SYMPLECTIC REDUCTION ON PSEUDOMANIFOLDS

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

The dissertation consists of symplectic reduction on a Frölicher space which is locally diffeomorphic to an Euclidean Frölicher subspaces of $\mathbb{R}^n$ of constant dimension equal to n. Such a space is called a Frölicher pseudomanifold or simply a pseudomanifold. The symplectic reduction under consideration in this work is an extension of the Marsden-Weinstein quotient (the reduced space) well-known for the finite-dimensional smooth manifold. Starting with a proper and free action of a Frölicher-Lie-group on a finite constant dimensional pseudomanifold, we study the smooth structure induced on a small subspace of the orbit space.
Aside the algebraic and geometric study of these new objects(pseudomanifolds), the work contains their topological fundamentals and symplectic structures, as well as an introduction to the geometric control theory.

I dedicate this work to my darling Gilles Mpumbu and to my kids Kadi, Christelle, Brice and Rosette for their patience and understanding. They were for a big support during this time of sacrifices.
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Contents

Chapter 1

Introduction

This dissertation contains a considerable amount of new mathematical concepts. The background is based on topological, geometric and algebraic fundamentals, so as to study the Marsden-Weinstein symplectic reduction process. This is a classical problem that we would solve on a new class of geometric objects provided with an unusual smooth structure. These modeling spaces are the locally Euclidean spaces which we call $\mathbb{F}$-pseudomanifolds or simply pseudomanifolds if there is no risk of confusion. [6]
To this end, we present the study through seven chapters. The first chapter is an Introduction as usual. The second chapter is devoted to topological foundations of the category of Frölicher spaces. It contains the following topics: a characterization of open and closed sets, smooth maps of Frölicher spaces, and the construction of initial (final) objects and morphisms (smooth maps) of this category. That is, the subspace, the product, the coproduct and quotient spaces. The third chapter restates the whole of Chapter 2 in a subcategory of Frölicher spaces called the category of pseudomanifolds. We present three classes of pseudomanifolds. The fourth chapter aims at the concepts of tangent, double tangent. In the fifth chapter we shall study the symplectic structure on linear and on more general non linear pseudomanifolds. After a comprehensive exposition of the exterior algebra in pseudomanifolds, we shall show that the cotangent bundle to a pseudomanifold is endowed with a canonical symplectic structure. The main chapter in this work is the sixth one. It goes from Lie-group, passes through integral curves, exponential maps, G-equivariance, adjoint and co-adjoint representation, moment map (momentum map in [56, 57, 52]), and ends by the symplectic reduction process. This is a first attempt in the category of $\mathbb{F}$-pseudomanifolds. The seventh chapter is a collection of basic concepts of geometric control theory.
Our investigation should specifically stands on four stages. First of all we shall refer the reader to [6, 19, 31, 44] for this section. There are given the relationship between a class of Sikorski differential spaces and the induced class of Frölicher spaces as in [6]. And also in the constant finite dimensional case, the previous relationship induces another one between differential spaces locally diffeomorphic to a subspace of a cartesian space and $\mathbb{F}$-pseudomanifolds as in

[6, 49, 50, 73, 74]. At this point what can be worth of attention from these references are the following results. A $\mathbb{F}$-structure induces a differential structure in the sense of Sikorski and if a given map is smooth map of differential spaces then it is a smooth map of $\mathbb{F}$-spaces. Secondly, we shall extend to quotient pseudomanifolds the natural construction of differential and symplectic geometry as in [6, 7, 19, 22, 18, 26, 34, 47, 62]. Thirdly, we shall deal with symplectic reduction algorithm on linear and on more general pseudomanifolds based on results in [34, 47, 53]. Finally, we shall present an introduction to the geometric control theory. We shall restrict to the first class of three classes announced in the abstract. This class gives the opportunity to extend concepts and notions from smooth manifolds to pseudomanifolds in the natural way. The difference in definitions and properties reside only on the smoothness of objects and morphisms. And also, the topology under consideration is generated by the sets $f^{-1}(0, \infty)$, forming its basis for all smooth functions, that is, the Frölicher topology. The symplectic reduction process was studied during the passed decades. For basics on this topic, we refer the reader to [24, 28, 4, 54, 16, 47, 15, 64, 65, 66, 52, 53]. The interest of the symplectic reduction can be withdrawn from the abundance of examples in various domains of applications available in the literature above. But, the interest is also purely geometric by constructing new nontrivial objects of the category endowed with smooth additional properties. It is worth noticing to quote P. Cherenack as in [19]:*" In physics an object is often known by various scalar fields(real valued functions), such as temperature and pressure defined on it"*. And, so to conclude that many physical concepts or technical phenomenon are curves (contours) or functions (scalar fields, or scalars for short). In [80, 82, 83] are stated conditions for a category to be able to host a control theory modeling. Our new category fulfills the late conditions: smooth exterior algebra, smooth differentiation theory and smooth transversality theory([7] for an attempt to define the concept). The foundation of theoretical studies and other potential applications of Frölicher spaces were launched in [44, 17, 31] and [6, 62, 85, 18, 19, 8, 22].

Chapter 2

Topologies on $\mathbb{F}$-spaces

2.1 Frölicher spaces

Definition 2.1.1
Let M be an non-empty set. Let $\mathcal{C}_M \subseteq M^{\mathbb{R}} := \{c \mid c : \mathbb{R} \to M\}$ and$\mathcal{F}_M \subseteq \mathbb{R}^M := \{f \mid f : M \to \mathbb{R}\}$. The pair $(\mathcal{C}_M, \mathcal{F}_M)$ is called a Frölicher structure on M if $\mathcal{C}_M$ and $\mathcal{F}_M$ are defined such that:

$$\mathcal{C}_M := \{c : \mathbb{R} \to M \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \textit{for all } f \in \mathcal{F}_M\} \tag{1}$$

and

$$\mathcal{F}_M := \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \textit{for all } c \in \mathcal{C}_M\}, \tag{2}$$

where $C^\infty(\mathbb{R}) := C^\infty(\mathbb{R}, \mathbb{R})$.

The identities (1) and (2) define the compatibility condition for the structure $(\mathcal{C}_M, \mathcal{F}_M)$ on M. They read $\Gamma\mathcal{F}_M = \mathcal{C}_M$ and $\Phi\mathcal{C}_M = \mathcal{F}_M$ in the literature. The pair $(\mathcal{C}_M, \mathcal{F}_M)$ satisfying the compatibility condition was originally called smooth structure (see [31, 20]), then Frölicher structure (see [44, 21]). In this text we state the basic concepts as follows.

Definition 2.1.2
Let M be an non-empty set. The pair $(\mathcal{C}_M, \mathcal{F}_M)$ is a $\mathbb{F}$-structure on M if it satisfies the compatibility condition that is $\mathcal{C}_M = \Gamma\mathcal{F}_M$ and $\mathcal{F}_M = \Phi\mathcal{C}_M$. The triple $(M, \mathcal{C}_M, \mathcal{F}_M)$ is called a Frölicher space (or $\mathbb{F}$-space, smooth space). While $\mathcal{C}_M$ is the set of structure curves and $\mathcal{F}_M$ is the set of structure functions. Some times we will say M is an $\mathbb{F}$-space, inferring the triple above. M is called a linear $\mathbb{F}$-space if it is a linear space whose $\mathbb{F}$-structure is compatible with the linear structure. That is, structure functions and structure curves are linear, also addition and scalar multiplication are smooth maps.

Lemma 2.1.1
Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be an* $\mathbb{F}$*-space. Then* $c \in \mathcal{C}_M$ *if, and only if* $\mathcal{F}_M \circ c \subset C^\infty(\mathbb{R})$ *and* $f \in \mathcal{F}_M$ *if, and only if* $f \circ \mathcal{C}_M \subset C^\infty(\mathbb{R})$*. Also,* $\mathcal{F}_M \circ \mathcal{C}_M \subset C^\infty(\mathbb{R})$*.*

Example 2.1.1
$(\mathbb{R}, C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))$ *is an* $\mathbb{F}$*-space, where* $C^\infty(\mathbb{R})$ *is the set of all differentiable functions from* $\mathbb{R}$ *to* $\mathbb{R}$*, in the usual sense. It is named the canonical* $\mathbb{F}$*-space and denoted by* $(\mathbb{R}, \mathcal{C}, \mathcal{F})$*.*

Example 2.1.2
$(\mathbb{R}^n, C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ *is the canonical* $\mathbb{F}$*-space. This structure follows from Boman's theorem [8].*

Example 2.1.3
Let M *be a differentiable manifold. The pair* $(C^\infty(\mathbb{R}, M), C^\infty(M, \mathbb{R}))$ *consisting of smooth curves into and smooth real-valued functions on* M *satisfies the compatibility condition. The set* M *together with this pair is therefore a Frölicher space (an* $\mathbb{F}$*-space).*

An $\mathbb{F}$-structure on a set M can be generated by either a subset $\mathcal{F}_0 \subset \mathbb{R}^M$ of functions or a subset $\mathcal{C}_0 \subset \mathbb{R}^M$ of curves as from Definition 2.1.1 and Definition 2.1.2. The properties given in Lemma 2.1.2 below are a straightforward consequence of the $\mathbb{F}$-structure generating process on M.

Lemma 2.1.2
Let $\mathcal{F}_0, \mathcal{F}_1 \subset \mathbb{R}^M$ *and* $\mathcal{C}_0, \mathcal{C}_1 \subset M^{\mathbb{R}}$*, where* M *is a non-empty set. The following hold.*

If $\mathcal{F}_0 \subseteq \mathcal{F}_1$ *then* $\Gamma\mathcal{F}_0 \supseteq \Gamma\mathcal{F}_1$*,* $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$ *and* $\Gamma\mathcal{F}_0 = \Gamma\Phi\Gamma\mathcal{F}_0$*.* (3)

If $\mathcal{C}_0 \subseteq \mathcal{C}_1$ *then* $\Phi\mathcal{C}_0 \supseteq \Phi\mathcal{C}_1$*,* $\mathcal{C}_0 \subseteq \Gamma\Phi\mathcal{C}_0$ *and* $\Phi\mathcal{C}_0 = \Phi\Gamma\Phi\mathcal{C}_0$*.* (4)

Proof.
The proof of this Lemma is straightforward. □

Definition 2.1.3
Let M *a non-empty set. Let a subset* $\mathcal{F}_o$ *of* $\mathbb{R}^M$ *be such that* $\Gamma\mathcal{F}_o = \mathcal{C}_M$ *and* $\Phi\Gamma\mathcal{F}_o = \mathcal{F}_M$*. The* $\mathbb{F}$*-structure* $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$ *on* M *is said to be generated by* $\mathcal{F}_o$ *or by functions in* $\mathcal{F}_o$*. Also, let a subset* $\mathcal{C}_o$ *of* $M^{\mathbb{R}}$ *be such that* $\Phi\mathcal{C}_o = \mathcal{F}_M$ *and* $\Gamma\Phi\mathcal{C}_o = \mathcal{C}_M$*. The* $\mathbb{F}$*-structure* $(\Gamma\Phi\mathcal{C}_o, \Phi\mathcal{C}_o)$ *on* M *is said to be generated by* $\mathcal{C}_o$ *or by curves in* $\mathcal{C}_o$*.*

Remark 2.1.1
We will state some consequences of Lemma 2.1.2 *and Definition* 2.1.3*. First of all, let* $\mathcal{P}(\mathbb{R}^M)$*,* $\mathcal{P}(M^{\mathbb{R}})$ *be the power sets of* $\mathbb{R}^M$ *and* $M^{\mathbb{R}}$*. The inclusion is an ordering relation in* $\mathcal{P}(\mathbb{R}^M)$ *and* $\mathcal{P}(M^{\mathbb{R}})$*. Thus,* Γ *and* Φ *are decreasing maps*

$\Gamma:\mathcal{P}(\mathbb{R}^M)\rightarrow\mathcal{P}(M^{\mathbb{R}})$ and $\Phi:\mathcal{P}(M^{\mathbb{R}})\rightarrow\mathcal{P}(\mathbb{R}^M)$ such that a small set of functions (curves) yields the largest set of curves (functions). That is,the smaller $\mathcal{F}_o$ is, the bigger $\Gamma\mathcal{F}_o$ is. Whereas, the smaller $\mathcal{C}_o$ is, the bigger $\Phi\mathcal{C}_o$ is. Hence, $\mathcal{C}_M$ and $\mathcal{F}_M$ are generating sets of the $\mathbb{F}$-structure $(\mathcal{C}_M,\mathcal{F}_M)$ since $\Phi\Gamma\mathcal{F}_M=\Phi\mathcal{C}_M=\mathcal{F}_M$ and $\Gamma\Phi\mathcal{C}_M=\Gamma\mathcal{F}_M=\mathcal{C}_M$. Thus, $\mathcal{C}_M,\mathcal{F}_M)=(\Gamma\mathcal{F}_M,\Phi\Gamma\mathcal{F}_M)=(\Gamma\Phi\mathcal{C}_M,\Phi\mathcal{C}_M)$. Lemma 2.1.2 *states that the structure functions (curves) set contains the generating functions (curves) set. It can be seen through the following examples that the generated $\mathbb{F}$-structures $(\Gamma\mathcal{F}_o,\Phi\Gamma\mathcal{F}_o)$ and $(\Gamma\Phi\mathcal{C}_o,\Phi\mathcal{C}_o)$ are not the same on M.*

Definition 2.1.4
Let $\mathcal{F}_o$ and $\mathcal{C}_o$ be respectively the generating functions set and the generating curves set of an $\mathbb{F}$-structure $(\mathcal{C}_M,\mathcal{F}_M)$. The $\mathbb{F}$-structure is finitely generated, countably generated or infinitely generated if $\mathcal{F}_o$ or $\mathcal{C}_o$ are respectively finite set, countable set or infinite set. The $\mathbb{F}$-structure is linearly generated if $\mathcal{F}_o$ or $\mathcal{C}_o$ is a set of linear functions or linear curves provided that M is a linear space.[31]

Example 2.1.4
Let M and M^* be a linear space and its algebraic dual. The $\mathbb{F}$-structure $(\Gamma\mathcal{F}_o,\Phi\Gamma\mathcal{F}_o)$, where $\mathcal{F}_o\subseteq M^*$ separates points in M. That is, for each two distinct elements $x,y\in M$ there exists $f\in\mathcal{F}_o\subseteq\Phi\Gamma\mathcal{F}_o$ such that $f(x)\neq f(y)$. Thus, this $\mathbb{F}$-structure is linearly generated.

Example 2.1.5
Let $M=\mathbb{R}$, $\mathcal{F}_o=\{id_{\mathbb{R}}\}$ and $\mathcal{C}_o=\{id_{\mathbb{R}}\}$. We want to show in this special case that $(\Gamma\Phi\mathcal{C}_o,\Phi\mathcal{C}_o)=(\Gamma\mathcal{F}_o,\Phi\Gamma\mathcal{F}_o)$. We could use the same technique for the characterization of structure curves and structure functions as in Example 2.1.1, *that is, $(\mathbb{R},C^\infty(\mathbb{R}),C^\infty(\mathbb{R}))$. On the one hand we have $\mathcal{C}=\Gamma\mathcal{F}_o=\{c:\mathbb{R}\rightarrow\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}),\ \text{for all } f=id_{\mathbb{R}}\in\mathcal{F}_o\}=\{c:\mathbb{R}\rightarrow\mathbb{R}\,|\,c\in C^\infty(\mathbb{R})\}=C^\infty(\mathbb{R})$. And $\mathcal{F}=\Phi\Gamma\mathcal{F}_o=\{f:\mathbb{R}\rightarrow\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}),\ \text{for all } c\in C^\infty(\mathbb{R})\}$ since $\mathcal{C}=C^\infty(\mathbb{R})$ on the other hand. We need to characterize $f\in\mathcal{F}$. Assume that $f\notin C^\infty(\mathbb{R})$ but $f\circ c\in C^\infty(\mathbb{R})$ for all $c\in C^\infty(\mathbb{R})$. In particular, if $c=id_{\mathbb{R}}\in C^\infty(\mathbb{R})$, then $f\circ id_{\mathbb{R}}=f\in C^\infty(\mathbb{R})$. This yields a contradiction. Thus $\Phi\Gamma\mathcal{F}_o=C^\infty(\mathbb{R})$. Finally, the $\mathbb{F}$-structure generated by $\mathcal{F}_o=\{id_{\mathbb{R}}\}$ on $\mathbb{R}$ is $(C^\infty(\mathbb{R}),C^\infty(\mathbb{R}))$. Now, if $\mathcal{C}_o=\{id_{\mathbb{R}}\}$ we have $\mathcal{F}=\Phi\mathcal{C}_o=\{f:\mathbb{R}\rightarrow\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}),\ \text{for all } c\in\mathcal{C}_o\}=C^\infty(\mathbb{R})$ and $\mathcal{C}=\Gamma\Phi\mathcal{C}_o==\{c:\mathbb{R}\rightarrow\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}),\ \text{for all } f\in C^\infty(\mathbb{R})\}$ since $\mathcal{F}=C^\infty(\mathbb{R})$. We need to characterize $c\in\mathcal{C}$. Assume that $c\notin C^\infty(\mathbb{R})$ but $f\circ c\in C^\infty(\mathbb{R})$ for all $f\in C^\infty(\mathbb{R})$. In particular, if $f=id_{\mathbb{R}}\in C^\infty(\mathbb{R})$, then $id_{\mathbb{R}}\circ c=c\in C^\infty(\mathbb{R})$. This yields a contradiction. Thus $\Gamma\Phi\mathcal{C}_o=C^\infty(\mathbb{R})$. Finally, the $\mathbb{F}$-structure generated by $\mathcal{C}_o=\{id_{\mathbb{R}}\}$ on $\mathbb{R}$ is $(C^\infty(\mathbb{R}),C^\infty(\mathbb{R}))$.*

Example 2.1.6
Let $\mathcal{F}_o=\{p_i:\mathbb{R}^2\rightarrow\mathbb{R}\,|\,p_i \text{ is the natural projection}\}$, $M=\mathbb{R}^2$. The set of smooth curves is $\mathcal{C}=\Gamma\mathcal{F}_o=\{c:\mathbb{R}\rightarrow\mathbb{R}^2\,|\,f\circ c\in C^\infty(\mathbb{R}),\ \text{for all } f\in\mathcal{F}_o\}=\mathcal{C}_{\mathbb{R}^2}$. Now, we need to characterize $\mathcal{C}=\Gamma\mathcal{F}_o=\mathcal{C}_{\mathbb{R}^2}$. The condition $f\circ c\in C^\infty(\mathbb{R})$ becomes $p_i\circ c\in C^\infty(\mathbb{R})$

since f is either p_1 or p_2. It requires that $c=(c_1,c_2)$, with $c_i \in C^\infty(\mathbb{R})$ for $i=1,2$. Thus, $c \in C^\infty(\mathbb{R},\mathbb{R}^2)$. One concludes that $\mathcal{C}=\mathcal{C}_{\mathbb{R}^2}$ is the set of all smooth vector-valued curves into $\mathbb{R}^2$. Thus, $\mathcal{C}=\{c:\mathbb{R}\to\mathbb{R}^2 \mid c=(c_1,c_2)$ with $c_1,c_2 \in C^\infty(\mathbb{R})\} = C^\infty(\mathbb{R},\mathbb{R}^2)=\mathcal{C}_{\mathbb{R}^2}$. These curves form a Frölicher structure on $\mathbb{R}^2$ together with the set of functions satisfying the compatibility condition above. That is,

$$\begin{aligned}
\mathcal{F} &= \Phi\Gamma\mathcal{F}_o \\
&= \{f:\mathbb{R}^2\to\mathbb{R} \mid f\circ c \in C^\infty(\mathbb{R}),\ \textit{for all } c\in C^\infty(\mathbb{R},\mathbb{R}^2)\} \\
&= \{f:\mathbb{R}^2\to\mathbb{R} \mid f\circ (c_1,c_2) \in C^\infty(\mathbb{R}),\ \textit{for all } c_1,c_2\in C^\infty(\mathbb{R})\} \\
&= C^\infty(\mathbb{R}^2,\mathbb{R})\}
\end{aligned}$$

following Boman's Theorem [8]. This result holds on $\mathbb{R}^n$.

Example 2.1.7
Let $M=\mathbb{Q}$, $\mathcal{F}_o=\{\iota:\mathbb{Q}\hookrightarrow\mathbb{R} \mid \iota=id_{\mathbb{R}|\mathbb{Q}}\}=\{\iota\}$.

$$\begin{aligned}
\mathcal{C}_{\mathbb{Q}} &= \Gamma\mathcal{F}_o \\
&= \{c:\mathbb{R}\to\mathbb{Q} \mid f\circ c\in C^\infty(\mathbb{R}) \textit{ for all } f\in\mathcal{F}_o\} \\
&= \{c:\mathbb{R}\to\mathbb{Q} \mid \iota\circ c = id_{\mathbb{Q}}\circ c = c \in C^\infty(\mathbb{R})\} \\
&= \{c:\mathbb{R}\to\mathbb{Q} \mid c\in C^\infty(\mathbb{R}) \textit{ and } c(\mathbb{R})\subset\mathbb{Q}\} \\
&= \mathbb{Q}^{\mathbb{R}}\cap C^\infty(\mathbb{R},\mathbb{R})
\end{aligned}$$

We want now to characterize $c\in\mathcal{C}_{\mathbb{Q}}$. Since $c\in\mathbb{Q}^{\mathbb{R}}$ implies $c\in\mathbb{R}^{\mathbb{R}}$. From the intersection above $c\in\mathcal{C}_{\mathbb{Q}}$ reads $c\in C^\infty(\mathbb{R},\mathbb{R})$, that is, c is continuous in the usual sense. Now, suppose $r,r'\in\mathbb{R}$ with $r<r'$ and $c(r)\neq c(r')$. Assume without loss of generality that $c(r)<c(r')$. It follows from the Intermediate Values Theorem that for each $s\in[c(r),c(r')]\subset\mathbb{R}$, there exists $t\in[r,r']$ such that $s=c(t)$. That is, c takes all real values (rationals and irrationals) between $c(r)$ and $c(r')$. By definition of c, we have $c(\mathbb{R})\subset\mathbb{Q}$ that is $t\mapsto c(t)\in\mathbb{Q}$ or equivalently, all $c(t)$ are rational numbers exclusively. This yields a contradiction with the conclusion above. As a consequence $c(r)=c(r')$, for all $r,r'\in\mathbb{R}$ with $r\neq r'$. That is to say, c is a constant curve. Finally, the generated curves in this structure are given by $\mathcal{C}_{\mathbb{Q}}=\{c:\mathbb{R}\to\mathbb{Q} \mid c$ constant $\}=\{c_k:\mathbb{R}\to\mathbb{Q} \mid c_k(t)=k,$ for all $t\in\mathbb{R},\ k\in\mathbb{Q}, k$ fixed$\}$. Thus, $\mathcal{F}_{\mathbb{Q}}=\Phi\Gamma\mathcal{F}_o=\Phi\mathcal{C}_{\mathbb{Q}}=\{f:\mathbb{Q}\to\mathbb{R} \mid f\circ c\in C^\infty(\mathbb{R}),$ for all $c\in\mathcal{C}_{\mathbb{Q}}\}$. therefore, $\mathcal{F}_{\mathbb{Q}}=\{f:\mathbb{Q}\to\mathbb{R} \mid f\circ c_k\in C^\infty(\mathbb{R}),$ such that $c_k(t)=k,$ for all $t\in\mathbb{R},\ k\in\mathbb{Q}$ fixed $\}$. Hence, $\mathcal{F}_{\mathbb{Q}}=\{f:\mathbb{Q}\to\mathbb{R} \mid f_k\in C^\infty(\mathbb{R}),\ f_k(t)=f(k),$ for all $t\in\mathbb{R},\ k\in\mathbb{Q}$ fixed $\}$. We need to characterize $\mathcal{F}_{\mathbb{Q}}\subset\mathbb{R}^{\mathbb{Q}}$, that is, the set of real-valued functions on $\mathbb{Q}$ such that $f\circ c_k\in C^\infty(\mathbb{R})$. For any $f\in\mathbb{R}^{\mathbb{Q}}$, for any $k\in\mathbb{Q}$, $f(k)$ determines a constant function

$$f_k:\mathbb{Q}\to\mathbb{R} \textit{ such that } f_k(t)=(f\circ c_k)(t)=f(c_k(t))=f(k). \qquad (5)$$

Thus, $f\circ c_k\in C^\infty(\mathbb{R})$. Hence $f\in\mathcal{F}_{\mathbb{Q}}$ if, and only if $\mathcal{F}_{\mathbb{Q}}=\mathbb{R}^{\mathbb{Q}}$. Therefore, $(\mathcal{C}_{\mathbb{Q}},\mathcal{F}_{\mathbb{Q}})$ is an $\mathbb{F}$-structure on $\mathbb{Q}$. That is, in the $\mathbb{F}$-structure generated on $\mathbb{Q}$ by the inclusion map, only constant curves are structure (smooth) curves. But all real-valued functions on $\mathbb{Q}$ are structure (smooth) functions.

Example 2.1.8
Let $M=\mathbb{R}$, $\mathcal{F}_o=\{f:\mathbb{R}\to\mathbb{R}\,|\,f \text{ is constant }\}$.

$$\begin{aligned}
\mathcal{C} &= \Gamma\mathcal{F}_o\\
&= \{c:\mathbb{R}\to\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}),\ \textit{for all } f\in\mathcal{F}_o\}\\
&= \{c:\mathbb{R}\to\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}),\ \textit{for all } f \textit{ constant }\}\\
&= \{c:\mathbb{R}\to\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}) \textit{ and } f(c(t))=k \textit{ for all } t \textit{ and } c(t)\ \in\mathbb{R}\}\\
&= \{c:\mathbb{R}\to\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R}) \textit{ for all } c\in\mathbb{R}^{\mathbb{R}}\}\\
&= \mathbb{R}^{\mathbb{R}}\\
&= \mathcal{C}_{\mathbb{R}}
\end{aligned}$$

$\mathcal{F}=\Phi\Gamma\mathcal{F}_o=\{f:\mathbb{R}\to\mathbb{R}\,|\,f\circ c\in C^\infty(\mathbb{R})$ *for all* $c\in\mathbb{R}^{\mathbb{R}}\}$. *We need a characterization of elements in* $\mathcal{F}$. *We borrow the technique from [39] to show that* $\mathcal{F}=\mathcal{F}_o$. *That is, all functions in* $\mathcal{F}$ *are constant. Let us assume* f *non constant on* $\mathbb{R}$. *That is,* $\mathbb{R}$ *contains an interval on which* f *is not constant. We will tackle the characterization in four steps as stated below.*

1. *To apply the Intermediate Values Theorem to* $f\circ c$, *where* c *is some particular curve into* $\mathbb{R}$ *and* $f\in\mathcal{F}$,
2. *to assume that* $f\circ c$ *has rationals images only,*
3. *to assume that* $f\circ c$ *has irrationals images only,*
4. *to assume that* $f\circ c$ *has both rationals and irrationals images.*

We need to show that each pairing of assumption (1) and another one among (2), (3), and (4) yields a contradiction so as to conclude that such a non constant function does not exist in the structure.
First assume $f(r)=q$ *for some* $r,q\in\mathbb{R}$. *Then there exists a small interval* $B(r,\epsilon)$ *centered at* r *with radius* ϵ *such that it does not contain any interval on which* f *would be constant. That is, there exists* $r'\in B(r,\epsilon)$ *such that* $r<r'$, $f(r')=q'$ *and* $q\neq q'$. *Now let,* $c:\mathbb{R}\to\mathbb{R}$ *be a curve into* $\mathbb{R}$, *defined by*

$$t\mapsto c(t)=(1-t)r+tr'=r+t(r'-r). \tag{6}$$

Thus, $c:[0,1]\to\mathbb{R}$ *maps* $[0,1]$ *on* $[r,r']$. *It follows that* $c[0,1]\subset B(r,\epsilon)$ *since* $c(0)=r$, $c(1)=r'$ *and* $r\leq c(t)\leq r'$. *Recall the fact that* $f\circ c\in C^\infty(\mathbb{R})$ *for all* $c\in\mathbb{R}^{\mathbb{R}}$, *this also holds on the interval* $[0,1]$ *that is* $f\circ c$ *is continuous function on* $[0,1]$ *onto* $[q,q']$. *For,*

$$[0,1]\to[r,r']\to[q,q'] \textit{ such that } t\mapsto c(t)\mapsto s=f(c(t)),$$

where $(f\circ c)(0)=f(r)=q$ *and* $(f\circ c)(1)=f(r')=q'$. *By the Intermediate Values Theorem it follows that for each* $s\in[q,q']$, *there exists*

$$t\in[0,1] \textit{ such that } s=(f\circ c)(t). \tag{7}$$

That is, $f \circ c$ takes all real values (rationals and irrationals) between q and q'. Secondly, assume (2) holds. That is $[q, q'] \subset \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is always an irrational between two rationals. This leads to a contradiction with Equation (7). Hence there does not exist such a function f that is f non constant taking only rationals images.
Thirdly, assume (3) holds. That is, $[q, q'] \subset \mathbb{R} - \mathbb{Q}$. Since $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$, there is always a rational between two irrationals. This is a contradiction with Equation (7). Thus there does not exist such a function f that is f non constant taking only irrationals images.
Fourthly, assume (4) holds. That is $f \circ c$ takes $q \in \mathbb{Q}$ and $q' \in \mathbb{R} - \mathbb{Q}$. This yields a partition of the image of c as follows. $A \cap B = \emptyset$ and $A \cup B = [r, r']$, where $A = \{u \in [r, r'] \mid f_{|A}(u)$ is a rational number in $[q, q']\}$ while $B = \{u \in [r, r'] \mid f_{|B}(u)$ is an irrational number in $[q, q']\}$. Also $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ do not contain $[q, q']$. From second and third steps above, such functions $f_{|A}$ and $f_{|B}$ do not exist. Therefore, there does not exist such non constant function f. Finally, $f \in \mathcal{F} = \Phi\Gamma\mathcal{F}_o = \mathcal{F}_o$, that is f must be constant.
The triple $(\mathbb{R}, \mathbb{R}^{\mathbb{R}}, \mathcal{F}_o)$ is an $\mathbb{F}$-space. In conclusion, the smooth structure generated on $\mathbb{R}$ by constant functions has all real functions as smooth curves and only constant functions are structure functions.

Example 2.1.9
Let $M = [0, 1] \subset \mathbb{R}$, $\mathcal{F}_o = \{\iota : [0, 1] \hookrightarrow \mathbb{R} \,|\, \iota$ is the inclusion map$\}$. Then $([0, 1], \mathcal{C}, \mathcal{F})$ is an $\mathbb{F}$-space. The generated curves for this structure are:

$$\begin{aligned} \mathcal{C} &= \Gamma\mathcal{F}_o \\ &= \{c : \mathbb{R} \to [0,1] \,|\, f \circ c \in C^\infty(\mathbb{R}), \text{ for all } f \in \mathcal{F}_o\} \\ &= \{c : \mathbb{R} \to [0,1] \,|\, \iota \circ c \in C^\infty(\mathbb{R})\} \\ &= \{c : \mathbb{R} \to [0,1] \,|\, c \in C^\infty(\mathbb{R}) \text{ such that } c(\mathbb{R}) \subset [0,1]\} \\ &= \mathcal{C}_M. \end{aligned}$$

Note that $\mathcal{C}_M \subset C^\infty(\mathbb{R})$ since $id_{\mathbb{R}} \in C^\infty(\mathbb{R})$ but $id_{\mathbb{R}} \notin \mathcal{C}_M$. The generated functions for this structure are:

$$\begin{aligned} \mathcal{F} &= \Phi\mathcal{C}_M \\ &= \{f : [0,1] \to \mathbb{R} \,|\, f \circ c \in C^\infty(\mathbb{R}), \text{ for all } c \in \mathcal{C}_M\} \\ &= \{f : [0,1] \to \mathbb{R} \,|\, f \circ c \in C^\infty(\mathbb{R}), \text{ for all } c \in C^\infty(\mathbb{R}), c(\mathbb{R}) \subset [0,1]\} \\ &= \{f : [0,1] \to \mathbb{R} \,|\, f = g_{|_{[0,1]}} \in C^\infty(\mathbb{R}), \text{ where } g \in \mathbb{R}^{\mathbb{R}}\} \end{aligned}$$

Thus $\mathcal{F} \supset \Phi C^\infty(\mathbb{R})$. The former and latter inclusions confirm the order reversing property of Φ and Γ.

Definition 2.1.5
An $\mathbb{F}$-structure $(\mathcal{C}_M, \mathcal{F}_M)$ on an $\mathbb{F}$-space M is discrete if $\mathcal{F}_M = \mathbb{R}^M$, that is, all real-valued functions on M are smooth. An $\mathbb{F}$-structure $(\mathcal{C}, \mathcal{F})$ on an $\mathbb{F}$-space M is finer than another $(\mathcal{C}', \mathcal{F}')$ on the same underlying set if $\mathcal{F}' \subseteq \mathcal{F}$. An $\mathbb{F}$-structure $(\mathcal{C}, \mathcal{F})$ on an $\mathbb{F}$-space M is coarser than another $(\mathcal{C}', \mathcal{F}')$ on the same underlying

set if $\mathcal{C}' \subseteq \mathcal{C}$. A discrete $\mathbb{F}$-structure is generated by an empty set of curves and in which all functions are smooth. [6]

A discrete $\mathbb{F}$-structure can be understood as one where $\mathcal{C}_o = \emptyset$, $\mathcal{F}_M = \Phi\mathcal{C}_o = \mathbb{R}^M$ and $\mathcal{C}_o \subset \Gamma\mathcal{F}_M = \mathcal{C}_M$. That is, the structure curves are all constant maps. While $\mathcal{F}_M = \Phi\mathcal{C}_o = \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}), \text{ for all } c \in \{\ \} = \emptyset\} = \mathbb{R}^M$ is the structure functions set. This is a straightforward consequence of the fact that false implies true in logic.

Example 2.1.10
$(\mathbb{Q}, \mathcal{C}_\mathbb{Q}, \mathcal{F}_\mathbb{Q})$ *in Example* 2.1.7*, the* $\mathbb{F}$*-structure* $(\mathcal{C}_\mathbb{Q}, \mathcal{F}_\mathbb{Q})$ *is discrete since* $\mathcal{F}_\mathbb{Q} = \mathbb{R}^\mathbb{Q}$.

Example 2.1.11
From Example 2.1.8*, we assume* $\mathcal{C}_o = \mathcal{F}_o$ *since in* $\mathbb{R}$*, curves and functions coincide. Thus,* $\mathcal{F}' = \Phi\mathcal{C}_o = \mathbb{R}^\mathbb{R}$ *and* $\mathcal{C}' = \Gamma\Phi\mathcal{C}_o = \Gamma\mathbb{R}^\mathbb{R} = \{c : \mathbb{R} \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}),\ for\ all\ f \in \mathbb{R}^\mathbb{R}\} = \{c : \mathbb{R} \to \mathbb{R} \mid c\ is\ constant\}$*. Therefore,* $(\mathbb{R}, \mathcal{C}', \mathcal{F}') = (\mathbb{R}, \mathcal{C}', \mathbb{R}^\mathbb{R})$ *yields the* $\mathbb{F}$*-structure* $(\mathcal{C}', \mathbb{R}^\mathbb{R})$ *that is discrete since* $\mathcal{F}_\mathbb{R} = \mathbb{R}^\mathbb{R}$*. Now we want to compare the canonical* $\mathbb{F}$*-structure on* $\mathbb{R}$ *that is* $(\mathcal{C}, \mathcal{F}) = (C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))$ *and the discrete* $\mathbb{F}$*-structure* $(\mathcal{C}', \mathcal{F}') = (\mathcal{C}', \mathbb{R}^\mathbb{R})$*, where* $\mathcal{C}'$ *is a set of constant curves, as built above. This yields the following conclusions.* $\mathcal{F} \subset \mathcal{F}'$*, that is,* $(\mathcal{C}', \mathcal{F}')$ *is finer than* $(\mathcal{C}, \mathcal{F})$ *and* $\mathcal{C}' \subset \mathcal{C}$*, that is,* $(\mathcal{C}, \mathcal{F})$ *is coarser than* $(\mathcal{C}', \mathcal{F}')$*. The above inclusions are consequences of the order reversing property of* Γ *and* Φ*, that is, being a finer or a coarser structure are dual concepts. We generalize the results in the following lemma.*

Lemma 2.1.3

1. *Let* $\mathcal{F}_o$ *be a generating set of the* $\mathbb{F}$*-structure* $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$ *on a set* M*. Let* $(\mathcal{C}_M, \mathcal{F}_M)$ *be another* $\mathbb{F}$*-structure on the same set* M *whose generating set* $\mathcal{F}_1$ *contains* $\mathcal{F}_o$*. Then* $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$ *is coarser than* $(\mathcal{C}_M, \mathcal{F}_M)$*.*
2. *Let* $\mathcal{C}_o$ *be a generating set of the* $\mathbb{F}$*-structure* $(\Gamma\Phi\mathcal{C}_o, \Phi\mathcal{C}_o)$ *on a set* M*. Let* $(\mathcal{C}_M, \mathcal{F}_M)$ *be another* $\mathbb{F}$*-structure on the same set* M *whose generating set* $\mathcal{C}_1$ *contains* $\mathcal{C}_o$*. Then* $(\Gamma\Phi\mathcal{F}_o, \Phi\mathcal{C}_o)$ *is finer than* $(\mathcal{C}_M, \mathcal{F}_M)$*.*

Proof.

1. $\mathcal{F}_1 \supseteq \mathcal{F}_o$ implies $\mathcal{C}_M = \Gamma\mathcal{F}_1 \subseteq \Gamma\mathcal{F}_o$.
2. $\mathcal{C}_1 \supseteq \mathcal{C}_o$ implies $\mathcal{F}_M = \Phi\mathcal{C}_1 \subseteq \Phi\mathcal{C}_o$. □

Lemma 2.1.4
Let $A, B \subseteq M^\mathbb{R}$*, where* M *is a non-empty set. The following statements hold:*

1. $\Phi(A) \cap \Phi(B) \subseteq \Phi(A) \cup \Phi(B)$

2. $\Phi(A \cup B) \subseteq \Phi(A \cap B)$

3. $\Phi(A) \cup \Phi(B) \subseteq \Phi(A \cap B)$

4. $\Phi(A \cup B) = \Phi(A) \cap \Phi(B)$

5. $\Phi(A \cup B) \subseteq \Phi(A) \cap \Phi(B) \subseteq \Phi(A) \cup \Phi(B) \subseteq \Phi(A \cap B)$

6. $\Phi(A \cap B) = \Phi(A) \cup \Phi(B)$

Proof.

1. True in set theory.

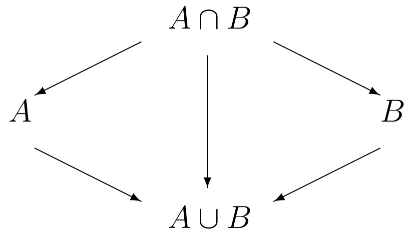
2. Since $A \cap B \subseteq A \cup B$ and Φ is order reversing, we have $\Phi(A \cup B) \subseteq \Phi(A \cap B)$.

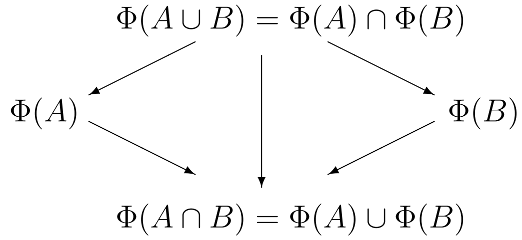
3. Since $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$, the order reversing property of Φ yields: $\Phi(A \cup B) \subseteq \Phi(A) \subseteq \Phi(A \cap B)$ and $\Phi(A \cup B) \subseteq \Phi(B) \subseteq \Phi(A \cap B)$. Hence,
$$\Phi(A \cup B) \subseteq \Phi(A) \cup \Phi(B) \subseteq \Phi(A \cap B). \tag{8}$$
The fore mentioned inclusion holds.

4. The chain of inclusions in (3) of the proof above yields again
$$\Phi(A \cup B) \subseteq \Phi(A) \cap \Phi(B) \subseteq \Phi(A \cap B). \tag{9}$$
To prove the converse inclusion that is $\Phi(A \cup B) \supseteq \Phi(A) \cap \Phi(B)$, we need to characterize the elements belonging to the set in both sides. By definition of Φ we have: $\Phi(A \cup B) = \{f : M \to \mathbb{R} \,|\, f \circ c \in C^\infty(\mathbb{R}) \text{ for all } c \in A \cup B\}$ and $\Phi(A \cap B) = \{f : M \to \mathbb{R} \,|\, f \circ c \in C^\infty(\mathbb{R}) \text{ for all } c \in A \cap B\}$. Now, assume $f \in \Phi(A) \cap \Phi(B)$. It follows that $f \in \Phi(A)$ and $f \in \Phi(B)$ that is $[f \circ a \in C^\infty(\mathbb{R})$ for all $a \in A]$ and $[f \circ b \in C^\infty(\mathbb{R})$ for all $b \in B]$. Thus $[f \circ a$ and $f \circ b \in C^\infty(\mathbb{R})$, for all $a \in A$ and for all $b \in B]$. Otherwise, that is: $[f \circ c \in C^\infty(\mathbb{R})$ for all $c \in A \cup B]$. Therefore $f \in \Phi(A \cup B)$ and consequently $\Phi(A \cup B) \supseteq \Phi(A) \cap \Phi(B)$. Finally, $\Phi(A \cup B) = \Phi(A) \cap \Phi(B)$ from (9) and the above inclusion.

5. Combining steps (4), (1) and (3) above yields the chain of inclusions in step (5) by the transitivity of the inclusion relation.

6. From Relation (8) in step (3) we have $\Phi(A) \cup \Phi(B) \subseteq \Phi(A \cap B)$. We need to prove the converse inclusion. Let us take the following diagrams:





We can consider the two diagrams below as Categories of trellis and Φ as a functor transforming the sup into inf and vice-versa because of the the order reversing property Φ. Since $sup(A, B) = A \cup B$ and $inf(A, B) = A \cap B$ are unique in these trellis, then it follows that $\Phi(A \cup B) = \Phi(A) \cap \Phi(B)$ and $\Phi(A \cap B) = \Phi(A) \cup \Phi(B)$. □

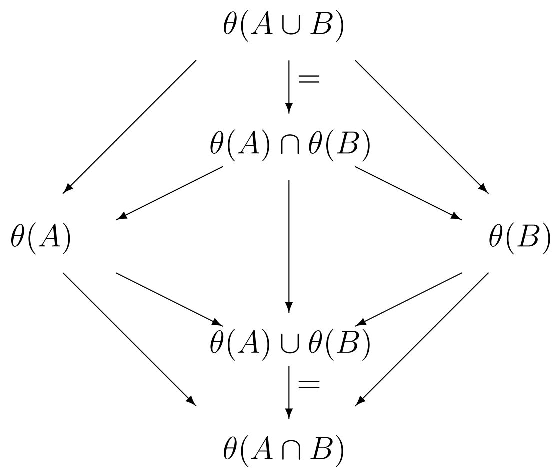
Lemma 2.1.5
Let $A, B \subseteq \mathbb{R}^M$, where M is a non-empty set. The following statements hold:

1. $\Gamma(A) \cap \Gamma(B) \subseteq \Gamma(A) \cup \Gamma(B)$
2. $\Gamma(A \cup B) \subseteq \Gamma(A \cap B)$
3. $\Gamma(A) \cup \Gamma(B) \subseteq \Gamma(A \cap B)$
4. $\Gamma(A \cup B) = \Gamma(A) \cap \Gamma(B)$
5. $\Gamma(A \cup B) \subseteq \Gamma(A) \cap \Gamma(B) \subseteq \Gamma(A) \cup \Gamma(B) \subseteq \Gamma(A \cap B)$
6. $\Gamma(A \cap B) = \Gamma(A) \cup \Gamma(B)$

Proof.
The proof is straightforward as from Lemma 2.1.4, substituting Φ by Γ, and $M^{\mathbb{R}}$ by $\mathbb{R}^M$. □

Remark 2.1.2

1. $\Phi(A) \cap \Phi(B) = \Phi(A) \cup \Phi(B)$ *if, and only if* $\Phi(A) = \Phi(B)$. *Hence* $\Phi(A) \cap \Phi(B) \subset \Phi(A) \cup \Phi(B)$ *is a strict inclusion in general.*
2. *The fore mentioned two lemmas can be represented by the diagram below, where* $\theta = \Phi$ *or* $\theta = \Gamma$



3. *Lemma* 2.1.4 *and Lemma* 2.1.5 *are powerful tools in building $\mathbb{F}$-structure from a generating set.*

Example 2.1.12
Let $\mathcal{F}_o=\{id_{\mathbb{R}}\}$ *and* $\mathcal{F}_1=\{id_{\mathbb{R}}\}\cup\{|\cdot|\}$. *Thus* $(\Gamma\mathcal{F}_o,\Phi\Gamma\mathcal{F}_o)=(C^\infty(\mathbb{R}),C^\infty(\mathbb{R}))$ *the canonical $\mathbb{F}$-structure on $\mathbb{R}$. Also* $\Gamma\mathcal{F}_1=\Gamma(\{id_{\mathbb{R}}\}\cup\{|\cdot|\})=\Gamma\{id_{\mathbb{R}}\}\cap\Gamma\{|\cdot|\}=C^\infty(\mathbb{R})\cap\Gamma\{|\cdot|\}$. *Thus,* $\Gamma\mathcal{F}_1\subseteq C^\infty(\mathbb{R})$ *and this inclusion involves curves. Hence the canonical $\mathbb{F}$-structure is coarser than* $(\Gamma\mathcal{F}_1,\Phi\Gamma\mathcal{F}_1)$ *on $\mathbb{R}$. Now* $\Phi\Gamma\mathcal{F}_1\supseteq C^\infty(\mathbb{R})$*: here we deal with functions therefore* $(\Gamma\mathcal{F}_1,\Phi\Gamma\mathcal{F}_1)$ *is finer than the canonical $\mathbb{F}$-structure. Finally, we have*

$$\Phi\Gamma\mathcal{F}_1=\Phi(C^\infty(\mathbb{R})\cap\Gamma\{|\cdot|\})\supseteq\Phi(C^\infty(\mathbb{R}))\cup\Phi\Gamma\{|\cdot|\}\supset C^\infty(\mathbb{R})$$

Remark 2.1.3
The structure functions set $\mathcal{F}_M$ endowed with addition(+), multiplication(.) and scalar multiplication(*) is a $\mathbb{R}$-algebra. That is, let $f,g\in\mathcal{F}_M$, $\lambda\in\mathbb{R}$, then $f+g, f\cdot g, \lambda * f\in\mathcal{F}_M$ Since the composition map, the evaluation map and insertion map are smooth then $(f+g)$, $f\cdot g$ and $\lambda * f$ are also smooth: $(f+g)\circ c=f\circ c+g\circ c$, $(f\cdot g)\circ c=(f\circ c)\cdot(g\circ c)$ and $(\lambda * f)\circ c=\lambda*(f\circ c)$ are smooth in the usual sense, where $c\in\mathcal{C}_M$.

2.2 Topologies underlying an $\mathbb{F}$-space

Definition 2.2.1
Let $(M,\mathcal{C}_M,\mathcal{F}_M)$ be an $\mathbb{F}$-space, $(\mathcal{C},\mathcal{F})=(C^\infty(\mathbb{R},\mathbb{R}),(C^\infty(\mathbb{R},\mathbb{R})))$ and $\tau_{\mathbb{R}}$, respectively, the canonical $\mathbb{F}$-structure and the canonical topology on $\mathbb{R}$. The topology induced by $\mathcal{F}_M$ on M, where all structure functions are continuous, is denoted by $\tau_{\mathcal{F}_M}:=\{f^{-1}(V)\mid V\in\tau_{\mathbb{R}},\ \text{for all } f\in\mathcal{F}_M)\}$.

Definition 2.2.2
Let $(M,\mathcal{C}_M,\mathcal{F}_M)$ be an $\mathbb{F}$-space, $(\mathcal{C},\mathcal{F})=(C^\infty(\mathbb{R},\mathbb{R}),(C^\infty(\mathbb{R},\mathbb{R})))$ and $\tau_{\mathbb{R}}$, re-

spectively, the canonical $\mathbb{F}$-structure and the canonical topology on $\mathbb{R}$. The topology induced by $\mathcal{C}_M$ on M, where all structure curves are continuous, is denoted by $\tau_{\mathcal{C}_M} := \{\mathcal{U} \mid c^{-1}(\mathcal{U}) \in \tau_{\mathbb{R}},\ \text{for all } c \in \mathcal{C}_M)\}$.

Definition 2.2.3 [*17*]
An $\mathbb{F}$-space M, where $\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M}$ is called a balanced space. M is Hausdorff if $\tau_{\mathcal{F}_M}$ and $\tau_{\mathcal{C}_M}$ are both Hausdorff. A compact Hausdorff balanced $\mathbb{F}$-space is called a base space.

It is known that the topology $\tau_{\mathcal{F}_M}$ is Hausdorff if for any two distinct points $x, y \in M$, there is $f \in \mathcal{F}_M$ such that $f(x) \neq f(y)$. Also, it is worth noticing that each $\mathbb{F}$-space can be associated to a Hausdorff space up to an equivalence relation as in [60, 6]. Henceforth we will deal with $\mathbb{F}$-spaces which are Hausdorff by assumption. A topology containing a Hausdorff topology is itself Hausdorff. In the sequel we only need to check whether $\tau_{\mathcal{F}_M}$ is Hausdorff and then conclude that M is Hausdorff. The following lemma shows that $\tau_{\mathcal{F}_M}$ is the weakest topology in which structure curves and functions are continuous. We therefore shall refer to it as the topology of a Frölicher spaces unless otherwise specified.

Lemma 2.2.1
The two natural topologies on an $\mathbb{F}$-space satisfy the property $\tau_{\mathcal{F}_M} \subset \tau_{\mathcal{C}_M}$

Proof.
Let $\mathcal{U} \in \tau_{\mathcal{F}_M}$. That is, $\mathcal{U} = \bigcup_{f \in \mathcal{F}_M} f^{-1}(V)$, where V is an open set in $\mathbb{R}$. Now, let $c \in \mathcal{C}_M$. Thus, $c^{-1}(\mathcal{U}) = c^{-1}(\bigcup_{f \in \mathcal{F}_M} f^{-1}(V)) = \bigcup_{f \in \mathcal{F}_M} (f \circ c)^{-1}(V) = W$ is an open set, as an union of open sets in $\mathbb{R}$. Thus $\mathcal{U}$ is a $\tau_{\mathcal{C}_M}$-open set, and the conclusion $\tau_{\mathcal{F}_M} \subset \tau_{\mathcal{C}_M}$ follows. □

We will see the worth of the construct we made in the following Lemma 2.2.2 in the proof of Proposition 2.2.1. The result was first stated by Dugmore [26], where he constructed the function ϕ defined as follows:

$$\phi(t) = \begin{cases} e^{-\frac{1}{t}} & , \text{if } t > 0 \\ 0 & , \text{if } t \leq 0 \end{cases}$$

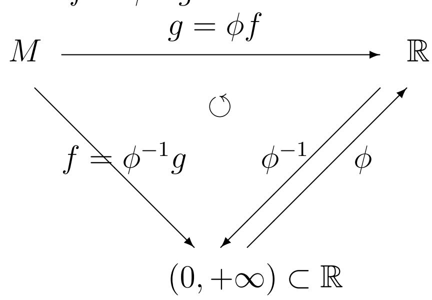
That is, ϕ is mapping $(0, +\infty)$ onto $(0, 1)$ and $(-\infty, 0]$ onto $\{0\}$. Thus ϕ is not a bijection on the whole $\mathbb{R}$ in spite of being a bijection on $(0, +\infty)$ onto (0,1), where $\phi : (0, +\infty) \cup (-\infty, 0] \longrightarrow \{0\} \cup (0, 1)$. We obtained here a good function ϕ that is smooth with smooth inverse in the usual sense.

Definition 2.2.4
Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher and $\tau_{\mathcal{F}_M}$ its topology. The set $\{f^{-1}(0, 1)\}_{f \in \mathcal{F}_M}$ is a subbasis of $\tau_{\mathcal{F}_M}$ and each $f^{-1}(0, 1)$ is a subbasic open set. [*31, 26*]

Lemma 2.2.2

Let M be an $\mathbb{F}$-space and $\mathbb{R}$ endowed with the canonical $\mathbb{F}$-structure. Then:

1. *The function $\phi : (0, +\infty) \to \mathbb{R}, t \mapsto \phi(t) = -t + \frac{1}{t}$ and its inverse ϕ^{-1} are bijective and smooth in the usual sense.*
2. *For any $g \in \mathcal{F}_M$ there exists a unique $f : M \to (0, +\infty)$ such that $f \in \mathcal{F}_M$, $g = \phi f$ and $f = \phi^{-1} g$ that is*



Proof.

1. The function ϕ does a partition of domain and co-domain in the way below:

$$\begin{array}{rcl} \phi : (0,+\infty) = (0,1) \cup [1,+\infty) & \longrightarrow & \mathbb{R} = (0,+\infty) \cup (-\infty, 0] \\ t & \longmapsto & \phi(t) = -t + \frac{1}{t} \\ t \to 0^+ & \longmapsto & \phi(t) \to +\infty \\ 0 < t < 1 & \longmapsto & 0 < \phi(t) < +\infty \\ t = 1 & \longmapsto & \phi(1) = 0 \\ 1 < t < +\infty & \longmapsto & -\infty < \phi(t) < 0 \\ t \to +\infty & \longmapsto & \phi(t) \to -\infty \end{array}$$

ϕ is continuous, and so are all its derivatives in the usual sense. It derivative is $\phi'(t) = -1 - \frac{1}{t^2} < 0$ for any $t \in (0, +\infty)$. Thus, ϕ is monotonic decreasing function. Its graph (curve) goes from $+\infty$ to $-\infty$ for $t \in (0, +\infty)$. All derivatives of ϕ can be found out by the formula below:

$$\phi^{(n)}(t) = \frac{d^n \phi(t)}{dt^n} = \frac{(-1)^n n!}{t^{n+1}}, \quad \text{for } n \geq 2.$$

For $n = 2$, $\frac{d^2\phi(t)}{dt^2} = \frac{2}{t^3} > 0$ for any $t \in (0, +\infty)$. It follows that the curve (graph) of ϕ is concave up. So the graph of ϕ is smooth (continuous without kink). Now, for any $t_1, t_2 \in (0, +\infty)$, assume $t_1 < t_2$ then $\phi(t_1) > \phi(t_2)$, that is $t_1 \neq t_2$ implies $\phi(t_1) \neq \phi(t_2)$. Thus, ϕ is injective. By horizontal parallels, we can provide for any $y \in \mathbb{R}$, a unique $t \in (0, +\infty)$ such that $y = \phi(t)$. Therefore ϕ is a bijection, which applies

$$(0,1) \ \text{ onto } \ (0,+\infty) \ \text{ and } \ [1,+\infty) \ \text{ onto } \ (-\infty, 0]. \qquad (10)$$

The derivative of $\phi^{-1} : \mathbb{R} \longrightarrow (0, +\infty)$ is defined such that $t = \phi^{-1}(y)$ if, and only if $y = \phi(t)$, that is $\phi^{-1} : y \longmapsto t$. Hence, the derivative is given by

$\frac{dt}{dy} = (\frac{dy}{dt})^{-1}$ if, and only if $\frac{d\phi^{-1}(y)}{dy} = (-1-\frac{1}{t^2})^{-1} = (\frac{-t^2-1}{t^2})^{-1} = -\frac{t^2}{1+t^2}$, which is continuous on $(0,+\infty)$. It is clear that, by expressing ϕ^{-1} in terms of y, we will get $\phi^{-1}(y) = \frac{1}{2}(-y+\sqrt{y^2+4})$ whose the derivative with respect to y is $\frac{d\phi^{-1}(y)}{dy} = \frac{1}{2}(-1+\frac{y}{\sqrt{y^2+4}})$. If we substitute y by $\frac{-t^2+1}{t}$ in these two expressions we get the result computed before. Now, $\dfrac{d^2\phi^{-1}(y)}{dy^2} = \dfrac{2}{\sqrt{y^2+4}^3}$,
$\dfrac{d^3\phi^{-1}(y)}{dy^3} = \dfrac{-6y}{\sqrt{y^2+4}^5}$, $\dfrac{d^4\phi^{-1}(y)}{dy^4} = \dfrac{24y^2-24}{\sqrt{y^2+4}^7}$, $\dfrac{d^5\phi^{-1}(y)}{dy^5} = \dfrac{-120y^3+6}{\sqrt{y^2+4}^9}$,
$\dfrac{d^6\phi^{-1}(y)}{dy^6} = \dfrac{720y^4-48y^2-1416}{\sqrt{y^2+4}^{11}}, \ldots$
Although we are not able to give the general form of $\frac{d^n\phi^{-1}(y)}{dy^n}$ for $n \geq 2$, we can nevertheless draw the following features: the polynomial in the numerator is of degree $n-2$, the coefficient of his term of high degree is $(-1)^n n!$ and the power of the denominator is $2n-1$. So its continuity is straightforward in $\mathbb{R}$. Therefore, ϕ^{-1} is smooth in the usual sense as well.

2. Let $g \in \mathcal{F}_M$, ϕ as defined above. Then, there exits a unique $f : M \longrightarrow (0,+\infty)$

   such that $f \in \mathcal{F}_M, \quad f = \phi^{-1}g \quad$ and $\quad g = \phi f.$ (11)

   Furthermore, (10) and (11) yield $\phi(0,1) = (0,+\infty)$ and $\phi^{-1}(0,+\infty) = (0,1)$. Thus, $g^{-1}(0,+\infty) = g^{-1}(\phi(0,1)) = f^{-1}(\phi^{-1}(\phi(0,1))) = f^{-1}(0,1)$. We have built a double bijection: $\mathcal{F}_M \longrightarrow \mathcal{F}_M$ and $\{g^{-1}(0,+\infty)\}_{g\in\mathcal{F}_M} \longrightarrow \{f^{-1}(0,1)\}_{f\in\mathcal{F}_M}$

   such that $g \longleftrightarrow f = \phi^{-1}g$ and $g^{-1}(0,+\infty) = f^{-1}(0,1)$. □ (12)

Proposition 2.2.1
Let M be an $\mathbb{F}$-space and $\mathbb{R}$ endowed with the canonical $\mathbb{F}$-structure. The family $B = \{g^{-1}(0,+\infty)\}_{g\in\mathcal{F}_M}$ is a basis for $\tau_{\mathcal{F}_M}$ that is each $g^{-1}(0,+\infty)$ is a $\tau_{\mathcal{F}_M}$-basic open set. Furthermore, $f^{-1}(0,1)$ is a $\tau_{\mathcal{F}_M}$-basic open set while $\bigcap_{i=1}^{n} f^{-1}(0,1)$ is a $\tau_{\mathcal{F}_M}$-basic open set.

Proof.
From Equations (10), (11) and (12) it follows that $\{g^{-1}(0,+\infty)\}_{g\in\mathcal{F}_M}$ is a subbasis of $\tau_{\mathcal{F}_M}$ that is it generates $\tau_{\mathcal{F}_M}$. With respect to the definition of a basis for a topology, we need now to prove the closeness of the given family under finite intersection. Without loss of generality, we assume $n = 2$, and let $g_1, g_2 \in \mathcal{F}_M$. Then $\mathcal{U}_1 = g_1^{-1}(0,+\infty)$ and $\mathcal{U}_2 = g_2^{-1}(0,+\infty)$ are arbitrary elements of B. Now, from (11) and (12) there exist smooth maps $f_1, f_2 : M \longrightarrow (0,+\infty)$ such that $g_1^{-1}(0,+\infty) = f_1^{-1}(0,1)$ and $g_2^{-1}(0,+\infty) = f_2^{-1}(0,1)$. Let $x \in \mathcal{U}_1 \cap \mathcal{U}_2$. It follows that $x \in f_1^{-1}(0,1)$ and $x \in f_2^{-1}(0,1)$ that is $f_1(x) \in (0,1)$, $f_2(x) \in (0,1)$. From Remarks 2.1.3: there exists $f_3 = f_1 \cdot f_2 : \mathcal{U}_1 \cap \mathcal{U}_2 \longrightarrow (0,1)$ such that

$0 < f_1(x) < 1$ and $0 < f_2(x) < 1$. Hence $0 < f_1(x)f_2(x) < 1$. Thus $f_3(x) \in (0,1)$ and $f_3 : M \longrightarrow (0,+\infty)$ is smooth. Therefore, there exists $g_3 \in \mathcal{F}_M$ such that $g_3^{-1}(0,+\infty) = f_3^{-1}(0,1)$. It follows that $x \in g_3^{-1}(0,+\infty)$, that is, $\mathcal{U}_1 \cap \mathcal{U}_2 \subset g_3^{-1}(0,+\infty)$. For any $x \in g_3^{-1}(0,+\infty) = f_3^{-1}(0,1)$, $g_3(x) \in (0,+\infty)$. Again, $f_1 \cdot f_2(x) = f_1(x)f_2(x) \in (0,1)$, where $g_1(x) \in (0,+\infty)$ and $g_2(x) \in (0,+\infty)$. In the sequel $x \in g_1^{-1}(0,+\infty) = \mathcal{U}_1$ and $x \in g_2^{-1}(0,+\infty) = \mathcal{U}_2$. Equivalently $x \in \mathcal{U}_1 \cap \mathcal{U}_2$. It yields $g_3^{-1}(0,+\infty) \subset \mathcal{U}_1 \cap \mathcal{U}_2$. We have proved that $g_1^{-1}(0,+\infty) \cap g_2^{-1}(0,+\infty) = g_3^{-1}(0,+\infty)$. From then on, we can extend the process to a general n by induction, and then conclude that B is a basis. That is, each $g^{-1}(0,+\infty)$ is a $\tau_{\mathcal{F}_M}$-basic open set. Now, with respect to Equation (12), it follows that $f^{-1}(0,1)$ is a $\tau_{\mathcal{F}_M}$-basic open set for each $f \in \mathcal{F}_M$. As a basis is closed under finite intersection, it follows: $\bigcap_{i=1}^{n} f_i^{-1}(0,1) = \bigcap_{i=1}^{n} g_i^{-1}(0,+\infty) \Longrightarrow h^{-1}(0,1) = k^{-1}(0,+\infty)$, where $\bigcap_{i=1}^{n} f_i^{-1}(0,1) = h^{-1}(0,1)$, $\bigcap_{i=1}^{n} g_i^{-1}(0,+\infty) = k^{-1}(0,+\infty)$, with $h, k \in \mathcal{F}_M$ satisfying Equation (12). □

In what follows we recall the characterization of open sets in terms of basis and subbasis in $\tau_{\mathcal{F}_M}$. After that we give some examples of Frölicher topologies.

Lemma 2.2.3
Let $B = \{f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$. B is a $\tau_{\mathcal{F}_M}$-basis if, and only if for each $\mathcal{U} \in \tau_{\mathcal{F}_M}$ and each $x \in \mathcal{U}$, there is $V \in B$ such that $x \in V \subset \mathcal{U}$.

Proof.
"$\Rightarrow$" Let $B = \{f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$ be a $\tau_{\mathcal{F}_M}$ -basis. For each $\mathcal{U} \in \tau_{\mathcal{F}_M}$, let $x \in \mathcal{U}$. Then $x \in \bigcup_{f \in \mathcal{F}_M} f^{-1}(0,+\infty)$. Hence there exists an $f \in \mathcal{F}_M$ such that $x \in V = f^{-1}(0,+\infty)$. Since $\mathcal{U} = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0,+\infty)$, then $x \in V \subset \mathcal{U}$ and $V \in B$.
"$\Leftarrow$" This is obvious. □

Corollary 2.2.1
$\mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open set if, and only if for each $x \in \mathcal{U}$, there is $f^{-1}(0,+\infty)$, where $f \in \mathcal{F}_M$, such that $x \in f^{-1}(0,+\infty) \subset \mathcal{U} = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0,+\infty)$.

Example 2.2.1
Let $(\mathbb{R}^n, \mathcal{C}, \mathcal{F})$ be the canonical $\mathbb{F}$-space. The topology $\tau_{\mathcal{C}_{\mathbb{R}^n}}$ coincides with the Euclidean topology, which is in turn equal to the Cartesian topology. $\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M}$ since $\mathbb{R}^n$ is a differentiable manifold with the canonical $\mathbb{F}$-structure. Thus M is a balanced space. [19]

Example 2.2.2
Let $(\mathbb{Q}, \mathcal{C}_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}})$ be as in Example 2.1.7, *that is, where the generating set is the*

set of constant curves then $\mathcal{C}_{\mathbb{Q}}=\{c:\mathbb{R}\to\mathbb{Q}\mid c$ *is constant* $\}$ *and* $\mathcal{F}_{\mathbb{Q}}=\mathbb{R}^{\mathbb{Q}}$. *Let c be any structure curve. It follows that* $c^{-1}(\mathbb{Q})=\mathbb{R}$, $c^{-1}(\emptyset)=\emptyset$ *for* $S=\mathbb{Q}$ *and* $S=\emptyset$ *respectively. Now for any* $S\in\mathcal{P}(\mathbb{Q})$, *such that* $\emptyset\subsetneqq S\subsetneqq\mathbb{Q}$,

$$c^{-1}(S)=\begin{cases}\emptyset & : \text{ for all } t\in\mathbb{R}, c(t)=a\notin S\\ \mathbb{R} & : \text{ for all } t\in\mathbb{R}, c(t)=a\in S\end{cases}$$

since $c(c^{-1}(S))=S\cap c(\mathbb{R})=S\cap\{a\}$, *where* $c(\mathbb{R})=\{a\}$. *This intersection yields the situation below:*

- $c(c^{-1}(S))=\emptyset$ *whenever* $a=c(t)\notin S$ *that is* $c(t)=a\in\mathbb{Q}-S$
- $c(c^{-1}(S))=\{a\}$ *whenever* $a=c(t)\in S$.

It follows that $c^{-1}(a)=c^{-1}(S)=\mathbb{R}$ *whenever* $a\in S$ *or* $c^{-1}(a)=c^{-1}(\mathbb{Q}-S)=\mathbb{R}$ *whenever* $a\in\mathbb{Q}-S$. *Thus* $\mathbb{R}=c^{-1}(\mathbb{Q})-c^{-1}(S)=\mathbb{R}-c^{-1}(S)$. *Since open sets in* $\tau_{\mathcal{C}_{\mathbb{Q}}}$ *are those subsets* $S\in\mathcal{P}(\mathbb{Q})$ *such that* $c^{-1}(S)$ *is an open set in* $\mathbb{R}$ *for an arbitrary* $c\in\mathcal{C}_{\mathbb{Q}}$. *But in this case* $c^{-1}(S)=\emptyset\in\tau_{\mathcal{R}}$. *One concludes that* $\tau_{\mathcal{C}_{\mathbb{Q}}}=\mathcal{P}(\mathbb{Q})$, *that is, a discrete topology. Recalling the inclusion* $\tau_{\mathcal{F}_{\mathbb{Q}}}\subset\tau_{\mathcal{C}_{\mathbb{Q}}}=\mathcal{P}(\mathbb{Q})$, *where* $\mathcal{F}_{\mathbb{Q}}=\mathbb{R}^{\mathbb{Q}}$. *It follows that for each x in* $\mathbb{Q}$, *there exists a unique structure curve* c_x *such that* $c_x(t)=x$ *and* $(f\circ c_x)(t)=f(c_x(t))=f(x)$ *for all* $t\in\mathbb{R}$ *and* $f\in\mathcal{F}_{\mathbb{Q}}$. *Thus* $f\circ c_x$ *is also constant. For any S where* $\emptyset\subsetneqq S\subsetneqq\mathbb{Q}$, *there exists* $f\in\mathcal{F}_{\mathbb{Q}}$ *such that f is constant on S and taking its unique value in* $(0,+\infty)$, *but f applies* $\mathbb{Q}-S$ *into* $(-\infty,0]$. *Thus,* $S=f^{-1}(0,+\infty)\in\tau_{\mathcal{F}_{\mathbb{Q}}}$. *So, each subset of* $\mathbb{Q}$ *is* $\tau_{\mathcal{F}_{\mathbb{Q}}}$*-open set, since* $\emptyset$ *and* $\mathbb{Q}$ *are open sets for any topology on* $\mathbb{Q}$. *Hence* $\tau_{\mathcal{F}_{\mathbb{Q}}}=\mathcal{P}(\mathbb{Q})=\tau_{\mathcal{C}_{\mathbb{Q}}}$ *that is* $\mathbb{Q}$ *is a balanced space. Furthermore,* $\mathbb{Q}$ *is a base space with the given* $\mathbb{F}$*-structure.*

Example 2.2.3
Let $(\mathbb{R},\mathcal{C}_{\mathbb{R}},\mathcal{F}_{\mathbb{R}})$ *be as in Example* 2.1.11, *that is, where the generating set is the set of constant curves and* $\mathcal{F}_{\mathbb{R}}=\mathbb{R}^{\mathbb{R}}$. *By a similar reasoning as in Example* 2.2.2, *we are dealing here with a discrete topology, a balanced space and a base space.*

Example 2.2.4 *[48]*
In $\mathbb{R}^2$ *each open ball can be inscribed in an open regular polygon and conversely. The family B of all open balls forms a basis for the usual topology on* $\mathbb{R}^2$, *say* τ_B . *But* $\tau_B=\tau_{\mathcal{F}_{\mathbb{R}^2}}=\tau_{\mathcal{C}_{\mathbb{R}^2}}$. *Thus, the basis B is equivalent to the basis* $\{f^{-1}(0,+\infty)\mid f\in\mathcal{F}_{\mathbb{R}^2}\}$ *since the topologies they generate are equal. Let P be the family of all open regular polygons of the same kind (that is equilateral triangles, or squares, or pentagons, or hexagons, or ...). Thus P is a subbasis generating a certain topology* τ_P. *Hence, P is a basis for the topology* τ_P. *Now, by Lemma* 2.2.3, *each* $\mathcal{U}\in\tau_B$ *is* $\mathcal{U}$ *is a union of some open balls. It follows that for each* $x\in\mathcal{U}$, *there is* $V\in P$ *such that* $x\in V\subset\mathcal{U}$ *that is x belongs also to an open ball of center x, which contains the polygon V and is one of the factors of the union which yields* $\mathcal{U}$. *Therefore,* $\tau_B\subset\tau_P$. *From the first statement of this example, we can say that* $\tau_P\subset\tau_B$. *Therefore* $\tau_B=\tau_P=\tau_{\mathcal{F}_{\mathbb{R}^2}}=\tau_{\mathcal{C}_{\mathbb{R}^2}}$. *Hence* P,B *and* $\{f^{-1}(0,+\infty)\mid f\in\mathcal{F}_{\mathbb{R}^2}\}$ *are equivalent bases. These concepts can be generalized for any n, to* $\mathbb{R}^n$, *with open balls and open regular polytopes.*

Example 2.2.5 [*48*]
Let S be a family of all open intervals $(a, +\infty)$, $(-\infty, b)$, where a,$b \in \mathbb{R}$ and $a<b$. We are interested in the finite intersections of these intervals. Let n be 2, then $(a, +\infty) \cap (-\infty, b) = (a, b)$. But all open intervals form a basis for the usual topology on $\mathbb{R}$. Thus, each finite intersection of elements of S is a τ-basic open. That is, S is a subbasis for the usual topology on $\mathbb{R}$.

2.3 Smooth maps between $\mathbb{F}$-spaces

Definition 2.3.1
Let $(M, \mathcal{C}_M, \mathcal{F}_M)$, and $(N, \mathcal{C}_N, \mathcal{F}_N)$ be two $\mathbb{F}$-spaces. A map $\varphi : M \to N$ is called a smooth map if $\mathcal{F}_N \circ \varphi \subseteq \mathcal{F}_M$ that is $g \circ \varphi \in \mathcal{F}_M$ whenever $g \in \mathcal{F}_N$. The smooth map φ is also called a map of $\mathbb{F}$-spaces or an $\mathbb{F}$-smooth map.

From generating set of $\mathbb{F}$-structure, the smoothness of φ reads $\mathcal{F}_{oN} \circ \varphi \subseteq \mathcal{F}_M$, that is $g \circ \varphi \in \mathcal{F}_{oM}$ whenever $g \in \mathcal{F}_{oN}$, where $\mathcal{F}_{oN}$ and $\mathcal{F}_{oM}$ generate the $\mathbb{F}$-structures on N and M respectively. Recall that an $\mathbb{F}$-structure involves smooth functions as well as smooth curves, we can state below the smoothness of φ in terms of smooth curves. We give these characterizations without proofs, for details see [6]. We will denote by $C^\infty(M, N) := \{\varphi : M \to N \mid \varphi \text{ is } \mathbb{F}\text{−smooth}\}$ the set of all smooth maps of $\mathbb{F}$-spaces M and N.

Lemma 2.3.1
The following statements are equivalent:
φ is a smooth map of $\mathbb{F}$-spaces $(M, \mathcal{C}_M, \mathcal{F}_M)$and $(N, \mathcal{C}_N, \mathcal{F}_N)$.
$\varphi \circ \mathcal{C}_M \subset \mathcal{C}_N$ (or $\varphi \circ \mathcal{C}_{oM} \subset \mathcal{C}_{oN}$) that is $\varphi \circ c \in \mathcal{C}_N$ whenever $c \in \mathcal{C}_M$ (or $\varphi \circ c \in \mathcal{C}_{oN}$ whenever $c \in \mathcal{C}_{oM}$, where $\mathcal{C}_{oN}$ and $\mathcal{C}_{oM}$ generate the $\mathbb{F}$-structures on N and M respectively).
$\mathcal{F}_N \circ \varphi \circ \mathcal{C}_M \subseteq C^\infty(\mathbb{R}, \mathbb{R})$ that is $f \circ \varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each $f \in \mathcal{F}_N$ and $c \in \mathcal{C}_M$.

Corollary 2.3.1
Let M, N be two $\mathbb{F}$-spaces. Let $\varphi \in C^\infty(M, N)$ and $\theta \in C^\infty(N, P)$. Then $\theta \circ \varphi \in C^\infty(M, P)$.

Proof.
φ and θ are smooth by assumption. That is, for every $c \in \mathcal{C}_M$, $\varphi \circ c \in \mathcal{C}_N$ and for every $h \in \mathcal{F}_P$, $h \circ \theta \in \mathcal{F}_N$. Thus for every $h \in \mathcal{F}_P$ and every $c \in \mathcal{C}_M$, $(h \circ \theta) \circ (\varphi \circ c) \in \mathcal{F}_N \circ \mathcal{C}_N$, that is $h \circ (\theta \circ \varphi) \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Therefore $\theta \circ \varphi$ is $\mathbb{F}$-smooth by Lemma 2.3.1. □

Corollary 2.3.2
Let M, N, P be $\mathbb{F}$-spaces. Assume $\varphi : M \to N$ is a set map and $\theta : N \to P$ an $\mathbb{F}$-smooth map. Then φ is an $\mathbb{F}$-smooth map if, and only if $\theta \circ \varphi$ is an $\mathbb{F}$-smooth map.

Proof.
"$\Longrightarrow$" Assume φ $\mathbb{F}$-smooth map. But θ is $\mathbb{F}$-smooth by assumption. Thus, by Corollary 2.3.1, $\theta \circ \varphi$ is $\mathbb{F}$-smooth as a composition of $\mathbb{F}$-smooth maps.
"$\Longleftarrow$" Assume that φ is not $\mathbb{F}$-smooth, but θ and $\theta \circ \varphi$ are $\mathbb{F}$-smooth maps. It follows from Lemma 2.3.1 that there exists $c' \in \mathcal{C}_M$ such that $\varphi \circ c' \notin \mathcal{C}_N$ and from Definition 2.3.1 for every $h \in \mathcal{F}_P$, that $h \circ \theta \in \mathcal{F}_N$. These yield $(h \circ \theta) \circ (\varphi \circ c') \notin C^\infty(\mathbb{R}, \mathbb{R})$. But, for every $c \in \mathcal{C}_M$ and $h \in \mathcal{F}_P$, $(h \circ \theta) \circ (\varphi \circ c) = h \circ (\theta \circ \varphi) \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ since $\theta \circ \varphi$ is $\mathbb{F}$-smooth. That is a contradiction with the particular $c' \in \mathcal{C}_M$. Thus φ is $\mathbb{F}$-smooth. □

Lemma 2.3.2
Let M, N be $\mathbb{F}$-spaces and $\varphi : M \to N$ a set map. The following properties are equivalent.

1. *φ is a smooth map of $\mathbb{F}$-spaces*
2. *the inverse image by φ of each closed set in N is a closed set in M.*
3. *If $g \in \mathcal{F}_N$ and $g^{-1}(0,1)$ is $\tau_{\mathcal{F}_N}$-subbasic ($\tau_{\mathcal{F}_N}$-basic) open set in N then $\varphi^{-1}(g^{-1}(0,1))$ is $\tau_{\mathcal{F}_M}$-subbasic ($\tau_{\mathcal{F}_M}$-basic) open set in M.*
4. *For each $p \in M$ and each open neighborhood $W_{\varphi(p)}$ of $\varphi(p)$ in N, there exists an open neighborhood V_p of p in M such that $\varphi(V_p) \subset W_{\varphi(p)}$.*
5. *The inverse image by φ of each $\tau_{\mathcal{F}_N}$-open set in N is a $\tau_{\mathcal{F}_M}$-open set in M.*

Proof.
(1) $\Longrightarrow$ (3) Assume φ $\mathbb{F}$-smooth, that is, for every $g \in \mathcal{F}_N$, $g \circ \varphi \in \mathcal{F}_M$. It follows that there exists $f \in \mathcal{F}_M$ such that $g \circ \varphi = f$. Thus $\varphi^{-1}(g^{-1}(0,1)) = (g \circ \varphi)^{-1}(0,1) = f^{-1}(0,1)$. We know that $g^{-1}(0,1)$ and $f^{-1}(0,1)$ are subbasic open sets. Hence the inverse image of each subbasic open set by an $\mathbb{F}$-smooth map is a subbasic open set. The basic open set case can be proved in a similar way.
(3) $\Longrightarrow$ (4) Assume (3) true. And let p be any element in M and $W_{\varphi(p)}$ any neighborhood of $\varphi(p)$ in N. For some functions g_ij running through $\mathcal{F}_N$, $g_{ij}^{-1}(0,1)$ are subbasic open sets so that $W_{\varphi(p)} = \bigcup_{i\in I}[\bigcap_{j=1}^{n}[g_{ij}^{-1}(0,1)]]$. Now $\varphi^{-1}(W_{\varphi(p)}) = \bigcup_{i\in I}[\bigcap_{j=1}^{n}[\varphi^{-1}g_{ij}^{-1}(0,1)]]$. From (3) and Proposition 2.2.1, $\varphi^{-1}(W_{\varphi(p)}) = V_p$ is a union of basic open sets containing p and then it is an open neighborhood of p in M. The existence of V_p is proved. In order to complete the proof we apply φ to both sides and this yields $\varphi(V_p) = \varphi\varphi^{-1}(W_{\varphi(p)}) = W_{\varphi(p)} \cap \varphi(M)$. As required $\varphi(V_p) \subset W_{\varphi(p)}$.
(4) $\Longrightarrow$ (2) Assume (4) true and let $W_{\varphi(p)} = \bigcup_{i\in I}[\bigcap_{j=1}^{n}[g_{ij}^{-1}(0,1)]]$ for some $g_{ij} \in \mathcal{F}_N$.

It follows that $N - W_{\varphi(p)} = N - \bigcup_{i\in I}[\bigcap_{j=1}^{n}[g_{ij}^{-1}(0,1)]]$ is $\tau_{\mathcal{F}_N}$-closed set in N. Therefore, $\varphi^{-1}(N - W_{\varphi(p)}) = \varphi^{-1}(N) - \varphi^{-1}(W_{\varphi(p)}) = M - \bigcup_{i\in I}[\bigcap_{j=1}^{n}[\varphi^{-1}g_{ij}^{-1}(0,1)]] = M - V_p$ is closed set in M for $\tau_{\mathcal{F}_M}$. Hence the inverse image of each $\tau_{\mathcal{F}_N}$-closed set in N, by φ, is a $\tau_{\mathcal{F}_M}$-closed set in M.

(2) $\Longrightarrow$ (5) Assume (2) true, that is, $\varphi^{-1}(F)$ is a $\tau_{\mathcal{F}_M}$-closed set in M whenever F is a $\tau_{\mathcal{F}_N}$-closed set in N. It follows that $\mathcal{U} = N - F$ is an open set in N. Thus, $\varphi^{-1}(\mathcal{U}) = \varphi^{-1}(N) - \varphi^{-1}(F) = M - \varphi^{-1}(F)$ is an open set in M as the complement of a closed set $\varphi^{-1}(F)$.

(5) $\Longrightarrow$ (1) Assume (5) true that is $\varphi^{-1}(g^{-1}(0,+\infty))$ is an open set in M for some $g \in \mathcal{F}_N$. Then, there is an basic open set $h^{-1}(0,+\infty)$, for some $h \in \mathcal{F}_M$ such that $\varphi^{-1}(g^{-1}(0,+\infty)) \supset h^{-1}(0,+\infty)$. But a basis $\{f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$ is closed under finite intersection, so there exists some $k_{i_0} \in \mathcal{F}_M$ such that $h^{-1}(0,+\infty) \subset k_{i_0}^{-1}(0,+\infty)$ and $\varphi^{-1}(g^{-1}(0,+\infty)) = \bigcap_{i=1}^{n} k_i^{-1}(0,+\infty)$. Therefore there exists $f \in \mathcal{F}_M$ such that $\varphi^{-1}(g^{-1}(0,+\infty)) = f^{-1}(0,+\infty)$, that is $g \circ \varphi = f$. Equivalently $g \circ \varphi \in \mathcal{F}_M$, that is, φ is smooth. □

The inverse image of a subbasic (basic) open set is not necessarily a subbasic (basic) open set in the general topology. We need to know the topological nature of the set formed by $\varphi^{-1}(g^{-1}(0,+\infty))$ for all $g \in \mathcal{F}_N$.

Corollary 2.3.3
Let $\varphi : M \to N$ be a smooth map of $\mathbb{F}$-spaces. Then

1. *The family $\{\varphi^{-1}(g^{-1}(0,+\infty)) \mid g \in \mathcal{F}_N\}$ is a basis for the topology $\tau_{\mathcal{F}_N \circ \varphi} \subset \tau_{\mathcal{F}_M}$.*
2. *If φ is an $\mathbb{F}$-diffeomorphism, then*
   - *φ induces an isomorphism of rings $\mathcal{F}_N \to \mathcal{F}_M$ such that $g \mapsto g \circ \varphi = \varphi^*(g)$.*
   - *$\{\varphi^{-1}(g^{-1}(0,+\infty)) \mid g \in \mathcal{F}_N\}$ and $\{f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$ are in bijective correspondence.*
   - *$\tau_{\mathcal{F}_N \circ \varphi} = \tau_{\mathcal{F}_M}$*

Proof.

1. Any family of subsets of M generates a unique, smallest topology containing it. So the family $\{\varphi^{-1}(g^{-1}(0,+\infty)) \mid g \in \mathcal{F}_N\}$ is a subbasis for the topology denoted by $\tau_{\mathcal{F}_N \circ \varphi}$ containing it. Since $\{g^{-1}(0,+\infty) \mid g \in \mathcal{F}_N\}$ forms an open covering of N then a covering of M is given by $\{\varphi^{-1}(g^{-1}(0,+\infty)) \mid g \in \mathcal{F}_N\}$. We need to show that this covering is closed under finite intersection.

In order to avoid a cumbersome manipulation of indices, without loss of generality, we restrict ourself to $n = 2$. So for any $g_1, g_2 \in \mathcal{F}_N$, there exists $g_3 \in \mathcal{F}_N$ such that $\varphi^{-1}(g_1^{-1}(0,+\infty)) \cap \varphi^{-1}(g_2^{-1}(0,+\infty)) = \varphi^{-1}(g_1^{-1}(0,+\infty) \cap g_2^{-1}(0,+\infty)) = \varphi^{-1}(g_3^{-1}(0,+\infty))$ since $\{g^{-1}(0,+\infty) \mid g \in \mathcal{F}_N\}$ is closed under finite intersection. The general case on n follows the foregoing by induction process. Hence $\{\varphi^{-1}(g^{-1}(0,+\infty)) \mid g \in \mathcal{F}_N\}$ is a basis for the topology $\tau_{\mathcal{F}_N \circ \varphi}$ on M. The smoothness of φ implies that $g \circ \varphi \in \mathcal{F}_M$ and the forward consequence is that $(g \circ \varphi)^{-1}(0,+\infty) = \cup f^{-1}(0,+\infty)$ for some f running through $\mathcal{F}_M$. Thus $(g \circ \varphi)^{-1}(0,+\infty) \supset f^{-1}(0,+\infty)$, where f is a one of the mentioned $f \in \mathcal{F}_M$ above. From the equivalence of bases, we obtain $\tau_{\mathcal{F}_N \circ \varphi} \subset \tau_{\mathcal{F}_M}$.

2. Since $g \circ \varphi = f$ and φ is an $\mathbb{F}$-diffeomorphism, we have $g = f \circ \varphi^{-1}$. Thus,
   - $g \mapsto \varphi^*(g) = g \circ \varphi$ and $f \mapsto (\varphi^{-1})^*(f) = f \circ \varphi^{-1}$ are inverse of each other. Furthermore $\varphi^*(g+h) = \varphi^*(g) + \varphi^*(h)$ and $\varphi^*(gh) = \varphi^*(g)\varphi^*(h)$ show that φ^* is an isomorphism of rings, that is $\mathcal{F}_N \simeq \mathcal{F}_M$.
   - To each g corresponds a unique f and conversely. Hence, there is a bijective correspondence between $\{\varphi^{-1}(g^{-1}(0,+\infty)) \mid g \in \mathcal{F}_N\}$ and $\{f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$.
   - It is easy to conclude that $\tau_{\mathcal{F}_N \circ \varphi} = \tau_{\mathcal{F}_M}$. □

Corollary 2.3.4
Let $\tau_{\mathbb{R}}$ be the canonical topology on $\mathbb{R}$. If φ is a smooth map of $\mathbb{F}$-spaces then φ is continuous in both $\tau_{\mathcal{C}_M}$ and $\tau_{\mathcal{F}_M}$.

Proof.
Recall that the topologies above are given by $\tau_{\mathcal{C}_M} = \{\mathcal{U} \subset M \mid c^{-1}(\mathcal{U}) \in \tau_{\mathbb{R}}$, for all $c \in \mathcal{C}_M\}$, and $\tau_{\mathcal{C}_N} = \{\mathcal{V} \subset N \mid d^{-1}(\mathcal{V}) \in \tau_{\mathbb{R}}$, for all $d \in \mathcal{C}_N\}$. Note that $\tau_{\mathbb{R}}$ is the standard topology of the real line. Assume that φ is an $\mathcal{F}_M$-smooth map, that is, $\varphi \circ c = d \in \mathcal{C}_N$ for all $c \in \mathcal{C}_M$. Now, let $\mathcal{V} \subset N$ be such that $\mathcal{V} \in \tau_{\mathcal{C}_N}$. We have $d^{-1}(\mathcal{V}) = (\varphi \circ c)(\mathcal{V}) = c^{-1}(\varphi^{-1}(\mathcal{V}))$ is an open set in $\mathbb{R}$. Hence $\varphi^{-1}(\mathcal{V})$ is a $\tau_{\mathcal{C}_M}$-open set in M. Thus φ is a $\tau_{\mathcal{C}_M}$-$\tau_{\mathcal{C}_N}$-continuous map. Also, assume $\mathcal{U} \subset N$ be such that $\mathcal{U} \in \tau_{\mathcal{F}_N}$. It follows that $\mathcal{U}$ is an arbitrary union of a finite intersection of subbasic open sets in $\tau_{\mathcal{F}_N}$. That is, $\mathcal{U} = \bigcup_{i \in I}[\bigcap_{j=1}^{n} g_{ij}^{-1}(0,1)]$, where $g_{ij} \in \mathcal{F}_N$. Now we need to show that $\varphi^{-1}(\mathcal{U})$ is a $\tau_{\mathcal{F}_M}$-open set in M since we have $\varphi^{-1}(\mathcal{U}) = \varphi^{-1}(\bigcup_{i \in I}\bigcap_{j=1}^{n} g_{ij}^{-1}(0,1)) = \bigcup_{i \in I}\bigcap_{j=1}^{n} \varphi^{-1}(g_{ij}^{-1})(0,1) = \bigcup_{i \in I}\bigcap_{j=1}^{n}(g_{ij} \circ \varphi)^{-1}(0,1) = \bigcup_{i \in I}\bigcap_{j=1}^{n} f_{ij}^{-1}(0,1)$, where $f_{ij} \in \mathcal{F}_M$. Hence $\varphi^{-1}(\mathcal{U})$ is a $\tau_{\mathcal{F}_M}$-open set in M. Thus φ is a $\tau_{\mathcal{F}_M}$-$\tau_{\mathcal{F}_N}$-continuous map. □

This corollary was proved in [17] using functions of compact support. Here we use the natural setting of $\mathbb{F}$-spaces. Note that the continuity of φ does not imply its smoothness. For, let φ be continuous, that is, $\varphi^{-1}(\mathcal{U})$ is a $\tau_{\mathcal{F}_M}$-open set in M with $\mathcal{U}$ a $\tau_{\mathcal{F}_N}$-open set in N. For some functions $g \in \mathcal{F}_N$, $\mathcal{U} = \bigcup_{g \in \mathcal{F}_N} g^{-1}(0, +\infty)$, thus $\varphi^{-1}(\mathcal{U}) = \bigcup_{g \in \mathcal{F}_N} \varphi^{-1}(g^{-1}(0, +\infty)) = \bigcup_{g \in \mathcal{F}_N} (g \circ \varphi)^{-1}(0, +\infty)$. Hence the existence of $f = g \circ \varphi \in \mathcal{F}_M$ is not granted.

Lemma 2.3.3
Let $\varphi : M \to N$ be a set map of $\mathbb{F}$-spaces. If $(\mathcal{U}_i)_{i \in I}$ is a $\tau_{\mathcal{C}_M}$-open covering of M such that for any i, the restriction of φ to $\mathcal{U}_i$ is smooth then φ is smooth.

Proof.
See in [17]. □

Definition 2.3.2
Let $\varphi : M \to N$ be a smooth map of $\mathbb{F}$-spaces. φ is open map if $\varphi(\mathcal{U})$ is a $\tau_{\mathcal{F}_N}$-open set in N whenever $\mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open set in M. φ is closed map if $\varphi(\mathcal{F})$ is a $\tau_{\mathcal{F}_N}$-closed set in N whenever $\mathcal{F}$ is a $\tau_{\mathcal{F}_M}$-closed in M.

Lemma 2.3.4
Let $\varphi : M \to N$ be a set map of $\mathbb{F}$-spaces. Then the following are equivalent:

1. *φ is an open map.*
2. *For any $f \in \mathcal{F}_M$ φ sends each $f^{-1}(0, +\infty)$ to a $\tau_{\mathcal{F}_N}$-open set in N.*
3. *For any $S \subset M$ $\varphi(Int(S)) \subseteq Int(\varphi(S))$.*
4. *If $p \in M$ and $\mathcal{U}_p \subset M$ is an open neighborhood in M at p, then there exists $W_{\varphi(p)} \subseteq \varphi(\mathcal{U}_p)$ such that $W_{\varphi(p)}$ is an open neighborhood in N at $\varphi(p)$.*

Proof.
(1) $\Longrightarrow$ (2) Let $f^{-1}(0, +\infty)$ be a $\tau_{\mathcal{F}_M}$-basic open set, then $\varphi(f^{-1}(0, +\infty))$ is an open set in N since $f^{-1}(0, +\infty)$ is a $\tau_{\mathcal{F}_M}$-open set in M and φ is an open map by assumption.
(2) $\Longrightarrow$ (3) Let $S \subset M$ be a subset of M. It follows that $Int(S) = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0, +\infty)$ since $Int(S)$ is $\tau_{\mathcal{F}_M}$-open set in M. Hence, $\varphi(Int(S)) = \bigcup_{f \in \mathcal{F}_M} \varphi(f^{-1}(0, +\infty)) \subseteq \varphi(S)$ by applying φ to both sides of the latter equality yields. From (2) above, $\varphi(Int(S))$ is $\tau_{\mathcal{F}_N}$-open set in N. But $Int(\varphi(S))$ is the largest open set in $\varphi(S)$. Hence $\varphi(Int(S)) \subseteq Int(\varphi(S))$.
(3) $\Longrightarrow$ (4) Let $p \in M$ and $\mathcal{U}_p \subset M$ be an open neighborhood in M at p. Assume $\varphi(Int(\mathcal{U}_p)) \subset Int(\varphi(\mathcal{U}_p))$ from (3). Furthermore $Int(\mathcal{U}_p) = \mathcal{U}_p$ since $\mathcal{U}_p$ is open set.

It follows that $\varphi(\mathcal{U}_p) \subseteq Int(\varphi(\mathcal{U}_p))$ in the one hand. But $Int(\varphi(\mathcal{U}_p)) \subseteq \varphi(\mathcal{U}_p)$ in the other hand, thus $\varphi(\mathcal{U}_p) = Int(\varphi(\mathcal{U}_p))$, by the definition of an interior. Therefore $W_{\varphi(p)} := Int\varphi(\mathcal{U}_p) \subseteq \varphi(\mathcal{U}_p)$
(4) $\Longrightarrow$ (1) $\varphi(\mathcal{U}_p) = Int\varphi(\mathcal{U}_p)$ implies that $\varphi(\mathcal{U}_p)$ is an open set. And so φ is an open map. □

Example 2.3.1
Let $S \subset \mathbb{R} \times \mathbb{R}$ defined by $S = \{(x, y) \,|\, xy = 1\} = \{(x, y) \,|\, y = \frac{1}{x}$ and $x \neq 0\}$ and $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the first canonical projection. That is, $\pi_1(x, y) = x$ with $x \neq 0$. The subset $\{0\}$ is a closed set in the canonical topology on $\mathbb{R}$. Therefore $\pi_1(S) = \{x \,|\, x \neq 0\} = \mathbb{R} - \{0\}$ is an open set. Hence, the map $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\varphi(x, y) = xy$ is $\mathbb{F}$-smooth with respect to the canonical structures on $\mathbb{R}^2$ and $\mathbb{R}$. For, let $c \in \mathcal{C}_{\mathbb{R}^2}$ then for all $t \in \mathbb{R}$, $c(t) = (c_1(t), c_2(t))$, where c_i are smooth functions on $\mathbb{R}$ and we have $(\varphi \circ c)(t) = \varphi(c_1(t), c_2(t)) = c_1(t) \cdot c_2(t) = (c_1 \cdot c_2)(t)$. Thus $\varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Hence, $\varphi^{-1}\{1\} = \{(x, y) \in \mathbb{R}^2 \,|\, \varphi(x, y) = 1\} = \{(x, y) \in \mathbb{R}^2 \,|\, xy = 1\} = S$. It follows that S is closed set, since $\{1\}$ is closed set in $\mathbb{R}$. One can observes that the image of a closed set S by π_1 is an open set $\mathbb{R} - \{0\}$, that is, π_1 is not a closed map.

Corollary 2.3.5
If $f : M \to \mathbb{R}$ is a structure function on an $\mathbb{F}$-space M, then $f^{-1}(0)$ is a $\tau_{\mathcal{F}_M}$-closed set in M if and only if $f^{-1}(\{t \,|\, t \neq 0\})$ is a $\tau_{\mathcal{F}_M}$-open set in M.

Proof.
Each structure function is continuous in $\tau_{\mathcal{F}_M}$. This ends the proof. Because that is a property satisfied by each real-valued continuous function on a topological space. So $f \in \mathcal{F}_M$ does. □

A map can be closed or open without being smooth or continuous.

2.4 Category of $\mathbb{F}$-spaces

The Sections 2.1, 2.2, and 2.3 provided the material for the construction of the category $\mathcal{FRL}$ of $\mathbb{F}$-spaces, where the Objects are $\mathbb{F}$ -spaces and the Morphisms are $\mathbb{F}$-smooth maps. Its study was initiated by Alfred Frölicher [30] and pursued by [31, 44] during the passed decades. It was enriched by the contribution of others researchers, among them we can name [18, 19, 20, 21, 8, 22, 17, 62] . This category is a full subcategory ([18, 19, 22]) of the category of differential spaces in the sense of Sikorsky. The main properties brought out in the literature are the following. The category is known to be Cartesian closed ([30, 31, 44, 19, 22]) in the sense that $C^\infty(M, N)$ can be endowed in a canonical way with an $\mathbb{F}$-structure. Another important feature is that the category $\mathcal{FRL}$ is topological over set ([31, 18, 19, 20]). That is, the forgetful functor $U : \mathcal{FRL} \longrightarrow \mathcal{SETS}$ is faithful and topological. Equivalently, this means that $\mathcal{FRL}$ behaves as the category of topological spaces. That is, objects are constructed in $\mathcal{SETS}$ and then are endowed with an $\mathbb{F}$-structure. Therefore, $\mathcal{FRL}$ has all limits and colimits lifted from the category $\mathcal{SETS}$ of sets and set maps. A category having limits and colimits is called complete and cocomplete ([31, 44, 18, 6, 62]). The straightforward consequence is the existence of initial objects (with initial $\mathbb{F}$-structure: product, subspace, ...) and also the existence of final objects (with final $\mathbb{F}$-structure: co-product, quotient,...) in $\mathcal{FRL}$ ([31, 18, 19, 8, 22, 6]). In [19] it is noticed the existence of coequalizer and in [20] the existence of the equalizer. We will restrict ourself in this dissertation to the study of subobject, product, coproduct and quotient in $\mathcal{FRL}$ because our work lies on these concepts. We are going to recall briefly their constructions. For more about the construction of final and initial objects in $\mathcal{FRL}$ we refer the reader to the detailed references above. Our contribution concerns a topological study of these objects in Sections 2.5 through 2.8.

2.5 $\mathbb{F}$-Subspaces

In what follows an $\mathbb{F}$-space is an initial object obtained by lifting process of the subobject S of an object M in $\mathcal{SETS}$ to the category $\mathcal{FRL}$. Let $F_{o\,M}$ be the set that generates the $\mathbb{F}$-structure $(\mathcal{C}_M, \mathcal{F}_M)$ on M. Let $S \subset M$. The $\mathbb{F}$-structure on Sgenerated by $\iota_S : S \hookrightarrow M$, the canonical inclusion. For this purpose, let $F_{o\,S} = \{f_{|S} \mid f_{|S} = f \circ \iota_S,\ f \in F_{o\,M}\} = F_{o\,M} \circ \iota_S = \iota_S^* F_{o\,M} = F_{o\,M|S}$ be a set generating the $\mathbb{F}$-structure on S. The structure curves set is given below with respect to the compatibility condition. $\mathcal{C}_S = \Gamma F_o = \{c' : \mathbb{R} \to S \mid f_{|S} \circ c' \in C^\infty(\mathbb{R})$ for all $f_{|S} \in F_{o\,S}\}$ or eqivalently $\mathcal{C}_S = \{c' : \mathbb{R} \to S \mid f \circ (\iota_S \circ c') \in C^\infty(\mathbb{R})$ for all $f \in F_{o\,M}\} = \{c' : \mathbb{R} \to S \mid \iota_S \circ c' \in C_M\}$. Also the following holds: $\mathcal{C}_S = \{c' : \mathbb{R} \to S \mid c'(\mathbb{R}) \subset S\}$. The compatibility condition yields the structure functions set as follows. $\mathcal{F}_S = \Phi \mathcal{C}_S = \{f' : S \to \mathbb{R} \mid f' \circ c' \in C^\infty(\mathbb{R})$ for all $c' \in \mathcal{C}_S\}$ or equivalently $\mathcal{F}_S = \{f' : S \to \mathbb{R} \mid f' \circ c' \in C^\infty(\mathbb{R}),\ c'(\mathbb{R}) \subset S\}$. Also the following holds: $\mathcal{F}_S = \{f' : S \to \mathbb{R} \mid f'(c'(\mathbb{R})) \subset f'(S)\}$. It follows that $\mathcal{F}_M \circ \iota_S \subset \mathcal{F}_S$. Therefore, $\mathcal{F}_S$ is not the restriction of $\mathcal{F}_M$ on S. That is the case when S is open or closed set. Hence ι_S is smooth if, and only if $\iota_S \circ \mathcal{C}_S \subset \mathcal{C}_M$ if, and only if $\mathcal{F}_M \circ \iota_S \subset \mathcal{F}_S$.

Definition 2.5.1
The $\mathbb{F}$-space $(S, \Gamma F_{o\,S}, \Phi\Gamma F_{o\,S}) = (S, \mathcal{C}_S, \mathcal{F}_S)$ is the $\mathbb{F}$-subspace of $(M, \mathcal{C}_M, \mathcal{F}_M)$. Also the pair $(\mathcal{C}_S, \mathcal{F}_S)$ is called the initial $\mathbb{F}$-structure on S induced by $(\mathcal{C}_M, \mathcal{F}_M)$, making ι_S a smooth map.

Every subset S of an $\mathbb{F}$-space M is canonically an $\mathbb{F}$-subspace with respect to the construction of the $\mathbb{F}$-structure on S done above.

Definition 2.5.2
The topologies $\tau_{\mathcal{F}_S}$ and $\tau_{\mathcal{C}_S}$ induced on S respectively by smooth functions and

curves are called $\mathbb{F}$-topologies on S and S is an $\mathbb{F}$-topological subspace of M or $\mathbb{F}$-subspace for short. That is the topologies, where all smooth functions and smooth curves are continuous.

Definition 2.5.3
Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an $\mathbb{F}$-space and $S \subseteq M$. The collections $\{g^{-1}(0,1) \mid g \in \mathcal{F}_S\}$ and $\{g^{-1}(0,+\infty) \mid g \in \mathcal{F}_S\}$ are subbasis and basis for $\tau_{\mathcal{F}_S}$, respectively.

The family $\{g^{-1}(0,1) \mid g \in \mathcal{F}_S\}$ is basis for $\tau_{\mathcal{F}_S}$ with respect the definition of a subbasis. It is easy to show that $\tau_{\mathcal{F}_S} \subset \tau_{\mathcal{C}_S}$ holds on $\mathbb{F}$-subspaces. In effect, let $g \in \mathcal{F}_S$, $I \in \tau_{\mathcal{F}_\mathbb{R}}$, we have $g^{-1}(I) \in \tau_{\mathcal{F}_S}$. Also, if $\mathcal{U} \in \tau_{\mathcal{C}_S}$ and $d \in \mathcal{C}_S$, then $d^{-1}(\mathcal{U}) \in \tau_{\mathcal{C}_\mathbb{R}} = \tau_{\mathcal{F}_\mathbb{R}}$. Now, $d^{-1}(g^{-1}(I)) = (g \circ d)^{-1}(I) \in \tau_{\mathcal{C}_\mathbb{R}} = \tau_{\mathcal{F}_\mathbb{R}}$ since $(g \circ d) \in \mathcal{F}_\mathbb{R} = \mathcal{C}_\mathbb{R}$. Therefore, $g^{-1}(I) \in \tau_{\mathcal{C}_S}$, $\tau_{\mathcal{F}_S} \subset \tau_{\mathcal{C}_S}$ as required. We want to show S carries another topology as an $\mathbb{F}$-subspace of M, that is, the trace topology or the relative topology.

Definition 2.5.4
Let $\tau_{\mathcal{F}_M}(S) = \{S \cap \mathcal{U} \mid \mathcal{U} \in \tau_{\mathcal{F}_M}\}$ and $\tau_{\mathcal{C}_M}(S) = \{S \cap \mathcal{U} \mid \mathcal{U} \in \tau_{\mathcal{C}_M}\}$ be the topologies defined on S. They are called the trace topologies or relative topologies. That is, the topologies on S for which an open (closed) set in S is the trace on S of any open (closed) set in M for $\tau_{\mathcal{F}_M}$ or $\tau_{\mathcal{C}_M}$ with respect to the context.

In what follows some results are standard and their proofs are in the literature of general topology. However, we decided to redo them, but in Frölicher space setting, and provide in a self-contained manner the concepts and main results of point-set topology in the setting of Frölicher spaces. Hence, it will be easy to introduce new smooth objects as products, coproducts, quotients, pseudomanifolds and subpseudomanifolds. In the next lemma, we will confirm that the usual results known from general topology hold true in the setting of $\mathbb{F}$-spaces. That is, the natural inclusion map is continuous, basis and subbasis are naturally characterized with respect to the trace topology of $\tau_{\mathcal{F}_M}$ on S.

Lemma 2.5.1
Let M be a $\mathbb{F}$-space, $S \subset M$ a subset and $f \in \mathcal{F}_M$. Then $S \cap f^{-1}(0,+\infty)$ is $\tau_{\mathcal{F}_M}(S)$-basic open set in S, $S \cap f^{-1}(0,1)$ is $\tau_{\mathcal{F}_M}(S)$-subbasis open set in S and ι_S is continuous in $\tau_{\mathcal{F}_M}(S)$.

Proof.
For the first assertion, consider the topology $\tau_{\mathcal{F}_M}(S)$ on S. Thus, an open set is of the form $S \cap V$, where V is any $\tau_{\mathcal{F}_M}$-open set in M that is $V = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0,+\infty)$. It follows that $S \cap V = S \cap (\bigcup_{f \in \mathcal{F}_M} f^{-1}(0,+\infty)) = \bigcup_{f \in \mathcal{F}_M} S \cap f^{-1}(0,+\infty)$. It remains to show that the family $\{S \cap f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$ is closed under finite intersections. Let us index $f \in \mathcal{F}_M$ by a finite number of indexes such that

the subset $\{f_i(0,+\infty) \mid 1 \leq i \leq n\} \subset \mathcal{F}_M\}$ is a finite set of $\tau_{\mathcal{F}_M}$-basic open sets. Now, recall that the collection $\{f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$ is closed under finite intersections. Thus, we have $\bigcap_{i=1}^{n} f_i^{-1}(0,+\infty) = g^{-1}(0,+\infty)$ with $g \in \mathcal{F}_M$. Therefore, $S \cap f_i^{-1}(0,+\infty)$ is a member of the family $\{S \cap f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$. We can show that this family is also closed under finite intersections since $\bigcap_{i=1}^{n}(S \cap f_i^{-1}(0,+\infty)) = S \cap (\bigcap_{i=1}^{n} f_i^{-1}(0,+\infty)) = S \cap g^{-1}(0,+\infty)$ is also a member of $\{S \cap f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$. Hence, $\{S \cap f^{-1}(0,+\infty) \mid f \in \mathcal{F}_M\}$ is a basis for $\tau_{\mathcal{F}_M}(S)$. That is, $S \cap f^{-1}(0,+\infty)$ is a $\tau_{\mathcal{F}_M}(S)$-basic open set in S. For the second assertion, let $V \in \tau_{\mathcal{F}_M}$. That is, $V = \bigcup_{j \in J} \bigcap_{i=1}^{n} f_{ij}^{-1}(0,1)$, where $f_{ij} \in \mathcal{F}_M$ and $S \cap V \in \tau_{\mathcal{F}_M}(S)$. It follows that $S \cap V = S \cap \bigcup_{j \in J} \bigcap_{i=1}^{n} f_{ij}^{-1}(0,1) = \bigcup_{j \in J}[\bigcap_{i=1}^{n}(S \cap f_{ij}^{-1}(0,1))]$. Therefore, $\bigcap_{i=1}^{n}(S \cap f_i^{-1}(0,1))$ is a $\tau_{\mathcal{F}_M}(S)$-basic open set. So $S \cap f_i^{-1}(0,1)$ is a $\tau_{\mathcal{F}_M}(S)$-subbasic open set. Below, we will prove the continuity of ι_S with respect to basis and subbasis respectively. First of all, we recall that ι_S should be continuous if, and only if for every $\mathcal{U} \in \tau_{\mathcal{F}_M}$, $\iota_S^{-1}(\mathcal{U}) \in \tau_{\mathcal{F}_M}(S)$ if, and only if the inverse image of each member of a subbasis (basis) of $\tau_{\mathcal{F}_M}$ is a member of a subbasis (basis) in S. Now, since ι_S is injective, thus we have $\iota_S^{-1}(\mathcal{U}) = S \cap \mathcal{U}$, where $\mathcal{U} \in \tau_{\mathcal{F}_M}$. So, $\iota_S^{-1}(\mathcal{U}) \in \tau_{\mathcal{F}_M}(S)$. From the first characterization of the continuity of ι_S above, one can conclude that ι_S is continuous. Using the second characterization above we get $\iota_S^{-1}(f^{-1}(0,+\infty))$ since ι_S is injective, and $\iota_S^{-1}(f^{-1}(0,+\infty))$ are members of $\tau_{\mathcal{F}_M}(S)$. With respect to two first assertions above in this lemma we have $\iota_S^{-1}(f^{-1}(0,+\infty)) = S \cap f^{-1}(0,+\infty)$ and $\iota_S^{-1}(f^{-1}(0,1))$ are $\tau_{\mathcal{F}_M}(S)$-basic open set and $\tau_{\mathcal{F}_M}(S)$-subbasic open sets respectively. Thus ι_S is continuous. □

Lemma 2.5.2 : *Transitivity principle*
Let P and N be $\mathbb{F}$-subspaces of a $\mathbb{F}$-space M such that $P \subset N \subset M$. If P and N are endowed with the trace topologies $\tau_{\mathcal{F}_N}(P))$ inherited from N and $\tau_{\mathcal{F}_M}(N))$ inherited from M, then P is also endowed with the trace topology $\tau_{\mathcal{F}_M}(P)$.

Proof.
Let $W \in \tau_{\mathcal{F}_N}(P)$. Thus $W = P \cap V$, where $V \in \tau_{\mathcal{F}_M}(N)$, that is, $V = N \cap \mathcal{U}$ with $\mathcal{U} \in \tau_{\mathcal{F}_M}$. Therefore $W = P \cap V = P \cap (N \cap \mathcal{U}) = (P \cap N) \cap \mathcal{U} = P \cap \mathcal{U}$, since $P \subset N$. Hence $W \in \tau_{\mathcal{F}_M}(P)$. This is a result of general topology: the subspace of a subspace is a subspace of the entire space. □

We want to characterize open and closed sets in the trace topologie.

Lemma 2.5.3
Let $\tau_{\mathcal{F}_M}(S)$ be the trace topology on an $\mathbb{F}$-subspace S of an $\mathbb{F}$-space M. Let $\mathcal{U} \subset S$

with $\mathcal{U}$ a $\tau_{\mathcal{F}_M}(S)$-open (closed) set. Then S is a $\tau_{\mathcal{F}_M}$-open (closed) set if, and only if $\mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open (closed) set.

Proof.

1. Open case:
   "$\Longrightarrow$" Let $\mathcal{U}$ be a $\tau_{\mathcal{F}_M}(S)$-open set and S a $\tau_{\mathcal{F}_M}$-open set. On the one hand, this implies that $\mathcal{U}=\bigcup_{i\in I}(S\cap f_i^{-1}(0,+\infty))=S\cap(\bigcup_{i\in I}(f_i^{-1}(0,+\infty)))$ for some functions $f_i\in\mathcal{F}_M$, and $S=\bigcup_{j\in J}(g_j^{-1}(0,+\infty)$, $g_j\in\mathcal{F}_M$ on the other hand. Putting all these together yields $\mathcal{U}=[\bigcup_{j\in J}(g_j^{-1}(0,+\infty))]\cap[\bigcup_{i\in I}(f_i^{-1}(0,+\infty))]=\bigcup_{(j,i)\in J\times I}[(g_j^{-1}(0,+\infty))\cap(f_i^{-1}(0,+\infty))]$. It follows that $\mathcal{U}=\bigcup_{k\in K}[h_k^{-1}(0,+\infty)]$ with $K=J\times I$ and $\{f^{-1}(0,+\infty)\,|\,f\in\mathcal{F}_M\}$ is closed under finite intersections. This yields $(h_k^{-1}(0,+\infty))=(g_j^{-1}(0,+\infty))\cap(f_i^{-1}(0,+\infty))$. Hence $\mathcal{U}\in\tau_{\mathcal{F}_M}$. That is, $\mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open set in M.
   "$\Longleftarrow$" Let $\mathcal{U}$ be both a $\tau_{\mathcal{F}_M}(S)$-open set and a $\tau_{\mathcal{F}_M}$-open set. We need to show that S is a $\tau_{\mathcal{F}_M}$-open set in M. We have $\mathcal{U}=\bigcup_{j\in J}(g_j^{-1}(0,+\infty))$ and $\bigcup_{j\in J}(g_j^{-1}(0,+\infty))\subset S$. Thus, there is $g_j\in\mathcal{F}_M$ such that $g_j^{-1}(0,+\infty)\subset S$. That is, S contains a $\tau_{\mathcal{F}_M}$-basic open set. So from Corollary 2.2.1, S is a $\tau_{\mathcal{F}_M}$-open set in M.

2. Closed case:
   "$\Longrightarrow$" We have by assumption: $\mathcal{U}\subset S$ a $\tau_{\mathcal{F}_M}(S)$-closed set and $S\subset M$ a $\tau_{\mathcal{F}_M}$-closed set. This implies $\mathcal{U}=S\cap[M-\bigcup_{i\in I}f_i^{-1}(0,+\infty)]$ and $S=M-\bigcup_{j\in J}g_j^{-1}(0,+\infty)$, with f_i, $g_j\in\mathcal{F}_M$. This yields $\mathcal{U}=[M-\bigcup_{j\in J}g_j^{-1}(0,+\infty)]\cap[M-\bigcup_{i\in I}f_i^{-1}(0,+\infty)]$. Equivalently, with respect to De Morgans Laws we obtain $\mathcal{U}=M-[(\bigcup_{j\in J}g_j^{-1}(0,+\infty))\cup(\bigcup_{i\in I}f_i^{-1}(0,+\infty))]$. Thus $\mathcal{U}$ is obviously a closed set in $\tau_{\mathcal{F}_M}$ as the complement of an open set in M.
   "$\Longleftarrow$" By assumptions the subspace S is a $\tau_{\mathcal{F}_M}(S)$-closed set and $\mathcal{U}$ is both $\tau_{\mathcal{F}_M}(S)$-closed set and $\tau_{\mathcal{F}_M}$-closed set. The deal is to show that S is $\tau_{\mathcal{F}_M}$-closed set. But, as a complement of an open set in $\tau_{\mathcal{F}_M}$, $\mathcal{U}=M-\bigcup_{i\in I}(f_i^{-1}(0,+\infty))\subset S\subset M$ for some $f_i\in\mathcal{F}_M$. Therefore, The ideal situation we can expect is one where $\mathcal{U}=S$ and thus S should be closed in $\tau_{\mathcal{F}_M}$. Assume $\mathcal{U}$ be a $\tau_{\mathcal{F}_M}(S)$-closed set. Then there exists a closed set $V=(M-\bigcup_{i\in I}(f_i^{-1}(0,+\infty)))\in\tau_{\mathcal{F}_M}$ that is, $\mathcal{U}=S\cap(M-\bigcup_{i\in I}(f_i^{-1}(0,+\infty)))$. It follows

that $\mathcal{U}=(S\cap M)-[S\cap\bigcup_{i\in I}(f_i^{-1}(0,+\infty)))]$. Also $\mathcal{U}=S-[S\cap\bigcup_{i\in I}(f_i^{-1}(0,+\infty)))]$. Finally, $\mathcal{U} = S-\bigcup_{i\in I}(f_i^{-1}(0,+\infty)$. Let $\emptyset = \bigcup_{i\in I}(f_i^{-1}(0,+\infty)) \in \tau_{\mathcal{F}_M}$. It follows that$\mathcal{U} = S - \emptyset = S$. Hence S is closed in $\tau_{\mathcal{F}_M}$ since $\mathcal{U}$ is closed by assumptions. □

Corollary 2.5.1 : Chain of open (closed) sets $\mathcal{U} \subset S \subset M$.
If $\mathcal{U}$ is a $\tau_{\mathcal{F}_M}(S)$-open (closed) set and S is a $\tau_{\mathcal{F}_M}$-open (closed) set then $\mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open (closed) set.

Proof.
It is straightforward consequence of Lemma 2.5.3 in (1) open case and (2) closed case for "$\Longrightarrow$". □

Lemma 2.5.4
Let $S \subset M$, M an $\mathbb{F}$-space, and τ any topology on M making the inclusion map $\iota_S : S \hookrightarrow M$ continuous. Then ι_S is an open (closed) map if, and only if S is an open (closed) set in M for the given topology τ.

Proof.
"$\Longrightarrow$" Since ι_S is smooth (so continuous) map from the subobject $(S, \mathcal{C}_S, \mathcal{F}_S)$ to , thus it maps back a member of a basis of $(M, \mathcal{C}_M, \mathcal{F}_M)$ to a member of a basis of $(S, \mathcal{C}_S, \mathcal{F}_S)$. That is,

$$\iota_S^{-1}(f^{-1}(0,+\infty)) = (f \circ \iota_S)^{-1}(0,+\infty) = f_{|S}^{-1}(0,+\infty).$$

But ι_S is an open map by assumption, thus $\iota_S(f_{|S}^{-1}(0,+\infty))$ is an open set in $\tau_{\mathcal{F}_M}$. It follows that

$$\iota_S^{-1}(M) = \iota_S^{-1}(\bigcup_{f\in\mathcal{F}_M)} f^{-1}(0,+\infty)) = \bigcup_{f\in\mathcal{F}_M)} f^{-1}{}_{|S}(0,+\infty) \tag{13}$$

and also

$$\iota_S^{-1}(M) = S \cap M = S \tag{14}$$

It follows from (13) and (14) that $S=\bigcup_{f\in\mathcal{F}} f^{-1}{}_{|S}(0,+\infty)$. Hence, $S \in \tau_{\mathcal{F}_M} \subset \tau_{\mathcal{C}_M}$.
"$\Longleftarrow$" Let S be open set in M for $\tau_{\mathcal{F}_M}$. From Corollary 2.5.1, for any $\tau_{\mathcal{F}_M}(S)$-open set $\mathcal{U}$ in S, $\mathcal{U}$ is also $\tau_{\mathcal{F}_M}$-open set in M, that is, $\mathcal{U}=\bigcup_{t\in T}[S\cap(\bigcup_{i\in I} f_{it}^{-1}(0,+\infty))]$, where $(\bigcup_{i\in I} f_i^{-1}(0,+\infty))$ are $\tau_{\mathcal{F}_M}$-open sets in M. But, $S = \bigcup_{j\in J} h_j^{-1}(0,+\infty)$ for $h_j \in \mathcal{F}_M$ is a $\tau_{\mathcal{F}_M}$-open set. Putting all together yields $\mathcal{U} = \bigcup_{t\in T}[(\bigcup_{j\in J} h_j^{-1}(0,+\infty))\cap$

$(\bigcup_{i\in I} f_{it}^{-1}(0,+\infty))] = \bigcup_{t\in T}[\bigcup_{(j,i)\in J\times I} \big(h_j^{-1}(0,+\infty)\cap f_{it}^{-1}(0,+\infty)\big)]$. It follows that $\mathcal{U} = \bigcup_{t\in T}[\bigcup_{(j,i)\in J\times I} g_{jit}^{-1}(0,+\infty)]$. Thus, $\mathcal{U} = \bigcup_{(j,i,t)\in J\times I\times T} g_{jit}^{-1}(0,+\infty)$. The result above follows from set theoretical properties of $\bigcup$, $\cap$ and the closeness property of the basis under finite intersection. So we have proved that, for an arbitrary $\mathcal{U}$ that is $\tau_{\mathcal{F}_M}(S)$-open set in S, $\mathcal{U}$ is $\tau_{\mathcal{F}_M}$-open as well. But $\mathcal{U} = \iota_S(\mathcal{U})$. Thus ι_S is an open map.

The proof of the closed case holds in the same way. □

We would like to compare $\mathbb{F}$-subspace topologies and the trace topology.

Proposition 2.5.1
Let S be an $\mathbb{F}$-subspace of an $\mathbb{F}$-space M. Then $\tau_{\mathcal{F}_M}(S) \subset \tau_{\mathcal{F}_S} \subset \tau_{\mathcal{C}_S}$ and $\tau_{\mathcal{F}_M}(S)$ is the smallest topology on S for which ι_S is continuous.

Proof.
For the first statement we let $\mathcal{U}\in\tau_{\mathcal{F}_M}(S)$. Thus $\mathcal{U}=S\cap V=\bigcup_{i\in I}(S\cap f_i^{-1}(0,+\infty))$ with $V\in\tau_{\mathcal{F}_M}$. This lies on the definition of the trace topology and with respect to Lemma 2.5.1. Thus, $\mathcal{U} = \bigcup_{f\in\mathcal{F}_M}(\iota_S^{-1}(f^{-1}(0,+\infty))) = \bigcup_{f\in\mathcal{F}_M}(f\circ\iota_S)^{-1}(0,+\infty) = \bigcup_{f\in\mathcal{F}_M}(f^{-1}{}_{|S}(0,+\infty)) \in \tau_{\mathcal{F}_S}$. Since $\mathcal{F}_{M|S}\subset f_{|S}$. Therefore, we have proved that $\tau_{\mathcal{F}_M}(S) \subset \tau_{\mathcal{F}_S}$. But $\tau_{\mathcal{F}_S} \subset \tau_{\mathcal{C}_S}$ holds true for the Frölicher topologies. Thus, $\tau_{\mathcal{F}_M}(S)\subset\tau_{\mathcal{F}_S}\subset\tau_{\mathcal{C}_S}$. For the second statement we let $V\in\tau_{\mathcal{F}_M}(S)$ and τ is any topology on S, where ι_S is continuous. These assumptions yield the following situation in S: $V=S\cap\mathcal{U}$ such that $\mathcal{U}\in\tau_{\mathcal{F}_M}$ by definition of $\tau_{\mathcal{F}_M}(S)$ and $\iota_S^{-1}(\mathcal{U})\in\tau$, since ι_S is continuous for τ. So, by the injectivity of ι_S, $\iota_S^{-1}(\mathcal{U})=S\cap\mathcal{U}=V$. Hence $V\in\tau$ and $\tau_{\mathcal{F}_M}(S)\subset\tau$ that is $\tau_{\mathcal{F}_M}(S)$ is the smallest topology on S for which ι_S is continuous. □

Proposition 2.5.2
Let $(M,\mathcal{C},\mathcal{F})$ be an $\mathbb{F}$-space and $S\subset M$ such that $\iota_S : S\hookrightarrow M$ is injective. If $S\in\tau_{\mathcal{F}_M}$, then $\tau_{\mathcal{F}_S}=\tau_{\mathcal{F}_M}(S)$. Also, if $S\in\tau_{\mathcal{C}_M}$, then $\tau_{\mathcal{C}_S}=\tau_{\mathcal{C}_M}(S)$

Proof.
First of all we assume $\mathcal{U}\in\tau_{\mathcal{F}_S}$ that is $\mathcal{U}=\bigcup_{i\in I}(f_i^{-1}{}_{|S}(0,+\infty))$, where $f_i^{-1}\in\mathcal{F}_M$. Since S is open and then ι_S is an open map with respect to Lemma 2.5.4. It follows that $\tau_{\mathcal{F}_S}\subset\tau_{\mathcal{F}_M}(S)$, as $\mathcal{U} = \iota_S^{-1}\iota_S(\mathcal{U}) = \iota_S^{-1}[\bigcup_{i\in I}\iota_S(f_i^{-1}{}_{|S}(0,+\infty))] = \iota_S^{-1}[\bigcup_{i\in I} f_i^{-1}(0,+\infty)] = \bigcup_{i\in I}[\iota_S^{-1}(f_i^{-1}(0,+\infty))]$. Thus, $\mathcal{U} = \bigcup_{i\in I}[S\cap f_i^{-1}(0,+\infty)] \in \tau_{\mathcal{F}_M}(S)$. The required equality follows as the reverse inclusion $\tau_{\mathcal{F}_M}(S)\subset\tau_{\mathcal{F}_S}$ was proved in Proposition 2.5.1. Secondly, assume $\mathcal{U}\in\tau_{\mathcal{C}_S}$. That is, $\mathcal{U}\in M$ such that

$d^{-1}(\mathcal{U})in\tau_{\mathcal{C}_{\mathbb{R}}}=\tau_{\mathcal{F}_{\mathbb{R}}}$, with $d\in\mathcal{C}_S$. The assumption $S\in\tau_{\mathcal{C}_M}$ yields $c^{-1}(S)\in\tau_{\mathcal{C}_{\mathbb{R}}}=\tau_{\mathcal{F}_{\mathbb{R}}}$, with $c\in\mathcal{C}_M$. Since ι_S is an $\mathbb{F}$-smooth map, we may assume $c=\iota_S\circ d$. Then, using the injectivity and the smoothness of ι_S and $\iota_S(\mathcal{U})=\mathcal{U}$ we have what follows $d^{-1}(\mathcal{U})=d^{-1}(\iota_S^{-1}\circ\iota_S)(\mathcal{U})=(\iota_S\circ d)^{-1}(\iota_S(\mathcal{U}))=c^{-1}(\mathcal{U})\in\tau_{\mathcal{C}_{\mathbb{R}}}=\tau_{\mathcal{F}_{\mathbb{R}}}$. Therefore, $\mathcal{U}\in\tau_{\mathcal{C}_M}$. That is, $\mathcal{U}\subset S\subset M$. Now $\mathcal{U}=\iota_S^{-1}(\iota_S(\mathcal{U}))=S\cap\iota_S(\mathcal{U})=S\cap\mathcal{U}\in\tau_{\mathcal{C}_M}(S)$ since $\mathcal{U}\in\tau_{\mathcal{C}_M}$. Therefore $\tau_{\mathcal{C}_M}(S)\supset\tau_{\mathcal{C}_S}$. The latter inclusion, together with Proposition 2.5.1 yield $\tau_{\mathcal{C}_S}=\tau_{\mathcal{C}_M}(S)$. □

2.6 $\mathbb{F}$-Product

The $\mathbb{F}$-product space in the category $\mathcal{FRL}$ is the initial object obtained by lifting the product in the category $\mathcal{SETS}$ to $\mathcal{FRL}$. Let $M^*:=\prod_{i\in I}M_i$ denotes the product in $\mathcal{SETS}$. The initial structure on M^* in $\mathcal{FRL}$ is the $\mathbb{F}$-structure generated by the family $(p_i:M^*\to M_i)_{i\in I}$ of the canonical projection maps in $\mathcal{SETS}$, with the universality condition given by $p_i\circ c=c_i$, where $c=(c_i)_{i\in I}:\mathbb{R}\to M^*$. The Frölicher structure is generated by a set $\mathcal{F}_o$ of functions, that is, $(\mathcal{C}_{M^*},\mathcal{F}_{M^*})$ such that $\mathcal{F}_o=\bigcup_{i\in I}\{f_i\circ p_i\,|\text{ for all }f_i\in\mathcal{F}_{oM_i}\}$, where $\mathcal{F}_{oM_i}$ generates the $\mathbb{F}$-structure $(\mathcal{C}_{M_i},\mathcal{F}_{M_i})$ on M_i for all i. It follows that the structure curves and functions are given by $\mathcal{C}_{M^*}=\Gamma\mathcal{F}_o=\{c:\mathbb{R}\to M^*\,|\,c=(c_i)_{i\in I}\text{ for all }c_i\in\mathcal{C}_{M_i},\text{ for all }i\in I\}$ and $\mathcal{F}_{M^*}=\Phi\mathcal{C}_{M^*}=\Phi\Gamma\mathcal{F}_o=\{f:M^*\to\mathbb{R}\,|\,f\circ(c_i)_{i\in I}\in C^\infty(\mathbb{R}),\ c_i\in\mathcal{C}_{M_i},\text{ for all }i\in I\}$.

Definition 2.6.1
The $\mathbb{F}$-space $(M^*,\mathcal{C}_{M^*},\mathcal{F}_{M^*})$ is called the $\mathbb{F}$-product of M_i or a product of $\mathbb{F}$-spaces $(M_i,\mathcal{C}_{M_i},\mathcal{F}_{M_i}$. Also, the pair $(\mathcal{C}_{M^*},\mathcal{F}_{M^*})$ is the initial $\mathbb{F}$-product structure (product structure for short) such that all p_i are smooth maps.

Definition 2.6.2
The topologies $\tau_{\mathcal{F}_{M^*}}$ and $\tau_{\mathcal{C}_{M^*}}$ induced by smooth functions and curves on M^* are called $\mathbb{F}$-topologies on M^* or $\mathbb{F}$-product topologies of M_i. They are the topologies where all smooth functions from M^* and all smooth curves into M^* are continuous.

Definition 2.6.3
Let $(M^*,\mathcal{C}_{M^*},\mathcal{F}_{M^*})$ be the product of $\mathbb{F}$-spaces. The families $S=\{f^{-1}(0,1)\,|\,f\in\mathcal{F}_{M^*}\}$ and $B=\{f^{-1}(0,+\infty)\,|\,f\in\mathcal{F}_{M^*}\}$ are respectively a subbasis and a basis for $\tau_{\mathcal{F}_{M^*}}$.

Observe that S is a basis for $\tau_{\mathcal{F}_{M^*}}$ since it is a subbasis. For all $f\in\mathcal{F}_{M^*}$, $O\in\tau_{\mathcal{F}_{\mathbb{R}}}$, $f^{-1}(O)\in\tau_{\mathcal{F}_{M^*}}$. Also, for all $\mathcal{U}\in\tau_{\mathcal{C}_{M^*}}$, and for all $c\in\mathcal{C}_{M^*}$, $c^{-1}(\mathcal{U})\in\tau_{\mathcal{C}_{\mathbb{R}}}=\tau_{\mathcal{F}_{\mathbb{R}}}$. Finally $\tau_{\mathcal{F}_{M^*}}\subset\tau_{\mathcal{C}_{M^*}}$. The product carries another topology as a topological product space of $(M_i,\tau_{\mathcal{F}_i})$. That is, the usual product topology.

Definition 2.6.4
The product topology on M^*, denoted by τ_Π is the usual product of $\mathbb{F}$-topological spaces M_i. That is also to say, the topology induced by the topologies $\tau_{\mathcal{F}_{M_i}}$ of factors, such that if $\mathcal{U}_i \in \tau_{\mathcal{F}_{M_i}}$ and $p_i : M^* \to M_i$ is the canonical projection then $p_i^{-1}(\mathcal{U}_i)$ is a subbasic open set in τ_Π.

It is important at this stage to understand the form of $p_i^{-1}(\mathcal{U}_i)$ which is a slab in $\prod_{i\in I} M_i$. That is, each factor of $\prod_{i\in I} M_i$ is M_j with $j \neq i$ except the i^{th} which is $\mathcal{U}_i$. In other words, we may use the notation: $p_i^{-1}(\mathcal{U}_i) := \mathcal{U}_i \times \prod_{k\neq i} M_k$ for means of $M_1 \times M_2 \times \cdots \times \mathcal{U}_i \times \cdots$, where i is fixed, $k \in I - \{i\}$, and $\mathcal{U}_i$ is ranging over all members of $\tau_{\mathcal{F}_{M_i}}$. Therefore we may write, for different values of i:

$$\begin{aligned}
i &= 1 : p_1^{-1}(\mathcal{U}_1) = \mathcal{U}_1 \times M_2 \times \cdots \times M_n \times \prod_{k\neq 1,\ldots,n} M_k \\
i &= 2 : p_2^{-1}(\mathcal{U}_2) = M_1 \times \mathcal{U}_2 \times M_3 \times \cdots \times M_n \times \prod_{k\neq 1,\ldots,n} M_k \\
&\vdots \\
i &= n : p_n^{-1}(\mathcal{U}_n) = M_1 \times M_2 \times \cdots \times M_{n-1} \times \mathcal{U}_n \times \prod_{k\neq 1,\ldots,n} M_k
\end{aligned}$$

The finite intersection takes the form below:
$$\bigcap_{i=1}^{n} p_i^{-1}(\mathcal{U}_i) = p_1^{-1}(\mathcal{U}_1) \cap p_2^{-1}(\mathcal{U}_2) \cap \cdots \cap p_n^{-1}(\mathcal{U}_n) = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n \times \prod_{k\neq 1,\ldots,n} M_k$$
The finite product yields the following
$p_1^{-1}(\mathcal{U}_1) = \mathcal{U}_1 \times M_2 \times M_3 \times \cdots \times M_n$, $p_2^{-1}(\mathcal{U}_2) = M_1 \times \mathcal{U}_2 \times M_3 \times \cdots \times M_n$, $\cdots$, $p_n^{-1}(\mathcal{U}_n) = M_1 \times M_2 \times M_3 \times \cdots \times \mathcal{U}_n$ and the finite intersection becomes
$$\bigcap_{i=1}^{n} p_i^{-1}(\mathcal{U}_i) = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n = \prod_{i=1}^{n} \mathcal{U}_i$$

Lemma 2.6.1
The set $B = \{\bigcap_{j=1}^{n} p_j^{-1}(\mathcal{U}_j) \,|\, \mathcal{U}_j \in \tau_{\mathcal{F}_{M_j}}, \forall\, j \in J\}$ is a basis of τ_Π.

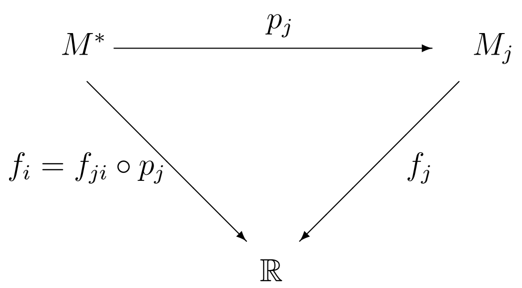
Proof.
The set B is closed under finite intersections since $\bigcap_{i=1}^{m}(\bigcap_{j=1}^{n} p_{ji}^{-1}(\mathcal{U}_{ji})) = \bigcap_{k=1}^{m\times n} p_k^{-1}(\mathcal{U}_k)$ by associativity. Thus B is a basis for τ_Π. □

Remark 2.6.1
Since the subbasis yields a unique and smallest topology that contains it, thus τ_Π is

the smallest topology such that $\tau_{\Pi} \supset \{p_i^{-1}(\mathcal{U}_i)\,|\,\mathcal{U}_i$ *ranging over* M_i*, for all* $i \in I\}$ *and* p_i *are continuous by definition. Therefore* $\tau_{\Pi} \subset \tau_{\mathcal{F}_{M^*}} \subset \tau_{\mathcal{C}_{M^*}}$*. Note that from the property of a subbasis in the topology* τ_{Π}*, an arbitrary union of* $p_i^{-1}(\mathcal{U}_i)$ *is a* τ_{Π}*-open set, and so is an arbitrary union of finite intersections* $\bigcap_{i=1}^{n} p_i^{-1}(\mathcal{U}_i)$*. In the finite product case, the basic open sets are all boxes* $\prod_{i=1}^{n} \mathcal{U}_i$*, where* $\mathcal{U}_i \in \tau_{\mathcal{F}_{M_i}}$*.*

Proposition 2.6.1
Let M^* *be a topological product of* $\mathbb{F}$*-spaces* M_i*,* $p_j : M^* \to M_j$ *the* j*th projection map and* $f_{ji} \in \mathcal{F}_{M_i}$*,* j *fixed. Let* $f_{ji}^{-1}(0,1)$ *be a subbasic open set in* $\tau_{\mathcal{F}_{M_j}}$*, for each* $i \in I$*. Then* $p_j^{-1}(f_{ji}^{-1}(0,1))$ *is also a subbasic open set in* τ_{Π} *for each* $i \in I$*.*

Proof.
Assume $\mathcal{U}_j = f_j^{-1}(0,1)$, where we denote by $\{f_{ji}\}$ a family of functions in $\mathcal{F}_{M_j}$. Thus $\bigcap_{i=1}^{n}(f_{ji}^{-1}(0,1))$ is a basic open set in M_j with $f_{ji} \in \mathcal{F}_{M_j}$, j fixed. It follows from the inverse image properties, that $p_j^{-1}[\bigcap_{i=1}^{n}(f_{ji}^{-1}(0,1))] = \bigcap_{i=1}^{n}[p_j^{-1}(f_{ji}^{-1}(0,1))] = \bigcap_{i=1}^{n}[(f_{ji} \circ p_j)^{-1}(0,1)] = \bigcap_{i=1}^{n} f_i^{-1}(0,1)$ which is a basic open set for τ_{Π} with respect to Lemma 2.6.1. And $f_{ji} \circ p_j = f_i$, since p_j is smooth, where $f_i \in \mathcal{F}_{M^*}$ is a generator of the structure on M^* following the commutativity of the diagram below.



Hence, $(f_{ji} \circ p_j)^{-1}(0,1) = (p_j^{-1} \circ f_{ji}^{-1})(0,1) = p_j^{-1}(f_{ji}^{-1}(0,1))$ is a subbasic open set in τ_{Π}. This confirms the inclusion $\tau_{\Pi} \subset \tau_{\mathcal{F}_{M^*}}$, that is to say that a subbasic open set in τ_{Π} has the form of subbasic open set in $\tau_{\mathcal{F}_{M^*}}$. □

Corollary 2.6.1
The canonical projection $p_j : \prod_{i \in I} M_i \to M_j$ *is continuous, onto and open map for* τ_{Π}*.*

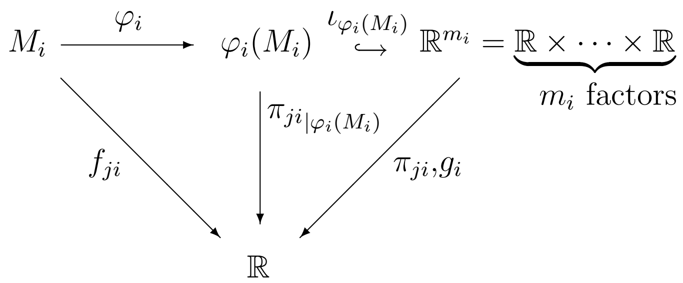
Proof.
As from Proposition 2.6.1, we have $\bigcap_{i=1}^{n} p_j^{-1}(f_{ji}^{-1}(0,1)) = \bigcap_{i=1}^{n}(f_i^{-1}(0,1))$ is a basic open set. Now, by definition, p_j is continuous, onto map. It follows that

$p_j\Big(\bigcap_{i=1}^{n}[f_i^{-1}(0,1)]\Big) = p_j\Big(p_j^{-1}[\bigcap_{i=1}^{n}[f_{ji}^{-1}(0,1)]]\Big) = \bigcap_{i=1}^{n}(f_{ji}^{-1}(0,1)) = \bigcap_{i=1}^{n} f_{ji}^{-1}(0,1)$. The result above follows from a set theoretic property of the surjective map p_j. Therefore, p_j sends each basic open set, element of τ_Π in M^* to a basic open set, that is an element of $\tau_{\mathcal{F}_{M_i}}$ in M_i. Equivalently p_j is an open map. □

Now, we want to make comparison of $\mathbb{F}$-product topologies and τ_Π.

Lemma 2.6.2 *(Finite product case).*
Let τ_Π be the product topology and $\tau_{\mathcal{F}_{M^*}}$ the $\mathbb{F}$-product topology generated by structure functions. Then $\tau_\Pi = \tau_{\mathcal{F}_{M^*}}$.

Proof. [71, 62, 20]
Observe that $\tau_\Pi \subset \tau_{\mathcal{F}_{M^*}}$ holds as from Remark 2.6.1. Now, assume that $\tau_{\mathcal{F}_{M_i}}$ is Hausdorff topology on each M_i. That is, there exists $f \in \mathcal{F}_{M_i}$ that separates points in M_i. Thus f is one-to-one. Hence, we have $\varphi_i = (f_{1i}, \ldots f_{m_i}) : M_i \to \mathbb{R}^{m_i}$ such that φ_i is one-to-one and induces a diffeomorphism between M_i and $\varphi_i(M_i) \subset \mathbb{R}^{m_i}$, where $\mathcal{F}_{oM_i} = \{f_{1_i}, \ldots, f_{m_i}\}$ is a generating set containing a separating points function, [62]. As a diffeomorphism, φ_i is smooth, continuous, open and bijective map. A summary of the context is given by the following diagram, where π_{ji} is the projection map, $\pi_{ji_{|\varphi_i(M_i)}} \circ \varphi_i = f_{ji}$ and the set $\{\pi_{ji_{|\varphi_i(M_i)}} \mid j = 1, \ldots, m\}$ is the generating set of $(\mathcal{C}_{\varphi_i(M_i)}, \mathcal{F}_{\varphi_i(M_i)})$. However, $(\mathcal{C}_{M_i}, \mathcal{F}_{M_i})$ is generated by $\{f_{1_i}, \ldots, f_{m_i}\} = \varphi_i^*\{\pi_{ji} \mid j = 1, \ldots, m\} = \{\pi_{ji_{|\varphi_i(M_i)}} \circ \varphi_i \mid j = 1, \ldots, m\}$.



The canonical Frölicher topology on $\mathbb{R}^{m_i}$ coincides with the natural product topology on $\mathbb{R}^{m_i} = \mathbb{R} \times \cdots \times \mathbb{R}$, m_i factors. That is, $\tau_{\mathcal{F}_{\mathbb{R}^{m_i}}} = \tau_{\mathbb{R}^{m_i}}$ and so $\tau_{\mathcal{F}_{\mathbb{R}^{m_i}}}(\varphi_i(M_i)) = \tau_{\mathbb{R}^{m_i}}(\varphi_i(M_i))$ is the trace topology of $\tau_{\mathbb{R}^{m_i}}$ on $\varphi_i(M_i)$. Let $\{g_i^{-1}(0, +\infty) \mid g_i \in \mathcal{F}_{\mathbb{R}^{m_i}}\}$ be the standard basis of $\tau_{\mathbb{R}^{m_i}}$. Thus, $g_i^{-1}(0, +\infty) \cap \varphi_i(M_i) = \iota^{-1}_{\varphi_{i(M_i)}}(g_i^{-1}(0, +\infty))$ is $\tau_{\mathbb{R}^{m_i}}(\varphi_i(M_i))$-basic open set in $\varphi_i(M_i)$, as the inverse image of the canonical injection (inclusion) map $\iota_{\varphi_{i(M_i)}}$. Since φ_i is a smooth map, it follows that the inverse image of a basic open set is a basic open set. We have $\varphi_i^{-1}\big(g_i^{-1}(0, +\infty) \cap \varphi_i(M_i)\big) = \varphi_i^{-1}(g_i^{-1}(0, +\infty)) \cap M_i = \varphi_i^{-1} g_i^{-1}((0, +\infty)) = f_i^{-1}(0, +\infty)$ is a basic open set in M_i, where $f_i \in \mathcal{F}_{M_i}$. From the latter identities, it follows that the push forward function of φ_i is a bijection between the basis of $\tau_{\mathcal{F}_{M_i}}$ and the basis of $\tau_{\mathcal{F}_{\varphi_i(M_i)}}$ in the following way. Consider the push forward map defined as follows. $\varphi_{i*} : \{f_i^{-1}(0, +\infty) \mid f_i \in \mathcal{F}_{M_i}\} \to \{g_i^{-1}(0, +\infty) \cap \varphi_i(M_i)) \mid g_i \in \mathcal{F}_{\mathbb{R}^{m_i}}\}$ such that $\varphi_{i*}(f_i^{-1}(0, +\infty)) = \varphi_i(f_i^{-1}(0, +\infty)) = (f_i \varphi_i^{-1})^{-1}(0, +\infty) = g_{i_{|\varphi_i(M_i)}}^{-1}(0, +\infty)$

$= \iota^{-1}_{\varphi_{i(M_i)}} g_i^{-1}(0,+\infty) \ = g_i^{-1}(0,+\infty)\cap\varphi_i(M_i)$ and $\varphi_{i*}^{-1}(g_i^{-1}(0,+\infty)\cap\varphi_i(M_i)) = \varphi_i^{-1}(g_i^{-1}(0,+\infty))$. Hence $\tau_{\mathcal{F}_{M_i}} \cong \tau_{\mathbb{R}^{m_i}}(\varphi_i(M_i))$. Let us point out that $\varphi_i : M_i \to \mathbb{R}^{m_i}$ is an open map (see the proof of Lemma 2.6.2). It follows that $\mathcal{F}_{\varphi(M_i)} = \{f_{i|\varphi(M_i)} \mid f_i \in \mathcal{F}_{\mathbb{R}^{m_i}}\}$ since $\varphi_i(V_i)$ is an open set in $\mathbb{R}^{m_i}$ and in particular $\varphi_i(M_i)$ is an open set in $\mathbb{R}^{m_i}$. Now, we can deal with $\tau_{\mathcal{F}_{M^*}}$ and τ_Π as shown below. Since the product topology τ_Π is Hausdorff if, and only if each factor M_i has a Hausdorff topology $\tau_{\mathcal{F}_{M_i}}$, we may construct $\varphi = \prod_{i=1}^{n} \varphi_i : M^* \to \mathbb{R}^m$, a $\mathbb{F}$-diffeomorphism $M^* \simeq \varphi(M^*) \subset \mathbb{R}^m$ in such a way as to send τ_Π-open sets of M^* to $\tau_{\mathbb{R}^m}$-open sets in $\mathbb{R}^m$ irrespective to $\tau_{\mathcal{F}_{\mathbb{R}^n}}(\varphi(M^*)) = \tau_{\mathbb{R}^n}(\varphi(M^*)) = \tau_{\mathcal{F}_{\varphi(M^*)}}$. From $\tau_{\mathbb{R}^n} = \tau_{\mathcal{F}_{\mathbb{R}^n}}$, the trace topology of the product topology of $\mathbb{R}^m$ on $\varphi(M^*)$ coincides with the $\mathbb{F}$-topology on $\varphi(M^*)$. Thus, the trace topology is the smallest topology on $\varphi(M^*)$ in bijective correspondence to the smallest topology on M^*. Hence,

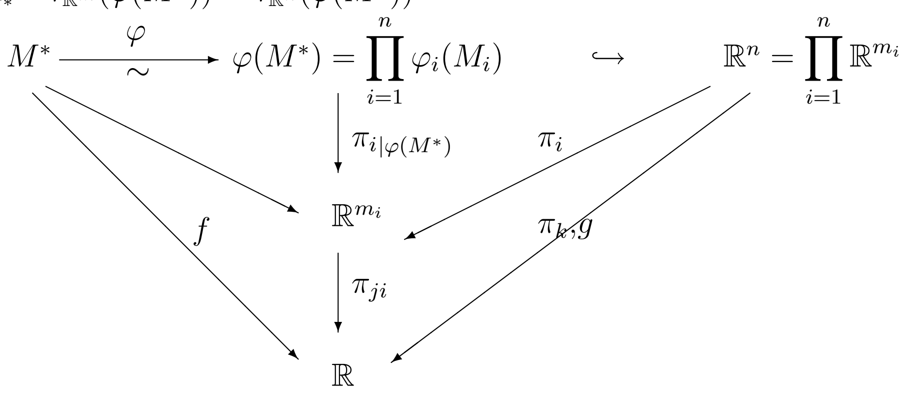
$$\tau_{\mathbb{R}^n}(\varphi(M^*)) \cong \tau_\Pi \qquad (15)$$

The diagram below, where $k = 1,\dots,m$, $m = m_1 + \cdots + m_n$, $\pi_{k|\varphi(M^*)} = \pi_{ji} \circ \pi_{i|\varphi(M^*)}$ shows that $\varphi : M^* \to \varphi(M^*)$ reads

$$\begin{aligned} \varphi &= \varphi_1 \times \ldots \times \varphi_i \times \ldots \times \varphi_n \\ &= (f_{1_1},\dots,f_{m_1},f_{1_2},\dots,f_{m_2},\dots,f_{1_i},\dots,f_{m_i},\dots,f_{1_n},\dots,f_{m_n}) \\ &= (f_1,\dots,f_k,\dots,f_m). \end{aligned}$$

By similar arguments as in the previous part of the proof we will construct bases $\mathcal{B}^*$ in $\tau_{\mathcal{F}_{M^*}}$ and $\mathcal{B}_\varphi$ in $\tau_{\varphi(M^*)}$. They are related by the bijective map defined as follows. $\varphi_* : \mathcal{B}^* \to \mathcal{B}_\varphi$, $f^{-1}(0,+\infty) \mapsto \iota^{-1}_{\varphi(M^*)}(\mathcal{U})$ with $\mathcal{U} = g_i^{-1}(0,+\infty) \subset \mathbb{R}^m$, such that $f = g \circ \varphi$. Thus. we can set $\varphi_*(f^{-1}(0,+\infty)) = \varphi(f^{-1}(0,+\infty)) = \mathcal{U} \cap \varphi(M^*)$ and with respect to an inverse image property $\varphi_*^{-1}(\mathcal{U}\cap\varphi(M^*)) = \varphi^{-1}(\mathcal{U}\cap\varphi(M^*)) = \varphi^{-1}(\mathcal{U}) \cap M^* = \varphi^{-1}(\mathcal{U}) = f^{-1}(0,+\infty)$. Since $\tau_{\mathcal{F}_{\mathbb{R}^m}}(\varphi(M^*)) = \tau_{\mathbb{R}^m}(\varphi(M^*))$. We have

$$\tau_{\mathcal{F}_{M^*}} \cong \tau_{\mathbb{R}^m}(\varphi(M^*)) = \tau_{\mathbb{R}^n}(\varphi(M^*)) \qquad (16).$$



Hence, $\tau_{\mathcal{F}_{M^*}} = \tau_\Pi$ from (15) and (16). □

This confirms the fact that in the set of topologies on M^* such that the natural projections are smooth, the topology generated by a subbasis (basis) needs to be the unique smallest one, containing the given subbasis (basis). Under an $\mathbb{F}$-smooth map, the inverse image sends a basic open set to a basic open set. The

canonical projection sends a basic open set to a basic open set. Thus,
$$\tau_{\prod} = \tau_{\mathcal{F}_{M^*}} \subset \tau_{\mathcal{C}_{M^*}}. \tag{17}$$

Corollary 2.6.2
An $\mathbb{F}$-space M_i is finitely generated by m_i real valued functions if, and only if there exists a map $\varphi_i : (M_i, \mathcal{C}_{M_i}, \mathcal{F}_{M_i}) \longrightarrow (\mathbb{R}^{m_i}, \mathcal{C}, \mathcal{F})$ for each i such that $1 \leq i \leq n$ and $\varphi_i^* : \mathcal{F}_{\varphi_i(M_i)} \longrightarrow \mathcal{F}_{M_i}$ is an isomorphism between rings, where $(\mathcal{C}, \mathcal{F})$ is the canonical $\mathbb{F}$-structure on $\mathbb{R}^{m_i}$.

Proof.
First of all, let us point out that the concept of finitely generated is understood as follows. In the one hand, as shown in [71], there exists a map $\varphi_i : M_i \to \mathbb{R}^{m_i}$ which is a diffeomorphism $M_i \simeq \varphi_i(M_i)$ if, and only if the generating set contains a points separating function [62]. In the other hand, $\varphi_i^* : \mathcal{F}_{\varphi_i(M_i)} \to \mathcal{F}_{M_i}$ $g \mapsto \varphi_i^*(g_i) = g_i \circ \varphi_i$. We have to show that such a map φ_i^* is a bijective homomorphism of the rings of functions $\mathcal{F}_{\varphi_i(M_i)}$ and $\mathcal{F}_{M_i}$. Let $g_i, h_i \in \mathcal{F}_{\varphi_i(M_i)}$ such that $\varphi_i^*(g_i) = \varphi_i^*(h_i)$. This implies $g_i \circ \varphi_i = h_i \circ \varphi_i$. Applying φ_i^{-1} to both sides yields $g_i = h_i$. Thus φ_i^* is injective. Let $f_i \in \mathcal{F}_{M_i}$, since φ_i and φ_i^{-1} are smooth, so is $f_i \circ \varphi_i^{-1} : \varphi_i(M_i) \to \mathbb{R}$. Hence for any $f_i \in \mathcal{F}_{M_i}$, there exists $g_i \in \mathcal{F}_{\varphi_i(M_i)}$ such that $\varphi_i^*(g_i) = f_i$. Therefore, φ_i^* is surjective. Now, we show that the bijective map φ_i^* is homomorphism of rings. Let $g_i, h_i \in \mathcal{F}_{\varphi_i(M_i)}$. It follows that $\varphi_i^*(g_i + h_i) = (g_i + h_i) \circ \varphi_i = g_i \circ \varphi_i + h_i \circ \varphi_i = \varphi^*(g_i) + \varphi^*(h_i)$ and also $\varphi_i^*(g_i \cdot h_i) = (g_i \cdot h_i) \circ \varphi_i = (g_i \circ \varphi_i) \cdot (h_i \circ \varphi_i) = \varphi^*(g_i) \cdot \varphi^*(h_i)$. Therefore, φ_i and φ_i^* are bijective maps with inverses φ_i^{-1} and φ_i^{*-1} respectively. So,

$$(\varphi_i^{-1})^* \circ \varphi_i^* = (\varphi_i \circ \varphi_i^{-1})^* = id_{\mathcal{F}_{\varphi(M_i)}}, \varphi_i^{*-1} \circ \varphi_i^* = id_{\mathcal{F}_{\varphi(M_i)}}. \tag{18}$$

The identities (18) yield $\varphi_i^{*-1} \circ \varphi_i^* = (\varphi_i^{-1})^* \circ \varphi_i^*$ which, in turn, give $(\varphi_i^*)^{-1} = (\varphi_i^{-1})^*$. Conversely, if φ_i^* is an isomorphism of rings, then φ_i is a diffeomorphism of M_i onto $\varphi_i(M_i)$. □

Lemma 2.6.3 *(Infinite product case)*
Let $\tau_{\mathcal{F}_{M^*}}$ be the $\mathbb{F}$-topology induced by structures functions on M^* and $\tau_{\prod}$ the usual topological product space. Then $\tau_{\mathcal{F}_{M^*}} = \tau_{\prod}$.

Proof.
Let $V \in \tau_{\mathcal{F}_{M^*}}$. Assume that V is $\tau_{\mathcal{F}_{M^*}}$-basic open set and $V = f^{-1}(0, +\infty)$, where $f \in \mathcal{F}_{M^*}$. In order to show that V is $\tau_{\prod}$-open set, we shall refer to Corollary 2.2.1 for characterization of open sets. For each $x \in V = f^{-1}(0, +\infty) \subset M^*$, $x \in f^{-1}(t)$, for some $t \in (0, +\infty)$ such that $f(x) = t$ and $x = (x_i)_{i \in I}$. It may exist an open set $\mathcal{U}_k$ such that $x_k \in \mathcal{U}_k \subset M_k$ with $\mathcal{U}_k \neq M_k$ for $k = 1, \ldots, n$ and $\mathcal{U}_j = M_j$ for $j \neq 1, \ldots, n$. That is, $(x_k) \in \prod_{k=1}^{n} \mathcal{U}_k$ and $(x_j)_{j \neq 1, \ldots, n} \in \prod M_j$. Thus, $f = f_i \circ p_i$ yields $t = f(x_i)_{i \in I} = (f_i \circ p_i)((x_i)_{i \in I}) = f_i(x_i)$. It follows that

$x=(x_i)_{i\in I}\in f^{-1}(t)=p_k^{-1}\circ f_k^{-1}(t)$ and even better, $x\in f^{-1}(0,+\infty)=p_k^{-1}\circ f_k^{-1}(0,+\infty)$ for $k=!,\ldots,n$. Therefore, there exists $\mathcal{U}\in\tau_{\prod}$, where $x\in\mathcal{U}=\prod_{k=1}^{n}\mathcal{U}_k\times\prod_{j\neq 1,\ldots,n}M_j$. From the definition of $\tau_{\prod}$-basic open set, $\mathcal{U}=\bigcap_{k=1}^{n}p_k^{-1}(\mathcal{U}_k)$ and $\mathcal{U}_k=f_k^{-1}(0,+\infty)$ since p_k is smooth, continuous and open. We can break down the infinite case, just by recalling that $\mathcal{U}=\bigcap_{k=1}^{n}p_k^{-1}f_k^{-1}(0,+\infty)=\bigcap_{k=1}^{n}(g_k^{-1}(0,+\infty))\subset f^{-1}(0,+\infty)=V$ since f is among the g_k. Therefore, $\mathcal{U}\subset V$, that is, V contains a basic open set $\mathcal{U}$ of $\tau_{\prod}$. Thus, $V\in\tau_{\prod}$ and $\tau_{\mathcal{F}_{M^*}}\subset\tau_{\prod}$. Since $\tau_{\prod}\subset\tau_{\mathcal{F}_{M^*}}$ the proof is completed by $\tau_{\mathcal{F}_{M^*}}=\tau_{\prod}$. □

2.7 $\mathbb{F}$-Coproduct Space

The $\mathbb{F}$-coproduct space in the category $\mathcal{FRL}$ is the final object obtained by lifting the coproduct in the category $\mathcal{SETS}$ to $\mathcal{FRL}$. Let $\bar{M}=\amalg_{i\in I}M_i$ denotes the coproduct in $\mathcal{SETS}$. The final structure on $\bar{M}$ in $\mathcal{FRL}$ is the $\mathbb{F}$-structure generated by the family $(s_i:M_i\to\bar{M})_{i\in I}$, where s_i are the inclusion maps with the universality condition given by $f\circ s_i=f_i$, where $f=(f_i)_{i\in I}$: $\bar{M}\to\mathbb{R}$. The Frölicher structure is generated by a set $\mathcal{C}_o$ of curves, that is, $(\mathcal{C}_{\bar{M}},\mathcal{F}_{\bar{M}})$ such that $\mathcal{C}_o=\bigcup_{i\in I}\{s_1\circ c_{1j},s_2\circ c_{2j},\ldots,s_n\circ c_{nj},\ldots\mid j$ running over $\mathcal{C}_{oM_i}$ with i fixed $\}$, where $\mathcal{C}_{oM_i}$ generates the $\mathbb{F}$-structure $(\mathcal{C}_{M_i},\mathcal{F}_{M_i})$ on M_i for all i. It follows that the structure functions and curves are given by $\mathcal{F}_{\bar{M}}=\Phi\mathcal{C}_o=\{f:\bar{M}\to\mathbb{R}\mid f_{|M_i}=f_i\in\mathcal{F}_{M_i}$ and $f=(f_i)_{i\in I},\forall\, i\in I\}$ and $\mathcal{C}_{\bar{M}}=\Gamma\mathcal{F}_{\bar{M}}=\Gamma\Phi\mathcal{C}_o=\{c:\mathbb{R}\to\bar{M}\mid c=s_i\circ c_i, c_i\in\mathcal{C}_{M_i}\}$.

Definition 2.7.1
The $\mathbb{F}$-space $(\bar{M},\mathcal{C}_{\bar{M}},\mathcal{F}_{\bar{M}})$ is called an $\mathbb{F}$-coproduct space of M_i or an $\mathbb{F}$-coproduct of $\mathbb{F}$-spaces M_i. Also the pair $(\mathcal{C}_{\bar{M}},\mathcal{F}_{\bar{M}})$ is the final $\mathbb{F}$-coproduct structure (coproduct structure for short) such that all s_i are smooth maps.

Note that $s_i\circ\mathcal{C}_{M_i}=\mathcal{C}_{s_i(M_i)}$ if, and only if $\mathcal{F}_{\bar{M}}\circ s_i=\mathcal{F}_{M_i}\simeq\mathcal{F}_{s_i(M_i)}=\mathcal{F}_{\bar{M}}|s_i(M_i)$. We want to study the topologies underlying an $\mathbb{F}$-coproduct space.

Definition 2.7.2
The topologies $\tau_{\mathcal{F}_{\bar{M}}}$ and $\tau_{\mathcal{C}_{\bar{M}}}$ induced by smooth functions and smooth curves are called $\mathbb{F}$-topologies on $\bar{M}$ or $\mathbb{F}$-coproduct topologies. That is, the topologies where all smooth functions are continuous. The family $\mathcal{S}=\{f^{-1}(0,1)\mid f\in\mathcal{F}_{\bar{M}}\}$ is a subbasis for the topology $\tau_{\mathcal{F}_{\bar{M}}}$ and the family $\mathcal{B}=\{f^{-1}(0,+\infty)\mid f\in\mathcal{F}_{\bar{M}}\}$ is a basis for the topology $\tau_{\mathcal{F}_{\bar{M}}}$.

The coproduct carries another topology as a topological coproduct space of $(M_i,\tau_{\mathcal{F}_{M_i}})$. That is the coproduct topology.

Definition 2.7.3
The coproduct topology on $\bar{M}$ is the coproduct of the family $(M_i)_{i\in I}$ in $\mathcal{SETS}$, endowed with the topology in which open sets are unions of $s_i(\mathcal{U}_i)$, $i \in I$ and, where $\mathcal{U}_i$ is an arbitrary $\tau_{\mathcal{F}_{M_i}}$-open set in M_i. That is the topology on $\bar{M}$ denoted by $\tau_{\amalg}$ and which is the coproduct topology of $\mathbb{F}$-topological spaces M_i.

Lemma 2.7.1
Let $\bar{M}$ be the coproduct of the family $(M_i)_{i\in I}$ and $\tau_{\amalg}$ its coproduct topology. Then:

1. *s_i is a continuous map for $\tau_{\amalg}$,*
2. *The family $\mathcal{B} = \{s_i(\mathcal{U}_i) \mid \mathcal{U}_i \in \tau_{\mathcal{F}_{M_i}},\ for\ all\ i \in I\}$ is a basis for $\tau_{\amalg}$.*
3. *$s_i(M_i)$ is a basic open set in $\tau_{\amalg}$ for all $i \in I$,*
4. *s_i is an open map for $\tau_{\amalg}$.*

Proof.

1. Let $\mathcal{U} \in \tau_{\amalg}$, that is, $\mathcal{U} = \bigcup_{j\in J} s_j(\mathcal{U}_j)$ with $\mathcal{U}_j = \bigcup_{f_j\in\mathcal{F}_{M_j}} f_j^{-1}(0,+\infty)$. Assume i being fixed and j varying in the sequel. Because the M_j form a partition of $\bar{M}$, it follows that:

$$\begin{aligned} s_i{}^{-1}(\mathcal{U}) &= s_i{}^{-1}[\bigcup_{j\in J} s_j(\mathcal{U}_j)] \\ &= s_i{}^{-1}[\bigcup_{j\in J}\bigcup_{f_j\in\mathcal{F}_{M_j}} s_j(f_j^{-1}(0,+\infty))] \\ &= [\bigcup_{j\in J}\bigcup_{f_j\in\mathcal{F}_{M_j}} (s_i{}^{-1}(s_j(f_j^{-1}(0,+\infty))))] \\ &= [\bigcup_{j\neq i}(s_i{}^{-1}s_j f_j^{-1}(0,+\infty))] \cup [\bigcup_{j=i}(s_i{}^{-1}s_j f_j^{-1}(0,+\infty))] \\ &= \emptyset \cup [\bigcup_{j=i} s_i{}^{-1}s_j f_j^{-1}(0,+\infty)] \\ &= [\bigcup_{f_i\in\mathcal{F}_{M_i}} f_i^{-1}(0,+\infty)] \in \tau_{\mathcal{F}_{M_i}} \end{aligned}$$

Since s_i is injective thus the composite with its inverse is the identity map on M_i for i = j. Now, whenever $i \neq j$ we have what follows:

$$\begin{aligned} s_i{}^{-1}[s_j f_j^{-1}(0,+\infty)] &= \{x \in M_i \mid s_i(x) \in s_j f_j^{-1}(0,+\infty) \subset s_j(M_j), i \neq j\} \\ &= \{x \in M_i \mid s_i(x) \in s_i(M_i) \cap s_j(M_j) = \emptyset\ , i \neq j\} \\ &= \emptyset. \end{aligned}$$

Hence s_i is a continuous map for $\tau_{\amalg}$.

2. That is a straightforward consequence of the definition of the coproduct topology above.

3. Since $M_i \in \tau_{\mathcal{F}_{M_i}}$ then $s_i(M_i) \in \mathcal{B}$ for all $i \in I$ by assumption. That is $s_i(M_i)$ is a $\tau_{\rm II}$-basic open set with respect to the result above.

4. Since $f_i^{-1}(0,+\infty)$ is a $\tau_{\mathcal{F}_{M_i}}$-basic open set, where $f_i \in \mathcal{F}_{M_i}$, thus it is also a $\tau_{\mathcal{F}_{M_i}}$-open. Therefore $s_i(f_i^{-1}(0,+\infty)) \in \mathcal{B}$. Hence s_i sends a basic open of $\tau_{\mathcal{F}_{M_i}}$ to a basic open of $\tau_{\rm II}$. Thus each $s_i(\mathcal{U}_i)$, where $\mathcal{U}_i \in \tau_{\mathcal{F}_{M_i}}$, is an open set in $\tau_{\rm II}$. We conclude that s_i is an open map.

Corollary 2.7.1
Let s_i be the canonical inclusion of M_i into $\bar{M}$. Then $s_i(M_i)$ is a $\tau_{\rm II}$-closed set in $\bar{M}$ for all $i \in I$.

Proof.
As $\bar{M}$ is partitioned by $\{s_i(M_i) \mid i \in I\}$, it follows that $\bar{M} - s_i(M_i) = C_{\bar{M}}^{s_i(M_i)} = \bigcup\limits_{j\neq i} s_j(M_j)$ is both closed and open set, since it is a closed set as a complement of an open set on the one hand and an open set as an arbitrary union of open sets on the other hand. Therefore $s_i(M_i)$ is both $\tau_{\rm II}$-closed set and $\tau_{\rm II}$-open set in $\bar{M}$ since $s_i(M_i) = \bar{M} - \bigcup\limits_{j\neq i} s_j(M_j) = C_{\bar{M}} \bigcup\limits_{j\neq i} s_j(M_j) = \bigcap\limits_{j\neq i} C_{\bar{M}}^{s_j(M_j)}$. That is a complement of a closed set and an arbitrary intersection of closed sets. □

Lemma 2.7.2
Let $\mathcal{U} \in \tau_{\rm II}$. Then $\mathcal{U}$ is a $\tau_{\rm II}$-open (closed) set in $\bar{M}$ if, and only if $\mathcal{U} \cap s_i(M_i)$ is a $\tau_{\rm II}$-open (closed) set in $\bar{M}$ for all $i \in I$.

Proof.
"$\Rightarrow$": We start with the open case. Let $\mathcal{U} \in \tau_{\rm II}$. That is, $\mathcal{U} = \bigcup\limits_{j\in J} s_j(U_j)$ with $\mathcal{U}_j \in \tau_{\mathcal{F}_{M_j}}$ for some j. But from Lemma 2.7.1 and the definition $\mathcal{B}$ more than one factor of the union may be in $s_i(M_i)$. We set i fixed and use partition on $\bar{M}$ as follows. Let $V = \mathcal{U} \cap s_i(M_i)$. Thus, $V = [\bigcup\limits_{j\in J} s_j(\mathcal{U}_j)] \cap s_i(M_i) = \bigcup\limits_{j\in J} [s_j(\mathcal{U}_j) \cap s_i(M_i)] = [\bigcup\limits_{j\neq i} (s_j(\mathcal{U}_j) \cap s_i(M_i))] \cup [\bigcup\limits_{j=i} s_i(\mathcal{U}_i) \cap s_i(M_i)]$. It follows that $V = \emptyset \cup (\bigcup\limits_{j=1} s_i(\mathcal{U}_i)) = \mathcal{V}_i$, where $\mathcal{V}_i = \bigcup\limits_{j=i} s_i(\mathcal{U}_i) \subset M_i$. Hence, V is a $\tau_{\rm II}$-open set in $\bar{M}$. Now, we prove the closed case. Let $\mathcal{U}$ be $\tau_{\rm II}$-closed in $\bar{M}$. However, $s_i(M_i)$ is $\tau_{\rm II}$-closed in $\bar{M}$, thus $\mathcal{U} \cap s_i(M_i)$ is $\tau_{\rm II}$-closed.
"$\Leftarrow$" Let $\mathcal{U} \cap s_i(M_i)$ be a $\tau_{\rm II}$-open (closed) set in $\bar{M}$. Since $s_i(M_i)$ is an open(closed) set in $\bar{M}$, it follows that $\mathcal{U} \cap s_i(M_i) \subset \mathcal{U}$ is open (closed) set for the trace topology of $\tau_{\rm II}$ on $\mathcal{U}$. That is, $\tau_{\rm II}(\mathcal{U})$. Referring to Lemma 2.5.3, it

follows from $\mathcal{U}\cap s_i(M_i)\subset\mathcal{U}\subset\bar{M}$ that $\mathcal{U}\cap s_i(M_i)$ is a $\tau_{\amalg}$-open (closed) set in $\bar{M}$ by assumption. Also $\mathcal{U}\cap s_i(M_i)$ is a $\tau_{\amalg}(\mathcal{U})$-open (closed) set. Thus $\mathcal{U}\in\tau_{\amalg}$ or $\mathcal{U}$ is a $\tau_{\amalg}$-closed set in $\bar{M}$. □

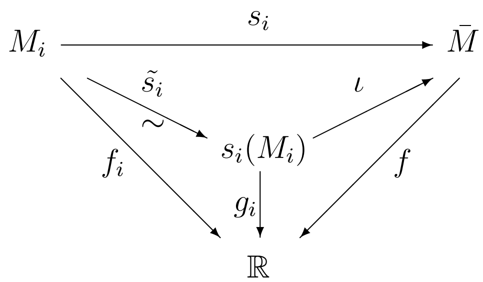
Now, we would like to compare the $\mathbb{F}$-coproduct topologies and standard coproduct topology.

Lemma 2.7.3
The topology $\tau_{\amalg}$ is the finest one in which all canonical inclusions $s_i : M_i \to \bar{M}$ are continuous, and also we have $\tau_{\mathcal{F}_{\bar{M}}}\subset\tau_{\mathcal{C}_{\bar{M}}}\subset\tau_{\amalg}$.

Proof.
Let τ be a topology on $\bar{M}$ for which all s_i are continuous, that is $s_i^{-1}(\mathcal{U})=\mathcal{U}_i$ is open set in M_i for all $i\in I$ and all $\mathcal{U}\in\tau$. Applying s_i to both sides yields $s_i(\mathcal{U}_i)=s_i(s_i^{-1}(\mathcal{U}))=\mathcal{U}\cap s_i(M_i)\in\tau_{\amalg}$. Observe that the latter equality holds as from a set theoretic property combining image and inverse image for a set map. But $s_i(\mathcal{U}_i)$ is a $\tau_{\amalg}$-basic open set. Hence from Lemma 2.7.2, it follows that $\mathcal{U}$ is a $\tau_{\amalg}$-open set. Thus $\tau\subset\tau_{\amalg}$. In Particular if $\tau=\tau_{\mathcal{F}_{\bar{M}}}$ or $\tau=\tau_{\mathcal{C}_{\bar{M}}}$, we have the inclusion $\tau_{\mathcal{F}_{\bar{M}}}\subset\tau_{\mathcal{C}_{\bar{M}}}\subset\tau_{\amalg}$. □

Lemma 2.7.4
Let $\tau_{\amalg}$ be the coproduct topology and $\tau_{\mathcal{F}_{\bar{M}}}$ the $\mathbb{F}$-topology on $\bar{M}$. Then $\tau_{\amalg}=\tau_{\mathcal{F}_{\bar{M}}}$ and $\bar{M}$ is a balanced $\mathbb{F}$-space.

Proof.
First of all, let us draw the diagram below:



This diagram commutes in all its components. It is worth noticing that M_i and $\bar{M}$ are endowed with their $\mathbb{F}$-structures, whereas $s_i(M_i)$ is equipped with the $\mathbb{F}$-subspace structure of $\bar{M}$. Moreover, the maps are related as follows. $f\circ\iota=g_i$, $g_i\circ\tilde{s}_i=f_i=f\circ s_i$ with f, g_i and f_i the structure functions, ι, s_i, $\tilde{s}_i$ smooth and injective maps such that $\tilde{s}_i$ is a diffeomorphism. Now let $\mathcal{U}\in\tau_{\amalg}$. So $\mathcal{U}=s_i(\mathcal{U}_i)$ with $\mathcal{U}_i\in\tau_{\mathcal{F}_{M_i}}$ for some $i\in I$. It follows that $\mathcal{U}_i=\bigcup_{j\in J} f_{ji}^{-1}(0,+\infty)$ with f_{ji} running over $\mathcal{F}_{M_i}$. That is i runs over M_i and j describes structures functions on M_i. Therefore, $\mathcal{U}=\bigcup_{i\in I} s_i(\bigcup_{f_{ij}\in\mathcal{F}_{M_i}} f_{ji}^{-1}(0,+\infty))=\bigcup_{i\in I}\bigcup_{f_{ij}\in\mathcal{F}_{M_i}}(s_i\, f_{ji}^{-1}(0,+\infty))$. Now, the associativity of the union yields $\mathcal{U}=\bigcup_{i\in I}\bigcup_{j\in J}(s_i(\tilde{s}_i^{-1}\circ g_{ji}^{-1}(0,+\infty)))=\bigcup_{(i,j)\in I\times J} s_i\,\tilde{s}_i^{-1}(g_{ji}^{-1}(0,+\infty))$.

Thus, $\mathcal{U} = \bigcup_{(i,j)\in I\times J} (g_{ji}^{-1}(0,+\infty)\cap s_i(M_i)) = \bigcup_{(i,j)\in I\times J} g_{ji}^{-1}(0,+\infty)$ since $g_{ji}^{-1}(0,+\infty)\subset s_i(M_i)$ for each $i\in I$. If we fixe i:

$$\bigcup_{j\in J} g_{ji}^{-1}(0,+\infty)\in\tau_{\mathcal{F}_{s_i(M_i)}} \qquad \textit{and} \tag{19}$$

$\bigcup_{j\in J} g_{ji}^{-1}(0,+\infty)=\bigcup_{j\in J}(\iota^{-1}\circ f_j{}^{-1}(0,+\infty))=\bigcup_{j\in J}[f_j{}^{-1}(0,+\infty)\cap s_i(M_i)]$. Henceforth,

$$\bigcup_{j\in J} g_{ji}^{-1}(0,+\infty)\in\tau_{\mathcal{F}_{\bar{M}}}(s_i(M_i)). \tag{20}$$

Thus, $\bigcup_{j} g_{ji}^{-1}(0,+\infty)\subset s_i(M_i)\subset\bar{M}$ and $\tau_{\mathcal{F}_{s_i(M_i)}}=\tau_{\mathcal{F}_{\bar{M}}}(s_i(M_i))$ with $f_j\in\mathcal{F}_{\bar{M}}$ such that $f_j\circ\iota=g_{ji}$. That is, $s_i(M_i)\in\tau_{\mathcal{F}_{\bar{M}}}$ and $\bigcup_{j} g_{ji}^{-1}(0,+\infty)\in\tau_{\mathcal{F}_{\bar{M}}}$. Hence $\tau_{\amalg}\subset\tau_{\mathcal{F}_{\bar{M}}}$. It follows from Lemma 2.7.3 that $\tau_{\amalg}=\tau_{\mathcal{F}_{\bar{M}}}$. So, $\tau_{\amalg}=\tau_{\mathcal{F}_{\bar{M}}}=\tau_{\mathcal{C}_{\bar{M}}}$ which means that $\bar{M}$ is a balanced space. □

Although M_i are not balanced, $\bar{M}$ is a balanced $\mathbb{F}$-space. $\tau_{\mathcal{F}_{\bar{M}}}=\tau_{\amalg}=\tau_{\mathcal{C}_{\bar{M}}}$ allows us to characterize open sets indistinctly being guided by the appropriate context we will deal with. Note that $s_i(f_i^{-1}(0,+\infty))=s_i(\tilde{s}_i{}^{-1}\circ g_i^{-1}(0,+\infty))=g_i^{-1}(0,+\infty)$, that is, s_i sends basic open sets of $\tau_{\mathcal{F}_{M_i}}$ to basic open sets of $\tau_{\mathcal{F}_{s_i(M_i)}}$. We have $\mathcal{F}_{s_i(M_i)}=\mathcal{F}_{\bar{M}_{|s_i(M_i)}}$ since $s_i(M_i)\in\tau_{\mathcal{F}_{\bar{M}}}$. The isomorphism $M_i\simeq s_i(M_i)$ induces an isomorphism of rings of functions between $\mathcal{F}_{s_i(M_i)}$ and $\mathcal{F}_{M_i}$ from Corollary 2.3.3. And also there is a bijection between bases of $\tau_{\mathcal{F}_{s_i(M_i)}}$ and $\tau_{\mathcal{F}_{M_i}}$.

Corollary 2.7.2
Let $\tau_{\amalg}$ be the coproduct topology and $\tau_{\mathcal{F}_{\bar{M}}}$ the $\mathbb{F}$ -topology on $\bar{M}$. Then, the following equalities hold $\tau_{\mathcal{F}_{s_i(M_i)}}=\tau_{\mathcal{F}_{\bar{M}}}(s_i(M_i))=\tau_{\amalg}(s_i(M_i))$

2.8 $\mathbb{F}$-Quotient Space

In what follows, an $\mathbb{F}$-Quotient space is a final object whose structure is obtained by the process of lifting from the category $\mathcal{SETS}$, to the category $\mathcal{FRL}$.

Definition 2.8.1
Let $f:M\to N$ be a smooth map of $\mathbb{F}$-spaces M into N. Consider an equivalence relation on M, denoted by $\sim_f$ and defined by: for any $x,y\in M, x\sim_f y$ if, and only if $f(x)=f(y)$. The relation $\sim_f$ is said to be consistent with the map f. It is called the kernel equivalence of f.

Now, we can define an equivalence relation consistent with all $f\in C^\infty(M,N)$ that is for all smooth maps on M into N as follows.

Definition 2.8.2
For any $x, y \in M, x \sim y$ if, and only if for any $f \in C^\infty(M,N)$ we have $f(x)=f(y)$. The set $[x] := \{y \in M \mid y \sim x\} = \{y \in M \mid f(y) = f(x),\ \text{for all}\ f \in C^\infty(M,N)\}$ is called the equivalence class of $x \in M$.

The quotient of M by $\sim$ denoted by $\tilde{M} := M_{/\sim}$ in $\mathcal{SETS}$ is given the final structure generated by $\pi : M \to \tilde{M}$, the canonical projection. Let $\mathcal{C}_o = \{\pi \circ c \mid c \in \mathcal{C}_{oM}\}$, where $\mathcal{C}_{oM}$ is the generates the $\mathbb{F}$-structure $(\mathcal{C}_{\tilde{M}}, \mathcal{F}_{\tilde{M}})$ on M. Then the structure curves and functions are given by $\mathcal{F}_{\tilde{M}} = \Phi\mathcal{C}_o = \{\tilde{g} : \tilde{M} \to \mathbb{R} \mid f = \tilde{g} \circ \pi, f \in \mathcal{F}_M\} = \{\tilde{g} : \tilde{M} \to \mathbb{R} \mid \tilde{g} = f \circ \pi^{-1}, f \in \mathcal{F}_M\}$ and $\mathcal{C}_{\tilde{M}} = \Gamma\Phi\mathcal{C}_o = \{\tilde{c} : \mathbb{R} \to \tilde{M} \mid \tilde{c} = \pi \circ c\ \text{ for all } c \in \mathcal{C}_M\}$. Thus $\mathcal{C}_{\tilde{M}} = \pi \circ \mathcal{C}_M$ if, and only if π is a smooth map. The smoothness of π reads $\mathcal{F}_{\tilde{M}} \circ \pi \subset \mathcal{F}_M$ if, and only if $\pi \circ \mathcal{C}_M \subset \mathcal{C}_{\tilde{M}}$. Let $\tilde{c} \in \mathcal{C}_{\tilde{M}}$. It follows that $\tilde{c}(t) = [x] = (\pi \circ c_x)(t)$, where $c_x : \mathbb{R} \to M$ is a constant curve such that $c_x(t) = x$ and $c_x \in \mathcal{C}_M$. And also $\tilde{c} = \pi \circ c_x$ is in $\pi \circ \mathcal{C}_M$. So, we have the equality $\mathcal{C}_{\tilde{M}} = \pi \circ \mathcal{C}_M$ as stated above. We can say that the projection of a curve in $\mathcal{C}_M$ by π is a curve in $\mathcal{C}_{\tilde{M}}$ Finally, $\mathcal{C}_{\tilde{M}} = \{\tilde{c} : \mathbb{R} \to \tilde{M} \mid \tilde{c} = \pi \circ c,\ c \in \mathcal{C}_M\}$, $\mathcal{F}_{\tilde{M}} = \{\tilde{g} : \tilde{M} \to \mathbb{R} \mid \tilde{g} \circ \pi = f,\ f \in \mathcal{F}_M\}$ and $\mathcal{F}_{\tilde{M}} \circ \mathcal{C}_{\tilde{M}} = \mathcal{F}_M \circ \mathcal{C}_M$, that is, $\tilde{g} \circ \tilde{c} = \tilde{g} \circ \pi \circ c = f \circ c \in C^\infty(\mathbb{R})$

Definition 2.8.3
The $\mathbb{F}$-space $(\tilde{M}, \mathcal{C}_{\tilde{M}}, \mathcal{F}_{\tilde{M}})$ is called an $\mathbb{F}$-quotient space of the $\mathbb{F}$-space M. The pair $(\mathcal{C}_{\tilde{M}}, \mathcal{F}_{\tilde{M}})$ is the final $\mathbb{F}$-quotient structure (quotient structure for short) in which π is a smooth map.

The injectivity of $\tilde{g}$ depends on the consistency of the relation $\sim$ with the smooth maps $f \in \mathcal{F}_M$: $[x] \neq [y]$ implies $f(x) \neq f(y)$ thus $\tilde{g}([x]) \neq \tilde{g}([y])$ or alternatively $\tilde{g}([x]) = \tilde{g}([y])$ reads f(x)=f(y). Thus, $f^{-1}(f(x)) = f^{-1}(f(y))$ yields $\{s \in M \mid f(s) = f(x)\} = \{t \in M \mid f(t) = f(y)\}$, that is, $[x] = [y]$. Without the consistency of $\sim$ with $f \in \mathcal{F}_M$, the inclusions $\pi \circ \mathcal{C}_M \subset \mathcal{C}_{\tilde{M}}$ and $\mathcal{F}_{\tilde{M}} \circ \pi \subset \mathcal{F}_M$ have to be strict. $[x] \in \tilde{M}$ if, and only if $\pi^{-1}([x]) = \{y \in M \mid f(y) = f(x)\}$ if, and only if $f(\pi^{-1}([x])) = \{f(x)\} = \{\tilde{g}([x])\}$ if, and only if $f \circ \pi^{-1} = \tilde{g}$.

Definition 2.8.4
The topologies $\tau_{\mathcal{F}_{\tilde{M}}}$ and $\tau_{\mathcal{C}_{\tilde{M}}}$ induced on $\tilde{M}$ by smooth functions and curves are called $\mathbb{F}$-quotient topologies. That is, the topologies making all smooth functions and smooth curves continuous the subbasis (respectively basis) of which are given by $\mathcal{S} = \{f^{-1}(0,1) \mid f \in \mathcal{F}_{\tilde{M}}\}$ (respectively $\mathcal{B} = \{f^{-1}(0,+\infty) \mid f \in \mathcal{F}_{\tilde{M}}\}$.)

As a topological quotient space of $(M, \tau_{\mathcal{F}_M})$, the $\mathbb{F}$-quotient space carries another topology, that is, the standard quotient topology.

Definition 2.8.5
The quotient topology on $\tilde{M}$ is the topology in which an open set in $\tilde{M}$ is a set of equivalence classes whose union of classes is an open set in M. Equivalently, we can say that V is an open set in $\tilde{M}$ if, and only if $\pi^{-1}(V) = \bigcup_{[x] \in V} [x] = \mathcal{U}$ lies in

$\tau_{\mathcal{F}_M}$, that is, when $\tau_\sim$ denotes the quotient topology on $\tilde{M}$ induced by $\tau_{\mathcal{F}_M}$, with $\tau_\sim = \{V \subset \tilde{M} \mid \pi^{-1}(V) = \mathcal{U} \in \tau_{\mathcal{F}_M}\}$ is the quotient topology of the $\mathbb{F}$-topological space M by the relation $\sim$.

The quotient topology is also called the standard quotient topology or the identification topology [27]. The identification topology is Hausdorff. For, let $\tilde{g} \in \mathcal{F}_{\tilde{M}}$ and let $[x] \neq [y]$. Thus, $\tilde{g}([x]) \neq \tilde{g}([y])$ since $\tilde{g}$ is injective. Hence $\tilde{g}$ separates points in $\tilde{M}$. Now, we recall some properties of the quotient topology $\tau_\sim$. Namely, $G \subset \tilde{M}$ is a $\tau_\sim$-closed set in $\tilde{M}$ if, and only if $\pi^{-1}(G) = F$ is a $\tau_{\mathcal{F}_M}$-closed set in M. For $\pi^{-1}(G) = \pi^{-1}(\tilde{M} - V) = \pi^{-1}(\tilde{M}) - \pi^{-1}(V) = M - \mathcal{U} = F$, where $V \in \tau_\sim$, $\mathcal{U} \in \tau_{\mathcal{F}_M}$. Also $\tau_\sim = \{\pi(\mathcal{U}) \subset \tilde{M} \mid \mathcal{U} \in \tau_{\mathcal{F}_M}\}$. For, $V \in \tau_\sim$, that is, $\pi^{-1}(V) = \mathcal{U}$. But, the surjectivity of π implies $V = \pi \circ \pi^{-1}(V) = \pi(\mathcal{U})$.

Lemma 2.8.1
The identification topology is the largest (finest) topology in $\tilde{M}$ for which π is continuous. So, $\tau_{\mathcal{F}_{\tilde{M}}} \subset \tau_\sim$.

Proof.
Let τ be another topology making π a continuous map on $\tilde{M}$. Let $V \in \tau$. It follows from the continuity of π, that $\pi^{-1}(V) \in \tau_{\mathcal{F}_M}$ that is $V \in \tau_\sim$. Hence $\tau \subset \tau_\sim$. And for $\tau = \tau_{\mathcal{F}_{\tilde{M}}}$, we get $\tau_{\mathcal{F}_{\tilde{M}}} \subset \tau_\sim$. □

Lemma 2.8.2
Let $\pi : M \to \tilde{M}$ be the canonical projection. Then π is open (closed) map from $\tau_{\mathcal{F}_M}$ to $\tau_\sim$.

Proof.
Assume $\pi^{-1}\pi\mathcal{U} \supset \mathcal{U}$ and $\mathcal{U}$ a $\tau_{\mathcal{F}_M}$-open set in M, where $\mathcal{U} = \bigcup_{j\in J} f_j^{-1}(0, +\infty)$. It follows that $\pi^{-1}(\pi\mathcal{U}) = \pi^{-1}[\bigcup_{j\in J}(\pi \circ f_j^{-1}(0, +\infty))] = \pi^{-1}[\bigcup_{j\in J}(f_j^{-1} \circ \pi^{-1})^{-1}(0, +\infty)]$. Thus, these equalities become $\pi^{-1}(\pi\mathcal{U}) = \pi^{-1}[\bigcup_{j\in J} \tilde{g}_j^{-1}(0, +\infty)] = [\bigcup_{j\in J}(\tilde{g}_j\pi)^{-1}(0, +\infty)] = [\bigcup_{j\in J} h_j^{-1}(0, +\infty) \in \tau_{\mathcal{F}_M}]$ with the formula $f_j \circ \pi^{-1} = \tilde{g}_j$ and $\tilde{g}_j \circ \pi = h_j$. Therefore, $\pi\mathcal{U} \in \tau_\sim$ by definition. Therefore π is an open map with respect to $\tau_{\mathcal{F}_M}$ and $\tau_\sim$. The proof is similar for a closed set. □

Lemma 2.8.3
Let $\pi : M \to \tilde{M}$ be the canonical projection. Let $\tilde{g} \in \mathcal{F}_{\tilde{M}}$ such that $\tilde{g} \circ \pi = f \in \mathcal{F}_M$. Then $\tilde{g}$ is open (closed) map from $\tau_\sim$ to $\tau_{\mathcal{F}_{\mathbb{R}}}$ if, and only if $f(\mathcal{U})$ is open (closed) set for each open (closed) set $\mathcal{U} = \pi^{-1}\pi\mathcal{U}$. Let us say that $\mathcal{U}$ is π-satured.

Proof.
"$\Longrightarrow$" Let $\tilde{g}$ be an open map with respect to $\tau_\sim$ and $\tau_{\mathcal{F}_{\mathbb{R}}}$, that is, $\tilde{g}(V) \in \tau_{\mathbb{R}}$ for any

$V \in \tau_\sim$. Hence $\pi^{-1}(V) = \mathcal{U}$ is a $\tau_{\mathcal{F}_M}$-open set in M by definition of $\tau_\sim$. Applying π to both sides yields naturally $V = \pi(\mathcal{U})$ by a set theoretic property of the surjective map π. Thus $\mathcal{U} = \pi^{-1}(V) = \pi^{-1}\pi\mathcal{U}$. It follows that $\tilde{g}(V) = \tilde{g}\pi(\mathcal{U}) = f(\mathcal{U})$ is an open set in $\tau_{\mathcal{F}_\mathbb{R}}$ such that $\mathcal{U} = \pi^{-1}\pi\mathcal{U}$ and $f \in \mathcal{F}_M$.
"$\Longleftarrow$" Let $f(\mathcal{U}) \in \tau_{\mathcal{F}_\mathbb{R}}$. That is, $f(\mathcal{U}) = f(\pi^{-1}\pi\mathcal{U}) = (f\pi^{-1})(\pi\mathcal{U}) = \tilde{g}(\pi\mathcal{U})$, with $\mathcal{U} = \pi^{-1}\pi\mathcal{U}$. Let $V \in \tau_\sim$. By the definition of $\tau_\sim$ and the surjectivity of π, it follows that $\pi^{-1}(V) = \mathcal{U} \in \tau_{\mathcal{F}_M}$ if, and only if $V = \pi(\mathcal{U})$. Therefore, $f(\mathcal{U}) = \tilde{g}(V)$ is a $\tau_{\mathcal{F}_\mathbb{R}}$-open set, with V any $\tau_\sim$-open set in $\tilde{M}$. Hence $\tilde{g}$ is an open map. It is no difficult to prove the closeness of π. □

Corollary 2.8.1
Let $\tau_{\mathcal{F}_{\tilde{M}}}$ and $\tau_{\mathcal{F}_M}$ be defined as usual. Then $\mathcal{B} = \{\pi\mathcal{U} \mid \mathcal{U} \in \tau_{\mathcal{F}_M}\}$ is a basis for $\tau_\sim$ and $\mathcal{B} = \{\pi(f^{-1}(0,+\infty)) \mid f \in \mathcal{F}_M\}$ is a basis for $\tau_{\mathcal{F}_{\tilde{M}}}$.

Proof.
Let $V \in \tau_\sim$. That is, $V = \pi\mathcal{U}$ with $\mathcal{U} \in \tau_{\mathcal{F}_M}$ by definition of $\tau_\sim$ and Lemma 2.8.3. Thus $\mathcal{B} = \tau_\sim$ is the trivial basis. Hence, with respect to the formula $f \circ \pi^{-1} = \tilde{g}$, we have $\pi(f^{-1}(0,+\infty)) = (f \circ \pi^{-1})^{-1}(0,+\infty) = \tilde{g}^{-1}(0,+\infty)$. Therefore, $\mathcal{B}$ is the standard basis of the $\mathbb{F}$-space $\tilde{M}$. □

In what follows, we are going to compare $\mathbb{F}$-topologies and $\tau\sim$ on $\tilde{M}$. In Lemma 2.8.1 was proven $\tau_{\mathcal{F}_{\tilde{M}}} \subset \tau_{\mathcal{C}_{\tilde{M}}} \subset \tau\sim$. Nevertheless we want to provide the proof in $\mathbb{F}$-space setting.

Proposition 2.8.1
Given the three topologies defined on $\tilde{M}$. Then $\tau_{\mathcal{F}_{\tilde{M}}} = \tau_{\mathcal{C}_{\tilde{M}}} = \tau\sim$.

Proof.
Let $V \in \tau_{\mathcal{F}_{\tilde{M}}}$. Hence $V = \bigcup \tilde{g}^{-1}(0,+\infty)$ with $\tilde{g} \in \mathcal{F}_{\tilde{M}}$ since $\tilde{g}^{-1}(0,+\infty)$ are basic open set for all $\tilde{g} \in \mathcal{F}_{\tilde{M}}$. Applying π^{-1} to both sides yields $\pi^{-1}(V) = \bigcup_{\tilde{g} \in \mathcal{F}_{\tilde{M}}} \pi^{-1}\tilde{g}^{-1}(0,+\infty) = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0,+\infty)$ with $f = \tilde{g} \circ \pi$. Thus, $\pi^{-1}(V) \in \tau_{\mathcal{F}_M}$. Therefore $V \in \tau_\sim$. So $\tau_{\mathcal{F}_{\tilde{M}}} \subset \tau_{\mathcal{C}_{\tilde{M}}}$. Now, let $V \in \tau_\sim$, that is, $\pi^{-1}(V) \in \tau_{\mathcal{F}_M}$. Thus, $\pi^{-1}(V) = \bigcup_{f \in \mathcal{F}_M} f^{-1}(0,+\infty)$. The composition with π in both sides yields

$$V = \pi\pi^{-1}V = \bigcup_{f \in \mathcal{F}_M} \pi f^{-1}(0,+\infty) = \bigcup_{f \in \mathcal{F}_M} (f\pi^{-1})^{-1}(0,+\infty) = \bigcup_{\tilde{g} \in \mathcal{F}_{\tilde{M}}} \tilde{g}^{-1}(0,+\infty),$$

with respect to the surjectivity of π and the formula $f\pi^{-1} = \tilde{g}$. Therefore $V \in \tau_{\mathcal{F}_{\tilde{M}}}$. Eventually, $\tau_\sim \subset \tau_{\mathcal{F}_{\tilde{M}}}$ and the equality $\tau_{\mathcal{F}_{\tilde{M}}} = \tau_{\mathcal{C}_{\tilde{M}}} = \tau\sim$ holds. □

In the sequel, because the three topologies coincide, we will freely use one of the three symbols $\tau\sim$ or $\tau_{\mathcal{F}_{\tilde{M}}}$ or $\tau_{\mathcal{C}_{\tilde{M}}}$ indiscriminately. In the same way we may make use of basis, or subbasis construction with respect of the topology chosen. It is important to study the topologies on $\mathbb{F}$-subspaces S of the $\mathbb{F}$-quotient space $\tilde{M}$.

Definition 2.8.6
Let S be a $\mathbb{F}$-subspace of $\tilde{M}$, $\pi : M \to \tilde{M}$ be the canonical projection and $\iota : S \to \tilde{M}$ be the natural inclusion. The trace topology on S is the smallest topology for which the inclusion is continuous. It is defined by $\tau_{\sim}(S) = \{\mathcal{U} \cap S \mid \mathcal{U} \in \tau_{\sim}\} = \tau_{\mathcal{F}_{\tilde{M}}}(S) = \tau_{\mathcal{C}_{\tilde{M}}}(S)$. The identification topology on S is the one determined by the surjection $\pi : \pi^{-1}(S) \to S$. That is, the largest topology for which π is continuous. It is defined by $\tau_{\sim S} = \{V \subset S \mid \pi^{-1}(V) \in \tau_{\mathcal{F}_M}(\pi^{-1}(S))\} = \tau_{\mathcal{F}_S} = \tau_{\mathcal{C}_S}$.

Lemma 2.8.4
Let S be a $\mathbb{F}$-subspace of $\tilde{M}$ endowed with the two topologies defined above. Then $\tau_{\sim}(S) \subset \tau_{\sim S}$.

Proof.
Let $V \in \tau_{\sim}(S)$, that is, $V = S \cap \mathcal{U} \subset S$ whenever $\mathcal{U} \in \tau_{\sim} = \tau_{\mathcal{F}_{\tilde{M}}} = \tau_{\mathcal{C}_{\tilde{M}}}$. But $\mathcal{U} = \bigcup_{\tilde{g} \in \mathcal{F}_{\tilde{M}}} \tilde{g}^{-1}(0, +\infty)$ for some $\tilde{g} \in \mathcal{F}_{\tilde{M}}$. Thus $V = S \cap [\bigcup_{\tilde{g} \in \mathcal{F}_{\tilde{M}}} \tilde{g}^{-1}(0, +\infty)] = \bigcup_{\tilde{g} \in \mathcal{F}_{\tilde{M}}} [S \cap \tilde{g}^{-1}(0, +\infty)]$. It follows that $\pi^{-1}(V) = \bigcup_{\tilde{g} \in \mathcal{F}_{\tilde{M}}} [\pi^{-1}(S) \cap \pi^{-1}\tilde{g}^{-1}(0, +\infty)] = \bigcup_{f \in \mathcal{F}_M} [\pi^{-1}(S) \cap f^{-1}(0, +\infty)] = \pi^{-1}(S) \cap [\bigcup_{f \in \mathcal{F}_M} f^{-1}(0, +\infty)]$ with $\tilde{g} \circ \pi = f$. Therefore, we have $\pi^{-1}(V) \in \tau_{\mathcal{F}_M}(\pi^{-1}(S)) \subset \tau_{\mathcal{F}_{\pi^{-1}(S)}}$, $\pi^{-1}(V) \in \tau_{\mathcal{F}_{\pi^{-1}(S)}}$ and $\pi^{-1}(V) = \bigcup h^{-1}(0, +\infty)$, where $h \in \mathcal{F}_{\pi^{-1}(S)}$ such that $\tilde{h} \circ \pi = h$ and $\tilde{h} = h \circ \pi^{-1}$ is injective. Now, $V = \pi\pi^{-1}(V) = \bigcup \pi h^{-1}(0, +\infty) = \bigcup (h\pi^{-1})^{-1}(0, +\infty) = \bigcup \tilde{h}^{-1}(0, +\infty)$ with $h \in \mathcal{F}_{\pi^{-1}(S)}$ and $\tilde{h} \in \mathcal{F}_S$. Therefore $V \in \tau_{\sim S} = \tau_{\mathcal{F}_S} = \tau_{\mathcal{C}_S}$. Hence $\tau_{\sim}(S) \subset \tau_{\sim S}$. $\square$

Lemma 2.8.5
Let $\pi : M \to \tilde{M}$ be the canonical projection and S an $\mathbb{F}$-subspace of $\tilde{M}$ such that S is open (closed) set in $\tilde{M}$ for $\tau_{\sim} = \tau_{\mathcal{F}_{\tilde{M}}} = \tau_{\mathcal{C}_{\tilde{M}}}$. Then $\tau_{\sim}(S) = \tau_{\sim S}$

Proof.
From Lemma 2.8.4, Proposition 2.8.1 and Definition 2.8.6 we have $\tau_{\sim}(S) = \tau_{\mathcal{F}_{\tilde{M}}}(S) \subset \tau_{\mathcal{F}_S} = \tau_{\sim S}$. Since S is a $\tau_{\mathcal{F}_{\tilde{M}}}$-open set in $\tilde{M}$, the inverse inclusion holds. Thus, $\tau_{\sim}(S) = \tau_{\mathcal{F}_{\tilde{M}}}(S) = \tau_{\mathcal{F}_S} = \tau_{\sim S}$. It is worth noticing that the proof for the closeness is to be done by analogy to the open case above by setting $F = \tilde{M} - S$, where S is a $\tau_{\sim}$-open set and F a $\tau_{\sim}$-closed set in $\tilde{M}$, or π is an open map with respect to $\tau_{\mathcal{F}_M}$ and $\tau_{\sim}$. $\square$

It seems that the condition for S to be open set is redundant since π is always an open map in our setting that is with respect to the consistency of $\sim$ with the $\mathbb{F}$-structure. For a general equivalence relation $\sim$, there are conditions for π to be an open map. That is π is an open map with respect to $\tau_{\mathcal{F}_M}$ and $\tau_{\sim}$ if, and only if $\mathcal{U}$ and $\pi\pi^{-1}\mathcal{U}$ belong to $\tau_{\mathcal{F}_M}$ if, and only if π sends a standard $\tau_{\mathcal{F}_M}$-basic-open set in M to a $\tau_{\sim}$-open set in $\tilde{M}$.

Definition 2.8.7
Let $\pi^{-1}(S)\subset M$ *and* $S\subset\tilde{M}$ *be* $\mathbb{F}$*-subspaces. Let* $\bar{\tilde{\sim}}$ *be an equivalence relation on* $\pi^{-1}(S)$ *defined by* $x\bar{\tilde{\sim}}y$ *if, and only if* $x\sim y$ *and* $[x]_\sim=[x]_{\bar{\tilde{\sim}}}\in S$*. The relation* $\bar{\tilde{\sim}}$ *on* $\pi^{-1}(S)$ *is called the equivalence relation induced by* $\sim$*.*

Here is how we can understand the equivalence relation $\bar{\tilde{\sim}}$. $S=\{[x]_\sim \mid [x]_\sim\in\tilde{M}\}$ and $\tilde{M}=\{[x]_\sim \mid x\in M\}$ $\pi^{-1}(S)=\{x\in M \mid [x]_\sim\in S\}\subset M$ $S=\pi\pi^{-1}(S)=\{[x]_\sim \mid x\in\pi^{-1}(S)\}$

Lemma 2.8.6
Let $\pi: M\to\tilde{M}$ *be the canonical projection and* $S\subset\tilde{M}$ *be either open or closed set. Then* S *is diffeomorphic to the space* $\pi^{-\bar{\tilde{1}}}(S):=\pi^{-1}(S)_{/\bar{\tilde{\sim}}}$*, where* $\bar{\tilde{\sim}}$ *is the relation given in Definition* 2.8.7*.*

Proof.
Since S is an open set, we have $\mathcal{F}_S=\mathcal{F}_{\tilde{M}|S}$ and $\pi^{-1}(S)$ is an open set in M from the definition of $\tau_\sim$ on $\tilde{M}$. Thus $\mathcal{F}_{\pi^{-1}(S)}=\mathcal{F}_{\tilde{M}|\pi^{-1}(S)}$. Also $\tau_\sim(S)=\tau_{\sim S}$ and $\tau_{\mathcal{F}_M}(\pi^{-1}(S))=\tau_{\mathcal{F}_{\pi^{-1}(S)}}$. Thus, $\bar{\tilde{\sim}}$ and $\sim$ on $\pi^{-1}(S)$ are consistent with $q\in C^\infty(\pi^{-1}(S),\pi^{-1}(S)_{/\bar{\tilde{\sim}}})$, and $s=\pi_{|\pi^{-1}(S)}\in C^\infty(\pi^{-1}(S),S)$, respectively the canonical projection and the canonical projection restricted to $\pi^{-1}(S)$. Then there exists a unique $\tilde{g}:S\to\pi^{-1}(S)$ such that $q=\tilde{g}s$. Also, there exists a unique $\tilde{h}:\pi^{-1}(S)\to S$ such that $s=\tilde{h}q$. Recall that $\tilde{g}$ and $\tilde{h}$ are smooth injections. Now $x\sim y$ if, and only if $q(x)=q(y)$ if, and only if $\tilde{g}s(x)=\tilde{g}s(y)$ if, and only if $\tilde{g}[x]_\sim=\tilde{g}[y]_{\underset{\sim}{}}$ and $x\bar{\tilde{\sim}}y$ if, and only if $s(x)=s(y)$ if, and only if $\bar{\tilde{h}}q(x)=\bar{\tilde{h}}q(y)$ if, and only if $\bar{\tilde{h}}[x]_{\bar{\tilde{\sim}}}=\bar{\tilde{g}}[y]_{\bar{\tilde{\sim}}}$. Thus, $\tilde{g}^{-1}=(qs^{-1})^{-1}=sq^{-1}=\bar{\tilde{h}}$, that is, $\tilde{g}$ and $\bar{\tilde{h}}=\tilde{g}^{-1}$ are smooth bijections. Hence $\tilde{g}$ is a diffeomorphism and so S is diffeomorphic to $\pi^{-\bar{\tilde{1}}}(S):=\pi^{-1}(S)_{/\bar{\tilde{\sim}}}$. $\square$

Chapter 3

Pseudomanifolds

3.1 $\mathbb{F}$-spaces locally diffeomorphic to $\mathbb{R}^n$

Definition 3.1.1
A Frölicher space M is called a pseudomanifold if M is locally diffeomorphic to $\mathbb{R}^n$ endowed with its canonical $\mathbb{F}$-structure, that is, there is an open cover $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of M such that for every $x\in M$, there exist an $\tau_{\mathcal{F}_M}$-open neighborhood $\mathcal{U}$ of x and an $\mathbb{F}$-diffeomorphism φ of $\mathcal{U}$ onto the $\mathbb{F}$-subspace $V:=\varphi(\mathcal{U})\subseteq\mathbb{R}^n$, $\varphi:\mathcal{U}\tilde{\rightarrow}\varphi(\mathcal{U})$, say.

In the definition above the subspace $\varphi(\mathcal{U})\subseteq\mathbb{R}^n$ can be either open or closed, or neither open nor closed $\mathbb{F}$-subspace of $\mathbb{R}^n$.

Example 3.1.1
$\mathbb{R}^n$ is a pseudomanifold.

1. *$(\mathbb{R}^n,\mathcal{C},\mathcal{F})$ is $\mathbb{F}$-space, where $(\mathcal{C},\mathcal{F})$ is the canonical $\mathbb{F}$-structure given by all C^∞ real valued functions and curves. It follows from Boman's theorem [8] that smooth functions in the smooth n-manifold $\mathbb{R}^n$ coincide with $\mathbb{F}$-smooth functions. In the sequel, smooth curves and smooth functions when $\mathbb{R}^n$ is a smooth manifold coincide with smooth curves and smooth functions when $\mathbb{R}^n$ is an $\mathbb{F}$-space.*

2. *Let $\mathbb{R}$ also having the canonical $\mathbb{F}$-structure. Let $\mathcal{U}\subset\mathbb{R}^n$ be an open set and $(\mathcal{C}_\mathcal{U},\mathcal{F}_\mathcal{U})$ be the $\mathbb{F}$-structure induced on $\mathcal{U}$ by maps $f_i:\mathcal{U}\rightarrow\mathbb{R}$, where $(i=1,\ldots,n)$. Assume that the map $\varphi:\mathcal{U}\rightarrow\mathbb{R}^n$ is a one-to-one map, given by $\varphi(x)=(f_1(x),f_2(x),\ldots,f_n(x))$. Hence, $\varphi=(f_1,\ldots,f_n)$ is a diffeomorphism onto the $\mathbb{F}$-subspace $\varphi(\mathcal{U})$ of $\mathbb{R}^n$ such that $\mathcal{F}_{o\mathbb{R}}\circ\iota_\mathcal{U}=\mathcal{F}_{o\mathcal{U}}$ and $\mathcal{F}_{o\mathbb{R}}$ contains f separating points. Thus, $f\circ\iota_\mathcal{U}$ is a separating point function on $\mathcal{U}$. From [62, p.80, Corollary 1.2] the set $\{f_1,\ldots,f_n\}$ is a generating set for $(\mathcal{C}_\mathcal{U},\mathcal{F}_\mathcal{U})$ and φ is a diffeomorphism onto the $\mathbb{F}$-subspace $\varphi(\mathcal{U})$ of $\mathbb{R}^n$, where $f_i=\pi_{i_{|\varphi(\mathcal{U})}}\circ\varphi$. Therefore, $\varphi:\mathcal{U}\rightarrow\varphi(\mathcal{U})$ is an $\mathbb{F}$-diffeomorphism. That is, $\varphi(\mathcal{U})$ is a $\mathbb{R}^n$-open set.*

3. *$\mathbb{R}^n$ is a pseudomanifold locally diffeomorphic to an open $\mathbb{F}$-subspace of $\mathbb{R}^n$. That is, $\varphi : \mathcal{U} \to \mathbb{R}^n$, where $\varphi(\mathcal{U}) = \mathcal{U}$ and $\varphi = id_{\mathbb{R}^n|\mathcal{U}}$.*

Example 3.1.2
$\varphi(\mathcal{U})$ neither open nor closed set.
Let $f_1 = sin : (0, \frac{\pi}{2}) \tilde{\to} (0,1)$, $f_2 : (0, \frac{\pi}{2}) \to \{0\}$ and $\varphi = (f_1, f_2) : (0, \frac{\pi}{2}) \tilde{\to} (0,1) \times \{0\} \subset \mathbb{R}^2$. It is known that $\{(x,0) \mid 0 \leq x \leq 1\} = [0,1] \times \{0\}$ is closed in $\mathbb{R}^2$ as a product of closed sets, whereas $\{(x,0) \mid 0 < x < 1\} = (0,1) \times \{0\}$ is neither closed nor open set. Therefore, its complement in $\mathbb{R}^2$ is computed by

$$\begin{aligned}
\complement[(0,1) \times \{0\}] &= \mathbb{R}^2 - (0,1) \times \{0\} \\
&= \{(x,y) \in \mathbb{R}^2 \mid x \notin (0,1) \ or \ y \neq 0\} \\
&= \{(x,y) \in \mathbb{R}^2 \mid x \in (-\infty, 0] \cup [1, +\infty) \ or \ y \in \mathbb{R}^*\} \\
&= \{(x,y) \in \mathbb{R}^2 \mid x \notin (0,1) \ or \ y \in \mathbb{R}\} \cup \{(x,y) \mid y \neq 0, \ x \in \mathbb{R}\} \\
&= ((-\infty, 0] \cup [1, +\infty)) \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}^*
\end{aligned}$$

This is a union of an open set $\mathbb{R} \times \mathbb{R}^* = \mathbb{R}^2 - \{(x,y) \mid y = 0\}$ and a closed set $[(-\infty, 0] \cup [1, +\infty)] \times \mathbb{R}$.

Following these examples, we note that Frölicher pseudomanifolds can be classified in three types.

Definition 3.1.2
An $\mathbb{F}$-pseudomanifold M is said to be of first kind if M is locally diffeomorphic to open $\mathbb{F}$-subspaces F_i of $\mathbb{R}^n$, of the second kind if F_i are closed and not all of constant dimension n, of third kind if F_i are closed and of constant maximal dimension with nonempty interior.

The rest of this work is devoted to $\mathbb{F}$-pseudomanifolds of constant dimension and of the first kind. From now on, we will restrict ourselves to pseudomanifolds of the first kind. We will say fort short, "pseudomanifolds of dimension n" or indiscriminately "n-pseudomanifold", that is pseudomanifolds of constant maximal dimension. The maximal dimension is related to subsets of $\mathbb{R}^n$ with non-empty interior. In the context of n-pseudomanifold, the dimension is one of all minimal submanifolds in $\mathbb{R}^n$, each of them containing an open neighborhood V of $\varphi(p)$, for each $p \in \mathcal{U} \subset M$, with $\mathcal{U}$ an open neighborhood and φ a local diffeomorphism from $\mathcal{U}$ to V. So the object of interest in practice should be $\varphi(\mathcal{U})$. The first kind is new concept, while the third was first studied by T.A Batubenge under the denomination of pseudomanifolds in [6]. There should exist a transfer of local features from $\mathbb{R}^n$ back to the $\mathbb{F}$-space M by a local diffeomorphism. The examples we shall deal with in the next Section lie in either finitely generated structures or graphs of smooth maps, or minimal submanifolds, as stated above. An n-pseudomanifold looks like $\mathbb{R}^n$ at two levels: $\mathbb{F}$-structure and $\mathbb{F}$-topology.

Lemma 3.1.1
Each n-pseudomanifold is locally finitely generated by n functions.

Proof.
The $\mathbb{F}$-substructure on an open set $\mathcal{U}$ in an $\mathbb{F}$-space M is the restriction of the $\mathbb{F}$-structure of$M$ to $\mathcal{U}$. Then the lemma holds from [62, *p*.80, Corollary 1.2]. $\square$

Example 3.1.3
$(\mathbb{R}^n, C^\infty(\mathbb{R},\mathbb{R}^n), C^\infty(\mathbb{R}^n,\mathbb{R}))$ is a natural model of pseudomanifolds of the first kind.

Example 3.1.4
A n-dimensional smooth manifold is an example of n-pseudomanifold of the first kind.

Example 3.1.5
Let $M := (0, 2\pi)$. With $\mathcal{U} = (0, \pi)$, $V = (\frac{\pi}{2}, \frac{3\pi}{2})$ and $W = (\pi, 2\pi)$, it is clear that $\mathcal{U} \cup V \cup W = (0, 2\pi)$. Let $f_1 := cos : (0, 2\pi) \to \mathbb{R}$ and $f_2 := exp : (0, 2\pi) \to \mathbb{R}$. Thus $\varphi = (f_1, f_2) : (0, 2\pi) \to \mathbb{R}^2$ is smooth and one-to-one since exp separates points in $(0, 2\pi)$. It follows that $\varphi(x) = \varphi(y)$ yields $(cos\, x, e^x) = (cos\, y, e^y)$. It follows from $x = \pm y + 2k\pi$ and $x = y$ that k must be 0 since $M = (0, 2\pi)$. Also $x, y > 0$. Since $x, y \in M$, so, $\varphi(0, 2\pi) = (-1, 1) \times (1, e^{2\pi})$ is an open set as finite product of open sets. Therefore, $(0, 2\pi) \simeq (-1, 1) \times (1, e^{2\pi})$. Hence, $\varphi_1 = \varphi_{|\mathcal{U}}$, $\varphi_2 = \varphi_{|V}$, and $\varphi_3 = \varphi_{|W}$ are local diffeomorphisms. In conclusion, dim $(0, 2\pi) =$ dim $(-1, 1)\times(1, e^{2\pi}) = 2$ for the $\mathbb{F}$-structure generated by $\{f_1, f_2\}$. But, dim $(0, 2\pi) = 1$ in the canonical $\mathbb{F}$-structure induced from $\mathbb{R}$.

Example 3.1.6
Let $N := (0, 2\pi)$ and $\varphi = (cos, sin)$ given by $\varphi(x) = (cos\, x, sin\, x)$ for all $x \in N$. It follows from the definition of φ that

$$\begin{array}{lll} \varphi(0, \frac{\pi}{2}] & = \mathcal{U}_1 & = [0, 1) \times (0, 1] = [0, 1] \times [0, 1] - \{(1, 0)\} \\ \varphi[\frac{\pi}{2}, \pi] & = \mathcal{U}_2 & = [-1, 0] \times [0, 1] \\ \varphi[\pi, \frac{3\pi}{2}] & = \mathcal{U}_3 & = [-1, 0] \times [-1, 0] \\ \varphi[\frac{3\pi}{2}, 2\pi) & = \mathcal{U}_4 & = [0, 1) \times [-1, 0) = [0, 1] \times [-1, 0] - \{(1, 0)\}. \end{array}$$

Thus, $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4 = S^1 - \{(1, 0)\} = \varphi(0, 2\pi)$ is $\mathbb{R}^2$-open set since $\{(0, 1)\}$ is $\mathbb{R}^2$-closed set. We want to define $\psi = \varphi \times id_{\mathbb{R}}$, that is, $\psi(x, y) = (\varphi(x), id_{\mathbb{R}}(y))$ such that

$$\begin{array}{rll} \psi : (0, 2\pi) \times \mathbb{R} \to & \varphi(0, 2\pi) \times \mathbb{R} & \hookrightarrow \mathbb{R}^3 \\ (x, y) \mapsto & (\varphi(x), y) & \mapsto (cos\, x, sin\, x, y) \end{array}$$

or equivalently $\psi : (0, 2\pi) \times \mathbb{R} \to (S^1 - \{(1, 0)\}) \times \mathbb{R} \hookrightarrow \mathbb{R}^3$. *Thus,*

$$\begin{array}{rcl} \varphi(0, 2\pi) & = & (S^1 - \{(1, 0)\}) \times \mathbb{R} \\ & = & \{(cos\, x, sin\, x, y) \mid 0 \le x \le 2\pi,\ y \in \mathbb{R}\} - \{(1, 0, y) \mid y \in \mathbb{R}\} \\ & = & \{(cos\, x, sin\, x, y) \mid 0 < x < 2\pi,\ y \in \mathbb{R}\}. \end{array}$$

Hence, $\underbrace{(S^1 - \{(1, 0)\}) \times \mathbb{R}} = S^1 \times \mathbb{R} - \underbrace{\{(1, 0)\} \times \mathbb{R}}$ *is an open set.*

product of open sets *product of closed sets*

We would like to retrace here the theoretical foundation of such a ψ:

$$\begin{array}{ccc} (0,2\pi)\times\mathbb{R} & \xrightarrow{\pi_{1|(0,2\pi)\times\mathbb{R}}} & (0,2\pi) \\ & {\scriptstyle f_1\circ\pi_{1|(0,2\pi)\times\mathbb{R}}}\searrow \quad \swarrow{\scriptstyle f_1 = cos} & \\ & [-1,1) = f_1(0,2\pi) = \{cos\, x \mid x\in(0,2\pi)\}, & \end{array}$$

that is $\hat{\pi}_1 = \pi_{1|(0,2\pi)\times\mathbb{R}}$

$$\begin{array}{ccc} (0,2\pi)\times\mathbb{R} & \xrightarrow{\pi_{1|(0,2\pi)\times\mathbb{R}}} & (0,2\pi) \\ & {\scriptstyle f_2\circ\pi_{1|(0,2\pi)\times\mathbb{R}}}\searrow \quad \swarrow{\scriptstyle f_2 = sin} & \\ & [-1,1) = f_2(0,2\pi) = \{sin\, x \mid x\in(0,2\pi)\}, & \end{array}$$

that is $\hat{\pi}_1 = \pi_{1|(0,2\pi)\times\mathbb{R}}$

$$\begin{array}{ccc} (0,2\pi)\times\mathbb{R} & \xrightarrow{\pi_{2|(0,2\pi)\times\mathbb{R}}} & \mathbb{R} \\ & {\scriptstyle id_{\mathbb{R}}\circ\pi_{2|(0,2\pi)\times\mathbb{R}},}\searrow \quad \swarrow{\scriptstyle id_{\mathbb{R}}} & \\ & \mathbb{R} & \end{array}$$

that is $\hat{\pi}_2 = \pi_{2|(0,2\pi)\times\mathbb{R}}$. *Let* $\psi := (f_1\circ\hat{\pi}_1, f_2\circ\hat{\pi}_1, id_{\mathbb{R}}\circ\hat{\pi}_2)$ *and* ψ *is one-to-one since one of its component separates points, that is,*

$$id_{\mathbb{R}}\circ\hat{\pi}_1(x_1,y_1) = id_{\mathbb{R}}\circ\hat{\pi}_1(x_2,y_2) \Rightarrow id_{\mathbb{R}}(y_1) = id_{\mathbb{R}}(y_2) \Rightarrow y_1 = y_2. \quad Thus,$$

$$\begin{aligned} \psi(x,y) &= (f_1\circ\hat{\pi}_1(x,y), f_2\circ\hat{\pi}_1(x,y), id_{\mathbb{R}}\circ\hat{\pi}_2(x,y)) \\ &= (f_1(x), f_2(x), id_{\mathbb{R}}(y)) \\ &= (cos\, x, sin\, x, y). \end{aligned}$$

Making use of the parity and symmetries of cos and sin functions we would say: if $x_1 \neq x_2$ *and are symmetric arcs with respect to x-axis (y-axis) then* $cos\, x_1 = cos\, x_2$ ($sin\, x_1 = sin\, x_2$) *and* $sin\, x_1$ ($cos\, x_1$) *is opposed to* $sin\, x_2$ ($cos\, x_2$). *Hence* $\psi(x_1,y_1) \neq \psi(x_2,y_2)$ *whenever* $x_1 \neq x_2$. *Therefore* ψ *is one-to-one and*

$(0,2\pi)\times\mathbb{R}\simeq\psi((0,2\pi)\times\mathbb{R})$:

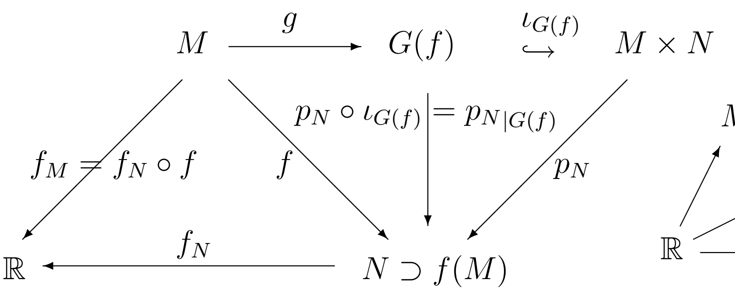
$$\begin{aligned}(0,2\pi)\times\mathbb{R} &\simeq \psi(0,2\pi)\times\mathbb{R}\\ &\simeq f_1(0,2\pi)\times f_2(0,2\pi)\times\mathbb{R}\\ &\simeq \{(\cos x,\sin x)\mid x\in(0,2\pi)\}\times\mathbb{R}\\ &\simeq [\{(s,t)\in\mathbb{R}^2\mid s^2+t^2=1\}-\{(1,0)\}]\times\mathbb{R}\\ &\simeq S^1\times\mathbb{R}-\{(1,0)\}\times\mathbb{R}\\ &\simeq [S^1-\{(1,0)\}]\times\mathbb{R}\end{aligned}$$

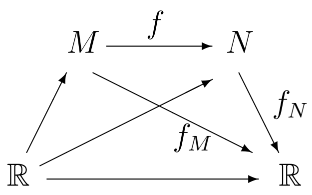
$dim\,(0,2\pi)\times\mathbb{R}=dim\,(S^1-\{(1,0)\})+dim\,\mathbb{R}=2$

3.2 Graph of smooth maps

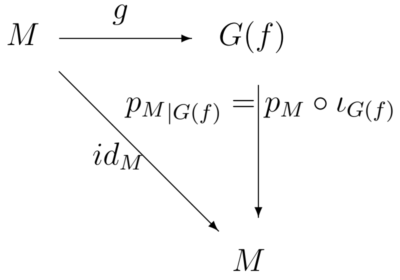
Lemma 3.2.1
Let $f:M\to N$ and $g:M\to G(f)$ such that $x\mapsto g(x)=(x,f(x))$ be set maps, where M, N are $\mathbb{F}$-spaces, and $G(f)=\{(x,f(x))\mid x\in M\}\subset M\times N$ is an $\mathbb{F}$-subspace of $M\times N$. Then f is smooth if, and only if g is diffeomorphism.

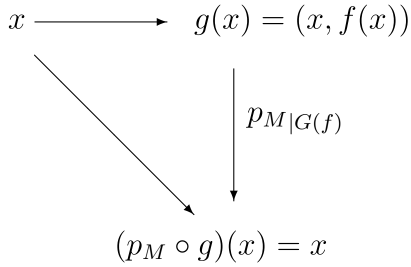
Proof.
"$\Longrightarrow$" Let f be a smooth map. That is f smooth if, and only if $\mathcal{F}_N\circ f\subset\mathcal{F}_M$ that is to say $f_N\circ f\in\mathcal{F}_M$. But, the following diagrams tell us more about the sequel of proof:



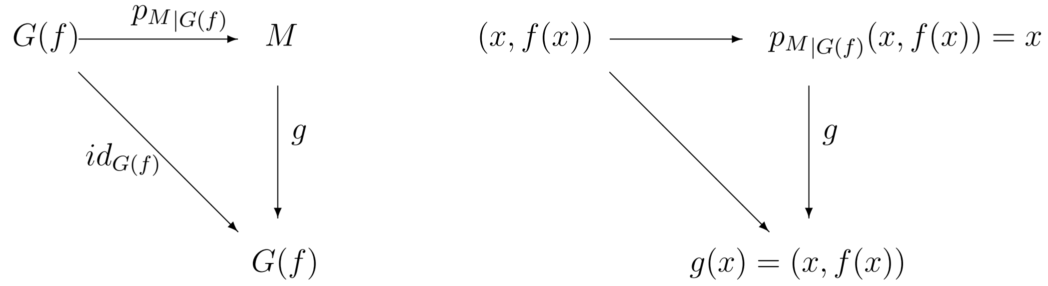


In fact $f=p_N\circ\iota_{G(f)}\circ g$ is smooth by assumption. And $f=p_{N|G(f)}\circ g$ with $p_{N|G(f)}$ smooth. Thus g is smooth with respect to Corollary 2.3.2. Now we have to show that g is a diffeomorphism.





Also, $p_{M|G(f)}$ is obviously a smooth bijective map from $p_{M|G(f)}\circ g=id_M$ and $g\circ p_{M|G(f)}=id_{G(f)}$. Therefore $g=p_{M|G(f)}^{-1}$ and $g^{-1}=p_{M|G(f)}$ are smooth bijective maps. Hence g is a diffeomorphism and $M \simeq G(f)$, such that the diagram below is commutative.



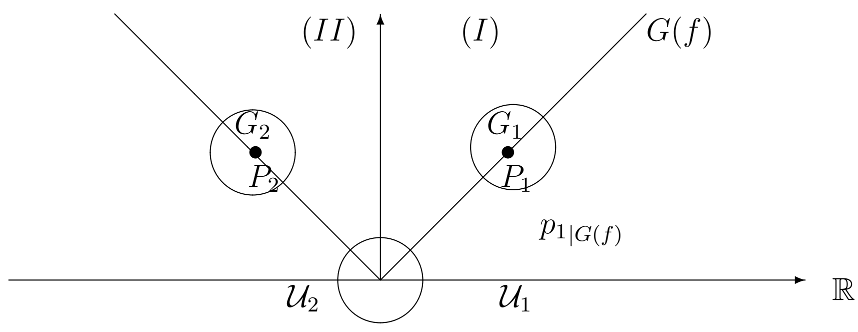
"$\Longleftarrow$" Let g be a diffeomorphism. So g is smooth. Hence $p_{N|G(f)} \circ g = f$ is smooth since it is a composition of smooth maps. □

Corollary 3.2.1
Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth map. Then $g : \mathbb{R} \to G(f) \subset \mathbb{R} \times \mathbb{R}$ defined as in Lemma 3.2.1 is a diffeomorphism, that is $\mathbb{R} \simeq G(f)$.

The series of examples below makes use of the diffeomorphism built on a graph of a smooth map.

Example 3.2.1
Let $f(x)=x^2$ and $G(f)=\{(x,x^2) \mid x\in\mathbb{R}\}=\{(x,y)\in\mathbb{R}^2 \mid y=x^2\}\subset\mathbb{R}^2$ a parabola. In this case $\mathbb{R}\simeq G(f)$. At each open set G of $G(f)$, we associate an open set $\mathcal{U}$ in $\mathbb{R}$ such that $G=\{(x,x^2) \mid x\in\mathcal{U}\}$ and $\mathcal{U}\simeq g(\mathcal{U})=G$. Finally $dim\, G(f)=dim\,\mathbb{R}=1$. Furthermore $G(f)$ is a 1-pseudomanifold of first kind.

Example 3.2.2
Let $G(f)$ be the graph of the real function $f = |\ \ |$, that is, $G(f)=\{(x,|x|) \mid x\in\mathbb{R}\}\subset\mathbb{R}\times\mathbb{R}$. Thus, $G(f)=\{(x,f(x))\ x\in\mathbb{R},\ f(x)=|x|\}=g(\mathbb{R})$. In $\mathbb{F}$-spaces setting, $f=|\ |:\mathbb{R}\to\mathbb{R}$ should be a smooth map for the $\mathbb{F}$-structure generated by $\{|\ |\}$. It is known that f is not smooth in the canonical $\mathbb{F}$-structure. Now, assume $G(f)$ endowed with the $\mathbb{F}$-structure generated by $\pi_{1G(f)}$ and $\pi_{2G(f)}$. With respect to Lemma 3.2.1, $\mathbb{R}\simeq G(f)$ and $p_{1|G(f)}$ are diffeomorphisms such that $g:\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ and $g=p_{1|G(f)}^{-1}$. The graph of f reveals two situations as shown on the figure below:



On the branches of $G(f)$ in (I) and (II) ,at each point p_1 or p_2 there exists an *open neighborhood G_1 or G_2 such that G_1 is mapped into $\mathcal{U}_1$ and G_2 into $\mathcal{U}_2$; $\mathcal{U}_1$ and $\mathcal{U}_2$ being an open sets in $\mathbb{R}$. In the sequel $dim\, G(f) = dim\, \mathbb{R}$, that is, $dim\, G(f) = 1$ on the two branches. At the origin, the open neighborhood G_3 in $G(f)$, is of dimension 2. Thus $G(f)$ is not of constant dimension at each point.*

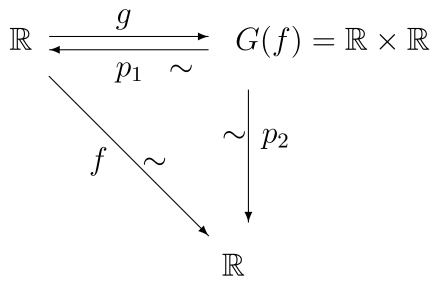
Example 3.2.3
Let $M = B^n = \{x = (x_1, \dots, x_n) \mid ||x|| \leq 1\}$, that is, the n-closed unit ball. Let the $\mathbb{F}$-structure on M be generated by functions $\hat{\pi}_i$, $f : M \to \mathbb{R}$, where $\hat{\pi}_i$ are restrictions of natural projections such that $\pi_i(x) = x_i$, $f(x) = \sqrt{1 - ||x||^2} = \sqrt{1 - \sum_{i=1}^{n} x_i^2}$ with $0 \leq f(x) \leq 1$, $1 \leq i \leq n$. Now, we define $g : M \to M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$ by $g(x) = (x, f(x))$, that is, $g = (\hat{\pi}_1, \dots, \hat{\pi}_n, f)$ and then $g(x) = (x_1, \dots, x_n, f(x)) = (\hat{\pi}_1(x), \dots, \hat{\pi}_n(x), f(x)) = (\hat{\pi}_1, \dots, \hat{\pi}_n, f)(x)$. So, $g(M) = \{(x, f(x)) \mid x \in M\} = G(f)$. By definition, $G(f)$ is the closed hemisphere viewed as a closed $\mathbb{F}$-subspace of $\mathbb{R}^{n+1}$. We have $M \simeq G(f) \subset M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$, from Lemma 3.2.1. *And so $dim\, M = dim\, G(f) = n < dim\, \mathbb{R}^{n+1}$. Furthermore, if we take $Int(B^n)$, with the same generating functions π_i, f, then $Int(B^n)$ is an open $\mathbb{F}$-subspace of $\mathbb{R}^{n+1}$ such that inclusion $Int(B^n) \hookrightarrow \mathbb{R}^{n+1}$ is a smooth map of $\mathbb{F}$-spaces. Therefore $Int(B^n)$ is diffeomorphic to the open top hemisphere $h(Int(B^n)) = \{(x, f(x)) \mid x \in Int(B^n)\}$, where $h = g_{|Int(B^n)}$. It follows that $h(Int(B^n)) = g_{|Int(B^n)}(Int(B^n)) = \{(x, f(x)) \mid x_1^2 + \dots + x_n^2 + f(x)^2 < 1$ and $0 < f(x) < 1\}$ and $Int(B^n) \simeq h(Int(B^n))$, that is, $Int(B^n)$ is n-pseudomanifold of the first kind.*

Example 3.2.4
Let $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (\cos x, \sin x)$, be a smooth map in the canonical $\mathbb{F}$-structures. Let $g : \mathbb{R} \to G(f)$ such that $g(x) = (x, f(x))$ and $G(f) \subset \mathbb{R} \times \mathbb{R}^2$. We have $\mathbb{R} \simeq G(f) = \{(x, \cos x, \sin x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^3 = \mathbb{R} \times S^1 \subset \mathbb{R}^3$ with respect to Lemma 3.2.1. *The graph $G(f)$ is a helix drawn on a unit cylinder whose axis and basis are respectively the x-axis and the unit circle S^1 in y-z-plane of $\mathbb{R}^3$. Now, any open neighborhood at any point q in $G(f)$ is mapped on an open neighborhood at $p_1(q)$ in $\mathbb{R}$, where $q = (x, f(x)) = g(x)$ and $p_1 : G(f) \to \mathbb{R}$ the canonical projection that is $p_1(x, f(x)) = x$. Hence $G(f)$ is a 1-pseudomanifold of first kind.*

Example 3.2.5
Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective set map defined by

$$f(x) = \begin{cases} \frac{x}{2} & \textit{if } x \in (0,1) \\ \frac{x}{2} - 1 & \textit{if } x \in (3,4) \\ x - 3 & \textit{and } x \in (-\infty, 0] \cup [1,3] \cup [4, +\infty) \end{cases}$$

In the sight of Lemma 3.2.1, *if we take f as a generating function for the $\mathbb{F}$-structure on $\mathbb{R}$, then f should be smooth. Therefore $g : \mathbb{R} \to G(f) \subset \mathbb{R} \times \mathbb{R}$ should, in turn, be a diffeomorphism, that is $g = p_{1|G(f)}^{-1}$ and $g^{-1} = p_{1|G(f)}$ are* *smooth bijective maps. Therefore, the inverse image of open and closed sets of* $G(f) = \mathbb{R} \times \mathbb{R}$ *by* g *are respectively open and closed sets in* $\mathbb{R}$. *That is the same for open and closed sets of* $f(\mathbb{R}) = \mathbb{R}$. *The diagram below summarize the informations gained*



that is, $f = p_2 \circ g$ *and* $f \circ p_1 = p_2$ *since* $g \circ p_1 = id_{\mathbb{R}\times\mathbb{R}}$. *Thus,* p_2 *is a diffeomorphism as composition of diffeomorphisms. We can conclude now, that* $\dim G(f) = 1$. *We would like to show what can be* $f^{-1}(0,+\infty)$ *for the given* f. *It follows from the definition of* f *that* $x \in f^{-1}(0,+\infty) = (0,1) \cup (3,4) \cup [4,\infty) = (0,1) \cup (3,\infty)$ *if, and only if* $y \in (0,\frac{1}{2}) \cup (\frac{1}{2},1) \cup [1,+\infty)$

$$\text{since } f^{-1}(y) = \begin{cases} x \in (0,1) & \text{if, and only if } y = \frac{x}{2} \in (0,\frac{1}{2}) \\ x \in (3,4) & \text{if, and only if } y = \frac{x}{2} - 1 \in (\frac{1}{2},1) \\ x \in [4,+\infty) & \text{if, and only if } y = x - 3 \in [1,+\infty). \end{cases}$$

3.3 Smooth maps between pseudomanifolds

Definition 3.3.1
Let M *be an* n*-pseudomanifold. Then at each point* $p \in M$, *there exists a pair* $(\mathcal{U},\varphi)$, *where* $\mathcal{U}$ *is an open neighborhood of* p *and* $\varphi : \mathcal{U} \to \varphi(\mathcal{U}) \subset \mathbb{R}^n$, *a local diffeomorphism. The pair* $(\mathcal{U},\varphi)$ *is called a local chart (or a coordinate neighborhood) at* $p \in M$, *that is, at each point* $p \in \mathcal{U}$ *correspond* n *coordinates* $x^1(p),\ldots,x^n(p)$ *of* $\varphi(p) \in \varphi(\mathcal{U}) \subset \mathbb{R}^n$, *where* $\varphi(p) = x^1(p),\ldots,x^n(p)$ *and* n *is constant for every point in* $\mathcal{U}$. *Each* $x^i(p)$ *is called the* i^{th} *coordinate, where* $x^i : \mathcal{U} \to \mathbb{R}$ *are smooth such that* $\varphi = (x^1,\ldots,x^n)$. *The open set* $\mathcal{U}$ *is called the domain of the chart.*

Definition 3.3.2
Let $(\mathcal{U},\varphi)$ *be a chart at* p *and* (V,ψ) *be chart at* q, *where* $p,q \in M$, $\varphi = (x^1,\ldots,x^n)$ *and* $\psi = (y^1,\ldots,y^n)$ *with* x^i, y^i *the* i^{th} *smooth coordinate functions. Let* $\mathcal{U} \cap V \neq \emptyset$. *The maps between open sets of* $\mathbb{R}^n$, $\psi \circ \varphi^{-1} : \varphi(\mathcal{U} \cap V) \to \psi(\mathcal{U} \cap V)$ *and* $\varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap V) \to \varphi(\mathcal{U} \cap V)$ *are called transition functions. The charts* $(\mathcal{U},\varphi)$ *and* (V,ψ) *are called* $\mathbb{F}$*-related (or* $\mathbb{F}$*-compatible) if* $\mathcal{U} \cap V \neq \emptyset$ *and the transition function* $\varphi \circ \psi^{-1}$, $\psi \circ \varphi^{-1}$ *are diffeomorphisms of the open sets* $\varphi(\mathcal{U} \cap V)$ *and* $\psi(\mathcal{U} \cap V)$ *in* $\mathbb{R}^n$.

That is an equivalence relation among charts.

Definition 3.3.3
Let M be an n-pseudomanifold and $(\mathcal{U},\varphi)$ a chart in M. A collection $\mathcal{A}$ of $\mathbb{F}$-related charts is called an $\mathbb{F}$-atlas if the domain of charts in $\mathcal{A}$ form an open covering for M. The chart $(\mathcal{U},\varphi)$ is $\mathbb{F}$-compatible ($\mathbb{F}$-related) with an atlas $\mathcal{A}$ if $(\mathcal{U},\varphi)$ is $\mathbb{F}$-related to each chart of the atlas $\mathcal{A}$.

Definition 3.3.4
Let $\mathcal{A}_1$, $\mathcal{A}_2$ be two $\mathbb{F}$-atlases in a n-pseudomanifold M. $\mathcal{A}_1$ is equivalent to $\mathcal{A}_2$ and denoted by $\mathcal{A}_1 \sim \mathcal{A}_2$ if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again an $\mathbb{F}$-atlas, that is each chart of one is $\mathbb{F}$-related to the other $\mathbb{F}$-atlas. The union of all equivalent $\mathbb{F}$-atlas is the maximal $\mathbb{F}$-atlas that is the biggest $\mathbb{F}$-atlas equivalent to all members of an equivalence class of $\mathbb{F}$-atlas. Each chart $(\mathcal{U},\varphi)$ in the maximal $\mathbb{F}$-atlas is called an admissible local chart.

Actually $\sim$ is an equivalence relation.

Definition 3.3.5
Let M, N be pseudomanifolds of dimensions m and n respectively. Let $\varphi : M \rightarrow N$ be a set map. φ is said to be smooth map of pseudomanifolds if for every $p \in M$, there is some chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ in M with $p \in \mathcal{U}_\alpha$ and $(\mathcal{V}_\beta, \psi_\beta)$ in N with $\varphi(p) \in \mathcal{V}_\beta$, α, β belonging to some set of indices for a covering of M, such that $\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] \rightarrow \psi_\beta[\varphi(\mathcal{U}_\alpha \cap \mathcal{V}_\beta)]$ (or equivalently, such that $\varphi(\mathcal{U}_\alpha) \subset \mathcal{V}_\beta$ and $\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha) \rightarrow \psi_\beta(\mathcal{V}_\beta)$) is a smooth map of $\mathbb{F}$-subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Note that $\varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta \neq \emptyset$ since $\varphi(p) \in \varphi(\mathcal{U}_\alpha)$ and $\varphi(p) \in \mathcal{V}_\beta$. But, $\varphi(p) \in \mathcal{V}_\beta$ implies $p \in \varphi^{-1}(\mathcal{V}_\beta)$, thus $\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta) \neq \emptyset$. Also, $\varphi[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta] \subseteq \varphi(\mathcal{U}_\alpha) \cap \varphi\varphi^{-1}(\mathcal{V}_\beta) = \varphi(\mathcal{U}_\alpha) \cap [\mathcal{V}_\beta \cap \varphi(M)] \subseteq \varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta$ as a consequence of a set theoretical property combining image and inverse image of a set map. It can be shown that $\varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta$ is an open set in $\varphi(\mathcal{U}_\alpha)$ and $\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)$ is an open set in $\varphi^{-1}(\mathcal{V}_\beta)$, both for $\mathbb{F}$-subspace topologies, since the given intersections are open sets for the respective trace topologies. But the trace topology is contained in the $\mathbb{F}$-subspace topology, so the claim holds. Now, the following diagram of restricted maps makes sense:

$$\begin{array}{ccc}
\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta) & \xrightarrow{\quad \varphi \quad} & \varphi[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] \\
\varphi_\alpha^{-1} \uparrow \quad \downarrow \varphi_\alpha & & \downarrow \psi_\beta \\
\varphi_\alpha[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] & \xrightarrow{\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1}} & \psi_\beta[\varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta]
\end{array}$$

where $\varphi[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] = \varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta$ with respect to Lemma a set theoretical property combining image and inverse image of a set map and $\varphi_\alpha[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] = \varphi_\alpha(\mathcal{U}_\alpha) \cap \varphi_\alpha[\varphi^{-1}(\mathcal{V}_\beta)]$ with respect to a characterization of injective set map.

Lemma 3.3.1
Let M be a pseudomanifold. Let $(\mathcal{U}_\alpha, \varphi_\alpha)$ and $(\mathcal{V}_\beta, \psi_\beta)$ be two charts at $p \in M$. The maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ in Definition 3.3.2 are smooth. They are inverse to each other. If $\mathcal{U}_\alpha \subset \mathcal{V}_\beta$ then the transition functions become $\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha) \to \psi_\beta(\mathcal{U}_\beta)$ and $\varphi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(\mathcal{U}_\beta) \to \varphi_\alpha(\mathcal{U}_\alpha)$.

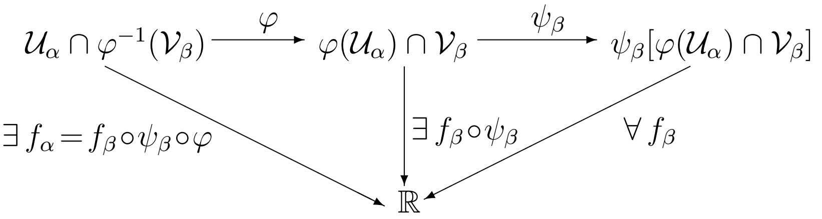
Proof.
First, since φ and ϕ are $\mathbb{F}$-diffeomorphisms, thus $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are $\mathbb{F}$-diffeomorphisms and $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$. Now, let $y^i(p) = h^i(x^1(p), \ldots, x^n(p))$ and $x^i(p) = g^i(y^1(p), \ldots, y^n(p))$. Thus, h^i and g^j are smooth functions as components of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$. These imply,

$$\begin{aligned} \psi \circ \varphi^{-1}(\varphi(p)) &= \psi(p) \\ &= (y^1(p), \ldots, y^n(p)) \\ &= (h^1(x^1(p), \ldots, x^n(p))), \ldots, h^n(x^1(p), \ldots, x^n(p))) \\ &= (h^1[g^1(y^1(p), \ldots, y^n(p)), \ldots, g^n(y^1(p), \ldots, y^n(p))] \ldots, \\ &\quad\; h^n[g^1(y^1(p), \ldots, y^n(p)), \ldots, g^n(y^1(p), \ldots, y^n(p))]), \end{aligned}$$

that is, $y^i(p) = h^i[g^1(y(p)), \ldots, g^n(y(p))]$ with $y(p) = (y^1(p), \ldots, y^n(p))$ for $i = 1, \ldots, n$. Therefore, by analogy to the previous equalities, $\varphi \circ \psi^{-1}(\psi(p)) = \varphi(p) = (x^1(p), \ldots, x^n(p))$ yields $x^j(p) = g^j[h^1(x(p)), \ldots, h^n(x(p))]$ with $x(p) = (x^1(p), \ldots, x^n(p))$ for $j = 1, \ldots, n$. Hence, the transition maps are dually invertible. Secondly, let $\mathcal{U}_\alpha \subset \mathcal{V}_\beta$. Thus, $\mathcal{U}_\alpha \cap \mathcal{V}_\beta = \mathcal{U}_\alpha$ and the charts $(\mathcal{U}_\alpha, \varphi_\alpha)$, $(\mathcal{U}_\alpha, \psi_{\beta|\mathcal{U}_\alpha})$ such as $\psi_{\beta|\mathcal{U}_\alpha} = \psi_\beta \circ \iota$ with $\iota : \mathcal{U}_\alpha \hookrightarrow \mathcal{V}_\beta$ the canonical inclusion. This ends the proof. □

Proposition 3.3.1
Let $\varphi : M \to N$ be a smooth map of pseudomanifolds, with $dim M = m$ and $dim N = n$. Then φ is $\mathbb{F}$-smooth map.

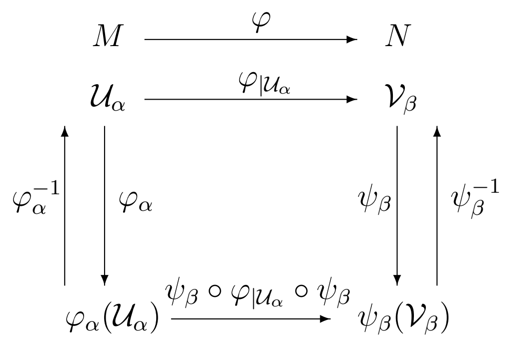
Proof.
Assume that φ is a smooth map of pseudomanifolds. It follows from Definition 3.3.5 that for every $p \in M$, there exists some chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ in M with $p \in \mathcal{U}_\alpha$ and some chart $(\mathcal{V}_\beta, \psi_\beta)$ in N with $\varphi(p) \in \mathcal{V}_\beta$ such that $\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha[\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)] \to \psi_\beta[\varphi(\mathcal{U}_\alpha) \cap \mathcal{V}_\beta]$ is smooth function of $\mathbb{F}$-subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. That is, $\psi_\beta \circ \varphi = (\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1}) \circ \varphi_\alpha$ is $\mathbb{F}$-smooth as the composite of smooth maps as shown in the diagram below:



In the light of Corollary 2.3.2, φ is smooth on $\mathcal{U}_\alpha \cap \varphi^{-1}(\mathcal{V}_\beta)$. Now, we have to show that the smoothness of φ does not depend on the choice of a chart. For, let $(\mathcal{U}'_\alpha, \varphi'_\alpha)$ be another chart at p in M. It is clear that $\mathcal{U}_\alpha \cap \mathcal{U}'_\alpha$ is non-empty and we can define the transition maps $\varphi_\alpha\circ\varphi'^{-1}_\alpha : \varphi'_\alpha(\mathcal{U}_\alpha\cap\mathcal{U}'_\alpha)\to\varphi_\alpha(\mathcal{U}_\alpha\cap\mathcal{U}'_\alpha)$, and $\varphi'_\alpha\circ\varphi^{-1}_\alpha : \varphi_\alpha(\mathcal{U}_\alpha\cap\mathcal{U}'_\alpha)\to\varphi'_\alpha(\mathcal{U}_\alpha\cap\mathcal{U}'_\alpha)$, which are $\mathbb{F}$-diffeomorphisms. In the sequel $\psi_\beta\circ\varphi\circ\varphi^{-1}_\alpha\circ(\varphi_\alpha\circ\varphi'^{-1}_\alpha)=\psi_\beta circ\varphi\circ\varphi'^{-1}_\alpha$ is a composition of $\mathbb{F}$-diffeomorphisms with $\psi_\beta\circ\varphi\circ\varphi'^{-1}_\alpha : \varphi'_\alpha[\mathcal{U}'_\alpha\cap\varphi^{-1}_\alpha(\mathcal{V}_\beta)]\to\psi_\beta[\varphi_\alpha(\mathcal{U}'_\alpha)\cap\mathcal{V}_\beta]$. Therefore, for any chart $(\mathcal{U}_\alpha,\varphi_\alpha)$, φ is smooth on $\mathcal{U}_\alpha\cap\varphi^{-1}_\alpha(\mathcal{V}_\beta)$. Without loss of generality, we may choose $\mathcal{U}_\alpha$ and $\mathcal{V}_\beta$ such that $\mathcal{U}_\alpha\subset\varphi^{-1}(\mathcal{V}_\beta)$ and $(\mathcal{U}_\alpha,\varphi_\alpha)_{\alpha\in A}$ is open covering of M for $\tau_{\mathcal{F}_M}$ and $\tau_{\mathcal{C}_M}$ since $\tau_{\mathcal{F}_M}\subset\tau_{\mathcal{C}_M}$. So φ is smooth on each $\mathcal{U}_\alpha$ member of a covering of M in $\tau_{\mathcal{C}_M}$. We should conclude, by means of Lemma 2.3.3, that φ is smooth on the whole set M. □

Proposition 3.3.2
Let M, N be pseudomanifolds with $\dim M=m$ and $\dim N=n$. Let $\varphi : M\to N$ be an $\mathbb{F}$-smooth map. Then φ is smooth map of pseudomanifolds.

Proof.
From Lemma 2.3.2 φ is $\mathbb{F}$-smooth if, and only if for every $p\in M$ and each neighborhood $W_{\varphi(p)}$ in N, there exists a neighborhood $\mathcal{V}_p$ containing p such that $\varphi(\mathcal{V}_p)\subset W_{\varphi(p)}$. Assume $\mathcal{U}_\alpha=\mathcal{V}_p$ and $\mathcal{V}_\beta=W_{\varphi(p)}$ with $\varphi(\mathcal{U}_\alpha)\subset\mathcal{V}_\beta$. Thus $\mathcal{U}_\alpha\subset\varphi^{-1}\varphi(\mathcal{U}_\alpha)\subset\varphi^{-1}(\mathcal{V}_\beta)$. It follows that $\mathcal{U}_\alpha\cap\varphi^{-1}(\mathcal{V}_\beta)=\mathcal{U}_\alpha$ and $\varphi(\mathcal{U}_\alpha)\cap\mathcal{V}_\beta=\varphi(\mathcal{U}_\alpha)$. Hence $\psi_\beta\circ\varphi\circ\varphi^{-1}_\alpha : \varphi(\mathcal{U}_\alpha)\to\psi_\beta(\mathcal{V}_\beta)$ is smooth as the composite of smooth maps. Furthermore φ is a smooth map of pseudomanifolds by Definition 3.3.5. The diagram below is related to the situation



□

Corollary 3.3.1
Let M, N be pseudomanifolds with $\dim M=m$ and $\dim N=n$. Let $\varphi : M\to N$ be a set map. φ is $\mathbb{F}$-smooth if, and only if φ is smooth map of pseudomanifolds.

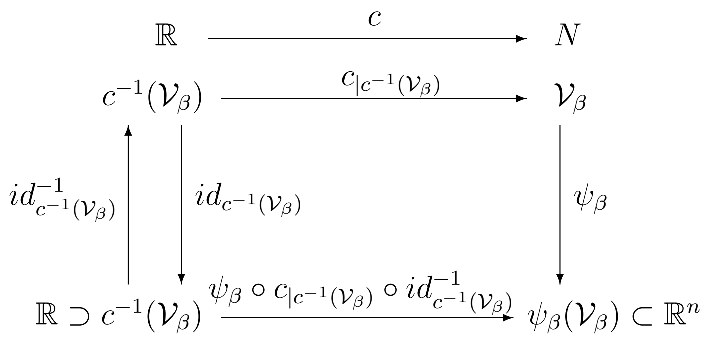
Proof.
That is a straightforward consequence of Proposition 3.3.1 and Proposition 3.3.2. □

Corollary 3.3.2
Let M be a n-pseudomanifold and $f : M\to\mathbb{R}$ a function. Then $f\in\mathcal{F}_M$ if, and only if for every $p\in M$ there is some chart $(\mathcal{U}_\alpha,\varphi_\alpha)$ at p in M so that $f\circ\varphi^{-1}_\alpha : \varphi_\alpha(\mathcal{U}_\alpha)\to\mathbb{R}$ is smooth, that is $f\circ\varphi^{-1}_\alpha\in\mathcal{F}_{\varphi_\alpha(\mathcal{U}_\alpha)}$, with $\varphi_\alpha(\mathcal{U}_\alpha)\subset\mathbb{R}^n$.

Proof.
"$\Longrightarrow$" Assume $f \in \mathcal{F}_M$. Then if $N = \mathbb{R}$ and $\varphi = f$ in Corollary 3.3.1, f is smooth map of pseudomanifolds. Thus, with respect to Definition 3.3.5; for every $p \in M$ there is some chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ at p and some chart (I_β, id_{I_β}) at $f(p)$ such that $f(\mathcal{U}_\alpha) \subset I_\beta$ and $id_{I_\beta} \circ \varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha) \to I_\beta$ is a smooth map of $\mathbb{F}$-subspaces, where $\mathcal{V}_\beta = I_\beta \subset \mathbb{R}$ and $\psi_\beta = id_{I_\beta}$. Therefore for every $p \in M$, there is some chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ at p so that $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha) \to \mathbb{R}$ is smooth, that is $f \circ \varphi_\alpha^{-1} \in \mathcal{F}_{\varphi_\alpha(\mathcal{U}_\alpha)}$.
"$\Longleftarrow$" Proposition 3.3.1 yields $N = \mathbb{R}$, $\varphi = f$. Thus f is smooth map of $\mathbb{F}$-spaces, that is $f \in \mathcal{F}_M$. □

Corollary 3.3.3
Let N be an n-pseudomanifold and $c : \mathbb{R} \to N$ be a curve. Then $c \in \mathcal{C}_N$ if, and only if for every $t \in \mathbb{R}$ there exists some chart $(\mathcal{V}_\beta, \psi_\beta)$ at $c(t)$ in N such that $\psi_\beta \circ c : c^{-1}(\mathcal{V}_\beta) \to \psi_\beta(\mathcal{V}_\beta)$ is smooth, that is $\psi_\beta \circ c \in \mathcal{C}_{\psi_\beta(\mathcal{V}_\beta)}$ with $c^{-1}(\mathcal{V}_\beta) \subset \mathbb{R}$, $\psi_\beta(\mathcal{V}_\beta) \subset \mathbb{R}^n$.

Proof.
"$\Longrightarrow$" Assume $c \in \mathcal{C}_N$, $M = \mathbb{R}$, $\varphi = c$ in Proposition 3.3.2. Thus c is a smooth map of pseudomanifolds. By Definition 3.3.5, one have, for every $t \in \mathbb{R}$ there exists some chart $(c^{-1}(\mathcal{V}_\beta), id_{c^{-1}(\mathcal{V}_\beta)})$ at t and some chart $(\mathcal{V}_\beta, \psi_\beta)$ such that $c(c^{-1}(\mathcal{V}_\beta)) = \mathcal{V}_\beta \cap c(\mathbb{R}) \subset \mathcal{V}_\beta$ and $\psi_\beta \circ c \circ id_{c^{-1}(\mathcal{V}_\beta)} : c^{-1}(\mathcal{V}_\beta) \to \psi_\beta(\mathcal{V}_\beta)$ is smooth map of $\mathbb{F}$-subspaces, where $\mathcal{U}_\alpha = c^{-1}(\mathcal{V}_\beta) \subset \mathbb{R}$ and $\varphi_\alpha = id_{c^{-1}(\mathcal{V}_\beta)}$. It follows a diagram of smooth maps:



Therefore, since $id^{-1}_{c^{-1}(\mathcal{V}_\beta)} = id_{c^{-1}(\mathcal{V}_\beta)}$, for every $t \in \mathbb{R}$, there is some chart $(\mathcal{V}_\beta, \psi_\beta)$ at $c(t)$ such that $\psi_\beta \circ c : c^{-1}(\mathcal{V}_\beta) \to \psi_\beta(\mathcal{V}_\beta)$ is smooth, that is,$\psi_\beta \circ c \in \mathcal{C}_{\psi_\beta(\mathcal{V}_\beta)}$.
"$\Longleftarrow$" Proposition 3.3.1 yields $M = \mathbb{R}$, $\varphi = c$, $p = t$. Thus c is smooth map of $\mathbb{F}$-spaces, that is $c \in \mathcal{C}_N$. □

Corollary 3.3.4
Let M, N be pseudomanifolds and $\varphi : M \to N$ a set map. Let c be any curve in $\mathcal{C}_M$. The following conditions are equivalent.

1. *φ is smooth.*
2. *$\varphi \circ c \in \mathcal{C}_N$.*

3. *$\psi_\beta \circ \varphi \circ c : c^{-1}(\varphi^{-1}(\mathcal{V}_\beta)) \to \psi_\beta(\mathcal{V}_\beta)$ is smooth, where $t \in \mathbb{R}$ and $(\mathcal{V}_\beta, \psi_\beta)$ a chart at $c(t)$.*

4. *$\varphi \circ c : c^{-1}(\varphi^{-1}(\mathcal{V}_\beta)) \to \mathcal{V}_\beta$ is smooth.*

Proof.
(1) $\Longrightarrow$ (2) Obvious from the definition of a smooth map.
(2) $\Longrightarrow$ (3) From Corollary 3.3.3, $\varphi \circ c \in \mathcal{C}_N$ if, and only if for every $t \in \mathbb{R}$ there exists $(\mathcal{V}_\beta, \psi_\beta)$ a chart at $(\varphi \circ c)(t)$ in N such that $\psi_\beta \circ (\varphi \circ c) : (\varphi \circ c)^{-1}(\mathcal{V}_\beta) \to \psi_\beta(\mathcal{V}_\beta)$ is smooth.
(3) $\Longrightarrow$ (4) Assume $\psi_\beta \circ \varphi \circ c : c^{-1}(\varphi^{-1}(\mathcal{V}_\beta)) \to \psi_\beta(\mathcal{V}_\beta)$ smooth. Note that $\varphi \circ c$ smooth implies $\varphi \circ c$ continuous for both $\tau_{\mathcal{C}_N}$ and $\tau_{\mathcal{F}_N}$, that is, $(\varphi \circ c)^{-1}(\mathcal{V}_\beta)$ is a $\mathbb{R}$-open set. Therefore $c^{-1}(\varphi^{-1}(\mathcal{V}_\beta))$ is a $\mathbb{R}$-open set. This yields $\varphi^{-1}(\mathcal{V}_\beta) \in \tau_{\mathcal{C}_N}$. As composite of ψ_β smooth and $\varphi \circ c$, it follows from Corollary 2.3.2 that $\varphi \circ c : c^{-1}(\varphi^{-1}(\mathcal{V}_\beta)) \to \mathcal{V}_\beta$ is smooth.
(4) $\Longrightarrow$ (5) Assume $c(\mathbb{R}) \subset \varphi^{-1}(\mathcal{V}_\beta)$. It follows that $c^{-1}(c(\mathbb{R})) \subset c^{-1}\varphi^{-1}(\mathcal{V}_\beta)$ and $\mathbb{R} \subset (\varphi \circ c)^{-1}(\mathcal{V}_\beta)$. Thus $\mathbb{R} = (\varphi \circ c)^{-1}(\mathcal{V}_\beta)$. So $\varphi \circ c : \mathbb{R} \to N$ is smooth curve.
(5) $\Longrightarrow$ (1) Straightforward consequence of the definition of a smooth map. Hence φ is smooth. □

Corollary 3.3.5
Let $\gamma : \mathbb{R} \to N$ be a set map. Let N be a pseudomanifold. If for every $t \in \mathbb{R}$, there exists $c \in \mathcal{C}_N$ and I_α a $\mathbb{R}$-open set, with $t \in I_\alpha$ such that $\gamma_{|I_\alpha} = c_{|I_\alpha}$ then $\gamma \in \mathcal{C}_N$.

Proof.
We may make use of Corollary 3.3.3. That is, $c \in \mathcal{C}_N$ if, and only if for any $t \in \mathbb{R}$ there exists some chart $(\mathcal{V}_\beta, \psi_\beta)$ at $c(t)$ in N so that $\psi_\beta \circ c : c^{-1}(\mathcal{V}_\beta) \to \psi_\beta(\mathcal{V}_\beta)$ is smooth. One can assume $I_\alpha = c^{-1}(\mathcal{V}_\beta)$, and so $t \in I_\alpha$ and $(I_\alpha)_{\alpha \in A}$ is a $\tau_{\mathcal{C}_\mathbb{R}} = \tau_{\mathcal{F}_\mathbb{R}}$ open covering of $\mathbb{R}$. Also $c : I_\alpha \to \mathcal{V}_\beta$ is smooth. Now, one can make use of the assumption $\gamma_{|I_\alpha} = c_{|I_\alpha}$. It follows that the set map $\gamma : \mathbb{R} \to N$, has its restriction smooth on each I_α, for any α, in the covering. With respect to Lemma 2.3.3 γ is smooth curve on the whole $\mathbb{R}$ for its structure of $\mathbb{F}$-space or pseudomanifold. □

3.4 Category of pseudomanifolds

From Sections 3.1 through 3.3, we may build a category formed by pseudomanifolds as objects and $\mathbb{F}$-smooth maps between them as morphisms. We denote this category by $\mathcal{PSF}$, the category of pseudomanifolds of the first kind. The category $\mathcal{PSF}$ is a full subcategory of $\mathcal{FRL}$ since smooth maps of pseudomanifolds are $\mathbb{F}$-smooth maps of $\mathbb{F}$-spaces. As shown later in Sections 3.5 through 3.8, the category $\mathcal{PSF}$ has all limits and colimits inherited from $\mathcal{FRL}$. Thus $\mathcal{PSF}$ is complete and cocomplete. The existence of initial objects and final objects is granted: $M = \{x\}$ admits $\emptyset$ and M as both $\tau_{\mathcal{F}_M}$-open set and $\tau_{\mathcal{F}_M}$-closed set, with $\tau_{\mathcal{F}_M} = \tau_{\mathcal{C}_M} = \{\emptyset, M\}$. We have $\mathcal{F}_M = \{f_a : M \to \mathbb{R} \mid f_a(x) = a\}$ and $\mathcal{C} = \{\varphi_x\}$ a singleton. There exists a bijective correspondence between $\mathcal{F}_M$ and $\mathbb{R}$. And so, for any $a \in \mathbb{R}$

$$f_a^{-1}(0,+\infty)\begin{cases}\emptyset & \text{if } a \notin (0,+\infty)\\ \{x\} & \text{if } a \in (0,+\infty).\end{cases}$$

It is possible to build a local diffeomorphism $\varphi : \{x\} \longrightarrow \{0\} = \mathbb{R}^0$ such that $\varphi(\{x\}) = \{0\}$ is a closed set in $\{0\} = \mathbb{R}^0$. Thus, $M = \{x\}$ is a pseudomanifold of dimension 0. Hence $\mathcal{PSF}$ admits terminal objects and products of two objects exists.

3.5 Subpseudomanifolds

Definition 3.5.1
Let $f : M \longrightarrow N$ be a smooth mapping of pseudomanifolds, with $dimM = m$ and $dimN = n$. The rank of f at $p \in M$ is the rank at $\varphi(p) \in \varphi(\mathcal{U})$ of the map $\hat{f} = \psi \circ f \circ \varphi^{-1} : \varphi(\mathcal{U}) \longrightarrow \psi(\mathcal{V})$, with $(\mathcal{U}, \varphi)$ a chart at p in M and $(\mathcal{V}, \psi)$ a chart at $f(p)$ in N, that is the rank at $\varphi(p)$ of the Jacobian matrix

$$\begin{bmatrix} \frac{\partial^1 f}{\partial x^1} & \cdots & \frac{\partial^1 f}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial^n f}{\partial x^1} & \cdots & \frac{\partial^n f}{\partial x^m} \end{bmatrix}$$

of the map $\hat{f}(x^1, \ldots, x^m) = (\psi \circ f \circ \varphi^{-1})(x^1, \ldots, x^m) = (f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m))$ expressing f in the local coordinates.

The rank must be independent of the choice of coordinates since the smoothness of f is independent of the choice of coordinates (charts). Let $M \xrightarrow{f} N$ be a smooth map. Thus, we get the following: $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$, $y = (f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m)) \in \mathbb{R}^n$. The important case for our study will be in which the rank is constant at each point $p \in M$.

Example 3.5.1 [*9, p.47*],
Let $f(x_1, x_2) = (x_1^2 + x_2^2, 2x_1x_2)$. Its Jacobian is given by

$$Df(x_1, x_2) = \begin{bmatrix} \frac{\partial(x_1^2+x_2^2)}{\partial x_1} & \frac{\partial(x_1^2+x_2^2)}{\partial x_2} \\ \frac{\partial 2x_1x_2}{\partial x_1} & \frac{\partial 2x_1x_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_2 & 2x_1 \end{bmatrix}$$

and $detDf(x_1, x_2) = 4x_1^2 + 4x_2^2$. First, $4x_1^2 + 4x_2^2 = 0 \Leftrightarrow x_1^2 + x_2^2 = 0 \Leftrightarrow x_1 = x_2 = 0$. Hence, $\Rightarrow Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, that is, $rankf = 0$ at $(0,0)$. Secondly, $4x_1^2 + 4x_2^2 \neq 0$ as a sum of squares $\Rightarrow x_1 \neq 0$ or $x_2 \neq 0$. Thus, $rankf = 2$ at $(x_1, x_2) \neq (0,0)$.

Let $f(x_1, x_2) = ((x_1)^2, 2x_1x_2)$. Then $Df(x_1, x_2) = \begin{bmatrix} \frac{\partial x_1^2}{\partial x_1} & \frac{\partial x_1^2}{\partial x_2} \\ \frac{\partial 2x_1x_2}{\partial x_1} & \frac{\partial 2x_1x_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 \\ 2x_2 & 2x_1 \end{bmatrix}$.

Thus, $detDf(x_1,x_2) = 4x_1^2$*. Then* $rankf = 0$ *at* $(0,0)$ *and* $rankf = 2$ *at* $(x_1,x_2) \neq (0,0)$.

The relationship between global diffeomorphisms and local diffeomorphisms on pseudomanifolds setting is given by Definition 3.3.5 and Corollary 3.3.1.

Definition 3.5.2 *Let* $f : M \longrightarrow N$ *be a smooth map of pseudomanifolds with* $dimM = m$, $dimN = n$*. The map* f *is said to be:*

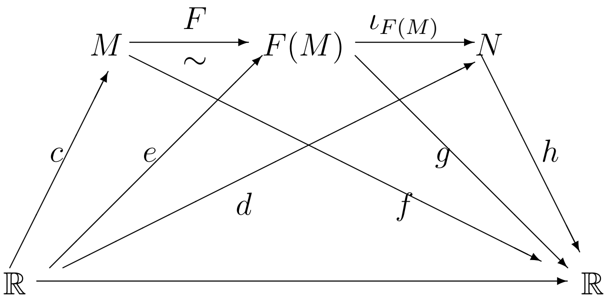
1. *a submersion if* $rankf = n$ *at every point* $p \in M$*, with* $n \geq m$.
2. *a immersion if* $rankf = m$ *at every point* $p \in M$*, with* $m \leq n$.
3. *a diffeomorphism if* f *maps* M *one-to-one onto* N *and* f^{-1} *is smooth.*
4. *a local diffeomorphism if* $dimM = dimN$ *and* f *a submersion (or equivalently,* f *is an immersion)*

We will deal with the concepts of substructure in the pseudomanifold setting, that is, in $\mathbb{F}$-setting.

Definition 3.5.3
Let $F : M \longrightarrow N$ *be a smooth map of pseudomanifolds and* $dimM = m$, $dimN = n$. $F(M) = (M, F)$ *is an immersed subpseudomanifold of* N *if, and only if* F *is an injective immersion.*

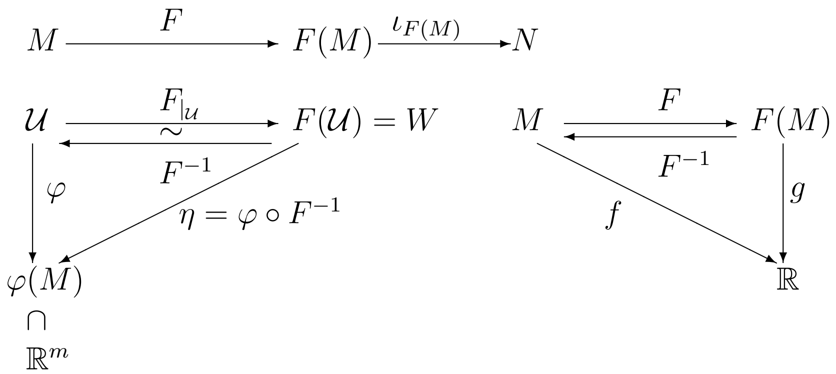
Remark 3.5.1

1. *Some authors, Frank W. Warner [86] among them, denote* $F(M) := (M, F)$ *to stress the fact that the structure lies on the nature of* F*. And changing* F *to* G*, , that is another map, should yield* $(M, G) \neq (M, F)$*. Also subpseudomanifold will mean immersed subpseudomanifold if no confusion is expected.*
2. *In Definition* 3.5.3, $F(M) := (M, F)$ *is endowed with topology and* $\mathbb{F}$*-structure which makes* $F : M \longrightarrow F(M)$ *an* $\mathbb{F}$*-diffeomorphism.*
3. *We will be aware of this* $\mathbb{F}$*-diffeomorphism: even when* F *is one-to-one immersion, it is not necessary an* $\mathbb{F}$*-diffeomorphism with* $F(M)$ *as an* $\mathbb{F}$*-subspace of* N *since the structure on* $F(M)$ *is generated by* $\mathcal{G}_{o\,F(M)} = \{f \circ F^{-1} = g : F(M) \to \mathbb{R} \mid f \in \mathcal{F}_M\}$*, that is, co-induced from that of* M*. The smoothness of* F *yields* $(f \circ F^{-1}) \circ F = f \in \mathcal{F}_M$*. But* $\mathcal{F}_{o\,F(M)} = \{h \circ \iota_{F(M)} = h_{|_{F(M)}} \mid h \in \mathcal{F}_N\} = \mathcal{F}_{N_{|_{F(M)}}}$ *generates the* $\mathbb{F}$*-substructure on* $F(M)$*, as show in the following diagram:*



$(F(M), \Gamma\mathcal{G}_{o\,F(M)}, \Phi\Gamma\mathcal{G}_{o\,F(M)}) := (F(M), \Gamma\mathcal{G}_{o\,F(M)}, \mathcal{G}_{F(M)})$
$(F(M), \Gamma\mathcal{F}_{o\,F(M)}, \Phi\Gamma\mathcal{F}_{o\,F(M)}) := (F(M), \Gamma\mathcal{F}_{o\,F(M)}, \mathcal{F}_{F(M)})$.

4. *Since $F : M \longrightarrow F(M) \subset N$ is one-to-one and onto, we do understand Definition* 3.5.3 *in the following way. By defining open set of $F(M)$ to be images of open sets of M and coordinate neighborhood (W, η) of $F(M)$ to be of the form $W = F(\mathcal{U})$, $\eta = \varphi \circ F^{-1}$, where $(\mathcal{U}, \varphi)$ is a coordinate neighborhood of M, we will carry over the topology and $\mathbb{F}$-structure of M to $F(M)$ as shown below*



The fact that $F : M \to N$ is continuous implies that, if $\mathcal{V} \in \tau_{\mathcal{F}_M}$ then $F^{-1}(\mathcal{V}) \in \tau_{\mathcal{F}_M}$ and also $F(F^{-1}(\mathcal{V})) = \mathcal{V} \cap F(M) = \iota^{-1}_{F(M)}(\mathcal{V})$ since $\iota_{F(M)}$ is the canonical inclusion. We recall that $\tau_{\mathcal{G}_{F(M)}} \supset \tau_{\mathcal{F}_N}(F(M))$, that is, $\mathbb{F}$-topology is finer than the relative topology. Thus, there may be open sets of $F(M)$ which are not of the form $\mathcal{V} \cap F(M)$. Nothing can allow us to state that all open sets in M are of the form $F^{-1}(\mathcal{V})$, thus there exists $\mathcal{U} \subset M$ such that $\mathcal{U} \neq F^{-1}(\mathcal{V})$ for any $\mathcal{V}$ open in N. Notice that $\mathcal{V} \subset N$ and not in $F(M)$.

Definition 3.5.4
Let $F : M \longrightarrow N$ be a smooth map of pseudomanifolds and $dimM = m$, $dimN = n$. $F(M) := (M, F)$ is an embedded subpseudomanifold of N if, and only if F is an injective immersion such that $F : M \to F(M)$ is an $\mathbb{F}$-diffeomorphism with $F(M)$ an $\mathbb{F}$ -subspace, but the topology is the trace topology of $\tau_{\mathcal{F}_N}$ on $F(M)$. Such a smooth map F is called an embedding of M to N.

Remark 3.5.2

1. *$\tau_{\mathcal{F}_N}(F(M))$ is the smallest topology on $F(M)$ for which $\iota_{F(M)}$ (the canonical inclusion) is continuous.*

2. *$F\colon M\to F(M)$ is an open map , that is $F(\mathcal{U})$ is open in $F(M)$ and is of the form $F(\mathcal{U})=\mathcal{V}\cap F(M)$ with any $\mathcal{V}$ open in N for $\tau_{\mathcal{F}_N}$ and $\mathcal{U}$ open in M.*

3. *An embedded subpseudomanifolds is a particular type of immersed subpseudomanifold.*

4. *We will denote $\tau_o=\tau_{\mathcal{F}_N}(F(M))$, $\tau_1=\tau_{\mathcal{F}_{F(M)}}$, the $\mathbb{F}$-subspace topology on $F(M)$, and τ_2 the co-induced topology from M to $F(M)$ as in Remark* 3.5.1 *(4). It is already known that $\tau_1\supset\tau_o$ and $\tau_2\supset\tau_o$. We need to know what is it about τ_1 and τ_2. From their definitions $\mathcal{G}_{F(M)}=\Phi\Gamma\mathcal{G}_{o\,F(M)}$ and $\mathcal{F}_{F(M)}=\Phi\Gamma F_{o\,F(M)}$. That is, $h\circ\iota_{F(M)}=h_{|_{F(M)}}\in F_{o\,F(M)}$ with for all $h\in\mathcal{F}_N$, $h\circ F=f\in\mathcal{F}_M$ since F is smooth also $F=\iota_{F(M)}\circ F$ by definition of F. Since $F: M\tilde{\to}F(M)$. Thus $h_{|_{F(M)}}=h\circ\iota_{F(M)}=h\circ\iota_{F(M)}\circ F\circ F^{-1}=h\circ F\circ F^{-1}=f\circ F^{-1}\in\mathcal{G}_{o\,F(M)}$. It yields $\mathcal{F}_{o\,F(M)}\subseteq\mathcal{G}_{o\,F(M)}$, that is, $\mathcal{F}_{F(M)}=\mathcal{G}_{F(M)}$. Now $B=\{h^{-1}_{|_{F(M)}}(0,+\infty)\,|\,h\in\mathcal{F}_N\}$ and $B'=\{g^{-1}(0,+\infty)\,|\,g\in\mathcal{G}_{F(M)}\}$. Thus, $B\subseteq B'$. That implies $\tau_o\subseteq\tau_1\subseteq\tau_2$ since for all $\mathcal{U}\in B$, there exists $\mathcal{V}\in B'$ such that $\mathcal{V}\subseteq\mathcal{U}$.*

Definition 3.5.5
Let N be an n-pseudomanifold. Let $M\subset N$ be a $\mathbb{F}$-subspace of N. M is said to be a regular m-subpseudomanifold of N with $0\le m\le n$ if, and only if for every point $p\in M$, there is a chart $(\mathcal{U},\varphi)$ in N with $p\in\mathcal{U}$ so that $\varphi(\mathcal{U}\cap M)=\varphi(\mathcal{U})\cap(\mathbb{R}^m\times\{\underbrace{(0,\dots,0)}_{n-m}\})$ and the topology on M is $\tau_{\mathcal{F}_M}=\tau_{\mathcal{F}_N}(M)$. The subset $\mathcal{U}\cap M$ is called a slice of $(\mathcal{U},\varphi)$ and the chart $(\mathcal{U},\varphi)$ is said to be adapted to M.

Definition 3.5.6
A subset M of an n-pseudomanifold N is a regular m-subpseudomanifold of N with $0\le m\le n$ if, and only if for each point $p\in M$, there exists a coordinate neighborhood $(\mathcal{U},\varphi)$ on N with $p\in\mathcal{U}$, with local coordinates $x^1,\dots,x^n$ such that

1. *$\varphi(p)=(0,\dots,0)\in\mathbb{R}^n$. That is, the coordinate system is centered at p.*

2. *$\varphi(\mathcal{U})=C^n_\epsilon(0)=C^n_\epsilon(0)=\{(x^1,\dots,x^n)\in\mathbb{R}^n\,|\,|x^i|<\epsilon$ for all $1\le i\le n\}$. That is, the open cube with sides of length 2ϵ and centered at the origin $(0,\dots,0)\in\mathbb{R}^n$.*

3. *$\varphi(\mathcal{U}\cap M)=\{x\in C^n_\epsilon(0)\,|\,x^{m+1}=\cdots=x^n=0\}$*

Definition 3.5.5 and Definition 3.5.6 are equivalent. See [69, 1.4 Submanifolds] for details on regular point and regular subpseudomanifold. A coordinate system $(\mathcal{U},\varphi)$ is called a cubic (cubical) coordinate system if $\varphi(\mathcal{U})=C^m_\epsilon(0)$.

Example 3.5.2
The sphere $S^2=\{x\in\mathbb{R}^3\,|\,||x||=1\}$ is a regular subpseudomanifold of $\mathbb{R}^3$. The examples given in [9, 12] are natural examples of regular subpseudomanifolds, where examples 4.9, 4.10 are not.

Definition 3.5.7
Let N be an n-pseudomanifold and $M \subset N$ a $\tau_{\mathcal{F}_N}$-closed. M is called a closed regular m-subpseudomanifold if, and only if for each $p \in M$, there exists $(\mathcal{U}, \varphi)$ a chart at p in N with $p \in \mathcal{U}$ such that $\varphi(\mathcal{U} \cap M) = \varphi(\mathcal{U}) \cap (\mathbb{R}^m \times \{(\underbrace{0, \ldots, 0}_{n-m})\})$. M is also called properly (or regularly) embedded subpseudomanifold of N.

Remark 3.5.3

1. *If M is a regular subpseudomanifold of N then $\iota_M \hookrightarrow N$ is a smooth map of pseudomanifolds such that $\iota_M : M \hookrightarrow \iota_M(M)$ is the identity map, that is injective and immersion ($\psi \circ \iota_M \circ \varphi^{-1} = \psi \circ \varphi^{-1}$ a diffeomorphism, thus rank $\iota_M = m$), that is M and $\iota_M(M)$ have the same $\mathbb{F}$-subspace structure and $\mathbb{F}$-subspace topology. Hence ι_M is naturally an embedding. Therefore M is an embedded subpseudomanifold of N.*

2. *If $F : M \to N$ is an embedding, then $F(M)$ is an embedded subpseudomanifold and $F : M \to F(M)$ is a diffeomorphism and the family of pairs $(\mathcal{U} \cap M, \varphi_{|\mathcal{U} \cap M})$, where $(\mathcal{U}, \varphi)$ ranges over the charts over any atlas for N, is an atlas for M, where M is given the topology $\tau_{\mathcal{F}_M} = \tau_{\mathcal{F}_N}(M)$.*

3. *Here are some Observations.*

   (a) *On $F(M)$: $\tau_o \subset \tau_1 \subset \tau_2$ as shown in Remark 3.5.2 (4).*

   (b) *$F(M)$ Embedded subpseudomanifold $\Rightarrow$ Immersed (with τ_2) and $\tau_o \Rightarrow \tau_2 = \tau_o \Rightarrow \tau_o \supset \tau_1 \wedge \tau_1 \supset \tau_2 \Rightarrow \tau_1 = \tau_o \Rightarrow \iota_{F(M)} : (F(M), \tau_o) \to (F(M), \tau_1)$ is an $\mathbb{F}$-diffeomorphism $\Rightarrow \iota_{F(M)}$ injective immersion such that such that $F(M)$ is an $\mathbb{F}$-subspace and $\tau_1 = \tau_o \Rightarrow F(M)$ is a regular subpseudomanifold of N.*

   (c) *$F(M)$ regular subpseudomanifold of $N \Longrightarrow$.*

      i. *$\iota_{F(M)}$ is an embedding $\Longrightarrow F(M)$ is an embedded subpseudomanifold of N.*

      ii. *Since $\mathcal{F}_{F(M)} \subset \mathcal{G}_{F(M)}$ and $\tau_o \subset \tau_1 \Longrightarrow \tau_1 \ni B = \{h^{-1}_{|_{F(M)}}(0, +\infty) \mid h \in \mathcal{F}_N\} \subset \{g^{-1}(0, +\infty) \mid g \in \mathcal{G}_{F(M)}\} = B' \in \tau_2 \Longrightarrow$ for all $\mathcal{U} \in B$ there exists $\mathcal{V} = \mathcal{U} \in B'$ such that $\mathcal{V} \subset \mathcal{U} \Longrightarrow \tau_1 \subset \tau_2$. Also, since F is an $\mathbb{F}$-diffeomorphism, any $\mathcal{V} = g^{-1}(0, +\infty) = (f \circ F^{-1})^{-1}(0, +\infty) = F \circ f^{-1}(0, +\infty) = F(f^{-1}(0, 0))$ is a τ_2-basic open in $F(M)$. But $F = \iota_{F(M)} \circ F$ and for all $h \in \mathcal{F}_N$ there $f_1 \in \mathcal{F}_M$ such that $h \circ F = f_1 \Rightarrow f_1^{-1}(0, +\infty) = (\iota_{F(M)} \circ F)^{-1} \circ h^{-1}(0, +\infty) \Rightarrow f^{-1}(0, +\infty) = F^{-1} \circ h^{-1}_{|_{F(M)}}(0, +\infty) = (h_{|_{F(M)}} \circ F)^{-1}(0, +\infty)$ basic open in M. Thus $F(f_1^{-1}(0, +\infty)) = FF^{-1}(h^{-1}_{|_{F(M)}})(0, +\infty) = h^{-1}_{|_{F(M)}}(0, +\infty)$ τ_1-basic open and τ_2-basic open in $F(M)$. From closeness under finite intersections $f^{-1}(0, +\infty) \cap f_1^{-1}(0, +\infty) = f_2^{-1}(0, +\infty)$, $f_2 \in \mathcal{F}_N$ and $F(f_2^{-1}(0, +\infty)) = Ff^{-1}(0, +\infty) \cap Ff_1^{-1}(0, +\infty) = g^{-1}(0, +\infty) \cap$*

$h_{|_{F(M)}}(0,+\infty) \supset \mathcal{V}$ is a τ_2-basic open and τ_1-basic open of $F(M)$. Hence for all $\mathcal{V} \in B'$ there exists $\mathcal{U} = g^{-1}(0,+\infty) \cap h_{|_{F(M)}}(0,+\infty) \in B$ such that $\mathcal{U} \subset \mathcal{V} \Longrightarrow \tau_2 \subset \tau_1$. Therefore $\tau_2 = \tau_1$ and $\tau_1 = \tau_o$, that is $\tau_2 = \tau_o \Longrightarrow F(M)$ immersed (with τ_2) and $\tau_o \Longrightarrow F$ embedding $\Longrightarrow F(M)$ embedded subpseudomanifold of N.

Example 3.5.3 *Examples of open and closed subpseudomanifolds*

1. *$\mathcal{U} = GL(n,\mathbb{R}) \subset M = \mu_n(\mathbb{R})$, $n \times n$ matrices over $\mathbb{R}$, which consists of all non-singular $n \times n$ matrices $\mathcal{U} = \{A \in \mu_n(\mathbb{R}) \mid det\, A \neq 0\}$ since $det\, A$ is a polynomial function of its entries a_{ij}, it is a continuous (smooth) function of its entries and of A in the topology of identification with $\mathbb{R}^{n^2}$. Thus $\mathcal{U} = GL(n,\mathbb{R})$ is an open set-the complement of the closed set of those A such that $det\, A \neq 0$, and we see that $\mathcal{U} = GL(n,\mathbb{R})$ is an open subpseudomanifold*

2. *S^2 is a closed, regular subpseudomanifold*

One can make the construction of a pseudomanifold from a given one by means of the following results borrowed from [9, 12].

Lemma 3.5.1 [*9*, Theorem 5.8]
Let M be an m-subpseudomanifold, N an n-pseudomanifold and $F : M \to N$ a smooth map. Suppose that F has constant rank k on M and that $q \in F(M)$. Then $F^{-1}(q)$ is closed, regular $(m-k)$-subpseudomanifold on M.

Corollary 3.5.1 [*9*, Corollary 5.9]
If $F : M \to N$ is a smooth map of pseudomanifolds with $dim\, N = n \leq dim\, M = m$ and if $rank\, F = n$ at every point of $F^{-1}(q)$, with $q \in M$ then $F^{-1}(q)$ is closed, regular subpseudomanifold of M.

3.6 Product of pseudomanifolds

Theorem 3.6.1
The finite product of pseudomanifolds is a pseudomanifold provided that each factor is endowed with a maximal atlas.

Proof.
Let $(\mathcal{U}_i, \varphi_i)$ be a local chart for $(M_i, \mathcal{C}_{M_i}, \mathcal{F}_{M_i})$, where M_i are pseudomanifolds of dimension m_i and $i = 1,\ldots,n$. thus the basic open sets in M^* have the following form $\mathcal{U}_1 \times \cdots \times \mathcal{U}_n = \bigcap_{i=1}^{n} p_i^{-1}(\mathcal{U}_i)$, where $\mathcal{U}_i$ are $\tau_{\mathcal{F}_{M_i}}$-open sets in M_i and $M_i \supset \mathcal{U}_i \simeq \varphi_i(\mathcal{U}_i) \subset \mathbb{R}^{m_i}$, with respect to the form of $p_i^{-1}(\mathcal{U}_i)$ shown after Definition 2.6.4. In the sequel, each open in M^* is an union of some basic open. Without

loss of the generality we will restrict ourself here to $n = 2$. The generality will follow the same way. Therefore, since M_1, M_2 are pseudomanifolds, we may make the $\mathbb{F}$-product $M_1 \times M_2$ a pseudomanifold by the following construction of charts: Given

$$\begin{array}{cccc} M_1 & M_2 & M_1 & M_2 \\ \mathcal{U}_1 & \mathcal{U}_2 & \mathcal{V}_1 & \mathcal{V}_2 \\ \downarrow \varphi_1 & \downarrow \varphi_2 & \downarrow \psi_1 & \downarrow \psi_2 \\ \varphi_1(\mathcal{U}_1) & \varphi_2(\mathcal{U}_2) & \psi_1(\mathcal{V}_1) & \psi_2(\mathcal{V}_2) \end{array}$$

where $(\mathcal{U}_i, \varphi_i)$ and $(\mathcal{V}_i, \psi_i)$ are charts with $i = 1, 2$. These yield $\varphi = (\varphi_1 \times \varphi_2) : \mathcal{U}_1 \times \mathcal{U}_2 \to \varphi(\mathcal{U}_1 \times \mathcal{U}_2) = \varphi_1(\mathcal{U}_1) \times \varphi_2(\mathcal{U}_2)$ and $\psi = (\psi_1 \times \psi_2) : \mathcal{V}_1 \times \mathcal{V}_2 \to \psi(\mathcal{V}_1 \times \mathcal{V}_2) = \psi_1(\mathcal{V}_1) \times \psi_2(\mathcal{V}_2)$ such that $(\mathcal{U}_1 \times \mathcal{U}_2, \varphi)$ and $(\mathcal{V}_1 \times \mathcal{V}_2, \psi)$ are charts and $\mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{V}_1 \times \mathcal{V}_2$ are basic open sets in $M_1 \times M_2$. Let $\mathcal{U} = (\mathcal{U}_1 \times \mathcal{U}_2) \cup (\mathcal{V}_1 \times \mathcal{V}_2) \subset (\mathcal{U}_1 \cup \mathcal{V}_1) \times (\mathcal{U}_2 \cup \mathcal{V}_2)$. Because M_1, M_2 satisfy to the maximality condition of atlas, that is, any union of charts is a chart in the maximal atlas. So, $\mathcal{U}_1 \cup \mathcal{V}_1 = W_1$, $\mathcal{U}_2 \cup \mathcal{V}_2 = W_2$ are again charts. The previous construction of product of charts allows us to let $W_1 \times W_2$ be a chart, this yields the following diagram:

$$\begin{array}{ccc} M_1 \times M_2 \supset \mathcal{U} & \xrightarrow{\ \iota\ } & W_1 \times W_2 \\ & \theta_{|\mathcal{U}} = \theta \circ \iota \searrow \quad \swarrow \theta = (\theta_1, \theta_2) & \\ & \theta(W_1 \times W_2) = \theta_1(W_1) \times \theta_2(W_2) & \end{array}$$

where (W_1, θ_1), (W_2, θ_2) are charts. Thereby $(\mathcal{U}, \theta_{|\mathcal{U}})$ is a chart. Among $\mathcal{U}$, built in this way, we may find out an open covering of $M_1 \times M_2$ so that $M_1 \times M_2$ is locally diffeomorphic to $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, that is $M_1 \times M_2$ is a pseudomanifold.

Finally for a general n, $M^* = \prod_{i=1}^{n} M_i$ is locally diffeomorphic to $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$, that is a pseudomanifold. □

Corollary 3.6.1
dim $\prod_{i=1}^{n} M_i = m_1 + \cdots + m_n$.

Example 3.6.1
$\mathbb{R}^n$.

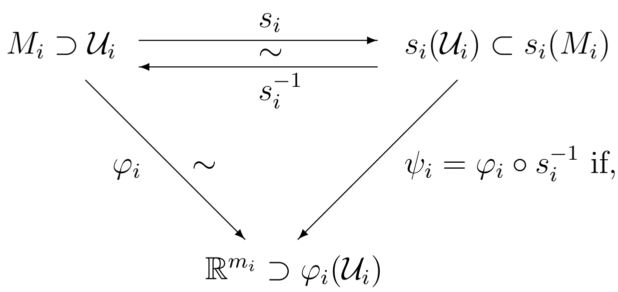
Example 3.6.2
$T^n = n$-*torus*

3.7 Coproduct of pseudomanifolds

Lemma 3.7.1
Let $(\mathcal{U}_i, \varphi_i)$ be a chart in M_i at point $p_i \in M_i$ with M_i an m_i-pseudomanifold for all $i \in I$. Then:

1. $(s_i(\mathcal{U}_i), \varphi_i \circ s_i^{-1})$ *is a chart of* $s_i(p_i)$ *in* $s_i(M_i)$ *with* $p_i \in M_i$.
2. $s_i(M_i)$ *is an* m_i*-pseudomanifold.*
3. $dim\, s_i(M_i) = dim\, M_i = m_i$.

Proof.
$s_i : M_i \to s_i(M_i) \subset \bar{M}$ is a diffeomorphism of $\mathbb{F}$-spaces, where the $\mathbb{F}$-structure on $s_i(M_i)$ is one induced by s_i , that is $g \in \mathcal{F}_{s_i(M_i)}$ if, and only if $f \circ s_i^{-1} = g$ with $f \in \mathcal{F}_{M_i}$. This structure coincides with one $s_i(M_i)$ inherits from $\bar{M}$ as an $\mathbb{F}$-subspace. We can see from the diagram below:



$\psi_i = \varphi_i \circ s_i^{-1}$ if, and only if $\varphi_i = \psi_i \circ s_i$.

that (1), (2) and (3) hold. □

Theorem 3.7.1
The finite coproduct of pseudomanifolds is a pseudomanifold.

Proof.
$\bar{M}$ is an $\mathbb{F}$-space by Definition 2.7.1. Let $\mathcal{U} \subset \bar{M}$ be an open neighborhood of an arbitrary q in $\bar{M}$. Thus $\mathcal{U} = \bigcup_{i \in I} s_i(\mathcal{U}_i)$ with $\mathcal{U}_i \subset M_i$, open. Also there exists a fixed $j \in I$, such that $q \in s_j(\mathcal{U}_j)$, where $\mathcal{U}_j \subset M_j$, open. Furthermore there exists $p \in \mathcal{U}_j$ such that $s_j(p) = q$ and $\mathcal{U}_j$ is an open neighborhood at p, since each $s_i : M_i \to \bar{M}$ is a smooth map of $\mathbb{F}$-spaces. Now, since $\{s_i(M_i)\,, i \in I\}$ is forming a partition of $\bar{M}$, it follows $\mathcal{U} = \coprod_{k \in K} s_k(\mathcal{V}_k)$, where $s_k(\mathcal{V}_k)$ is a union of all $s_i(\mathcal{U}_i)$ included in a particular $s_k(M_k)$. Afterwards, using $\psi_k \circ s_k = \varphi_k$ from Lemma 3.7.1, one should build a local diffeomorphism by means of the diagram below:

$$\begin{array}{ccccc}
\mathcal{V}_k & \underset{s_k^{-1}}{\overset{s_k}{\rightleftarrows}} \sim & s_k(\mathcal{V}_k) & \xrightarrow{\quad \iota \quad} & \mathcal{U} = \coprod_{k\in K} s_k(\mathcal{V}_k) \\
& \varphi_k \searrow \sim & \sim \Big\downarrow \psi_k = \varphi_k \circ s_k^{-1} & & \sim \Big\downarrow \psi = (\psi_k)_{k\in K} \\
& & \varphi_k(\mathcal{V}_k) & \xrightarrow{\quad s'_k \quad} & \psi(\mathcal{U}) = \coprod_{k\in K} s'_k(\varphi_k(\mathcal{V}_k))
\end{array}$$

where $\psi_{|_{s_k(\mathcal{V}_k)}} = \psi \circ \iota = s'_k \circ \psi_k = s'_k \circ \varphi_k \circ s_k^{-1}$, $\mathcal{V}_k \subset M_k$, $s_k(\mathcal{V}_k) \subset s_k(M_k)$, $\varphi_k(\mathcal{V}_k) \subset \mathbb{R}^{m_k}$, $\mathcal{U} \subset \mathbb{R}^m$ and ψ is a local diffeomorphism by construction. Thus, $(\mathcal{U}, \psi)$ is a chart at q in $\bar{M}$ with $\mathcal{U} = \coprod_{k\in K} s_k(\mathcal{V}_k)$. Finally, $\bar{M}$ is a pseudomanifold. □

Lemma 3.7.2
$s_i(M_i)$ is open, closed regular subpseudomanifold of $\bar{M}$ such that $dim\, s_i(M_i) = dim\, M_i = m_i$.

Proof.
Since $\tau_{\sqcup} = \tau_{\mathcal{F}_{\bar{M}}}$, thus the trace topology on $s_i(M_i)$, induced from $\bar{M}$, the $\mathbb{F}$-subspace topology, and the induced topology from M_i coincide. Also $s_i : M_i \to s_i(M_i)$ is an $\mathbb{F}$-diffeomorphism. $(\mathcal{V}_i, \varphi_i)$, and $(s_i(\mathcal{V}_i), \psi_i)$ are charts respectively in M_i and $s_i(M_i)$. Therefore, $\psi_i \circ s_i \circ \varphi_i : \varphi_i(\mathcal{V}_i) \to \psi_i(s_i(\mathcal{V}_i))$ is a diffeomorphism as a composite of diffeomorphisms. In the sequel $rank\, s_i = rank\, (\psi_i \circ s_i \circ \varphi_i) = m_i$ and s_i is an injective immersion. Hence $s_i(M_i)$ is an open closed, regular subpseudomanifold of $\bar{M}$ such that $dim\, s_i(M_i) = dim\, M_i = m_i$. □

Corollary 3.7.1
If $\mathcal{U} = \coprod_{i\in I} \mathcal{V}_i$ with $\mathcal{V}_i \subset M_i$ open then $\mathcal{U}$ is an open regular subpseudomanifold of $\bar{M}$.

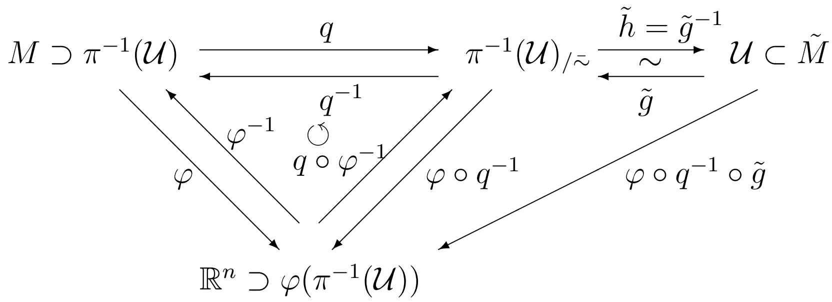
Example 3.7.1
Let $M_1 = \mathbb{R}$, $M_2 = \mathbb{R}$. Thus, $M_1 \sqcup M_2 = \{x \mid x \in M_1$ or $x \in M_2$ such that $M_1 \cap M_2 = \emptyset\}$. The open sets are $\emptyset$, M_1, M_2, $M_1 \sqcup M_2$ and $\mathcal{U} = \mathcal{U}_1 \sqcup \mathcal{U}_2$, where $\mathcal{U}_1$, $\mathcal{U}_2$ are open respectively in M_1 and M_2 such that $\mathcal{U}_1 \subset M_1$ and $\mathcal{U}_2 \subset M_2$, that is $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$.

3.8 Quotient of a pseudomanifold

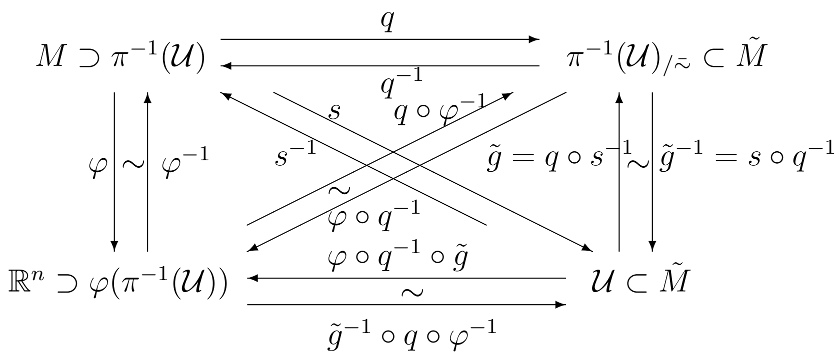
Theorem 3.8.1
The quotient of a pseudomanifold is a pseudomanifold.

Proof.
Let $\pi : M \to \tilde{M}$ be the canonical projection. Let $\mathcal{U}$ be an open neighborhood in

$\tilde{M}$, thus $\tilde{h}:\pi^{-1}(\mathcal{U})_{/\tilde{\sim}} \to \mathcal{U}$ is a diffeomorphism from Lemma 2.8.6 and $\pi^{-1}(\mathcal{U})$ is an open neighborhood in M. Assume $(\pi^{-1}(\mathcal{U}), \varphi)$ be a local chart in M , that is, $\varphi : \pi^{-1}(\mathcal{U}) \to \varphi(\pi^{-1}(\mathcal{U})) \subset \mathbb{R}^n$. We can construct a commutative diagram of smooth maps as follows with respect to Lemma 2.8.6:



It follows that $(\varphi \circ q^{-1}[x]_{/\tilde{\sim}} = \varphi \circ q^{-1}[y]_{/\tilde{\sim}})$ yields $(q^{-1}[x]_{/\tilde{\sim}} = q^{-1}[y]_{/\tilde{\sim}})$ since φ is a diffeomorphism. The surjectivity of q yields $(qq^{-1}[x]_{/\tilde{\sim}} = qq^{-1}[y]_{/\tilde{\sim}})$ implies $[x]_{/\tilde{\sim}} = [y]_{/\tilde{\sim}}$. Thus $\varphi \circ q^{-1}$ is injective smooth map. Also $(\varphi \circ q^{-1})^{-1} = q \circ \varphi^{-1}$ is a surjective smooth map. Then, $\varphi \circ q^{-1}$ is a bijective smooth map , that is, a diffeomorphism. Now, in the sequel $(\varphi \circ q^{-1}) \circ \tilde{g}$ is a diffeomorphism as a composition of diffeomorphisms. To summarize the situation, we give the following diagram



Finally, $\tilde{M}$ is a n-pseudomanifold. $\square$

Chapter 4

Tangent structures on pseudomanifolds

4.1 Tangent structures

There exist two kinds of tangent vector on an $\mathbb{F}$-space. The first one is called operational tangent vector (as in [6]) and is defined from structure functions. Instead, the second kind is called the kinematic tangent vector (as in [6]) and is defined from structure curves. We would like to provide, in the natural way, the set of operational tangent vectors and the set of kinematic tangent vectors with an $\mathbb{F}$-structure. After that, the objects of interest will be the tangent bundles, the double tangent bundles and others concepts as of vector fields, dimension, tangent maps.

Definition 4.1.1
Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an n-pseudomanifold. An operational tangent vector v to M at the point $p \in M$ is a smooth derivation (linear operator) of the algebra $\mathcal{F}_M$ at p. That is, $v := d_p = ev_p \circ d : \mathcal{F}_M \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}_M$, $\alpha \in \mathbb{R}$ we have $v(f + \alpha g) = v(f) + \alpha v(g)$ and $v(fg) = f(p)v(g) + g(p)v(g)$, the so-called Leibniz condition (or rule).

We denote by $Der(M) := \{d : \mathcal{F}_M \to \mathcal{F}_M \mid d$ is a smooth derivation of $\mathcal{F}_M$ on $M\}$ the $\mathcal{F}_M$-module containing all smooth derivations. The operational tangent vector v is also called the contravariant tangent vector or the derivative.

Lemma 4.1.1 [*6*]
Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an n-pseudomanifold and $p \in M$. Let $v : \mathcal{F}_M \to \mathbb{R}$ be a linear map. Then v is an operational tangent vector to M at p, if and only if v satisfies the following conditions: $v(f) = 0$ if f is constant, and $\mathcal{V}_{|_{\alpha_p^2}} = 0$, where $\alpha_p^2 := \{(f - f(p))(g - g(p)) \mid f, g \in \mathcal{F}_M\}$.

Definition 4.1.2
Let $(M,\mathcal{C}_M,\mathcal{F}_M)$ be an n-pseudomanifold and $p\in M$. The set $T_pM\subseteq C^\infty(\mathcal{F}_M,\mathbb{R})$ of all operational tangent vectors at p is called the operational tangent space at p on M.

Lemma 4.1.2 [6, 85]
Let $(M,\mathcal{C}_M,\mathcal{F}_M)$ be an n-pseudomanifold and $p\in M$. The operational tangent space T_pM at the point p of M is a linear Frölicher space of dimension n, a linear n-pseudomanifold, say.

Remark 4.1.1
The set $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\}$ is a basis for T_pM, where $(x_1,\dots,x_n)$ is a standard local coordinate system on M. The $\mathbb{F}$-structure on T_pM is generated by the set of functions, $\mathcal{F}_o=\{(df)_p\,|\,(df)_p:T_pM\to\mathbb{R}, v\mapsto(df)_p(v):=v(f)\}$, where $d\in Der(M)$ and $f\in\mathcal{F}_M$. Let $\mathcal{U}\subset M$ be an open neighborhood of $p\in M$. Since M is an n-pseudomanifold, thus there exists a local diffeomorphism $\varphi:\mathcal{U}\to\varphi(\mathcal{U})\subseteq\mathbb{R}^n$ such that $n=dimM=dim\mathcal{U}=dim\varphi(\mathcal{U})=dim\mathbb{R}^n=dimT_{\varphi(p)}\mathbb{R}^n=dim\,T_{\varphi(p)}\varphi(\mathcal{U})$.

Definition 4.1.3
Let $\varphi:M\to N$ be a smooth map of pseudomanifolds. Let $p\in M$ and $v\in T_pM$. The tangent map associated to φ at p is the map $\varphi_{*p}:=T_p\varphi:T_pM\to T_{\varphi(p)}N$ defined by $\varphi_{*p}(v):=v(g\circ\varphi)$ for all $g\in\mathcal{F}_N$.

Lemma 4.1.3 [6]
Let $\varphi:M\to N$ be a smooth map of pseudomanifolds. Let $p\in M$ and $v\in T_pM$. Then The tangent map $\varphi_{*p}:=T_p\varphi:T_pM\to T_{\varphi(p)}N$ defined above is linear and $\mathbb{F}$-smooth. The pair (v,φ) determines an operational tangent vector of $\mathcal{F}_N$ in a neighborhood of $\varphi(p)$ defined by $\varphi_{*p}v:\mathcal{F}_N\to\mathbb{R}$ such that $\varphi_{*p}v(g)=(gcirc\varphi)$.

Lemma 4.1.4 [6]
Let M be an n-pseudomanifold. The following conditions are equivalent:

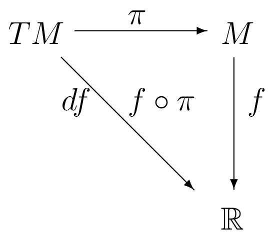
1. *n tangent vectors are linearly independent;*
2. *For all smooth functions $f\in\mathcal{F}_M$, the map $\theta:=(v_1,\dots,v_n):\mathcal{F}_M\to\mathbb{R}^n$ is a surjective map;*
3. *There exists n smooth functions $f_1,\dots,f_n\in\mathcal{F}_M$ such that $v_i(f_j)=\delta_{ij}$, where δ_{ij} is the Krönecker symbol;*
4. *There exists n smooth functions $f_1,\dots,f_n\in\mathcal{F}_M$ such that $det\,(v_i(f_j))\neq 0$.*

Definition 4.1.4
Let M be an n-pseudomanifold and $p\in M$. The algebraic dual of the operational tangent space at p, T_pM, denoted by $T_p^*M=\{\theta:T_pM\to\mathbb{R}\,|\,\theta$ is smooth linear$\}$ is called the operational cotangent space at $p\in M$. The elements of T_p^*M are called covariant tangent vectors or covectors for short.

Note that T_p^*M is a linear $\mathbb{F}$-space of dimension n, with respect to the Cartesian closedness of $\mathcal{F}RL$. Its basis is $\{dx_1, \ldots, dx_n\}$ where $\{x_1, \ldots, x_n\}$ is a local coordinate system of $p \in M$. From linear algebra theory we have $n = dimT_pM = dimT_p^*M$.

Definition 4.1.5
Let M be an n-pseudomanifold and p is running through M. Let the set denoted by $TM := \coprod_{p\in M}\{p\} \times T_pM = M \times (\coprod_{p\in M} T_pM) = \{(p, v_p) \mid p \in M,\ v_p \in T_pM\}$. That is, $TM \subseteq M \times Der(M) \subseteq M \times C^\infty(\mathcal{F}_M, \mathbb{R})$. Let $T^*M = \{(p, \theta_p) \mid p \in M, \theta_p \in T_p^*M\} = \coprod_{p\in M}\{p\}\times T_p^*M = M\times(\coprod_{p\in M} T_p^*M)$. Then TM is called the operational tangent bundle on M, and T^*M is called the operational cotangent bundle on M.

Remark 4.1.2
There exists natural projections defined as follows: $\pi : TM \to M, (p, v_p) \mapsto p$ and $\tau : T^*M \to M$, $(p, \theta_p) \mapsto p$. The family $(s_i)_{i\in I}$ of inclusion maps in Sections 2.7, 3.7, is here replaced by the families $(\iota_p)_{p\in M}$ and $(\tilde{\iota}_p)_{p\in M}$ of canonical inclusion maps $\iota_p : T_pM \hookrightarrow TM$ and $\tilde{\iota}_p : T_p^*M \hookrightarrow T^*M$. At each point $p \in M$, $d : \mathcal{F}_M \to \mathcal{F}_M$, induces a map $d_p : \mathcal{F}_M \to \mathbb{R}$ such that, for all $f \in \mathcal{F}_M$, $d_p(f) = (df)_p = ev_p(df) = (ev_p \circ d)(f)$ with ev_p the evaluation map at p. It follows that $d_p = ev_p \circ d$ is a smooth linear map and a derivation. As $(df)_p$ is defined for each $p \in M$. Then it determines globally a smooth map $df : TM \to \mathbb{R}$ such that $(df)_{|T_pM} = (df)_p = d_p(f)$. Also, $\pi^{-1}(p) = \{v \in TM \mid \pi(v) = p\} = T_pM$ is the fiber of TM at p and $\tau^{-1}(p) = T_p^*M$ is the fiber of T^*M at p. Let M be an n-pseudomanifold. Let $(\mathcal{U}, \varphi)$ be local chart. T_pM and T_p^*M are linear n-pseudomanifolds diffeomorphic to $\mathbb{R}^n$ with respective basis $\{\frac{\partial}{\partial x_i}\}$ and $\{dx_i\}$, where (x_i) are local coordinates of $p \in \mathcal{U} \subset M$ such that $\varphi(p) = (x_1, \ldots, x_n)$. $(x, v) \in TM$ is given in local coordinates by $(x_i, \frac{\partial}{\partial x_i})$ whereas $(x, \theta) \in T^*M$ is given by (x_i, dx_i). Thus TM and T^*M are both $2n$-pseudomanifolds. The $\mathbb{F}$-structure on TM is generated by the set of functions $\mathcal{F}_o = \{df \mid f \in \mathcal{F}_M\} \cup \{f \circ \pi \mid f \in \mathcal{F}_M\}$.



Thus, $(TM, \Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o) := (TM, T\mathcal{C}_M, T\mathcal{F}_M)$.

Definition 4.1.6
Let $\varphi : M \to N$ be a smooth map of pseudomanifolds and $p \in M$.

1. *φ is an immersion if for any $p \in M$, $d_p\varphi = \varphi_{*p} : T_pM \to T_{\varphi(p)}N$ is a monomorphism.*

2. *φ is an embedding if φ is an injective immersion.*

3. *$\varphi_* : TM \to TN$ defined by $\varphi_*(p, v_p) = (\varphi(p), \varphi_{*p}(v_p))$ is called the tangent map to φ.*

Lemma 4.1.5 [*85*]
Let $\varphi : M \to N$ be a smooth map of pseudomanifolds. Then $\varphi_* : TM \to TN$ defined by $\varphi_*(p, v_p) = (\varphi(p), \varphi_{*p}(v_p))$ is a smooth map. Moreover, it is one-to-one, onto and diffeomorphism if φ is so.

Definition 4.1.7
Let M be an n-pseudomanifold and $p \in M$. A map $\chi : M \to TM$ defined by $p \mapsto v_p \in T_pM$ such that $\pi \circ \chi = id_M$ is called a tangent vector field to M (or a section of π). That is, $\chi(p) : \mathcal{F}_M \to \mathbb{R}$ with $f \mapsto \chi(p)(f) = (\chi f)(p) = v_p(f)$ and $\chi f \in \mathbb{R}^M$, for any $f \in \mathcal{F}_M$. χ is a smooth tangent vector field if $\chi f = \chi(f) \in \mathcal{F}_M$. That is, $\chi : \mathcal{F}_M \to \mathcal{F}_M$ is a smooth derivation.

Remark 4.1.3
Definition 4.1.7 *gives both local and global interpretation of the concept of tangent vector field. The evaluation of χ at p can be understood as follows. $ev_p \circ \chi : \mathcal{F}_M \to \mathcal{F}_M \to \mathbb{R}$, $f \mapsto \chi(f) \mapsto (ev_p \circ \chi)(f) = \chi(p)(f) = \chi(f)(p)$. The set of all smooth tangent vector fields on M is denoted by $\mathfrak{X}(M)$.*

Lemma 4.1.6 [*6*]
Let M be an n-pseudomanifold and $f \in \mathcal{F}_M$. Let χ be a tangent vector field on M. Then χ is smooth if, and only if $(f \circ \pi) \circ \chi \in \mathcal{F}_M$ and $df \circ \chi \in \mathcal{F}_M$. There exists $\chi^* : T^*M \to \mathbb{R}$ defined by $\chi^*(\theta) = \theta(\chi(\tau(\theta)))$, where $\theta \in T^*M$ and τ is the canonical projection $\tau : T^*M \to M$.

Remark 4.1.4
Let M be an n-pseudomanifold. The cotangent bundle T^*M on M has the natural structure generated by the set of functions $G_o = \{\chi^* \mid \chi \in \mathfrak{X}(M)\} \cup \{f \circ \tau \mid f \in \mathcal{F}_M\}$. Let $\varphi : M \to N$ be a diffeomorphism of pseudomanifolds. The following diagrams are commutative:

$$
\begin{array}{ccc}
TM & \underset{(\varphi_*)^{-1}}{\overset{\varphi_*}{\rightleftarrows}} & TN \\
{\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\
M & \xrightarrow{\ \varphi\ } & N
\end{array}
\qquad
\begin{array}{ccc}
T^*M & \underset{\varphi^*}{\overset{(\varphi^*)^{-1}}{\rightleftarrows}} & T^*N \\
{\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau_N} \\
M & \xrightarrow{\ \varphi\ } & N
\end{array}
$$

with $T^*\varphi := \left((\varphi)^{-1}\right)^* = \left(\varphi^*\right)^{-1}$ and $\varphi^* = (T^*\varphi)^{-1}$ such that $\varphi^*(\theta) := \theta \circ \varphi_* = \alpha$ if, and only if $(\varphi^*)^{-1}(\alpha) := \alpha \circ \varphi_*^{-1} = \theta$.

Lemma 4.1.7
Let M be an n-pseudomanifold and $\pi : TM \to M$ the canonical projection.

1. *If $\mathcal{U}$ is an open subset of M, then*

   (a) *$T\mathcal{U} := \pi^{-1}(\mathcal{U}) \subset TM \subset M \times (\bigcup_{p\in M} T_pM)$ is $\mathbb{F}$-diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$.*

   (b) *$T_p\mathcal{U} \simeq T_pM$, that is, $dim\, T_p\mathcal{U} = dim\, \mathbb{R}^n = n = dim T_pM$*

2. *The $\mathbb{F}$-substructure on $\pi^{-1}(p) = T_pM \subset TM$ coincides with the $\mathbb{F}$-structure generated by $\mathcal{G}_o = \{d_p(f) \mid f \in \mathcal{F}_M, d \in Der(M)\}$*

Proof.
The proof of the first part is given in [6]. Now, for the second part recall that T_pM open and closed in TM. Thus $(df)_{|T_pM} = (df)_p = d_p(f)$ and $f \circ \pi_{|T_pM} : T_pM \to M \to \{f(p) \mid f \in \mathcal{F}_M\} \subset \mathbb{R}$ is constant map, for every $f \in \mathcal{F}_M$. Let $\mathcal{F}_p$ be the set of these constant functions. From Remark 4.1.2 we have $\mathcal{F}_{o|T_pM} = \mathcal{G}_o \cup \mathcal{F}_p$ and $T\mathcal{F}_{M_{|T_pM}} = \Phi\Gamma\mathcal{F}_{o|T_pM} = \Phi\Gamma\mathcal{G}_o \cup \Phi\Gamma\mathcal{F}_p$ with respect to Lemmas 2.1.4 and 2.1.5. Note that each structure functions set contains constant functions. Thus, $\mathcal{F}_p \subset \Phi\Gamma\mathcal{G}_o$. It follows that $\Phi\Gamma\mathcal{F}_p \subset \Phi\Gamma\mathcal{G}_o$ with respect to Lemma 2.1.2. Hence, $T\mathcal{F}_{M|T_pM} = \Phi\Gamma\mathcal{G}_o$. □

Definition 4.1.8
Let M be an $\mathbb{F}$-space. M is said to be of constant dimension n if either

1. *$dim\, T_pM = dim\, T_qM = n$ for any $p, q \in M$, with $p \neq q$ and for all $v \in T_pM$, there exists $\chi \in \mathfrak{X}(M)$ such that $\chi(p) = v$; or*

2. *for each $p \in M$, there exists an open neighborhood $\mathcal{U}$ of p in M and a local basis of vector fields over $\mathcal{U}$ making $\mathfrak{X}(\mathcal{U})$ a free module on $\mathcal{F}_M$.*

Definition 4.1.9
Let M be an n-pseudomanifold. Let $C_M^{a,p}$ be the set of all structure curves $c : \mathbb{R} \to M$ such that $c(a) = p$, with $a \in \mathbb{R}$ and $p \in M$. That is, the set of curves passing through p, with the foot point a. The kinematic tangent vector to the space M, with foot point a, $c \in C_M^{a,p}$ and $f \in \mathcal{F}_M$, is a derivation $\chi_{c,a} : \mathcal{F}_M \to \mathbb{R}$ defined by $\chi_{c,a}(f) := \frac{d}{dt}(f \circ c)_{|t=a} = (df)(c(a)) = (df)(p) = d_p(f)$, where $d_p \in T_pM$. The tangent cone space at $p \in M$, denoted by $T_pCM = \{\chi_{c,a} \mid c \in C_M^{a,p}\}$, is the set of all kinematic vectors at $p \in M$. If T_pCM is linear then the cotangent cone space at $p \in M$, denoted by T_p^*CM, is the algebraic dual of T_pCM.

T_pCM can fail to be linear in a general $\mathbb{F}$-space. In Proposition 4.1.1 we will prove its linearity.

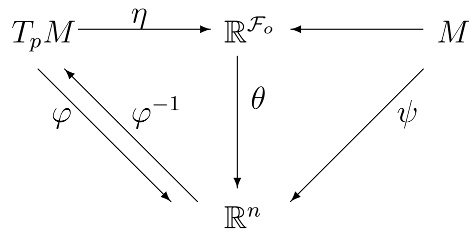
Definition 4.1.10
Let M be an n-pseudomanifold and $p \in M$. The disjoint union of tangent cone spaces T_pCM at each $p \in M$ is called the tangent cone bundle and denoted by TCM. The algebraic dual of TCM, denoted by T^*CM, is called the cotangent cone bundle.

The straightforward consequence of the definition above is that $T_pCM \subset T_pM$ by Definition 4.1.9.

Lemma 4.1.8 [6]
Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an n-pseudomanifold. Let $\{f_1, \ldots, f_n\}$ be a generating set of $\mathbb{F}$-structure on M such that the map given by $\psi(p) = (f_1(p), \ldots, f_n(p))$ for all $p \in M$ is one-to-one. Then the associated tangent map $\psi_{*p} : T_pM \rightarrow T_{\psi(p)}\psi(M)$ is an isomorphism of linear spaces.

Lemma 4.1.9
Let M be an $\mathbb{F}$-space and $p \in M$ and $\mathcal{F}_o = \{f_1, \ldots, f_n\} \subset \mathcal{F}_M$ the generating set of $\mathbb{F}$-structure on M such that $\psi = (f_1, \ldots, f_n)$ is one-to-one. Then the map $\eta : T_pM \longrightarrow \mathbb{R}^{\mathcal{F}_o}$ defined by $\eta(v) := v_{|\mathcal{F}_o}$ is an isomorphism of linear spaces.

Proof.
$M \simeq^{\psi} \psi(M) \subset \mathbb{R}^n$. Thus $dim\, M = dim\, \psi(M) = dim\, T_pM = dim\, T_{\psi(p)}\psi(M)$. It is known that $\mathbb{R}^n \simeq^{\theta} \mathbb{R}^{\mathcal{F}_o}$. It is known that $\mathbb{R}^n \simeq^{\varphi} T_pM$. Thus, the diagram below



commutes. That is $\eta = \theta^{-1} \circ \varphi$ and $\eta^{-1} = \varphi^{-1} \circ \theta$. It follows that η is an isomorphism of linear space as the composition of isomorphisms with $\eta(v) = \theta^{-1}(\varphi(v))$. Since $\varphi(v) = (v_1, \cdots, v_n) \in \mathbb{R}^n$, and $v_{|\mathcal{F}_o} := (v(f_1), \cdots, v(f_n))$, we set $\theta(v(f_1), \cdots, v(f_n)) = (v_1, \cdots, v_n) = \varphi(v)$. Therefore, $\theta^{-1}(\varphi(v)) = \theta^{-1}(v_1, \cdots, v_n) = (v(f_1), \cdots, v(f_n)) = v_{|\mathcal{F}_o}$. Hence $\eta(v) = (\theta^{-1} \circ \varphi)(v) = v_{|\mathcal{F}_o}$. □

Corollary 4.1.1
Let M be an n-pseudomanifold. Let $\mathcal{U}$ be an open neighborhood of $p \in M$. Then there exists n smooth functions $f_1, \cdots, f_n \in \mathcal{F}_M$ such that $\{(df_1)_p, \cdots, (df_n)_p\}$ is a basis on T_p^*M corresponding to a basis $\{v_1, \cdots, v_n\}$ of T_pM.

Proof.
Assume M an n-pseudomanifold. That implies $dim\, M = dim\, T_pM = n$ for all $p \in M$. From Lemma 4.1.4, there exists n tangent vectors $v_1, \cdots, v_n$ linearly independent and forming a basis on T_pM. The dual basis $\{v^{1*}, \cdots, v^{n*}\}$ on T_p^*M

is defined by $v_i(v^{j*})=\delta_{ij}$, where $v^{j*}:T_pM\longrightarrow\mathbb{R}$ and δ_{ij} is the Krönecker symbol. That is, $v^{j*}=f_j\circ\pi$. It is the same to say that for every $f\in\mathcal{F}_M$, the map $\theta:=(v_1,\cdots,v_n):\mathcal{F}_M\longrightarrow\mathbb{R}^n$ defined by $\theta(f):=(v_1(f),\cdots,v_n(f))$ is a surjective map. Now, $v_i(v^{j*})=v_i(f_j\circ\pi)=d_p(f_j\circ\pi)=d(f_j(p))=v_i(f_j)=\delta_{ij}$. Equivalently, there exists n functions $f_1,\cdots,f_n\in\mathcal{F}_M$ such that $\{(df_1)_p,\cdots,(df_n)_p\}$ is a basis on T_p^*M since $v^{j*}=(df_j)_p$. □

Definition 4.1.11
Let $(M,\mathcal{C}_M,\mathcal{F}_M)$ be an n-pseudomanifold and $p\in M$. Let $\{f_1,\cdots,f_n\}\subset\mathcal{F}_M$ generates the $\mathbb{F}$-structure on M, such that $\varphi(p):=(f_1(p),\cdots,f_n(p))$ is an $\mathbb{F}$-diffeomorphism on a neighborhood of p onto an $\mathbb{F}$-subspace of $\mathbb{R}^n$ endowed with the canonical $\mathbb{F}$-structure. Two curves c and d in $\mathcal{C}_M^{o,p}$ are said to be tangent at p and that is denoted by $c\sim d$, if $d(\varphi\circ c)_{|_o}=d(\varphi\circ d)_{|_o}$.

Clearly, $\sim$ does not depend on the choice of generating sets. Also $\sim$ is an equivalent relation. The equivalence class of a curve c is denoted by $\hat{c}$. Therefore, $T_pCM=\{\hat{c}\mid c\in\mathcal{C}_M^{o,p}\}$.

Proposition 4.1.1
Let M be an n-pseudomanifold and $p\in M$. Let $\mathcal{U}$ be an open neighborhood of p in M and $\hat{c}$ the equivalence class of all structure curves at p. Then:

1. *$\theta:\mathbb{R}^n\longrightarrow T_pCM$ given by $v\longmapsto\theta(v)=\hat{c}$ is an isomorphism of linear spaces.*
2. *$T_pCM=T_pM$*
3. *$dim\,TCM=dim\,TM=2n$*

Proof.

1. Since M is an n-pseudomanifold, then for every $p\in M$, there exists ψ an $\mathbb{F}$-diffeomorphism $\psi:\mathcal{U}\longrightarrow\psi(\mathcal{U})\subset\mathbb{R}^n$ onto an open $\mathbb{F}$-subspace of $\mathbb{R}^n$, where $\psi(p)+tv\in\mathbb{R}^n$ for any $t\in\mathbb{R}$ and $v\in\mathbb{R}^n$. We set $c_v(t)=\psi^{-1}(\psi(p)+tv)\subset M$ is a smooth curve which passes through $p\in M$. Let $v,w\in\mathbb{R}^n$. We set $\theta(v)=\theta(w)$. That is, $\hat{c}_v=\hat{c}_w$ by definition of θ and c_v is tangent to c_w at p. That means $(d(\psi\circ c_v))_o=(d(\psi\circ[\psi^{-1}(\psi(p)+tv)]))=(d(\psi(p)+tv))_o=d_o(\psi(p))+d_o(tv)=o+d_o(tv)=\frac{d(vt)}{dt}=v_{|_o}=v$. It will similarly be proven that $(d(\psi\circ c_w))_o=w$. It follows that θ is injective. Let $\hat{c}\in T_pCM$ and c is its representative. Let $v=(d(\psi cir c))_o$ be a vector in $\mathbb{R}^n$. Then $v=(d(\psi\circ c_v))_o=(d(\psi\circ c))_o$. This means that c_v is tangent to c at p. Thus, $\hat{c}=\hat{c}_v=\theta(v)$. Therefore, for all $\hat{c}\in T_pCM$, there exists $v\in\mathbb{R}^n$ such that $\hat{c}=\theta(v)$. Hence θ is surjective. Finally, θ induces a linear structure on T_pCM the one of $\mathbb{R}^n$ by setting $\hat{c}+t\hat{d}=\theta(v)+t\theta(w):=\theta(v+tw)$. It follows that $\theta(\mathbb{R}^n)=T_pCM$ is a linear space isomorphic to $\mathbb{R}^n$.
2. $\theta\circ\varphi:T_pM\longrightarrow T_pCM$ is an isomorphism of linear spaces. Also $T_pCM\subset T_pM$. Thus $T_pM=T_pCM$.

3. From (2) $TCM = TM$. Thus $dim\, TCM = dim\, TM = 2n$. □

4.2 Double tangent and cotangent structures.

In what follows we suppose the reader is familiar with the concept of bundle and its pullback in differentiable manifolds [45, 43], and in the differential spaces [38, 37]. More on the concept of bundle in $\mathbb{F}$-spaces can be found in [62, 85]. So the notions of tangent $\mathbb{F}$-bundle and cotangent $\mathbb{F}$-bundle on an n-pseudomanifold look like those of tangent and cotangent bundles on an n-dimensional smooth manifold.

Definition 4.2.1
Let E, M be $\mathbb{F}$-spaces and $\pi : E \to M$ a smooth surjective map. The $\mathbb{F}$-bundle (E, π, M) is the bundle in the category $\mathcal{FRL}$ of $\mathbb{F}$-spaces where E is called the total space, M the base space and π the projection (or submersion) of the $\mathbb{F}$-bundle. Moreover, for any $p \in M$, $E_p := \pi^{-1}(p)$ is called the fiber over p of the $\mathbb{F}$-bundle.

Example 4.2.1
$(\mathbb{R}^n, \pi_i, \mathbb{R})$ is an $\mathbb{F}$-bundle where $\mathbb{R}^n$ is the total space, $\mathbb{R}$ the base space, $\pi_i : \mathbb{R}^n \to \mathbb{R}$ the projection and $\pi^{-1}(x_i) := \{x = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n \mid \pi_i(x) = x_i\}$ is the fiber over $x_i \in \mathbb{R}$ of the $\mathbb{F}$-bundle.

Example 4.2.2
Let M_i be a finite dimensional pseudomanifolds with $i = 1, \ldots, m$. (M^*, π_i, M_i) is an $\mathbb{F}$-bundle where the product M^* is the total space, each M_i the base space, and $\pi_i : M^* \to M_i$ the canonical projection. The fiber over $x_i \in M_i$ is given by $\pi^{-1}(x_i) := \{x = (x_1, \ldots, x_i, \ldots, x_n) \in M^* \mid \pi_i(x) = x_i\}$.

Example 4.2.3
$(M, \pi, \tilde{M})$ is an $\mathbb{F}$-bundle where M is an n-pseudomanifold, $\tilde{M}$ the quotient pseudomanifold and $\pi : M \to \tilde{M}$ the canonical surjection. $\pi^{-1}([p]) := \{x \in M \mid x \sim p\}$ is the fiber over $[p] \in \tilde{M}$ of the $\mathbb{F}$-bundle.

Example 4.2.4
Obviously, from Definition 4.1.5 and Remark 4.1.2, (TM, π, M) and (T^*M, τ, M) are $\mathbb{F}$-bundles.

Definition 4.2.2
Let (E, π, M) be an $\mathbb{F}$-bundle and $E' \subset E$, $N \subset M$ are $\mathbb{F}$-subspaces. The triple (E', τ, N) with $\tau = \pi_{|_{E'}}$ is called an $\mathbb{F}$-subbundle of the $\mathbb{F}$-bundle (E, π, M).

Definition 4.2.3
Let (E, π, M) and (E', τ, N) be $\mathbb{F}$-bundles. Let $H : E \to E'$ and $h : M \to N$ two $\mathbb{F}$-smooth maps. The pair (H, h) is called an $\mathbb{F}$-bundle morphism of (E, π, M) and (E', τ, N) if $\tau \circ H = h \circ \pi$.

It is also said that H is an $\mathbb{F}$-bundle morphism over h instead of the pair (H, h) is $\mathbb{F}$-bundle morphism. Finally, $H(\pi^{-1}(p)) = \tau^{-1}(h(p))$ where $p \in M$, $h(p) \in N$.

Example 4.2.5
If (E', τ, N) is an $\mathbb{F}$-subbundle of (E, π, M) then $\pi \circ \mathfrak{I} = \iota \circ \tau$, where $\mathfrak{I} : E' \to E$ and $\iota : N \to M$, two canonical injections. Thus $(\mathfrak{I}, \iota)$ is an $\mathbb{F}$-bundle morphism. That is, $\mathfrak{I}$ is $\mathbb{F}$-bundle morphism over ι.

Example 4.2.6
Let M, N be two $\mathbb{F}$-spaces. Let $\varphi : M \to N$ be an $\mathbb{F}$-smooth map. The following diagram shows the examples of $\mathbb{F}$-morphisms over φ.

$$\begin{array}{ccc} TM & \xrightarrow{\quad T\varphi \quad} & TN \\ \Big\downarrow \pi_M & & \Big\downarrow \pi_N \\ M & \xrightarrow{\quad \varphi \quad} & N \end{array}$$

with $\pi_N \circ T\varphi = \varphi \circ \pi_M$

$$\begin{array}{ccc} T^*M & \xrightarrow{\ T^*\varphi = [(T\varphi)^{-1}]^* \ } & T^*N \\ \Big\downarrow \tau_M & & \Big\downarrow \tau_N \\ M & \xrightarrow{\quad \varphi \quad} & N \end{array}$$

with $\tau_N \circ T^*\varphi = \varphi \circ \tau_M$ if φ is an $\mathbb{F}$-diffeomorphism.

$$\begin{array}{ccc} T^*N & \xrightarrow{\ (T\varphi)^* = (T^*\varphi)^{-1} \ } & T^*M \\ \Big\downarrow \tau_N & & \Big\downarrow \tau_M \\ N & \underset{\varphi}{\overset{\varphi^{-1}}{\rightleftarrows}} & M \end{array}$$

with $\tau_M \circ (T^*\varphi)^{-1} = \varphi^{-1} \circ \tau_N$ if φ is an $\mathbb{F}$-diffeomorphism.
In what above we have made use of $(T\varphi)^{-1}$. This is a straightforward application of the facts that: $T\varphi$ is an injective smooth map, a surjective smooth map or an $\mathbb{F}$-diffeomorphism if φ is is an injective smooth map, a surjective smooth map or an $\mathbb{F}$-diffeomorphism.

It is worth noticing that if H and h in Definition 4.2.3 are $\mathbb{F}$-diffeomorphisms then the $\mathbb{F}$-bundle (E,π,M) and (E',τ,N) are said to be $\mathbb{F}$-diffeomorphic. Moreover, if $M=N$ and $h=id_M$ then H is an $\mathbb{F}$-diffeomorphism and the $\mathbb{F}$-bundles above are said to be equivalent. The class of n-pseudomanifolds is closed to all constructions made in Section 4.2. Also we can say that these constructions look like those defined on the n-dimensional smooth manifolds.

Definition 4.2.4 [*5*, *23*, *87*, *59*]
Let M be an n-pseudomanifold. The set $TM_o=\{(p,y)\in TM \mid p\in M, y\in T_pM,\ y\neq 0\}$ is called a slit tangent bundle over M.

Since $T_pM_o = T_pM - \{0\} \subset T_pM \subset TM$ and $TM = \bigsqcup_{p\in M}\{p\}\times T_pM$, then TM is a balanced space and the coproduct topology coincides with the underlying $\mathbb{F}$-topologies with respect to Section 2.7. It follows that T_pM_o is an open set in T_pM. Thus, $dim\,T_pM_o = n$, $TM_o = \bigsqcup_{p\in M}\{p\}\times T_pM_o$ is an open set in TM, and so $dim\,TM_o = 2n$. That is T_pM_o and TM_o are respectively n-pseudomanifold and $2n$-pseudomanifold. We are able to construct the tangent and the cotangent $\mathbb{F}$-bundles of both TM and T^*M denoted by $T(TM)$, $T^*(TM)$, $T(T^*M)$ and $T^*(T^*M)$.

Definition 4.2.5
$T(TM)$ is the tangent $\mathbb{F}$-bundle of TM called the second tangent bundle. $T(T^*M)$ is the tangent $\mathbb{F}$-bundle of T^*M. $T(TM)$ is the cotangent $\mathbb{F}$-bundle of TM. $T^*(T^*M)$ is the cotangent $\mathbb{F}$-bundle of T^*M.

Definition 4.2.6
Let E', N, M be pseudomanifolds of finite constant dimensions. Let (E',τ,N) be an $\mathbb{F}$-bundle and $h:M\to N$ a $\mathbb{F}$-smooth map. Let $E\subset M\times E'$ be an $\mathbb{F}$-subspace defined by $E=\{(p,x)\in M\times E' \mid h(p)=\tau(x)\}$, the projection $\pi:E\to M$ defined by $\pi(p,x)=p$ and $H:E\to E'$ defined by $H(p,x)=x$. The $\mathbb{F}$-bundle (E,π,M) is called the pullback $\mathbb{F}$-bundle over M of the $\mathbb{F}$-bundle (E',τ,N) by h and denoted by $h^*(E')$ or sometimes by $h^*(E',\tau,N)$. That is, $(E,\pi,M) \coloneqq (h^*(E'),h^*(\tau),M)$.

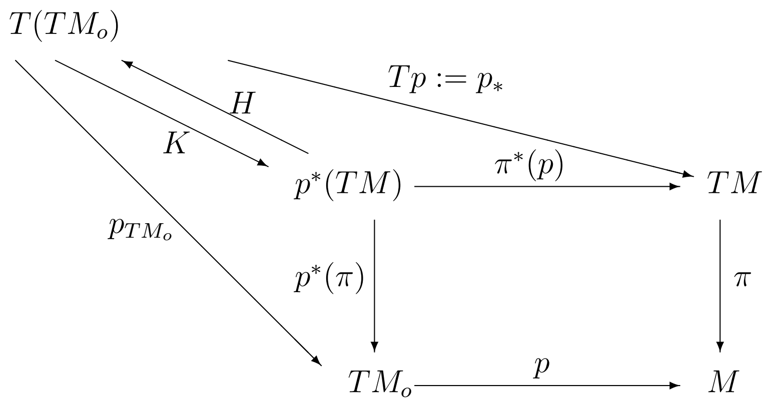
Definition 4.2.7
Let (x_i,y_i) be a standard local coordinate system in TM, where (x_i) is a local coordinate system in M and (y_i) is a global system of components of $y\in T_xM$ such that $y=\sum y_i\frac{\partial}{\partial x_i}$ or $y=y_i\frac{\partial}{\partial x_i}$ by Einstein Convention. Naturally $\{\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_i}\}$ and $\{dx_i,dy_i\}$ are local coordinate systems for $T(TM_o)$ and $T^*(TM_o)$ respectively.

Example 4.2.7 [*32*, *59*, *23*, *51*]

1. *The pullback of the $\mathbb{F}$-bundles (TM,π,M) and (T^*M,τ,M) yield the pullback of the $\mathbb{F}$-bundles $(p^*(TM),p^*(\pi),TM_o)$ and $(p^*(T^*M),p^*(\tau),TM_o)$*

over TM_o, respectively called the pullback tangent $\mathbb{F}$-bundle and the pullback cotangent $\mathbb{F}$-bundle. The diagrams below describe objects and maps related to the pullback concepts. Firstly,

2. *We need to understand this diagram, where*

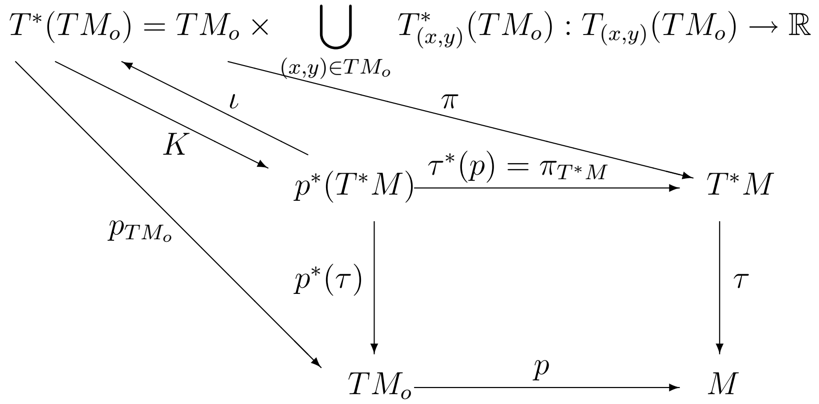


Let $\mathcal{U}$ an open set in M. Thus $\mathcal{U} \to \varphi(\mathcal{U}) \subset \mathbb{R}^n$ such that $q \mapsto (q_1, \ldots, q_n)$ is the coordinate system. $T(TM_o) \subset T(TM)$ is the tangent $\mathbb{F}$-bundle of TM_o and it is a subpseudomanifold of the pseudomanifold $T(TM)$. p is the restriction of π to $TM_o \subset TM$ such that $\pi^{-1}(\mathcal{U})$ is an open set in TM for the identification topology and $p : (q_i, \hat{q}_i) \mapsto q_i$. p_{TM_o} is the projection of $T(TM_o)$ on TM_o. That is the restriction of the projection p_{TM} of $T(TM)$ on TM such that $p_{TM}^{-1} p^{-1}(\mathcal{U}) = (T_p)^{-1}\pi^{-1}(\mathcal{U})$ is an open set in $T(TM)$. Also $p_{TM} : ((q_i, \hat{q}_i), (dq_i, d\hat{q}_i)) \mapsto (q_i, \hat{q}_i)$ and $p^*(TM) := \bigcup_{q \in M} p^{-1}(q) \times T_q M \subset TM_o \times TM$. $p^*(\pi)$ is the restriction of the projection of $TM_o \times TM$ on TM_o to $p^*(TM)$ such that $p^*(\pi) : (q_i, \hat{q}_i, dq_i) \mapsto (q_i, \hat{q}_i)$ and $(p^*(\pi))^{-1} p^{-1}(\mathcal{U})$ is an open set in $p^*(TM)$. $p^*(\pi)$ is the restriction of the projection of $TM_o \times TM$ on TM to $p^*(TM)$ such that $p^*(\pi) : (q_i, \hat{q}_i, dq_i) \mapsto (q_i, dq_i)$. T_p is the tangent map associated to p such that $T_p = p_{TM}$. H is an injection given by $(q_i, \hat{q}_i, dq_i) \mapsto (q_i, \hat{q}_i, 0, dq_i)$. $K : (q_i, \hat{q}_i, dq_i, d\hat{q}_i) \mapsto (q_i, \hat{q}_i, dq_i)$. is a surjection. Let (q_i) be a local coordinates system on $\mathcal{U} \subset M$. Thus $(q_i, \hat{q}_i)$ and $(q_i, \hat{q}_i, dq_i, d\hat{q}_i)$ are respectively the local coordinates system on $\pi^{-1}(\mathcal{U})$ in $T(M)$ and $p_{TM}^{-1} p^{-1}(\mathcal{U})$ in $T(TM)$.

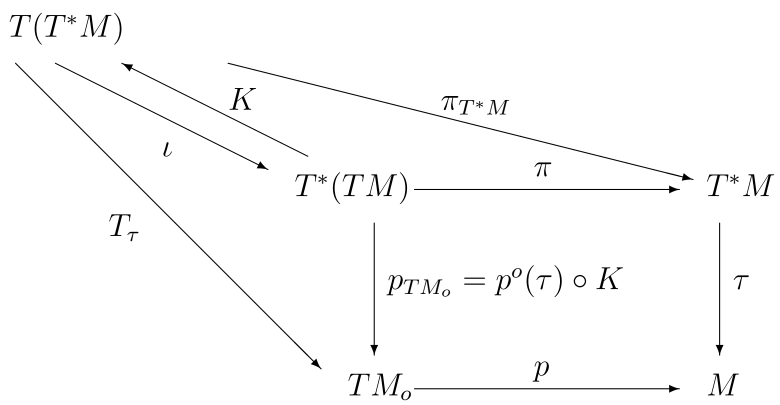
3. *For derivation of the form of elements in p^*TM and p^*T^*M we need to define the maps (p_1, T_p) and (τ_1, π_2). By using the characterization of elements in $T(TM_o) = TM_o \times \bigcup_{(x,y) \in TM_o} T_{(x,y)}(TM_o) = \{((x,y), V) \mid (x,y) \in TM_o, V \in T_{(x,y)}(TM_o)\} = \{(x, y, V) \mid x \in M, y \in T_x M_o, V \in T_{(x,y)}(TM_o)\}$. Now $V \in T_{(x,y)}(TM_o)$ if, and only if $V : \mathcal{F}_{TM_o} \to \mathbb{R}$. The diagram reveals (p_1, p_*) where $p_1 : T(TM_o) \to TM_o$ is the projection and $p_* : T(TM_o) \to TM$ is the tangent bundle map. It follows that the tangent map at $(x,y) \in TM_o$*

is given by $p_*(x,y) : T_{(x,y)}(TM_o) \to T_{p(x,y)}M \simeq T_xM$ *such that* $p_{*(x,y)} : \mathcal{F}_M \to \mathbb{R}$ *and* $W = p_{*(x,y)}V \in T_xM$. *Therefore,* $(p_1, p_{*(x,y)})((x,y),V) = (p_1((x,y),V), p_{*(x,y)}((x,y),V)) = ((x,y),W) \in TM_o \times T_xM \subset p^*TM_{|(x,y)}$. *Since* $(p_1, p_{*(x,y)})$ *is surjective* $p^*TM_{|(x,y)} = im\,(p_1, p_{*(x,y)})$*, with "im" for image.*

4. *The second diagram is also worthy of explanations;*



$T^*(TM_o) \subset T^*(TM)$ *is the cotangent* $\mathbb{F}$*-bundle of* TM_o *and it is a subpseudomanifold of the pseudomanifold* $T^*(TM)$. p *is as in (2).* p_{TM_o} *is the projection of* $T^*(TM_o)$ *on* TM_o. *That is the restriction of the projection* p_{TM} *of* $T^*(TM)$ *on* TM. $p^*(T^*M) = (p^*(TM))^* := \bigcup_{q\in M} p^{-1}(q) \times \tau^{-1}(q) \subset TM_o \times T^*M$. $p^*(\tau)$ *is the restriction of the projection of* $TM_o \times T^*M$ *on* TM *to* $p^*(T^*M)$. $\tau^*(p) = \pi_{T^*M}$ *is the restriction of the projection of* $TM_o \times T^*M$ *on* T^*M *to* $p^*(T^*M)$. $\pi : ((x,y),(\alpha,\beta)) \mapsto (x,\alpha)\,; (x,y) \in TM_o, (\alpha,\beta) : TM_o \to \mathbb{R}$. $\iota : ((x,\alpha),(y,\beta)) \mapsto ((x,y),(\alpha,\beta))$. $K : ((x,y),(\alpha,\beta)) \mapsto ((x,\alpha),(y,\beta))$. *Thus* $\iota = K^{-1}$ *and* $p^*(T^*M) \simeq T^*(TM_o)$. *Thirdly, the diagram above changes to:*



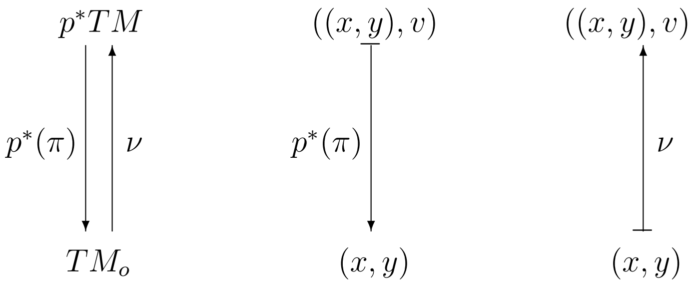
5. *The fibers at* $(x,y) \in TM_o$ *are given by* $p^*(TM)_{|(x,y)} := \{((x,y),v) \mid v \in T_xM\} \simeq T_xM$ *and* $p^*(T^*M)_{|(x,y)} := \{((x,y),\alpha) \mid \alpha \in T^*_xM\} \simeq T^*_xM$. *Thus these fibers are of dimension* n.

6. *The dual relationship between $p^*(TM)$ and $p^*(T^*M)$ is brought out by setting, in the light of the identification in (4): $(x,y,\alpha) \in p^*(T^*M)$ if, and only if $(x,y,\alpha)\colon p^*(TM) \to \mathbb{R}$ with $(x,y,\alpha)(x,y,v) = \alpha(v)$ such that $(\alpha, v) \in T_x^*M \times T_xM$.*

7. *The constructions done above lie on the following principles. Each $(x,y) \in TM_o$ provides a fiber of dimension n diffeomorphic to T_xM. Elements of the form $(x,\lambda y) \in TM_o$, with $\lambda > 0$ and x fixed, produce others fibers diffeomorphic to T_xM.*

Definition 4.2.8
Let M be an n-pseudomanifold. The subpseudomanifold of $T(TM_o)$ defined and denoted by $VTM := span\{\frac{\partial}{\partial y_i}\}$ is called Vertical tangent bundle of M. Whereas the subpseudomanifold of $T^*(TM_o)$ defined and denoted by $HT^*M := span\{dx_i\}$ is called Horizontal cotangent bundle of M.

Remark 4.2.1
Note that p^*T^*M can be identified with HT^*M. That is, there are both dual of p^*TM. The local coordinate system for p^*TM can be denoted by $\{\partial_i\}$ with $\partial_i := (x, y_i \frac{\partial}{\partial x_i})$. So one has:



where $p^*(\pi) \circ \nu = id_{TM_o}$. Thus ν is a vector field defined by $\nu(x,y) := ((x,y),y) = \nu_{(x,y)}$. That is, ν is a canonical section of $p^*(\pi)$ or a section on p^*TM. Finally, $\nu = y_i \partial_i$, for $y = y_i \frac{\partial}{\partial x_i} \in T_xM$, is defined locally in x and globally in y. By analogy, to Example 4.2.7 *we can yield the dual treatment for p^*T^*M.*

Definition 4.2.9
Let $(x,y), (\bar{x},\bar{y}) \in TM_o$. We define a relation on TM_o by $(x,y) \sim (\bar{x},\bar{y})$ if, and only if there exists a real $\lambda > 0$ such that $x = \bar{x}$ and $y = \lambda \bar{y}$.

Lemma 4.2.1
The relation $\sim$ is an equivalence relation on TM_o

Proof.
The relation $\sim$ is reflexive: let $(x,y) \in TM_o$. Thus $x = x$ and $y = 1.y$. It follows

that $\lambda = 1$, that is to say $(x, y) \sim (x, y)$. The relation $\sim$ is symmetric: let $(x, y), (\bar{x}, \bar{y}) \in TM_o$. Assume $(x, y) \sim (\bar{x}, \bar{y})$, that is $x = \bar{x}$ and $y = \lambda\bar{y}$. It follows that $x = \bar{x}$ and $\bar{y} = \gamma y$ with $\gamma = \frac{1}{\lambda}$. Thus $(\bar{x}, \bar{y}) \sim (x, y)$. Finally, the relation $\sim$ is transitive: let $(x, y), (\bar{x}, \bar{y}), (\hat{x}, \hat{y}) \in TM_o$. Assume $(x, y) \sim (\bar{x}, \bar{y})$ and $(\bar{x}, \bar{y}) \sim (\hat{x}, \hat{y})$, that is $(x = \bar{x}$ and $y = \lambda\bar{y})$ and $(\bar{x} = \hat{x}$ and $\bar{y} = \gamma\hat{y})$ with $\lambda, \gamma > 0$. It follows that $x = \hat{x}$ and $y = \delta\hat{y}$ with $\delta = \lambda\gamma > 0$. Thus $(x, y) \sim (\hat{x}, \hat{y})$. □

Definition 4.2.10
Let M be an n-pseudomanifold. The equivalence class of $(x, y) \in TM_o$ is of the form $(x, [y]) := \{(x, \lambda y) \mid \lambda > 0,\ (x, y) \in TM_o\}$, where $[y] = \{\bar{y} = \lambda y \mid \lambda > 0\}$. It is called a ray or a direction. The quotient pseudomanifold $TM_{o/\sim}$ is called the projective sphere bundle denoted by $TM_{o/\sim} := SM = \{(x, [y]) \mid (x, y) \in TM_o\}$.

Note that each T_xM is partitioned by the equivalence classes. SM is a $(2n-1)$-subpseudomanifold of TM. The fibers at $(x, [y]) \in SM$, denoted by $S_xM := \tau^{-1}(x)$ and $S^*_xM := \Gamma^{-1}(x)$, where $\Gamma : S^*M \to M$ is the canonical projection, are diffeomorphic to $(n-1)$-subpseudomanifolds in T_xM and T^*_xM respectively. Thus, S_xM and S^*_xM are diffeomorphic to S^{n-1}. There are called projective spheres at x.

Chapter 5

Symplectic pseudomanifolds.

Under this title we would like to define symplectic structures on pseudomanifolds. Firstly, on the linear and general pseudomanifolds. Secondly, on the cotangent and the slit cotangent bundles. That is, the canonical symplectic structures. Finally, on the tangent and the slit tangent bundles. It is known that the category of smooth manifolds which we denote by $\mathcal{MFD}$ is a subcategory of the category $\mathcal{FRL}$ of $\mathbb{F}$-spaces as remarked in [19, 6]. Also in the category $\mathcal{MFD}$ there exists a canonical symplectic structure on T^*M, but no such canonical symplectic structure is known for TM [25]. It has been proved in [6] that for any $\mathbb{F}$-spaces M, its cotangent bundle T^*M, as an $\mathbb{F}$-space, is endowed with the canonical symplectic structure ω_o, while the tangent bundle TM, as an $\mathbb{F}$-space, is endowed with a symplectic structure $\omega := \mathcal{L}^*\omega_o$, where $\mathcal{L}^*$ is the pullback of the Legendre transform $\mathcal{L}: TM \longrightarrow T^*M$. This leads to the same conclusion on the nonexistence of a canonical symplectic structure on TM considered as an $\mathbb{F}$-space.

5.1 Exterior algebra of differential forms.

In this section we mainly use the material from [47] and [1]. Otherwise the source will be mentioned.

Definition 5.1.1 *[32]*
Let V be a real linear space of dimension n and V^* its dual space. Let $x \in V$, $\alpha \in V^*$ and a linear subspace $W \subset V$. The map $<,>: V \times V^* \longrightarrow \mathbb{R}$ defined by $(\alpha, x) \longmapsto < \alpha, x > := \alpha(x)$ is called the canonical bilinear form on $V \times V^*$, that is, the evaluation of the form alpha at x. The linear subspace of V^*, $W^o := \{\alpha \in V^* \mid \alpha(x) = 0,\ for\ all\ x \in W\}$ is called the annihilator of W.

Remark 5.1.1
Some authors denote $W^o := W^\perp$ and call it the orthogonal of W, [32]. The dimension of W and W^o are related by the identity: $dim\, W + dim\, W^\perp = dim\, V = n$. The dual of the quotient space V/W, that is, $(V/W)^*$ is defined such that $(V/W)^* \simeq$

$W^{\perp}$. The form $\alpha \in V^*$, that is a linear function $\alpha : V \longrightarrow \mathbb{R}$ called a 1-form. And by analogy, $\alpha \in LIN^2(V, \mathbb{R})$ is called a bilinear form (or a 2-form for short) on V. That is, 2 copies of V. Finally, continuing the process up to k copies of V yields $LIN^k(V, \mathbb{R})$ the set of k-linear forms. The set denoted by $\bigwedge^k(V) := LIN^k_{alt}(V, \mathbb{R})$ is the linear space of skew symmetric k-linear forms (that is, k copies of V), or alternating k-linear forms or exterior k-forms on V. So, $\bigwedge^0(V) := LIN^0_{alt}(V, \mathbb{R}) = \mathbb{R}$, $\bigwedge^1(V) := LIN^1_{alt}(V, \mathbb{R}) = V^*$ and $\bigwedge^2(V) := LIN^2_{alt}(V, \mathbb{R}) = LIN_{alt}(V, V, \mathbb{R})$. The set $\bigwedge^k(V)$ has the algebra structure and it is called the exterior algebra of V.

Definition 5.1.2
Let V be a linear space and $dim\, V = n < \infty$ and η a nonzero exterior k-form on V. Let $x \in V$. The interior (inner)) product of η and x is the exterior $(k-1)$-form which satisfies the following relation: $\iota_x \eta(x_1, \ldots, x_{k-1}) = \eta(x, x_1, \ldots, x_{k-1})$ for all $x_1, \ldots, x_{k-1} \in V$. Some authors denote it by $\iota_x \eta = x \lrcorner\, \eta$ and call it the left inner product. The kernel $Ker\eta := Ker f_\eta$, where $Ker f_\eta$ is the kernel of the linear map $f_\eta : V \longrightarrow \bigwedge^{k-1}(V^*)$ defined by $x \longmapsto f_\eta(x) = \iota_x \eta$. The rank of f_η is called the rank of η, that is, $rank(\eta) := rank(f_\eta)$.

Remark 5.1.2
For an exterior 2-form denoted by ω, the induced linear map $f_\omega := \omega^\flat$ is defined by $\omega^\flat(x) = \iota_x \omega = \omega(x, .)$, for any $x \in V$. That is, $\omega^\flat(x)(y) = (\iota_x \omega)(y) = \omega(x, y)$, for all $y \in V$. The map $\omega^\flat$ is an isomorphism if, and only if ω is non degenerate if, and only if $rank(\omega) = dim\, V^* = n$.

Lemma 5.1.1
Let $\omega^\flat$ as defined above. The following hold:

1. *If we set the kernel of $\omega^\flat$ by $Ker\omega^\flat := N$ then the image of V, that is, $\omega^\flat(V) = N^o$ is the annihilator of N.*
2. *If $Ker\omega^\flat$ is nonzero, then there exists an isomorphism $\overline{\omega}^\flat : V/N \xrightarrow{\sim} \omega^\flat(V)$.*
3. *The 2-form $\overline{\omega}$ defined on V/N by $\overline{\omega}([x], [y]) = \omega(x, y)$ for all $x, y \in V$, is non degenerate.*

Proof.

1. It follows from the definition of the kernel that $N = Ker\omega^\flat = \{x \in V \mid \omega^\flat(x) = \iota_x \omega = 0\}$. Thus, $x \in N$ if, and only if $\omega^\flat(x)(y) = \iota_x \omega(y) = 0$, for all $y \in V$. Now, from the definition of the annihilator of N, we have $N^o = \{\alpha \in V^* \mid \alpha(x) = 0$ for all $x \in N\}$. Hence, $\alpha \in N^o$ if, and only if $0 = \alpha(x) = \iota_x \omega(y) = \omega(x, y) = -\omega(y, x) = \omega(-y, x) = \iota_{-y} \omega(x)$ for all $y \in V$, $x \in N$. Therefore, $N^o = \omega^\flat(V)$.

2. Let $\pi : V \longrightarrow V/N$ be the canonical projection. Thus, a theorem of isomorphism of linear spaces asserts that: $\omega^\flat = \overline{\omega}^\flat \circ \pi$ and $\overline{\omega}^\flat$ is onto if and only if $\omega^\flat : V \longrightarrow \omega^\flat(V)$ is onto. Also, it is one-to-one if and only if $Ker\omega^\flat = N$. Hence, it is an isomorphism under the given assumptions.

3. The dual of the quotient space in Remark 5.1.1 and the isomorphism above yield following consequences. First, the isomorphism $\overline{\omega}^\flat$ is constant on each equivalence class, that is, for $x, y \in V$: $[x] = [y]$ if, and only if $x - y \in Ker\omega^\flat = N$ if, and only if $\omega^\flat(x-y) = 0$ if, and only if $\omega^\flat(x) = \omega^\flat(y)$. We can deduce a well defined 2-form $\overline{\omega}$ on V/N given by $\overline{\omega}([x],[y]) = \omega(u,v)$, where u, v are representatives of the equivalence classes. That is, $\overline{\omega}$ is independent of the choice of representatives. So, $\overline{\omega}^\flat(V/N) = \overline{\omega}^\flat(\pi(V)) = (\overline{\omega}^\flat \circ \pi)(V) = \omega^\flat(V) \simeq N^\perp \simeq (V/N)^*$ confirms the latter. Finally, assume for all $[y] \in V/N$ that $0 = \overline{\omega}([x],[y])$. Hence, $0 = \overline{\omega}^\flat([x])([y]) = \omega^\flat(x)(y)$ for all $y \in V$. Thus, $0 = \omega^\flat(x)$. That is, $x \in Ker\omega^\flat = N = [0] = 0 \in V/N$. Therefore, $\overline{\omega}$ is a non degenerate 2-form on V/N.

5.2 Symplectic linear pseudomanifold.

It is worth noticing that in this section we will deal with the finite dimensional linear pseudomanifolds, which are in the sight of Definition 2.1.2 and Example 2.1.4, both finite dimensional linear spaces, endowed with linear $\mathbb{F}$-structures compatible with the addition and scalar multiplication and the linear structure functions are separating points. Hence, each finite dimensional $\mathbb{F}$-space is naturally a linear pseudomanifold since it is isomorphic globally (also locally) to $\mathbb{R}^n$. The symplectic framework in the category $\mathcal{FRL}$, was introduced in [6] and symplectic structures on $\mathbb{F}$-cotangent bundle and pseudomanifolds were investigated in [85]. So, there are our main references for the purpose of symplectic properties. The reader will be often referred to them for details of proofs. The main references on symplectic linear spaces and symplectic manifolds will be [1, 47].

Definition 5.2.1
Let M be a finite dimensional linear $\mathbb{F}$-space and ω an $\mathbb{F}$-smooth 2-form on M. The form ω is called a symplectic form or a symplectic structure on M if it is both, skew-symmetric and non degenerate. That is, $\omega(x,y) + \omega(y,x) = 0$ for all $x, y \in M$ and for all $y \in M$, $\omega(x,y) = 0$ implies that $x = 0$.

Definition 5.2.2
Let M be a finite dimensional linear $\mathbb{F}$-space and ω a symplectic form (or a symplectic structure) on M. The pair (M, ω) is called a symplectic linear $\mathbb{F}$-space.

Remark 5.2.1
The non degeneracy of the symplectic form ω is equivalent to the following statements.

1. *The linear map* $\omega^\flat : M \longrightarrow M^*$*, as defined in Remark* 5.1.2*, is a smooth isomorphism of linear* $\mathbb{F}$*-spaces.*

2. *The transpose* $\omega^t = -\omega$ *is non degenerate.*

3. *The dimension* $dim\, M = dim\, \omega^\flat(M) = rank(\omega) = 2p$*, that is maximal even integer, where* p *is independent of the choice of a basis in* M*.*

4. *There is a basis* $\{u_1,\ldots,u_p,v_1,\ldots,v_p\}$ *in* M*, such that* $\omega(u_i,u_j) = \omega(v_i,v_j) = 0$ *and* $\omega(u_i,v_j) = \delta_{ij}$*, where* $i,j \in \{1,2,\ldots,p\}$ *and* δ_{ij} *is the Kronecker symbol. This basis is called the canonical or symplectic basis.*

5. *Let* $(\omega_{ij})_{1<i,j<p}$ *be the matrix of* ω *in any basis and* $(\omega^t_{ij})_{1<i,j<p}$ *its transpose. It follows that* $det(\omega_{ij})_{1<i,j<p} \neq 0$ *and also* $det(\omega^t_{ij})_{1<i,j<p} \neq 0$*. Moreover* $rank(\omega_{ij})_{1<i,j<p} = 2p$*.*

6. $\omega \wedge \omega \wedge \cdots \wedge \omega = \omega^n$*(*$n$ *copies of* ω*) is the volume form, that is, nowhere vanishing.*

Lemma 5.2.1
Let (M,ω) *be a symplectic linear* $\mathbb{F}$*-space of dimension 2n and* M^* *its dual space. The 2-form on* M^* *denoted by* Λ *such that* $\Lambda(\varphi,\psi) = \omega(\omega^\sharp\varphi,\omega^\sharp\psi)$ *defines a symplectic structure on* M^**, such that* $\Lambda(\omega^\flat(x),\omega^\flat(y)) = \omega(x,y)$*. Furthermore,* $\Lambda^\sharp = \omega^\sharp$ *as smooth isomorphisms of* M^* *onto* M*.*

Definition 5.2.3
Let (M,ω) *be a symplectic linear* $\mathbb{F}$*-space of dimension* n*. Let* W *and* $W^{'}$ *be two linear subspaces of* M*. Two vectors* x *and* y *in* M *are called orthogonal with respect to* ω *(or* ω*-orthogonal) if* $\omega(x,y) = 0$*. The linear subspaces* W *and* $W^{'}$ *are called* ω*-orthogonal if every* $x \in W$ *is* ω*-orthogonal to every* $y \in W^{'}$*. The set* $\{x \in M \mid \omega(x,y) = 0 \; for\ every\ y \in W\} := orth_\omega W := W^\perp$ *is called the* ω*-orthogonal of* W *and it is the maximal element in the set of all linear subspaces of* M *which are* ω*-orthogonal to* W*.*

Definition 5.2.4
Let (M,ω) *be a symplectic linear* $\mathbb{F}$*-space of dimension* n*,* F *and* $F^{'}$ *its linear subspaces and* $\varphi,\ \psi \in M^*$*. The forms* $\varphi,\ \psi$ *are called orthogonal with respect to* Λ *or in involution with respect to* ω *if* $\Lambda(\varphi,\psi) = \omega(\omega^\sharp(\varphi),\omega^\sharp(\psi)) = 0$*, where* $\omega^\sharp : M^* \longrightarrow M$ *is the inverse smooth isomorphism of* $\omega^\flat$*. The linear subspaces* F *and* $F^{'}$ *are called orthogonal if every form* $\varphi \in F$ *is in involution (orthogonal) with every form* $\psi \in F^{'}$*. The orthogonal of* F *is the set* $\{\varphi \in M^* \mid \Lambda(\varphi,\psi) = \omega(\omega^\sharp(\varphi),\omega^\sharp(\psi)) = 0 \; for\ every\ \psi \in F\} := orth_\omega F := F^\perp$*.*

Remark 5.2.2
By dual viewpoint, we may transpose the properties of exteriors forms on a finite dimensional linear $\mathbb{F}$*-space* M*. If we set* $N = M^*$ *then* $N^* = M$ *since* $M^{**} \simeq M$*.*

The exterior forms on $N=M^*$ are p-vectors, that is, the elements of the exterior algebra $\bigwedge M$. We have $N=M^*\longrightarrow\bigwedge^p N^*=\bigwedge^p M$. In this case, the left inner product and the pullback are replaced respectively by the right inner product and the direct image [47]. The smooth isomorphism $\omega^\flat : M\longrightarrow M^*$ may be extended to a smooth isomorphism from the exterior algebra $\bigwedge M$ of M onto the exterior algebra $\bigwedge M^*$ of M^*. Thus, one has the following commutative diagram:

$$\begin{array}{ccc} M & \underset{\omega^\sharp}{\overset{\omega^\flat}{\rightleftarrows}} & M^* \\ \downarrow f_\omega & & \downarrow g_\omega \\ \bigwedge^p M^* & \underset{\hat{\omega}^\sharp}{\overset{\hat{\omega}^\flat}{\leftrightarrows}} & \bigwedge^p M \end{array}$$

where $\hat{\omega}^\sharp$ transforms a p-form into a p-vector.

Proposition 5.2.1 [47]
Let (M,ω) be a symplectic linear $\mathbb{F}$-space of dimension 2n, W,V its linear subspaces and W^o, V^o their respective annihilators. Then

1. $orth_\omega(orth_\omega W)=W$ *and* $orth_\Lambda(orth_\Lambda W^o)=W^o$.
2. $dim\, W + dim\, orth_\omega W = dim\, M = 2n$ *and* $dim\, W^o + dim\, (orth_\Lambda W^o = dim\, M^* = 2n$.
3. $\omega^\flat(orth_\omega W)=W^o$ *and* $\omega^\flat(W)=(orth_\omega W)^o$.
4. $\Lambda^\sharp(W^o)=orth_\omega W$ *and* $\Lambda^\sharp(orth_\Lambda W^o)=W$.
5. $(orth_\omega W)^o = orth_\Lambda W^o$
6. *The inclusion $W\subset V$ is equivalent to $orth_\omega W \supset orth_\omega V$.*
7. $orth_\omega(W\cap V)=orth_\omega W + orth_\omega V$ *and* $orth_\Lambda(W^o\cap V^o)=orth_\Lambda W^o + orth_\Lambda V^o$.

Corollary 5.2.1 [47]
Let (M,ω) be a symplectic linear $\mathbb{F}$-space of dimension 2n, W and V its linear subspaces.

1. $dim\, W\cap V - dim\, orth_\omega W \cap orth_\omega V = dim\, W + dim\, V - 2n$
2. *If* $W = orth_\omega W$ *then* $dim\, W\cap V - dim\, orth_\omega W\cap orth_\omega V = \frac{1}{2}(dim\, V - dim\, orth_\omega V)$.

Definition 5.2.5
Let (M,ω) be a symplectic linear $\mathbb{F}$-space of dimension 2n and N its linear $\mathbb{F}$-subspace of dimension s. Let $\omega_N : N\times N \longrightarrow \mathbb{R}$, be the restriction of ω on $N\times N$. The kernel of ω_N is one of the restriction $\omega^\flat|_N$ of the map $\omega^\flat$ on N. That is, $Ker\omega_N = Ker\omega^\flat_N = \{x\in N \mid \omega^\flat(x) = \iota_x\omega = 0\} = N\cap N^\perp$. The rank of ω_N is called the symplectic rank of N. It is equal to the co-dimension of $Ker\omega_N = N\cap N^\perp$ in N, where $rank(\omega_N) - rank(\omega_{N^\perp}) = 2(s-n)$.

The kernel of ω_N is not necessarily equal to $\{0\}$. It rises a need of characterization among linear $\mathbb{F}$-subspaces of M with respect to ω_N.

Definition 5.2.6
Let (M,ω) be a symplectic linear $\mathbb{F}$-space of dimension 2n and N its linear $\mathbb{F}$-subspace of dimension s. The linear $\mathbb{F}$-subspace N is called symplectic if ω_N is a symplectic structure on N defined by $\omega_N := \iota_N^*\omega$, where ι_N is the canonical inclusion of N into M. That is, if $N\cap orth_\omega N = \{0\}$. The linear $\mathbb{F}$-subspace N is called isotropic if $\omega_N = 0$. That is, if $N\subset orth_\omega N$. The linear $\mathbb{F}$-subspace N is called co-isotropic if $\omega_{orth_\omega N} = 0$. That is, if $orth_\omega N \subset N$. The linear $\mathbb{F}$-subspace N is called Lagrangian if N is both isotropic and co-isotropic. That is, $N = orth_\omega N$.

Lemma 5.2.2 [47]
Let $V\subset W$ be two linear $\mathbb{F}$-subspaces of M, a linear $\mathbb{F}$-space of dimension m. If W is isotropic then V is isotropic. The vectors subspaces $orth_\omega(W\cap orth_\omega W) = orth_\omega W + W$ and $(W\cap orth_\omega W)$ are coisotropic and isotropic respectively.

Proposition 5.2.2
Let M be a linear $\mathbb{F}$-space of dimension m and ω any 2-form (skew symmetric) on M with $N = Ker\omega$. The form ω is the pullback $\pi^*\overline{\omega}$ of a symplectic form $\overline{\omega}$ on the linear $\mathbb{F}$-space M/N, where π is the canonical projection of M onto the quotient.

Proof.
The non degeneracy of $\overline{\omega}$ is a straightforward consequence of Lemma 5.1.1. It is also skew symmetric. For, let $[x],[y]\in M/N$. We have $\pi^*\overline{\omega}([x],[y]) := \omega(x,y) = -\omega(y,x) = -\pi^*\overline{\omega}([y],[x])$ from the definition of $\overline{\omega}$. Thus $\overline{\omega}([x],[y]) = -\overline{\omega}([y],[x]))$ since π^* is a one-to-one linear map. That is, $\overline{\omega}$ is skew symmetric. Therefore, $\overline{\omega}$ is a symplectic form. □

Proposition 5.2.3 [47]
Let (M,ω) be a symplectic linear $\mathbb{F}$-space of dimension 2n and N its linear $\mathbb{F}$-subspace. The formula $\omega_N := \iota_N^*\omega = \pi_N^*\overline{\omega}_N$ defines a symplectic form, on the quotient linear $\mathbb{F}$-space $\overline{N} = N/(N\cap orth_\omega N)$, induced by ω, where π is the canonical projection of M onto the quotient and ι_N the canonical inclusion of N into M.

Proof.
From Definition 5.2.5, $\omega_N := \iota_N^* \omega$ is the restriction of the symplectic form ω to $N \subset M$, with $Ker\omega_N = N \cap orth_\omega N)$ and where ι_N is the canonical inclusion of N into M. It follows that ω_N is a skew symmetric 2-form since for all $x, y \in N$ we have $\omega_N(y,x) := \iota_N^* \omega(y,x) = \omega \circ (\iota_N, \iota_N)(y,x) = \omega(\iota_N(y), \iota_N(x)) = -\omega(\iota_N(x), \iota_N(y)) = -\iota_N^* \omega(x,y) = -\omega_N(x,y)$. Now, let π_N^* be the restriction of the canonical projection of M onto the quotient $\overline{N} = N/(N \cap orth_\omega N)$. Thus, from Lemma 5.1.1, there exists a non degenerate 2-form $\overline{\omega}_N$ on the quotient $\overline{N} = N/(N \cap orth_\omega N)$ such that for all $x, y \in N$ we have $\omega_N(x,y) = \overline{\omega}_N([x],[y]) = \overline{\omega}_N(\pi_N(x), \pi_N(y)) = \overline{\omega}_N \circ (\pi_N, \pi_N)(x,y) = \pi_N^* \overline{\omega}_N(x,y)$, where $[x],[y] \in \overline{N}$. Thus, $\omega_N = \iota_N^* \omega = \pi_N^* \overline{\omega}_N$. Hence, from Proposition 5.2.2 $\overline{\omega}_N$ is a symplectic form on $\overline{N} = N/(N \cap orth_\omega N)$ induced by ω. □

Definition 5.2.7
The symplectic linear $\mathbb{F}$-space $(\overline{N}, \overline{\omega}_N)$ is called the reduced symplectic linear $\mathbb{F}$-space associated to N.

Propositions 5.2.2 and 5.2.3, are used in analytical mechanics to reduce the number of degrees of freedom of a Hamiltonian system by means of first integrals.

Proposition 5.2.4 [*17*]
Let (M, ω) be a symplectic linear $\mathbb{F}$-space of dimension 2n and N its linear $\mathbb{F}$-subspace of dimension s. If N is isotropic, that is, $\omega_N \equiv 0$. Then, ω induces a canonical symplectic form $\overline{\omega}_N$ on N^ω/N.

Proof.
Let $u, v \in N^\omega$ and $[u],[v] \in N^\omega/N$. Defines $\overline{\omega}_N([u],[v]) = \omega(u,v)$. We will show below that $\overline{\omega}_N$ is a well-defined symplectic form. Let $u' = u + x$, $y' = v + y$, with $x, y \in N$. Thus, $\overline{\omega}_N([u'],[v']) = \omega(u',v') = \omega(u+x, v+y) = \omega(u,v) + \omega(u,y) + \omega(x,v) + \omega(x,y) = \omega(u,v)$, since $\omega(u,y) = \omega(x,v) = 0$ because $u, v \in N^\omega$ with $x, y \in N$. So, by isotropy of N, we have $N \subset N^\omega = orth_\omega N$. That is, $\omega(x,y) = \omega_N(x,y) = 0$. Hence, $\overline{\omega}_N$ is well-defined. Suppose $u \in N^\omega$ and $\omega(u,v) = 0$ for all $v \in N^\omega$. It follows that $u \in (N^\omega)^\omega = N$, that is, $[u] = 0$. Thus, $\overline{\omega}_N([u],[v]) = \omega(u,v) = 0$ for all $[v] \in N^\omega/N$ implies $[u] = 0$. Hence, $\overline{\omega}_N$ is non degenerate. Thus, since $\overline{\omega}_N([u],[v]) = \omega(u,v) = -\omega(v,u) = -\overline{\omega}_N([v],[u])$ for all $[u],[v] \in N^\omega/N$, $\overline{\omega}_N$ is skew-symmetric. Therefore, $\overline{\omega}_N$ is a canonical symplectic form. □

Definition 5.2.8
Let φ be a linear smooth map of symplectic linear $\mathbb{F}$-spaces, from (M, ω) to (N, η). The map φ is called a symplectic $\mathbb{F}$-smooth map if it preserves the symplectic structures in the sense that $\varphi^* \eta = \omega$, that is, for all $u, v \in M$, one has: $\varphi^* \eta(u,v) = \omega(\varphi(u), \varphi(v)) = (\omega \circ \varphi)(u,v)$. A symplectic linear transformation of (M, ω) is called a linear symplectomorphism.

The set of all symplectomorphisms on the symplectic linear $\mathbb{F}$-spaces (M, ω) is denoted by $Sympl(M)$. From some results in [6, 62, 85], we note what follows. It was shown that the set $Sympl(M)$ is a group for the composition of maps. Moreover, it is even an $\mathbb{F}$-Lie group. We will come back to this concept in the next chapter. Two symplectic linear $\mathbb{F}$-spaces (M,ω) and (N,η) of the same dimension are symplectically isomorphic, that is, there exists an $\mathbb{F}$-smooth isomorphism $\varphi : M \longrightarrow N$, such that $\varphi^*\eta = \omega$. All $2n$-dimensional symplectic linear $\mathbb{F}$-spaces (M,ω) are symplectically isomorphic to $(\mathbb{R}^{2n},\omega_0)$, where ω_0 is the canonical symplectic form defined by $\omega_0(x,y) = \sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i}y_i)$ for $x = (x_1,\ldots,x_{2n})$, $y = (y_1,\ldots,y_{2n}) \in \mathbb{R}^{2n}$. Let us choose the canonical basis $(u_1,\ldots,u_n; v_1,\ldots,v_n)$ on (M,ω) such that $\omega(u_i,u_j) = \omega(v_i,v_j) = 0$ and $\omega(u_i,v_j) = \delta_{ij}$, where $i,j \in \{1,2,\ldots,n\}$ and δ_{ij} is the Kronecker symbol. It follows that there exists a symplectic $\mathbb{F}$-smooth isomorphism $\varphi : \mathbb{R}^{2n} \longrightarrow M$ such that $\varphi^*\omega = \omega_0$, that is, $\omega \circ \varphi = \omega_0$. In the sequel, all symplectic linear $\mathbb{F}$-spaces of the same dimension are symplectically isomorphic. That is, they all look alike.

5.3 Symplectic pseudomanifold.

We had defined pullback and the differential in Chapter 4, k-forms in Section 5.1 and interior product in Section 5.2. Now we are going to define new smooth operations as exterior product, exterior derivative, pullback, Lie bracket, interior product and Lie derivative. Also, we will give some of their properties in the setting of pseudomanifolds. The Lie derivative will be link to the flow of a vector field in the next chapter. Smoothness of operators announced above can be proved by the Cartesian closedness in the category of $\mathbb{F}$-spaces and the characterization of smooth maps in the category of $\mathbb{F}$-spaces. It is assumed where not stated in this section. The proofs of others properties are similar to those done in the smooth manifold setting.

Definition 5.3.1
Let M be an n-pseudomanifold. A k-form on M or a k-form of degree k is a section of the $\mathbb{F}$-bundle $\bigwedge^{k} T^*M = \bigsqcup_{x\in M} \bigwedge^{k} T_x^*M$ with base space M and fibers $\bigwedge^k T_x^*M$. The set of all k-forms on M is denoted by $\Omega^k(M) := (M, \bigwedge^{k} T^*M)$. It is a module on the algebra $\mathcal{F}_M$.

With respect to Definition 4.2.1, the fact that the coproduct of pseudomanifolds Remark 4.1.4 and the cotangent bundle are pseudomanifolds, one can conclude that $\bigwedge^{k} T^*M$ is a pseudomanifold as in [86, 85, 62]. Note that we are dealing with smooth vector fields with respect to Definitions 4.1.7 and Lemma 4.1.6. Moreover, the sections (k-forms) of the $\mathbb{F}$-bundle $\bigwedge^{k} T^*M$ are smooth with respect to Corollary 2.3.2. For k=0,1,2, we have: $\bigwedge^{0} T_x^*M = \mathbb{R}$, $\Omega^0(M) = \mathcal{F}_M$,

$\bigwedge^1 T_x^*M = T_x^*M$, and $\bigwedge^2 T_x^*M$ is the set of all 2-linear alternating functions $\omega : T_xM \times T_xM \longrightarrow \mathbb{R}$, with $\Omega^2(M) := (M, \bigwedge^2 T^*M)$. If $X_1, X_2, \ldots, X_k$ are k smooth vectors fields on M, then $\omega(X_1, X_2, \ldots, X_k)(x) = \omega(x)(X_1(x), X_2(x), \ldots, X_k(x)) = \omega_x(X_1(x), X_2(x), \ldots, X_k(x))$, where $\omega(x) := \omega_x$ is a smooth function for all $x \in M$. That is, the k-form ω on M is a collection of smoothly varying k-linear alternating maps $\omega_x \in \bigwedge^k T^*M$, [79]. In local coordinate any 1-form ω and any vector field Z on M are given by, $\omega = \sum_{i=1}^{n} h_i(x) dx^i$ and $Z = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x^i}$, where $x^1, x^2, \ldots, x^n$, $h_i, \xi_i \in \mathcal{F}_M$. Thus, $< \omega, Z > = \sum_{i=1}^{n} h_i(x)\xi_i(x)$ is a smooth function.

Definition 5.3.2 [*68, 79, 41*]
Let M be an n-dimensional pseudomanifold. The operator $\wedge : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, called the exterior product (also wedge or Grassmann product), satisfies the following conditions.

1. *The exterior product is an $\mathbb{F}$-smooth multilinear and alternating map.*
2. *Let $\alpha \in \bigwedge^k T_x^*M$ and $\beta \in \bigwedge^l T_x^*M$. The (k+l)-form $\alpha \wedge \beta : M \longrightarrow \bigwedge^{k+l} T_x^*M$ is their exterior product.*
3. *Given k 1-forms $\omega_1, \omega_2 \ldots \omega_k$ then $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k$ is a k-form defined, as a determinant of order k, by $< \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k; Z_1(x), Z_2(x), \ldots, Z_k(x) > := det(< \omega_i, Z_i(x) >)_{1 \leq i,j \leq k}$, where $Z_i(x)$ is any vector of T_xM. This is, a smooth real valued function on $T_xM \times T_xM \times \ldots \times T_xM$, with k factors.*
4. *$\bigwedge^k T_x^*M$ is spanned by the basic k-forms $dx^I := dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, with I running over all strictly increasing multi-indexes $1 \leq i_1 < i_2 < \ldots < i_k \leq dim\, M$. Thus, any k-form ω on M has the local coordinate expression $\omega = \sum_{i=1}^{n} h_I(x) dx^I$, where h_I is a smooth function, dx^I a k-form as the exterior product of k 1-forms $dx^{i_1}, \ldots, dx^{i_k}$.*

For more on the exterior product the following references [68, 79] are useful.

Definition 5.3.3 [*85, 79, 41*]
Let M be an n-dimensional pseudomanifold. The operator $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, called the exterior derivative, satisfies the following. $d : \bigwedge^k T_x^*M \longrightarrow \bigwedge^{k+1} T_x^*M$ is a linear map that takes each k-form to a (k+1)-form, such that $df(Z) = Z(f)$ for

$f\in\bigwedge^0 T_x^*M$, $df\in\bigwedge^1 T_x^*M$ *and* $Z\in\mathfrak{X}(M)$. *That is, the differential we encountered in Chapter 4. If* $k\geq 1$ *and* $\omega\in\Omega^k(M)$, *then for any* $Z_1, Z_2,\ldots, Z_k, Z_{k+1}\in\mathfrak{X}(M)$, *where* $\widehat{Z}_i$ *and* $\widehat{Z}_j$ *are omitted, we call* $d\omega$ *the exterior derivative(differential) of the k-form* ω *such that*

$$d\omega(Z_1, Z_2, \ldots, Z_k, Z_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}Z_i(\omega(Z_1, Z_2, \ldots, \widehat{Z}_i, \ldots, Z_k, Z_{k+1}))$$

$$+\sum_{1\leq i<j\leq k+1}(-1)^{i+j}\omega([Z_i, Z_j], Z_1, Z_2, \ldots, \widehat{Z}_i, \ldots, \widehat{Z}_j, \ldots, Z_k, Z_{k+1}).$$

For α *a k-form and* β *a l-form,* $d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta$ *and* $d(d\alpha)=0$. $\alpha\in\Omega(M)$ *is called a closed form if* $d\alpha=0$. *In local coordinate if* $\omega=\sum_I h_I(x)dx^I$ *for any k-form* ω *then* $d\omega=\sum_I dh_I\wedge dx^I=\sum_{i_1<\cdots<i_k} dh_{i_1\cdots k}\wedge dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_k}$.

Example 5.3.1 [*10*, *85*, *68*, *41*]
The exterior derivative satisfies the following properties. If ω *is a 1-form then* $d\omega$ *is a 2-form defined by* $d\omega(X,Y)=X\omega(Y)-Y\omega(X)-\omega([X,Y])$ *for* X, Y *vector fields on* M. *If* ω *is a 2-form then* $d\omega(X,Y,Z)=\circlearrowleft Z\omega(Y,Z)-\circlearrowleft\omega([X,Y],Z)$ *is a 3-form, where* $\circlearrowleft$ *means the summation over cyclic permutations of* $X, Y, Z\in\mathfrak{X}(M)$.

Definition 5.3.4 [*85*, *10*, *68*, *79*]
Let $\varphi: M\longrightarrow N$ *be an* $\mathbb{F}$*-smooth map of finite dimensional pseudomanifolds. The pullback* $\varphi^*:\Omega(N)\longrightarrow\Omega(M)$ *is a smooth morphism of algebra which pulls back k-forms on* N *to k-form on* M, *and satisfies three requirements as below.* $\varphi^*:\bigwedge^k T^*_{\varphi(x)}N\longrightarrow\bigwedge^k T_x^*M$ *is the restriction of* φ^* *above. For each* $f\in\mathcal{F}_M$, *that is, for each 0-form one has* $\varphi^*f=f\circ\varphi$. *For* $k>0$, $\varphi^*\omega=\omega\circ\varphi_*$ *is* $\mathbb{F}$*-smooth and induces a k-form on* M, *for each k-form on* N, *such that* $\varphi^*\omega(v_1, v_2,\ldots, v_k)=\omega_{\varphi(x)}(\varphi_{*x}(v_1), \varphi_{*x}(v_2),\ldots, \varphi_{*x}(v_k))$ *for* $v_1, v_2, \ldots, v_k\in T_xM$.

Proposition 5.3.1 [*10*, *85*, *79*, *68*]
Let $f\in\bigwedge^0 T^*_{\varphi((x)}M$ *and* $\alpha,\beta\in\bigwedge^1 T^*_{\varphi(x)}N$. *Let* $\varphi: M\longrightarrow N$ *and* $\psi: P\longrightarrow M$ *be two* $\mathbb{F}$*-smooth maps. Then the pullbacks* φ^* *and* ψ^* *have the following properties.*

1. $\varphi^*(\alpha+\beta)=\varphi^*\alpha+\varphi^*\beta$.

2. $\varphi^*(\alpha\wedge\beta)=\varphi^*\alpha\wedge\varphi^*\beta$.

3. $\varphi^*(d\alpha)=d(\varphi^*\alpha)$, *that is,* φ^* *commutes with* d.

4. $\varphi^*(df) = df \circ \varphi_* = d(f \circ \varphi)$.

5. $\varphi^*(f\alpha) = \varphi^*(f)\varphi^*\alpha = (f \circ \varphi)\varphi^*\alpha$.

6. $(\varphi \circ \psi)^*\alpha = (\psi^* \circ \varphi^*)\alpha$.

7. *If φ is an $\mathbb{F}$-diffeomorphism, then φ^* is a an $\mathbb{F}$-isomorphism of $\Omega(N)$ onto $\Omega(M)$ and $(\varphi^*)^{-1} = (\varphi^{-1})^*$.*

8. *If φ is the identity map of a finite dimensional pseudomanifold M then φ^* is the identity of $\Omega(M)$.*

Definition 5.3.5
Let M be an n-dimensional pseudomanifold. Let $[,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ be an $\mathbb{F}$-smooth map denoted by $(X, Y) \mapsto [X, Y]$ and satisfying for any $f \in \mathcal{F}$ and for all $X, Y, Z \in \mathfrak{X}(M)$ the properties below:

1. *Closure:* $[X, Y] := XY - YX \in \mathfrak{X}(M)$.

2. *Bilinearity:* $[X, Y + Z] = [X, Y] + [X, Z]$ *and* $[X, fY] = (X.f)Y + f[X, Y]$. *That is, the linearity in both two components.*

3. *Antisymmetry:* $[X, Y] = -[Y, X]$.

4. *Derivation property, known as the Jacobi identity:* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The map $[,]$ is called the commutator or the Lie-bracket. The $\mathcal{F}_M$-module $\mathfrak{X}(M)$ together with the Lie-bracket is called a $\mathbb{F}$-Lie algebra of vector fields on M.

Definition 5.3.6
Let ω be a k-form and Z a vector field on a finite dimensional pseudomanifold M. The interior product (inner product or contraction) of Z and ω is a (k-1)-form denoted by $\iota_Z\omega := Z \lrcorner\, \omega$, whose evaluation at every $Z_1, Z_2, \ldots, Z_{k-1} \in \mathfrak{X}(M)$ is given by $< Z \lrcorner\, \omega; Z_1, Z_2, \ldots, Z_{k-1} > = < \omega; Z, Z_1, Z_2, \ldots, Z_{k-1} >$. The inner product $Z \lrcorner\, \omega$ satisfies the following properties.

1. $Z \lrcorner\, f = 0$, *for any 0-form f.*

2. $< Z; \omega > = \iota_Z\omega = Z \lrcorner\, \omega$ *is a 0-form, for any 1-form ω.*

3. $< \omega; Z_1, Z_2, \ldots, Z_{k-1}, Z_k > = Z_k \lrcorner\, Z_{k-1} \lrcorner \ldots \lrcorner\, Z_2 \lrcorner\, Z_1 \lrcorner\, \omega$ *for all $Z_1, Z_2, \ldots, Z_{k-1}, Z_k$ vector fields and ω any k-form on M.*

4. $Z \lrcorner\, (Z \lrcorner\, \omega) = (Z \lrcorner\, Z) \lrcorner\, \omega) = 0$ *for any Z a vector field and ω a k-form on M.*

5. $Z \lrcorner (\alpha \wedge \beta) = (Z \lrcorner\, \alpha) \wedge \beta + (-1)^k \alpha \wedge (Z \lrcorner \beta)$ *for α a k-form, β l-forms and Z a vector field on M.*

6. $Z \lrcorner Y \lrcorner \omega = -Y \lrcorner Z \lrcorner \omega$ *for* Z, Y *vector fields and* ω *a k-form on* M.

Proposition 5.3.2 [*79*, *68*, *10*]
Let $\mathcal{U}$ *be an open neighborhood in* M, Z, Y *vector fields,* ω *a k-form and* f *a 0-form on* M. *Then, the interior product satisfies the following properties.*

1. *There exists* $\iota : \mathfrak{X}(M) \times \Omega(M) \longrightarrow \Omega(M)$ *defined by* $(Z, \omega) \mapsto \iota_Z \omega$ *where* $\iota_Z : \Omega^k \longrightarrow \Omega^{k-1}$ *is an operator that is,* $\mathbb{R}$*-linear map.*
2. ι_Z *is a local operator, that is,* $\iota_{Z_{|\mathcal{U}}} \omega_{|\mathcal{U}} = \iota_Z \omega_{|\mathcal{U}}$.
3. $\iota_{Z+Y} = \iota_Z + \iota_Y$ *and* $\iota_{fZ} = f \iota_Z$.
4. $\iota_Z^2 = \iota_Z \circ \iota_Z = 0$.
5. *In local coordinate, if* $\omega = \displaystyle\sum_{i_1 < \cdots < i_k} h_{i_1 .. i_k}(x) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$, *then*
$$\iota_Z \omega = \sum_{\substack{i_1 < .. < i_k \\ 1 \leq l \leq k}} Z^{i_l} h_{i_1 .. i_l .. i_k} dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_l}} \wedge \ldots \wedge dx^{i_k} \text{ with } Z = \sum_{i=1}^{dim\, M} Z^i \frac{\partial}{\partial x^i}.$$
6. $\iota_Z(\varphi^* \omega) = \varphi^*(\iota_{\varphi_* Z} \omega)$, *that is,* $\iota_Z(\varphi^*) = \varphi^*(\iota_{\varphi_* Z})$. *In particular, If* $\varphi_* Z = Z$, *that is,* Z *is invariant then* $\iota_Z(\varphi^*) = \varphi^*(\iota_Z)$.

Definition 5.3.7 [*10*, *68*, *79*]
The Lie derivative of a k-form ω *with respect to a vector field* Z *is given in terms of* ι_Z, *the interior product and of d, the exterior derivative by the formula called the Cartan identity, that is,* $\mathfrak{L}_Z \omega = \iota_Z(d\omega) + d(\iota_Z \omega)$, *such that the following hold.*

1. *The operation* $\mathfrak{L} : \mathfrak{X}(M) \times \Omega^k(M) \longrightarrow \Omega^k(M)$ *is compatible with the* $\mathbb{F}$*-structure, that is, it is* $\mathbb{F}$*-smooth* $\mathbb{R}$*-bilinear map.*
2. $\mathfrak{L}_Z = \iota_Z \circ d + d \circ \iota_Z$ *is obviously an* $\mathbb{F}$*-smooth map, since* ι_Z *and d are* $\mathbb{F}$*-smooth maps.*
3. $\mathfrak{L}_Z f = Z(f)$ *with* $\mathfrak{L}_Z(c) = 0$ *for* f *a 1-form and* $f = c$ *a constant.*
4. $\mathfrak{L}_Z$, *applying a k-form to a k-form, is a* $\mathbb{R}$*-linear map and a local operator.*
5. $\mathfrak{L}_Z(Y) = [Z, Y]$

Proposition 5.3.3 [*10*, *68*, *79*]
Let $Z, Y \in \mathfrak{X}(M)$, f, g *two 0-forms,* ω *a k-form ,and* $a \in \mathbb{R}$. *The Lie derivative satisfies the following properties.*

1. $\mathfrak{L}_{Z+Y} = \mathfrak{L}_Z + \mathfrak{L}_Y$.
2. $\mathfrak{L}_{aZ} = a \mathfrak{L}_Z$.

3. $\mathfrak{L}_{fZ} \neq f\mathfrak{L}_Z$.

4. $\mathfrak{L}_Z(f.g) = f\mathfrak{L}_g + g\mathfrak{L}_f$.

Proof.

1. The property holds with respect to the linearity of the interior product.

2. The same argument yields the property.

3. Since $\mathfrak{L}_Z$ is a derivation, it follows as a consequence of following computations. On the one hand side, $\mathfrak{L}_{fZ}\omega = df \wedge (\iota_Z\omega) + f(\mathfrak{L}_Z\omega)$. On the other hand, $\mathfrak{L}_Z(f\omega) = \mathfrak{L}_Z(f)\omega + f(\mathfrak{L}_Z\omega) = (Z(f))\omega + f(\mathfrak{L}_Z\omega)$. Thus, the inequality holds.

4. $\mathfrak{L}_Z(f.g) = Z(f.g) = fZ(g) + gZ(f) = f(\mathfrak{L}_Z g) + g(\mathfrak{L}_Z f)$. □

Proposition 5.3.4 [*68*, *79*, *10*]
The Lie derivative has the following properties.

1. *Let α be a k-form, β a l-form and $Z \in \mathfrak{X}(M)$. Then, the Lie derivative of $\alpha \wedge \beta$ is given by $\mathfrak{L}_Z(\alpha \wedge \beta) = (\mathfrak{L}_Z\alpha) \wedge \beta + \alpha \wedge (\mathfrak{L}_Z\beta)$.*

2. *Let α be any exterior form and Z any vector field. Then the operators d and $\mathfrak{L}_Z$ commute, that is, $\mathfrak{L}_Z(d\alpha) = d(\mathfrak{L}_Z\alpha)$.*

3. *$(\mathfrak{L}_Z)\varphi^* = \varphi^*(\mathfrak{L}_{\varphi_* Z})$. In particular, $(\mathfrak{L}_Z)\varphi^* = \varphi^*(\mathfrak{L}_Z)$ if $\varphi_* Z = Z$, that is, Z is invariant.*

4. *$\mathfrak{L}_{[Z,Y]}\alpha = [\mathfrak{L}_Z, \mathfrak{L}_Y]\alpha$.*

5. *$\mathfrak{L}_Z(Y \lrcorner\, \alpha) = \mathfrak{L}_Z Y \lrcorner\, \alpha \;+\; Y \lrcorner\, \mathfrak{L}_Z\alpha$.*

6. *$\iota_{[Z,Y]} = \mathfrak{L}_Z \circ \iota_Y \;-\; \iota_Y \circ \mathfrak{L}_Z = [\mathfrak{L}_Z, \iota_Y]$.*

7. *$\mathfrak{L}_Z(\varphi^*\alpha) = \varphi^*(\iota_{\varphi_* Z}\alpha)$.*

Note that the non degeneracy and the skew-symmetry of the symplectic form were both purely algebraic conditions. Now, we will restate these conditions in the setting of pseudomanifold. The non degeneracy remains algebraic condition while the skew symmetry gives rise to the closedness, which is a geometric condition, as it is related to the smooth structure on the pseudomanifold.

Definition 5.3.8
Let M be a finite dimensional pseudomanifold and $\omega \in \Omega^2(M)$. The 2-form ω is a symplectic structure on M if it is a non degenerate closed 2-form. That is, $\omega \in \Omega^2(M)$ satisfies,

1. *it is closed if* $d\omega = 0$.

2. *it is non degenerate, that is,* $\omega(X,Y)=0$ *for all* Y *implies* $X=0$, *where* $X,Y\in\mathfrak{X}(M)$.

We say that the pair (M,ω) is a symplectic pseudomanifold.

Remark 5.3.1 [*6, 47, 85*]
The non degeneracy of the 2-form ω *does equivalently say:*

1. *For all* $x \in M$ *and* $Y_x \in T_xM$, *if* $\omega_x(X_x, Y_x) = 0$, *then* $X_x = 0$, *where,* $\omega_x := \omega|_{T_xM}$ *is a skew-symmetric smooth bilinear form associated with the exterior form* ω *at the point* x. *That is, for all* $x\in M$, *the pair* (T_xM,ω_x) *is a symplectic linear* $\mathbb{F}$*-space, and its dimension is even.*

2. *The* $\mathbb{F}$*-smooth map* $\omega^\flat : TM \longrightarrow T^*M$ *is a smooth isomorphism of vector bundles, that is,* $\omega^\flat_x : T_xM \longrightarrow T^*_xM$ *is a smooth isomorphism of linear spaces such that* $\omega^\flat_x(v)=\iota_v\omega_x$ *for every* $x\in M$ *and every* $v\in T_xM$. *That is,* $\omega^\flat_x(v)$ *is the unique (1-form) element of* T^*_xM *such that for every* $u\in T_xM$ *one has* $<\omega^\flat_x(v), u>=\omega_x(v,u)$.

3. *The* $\mathbb{F}$*-smooth map* $\omega^\flat:\mathfrak{X}(M)\longrightarrow\Omega^1(M)$ *is an isomorphism of* $\mathcal{F}_M$*-modules. In the latter case, using analogy in notations with the linear spaces setting, the inverse of* $\omega^\flat$ *will be denoted by* $\omega^\sharp$. *Hence,* $\omega^\flat(X)=\omega(X,.)=\iota_X\omega=\alpha\in\Omega^1(M)$ *if, and only if* $\omega^\sharp(\alpha)=X=X_\alpha\in\mathfrak{X}(M)$ *if, and only if* $\iota_{\omega^\sharp(\alpha)}\omega=\alpha$. *That is, the vector field* $X\in\mathfrak{X}(M)$ *and the 1-form (Pfaffian form)* $\alpha\in\Omega^1(M)$ *are related in a bijective correspondence.*

Definition 5.3.9
Let (M,ω), (N,σ) *be two finite dimensional symplectic pseudomanifolds. An* $\mathbb{F}$*-map* $\varphi:M\longrightarrow N$ *is called symplectic if* $\varphi^*\sigma=\omega$. *Moreover, if* φ *is a symplectic* $\mathbb{F}$*-diffeomorphism, it is called a symplectomorphism.*

Proposition 5.3.5 [*85*]
Let M *be a finite dimensional pseudomanifold and* (N,ω) *be a finite dimensional symplectic pseudomanifold. Let* $\varphi : M \longrightarrow N$ *be an* $\mathbb{F}$*-map. If* φ *is an* $\mathbb{F}$*-diffeomorphism, then* $\varphi^*\omega$ *is a symplectic form on* M.

Lemma 5.3.1 [*6, 85*]
Let (M,ω) *be a symplectic pseudomanifold of dimension 2n and* N *a subpseudomanifold of maximal constant dimension, that is, dim* $N=$ *dim* $M=2n$. *Then*

1. *There exists on* N *a symplectic structure induced by* ω *such that* ${\iota_N}^*\omega=\omega_N$, *where* ι_N *is the canonical inclusion of* N *into* M. *That is,* ι_N *is a symplectomorphism.*

2. *For every $x \in M$ there exists an open neighborhood $\mathcal{U}$ of $x \in M$ and $2n$ smooth functions $q^1,\ldots,q^n;p_1,\ldots,p_n \in \mathcal{G}_x$, the germ of the $\mathbb{F}$-smooth functions at $x \in \mathcal{U}$ such that $\omega|_{\mathcal{U}} = \sum_{i=1}^{n} dq^i \wedge dp_i$. This is the Darboux's theorem in $\mathbb{F}$-spaces setting.*

3. *Every local basis of smooth vector fields $\{W_1,\ldots,W_{2n}\} \subset \mathfrak{X}(M)$ induces a local basis of smooth vector fields $\{V_1,\ldots,V_{2n}\} \subset \mathfrak{X}(N)$.*

4. *Let $x \in M$ and $\varphi = (x^1,x^2,\ldots,x^{2n})$ a coordinate system of M at x with domain $\mathcal{U}$. Then,*
   - *every local basis over $\mathcal{U}$ induces a basis on the open subpseudomanifold $\varphi(\mathcal{U}) \subseteqq \mathbb{R}^{2n}$;*
   - *moreover, the symplectic structure on $\mathcal{U}$ induces a symplectic structure on $\varphi(\mathcal{U})$ with respect to the chart $(\mathcal{U},\varphi)$.*

Definition 5.3.10 [47]
Let (M,ω) be a symplectic pseudomanifold. Let $\varphi : N \longrightarrow M$ be a smooth map from a pseudomanifold N into the symplectic pseudomanifold M. Let $x \in M$. Assume that the map φ is an immersion at x, that is, the tangent map $T_x\varphi : T_xN \longrightarrow T_{\varphi(x)}M$ is injective. Let $T_{\varphi(x)}\varphi(T_xN)$ be the linear subspace of the symplectic linear space $(T_{\varphi(x)}M, \omega_{\varphi(x)})$. The map φ is isotropic, co-isotropic, Lagrangian or symplectic immersion at x, if $(T_{\varphi(x)}\varphi(T_xN)$ is respectively isotropic, co-isotropic, Lagrangian or symplectic in $(T_{\varphi(x)}M, \omega_{\varphi(x)})$. Let φ be the canonical inclusion, then N is isotopic, co-isotropic, Lagrangian or symplectic at x, if T_xN is respectively isotopic, co-isotropic, Lagrangian or symplectic in (T_xM, ω_x). The map φ is isotropic, co-isotropic, Lagrangian or symplectic immersion on N, if φ is isotropic, co-isotropic, Lagrangian or symplectic at every point $x \in N$. In particular, let $N \subset M$, N is isotropic, co-isotropic, Lagrangian or symplectic subpseudomanifold of (M,ω), if N possesses the property at every point $x \in N$.

Lemma 5.3.2 [47]
Let N be a pseudomanifold of dimension n in the symplectic pseudomanifold (M,ω) of dimension $2m$. Let $\iota_N : N \longrightarrow M$ be its canonical inclusion. Then,

1. *The 2-form $\omega_N = {\iota_N}^*\omega$, induced by ω on N, has its kernel at a point x of N defined by $Ker_x\omega_N = T_xN \cap orth(T_xN)$, where $orth(T_xN)$ is the orthogonal of T_xN in the symplectic linear space (T_xM, ω_x).*

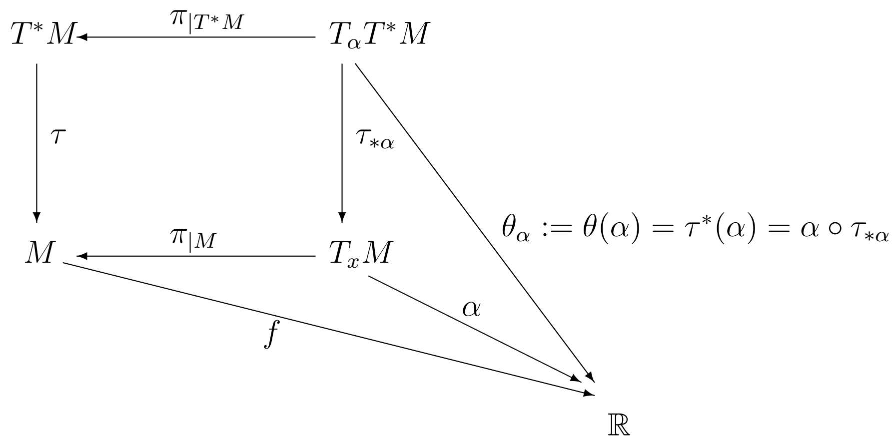
2. *The rank of ω_N at the point $x \in N$ is an even integer $2p(x)$, equal to the co-dimension of $Ker_x\omega_N$ such that it satisfies the inequalities $sup(0, 2(n-m)) \leq 2p(x) \leq n$.*

These inequalities come from the fact that $dim\, Ker_x\omega_N$ is positive and bounded by $dim\, T_xN$ and $dim\, orth(T_xN)$. The rank of ω_N reaches its least possible value in the inequalities (that is, $sup(0, 2(n-m))$ if, and only if the subpseudomanifold N is either co-isotropic (that is, $n \geq m$), isotropic (that is, $n \leq m$) or Lagrangian (that is, $n = m$) at $x \in N$. The rank of ω_N reaches its greatest possible value in the inequalities (that is, n) if, and only if the subpseudomanifold N is even-dimensional and symplectic at $x \in N$. Similar consequences can be drawn in the case of an immersion of a subpseudomanifold N into a symplectic pseudomanifold (M, ω).

5.4 Symplectic structures on T^*M and TM.

Definition 5.4.1
Let M be an n-pseudomanifold and T^*M its cotangent bundle considered as a $2n$-pseudomanifold. Let $\alpha \in T^*M$ be a 1-form on M, that is, $\alpha : TM \longrightarrow \mathbb{R}$. Let $\tau : T^*M \longrightarrow M$ the canonical projection and $\tau_{*\alpha} : T_\alpha(T^*M) \longrightarrow T_{\tau(\alpha)} = T_m M$ its tangent map with $\tau(\alpha) = m$. Given a 1-form $\theta : T_\alpha T^*M \longrightarrow \mathbb{R}$ on T^*M, defined by $\theta(\alpha) := \tau^*(\alpha)$, where $\theta(\alpha)(u) = \alpha(\tau_{*\alpha}(u)) = (\alpha \circ \tau_{*\alpha})(u)$, for all $u \in T_\alpha(T^*M)$. The form θ is called the Liouville 1-form as in [6] or Poincaré 1-form as in [25].

This is a free coordinate expression of θ. The above definition says that the diagram below is commutative:



where $\tau(\alpha) = x$ and $\theta : TT^*M \longrightarrow \mathbb{R}$, with $\theta(\alpha, u) = \theta(\alpha)(u)$, $\alpha \in T^*M$ and $u \in T_\alpha T^*M$. Thus, $\theta \in T^*T^*M = C^\infty\big(TT^*M, \mathbb{R}\big)$. With respect to the Cartesian closedness of the category $\mathcal{FRL}$, we have $C^\infty\big(TT^*M, \mathbb{R}\big) \simeq C^\infty\big(T^*M \times \bigsqcup_{\alpha \in T^*M} T_\alpha T^*M, \mathbb{R}\big) \simeq C^\infty\Big(T^*M, C^\infty\big(\bigsqcup_{\alpha \in T^*M} T_\alpha T^*M, \mathbb{R}\big)\Big)$. This identification means what follows: $\theta : TT^*M \longrightarrow \mathbb{R}$, $(\alpha, u) \longmapsto \theta(\alpha, u) \cong T^*M \longrightarrow C^\infty\big(\bigsqcup_{\alpha \in T^*M} T_\alpha T^*M, \mathbb{R}\big)\Big)$, $\alpha \longmapsto \theta_\alpha = \theta(\alpha) = \theta(\alpha, .)$, where $\theta_\alpha : \bigsqcup_{\alpha \in T^*M} T_\alpha T^*M \longrightarrow \mathbb{R}$, $u \longmapsto \theta_\alpha(u) = \theta(\alpha, u)$. The result above is a particular case of the general one stated as follows. The fact that the smoothness in the category $\mathcal{PSF}$ implies the smoothness in $\mathcal{FRL}$ with respect to Section 3.3, and that the inverse statement is true only if all the objects are restricted to $\mathcal{PSF}$. Thus the Cartesian closedness is satisfied in the category $\mathcal{PSF}$. Therefore, $C^\infty(M,N)\in\mathcal{PSF}$ if, and only if M, N are both pseudomanifolds.

Definition 5.4.2 [*25*]
The Liouville 1-form $\theta\in\Omega^1(T^*M)$ *is defined in local coordinates* (x^i,α_i) *on* T^*M *by* $\theta=-\alpha_i dx^i$*, where* $\alpha=\alpha_i dx^i|_{\tau(\alpha)}$*,* $u=\xi^i\frac{\partial}{\partial x^i}|_\alpha+\beta^i\frac{\partial}{\partial\alpha_i}|_\alpha$*, and* $\tau_{*\alpha}(u)=\xi^i\frac{\partial}{\partial x^i}|_{\tau(\alpha)}$*, with respect to Definition* 5.4.1*.*

Let $(\tilde{x}^i,\tilde{\alpha}_i)$ be another local coordinates for T^*M. It follows that $\alpha_i=\frac{\partial\tilde{x}^k}{\partial x^i}\tilde{\alpha}_k$, $dx^i=\frac{\partial x^i}{\partial\tilde{x}^j}d\tilde{x}^j$ and $\alpha_i dx^i=\frac{\partial\tilde{x}^k}{\partial x^i}\tilde{\alpha}_k\cdot\frac{\partial x^i}{\partial\tilde{x}^j}d\tilde{x}^j=\tilde{\alpha}_i d\tilde{x}^i$. Hence, θ does not depend on the choice of coordinate system.

Theorem-Definition 5.4.1
Let T^*M *be the cotangent bundle of an n-pseudomanifold* M *and* $\theta:T^*M\longrightarrow\mathbb{R}$ *the Liouville 1-form. Then* $\omega_0:=d\theta$ *is a symplectic form on* T^*M*, called the canonical symplectic structure on the cotangent bundle.*

Proof.
The definition of ω_0, the defining properties of $\wedge$, the linearity of d and the fact that $d^2=0$, yield $\omega_0=d\theta=d(-\alpha_i dx^i)=d(-\alpha_i)\wedge dx^i+(-1)^{deg\alpha_i}\alpha_i\wedge d(dx^i)=-d\alpha_i\wedge dx^i-\alpha_i\wedge d^2(x^i)=-d\alpha_i\wedge dx^i$. That is, $\omega_0=dx^i\wedge d\alpha_i$ is a 2-form. Since $d\omega_0=dd\theta=0$, thus, ω_0 is a closed form. Now, we want to show that ω_0 is non degenerate. For, let $X,Y\in\mathfrak{X}(T^*M)$, where $X=a^i\frac{\partial}{\partial x^i}+b^i\frac{\partial}{\partial\alpha_i}$ is fixed and for any $Y=p^i\frac{\partial}{\partial x^i}+q^i\frac{\partial}{\partial\alpha_i}$. Recall that $\omega_0(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^i})=\omega_0(\frac{\partial}{\partial\alpha_i},\frac{\partial}{\partial\alpha_i})=0$. And, also $-\omega_0(\frac{\partial}{\partial\alpha_i},\frac{\partial}{\partial x^i})=\omega_0(\frac{\partial}{\partial x^i},\frac{\partial}{\partial\alpha_i})=dx^i\wedge d\alpha_i(\frac{\partial}{\partial x^i},\frac{\partial}{\partial\alpha_i})=dx^i\frac{\partial}{\partial x^i}d\alpha_i\frac{\partial}{\partial\alpha_i}-dx^i\frac{\partial}{\partial\alpha_i}d\alpha_i\frac{\partial}{\partial x^i}=1.1-0.0=1$, where the minus comes from the signature of permutations $\sigma\in S_2=\{(1\ \ 2),\ (2\ \ 1)\}$. Thus, $0=\omega_0(X,Y)=a^ip^i\omega_0(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^i})+a^iq^i\omega_0(\frac{\partial}{\partial x^i},\frac{\partial}{\partial\alpha_i})+b^ip^i\omega_0(\frac{\partial}{\alpha_i},\frac{\partial}{\partial x^i})+b^iq^i\omega_0(\frac{\partial}{\partial\alpha_i},\frac{\partial}{\partial\alpha_i})=a^iq^i-b^ip^i$. It follows that $a^iq^i=b^ip^i$ for all q^i and p^i. Hence, $a^i=b^i=0$. Therefore, $X=0$. That is, ω_0 is non degenerate 2-form. We have shown that ω_0 is a symplectic form on T^*M. $\square$

Lemma 5.4.1
Let $L\in C^\infty(TM,\mathbb{R})$*, where* M *is an n-pseudomanifold and* L *one-to-one. Let* $c(t)=y+vt$*, where* $y,v\in T_xM$*, with* y *fixed and* v *any vector, that is,* c *is a smooth curve on* TM*, with foot point* y*, and* $L\circ c\in C^\infty(\mathbb{R},\mathbb{R})$*. Then there exist* $\mathbb{F}$*-diffeomorphisms,* $\mathcal{L}:TM\longrightarrow T^*M$ *,* $\mathcal{H}:T^*M\longrightarrow T^{**}M$*, and* $\mathcal{I}:TM\longrightarrow T^{**}M$ *such that* $\mathcal{H}\circ\mathcal{L}=\mathcal{I}$*,* $\mathcal{L}^{-1}=\mathcal{I}^{-1}\circ\mathcal{H}$*, and* $\mathcal{H}^{-1}=\mathcal{L}\circ\mathcal{I}^{-1}$*.*

Proof.
Recall that $C^\infty(TM,T^*M)=C^\infty(TM,C^\infty(TM,\mathbb{R}))\simeq C^\infty(TM\times TM,\mathbb{R})$, by Cartesian closedness. We have $C^\infty(T_xM,T_x^*M)\simeq C^\infty(T_xM\times T_xM,\mathbb{R})$, on spaces of linear maps, for a particular $x\in M$. For $\widehat{\mathcal{L}}\in C^\infty(T_xM\times T_xM,\mathbb{R})$, such that for

all $y, v \in T_xM$, the formula $\widehat{\mathcal{L}}(y,v) := \frac{d}{dt}(L \circ c)|_{t=0}$, with respect to Definition 4.1.9 and Proposition 4.1.1, defines an $\mathbb{F}$-smooth linear function. There exists a unique associated smooth map $\mathcal{L} \in C^\infty(TM, T^*M)$ such that $\mathcal{L}(y)(v) := \widehat{\mathcal{L}}(y,v)$ viewed as the evaluation of $\mathcal{L}(y)$ at v, with $y \longmapsto \mathcal{L}(y)$. But y is not a function of t. Thus, by the chain rule we have $\widehat{\mathcal{L}}(y,v) = \frac{d}{d(y+vt)}(L(y+vt))|_{t=0} \frac{d((y+vt))}{dt}|_{t=0} = \frac{dL}{dy}(y).v = L(v)$, since the restriction $L|_{T_xM}$ is linear. Now, assume $\mathcal{L}(y)(u) = \mathcal{L}(y)(v)$. This implies $u = v$ since L is one-to-one. Thus $\mathcal{L}(y)$ is one-to-one. Moreover, $\widehat{\mathcal{L}}$ is one-to-one since $L \circ c$ is one-to-one. It can be proven that $\mathcal{L}$ is linear map since $\widehat{\mathcal{L}}$ is linear. Assume $\mathcal{L}(y) = \mathcal{L}(z)$, for all $y, z \in T_xM$. It yields $\mathcal{L}(y)(v) = \mathcal{L}(z)(v)$ for all $v \in T_xM$. Thus, $(y,v) = (z,v)$ since $\widehat{\mathcal{L}}$ is one-to-one. Hence,$y = z$ and it follows that the restriction of $\mathcal{L}$ to T_xM, denoted also by $\mathcal{L} : T_xM \longrightarrow T_x^*M$, is an isomorphism of n-dimensional linear spaces. Recall that TM and T^*M are disjoint unions of linear spaces T_xM and T_x^*M, respectively. Therefore, the global $\mathcal{L} : TM \longrightarrow T^*M$, is an $\mathbb{F}$-diffeomorphism. It would be worth noticing that, if we substitute T_xM for T_x^*M and T_x^*M for $T_x^{**}M$ in all above, we will obtain $\mathcal{H}$ and $\widehat{\mathcal{H}}$, such that $\widehat{\mathcal{H}}(\alpha,\theta) = \mathcal{H}(\alpha)(\theta)$, with $\mathcal{H} : T_x^*M \longrightarrow T^{**}M$, an injective linear map between two n-dimensional linear spaces. That is, $\mathcal{H}$ is an isomorphism of linear spaces. Also, assume that $\{\frac{\partial}{\partial x^i}\}, \{dx^i\}$ and $\{\triangle_i\}$ are bases respectively for T_xM, T_x^*M and $T_x^{**}M$, with$\{dx^i\}$ dual of $\{\frac{\partial}{\partial x^i}\}$ and $\{\triangle_i\}$ dual of $\{dx^i\}$. Thus, it follows from linear algebra that, $\frac{\partial}{\partial x^i} \overset{\mathcal{L}}{\longmapsto} dx^i \overset{\mathcal{H}}{\longmapsto} \triangle_i$, gives $\frac{\partial}{\partial x^i} \overset{\mathcal{I}}{\longmapsto} \triangle_i$, the canonical isomorphism (identification) of T_xM onto $T_x^{**}M$, such that $\mathcal{H} \circ \mathcal{L} = \mathcal{I}$. Finally, $\mathcal{I}^{-1} = \mathcal{L}^{-1} \circ \mathcal{H}^{-1}$ yields $\mathcal{L} \circ \mathcal{I}^{-1} = \mathcal{H}^{-1}$, $\mathcal{I}^{-1} \circ \mathcal{H} = \mathcal{L}^{-1}$. □

This Legendre transform comes from an arbitrary smooth function L on TM. The function L is called hyperregular if $\mathcal{L}$ is an $\mathbb{F}$-diffeomorphism. [6]

Corollary 5.4.1
Let $\mathcal{L}$ be the Legendre transform constructed in Lemma 5.4.1. Then the following hold: The pullback of ω_0 under $\mathcal{L}$, is a symplectic structure on TM, that is, $\omega = \mathcal{L}^*\omega_0$. The restriction $\omega_0|_{T^*M_0}$ of ω_0 is the canonical symplectic structure to the open symplectic subpseudomanifold T^*M_0 of T^*M. The restriction $\omega|_{TM_0}$ of ω is the non canonical symplectic structure to the open symplectic subpseudomanifold TM_0 of TM.

Proof.
Let $u, v \in TM$, thus, $\omega_0(\mathcal{L}(u), \mathcal{L}(v)) = (\omega_0 \circ \mathcal{L})(u,v) = (\mathcal{L}^*\omega_0)(u,v) = \omega(u,v)$. It follows that $\omega = \mathcal{L}^*\omega_0$ is a 2-form on TM. It is non degenerate since the Legendre transform is a diffeomorphism. It is closed since $d\omega = \mathcal{L}^* d\omega_0 = 0$. Hence, ω is a symplectic structure. The others two statements come from the restriction of a smooth map on an open subset. □

The structure functions on the open subpseudomanifolds T^*M_0 and TM_0 are respectively the restrictions of $\mathcal{F}_{T^*M}$ and $\mathcal{F}_{TM}$ to that sets. The pullback of the Legendre transform is an $\mathbb{F}$-diffeomorphism. Thus, the following properties hold. For any $L \in C^\infty(TM, \mathbb{R})$, there exists a unique $H \in C^\infty(T^*M, \mathbb{R})$ such that $H \circ \mathcal{L} = L$. Moreover, let c be a curve into TM. Thus, $d = \mathcal{L} \circ c$ is a structure curve into T^*M and $L\circ c\in C^\infty(\mathbb{R},\mathbb{R})$. So, $H\circ d=L\circ c$. If τ and π are the canonical projections of T^*M_0 and TM on M respectively then $\tau\circ\mathcal{L}=\pi$ since the $\mathcal{L}$ maps T_xM to T_x^*M for $x\in M$. It can be observed that for any tangent vector field X on TM there exists a unique vector field X^* on T^*M such that $\mathcal{L}_*\circ X=X^*\circ\mathcal{L}$, that is, they are $\mathcal{L}$-related. The main ingredient in the proof being the equation $\omega=\mathcal{L}^*\omega_0$, the non degeneracy of symplectic form and $H\circ\mathcal{L}=L$ as above. In mechanical setting, $L\in C^\infty(TM,\mathbb{R})$ is a Lagrangian and $H\in C^\infty(T^*M,\mathbb{R})$ is the Hamiltonian, both defined by $X \lrcorner\ \omega=dL$ and $X^* \lrcorner\ \omega_0=dH$. Thus, we set $X:=X_L$ and $X^*:=X_H^*$.

Chapter 6

Symplectic reduction on pseudomanifolds.

6.1 Basic concepts of group actions

Definition 6.1.1 [*62*]
Let G be an $\mathbb{F}$-space and a group with identity element e. The Triple $(G, \mathcal{C}_G, \mathcal{F}_G)$ is called a Frölicher-Lie group or $\mathbb{F}$-Lie group for short if the multiplication map $\sigma : G \times G \longrightarrow G$ given by $\sigma(g, h) = gh$ is $\mathbb{F}$ -smooth and, the map $\theta : G \longrightarrow G$ given by $\theta(g) = g^{-1}$ is $\mathbb{F}$-smooth. Equivalently, that is, the map $\varsigma : G \times G \longrightarrow G$ given by $\varsigma(g, h) = gh^{-1}$ is $\mathbb{F}$-smooth.

Let $H \subset G$. The subset H is called an $\mathbb{F}$-Lie subgroup of the group G if H is a subgroup of G which is a subpseudomanifold. The unit element of G will be denoted by e.

Definition 6.1.2
Let G, H be two $\mathbb{F}$-Lie groups, that is, G and H are finite dimensional pseudomanifolds and groups also. The map $\varphi : G \longrightarrow H$ is an $\mathbb{F}$-Lie group map if it is a smooth map of pseudomanifolds on the one hand and a homomorphism of groups on the other hand.

Definition 6.1.3
Let M be an n-pseudomanifold and G an $\mathbb{F}$-Lie group. Assume that for each $g \in G$ the maps defined by $\sigma : G \times M \longrightarrow M$, $(g, x) \mapsto \sigma(g, x) := g.x$ and $\delta : M \times G \longrightarrow M$, $(x, g) \mapsto \delta(x, g) := x.g$ are smooths maps of pseudomanifolds such that the induced maps $\sigma_g : M \longrightarrow M$, $x \mapsto \sigma_g(x) := \sigma(g, x)$ and $\delta_g : M \longrightarrow M$, $x \mapsto \delta_g(x) := \delta(g, x)$ are diffeomorphisms of the pseudomanifold M. The map σ is called a left action of G on M if $(\sigma_g \circ \sigma_h)(x) = (\sigma_{gh})(x)$ and $\sigma_e = id_M$, for all $g, h \in G$, $x \in M$. The map δ is called a right action of G on M if $(\delta_h \circ \delta_g)(x) = (\delta_{gh})(x)$ and $\delta_e = id_M$, for all $g, h \in G$, $x \in M$.

Remark 6.1.1
The equation $\sigma_{gh}(x)=(\sigma_g \circ \sigma_h)(x)$ *reads* $\sigma(gh,x)=\sigma_g((\sigma_h)(x))=\sigma(g,\sigma(h,x))$*, that is,* $(gh).x=g.(h.x)$*, for all* $g,h \in G$*,* $x \in M$*. Likewise,* $(\delta_{gh})(x)=(\delta_h \circ \delta_g)(x)$ *reads* $\delta(x,gh)=\delta_h((\delta_g)(x))=\delta(h,\delta(g,x))$*, that is,* $x.(gh)=(x.g).h$*, for all* $g,h \in G$*,* $x \in M$*. It follows that* $\sigma_{gh}=\sigma_g \circ \sigma_h$*,* $\delta_{gh}=\delta_h \circ \delta_g$ *and* $\sigma_e=\delta_e=id_M$*. In what follows, we will say actions for means of left group actions. Whenever the right actions will be concerned the distinction will be stated. The set of all diffeomorphisms of the pseudomanifold* M *is denoted by* $\mathfrak{Diff}(M)$ *and it is a group for the composition of maps. It is called "group of diffeomorphisms of* M*".*

Lemma 6.1.1
Let M *and* G *be pseudomanifolds, where* G *is an* $\mathbb{F}$*-Lie group. Let* $\sigma : G \times M \longrightarrow M$ *a left action of* G *on* M*.*

1. *The set* $GM := \{\sigma_g \mid g \in G\}$ *is an* $\mathbb{F}$*-Lie group of transformations of* G *on* M *and* $GM \subset \mathfrak{Diff}(M)$*.*
2. *The map* $\rho : G \longrightarrow \mathfrak{Diff}(M)$*,* $g \longmapsto \rho g := \sigma_g$ *is an* $\mathbb{F}$*-smooth map, an injective homomorphism of abstract groups and* $\rho(G)=GM$*.*

Proof.

1. Let σ be a left action. That is, $(\sigma_g \circ \sigma_h)(x)=(\sigma_{gh})(x)$ and $\sigma_e=id_M$, for all $g,h \in G$, $x \in M$. Let $\sigma_g(x)=\sigma_g(y)$ for any $x,y \in M$. Since $\sigma_{g^{-1}}$ exists, then $\sigma_{g^{-1}}((\sigma_g)(x))=\sigma_{g^{-1}}((\sigma_g)(y))$. So, $x=y$, since $(\sigma_{g^{-1}} \circ \sigma_g)=(\sigma_{g^{-1}g})=\sigma_e=id_M$. Therefore, σ_g is injective for any $g \in G$. Now, for each $y \in M$ there exists $x=g^{-1}.y$ such that $(\sigma_g)(x)=y$ and $g^{-1}.y=\sigma_{g^{-1}}(y) \in M$ since $\sigma_g(g^{-1}.y)=\sigma_g((\sigma_{g^{-1}})(y))=(\sigma_g \circ \sigma_{g^{-1}})(y)=(\sigma_{gg^{-1}}(y))=\sigma_e(y)=id_M(y)=y$. Hence, σ_g is onto. Therefore, σ_g is a bijective map. Finally, we need to show the smoothness of σ_g and $(\sigma_g)^{-1}$. But, σ is a smooth map of pseudomanifolds, then an $\mathbb{F}$-smooth map in all $(g,x) \in G \times M$. It follows that σ_g is a smooth map in all $x \in M$, for all $g \in G$. This is true in particular for $h=g^{-1}$. Hence, $\sigma_{g^{-1}}$ is $\mathbb{F}$-smooth. We need to show that $(\sigma_g)^{-1}=\sigma_{g^{-1}}$. For, $(\sigma_g)^{-1} \circ \sigma_g = id_M = \sigma_e = \sigma_{g^{-1}g} = \sigma_{g^{-1}} \circ \sigma_g$. Thus, we have $(\sigma_g)^{-1}=\sigma_{g^{-1}}$ which is an $\mathbb{F}$-smooth map. Therefore, σ_g is a diffeomorphism of M. That is, $GM \subset \mathfrak{Diff}(M)$.
2. Let $g,h \in G$. Thus $\rho(gh)=\sigma_{gh}=\sigma_g \circ \sigma_h=\rho(g) \circ \rho(h)$. Hence, ρ is a homomorphism of abstract groups. To prove the injectivity of ρ, we set $\rho(g)=\rho(h)$. That is, $\sigma_g=\sigma_h$. So, $id_M=\sigma_g \circ \sigma_{g^{-1}}=\sigma_h \circ \sigma_{g^{-1}}$ and $\sigma_e=\sigma_{hg^{-1}}$. It follows that $e=hg^{-1}$. This implies $g=h$. Therefore, ρ is injective. Now, from the Cartesian closedness of the Category $\mathcal{FRL}$, we have $\sigma \in C^\infty(G \times M, M)$ if, and only if $\rho \in C^\infty(G, C^\infty(M,M))$. Hence, ρ is smooth. □

Remark 6.1.2
Let M *be an* n*-pseudomanifold and* G *an* $\mathbb{F}$*-Lie group. The map* ρ *as given in*

Lemma 6.1.1 *is called a realization of* G. *The realization* $\rho : G \longrightarrow GM = Aut(M)$ *is a a representation of* G *when transformations* $\sigma_g : M \longrightarrow M$ *are linear transformations of the linear space* M. *It follows that a realization of* G *defines a left action of* G *on* M *and vice-versa.*

Definition 6.1.4
Let M *be an* n*-pseudomanifold and* G *an* $\mathbb{F}$*-Lie group. Let* $x \in M$ *be a fixed element and* $\sigma : G \times M \longrightarrow M$ *a left action of* G *on* M. *The image of the map* $\sigma_x : G \longrightarrow M$, $g \mapsto \sigma_x(g) := \sigma(g, x)$, *denoted by* $G.x := \sigma_x(G) \subset M$, *is called the orbit (of) through* x *for the action* σ. *That is,* $G.x = \{\sigma_x(g) = g.x \mid g \in G.\}$. *The subset of* G *given by* $G_x := \{g \in G \mid g.x = x\}$ *is called the stabilizer (or the isotropy group) of* $x \in M$.

Let M be an n-pseudomanifold. A set map $\varphi : M \longrightarrow M$ is called a transformation of M if φ is an $\mathbb{F}$-diffeomorphism.

Definition 6.1.5
Let M *be an* n*-pseudomanifold and* $\sigma \mathbb{R} \times M \longrightarrow M$ *an* $\mathbb{F}$*-smooth map.*
The map σ *is a one-parameter group of transformations of* M *if it has the following properties:*

1. *For each* $t \in \mathbb{R}$, $\sigma_t : M \longrightarrow M$, *is an* $\mathbb{F}$*-diffeomorphism (a transformation) of* M *such that* $\sigma_t(x) = \sigma(t, x)$, *for all* $x \in M$.
2. *For each* $x \in M$, $\sigma_x : \mathbb{R} \longrightarrow M$ *is an* $\mathbb{F}$*-smooth curve on* M *going through* x *such that* $\sigma_x(t) = \sigma(t, x)$, *for all* $t \in \mathbb{R}$ *and* $\sigma_x(0) = \sigma(0, x) = x$.
3. *For all* $t, s \in \mathbb{R}$, $\sigma_{t+s} = \sigma_t \circ \sigma_s$.

Lemma 6.1.2
Let M *be an* n*-pseudomanifold and* $\sigma : \mathbb{R} \times M \longrightarrow M$ *a one-parameter group of transformations of* M. *Then, there exists* $X \in \mathfrak{X}(M)$ *such that* $X = (x, X_x)_{x \in M}$ *and* $X_x(f) = \frac{d}{dt}(f \circ \sigma_x)(t)\big|_{t=0}$ *for some* $f \in \mathcal{F}_M$.

Proof.
The right-hand side is a limit, which exists since f and σ_x are smooth by assumption. Combining (1) and (2) in Definition 6.1.5 yields a tangent vector at each x. A vector field is therefore induced globally. It is the so-called infinitesimal generator of the one-parameter group of transformation σ on M. □

Definition 6.1.6
Let M *be an* n*-pseudomanifold. A local one-parameter group of local transformations of* M *is defined by the following conditions: for all* $x \in M$, *there exists an open neighborhood* $\mathcal{U}$ *containing* x, $\epsilon \in \mathbb{R}$, *with* $\epsilon > 0$, *and* $\phi : (-\epsilon, \epsilon) \times \mathcal{U} \longrightarrow M$ *such that*

1. *For each* $t \in (-\epsilon, \epsilon)$, ϕ_t *is an* $\mathbb{F}$*-diffeomorphism of* $\mathcal{U}$ *onto* $\phi_t(\mathcal{U})$.
2. *For each* $y \in \mathcal{U}$, $\phi_y : (-\epsilon, \epsilon) \longrightarrow \mathcal{U}$ *is an* $\mathbb{F}$*-smooth curve on* $\mathcal{U}$*, going through* y *such that* $\phi_y(t) = \phi(t, y)$ *for all* $t \in (-\epsilon, \epsilon)$ *and* $\phi_y(0) = \phi(0, y) = y$.
3. *We have the image* $\phi_t(\mathcal{U}) \subset \mathcal{U}$*, for all* $t, s \in (-\epsilon, \epsilon)$ *such that one has* $t + s \in (-\epsilon, \epsilon)$*, that is equivalently* $|t + s| < \epsilon$.

Corollary 6.1.1
Let M *be an n-pseudomanifold and* $X \in \mathfrak{X}(M)$ *a vector field. Then the vector field* X *induces a local one-parameter group of local transformations of* M*. In turn the latter induces a vector fields where the local one-parameter group of transformations has been induced.*

Definition 6.1.7
Let G *be an* $\mathbb{F}$*-Lie group. A one-parameter subgroup of* G *is a smooth curve* $\gamma : \mathbb{R} \longrightarrow G$, $g \mapsto \gamma(t)$ *satisfying the following conditions:* $\gamma(t + s) = \gamma(t)\gamma(s)$*, for all* $t, s \in \mathbb{R}$ *and* $\gamma(0) = e$*, where* e *is the unit element of* G.

Remark 6.1.3
The above properties in Lemma 6.1.2 *and Corollary* 6.1.1 *related the 1-dimensional group of transformations* $\{\sigma_t \mid t \in \mathbb{R}\}$ *to the vector field that generates it. We can derive similar properties to those of* $\{\sigma_t \mid t \in \mathbb{R}\}$ *for the group of transformations* $\{\sigma_{\gamma(t)} \mid t \in \mathbb{R}\}$*, where* $\gamma(\mathbb{R}) = \{\gamma(t) \mid t \in \mathbb{R}\}$ *is a one-parameter subgroup of* G*. Let* $\gamma : \mathbb{R} \longrightarrow G$ *be a curve on* G*, and* $\sigma : G \times M \longrightarrow M$*, a left action of* G *on* M*. Then* $\gamma(\mathbb{R}) \times M \longrightarrow M$ *yields a transformations group, for* $t \in \mathbb{R}$, $\sigma_{\gamma(t)} : M \longrightarrow M$*. Thus, we can discuss about the infinitesimal generator which generates the group of transformations* $\{\sigma_{\gamma(t)} \mid t \in \mathbb{R}\}$*. The following transformations of* G *onto itself play a central role in the theory of* $\mathbb{F}$*-Lie groups.*

Definition 6.1.8
Let G *be an* $\mathbb{F}$*-Lie group and* $g \in G$*, a fixed element. Let* $h \in G$*, be any element. The transformation* $L_g : G \longrightarrow G$*, defined by* $L_g(h) := gh$ *is called the left translation, that is, a left multiplication by* g*. The transformation* $R_g : G \longrightarrow G$*, defined by* $R_g(h) := hg$ *is called the right translation, that is, a right multiplication by* g*. The transformation* ${R_g}^{-1} : G \longrightarrow G$*, defined by* $R_g^{-1}(h) := R_{g^{-1}}(h) = hg^{-1}$ *is called the inverse right translation, that is, a right multiplication by* g^{-1}*. The transformation* $L_g {R^{-1}}_g : G \longrightarrow G$*, called the inner automorphism or the conjugation, is defined by* $L_g {R^{-1}}_g(h) := ghg^{-1}$*, that is the composition map* $(L_g \circ {R^{-1}}_g)(h) = L_g({R^{-1}}_g(h)) = L_g(hg^{-1}) = g(hg^{-1}) = ghg^{-1}$.

Remark 6.1.4
It can be easily shown that:

1. L_g, R_g, $R^{-1}{}_g$, *and* $L_g R^{-1}{}_g$ *are* $\mathbb{F}$*-diffeomorphisms. Since* $L: G\times G \longrightarrow G$, *such that* $L(g,h)=gh$ *is the group multiplication on* G *and* $\theta: G\longrightarrow G$ *is the inversion map, both being* $\mathbb{F}$*-smooth maps by the Definition* 6.1.1. *Thus,* L_g, R_g, $R^{-1}{}_g$, $L_g R^{-1}{}_g$ *are* $\mathbb{F}$*-smooth maps. It is obvious that they are bijective and as well as their inverse maps. For, let* $h,k\in G$ *be any elements, assume* $L_g(h)=L_g(k)$. *It follows* $gh=gk$. *The left multiplication by* g^{-1} *in both sides yields* $h=k$. *So,* L_g *is an injection. By similar argument,* R_g, $R^{-1}{}_g$ *are injections. Hence,* $L_g R^{-1}{}_g$ *is also an injective map. Now, for each* $k\in G$, $k=ek=gg^{-1}k=g(g^{-1}k)=L_g(g^{-1}k)$. *Hence,* L_g *is surjective. So,* R_g, $R^{-1}{}_g$, *and* $L_g R^{-1}{}_g$ *are surjective maps. Therefore, we are dealing with* $\mathbb{F}$*-diffeomorphisms.*
2. $R_g(h)=L_h(g)$ *by definition of* L_g *and* L_h.
3. $dL_g(h): T_hG\longrightarrow T_{gh}G$. *Since* $L_g: G\longrightarrow G$, $h\mapsto L_g(h)$, *then* $dL_g: TG\longrightarrow TG$, $(h,v_h)\mapsto (L_g(h), v_{L_g(h)})=(gh,v_{gh})$. *It follows that* $dL_g(h)=dL_g|_{T_hG}$, *such that* $dL_g(h): T_hG\longrightarrow T_{L_gh}G=T_{gh}G$.
4. $dR_g(h): T_hG\longrightarrow T_{R_gh}G=T_{hg}G$. *By similar argument as in part (3.) above.*
5. L_g, R_g, $R^{-1}{}_g$, *and* $L_g R^{-1}{}_g$ *are actions of* G *on* G *satisfying the following.* $L_g\circ L_h=L_{gh}$ *and* $L_e=id_G$, *for all* $g,h\in G$. $R_g\circ R_h=R_{hg}$ *and* $R_e=id_G$, *for all* $g,h\in G$. $R_{g^{-1}}\circ R_{h^{-1}}=R_{h^{-1}g^{-1}}$ *and* $R_{e^{-1}}=R_e=id_G$, *for all* $g,h\in G$. $(L_gR_{g^{-1}}\circ L_hR_{h^{-1}})(k)=g(L_hR_{h^{-1}}(k))g^{-1}=(gh)k(h^{-1}g^{-1})=(gh)k(gh)^{-1}=(L_{gh}R_{(gh)^{-1}})(k)$ *and* $L_eR_e=id_G\circ id_G=id_G$, *for all* $g,h,k\in G$.
6. $L_gR^{-1}{}_g=L_gR_{g-1}=ghg^{-1}$ *is the conjugation action of* G *on* G.

Definition 6.1.9
Let $\mathcal{L}$ *be a real linear space. Let* $[,]:\mathcal{L}\times\mathcal{L}\longrightarrow\mathcal{L}$ *be an* $\mathbb{F}$*-smooth map denoted by* $(X,Y)\mapsto[X,Y]$ *and satisfying for all* $X,Y,Z\in\mathcal{L}$ *and* $a,b\in\mathbb{R}$ *the properties below:*

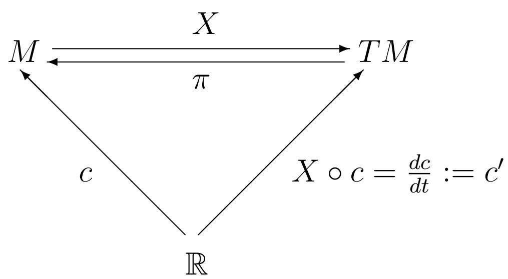
1. *Closure:* $[X,Y]\in\mathcal{L}$.
2. *Bilinearity:* $[X,aY+bZ]=a[X,Y]+b[X,Z]$. *That is, the linearity in both two components.*
3. *Antisymmetry:* $[X,Y]=-[Y,X]$.
4. *Derivation property, known as the Jacobi identity:* $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$.

The map $[,]$ *is called the commutator or the Lie-bracket. The linear space* $\mathcal{L}$ *endowed with the Lie-bracket is called a real* $\mathbb{F}$*-Lie algebra.*

It is worth noticing that $[X,Y] = -[Y,X]$ if, and only if $[X,X]=0$. Setting $Y=X$ in $[X,Y]=-[Y,X]$ yields $[X,X]=-[X,X]=0$. Conversely, assume $[X,X]=0$. It follows that $0=[X+Y,X+Y]$. Thus, $0=[X,X]+[X,Y]+[Y,X]+[Y,Y]=[X,Y]+[Y,X]$ by the bilinear condition. Thus, $[X,Y]=-[Y,X]$. [15]

6.2 Integral curve and Exponential map

Definition 6.2.1
Let M be an n-pseudomanifold. Let $X \in \mathfrak{X}(M)$ and $c : \mathbb{R} \longrightarrow M$ a smooth curve. The curve c is the integral curve of the vector field X if $c_{*r}(\frac{d}{dt}|_{t=r}) = dc(\frac{d}{dt}|_{t=r}) = X(c(r)) := X_{c(r)} := X_x$, where $t \in \mathbb{R}$, $c(r) = x \in M$ and $(\frac{d}{dt}|_{t=r}) = X(c(r))$ a vector. That is, $\pi \circ X = id_M$ and the following diagram is commutative



Definition 6.2.2
Let G be an $\mathbb{F}$-Lie group and $X \in \mathfrak{X}(G)$ a vector field on G. The vector field X is called left invariant if $dL_g(X(h)) = X(L_g)(h) = X(gh)$, for all $g, h \in G$.

Proposition 6.2.1
Let G be an $\mathbb{F}$-Lie group and $e \in G$ its unit element. Let T_eG be the set of tangent vectors to G at e. Let $\mathcal{G}$ be the set of left invariant vector fields on G. Then, we have,

1. *$X \in \mathcal{G}$ if and only if $dL_g(X(e)) = X(g)$.*
2. *$\mathcal{G}$ is a real linear space.*
3. *The map $\alpha : \mathcal{G} \longrightarrow T_eG$, defined by $\alpha(X) = X(e)$, is a linear $\mathbb{F}$-diffeomorphism of n-pseudomanifolds. Thus, $dim\, \mathcal{G} = dim\, T_eG = dim\, G$.*
4. *$\mathcal{G}$ is an $\mathbb{F}$-Lie algebra under the Lie bracket operation on vector fields.*

Proof.

1. " $\Longrightarrow$ " Let $X \in \mathcal{G}$. Thus, $dL_g(X(h)) = X(L_g)(h) = X(gh)$, for all $g, h \in G$, from Definition 6.2.2. It follows that for $h = e$, the relation above becomes $dL_g(X(e)) = X(L_g)(e) = X(ge) = X(g)$.

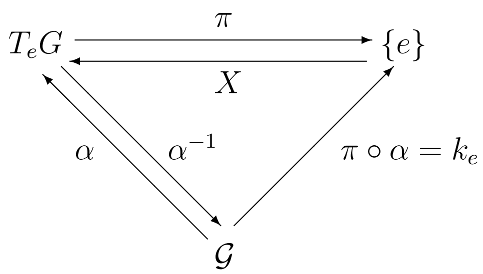
" $\Longleftarrow$ " Conversely, let $X \in \mathfrak{X}(G)$. Assume $dL_g(X(e)) = X(g)$. We have $X(gh) = X(L_g)(h) = dL_{gh}(X(e)) = d(L_g \circ L_h)(X(e))$ for all $h \in G$ by the definition of left translation. The action of d on the composite yields $X(gh)=(dL_g \circ dL_h)(X(e))=dL_g(dL_h(X(e)))=dL_g(X(h))$.

2. $\mathcal{G}$ is a real linear subspace of $\mathfrak{X}(G)$. In fact, for all $X, Y \in \mathcal{G}$, for all $a, b \in \mathbb{R}$, and for all $g \in G$ we have:$X(g) = dL_g(X(e))$ and $Y(g) = dL_g(Y(e))$. So $(aX + bY)(g) = aX(g) + bY(g) = adL_g(X(e)) + bdL_g(Y(e)) = d(L_g(aX(e) + bY(e)) = dL_g((aX + bY)(e))$. Therefore, $aX + bY \in \mathcal{G}$. Hence $\mathcal{G}$ is a real linear space.

3. From Part 1. in this proof, namely $X(g)=dL_g(X(e))$, there is a bijective correspondence $\alpha : \mathcal{G} \longrightarrow T_eG$, defined by $\alpha(X) = X(e)$, since $X(e)$ is unique and for any element $\xi \in T_eG$, there exists a unique $X \in \mathcal{G}$ such that $X(e)=\xi$. Now, $\alpha(aX + bY) = (aX + bY)(e) = aX(e) + bY(e) = a\alpha(X) + b\alpha(Y)$ for all $X, Y \in \mathcal{G}$, and for all $a, b \in \mathbb{R}$. This proves the linearity of α. Finally, we need to prove the smoothness of α in the $\mathbb{F}$-setting. By combining the assumption and the definition of α, we obtain:

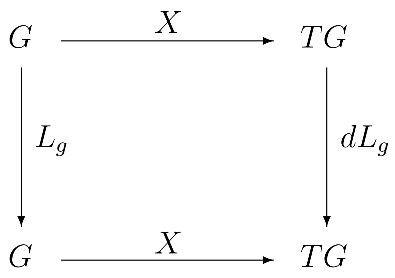
$$X(g)=dL_g(X(e))=dL_g(\alpha(X))=(dL_g \circ \alpha)(X).$$

Thus, α is $\mathbb{F}$-smooth with respect to Corollary 2.3.2 since the composite $dL_g \circ \alpha$ and dL_g are $\mathbb{F}$-smooth maps. Otherwise: the following diagram depicts the situation.



It reads: for any $X \in \mathcal{G}$, we have $(\pi \circ \alpha)(X) = \pi(\alpha(X)) = \pi(e, X_e) = e$. Thus, $\pi \circ \alpha = k_e$ is a smooth map since k_e is a constant map. So, from Corollary 2.3.2, α is a smooth map. Now, the following maps are naturally smooth, $\pi = k_e \circ \alpha^{-1}$ or $dL_g = (dL_g \circ \alpha) \circ \alpha^{-1}$. Hence, α^{-1} is a smooth map with respect to the same reference above. Therefore, α is a linear $\mathbb{F}$-diffeomorphism. Consequently, $\dim \mathcal{G} = \dim T_eG = \dim G$.

4. Let $X, Y \in \mathcal{G}$ and $[X, Y] = XY - YX$, their Lie-bracket. Since $\mathcal{G} \subset \mathfrak{X}(G)$ we only need to show that $[X, Y] \in \mathcal{G}$. In order to do that, we will first use Definition 6.2.2 for general purpose and secondly by setting $h=e$, we will deal with the characterization stated in Part 1. of this proposition. The commutative diagram below will play a central role in the proof:



This reads, $X \circ L_g = dL_g \circ X$ and refers to related vector fields in Chapter 4. Now, let $g, h \in G$. The left invariance of X and Y yields:

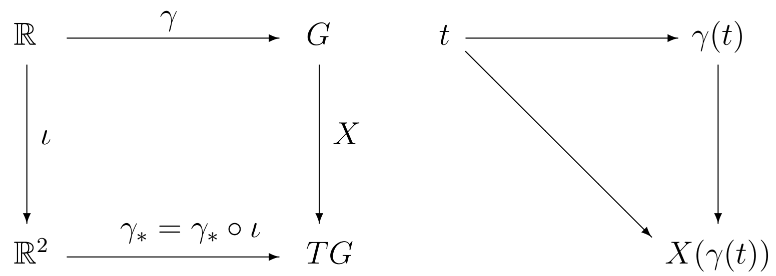
$$\begin{array}{rcl}(XY - YX)(gh) & = & X(Y(gh)) - Y(X(gh)) \\ & = & X(dL_g(Y(h))) - Y(dL_g(X(h))) \\ & = & X((dL_g \circ Y)(h)) - Y((dL_g \circ X)(h)) \\ & = & X((Y \circ dL_g)(h)) - Y((X \circ dL_g)(h)) \\ & = & (XY)(L_g(h)) - (YX)(L_g(h)) \\ & = & (XY - YX)(L_g(h)) \\ & = & ((XY - YX) \circ L_g)(h) \\ & = & (dL_g \circ (XY - YX))(h).\end{array}$$

Therefore, $XY - YX = [X, Y] \in \mathcal{G}$ for all $X, Y \in \mathcal{G}$. □

But, by setting $h = e$ in all the computations above, we will rewrite a proof which lies on the characterization stated in Part 1. of this Proposition. It follows that $(XY - YX)(g) = dL_g((XY - YX)(e))$ for all $g \in G$. That is, $[X, Y]$ is left invariant vector field if X and Y are.

Definition 6.2.3
Let G be an n-$\mathbb{F}$-Lie group. The $\mathbb{F}$-Lie algebra $\mathcal{G}$ of left invariant vector fields on G is called the n-$\mathbb{F}$-Lie algebra of the n-$\mathbb{F}$-Lie group, such that every $X \in \mathcal{G}$ is characterized by $X = X_\xi$ with $\xi = X(e)$. That is, the vector fields X_ξ is invariant under left translation by any element of G.

Definition 6.2.4
Let G be an n-$\mathbb{F}$-Lie group, $\mathcal{G}$ the $\mathbb{F}$-Lie algebra of G and $X \in \mathcal{G}$. Let the commutative diagram



, where $\iota(t) = (t, 0)$, $\gamma(t+s) = \gamma(t)\gamma(s)$ and $\gamma_*(s) = X(\gamma(s))$ for all $t, s \in \mathbb{R}$, with

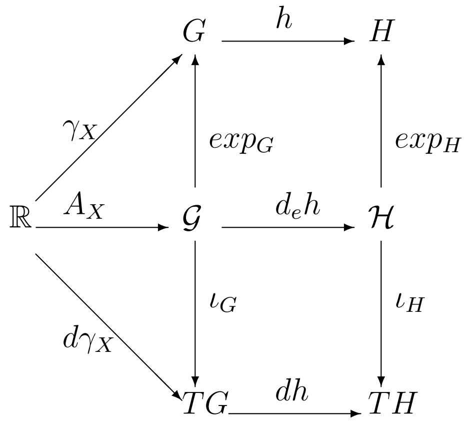
γ a curve on M and γ_* its tangent. The curve $\gamma(\mathbb{R})$ is called the one-parameter subgroup of G corresponding to X or generated by X. This curve is the integral curve of X which passes through e.

Definition 6.2.5

Let G be an $\mathbb{F}$-Lie group, $\mathcal{G}$ the $\mathbb{F}$-Lie algebra of G and $X \in \mathcal{G}$. Let $\gamma_X : \mathbb{R} \longrightarrow G$ be the curve integral of X starting at the identity, that is, $\frac{d}{dt}\gamma_X(t)|_{t=0} = X(e)$. The map $exp : \mathcal{G} \equiv T_eG \longrightarrow G$ defined by $X \longmapsto exp(X) = \gamma_X(1)$ is called the exponential map.

Remark 6.2.1

1. *One can derive $exp(X) = \gamma_X(1)$ as follows. Consider the smooth maps $A_X : \mathbb{R} \longrightarrow \mathcal{G}$, $A_t : \mathcal{G} \longrightarrow \mathcal{G}$ and $exp : \mathcal{G} \longrightarrow G$ such that $t \longmapsto A_X(t) = tX$, $X \longmapsto A_t(X) = tX$ and $tX \longmapsto exp(tX) = \gamma_X(t)$. They are related by $(exp \circ A_t)(X) = (exp \circ A_X)(t)$. Therefore, one has a new map, $\gamma_X : \mathbb{R} \longrightarrow G$, defined by $t \longmapsto exp(tX) = \gamma_X(t)$. In particular, $t = 0$ and $t = 1$ yield $exp(0) = \gamma_X(0) = e$ and $exp(X) = \gamma_X(1)$.*
2. *Since $X : G \longrightarrow TG$, then the curve γ_X satisfies $\dot{\gamma_X} = X \circ \gamma_X$.*
3. *At this stage,we can summarize this discussion as follows. $\gamma_X(1) = exp(X)$, $e = \gamma_X(0)$ and $X(e) = X(\gamma_X(0)) = \dot{\gamma_X}(0)$.*
4. *It is worth noticing that the linear map $\dot{\gamma_X} : \mathbb{R} \longrightarrow T_eG \equiv \mathcal{G}$ is a curve passing through 0 in the linear space $\mathcal{G}$. In the contrary, the homomorphism of groups $\gamma_X : \mathbb{R} \longrightarrow G$ is a curve passing through e in G, where G is not necessarily a linear space but a smooth space.*
5. *We are going to state some properties of the exponential map.[68, pp.154 − 157]*
   - *$exp(tX)exp(sX) = exp((t + s)(X))$. Then $exp(0) = \gamma_X(0) = e$. Since $exp(0) = exp(0 + 0) = exp(0)exp(0)$. Here is the reason which justifies the word "exponential" map.*
   - *$exp(-tX) = (exp(tX))^{-1}$, if $s = -t$ in the exponential relation above.*
   - *$exp(s(tX)) = s\ exp(tX)$.*
   - *$exp(tX)exp(tY) \neq exp(t(X + Y))$ in general. The equality holds if $[X, Y] = 0$, that is, the algebra $\mathfrak{X}(M)$ is commutative.*
   - *Let $h : G \longrightarrow H$ be a smooth map of Lie groups. The following equality, that is, $h \circ exp_G = exp_H \circ dh$ holds. This result is readable from the diagram below:*



- *One has* $(h\circ exp)(X)=h(exp(X))=h(\gamma_X(1))=(h\circ\gamma_X)(1)$, *for all* $h\in\mathcal{F}_G$. *Then* $h\circ exp=\mathfrak{h}$ *is a smooth map since* $h\circ\gamma_X\in C^\infty(\mathbb{R})$. *It implies that exp is a smooth map with respect to Corollary* 2.3.2.

6. *Every left invariant vector field on G is complete , that is, the flow* σ *associated to X has* $\mathbb{R}\times G$ *as domain. This is not the case for general* $X\in\mathfrak{X}(G)$ *which yields a local one parameter group of local transformations.* [*15*]

Definition 6.2.6
A vector field $X\in\mathfrak{X}(G)$ *is right invariant vector field on an* $\mathbb{F}$*-Lie group G if* $X=X_\xi$ *and* $\xi=X(e)$, *where e is the unit element and* ξ *a tangent vector to G at e. The invariance of X is taken with respect to the right translation by any element of G. The set of such vector fields will be denoted by* $\mathcal{G}^{opp}$, *to say the opposite algebra of* $\mathcal{G}$.

Remark 6.2.2

1. *The Lie-bracket* $[,]$ *on* $\mathcal{G}$ *is the first derivation of the Lie-group multiplication.* [*15*, *p*.21]

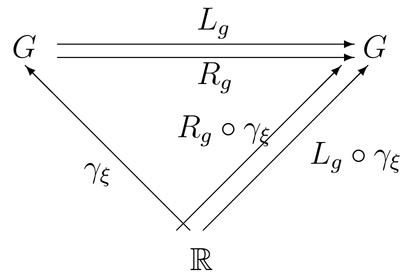
2. *As for right and left actions, there is a way to go from* $\mathcal{G}^{opp}$ *to* $\mathcal{G}$. *For details see* [*68*, *pp*.148, 149]. *We state and comment this reversible process.*

   - *Let* $X\in\mathcal{G}$ *or* $X\in\mathcal{G}^{opp}$ *and Y be the vector field defined by* $Y:G\longrightarrow TG$, *with* $g\longmapsto Y(g)=d(inv)(X(g^{-1}))$, *where* $inv:G\longrightarrow G;\ g\longmapsto g^{-1}$. *Then, it follows that* $dL_gX+dR_{g^{-1}}Y=0$ *and* $dL_gY+dR_{g^{-1}}X=0$, *where* $dL_g:TG\longrightarrow TG$ *and* $dL_g(h):T_gG\longrightarrow T_{gh}G$.
   - *The invariance of X and Y may be read as follows. If* $X\in\mathcal{G}$, *then* $dL_gX(e)+dR_{g^{-1}}Y(e)=0$ *and* $dL_gY(e)+dR_{g^{-1}}X(e)=0$. *Therefore,* $X(g)=-dR_{g^{-1}}$ *and* $Y(g^{-1})=d(inv)(X(g))=-X(g)$. *Hence, we have* $Y(g^{-1})=X(e)+dR_{g^{-1}}Y(e)=0$ *and* $dL_gY(e)+dR_{g^{-1}}X(e)=0$. *Thus,* $X\in\mathcal{G}$ *implies* $Y\in\mathcal{G}^{opp}$. *In particular,* $Y(g^{-1})=-X(g)$ *becomes* $Y(e)=-X(e)$ *for* $g=e$. *This yields* $Y(e)+X(e)=0$, *where* $X(e)=\xi$. *Thus,* $Y(e)=-\xi$. *The same arguments yield* $X\in\mathcal{G}^{opp}$ *implies* $Y\in\mathcal{G}$.

3. *Since* $inv : G \longrightarrow G$ *is a bijective map, then* $d(inv)$ *is an isomorphism. The Lie-bracket* $[,]$ *on* $\mathcal{G}$ *induces a related Lie-bracket on* $\mathcal{G}^{opp}$*, defined by* $[X^{opp}, Y^{opp}]^{opp} = -[X, Y]$ *for all* $X^{opp}, Y^{opp} \in \mathcal{G}^{opp}$ *and* $X, Y \in \mathcal{G}$ *with* $X^{opp}(g) = d(inv)(X(g^{-1}))$ *and* $Y^{opp}(g) = d(inv)(Y(g^{-1}))$*. Actually, there is an anti-homomorphism of Lie-algebras between* $\mathcal{G}$ *and* $\mathcal{G}^{opp}$*.* [*68*]

Definition 6.2.7 [*15*, *p.*19]
Let G *be an* $\mathbb{F}$*-Lie group and* $\mathcal{G}$*, its* $\mathbb{F}$*-Lie algebra. Let* $\xi \in \mathcal{G}$*. The flow of the left invariant vector field* X_ξ *on* G *is denoted by* $\Phi : \mathbb{R} \times G \longrightarrow G$*, and defined by* $(t, g) \longmapsto \Phi_\xi(t, g) = gexp(t\xi)$*. The flow of the right invariant vector field* X_ξ *on* G *is denoted by* $\Psi : G \times \mathbb{R} \longrightarrow G$*, and defined by* $(g, t) \longmapsto \Psi_\xi(g, t) = exp(t\xi)g$*.*

The reason of these notations can be explained by the following diagram:



The diagram reads:

$$L_g(\gamma_\xi)(t) = L_g(exp(t\xi)) = gexp(t\xi)$$

$$R_g(\gamma_\xi)(t) = R_g(exp(t\xi)) = exp(t\xi)g$$

Definition 6.2.8
Let G *be an* $\mathbb{F}$*-Lie group and* M *an* m*-pseudomanifold. Suppose* G *acts smoothly on* M *by the action map* $\sigma : G \times M \longrightarrow M$ *such that* $(g, m) \longmapsto \sigma(g, m) = \sigma_g(m) = g.m$*. Let* $\mathcal{G} = T_eG$ *be the set of all left invariant vector fields on* G*. Let* $\mathcal{A} : \mathcal{G} \times M \longrightarrow TM$ *be the map defined by* $(X, m) \longmapsto \mathcal{A}(X, m) = \mathcal{A}(X)(m) = \mathcal{A}_X(m) = X_m \in T_mM$*, where* $X_m := (\frac{d}{dt}exp(tX)|_{t=0}).m$ *with* $exp(tX) \in G$*, the one-parameter group generated by* X*, and* $\frac{d}{dt}exp(tX)|_{t=0} \in \mathcal{G}$*. The map* $\mathcal{A}$ *is called the infinitesimal action of* $\mathcal{G}$ *on* M *associated to the action* σ *of* G *on* M*.*

Remark 6.2.3
As for $\sigma : G \times M \longrightarrow M$*, we can define the infinitesimal analogous of* σ_g *and* σ_m*.*

1. *The map* $\mathcal{A}_X : M \longrightarrow TM$*,* $m \longmapsto X_m \in T_mM$*, for all* $m \in M$*, is defined in such a way that* $\mathcal{A}_X(M) = \{X_m \in T_mM \mid m \in M\} = (m, X_m)_{m \in M}$*. It follows that every* $X \in \mathcal{G}$ *determines a vector field on* M *denoted by* X_M*. Therefore,* $X_M := (m, X_m)_{m \in M} : M \longrightarrow TM$*, is a smooth map defined by* $m \longmapsto X_M(m) = (m, X_m) = (m, (\frac{d}{dt}exp(tX)|_{t=0}).m)$*. Thus,* $X_M = \mathcal{A}_X \in \mathfrak{X}(M)$*. Hence, for* $a \in \mathbb{R}$*,* $X, Y \in \mathcal{G}$*, the map* $\alpha : \mathcal{G} \longrightarrow \mathfrak{X}(M)$*,* $X \longmapsto X_M = \mathcal{A}_X$*, is an anti-homomorphism of* $\mathbb{F}$*-Lie algebras. That is,* $X + Y \longmapsto (X + Y)_M = X_M + Y_M$*,* $aX \longmapsto (aX)_M = aX_M$*,* $[X, Y] \longmapsto [X, Y]_M = -[X_M, Y_M]$*.*

2. *The map* $\mathcal{A}_m : \mathcal{G} \longrightarrow TM$*,* $X \longmapsto X_m \in T_mM$*, for all* $X \in \mathcal{G}$ *is the orbit map at* m *under the infinitesimal action* $\mathcal{A}$ *of* $\mathcal{G}$ *on* M*. That is, the orbit set of* $\mathcal{A}_m$ *is given by its image* $\mathcal{A}_m(\mathcal{G}) = \{\ \mathcal{A}_m(X) = X_m \mid X \in \mathcal{G}$ *and* m *is fixed* $\}$*.*

3. *The group multiplication on G transforms in the group addition in $\mathcal{G}$. So, let $inv : G \longrightarrow G$, $g \longmapsto g^{-1}$ be the inversion map. It follows that, its tangent map $inv_* = d(inv) : \mathcal{G} \longrightarrow \mathcal{G}$ is defined such that $X \longmapsto -X$.*

4. *It is known that to each flow $\sigma : \mathbb{R}\times M \longrightarrow M$ is associated a vector field on M which is its generator. This correspondence applies to a general group left action $\sigma : G\times M \longrightarrow M$ in the following way: For each $\xi \in \mathcal{G}$, there is a flow on M, denoted by $\Psi_\xi := \Psi_\xi^\sigma$, such that $\Psi_\xi^\sigma(t,m) = exp(t\xi).m \in M$, where $\{exp(t\xi) \in G \mid t \in \mathbb{R}\} \subset G$and $m \in M$. The associated vector field on M will be denoted by Y_ξ^σ. This yields the map $\xi \longmapsto Y_\xi^\sigma$. That is, $X = X_\xi \longmapsto Y_\xi^\sigma$. Since each $X \in \mathcal{G}$ is characterized by $X(e) = \xi$. This is the map $\alpha : \mathcal{G} \longrightarrow \mathfrak{X}(M)$, $X = X_\xi \longmapsto X_M = Y_\xi^\sigma$, defined in Part 1. above. We can look now at the properties of this map α.*

Proposition 6.2.2 [*16*, Section 18.7 and Exercice], [*15*], [*33*, *pp.*167, 177],
Let M be a pseudomanifold and $\mathcal{G}$ the Lie algebra of the Lie group G acting on M. The map $\alpha : \mathcal{G} \longrightarrow \mathfrak{X}(M)$ defined by $\alpha(X) = X_M = \mathcal{A}_X$ is an anti-homomorphism of $\mathbb{F}$-Lie algebras.

Proof. [15, *p.*44, Proposition 1]
From Definition 6.2.3, for every $X \in \mathcal{G}$, $X = X_\xi$ with $\xi = X(e) \in \mathcal{G}$. Let Y_ξ be the right invariant vector field on G, that is, $Y(e) = \xi$. From Definition 6.2.7, the flows of Y_ξ and X_ξ on G are $\Psi_\xi(t,g) = \Psi_\xi(t)(g) = exp(t\xi)g = R_g(exp(t\xi))$ and $\Phi_\xi(t,g) = \Phi_\xi(t)(g) = gexp(t\xi) = L_g(exp(t\xi))$, respectively. Now, consider the inverse map $inv : G \longrightarrow G$, $g \longmapsto g^{-1}$. It follows that $(gexp(t\xi))^{-1} = exp(t\xi)^{-1}g^{-1} = exp(-t\xi)g^{-1}$. That is equivalent to $(\Phi_\xi(t,g))^{-1} = \Psi_{-\xi}(t,g^{-1})$, where $\Phi_\xi(t) : \mathbb{R}\times G \longrightarrow G$ and $\Psi_{-\xi} : G\times\mathbb{R} \cong \mathbb{R}\times G \longrightarrow G$, are the left and right actions of the flows $\Phi_\xi(t) = exp(t\xi)$ and $\Psi_{-\xi}(t) = exp(-t\xi)$. That is, $inv(\Phi_\xi(t,g)) = \Psi_{-\xi}(t,g^{-1})$ or symmetrically $\Phi_{-\xi}(t,g^{-1}) = inv(\Psi_\xi(t,g))$. Therefore, to the infinitesimal level these relations can be combined to Remark 6.2.1 (2), mainly, $X_{-\xi} = -X_\xi \longmapsto Y_\xi$ and $X_\xi \longmapsto Y_{-\xi} = -Y_\xi$. They yield, with respect to Definition 6.2.5 and Remark 6.2.1 (2) and (3), the following:

$$
\begin{array}{rcll}
inv_*(\dot{\Phi}_{-\xi}(t)(g^{-1})) &=& \dot{\Psi}_\xi(t)(g) & \\
inv_*\Big((X_{-\xi}\circ\Phi_{-\xi}(t))(g^{-1})\Big) &=& \Big(Y_\xi\circ\Psi_\xi(t)\Big)(g) & \\
inv_*\Big(X_{-\xi}(\Phi_{-\xi}(t,g^{-1}))\Big) &=& Y_\xi(\Psi_\xi(t,g)) & \\
inv_*\Big(X_{-\xi}(inv(\Psi_\xi(t,g)))\Big) &=& Y_\xi(\Psi_\xi(t,g)) & \\
inv_*\circ X_{-\xi}\circ inv &=& Y_\xi & \text{with } X_{-\xi}\in\mathcal{G},\ Y_\xi\in\mathcal{G}^{opp} \\
(inv_*\circ X_{-\xi}\circ inv)(e) &=& Y_\xi(e) & \\
(inv_*(X_{-\xi})(e)) &=& Y_\xi(e) & \\
inv_*(-\xi) &=& \xi & \text{thus} \\
inv_*(\xi) &=& -\xi & \text{with } inv* : \mathfrak{X}(\mathcal{G}) \longrightarrow \mathfrak{X}(\mathcal{G}).
\end{array}
$$

The Lie bracket of $Y_\xi,\ Y_\eta \in \mathcal{G}^{opp}$ is computed as below:

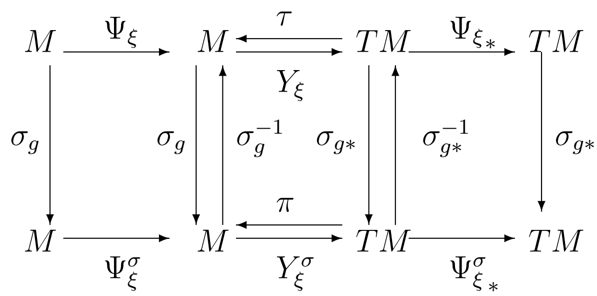
$$\begin{array}{rcl} [Y_\xi,\ Y_\eta] &=& [inv_* \circ X_{-\xi} \circ inv, inv_* \circ X_{-\eta} \circ inv] \\ &=& inv_* \circ [X_{-\xi}, X_{-\eta}]. \circ inv. \\ \text{Thus, } [Y_\xi,\ Y_\eta](e) &=& (inv_* \circ [X_{-\xi}, X_{-\eta}] \circ inv)(e) \\ &=& (inv_* \circ [X_{-\xi}, X_{-\eta}])(e) \\ &=& inv_* \Big((-\xi)(-\eta) - (-\eta)(-\xi) \Big) \\ &=& inv_*[\xi, \eta] \\ &=& -[\xi, \eta]. \\ \text{Therefore, } [Y_\xi,\ Y_\eta] &=& [inv_* \circ X_{[\xi,\eta]} \circ inv \\ &=& Y_{-[\xi,\eta]} \\ &=& -Y_{[\xi,\eta]} \qquad\qquad \text{for all } \xi,\ \eta \in \mathcal{G}. \end{array}$$

We need to extend Y_ξ and Ψ_ξ to $G \times M$ by the formula $\Psi_\xi(g, m) := (exp(t\xi)\ g, m)$. The following diagrams can help in the understanding of the given formula:

$$\begin{array}{ccc} G \times M & \xrightarrow{\Psi_\xi} & G \times M \\ \downarrow \sigma & & \downarrow \sigma \\ M & \xrightarrow{\Psi^\sigma_\xi} & M \end{array} \qquad \begin{array}{ccc} (g, m) & \longrightarrow & (exp(t\xi)\ g, m) \\ \downarrow & & \downarrow \\ g.m & \longrightarrow & (exp(t\xi)\ g).m = \\ & & (exp(t\xi)).(g.m) \end{array}$$

This diagram is commutative in the sight of Remark 6.2.3, (4). It follows that $\sigma \circ \Psi_\xi = \Psi^\sigma_\xi \circ \sigma$. At the infinitesimal level it yields: $\sigma_* \circ \Psi_{\xi_*} = \Psi^\sigma_{\xi_*} \circ \sigma_*$. We need a defining relation between Y_ξ and Y^σ_ξ. First of all, we have to show that τ and π are σ-related. That is, $\sigma \circ \tau = \pi \circ \sigma_*$. Now, for all $(g, m) \in G \times M$, $v(g, m) \in T_{(g,m)} G \times M$ we have what follows: $\sigma \circ \tau((g, m), v(g, m)) = \sigma(g, m) = \sigma(g, m) = g.m$. It can be shown that the identity $\sigma \circ \tau = \pi \circ \sigma_*$ holds, because $\pi \circ \sigma_*((g, m), v(g, m)) = \pi\Big(\sigma_*\big((g, m), v_{(g,\ m)}\big)\Big) = \pi\Big(\sigma\big((g, m), \sigma_{*(g,\ m)}(v_{(g,\ m)})\big)\Big) = \pi(g.m, \sigma_{*(g,\ m)}(v_{(g,\ m)})) = g.m$. So, $\tau \circ {\sigma_*}^{-1} = \sigma^{-1} \circ \pi$. Also, since the vector fields Y^σ_ξ and Y_ξ are smooth sections then we have $\pi \circ Y^\sigma_\xi = id_M$ and $\tau \circ Y_\xi = id_M$. This follows for any fixed $g \in G$ that, $\sigma_* \circ Y_\xi \circ \sigma^{-1} \circ \sigma \circ \tau \circ {\sigma_*}^{-1} = id_{TM}$. Then, $\sigma_* \circ Y_\xi \circ \underbrace{\tau \circ {\sigma_*}^{-1} \circ Y^\sigma_\xi \circ \sigma} = Y^\sigma_\xi \circ \sigma$.

Note that $\tau \circ {\sigma_*}^{-1} \circ Y^\sigma_\xi \circ \sigma = \sigma^{-1} \circ \pi \circ Y^\sigma_\xi \circ \sigma = \sigma^{-1} \circ id_M \circ \sigma = \sigma^{-1} \circ \sigma = id_M$. That is, $Y^\sigma_\xi \circ \sigma$ is a section for the surjective map $\tau \circ {\sigma_*}^{-1}$. Finally, for any fixed $g \in G$, we will have $Y^\sigma_\xi \circ \sigma_g = \sigma_{g*} \circ Y_\xi$. Therefore, $\sigma_{g*}(\xi) = Y^\sigma_\xi = \sigma_{g*} \circ Y_\xi \circ \sigma_g^{-1}$. Below is given a commutative diagram for the infinitesimal part of the proof.



We would like to compute $\sigma_{g*}([\xi,\eta])$ as shown below:

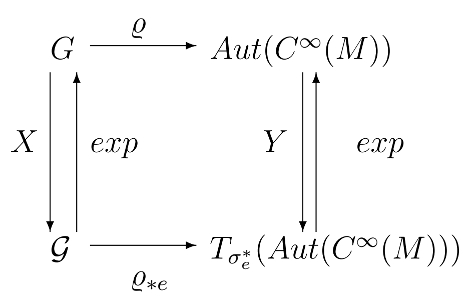
$$\begin{aligned}
[Y_\xi^\sigma, Y_\eta^\sigma] &= [\sigma_{g*}\circ Y_\xi\circ\sigma_g^{-1}, \sigma_{g*}\circ Y_\eta\circ\sigma_g^{-1}]\\
&= \sigma_{g*}\circ[Y_\xi, Y_\eta]\circ\sigma_g^{-1}\\
&= \sigma_{g*}\circ(-Y_{[\xi,\eta]})\circ\sigma_g^{-1}\\
&= -Y_{[\xi,\eta]}^\sigma\\
&= -\sigma_{g*}([\xi,\eta])
\end{aligned}$$

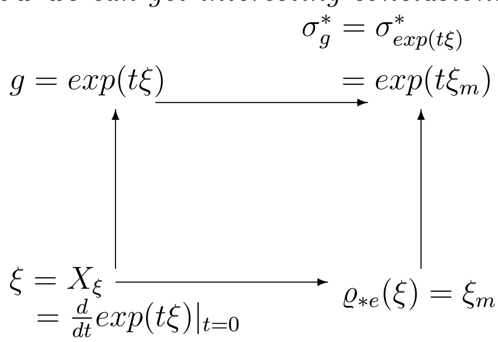
Hence, $\sigma_{g*}([\xi,\eta]) = -[Y_\xi^\sigma, Y_\eta^\sigma]$. Recall that, with respect to Remark 6.2.3, (1) (4) , where $\sigma_{g*}=\alpha$, we have: $\sigma_{g*}(\xi,\eta) = Y_{\xi+\eta}^\sigma = Y_\xi^\sigma + Y_\eta^\sigma$ and $\sigma_{g*}(a\xi) = Y_{a\xi}^\sigma = aY_\xi^\sigma$. Therefore, we have proved that $\sigma_{g*}:\mathcal{G}\longrightarrow\mathfrak{X}(M)$, $\xi\longmapsto Y_\xi^\sigma=\xi_M$ is actually an anti-homomorphism of Lie algebras. □

Remark 6.2.4

1. *Note that $Y_\xi^\sigma(m)=\xi_M(m)=(m,\xi_M(m))=\xi_m$. This yields a smooth map $m\longmapsto\sigma_g^{-1}(m)=g^{-1}.m\longmapsto Y_\xi(\sigma_g^{-1}(m))=\sigma_{g*}^{-1}(Y_\xi^\sigma(m))$ as a composition of smooth maps.*

2. *From [33, pp.167, 177], and [15, p.53], we can define an action of G on $C^\infty(M)$ by pullback. Let $\sigma: G\times M\longrightarrow M$ be a group action of an $\mathbb{F}$-Lie group G on a pseudomanifold M. The map ϱ given by $g\longmapsto\varrho(g)=\sigma_g^*$ ensures the rules below: $gh\longmapsto(\sigma_{gh})^*=(\sigma_g\circ\sigma_h)^*=\sigma_h^*\circ\sigma_g^*$, $\sigma_g^*(f+h)=(f+h)\circ\sigma_g=\sigma_g^*(f)+\sigma_g^*(h)$, $\sigma_g^*(af)=(af)\circ a\sigma_g^*(f)$ and $\varrho(e)=\sigma_e^*$, with $e\in G$ the unit element. We can conclude that ϱ is an injective anti-morphism of groups. Finally, ϱ is an anti-representation since the map $\sigma_g^*\in Aut(C^\infty(M))$ is linear map. From the diagram below we can get interesting conclusions:*





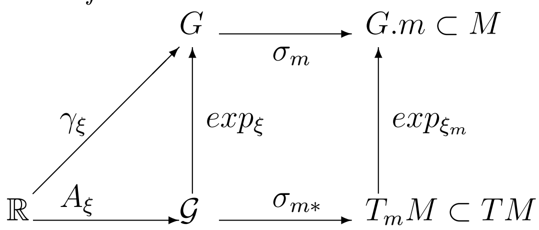
where X and Y are smooth vector fields, with $X_g=X(g)=dL_g(X(e))=dL_g(\xi)\in T_gG$ for $X\in\mathcal{G}$. The linearity of ϱ_{*e} yields the following: $\varrho_{*e}(sX_\xi)=$

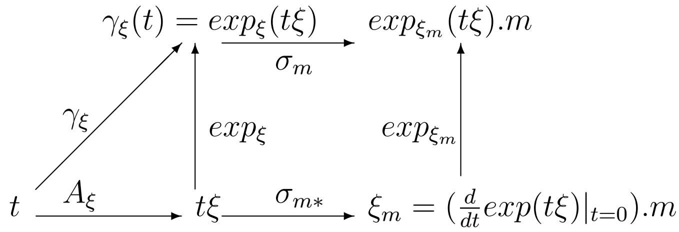
$s(X_\xi)_m = s\varrho_{*e}(X_\xi)$ and $\varrho_{*e}(X_\xi + X_\eta) = \varrho_{*e}(X_{\xi+\eta}) = (X_{\xi+\eta})_m = X_{\xi|m} + X_{\eta|m} = \varrho_{*e}(X_\xi) + \varrho_{*e}(X_\eta)$. Now, our concern will be the action of ϱ_{*e} on the Lie Bracket, that is, for every $[X_g, X_h] = [X_\xi, X_\eta] \in \mathcal{G} \longmapsto \varrho_{*e}([X_\xi, X_\eta]) \in T_{\sigma_g^*}(Aut(C^\infty(M)))$, for all $g,\ h \in G$ and $X \in \mathcal{G}$. Thus,

$$\begin{aligned}
\varrho_{*e}([X_\xi, X_\eta]) &= \varrho_{*e}(-X_{[\xi,\eta]}) \\
&= -\varrho_{*e}(X_{[\xi,\eta]}) \\
&= -(X_{[\xi,\eta]})_m \\
&= -[X_\xi|_m, X_\eta|_m] \\
&= -([X_\xi, X_\eta])_m \\
&= -[\varrho_{*e}(X_\xi), \varrho_{*e}(X_\eta)]
\end{aligned}$$

Hence, we have proved that $\varrho_{*e} : \mathcal{G} \longrightarrow \mathfrak{X}(M)$ is an anti-homomorphism of Lie algebras. □

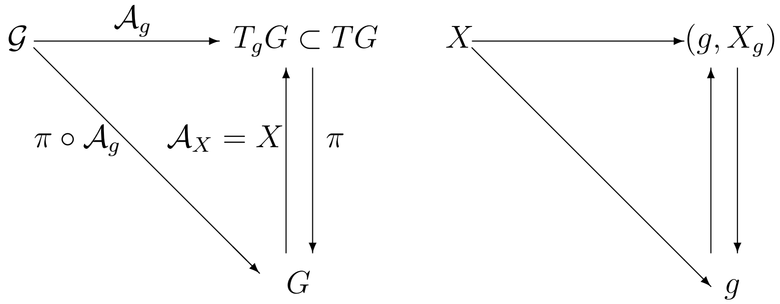
3. *The arguments used above lie on the diagrams below, which are an adaptation of Remark* 6.2.1*:*



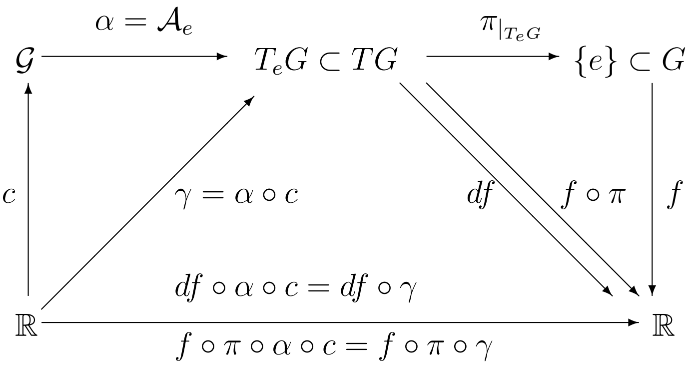


4. *Note that if $\mathcal{A} : \mathcal{G} \times M \longrightarrow TM$ is the infinitesimal action of $\mathcal{G}$ on M, then for all $X,\ Y \in \mathcal{G},\ m \in M$ and $X_m,\ Y_m \in T_mM$ one has the following: $\mathcal{A}(X+Y, m) = \mathcal{A}_{X+Y}(m) = (X+Y)_m = X_m + Y_m = \mathcal{A}_X(m) + \mathcal{A}_Y(m)$. Thus, $\mathcal{A}(X+Y, m) = (\mathcal{A}_X + \mathcal{A}_Y)(m)$. Let $\theta \in \mathcal{G}$ be defined by $\theta : G \longrightarrow TG$ such that $\theta(g) = (g, \theta_g) = (g, 0_g)$, that is, θ is the nil vector field, with 0_g the zero vector of T_gG. It follows that $\theta(G)$ is the zero section. Therefore, $\mathcal{A}(\theta, m) = (\mathcal{A}_\theta(m) = \theta_m = (m, 0) \in T_mM$. Hence, $\mathcal{A}_\theta(m) = (e, 0_e)_m.m = m$, since θ is determined by its value at e.*

5. *If we set $M = G$, then the infinitesimal action $\mathcal{A} : \mathcal{G} \times G \longrightarrow TG$ is determined by $(X, g) \longmapsto (g, X_g) = X(g) = \mathcal{A}_X(g)$. It follows the commutativity of the diagram below:*



Now, let $c \in \mathcal{C}_{\mathcal{G}}$. Then $c(t) = X \in \mathcal{G}$ with $X(g) = dL_gX(e)$. From Proposition 6.2.2, *the isomorphism $\alpha : \mathcal{G} \longrightarrow T_eG \subset TG$ is clearly equal to $\mathcal{A}_e$. This yields the following diagram:*



As from the characterization of smooth maps, α is a smooth map if, and only if $\gamma = \alpha \circ c$ is a smooth curve. Also, it follows from Corollary 2.3.2 *that γ is smooth if, and only if $df \circ \gamma,\ f \circ \pi \circ \gamma \in C^\infty(\mathbb{R})$.*

6. *The map $A : \mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$ defined by $A(t, X) = tX$ is an action of $\mathbb{R}$ on $\mathcal{G}$.*

7. *From [28] the infinitesimal action of $X \in \mathcal{G}$ on M with value in $\mathfrak{X}(M)$ yields the following $(\sigma_m \circ exp_G \circ A)(X) = (exp_{T_mM} \circ \sigma_{m*} \circ A)(X)$ with respect to Part (3) and the action above, that is, $\sigma_m(exp_G(tX)) = exp_{T_mM}(\sigma_{m*}(tX))$. Hence, $exp_{T_mM}(tX_m) = exp_G(tX).m$ at $m \in M$. Therefore, for all $m \in M$ we have $(exp_{T_mM}(t(X_M(m))))_{m \in M} = (exp_G(tX).m)_{m \in M}$. Finally, it follows that we can globally define $exp_{TM}(tX_M) := exp_G(tX).M$.*

8. *The rule transforming left invariant to right invariant vector fields is the following: a left action σ relates a right invariant vector field $Y_\xi \in \mathcal{G}^{opp}$ to a vector field $\sigma_*(\xi) \in \mathfrak{X}(M)$ with $\xi \in \mathcal{G}$.*

6.3 G-equivariance, Adjoint and Co-adjoint representations

Definition 6.3.1 [*16*, *15*, *54*]
Let (M,ω) be a symplectic pseudomanifold, G an $\mathbb{F}$-Lie group acting on M by an action σ. Let $Sympl(M)$ be the group of symplectomorphisms on M. The symplectic form ω is invariant under the action σ of G on M if G acts by symplectomorphisms. That is, the map $\rho: G \longrightarrow Sympl(M)$ such that for each $g \in G$, $\rho(g) := \sigma_g : M \longrightarrow M$ is a symplectomorphism on M. In other words, $\sigma_g^*\omega := \omega$. Such an action σ of G on M is called a symplectic action.

Remark 6.3.1
As from the definition of σ_g^*, for all X, $Y \in \mathfrak{X}(M)$ we have the following defining equalities $\sigma_g^*\omega(X,Y) = \omega(d\sigma_g(X), d\sigma_g(Y)) = \omega(X \circ \sigma_g, Y \circ \sigma_g) = \omega(X,Y)$, since $X(\sigma_g(M)) = X(M)$. Note that $d\sigma_g = \sigma_{g*}$ is the tangent map associated to σ_g. Also, the infinitesimal action of $\mathcal{G} = T_eG$ on M is $\mathcal{A} : \mathcal{G} \times M \longrightarrow TM$, $(\xi, m) \longmapsto \xi_m = \mathcal{A}(\xi)(m) = (\frac{d}{dt}exp(t\xi)|_{t=0}).m \in T_mM$. If we set $G = \mathbb{R}$ in the Definition 6.3.1 *then this yields a smooth map $\rho: \mathbb{R} \longrightarrow Sympl(M)$ such that for each $t \in \mathbb{R}$, $\rho(t) := \rho_t : M \longrightarrow M$ is a symplectomorphism. Let (M,ω) be a symplectic pseudomanifold, G an $\mathbb{F}$-Lie group acting on M by symplectomorphisms, $\mathcal{G}$ the $\mathbb{F}$-Lie algebra of G and $\mathcal{G}^*$ the dual of G. The action of G on M induces a map $\alpha : \mathcal{G} \longrightarrow \mathfrak{X}(M)$, such that $\rho_t(m) = exp(tX).m = \gamma_X(t).m$, where ρ_t is the flow of $\alpha(X) = X_M$.*

Definition 6.3.2 [*16*]
Let (M,ω) be a symplectic pseudomanifold. Let X be a vector field on M preserving ω, that is, $\mathfrak{L}_X\omega = 0$. Such a vector field is called symplectic vector field. The space of symplectic vector fields on M is denoted by $\mathcal{S}\mathfrak{p}(\omega)$.

Lemma 6.3.1
Let (M,ω) be a symplectic pseudomanifold, σ an action of G, an $\mathbb{F}$-Lie group acting by symplectomorphisms, on M. The symplectic form ω on M is invariant under the action of G if, and only if the one-form $\iota_X\omega = X \lrcorner\, \omega$ is closed for all $X := X_M \in \mathfrak{X}(M)$ with respect to Remark 6.3.1.

Proof.
Recall the definition of a symplectomorphism: $\sigma_g \in Sympl(M)$ if $\sigma_g^*\omega = \omega$, with the map $\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathbb{R}$ such that $X \lrcorner\, \sigma_g^*\omega = X \lrcorner\, \omega$ and $d(X \lrcorner\, \sigma_g^*\omega) = d(X \lrcorner\, \omega)$. " $\Longrightarrow$ " Let $\sigma_g^*\omega = \omega$ for all $g \in G$ by assumption. Then the flow associated to the vector field $X = X_M \in \mathfrak{X}(M)$ is $\rho_t : M \longrightarrow M$, $m \longmapsto \rho_t(m) = exp(t\xi).m = exp(tX_m)$, with respect to Remark 6.2.4 (7). Thus, $\rho_t \in G$ are symplectomorphisms for all $t \in \mathbb{R}$. That is, $\rho_t^*\omega = \omega$. [16, Section 18.1.]. As from the definition of the flow one has $\rho_t(m) = \sigma_{exp(t\xi)}(m) := exp(tX_m)$. Hence, $\rho_t^*\omega = \sigma_{exp(t\xi)}^*\omega := exp^*(tX)\omega$, with respect to [15, 28, 68, 16]. This yields, $\mathfrak{L}_X\omega = \frac{d}{ds}\rho_t^*\omega|_{s=0} = \frac{d}{ds}(exp^*(sX))|_{s=0}$.

From the assumption and Remark 6.3.1, for all $t\in\mathbb{R}$, the curve $t\mapsto\sigma^*_{exp(tX)}\omega=\omega$ is constant. That is, it takes the same value for all $t\in\mathbb{R}$ as for $t=0$, It follows that $\sigma^*_{exp(tX)}\mathfrak{L}_X\omega=\frac{d}{dt}\omega=0$. Therefore, $\mathfrak{L}_X\omega=0$ since $\sigma^*_{exp(tX)}$ is a linear isomorphism. From Cartan identity we draw $d(\iota_X\omega)=0$, that is, $\iota_X\omega$ is closed.
" $\Longleftarrow$ " Assume $\iota_X\omega$ is closed. Thus, from Cartan identity, we draw $\mathfrak{L}_X\omega=0$ which means the vector field X preserves ω. That is, $\rho_t^*\omega=\omega$. This equality can be extended to a general $g\in G$ with respect to Remark 6.2.3. Therefore, ω is invariant under the action of G. $\square$

Definition 6.3.3
Let (M,ω) be a symplectic pseudomanifold, and $H:M\longrightarrow\mathbb{R}$ any structure function. A vector field on M denoted by X_H and defined by $\iota_{X_H}\omega=dH$ is called the Hamiltonian vector field associated to H and H is called the Hamiltonian function.

The set of all Hamiltonian vector fields on M is denoted by $\mathfrak{h}(\omega)$. In other words the 1-form ι_{X_H} is an exact 1-form and H is the primitive of ι_{X_H} with respect to ω defined from $\mathfrak{X}(M)\times\mathfrak{X}(M)$ into $C^\infty(M)$. The symplectic and Hamiltonian vector fields are both related to 1-forms. We want to explain below these relationships.

Lemma 6.3.2
Let (M,ω) be a symplectic pseudomanifold and $\omega^\flat:\mathfrak{X}(M)\longrightarrow\Omega^1(M)$ a map defined such that $X\longmapsto\omega^\flat:=\iota_X\omega=\alpha=\omega(X,.)$. The map $\omega^\flat$ is an isomorphism of $C^\infty(M)$-modules.

Proof.
The map $\omega^\flat$ is linear. Indeed, for all $X,Y\in\mathfrak{X}(M)$ and $f\in C^\infty(M)$ the following holds: $\omega^\flat(X+Y)=\iota_{X+Y}\omega=(X+Y)\lrcorner\,\omega=X\lrcorner\,\omega+Y\lrcorner\,\omega=\omega^\flat(X)+\omega^\flat(Y)$, $\omega^\flat(fX)=\iota_{fX}\omega=(fX)\lrcorner\,\omega=f(X\lrcorner\,\omega)=f(\omega^\flat(X))$. After the linearity has been proven, we need now to show that $\omega^\flat$ one-to-one and onto. Recall that ω is non-degenerate. Now, let $X,Y\in\mathfrak{X}(M)$. If $\omega(X,Y)=0$ for all Y, then we have $\omega(X,Y)=\omega^\flat(X)(Y)=(\iota_X\omega)(Y)=0$. That is, $X=0$ and $Ker\omega^\flat=\{0\}$. It follows that the linear map $\omega^\flat$ is one-to-one since. But, $\omega^\flat$ can be equivalently considered as a map of TM into T^*M, which have the same dimension. Then from the rank theorem, if $Ker\omega^\flat=\{0\}$, then $\omega^\flat$ is an isomorphism. $\square$

It follows that $\omega^\flat:TM\longrightarrow T^*M$ is a bijection and also a smooth map since it is smooth into all its component $\omega^\flat_m$. Since to each 1-form α on M corresponds a unique vector field X on M such that $\omega^\flat(X)=\alpha$, we can define the inverse map $(\omega^\flat)^{-1}:=\omega^\sharp:\Omega^1(M)\longrightarrow\mathfrak{X}(M)$, such that $\omega^\sharp(\alpha)=X$ with $\iota_X\omega=\alpha$.

Corollary 6.3.1
Let $\omega\in\Omega^2(M)$. The map $\omega^\flat$ is an isomorphism of $C^\infty(M)$-modules if, and only if ω is non-degenerate.

Definition 6.3.4
Let (M,ω) be a symplectic pseudomanifold. Let $\alpha,\beta\in\Omega^1$ and ρ_t be the flow of X_α such that X_α is uniquely associated to α by $X_\alpha\longmapsto\omega^\flat:=\iota_{X_\alpha}\omega=\alpha=\omega(X_\alpha,.)=dH_\beta$. One calls Poisson bracket of α and β the one-form $\{\alpha,\beta\}:=-\iota_{[X_\alpha,X_\beta]}\omega$ on M with $[X_\alpha,X_\beta]=\lim\limits_{t\to 0}\dfrac{1}{t}\big(X_\beta-d\rho_t\circ H_\beta\big)$.

Corollary 6.3.2 [*15*], [*16*, Section 9.3],
Let $\mathcal{Z}^1(M)$ be the $C^\infty(M)$-submodule of $\Omega^1(M)$ containing closed 1-forms on M and $\mathcal{B}^1(M)$ the $C^\infty(M)$-submodule of $\mathcal{Z}^1(M)$ containing exact 1-forms on M, that is, $\mathcal{B}^1(M)=d(C^\infty(M))$. Then,

1. $\mathcal{S}\mathfrak{p}(\omega)=\omega^\sharp(\mathcal{Z}^1(M))$, *that is,* $\omega^\flat(\mathcal{S}\mathfrak{p}(\omega))=\mathcal{Z}^1(M)$.
2. $\mathfrak{h}(\omega)=\omega^\sharp(\mathcal{B}^1(M))$, *that is,* $\omega^\flat(\mathfrak{h}(\omega))=\mathcal{B}^1(M)$.
3. $T_{id}Sympl(M)\simeq\{\alpha\in\Omega^1(M)\mid d\alpha=0\}=\mathcal{Z}^1(M)$.
4. $T_{id}Ham(M)\simeq\{\alpha=h\mid h\in C^\infty(M)\}=\mathcal{B}^1(M)$.
5. $[\mathcal{S}\mathfrak{p}(\omega),\mathcal{S}\mathfrak{p}(\omega)]\subset\mathfrak{h}(\omega)$.

Proof.
The proof of the first four identity is straightforward of definitions. For the last identity, we refer the reader to [15, *p*.90, Proposition 4]. For $X,Y\in\mathcal{S}\mathfrak{p}(\omega)$, we have $[X,Y]=X_{\omega(X,Y)}$, where $\omega(X,Y)\in C^\infty(M)$. Thus, $[X,Y]\in\mathcal{S}\mathfrak{p}(\omega)$. In particular, since the Poisson bracket of $f,g\in C^\infty(M)$ is defined by $\{f,g\}=\omega(X_f,X_g)$, it follows that $\{X_f,X_g\}=X_{\{f,g\}}$, with respect to Definition 6.3.4. □

A symplectic vector field is of the form $X=\omega^\sharp(\alpha)$, with $\alpha\in\Omega^1(M)$ and $d\alpha=0$, that is, $\omega^\flat(X)=\alpha$. A Hamiltonian vector field is of the form $X_H=X=\omega^\sharp(dH)$ with $H\in C^\infty(M)$, $dH\in\Omega^1(M)$, that is, $\omega^\flat(X_H)=dH$. It follows that $\mathfrak{h}(\omega)$ is an ideal of the Lie subalgebra $\mathcal{S}\mathfrak{p}(\omega)$ of $\mathfrak{X}(M)$ with respect to the inclusion $[\mathcal{S}\mathfrak{p}(\omega),\mathcal{S}\mathfrak{p}(\omega)]\subset\mathfrak{h}(\omega)$. Thus, we have the following chain of inclusions of Lie algebras: $\mathfrak{h}(\omega)\subset\mathcal{S}\mathfrak{p}(\omega)\subset\mathfrak{X}(M)$. The 1-forms counterpart of the inclusions above is: $\mathcal{B}^1(M)\subset\mathcal{Z}^1(M)\subset\Omega^1(M)$. The set diffeomorphism counter part of these inclusions is: $Ham(M)\subset Sympl(M)\subset\mathfrak{Diff}(M)$.

Proposition 6.3.1
Let (M,ω) be a symplectic pseudomanifold and $\{\alpha,\beta\}$ the Poisson bracket of one-forms as given in Definition 6.3.4.

1. $(\Omega^1,\{,\})$ *is a Lie algebra on* $\mathbb{R}$.
2. $\{\alpha,\beta\}=-\mathfrak{L}_{X_\alpha}(\beta)+\mathfrak{L}_{X_\beta}(\alpha)+d(\iota_{X_\alpha}\circ\iota_{X_\beta}\omega)$.
3. *If α and β are closed one-forms, then $\{\alpha,\beta\}$ is an exact form.*

4. *If α and β are exact one-forms, then $\{\alpha, \beta\}$ is an exact form.*

5. *The $\omega^\flat$ is a smooth bijective antimorphism of Lie algebras.*

Proof.

1. The definition links Lie bracket and interior product to Poisson bracket. Thus, The Poisson bracket satisfies the same properties than the Lie bracket.

2. Another mixed property involving the Lie derivative, the interior product and the Poisson bracket is given by the formula $\iota_{[X_\alpha, X_\beta]} = [\mathfrak{L}_{X_\alpha}, \iota_{X_\beta}]$. Combining the formula above with the Cartan magic identity, we break out with the proof by using the closedness of the symplectic form.

3. That is the straightforward consequence of previous item.

4. We are done, if we use the definition of an exact form for α and β and the previous item.

5. Let $\omega^\flat(X) = \alpha$ and $\omega^\flat(Y) = \beta$. By the definition of the Poisson bracket we have $\{\omega^\flat(X), \omega^\flat(Y)\} = \{\alpha, \beta\} = -\iota_{[X,Y]} = -\omega^\flat([X, Y])$.

Definition 6.3.5
Let $\varphi : M \longrightarrow N$ be a smooth map of pseudomanifolds, G an $\mathbb{F}$-Lie group acting on M and N on the left. Let σ and δ be these actions respectively on M and N. The map φ is called G-equivariant if $\varphi(g.m) = g \bullet \varphi(m)$ for all $m \in M$ and $g \in G$, where $\sigma(g, m) = g.m$ and $\delta(g, \varphi(m)) = g \bullet \varphi(m)$. In other words, we say that φ preserves the actions σ and δ, or the diagram below is commutative:

$$\begin{array}{ccc} G \times M & \xrightarrow{\sigma} & M \\ \downarrow{\scriptstyle id_G \times \varphi} & & \downarrow{\scriptstyle \varphi} \\ G \times N & \xrightarrow{\delta} & N \end{array} \qquad \begin{array}{ccc} m & \xrightarrow{\sigma_g} & g.m \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \varphi(m) & \xrightarrow{\delta_g} & (\varphi \circ \sigma_g)(m) = (\delta_g \circ \varphi)(m) \end{array}$$

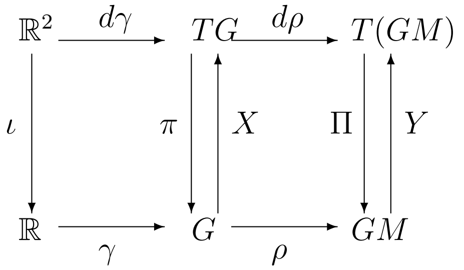
It will be worth noticing that $\varphi \circ \sigma_g$ and $\delta_g \circ \varphi$ are not equal in general.

Definition 6.3.6
Let M be a pseudomanifold. Let G be an $\mathbb{F}$-Lie group acting on the left and $\sigma : G \times M \longrightarrow M$ the action map. An element $m_0 \in M$ is a fixed point for σ if $\sigma_g(m_0) = m_0$ for each $g \in G$, where σ_g is a transformation on M.

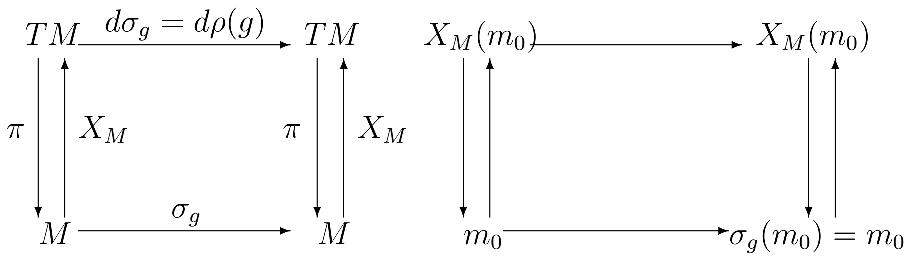
Lemma 6.3.3
Let M be a pseudomanifold and G an $\mathbb{F}$-Lie group acting on M on the left by

$\sigma : G \times M \longrightarrow M$ the action map. Assume that $m_0 \in M$ is a fixed point for σ. Then the map $\psi : G \longrightarrow Aut(T_{m_0 M})$ defined by $\psi(g) := d\sigma_{g|_{T_{m_0}M}}$ is a representation of G in the linear space $Aut(T_{m_0}M)$.

Proof.
Let ρ be as defined in Lemma 6.1.1 and m_0 is a fixed point for σ. Then $\psi(g) = d\sigma_{g|_{T_{m_0}M}} = d\rho(g)_{|_{T_{m_0}M}} = d\rho(g)_{|_{T_{\sigma_g(m_0)}M}} = d\rho(g)_{|_{T_{\rho(g)(m_0)}M}}$. We need to show that ψ is an $\mathbb{F}$-smooth injective linear map. Let γ be any structure curve on G. Thus, for all $t \in \mathbb{R}$ we have $T_{m_0}M = T_{\sigma_{\gamma(t)}(m_0)}M = T_{\rho(\gamma(t))(m_0)}M$. From Definition 6.2.4 and $X \in \mathcal{G}$ we have $(\psi \circ \gamma)(t) = d\sigma_{\gamma(t)|_{T_{m_0}M}} = d\rho(\gamma(t)|_{T_{m_0}M}) = d(\rho \circ \gamma)(t)|_{T_{m_0}M}$. Now, $(\psi \circ \gamma)(t) = (d\rho \circ d\gamma)(t)|_{T_{m_0}M} = (d\rho \circ X \circ \gamma)(t)|_{T_{m_0}M}$. from the action of the operator d on the composite. Hence, $\iota(t) := (t, 0)$ and $d\gamma \circ \iota := d\gamma$. Recall that $GM := \{\sigma_g : M \longrightarrow M \mid g \in G\}$ with respect to Lemma 6.1.1 (1). It follows that $T(GM) = \mathfrak{Diff}(TM)$, with respect to Corollary 6.3.2. So, $T_{\sigma_g}(GM) = Aut(T_{m_0}M)$. The arguments above lie on the commutative diagram below:



Thus, $\psi \circ \gamma = d\rho \circ X \circ \gamma$ and finally $\psi = d\rho \circ X$. Therefore, ψ is an injective smooth map since $d\rho$ and X are so. Now, from the equality $T_{\sigma_g}(GM) = Aut(T_{m_0}M)$ it follows at the infinitesimal level the following: $d\rho(g) \circ X_M = d\sigma_g \circ X_M = X_M \circ \sigma_g = X_M \circ \rho(g)$. The diagram below is commutative, say.



where $X_M(m_0) \in T_{m_0}M = T_{\sigma_g(m_0)}M$ since m_0 is a fixed point for σ. But, $\rho : G \longrightarrow GM$ and $\sigma_g : M \longrightarrow M$ are $\mathbb{F}$-diffeomorphisms. Thus their tangent maps $d\rho = \rho_*$ and $d\sigma_g = \sigma_{g*}$ are linear $\mathbb{F}$-diffeomorphisms too. It follows that $\rho_{*g} = d\rho|_{T_gG} : T_gG \longrightarrow T_{\rho(g)}(GM) = T_{\sigma_g}(GM)$ and $\rho(g)_{*m_0} = \sigma_{g*m_0} : T_{m_0}M \longrightarrow T_{\sigma_g(m_0)}M = T_{m_0}M$ are $\mathbb{F}$-smooth isomorphisms of linear spaces. Hence, $\psi(g) = d\sigma_g|_{T_{m_0}M} = \sigma_{g*m_0} \in Aut(T_{m_0}M)$. Therefore, ψ is a representation of G with respect to Definition 6.1.2. □

Definition 6.3.7
Let G be an $\mathbb{F}$-lie group acting on itself by inner automorphisms, that is, $L_gR_{g^{-1}}$:

$G \longrightarrow G$ such that $\mathfrak{a}_g(h) := L_g R_{g^{-1}}(h) = ghg^{-1}$. This conjugation action on G induces a map $\rho : G \longrightarrow Aut(G) \subset \mathfrak{Diff}(M)$ defined by $\rho(g)(h) = \mathfrak{a}_g(h)$ for any $h \in G$ and a fixed element $g \in G$.

Definition 6.3.8
Let G be an $\mathbb{F}$-lie group acting on itself by inner automorphisms, that is, by $\mathfrak{a}_g$. Let $Lin(\mathcal{G},\mathcal{G})$ be the space of linear maps from $\mathcal{G}$ to $\mathcal{G}$. Let the map denoted by $Ad : G \tilde{\longrightarrow} Aut(\mathcal{G}) \subset Lin(\mathcal{G},\mathcal{G})$, such that $g \tilde{\longmapsto} Ad(g) = d\mathfrak{a}_g|_{T_eG} = \mathfrak{a}_{g*e}$ with $T_eG \simeq \mathcal{G}$. The equality $Ad(g)(X) = gXg^{-1}$ defines the action of G on $\mathcal{G}$. The map Ad is called the adjoint representation of G into $Aut(\mathcal{G})$, or the adjoint representation of G on $\mathcal{G}$ following the action of G on $\mathcal{G}$.

Notice that, the unit element $e \in G$ is a fixed point for the conjugation action. That is, $\mathfrak{a}_e(h) = h$ for each $h \in G$. The adjoint representation $Ad : G \tilde{\longrightarrow} Aut(\mathcal{G})$ is actually $\psi : G \longrightarrow Aut(T_{m_0}M)$, where $Aut(T_{m_0}M) = Aut(T_eG) = Aut(\mathcal{G})$ when $M = G$ and $m_0 = e$. Usually in the literature $Ad(g) := Ad_g$. The differential $d(Ad)$ of Ad is denoted by ad, that is, $ad := d(Ad)$. Now the map $ad(X) := ad_X$ is the associated tangent map to Ad_g with $X \in \mathcal{G}$ when $g \in G$.

Definition 6.3.9
Let G be an $\mathbb{F}$-lie group acting on itself by conjugation action $\mathfrak{a}_g : G \longrightarrow G$. Let $\mathcal{G}$ and $\mathcal{G}^*$ be the $\mathbb{F}$-Lie algebra of invariant vector fields and its dual. Let $Ad : G \longrightarrow Aut(\mathcal{G})$ be the adjoint representation of G. The action of G on $\mathcal{G}^*$ denoted by Ad^* and given by $< Ad^*(g)\zeta, X > = < \zeta, \ Ad(g^{-1})(X) >$ for all $g \in G$, $\zeta \in \mathcal{G}^*$ and $X \in \mathcal{G}$ is called the co-adjoint action (co-adjoint representation). The corresponding infinitesimal action of $\mathcal{G}$ on $\mathcal{G}^*$ denoted by ad^* is given by $< ad^*(X)\zeta, Y > = < \zeta, -[X,Y] >$ for all $X, Y \in \mathcal{G}$ and $\zeta \in \mathcal{G}^*$.

Remark 6.3.2
The inverse of g in the definition of $Ad^*\zeta$ is taken in order to deal with a group homomorphism (left representation), instead of a group anti-homomorphism (right representation). Note that $Ad^* : \mathcal{G}^* \longrightarrow \mathcal{G}^*$ is not the pullback, which we would denote by $(Ad)^*$, of $Ad : \mathcal{G} \longrightarrow \mathcal{G}$. But the defining condition of Ad^* may state: $(Ad^*(g)\zeta)(X) = (\zeta(Ad(g^{-1})))(X)$, for all $g \in G$, $\zeta \in \mathcal{G}^*$ and $X \in \mathcal{G}$. It follows that

$$(Ad^*(g))(\zeta) = (Ad(g^{-1})^*)(\zeta) \text{ and } Ad^*(g) = (Ad(g^{-1}))^* = ((Ad(g))^{-1})^* = ((Ad(g))^*)^{-1},$$

so that $Ad^*(g)$ will not be confused with the pullback of $Ad(g)$, that is $(Ad(g))^*$. We would like to understand how the definition of Ad^* is derived. Firstly, we have: $Ad^*(g) : \mathcal{G}^* \longrightarrow \mathcal{G}^*$, $\zeta \longmapsto Ad^*(g)\zeta$ and $Ad^*(g)\zeta : \mathcal{G} \longrightarrow \mathbb{R}$, $X \longmapsto (Ad^*(g)\zeta)(X)$. Secondly, we have:

$$\begin{array}{ccccc} \mathcal{G} & \xrightarrow{Ad(g^{-1}) = (Ad(g))^{-1}} & \mathcal{G} & \xrightarrow{\quad\zeta\quad} & \mathbb{R} \\ X & \longrightarrow & Ad(g^{-1})(X) & \longrightarrow & \zeta(Ad(g^{-1})) \end{array}$$

And finally, we set $(Ad^*(g)\zeta)(X) = (\zeta(Ad(g^{-1})))(X)$, for all $g \in G$, $\zeta \in \mathcal{G}^*$ and $X \in \mathcal{G}$.

Proposition 6.3.2
Let ad_X be the associated tangent map to Ad_g with $X \in \mathcal{G}$ and $g \in G$. Then $ad_X Y = -[X,Y]$.

Proof.
The proof is a straightforward consequence of Definition 6.3.9. □

6.4 Moment map

From the definition of the anti-homomorphism $\sigma_* : \mathcal{G} \longrightarrow \mathfrak{X}(M)$ of Lie-algebras it follows that $\sigma_*(X) \in \mathcal{S}\mathfrak{p}(\omega)$ and $\sigma_*[X,Y] = -[\sigma_*(X), \sigma_*(Y)]$ with $X, Y \in \mathcal{G}$ if the left action of G on M, $\sigma : G \times M \longrightarrow M$ is symplectic. The inclusion $[\mathcal{S}\mathfrak{p}(\omega), \mathcal{S}\mathfrak{p}(\omega)] \subset \mathfrak{h}(\omega)$ implies that $\sigma_*[X,Y] = -[\sigma_*(X), \sigma_*(Y)] \in \mathfrak{h}(\omega)$ for all $X, Y \in \mathcal{G}$. It follows that $\sigma_*[\mathcal{G},\mathcal{G}] \subset \mathfrak{h}(\omega)$. [16, Theorem 18.3].

Definition 6.4.1
A symplectic action $\sigma : G \times M \longrightarrow M$ is called Hamiltonian if $\sigma_*(\mathcal{G}) \subset \mathfrak{h}(\omega)$. That is, each left invariant vector field ξ is associated to a Hamiltonian vector field X_ξ on M such that $X_\xi(m) = X_M(m) = (\frac{d}{dt} exp(t\xi)|_{t=0}).m$, where $X \in \mathcal{G}$ and $X(e) = \xi$.

By the characterization of Hamiltonian vector fields, it follows that $X_\xi = X_{\mu_\xi}$ such that $X_{\mu_\xi} \lrcorner\, \omega = d\mu_\xi$, where $\mu_\xi \in C^\infty(M)$. Hence, we can define a map $\mu^* : \mathcal{G} \longrightarrow C^\infty(M)$, which will be an anti-homomorphism of Lie-algebras $(\mathcal{G}, [,])$ and $(C^\infty(M), \{,\})$, such that$\mu^*(X) = \mu_X$ and $\mu^*[X,Y] = -\{\mu^*(X), \mu^*(Y)\}$. The map μ^* is called the comoment map of the action σ of G on M.

Definition 6.4.2
Let (M, ω) be a symplectic pseudomanifold, G an $\mathbb{F}$-Lie group acting on M by a symplectic action σ, $\mathcal{G}$ its Lie algebra and $\mathcal{G}^*$ the dual of $\mathcal{G}$. The moment map for the action σ of G on M is the map $\mu : M \longrightarrow \mathcal{G}^*$ defined by the following property: for all $\xi \in \mathcal{G}$ we have $<\mu(m), \xi> = \mu_\xi(m)$ and $d\mu_\xi = X_{\mu_\xi} \lrcorner\, \omega$, where $<,>$ is the duality pairing (bracket) on $\mathcal{G}^* \times \mathcal{G}$. That is the evaluation of $\mu(m)$ at ξ: $\mu(m)(\xi) = \mu_\xi(m)$, with $\mu(m) \in \mathcal{G}^*$, $\mu_\xi \in C^\infty(M)$.

Definition 6.4.3
The moment map $\mu : M \longrightarrow \mathcal{G}^*$ is G-equivariant if we have $(\mu \circ \sigma_g)(m) = (Ad^*(g) \circ \mu)(m)$ for every $(g,m) \in G \times M$. That is, with $Ad^* = (Ad(g^{-1}))^*$ as shown in Remark 6.3.2 and for $X \in \mathcal{G}$, $\big((\mu \circ \sigma_g)(m)\big)(X) = \big((Ad^*(g) \circ \mu)(m)\big)(X) = \big((Ad(g^{-1})^*)(\mu(m))\big)(X) = (\mu(m)(Ad(g^{-1})))(X)$.

Lemma 6.4.1
The moment map is G-equivariant if, and only if $\mu_X \circ \sigma_g = \mu_{g^{-1}Xg}$ for every $g \in G$ and $X \in \mathcal{G}$.

Proof.
The proof is a straightforward consequence of the second statement of Definition 6.4.3. Let $g \in G$, $X \in \mathcal{G}$ and $m \in M$. We have, $\big((\mu \circ \sigma_g)(m)\big)(X) = (\mu(m)(Ad(g^{-1})))(X)$ if, and only if $\mu_X(\sigma_g)(m)) = \mu_{Ad(g^{-1})X}(m)$ if, and only if $\mu\circ\sigma_g = \mu_{g^{-1}Xg}$. Recall that $Ad(g)X = gXg^{-1}$, with $gXg^{-1} \in \mathcal{G}$. Also $\mu\circ\sigma_g : M \longrightarrow \mathbb{R}$ and $\mu_{g^{-1}Xg} : M \longrightarrow \mathbb{R}$. □

In [33, *p*.15, Definition 1.1], μ_X is defined by the following property: for every tangent vector η we have $\eta\mu_X = \omega(X_M, \eta)$, where η is taken as a derivation on the smooth function $\mu_X \in C^\infty(M)$.

Definition 6.4.4 [*16*, Definition 22.1]
Let G be an $\mathbb{F}$-Lie group acting on a symplectic pseudomanifold (M, ω) and leaving ω invariant, that is, acting by symplectomorphisms. The action is called Hamiltonian if there is a G-equivariant moment map of the action.

Remark 6.4.1

1. *Definitions* 6.4.1 *and* 6.4.4 *are equivalent. Since the later (second one) says: there exists a map $\mu : M \longrightarrow \mathcal{G}^*$, such that*
   - *For each $X \in \mathcal{G}$, $\mu_X \in C^\infty(M)$, defined by $\mu_X(m) =< \mu(m), X >$ is the component of μ along X.*
   - *The Hamiltonian vector field X_M on M, generated by the one-parameter subgroup $\{exp(tX) \mid t \in \mathbb{R}\} \subset Ham(M) \subset G$, is equal to X_{μ_X} and $d\mu_X = X_{\mu_X} \lrcorner\, \omega$. That is, μ_X is a Hamiltonian function for X_M.*
   - *For each $g \in G$, we have $\mu \circ \sigma_g = Ad^*_g \circ \mu$.*
2. *The G-equivariance of the moment map $\mu : M \longrightarrow \mathcal{G}^*$ means that it intertwines the action of G on M and the co-adjoint action of G on $\mathcal{G}^*$.* [*28*, *p*.29].
3. *At the infinitesimal level, the moment map intertwines the infinitesimal action of $\mathcal{G}$ on M with the infinitesimal co-adjoint action of $\mathcal{G}$ on $\mathcal{G}^*$. That is, $ad^*(X) : \mathcal{G}^* \longrightarrow \mathcal{G}^*$, $< ad^*{}_X\zeta, Y >=< \zeta, -[X, Y] >$, for $\zeta \in \mathcal{G}^*$, $X, Y \in \mathcal{G}$, such that $Y \longmapsto ad_X(Y) = -[X, Y]$. It follows $ad^*{}_X\mu = -\mathcal{L}_{X_M}\mu$.*
4. *Let us take $\xi \in \mathcal{G}$. By differentiating $\mu_\xi Ad(g^{-1}) = \sigma^*_g\mu_\xi$, the co-adjoint equivariance will be equivalent to $\mu_\xi[X, Y] = \{\mu_\xi(X), \mu_\xi(Y)\}$, for any $X, Y \in \mathcal{G}$. That is, a Lie algebra homomorphism of $\mathcal{G}$ into $C^\infty(M)$* [*57*, *p*.195]. *In co-moment setting, we have $\mu^*[X, Y] = \{\mu^*(X), \mu^*(Y)]\}$ and the proof is similar to the one provided in* [*35*, *p*.4]. *The relation $d\mu_\xi = \xi \lrcorner\, \omega$ and the point 3. above can be taken into account.*
5. *We need to know the nature of $d\mu_m$. We recall that the moment map is defined by $\mu : M \longrightarrow \mathcal{G}^*$, $m \longmapsto \mu(m) = \mu_m$ with $\mu(m) : \mathcal{G} \longrightarrow \mathbb{R}$, and $X \longmapsto \mu(m)(X) = \mu_m(X)$. Thus, for all $m \in M$, the differential of μ at m is $(d\mu)_m = \mu_{*m} : T_mM \longrightarrow T_{\mu(m)}\mathcal{G}^*$. While for all$X \in \mathcal{G}$, the differential of μ_m at X is $(d\mu_m)_X : T_X\mathcal{G} \longrightarrow \mathbb{R}$.*

6. *The diagram below is commutative,*

$$\begin{array}{ccccccc} \mathcal{G} & \xrightarrow{\alpha} & T\mathcal{G} & & X & \xrightarrow{\alpha} & \alpha(X) \\ \downarrow exp & & \downarrow dexp & & \downarrow exp & & \downarrow dexp \\ G & \xrightarrow{\beta} & TG & & exp(tX) & \xrightarrow{\beta} & \beta(exp(tX)) = (dexp)(\alpha(X)) \end{array}$$

Where $\alpha(X) \in T_X\mathcal{G}$ *and* $\beta(exp(tX)) = (dexp)(\alpha(X)) \in T_{exp(tX)}G$, *with* α *and* β *being vector fields on* $\mathcal{G}$ *and* G *respectively.*

7. *We will consider the case where the existence of a* G*-equivariant moment map is granted. There exist in the literature some results concerning this purpose. But there are using cohomological arguments see for example [56, 57] for details. That topics are beyond those considered in this dissertation.*

Lemma 6.4.2
Let (M, ω) *be a symplectic pseudomanifold,* G *an* $\mathbb{F}$*-Lie group acting on* M *by symplectomorphisms. The flow of a symplectic vector field is a one-parameter subgroup of* $Sympl(M)$ *in* G*. The flow of a Hamiltonian vector field is a one-parameter subgroup of* $Ham(M)$ *in* $Sympl(M)$*.*

Definition 6.4.5
Let $\mathcal{G}$ *be the Lie algebra of a Lie group* G *acting on a pseudomanifold* M*. Let* $m \in M$, $\xi \in \mathcal{G}$ *and* $\mathcal{A}: \mathcal{G} \times M \longrightarrow TM$ *be the infinitesimal action of* $\mathcal{G}$ *on* M *such that* $\mathcal{A}(\xi) = \frac{d}{dt}exp(t\xi)|_{t=0} : M \longrightarrow TM$, $\mathcal{A}(\xi)(m) = (\frac{d}{dt}exp(t\xi)|_{t=0}).m = (\frac{d}{dt}exp(t\xi).m)|_{t=0}$. *The Lie subalgebra of* $\mathcal{G}$ *defined by* $\mathcal{G}_m := \{\xi \in \mathcal{G} \mid \mathcal{A}(\xi)(m) = \mathcal{A}(\xi, m) = 0\} = \{\xi \in \mathcal{G} \mid X^{\xi}_M(m) = 0\}$ *is called the isotropy (symmetry) subalgebra of* m*.*

Definition 6.4.6 [15, 16]
Let σ *be the Lie group action of* G *on a pseudomanifold* M*. The action* σ *is called free if* $g \neq e$ *implies that* $g.m \neq m$*, for all* $m \in M$*. That is,* $G_m = \{e\}$*, for every* $m \in M$*. Equivalently, all stabilizers are equal to the trivial subgroup* $\{e\}$*. The action* σ *is called locally free if* $\mathcal{G}_m = \{0\}$*, for every* $m \in M$*. That is, all stabilizers are discrete.*

Definition 6.4.7 [49, 64, 66]
Let σ *be the Lie group action of* G *on a pseudomanifold* M*. The action* σ *is called proper action if for every two convergent sequence* $\{m_n\}_{n\in\mathbb{N}}$ *and* $\{g_n.m_n\}_{n\in\mathbb{N}}$ *in* M*, there exists a convergent subsequence* $\{g_{nk}\}$ *of* $\{g_n\}_{n\in\mathbb{N}}$ *in* G *such that*

$$\lim_{k\to\infty}(g_{nk}.m_{nk}) = (\lim_{k\to\infty} g_{nk})(\lim_{k\to\infty} m_{nk}).$$

or equivalently, as in [4],

The action σ is called proper action if the map $\Phi : G \times M \longrightarrow M \times M$ defined by $(g, m) \longmapsto (m, g.m)$, where $g.m = \sigma(g, m) = \sigma_g(m) = \sigma_m(g)$, is a proper map. That is, the pre-image of any compact set is a compact set with respect to $\tau_{\mathcal{F}_{M\times M}}$ and $\tau_{\mathcal{F}_{G\times M}}$, where $M\times M$ and $G\times M$ are finite products of pseudomanifolds, thus they are also pseudomanifolds.

Lemma 6.4.3
The properness of an action implies the compactness of G_m and the closeness of Φ defined above.

Proof.
First of all, we claim that Φ is a smooth map with respect to Lemma 3.2.1, since σ_g is smooth if, and only if $\Phi_g : M \longrightarrow G(\sigma_g) \subset M \times M$ is a diffeomorphism into the graph of σ_g. Hence, Φ is smooth since it is so in its all G-components Φ_g. We could also say, since $\Phi = \imath \circ (\sigma_g \times id_M)$, where $\imath : M \times M \longrightarrow M \times M$ is given by $\imath(\sigma_g(m), m) = (m, \sigma_g(m))$. Thus, Φ is smooth as the composite of smooth maps and in the sequel a continuous map. Now, we need to derive $\Phi^{-1}\{(m, m)\} = \Phi^{-1}(\{m\} \times \{m\})$. For, the following equalities hold:

$$\begin{aligned}
\Phi^{-1}(\{(m, m)\}) &= \{(g, m) \in G \times \{m\} \mid \Phi(g, m) = (m, m)\} \\
&= \{(g, m) \in G \times \{m\} \mid (m, g.m) = (m, m)\} \\
&= \{(g, m) \in G \times \{m\} \mid g.m = m\} \\
&= \{(g, m) \in G \times \{m\} \mid g. \in G_m\} \\
&= G_m \times \{m\}.
\end{aligned}$$

From the diffeomorphism $\Phi_g : M \longrightarrow G(\sigma_g) \subset M \times M$, it follows that for each $g \in G_m$, we have $\Phi_g(m) = (m, g.m) = (m, m)$. Hence, $\Phi(G_m \times \{m\}) = \{(m, m)\}$. We should note that $\{(m, m)\} = (\{m\} \times \{m\})$ is compact, since from general topology, we draw the following results. In any topological space, all finite subsets and $\emptyset$ are compact subsets. [27, p.222, Example 2]. The product of a family of topological spaces is compact if, and only if each factor of the product is a compact set. It follows from the properness of Φ that $G_m \times \{m\} \subset G \times M$ is compact as the pre-image of a compact set. Therefore, G_m is a compact set as a factor of a product of compact sets. It will be noted that the continuous image of a compact set is compact. This confirms the compactness of G_m shown above. Since the canonical projection $\pi : G \times M \longrightarrow G$ is smooth, thus continuous, it follows that $\pi(G_m \times \{m\}) = G_m$ is a compact set in G. [27, p.224, Theorem 1.4]. Finally, a compact set in a Hausdorff space is closed and the Cartesian product of closed sets is always a closed set. It follows that $\{m\}, \{(m, m)\}, G_m$ are closed set, with $m \in M$ and G_m. From the definition of a proper action, one can see that Φ is a closed map. □

Remark 6.4.2
Let $\varphi : M \longrightarrow N$ be a smooth map of pseudomanifolds with $dim\, M = m$ and

$dim\, N = n$. If φ has constant rank k on N and $y = \varphi(x)$, where $y \in N$ and $x \in M$. Then,

1. *The set $\varphi^{-1}(y)$ is a closed, regular subpseudomanifold of N of dimension $m - k$, that is, of codimension k.*
2. *For any point $x \in M$, we have $T_x(\varphi^{-1}(y)) = Ker(d_x\varphi)$.*
3. *For any point $x \in M$, for any $\mathcal{U}_x$, a sufficiently small neighborhood of x, the image $\varphi(\mathcal{U}_x)$ is a k-dimensional subpseudomanifold in N.*
4. *For any point $x \in M$, we have $T_{\varphi(x)}\varphi(\mathcal{U}_x) = Im(d_x\varphi)$.*
5. *If $\varphi(M)$ is a subpseudomanifold of N then $dim\, \varphi(M) = k$.*

Lemma 6.4.4
Let σ be an action of an $\mathbb{F}$-Lie group G on a pseudomanifold M. Then,

1. *The orbit map $\sigma_m : G \longrightarrow M$, where $\sigma_m(g) = g.m$, is a smooth map of pseudomanifolds. Its rank is constant, that is, for any $g \in G$, $rank(\sigma_m) = rank(d_g\sigma_m) = k$.*
2. *$T_e(G_m) = Ker(d_e\sigma_m)$.*
3. *The stabilizer G_m is a closed Lie subgroup of G such that $dim\, G_m = dim\, G - rank(\sigma_m) = dim\, G - k$.*
4. *For any sufficiently small neighborhood $\mathcal{U}_e$ of the unit element of G, the subset $\sigma_m(\mathcal{U}_e) = \mathcal{U}_e.m$ is a subpseudomanifold of dimension k in M.*
5. *$T_m(\mathcal{U}_e.m) = im\, d_e\sigma_m$.*
6. *If the orbit $G.m$ is a subpseudomanifold in M then $dim\, G.m = k$.*

Proof.
The proof is done with respect to following references: [63, *pp*.6, 17], [12], [15, *p*.41] and [9, *p*.79 Theorem 5.8, p.82 Theorem 6.6].

1. If one gives a glance to the following commutative diagrams, the constant rank of σ_m followed:

$$\begin{array}{ccc} G & \xrightarrow[\sim]{L_g} & G \\ \downarrow{\sigma_m} & & \downarrow{\sigma_m} \\ M & \xrightarrow[\sim]{\sigma_g} & M \end{array} \qquad \begin{array}{ccc} h & \xrightarrow[\sim]{L_g} & L_g(h) = gh \\ \downarrow{\sigma_m} & & \downarrow{\sigma_m} \\ \sigma_m(h) = h.m & \xrightarrow[\sim]{\sigma_g} & \sigma_g(\sigma_m(h)) = \sigma_m(L_g(h)) \end{array}$$

Indeed, $\sigma_g(\sigma_m(h)) = \sigma_g(h.m) = \sigma_{gh}(m) = \sigma_m(gh) = \sigma_m(L_g(h))$. This yields:

$$\begin{array}{ccc} T_hG & \xrightarrow[\sim]{d_hL_g} & T_{gh}G \\ \Big\downarrow d_h\sigma_m & & \Big\downarrow d_{gh}\sigma_m \\ T_{\sigma_m(h)}M & \xrightarrow[\sim]{d_{\sigma_m(h)}\sigma_g} & T_{\sigma_g(h.m)}M = T_{\sigma_m(gh)}M \end{array}$$

Thus, for $h = e$:

$$\begin{array}{ccc} T_eG = \mathcal{G} & \xrightarrow[\sim]{d_eL_g} & T_gG \\ \Big\downarrow d_e\sigma_m = \mathcal{A}_m & & \Big\downarrow d_g\sigma_m \\ T_mM & \xrightarrow[\sim]{d_m\sigma_g} & T_{g\,.\,m}M \end{array}$$

Note that both d_eL_g and $d_m\sigma_g$ are isomorphisms of linear spaces, where $d_g\sigma_m \circ d_eL_g = d_m\sigma_g \circ d_e\sigma_m$. It follows that $d_g\sigma_m = d_m\sigma_g \circ d_e\sigma_m \circ (d_eL_g)^{-1}$. Hence,

$$\begin{aligned} Ker d_g\sigma_m &= (d_g\sigma_m)^{-1}(0) \\ &= \big(d_m\sigma_g \circ d_e\sigma_m \circ (d_eL_g)^{-1}\big)^{-1}(0) \\ &= \big(d_eL_g \circ (d_e\sigma_m)^{-1} \circ (d_m\sigma_g)^{-1}\big)(0) \\ &= d_eL_g\big((d_e\sigma_m)^{-1}(0)\big), \quad \text{since } d_m\sigma_g \text{ } is \text{ a linear isomorphism.} \end{aligned}$$

We need to investigate the nature of $(d_e\sigma_m)^{-1}(0)$ for a fixed $m \in M$:

$$\begin{aligned} (d_e\sigma_m)^{-1}(0) &= Ker d_e\sigma_m \\ &= \{X \in \mathcal{G} \mid d_e\sigma_m(X) = 0\}, \quad but \;\; \mathcal{A}(m) = d_e\sigma_m \\ &= \{X \in \mathcal{G} \mid \mathcal{A}(m)(X) = \mathcal{A}(X)(m) = X_m = 0\} \\ &= \mathcal{G}_m. \end{aligned}$$

It follows that $Ker d_g\sigma_m = d_eL_g\big(Ker d_e\sigma_m\big) = d_eL_g\big(\mathcal{G}_m\big)$. Therefore, $X \in \mathcal{G}_m$ corresponds to $\mathcal{A}(m)(X) = d_e\sigma_m(X) = 0$. That is, σ_m is a constant map for some $h \in G$.
This means that $h.m = \sigma_m(h) = \sigma_h(m) = m$. That is equivalent to saying that $h \in G_m$. and T_eG_m. Now, considering G_m and $\mathcal{G}$ as linear spaces, we have:

$$\begin{aligned} dim\, \mathcal{G} &= dim\, Ker d_e\sigma_m + dim\, im\, d_e\sigma_m \\ &= dim\, \mathcal{G}_m + rank(d_e\sigma_m) \\ &= dim\, d_eL_g\mathcal{G} \\ &= dim\, T_gG \end{aligned}$$

Hence, for all $g \in G$, we have $rank(d_g\sigma_m) = rank(d_e\sigma_m)$. That is, the rank of $(d_g\sigma_m)$ is constant for all $g \in G$. Therefore, σ_m has a constant rank, $rank(\sigma_m) = k$, say.

2. From Part 1. in the proof, $T_eG_m = \mathcal{G}_m = Ker d_e\sigma_m$.

3. Since $G_m = \{g \in G \mid \sigma_m(g) = m\} = \sigma_m - 1(m)$ and $\{m\}$ is a closed set in the Hausdorff space M, thus, G_m is closed set in G by the smoothness of σ_m. Let $g, h \in G_m$. Then $(gh).m = g.(h.m) = g.m = m$, that is, $gh \in G_m$. Also, $g.m = m$ implies $m = g^{-1}.m$, that is, $g^{-1} \in G_m$. Hence, G_m is a closed subgroup of G. From Remark 6.4.2, (1), G_m is a closed, regular subpseudomanifold of G. Therefore, G_m is a Lie subgroup of G. From Part (2) above in the proof we have,

$$\begin{aligned} dim\, G_m &= dim\, T_eG_m \\ &= dim\, Ker d_e\sigma_m \\ &= dim\, G - rank(\sigma_m) \\ &= dim\, G - k. \end{aligned}$$

4. Since $\mathcal{U}_e$ is an open set in G then $T_g\mathcal{U}_e = T_gG$ for all $g \in \mathcal{U}_e$ with respect to Lemma 4.1.7. It follows that $\sigma_m(\mathcal{U}_e) = \mathcal{U}_e.m$ is a subpseudomanifold of dimension k with respect to Lemma 6.4.4, (4) and the computation of $dim\,\mathcal{G}$ in Part (1) of the proof above.

5. We have $T_m\sigma_m(\mathcal{U}_e) = T_m(\mathcal{U}_e.m) = im\, d_e\sigma_m$ from Part (1) above in the proof.

6. Recall that the orbit $G.m = \sigma_m(G)$ and $T_m\sigma_m(G) = im\, d_e\sigma_m$. Therefore, $dim\, G.m = dim\, im d_e\sigma_m = rank(\sigma_m) = k$. □

Remark 6.4.3

1. *We would like to link the quotient $M/G = \{G.m \mid m \in M\}$ to the construction made in Sections* 2.8 *and* 3.8.

2. *It is obvious that the action of a compact group on a pseudomanifold is naturally proper.*

3. *If the action is proper the the orbit map $\sigma_m : G \longrightarrow M$ is proper for each $m \in M$.* [*33*, *p.*174, Lemma B.3].

4. *Note that if σ : GtimesM $\longrightarrow M$, $(g, m) \longmapsto g.m$ is a proper map then $\Phi : G \times M \longrightarrow M \times M$ is a proper map. That is, the action σ is proper.*

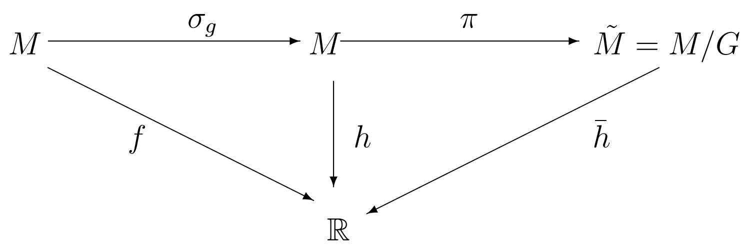
5. *The reciprocal statement is not true in general.* [*33*, *p.*174, Remark B.4].

6. *From Part (6) in the proof of Lemma* 6.4.4, *one concludes that the tangent space $G.m$ at m is $T_m(G.m) = span(im\,\mathcal{A}_m) = span(\{\mathcal{A}_m(X) \mid X \in \mathcal{G}\}) = span(\{X_m \mid X \in \mathcal{G}\})$, with $T_m(G.m) \subset T_mM$.*

Lemma 6.4.5
For each $f \in C^\infty(M)$ and for each $g \in G$, there exists a unique $h \in C^\infty(M)$, such that:

1. $f = h \circ \sigma_g$.
2. $\pi \circ \sigma_g = \pi$.
3. *The quotient structure is compatible with the $\mathbb{F}$-structure.*

Proof.
The diagram below shows the position of the problem.



1. Since the pullback $\sigma_g^* : C^\infty(M) \longrightarrow C^\infty(M)$ is one-to-one and onto, then h exists and is unique. And $h = f \circ \sigma_{g^{-1}}$ is a smooth function as the composite of smooth maps.
2. The equivalence relation induced by the action σ of G on M is defined by: Given $x, y \in M$, $x \sim y$ if, and only if $y \in G.x$ or $G.x = G.y$ or there exists $g \in G$, such that $y = g.x = \sigma_g(x) = \sigma_x(g) = \sigma(g, x)$. It follows that $G.y = G.(g.x) = (Gg).x = G.x$ if, and only if $\pi(y) = \pi(g.x) = \pi(x)$ if, and only if $\pi \circ \sigma_g(x) = \pi(x)$ if, and only if $\sigma_g^* \pi = \pi \circ \sigma_g = \pi$ if, and only if π is invariant under the action σ of G on M. Note that $Gg = R_g(G) = \{hg \mid h \in G\} = G$ since $R_g : G \longrightarrow G$ is a diffeomorphism.
3. Now, we want to show the compatibility of the relation above with the structure functions as in Sections 2.8 and 3.8. From the definition, one has $x \sim y$ if, and only if $y = g.x$ for some $g \in G$. We get $\bar{h}(G.y) = \bar{h}(G.x)$ if and only if, $\bar{h}(\pi(y)) = \bar{h}(\pi(x))$ if, and only if $h(y) = h(x)$ since $h = \bar{h} \circ \pi$, which is equivalently the formula $\bar{h} = h \circ \pi^{-1}$, where $\bar{h}$ one-to-one. It follows that h is constant on each orbit the equivalence class. That is, $h \circ \sigma_g = h$ if, and only if h is invariant under the action of σ of G on M if, and only if $h \in C^\infty(M)^G$, the spaces of smooth invariant functions on M. Linking this to Part 1. above, we draw the following consequence, $f = h \circ \sigma_g = h$, that is, $C^\infty(M)^G \subseteq C^\infty(M)$. Therefore, the is compatibility with invariant smooth functions. $\square$

Remark 6.4.4

1. *The equality* $C^\infty(M)^G = C^\infty(M)$ *would be a consequence of the definition of the equivalence relation compatible with the* $\mathbb{F}$*-structure functions. That is a strong condition providing the quotient with a Hausdorff topology. We may build the quotient, by assuming it to be Hausdorff space, with respect to one function. Thus, by extension to invariant smooth functions. In this case, we get a strict inclusion* $C^\infty(M)^G \subset C^\infty(M)$*, where also, the constant functions are invariant under the action.*

2. *The construction of quotient pseudomanifold in the sight of Sections* 2.8 *and* 3.8 *yields the fact that the orbit set* $M/G = \{G.x \mid x \in M\}$ *is the quotient pseudomanifold of* M *by the relation* $\sim$ *induced from the action* σ *of* G *on* M*.*

3. *It seems that in pseudomanifold setting, there is no need for* σ *to be proper for* M/G *being a pseudomanifold since we do not use this assumption in the construction of the quotient pseudomanifold.*

4. *But, we will need the properness of* σ *to get the compactness of* G_m *and consequently its closeness. Thus, the restriction of the action to* G_m *will be proper.*

5. *If* $y \in G.x$ *then* $G_y = gG_xg^{-1}$*, for all* $g \in G$. [*15*, *p.*39]. *The stabilizers* G_x *and* G_y *are defined by:* $G_x = \{h \in G \mid h.x = x\}$ *and* $G_y = \{k \in G \mid k.x = x\}$*. But,* $y = g.x$*, if* $g \in G$*. Thus,* $G_{g.x} = \{k \in G \mid k.(g.x) = g.x\} = \{k \in G \mid (kg).x = g.(h.x)$ *for all* $h \in G_x$*,* g *fixed in* $G\}$*. It follows* $kg = gh$ *for all* $k \in G_{g.x}$ *and for all* $h \in G_x$*. Thus,* $k = ghg^{-1}$*. Hence,* $G_{g.x} = gG_xg^{-1}$*. That is, if* y *is in the orbit of* x*, then* G_y *and* G_x *are conjugate subgroups of* G*.*

6. *Otherwise stated: A conjugate* gG_xg^{-1} *of the stabilizer* G_x *is also a stabilizer.*

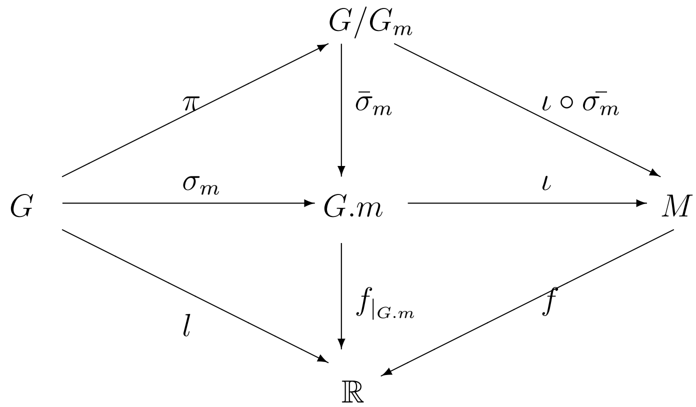
7. *Recall that the canonical projection* $\pi : M \longrightarrow M/G$ *is a smooth map then a continuous map. Thus, the continuous image of a compact is compact and closed too, since* M/G *is Hausdorff topological space.*

8. *If* $h \in C^\infty(M)^G$ *then* X_h *is* G*-invariant, that is,* $T_m\sigma_g X_h(m) = X_h(\sigma_g(m))$*.* [*24*]

Lemma 6.4.6 [*33*, *p.*174, Lemma B.3].
If the action is proper, its restriction to any closed subgroup $H \subsetneqq G$ *is proper* H*-action on* M*, and its restriction to any invariant subset* S *of* M *is a proper* G*-action on* S*.*

Lemma 6.4.7 [*15*, *pp.*39 – 42, Theorem 1], [*33*, *p.*175, Proposition B.8].
If the action is proper, every orbit is closed subset of M *and a subpseudomanifold with* $\dim G.m = rank(\sigma_m)$*.*

Proof.
Let $\sigma_m : G \longrightarrow M$ be the orbit map. From Lemma 6.4.4 and Remark 6.4.3 (3), the orbit map is proper. Hence a closed map. Therefore, $\sigma_m(G) = G.m \subset M$ is a closed subset of M since G is a closed set. Now, le σ_m be considered into its image $\sigma_m(G) = G.m$. That is, ${\sigma_m}^{-1}(m) = \{g \in G \mid \sigma_m(g) = g.m = m\}$, where $\sigma_m : G \longrightarrow G.m$ is onto. It follows that g and h belong to ${\sigma_m}^{-1}(m)$ if, and only if $\sigma_m(g) = \sigma_m(h) = m$ if, and only if $g.m = h.m$ if, and only if $h^{-1}gm = m$ if, and only if $h^{-1}g \in G_m$ if, and only if there exists $k \in G_m$ such that $h^{-1}g = k$ if, and only if $g = hk$ if, and only if $gk^{-1} = h$ if, and only if $h \in gG_m$ if, and only if $g \sim h$, (where $\sim$ is the orbit relation) if, and only if the fibers of σ_m are left G_m-cosets in G. Thus, there exists a bijection denoted by $\bar{\sigma}_m : G/G_m \overset{\sim}{\longrightarrow} G.m$, with $gG_m \longmapsto g.m$ such that, if ι is the canonical inclusion of $G.m$ into M then the following diagram is commutative:



That is,$\sigma_m = \bar{\sigma}_m \circ \pi$, $l = f_{|G.m} \circ \sigma_m$ and $l = f_{|G.m} \circ \bar{\sigma}_m \circ \pi$, with $f \in C^\infty(M)$, $l \in C^\infty(G)$, and $f_{|G.m} \in C^\infty(G.m) = C^\infty(M)_{|G.m}$, by the closeness of $G.m$ in M. But, $l\pi^{-1} = f_{|G.m} \circ \bar{\sigma}_m$ is smooth from the construction of the quotient pseudomanifold G/G_m. From Corollary 2.3.2, it follows that $\bar{\sigma}_m$ is a diffeomorphism of pseudomanifolds. Thus, $G.m$ is a closed regular subpseudomanifold of M with respect to Section 3.5. From Lemma 6.4.4 (6), one can conclude that $dim\, G.m = rank(\sigma_m)$. □

Remark 6.4.5
The map $\iota : G.m \longrightarrow M$ is a one-to-one smooth immersion such that the composite map $\iota \circ \bar{\sigma}_m : G/G_m \longrightarrow M$ is a one-to-one smooth immersion. The orbit map $\sigma_m : G \longrightarrow G.m$ is a surjective smooth submersion. The orbit $G.m$ is open and closed set since the orbits form a partition of M. The map $\iota \circ \bar{\sigma}_m : G/G_m \longrightarrow M$ is an open and closed map.

Lemma 6.4.8 [*64*], [*15*, *p.*41].
Let $\mathcal{G}$ be the Lie algebra of a Lie group acting on a pseudomanifold M. If the Lie algebra action is given by the infinitesimal generators, then

1. *The algebra $\mathcal{G}_m$ is a closed Lie subalgebra of $\mathcal{G}$ as the Lie algebra of G_m. Thus, $dim\, \mathcal{G}_m = dim\, G_m$.*

2. *If $G_m = \{e\}$, that is, the action is free, then the orbit map $\sigma_m : G \longrightarrow M$ is a one-to-one immersion.*

Proof.

1. From Lemma 6.4.4, (2), the set $\mathcal{G}_m$ is a linear subspace of $\mathcal{G}$ as the kernel of the linear map $d_e\sigma_m$ and $dim\, \mathcal{G}_m = dim\, T_e G_m = dim\, G_m$. The fact that $\mathcal{G}_m = (d_e\sigma_m)^{-1}(0)$ implies that $\mathcal{G}_m$ is a closed subpseudomanifold in $\mathcal{G}$. Now, let $X, Y \in \mathcal{G}_m$. Thus, $[X,Y]_m = -[X_m, Y_m] = -[0,0] = 0$. One concludes that $[X,Y] \in \mathcal{G}_m$ with respect to Remark 6.2.2, (1),(2). It follows that $\mathcal{G}_m$ is a closed Lie subalgebra of $\mathcal{G}$ with its dimension given by $dim\, G_m = dim\, \mathcal{G}_m$.

2. Let us assume that $G_m = \{e\}$. It follows from the diagram drown in the proof of Lemma 6.4.7 that the diffeomorphism $\bar{\sigma}_m : G/G_m \longrightarrow G.m$ changes to $\bar{\sigma}_m = \sigma_m : G/\{e\} \longrightarrow G.m$. Therefore, $\sigma_m : G \longrightarrow M$ is a one-to-one immersion. □

Remark 6.4.6
The Part 2. in Lemma 6.4.8 *is a characterization of a free action. In what follows we will be dealing with regular elements in the pseudomanifold setting by borrowing the definitions from subcartesian context as in* [*50, p.*4] *and from the regularity in smooth manifolds. We will show that these two definitions are equivalent.*

Definition 6.4.8 [*35*]
Let M be a pseudomanifold, $\mu : M \longrightarrow \mathcal{G}^*$ a moment map associated to a Hamiltonian action of G on M. An element $m \in M$ is called a regular point(element) of the moment map μ if the tangent map of μ, that is, $\mu_{*m} : T_m M \longrightarrow T_{\mu(m)}\mathcal{G}^* = \mathcal{G}^*$ is onto. An element $\theta \in \mathcal{G}^*$ is called a regular value of the moment map μ if all elements in the inverse image $\mu^{-1}(\theta)$ are regular elements of μ.

Definition 6.4.9 [*50*]
Let M be a pseudomanifold, $\mu : M \longrightarrow \mathcal{G}^*$ a moment map associated to a Hamiltonian action of G on M. An element $m \in M$ is called a regular point of the moment map μ if there exists an open neighborhood $\mathcal{U}$ of m in M such that $dim\, T_m M = dim\, T_p M$ for all $p \in \mathcal{U}$.

Lemma 6.4.9
Definitions 6.4.5 *and* 6.4.6 *are equivalent.*

Proof.
Let $\mathcal{U}$ and $\mathcal{V}$ be open neighborhoods of $m \in M$ and of $\mu(m) \in \mathcal{G}^*$. Then $T_m\mathcal{U} \simeq T_m M$ and $\mathcal{G}^* = T_{\mu(m)}\mathcal{G}^* \simeq T_{\mu(m)} V$ with respect to Lemma 4.1.7. In the sequel the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad\mu\quad} & \mathcal{V} \\ \Big\downarrow X_{|\mathcal{U}} & & \Big\downarrow X_{|\mathcal{V}} \\ T_m\mathcal{U} \simeq T_mM & \xrightarrow{\quad\mu_{*m}\quad} & T_{\mu(m)}\mathcal{V} \simeq T_{\mu(m)}\mathcal{G}^* \end{array}$$

Now, from $T_m\mathcal{U} \simeq T_mM$ one yields $T_p\mathcal{W} \simeq T_pM$, where $p \in \mathcal{U}$ and $\mathcal{W}$ is an open neighborhood of $p \in M$. Hence, $T_mM = T_pM$. Recall that M is assumed to be a constant dimensional pseudomanifold. Finally, μ_{*m} is onto means that for every $\xi \in \mathcal{G}^*$, there exists $v_m \in T_mM \simeq T_pM$ such that $\mu_{*m}(v_m) = \xi = \mu_{*m}(w_p)$, where the isomorphism $\simeq$ maps v_m onto w_p. That is, $dim\, T_mM = dim\, T_pM$. If we work backwards the arguments above, then we are done with the equivalence we were seeking for. □

6.5 Symplectic reduction on pseudomanifolds

Theorem 6.5.1
Let (M, ω) be a symplectic pseudomanifold and $\mu : M \longrightarrow \mathcal{G}^*$ a moment map associated to a Hamiltonian G-action on M, with $dim\, G = n$, $dim\, M = q$ and $q \geq n$. Let $\theta \in \mathcal{G}^*$ such that $\mu^{-1}(\theta)$ is nonempty and θ a regular value of μ. Let $\iota_\theta : \mu^{-1}(\theta) \hookrightarrow M$ be the canonical inclusion. Then the subset $\mu^{-1}(\theta)$ is a closed embedded subpseudomanifold of M and its dimension is given by $dim\, \mu^{-1}(\theta) = dim\, M - dim\, G = q - n$.

Proof.
From Remark 6.4.2, $\mu^{-1}(\theta)$ is a closed embedded (regular) subpseudomanifold with $dim\, \mu^{-1}(\theta) = dim\, M - rank_m\, \mu$, where $m \in M$ such that $\mu(m) = \theta$, since the inclusion ι_θ is an injective immersion because of the closeness of $\mu^{-1}(\theta)$. As a consequence of the regularity of θ, it follows that μ_{*m} is a surjective map (or μ is a submersion). Thus, $rank\, \mu = dim\, \mathcal{G}^* = n$. Therefore, $dim\, \mu^{-1}(\theta) = q - n$. □

Definition 6.5.1
Let $\mathcal{G}$ be the Lie algebra of a Lie group G and $\theta \in \mathcal{G}^*$. The subgroup of G denoted by $G_\theta = \{g \in G \mid Ad^*_g\theta = \theta\}$ is called the isotropy subgroup of θ with respect to the co-adjoint action of G on $\mathcal{G}^*$. The set $G.\theta = \{Ad^*_g\theta \mid g \in G\} \subset \mathcal{G}^*$ is the orbit of the co-adjoint action of G on $\mathcal{G}^*$. The Lie algebra of G_θ, denoted by $\mathcal{G}_\theta = \{\xi \in \mathcal{G} \mid ad^*(\xi)\theta = \theta\}$, is the isotropy subalgebra of θ.

Lemma 6.5.1
Let $\mathcal{G}$ be the Lie algebra of an $\mathbb{F}$-Lie group G and $\theta \in \mathcal{G}^*$, a regular value of a moment map $\mu : M \longrightarrow \mathcal{G}^*$ associated to a Hamiltonian G-action on a symplectic pseudomanifold (M, ω). Assume the G-action free and proper. Then

1. *The subgroup G_θ is a compact (thus, closed) set in G, acting smoothly on $\mu^{-1}(\theta)$.*

2. *$G_m \subset G_\theta$ for all $m \in \mu^{-1}(\theta)$.*

3. *$\mu^{-1}(\theta)$ is invariant under the restricted action of G_θ.*

4. *Every $\alpha \in G.\theta = \{Ad_g^*\theta \mid g \in G\} \subset \mathcal{G}^*$ is a regular value of the moment map μ.*

5. *G_θ acts freely and properly on the subpseudomanifold $\mu^{-1}(\theta)$. [47].*

Proof.

1. The subgroup G_θ is compact and so closed following Lemma 6.4.3. It restricts the G-action. Thus, it acts on the entire M smoothly. It does the same on $\mu^{-1}(\theta)$.

2. Let $g \in G_m$. That is, $\sigma(m) = m$. Thus, $\mu(m) = \mu(\sigma(m))$. But, μ is equivariant, $m \in \mu^{-1}(\theta)$ and $\theta \in \mathcal{G}^*$ is a regular value of μ, it follows that $\theta = \mu(m) = \mu(\sigma(m)) = Ad_g^*\mu(m) = Ad_g^*\theta$. Hence, $g \in G_\theta$. Therefore, $G_m \subset G_\theta$.

3. From Part 2. above, we have $\theta = Ad_g^*\theta$ for all $g \in G_\theta$ if, and only if $\mu(m) = Ad_g^*\mu(m)$ for all $g \in G_\theta$, for all $m \in \mu^{-1}(\theta)$ if, and only if $\sigma_g(m) = m$ for all $g \in G_\theta$, for all $m \in \mu^{-1}(\theta)$ if, and only if G_θ preserves $\mu^{-1}(\theta)$.

4. This is again a consequence of the equivariance of μ. Let $\alpha \in G.\theta$, that is, there exists $g \in G$, such that $g.\theta = \alpha$ or $Ad_g^*\theta = \alpha$. Hence, for all $m \in \mu^{-1}(\theta)$, we have $\alpha = Ad_g^*\theta = Ad_g^*\mu(m) = \mu(\sigma_g(m)) = \mu(g.m)$. Thus, $\mu^{-1}(\theta) = \sigma_g^{-1}(\mu^{-1}(Ad_g^*\theta)) = \sigma_g^{-1}(\mu^{-1}(\alpha))$, since the moment map is defined by $\mu = (Ad_g^*)^{-1} \circ \mu \circ \sigma_g$. But, $\sigma_g^{-1} : \mu^{-1}(\theta) \longrightarrow \mu^{-1}(\alpha)$ is a diffeomorphism of pseudomanifolds. Hence, we are done.

5. This is an obvious consequence of Parts 2. ,3. and 4. above. □

Lemma 6.5.2
Let (M, ω) be a symplectic pseudomanifold and G an $\mathbb{F}$-Lie group. Assume that the G-action on M is Hamiltonian. Let $\mu : M \longrightarrow \mathcal{G}^*$ be its moment map. For every $\xi \in \mathcal{G}$, every $g \in G$ and every $m \in M$ the following equality holds: $T_m\mu(\xi_M(m)) = \xi_{\mathcal{G}^*}(\mu(m))$

Proof.
Since the G-action is Hamiltonian, then the moment map μ is co-adjoint equivariant. That is, $\mu \circ \sigma_g = Ad_g^*\mu$ or the diagrams below are commutative:

$$\begin{array}{ccc} M & \xrightarrow[\sim]{\sigma_g} & M \\ \downarrow \mu & & \downarrow \mu \\ \mathcal{G}^* & \xrightarrow[\sim]{Ad^*_g} & \mathcal{G}^* \end{array} \qquad \begin{array}{ccc} T_mM & \xrightarrow[\sim]{T_m\sigma_g} & T_mM \\ \downarrow T_m\mu & & \downarrow T_m\mu \\ \mathcal{G}^* & \xrightarrow[\sim]{T_m\sigma_g} & \mathcal{G}^* \end{array}$$

The arguments above are true if we restrict the action to the one-parameter group $\sigma_{exp(t\xi)}$, so, one gets $\mu(\sigma_{exp(t\xi)}(m)) = Ad^*(\sigma_{exp(t\xi)})(\mu(m))$. The infinitesimal version reads $T_m\mu(\xi_M(m)) = \frac{d}{dt}exp(t\xi)|_{t=0} = \frac{d}{dt}Ad^*(\sigma_{exp(t\xi)})(\mu(m))|_{t=0} = \xi_{\mathcal{G}^*}(\mu(m))$, as required. □

Lemma 6.5.3 [*16, 15, 24*]
Let G be an $\mathbb{F}$-Lie group. Let $\mu: M \longrightarrow \mathcal{G}^*$ be the moment map associated to a Hamiltonian G-action on a symplectic pseudomanifold (M,ω). If the G-action is free then $KerT_m\mu = T_m\mu^{-1}(\theta)$ and $im\,T_m\mu = \mathcal{G}^o_m = \mathcal{G}^{\omega_m}_m$, where $m \in \mu^{-1}(\theta)$, $\theta \in \mathcal{G}^*$, is a regular value of the moment map μ and $\mathcal{G}^o_m = \mathcal{G}^{\omega_m}_m$ is the annihilator of the Lie algebra $\mathcal{G}_m$ of the stabilizer of m, with respect to ω.

Proof.
In the sight of Remark 6.4.2, one has, $dim\,M = q$, $dim\,\mathcal{G}^* = n$, and the moment map $\mu: M \longrightarrow \mathcal{G}^*$ is smooth. Also, since θ is a regular value of μ then $m \in \mu^{-1}(\theta)$ is a regular element of μ. That is, $T_m\mu$ is surjective at each $m \in \mu^{-1}(\theta)$. That is, $rank\,(T_m\mu) = rank\,(\mu) = k$. Thus, $T_m\mu^{-1}(\theta) = KerT_m\mu$. But, we know from Linear Algebra that $dim\,T_mM = dim\,KerT_m\mu + dim\,im\,T_m\mu$ (the rank theorem for a linear map). From Definition 5.1.1 and Remark 5.1.1, (1), and (2), we had $dim\,V = dim\,W + dim\,W^\perp$ and $(V/W)^* = W^{\omega_m} = W^\perp$. Now, we can set $W = KerT_m\mu$. Then, $KerT_m\mu^{\omega_m} = (T_mM/KerT_m\mu)^* \simeq (im\,T_m\mu)^* \simeq im\,T_m\mu$. Recall that the tangent map $T_m\mu: T_mM \longrightarrow \mathcal{G}^*$, $v \longmapsto T_m\mu(v) = \alpha$, for $v \in T_mM$ and $\alpha \in \mathcal{G}^*$ since $T_{\mu(m)}\mathcal{G}^* \simeq \mathcal{G}^*$ from Linear Algebbra. From Lemma 6.4.8 the map $\mathcal{A}_m : \mathcal{G} \longrightarrow T_mM$, $\xi \longmapsto \xi_M(m)$ is one-to-one, since the action is free. That is, $\mathcal{G}.m \subset T_mM$ and the following holds: $\mathcal{G} \simeq im\,\mathcal{A}_m = \mathcal{A}_m(\mathcal{G}) = T_m(G.m) = \mathcal{G}.m$. Since $\mathcal{A}_m$ is a linear map, it follows that $Ker\mathcal{A}_m = \{\xi \in \mathcal{G} \mid \xi_M(m) = 0\} = \mathcal{G}_m \subset \mathcal{G}$. By the same arguments as above we can write:

$$im\,\mathcal{A}_m{}^{\omega_m} = (T_mM/im\,\mathcal{A}_m)^* \simeq (Ker\mathcal{A}_m)^* \simeq Ker\mathcal{A}_m = \mathcal{G}_m.$$

We claim that $im\,\mathcal{A}_m{}^{\omega_m} = KerT_m\mu \subset T_mM$. For,

$$\begin{array}{rcl} KerT_m\mu & = & \{v \in T_mM \mid T_m\mu(v) = \alpha = 0,\ \alpha = \iota_{\xi_M(m)}\omega_m,\ \text{for all } \xi \in \mathcal{G}\} \\ & = & \{v \in T_mM \mid T_m\mu(v)(\xi) = \alpha(\xi) = 0, \alpha = \iota_{\xi_M(m)}\omega_m,\ \text{for all } \xi \in \mathcal{G}\} \\ & = & \{v \in T_mM \mid \ < T_m\mu(v), \xi >= d_m\mu_\xi(v) = 0,\ \text{for all } \xi \in \mathcal{G}\} \\ & = & \{v \in T_mM \mid \omega_m(\xi_M(m), v) = 0,\ \text{for all } \xi \in \mathcal{G}\} \\ & = & \{v \in T_mM \mid v \ \perp\ \xi_M(m),\ \text{for all } \xi \in \mathcal{G}\} \\ & = & \{v \in T_mM \mid v \ \perp\ im\,\mathcal{A}_m)\} \\ & = & \{v \in T_mM \mid v \ \in im\,\mathcal{A}_m{}^{\omega_m}\} \\ & = & im\,\mathcal{A}_m{}^{\omega_m}. \end{array}$$

From Definition 5.2.5 and the duality between the kernel and the range of a linear map, we have:

$$KerT_m\mu = im\,\mathcal{A}_m{}^{\omega_m} \simeq Ker\mathcal{A}_m = \mathcal{G}_m \text{ and } im\,\mathcal{A}_m = KerT_m\mu^{\omega_m} \simeq im\,T_m\mu.$$

It follows that $KerT_m\mu^{\omega_m} \simeq Ker\mathcal{A}_m{}^{\omega_m} = \mathcal{G}_m^o$. Therefore, $im\,T_m\mu = \mathcal{G}_m^o$. □

Lemma 6.5.4
Let $\mathcal{G}$ be the Lie algebra of a Lie group acting on a pseudomanifold M by infinitesimal generators. Then The range $im\,\mathcal{A}_m = im\,d_e\sigma_m$ is spanned by $\{\xi_M(m) \mid \xi \in \mathcal{G}\}$. Furthermore, if the G-action is free then $\mathcal{G}.m = T_m(G.m) = \{\xi_M(m) \mid \xi \in \mathcal{G}\} \simeq \mathcal{G}$.

Proof.
We know from linear Algebra that $im\,\mathcal{A}_m = span\{\xi_M(m) \mid \xi \in \mathcal{G}\}$. So, the first statement is a straightforward consequence of the linearity of $\mathcal{A}_m$. The second one follows from the characterization of a free group action stated in Remark 6.4.6. □

Corollary 6.5.1
We have with respect to the assumptions of Lemma 6.5.3 *and Lemma* 6.5.4 *both*

$$\mathcal{G}.m = T_m(G.m) = KerT_m\mu^{\omega_m} = T_m\mu^{-1}(\theta)^{\omega_m}$$

and

$$\mathcal{G}.m^{\omega_m} = T_m(G.m)^{\omega_m} = KerT_m\mu = T_m\mu^{-1}(\theta).$$

Remark 6.5.1

1. *The set $\mathcal{G}.m = T_m(G.m)$ is the tangent space at m to the orbit $G.m$. While $\mathcal{G}.m^{\omega_m}$ is the symplectic orthogonal complement space to $\mathcal{G}.m$ in the symplectic linear space (T_mM, ω_m). The relation $KerT_m\mu = \mathcal{G}.m^{\omega_m} = T_m(G.m)^{\omega_m}$ is called the bifurcation Lemma since it establishes a link between the symmetry of a point and the rank of the moment map at that point.* [64].

2. *$Ker(T_m\mu \circ \mathcal{A}_m) = \{\xi \in \mathcal{G} \mid (T_m\mu \circ \mathcal{A}_m)(\xi) = 0\} = \{\xi \in \mathcal{G} \mid T_m\mu(\mathcal{A}_m(\xi)) = 0\} = \{\xi \in \mathcal{G} \mid \mathcal{A}_m(\xi) \in KerT_m\mu\}$ since $T_m\mu$ and $\mathcal{A}_m$ are linear maps. This implies that $v \in im\,\mathcal{A}_m$ if, and only if $\mathcal{A}_m(\xi) = v$ for some $\xi \in \mathcal{G}$ if, and only if $\xi_M(m) = v$ for some $\xi \in \mathcal{G}$ if, and only if $< T_m\mu v, \xi >= d_m\mu_\xi(v) = \omega_m(\xi_M(m), v) = 0$ for some $\xi \in \mathcal{G}$ if, and only if $T_m\mu v\xi = 0$ for some $\xi \in \mathcal{G}$.*

3. *$im\,T_m\mu = T_m\mu(T_mM) = \{T_m\mu(v) = \alpha \in \mathcal{G}^* \mid v \in T_mM\}$. Equivalently, $< T_m\mu v, \xi >= \iota_v\omega_m(\xi_M(m)) = \alpha(\xi)$, for all $\xi \in \mathcal{G}$*

4. *$\mathcal{G}_m = T_m\mathcal{G}_m$ and $\mathcal{G}_m^0 = T_m\mathcal{G}_m{}^{\perp}$*

5. *$Ker\iota_v\omega_m = Ker\omega_m = KerT_m\mu = T_m\mu^{-1}(\theta)$*

Lemma 6.5.5 [*15, 53*]
Let G be an $\mathbb{F}$-Lie group acting on a symplectic pseudomanifold (M,ω) by a Hamiltonian action σ. Let $\theta \in \mathcal{G}^*$ be a regular value of $\mu : M \longrightarrow \mathcal{G}^*$, the moment map of the action. For all $m \in \mu^{-1}(\theta)$, we have:

1. $T_m(G_\theta.m) = T_m(G.m) \ \cap \ T_m\mu^{-1}(\theta)$.
2. $T_m(G_\theta.m) = KerT_m\mu^{\omega_m} \ \cap \ KerT_m\mu$.
3. $\mathcal{G}_\theta.m = \mathcal{G}.m \ \cap \ \mathcal{G}.m^{\omega_m}$.

Proof.

1. Let $m \in \mu^{-1}(\theta)$, and the G-action free. Thus, $G \simeq G.m$ from Lemma 6.4.8. It follows that $G_\theta \simeq G_\theta.m$ and $\mathcal{G}_\theta = T_eG_\theta \simeq \mathcal{G}_\theta.m = T_m(G_\theta.m)$. The properness and freeness of the G-action induce a free and proper action of G_θ on $\mu^{-1}(\theta)$, as from Lemma 6.5.1, (5). The co-adjoint-equivariance of the moment map $\mu : M \longrightarrow \mathcal{G}^*$,implies $\mu \circ \sigma_g = Ad_g^*\mu$ for all $m \in \mu^{-1}(\theta)$. Thus, from Lemma 6.5.1, (3) $\mu(g.m) = Ad_g^*\theta = \theta$ for all $m \in \mu^{-1}(\theta)$, for all $g \in G_\theta$. In the sequel we have, $m \in \mu^{-1}(\theta) \cap G.m = G_\theta.m$. It follows that $T_m(\mu^{-1}(\theta) \cap G.m) = T_m(G_\theta.m) \subset T_m\mu^{-1}(\theta) \cap T_m(G.m)$.Now, for the reverse inclusion we let $v \in T_m\mu^{-1}(\theta) \cap T_m(G.m)$. That is, $v \in T_m\mu^{-1}(\theta)$ and $v \in T_m(G.m)$. Thus, $v = \xi_M(m)$ and $v \in KerT_m\mu = T_m\mu^{-1}(\theta)$, for some $\xi \in \mathcal{G}$, since the moment map μ is G-equivariant and θ is its regular value. So,

$$0 = T_m(v) = T_m(\xi_M(m)) = \xi^*_{\mathcal{G}}(\mu(m)) = \big((Ad_g^*)_*(\xi)\big)(\mu(m)) = \big((Ad_g^*)_*(\xi)\big)(\theta)$$

, in light of Lemma 6.5.2. The definition of $\mathcal{G}_\theta$ gives $\xi \in \mathcal{G}_\theta$. Thus, we are done with the reverse inclusion since $v = \mathcal{A}_\xi(m) \in T_m(G_\theta.m) = \mathcal{G}_\theta.m$. We conclude that $T_m(G_\theta.m) = T_m\mu^{-1}(\theta) \cap T_m(G.m)$. [15, 24, 53]

2. Since $T_m(G.m) = KerT_m\mu^{\omega_m}$ and $T_m\mu^{-1}(\theta) = KerT_m\mu$ the conclusion in (1) above becomes $T_m(G_\theta.m) = KerT_m\mu \cap KerT_m\mu^{\omega_m}$.

3. Naturally, $\mathcal{G}_\theta.m = \mathcal{G}.m \cap \mathcal{G}.m^{\omega_m}$, as a consequence of the equalities below: $\mathcal{G}_\theta.m = T_m(G_\theta.m)$, $\mathcal{G}.m = T_m(G.m)$ and $\mathcal{G}_\theta.m^{\omega_m} = T_m(G_\theta.m)^{\omega_m} = KerT_m\mu$. □

Lemma 6.5.6 [*24, 47*], [*15, p.*123]
Let (M,ω) be a symplectic pseudomanifold and $\mu : M \longrightarrow \mathcal{G}^*$, the moment map of a Hamiltonian G-action on M, with $\dim G = n$ and $\dim M = q$. Let $\theta \in \mathcal{G}^*$ be a regular value of μ such that $\mu^{-1}(\theta)$ is nonempty. If $\iota_\theta : \mu^{-1}(\theta) \hookrightarrow M$ is the canonical inclusion, then the induced 2-form $\omega_{|\mu^{-1}(\theta)} := (\iota_\theta^*\omega)$ has a constant rank.

Proof.
The 2-form $\omega_\theta := \iota_\theta^*\omega = \omega_{|\mu^{-1}(\theta)}$ was constructed in Theorem 6.5.1, (2). Now, we

have $Ker_m\omega_{|\mu^{-1}(\theta)} = T_m\mu^{-1}(\theta) \cap T_m\mu^{-1}(\theta)^{\omega_m} = T_m\mu^{-1}(\theta) \cap T_m(G.m) = T_m(G_\theta.m)$ with respect to Lemma 5.3.1, where $N := \mu^{-1}(\theta)$, $x := m \in \mu^{-1}(\theta)$. As in [47, chapterIII. Remark 2.3], we can state: the rank of $\omega_{|\mu^{-1}(\theta)}$ at the point m is an even integer $k = 2p(m)$ equals to the co-dimension of $Ker_m\omega_{|\mu^{-1}(\theta)}$ such that it satisfies the inequalities $sup(0, 2(n-q)) \leq 2p(m) \leq n$. Recall that $dim\, Ker_m\omega_{|\mu^{-1}(\theta)}$ is both non negative and bounded by the dimensions of $T_m\mu^{-1}(\theta)$, where $T_m\mu^{-1}(\theta) = kerT_m\mu$ and $im\, T_m\mu = T_m(G.m) = T_m\mu^{-1}(\theta)^{\omega_m} = KerT_m\mu^{\omega_m}$. It follows that $Ker_m\omega_{|\mu^{-1}(\theta)}$ is of (maximal) constant dimension, since $G.m$, $T_m(G.m)$ and $KerT_m\mu$ are of constant dimensions. Hence, $\omega_{|\mu^{-1}(\theta)}$ has constant rank on $\mu^{-1}(\theta)$. □

Corollary 6.5.2
Under the assumptions of Lemma 6.5.6 and let $m \in \mu^{-1}(\theta)$; we have: $T_m\mu^{-1}(\theta)$ and $T_m(G.m)$ are orthogonal complement in the symplectic linear space (T_mM, ω_m). $T_m(G.m)$ is an isotropic linear subspace of the symplectic linear space (T_mM, ω_m). That is, $T_m(G.m) \subset T_m(G.m)^\omega = Kerd\mu_m = T_m\mu^{-1}(\theta)$. $Ker_m\omega_{|\mu^{-1}(\theta)} = T_m(G_\theta.m)$ is an isotropic linear subspace of $T_m\mu^{-1}(\theta)$.

Proof.

1. Since they are symplectic orthogonal to each other, then the conclusion follows.

2. From Definition 5.2.6 of isotropic linear subspace, it is enough to show that $\omega_{|T_m(G\ .\ m)} = 0$. For, let $m \in \mu^{-1}(\theta)$ and $\xi,\ \eta \in \mathcal{G}$ be any left invariant vector fields. It follows that $T_m\mu^{-1}(\theta) = kerT_m\mu \simeq im\,\mathcal{A}_m{}^{\omega_m}$ and $T_m(G\ .\ m) = im\,\mathcal{A}_m \simeq im\,T_m\mu$. from the proof of Lemma 6.5.3. Therefore, $T_m\mu(\xi_M(m)) = T_m\mu(\eta_M(m)) = 0$ and $\xi_M(m) \perp \eta_M(m)$. Equivalently, $\omega_m(\xi_M(m), \eta_M(m)) = 0$. Hence, $\omega_{|T_m(G\ .\ m)} = 0$.

3. We have $Ker_m\omega_{|\mu^{-1}(\theta)} = T_m(G_\theta.m)$, and $\omega_{|\mu^{-1}(\theta)_m} := (\iota_\theta^*\omega)_m$, with respect to Lemma 6.5.6 . It follows that $((\iota_\theta^*\omega)_m)^{-1}(0) = T_m(G_\theta.m)$. It follows that $(\iota_\theta^*\omega)_m(T_m(G_\theta.m)) = \{0\}$, that is, $(\iota_\theta^*\omega)_m(u,v) = 0$, for all $u, v \in T_m(G_\theta.m)$. But $T_m(G_\theta.m) = T_m\mu^{-1}(\theta) \cap T_m\mu^{-1}(\theta)^{\omega_m}$, hence it is an isotropic linear subspace of $T_m\mu^{-1}(\theta)$. □

Lemma 6.5.7
Let G be an $\mathbb{F}$-Lie group and $\mu : M \longrightarrow \mathcal{G}^*$, the moment map associated to a Hamiltonian G-action on a symplectic pseudomanifold (M, ω). Then, there exists an induced $\mathbb{F}$-smooth map $\overline{\mu} : M/G \longrightarrow \mathcal{G}^*/G$.

Proof. $[15, pp.121 - 123]$
The existence of $\overline{\mu}$ is a consequence of the G-equivariance of the moment map μ. For, we have $\pi_{\mathcal{G}^*} \circ Ad_g^* \circ \mu = \pi_{\mathcal{G}^*} \circ \mu \circ \sigma_g$. We set $\overline{\mu}([m]) := \pi_{\mathcal{G}^*}([\mu(m)])$ by definition. Thus, $\overline{\mu} \circ \pi_M = \pi_{\mathcal{G}^*} \circ \mu$. Let $n \in [m]$, that is $n = \sigma_g(m)$. It follows that $(\overline{\mu}[n] = (\overline{\mu} \circ \pi_M)(n) = (\overline{\mu} \circ \pi_M \circ \sigma_g)(m) = (\overline{\mu} \circ \pi_M(m) = \overline{\mu}[m]$. Therefore, $\overline{\mu}$ is well-defined and smooth map. □

Lemma 6.5.8 [*24*, p.15, Section 2.2]
If the Hamiltonian G-action on a symplectic pseudomanifold (M,ω) is free and proper, then, every $\theta\in\mathcal{G}^*$ is a regular value of the moment map $\mu: M\longrightarrow\mathcal{G}^*$ associated to the G-action.

Remark 6.5.2

$T_{[m]}\big(\mu^{-1}(\theta)/G_\theta.m\big) = T_m\mu^{-1}(\theta)/T_mG_\theta.m$

Theorem 6.5.2 [*33*, *53*, *24*, *64*]
Let $\mu: M\longrightarrow\mathcal{G}^*$ the moment map associated to a Hamiltonian, free and proper G-action on a symplectic pseudomanifold (M,ω). Let θ be a regular value of μ. Let $\pi: M\longrightarrow M/G$ be the canonical projection and $\pi_\theta=\pi|_{\mu^{-1}(\theta)}$ the restriction of the canonical projection to $\mu^{-1}(\theta)$. Let $\iota_\theta=\iota|_{\mu^{-1}(\theta)}:\mu^{-1}(\theta)\longrightarrow M$ be the canonical inclusion of $\mu^{-1}(\theta)$ to M and $\omega_\theta=\omega|_{\mu^{-1}(\theta)}$ the restriction of ω to $\mu^{-1}(\theta)$.

1. *The reduced space $M_\theta=\pi(\mu^{-1}(\theta))=\mu^{-1}(\theta)/G_\theta$ is a symplectic subpseudomanifold of $\overline{M}=M/G$ with the symplectic form $\overline{\omega_\theta}$ defined by $\pi_\theta^*\overline{\omega_\theta}=\iota_\theta^*\omega$.*

2. *Let $h\in C^\infty(M)^G$ be a G-invariant Hamiltonian. Then, the flow φ_t of the Hamiltonian vector field X_h leaves the connected component of $\mu^{-1}(\theta)$ invariant and commutes with the G-action, so it induces a flow φ_t^θ on M_θ defined by $\pi_\theta\circ\varphi_t\circ\iota_\theta=\varphi_t^\theta\circ\pi_\theta$.*

3. *The vector field generated by the flow φ_t^θ on (M_θ,ω_θ) is Hamiltonian with its associated Hamiltonian function $h_\theta\in C^\infty(M_\theta)$ defined by $h_\theta\circ\pi_\theta=h\circ\iota_\theta$. Moreover, the vector field X_h and X_{h_θ} are π_θ-related.*

4. *Let $k\in C^\infty(M)^G$ be another G-invariant function. Then $\{h,k\}$ is also G -invariant and $\{h,k\}_\theta=\{h_\theta,k_\theta\}_{M_\theta}$, where $\{.,.\}_{M_\theta}$ denotes the Poisson bracket associated to the symplectic form ω_θ on M_θ.*

Proof.
Theorem 6.5.1, Lemma 6.5.5 together with Propositions 5.2.2, 5.2.3 and 5.2.4 allow the construction of $\overline{\omega_\theta}$ defined by $\pi_\theta^*\overline{\omega_\theta}=\iota_\theta^*\omega$. It is well-defined, non-degenerate. Also, it can be shown to be a closed 2-form. For, let $Ker\pi^*=\{\alpha\in\Omega(M_\theta)\mid\pi^*(\alpha)=0_{\Omega(\mu^{-1}(\theta))}\}$, where $\pi^*:\Omega(M_\theta)\to\Omega(\mu^{-1}(\theta))$ and $\alpha\circ\pi_*=0_{\Omega(\mu^{-1}(\theta))}$. Since the tangent map of π at $m\in\mu^{-1}(\theta)$ is surjective, one has $\alpha\circ\pi_*\circ\pi_*^{-1}=0_{\Omega(\mu^{-1}(\theta)}$ at each $m\in\mu^{-1}(\theta)$. It follows that $\alpha=0$. Hence, the pullback of π is a one-to-one map. Therefore, $\pi^*d\overline{\omega_\theta}=d\pi^*\overline{\omega_\theta}=d\iota_\theta^*\omega=\iota_\theta^*d\omega=0$. So, $d\overline{\omega_\theta}=0$. That is, $\overline{\omega_\theta}$ is closed. Obviously, the remaining statements are consequent of the first one. □

The introduction of basic concepts of geometric control theory is based on Sussmann's claim concerning a general differential structure susceptible to host a control theory model formulation depending on the existence of three criteria: smooth exterior algebra, smooth differentiation theory and smooth transversality theory as in [80, 81, 82, 83, 84].

Chapter 7

Introduction to geometric optimal control on pseudomanifolds

7.1 A historical viewpoint of control theory.

[3, 80, 84]
This section aims to show, by a review of the literature [3, 80, 84], that the evolution of the concept of control theory through several centuries can be subdivided in five stages as follows.
The first period starts back at Aristotle's times, as witnessed in [3]. So Aristotle (384-322, BC), in his book *"Politics"*, had written: *... if every instrument could accomplish its own work, obeying or anticipating the will of others ..., if the shuttle weaved and the pick touched the lyre without a hand to guide them, Chief workman would not need servants, nor masters slaves.* That is, simply Aristotle said, whether he was in our times, the purpose of control theory is to automatize processes in such a way to get workmen left free while processes execute the job for which they were built. This period is characterized in [3] as the primacy of existence, or the primacy of the laws.

The second period concerns the Calculus of Variations of variation as the summum of a metamorphosis process from the first attempt of mathematization of science by Fibonacci (1170-1250) in his *Liber abaci*. And so, three centuries after, came the launch of the transformation of the foundations of science under the influence of works due to some renowned scientists. The cornerstones and their discoveries are listed below. The mathematical models involving differential equations as a synthesis of the *Method* by Réné Descartes (1596-1650), and the *Calculus* by Sir Isaac Newton (1642-1727) and Baron Gottfried Leibniz (1646-1716). The physical experiments by Galileo Galilei (1564-1642). That is a mutation to the primacy of mathematics in science. This concept is still in use today. We want to borrow some historical facts from [80] and [84] as related(stated) inside to describe the second period.

- Queen Dido needed to build the prestigious Carthage along the sea. She determined the land by enclosing it between a thread long of more than two-and-a-half miles and the sea. By doing so, she was facing the so-called *isopermetric (isoperimetrical) problem.* Mathematically, the thread can be thought of as a curve (trajectory). She had faced a fixed-endpoint minimum-time optimal control problem in modern statement. Because *"given the area, to minimize the length of the boundary of the region"* is equivalent to *"given the length of the boundary, to maximize the area of the region".*

- The reflection of the light obeys to the shortest path principle. While the rays of propagation of the light in the medium are the minimum-time paths. The first principle assumes the motion along shortest paths. The second one considers the travel time along the curve. Thus, This is a time minimizing problem by Fermat (1601-1665).

- The solid of revolution of least resistance, presented by Newton in 1686, aimed to find a function y:=y(u) minimizing the integral:

$$I = \int_a^b \frac{x}{1+y'(x)} dx \tag{7.1}$$

where $a,\ b,\ y(a) = y_a$ and $y(b) = y_b$ are given such that $a < b$. That is, $y := y(x)$ is a curve.

- The brachystochrone problem, presented in 1696 by Johann Bernoulli (1667-1748), concerns a family of curves linking a point A to a point B in a vertical plane. Assume a particle is failing along that curves under the resultant action of the gravitational force and a virtual force, keeping the particle on the path. Then one needs to determine the curve where the particle will reach B from A in a minimum time. This problem belongs to Optimal control theory. Within 1696-1697, six answers were given to the problem by Leibniz, Newton, l'Hôpital, Tschirnhaus, Jako Bernoulli, and John Bernoulli. The solution was a curve named cycloid.

- The Calculus of Variations deals with methods of minimizing problem of the form :

$$I = \int_a^b L(x(t), \dot{x}(t))dt \ constrained\ to\ x(a) = q_a\ and\ x(b) = q_b \tag{7.2}$$

with L called the Lagrangian. That is, the calculus of variations.

  This problem has marked the launch of this topic, named so by Euler (1707-1783) in 1755. But the method was proposed by Lagrange (1736-1813). This method gives rise to four important concepts in mathematics.

1. The Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(x(t), \dot{x}(t)) = \frac{\partial L}{\partial q}(x(t), \dot{x}(t)) \tag{7.3}$$

such that $x : [a, b] \longrightarrow \mathcal{R}^n, t \mapsto x(t),$ *is the necessary condition* for *a curve* x *to be a solution* of *variations problem.*

   where x:$[a, b] \to \mathbb{R}^n$, $t \mapsto x(t)$. That is the necessary condition for a curve x to be a solution of a Calculus of Variations problem.

2. The invariance of the above equation under arbitrary nonlinear changes of coordinates as established by Lagrange.
3. The geometrization of physics and the birth of differential geometry.
4. The transformation of the second-order Euler-Lagrange equation into first-order Hamiltonian equations. In 1833-1834 and 1835, W.R. Hamilton (1805-1865) established the way for doing it and proved that this transformation conserves the invariance property under nonlinear changes of coordinates.

- Differential geometry growths from the work of B. Riemann(1826-1866), whose ideas stands for the foundation of this discipline. he called the line segment a geodesic. The length of an arbitrary curve is measured by the use of the arc-length element. the former is defined by a tensor called the Riemann metric. The tensor, the vector and the covector are the so-called covariant quantities whose nonlinear changes obey to a specific transformation law. That is the birth of Riemann geometry. The concept of differentiable manifold as a phase space in mechanic was introduced by Poincaré. This even dimensional manifold is called a symplectic manifold since it is endowed with a symplectic structure together with a Hamiltonian vector field. This vector field is induces a system of first-order differential equations that preserves the symplectic structure. The growth of the discipline gives rise to others type of new geometries such as pseudo-Riemannian geometry, Minkowski geometry, and Finsler geometry named so after Paul Finsler (1894-1970). And the consequence of this intensive scientific activity is the reciprocal influence which took place between calculus of variations and differential geometry.

The third period, covering the 1950's and 1960's, is characterized by the fact that the control theory has acquired its independence from the Calculus of Variations, and so, it is today a separate branch of mathematics. It covers the topics such as *Controllability, Reachability, Optimality, Observability and Stabilizability* of linear and nonlinear systems. The important tools of its study are the *maximum principle* given in 1962 by Lev Pontriagin (1908-1988) and the *equation* of Richard Bellman (1920-1984) when one needs to solve a minimizing problem. Both of them are necessary conditions for optimality and the later is the analytic relation derived from the Dynamic Programming and tends sometimes to sufficient conditions for optimality. Together they have established the foundations of the modern control theory which contains the Calculus of Variations as a special case.

The fourth period comes from the use of computer in physical experiments. Here we want to analyze the influence of Computer Science and Informatics in the progress of control theory. Since a computer is a digital device the numerical experiments of mathematical models yield the discretization of continuous systems. And then comes into the scene the birth of Computer (digital) Control, where one has to convert position and velocity to digital form or sampled data for calculating the necessary controls. In turn, the calculations are computed by a program. The transformation of mathematical models in algorithms, the implementation of algorithms in computing programs constitutes the definition (purpose) of Informatics. There is a mutual enrichment between control theory and others Sciences, say, Biology, Economy, ..., and also the rest of Mathematics.

The fifth period, which took place after 1960, has been devoted to geometric control theory initiated by an idea of Roger Brockett concerning the use in control theory of concepts and tools from differential geometry [13]. The main result in optimal control is for considering a control system as a family of vector fields. And then the geometric approach consists in carrying the algebra structure on the set of vector fields by the means of Lie bracket. With this geometric approach, by means of either Poisson brackets or connections along curves, the finite dimensional Maximum Principle has been reformulated in terms of Reachability of separable sets and also as the invariance under arbitrary changes of coordinates. For this purpose, when dealing with nonlinear systems, one needs vector fields, differential forms, Lie bracket, exterior multiplication, exterior differentiation and geodesics. The relevant consequence should be of expressing the coordinates invariant properties, the maximum principle and the time-optimal trajectories in terms of Lie brackets.

Actually, after this travel towards the past, as claimed in [67], while the mid-seventies the geometrical control theory has emphasized the crucial role of Lie group theory, differential geometry and global analysis. Hence, the engineering problem solving theories have to borrow tools from the mixing methods of geometry, topology and global analysis. For completing this review, one can says that the optimal control theory is an interdisciplinary branch of research where are synthesized the Geometric Theory of Differential Equation Systems, the Maximum Principle and the Dynamic Programming Equation, both of them viewed as necessary conditions of optimality enriched by Symplectic Geometry.
At this stage , what we can know is that a control system is a system under control or a controlled system which achieves a given task. It can be anything around us such as an automobile, a underwater vehicle, an airplane autopilots, Watt's stream engine governor, a CD-player, a microprocessor controller, an alarm system, a robot, a biological system, or a financial system. A planet, the solar

system, the weather system are among them. Furthermore, to control a system means to influence or observe its behavior while it is achieving the assigned task. In this work we will focus on the geometric optimal control theory as especially on hybrid systems. Our ultimate aim is to establish a link between hybrid systems and the geometrical setting of a special class of Frölicher spaces, named pseudomanifolds. The hybrid systems are evolving on them instead of differentiable manifolds. The branch of hybrid systems is brought into scene recently in the decade 1991-2000 as stated in [42]. At its origin the hybrid system was an interface between control theory, electrical engineering and computer science. But now, it is extending its influence to all others domains of sciences where optimal control problems are studied. It becomes an interdisciplinary science both in theoretical approach and application-oriented studies.
In general, a hybrid system is a mixture of continuous and discrete systems, both considered as embed special systems. This mixture involves both computer science topics and ones of control theory. The topics in the first case are formal methods, digital control and discrete methods (for example: difference equation). While the second case consists of optimal control, continuous dynamical system, geometric properties of differential equations. The transformation of data from a continuous system to a discrete one , and vice-versa, is one of the important things sustaining any approach for solving a hybrid system problem. It will be worth noticing that the hybrid system concept comes to fill the lack in standard methods known before where the continuous system models and the discrete system models tended to ignore each other. However, the exclusive use of the aforementioned models is not appropriate to handle optimal control laws in some systems where the inputs, outputs or dynamics are continuous and discrete.
Several authors in the literature on the subject and specially in [11] show that we are always facing hybrid systems everywhere around us. The given real-world examples below are for convincing us about the natural existence of hybrid systems:

- on the ground: vehicle, automated highway robot systems, vehicle power trains, others motion controllers;
- under water: underwater vehicle;
- in the sky: flight control and management systems, satellite;
- in the office: computer drive disk, Internet network management, data transmission protocol;
- in the house: control thermostat to compare actual to desired room temperature;
- in factories: enterprise control system, stepper motors constrained robotic systems, power distribution;

- in biology: nervous system, DNA, ..., are the biological systems with more networks than single dynamical systems to be modeled monolithically as stated in [58].

7.2 Basic concepts of control theory.

[72, 40, 46, 2]

Definition 7.2.1 [*72*]
Let M be an n-pseudomanifold and $\mathcal{U} \subset \mathbb{R}^m$ with $m > 0$. Let $f : M \times \mathcal{U} \to TM$ be a smooth map of pseudomanifolds where $x \in M$, $u \in \mathcal{U}$, and TM the tangent bundle of M. A control system is an ordinary differential equation (ODE in short) of the form:

$$\dot{x} := \frac{dx}{dt} = f(x, u). \tag{7.4}$$

- *M is called the state space and x the state of the system;*
- *$\mathcal{U}_x = \{u \in \mathcal{U} \mid u$ is related to a fixed x by $\dot{x} = f(x, u)\}$ is called the state dependent input set and $u \in \mathcal{U}_x$ is called a control (input);*
- *$\mathcal{U} = \bigcup_{x \in M} \mathcal{U}_x$ is called the control bundle;*
- *f is called a system map.*

Definition 7.2.2 [*72*]
Let $T \in [0, +\infty)$ and $t \in [0, T]$. Consider $u : [0, T] \to \mathcal{U}$ and $x : [0, T] \to M$ be some (piecewise) continuous or smooth functions. The functions x are called curves in the state space. The functions u are called admissible (measurable) control functions and their set is denoted by $\mathbb{U} := \{u : [0, T] \to \mathcal{U} \mid u$ is admissible control$\}$.

Remark 7.2.1
If M is a linear n-pseudomanifold, suppose $M = \mathbb{R}^n$, f a linear function in x and u. That is, f is bilinear form. Then the control system in 7.4 becomes

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i = Ax + Bu, \tag{7.5}$$

where A is a real $n \times n$ matrix, and B a real $n \times m$ matrix with $b_1, \ldots, b_m \in \mathbb{R}^n$ forming its columns. Thus the equation (7.5) is called linear control system. But, if M is a nonlinear n-pseudomanifold then we need a linearization of M using T_xM, the tangent space to M at x, and TM, the tangent bundle of M.

Definition 7.2.3 [*72, 46*]
A nonlinear (affine) control system is a differential equation of the following type

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i X_i(x), \tag{7.6}$$

where $T \in [0, +\infty)$*,* $x : [0, T] \to M$ *is a curve into the state space, and* $X_1, \ldots, X_m$ *are smooth vector fields on* M*.* X_0 *is called the drift vector field. It describes the dynamics of the control-free system.* $X_1, \ldots, X_m$ *are called the control vector fields (input vector fields). That is to say, the controlled vector fields. A nonlinear system where* $X_0 = 0$ *is called drift-free.*

Definition 7.2.4 [*46*]
Let M *be an n-pseudomanifold,* $x : [0, T] \to M$ *a curve into the state space* M*, where* $T \in [0, +\infty)$*, and* $X_1, \ldots, X_m$ *are smooth vector fields on* M*. The curve* x *is called a control trajectory if there exists* $u \in \mathbb{U}$ *such that for all* $t \in [0, T]$*,* $u(t) \in \mathcal{U}_x$*,*

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)). \tag{7.7}$$

Remark 7.2.2 [*46*]
The existence and the uniqueness of solution for ODE with initial condition yield a unique $x : [0, T] \to M$ *for* $u \in \mathbb{U}$ *and* $x(0) = p$*, where* $p \in M$*. At this point, we are not assuming* $X_1, \ldots, X_m$ *linearly independent. But,* $\mathcal{U}$ *is still some subset of* $\mathbb{R}^m$*. We will notice that the particular* $\mathcal{U}$*'s,* $\mathcal{U} = \mathbb{R}^m$*,* $\mathcal{U} = [-1, 1]^m$ *and* $\mathcal{U} = \{u \in \mathbb{R}^m \mid ||u|| = 1\}$*, play an important role in control theory. In this section and in all others that follow* M *is an n-pseudomanifold even no explicitly stated. Also all maps are smooth maps of pseudomanifolds unless otherwise stated. We want to emphasize the fact that any nonlinear control system is characterized by three object as given in the definition below.*

Definition 7.2.5
A nonlinear control system on an n-pseudomanifold M *is a triple* $\sum := (M, \chi, \mathcal{U})$*, where ,* $\chi := \{X_0, X_1, \ldots, X_m\}$ *is a finite family of smooth vector fields on* M *and* $\mathcal{U}$ *is a control set. The pair* (x, u) *as defined in Definition* 7.2.4 *is called a control trajectory pair.*

Definition 7.2.6
Let M *be an n-pseudomanifold and* $p, q \in M$*. Let* (x, u) *be a control trajectory starting at* $x(0) = p$ *to* $x(T) = q$*. The point* q *is called reachable or accessible*

from p. The set of all points accessible from p is denoted by

$$\Re(p,T) = \{x(T) \mid (x,u) \text{ is a control trajectory and } x(0) = p\}. \tag{7.8}$$

And we can denote the following sets:

$$\Re(p,t) = \{x(t) \mid (x,u) \text{ is a control trajectory and } x(0) = p\}, \tag{7.9}$$

$$\Re(p,t,T) = \bigcup_{t\in[0,T]} \Re(p,t), \tag{7.10}$$

$$\Re(p) = \bigcup_{t\in[0,+\infty)} \Re(p,t). \tag{7.11}$$

Definition 7.2.7 [*72, 46*]
Let $\sum := (M,\chi,\mathcal{U})$ be a nonlinear control system and $p\in M$. Let $\tau_{\mathcal{F}_M}$ be the $\mathbb{F}$-topology on M. $\sum$ is locally accessible from (at) p if the interior of $\Re(p)$ with respect to $\tau_{\mathcal{F}_M}$ is nonempty. $\sum$ is locally accessible in M if it is locally accessible from (at) each p. $\sum$ is strongly accessible from (at) p if the interior of $\Re(p,T)$ with respect to $\tau_{\mathcal{F}_M}$ is nonempty for each $T>0$. $\sum$ is locally controllable from (at) p if x belongs to the interior of $\Re(p)$. $\sum$ is small-time locally controllable from (at) p if there exists $T>0$ so that x belongs to the interior of $\Re(p,t,T)$ for each $t\in[0,T]$, with respect to $\tau_{\mathcal{F}_M}$. $\sum$ is globally controllable from (at) p if $\Re(p)=M$ for some (and therefore all) p.

7.3 Accessibility or Reachability conditions.

The accessibility conditions constitute the first stage for defining the controllability conditions. The geometric approach in nonlinear control leads to the use of vector fields, differential forms, Lie bracket, Lie derivative, interior product, exterior product and exterior differentiation. That is, the differential geometry in Frölicher setting. Here accessibility will be studied by means of Lie bracket. For this purpose, we need to remember the more general differentiation theory as stated in Chapters 4 and 5. The integral curve and the exponential map, in Chapter 6 and in [73, 74] will play a crucial role. We recall that $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M, an n-pseudomanifold. That is, $\mathfrak{X}(M)=\{X:M\to TM \mid X(p)\in T_pM\}$. Let $\mathcal{V}\subset\mathfrak{X}(M)$ be an arbitrary family of smooth vector fields on M. Let $x:[0,T]\to M$ be an integral curve for a vector field $X\in\mathcal{V}$, that is, $\dot{x}(t)=X(x(t))$ for all $t\in[0,T]$. Let $exp_X:\mathbb{R}\times M\to M$ be the flow of $X\in\mathcal{V}$, under the assumption that all X are complete, such that

$$x(t) = exp_X(t,p) := e^{tX}(p), \tag{7.12}$$

where x is the integral curve for X satisfying $x(0)=p$.

Definition 7.3.1 [*46*]
Let $\sum:=(M,\chi,\mathcal{U})$ be a given nonlinear control system. The family

$$\mathcal{V}_\chi = \{X_0 + \sum_{i=1}^{m} u_iX_i \mid u_i \in \mathcal{U}\} \tag{7.13}$$

is called the associated family of vector fields to $\sum$*, where* χ *is given in Definition 7.2.5.*

Definition 7.3.2 [46]
Let $\mathfrak{Diff}(M)$ *be the group of diffeomorphisms on* M.

$$\mathfrak{Diff}(\mathcal{V}) := span\{e^{t_1X_1} \circ \ldots \circ e^{t_kX_k}(p) \mid t_1, \ldots, t_k \in \mathbb{R}; X_1, \ldots, X_k \in \mathcal{V}; \\ k \in \mathbb{N} \text{ and } p \in M\} \quad (7.14)$$

is a subgroup of the group $\mathfrak{Diff}(M)$.

$$\mathfrak{Diff}_0(\mathcal{V}) \subset \mathfrak{Diff}(\mathcal{V}) \quad (7.15)$$

$$\text{such that } \sum_{i=1}^{k} t_i = 0 \text{ is a normal subgroup of } \mathfrak{Diff}(\mathcal{V}).$$

$$\mathfrak{Diff}_T(\mathcal{V}) \subset \mathfrak{Diff}_0(\mathcal{V}), \quad (7.16)$$

$$\text{such that } \phi \circ e^{TX} \in \mathfrak{Diff}_T(\mathcal{V}), \text{ where } \phi \in \mathfrak{Diff}_0(\mathcal{V})$$

$$\text{and } \sum_{i=1}^{k} t_i = T, \text{ is the coset of } e^{TX} \in \mathfrak{Diff}_0(\mathcal{V}).$$

Definition 7.3.3 [46]
Let $\mathfrak{Diff}(M)$ *be the group of diffeomorphisms on* M.

$$\mathfrak{Diff}^+(\mathcal{V}) \subset \mathfrak{Diff}(\mathcal{V}), \quad (7.17)$$

$$\text{such that } t_1, \ldots, t_k \geq 0, \text{ is a semi} - \text{group of } \mathfrak{Diff}(\mathcal{V})$$

for positive times.

$$\mathfrak{Diff}_T^+(\mathcal{V}) \subset \mathfrak{Diff}^+(\mathcal{V}), \quad (7.18)$$

$$\text{such that } \sum_{i=1}^{k} t_i = T \geq 0, \text{ is a semi} - \text{group of } \mathfrak{Diff}^+(\mathcal{V}).$$

Definition 7.3.4 [46]
Let $\mathfrak{Diff}(M)$ *be the group of diffeomorphisms on* M *and* $p \in M$.

$$\mathcal{O}(p, \mathcal{V}) := \{\phi(p) \mid \phi \in \mathfrak{Diff}(\mathcal{V})\} \quad (7.19)$$

is the $\mathcal{V}$ *– orbit through* p*, that is, the orbit of the family* $\mathcal{V}$ *of vector fields on* M.

$$\mathcal{O}_T(p, \mathcal{V}) := \{\phi(p) \mid \phi \in \mathfrak{Diff}_T(\mathcal{V})\} \quad (7.20)$$

is the $(\mathcal{V}, T)$ *– orbit through* p.

$$\mathcal{O}^+(p, \mathcal{V}) := \{\phi(p) \mid \phi \in \mathfrak{Diff}^+(\mathcal{V})\} \text{ and} \quad (7.21)$$

$$\mathcal{O}_T^+(p, \mathcal{V}) := \{\phi(p) \mid \phi \in \mathfrak{Diff}_T^+(\mathcal{V})\} \quad (7.22)$$

are subsets of $\mathcal{O}(p, \mathcal{V})$ *restricted to positive times.*

Definitions 7.3.1 trough 7.3.4 will serve as characterizations to the concept of attainability of the family $\mathcal{V}_\chi$ of vectors fields associated to a nonlinear control system $\sum := (M, \chi, \mathcal{U})$. In what follows, $\mathcal{V}$ represents indistinctly $\mathcal{V}_\chi$ or an arbitrary family of vector fields on an n-pseudomanifold M. Furthermore, only otherwise stated, the topology considered is $\tau_{\mathcal{F}_M}$, the $\mathbb{F}$-topology induced by structure functions on M.

Definition 7.3.5 [*46*]
$\mathcal{V}$ is attainable from p if the interior of $\mathcal{O}^+(p, \mathcal{V})$ is nonempty. $\mathcal{V}$ is strongly attainable from p if the interior of $\mathcal{O}_T^+(p, \mathcal{V})$ is nonempty for each $T > 0$.

Definition 7.3.6

$$\mathcal{L}(\mathcal{V}) := span(\mathcal{V}) \tag{7.23}$$

is the smallest Lie subalgebra of $\mathfrak{X}(M)$ that contains $\mathcal{V}$.

$$\mathcal{L}(\mathcal{V})_p := \{X(p) \mid X \in \mathcal{L}(\mathcal{V})\} \tag{7.24}$$

is a subspace of T_pM.

Remark 7.3.1
$\mathcal{L}(\mathcal{V})_p$ describes the set of directions along which the system can evolve. Thus, the evolution of the whole given system $\sum := (M, \chi, \mathcal{U})$ is described by a collection of $\mathcal{L}(\mathcal{V})_p$ for all $p \in M$. Now, we will study the nature and the role of this collection $\mathcal{L}(\mathcal{V}) := \bigcup_{p \in M} \mathcal{L}(\mathcal{V})_p$ in the accessibility property of the given system $\sum$. Also, we will show how the Lie bracket on $\mathcal{L}(\mathcal{V})$ is more relevant for the study of the controllability of a given system under some assumptions.

Definition 7.3.7
Let $\sum := (M, \chi, \mathcal{U})$ be a nonlinear control system. $\mathcal{L}(\mathcal{V})$ is called a control distribution of the given system $\sum$. That is, an assignment of the linear subspace $\mathcal{L}(\mathcal{V})_p$ to each tangent space T_pM.

Definition 7.3.8
Let M be an n-pseudomanifold and N an immersed m-pseudomanifold with $m \leq n$. N is an integral pseudomanifold for $\mathcal{L}(\mathcal{V})$ if $T_pN \subset \mathcal{L}(\mathcal{V})_p$ for each $p \in N$. N is the maximal integral pseudomanifold for $\mathcal{L}(\mathcal{V})$ through p if it is an integral pseudomanifold containing $p \in M$ such that it contains any other integral pseudomanifold containing p.

Proposition 7.3.1 [*46*, Theorem 2.1]
Let M be an n-pseudomanifold and $p \in M$. Let $\mathcal{V}$ be a family of complete smooth vector fields on M. Then the following statements hold. $\mathcal{O}(p, \mathcal{V})$ is an immersed subpseudomanifold of M. $T_q(\mathcal{O}(p, \mathcal{V})) \supset \mathcal{L}(\mathcal{V})_q$ for each $q \in \mathcal{O}(p, \mathcal{V})$. The family $\{\mathcal{O}(p, \mathcal{V}) \mid p \in M\}$ forms a partition on M.

Remark 7.3.2
In [46] the proof for Proposition 7.3.1 *is based on linear algebraic arguments and does not need any change for our setting. This is also applicable to a list of results given below without proof. However, in each case the reference is provided for the proof. The equality* $T_q(\mathcal{O}(p,\mathcal{V})) = \mathcal{L}(\mathcal{V})_p$ *is valid in the case of analytic vector fields. This concept is beyond the scope of our study, where only the real setting is considered. It should be an appealing subject for further researches. By convention, we will write* $dim\,\mathcal{V} := max_{p\in M} dim\,\mathcal{O}(p,\mathcal{V})$. $dim\,\mathcal{O}(p,\mathcal{V}) \leq dim\,\mathcal{V}$ *in general. If* M *is connected, the set* $\{p\in M \mid dim\,\mathcal{O}(p,\mathcal{V}) = dim\,\mathcal{V}\}$ *is an open set under assumptions of Proposition* 7.3.1.

Proposition 7.3.2 [46, Theorem 2.4]
Let M *be an* n*-pseudomanifold and* $p \in M$. *Let* $\mathcal{V}$ *be a family of smooth vector fields on* M. *Then the following implication holds.* $\mathcal{L}(\mathcal{V})_p = T_pM \quad \Rightarrow \quad \mathcal{V}$ *is attainable from* $p \in M$.

The inverse implication is valid in the case of analytic vector fields. The Remark 7.3.2 is applicable here too. Now, we can look at the strong attainability (strongly attainable) condition. For this purpose we proceed as follows. First of all, we recall a characterization of what is called a derived algebra $\mathcal{G}'$ of a given algebra $\mathcal{G}$. Thus,

Definition 7.3.9 [46]
Let $\mathcal{G}$ *be a given Lie algebra and* $\mathcal{L}(\mathcal{V})$ *a control distribution of the given nonlinear control system* $\sum := (M, \chi, \mathcal{U})$. *The set* $\mathcal{G}'$ *is the Lie subalgebra (called the derived algebra) of* $\mathcal{G}$ *generated by vector fields of the form bellow:*

$$X_i, \quad [X_i, X_j], \quad [X_i, [X_j, X_k]\,], \quad \ldots, \quad . \tag{7.25}$$

$$\mathcal{V}_0 := \{\sum_{j=1}^{m} \lambda_j X_j \mid \sum_{j=1}^{m} \lambda_j = 0, \; X_j \in \mathcal{V}\}. \tag{7.26}$$

$$\mathcal{L}'(\mathcal{V}) \textit{ is the derived subalgebra of } \mathcal{L}(\mathcal{V}). \tag{7.27}$$

$$\Im(\mathcal{V}) := \{X = Y + Z \mid Y \in \mathcal{V}_0, Z \in \mathcal{L}'(\mathcal{V})\} \textit{ is an ideal of } \mathcal{L}(\mathcal{V}). \tag{7.28}$$

$$\Im(\mathcal{V})_p := \{X(x) \mid X \in \Im(\mathcal{V})\}. \tag{7.29}$$

Remark 7.3.3 [46]
If $\mathcal{V} = \{X_1, \ldots, X_m\}$ *is a finite set then each vector field of* $\mathcal{L}(\mathcal{V})$ *is a* $\mathbb{R}$*-linear combination of vector fields of the form in Equation* 7.25. *In [46], it is indicated that* $\Im(\mathcal{V})$ *is the contribution of [83], whereas in [61] it is pointed out the fact that* $\Im(\mathcal{V})$ *should possess a maximal pseudomanifold at any point* p, *and the tangent space of this integral pseudomanifold at any point* p *is* $\Im(\mathcal{V})_p$. *The later result is a generalization of the Frobenius Theorem given in* 7.4.1 *with the advantage that it holds even in the non constant dimension case.* $\Im(\mathcal{V})$ *is related to* $\mathcal{O}_T(p,\mathcal{V})$ *as* $\mathcal{L}(\mathcal{V})$ *to* $\mathcal{O}(p,\mathcal{V})$.

Definition 7.3.10
Let $\mathcal{D}$ be a distribution on M. $\mathcal{D}$ is said involutive if for every $X, Y \in \mathcal{D}$ the Lie-bracket $[X,Y] \in \mathcal{D}$ with $X(p), Y(p) \in \mathcal{D}_p$ for all $p \in M$. The distributions $\mathcal{L}(\mathcal{V})$ and $\Im(\mathcal{V})$ are involutive.

Definition 7.3.11
Let $\mathcal{D}$ be a distribution on M. $\mathcal{D}$ is said integrable if the maximal integral pseudomanifold N through $p \in M$ is such that $\mathcal{D}_q = T_q N$ for each $q \in N$. $\mathcal{D}$ is said Lie-bracket generating if the iterated lie-brackets in Equation 7.25 span the tangent space of M at every point.

Proposition 7.3.3 [*46*, Theorem 2.5]
Let M be an n-pseudomanifold and $p \in M$. Let $\mathcal{V}$ be a family of complete smooth vector fields on M. Then the following statements hold. $\mathcal{O}_T(p,\mathcal{V})$ is an immersed subpseudomanifold of M. $T_q(\mathcal{O}_T(p,\mathcal{V})) \supset \Im(\mathcal{V})_q$ for each $q \in \mathcal{O}_T(p,\mathcal{V})$. The family $\{\mathcal{O}_T(p,\mathcal{V}) \mid p \in M\}$ forms a partition on M.

The equality $T_q(\mathcal{O}(p,\mathcal{V})) = \Im(\mathcal{V})_q$ is valid in the case of analytic vector fields. The Remark 7.3.2 is applicable here too.

Proposition 7.3.4 [*46*, Theorem 2.6]
Let M be an n-pseudomanifold and $p \in M$. Let $\mathcal{V}$ be a family of smooth vector fields on M. Then the following implication holds. $\Im(\mathcal{V})_p = T_p M \quad \Rightarrow \quad \mathcal{V}$ is attainable from $p \in M$.

The inverse implication is valid in the case of analytic vector fields. The Remark 7.3.2 is applicable here too. We need to establish the link between attainability and accessibility. Namely, to find out how to relate $\chi := \{X_0, X_1, \ldots, X_m\}$ to $\mathcal{V}_\chi = \{X_0 + \sum_{i=1}^{m} u_i X_i \mid u_i \in \mathcal{U}\}$ for a given nonlinear control system $\sum := (M, \chi, \mathcal{U})$. And subsequently, which relationship links $\mathcal{L}(\mathcal{V}_\chi)$ to $\mathcal{L}(\chi)$.

Definition 7.3.12
Let V be a $\mathbb{R}$-linear space, $S \subset V$ a subset and $U \subset V$ a subspace.

- *S is convex in V if $(1-t)x + ty \in S$ for all $t \in [0,1]$, and for all $x, y \in S$.*
- *Let $x_i \in S$ with $i \in [1,k] \subset \mathbb{N}$. A convex combinations of vectors x_i is a linear combination $\sum_{i=1}^{k} \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. That is, S is convex if, and only if it contains all convex combinations of its elements.*
- *A convex hull of S, denoted by $conv(S)$, is the smallest convex set containing S. That is, $conv(S)$ is the union of all convex combinations of elements of S.*

- *An affine subspace of V, is a shifted subspace. That is, a subset of the form* $\{x+u \mid u\in U\}$.
- *An affine hull of S, denoted $aff(S)$, is the smallest affine subspace of V containing S. That is, $aff(S)$ is the set whose elements are of the form* $\sum_{i=1}^{k}\lambda_i x_i$, *where* $\lambda_i\in\mathbb{R}$, $\sum_{i=1}^{k}\lambda_i=1$, *and* $i\in[1,k]\subset\mathbb{N}$.

The following results are proved in [46] using only linear algebra properties and with respect to concepts in Definition 7.3.12. We are going to state them without proofs.

Lemma 7.3.1 [*46*, Theorem 2.7]
Let M be an n-pseudomanifold and $\sum:=(M,\chi,\mathcal{U})$ a nonlinear control system. If $0\in conv(\mathcal{U})$ and $aff(\mathcal{U})=\mathbb{R}^m$ then $span_{\mathbb{R}}(\chi)=span_{\mathbb{R}}(\mathcal{V}_\chi)$ which yields $\mathcal{L}(\chi)=\mathcal{L}(\mathcal{V}_\chi)$.

Definition 7.3.13
Let M be an n-pseudomanifold and $\sum:=(M,\chi,\mathcal{U})$ a nonlinear control system. $\mathcal{U}$ is called almost proper in $\mathbb{R}^m$ if $0\in conv(\mathcal{U})$ and $aff(\mathcal{U})=\mathbb{R}^m$. $\mathcal{U}$ is called proper in $\mathbb{R}^m$ if $0\in Int(conv(\mathcal{U}))$ and $aff(\mathcal{U})=\mathbb{R}^m$.

Proposition 7.3.5 [*46*, Theorem 2.8]
Let M be an n-pseudomanifold and $p\in M$. Let $\sum:=(M,\chi,\mathcal{U})$ be a nonlinear control system and $\mathcal{U}$ almost proper in $\mathbb{R}^m$. If $\mathcal{L}(\chi)_p=T_pM$ then $\sum$ is accessible from p.

Here again the converse implication is not in general true since it depends on the analyticity of vector fields. The Remark 7.3.2 is still applicable. However, there exist accessible systems with $\mathcal{L}(\chi)_p\subsetneq T_pM$.

Lemma 7.3.2 [*46*, Lemma 2.9]
Let M be an n-pseudomanifold and $\sum:=(M,\chi,\mathcal{U})$ a nonlinear control system. Let $\mathcal{L}_0(\chi)$ be the smallest subalgebra of $\mathfrak{X}(M)$ containing $\{X_1,\ldots,X_m\}$ and which is invariant under X_0, that is, $[X_0,X]\in\mathcal{L}_0(\chi)$ for each $X\in\mathcal{L}_0(\chi)$. The following statements hold.

$$\begin{aligned}&\mathcal{L}_0(\chi)\ \textit{is generated as a}\ \mathbb{R}\ \textit{vector space by vectors of the form below.}\\&[X_{i1},[X_{i2},\ldots,[X_{ik-1},X_i],\ldots,]],\\&\textit{where}\ i1,\ldots,ik-1\in\{0,1,\ldots,m\}\ \textit{and}\ i\in\{1,\ldots,m\}.\\&\textit{If}\ \mathcal{U}\ \textit{is almost proper then}\ \mathcal{L}_0(\chi)=\Im(\mathcal{V}_\chi).\end{aligned} \tag{7.30}$$

Proposition 7.3.6 [*46*, Theorem 2.10]
Let M be an n-pseudomanifold and $p\in M$. Let $\sum:=(M,\chi,\mathcal{U})$ be a nonlinear control system with $\mathcal{U}$ almost proper in $\mathbb{R}^m$. The following statement holds.
$\mathcal{L}_0(\chi)_p=\{X(p)\mid X\in\mathcal{L}_0(\chi)\}=T_pM \quad\Rightarrow\quad \sum$ *is strongly accessible from p.*

The Remark 7.3.2 is still valid here. The inverse implication depends on the analyticity of vector fields.

Remark 7.3.4 [*88*]

- *We may say that the reachability condition is the geometric analogue of the optimality condition. That is, the object of interest in an optimal problem is a control u. So, given two states a and b. Two situations arise in optimality questions.*
  - *If* $[0,T]$ *is known, then one is looking for a control u which minimizes the integral $\int_0^T g(\frac{dx}{dt})dt$, where g is a given function of $\frac{dx}{dt} = f(x(t), u(t))$.*
  - *If $\int_0^{Tx} g(\frac{dx}{dt})dt$ is given and Tx depends on x, then one is looking for a control u which does the transfer from a to b in a minimal time T.*
- *A control system equation $\frac{dx}{dt} = f(x(t), u(t))$ contains more qualitative informations when one changes the object of interest in the study as shown below:*
  - *Initial condition or endpoint a for Observability;*
  - *Solution $x(t)$ and its functions for Realization;*
  - *A particular control trajectory $(\tilde{x}, \tilde{u})$ for stabilizability;*
  - *Terminal condition or endpoint b for Controllability.*
- *The next section will be devoted to the controllability. For other topics above, the reader may see in the literature.*

7.4 Controllability conditions.

It is a general notice from the literature on controllability that many people worked on problem of local controllability. It is indicated in [46] a non exhaustive list of researchers who had tackled the problem. Now, about a test of non controllability, it is shown in [72] that if $\mathcal{L}(\chi)$ is integrable then $\sum := (M, \chi, \mathcal{U})$ is non controllable. The following results on controllability are given in the reference above by means of linear algebra properties and concepts stated in Definition 7.3.12.

Theorem 7.4.1 *(Frobenius)* [*72*, Theorem 2.4]
If $\mathcal{L}(\chi)$ is involutive and has a constant dimension k then $\mathcal{L}(\chi)$ is integrable.

Remark 7.4.1
Equivalently Theorem 7.4.1 *says:*

- *Through every point* $p \in M$ *there passes a* k*-dimensional immersed subpseudomanifold* $\mathcal{O}(p, \mathcal{V})$ *of* M *which is everywhere tangent to* $\mathcal{L}(\chi)$. *Also as consequence of the discussion in Section* 7.3 *the* χ *-orbit form a partition on* M.

- *Every point lies in a local chart such that* $\mathcal{L}(\chi)_p = span\{e_1, \ldots, e_k\}$ *with* 1 *in the* i^{th} *spot,*

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- *If* $\mathcal{L}(\chi)$ *is the control distribution of a nonlinear drift-free (see Definition* 7.2.3*) control system, then the set of reachable points coincides with* $\mathcal{L}(\chi)$.

Theorem 7.4.2 *(Chow - Rashevskii)* [*72*, Theorem 2.6]
If $\mathcal{L}(\chi)$ *is Lie-bracket generating then every two points can be connected by a path which is almost everywhere tangent to* $\mathcal{L}(\chi)$. *Furthermore, the path can be chosen to be piecewise smooth curves, consisting of arcs of integral curves of* $X_1, \ldots, X_m \in \chi$.

Corollary 7.4.1 [*72*, Corollary 2.7]
If $\mathcal{L}(\chi)$ *is Lie-bracket generating then the system* $\sum := (M, \chi, \mathcal{U})$ *is controllable.*

Remark 7.4.2
This corollary is a test of controllability for non linear control systems. The straightforward consequence from Theorems 7.4.1 *and* 7.4.2, *and the Corollary* 7.4.1 *is the chains of implications given below.*

$$\mathcal{L}(\chi) \ \text{ is involutive and with constant dimension .}$$
$$\Downarrow$$
$$\mathcal{L}(\chi) \text{ is integrable .}$$
$$\Downarrow$$
$$\sum := (M, \chi, \mathcal{U}) \ \text{ is non controllable .}$$
$$\Downarrow$$
$$\mathcal{L}(\chi) \ \text{ is non Lie-bracket generating.}$$

And the dual chain obtained by logical transposition:

$$\mathcal{L}(\chi) \text{ is non involutive or with non constant dimension .}$$

$$\Uparrow$$

$$\mathcal{L}(\chi) \text{ is non integrable .}$$

$$\Uparrow$$

$$\sum := (M, \chi, \mathcal{U}) \text{ is controllable .}$$

$$\Uparrow$$

$$\mathcal{L}(\chi) \text{ is Lie-bracket generating.}$$

In [46] is given a review of some known conditions for the controllability of small-time locally controllable system for k^{th}-order Lie-bracket iteration where $k \in \{o, 1, 2\}$. In [36] the controllability and observability of nonlinear systems are studied. It is shown that the Lie-bracket is related to the controllability, whereas the Lie-derivative is to the observability. Some examples of controllability are given there. Both [72, 46, 36] give the necessary condition for local controllability. But [14] give a sufficient condition for local controllability of nonlinear systems along closed orbits. The definition of Lie-bracket generating distribution can be geometrically interpreted as an opportunity to get more directions for the evolution of the system than what we have at our disposal through the given distribution by sufficiently many Lie-brackets in $span\{X_1, \ldots, X_m\}$. Lie-bracket generating is independent of the choice of generators of the distribution.

7.5 Open and closed loops - feedback.

There are generally two types of control functions. An open loop control and a closed loop control whose definitions are given below.

Definition 7.5.1 [*88, 87*]
Let M be a n-pseudomanifold and $\dot{x} = \frac{dx}{dt} = f(x(t), u(t))$ a control system equation with respect to Definition 7.2.4*. An open loop control is a control function $u: [0, +\infty) \to \mathcal{U} \subset \mathbb{R}^m$ such that the control system equation has a determined solution x for the initial condition $x(0) = x_0$.*

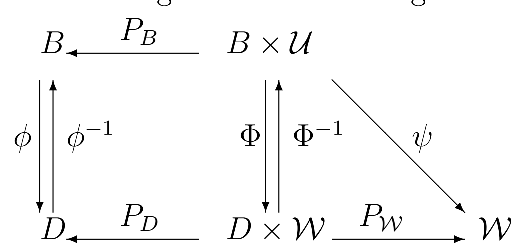
Definition 7.5.2 [*88, 87*]
Let M be a n-pseudomanifold and $\dot{x} = \frac{dx}{dt} = f(x(t), u(t))$ a control system equation with respect to Definition 7.2.4*. An closed loop control is a control function $u: M \to \mathcal{U} \subset \mathbb{R}^m$ such that the control system equation has a determined solution x for $t \in [0, +\infty)$ and the initial condition $x(0) = x_0$. The control function $u: M \to \mathcal{U}$ is called static state feedback or feedback for short by opposition to dynamic feedback.*

Remark 7.5.1 [*88, 87*]
An open loop control refers to a control function as an action of a human operator. An closed loop refers to a control function as an automatic process which is predetermined off-line. In a closed loop control, the control system equation $\dot{x}=\frac{dx}{dt}=f(x(t),u(t))$ *is an autonomous system of equations. A local study using an open set* $B\subset M$ *instead of the whole* M*, can be done.*

Definition 7.5.3 [*87*]
Let M *be an* n*-pseudomanifold. Let* $B,D\subset M$ *be two open sets and* $\mathcal{U},\mathcal{W}\subset\mathbb{R}^m$*. A feedback transform is a local diffeomorphism* $\Phi : B\times\mathcal{U} \overset{\sim}{\longrightarrow} D\times\mathcal{W}$ *such that* $(x,u)\mapsto(y,w)$*, where* $\phi : B\overset{\sim}{\longrightarrow}D$ *defined by* $y=\phi(x)$ *is a state spaces diffeomorphism and* $\psi : B\times\mathcal{U}\longrightarrow\mathcal{W}$ *defined by* $\psi(x,u)=w$*, that is,*

$$\Phi(x,u) := (\phi(x),\psi(x,u)) = (y,w). \tag{7.31}$$

This definition yields the following commutative diagram



(7.32)

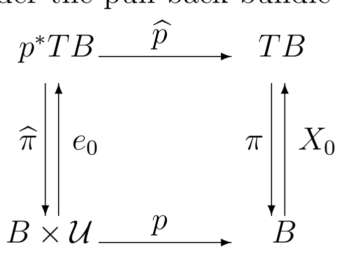
where P_B, P_D and $P_{\mathcal{W}}$ are natural projection maps.

Definition 7.5.4 [*87*]
Let Φ *be the feedback transform defined in* 7.31 *and*

$$\dot{y} := g(y,w). \tag{7.33}$$

a control system on $D\times\mathcal{W}$*. The control system* $\dot{x}=f(x(t),u(t))$ *is a feedback equivalent of* $\dot{y}:=g(y,w)$ *if every control trajectory* (x,u) *maps to a control trajectory* (y,w)*.*

Now, let us consider the pull-back bundle diagram,



(7.34)

where objects are defined as follows: B with respect to Definition 7.5.3; $P_B:=p$ with respect to the diagram in Equation 7.32; $\widehat{p}:=\pi^*(p)$ and $\widehat{\pi}=p^*(\pi)$ with respect to Definition 4.2.6 and Definition 4.2.7; e_0 is a section of $\widehat{\pi}$ defined by $\widehat{\pi}\circ e_0=id_{B\times\mathcal{U}}$; X_0 is a section of π defined by $\pi\circ X_0=id_B$ and $v=\widehat{p}\circ e_0=X_0\circ p$.

Definition 7.5.5 [*87*]
Let ∂_i be the unique local coordinate frame of p^*T_xB and $\frac{\partial}{\partial x^i}$ the global coordinate frame of T_xB with respect to Remarks 4.1.1 *and* 4.2.1.

$$\widehat{\pi}(\partial_i) = \frac{\partial}{\partial x^i}, \tag{7.35}$$

$$v(x,u) = \sum_{k=1}^{n} f^k(x,u)\frac{\partial}{\partial x^k}, \tag{7.36}$$

with $f = (f^k)_{1\leq k\leq n}$, $(\pi \circ v)(x,u) = p(x,u)$, *and* $im\, v \subset TB$.

$$e_0(x,u) = \sum_{k=1}^{n} f^k(x,u)\partial_k. \tag{7.37}$$

For each $x \in B$, $v_x : U \longrightarrow T_xB$ is defined by$v_x(u) := v(x,u) \in T_xB$. That is under the assumption that rank$(v_x) = m$ and $v_x(\mathcal{U})$ is a regular m-subpseudomanifold of T_xB.

Remark 7.5.2 [*87*]
If rank$v_x = m$ then all of the control are essential. Definitions and results below use only the algebraic side of tangent spaces and 1-forms on $B \times \mathcal{U}$. Therefore, we will state the results without proofs since the reader is referred to [87] for details.

Definition 7.5.6 [*87*]
For each $(x,u) \in B \times \mathcal{U}$, we assume $f(x,u) \neq 0$. A Pfaffian system on $B \times \mathcal{U}$ associated to the control system on $B \times \mathcal{U}$ is the set

$$I_{|(x,u)} := p^*\{\eta \in T_x^*B \mid f(x,u) \lrcorner\, \eta = 0\}. \tag{7.38}$$

The dimension of $I_{|(x,u)}$ is $n-1$ and it is constant at each point (x,u).
The affine translate of $I_{|(x,u)} \subset p^*T^*B$ is defined by

$$J_{|(x,u)} := p^*\{\varphi \in T_x^*B \mid f(x,u) \lrcorner\, \varphi = 1\}. \tag{7.39}$$

Proposition 7.5.1 [*87,* Proposition 2.4]
If $\gamma(t) = (x(t), u(t))$ is a smooth curve in $B \times \mathcal{U}$ then $\gamma(t)$ is an integral curve of the control system $\dot{x} = f(x(t), u(t))$ if, and only if for every $\varphi \in J_{|\gamma(t)}$ we have $\dot{\gamma(t)} \lrcorner\, \varphi = 1$.

Corollary 7.5.1 [*87,* Corollary 2.5]]
If $\gamma(t) = (x(t), u(t))$ is an integral of $\dot{x} = f(x(t), u(t))$ then $\gamma(t) = (x(t), u(t))$ is also an integral curve of $I = \bigcup_{(x,u)\in B\times\mathcal{U}} I_{|(x,u)}$. Moreover, if $\gamma(t) = (x(t), u(t))$ does not annihilate J, then it can be reparametrized to be an integral curve of $\dot{x} = f(x(t), u(t))$.

Remark 7.5.3 [*87*]
Note from Corollary 7.5.1 *that an integral curve of the control system is completely determined by its corresponding affine system* $I = \bigcup_{(x,u)\in B\times\mathcal{U}} I_{|(x,u)}$. *The set* $\{\eta^1, \ldots, \eta^{n-1}\}$, *formed by* $(n-1)$ 1*-forms, which are sections of* I, *is a basis of* $I_{|(x,u)} \subset p^*T^*B$. *Whereas, the set* $\{\varphi, \eta^1, \ldots, \eta^{n-1}\}$ *is a basis of* $J_{|(x,u)}$, *formed by* n 1*-forms, with* φ *a section of* $J_{|(x,u)}$. *Its dual* $\{e_0, e_1, \ldots, e_{n-1}\}$ *is a basis of* p^*T_xB. *The immediate consequence is that* p^*T_xB *is isomorphic to* $J_{|(x,u)}$.

In the control setting, the affine space is $J = \varphi + I$, *where* φ *is well defined (mod* I*). We have a time-optimal control problem obtained by Proposition* 7.5.1. *If* $\gamma(t)$ *is an integral curve then* $\gamma^*\varphi = dt$ *so* $int_\gamma \varphi = t_1 - t_0$. *That is, the time taken to cover the trajectory from initial time* t_0 *to* t_1. *To every control system we do associate a natural time optimal control problem.*

Conclusion

The symplectic reduction process in its essence is a customer of mathematical concepts in different fields as it might be seen along this study. In spite of that, it holds its intrinsic interest in many applications. In chapter 2 we have compared the initial and final topologies to Frölicher topologies on initial and final objects of the category of Frölicher spaces, using a characterization of open sets in Frölicher topologies. We emphasize here that the Frölicher topologies are not taken a priori, but they are induced by the $\mathbb{F}$ -structure functions or curves. It is fortunately proved in Corollary 2.3.4 of this work that these $\mathbb{F}$-smooth maps are continuous irrespective to the underlying topologies. We have given in chapters 2 and 3 several worked examples of Frölicher spaces, pseudomanifolds and we have constructed the initial (final) objects and structures. That is, subpseudomanifold, product, co-product and quotient in the category of pseudomanifolds. In chapter 2, we have proved that the intersection (union) of two generating functions sets yields the union (intersection) of two smooth structure curves sets, while the union (intersection) of two generating curves sets yields in turn the intersection (union) of two smooth structure functions sets. In chapter 3 we have defined three different classes of pseudomanifolds. Note that our work is completely devoted to the first class of pseudomanifolds.

The chapter 4 is devoted to tangent structures in the pseudomanifold category, that is the tangent map, tangent and cotangent bundles, the pullback of bundle over a given pseudomanifold, the double tangent and cotangent bundles. We have induced the canonical symplectic structure on the cotangent bundle and constructed a non canonical structure on the tangent bundle in chapter 5 as in [6, 25, 5, 59]. Finally, a Legendre transform was built for symplectic structures on the aforementioned bundles as in [5, 6, 25, 32]. It appears that the symplectic reduction process on a pseudomanifold is similar to the one on a smooth manifold. It will be possible to do reduction on any set (not smooth in general) endowed with a generated $\mathbb{F}$-structure and locally diffeomorphic to $\mathbb{R}^n$ of constant dimension equal to n. In the sequel, the first examples of symplectic reduction can be borrowed from the smooth manifolds setting as in [35, Remark 1.21, 1.22, Theorem 1.23] and [15, p.124]. From section 23.3 and section 24.4 in [16] and [15, p.127] the reduction at a general regular value $\theta \in \mathcal{G}$ can be related to a reduction at $0 \in \mathcal{G}$. This is what is called the shifting trick in the literature. That is, there exists a natural identification (a symplectomorphism) of the two reduced spaces.

Recall that all concepts, objects, algorithms were similar to ones on smooth manifolds. But, the difference resides on the smoothness of objects and morphisms, with an advantage here we were able to construct the differential structure using any curves or any functions set on the underlying modeling set. The characterization of the topology of structure curves, $\tau_{\mathcal{C}_M}$, is not yet issued, at our own knowledge. By the strength of Cherenack's observation on the nature of problems in practice being either functions, curves or their differential, we are able to construct appropriate differential structure even when the function or the curve is not smooth in the usual setting. As other problems in practice refer to a smooth differentiation theory, it will be possible to make extension of any process modeled on smooth manifolds to pseudomanifolds. We can take advantage by building a topology compatible with the generated differential structure.

There still are exiting questions we did not study in this work and among them we have for instance: the smooth structure and topologies on the group of diffeomorphisms of an $\mathbb{F}$-pseudomanifold, the cohomology side in this category as the De Rham theorem and connections; the algebraic topology in this category; the reduction by stages and analytic vector fields. By the strength of materials available up to now, it shall be possible to construct a Finsler geometry on $\mathbb{F}$-pseudomanifolds as in [25, 59, 23, 5, 70]. Following [59, 5, 25], canonical symplectic structures induced from a Finsler structure can be constructed on the slit tangent bundle (that is, tangent bundle without zero-section) and the slit cotangent bundle of a given $\mathbb{F}$-pseudomanifold. A Legendre transform can be defined such that it will be possible to pullback a co-Finsler to a Finsler structure and the induced symplectic structure on the slit cotangent bundle to the slit tangent bundle. It can be proven that the metric topology induced from a Finsler structure coincides with the $\mathbb{F}$-topology induced by the structure functions as in [5].

The field of potential applications may cover various domains, namely, Hamiltonian differential equations, geodesics, spray vector fields, Hamiltonian mechanic, Radon transform and its applications to seismology and imagery. Cherenack's contribution can be cited for application to cosmology as in [18, 19, 22], and the construction of Riemannian and pseudo-Riemannian metrics as in [6]. So far we have dealt with only the mathematical concepts on pseudomanifolds. Just giving a glance to potential applications areas, it is obviously clear that there is a real need for the future to know some physical and technical features of these concepts. The practical side of further researches should concern a detailed exposition of each potential applications identified above, more particularly, the following: a detailed solution of Hamiltonian differential equations by Lie techniques; a symplectic approach of geometric control theory [2, 75, 80, 84, 40]; hybrid systems with Finsler dynamic and discretization in pseudomanifolds category [42, 11, 14] and [76, 77, 78]; the Radon transform and its applications as scanner, radar, seismology; plasma physics and cosmology as stated in [6, 18, 19, 22].

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