phi^B(x_B)}\ln\Delta_B^2=2⨍ dz\frac{\phi^A(z)}{x_A-z}.$$

The ∂_{x_A} will pick out the term in the sum corresponding to A and thus we have exactly the same equation as in the single matrix case, which we know to be correct. Therefore we have seen that taking the initial collective field coordinates to be a product of traces corresponds to the correct Jacobian.

The contribution to the effective potential for the multiple matrix case is

$$\frac{1}{8}\int dx_1,...,dx_D \int dx'_1,...,dx'_D\, \omega(x_1,...,x_D)\Omega^{-1}(x_1,...,x_D,x'_1,...,x'_D)\omega(x'_1,...,x'_D)$$

$$= \frac{1}{8}\int dx_1,...,dx_D\, \omega(x_1,...,x_D)\frac{\partial \ln J}{\partial\phi(x_1,...,x_D)}$$

$$= \frac{1}{8}\int dx_1,...,dx_D \sum_A \left(2\partial_{x_A}\phi^A(x_A) ⨍ dz\frac{\phi^A(z)}{x_A - z}\prod_{B\neq A}\phi^B(x_B)\right)$$
$$\times\left(\frac{2}{N^{D-1}} ⨍ dz\phi^A(z)\ln(x_A - z)\right)$$

Similarly to before the integrals that don't involve the index A annihilate the observers into a factor of N^{D-1} which cancels, then integrating by parts with respect to x_A gives us

$$\frac{1}{2}\sum_A \int dx_A\phi^A(x_A)\left(⨍ dz\frac{\phi^A(z)}{x_A - z}\right)^2$$

which in the same way as the single matrix case becomes

$$\frac{\pi}{6}\sum_A \int dx_A\big(\phi^A(x_A)\big)^3.$$

Because of the ln in the Jacobian equation we did also consider defining the invariants in terms of a sum traces rather than a product, but this proved to be incorrect. It did however come close enough to be of interest and thus has been included in appendix A.

4.3 Multiple Matrices: Second Method

We have seen that the collective field invariants for the multiple matrix case can not be written as single traces of the original variables. There is however a way in which the invariants may be written that is both more elegant and brings out more clearly the eigenvalue dynamics of the system. Consider collective field coordinates defined as follows:

$$\phi_{k_1,\ldots,k_D} = \sum_{i=1}^{N} e^{ik_1\lambda_i^1}\ldots e^{ik_D\lambda_i^D} = \sum_{i=1}^{N} e^{i\vec{k}\cdot\vec{\lambda}_i}$$

where $\vec{k}$ is a vector representing the indices and $\vec{\lambda}_i$ is a vector representing the ith eigenvalues of the matrices. Another advantage of this formulation is that it involves only one sum, as opposed to having a sum for each trace as we had before.

Taking the fourier transform results in

$$\phi(x_1,\ldots,x_D) = \sum_{i=1}^{N} \delta(x_1-\lambda_i^1)\ldots\delta(x_D-\lambda_i^D)$$

which is just the definition of the vector delta function, so

$$\phi(x_1,\ldots,x_D) = \sum_{i=1}^{N} \delta(\vec{x}-\vec{\lambda}_i) \equiv \phi(\vec{x}).$$

In order to find the joining and splitting operators we need the derivatives in terms of the eigenvalues. They are as follows:

$$\frac{\partial}{\partial M_{ij}^A} = \sum_a U_{ai}^A (U^A)^\dagger_{ja} \frac{\partial}{\partial \lambda_a^A} + \sum_{a\neq b}\sum_d \frac{U_{ai}^A (U^A)^\dagger_{jb} U_{bd}^A}{\lambda_a^A - \lambda_b^A} \frac{\partial}{\partial U_{ad}^A}.$$

A proof of this can be found in appendix B. Since $\phi_{k_1,\ldots,k_D}$ depends only on the eigenvalues, the terms involving the derivative with respect to U^A will disappear. Therefore

$$\frac{\partial \phi_{k_1,\ldots,k_D}}{\partial M_{ij}^A} = \sum_a U_{ai}^A (U^A)^\dagger_{ja} \frac{\partial}{\partial \lambda_a^A} \sum_p e^{ik_1\lambda_p^1}\ldots e^{ik_D\lambda_p^D}$$

$$= \sum_a \sum_p U^A_{ai}(U^A)^\dagger_{ja} i k_A \delta_{pa} e^{ik_A\lambda^A_p} \prod_{B\neq A} e^{ik_B\lambda^B_p}$$

$$= \sum_a U^A_{ai}(U^A)^\dagger_{ja} i k_A e^{ik_A\lambda^A_a} \prod_{B\neq A} e^{ik_B\lambda^B_a}$$

The joining operator is then

$$\Omega_{k_1,\ldots,k_D,k'_1,\ldots,k'_D} = \sum_A \sum_{ij} \frac{\partial \phi_{k_1,\ldots,k_D}}{\partial M^A_{ij}} \frac{\partial \phi_{k'_1,\ldots,k'_D}}{\partial M^A_{ji}}.$$

$$= \sum_A \sum_{ij} \left(\sum_a U^A_{ai}(U^A)^\dagger_{ja} i k_A e^{ik_A\lambda^A_a} \prod_{B\neq A} e^{ik_B\lambda^B_a} \right) \left(\sum_b U^A_{bj}(U^A)^\dagger_{ib} i k'_A e^{ik'_A\lambda^A_b} \prod_{C\neq A} e^{ik'_C\lambda^C_b} \right)$$

$$= -\sum_A k_A k'_A \sum_a \sum_b \delta_{ab}\delta_{ab} e^{i(k_A+k'_A)\lambda^A_a} \left(\prod_{B\neq A} e^{ik_B\lambda^B_a} \right) \left(\prod_{C\neq A} e^{ik'_C\lambda^C_b} \right)$$

$$= -\sum_A k_A k'_A \sum_a e^{i(k_A+k'_A)\lambda^A_a} \prod_{B\neq A} e^{ik_B\lambda^B_a} e^{ik'_B\lambda^B_a}$$

$$= -\sum_A k_A k'_A \phi_{k_1+k'_1,\ldots,k_D+k'_D}$$

and its fourier transform is

$$\Omega(x_1,\ldots,x_D,x'_1,\ldots,x'_D) = \sum_A \partial_{x_A}\partial_{x'_A} \sum_a \delta(x_1-\lambda^1_a)\delta(x'_1-\lambda^1_a)\ldots\delta(x_D-\lambda^D_a)\delta(x'_D-\lambda^D_a)$$

$$= \sum_A \partial_{x_A}\partial_{x'_A} \sum_a \delta(x_1-x'_1)\delta(x_1-\lambda^1_a)\ldots\delta(x_D-x'_D)\delta(x_D-\lambda^D_a)$$

$$= \sum_A \partial_{x_A}\partial_{x'_A} \delta(\vec{x}-\vec{x}\,')\phi(\vec{x}).$$

The splitting operator will require the use of second derivatives with respect to the matrices. In the single matrix case this would look like

$$\sum_{ij} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} \phi_k.$$

This derivative can be written in terms of the eigenvalues and the "angular" matrices U, but it turns out that when this acts on a function that depends only on the eigenvalues of the matrix, all of the angular components disappear, and what remains is

$$\sum_a \frac{\partial^2}{\partial \lambda_a^2} + \sum_{a\neq b} \frac{2}{\lambda_a-\lambda_b}\frac{\partial}{\partial\lambda_a},$$

a proof of which can be found in appendix B. In our formulation each matrix is taken to be independent and thus the derivative will act only on the element of the product that corresponds to the same matrix, which itself is a function only of the eigenvalues. Therefore the joining operator in this instance will be

$$\omega_{k_1,\dots,k_D} = \sum_A \left(\sum_a \frac{\partial^2}{\partial {\lambda_a^A}^2} + \sum_{a\neq b} \frac{2}{\lambda_a^A - \lambda_b^A} \frac{\partial}{\partial \lambda_a^A} \right) \phi_{k_1,\dots,k_D}$$

$$= -\sum_A \sum_a k_A^2 e^{ik_A\lambda_a^A} \prod_{B\neq A} e^{ik_B\lambda_a^B} + ik_A \sum_A \sum_{a\neq b} \frac{2e^{ik_A\lambda_a^A}}{\lambda_a^A - \lambda_b^A} \prod_{B\neq A} e^{ik_B\lambda_a^B}.$$

The first term can be ignored for the same reasons as its equivalent in the previous section. Taking the fourier transform then yields

$$\omega(x_1,\dots,x_D) = 2\sum_A \sum_{a\neq b} \partial_{x_A} \frac{\delta(x_A - \lambda_a^A)}{\lambda_a^A - \lambda_b^A} \prod_{B\neq A} \delta(x_B - \lambda_a^B)$$

$$= 2\sum_A \partial_{x_A} \sum_{ab} \fint dy_A \frac{\delta(x_A - \lambda_a^A)\delta(y_A - \lambda_b^A)}{x_A - y_A} \prod_{B\neq A} \delta(x_B - \lambda_a^B)$$

$$= 2\sum_A \partial_{x_A}\phi(\vec{x}) \sum_b \fint dy_A \frac{\delta(y_A - \lambda_b^A)}{x_A - y_A}$$

$$= 2\sum_A \partial_{x_A}\phi(\vec{x}) \sum_b \fint dy_1 \dots dy_D \frac{\delta(y_1 - \lambda_b^1)\dots\delta(y_D - \lambda_b^D)}{x_A - y_A}$$

$$= 2\sum_A \partial_{x_A}\phi(\vec{x}) \fint d\vec{y}\, \frac{\phi(\vec{y})}{x_A - y_A}$$

where in the fourth line we have used the fact that for any B we have

$$\int dx_B\, \delta(x_B - \lambda_i^B) = 1. \tag{4.3}$$

The Jacobian equation is

$$\int dx_1' \dots dx_D' \Omega(x_1,\dots,x_D,x_1',\dots,x_D') \frac{\partial \ln J}{\partial \phi(x_1',\dots,x_D')} = \omega(x_1,\dots,x_D)$$

or more compactly

$$\int d\vec{x}\,'\, \Omega(\vec{x},\vec{x}\,') \frac{\partial \ln J}{\partial \phi(\vec{x}\,')} = \omega(\vec{x}). \tag{4.4}$$

We will now check that the Jacobian (4.1) is indeed a solution to (4.4). This can be done as follows:

$$\ln J = \ln \prod_A \prod_{i<j} (\lambda_i^A - \lambda_j^A)^2 = \ln \prod_A \prod_{i \neq j} |\lambda_i^A - \lambda_j^A| = \sum_A \sum_{i \neq j} \ln |\lambda_i^A - \lambda_j^A|$$

$$= \sum_A \sum_{ij} \int dx_A \fint dy_A \, \delta(x_A - \lambda_i^A) \delta(y_A - \lambda_j^A) \ln |x_A - y_A|.$$

Using equation (4.3) we can freely insert integrals into the equation above, resulting in

$$\sum_A \sum_{ij} \int dx_1 ... dx_D \fint dy_1 ... dy_D \delta(x_1 - \lambda_i^1) ... \delta(x_D - \lambda_i^D) \delta(y_1 - \lambda_j^1) ... \delta(y_D - \lambda_j^D) \ln |x_A - y_A|$$

$$= \sum_A \int d\vec{x} \fint d\vec{y} \, \phi(\vec{x}) \phi(\vec{y}) \ln |x_A - y_A|.$$

We can now approach the derivative:

$$\frac{\partial \ln J}{\partial \phi(\vec{x})} = \frac{\partial}{\partial \phi(\vec{x})} \sum_A \int d\vec{z} \fint d\vec{y} \, \phi(\vec{z}) \phi(\vec{y}) \ln |z_A - y_A|$$

$$= \sum_A \left(\int d\vec{z} \fint d\vec{y} \, \delta(\vec{z} - \vec{x}) \phi(\vec{y}) \ln |z_A - y_A| + \int d\vec{z} \fint d\vec{y} \, \phi(\vec{z}) \delta(\vec{y} - \vec{x}) \ln |z_A - y_A| \right)$$

$$= 2 \sum_A \int d\vec{z} \fint d\vec{y} \, \delta(\vec{z} - \vec{x}) \phi(\vec{y}) \ln |z_A - y_A|$$

$$= 2 \sum_A \int dz_1 ... dz_D \fint d\vec{y} \, \delta(z_1 - x_1) ... \delta(z_1 - x_1) \phi(\vec{y}) \ln |z_A - y_A|$$

$$= 2 \sum_A \fint d\vec{y} \, \phi(\vec{y}) \ln |x_A - y_A|.$$

Inserting this into the left hand side of equation (4.4) gives us

$$\int d\vec{x}\,' \sum_A \left(\partial_{x_A} \partial_{x'_A} \delta(\vec{x} - \vec{x}\,') \phi(\vec{x}) \right) 2 \sum_B \fint d\vec{y} \, \phi(\vec{y}) \ln |x'_B - y_B|.$$

Integration by parts on x'_A allows the derivative to pick out the A term from the second sum:

$$= 2 \sum_A \partial_{x_A} \int d\vec{x}\,' \, \delta(\vec{x} - \vec{x}\,') \phi(\vec{x}) \fint d\vec{y} \, \frac{\phi(\vec{y})}{x'_A - y_A}$$

$$= 2 \sum_A \partial_{x_A} \phi(\vec{x}) \fint d\vec{y} \, \frac{\phi(\vec{y})}{x_A - y_A}$$

which is the same as $\omega(\vec{x})$. Thus we have shown that equation (4.4) is satisfied and taking the collective field invariants in this way does indeed return the expected Jacobian.

Chapter 5

Radial Sector

5.1 Overview

The fact that the method used in the previous section resulted in the correct Jacobian is encouraging, and in this chapter we will apply it to a system of complex matrices in order to see what can be learned. Before that however we will examine the current understanding of this context as illustrated in [14]. The system under investigation is based on an even number of hermitian matrices M^A, $A = 1, ..., d = 2m$, complexified as follows:

$$Z^A = M^{2A-1} + iM^{2A}.$$

Rather than considering the eigenvalues of each hermitian matrix individually, we consider the eigenvalues of a sum over these complex matrices:

$$\sum_A^m (Z^A)^\dagger Z^A.$$

This quantity is positive definite and its eigenvalues, denoted by ρ_i or r_i^2, $i = 1, ..., N$, can be interpreted as the eigenvalues of a matrix valued radial coordinate.

5.2 Single Complex Matrix

First we will examine the procedure for a single complex matrix

$$Z = M^1 + iM^2.$$

The collective field coordinates we begin with are

$$\phi_k = Tre^{ikZ^\dagger Z} = \sum_i e^{ik\rho_i}$$

and

$$\phi(x) = \int dke^{-ikx}\phi_k = \sum_i \delta(x - \rho_i).$$

Identity (3.3) now becomes

$$\frac{\partial}{\partial Z_{ij}}\left(e^{ikZ^\dagger Z}\right)_{ab} = ik\int_0^1 d\alpha Z^\dagger_{ai}\left(e^{i\alpha kZ^\dagger Z}\right)_{jc}\left(e^{i(1-\alpha)kZ^\dagger Z}\right)_{cb}$$

and

$$\frac{\partial}{\partial Z^\dagger_{ij}}\left(e^{ikZ^\dagger Z}\right)_{ab} = ik\int_0^1 d\alpha\left(e^{i\alpha kZ^\dagger Z}\right)_{ai}Z_{jc}\left(e^{i(1-\alpha)kZ^\dagger Z}\right)_{cb}.$$

From these we obtain

$$\frac{\partial\phi_k}{\partial Z^\dagger_{ij}} = ik\left(Ze^{ikZ^\dagger Z}\right)_{ji}, \frac{\partial\phi_k}{\partial Z_{ij}} = ik\left(e^{ikZ^\dagger Z}Z^\dagger\right)_{ji}$$

and

$$\frac{\partial^2\phi_k}{\partial Z^\dagger_{ij}\partial Z_{ji}} = \frac{\partial}{\partial Z^\dagger_{ij}}ik\left(e^{ikZ^\dagger Z}Z^\dagger\right)_{ij}$$

$$= (ik)^2\int_0^1 d\alpha\left(e^{i\alpha kZ^\dagger Z}\right)_{ii}Z_{jc}\left(e^{i(1-\alpha)kZ^\dagger Z}\right)_{cq}Z^\dagger_{qj} + ik\left(e^{ikZ^\dagger Z}\right)_{jq}\delta_{ii}\delta_{qj}$$

$$= -k\int_0^k dk'\left(Tre^{ik'Z^\dagger Z}\right)\left(TrZ^\dagger Ze^{i(k-k')Z^\dagger Z}\right) + ikN\left(Tre^{ikZ^\dagger Z}\right).$$

We can now find the joining and splitting operators:

$$\Omega_{kk'} = \frac{\partial\phi_k}{\partial Z^\dagger_{ij}}\frac{\partial\phi_{k'}}{\partial Z_{ji}} = -kk'TrZ^\dagger Ze^{i(k+k')Z^\dagger Z},$$

and

$$\omega_k = \frac{\partial^2 \phi_k}{\partial Z^\dagger_{ij} \partial Z_{ji}} = -k \int_0^k dk' \phi_{k'} \left(TrZ^\dagger Z e^{i(k-k')Z^\dagger Z} \right) + ikN\phi_k$$

$$= -k \sum_i \sum_j \int_0^k dk' e^{ik'\rho_i} \rho_j e^{i(k-k')\rho_j} + ikN \sum_i e^{ik\rho_i}$$

$$= -k \sum_{i \neq j} \rho_j \int_0^k dk' e^{ik'\rho_i} e^{i(k-k')\rho_j} - k^2 \sum_i \rho_i e^{ik\rho_i} + ikN \sum_i e^{ik\rho_i}$$

$$= ik \sum_{i \neq j} \rho_j \frac{e^{ik\rho_i} - e^{ik\rho_j}}{\rho_i - \rho_j} - k^2 \sum_i \rho_i e^{ik\rho_i} + ikN \sum_i e^{ik\rho_i}$$

$$= -2ik \sum_{i \neq j} \frac{\rho_j e^{ik\rho_j}}{\rho_i - \rho_j} - ik(N-1) \sum_i e^{ik\rho_i} - k^2 \sum_i \rho_i e^{ik\rho_i} + ikN \sum_i e^{ik\rho_i}$$

$$= -2ik \sum_{i \neq j} \frac{\rho_j e^{ik\rho_j}}{\rho_i - \rho_j} - k^2 \sum_i \rho_i e^{ik\rho_i} + ik \sum_i e^{ik\rho_i}$$

as well as their coordinate representation counterparts:

$$\Omega(x, x') = \int dk \int dk' e^{-ikx} e^{-ik'x'} \Omega_{kk'}$$

$$= \partial_x \partial_{x'} (x\phi(x)\delta(x - x'))$$

and

$$\omega(x) = \int dk e^{-ikx} \omega_k$$

$$= \partial_x \left(2x\phi(x) \fint dy \frac{\phi(y)}{x - y} + \phi(x) - \partial_x (x\phi(x)) \right)$$

$$= \partial_x \left(x\phi(x) \left(2 \fint dy \frac{\phi(y)}{x - y} - \frac{\partial_x \phi(x)}{\phi(x)} \right) \right).$$

Substituting Ω and ω into the equation for the Jacobian and ignoring terms not of leading order in N yields

$$\partial_x \frac{\partial \ln J}{\partial \phi(x)} = 2 \fint dy \frac{\phi(y)}{x - y}$$

the solution to which is

$$J = \prod_{i<j} (\rho_i - \rho_j)^2$$

in agreement with what was found in [14] but for a prefactor pertaining to a change of variables from ρ_i to $r_i = \sqrt{\rho_i}$.

The contribution to the potential (the second term in (3.2)) is given by

$$\frac{1}{8}\int dx \int dx' \omega(x)\Omega^{-1}(x,x')\omega(x')$$

$$= \frac{1}{8}\int dx \omega(x) \frac{\partial \ln J}{\partial \phi(x)}$$

$$= \frac{1}{2}\int dx \partial_x \left(x\phi(x) \left(⨍ dy \frac{\phi(y)}{x-y} \right) \right) \left(⨍ dz \phi(z) \ln|x-z| \right)$$

$$= \frac{1}{2}\int dx x\phi(x) \left(⨍ dy \frac{\phi(y)}{x-y} \right)^2 .$$

This can be simplified by introducing a new density of eigenvalues $\Phi(r) = 2r\phi(r^2)$ with the condition that $\Phi(-r) = \Phi(r)$ and then changing the variables of integration:

$$x = r^2, y = q^2$$

$$dx = 2rdr, dy = 2qdq.$$

Now

$$\frac{1}{2}\int_0^\infty dx x\phi(x) \left(⨍_0^\infty dy \frac{\phi(y)}{x-y} \right)^2$$

$$= \frac{1}{2}\int_0^\infty dr 2r r^2 \phi(r^2) \left(⨍_0^\infty dq 2q \frac{\phi(q^2)}{r^2-q^2} \right)^2$$

$$= \frac{1}{2}\int_0^\infty dr r^2 \Phi(r) \left(⨍_0^\infty dq \frac{\Phi(q)}{r^2-q^2} \right)^2$$

$$= \frac{1}{2}\int_0^\infty dr \Phi(r) \left(⨍_0^\infty dq \Phi(q) \frac{r}{r^2-q^2} \right)^2$$

$$= \frac{1}{8} \int_0^\infty dr \Phi(r) \Big(⨍_0^\infty dq \Phi(q) \frac{2r + q - q}{(r-q)(r+q)} \Big)^2$$

$$= \frac{1}{8} \int_0^\infty dr \Phi(r) \Big(⨍_0^\infty dq \Phi(q) \frac{1}{(r-q)} + ⨍_0^\infty dq \Phi(q) \frac{1}{(r+q)} \Big)^2$$

$$= \frac{1}{8} \int_0^\infty dr \Phi(r) \Big(⨍_0^\infty dq \Phi(q) \frac{1}{(r-q)} + ⨍_{-\infty}^0 dq \Phi(q) \frac{1}{(r-q)} \Big)^2$$

$$= \frac{1}{16} \int_{-\infty}^\infty dr \Phi(r) \Big(⨍_{-\infty}^\infty dq \frac{\Phi(q)}{(r-q)} \Big)^2.$$

Using the same identity as before we find then that the contribution is

$$\frac{\pi^2}{48} \int_{-\infty}^\infty dr \Phi^3(r).$$

5.3 Multiple Complex Matrices

For the general case of $m > 1$ the collective field coordinates we begin with are

$$\phi_k = Tr e^{ik \sum_A^m (Z^A)^\dagger Z^A} = \sum_i e^{ik\rho_i} \,,$$

$$\phi(x) = \int dk e^{-ikx} \phi_k = \sum_i \delta(x - \rho_i).$$

The derivatives are

$$\frac{\partial \phi_k}{\partial (Z^A)^\dagger_{ij}} = ik \Big(Z^A e^{ik \sum_B (Z^B)^\dagger Z^B} \Big)_{ji} \,, \; \frac{\partial \phi_k}{\partial Z^A_{ij}} = ik \Big(e^{ik \sum_B (Z^B)^\dagger Z^B} (Z^A)^\dagger \Big)_{ji}.$$

The joining and splitting operators are

$$\Omega_{kk'} = \sum_A \frac{\partial \phi_k}{\partial (Z^A)^\dagger_{ij}} \frac{\partial \phi'_k}{\partial Z^A_{ji}} = -kk' Tr \Big((Z^A)^\dagger Z^A e^{i(k+k') \sum_B (Z^B)^\dagger Z^B} \Big),$$

$$\Omega(x, x') = \int dk \int dk' e^{-ikx} e^{-ik'x'} \Omega_{kk'} = \partial_x \partial_{x'} (x \phi(x) \delta(x - x'))$$

and

$$\omega_k = \sum_A \frac{\partial^2 \phi_k}{\partial (Z^A)^\dagger_{ij} \partial Z^A_{ji}} = -k \sum_{ij} \int_0^k dk' e^{ik'\rho_i} e^{i(k-k')\rho_j} + ikmN \sum_i e^{ik\rho_i}$$

which, through a process similar to that in section 4.2, results in

$$\omega(x) = \partial_x \Bigg(x\phi(x) \Bigg(2 \fint dy \frac{\phi(y)}{x-y} + \frac{N(m-1)}{x} \Bigg) \Bigg).$$

The equation for the Jacobian is then

$$\partial_x \frac{\partial \ln J}{\partial \phi(x)} = 2 \fint dy \frac{\phi(y)}{x-y} + \frac{N(m-1)}{x}$$

$$\begin{aligned} \ln J &= 2 \int dx \fint dy \phi(x)\phi(y) \ln|x-y| + \int dx \phi(x) N(m-1) \ln|x| \\ &= 2 \sum_{i \neq j} \ln|\rho_i - \rho_j| + N(m-1) \sum_i \ln(\rho_i) \\ &= \ln \prod_{i<j} (\rho_i - \rho_j)^2 + \ln \prod_i (\rho_i)^{N(M-1)} \end{aligned}$$

thus the Jacobian for this case is

$$J = \Bigg(\prod_{i<j} (\rho_i - \rho_j)^2 \Bigg) \Bigg(\prod_i (\rho_i)^{N(M-1)} \Bigg).$$

The contribution to the potential is

$$\frac{1}{8} \int dx \omega(x) \frac{\partial \ln J}{\partial \phi(x)}$$

$$= \frac{1}{8} \int dx \partial_x \Bigg(x\phi(x) \Bigg(2 \fint dy \frac{\phi(y)}{x-y} + \frac{N(m-1)}{x} \Bigg) \Bigg) \Bigg(2 \fint dz \phi(z) \ln|x-z| + N(m-1) \ln|x| \Bigg)$$

$$= \frac{1}{2} \int dx\, x\phi(x) \Bigg(\fint dy \frac{\phi(y)}{x-y} + \frac{N(m-1)}{x} \Bigg)^2$$

$$= \frac{\pi^2}{12} \int dx\, x\phi^3(x) + \frac{N(m-1)}{2} \int dx \phi(x) \Bigg(\fint dy \frac{\phi(y)}{x-y} \Bigg) + \frac{N^2(m-1)^2}{2} \int dx \frac{\phi(x)}{x}$$

The second term seems ungainly but it turns out that it does not contribute when one does the minimisation of the effective potential. This can be seen as follows:

$$\frac{\partial}{\partial \phi(x')} \Bigg(\frac{N(m-1)}{2} \int dx \phi(x) \Bigg(\fint dy \frac{\phi(y)}{x-y} \Bigg) \Bigg)$$

$$= \frac{N(m-1)}{2} \int dx \delta(x-x') \left(\fint dy \frac{\phi(y)}{x-y} \right) + \frac{N(m-1)}{2} \int dx \phi(x) \left(\fint dy \frac{\delta(y-x')}{x-y} \right)$$
$$= \frac{N(m-1)}{2} \fint dy \frac{\phi(y)}{x'-y} + \frac{N(m-1)}{2} \int dx \fint dy \frac{\phi(x)\delta(y-x')}{x-y}$$
$$= \frac{N(m-1)}{2} \fint dy \frac{\phi(y)}{x'-y} + \frac{N(m-1)}{2} \int dy \fint dx \frac{\phi(x)\delta(y-x')}{x-y}$$
$$= \frac{N(m-1)}{2} \fint dy \frac{\phi(y)}{x'-y} + \frac{N(m-1)}{2} \fint dx \frac{\phi(x)}{x-x'}$$
$$= \frac{N(m-1)}{2} \fint dy \frac{\phi(y)}{x'-y} - \frac{N(m-1)}{2} \fint dx \frac{\phi(x)}{x'-x} = 0.$$

Therefore the effective contribution will be

$$\frac{\pi^2}{48} \int dr \Phi^3(r) + \frac{N^2(m-1)^2}{2} \int dr \frac{\Phi(r)}{r^2}$$

wherein the change from $\phi(x)$ to $\Phi(r)$ happens in the same way as in the previous section. Note that there appears a $\frac{1}{r^2}$ potential when $m > 1$ ($d > 2$).

5.4 Angular Considerations

We have examined the radial dynamics of the complex matrix formulation, which is effectively a one dimensional formulation. In this section we will investigate a two dimensional formulation, with both a radial and an angular component. This is by no means a simple matter, but the method used in section 4.3 does present an interesting approach that has not been used before for this purpose as far as we know. We will use it to attempt to find a Jacobian that encapsulates both the radial and angular dynamics.

We will write the complex matrix Z as RU, where R is a hermitian matrix representing the radial dimension and U is a unitary matrix representing the angular dimension. This is analogous to normal complex numbers where $z = re^{i\theta}$. Presently the only sectors that have been considered are those that depend only on the eigenvalues of R, but we will restrict our investigation to a sector that depends only on the eigenvalues of R and U. These can be diagonalised such that $R = V^\dagger r V$ and $U = W^\dagger q W$ where r and q are diagonal matrices of the eigenvalues of R and U respectively, and $q_i = e^{i\theta_i}$. We need an expression for $\frac{\partial}{\partial Z}$ in terms of R and U that satisfies the condition

$$\frac{\partial}{\partial Z_{ij}} Z_{cd} = \delta_{ci}\delta_{dj}.$$

This will be

$$\frac{\partial}{\partial Z_{ij}} = \frac{1}{2}\big(U^\dagger_{ja}\frac{\partial}{\partial R_{ia}} + R^{-1}_{ai}\frac{\partial}{\partial U_{aj}}\big).$$

Therefore

$$\begin{aligned} &\frac{\partial}{\partial Z_{ij}} Z_{cd} \\ &= \frac{1}{2}\big(U^\dagger_{ja}\delta_{ci}\delta_{ea}U_{ed} + R^{-1}_{ai}R_{ce}\delta_{ea}\delta_{dj}\big) \\ &= \frac{1}{2}\big(U^\dagger_{ja}\delta_{ci}U_{ad} + R^{-1}_{ai}R_{ca}\delta_{dj}\big) \\ &= \frac{1}{2}\big(\delta_{dj}\delta_{ci} + \delta_{ci}\delta_{dj}\big) \\ &= \delta_{ci}\delta_{dj}. \end{aligned}$$

Similarly for $Z^\dagger = U^\dagger R$ we can write

$$\frac{\partial}{\partial Z^\dagger_{ij}} = \frac{1}{2}\big(U_{ai}\frac{\partial}{\partial R_{aj}} + R^{-1}_{ja}\frac{\partial}{\partial U^\dagger_{ia}}\big).$$

This will result in

$$\begin{aligned} &\frac{\partial}{\partial Z^\dagger_{ij}} Z^\dagger_{cd} \\ &= \frac{1}{2}\big(U_{ai}U^\dagger_{ce}\delta_{ea}\delta_{dj} + R^{-1}_{ja}\delta_{ci}\delta_{ea}R_{ed}\big) \\ &= \frac{1}{2}\big(U_{ei}U^\dagger_{ce}\delta_{dj} + R^{-1}_{je}\delta_{ci}R_{ed}\big) \\ &= \frac{1}{2}\big(\delta_{ci}\delta_{dj} + \delta_{dj}\delta_{ci}\big) \\ &= \delta_{ci}\delta_{dj}. \end{aligned}$$

The derivatives can be written in terms of these new variables using the following formulas obtained from appendix B:

$$\frac{\partial}{\partial R_{ij}} = \sum_p V_{pi} V^\dagger_{jp} \frac{\partial}{\partial r_p} + \sum_{pk} \sum_{b \neq p} \frac{V_{pi} V^\dagger_{jb} V_{bk}}{(r_p - r_b)} \frac{\partial}{\partial V_{pk}}$$

$$\frac{\partial}{\partial U_{ij}} = \sum_p W_{pi} W^\dagger_{jp} \frac{\partial}{\partial q_p} + \sum_{pk} \sum_{b \neq p} \frac{W_{pi} W^\dagger_{jb} W_{bk}}{(q_p - q_b)} \frac{\partial}{\partial W_{pk}}$$

$$\frac{\partial}{\partial U^\dagger_{ij}} = \sum_p W_{pi} W^\dagger_{jp} \frac{\partial}{\partial q^*_p} + \sum_{pk} \sum_{b \neq p} \frac{W_{pi} W^\dagger_{jb} W_{bk}}{(q^*_p - q^*_b)} \frac{\partial}{\partial W_{pk}}$$

In practice these derivatives will act on terms that depend only on q and r, so the right hand terms can be ignored. Substituting these back into the previous expressions yields

$$\frac{\partial}{\partial Z_{ij}} = \frac{1}{2}\big(U^\dagger_{ja} \sum_p V_{pi} V^\dagger_{ap} \frac{\partial}{\partial r_p} + R^{-1}_{ai} \sum_p W_{pa} W^\dagger_{jp} \frac{\partial}{\partial q_p}\big)$$

$$= \frac{1}{2}\big(\sum_{pl} W^\dagger_{jl} q^*_l W_{la} V_{pi} V^\dagger_{ap} \frac{\partial}{\partial r_p} + \sum_{pl} V^\dagger_{al} r^{-1}_l V_{li} W_{pa} W^\dagger_{jp} \frac{\partial}{\partial q_p}\big)$$

and

$$\frac{\partial}{\partial Z^\dagger_{ij}} = \frac{1}{2}\big(U_{ai} \sum_p V_{pa} V^\dagger_{jp} \frac{\partial}{\partial r_p} + R^{-1}_{ja} \sum_p W_{pi} W^\dagger_{ap} \frac{\partial}{\partial q^*_p}\big)$$

$$\frac{\partial}{\partial Z^\dagger_{ij}} = \frac{1}{2}\big(\sum_{pl} W^\dagger_{al} q_l W_{li} V_{pa} V^\dagger_{jp} \frac{\partial}{\partial r_p} + \sum_{pl} V^\dagger_{jl} r^{-1}_l V_{la} W_{pi} W^\dagger_{ap} \frac{\partial}{\partial q^*_p}\big)$$

The collective field coordinate is chosen to be

$$\phi_{k,m} = \sum_{i=1}^{N} e^{ikr_i} q_i^m = \sum_{i=1}^{N} e^{ikr_i} e^{im\theta_i}$$

and the quantity of interest is

$$\Omega_{k,m,k',m'} = \sum_{ij} \frac{\partial \phi_{k,m}}{\partial Z^\dagger_{ij}} \frac{\partial \phi_{k',m'}}{\partial Z_{ji}}.$$

This will consist of four terms which we will examine one at a time. In the following each index other than k, k', m and m' is summed over and we have omitted the sums. We have also used the fact that

$$\frac{\partial}{\partial q_j^*} q_i = -q_i^2 \delta_{ij}.$$

The first term is

$$\frac{1}{4}\left(W^\dagger_{al} q_l W_{li} V_{pa} V^\dagger_{jp} ike^{ikr_p} q_p^m\right)\left(W^\dagger_{in} q_n^* W_{nb} V_{sj} V^\dagger_{bs} ik' e^{ik' r_s} q_s^{m'}\right)$$

$$= \frac{1}{4}\left(W^\dagger_{al} q_l \delta_{ln} V_{pa} ike^{ikr_p} q_p^m q_n^* W_{nb} \delta_{ps} V^\dagger_{bs} ik' e^{ik' r_s} q_s^{m'}\right)$$

$$= \frac{1}{4}\left(W^\dagger_{al} q_l V_{pa} ike^{ikr_p} q_p^m q_l^* W_{lb} V^\dagger_{bp} ik' e^{ik' r_p} q_p^{m'}\right)$$

$$= -kk'\frac{1}{4}\left(W^\dagger_{al} V_{pa} W_{lb} V^\dagger_{bp} e^{i(k+k')r_p} q_p^{m+m'}\right)$$

$$= -kk'\frac{1}{4}\left(\delta_{ab} V_{pa} V^\dagger_{bp} e^{i(k+k')r_p} q_p^{m+m'}\right)$$

$$= -kk'\frac{1}{4}\left(e^{i(k+k')r_p} q_p^{m+m'}\right)$$

The second term is

$$\frac{1}{4}\left(W^\dagger_{al} q_l W_{li} V_{pa} V^\dagger_{jp} ike^{ikr_p} q_p^m\right)\left(V^\dagger_{bn} r_n^{-1} V_{nj} W_{sb} W^\dagger_{is} m' e^{ik' r_s} q_s^{m'-1}\right)$$

$$= \frac{1}{4}\left(W^\dagger_{al} q_l \delta_{ls} V_{pa} \delta_{np} ike^{ikr_p} q_p^m V^\dagger_{bn} r_n^{-1} W_{sb} m' e^{ik' r_s} q_s^{m'-1}\right)$$

$$= \frac{1}{4}\left(W^\dagger_{al} q_l V_{pa} ike^{ikr_p} q_p^m V^\dagger_{bp} r_p^{-1} W_{lb} m' e^{ik' r_l} q_l^{m'-1}\right)$$

$$= \frac{1}{4}\frac{ikm'}{r_p}\left(W^\dagger_{al} V_{pa} V^\dagger_{bp} W_{lb} e^{ikr_p} q_p^m e^{ik' r_l} q_l^{m'}\right)$$

The third term is

$$-\frac{1}{4}\left(V^\dagger_{jl} r_l^{-1} V_{la} q_p W_{pi} W^\dagger_{ap} q_p m e^{ikr_p} q_p^{m-1}\right)\left(W^\dagger_{in} q_n^* W_{nb} V_{sj} V^\dagger_{bs} ik' e^{ik' r_s} q_s^{m'}\right)$$

$$= -\frac{1}{4}\left(\delta_{ls} r_l^{-1} V_{la} q_p \delta_{pn} W^\dagger_{ap} q_p m e^{ikr_p} q_p^{m-1} q_n^* W_{nb} V^\dagger_{bs} ik' e^{ik' r_s} q_s^{m'}\right)$$

$$= -\frac{1}{4}\left(r_l^{-1} V_{la} q_p W_{ap}^{\dagger} q_p m e^{ikr_p} q_p^{m-1} q_p^* W_{pb} V_{bl}^{\dagger} ik' e^{ik' r_l} q_l^{m'} \right)$$

$$= -\frac{1}{4}\frac{ik'm}{r_l}\left(V_{la} W_{ap}^{\dagger} W_{pb} V_{bl}^{\dagger} e^{ikr_p} q_p^{m} e^{ik' r_l} q_l^{m'} \right)$$

The fourth term is

$$-\frac{1}{4}\left(V_{jl}^{\dagger} r_l^{-1} V_{la} q_p W_{pi} W_{ap}^{\dagger} q_p m e^{ikr_p} q_p^{m-1} \right)\left(V_{bn}^{\dagger} r_n^{-1} V_{nj} W_{sb} W_{is}^{\dagger} m' e^{ik' r_s} q_s^{m'-1} \right)$$

$$= -\frac{1}{4}\left(\delta_{ln} r_l^{-1} V_{la} q_p \delta_{ps} W_{ap}^{\dagger} q_p m e^{ikr_p} q_p^{m-1} V_{bn}^{\dagger} r_n^{-1} W_{sb} m' e^{ik' r_s} q_s^{m'-1} \right)$$

$$= -\frac{1}{4}\left(r_l^{-1} V_{la} q_p W_{ap}^{\dagger} q_p m e^{ikr_p} q_p^{m-1} V_{bl}^{\dagger} r_l^{-1} W_{pb} m' e^{ik' r_p} q_p^{m'-1} \right)$$

$$= -\frac{1}{4}\frac{mm'}{r_l^2}\left(V_{la} W_{ap}^{\dagger} V_{bl}^{\dagger} W_{pb} e^{ikr_p} q_p^{m} e^{ik' r_p} q_p^{m'} \right)$$

Thus Ω is (after swapping the indices p and l in terms 3 and 4)

$$-kk'\frac{1}{4}\left(e^{i(k+k')r_p} q_p^{m+m'} \right)$$

$$+\frac{1}{4}\frac{W_{al}^{\dagger} V_{pa} V_{bp}^{\dagger} W_{lb}}{r_p}\left(ikm' e^{ikr_p} q_p^{m} e^{ik' r_l} q_l^{m'} - ik'm e^{ikr_l} q_l^{m} e^{ik' r_p} q_p^{m'} - \frac{mm'}{r_p} e^{ikr_l} q_l^{m} e^{ik' r_l} q_l^{m'} \right)$$

The fourier transform of the collective field coordinate is

$$\phi(x,y) = \int dk \int dm\, e^{-ikx} e^{-imy} \sum_{i=1}^{N} e^{ikr_i} q_i^m = \int dk \int dm\, e^{-ikx} e^{-imy} \sum_{i=1}^{N} e^{ikr_i} e^{im\theta_i}$$

$$= \sum_{i=1}^{N} \delta(x - r_i)\delta(y - \theta_i).$$

Thus the fourier transform of the joining operator is

$$\Omega(x,y,x',y') = \int dk \int dm\, e^{-ikx} e^{-imy} \int dk' \int dm'\, e^{-ik'x'} e^{-im'y'} \Omega_{k,m,k',m'}$$

$$= \frac{1}{4}\partial_x\partial'_x\delta(x-r_p)\delta(y-\theta_p)\delta(x'-r_p)\delta(y'-\theta_p)$$

$$-\frac{1}{4}\frac{W^\dagger_{al}V_{pa}V^\dagger_{bp}W_{lb}}{r_p}\Bigg(i\partial_x\partial_{y'}\delta(x-r_p)\delta(y-\theta_p)\delta(x'-r_l)\delta(y'-\theta_l)$$

$$-i\partial_{x'}\partial_y\delta(x-r_l)\delta(y-\theta_l)\delta(x'-r_p)\delta(y'-\theta_p)$$

$$-\frac{\partial_y\partial_{y'}}{r_p}\delta(x-r_l)\delta(y-\theta_l)\delta(x'-r_l)\delta(y'-\theta_l)\Bigg).$$

The quantity in the numerator can be written as $(VW^\dagger)_{pl}(WV^\dagger)_{lp}$. $VW^\dagger$ is the same thing that appears in the middle of RU when written in diagonalised terms: $RU = V^\dagger rVW^\dagger qW$. The fact that this does not cancel means that this operator does not close loops in a succinct manner, and thus the Jacobian cannot be found as easily as was hoped. For the sake of completeness we will still investigate the splitting operator, which is defined as

$$\omega_{k,m} = \sum_{ij}\frac{\partial^2\phi_{km}}{\partial Z^\dagger_{ij}\partial Z_{ji}}.$$

For the Z derivative we can once again ignore the derivatives other than r and q, but this is not the case for the $Z^\dagger$ derivative. Its full form will be

$$\frac{\partial}{\partial Z^\dagger_{ij}} = \frac{1}{2}\Bigg(W^\dagger_{ae}q_eW_{ei}V_{pa}V^\dagger_{jp}\frac{\partial}{\partial r_p} + W^\dagger_{ae}q_eW_{ei}\frac{V_{pa}V^\dagger_{jb}V_{bk}}{(r_p-r_b)}\frac{\partial}{\partial V_{pk}}$$

$$+V^\dagger_{jd}r_d^{-1}V_{da}W_{pi}W^\dagger_{ap}\frac{\partial}{\partial q^*_p} + V^\dagger_{jd}r_d^{-1}V_{da}\frac{W_{pi}W^\dagger_{ab}W_{bk}}{(q^*_p-q^*_b)}\frac{\partial}{\partial W_{pk}}\Bigg).$$

Thus the splitting operator will be

$$\frac{1}{2}\Bigg(W^\dagger_{ae}q_eW_{ei}V_{pa}V^\dagger_{jp}\frac{\partial}{\partial r_p} + W^\dagger_{ae}q_eW_{ei}\frac{V_{pa}V^\dagger_{jb}V_{bk}}{(r_p-r_b)}\frac{\partial}{\partial V_{pk}}$$

$$+V^\dagger_{jd}r_d^{-1}V_{da}W_{pi}W^\dagger_{ap}\frac{\partial}{\partial q^*_p} + V^\dagger_{jd}r_d^{-1}V_{da}\frac{W_{pi}W^\dagger_{ab}W_{bk}}{(q^*_p-q^*_b)}\frac{\partial}{\partial W_{pk}}\Bigg)$$

$$\Big(W^\dagger_{in}q^*_nW_{nb}V_{sj}V^\dagger_{bs}ik'e^{ik'r_s}q_s^{m'} + V^\dagger_{bn}r_n^{-1}V_{nj}W_{sb}W^\dagger_{is}m'e^{ik'r_s}q_s^{m'-1}\Big)$$

which, after some manipulation, results in

$$\frac{1}{2}\Bigg(-k'^2 e^{ik'r_p} q_p^{m'} - W^\dagger_{as} V_{pa} V^\dagger_{bp} r_p^{-2} W_{sb} m' e^{ik'r_s} q_s^{m'} + ik' W^\dagger_{ap} V_{pa} V^\dagger_{bp} r_p^{-1} W_{pb} m' e^{ik'r_p} q_p^{m'}$$

$$\frac{V_{pb}V^\dagger_{bp}}{(r_p - r_b)} ik' e^{ik'r_p} q_p^{m'} - \frac{V_{pb}V^\dagger_{bp}}{(r_p - r_b)} ik' e^{ik'r_b} q_b^{m'} - W^\dagger_{as} \frac{V_{pa}V^\dagger_{bp}}{(r_p - r_b)} r_b^{-1} W_{sb} m' e^{ik'r_s} q_s^{m'}$$

$$+W^\dagger_{as} \frac{V_{pa}V^\dagger_{bp}}{(r_p - r_b)} r_p^{-1} W_{sb} m' e^{ik'r_s} q_s^{m'} + r_s^{-1} ik' e^{ik'r_s} q_s^{m'} + m' r_p^{-1} V_{pa} W^\dagger_{ap} W_{pb} V^\dagger_{bp} ik' e^{ik'r_p} q_p^{m'-2}$$

$$-(m'-1) r_n^{-2} V_{na} W^\dagger_{ap} V^\dagger_{bn} W_{pb} m' e^{ik'r_p} q_p^{m'} - r_s^{-1} V_{sa} \frac{W^\dagger_{ab} W_{bb}}{(q_p^* - q_b^*)} q_b^* V^\dagger_{bs} ik' e^{ik'r_s} q_s^{m'}$$

$$+r_s^{-1} V_{sa} \frac{W^\dagger_{ab} W_{bb}}{(q_p^* - q_b^*)} q_p^* V^\dagger_{bs} ik' e^{ik'r_s} q_s^{m'} + r_n^{-2} V_{na} \frac{W^\dagger_{ab} W_{bb}}{(q_p^* - q_b^*)} V^\dagger_{bn} m' e^{ik'r_p} q_p^{m'-1}$$

$$-r_n^{-2} V_{na} \frac{W^\dagger_{ab} W_{bb}}{(q_p^* - q_b^*)} V^\dagger_{bn} m' e^{ik'r_b} q_b^{m'-1} \Bigg).$$

Once again there is no obvious simplification to remove the matrix components.

Chapter 6

Conclusion

In this dissertation we set out to determine if the methods of the collective field theory could be applied to consistently describe a large N formulation of multiple matrix models depending only on the eigenvalues of the matrices, and subsequently to see if the insights gained from doing so could be applied to a more general system dependent on matrix-valued curvilinear coordinates.

In the case of multiple matrices we found that the collective field theory does in fact prove an effective tool for investigating such systems, although its ease of use is somewhat complicated by the need to find a suitable set of invariants. We have shown that finding these invariants can be aided by understanding the Jacobian of the system. Additionally we have shown that the invariants do not necessarily have to be single trace operators in terms of the original variables for the method to work, as the two approaches considered did reveal the Jacobian, both for a single hermitian matrix and then generalised to multiple hermitian matrices. Finally, we also noted that the repulsive potential associated with emergent geometry appeared in our formulation.

In the case of a more general system dependent on matrix-valued curvilinear coordinates, we found that the set of starting invariant operators does not close under joining, and thus the Jacobian cannot be found as easily as was hoped. For completeness of the approach the splitting operator was also derived.

A more complete understanding of emergent geometry from multiple matrix systems would likely require both a radial and an angular dimension. This particular attempt to investigate this did not fulfil that aim, but we would suggest that the insight that collective field invariants need not necessarily be single trace operators could open up new avenues for approaching this problem in the future.

Finally, the hints of emergent geometry suggest that there might well be a gravity theory with which these systems are associated, as per the AdS/CFT correspondence, and that this could also be a direction for further investigation.

Appendix A

Initial Attempt at Multiple Hermitian Matrices

This appendix details the first method we attempted to apply the collective field method to a system of Hermitian matrices wherein the invariants were defined as a sum of traces rather than a product. Consider the following collective field coordinate:

$$\phi_k = \sum_A \phi_k^A = \sum_A Tr e^{ikM^A}$$

and its fourier transform

$$\phi(x) = \int dk e^{-ikx} \phi_k = \sum_A \sum_i \delta(x - \lambda_i^A) = \sum_A \phi^A(x).$$

Using identity (3.3) we find that

$$\frac{\partial \phi_k}{\partial M_{ij}^A} = ik(e^{ikM^A})_{ji},$$

$$\frac{\partial^2 \phi_k}{\partial M_{ij}^A \partial M_{ji}^A} = -k \int_0^k dk' \phi_{k'}^A \phi_{k-k'}^A.$$

The joining and splitting operators are

$$\Omega_{kk'} = \sum_A \frac{\partial \phi_k}{\partial M_{ij}^A} \frac{\partial \phi_k'}{\partial M_{ji}^A}$$

$$= -kk' \sum_A Tr(e^{ikM^A} e^{ik'M^A})$$

$$= -kk' \sum_A \sum_i e^{i(k+k')\lambda_i^A},$$

and

$$\omega_k = \sum_A \frac{\partial^2 \phi_k}{\partial M_{ij}^A \partial M_{ji}^A} = -\sum_A k \int_0^k dk' \phi_{k'}^A \phi_{k-k'}^A$$

$$= -k \sum_A \sum_{ij} e^{ik\lambda_j^A} \int_0^k dk' e^{ik'(\lambda_i^A - \lambda_j^A)}$$

$$= -k \sum_A \sum_{i \neq j} e^{ik\lambda_j^A} \int_0^k dk' e^{ik'(\lambda_i^A - \lambda_j^A)} - k^2 \sum_A \sum_i e^{ik\lambda_i^A}$$

$$= ik \sum_A \sum_{i \neq j} e^{ik\lambda_j^A} \left[\frac{e^{ik'(\lambda_i^A - \lambda_j^A)}}{\lambda_i^A - \lambda_j^A} \right]_0^k - k^2 \sum_A \sum_i e^{ik\lambda_i^A}$$

$$= ik \sum_A \sum_{i \neq j} \frac{e^{ik\lambda_i^A} - e^{ik\lambda_j^A}}{\lambda_i^A - \lambda_j^A} - k^2 \sum_A \sum_i e^{ik\lambda_i^A}$$

$$= -2ik \sum_A \sum_{i \neq j} \frac{e^{ik\lambda_i^A}}{\lambda_i^A - \lambda_j^A} - k^2 \sum_A \sum_i e^{ik\lambda_i^A}.$$

In the coordinate representation they become

$$\Omega(x, x') = \int dk \int dk' e^{-ikx} e^{-ik'x'} \Omega_{kk'}$$

$$= \partial_x \partial_{x'} \sum_A \sum_i \delta(x - \lambda_i^A) \delta(x' - \lambda_i^A)$$

$$\partial_x \partial_{x'} \delta(x - x') \sum_A \sum_i \delta(x - \lambda_i^A)$$

$$\partial_x \partial_{x'} \delta(x - x') \phi(x)$$

and

$$\omega(x) = \int dk e^{-ikx} \omega_k$$

$$= 2\partial_x \sum_A \sum_{i \neq j} \frac{\delta(x - \lambda_i^A)}{\lambda_i - \lambda_j} + \partial_x^2 \phi(x)$$

$$= 2\partial_x \sum_A \sum_{i \neq j} \frac{\delta(x - x_i^A)}{x_i^A - x_j^A} + \partial_x^2 \phi(x)$$

$$= 2\partial_x \sum_A \sum_{ij} \int dy ⨍ dz \frac{\delta(x-y)\delta(y - x_i^A)\delta(z - x_j^A)}{y - z} + \partial_x^2 \phi(x)$$

$$= 2\partial_x \sum_A \sum_{ij} ⨍ dz \frac{\delta(x - x_i^A)\delta(z - x_j^A)}{x - z} + \partial_x^2 \phi(x)$$

$$= 2\partial_x \sum_A ⨍ dz \frac{\phi^A(x)\phi^A(z)}{x - z} + \partial_x^2 \phi(x).$$

The term on the right is associated with an $i = j$ regularisation and is not of leading order in N and can be ignored. Substituting into equation (3.4) gives us

$$\phi(x)\partial_x \frac{\partial \ln J}{\partial \phi(x)} = 2 \sum_A ⨍ dy \frac{\phi^A(x)\phi^A(y)}{x - y}$$

or

$$\left(\sum_A \phi^A(x) \right) \partial_x \frac{\partial \ln J}{\partial \phi(x)} = 2 \sum_A ⨍ dy \frac{\phi^A(x)\phi^A(y)}{x - y}. \tag{A.1}$$

Now consider the following:

$$\sum_A \phi^A(x)\partial_x \frac{\partial \ln J}{\partial \phi^A(x)}. \tag{A.2}$$

Substituting in the Jacobian we found earlier yields

$$\sum_A \phi^A(x)\partial_x \frac{\partial}{\partial \phi^A(x)} \left(\sum_B \int dx' ⨍ dy \sum_i \delta(x' - \lambda_i^B) \sum_j \delta(y - \lambda_j^B) \ln|x' - y| \right)$$

$$= \sum_A \phi^A(x)\partial_x \frac{\partial}{\partial \phi^A(x)} \left(\sum_B \int dx' ⨍ dy \phi^B(x')\phi^B(y) \ln|x' - y| \right)$$

$$= \sum_A \phi^A(x)\partial_x \sum_B \left(\int dx' ⨍ dy \delta_{BA}\delta(x - x')\phi^B(y) \ln|x' - y| \right.$$

$$\left. + \int dx' ⨍ dy \phi^B(x')\delta_{BA}\delta(x - y) \ln|x' - y| \right)$$

$$= \sum_A \phi^A(x)\partial_x \left(2 \fint dy \phi^A(y) \ln |x-y| \right)$$

$$= 2 \sum_A \fint dy \frac{\phi^A(x)\phi^A(y)}{x-y}$$

which is precisely what we have on the right hand side of equation (A.1).

So it seems to be the case that choosing the collective field coordinates in this way only corresponds to the correct Jacobian if the left hand side of equation (A.1) is the same as (A.2), or

$$\left(\sum_A \phi^A(x) \right) \partial_x \frac{\partial \ln J}{\partial \phi(x)} = \sum_A \phi^A(x) \partial_x \frac{\partial \ln J}{\partial \phi^A(x)}. \tag{A.3}$$

If this equation is true then it suggests that the collective field invariants can be either a sum or a product of traces. If that is the case then it might provide some insight into applying the collective field method to other problems in the future.

Appendix B

Diagonalised Derivatives

For a hermitian matrix M that can be diagonalised into $U^\dagger \lambda U$, we are interested in the quantity $\frac{\partial}{\partial M}$ in terms of λ and U. Using the chain rule this can be written in component form as

$$\frac{\partial}{\partial M_{ij}} = \sum_p \frac{\partial \lambda_p}{\partial M_{ij}} \frac{\partial}{\partial \lambda_p} + \sum_{pq} \frac{\partial U_{pq}}{\partial M_{ij}} \frac{\partial}{\partial U_{pq}}.$$

In order to find the coefficients we use the following:

$$dM_{ij} = \sum_k dU^\dagger_{ik} \lambda_k U_{kj} + U^\dagger_{ik} d\lambda_k U_{kj} + U^\dagger_{ik} \lambda_k dU_{kj}$$

We know from chapter three that

$$dM = U^\dagger (d\lambda + [\lambda, dS])U$$

where $dS = dUU^\dagger$. Multiplying by U on the left and $U^\dagger$ on the right yields

$$\sum_{ij} U_{ai} dM_{ij} U^\dagger_{jb} = (d\lambda + [\lambda, dS])_{ab}$$

$$= d\lambda_{ab} + \lambda_{ap} dS_{pb} - dS_{aq} \lambda_{qb}$$

$$= \delta_{ab} d\lambda_a + (\lambda_a - \lambda_b) dS_{ab}$$

since λ and $d\lambda$ are diagonal.

We now have two equations to work with, one for which $a = b$ and one for which $a \neq b$. For the case where $a = b$ we have

$$\sum_{ij} U_{ai} dM_{ij} U^{\dagger}_{ja} = d\lambda_a$$

and therefore

$$\frac{\partial \lambda_a}{\partial M_{ij}} = U_{ai} U^{\dagger}_{ja}.$$

For the case where $a \neq b$ we have

$$\sum_{ij} U_{ai} dM_{ij} U^{\dagger}_{jb} = (\lambda_a - \lambda_b) dS_{ab}$$

and therefore

$$\frac{\partial S_{ab}}{\partial M_{ij}} = \frac{U_{ai} U^{\dagger}_{jb}}{\lambda_a - \lambda_b}.$$

Now all that remains to be found is $\frac{\partial U}{\partial S}$ which can be obtained from the definition of dS

$$dS_{ab} = dU_{aj} U^{\dagger}_{jb}.$$

Multiplying by U on the right yields

$$dU_{ac} = dS_{ab} U_{bc} = \delta_{ad} dS_{db} U_{bc}$$

from which we can see that

$$\frac{\partial U_{ac}}{\partial S_{db}} = \delta_{ad} U_{bc}.$$

The quantity we need is

$$\frac{\partial U_{pq}}{\partial M_{ij}} = \frac{\partial U_{pq}}{\partial S_{ab}} \frac{\partial S_{ab}}{\partial M_{ij}} = \delta_{pa} U_{bq} \frac{U_{ai} U^{\dagger}_{jb}}{\lambda_a - \lambda_b}$$

We can now see that the original derivative becomes

$$\frac{\partial}{\partial M_{ij}} = \sum_p U_{pi} U^\dagger_{jp} \frac{\partial}{\partial \lambda_p} + \sum_{pq} \sum_{a \neq b} \delta_{pa} U_{bq} \frac{U_{ai} U^\dagger_{jb}}{\lambda_a - \lambda_b} \frac{\partial}{\partial U_{pq}}.$$

$$= \sum_p U_{pi} U^\dagger_{jp} \frac{\partial}{\partial \lambda_p} + \sum_{q} \sum_{a \neq b} U_{bq} \frac{U_{ai} U^\dagger_{jb}}{\lambda_a - \lambda_b} \frac{\partial}{\partial U_{aq}}$$

It is also important to see what happens when we have the trace of the second derivative acting on a function that depends only on the eigenvalues, i.e.

$$\sum_{ij} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} f(\lambda).$$

We can immediately drop the ∂U term in the first application of the derivative since the function only depends on the eigenvalues and thus we will have

$$\sum_{ij} \left(\sum_p U_{pi} U^\dagger_{jp} \frac{\partial}{\partial \lambda_p} + \sum_q \sum_{a \neq b} \frac{U_{ai} U^\dagger_{jb} U_{bq}}{(\lambda_a - \lambda_b)} \frac{\partial}{\partial U_{aq}} \right) \sum_k U_{kj} U^\dagger_{ik} \frac{\partial}{\partial \lambda_k}$$

$$= \sum_{ijkp} U_{pi} U^\dagger_{jp} U_{kj} U^\dagger_{ik} \frac{\partial}{\partial \lambda_p} \frac{\partial}{\partial \lambda_k}$$

$$+ \sum_{ijqka \neq b} \frac{U_{ai} U^\dagger_{jb} U_{bq}}{(\lambda_a - \lambda_b)} \delta_{ka} \delta_{jq} U^\dagger_{ik} \frac{\partial}{\partial \lambda_k}$$

$$- \sum_{ijqka \neq b} \frac{U_{ai} U^\dagger_{jb} U_{bq}}{(\lambda_a - \lambda_b)} U_{kj} U^\dagger_{ia} U^\dagger_{qk} \frac{\partial}{\partial \lambda_k}.$$

There is another term but it has a ∂U on the right and so will disappear when acting on a function of only the eigenvalues. Dealing with all the Us and $U^\dagger$s that cancel we end up with

$$= \sum_{kp} \delta_{pk} \delta_{kp} \frac{\partial}{\partial \lambda_p} \frac{\partial}{\partial \lambda_k}$$

$$+\sum_{b\neq k}\frac{1}{(\lambda_k-\lambda_b)}\frac{\partial}{\partial\lambda_k}$$

$$-\sum_{k\neq a}\frac{1}{(\lambda_a-\lambda_k)}\frac{\partial}{\partial\lambda_k}.$$

$$=\sum_{k}\frac{\partial^2}{\partial\lambda_k^2}+\sum_{k\neq a}\frac{2}{(\lambda_k-\lambda_a)}\frac{\partial}{\partial\lambda_k}.$$

Bibliography

[1] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions", Nucl. Phys. B **72**, 461 (1974).

[2] J. Maldacena, "TASI 2003 Lectures on AdS/CFT" [arXiv:hep-th/0309246v5].

[3] J. Maldacena, "AdS/CFT Correspondence, Part 1" (2010), retrieved from https://www.youtube.com/watch?v=sYqtGXN59uE&

[4] J. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200].

[5] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, "Gauge Theory Correlators from Non-Critical String Theory", Phys. Lett. **B428**, 105-114 (1998) [arXiv:hep-th/9802109].

[6] A. Jevicki and B. Sakita, "The Quantum Collective Field Method and its Application to the Planar Limit", Nucl. Phys. B **165**, 511 (1980).

[7] I. R. Klebanov, "String Theory in Two Dimensions" [arXiv:hepth/ 9108019].

[8] D. Berenstein, "Large N BPS States and Emergent Quantum Gravity" [arXiv:hep-th/0507203].

[9] D. Berenstein, D. H. Correa and S. E. Vazquez, "All Loop BMN State Energies From Matrices" [arXiv:hep-th/0509015].

[10] D. Berenstein and R. Cotta, "Aspects of Emergent Geometry in the AdS/CFT Context" [arXiv:hep-th/0605220].

[11] D. E. Berenstein, M. Hanada and S. A. Hartnoll, "Multi-matrix Models and Emergent Geometry", JHEP **0902**, 010 (2009) [arXiv:0805.4658 [hep-th]].

[12] R. de Mello Koch and J. Murugan, "Emergent Spacetime" [arXiv:0911.4817 [hep-th]].

[13] M. Masuku and J. A. P. Rodrigues, "De Alfaro, Fubini and Furlan from Multi Matrix Systems" [arXiv:1509.06719v2 [hep-th]].

[14] M. Masuku and J. A. P. Rodrigues, "Laplacians in Polar Matrix Coordinates and Radial Fermionization in Higher Dimensions", J. Math. Phys. **52**, 032302 (2011) [arXiv:0911.2846 [hep-th]].

[15] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams", Commun. Math. Phys. **59**, 35 (1978).

[16] M. Masuku, "Matrix Polar Coordinates" (Masters Dissertation), University of the Witwatersrand (2010), retrieved from http://wiredspace.wits.ac.za/handle/10539/9240.

[17] A. Jevicki, "Non-perturbative collective field theory", Nucl. Phys. B **376**, 75 (1992).

[18] J. Shapiro, "A Test of the Collective Field Method for the $N \to \infty$ Limit", Nucl. Phys. B **184**, 218 (1981).

[19] M. Green, J. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction", Cambridge University Press (2012).

[20] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M Theory as a Matrix Model: A Conjecture”, Phys. Rev. D **55**, 5112 (1997) [arXiv:hep-th/9610043].

[21] F. Bergeron, “Algebraic Combinatorics and Coinvariant Spaces”, Wellesley, Mass. A K Peters (2009).

[22] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N Field Theories, String Theory and Gravity”, Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[23] E. D’Hoker and D. Z. Freedman, “Supersymmetric Gauge Theories and the AdS/CFT Correspondence” [arXiv:hep-th/0201253].

[24] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in Flat Space and PP Waves From N = 4 Super Yang Mills”, JHEP 0204, 013 (2002) [arXiv:hep-th/0202021].

[25] M. Spradlin and A. Volovich, “Light-cone String Field Theory in a Plane Wave” [arXiv:hepth/0310033].

[26] R. R. Metsaev, “Type IIB Green-Schwarz Superstring in Plane Wave Ramond-Ramond Background,” Nucl. Phys. B **625**, 70 (2002) [arXiv:hep-th/0112044].