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A detailed study of a class of recurrence equations with a generalized order

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Abstract

In this paper, we study some family of difference equations. The study involves the use of symmetries to find exact solutions of difference equations with the aim of extending the studies that have been done in the literature. We also investigate the periodic nature and behavior of the solutions in some cases. Finally, some numerical examples illustrating our findings are presented.

Keywords: Difference equation, symmetry, reduction, exact solution.

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1. Introduction

During the nineteenth century, a prominent Norwegian mathematician, Sophus Lie (1842-1899) established remarkable work that became an important part to the theory of groups of transformations (continuous) that leave a differential equation invariant [14]. Lie aimed to create a theory of integrating ordinary differential equations that is equivalent to the Abelian theory of computing algebraic equations. He was inspired by Abel and Galois' theory. He observed that the procedure in all exceptional cases of a universal integration on differential equations is centered on the invariance of the differential equation under continuous symmetries. It is important to note that Lie's group analysis classifies ordinary differential equations in terms of the symmetry group associated with them.

Shigeru Maeda in 1987 showed that Lie's method can be extended to also solve ordinary difference equations. He showed that a set of functional equations amounted from the linearized symmetry condition of ordinary difference equations [15]. The philosophy of difference equations and their applications have cemented a central importance in applicable analysis. Later, several authors studied ordinary difference equations and have obtained some interesting results, see [1–6, 8–13, 16, 17]. Maeda [15] showed how to use symmetry methods to obtain the solution of a system of first-order difference equations. It is now known that symmetries can be used to solve higher-order difference equations.

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In [18], the authors investigated the solutions of the fifth-order difference equation

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(\pm 1 + \pm x_{n-2}x_{n-3}x_{n-4})}, n \in \mathbb{N}_0. \tag{1.1}$$

In [10], the authors investigated the solutions and behavior of solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1}x_{n-3}\left(\lambda + \mu x_n x_{n-2}x_{n-4}\right)}, \tag{1.2}$$

where λ and μ are real constants. In [7], the authors investigated the solutions and the properties of the difference equation

$$x_{n+1} = \frac{x_{n-3k}x_{n-4k}x_{n-5k}}{x_{n-k}x_{n-2k}\left(\pm 1 \pm x_{n-3k}x_{n-4k}x_{n-5k}\right)}. \tag{1.3}$$

We show in this paper that the left-hand side in the above equation is x_{n+1} and not x_n as seen in [7]. Clearly, equations (1.1)-(1.3) are all special cases of

$$x_{n+1} = \frac{x_{n-3k}x_{n-4k}x_{n-5k}}{x_{n-k}x_{n-2k}\left(a_n + b_n x_{n-3k}x_{n-4k}x_{n-5k}\right)}, \tag{1.4}$$

for some arbitrary real sequences a_n and b_n.

A symmetry based method will be employed to solve the generalized case (1.4) and compare the solutions of the corresponding special cases to those of [7, 10, 18]. To achieve this, for the sake of definitions, we will derive the solutions of the equivalent difference equation

$$x_{n+5k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+3k}x_{n+4k}\left(A_n + B_n x_n x_{n+k}x_{n+2k}\right)}, \tag{1.5}$$

where $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are non-zero real random sequences, using Lie group analysis technique. Eventually, invariants of (1.5) are derived and a relationship between these invariants and the similarity variables is given.

The paper is organized in the following manner. In Section 2, we revise some essential ideas that are required for computing symmetries of difference equations and order reduction. In Section 3, symmetries and solutions of (1.5) are obtained and a detailed analysis of some special cases is conducted. In Section 4, we study the periodicity and behavior of the solutions of (1.5).

2. Definitions and notation

The definitions and notations in this paper are similar to those that Hydon adopted in [13]. We consider the general form of the ordinary difference equation

$$x_{n+5k} = \omega\left(n, x_n, x_{n+k}, x_{n+2k}, x_{n+3k}, x_{n+4k}\right), \tag{2.1}$$

for some function ω with $k \in \mathbb{N}$.

Definition 2.1. We define S to be the shift operator acting on n as

$$S : n \to n + 1.$$

Consider a one-parameter Lie group of point transformations given below

$$\Psi_\epsilon : (n, x_n) \to (n, x_n + \epsilon\xi(n, x_n)), \tag{2.2}$$

for the continuous characteristic function $\xi = \xi(n, x_n)$. It is known that the action of the Lie group can be recovered from the corresponding infinitesimal generators.

Definition 2.2. The symmetry generator, denoted by X, is given by

$$X = \xi(n, x_n)\frac{\partial}{\partial x_n}. \tag{2.3}$$

The linearized symmetry condition [13] is given by

$$S^{5k}\xi - \hat{X}\omega = 0, \tag{2.4}$$

provided (2.1) holds. Note that $\hat{X}$ denotes the prolongation of X to all shifts of x_n appearing in the equation and is given by

$$\hat{X} = \xi\frac{\partial}{\partial x_n} + S^k\xi\frac{\partial}{\partial x_{n+k}} + \cdots + S^{4k}\xi\frac{\partial}{\partial x_{n+4k}}. \tag{2.5}$$

Definition 2.3. A function ν is an invariant under the group of transformation (2.2) if and only if $X\nu = 0$.

The generator in (2.3) can be used to derive the canonical coordinate which in turn can be used to obtain the invariant functions. The method for finding symmetries is explained at length in [13].

3. Symmetry analysis and exact solutions

Consider the difference equation (1.5). So, in this case, the function ω is given by

$$\omega = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+3k}x_{n+4k}\left(A_n + B_n x_n x_{n+k} x_{n+2k}\right)}.$$

Assuming that the prolonged symmetry generator takes the form in (2.5), the linearized symmetry condition (2.4) on (1.5) gives

$$\begin{aligned} &\xi(n+5k, x_{n+5k}) - \xi(n, x_n)\frac{\partial\omega}{\partial x_n} - \xi(n+k, x_{n+k})\frac{\partial\omega}{\partial x_{n+k}} \\ &\quad - \xi(n+2k, x_{n+2k})\frac{\partial\omega}{\partial x_{n+2k}} - \xi(n+3k, x_{n+3k})\frac{\partial\omega}{\partial x_{n+3k}} - \xi(n+4k, x_{n+4k})\frac{\partial\omega}{\partial x_{n+4k}} = 0, \end{aligned}$$

that is to say,

$$\begin{aligned} &\xi(n+5k, x_{n+5k}) - \frac{\xi(n, x_n)A_n x_{n+k}x_{n+2k}}{x_{n+3k}x_{n+4k}(A_n + B_n x_n x_{n+k}x_{n+2k})^2} - \frac{\xi(n+k, x_{n+k})A_n x_n x_{n+2k}}{x_{n+3k}x_{n+4k}(A_n + B_n x_n x_{n+k}x_{n+2k})^2} \\ &+ \frac{\xi(n+3k, x_{n+3k})x_n x_{n+k}x_{n+2k}}{x_{n+3k}^2 x_{n+4k}(A_n + B_n x_n x_{n+2}x_{n+2k})} + \frac{\xi(n+4k, x_{n+4k})x_n x_{n+k}x_{n+2k}}{x_{n+3k}x_{n+4k}^2(A_n + B_n x_n x_{n+k}x_{n+2k})} \\ &- \frac{\xi(n+2k, x_{n+2k})A_n x_{n+k}}{x_{n+3k}x_{n+4k}(A_n + B_n x_n x_{n+k}x_{n+2k})^2} = 0. \end{aligned} \tag{3.1}$$

We apply the operator $\frac{\partial}{\partial x_n} + \frac{A_n x_{n+3k}}{x_n(A_n + B_n x_n x_{n+k}x_{n+2k})}\frac{\partial}{\partial x_{n+3k}}$ on (3.1). After clearing fractions and then differentiating thrice with respect to x_n, we obtain the following:

$$2(A_n + 2B_n x_n x_{n+k}x_{n+2k})\xi^{(3)}(n, x_n) + (A_n + B_n x_n x_{n+k}x_{n+2k})x_n\xi^{(4)}(n, x_n) = 0.$$

Now we separate the above, since ξ depends only on x_n, to get

$$x_{n+k}x_{n+2k}x_{n+3k} : x_n\xi^{(4)}(n, x_n) + 4\xi^{(3)}(n, x_n) = 0, \qquad x_{n+3k} : x_n\xi^{(4)}(n, x_n) + 2\xi^{(3)}(n, x_n) = 0,$$

whose solution is given by

$$\xi(n, x_n) = \beta_n x_n^2 + \gamma_n x_n + \lambda_n \tag{3.2}$$

for some functions β_n, γ_n, and λ_n of n. Next, we substitute (3.2) into (3.1) and then separate the resulting equation by the coefficients of products of shifts of u_n and then setting them to zero. It turns out that $\lambda_n = \beta_n = 0$ and γ_n must satisfy the following linear difference equation:

$$\gamma_n + \gamma_{n+k} + \gamma_{n+2k} = 0.$$

Solving the above equation yields

$$\gamma_{n_1}(m) = \exp\left\{i\left(-\frac{2n\pi}{3k} + \frac{2m\pi n}{k}\right)\right\}, \ \gamma_{n_2}(m) = \exp\left\{i\left(\frac{2n\pi}{3k} + \frac{2m\pi n}{k}\right)\right\},$$

where $m = 0, 1, \dots, k-1$. From (3.2), we have the characteristics $\xi_1 = \gamma_{n_1}(m)x_n$ and $\xi_2 = \gamma_{n_2}(m)x_n$, $m = 0, 1, \dots, k-1$. Hence, we obtain the following 2k symmetries:

$$X_{1m} = \gamma_{n_1}(m)x_n\frac{\partial}{\partial x_n} \text{ and } X_{2m} = \gamma_{n_2}(m)x_n\frac{\partial}{\partial x_n}, \ m = 0, 1, \dots, k-1.$$

The canonical coordinate that linearizes (1.5) is given by

$$S_n = \int \frac{dx_n}{\xi(n, x_n)} = \frac{1}{\gamma_n}\ln|x_n|$$

and the function given by

$$\tilde{V}_n = \gamma_n S_n + \gamma_{n+k}S_{n+k} + \gamma_{n+2k}S_{n+2k}$$

is invariant under the group of transformations admitted by (1.5) since $\hat{X}(\tilde{V}_n) = 0$. For the sake of convenience, we will use the invariant $V_n = \exp(-\tilde{V}_n)$ and one can easily verify that $\hat{X}(V_n) = 0$. It happens that,

$$V_n = \frac{1}{x_n x_{n+k} x_{n+2k}} \tag{3.3}$$

and applying the forward shift of 3k on V_n (and substituting x_{n+5k} by its expression given in (1.5)) yields

$$V_{n+3k} = A_n V_n + B_n. \tag{3.4}$$

By iterating (3.4), we obtain its solution in closed form

$$V_{3kn+i} = V_i\left(\prod_{m_1=0}^{n-1} A_{3km_1+i}\right) + \sum_{l=0}^{n-1}\left(B_{3kl+i}\prod_{m_2=l+1}^{n-1} A_{3km_2+i}\right), \tag{3.5}$$

where $i = 0, 1, 2, \dots, 3k-1$. It follows from (3.3) that

$$x_{n+3k} = \frac{V_n}{V_{n+k}}x_n$$

and by iterating the above equation, we have that

$$x_{3kn+i} = x_i\left(\prod_{s=0}^{n-1}\frac{V_{3ks+i}}{V_{3ks+i+k}}\right), \tag{3.6}$$

where $i = 0, 1, 2, \dots, 3k-1$. To avoid any possible confusion, we rewrite (3.6) in the following forms:

$$x_{3kn+i} = x_i\left(\prod_{s=0}^{n-1}\frac{V_{3ks+i}}{V_{3ks+i+k}}\right), i = 0, \dots, 2k-1 \tag{3.7}$$

and

$$x_{3kn+i} = x_i \left(\prod_{s=0}^{n-1} \frac{V_{3ks+i}}{V_{3k(s+1)+i-2k}} \right), i = 2k, \dots, 3k-1. \tag{3.8}$$

Using (3.5) in (3.7) and (3.8), remembering that $V_i = 1/(x_i x_{i+k} x_{i+2k})$, we have the following solutions of (1.5):

$$x_{3kn+i} = \frac{x_{i+3k}^n}{x_i^{n-1}} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{m_1=0}^{s-1} A_{3km_1+i} \right) + x_i x_{i+k} x_{i+2k} \sum\limits_{l=0}^{s-1} \left(B_{3kl+i} \prod\limits_{m_2=l+1}^{s-1} A_{3km_2+i} \right)}{\left(\prod\limits_{m_1=0}^{s-1} A_{3km_1+k+i} \right) + x_{i+k} x_{i+2k} x_{i+3k} \sum\limits_{l=0}^{s-1} \left(B_{3kl+k+i} \prod\limits_{m_2=l+1}^{s-1} A_{3km_2+k+i} \right)}, \tag{3.9}$$

where $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn+i} = \frac{x_i x_{i-2k}^n x_{i-k}^n}{x_{i+k}^n x_{i+2k}^n} \times \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{m_1=0}^{s-1} A_{3km_1+i} \right) + x_i x_{i+k} x_{i+2k} \sum\limits_{l=0}^{s-1} \left(B_{3kl+i} \prod\limits_{m_2=l+1}^{s-1} A_{3km_2+i} \right)}{\left(\prod\limits_{m_1=0}^{s} A_{3km_1+i-2k} \right) + x_{i-2k} x_{i-k} x_i \sum\limits_{l=0}^{s} \left(B_{3kl+i-2k} \prod\limits_{m_2=l+1}^{s} A_{3km_2+i-2k} \right)}, \tag{3.10}$$

where $i = 2k, 2k+1, \dots, 3k-1$. We derive the solution of (1.4) by back shifting (3.9) and (3.10) $5k-1$ times. This yields

$$x_{3kn-5k+1+i} = \frac{x_{i-2k+1}^n}{x_{i-5k+1}^{n-1}} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{m_1=0}^{s-1} a_{3km_1+i} \right) + x_{i-5k+1} x_{i-4k+1} x_{i-3k+1} \sum\limits_{l=0}^{s-1} \left(b_{3kl+i} \prod\limits_{m_2=l+1}^{s-1} a_{3km_2+i} \right)}{\left(\prod\limits_{m_1=0}^{s-1} a_{3km_1+k+i} \right) + x_{i-4k+1} x_{i-3k+1} x_{i-2k+1} \sum\limits_{l=0}^{s-1} \left(b_{3kl+k+i} \prod\limits_{m_2=l+1}^{s-1} a_{3km_2+k+i} \right)}, \tag{3.11}$$

for $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn-5k+1+i} = \frac{x_{i-5k+1} x_{i-7k+1}^n x_{i-6k+1}^n}{x_{i-4k+1}^n x_{i-3k+1}^n} \times \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{m_1=0}^{s-1} a_{3km_1+i} \right) + x_{i-5k+1} x_{i-4k+1} x_{i-3k+1} \sum\limits_{l=0}^{s-1} \left(b_{3kl+i} \prod\limits_{m_2=l+1}^{s-1} a_{3km_2+i} \right)}{\left(\prod\limits_{m_1=0}^{s} a_{3km_1+i-2k} \right) + x_{i-7k+1} x_{i-6k+1} x_{i-5k+1} \sum\limits_{l=0}^{s} \left(b_{3kl+i-2k} \prod\limits_{m_2=l+1}^{s} a_{3km_2+i-2k} \right)}, \tag{3.12}$$

for $i = 2k, 2k-1, \dots, 3k-1$.

3.1. *The case where a_n and b_n are constant*

Here, let $a_n = a$ and $b_n = b$, where $a, b \in \mathbb{R}$. Thus, equations (3.11) and (3.12) reduce to

$$x_{3kn-5k+i+1} = \frac{x_{i-2k+1}^n}{x_{i-5k+1}^{n-1}} \prod_{s=0}^{n-1} \frac{a^s + x_{i-5k+1} x_{i-4k+1} x_{i-3k+1} b \sum\limits_{l=0}^{s-1} a^l}{a^s + x_{i-4k+1} x_{i-3k+1} x_{i-2k+1} b \sum\limits_{l=0}^{s-1} a^l},$$

for $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn-5k+i+1} = \frac{x_{i-5k+1}x^n_{i-7k+1}x^n_{i-6k+1}}{x^n_{i-4k+1}x^n_{i-3k+1}} \prod_{s=0}^{n-1} \frac{a^s + x_{i-5k+1}x_{i-4k+1}x_{i-3k+1}b\sum_{l=0}^{s-1} a^l}{a^{s+1} + x_{i-7k+1}x_{i-6k+1}x_{i-5k+1}b\sum_{l=0}^{s} a^l},$$

for $i = 2k, 2k+1, \dots, 3k-1$.

3.1.1. The case where $a = 1$

We have

$$x_{3kn-5k+i+1} = \frac{x^n_{i-2k+1}}{x^{n-1}_{i-5k+1}} \prod_{s=0}^{n-1} \frac{1 + x_{i-5k+1}x_{i-4k+1}x_{i-3k+1}bs}{1 + x_{i-4k+1}x_{i-3k+1}x_{i-2k+1}bs}, \tag{3.13}$$

for $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn-5k+i+1} = \frac{x_{i-5k+1}x^n_{i-7k+1}x^n_{i-6k+1}}{x^n_{i-4k+1}x^n_{i-3k+1}} \prod_{s=0}^{n-1} \frac{1 + x_{i-5k+1}x_{i-4k+1}x_{i-3k+1}bs}{1 + x_{i-7k+1}x_{i-6k+1}x_{i-5k+1}b(s+1)}, \tag{3.14}$$

for $i = 2k, 2k+1, \dots, 3k-1$.

Remark 3.1. The results in [7] (see Theorems 2.1.1 and 2.2.1) are special cases of ours. In fact,

$$\begin{aligned} x_{3kn+i} = x_{3k(n+1)-5k+2k+i}, \quad i = 1, 2, \dots, k &= \frac{x_{i-3k}x^{n+1}_{i-5k}x^{n+1}_{i-4k}}{x^{n+1}_{i-2k}x^{n+1}_{i-k}} \prod_{s=0}^{n} \frac{1 + x_{i-3k}x_{i-2k}x_{i-k}bs}{1 + x_{i-5k}x_{i-4k}x_{i-3k}b(s+1)} \\ &= x_{i-3k} \prod_{s=0}^{n} \frac{x_{i-5k}x_{i-4k} + x_{i-5k}x_{i-4k}x_{i-3k}x_{i-2k}x_{i-k}bs}{x_{i-2k}x_{i-k} + x_{i-5k}x_{i-4k}x_{i-3k}x_{i-2k}x_{i-k}b(s+1)} \end{aligned}$$

and similarly,

$$\begin{aligned} x_{3kn+i} &= x_{3k(n+2)-5k-k+i}, \quad i = k+1, 2, \dots, 3k-1 \\ &= \frac{x^{n+2}_{i-3k}}{x^{n+1}_{i-6k}} \prod_{s=0}^{n+1} \frac{1 + x_{i-6k}x_{i-5k}x_{i-4k}bs}{1 + x_{i-5k}x_{i-4k}x_{i-3k}bs} = x_{i-3k} \prod_{s=0}^{n} \frac{x_{i-3k} + x_{i-6k}x_{i-5k}x_{i-4k}x_{i-3k}b(s+1)}{x_{i-6k} + x_{i-6k}x_{i-5k}x_{i-4k}x_{i-3k}b(s+1)}. \end{aligned}$$

Consequently, Corollaries 3.1.1 and 3.2.1 are easily recovered from (3.13) and (3.14) by setting $k = 2$.

3.1.2. The case where $a \neq 1$

We have

$$x_{3kn-5k+i+1} = \frac{x^n_{i-2k+1}}{x^{n-1}_{i-5k+1}} \prod_{s=0}^{n-1} \frac{a^s + x_{i-5k+1}x_{i-4k+1}x_{i-3k+1}b\left(\frac{1-a^s}{1-a}\right)}{a^s + x_{i-4k+1}x_{i-3k+1}x_{i-2k+1}b\left(\frac{1-a^s}{1-a}\right)},$$

for $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn-5k+i+1} = \frac{x_{i-5k+1}x^n_{i-7k+1}x^n_{i-6k+1}}{x^n_{i-4k+1}x^n_{i-3k+1}} \prod_{s=0}^{n-1} \frac{a^s + x_{i-5k+1}x_{i-4k+1}x_{i-3k+1}b\left(\frac{1-a^s}{1-a}\right)}{a^{s+1} + x_{i-7k+1}x_{i-6k+1}x_{i-5k+1}b\left(\frac{1-a^{s+1}}{1-a}\right)},$$

for $i = 2k, 2k+1, \dots, 3k-1$.

3.1.3. The case where $k = 1$

Assume $a_n = \lambda$ and $b_n = \mu$, where $\lambda, \mu \in \mathbb{R}$. Thus, equations (3.11) and (3.12) simplify to

$$x_{3n-4} = \frac{x_{-1}^n}{x_{-4}^{n-1}} \prod_{s=0}^{n-1} \frac{\lambda^s + x_{-4}x_{-3}x_{-2}\mu \sum_{l=0}^{s-1} \lambda^l}{\lambda^s + x_{-3}x_{-2}x_{-1}\mu \sum_{l=0}^{s-1} \lambda^l},$$

$$x_{3n-3} = \frac{x_0^n}{x_{-3}^{n-1}} \prod_{s=0}^{n-1} \frac{\lambda^s + x_{-3}x_{-2}x_{-1}\mu \sum_{l=0}^{s-1} \lambda^l}{\lambda^s + x_{-2}x_{-1}x_0\mu \sum_{l=0}^{s-1} \lambda^l},$$

$$x_{3n-2} = \frac{x_{-2}x_{-4}^n x_{-3}^n}{x_{-1}^n x_0^n} \prod_{s=0}^{n-1} \frac{\lambda^s + x_{-2}x_{-1}x_0\mu \sum_{l=0}^{s-1} \lambda^l}{\lambda^{s+1} + x_{-4}x_{-3}x_{-2}\mu \sum_{l=0}^{s} \lambda^l}.$$

For $\lambda = 1$, using equations (3.11) and (3.12), we have that

$$\begin{aligned} x_{3n-4} &= \frac{x_{-1}^n}{x_{-4}^{n-1}} \prod_{s=0}^{n-1} \frac{1 + x_{-4}x_{-3}x_{-2}\mu s}{1 + x_{-3}x_{-2}x_{-1}\mu s}, \\ x_{3n-3} &= \frac{x_0^n}{x_{-3}^{n-1}} \prod_{s=0}^{n-1} \frac{1 + x_{-3}x_{-2}x_{-1}\mu s}{1 + x_{-2}x_{-1}x_0\mu s}, \\ x_{3n-2} &= \frac{x_{-2}x_{-4}^n x_{-3}^n}{x_{-1}^n x_0^n} \prod_{s=0}^{n-1} \frac{1 + x_{-2}x_{-1}x_0\mu s}{1 + x_{-4}x_{-3}x_{-2}\mu(s+1)}. \end{aligned} \tag{3.15}$$

The results in (3.15) were obtained by Yazlik in [18] (see Theorems 5 and 9). For $\lambda \neq 1$, equations (3.11) and (3.12) become

$$x_{3n-4} = \frac{x_{-1}^n}{x_{-4}^{n-1}} \prod_{s=0}^{n-1} \frac{\lambda^s + x_{-4}x_{-3}x_{-2}\mu\left(\frac{1-\lambda^s}{1-\lambda}\right)}{\lambda^s + x_{-3}x_{-2}x_{-1}\mu\left(\frac{1-\lambda^s}{1-\lambda}\right)},$$

$$x_{3n-3} = \frac{x_0^n}{x_{-3}^{n-1}} \prod_{s=0}^{n-1} \frac{\lambda^s + x_{-3}x_{-2}x_{-1}\mu\left(\frac{1-\lambda^s}{1-\lambda}\right) l}{\lambda^s + x_{-2}x_{-1}x_0\mu\left(\frac{1-\lambda^s}{1-\lambda}\right)},$$

$$x_{3n-2} = \frac{x_{-2}x_{-4}^n x_{-3}^n}{x_{-1}^n x_0^n} \prod_{s=0}^{n-1} \frac{\lambda^s + x_{-2}x_{-1}x_0\mu\left(\frac{1-\lambda^s}{1-\lambda}\right)}{\lambda^{s+1} + x_{-4}x_{-3}x_{-2}\mu\left(\frac{1-\lambda^{s+1}}{1-\lambda}\right)}.$$

In particular, when $\lambda = -1$, we have

$$\begin{aligned} x_{6n-4} &= \frac{x_{-1}^{2n}}{x_{-4}^{2n-1}} \left(\frac{-1 + x_{-4}x_{-3}x_{-2}\mu}{-1 + x_{-3}x_{-2}x_{-1}\mu}\right)^n, & x_{6n-3} &= \frac{x_0^{2n}}{x_{-3}^{2n-1}} \left(\frac{-1 + x_{-3}x_{-2}x_{-1}\mu}{-1 + x_{-2}x_{-1}x_0\mu}\right)^n, \\ x_{6n-2} &= \frac{x_{-2}x_{-4}^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_0^{2n}} \left(\frac{-1 + x_{-2}x_{-1}x_0\mu}{-1 + x_{-4}x_{-3}x_{-2}\mu}\right)^n, & x_{6n-1} &= \frac{x_{-1}^{2n+1}}{x_{-4}^{2n}} \left(\frac{-1 + x_{-4}x_{-3}x_{-2}\mu}{-1 + x_{-3}x_{-2}x_{-1}\mu}\right)^n, \\ x_{6n} &= \frac{x_0^{2n+1}}{x_{-3}^{2n}} \left(\frac{-1 + x_{-3}x_{-2}x_{-1}\mu}{-1 + x_{-2}x_{-1}x_0\mu}\right)^n, & x_{6n+1} &= \frac{x_{-2}x_{-4}^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_0^{2n+1}} \frac{(-1 + x_{-2}x_{-1}x_0\mu)^n}{(-1 + x_{-4}x_{-3}x_{-2}\mu)^{n+1}}. \end{aligned} \tag{3.16}$$

The results in (3.16) were obtained by Yazlik in [18] (see Theorems 7 and 11). Also, by replacing n with $2n$ or $2n+1$ in (3.11) and (3.12), we recover the results in equation (67) of [10].

3.2. The case when $k = 2$

When $k = 2$, thanks to (3.11) and (3.12), the solution of

$$x_{n+1} = \frac{x_{n-6}x_{n-8}x_{n-10}}{x_{n-2}x_{n-4}\left(a_n + b_n x_{n-6}x_{n-8}x_{n-10}\right)}$$

is given by

$$x_{6n-9+i} = \frac{x_{i-3}^{n}}{x_{i-9}^{n-1}} \prod_{s=0}^{n-1} \frac{\left(\prod_{m_1=0}^{s-1} a_{6m_1+i}\right) + x_{i-9}x_{i-7}x_{i-5}\sum_{l=0}^{s-1}\left(b_{6l+i}\prod_{m_2=l+1}^{s-1} a_{6m_2+i}\right)}{\left(\prod_{m_1=0}^{s-1} a_{6m_1+2+i}\right) + x_{i-7}x_{i-5}x_{i-3}\sum_{l=0}^{s-1}\left(b_{6l+2+i}\prod_{m_2=l+1}^{s-1} a_{6m_2+2+i}\right)},$$

for $i = 0, 1, 2, 3$; and

$$x_{6n-9+i} = \frac{x_{i-9}x_{i-13}^{n}x_{i-11}^{n}}{x_{i-7}^{n}x_{i-5}^{n}} \prod_{s=0}^{n-1} \frac{\left(\prod_{m_1=0}^{s-1} a_{6m_1+i}\right) + x_{i-9}x_{i-7}x_{i-5}\sum_{l=0}^{s-1}\left(b_{6l+i}\prod_{m_2=l+1}^{s-1} a_{6m_2+i}\right)}{\left(\prod_{m_1=0}^{s} a_{6m_1+i-4}\right) + x_{i-13}x_{i-11}x_{i-9}\sum_{l=0}^{s}\left(b_{6l+i-4}\prod_{m_2=l+1}^{s} a_{6m_2+i-4}\right)},$$

for $i = 4, 5$.

3.2.1. The case where $a_n = a$ *and* $b_n = b$ *are constant*

The case where $a = 1$ we have

$$x_{6n-9+i} = \frac{x_{i-3}^{n}}{x_{i-9}^{n-1}} \prod_{s=0}^{n-1} \frac{1 + x_{i-9}x_{i-7}x_{i-5}bs}{1 + x_{i-7}x_{i-5}x_{i-3}bs},$$

for $i = 0, 1, 2, 3$; and

$$x_{6n-9+i} = \frac{x_{i-9}x_{i-13}^{n}x_{i-11}^{n}}{x_{i-7}^{n}x_{i-5}^{n}} \prod_{s=0}^{n-1} \frac{1 + x_{i-9}x_{i-7}x_{i-5}bs}{1 + x_{i-13}x_{i-11}x_{i-9}b(s+1)}.$$

for $i = 4, 5$. More explicitly,

$$x_{6n-9} = \frac{x_{-3}^{n}}{x_{-9}^{n-1}} \prod_{s=0}^{n-1} \frac{1 + x_{-9}x_{-7}x_{-5}bs}{1 + x_{-7}x_{-5}x_{-3}bs}, \qquad x_{6n-8} = \frac{x_{-2}^{n}}{x_{-8}^{n-1}} \prod_{s=0}^{n-1} \frac{1 + x_{-8}x_{-6}x_{-4}bs}{1 + x_{-6}x_{-4}x_{-2}bs},$$

$$x_{6n-7} = \frac{x_{-1}^{n}}{x_{-7}^{n-1}} \prod_{s=0}^{n-1} \frac{1 + x_{-7}x_{-5}x_{-3}bs}{1 + x_{-5}x_{-3}x_{-1}bs}, \qquad x_{6n-6} = \frac{x_{0}^{n}}{x_{-6}^{n-1}} \prod_{s=0}^{n-1} \frac{1 + x_{-6}x_{-4}x_{-2}bs}{1 + x_{-4}x_{-2}x_{0}bs},$$

$$x_{6n-5} = \frac{x_{-5}x_{-9}^{n}x_{-7}^{n}}{x_{-3}^{n}x_{-1}^{n}} \prod_{s=0}^{n-1} \frac{1 + x_{-5}x_{-3}x_{-1}bs}{1 + x_{-9}x_{-7}x_{-5}b(s+1)}, \qquad x_{6n-4} = \frac{x_{-4}x_{-8}^{n}x_{-6}^{n}}{x_{-2}^{n}x_{0}^{n}} \prod_{s=0}^{n-1} \frac{1 + x_{-4}x_{-2}x_{0}bs}{1 + x_{-8}x_{-6}x_{-4}b(s+1)}.$$

Setting $b = \pm 1$ and replacing n with $n+1$ or $n+2$ in the above equations, we recover the results in [7] (see Corollaries 3.1.1 and 3.2.1). We note some typos in the formulas for x_{6n+3} (d^n should be d^{n+2}) in Corollaries 3.1.1 and 3.2.1. In fact, $x_{6n+3} = x_{6(n+2)-9}$ and it follows from the above expressions that the power of x_{-3} must then be $n+2$. Another way to confirm that it should be $n+2$ is to set $k = 2$ in equations (2.1.3) and (2.2.3) of [7].

For the case where $a \neq 1$ we have

$$x_{6n-9+i} = \frac{x_{i-3}^{n}}{x_{i-9}^{n-1}} \prod_{s=0}^{n-1} \frac{a^s + x_{i-9}x_{i-7}x_{i-5}b\left(\frac{1-a^s}{1-a}\right)}{a^s + x_{i-7}x_{i-5}x_{i-3}b\left(\frac{1-a^s}{1-a}\right)},$$

for $i = 0, 1, 2, 3$; and

$$x_{6n-9+i} = \frac{x_{i-9}x_{i-13}^{n}x_{i-11}^{n}}{x_{i-7}^{n}x_{i-5}^{n}} \prod_{s=0}^{n-1} \frac{a^s + x_{i-9}x_{i-7}x_{i-5}b\left(\frac{1-a^s}{1-a}\right)}{a^{s+1} + x_{i-13}x_{i-11}x_{i-9}b\left(\frac{1-a^{s+1}}{1-a}\right)},$$

for $i = 4, 5$. More explicitly,

$$x_{6n-9} = \frac{x_{-3}^{n}}{x_{-9}^{n-1}} \prod_{s=0}^{n-1} \frac{a^s + x_{-9}x_{-7}x_{-5}b\left(\frac{1-a^s}{1-a}\right)}{a^s + x_{-7}x_{-5}x_{-3}b\left(\frac{1-a^s}{1-a}\right)}, \quad x_{6n-8} = \frac{x_{-2}^{n}}{x_{-8}^{n-1}} \prod_{s=0}^{n-1} \frac{a^s + x_{-8}x_{-6}x_{-4}b\left(\frac{1-a^s}{1-a}\right)}{a^s + x_{-6}x_{-4}x_{-2}b\left(\frac{1-a^s}{1-a}\right)},$$

$$x_{6n-7} = \frac{x_{-1}^{n}}{x_{-7}^{n-1}} \prod_{s=0}^{n-1} \frac{a^s + x_{-7}x_{-5}x_{-3}b\left(\frac{1-a^s}{1-a}\right)}{a^s + x_{-5}x_{-3}x_{-1}b\left(\frac{1-a^s}{1-a}\right)}, \quad x_{6n-6} = \frac{x_{0}^{n}}{x_{-6}^{n-1}} \prod_{s=0}^{n-1} \frac{a^s + x_{-6}x_{-4}x_{-2}b\left(\frac{1-a^s}{1-a}\right)}{a^s + x_{-4}x_{-2}x_{0}b\left(\frac{1-a^s}{1-a}\right)},$$

$$x_{6n-5} = \frac{x_{-5}x_{-9}^{n}x_{-7}^{n}}{x_{-3}^{n}x_{-1}^{n}} \prod_{s=0}^{n-1} \frac{a^s + x_{-5}x_{-3}x_{-1}b\left(\frac{1-a^s}{1-a}\right)}{a^{s+1} + x_{-9}x_{-7}x_{-5}b\left(\frac{1-a^{s+1}}{1-a}\right)}, \quad x_{6n-4} = \frac{x_{-4}x_{-8}^{n}x_{-6}^{n}}{x_{-2}^{n}x_{0}^{n}} \prod_{s=0}^{n-1} \frac{a^s + x_{-4}x_{-2}x_{0}b\left(\frac{1-a^s}{1-a}\right)}{a^{s+1} + x_{-8}x_{-6}x_{-4}b\left(\frac{1-a^{s+1}}{1-a}\right)}.$$

For $a = -1$, the formulas reduce to

$$x_{12n-9} = \frac{x_{-3}^{2n}}{x_{-9}^{2n-1}} \left(\frac{-1 + x_{-9}x_{-7}x_{-5}b}{-1 + x_{-7}x_{-5}x_{-3}b}\right)^{n}, \quad x_{12n-8} = \frac{x_{-2}^{2n}}{x_{-8}^{2n-1}} \left(\frac{-1 + x_{-8}x_{-6}x_{-4}b}{-1 + x_{-6}x_{-4}x_{-2}b}\right)^{n},$$

$$x_{12n-7} = \frac{x_{-1}^{2n}}{x_{-7}^{2n-1}} \left(\frac{-1 + x_{-7}x_{-5}x_{-3}b}{-1 + x_{-5}x_{-3}x_{-1}b}\right)^{n}, \quad x_{12n-6} = \frac{x_{0}^{2n}}{x_{-6}^{2n-1}} \left(\frac{-1 + x_{-6}x_{-4}x_{-2}b}{-1 + x_{-4}x_{-2}x_{0}b}\right)^{n},$$

$$x_{12n-5} = \frac{x_{-5}x_{-9}^{2n}x_{-7}^{2n}}{x_{-3}^{2n}x_{-1}^{2n}} \left(\frac{-1 + x_{-5}x_{-3}x_{-1}b}{-1 + x_{-9}x_{-7}x_{-5}b}\right)^{n}, \quad x_{12n-4} = \frac{x_{-4}x_{-8}^{2n}x_{-6}^{2n}}{x_{-2}^{2n}x_{0}^{2n}} \left(\frac{-1 + x_{-4}x_{-2}x_{0}b}{-1 + x_{-8}x_{-6}x_{-4}b}\right)^{n},$$

$$x_{12n-3} = \frac{x_{-3}^{2n+1}}{x_{-9}^{2n}} \left(\frac{-1 + x_{-9}x_{-7}x_{-5}b}{-1 + x_{-7}x_{-5}x_{-3}b}\right)^{n}, \quad x_{12n-2} = \frac{x_{-2}^{2n+1}}{x_{-8}^{2n}} \left(\frac{-1 + x_{-8}x_{-6}x_{-4}b}{-1 + x_{-6}x_{-4}x_{-2}b}\right)^{n},$$

$$x_{12n-1} = \frac{x_{-1}^{2n+1}}{x_{-7}^{2n}} \left(\frac{-1 + x_{-7}x_{-5}x_{-3}b}{-1 + x_{-5}x_{-3}x_{-1}b}\right)^{n}, \quad x_{12n} = \frac{x_{0}^{2n+1}}{x_{-6}^{2n}} \left(\frac{-1 + x_{-6}x_{-4}x_{-2}b}{-1 + x_{-4}x_{-2}x_{0}b}\right)^{n},$$

$$x_{12n+1} = \frac{x_{-5}x_{-9}^{2n+1}x_{-7}^{2n+1}}{x_{-3}^{2n+1}x_{-1}^{2n+1}} \frac{(-1 + x_{-5}x_{-3}x_{-1}b)^{n}}{(-1 + x_{-9}x_{-7}x_{-5}b)^{n+1}}, \quad x_{12n+2} = \frac{x_{-4}x_{-8}^{2n+1}x_{-6}^{2n+1}}{x_{-2}^{2n+1}x_{0}^{2n+1}} \frac{(-1 + x_{-4}x_{-2}x_{0}b)^{n}}{(-1 + x_{-8}x_{-6}x_{-4}b)^{n+1}}.$$

If we set $b = \pm 1$ and we replace n with $n + 1$ in the above equations, we recover the results [7] (see Corollaries 3.3.1 and 3.4.1). We note that the x_{12n+5} in the last equation in Corollaries 3.3.1 and 3.4.1 should be x_{12n+11}.

4. Periodic nature and behavior of the solutions

Theorem 4.1. *Let x_n be a solution of*

$$x_{n+5k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+3k}x_{n+4k}(A + Bx_n x_{n+k} x_{n+2k})} \tag{4.1}$$

for some constants $A \neq 1$ and B. If the initial conditions x_i, $i = 0, \ldots, 5k-1$, are such that $x_i^3 = x_{i+k}^3 = (1 - A)/B$, then $x_n = x = [(1 - A)/B]^{1/3}$ for all n.

Proof. Suppose the initial conditions are such that $x_i = x_{i+k}$ and $x_i^3 = (1-A)/B$ for $i = 0, \dots, k-1$. From (3.9) and (3.10), we have that

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{A^s + x_i^3 B \sum_{l=0}^{s-1} A^l}{A^s + x_i^3 B \sum_{l=0}^{s-1} A^l} = x_i,$$

where $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{A^s + x_i^3 B \sum_{l=0}^{s-1} A^l}{A^{s+1} + x_i^3 B \sum_{l=0}^{s} A^l} = \frac{x_i}{A^n + x_i^3 B \sum_{l=0}^{n-1} A^l} = x_i,$$

where $i = 2k, 2k+1, \dots, 3k-1$. That is, $x_{3kn+i} = x_i$, $i = 0, \dots, 3k-1$, and $x_{3kn+i+k} = x_i$, for all k. □

Figure 1 illustrates Theorem 4.1. Note that x in Theorem 4.1 is a fixed point of (4.1). This theorem is interesting in the sense that if any of the initial condition does not satisfy $x_i^3 = (1-A)/B$, x_n can neither be a constant nor periodic even if the initial conditions are all the same (see Figure 2).

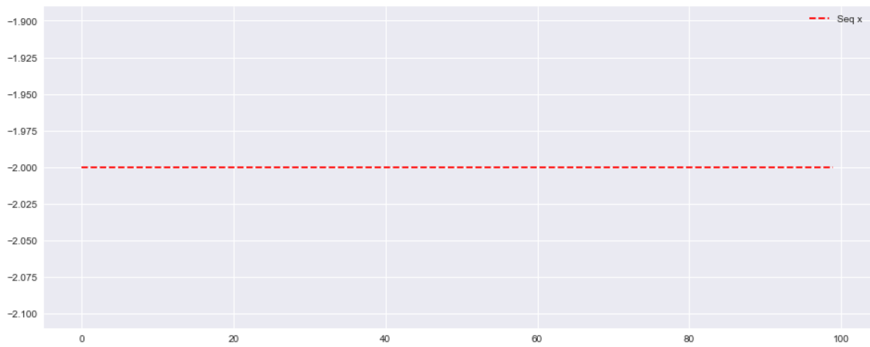

Figure 1: Graph of $x_{n+10} = \dfrac{x_n x_{n+4} x_{n+2}}{x_{n+6} x_{n+8}(3 + 0.25 x_n x_{n+2} x_{n+4})}$, where $x_0 = x_1 = \dots = x_9 = -2 = ((1-A)/B)^{1/3}$.

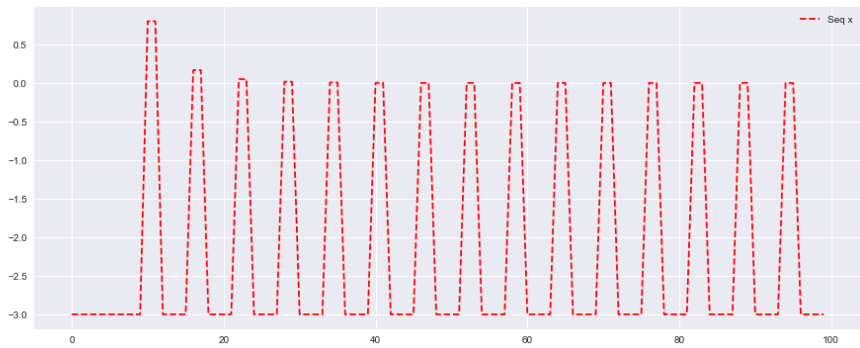

Figure 2: Graph of $x_{n+10} = \dfrac{x_n x_{n+4} x_{n+2}}{x_{n+6} x_{n+8}(3 + 0.25 x_n x_{n+2} x_{n+4})}$, where $x_0 = x_1 = \dots = x_9 = -3 \neq ((1-A)/B)^{1/3}$.

Theorem 4.2. *Let x_n be non-zero solutions of*

$$x_{n+5k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+3k} x_{n+4k}(1 + B x_n x_{n+k} x_{n+2k})}$$

for some constant B*. If the initial conditions* $x_i,\ i = 0, \dots, 5k-1$*, are such that* $x_i = x_{i+k}$*, then the solution can not be periodic. Furthermore, the limit of* x_n*, as* $n \to \infty$*, does not exist.*

Proof. Suppose the non-zero initial conditions are such that $x_i = x_{i+k}$. From (3.9) and (3.10), we have that

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{1 + x_i^3 Bs}{1 + x_i^3 Bs} = x_i,$$

where $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{1 + x_i^3 Bs}{1 + x_i^3 B(s+1)} = \frac{x_i}{1 + x_i^3 Bn} \neq x_i,$$

where $i = 2k, 2k+1, \dots, 3k-1$. It follows that $\lim_{n\to\infty} x_{3kn+i} = x_i$ for $i = 0, \dots, 2k-1$; and $\lim_{n\to\infty} x_{3kn+i} = 0$ for $i = 2k, \dots, 3k-1$. Thus, the limit does not exist. □

Figure 3 illustrates Theorem 4.2.

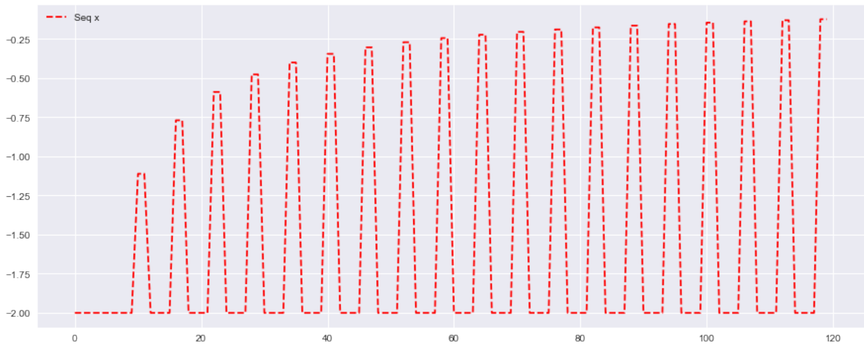

Figure 3: Graph of $x_{n+10} = \dfrac{x_n x_{n+4} x_{n+2}}{x_{n+6} x_{n+8}(1 + x_n x_{n+2} x_{n+4})}$, where $x_0 = x_1 = \cdots = x_9 = -2$.

Theorem 4.3. *Let* x_n *be a solution of*

$$x_{n+5k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+3k} x_{n+4k}(A + B x_n x_{n+k} x_{n+2k})}$$

for some constants $A \neq 1$ *and* B*. If the initial conditions* $x_i,\ i = 0, \dots, 5k-1$ *are such that* $x_i = x_{i+3k}$*, then the solution is periodic with period* $3k$ *if and only if* $x_i x_{i+k} x_{i+2k} = (1-A)/B$*.*

Proof. Suppose the initial conditions are such that $x_i = x_{i+3k}$ and $x_i x_{i+k} x_{i+2k} = (1-A)/B$. From (3.9) and (3.10), we have that

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{A^s + x_i x_{i+k} x_{i+2k} B \sum_{l=0}^{s-1} A^l}{A^s + x_i x_{i+k} x_{i+2k} B \sum_{l=0}^{s-1} A^l} = x_i,$$

where $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{A^s + x_i x_{i+k} x_{i+2k} B \sum_{l=0}^{s-1} A^l}{A^{s+1} + x_i x_{i+k} x_{i+2k} B \sum_{l=0}^{s} A^l} = \frac{x_i}{A^n + x_i x_{i+k} x_{i+2k} B \sum_{l=0}^{n-1} A^l} = x_i,$$

where $i = 2k, 2k+1, \dots, 3k-1$. That is, $x_{3kn+i} = x_i,\ i = 0, \dots, 3k-1$, and $x_{3kn+i+3k} = x_i$, for all k. □

Figures 4 and 5 illustrate Theorem 4.3.

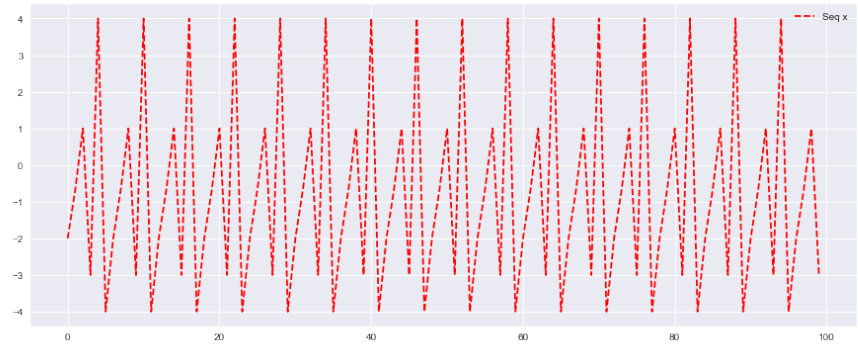

Figure 4: Graph of $x_{n+10} = \dfrac{x_n x_{n+4} x_{n+2}}{x_{n+6} x_{n+8}(3 + 0.25 x_n x_{n+2} x_{n+4})}$, where $x_0 = x_6 = -2, x_1 = x_7 = -2/3, x_2 = x_8 = 1, x_3 = x_9 = -3, x_4 = 4, x_5 = -4$ and are such that $x_0x_2x_4 = x_1x_3x_5 = (1 - A)/B$.

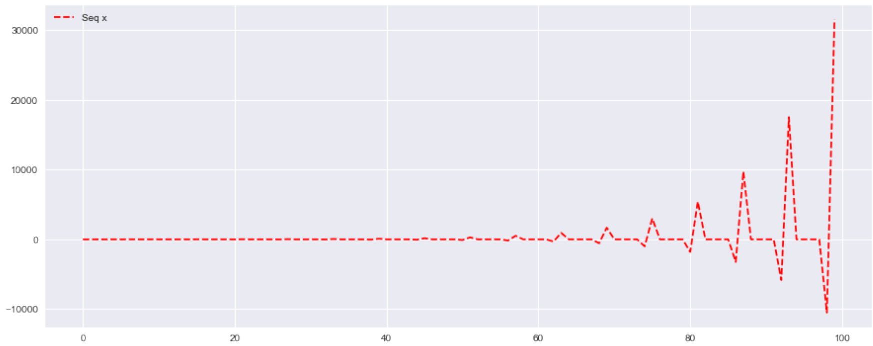

Figure 5: Graph of $x_{n+10} = \dfrac{x_n x_{n+4} x_{n+2}}{x_{n+6} x_{n+8}(3 + 0.25 x_n x_{n+2} x_{n+4})}$, where $x_0 = -x_6 = -7, x_1 = -x_7 = -7/3, x_2 = -x_8 = 1, x_3 = -x_9 = -3, x_4 = 4, x_5 = -4$ and are such that $x_0x_2x_4 \neq x_1x_3x_5 \neq (1 - A)/B$.

Theorem 4.4. *Let x_n be a non-zero solution of*

$$x_{n+5k} = \frac{x_n x_{n+k} x_{n+2k}}{x_{n+3k} x_{n+4k}(1 + B x_n x_{n+k} x_{n+2k})}$$

for some constant B. If the initial conditions x_i, $i = 0, \dots, 5k-1$ are such that $x_i = x_{i+3k}$, then the solution can not be periodic. Furthermore, the limit of x_n, as $n \to \infty$, does not exist.

Proof. Suppose the non-zero initial conditions are such that $x_i = x_{i+3k}$. From (3.9) and (3.10), we have that

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{1 + x_i x_{i+k} x_{i+2k} Bs}{1 + x_i x_{i+k} x_{i+2k} Bs} = x_i,$$

where $i = 0, 1, 2, \dots, 2k-1$; and

$$x_{3kn+i} = x_i \prod_{s=0}^{n-1} \frac{1 + x_i x_{i+k} x_{i+2k} Bs}{1 + x_i x_{i+k} x_{i+2k} B(s+1)} = \frac{x_i}{1 + x_i x_{i+k} x_{i+2k} Bn} \neq x_i,$$

where $i = 2k, 2k+1, \dots, 3k-1$. It follows that $\lim_{n\to\infty} x_{3kn+i} = x_i$ for $i = 0, \dots, 2k-1$; and $\lim_{n\to\infty} x_{3kn+i} = 0$ for $i = 2k, \dots, 3k-1$. Hence the limit does not exist. □

Figure 6 illustrates Theorem 4.4.

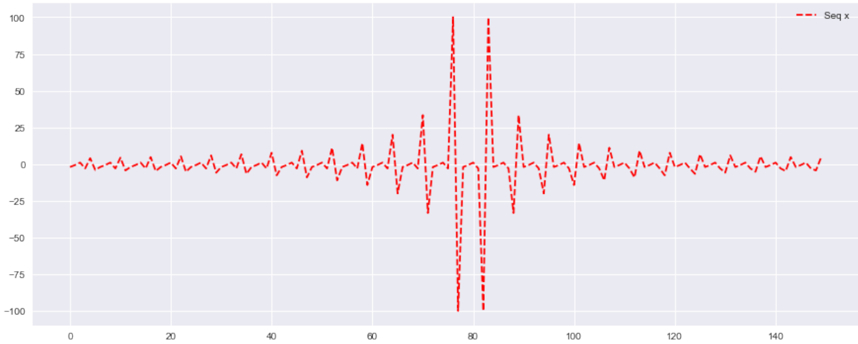

Figure 6: Graph of $x_{n+10} = \frac{x_n x_{n+4} x_{n+2}}{x_{n+6} x_{n+8}(1+0.01 x_n x_{n+2} x_{n+4})}$, where $x_0 = x_6 = -2, x_1 = x_7 = -2/3, x_2 = x_8 = 1, x_3 = x_9 = -3, x_4 = 4, x_5 = -4$.

5. Conclusion

We investigated the difference equation (1.4) by finding the symmetry generators and we used the canonical coordinates to find its invariants which led to the solutions in closed form. We showed that the findings in [7, 10, 18] are special cases of our results and we pointed out some errors in [7]. As a matter of fact, all the formulas solutions found in [7] are solutions of (1.4), when $a_n = b_n = 1$, and not (1.3). Finally, we studied the periodic nature and behavior of the solutions in some cases.

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