Symmetries and Conservation Laws of Certain Classes Of Complex Partial Differential Equations

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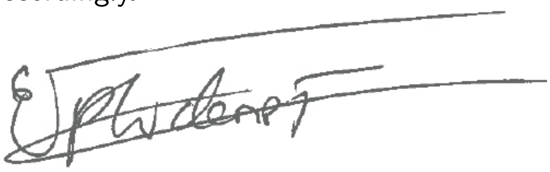
Declaration

I, the undersigned, hereby declare that the work contained in this Dissertation is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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Dedication

Dzindivhuwo kha Nwali-Mutumbukavhathu, Dzindivhuwo kha mmeanga Vho Dovhani Annah Ragimana, khotsimunene wanga vho Dr Mulalo Godfrey Phidane vhone vhane vhanga vhone vhane vhaamba nda tetemela ngauri zwevha ntonda ngazwo thikoni nau zwivhala. Ndisahangwi makhuluwanga vho Amos Pandeani Ragimana, Mifhululu khai lile ngauri nwana wa vhatavhatsindi vhaha Ramalowa vha sina ndevhe vha tshi vhona vhaha tivha li sa dali li dalaho nga mavhilivhili vhabva Thengwe o thaphudza pfunzo, Muduhuludulwana wa vhatwanamba vhaha Tshivhula tsha matshokotiko tsha gumbo lo fhelaho nga mipfa vhone vha ha tshinoni tsha pinda vhusiku vha sala vha tshi vhona ngwala lo fhira, Muduhuludulwana wa makwinda a Thathe minenzeni miri i doliwaho nga vhakegulu na vhakalaha, Muduhulu wa vhanyai vhakalanga vhaila thende vhaha nyatshigwa tshivhokodzi tshino vhokodza tshi nwavhoni vhone vhane vhari lalani ra lala vhari vuwani ra vuwa, Muduhulu dulwana wa Vho Nyadzanga Maemu Gadavha Mpande (-2007) mukegulu weavha asa tendi u dzula nga nala o lelaho vhana vhawe na vhaduhulu vhavho ene ane wa tevhedza ndayo dzawe wasa lale na ndala mukegulu museta wa muhaga ,Muduhulu wa Thuseni Agnes Mpande Mawewe mukegulu , Nwana wa mutahabvu vho Dr Maanda Patrick Phidane (1970-2013) munna weavha atshi funa pfunzo vhukuma ane anga nidzima zothe fhedzi tshikolo udo ni badelela, Nwana wa mukomana wa Rudzani na Lavhelesani Nenzhelele ndi ri matongoni kha li lale. Dzindivhuwo kha mmea a nwananga Mufunwa Tshibubudze, vhatutuwedzi mukalaha vho Silas Mawelewele Nemakonde,Rakgwahla Phalafala,Pandelani Nekhumbe,Thompho Rambuda,Dineo Makoro,Matodzi Radzhadzhi, Vho Sarah Nditsheni Munyai Negodima,Tsivhudzo Nangammbi, Ambuwani Mahamba, Tendani Nemakonde, Theophilus Phungo, Dembe Netshipale, Luruli Ndivhuwo,Rinae Ramagoma,Kundani Nethonzhe Netshishivhe, David, Lutendo na Dakalo Nemakonde, Mbudziseni Muthubi, Mudinda Daba.Dzindivhuwo khavho Lucky khumela navho mahada. Dzindivhuwo kha mubva kanwe na nne lupedzi lwa mmeanga ene zwilitsheni zwothe zwimele ri do zwivhona musi wa Khano Phidane ene ane a nnyimisa nga zwikunwe misi yothe.Dzindivhuwo kha khaladzi anga ene wenda mama nda siela ene ane ovha atshi nkuvhela Gudani Ragimana. Vhakegulu lidzani mifhululu zwipfale Makonde na Khubvi uri ula nwana wa Dovhani olwa na pfunzo a kunda, Dzindivhuwo kha mmemuhulu wanga vho Kundisani Ragimana Kwinda na mukalaha wavho, Kha mukhulu vho Ntodeni Wilson Nekhubvi ngauri mudini wavho hovha hutshinga haya hanga ha vhuvhili ndo fhiamelwa

Abstract

Lie symmetry analysis is an established method for generating symmetries of differential equations. We apply this method together with fundamental theorem of double reduction. In particular, Noether symmetries and some associated conservation laws are constructed in our investigation to find exact solutions of higher order partial differential equations and complex partial differential equations.

Contents

1. Introduction

In this dissertation we will be investigating higher order partial differential equations and complex partial differetial equations, in particular, the Swift Hohenberg partial differential equation, the Ohta-Kawasaki partial differential equation and Cubic non-linear Schrödinger Partial Differential equation(NLSE). The main objective of this Dissertation is to use Lie-point Symmetries and conservation laws of this partial differential equations to reduce their order using fundametal theorem of double reduction.

The Swift Hohenberg partial differential equation models the existence of standing waves between rolls and hexagonal patterns of the two-dimensional pattern formation[7, 8, 11].

The OhtaKawasaki equation models the evolution of di-block copolymers. Depending on the value of the parameter δ which represent a measure of the ratio of the mixture of the polymers and k, the incompatibility of the polymer types, there is a range of stationary states with a three-dimensional geometry. These have been analyzed and presented using numerical techniques[11].

We get the Cubic nonlinear Schrödinger Partial Differential equation(NLSE) from experiments on compressional dispersive Alven(CDA) waves [21].

Each chapter begins with an introduction and description of what is to follow. In chapter 2 we present a literature review of results and definitions pertinent to the work that follows. In chapter 3 study symmetry properties and conservation laws of Swift Hohenberg partial differential equation. We also show how knowledge of this lead to various reductions of pde. In chapter 4 study symmetry properties and multipliers of OhtaKawasaki partial differential equation. We also show how the knowledge of this can help us to find conserved densities of pde. In chapter 5 we study multipliers, symmetries and conservation laws of Schrödinger Partial Differential equation and show how the knowledge of this lead to reduction of pde using double reduction.

Contents of chapter 3 and chapter 4 where combined and published in[23]. Contents of chapter 5 will be sent for possible publication.

2. Preliminaries

2.1 Introduction

In this chapter we will introduce all the definitions, notations and theorems that we will need analyse the differential equations.

2.2 Definitions and Theorems

2.2.1 Definition. [1] Given $g(x, v^{(n)})$, the q-th total derivative of g has the general form

$$D_q g = \frac{\partial g}{\partial x^q} + \sum_{\alpha=1}^{i} \sum_{J} v^{\alpha}_{J,i} \frac{\partial g}{\partial v^{\alpha}_{J}} \tag{2.1}$$

where, for $J = (j_1, ..., j_l)$,

$$v^{\alpha}_{\tau,q} = \frac{\partial v^{\alpha}_{\tau}}{\partial x^q} = \frac{\partial^{l+1} v^{\alpha}}{\partial x^q \partial x^{\tau_1 \dots} \partial x^{\tau_l}}. \tag{2.2}$$

In(2.1) the sum is over all J's of order $0 \leqslant |J| \leqslant n$, where n is the highest order derivative in g.

2.2.2 Definition. A function $g(x, v, v_{(1)}, ..., v_{(k)})$ whose variables are finite is called a differential function of order k. Where $v_{(t)}$ denotes the collection of all tth-order partial derivatives, $v^{\beta}_{s} = D_s(v^{\beta})$ and $v^{\beta}_{st} = D_t D_s(v^{\beta}), \dots$ respectively.

The universal vector space of differential functions of finite order will be denoted by $\mathcal{U}$

2.2.3 Definition. The operator,

$$E_{u^{\alpha}} = \frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^{\alpha}_{i_1}}, \qquad \alpha = 1, ..., m \tag{2.3}$$

is called Euler operator for each α (it is also called the Euler-Lagrange operator).

2.2.4 Definition. [2, 3, 4, 5] Consider a system **R**$\{x; u\}$ of N partial differential equations of order k given by

$$R^{\sigma} = R^{\sigma}(x, u, \partial u, ..., \partial^k u) = 0, \ \sigma = 1, 2, ..., N, \tag{2.4}$$

with n independent variables $x = (x^1, ..., x^n)$ and m dependent variables $u = (u^1, ..., u^m)$. Local conservation law of PDE system (2.4) is a divergence expression

$$D_i F^i[u] = D_1 F^1[u] + D_2 F^2[u] + \dots + D_n F^n[u] = 0 \tag{2.5}$$

holding for all solutions of PDE system (2.4). In (2.5), $F^i[u] = F^i(x, u, \partial u, ..., \partial^r u), \ \ i = 1, 2, ..., n,$ are called the $fluxes$ of the conservation law and the highest order derivative (r) present in the fluxes $F^i[u]$ is called the (differential)$order\ of\ a\ conservation\ law.$

2.2.5 Definition. [2, 3, 4, 5] A set of multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N = \{\Lambda_\sigma(x, U, \partial U, ..., \partial^l U)\}_{\sigma=1}^N$ yields a divergence expression for PDE system (2.4) if the identity

$$\Lambda_\sigma[U]R^\sigma[U] = D_i F^i[U] \tag{2.6}$$

is satisfied for all the solutions of (2.4).

2.2.6 Theorem. *[2, 3, 4, 5]A set of non-singular multipliers*

$$\{\Lambda_\sigma[U]\}_{\sigma=1}^N = \{\Lambda_\sigma(x, U, \partial U, ..., \partial^l U)\}_{\sigma=1}^N$$

yields a local conservation law of the given system of differential equations (2.4) *if and only if*

$$E_{u^\alpha}(\Lambda_\sigma(x, U, \partial U, ..., \partial^l U)R^\sigma(x, U, \partial U, ..., \partial^k U)) = 0 \tag{2.7}$$

for all the solutions of (2.4).

2.2.7 Definition. The Lie-Bäcklund symmetry operator is given by

$$X = \tau^i \frac{\partial}{\partial x^i} + \xi^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \tau^i, \xi^\alpha \in \mathcal{U} \tag{2.8}$$

The operator is an abbreviated form of the following infinite sum

$$X = \tau^i \frac{\partial}{\partial x^i} + \xi^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{m \geq 1} \eta^\alpha_{i_1,...,i_m} \frac{\partial}{\partial u^\alpha_{i_1,...,i_m}} \tag{2.9}$$

where

$$\eta^\alpha_i = D_i(M^\alpha) + \tau^j u^\alpha_{ij}, \tag{2.10}$$

$$\eta^\alpha_{i_1,...,i_m} = D_{i_1}...D_{i_m}(M^\alpha) + \tau^j u^\alpha_{ji_1,...,i_m}, \qquad m > 1. \tag{2.11}$$

In (2.10) and (2.11),M^α is the Lie characteristic function given by

$$M^\alpha = \xi^\alpha - \tau^j u^\alpha_j. \tag{2.12}$$

2.2.8 Definition. [9] The Lie-Bäcklund symmetry generator X of the form of (2.9) is associated with the $fluxes$ of the conservation law F^ι, if it satisfies the following equation

$$X(F^\iota) + F^\iota D_k(\tau^k) - F^k D_k(\tau^\iota) = 0 \qquad \iota = 1, ..., n. \tag{2.13}$$

2.2.9 Theorem. *[9] Suppose that X is any Lie-Bäcklund symmetry of* (2.4) *and F^ι, $\iota = 1, ..., n$, are the $fluxes$ of the conservation law of* (2.4)*. Then*

$$F^{*\iota} = [F^\iota, X] = X(F^\iota) + F^\iota D_j \tau^j - F^j D_j \tau^\iota \qquad \iota = 1, ..., n. \tag{2.14}$$

constitute the components of conserved vector of (2.4).

2.2.10 Theorem. *[10] Suppose that $D_iF^i = 0$ is a conservation law of a PDE system (2.4). Then, under contact transformation, the exist a function $\widetilde{F}^i$ such that $\Theta D_iF^i = \widetilde{D}_i\widetilde{F}^i$, where $\widetilde{F}^i$ is given as*

$$\begin{pmatrix} \widetilde{F}^1 \\ \widetilde{F}^2 \\ . \\ . \\ . \\ \widetilde{F}^n \end{pmatrix} = \Theta(\Psi^{-1})^T \begin{pmatrix} F^1 \\ F^2 \\ . \\ . \\ . \\ F^n \end{pmatrix}, \tag{2.15}$$

$$\Theta \begin{pmatrix} F^1 \\ F^2 \\ . \\ . \\ . \\ F^n \end{pmatrix} = \Psi^T \begin{pmatrix} \widetilde{F}^1 \\ \widetilde{F}^2 \\ . \\ . \\ . \\ \widetilde{F}^n \end{pmatrix} \tag{2.16}$$

in which

$$\Psi = \begin{pmatrix} \widetilde{D}_1x_1 & \widetilde{D}_1x_2 & ... & \widetilde{D}_1x_n \\ \widetilde{D}_2x_1 & \widetilde{D}_2x_2 & ... & \widetilde{D}_2x_n \\ . & . & . & . \\ \\ . & . & . & . \\ . & . & . & . \\ \widetilde{D}_nx_1 & \widetilde{D}_nx_2 & ... & \widetilde{D}_nx_n \end{pmatrix}, \Psi^{-1} = \begin{pmatrix} D_1\widetilde{x}_1 & D_1\widetilde{x}_2 & ... & D_1\widetilde{x}_n \\ D_2\widetilde{x}_1 & D_2\widetilde{x}_2 & ... & D_2\widetilde{x}_n \\ . & . & . & . \\ \\ . & . & . & . \\ . & . & . & . \\ D_n\widetilde{x}_1 & D_n\widetilde{x}_2 & ... & D_n\widetilde{x}_n \end{pmatrix} \tag{2.17}$$

and Θ= det(Ψ).

2.2.11 Theorem. *[10]* **(Fundamental theorem of double reduction)**

Suppose that $D_iF^i = 0$ is a conservation law of a PDE system (2.4). Then, under a similarity transformation of a symmetry X of the form (2.9) for the PDE, there exist functions $\widetilde{F}^i$ such that X is still a symmetry for the PDE $\widetilde{D}_i\widetilde{F}^i = 0$ and

$$\begin{pmatrix} X\widetilde{F}^1 \\ X\widetilde{F}^2 \\ . \\ . \\ . \\ X\widetilde{F}^n \end{pmatrix} = \Theta(\Psi^{-1})^T \begin{pmatrix} [F^1, X] \\ [F^2, X] \\ . \\ . \\ . \\ [F^n, X] \end{pmatrix} \tag{2.18}$$

in which

$$\Psi = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & ... & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & ... & \widetilde{D}_2 x_n \\ . & . & . & . \\ \\ . & . & . & . \\ . & . & . & . \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & ... & \widetilde{D}_n x_n \end{pmatrix}, \Psi^{-1} = \begin{pmatrix} D_1 \widetilde{x}_1 & D_1 \widetilde{x}_2 & ... & D_1 \widetilde{x}_n \\ D_2 \widetilde{x}_1 & D_2 \widetilde{x}_2 & ... & D_2 \widetilde{x}_n \\ . & . & . & . \\ \\ . & . & . & . \\ . & . & . & . \\ D_n \widetilde{x}_1 & D_n \widetilde{x}_2 & ... & D_n \widetilde{x}_n \end{pmatrix} \tag{2.19}$$

and Θ= *det*(Ψ)*.*

Our original system is equivalent to

$$sys_1 = \begin{cases} \Lambda_1 R^1 + \Lambda_2 R^2 = 0, \\ \Lambda_1 R^1 - \Lambda_2 R^2 = 0. \end{cases}$$

This system can be rewritten as

$$\begin{aligned} D_t F^t + D_x F^x + D_y F^y = 0, \\ \Lambda_1 R^1 - \Lambda_2 R^2 = 0. \end{aligned} \tag{2.21}$$

where R^1 is the first equation of our system of partial differential equations(2.4),R^2 is the second equation of our system of partial differential equations (2.4), Λ_1 and Λ_2 are multipliers of our system of partial differential equations(2.4).

2.2.12 Theorem. *[4, 13]* **(Noether's theorem)**

We consider the field functions u:$\Omega \subset$*R*$^n \mapsto \mathbb{R}^m$, (x^i) $\mapsto$ ($u^\alpha(x)$) *and a functional of the form*

$$J(u) = \int_\Omega L(x^i, u^\alpha(x), u^\alpha_{x^i}(x), u^\alpha_{x^i x^j}(x))dx, \tag{2.22}$$

where L satisfies the Euler-Lagrange equation:

$$\sum_{k\leq l} D_{x^k} D_{x^l}\Big(\frac{\partial L}{\partial u^\alpha_{x^k x^l}}\Big) - D_{x^k}\Big(\frac{\partial L}{\partial u^\alpha_{x^k}}\Big) + \frac{\partial L}{\partial u^\alpha} = 0. \tag{2.23}$$

We consider the infinitesimal transformation of the form:

$$\begin{aligned}\bar{x}^i &= x^i + \varepsilon\tau^i(x,u),\\ \bar{u}^i &= u^\alpha + \varepsilon^\alpha(x,u),\end{aligned}$$

with its prolongation to:

$$\bar{u}^\alpha_{\bar{x}^k} = u^\alpha_{\bar{x}^k} + \varepsilon\eta^\alpha_k, \tag{2.24}$$

$$\bar{u}^\alpha_{\bar{x}^k\bar{x}^l} = u^\alpha_{x^k x^l} + \varepsilon\eta^\alpha_{kl}, \tag{2.25}$$

where η^α_k, η^α_{kl} are as defined in (2.10) and (2.11).

Such an infinitesimal transformation is said to be Noether Symmetry with respect to L if for all field functions $u^\alpha(x)$ and for all subdomains $\mathcal{D} \subset \Omega$, there exist functions $f^i(x,u)$ such that

$$\begin{aligned}\int_{\bar{\mathcal{D}}} L(\bar{x}^i, \bar{u}^\alpha(\bar{x}), \bar{u}^\alpha_{\bar{x}^i}(\bar{x})\bar{u}_{\bar{x}^i\bar{x}^j}(\bar{x})d\bar{x} = &\int_{\mathcal{D}} L(x^i, u^\alpha(x), u^\alpha_{x^i}(x)u_{x^i x^j}(x))dx\\ &+ \varepsilon\int_{\mathcal{D}} D_{x^i}(f^i(x,u(x)))dx + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{2.26}$$

where $f^i(x,u)$ satisfies Noether identity:

$$\tau^i\frac{\partial L}{\partial x^i} + \xi^\alpha\frac{\partial L}{\partial u^\alpha} + \eta^\alpha_k\frac{\partial L}{\partial u^\alpha_{x^k}} + \sum_{k\leq l}\eta^\alpha_{kl}\frac{\partial L}{\partial u^\alpha_{x^k x^l}} + LD_{x^i}\tau^i = D_{x^i}f^i. \tag{2.27}$$

The $fluxes$ of the conservation law of differential equation for fixed k will be given by the following equation:

$$F^k = L\tau^k + (\xi^\alpha - u^\alpha_{x^j}\tau^j)\Big(\frac{\partial L}{\partial u^\alpha_{x^k}} - \sum_{l=1}^{k} D_{x^l}\Big(\frac{\partial L}{\partial u^\alpha_{x^l x^k}}\Big)\Big) + \sum_{l=k}^{n}(\eta^\alpha_l - u_{x^l x^j}\tau^j)\frac{\partial L}{\partial u^\alpha_{x^k x^l}} - f^k. \tag{2.28}$$

3. Swift Hohenberg equation

3.1 Introduction

In this chapter, we analyze some fourth-order (1+2) partial differential equations (PDEs) that model propagation of hexagonal patterns and the microphase separation of di-block copolymers. The existence of standing waves between rolls and hexagonal patterns of the two-dimensional pattern formation Partial Differential Equation (pde) model is given in [7, 8, 11] as the evolutionary equation:

$$u_t = -(1+\Delta)^2 u + \mu u - \beta|\nabla u|^2 - \alpha u^3, \tag{3.1}$$

where $u = u(x, y, t)$ and Δ is the two-dimensional Laplacion. This is a generalization of the Swift Hohenberg equation given in [12]. The term $\beta|\nabla u|^2$ contributes to a break in symmetry[8]; other finer details of the model are spelt out in the references above, particularly in [7]. In this section we study the invariance (symmetry properties) and conservation laws of the model (3.1) and other related models that display a broader spectrum of conservation laws. We also show how a knowledge of this lead to various reductions of the pde.

The contents of this chapter have been published in [23].

3.2 Symmetries and conservation laws

In the first case we write (3.1) as

$$u_t = \mu\, u - \alpha\, u^3 - u - 2\, u_{xx} - 2\, u_{yy} - u_{xxxx} - 2\, u_{xxyy} - u_{yyyy} - \beta(u_x^2 + u_y^2). \tag{3.2}$$

The infinitesimal generator is prolonged to derivatives by adding all terms of the form $\eta_J \partial_{u_J}$ up to the desired order[1]. For example,

$$X^{[1]} = \tau^t \frac{\partial}{\partial t} + \tau^x \frac{\partial}{\partial x} + \tau^y \frac{\partial}{\partial y} + \xi \frac{\partial}{\partial u} + \eta_t \frac{\partial}{\partial u_t} + \eta_x \frac{\partial}{\partial u_x} + \eta_y \frac{\partial}{\partial u_y},$$

$$X^{[2]} = X^{[1]} + \eta_{xt} \frac{\partial}{\partial u_{xt}} + \eta_{xx} \frac{\partial}{\partial u_{xx}} + \eta_{xy} \frac{\partial}{\partial u_{xy}} + \eta_{ty} \frac{\partial}{\partial u_{ty}} + \eta_{tt} \frac{\partial}{\partial u_{tt}} + \eta_{yy} \frac{\partial}{\partial u_{yy}},$$

$\eta_J \partial_{u_J}$ is as defined in (2.10) and (2.11).

The criteria that yields the Lie point symmetries of (3.2) is given by the invariance condition :

$$X^{[4]}(u_t - \mu\, u + \alpha\, u^3 + u + 2\, u_{xx} + 2\, u_{yy} + u_{xxxx} + 2\, u_{xxyy} + u_{yyyy} + \beta(u_x^2 + u_y^2)) = 0.[1] \quad (3.3)$$

The vector fields, corresponding to the one parameter Lie group of transformations, that leave (3.2) invariant is found by solving (3.3) by method presented in [1] and get principal algebra with basis,

$$X_1 = \partial_x, \quad (3.4)$$
$$X_2 = \partial_y, \quad (3.5)$$
$$X_3 = \partial_t, \quad (3.6)$$
$$X_4 = -y\partial_x + x\partial_y. \quad (3.7)$$

The algebra is extended for the following cases of the parameters,

Case 1. $\alpha = 0,\ \beta = 0$:

$$X_5 = u\partial_u, \quad (3.8)$$
$$X_6 = f(x, y, t)\partial_u, \quad (3.9)$$
$$(3.10)$$

where $f(x, y, t)$ satisfies the pde (3.2).

Case 2. $\mu = 1,\ \alpha = 0,\ \beta \neq 0$:

$$X_7 = \partial_u, \quad (3.11)$$
$$X_8 = \frac{1}{2\beta} x\partial_u + t\partial_x, \quad (3.12)$$
$$X_9 = \frac{1}{2\beta} y\partial_u + t\partial_y. \quad (3.13)$$

Case 3. $\mu \neq 1,\ \alpha = 0,\ \beta \neq 0$:

$$X_{10} = e^{t(\mu-1)}\partial_u, \quad (3.14)$$
$$X_{11} = \frac{1}{2\beta} e^{t(\mu-1)} x\partial_u + \frac{1}{\mu - 1} e^{t(\mu-1)}\partial_x, \quad (3.15)$$
$$X_{12} = \frac{1}{2\beta} e^{t(\mu-1)} y\partial_u + \frac{1}{\mu - 1} e^{t(\mu-1)}\partial_y. \quad (3.16)$$

Using Maple we see that 3.2 does not admit any conservation laws for $\beta \neq 0$ despite having the Lie point symmetry algebra.

The steady state solutions are obtained by using time translation invariance, $X_3 = \partial_t$ and reduce (3.2) to get:

$$(\mu-1)u - \alpha\, u^3 - 2\, u_{xx} - 2\, u_{yy} - u_{xxxx} - 2\, u_{xxyy} - u_{yyyy} - \beta(u_x^2 + u_y^2) = 0, \qquad (3.17)$$

where $u = u(x, y, t)$. We perform two symmetry reductions of this pde below.

(i). Firstly, a linear combination of X_1 and X_2, $X = -b\partial_x + \partial_y$ lead to the invariants $z = x + by$ and $U = u$, where $U = U(z)$. The transformation leads to the ordinary differential equation (ode):

$$(\mu-1)U - \alpha\, U^3 - 2(1+b^2)U'' - (1+b^2)^2 U'''' - \beta(1+b^2)U'^2 = 0. \qquad (3.18)$$

Equation (3.18) has a Lagrangian for $\beta = 0$, viz.,

$$L = \frac{1}{2}(\mu-1)U^2 - \frac{1}{4}\alpha\, U^4 - (1+b^2)U'^2 - \frac{1}{2}(1+b^2)^2 U''^2, \qquad (3.19)$$

admitting a single variational symmetry ∂_z and corresponding first integral:

$$-\frac{1}{2}(1+b^2)^2 U''^2 + (1+b^2)^2 U' U''' - (1+b^2)U'^2 - \frac{1}{2}\mu U^2 + \frac{1}{2}U^2 + \frac{1}{4}\alpha\, U^4,$$

so that the ode becomes

$$-\frac{1}{2}(1+b^2)^2 U''^2 + (1+b^2)^2 U' U''' - (1+b^2)U'^2 - \frac{1}{2}\mu U^2 + \frac{1}{2}U^2 + \frac{1}{4}\alpha\, U^4 = k,$$

with an inherited symmetry ∂_z, can be reduced to a second-order ode being a double reduction of (3.18) for $\beta = 0$. That is, with $U = a$ and $U' = B$ (where $B = B(a)$), we can show that the above ode reduces to

$$B^3 B'' + \frac{1}{2}B^2 B'^2 - \frac{1}{(1+b^2)}B^2 - f(a) = 0,$$

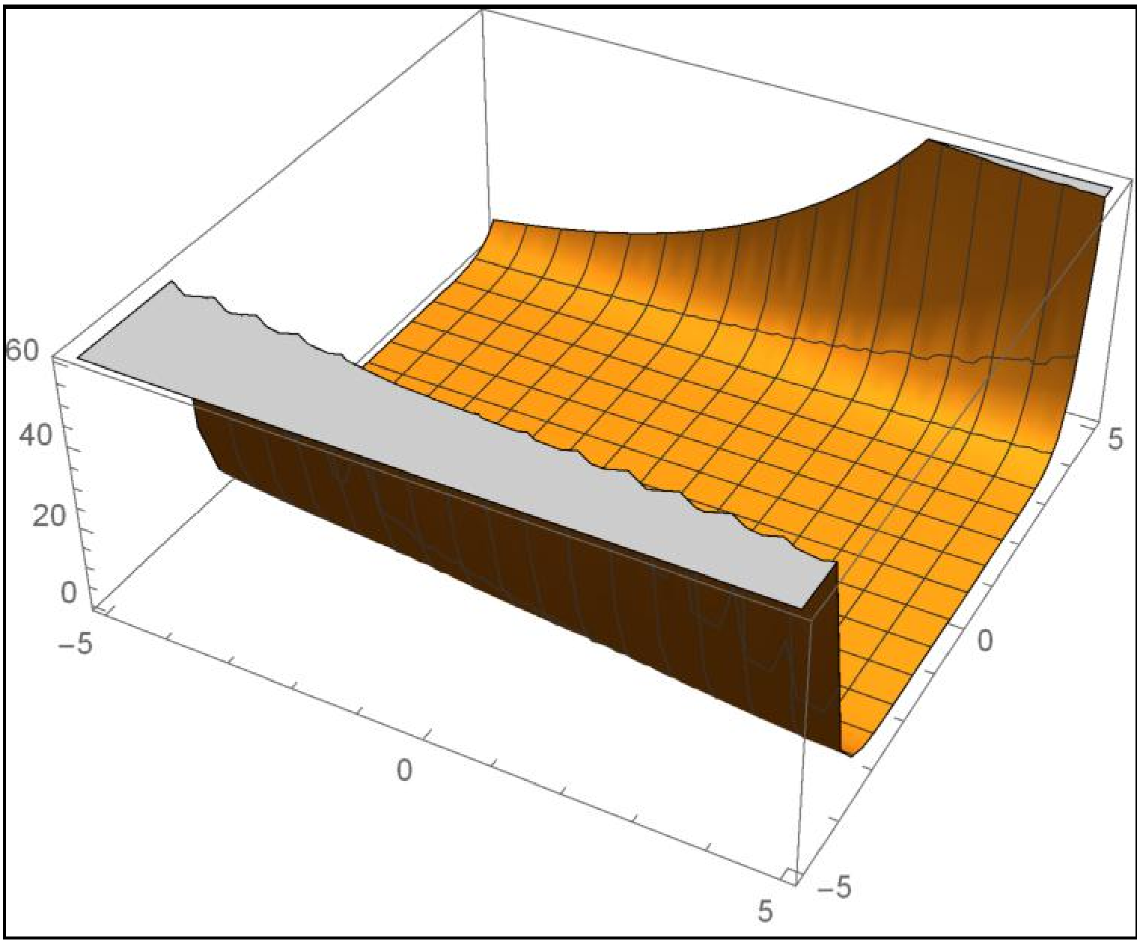
where $-f(a) = -\frac{1}{2}(\mu-1)u^2 + \frac{\alpha}{4}u^4 - k$.

Note. The fourth-order ode (3.18) can be solved directly for $\alpha = \beta = 0$ using Maple or Mathematica

$$U = c_1\, \mathrm{e}^{-\frac{\sqrt{-(1+b^2)(1+\sqrt{\mu})}}{1+b^2} z} + c_2\, \mathrm{e}^{\frac{\sqrt{-(1+b^2)(1+\sqrt{\mu})}}{1+b^2} z} + c_3\, \mathrm{e}^{-\frac{\sqrt{(1+b^2)(-1+\sqrt{\mu})}}{1+b^2} z} + c_4\, \mathrm{e}^{\frac{\sqrt{(1+b^2)(-1+\sqrt{\mu})}}{1+b^2} z}.$$

As an example, if we choose $c_1 = c_2 = 0$, $c_3 = 1/10^5$, $c_4 = 1/10^3$ and $b = 10$, $\sqrt{\mu} = 10$, we get the following profile of

$$u = c_1\, \mathrm{e}^{-\frac{\sqrt{-(1+b^2)(1+\sqrt{\mu})}}{1+b^2}(x+by)} + c_2\, \mathrm{e}^{\frac{\sqrt{-(1+b^2)(1+\sqrt{\mu})}}{1+b^2}(x+by)} + c_3\, \mathrm{e}^{-\frac{\sqrt{(1+b^2)(-1+\sqrt{\mu})}}{1+b^2}(x+by)} + c_4\, \mathrm{e}^{\frac{\sqrt{(1+b^2)(-1+\sqrt{\mu})}}{1+b^2}(x+by)}.$$


60
40
20
0
−5
0
5
5
0
−5

(ii). The rotation in $x-y$, symmetry X_4, $-y\partial_x + x\partial_y$ lead to the invariants $z = x^2+y^2$ and $U = u$, where $U = U(z)$. In this case, the transformed ode is:

$$(\mu-1)U - \alpha U^3 - 8(zU'' + U') - 4(4z^2U'''' + 11zU''' + 3U'') - 4\beta zU'^2 = 0, \tag{3.20}$$

or

$$(\mu-1)U - \alpha U^3 - 8D(zU') - 4D(4z^2U''' + 3zU'') - 4\beta zU'^2 = 0, \tag{3.21}$$

where D is the total derivative with respect to z.

3.3 Explicit power series solutions

Here we use the linear combination symmetries $X_1 = \partial_x$, $X_2 = \partial_y$ and $X_3 = \partial_t$ to reduce our pde (3.2) into an odinary differential equation and then use series solution method presented in [6, 17, 24] to solve our differential equation.

The linear combination of X_1, X_2 and X_3, $X = \partial_x + c\partial_y + \partial_t$ lead to invariants $\xi = x + y - ct$ and $u = u(\xi)$. The transformation leads to the ode:

$$(\mu-1)u + cu' - \alpha u^3 - 4u'' - 4u'''' - 2\beta u'^2 = 0. \tag{3.22}$$

By [6, 17, 24] u can be written as:

$$u(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{3.23}$$

Substituting (3.23) into (3.22) we get the following equation:

$$\begin{aligned}
&(\mu-1)\left(c_0 + \sum_{n=1}^{\infty} c_n\xi^n\right) + cc_1 + c\sum_{n=1}^{\infty}(n+1)c_{n+1}\xi^n - \alpha c_0^3 - \alpha \sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k}\xi^n \\
&- 96c_4 - 4\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}\xi^n - 8c_2 - 4\sum_{n=1}^{\infty}(n+1)(n+2)c_{n+2}\xi^n \\
&- 2\beta c_1^2 - 2\beta\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n+1-k)(k+1)c_{k+1}c_{n+1-k}\xi^n = 0,
\end{aligned} \tag{3.24}$$

We investigate the general case fo $n \geq 1$, for this case, we get:

$$\begin{aligned}
c_{n+4} = &\frac{1}{4(n+1)(n+2)(n+3)(n+4)}\Bigg((\mu-1)c_n + c(n+1c_{n+1}) - \alpha\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
&- 4(n+1)(n+2)c_{n+2} - 2\beta\sum_{k=0}^{n}(n+1-k)(k+1)c_{k+1}c_{n+1-k}\Bigg).
\end{aligned} \tag{3.25}$$

When $n = 0$, we get:

$$c_4 = \frac{(\mu - 1)c_0 + cc_1 - \alpha c_0^3 - 8c_2 - 2\beta c_1^2}{96}. \tag{3.26}$$

In this way, we have

$$u(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \sum_{n=1}^{\infty} c_{n+4}\xi^{n+4} \tag{3.27}$$

$$\begin{aligned}
=& c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \frac{(\mu - 1)c_0 + cc_1 - \alpha c_0^3 - 8c_2 - 2\beta c_1^2}{96}\xi^4 \\
&+ \sum_{n=1}^{\infty}\left(\frac{1}{4(n+1)(n+2)(n+3)(n+4)}\left((\mu-1)c_n + c(n+1c_{n+1}) - \alpha\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k}\right.\right. \\
&\left.\left. - 4(n+1)(n+2)c_{n+2A} - 2\beta\sum_{k=0}^{n}(n+1-k)(k+1)c_{k+1}c_{n+1-k}\right)\right)\xi^{n+4}.
\end{aligned} \tag{3.28}$$

At last, the explicit solutions of (3.2) are

$$u(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \sum_{n=1}^{\infty} c_{n+4}\xi^{n+4} \tag{3.29}$$

$$\begin{aligned}
=& c_0 + c_1(x+y-ct) + c_2(x+y-ct)^2 + c_3(x+y-ct)^3 \\
&+ \frac{(\mu - 1)c_0 + cc_1 - \alpha c_0^3 - 8c_2 - 2\beta c_1^2}{96}(x+y-ct)^4 \\
&+ \sum_{n=1}^{\infty}\left(\frac{1}{4(n+1)(n+2)(n+3)(n+4)}\left((\mu-1)c_n + c(n+1c_{n+1}) - \alpha\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k}\right.\right. \\
&\left.\left. - 4(n+1)(n+2)c_{n+2} - 2\beta\sum_{k=0}^{n}(n+1-k)(k+1)c_{k+1}c_{n+1-k}\right)\right)(x+y-ct)^{n+4},
\end{aligned} \tag{3.30}$$

where $c_i(i = 0, 1, 2, 3)$ are arbitrary constants, the other coefficients $c_n(n \geq 4)$ should be derived from the similar manner.

In general, the approximate form maybe more useful, i.e.,

$$\begin{aligned}
u(x,y,t) =& c_0 + c_1(x+y-ct) + c_2(x+y-ct)^2 + c_3(x+y-ct)^3 \\
&+ \frac{(\mu - 1)c_0 + cc_1 - \alpha c_0^3 - 8c_2 - 2\beta c_1^2}{96}(x+y-ct)^4 + \cdots,
\end{aligned} \tag{3.31}$$

and (3.31) converges for $|(x+y-ct)| < 1$.

3.4 Wave form of Swift Hohenberg equation

In this section we consider the wave-like properties of the model instead of the evolutionary version discussed above for the case $\beta = 0$, viz.,

$$u_{tt} = \mu\, u - \alpha\, u^3 - u - 2\, u_{xx} - 2\, u_{yy} - u_{xxxx} - 2\, u_{xxyy} - u_{yyyy} \tag{3.32}$$

which is variational with Lagrangian

$$L = -{u_x}^2 - {u_y}^2 - 1/2\, {u_t}^2 + 1/2\, {u_{xx}}^2 + 1/2\, {u_{yy}}^2 + {u_{xy}}^2 - 1/2\, \mu\, u^2 + 1/4\, \alpha\, u^4 + 1/2\, u^2.$$

The conservation laws may be determined by the multiplier approach or via the variational symmetries and Noether's theorem. Both of these are enumerated below. First, the variational symmetries $X = \tau^x \partial_x + \tau^t \partial_t + \tau^y \partial_y + \xi \partial_u$ are given and are used to calculate $fluxes$ of the conservation law via Nother's theorem and use this $fluxes$ of the conservation law to find multipliers of our pde. Since the pde is variational, each multiplier is a symmetry and has the form $Q = \xi - u_x \tau^x - u_t \tau^t - u_y \tau^y$. Second we use multiplier that we have found to calculate $fluxes$ of conservation law using Maple and homotopy formula explained in detail in[2]. The two forms of the conserved flows are equivalent up to divergence terms.

Equation (3.32) has the following variational symmetries

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_y, \\ X_3 &= \partial_t, \\ X_4 &= -y\partial_x + x\partial_y. \end{aligned}$$

Here, we use the above Noether symmetries to calculate conservation laws using Noether's theorem .

We have three independant variables and one dependent variable, from Noether's theorem F^1, F^2 and F^3 will be given by following equations:

$$\begin{aligned} F^1 =& L\tau^1 + (\xi^\alpha - u^\alpha_{x^j}\tau^j)\left[\frac{\partial L}{\partial u^\alpha_{x^1}} - D_{x^1}\Big(\frac{\partial L}{\partial u^\alpha_{x^1x^1}}\Big)\right] + (\eta^\alpha_1 - u^\alpha_{x^1x^j}\tau^j)\frac{\partial L}{\partial u^\alpha_{x^1x^1}} + (\eta^\alpha_2 - u^\alpha_{x^2x^j}\tau^j)\frac{\partial L}{\partial u^\alpha_{x^1x^2}} \\ &+ (\eta^\alpha_3 - u^\alpha_{x^3x^j}\tau^j)\frac{\partial L}{\partial u^\alpha_{x^1x^3}} - f_1, \\ F^2 =& L\tau^2 + (\xi^\alpha - u^\alpha_{x^j}\tau^j)\left[\frac{\partial L}{\partial u^\alpha_{x^2}} - D_{x^1}\Big(\frac{\partial L}{\partial u^\alpha_{x^1x^2}}\Big) - D_{x^2}\Big(\frac{\partial L}{\partial u^\alpha_{x^2x^2}}\Big)\right] + (\eta^\alpha_2 - u^\alpha_{x^2x^j}\tau^j)\frac{\partial L}{\partial u^\alpha_{x^2x^2}} \\ &+ (\eta^\alpha_3 - u^\alpha_{x^3x^j}\tau^j)\frac{\partial L}{\partial u^\alpha_{x^2x^3}} - f_2, \\ F^3 =& L\tau^3 + (\xi^\alpha - u^\alpha_{x^j}\tau^j)\left[\frac{\partial L}{\partial u^\alpha_{x^3}} - D_{x^1}\Big(\frac{\partial L}{\partial u^\alpha_{x^1x^3}}\Big) - D_{x^2}\Big(\frac{\partial L}{\partial u^\alpha_{x^2x^3}}\Big) - D_{x^3}\Big(\frac{\partial L}{\partial u^\alpha_{x^3x^3}}\Big)\right] \\ &+ (\eta^\alpha_3 - u^\alpha_{x^3x^j}\tau^j)\frac{\partial L}{\partial u^\alpha_{x^3x^3}} - f_3, \end{aligned}$$

and $F^1 = F^t$,$F^2 = F^x$ and $F^3 = F^y$. We then calculate conservation laws for all symmetries X_1, X_2, X_3 and X_4.

3.4.1 Conservation laws using space translation invariance $X_1 = \partial_x$.

Here $\tau^1 = \tau^3 = 0$, $\tau^2 = 1$ and $\xi = 0$, resulting in $\eta_1 = 0, \eta_2 = 0$ and $\eta_3 = 0$. From Noether's identinty (2.27) $f_t = f_x = f_y = 0$.

By Noether's theorem the $fluxes$ of the conservation law will be:

$$F^t = u_x u_t, \tag{3.33}$$

$$F^x = u_x^2 - u_y^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 - u_{xy}^2 - \frac{\mu}{2}u^2 + \frac{\alpha}{4}u^4 + \frac{u^2}{2} + u_x u_{xxx}, \tag{3.34}$$

$$F^y = 2u_x u_y + 2u_x u_{xxy} + u_x u_{yyy} - u_{xy}u_{yy}, \tag{3.35}$$

$$\begin{aligned} D_tF^t + D_xF^x + D_yF^y =& D_t(u_xu_t) + D_x(u_x^2 - u_y^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 - u_{xy}^2 - \frac{\mu}{2}u^2 + \frac{\alpha}{4}u^4 + \frac{u^2}{2} + u_x u_{xxx}) \\ &+ D_y(2u_xu_y + 2u_xu_{xxy} + u_xu_{yyy} - u_{xy}u_{yy}) \\ =& u_x(u_{tt} + 2u_{xx} - \mu u + \alpha u^3 + u + u_{xxxx} + 2u_{yy} + 2u_{xxyy} + u_{yyyy}) = 0 \end{aligned}$$

along the solutions of (3.32).Therefore u_x is the multiplier of (3.32).

Lastly we use multiplier u_x to calculate $fluxes$ of the conservation law using Maple and homotopy formula explained in detail in[2] and get the following $fluxes$ of the conservation law:

$$\begin{aligned} F^t =& \ 1/2\, u_t u_x - 1/2\, u u_{xy}, \\ F^y =& \ 1/2\, u_{yyy} u_x + 5/6\, u_x u_{xxy} - 1/2\, u_{yy} u_{xy} - 1/2\, u_{xx} u_{xy} + 1/2\, u_y u_{xyy} + 1/6\, u_y u_{xxx} \\ & + u_y u_x - 1/2\, u u_{xyyy} - 1/2\, u u_{xxxy} - u u_{xy}, \\ F^x =& \ 1/4\, \alpha\, u^4 + 2/3\, u_{xyy} u_x + u_{xxx} u_x - 1/6\, u_{yy} u_{xx} - 1/3\, {u_{xy}}^2 - 1/2\, {u_{xx}}^2 + 1/3\, u_y u_{xxy} + {u_x}^2 \\ & + 1/2\, u u_{xxyy} - 1/2\, \mu\, u^2 + 1/2\, u u_{yyyy} + 1/2\, u u_{tt} + 1/2\, u^2 + u u_{yy}. \end{aligned}$$

3.4.2 Conservation laws using Noether symmetry $X_2 = \partial_y$.

Here $\tau^1 = \tau^2 = 0$, $\tau^3 = 1$ and $\xi = 0$, resulting in $\eta_1 = 0, \eta_2 = 0$ and $\eta_3 = 0$. From Noether's identinty (2.27) $f_t = f_x = f_y = 0$.

By Noether's theorem the $fluxes$ of the conservation law will be:

$$\begin{aligned} F^t =& u_y u_t, \\ F^x =& 2u_x u_y + u_y u_{xxx} - u_{xy} u_{xx} - 2u_{yy} u_{xy}, \\ F^y =& u_y^2 - u_x^2 - \frac{u_t^2}{2} + \frac{u_{xx}^2}{2} - \frac{u_{yy}^2}{2} + u_{xy}^2 - \frac{\mu}{2}u^2 + \frac{\alpha}{4}u^4 + \frac{u^2}{2} + 2u_y u_{xxy} + u_y u_{yyy}, \end{aligned}$$

$$\begin{aligned}D_tF^t + D_xF^x + D_yF^y =& D_t(u_yu_t) + D_x(2u_xu_y + u_yu_{xxx} - u_{xy}u_{xx} - 2u_{yy}u_{xy})\\ &+ D_y(u_y^2 - u_x^2 - \frac{u_t^2}{2} + \frac{u_{xx}^2}{2} - \frac{u_{yy}^2}{2} + u_{xy}^2 - \frac{\mu}{2}u^2 + \frac{\alpha}{4}u^4 + \frac{u^2}{2} + 2u_yu_{xxy}\\ &+ u_yu_{yyy})\\ =& u_y(u_{tt} + 2u_{xx} - \mu u + \alpha u^3 + u + u_{xxxx} + 2u_{yy} + 2u_{xxyy} + u_{yyyy}) = 0\end{aligned}$$

along the solutions of (3.32).Therefore u_y is the multiplier of (3.32).

Lastly we use multiplier u_y to calculate $fluxes$ of the conservation law using Maple and homotopy formula explained in detail in[2] and get the following $fluxes$ of the conservation law:

$$\begin{aligned}F^t =& \; 1/2\, u_tu_y - 1/2\, uu_{yt},\\ F^y =& \; 1/4\, \alpha\, u^4 + u_{yyy}u_y + 2/3\, u_yu_{xxy} - 1/2\, {u_{yy}}^2 - 1/3\, {u_{xy}}^2 - 1/6\, u_{yy}u_{xx} + {u_2}^2 + 1/3\, u_{xyy}u_x\\ &+1/2\, uu_{xxyy} - 1/2\, \mu\, u^2 + 1/2\, uu_{xxxx} + 1/2\, uu_{tt} + 1/2\, u^2 + uu_{xx},\\ F^x =& \; 5/6\, u_yu_{xyy} + 1/2\, u_yu_{xxx} - 1/2\, u_{yy}u_{xy} - 1/2\, u_{xx}u_{xy} + 1/6\, u_{yyy}u_x\\ &+1/2\, u_xu_{1,1,2} + u_yu_x - 1/2\, uu_{xyyy} - 1/2\, uu_{xxxy} - uu_{xy}.\end{aligned}$$

3.4.3 Conservation laws using Noether symmetry $X_3 = \partial_t$.

Here $\tau^1 = 1$, $\tau^2 = \tau^3 = 0$ and $\xi = 0$, resulting in $\eta_1 = 0, \eta_2 = 0$ and $\eta_3 = 0$.From Noether's identity (2.27) $f_t = f_x = f_y = 0$.

By Noether's theorem the $fluxes$ of the conservation law will be:

$$\begin{aligned}F^t =& - u_x^2 - u_y^2 + \frac{u_t^2}{2} + \frac{u_{xx}^2}{2} + \frac{u_{yy}^2}{2} + u_{xy}^2 - \frac{\mu u}{2} + \frac{\alpha u^4}{4} + \frac{u^2}{2},\\ F^x =& 2u_xu_t + u_tu_{xxx} - u_{xt}u_{xx} - 2u_{yt}u_{xy},\\ F^y =& 2u_tu_y + 2u_tu_{xxy} + u_tu_{yyy} - u_{yt}u_{yy}.\end{aligned}$$

$$\begin{aligned}D_tF^t + D_xF^x + D_yF^y =& D_t(-u_x^2 - u_y^2 + \frac{u_t^2}{2} + \frac{u_{xx}^2}{2} + \frac{u_{yy}^2}{2} + u_{xy}^2 - \frac{\mu u}{2} + \frac{\alpha u^4}{4} + \frac{u^2}{2})\\ &+ D_x(2u_xu_t + u_tu_{xxx} - u_{xt}u_{xx} - 2u_{yt}u_{xy})\\ &+ D_y(2u_tu_y + 2u_tu_{xxy} + u_tu_{yyy} - u_{yt}u_{yy})\\ =& u_t(u_{tt} + 2u_{xx} - \mu u + \alpha u^3 + u + u_{xxxx} + 2u_{yy} + 2u_{xxyy} + u_{yyyy}) = 0\end{aligned}$$

along the solutions of (3.32). Therefore u_t is the multiplier of (3.32).

Lastly we use multiplier u_t to calculate $fluxes$ of the conservation law using Maple and homotopy formula explained in detail in[2] and get the following $fluxes$ of the conservation law:

$$\begin{aligned}F^t \;&= 1/4\, \alpha\, u^4 + 1/2\, {u_t}^2 + 1/2\, u^2 + uu_{1,1} + uu_{yy} + 1/2\, uu_{xxxx} + uu_{xxyy} + 1/2\, uu_{yyyy} - 1/2\, \mu\, u^2,\\ -F^y \;&= -1/2\, u_{yyy}u_t - 1/2\, u_{xxy}u_t + 1/2\, u_{yy}u_{yt} + 1/3\, u_{xy}u_{xz} + 1/6\, u_{xx}u_{yt} - 1/2\, u_yu_{yyt} - 1/6\, u_yu_{xxt}\\ &\quad -u_tu_y - 1/3\, u_xu_{1,2,3} + 1/2\, uu_{yyyt} + 1/2\, uu_{xxyt} + uu_{yt},\\ F^x \;&= 1/2\, u_{xyy}u_t + 1/2\, u_{xxx}u_t - 1/6\, u_{yy}u_{xt} - 1/3\, u_{xy}u_{yt} - 1/2\, u_{xx}u_{xz} + 1/3\, u_yu_{xyt} + 1/6\, u_xu_{yyt}\\ &\quad +1/2\, u_xu_{xxt} + u_tu_x - 1/2\, uu_{xyyt} - 1/2\, uu_{xxxt} - uu_{xz}.\end{aligned}$$

3.4.4 Conservation laws using Noether symmetry $X_4 = -y\partial_x + x\partial_y$.

Here $\tau^1 = 0$, $\tau^2 = -y$, $\tau^3 = x$ and $\xi = 0$, resulting in $\eta_1 = 0, \eta_2 = -u_y$ and $\eta_3 = u_x$.From Noether's identity (2.27) $f_t = f_x = f_y = 0$.

By Noether's theorem the $fluxes$ of the conservation law will be:

$$
\begin{aligned}
F^t =& xu_y u_t - yu_x u_t,\\
F^x =& yu_x^2 + yu_y^2 + \frac{yu_t^2}{2} - \frac{yu_{xx}^2}{2} - \frac{yu_{yy}^2}{2} - yu_{xy}^2 + \frac{\mu y u^2}{2} - \frac{\alpha y u^4}{4} - \frac{yu^2}{2} - 2yu_x^2 - yu_x u_{xxx}\\
&+ 2xu_y u_x + xu_y u_{xxx} - u_y u_{xx} + yu_{xx}^2 - xu_{xy}u_{xx} + 2u_x u_{xy} + 2yu_{xy}^2 - 2xu_{xy}u_{yy},\\
F^y =& 2u_t u_y + 2u_t u_{xxy} + u_t u_{yyy} - u_{yt}u_{yy}.
\end{aligned}
$$

$$
\begin{aligned}
D_tF^t + D_xF^x + D_yF^y =& D_t(xu_y u_t - yu_x u_t) + D_x(yu_x^2 + yu_y^2 + \frac{yu_t^2}{2} - \frac{yu_{xx}^2}{2} - \frac{yu_{yy}^2}{2} - yu_{xy}^2 + \frac{\mu y u^2}{2}\\
&- \frac{\alpha y u^4}{4} - \frac{yu^2}{2} - 2yu_x^2 - yu_x u_{xxx} + 2xu_y u_x + xu_y u_{xxx} - u_y u_{xx} + yu_{xx}^2\\
&- xu_{xy}u_{xx} + 2u_x u_{xy} + 2yu_{xy}^2 - 2xu_{xy}u_{yy}) + D_y(2u_t u_y + 2u_t u_{xxy} + u_t u_{yyy}\\
&- u_{yt}u_{yy})\\
=& (xu_y - yu_x)(u_{tt} + 2u_{xx} - \mu u + \alpha u^3 + u + u_{xxxx} + 2u_{yy} + 2u_{xxyy} + u_{yyyy}) = 0
\end{aligned}
$$

along the solutions of (3.32). Therefore $xu_y - yu_x$ is the multiplier of (3.32).

Lastly we use multiplier $xu_y - yu_x$ to calculate $fluxes$ of the conservation law using Maple and homotopy formula explained in detail in[2] and get the following $fluxes$ of the conservation law:

$$
\begin{aligned}
F^t \quad &= 1/2\, u_t x u_y - 1/2\, u_t y u_x + 1/2\, u y u_{xz} - 1/2\, u x u_{yt},\\
-F^y \quad &= 1/6\, u_{xx} x u_{yy} + 1/2\, u_y y u_{xyy} + 1/2\, x\mu\, u^2 - u_{yyy} x u_y - 1/2\, u x u_{xxyy} - u x u_{xx} + 1/2\, u_{yyy} y u_x\\
&-1/2\, u x u_{xxxx} - 1/2\, u_{yy} y u_{1,2} - u y u_{xy} - 1/2\, u y u_{xxxy} - 1/3\, u_{xyy} u_x x - 1/2\, u y u_{xyyy}\\
&+5/6\, u_{xxy} y u_x - 2/3\, u_{xxy} x u_2 - 1/2\, u x u_{tt} + 1/6\, u_y y u_{xxx} - 1/4\, x\alpha\, u^4 + u_{xy} u_y - x {u_y}^2 - u u_x\\
&-5/6\, u_{yy} u_x + 1/2\, x {u_{yy}}^2 - 1/2\, u_{xy} y u_{xx} + u_y y u_1 + 1/3\, x {u_{xy}}^2 + 1/6\, u_{xx} u_x - 1/2\, u u_{xyy}\\
&-1/2\, u u_{xxx} - 1/2\, x u^2,\\
F^x \quad &= 1/2\, u_x x u_{xxy} + 1/6\, u_x x u_{yyy} - 1/2\, u x u_{xxxy} - u y u_{yy} - 1/2\, u_{xx} x u_{xy} - 1/3\, u_y y u_{xxy}\\
&-1/2\, u y u_{xxyy} + 1/6\, u_{yy} y u_{xx} - 1/2\, u_{yy} x u_{1,2} - u x u_{xy} - 2/3\, u_{xyy} y u_x - 1/2\, y u^2 - y {u_x}^2\\
&-u u_y + u_{xy} u_x - 1/2\, u u_{xxy} - 1/2\, u u_{yyy} + 5/6\, u_{xyy} x u_y + u_x x u_y - 1/2\, u y u_{tt} - u_{xxx} y u_x\\
&+1/2\, u_{xxx} x u_y - 1/2\, u u_{xyyy} x - 1/2\, u y u_{yyyy} + 1/2\, y\mu\, u^2 - 1/4\, y\alpha\, u^4 + 1/6\, u_{yy} u_y + 1/3\, y {u_{xy}}^2\\
&-5/6\, u_{xx} u_y + 1/2\, y {u_{xx}}^2.
\end{aligned}
$$

3.5 Double reduction of the wave form of Swift Hohenberg equation

Here we will apply the fundamental theorem of double reduction to solve our pde(3.32) using the following $fluxes$ of the conservation law $F = (F^t, F^x, F^y)$:

$$\begin{aligned}
F^t =& u_x u_t, \\
F^x =& u_x^2 - u_y^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2 - u_{xy}^2 - \frac{\mu}{2}u^2 + \frac{\alpha}{4}u^4 + \frac{u^2}{2} + u_x u_{xxx}, \\
F^y =& 2u_x u_y + 2u_x u_{xxy} + u_x u_{yyy} - u_{xy} u_{yy},
\end{aligned}$$

that we calculated in (3.33),(3.34),(3.35) and symmetries X_1, X_2, X_3 to solve our partial differential equation.

In order to use fundamental theorem of double reduction to solve our partial differential equation, the symmetries that we want to use must be associated to the conservation laws that we want to use to reduce the partial differential equation.

To check whether conserved vectors are associated with symmetry we use the following version of (2.13):

$$F^* = X \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t & D_y\xi^t \\ D_t\xi^x & D_x\xi^x & D_y\xi^x \\ D_t\xi^y & D_x\xi^y & D_y\xi^y \end{pmatrix} \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} + (D_t\xi^t + D_x\xi^x + D_y\xi^y) \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix}. \tag{3.36}$$

For $X_1 = \partial_x$,

$$\begin{aligned}
F^* =& X_1 \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} + (0) \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} \\
=& \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

as required.

For $X_2 = \partial_y$

$$\begin{aligned}
F^* =& X_2 \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} + (0) \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} \\
=& \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

as required.

For $X_3 = \partial_t$

$$\begin{aligned} F^* =& X_3 \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix} + (0) \begin{pmatrix} F_1^t \\ F_1^x \\ F_1^y \end{pmatrix} \\ =& \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

as required.

Therefore F is associated with X_1, X_2 and X_3.

We then consider $\Gamma = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}$ (ι and c are constants), and this generator is then transformed into its canonical form $T = \frac{\partial}{\partial k}$, where we assume that T is of the form:

$$T = \Gamma(\tau)\frac{\partial}{\partial \tau} + \Gamma(k)\frac{\partial}{\partial k} + \Gamma(w)\frac{\partial}{\partial w} + \Gamma(\rho)\frac{\partial}{\partial \rho} = 0\frac{\partial}{\partial \tau} + \frac{\partial}{\partial k} + 0\frac{\partial}{\partial w} + 0\frac{\partial}{\partial \rho}. \tag{3.37}$$

From the above equation we get $\Gamma(\tau) = 0$,$\Gamma(k) = 1$,$\Gamma(w) = 0$, and $\Gamma(\rho) = 0$. The invariants will be found by solving the following equation [19]:

$$\frac{d\rho}{0} = \frac{dt}{1} = \frac{dk}{1} = \frac{dx}{c} = \frac{dy}{\iota} = \frac{du}{0} = \frac{d\tau}{0} = \frac{dw}{0}. \tag{3.38}$$

The invariants of (3.38) are summarized in the table below,

Invariants of $\Gamma = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}$	
$\frac{dt}{1} = \frac{dx}{c}$	$b_1 = x - ct$
$\frac{dy}{m} = \frac{dt}{1}$	$b_2 = y - mt$
$\frac{dk}{1} = \frac{dt}{1}$	$b_3 = k - t$
$\frac{du}{0}$	$b_4 = u$
$\frac{d\tau}{0}$	$b_5 = \tau$
$\frac{d\rho}{0}$	$b_6 = \rho$
$\frac{dw}{0}$	$b_7 = w$

Table 3.1: invariants table

By choosing $b_4 = b_7$, $b_3 = 0$, $b_5 = b_1$ and $b_6 = b_2$ we get the following canonical coordinates

$$\begin{aligned} w &= u, \\ s &= t, \\ \tau &= x - cs, \\ \rho &= y - \iota t, \end{aligned}$$

where $w = w(r, p)$.

The matrices Ψ and Ψ^{-1} will be constructed using the above canonical coordinates.

$$\Psi = \begin{pmatrix} D_\tau t & D_\tau x & D_\tau y \\ D_s t & D_s x & D_s y \\ D_\rho t & D_\rho x & D_\rho y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & c & \iota \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.39}$$

$$\Psi^{-1} = \begin{pmatrix} D_t\tau & D_t s & D_t\rho \\ D_x\tau & D_x s & D_x\rho \\ D_y\tau & D_y s & D_y\rho \end{pmatrix} = \begin{pmatrix} -c & 1 & -\iota \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.40}$$

$$(\Psi^{-1})^T = \begin{pmatrix} -c & 1 & 0 \\ 1 & 0 & 0 \\ -\iota & 0 & 1 \end{pmatrix}, \tag{3.41}$$

and $\Theta = \det(\Psi) = -1$. The derivatives of u in terms of w are,

$$\begin{aligned}
u_y &= w_\rho, \\
u_x &= w_\tau, \\
u_{xx} &= w_{\tau\tau}, \\
u_{xy} &= w_{\tau\rho}, \\
u_{yy} &= w_{\rho\rho}, \\
u_{xxx} &= w_{\tau\tau\tau}, \\
u_{xxy} &= w_{\tau\tau\rho}, \\
u_{yyy} &= w_{\rho\rho\rho}, \\
u_t &= -cw_\tau - \iota w_\rho.
\end{aligned}$$

The reduced conserved form will be:

$$\begin{pmatrix} F^s \\ F^\tau \\ F^\rho \end{pmatrix} = \Theta(\Psi^{-1})^T \begin{pmatrix} F^t \\ F^x \\ F^y \end{pmatrix}. \tag{3.42}$$

Now by substituting the first, second and third derivatives of u in (3.42) we obtain,

$$\begin{pmatrix} F^s \\ \\ F^\tau \\ \\ F^\rho \end{pmatrix} = \begin{pmatrix} (-c^2-1)w_\tau^2 - c\iota w_\tau w_\rho + w_p^2 + \frac{1}{2}w_{\tau\tau}^2 + w_{\tau\rho} + \frac{\mu}{2}w^2 - \frac{\alpha}{4}w^4 - \frac{1}{2}w^2 - w_\tau w_{\tau\tau\tau} \\ \\ -c^2 w_\tau^2 - c\iota w_\tau w_\rho \\ \\ -\iota c w_\tau^2 - \iota^2 w_\tau w_\rho - 2w_\tau w_\rho - 2w_\tau w_{\tau\tau\rho} - w_\tau w_{\rho\rho\rho} + w_{\tau\rho}w_{\rho\rho} \end{pmatrix}, \tag{3.43}$$

the first equation of the system (3.43) can also be given as,

$$D_s F^s = 0. \tag{3.44}$$

Intergrating (3.44) with respect to s we get the following equation:

$$(-c^2-1)w_\tau^2 - c\iota w_\tau w_\rho + w_\rho^2 + \frac{1}{2}w_{\tau\tau}^2 + w_{\tau\rho} + \frac{\mu}{2}w^2 - \frac{\alpha}{4}w^4 - \frac{1}{2}w^2 - w_\tau w_{\tau\tau\tau} = k, \tag{3.45}$$

where k is constant.

The second equation of the system can also be given as $D_\tau F^\tau = 0$, so by intergrating with respect to τ we can get:

$$-c^2 w_\tau^2 - c\iota w_\tau w_\rho = j, \tag{3.46}$$

where j is constant.

Substituting (3.46) into (3.45) we get:

$$-w_\tau^2 + w_\rho^2 + \frac{1}{2}w_{\tau\tau}^2 + w_{\tau\rho} + \frac{\mu}{2}w^2 - \frac{\alpha}{4}w^4 - \frac{1}{2}w^2 - w_\tau w_{\tau\tau\tau} = \epsilon, \tag{3.47}$$

where $\epsilon = k - j$.

It is easy to see that $\Phi_1 = \partial_\rho$ and $\Phi_2 = \partial_\tau$ are symmetries of (3.47). We then consider:

$$\Phi = \partial_\tau + h\partial_\rho, \tag{3.48}$$

h is constant To find invariants of (3.48) we solve the following equation:

$$\frac{d\tau}{1} = \frac{d\rho}{h} = \frac{dw}{0}. \tag{3.49}$$

The above equation gives us:

$$m = \rho - h\tau,$$

$w = w(m)$

The derivatives of w are,

$$\begin{aligned} w_\tau &= w^{'}(-h), \\ w_\rho &= w^{'}(1), \\ w_{\tau\tau} &= w^{''}(-h)^2, \\ w_{\tau\rho} &= w^{''}(-h), \\ w_{\tau\tau\tau} &= w^{'''}(-h)^3. \end{aligned}$$

Substituting first, second and third order derivatives of w into (3.47) we get:

$$(-h^2+1)w'^2 + \frac{h^4}{2}w''^2 - hw'' - h^4 w' w''' + \frac{\mu - 1}{2}w^2 - \frac{\alpha}{4}w^4 = \epsilon.$$

The above equation can be solved for $\alpha = h = 0$ and get:

$$w(z) = \frac{ie^{-\frac{1}{2}i\sqrt{1-\mu}\left(2c_1+\sqrt{2}z\right)}\left(8\epsilon e^{2ic_1\sqrt{1-\mu}} + e^{i\sqrt{2-2\mu}z}\right)}{4\sqrt{1-\mu}} \tag{3.50}$$

or

$$w(z) = \frac{ie^{-\frac{1}{2}i\sqrt{1-\mu}\left(2c_1+\sqrt{2}z\right)}\left(1 + 8\epsilon e^{i\sqrt{1-\mu}\left(2c_1+\sqrt{2}z\right)}\right)}{4\sqrt{1-\mu}}. \tag{3.51}$$

Substituting back the canonical coordinates of z and w into (3.50) and (3.51) we get:

$$u = \frac{ie^{-\frac{1}{2}i\sqrt{1-\mu}\left(2c_1+\sqrt{2}(y-\iota t)\right)}\left(8\epsilon e^{2ic_1\sqrt{1-\mu}} + e^{i\sqrt{2-2\mu}(y-\iota t)}\right)}{4\sqrt{1-\mu}} \tag{3.52}$$

or

$$u = \frac{ie^{-\frac{1}{2}i\sqrt{1-\mu}\left(2c_1+\sqrt{2}(y-\iota t)\right)}\left(1 + 8\epsilon e^{i\sqrt{1-\mu}\left(2c_1+\sqrt{2}(y-\iota t)\right)}\right)}{4\sqrt{1-\mu}}. \tag{3.53}$$

3.6 Conclusion

In this chapter we have shown that Swift Hohenberg equation has nontrivial Lie point symmetries which can be used to reduce it to an ordinary differential equation. We have showed how to use explicit power solutions to solve Swift-Hohenberg equation. We then converted the Swift-Hohenberg equation into its wave form for β=0 and used Noether's theorem to find its conservation laws. The conservation laws, together with multipliers were used to calculate further conservation laws. We have shown that if symmetries are associated with conservation law, we can use the fundamental theorem of double reduction to reduce our partial diffrential equation into an ordinary differential equation. These may be solved by Maple or Mathematica when $\alpha = 0$.

4. Ohta-Kawasaki equation

4.1 Introduction

The Ohta-Kawasaki equation

$$u_t + \Delta(\delta\Delta u + u - u^3) - u + k = 0 \tag{4.1}$$

models the evolution of di-block copolymers [14, 15, 16, 18]. Depending on the values of the parameter δ which represents a measure of the ratio of the mixture of the polymers, and k, the incompatibility of the polymer types, there is a range of stationary states with a three-dimensional geometry. These have been analysed and presented using numerical techniques[11]. In [14, 16], for $k = 0$, an energy functional is given by

$$E = \int [\frac{1}{2}(u_x^2 + u_y^2) + \frac{1}{4}(1 - u^2)^2]dx + \frac{1}{2}u^2. \tag{4.2}$$

The invariance properties and conservation laws of (4.1) are analysed in this chapter.

4.2 Symmetries and conservation laws

We first expand (4.1) into:

$$u_t + \delta(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) - 6(uu_x^2 + uu_y^2) - 3(u^2u_{xx} + u^2u_{yy}) + u_{xx} + u_{yy} - u + k = 0. \tag{4.3}$$

The criteria that yields the Lie point symmetries of (4.3) is given by the invariance condition :

$$X^{[4]}(u_t + \delta(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) - 6(uu_x^2 + uu_y^2) - 3(u^2u_{xx} + u^2u_{yy}) + u_{xx} + u_{yy} - u + k) = 0.[1] \tag{4.4}$$

The vector fields, corresponding to the one parameter Lie group of transformations, that leave (4.3) invariant is found by solving (4.4) by method presented in [1] and get principal algebra with basis,

$$X_1 = \partial_x, \tag{4.5}$$
$$X_2 = \partial_y, \tag{4.6}$$
$$X_3 = \partial_t, \tag{4.7}$$
$$X_4 = -y\partial_x + x\partial_y. \tag{4.8}$$

Using Maple we can be show that (4.3) generates infinitely many conservation laws based on multipliers dependent on the basis variables, viz.,

$$Q_1 = f(y - ix)e^t, \qquad Q_2 = g(y + ix)e^t, \qquad (i^2 = -1) \tag{4.9}$$

where f and g are arbitrary functions in the argument. No fibre or jet dependent multipliers exist. Some multipliers and the corresponding conserved densities are given for different cases below.

Case 1. $f(y - ix) = 1$:

$$Q = e^t,$$

we then use multiplier $Q = e^t$ to calculate conserved densities using Maple and homotopy formula explained in detail in[2] and get the following conserved density:

$$F^t = \frac{1}{t^2}(-2\,k + 2\,\mathrm{e}^t k - 2\,t\mathrm{e}^t k + k\mathrm{e}^t t^2 + \mathrm{e}^t u t^2).$$

Case 2. $f(y - ix) = (x + ay)$:

$$Q = (x + ay)e^t,$$

we then use multiplier $Q = (x+ay)e^t$ to calculate conserved densities using Maple and homotopy formula explained in detail in[2] and get the following conserved density:

$$F^t = \frac{1}{t^3}\,(x + ay)\left(6\,k - 6\,\mathrm{e}^t k + 6\,t\mathrm{e}^t k - 3\,k\mathrm{e}^t t^2 + \mathrm{e}^t k t^3 + \mathrm{e}^t u t^3\right).$$

Case 3 $f(y - ix) = (y - ix)$:

$$Q = (y - ix)e^t,$$

we then use multiplier $Q = (y-ix)e^t$ to calculate conserved densities using Maple and homotopy formula explained in detail in[2] and get the following conserved density:

$$\begin{aligned} F^t = -\frac{1}{t^3}(&-6\,yk + 6\,ikx + 6\,k\mathrm{e}^t y - 6\,i\mathrm{e}^t kx - 6\,\mathrm{e}^t kyt + 6\,i\mathrm{e}^t ktx + 3\,\mathrm{e}^t kyt^2 - 3\,i\mathrm{e}^t kt^2 x \\ &- \mathrm{e}^t kt^3 y + i\mathrm{e}^t kt^3 x - u\mathrm{e}^t t^3 y + iu\mathrm{e}^t t^3 x). \end{aligned}$$

Examples of other multipliers are $(e^{(\pm x)}\sin y)e^t$ and $\cosh x \sin y$.

4.3 Wave form of Ohta-Kawasaki equation

If u_t is replaced with u_{tt}, giving the equation a wave structure.

$$u_{tt} + \Delta(\delta \Delta u + u - u^3) - u + k = 0, \tag{4.10}$$

we then expand (4.10) into:

$$u_{tt} + \delta(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) - 6(uu_x^2 + uu_y^2) - 3(u^2 u_{xx} + u^2 u_{yy}) + u_{xx} + u_{yy} - u + k = 0. \tag{4.11}$$

The vector fields, corresponding to the one parameter Lie groups of transformation, that leave (4.3) invariant generate a principal algebra with basis,

$$X_1 = \partial_x, \tag{4.12}$$
$$X_2 = \partial_y, \tag{4.13}$$
$$X_3 = \partial_t, \tag{4.14}$$
$$X_4 = -y\partial_x + x\partial_y. \tag{4.15}$$

Using Maple we get the following multipliers that give rise to the conservation laws:

$$q_1 = (f_1(y - ix) + g_1(y + ix)) \cos t, \qquad q_2 = (f_2(y - ix) + g_2(y + ix)) \sin t.$$

where f_1, f_2, g_1 and g_2 are arbitrary functions in the argument. Some multipliers and the corresponding conserved densities are given for different cases below.

Case 1. $f_1 = g_1$ and $f_2 = g_2 = 0$:

$$q = \cos t,$$

we then use multiplier $q = \cos t$ to calculate conserved densities using Maple and homotopy formula explained in detail in[2] and get the following conserved density:

$$F^t = \frac{1}{t^2}[kt^2 \sin(t) - 2k \sin(t) + 2t \cos(t) k + u_3 \cos(t) t^2 + u \sin(t) t^2]$$

Case 2. $f_1 = (y - ix)$ and $f_2 = g_1 = g_2 = 0$:

$$q = (y - ix) \cos t,$$

we then use multiplier $q = (y - ix) \cos t$ to calculate conserved densities using Maple and homotopy formula explained in detail in[2] and get the following conserved density:

$$
\begin{aligned}
F^t =& \frac{1}{t^3}[-6\,yk + 6\,ikx + 6\,kt\sin(t)\,y - 6\,ikt\sin(t)\,x + 6\,k\cos(t)\,y - 6\,ik\cos(t)\,x - kt^3\sin(t)\,y \\
&+ ikt^3\sin(t)\,x - 3\,kt^2\cos(t)\,y + 3\,ikt^2\cos(t)\,x - t^3yu_3\cos(t) - t^3yu\sin(t) + it^3xu_3\cos(t) \\
&+ it^3xu\sin(t)].
\end{aligned}
$$

Note. In each case, the *conserved quantities* are given by $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^t dxdy$ if u and its derivatives converge when $x, y \to \pm\infty$.

4.4 Conclusion

In this chapter we have calculated the multipliers and conseved densities of Ohta-Kawasaki equation.

5. Cubic Schrödinger Partial Differential equation

5.1 Introduction

In this chapter we use the method of double reduction to find the exact solutions of a Cubic nonlinear Schrödinger Partial Differential equation(NLSE). Experiments on compressional dispersive Alven(CDA) waves have been done before in [21] and [22] where the relationship between CDA waves and pertubations were analyzed. Further, in [22], the amplitude of the waves in a magnetic electron-positron plasma was discussed. In both experiments it was observed that the system of equations under investigation can be written as the following equation:

$$\Upsilon_{tt} - (3a^2 + c^2)\Upsilon_{xx} - \delta^2\Upsilon_{xxxx} - \delta^2\Upsilon_{xxtt} = 0. \tag{5.1}$$

This equation was analyzed in [20].

The contents of this chapter have been sent for possible publication.

If compressional dispersive Alven pump came into contact with a quasi stationary compressional magnetic field we get an envelope of compressional dispersive Alven waves[21]. When the CDA envelope evolves we get the following (NLSE):

$$i\psi_t + \beta i\psi_x - \gamma\psi_{xx} + \delta\psi|\psi|^2 = 0, \tag{5.2}$$

where ψ is the complex valued function, and γ is the coefficient of the group velocity dispersion. δ is the self-phase modulation (SPM) owing to kerr law, and β is the inter-modal dispersion(IMD).γ, δ and β are constants. If self-phase modulation and group velocity dispersion is balanced we get solitons.

To create a system of equations, we substitute $\psi = \theta + i\upsilon$ into (5.2), $\theta = \theta(x, t)$, $\upsilon = \upsilon(x, t)$. Then separate into real and imaginary parts to obtain:

$$\begin{aligned} R^1 &= \theta_t + \beta\theta_x - \gamma\upsilon_{xx} + \delta\upsilon(\theta^2 + \upsilon^2) = 0, \\ R^2 &= -\upsilon_t - \beta\upsilon_x - \gamma\theta_{xx} + \delta(\theta^2 + \upsilon^2) = 0. \end{aligned} \tag{5.3}$$

5.2 Conservation laws and conserved quantities of Cubic Schrödinger Partial Differential equation

To find multipliers and $fluxes$ of the conservation law, the condition below must hold.

$$\Lambda_1 \cdot (\theta_t + \beta\theta_x - \gamma\upsilon_{xx} + \delta\upsilon(\theta^2+\upsilon^2)) + \Lambda_2 \cdot (-\upsilon_t - \beta\upsilon_x - \gamma\theta_{xx} + \delta(\theta^2+\upsilon^2)) = D_t F^t + D_x F^x, \quad (5.4)$$

where Λ_1, Λ_2 are multipliers of (5.2) and F^t, F^x are the $fluxes$ of the conservation law.

We get multipliers by noting that the Euler operator annihilates total divergences,

$$\frac{\delta}{\delta\theta}[q^1(\theta_t + \beta\theta_x - \gamma\upsilon_{xx} + \delta\upsilon(\theta^2+\upsilon^2)) + q^2(-\upsilon_t - \beta\upsilon_x - \gamma\theta_{xx} + \delta(\theta^2+\upsilon^2))] = 0. \quad (5.5)$$

First we use Maple and Mathematica to solve (5.5) and get multipliers of (5.2). Second we substitute the multipliers that we have found into (5.4) and get $fluxes$ of the conservation law using Maple and homotopy formula explained in detail in[2] and get the following results:

(i) $(\Lambda_1, \Lambda_2) = (\upsilon_x, \theta_x)$

$$\begin{aligned} F_1^x &= \frac{1}{4}(\delta\theta^4 + 2\delta\theta^2\upsilon^2 + \delta\upsilon^4 + 2\upsilon\theta_t - 2\theta\upsilon_t - 2\gamma(\theta_x^2 + \upsilon_x^2)), \quad (5.6)\\ F_1^t &= \frac{1}{2}(-\upsilon\theta_x + \theta\upsilon_x). \end{aligned}$$

(ii) $(\Lambda_1, \Lambda_2) = (\theta, -\upsilon)$

$$F^x = \frac{1}{2}(\beta\theta^2 + \upsilon(\beta\upsilon + 2\gamma\theta_x) - 2\gamma\theta\upsilon_x), \quad (5.7)$$

$$F^t = \frac{1}{2}(\theta^2 + \upsilon^2). \quad (5.8)$$

(iii) $(\Lambda_1, \Lambda_2) = (\theta_t, \upsilon_t)$

$$\begin{aligned} F_3^x &= \frac{1}{2}(-\gamma(\theta_t\theta_x + \upsilon_x\upsilon_t) + \theta(\beta\upsilon_t + \gamma\theta_{xt}) + \upsilon(-\beta\theta_t + \gamma\upsilon_{xt})),\\ F_3^t &= \frac{1}{4}(\delta\theta^4 + 2\delta\theta^2\upsilon^2 - 2\theta(\beta\upsilon_x + \gamma\theta_{xx}) + \upsilon(\delta\upsilon^3 + 2\beta\theta_x - 2\gamma\upsilon_{xx})). \quad (5.9) \end{aligned}$$

(iv) $(\Lambda_1, \Lambda_2) = (x\theta + 2\gamma t\upsilon_x - \beta\theta, 2\gamma t\theta_x + \beta t\upsilon - x\upsilon)$

$$\begin{aligned} F_4^x =& \frac{1}{2}t\gamma\delta\theta^4 + \frac{1}{2}\beta x\upsilon^2 - \frac{1}{2}t\beta^2\upsilon^2 + \frac{1}{2}t\gamma\delta\upsilon^4 + \frac{1}{2}\theta^2\beta x - \frac{1}{2}\theta^2\beta^2 t \\ &+ t\gamma\delta\upsilon^2\theta^2 + \gamma\upsilon x\theta_x - \gamma\upsilon t\beta\theta_x - \gamma\theta t\upsilon_t - \gamma\theta x\upsilon_x + \gamma\beta\theta t\upsilon_x - t\gamma^2\theta_x^2 - t\gamma^2\upsilon_x^2, \\ F_4^t =& \frac{1}{2}x\theta^2 - \frac{1}{2}t\beta\theta^2 + \frac{1}{2}x\upsilon^2 - \frac{1}{2}\beta t\upsilon^2 - t\gamma\theta_x\upsilon + t\gamma\theta\upsilon_x \end{aligned} \tag{5.10}$$

5.3 Symmetries of Cubic Schrödinger Partial Differential equation

A one-parameter Lie group of transformations that leave (5.2) invariant will be written as a vector field

$$X = \tau^t(x,t,\theta,\upsilon)\partial_t + \tau^x(x,t,\theta,\upsilon)\partial_x + \xi^\theta(x,t,\theta,\upsilon)\partial_\theta + \xi^\upsilon(x,t,\theta,\upsilon)\partial_\upsilon \tag{5.11}$$

Equation (5.2) has the following variational symmetries:

$$\begin{aligned} X_1 =& \partial_t, \\ X_2 =& \partial_x, \\ X_3 =& \theta\partial_\upsilon - \upsilon\partial_\theta, \\ X_4 =& 2\gamma t\partial_x + (\beta t - x)\theta\partial_\upsilon + (-\beta t + x)\upsilon\partial_\theta, \\ X_5 =& 2t\partial_t + (\beta t + x)\partial_x - \theta\partial_\theta - \upsilon\partial_\upsilon. \end{aligned}$$

5.4 Double reduction of Cubic Schrödinger Partial Differential equation

Here we will apply the fundamental theorem of double reduction to solve our pde(5.2) using the following $fluxes$ of the conservation law $F = (F^t, F^x)$:

$$\begin{aligned} F^t =& \frac{1}{2}(\theta^2 + \upsilon^2), \\ F^x =& \frac{1}{2}(\beta\theta^2 + \upsilon(\beta\upsilon + 2\gamma\theta_x) - 2\gamma\theta\upsilon_x), \end{aligned}$$

that we calculated in (5.7),(5.8) and symmetries X_1, X_3 to solve our complex partial differential equation.

To check whether conserved vectors are associated with symmetry we use the following version of (2.13):

$$F^* = X\begin{pmatrix} F^t \\ F^x \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t \\ D_t\xi^x & D_x\xi^x \end{pmatrix}\begin{pmatrix} F^t \\ F^x \end{pmatrix} + (D_t\xi^t + D_x\xi^x)\begin{pmatrix} F^t \\ F^x \end{pmatrix}.$$

For X_1:

$$\begin{pmatrix} F_2^{*t} \\ F_2^{*x} \end{pmatrix} = X_1^{[1]} \begin{pmatrix} F_2^t \\ F_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F_2^t \\ F_2^x \end{pmatrix} + (0) \begin{pmatrix} F_2^t \\ F_2^x \end{pmatrix}$$

$$= X_1^{[1]} \begin{pmatrix} \frac{1}{2}(\theta^2 + \upsilon^2) \\ \frac{1}{2}(\beta\theta^2 + \upsilon(\beta\upsilon + 2\gamma\theta_x) - 2\gamma\theta\upsilon_x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For X_3:

$$\begin{pmatrix} F_2^{*t} \\ F_2^{*x} \end{pmatrix} = X_3^{[1]} \begin{pmatrix} F_2^t \\ F_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F_2^t \\ F_2^x \end{pmatrix} + (0) \begin{pmatrix} F_2^t \\ F_2^x \end{pmatrix}$$

$$= X_3^{[1]} \begin{pmatrix} \frac{1}{2}(\theta^2 + \upsilon^2) \\ \frac{1}{2}(\beta\theta^2 + \upsilon(\beta\upsilon + 2\gamma\theta_x) - 2\gamma\theta\upsilon_x) \end{pmatrix}$$

$$= \begin{pmatrix} \theta\frac{1}{2}2\upsilon - \upsilon\frac{1}{2}2\theta \\ \theta\frac{1}{2}(2\beta\upsilon + 2\gamma\theta_x) - \upsilon\frac{1}{2}(2\beta\theta + 2\gamma\upsilon_x) + \theta_x\frac{1}{2}(-2\gamma\theta) + \upsilon_x(\frac{1}{2}2\gamma\upsilon) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\begin{aligned} X_1^{[1]} &= \partial_t, \\ X_3^{[1]} &= \theta\partial_\upsilon - \upsilon\partial_\theta + \theta_t\partial_{\upsilon_t} + \theta_x\partial_{\upsilon_x} - \upsilon_t\partial_{\theta_t} - \upsilon_x\partial_{\theta_x}. \end{aligned}$$

Thus F_2 is associated with both X_1 and X_3.

We then consider the linear combination of X_1 and X_3:

$$\Gamma = \partial_t + c\theta\partial_\upsilon - c\upsilon\partial_\theta,$$

and this generator is then transfprmed into its canonical form $T = \frac{\partial}{\partial_k}$, where we assume that T is of the form

$$T = \Gamma(k)\frac{\partial}{\partial_k} + \Gamma(\tau)\frac{\partial}{\partial_\tau} + \Gamma(\omega)\frac{\partial}{\partial_\omega} + \Gamma(m)\frac{\partial}{\partial_m} = \frac{\partial}{\partial_k} + 0\frac{\partial}{\partial_\tau} + 0\frac{\partial}{\partial_\omega} + 0\frac{\partial}{\partial_m}.$$

From $\Gamma(\tau) = 0$, $\Gamma(k) = 1$, $\Gamma(\omega) = 0$, and $\Gamma(\rho) = 0$, we obtain[19]:

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\theta}{-c\upsilon} = \frac{dk}{1} = \frac{d\tau}{0} = \frac{d\omega}{0} = \frac{dm}{0} = \frac{d\upsilon}{c\theta}. \tag{5.12}$$

The invariants of (5.12) are summarized in a table below,

Invariats of $\Gamma = \frac{\partial}{\partial_t} + c(\theta\partial_\upsilon - \upsilon\partial_\theta)$	
$\frac{dt}{1} = \frac{dk}{1}$	$b_1 = k - t$
$\frac{d\theta}{-c\upsilon} = \frac{d\upsilon}{k\theta}$	$b_2 = \theta^2 + \upsilon^2$
$\frac{d\upsilon}{c\theta} = \frac{dt}{1}$	$b_3 = \arctan(\frac{\upsilon}{\theta}) - ct$
$\frac{d\tau}{0}$	$b_4 = \tau$
$\frac{dm}{0}$	$b_5 = m$
$\frac{d\omega}{0}$	$b_6 = \omega$
$\frac{dx}{0}$	$b_7 = x$

Table 5.1: invariants table

By choosing $b_1 = 0$, $b_4 = b_7$, $b_6 = \sqrt{b_2}$, and $b_3 = b_5$ we get

$$\tau = x,$$
$$k = t,$$
$$\omega = \sqrt{\theta^2 + \upsilon^2},$$
$$m = \arctan\left(\frac{\upsilon}{\theta}\right) - ct.$$

The inverse canonical coordinates are presented below

$$x = \tau,$$
$$t = k,$$
$$\theta = \omega\cos(m + ck),$$
$$\upsilon = \omega\sin(m + ck).$$

The matrices Ψ and Ψ^{-1} can be computed using the canonical coordinates above

$$\Psi = \begin{pmatrix} D_k t & D_k x \\ D_\tau t & D_\tau x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\Psi^{-1})^T \tag{5.13}$$

and $\Theta = \det(\Psi) = 1$.

The partial derivatives of θ and υ in terms of new dependent variables ω and m are,

$$\begin{aligned}
\theta_x =& \omega_\tau \cos(m+ck) - \omega m_\tau \sin(m+ck),\\
\theta_{xx} =& \omega_{\tau\tau} \cos(m+ck) - 2\omega_\tau m_\tau \sin(m+ck) - \omega m_\tau^2 \cos(m+cs) - \omega m_{\tau\tau} \sin(m+cs),\\
\theta_t =& -c\omega \sin(m+ck),\\
\upsilon_x =& \omega_\tau \sin(m+ck) + \omega m_\tau \cos(m+ck),\\
\upsilon_{xx} =& \omega_{\tau\tau} \sin(m+ck) + 2\omega_\tau m_\tau \cos(m+ck) + \omega m_{\tau\tau} \cos(m+ck) - \omega m_\tau^2 \sin(m+ck),\\
\upsilon_t =& c\omega \cos(m+ck).
\end{aligned}$$

The reduced conserved form will be:

$$\begin{aligned}
\begin{pmatrix} F_2^k \\ \\ F_2^\tau \end{pmatrix} =& \Theta(\Psi^{-1})^T \begin{pmatrix} F_2^t \\ \\ F_2^x \end{pmatrix} \\
=& \begin{pmatrix} \frac{1}{2}(\theta^2+\upsilon^2) \\ \\ \frac{1}{2}(\beta\theta^2 + \upsilon(\beta\upsilon + 2\gamma\theta_x) - 2\gamma\theta\upsilon_x) \end{pmatrix}.
\end{aligned} \tag{5.14}$$

Substituting the first and second partial derivatives of θ and υ into (5.14) we get

$$\begin{pmatrix} F_2^k \\ \\ F_2^\tau \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\omega(\tau)^2 \\ \\ \frac{1}{2}(2\beta\omega(\tau)^2 + \gamma\omega(\tau)\omega_\tau(\tau)\sin(2(m+ck)) + 2\gamma\omega(\tau)^2\cos(2(m+ck)) \end{pmatrix}. \tag{5.15}$$

The first equation of the system (5.15) can also be given as:

$$D_k F_2^k = 0. \tag{5.16}$$

Integrating (5.16) with respect to k we get the following equation:

$$\omega(\tau)^2 = \epsilon, \tag{5.17}$$

where ϵ is constant, or equivalently,

$$\omega(\tau) = \pm\sqrt{\epsilon}. \tag{5.18}$$

Differentiating (5.18) implicitly with respect to τ we get:

$$\frac{d}{d\tau}\omega(\tau) = 0. \tag{5.19}$$

The second equation of sys_1 from (2.21) can be written as:

$$\theta(\theta_t + \beta\theta_x - \gamma\upsilon_{xx} + \delta\upsilon(\theta^2 + \upsilon^2)) - (-\upsilon)(-\upsilon_t - \beta\upsilon_x - \gamma\theta_{xx} + \delta(\theta^2 + \upsilon^2)) = 0. \tag{5.20}$$

Substituting (5.18),(5.19) and partial derivatives of θ and υ in terms of $\omega(\tau)$ and $m(\tau)$ into (5.20) we get:

$$\begin{aligned} &- c\epsilon\sin(2m(\tau) + 2ck) - \beta\epsilon m'(\tau)\sin(2m(\tau) + 2ck) - \gamma\epsilon m''(\tau)\cos(2m(\tau) + 2ck) \\ &+ \gamma\epsilon m'(\tau)^2\sin(2m(\tau) + 2ck) + \delta\epsilon^2\sin(2m(\tau) + 2ck) = 0. \end{aligned} \tag{5.21}$$

When computing the final solution to equation (5.21) we get an indirect solution. We then compute the solution of (5.21) for the following cases.

Case 1. $c = \gamma = 0$:

Substituting $c = \gamma = 0$ into (5.21) and solve (5.21) by Maple and Mathematica we get the following solutions of (5.21):

$$m(\tau) = 0, \tag{5.22}$$

$$m(\tau) = -\frac{\pi}{2}, \tag{5.23}$$

$$m(\tau) = \frac{\pi}{2}, \tag{5.24}$$

$$m(\tau) = \frac{\tau\delta\epsilon}{\beta} + c_1. \tag{5.25}$$

We then solve for our ψ using all the values of $m(\tau)$ calculated above.

From (5.22), $m(\tau) = 0$, we get the following values of θ and υ:

$$\begin{aligned} \theta &= \pm\sqrt{\epsilon}, \\ \upsilon &= 0, \end{aligned}$$

Substituting θ and υ into ψ, we get:

$$\psi = \pm\sqrt{\epsilon}.$$

From (5.23), $m(\tau) = -\frac{\pi}{2}$, we get the following values of θ and υ:

$$\theta = 0,$$
$$\upsilon = \mp\sqrt{\epsilon},$$

Substituting θ and υ into ψ we get:

$$\psi = \mp i\sqrt{\epsilon}.$$

From (5.24), $m(\tau) = \frac{\pi}{2}$, we get the following values of θ and υ

$$\theta = 0,$$
$$\upsilon = \pm\sqrt{\epsilon},$$

Substituting θ and υ into ψ we get:

$$\psi = \pm i\sqrt{\epsilon}.$$

From (5.25), $m(\tau) = \frac{\tau\delta\epsilon}{\beta} + c_1$, we get the following values of θ and υ:

$$\theta = \pm\sqrt{\epsilon}\cos\left(\frac{x\delta\epsilon}{\beta} + c_1\right),$$
$$\upsilon = \pm\sqrt{\epsilon}\sin\left(\frac{x\delta\epsilon}{\beta} + c_1\right).$$

Substituting θ and υ into ψ we get:

$$\psi = \pm\sqrt{\epsilon}e^{i(\frac{x\delta\epsilon}{\beta}+c_1)} \tag{5.26}$$

Case 2. $c = \beta = 0$:

Substituting $c = \beta = 0$ into (5.21) and solve (5.21) by Maple and Mathematica we get the following solutions of (5.21):

$$m(\tau) = 0, \tag{5.27}$$
$$m(\tau) = -\frac{\pi}{2}, \tag{5.28}$$
$$m(\tau) = \frac{\pi}{2}, \tag{5.29}$$
$$m(\tau) = c_1. \tag{5.30}$$

We then solve for our ψ using all the values of $m(\tau)$ calculated above.

From (5.30), $m(\tau) = c_1$, we get the following values of θ and υ:

$$\theta = \sqrt{\epsilon}\cos(c_1),$$
$$\upsilon = \sqrt{\epsilon}\sin(c_1),$$

Substituting θ and υ into ψ we get:

$$\psi = \sqrt{\epsilon}e^{ic_1}.$$

Case 3. $\delta = \gamma = 0$:

Substituting $\delta = \gamma = 0$ into (5.21) and solve (5.21) by Maple and Mathematica we get the following solutions of (5.21):

$$m(\tau) = -\,ck, \tag{5.31}$$
$$m(\tau) = -\,\frac{\pi}{2} - ck, \tag{5.32}$$
$$m(\tau) = \frac{\pi}{2} - ck, \tag{5.33}$$
$$m(\tau) = -\,\frac{c\tau}{\beta} + c_1. \tag{5.34}$$

From (5.31) $m(\tau) = -\frac{c\tau}{\beta} + c_1$ e get the following values of θ and υ:

$$\theta = \sqrt{\epsilon}\cos(\frac{c(\beta k - \tau)}{\beta} + c_1),$$
$$\upsilon = \sqrt{\epsilon}\sin(\frac{c(\beta k - \tau)}{\beta} + c_1).$$

Substituting θ and υ into ψ we get:

$$\psi = \sqrt{\epsilon}e^{i(\frac{c(\beta t - x)}{\beta} + c_1)}. \tag{5.35}$$

5.5 Conclusion

In this chapter we calculated multipliers and conservation laws of the Cubic Schrödinger partial differential equation. Further, we showed that if a symmetry is associated with a conservation law, we can use the fundamental theorem of double reduction to an ordinary differential equation. Consequently, we used Maple and Mathematica to calculate exact solutions for special cases.

6. Conclusion

In this dissertation, we have used the method of symmetry analysis and showed that Swift Hohenberg equation has Lie point symmetries and this symmetries can be used to reduce it into an ordinary differential equation. We have also shown that Swift-Hohenberg equation does not admit any conservation law despite having the Lie point symmetry. We have also shown how to use explicit power solutions to solve our Swift-Hohenberg equation. Furthermore, we converted the Swift-Hohenberg equation into its wave form for β=0 and used Noether's theorem to find the conservation laws, use this conservation laws to find multipliers and use this multipliers to calculate conservation laws again. We used the double reduction theorem together with Maple and Mathematica, to obtain solutions for particular case $\alpha = 0$. The procedure was repeated for the Ohta -Kawasaki equation. The symmetries in the cubic Schrodinger equation were not Noether symmetries, this didn't deter us from obtaining exact solutions in a similar method as the Ohta-Kawasaki equation. We have calculated multipliers of Cubic Schrödinger Partial Differential equation and then use this multipliers to find conservation laws.We then show that if a symetry is associated with conservation law we can use the fundamental theorem of doule reduction to convert Cubic Schrödinger Partial Differential equation into an ordinary differential equation which can be easily solved by Mapple or Mathematica for special cases. Although the findings presented in this dissertation are by no means exhaustive, at the very least my hope is that they may inspire further work in this area.

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