DARBOUX-CRUM TRANSFORMATIONS OF ORTHOGONAL POLYNOMIALS AND ASSOCIATED BOUNDARY CONDITIONS

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Abstract

Linear second order ordinary differential boundary value problems feature prominently in many scientific field, such as physics and engineering. Solving these problems is often riddled with complications though a myriad of techniques have been devised to alleviate these difficulties. One such method is by transforming a problem into a more readily solvable form or a problem which behaves in a manner which is well understood. The Darboux-Crum transformation is a particularly interesting transformation characterised by some surprising properties, and an increase in the number of works produced in the last few years related to this transformation has prompted this investigation. The classical orthogonal polynomials, namely those of Jacobi, Legendre, Hermite and Laguerre, have been nominated as test candidates and this work will investigate how these orthogonal families are affected when transformed via Darboux-Crum transformations.

Aknowledgements

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Maryke Carleen Rademeyer

Signed on this the 14th day of May 2013, at Johannesburg, South Africa.

Contents

Chapter 1

Introduction

The first of the classical orthogonal polynomials was introduced in the nineteenth century by Legendre. Subsequent work done by Chebyshev, as well as Markov and Stieltjes, resulted in the creation of a formal theory for orthogonal polynomials. In recent years, the area of mathematics dealing with classical orthogonal polynomials has been greatly extended, simply because of how the classical orthogonal polynomials tend to generate elegant results that can be neatly described and efficiently analysed. Most notably, the work of [1] and [3], as well as [35], has established a well-constructed foundation for research in this field.

The classical orthogonal polynomials consist of the Jacobi polynomials, the Laguerre polynomials and the Hermite polynomials. Special cases of the Jacobi polynomials, namely the Legendre polynomials, Chebyshev polynomials and the ultraspherical, or Gegenbauer, polynomials, are sometimes discussed separately.

Given that the classical orthogonal polynomials are so well behaved, it is often convenient to use these functions as samples for testing what phenomena emerge when the polynomials undergo different types of transformations. One may be tempted to ask why the transformation of a problem set would be of any interest and the answer to this is that one would like to know whether solving the transformed problem would solve the initial problem as well. Transformations allow for a problem which may be difficult to solve, to be transformed into a more tractable problem. In particular, a singular boundary value problem (i.e. one whose boundary conditions contains singularities) might be transformed into a regular problem.

Darboux transformations of Sturm-Liouville problems have been of some interest since the early 1980s, with extensive work done in recent years by Binding, Browne and Watson [7], [8] and [9]. They studied the general transformation of the differential equation but with particular focus on the changes undergone by the boundary conditions and how these would affect the new problem.

A comprehensive account of the events that gave rise to what is now understood to be the Darboux-Crum transformation is given in [7], of which only a few key points will be highlighted here. In a paper concerning Sturm-Liouville substitutions, published in 1882, Darboux showed that if y is a solution to the Sturm-Liouville equation

$$-y'' + qy = \lambda y, \tag{1.0.1}$$

where $' = \frac{d}{dx}$ and z is a solution to (1.0.1) for $\lambda = 0$, then

$$\tilde{y} = y' - \frac{z'}{z} y \tag{1.0.2}$$

is a solution to the "transformed equation"

$$-\tilde{y}'' + z \left(\frac{1}{z} \right)'' \tilde{y} = \lambda \tilde{y}.$$

Darboux's result is essentially the first application of the Darboux-Crum transformation with z as base function. As pointed out in [7], no boundary conditions were taken into account.

Crum extended Darboux's proposed transformation by selecting as base functions the first n eigenfunctions for use in n successive Darboux-like transformations and went further by including boundary conditions. A pivotal concept here is that z

does not change sign on the interval. This method for constructing operators of the form $-\frac{d^2}{dx^2} + q$ makes it possible to attain a full description of the spectrum of the operator. That is, the original potential function's known wave functions are utilized to create the new potential function and wave functions. The original operator and the transformed operator have near identical spectral characteristics.

Binding, Browne and Watson ([7], [8] and [9]) go into great detail in analysing the transformation of eigendependent boundary values. In [8], a singular problem with eigendependent boundary conditions is transformed into a regular problem through a modified multi-application of the Darboux-Crum transformation. There too the original and new operators are "almost" isospectral, altered by at most two eigenvalues. Their modified transformation makes use of two base functions which may or may not be eigenfunctions.

Having noted the benefits of using classical orthogonal polynomials as test subjects for generating observations, it would be appropriate to employ them for studying the effects of the Darboux-Crum transformation on differential equations of second order. The classical orthogonal polynomials satisfy equations of the form

$$-(py')' = \lambda ry \tag{1.0.3}$$

with their respective boundary conditions.

There are several questions that this investigation will explore and, hopefully, answer.

1. How are the boundary conditions transformed by the Darboux-Crum transformation?
2. Do the transformed equations form part of a known class of equations with well-studied solutions?
3. Is orthogonality preserved by the transformation and, if so, in what sense?

4. How does the spectrum of the transformed operator compare with that of the original operator?

5. Can singular problems be transformed into regular ones by the Darboux-Crum transformation?

This investigation can be regarded as a work in three parts. Part I looks at the Darboux-Crum transformation, the theory that validates it and further results that were motivated by it. The classical orthogonal polynomials are discussed in detail in Part II. Finally, applications of the transformation theory are presented in Part III.

Chapter 2

Transformation Theory

2.1 Introduction

The origins of Sturm-Liouville theory can be traced back to a series of publications concerning mathematical analysis produced by Sturm and Liouville in 1836-37. The standard general Sturm-Liouville boundary value problem is given by the general second order linear differential equation

$$L(y) = -[p(x)y'(x)]' + q(x)y(x) = \lambda r(x)y(x), \qquad x \in [a, b],$$

with the boundary conditions

$$y(a)\cos\alpha - p(a)y'(a)\sin\alpha = 0,$$
$$y(b)\cos\beta - p(a)y'(b)\sin\beta = 0,$$

and $q(x), p(x), r(x) \in \mathcal{L}^1(a, b)$. α and β are given constants and λ is a parameter. Only those solutions that are non-trivial for certain values of λ are allowed. These values of λ are determined by the characteristic equation $h(\lambda) = 0$ and are known as *eigenvalues*. The corresponding non-trivial solutions are called *eigenfunctions*.

Sturm and Liouville investigated three aspects of eigenvalue problems, the first being the properties of the eigenvalues. Secondly, they looked at the qualitative description of eigenfunction behaviour and, lastly, the expansions of arbitrary functions as infinite series of eigenfunctions. In 1824 and 1835, Cauchy formulated and proved an existence theorem for eigenfunctions. Sturm-Liouville theory was the first to consider the qualitative nature of differential equations for which no tractable explicit solutions could be found. For further details on the history of Sturm-Liouville theory the interested reader is referred to [30].

2.2 The Theory of Ordinary Differential Equations

From Ince [24] concerning the theory of ordinary differential equations comes the following important properties of linear differential equations.

Let $L = p_0 D^n + p_1 D^{n-1} + \cdots + p_{n-1} D + p^n$ denote a linear differential operator of order n and suppose that $u_1, u_2, \ldots, u_n$ are n solutions to the equation, [24, p.144],

$$L(u) = 0.$$

We define the Wronskian of the functions $u_1, u_2, \ldots, u_n$ to be the determinant, [24, pg116],

$$W(u_1, u_2, \ldots, u_n) \equiv \begin{vmatrix} u_1, & u_2, & \ldots & u_n \\ u_1', & u_2', & \ldots & u_n' \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}, & u_2^{(n-1)}, & \ldots & u_n^{(n-1)} \end{vmatrix}.$$

For solutions of L it is necessary and sufficient to show that the Wronskian does not vanish to prove that the functions $u_1, u_2, \ldots, u_n$ are linearly independent.

A *fundamental set* of solutions is a maximal linearly independent set of solutions, [24, p.119], in particular a fundamental set of solutions to the equation $L(u) = 0$ consists of n functions.

If the operator L is factorised into a composition of first order operators then, in general, when the factors of the differential operator are permuted, the operator is not preserved [24, p.121]. In particular, for functions α_i with $i \neq j$, we have that

$$(D - \alpha_i)(D - \alpha_j) \neq (D - \alpha_j)(D - \alpha_i).$$

It is also worth pointing out that the order of an equation may be reduced if a solution of the equation is known. More specifically, if the order of the equation $L(u) = 0$ is n but r independent solutions to this equation are known, then it is possible for the equation to have its order reduced to $n - r$, as stated in [24, p.121].

A brief discussion is provided in [24, p.122] regarding the solution of a non-homogeneous equation of the form

$$L(y) = r(x). \tag{2.2.1}$$

Suppose that the reduced equation

$$L(u) = 0 \tag{2.2.2}$$

has a known fundamental set of solutions given by

$$u_1, u_2, \ldots, u_n,$$

so that (2.2.2) is solved by the general solution

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n,$$

with $c_1, c_2, \ldots, c_n$ representing arbitrary constants. A general solution to (2.2.1) is found by the *variation of parameters* method. Suppose $V_1, V_2, \ldots, V_n$ are functions in x satisfying (2.2.1) such that

$$y = V_1 u_1 + V_2 u_2 + \cdots + V_n u_n.$$

Finding explicit representations for the functions $V_1, V_2, \ldots, V_n$ becomes the primary objective in solving the problem (2.2.1). In the particular case in which (2.2.1) is of degree two, the functions V_1 and V_2 are given by

$$V_1 = \int \frac{u_2(x)}{W(u_1, u_2)} r(x)\, dx \qquad V_2 = \int \frac{u_1(x)}{W(u_1, u_2)} r(x)\, dx.$$

Suppose we have the formal differential operators, [24, p.123],

$$L(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \cdots + p_{n-1}\frac{du}{dx} + p_n u,$$

and

$$L^{\dagger}(v) = (-1)^n \frac{d^n(p_0 v)}{dx^n} + (-1)^{n-1}\frac{d^{n-1}(p_1 v)}{dx^{n-1}} + \cdots - \frac{d(p_{n-1}v)}{dx} + p_n v.$$

The equation $L^{\dagger}(v) = 0$ is called the *formal adjoint equation* to $L(u) = 0$, [24, p.124], so that we now have the *Lagrange identity*

$$vL(u) - uL^{\dagger}(v) = \frac{d}{dx}\{P(u,v)\} \tag{2.2.3}$$

where the *bilinear concomitant* $P(u,v)$ is linear and homogeneous in u and v and their first $n-1$ derivatives, respectively. It is necessary and sufficient for v to satisfy $L^{\dagger}(v) = 0$ (i.e. the adjoint equation) in order for v to be an integrating factor for $L(u)$.

The formal operator L is said to be *formally self-adjoint* if $L = L^{\dagger}$, [24, p.125], and in this case we say that a formally self-adjoint differential equation $L(u) = 0$ of even order $2m$ may be expressed in a factorised form as

$$\frac{d}{v_1 dx}.\frac{d}{v_2 dx}.\cdots.\frac{d}{v_m dx}.\frac{d}{v_{m+1}dx}.\frac{d}{v_m dx}.\cdots.\frac{d}{v_2 dx}.\frac{u}{v_1} = 0$$

while, for an equation of odd order $2m+1$ the factorisation, [24, p.126], is

$$\frac{d}{v_1 dx}.\frac{d}{v_2 dx}.\cdots.\frac{d}{v_m dx}.\frac{d}{v_m dx}.\cdots.\frac{d}{v_2 dx}.\frac{u}{v_1} = 0.$$

It is true that any differential equation of the form

$$y'' + 2py' + qy = 0 \tag{2.2.4}$$

can be expressed, as shown in [24, p.128], as the product of two factors

$$(D + a_2(x))(D + a_1(x))y = 0. \tag{2.2.5}$$

Performing the operations above and then equating coefficients give that a_1 and a_2 can be solved by the equations

$$a_1(x) + a_2(x) = 2p \qquad a_1(x)a_2(x) + a_1'(x) = q.$$

Furthermore, we have that the general solution of the equation $(D+a_1(x))y = 0$ satisfies (2.2.4). This is, however, not necessarily true for the solutions of $(D+a_2(x))y = 0$. If, and only if, the two factors in (2.2.5) are *commutative*, will the general solutions to both factorisation equations satisfy (2.2.4). The factors commute under condition that

$$a_2'(x) = a_1'(x) \quad \text{gives} \quad a_2(x) = a_1(x) + A$$

for some arbitrary constant A. This condition is both necessary and sufficient.

The aforementioned may be extended to the general case. For, supposing P is an operator of order n and Q an operator of order m, then to show that P and Q permute, [24, p.129] suggests that it is sufficient to show that

$$P = \{D + a(x) + A_1\} \ldots \{D + a(x) + A_m\},$$
$$Q = \{D + a(x) + A_{m+1}\} \ldots \{D + a(x) + A_{m+n}\}. \tag{2.2.6}$$

However, a problem arises in that two operators may not be expressible in the product-form stated above, even though they are permutable. Thus, [24] suggests the following condition for permutability of two linear differential operators P and Q, of orders n and m respectively. If, for some arbitrary constant h, it is possible to permute P and Q, then the following relation holds

$$(P - h)Q = Q(P - h).$$

From this condition, we have that if a fundamental set of solutions to $P(y) - hy = 0$ is given by $y_1, y_2, \ldots, y_m$, then [24, p.129] suggests that the same differential equation is satisfied by the set of solutions $Q(y_1), Q(y_2), \ldots, Q(y_n)$. A constant k may be determined from the determinant of the matrix of coefficients generated by the relations between the two sets of solutions, so that $Q(Y) = kY$. After some working, [24, p.131] states that the equation of order mn given by

$$L(y) = F(P, Q)y = 0$$

is satisfied by y, where $F(P, Q) = 0$ is an algebraic expression relating P and Q.

Subsequently, we have a fundamental theorem, as stated in [24, p.131], regarding the permutability of differential operators.

Theorem 2.2.1. *If P and Q are permutable operators of orders n and m respectively, then they identically satisfy an algebraic relation of the form*

$$F(P, Q) = 0$$

of degree n in P and m in Q.

Infeld and Hull [25] achieve factorisations of many widely used and well-known differential equations via transformations, though they do not look at what happens to the boundary values of the transformed equations.

2.3 Preliminary Results

2.3.1 Sturm-Liouville Theory

As an introductory example, Coddington and Levinson [14, pp.186-187] consider the second order linear differential equation

$$Ly = -y'' = \lambda y \qquad y(0) = y(1) = 0 \tag{2.3.7}$$

on the interval $[0, 1]$. In this example, y is scalar-valued and parameter λ is complex. Solutions are $y = c\sin(\sqrt{\lambda}x)$ suggesting that a non-trivial solution (i.e. not identically zero) to the problem exists if, and only if, $\sin(\sqrt{\lambda}) = 0$, which gives $\lambda = \pi^2 k^2$ for $k = 1, 2, \ldots$. The eigenvalues λ have corresponding eigenfunctions given by

$$\phi_k(x) = \sqrt{2}\sin(k\pi x), \quad k = 1, 2, \ldots. \tag{2.3.8}$$

Orthogonality holds in this example so that $\int_0^1 \phi_j \phi_k dx = \delta_{jk}$. The function $\phi_k(x)$ is a representation of the Fourier sine series and, as it happens, a large number of functions can be written as a series in ϕ_k.

A sequence of functions satisfying $\int_0^1 \phi_j \bar{\phi}_k dx = \delta_{jk}$, where $\bar{\phi}$ denotes complex conjugate, is termed an *orthonormal* sequence. For example

$$Ly = iy' = \lambda y \qquad y(0) - y(1) = 0, \tag{2.3.9}$$

has eigenvalues $\lambda = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, and eigenfunctions $\phi_k(x) = e^{-2\phi i k x}$.

From [14, p.197] we source an expansion and a completeness theorem. These results are used in Crum's proof of completeness, as is demonstrated later.

Theorem 2.3.1. *Let $f \in \mathcal{C}^n$ on $[a, b]$ and satisfy boundary conditions $U(f) = 0$ with eigenfunctions $\phi_k(x)$ as in* (2.3.8)*. Then on $[a, b]$*

$$f = \sum_{k=0}^{\infty} \langle f, \phi_k \rangle \phi_k \tag{2.3.10}$$

where the series converges uniformly on $[a, b]$.

Corollary 2.3.2. *If f is as in Theorem 2.3.1, then we have* Parseval's equality

$$\|f\|^2 = \sum_{k=0}^{\infty} |\langle f, \phi_k \rangle|^2,$$

also known as the completeness relation.

The interval (a, b) is often considered finite. However, working on infinite intervals gives rise to singular cases where singular behaviour is exhibited by the differential operator coefficients at either of the boundary points a and b. All of the problems we will be looking at are singular.

If functions p, p', q are assumed to be continuous and real on some real x interval, with $p(x) > 0$, then a formally self-adjoint differential operator, denoted by L, can be defined (see [14, p.224]) as

$$Ly = -(py')' + qy.$$

Consider the interval $[x_1, x_2]$ in which L is defined. Let f and g be any two functions such that Lf and Lg exists, then, as L is self-adjoint, the *Green's formula* is given by

$$\int_{x_1}^{x_2} (\bar{g}Lf - f\overline{Lg})dx = [f, \bar{g}](x_2) - [f, \bar{g}](x_1) \tag{2.3.11}$$

where

$$[f, \bar{g}](x) = p(x)(f(x)\bar{g}'(x) - f'(x)\bar{g}(x)).$$

For $\lambda \in \mathbb{C}$, applying the Green's formula to function f and the complex conjugate of g, in the case where f and g both solve equation $Ly = \lambda y$, results in the Wronskian $W = [f, \bar{g}](x)$ having a value independent of x, and as such, W is constant. Hence, writing $W = [f, \bar{g}]$ only is permissible.

In [14, p.224], it is stated that, when working in the interval $[a, b)$, $-\infty < a < b \leq \infty$, where singular behaviour is exhibited by coefficients of operator L at boundary point b, one may use the results obtained from the case in which operator L is considered on $[0, \infty)$. Similarly, results for the case in $(-\infty, \infty)$ are valid when dealing with a situation where the coefficients of L behave singularly at both a and b.

2.3.2 Results from Functional Analysis

Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{D}(A) \subset \mathcal{H} \mapsto X$ be a densely defined operator in $\mathcal{H}$. We say that y is in the domain of the adjoint of A, A^*, if there is a $z \in \mathcal{H}$ so that

$$\langle Ax, y \rangle = \langle x, z \rangle$$

for all $x \in \mathcal{D}(A)$. In this case $A^*y := z$. If $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $Ax = A^*x \quad \forall x \in \mathcal{D}(A) = \mathcal{D}(A^*)$, we say that A is *self-adjoint*.

We refer to [18, p.4] for an explanation on the spectral decomposition of an operator $A : \mathcal{D}(A) \subset \mathcal{H} \mapsto \mathcal{H}$, where $\mathcal{D}(A)$ is dense in $\mathcal{H}$. The *resolvent set* of A is denoted by $\rho(A)$ and defined as the set of all $\lambda \in \mathbb{C}$ such that the operator $(A - \lambda I)^{-1} : \mathcal{H} \mapsto \mathcal{H}$ exists and is a bounded operator on $\mathcal{H}$. The *spectrum* of A, denoted $\sigma(A)$, is formed by all $\lambda \in \mathbb{C} \setminus \rho(A)$ and is composed of three subsets.

(i) The *point spectrum* $\sigma_p(A)$ is the set consisting of all those $\lambda \in \sigma(A)$ such that $(A - \lambda I)$ is not injective. The eigenvalues of A are the elements of this set.

(ii) The *continuous spectrum* $\sigma_c(A)$ is the set $\lambda \in \sigma(A)$ for which $(A - \lambda I)$ is injective and $(A - \lambda I)\mathcal{D}(A)$ is dense in $\mathcal{H}$, though not equal to $\mathcal{H}$.

(iii) The *residual spectrum* $\sigma_r(A)$ is formed by those $\lambda \in \sigma(A)$ for which $(A - \lambda I)$ is injective but $(A - \lambda I)\mathcal{D}(A)$ is not dense in $\mathcal{H}$.

The three subsets of $\sigma(A)$ given above are disjoint and, in general, their union does not specify the whole of $\sigma(A)$. It is possible that for $\lambda \in \sigma(A)$ we have that $(A - \lambda I)^{-1}$ exists as an unbounded operator with domain $\mathcal{H}$. However, this cannot happen when A is a closed operator. Here an operator $A : \mathcal{D}(A) \subset \mathcal{H} \mapsto \mathcal{H}$ is closed if the graph $\mathcal{G}(A) := \{(x, Ax) | x \in \mathcal{D}(A)\}$ is a closed set in $\mathcal{H} \times \mathcal{H}$. We note that every self-adjoint operator in a Hilbert space is closed.

If A is self-adjoint with y_1 and y_2 eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2 \in \sigma_p$ for $\lambda_1 \neq \lambda_2$, then y_1 is orthogonal to y_2.

Definition 2.3.3. [18, p.98] *A subspace $\mathcal{E}$ of $\mathcal{H}$ is said to be a* core *of a closed linear operator $A : \mathcal{D}(A) \subset \mathcal{H}$ if $\mathcal{E}$ is dense in $\mathcal{D}(A)$. Equivalently $\mathcal{E}$ is a core if the restriction of $\mathcal{A}$ to $\mathcal{E}$ has closure A.*

Suppose A is a densely defined operator in a Hilbert space $\mathcal{H}$ [31, 219]. Then A is termed *symmetric* if $A = A^*$ or if, and only if,

$$\langle Ax, y \rangle = \langle x, Ay \rangle \qquad \forall x, y \in \mathcal{D}(A).$$

In particular,

$$\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle} \qquad \forall x \in \mathcal{D}(A)$$

for each symmetric $A \in \mathcal{H}$. Also, $\langle Ax, x\rangle \in \mathbb{R}$ for each $x \in \mathcal{D}(A)$. Symmetric operator A is closable and its closure $\overline{A}$ is symmetric in $\mathcal{H}$ as well.

Lemma 2.3.4. *For each symmetric, closed operator $A \in \mathcal{H}$ and each $z \in \mathbb{C} \setminus \mathbb{R}$, $zI - A$ is injective and $R(zI - A)$ is closed.*

Theorem 2.3.5. *Let $\mathcal{H}$ be a Hilbert space, and suppose that $L : \mathcal{H} \mapsto \mathcal{H}$ is a closed linear operator with $\mathcal{D}(L) \in \mathcal{H}$. Then L is bounded.*

Definition 2.3.6. *The set $\{f_\alpha\}_{\alpha\in\Lambda}$ is complete in a Hilbert space $\mathcal{H}$ if $x \perp f_\alpha \quad \forall\alpha \in \Lambda$ implies $x = 0$.*

A sequence $\{x_i\}_{i=1}^{\infty}$ is a basis for the Hilbert space $\mathcal{H}$ if, for each $x \in \mathcal{H}$, there exists a unique sequence (a_i) so that $\sum_{i\geq 1} a_i x_i$ converges to x. Furthermore, in a Hilbert space $\mathcal{H}$, each complete orthonormal system $\{e_i\}_{i=1}^{\infty}$ is a basis.

2.3.3 Limit-point and Limit-circle Classification

Consider the second-order linear differential equation

$$Ly = -(py')' + qy \tag{2.3.12}$$

where $p > 0$ and both functions p and q are real-valued on the interval $[0, \infty)$. Operators for problems such as (2.3.12) are classified as being either in the limit-circle, where L is such that all of the solutions to the equation $Ly = 0$ are in $\mathcal{L}^2(0, \infty)$, or in the limit-point, which includes all other possible expressions for L. When in the limit-point case, for any non-real λ, there exists precisely one linearly independent solution in $\mathcal{L}^2(0, \infty)$ to the differential equation $Ly = \lambda y$.

A formal statement of these notions is given in [14, p.225], where the case on $[0, \infty)$ is explained and it reads as follows.

Definition 2.3.7. *Operator L is of the* limit-circle *type at infinity, if every solution ϕ of the differential equation*

$$Ly = \lambda_0 y$$

is square integrable on $(0, \infty)$ for a specific $\lambda_0 \in \mathbb{C}$. Else, L is of the limit-point *type at infinity.*

The particular λ_0 selected does not determine to which category L belongs, as the following theorem, taken from [14, p.225], shows.

Theorem 2.3.8. *If every solution of $Ly = \lambda_0 y$ is of class $\mathcal{L}^2(0, \infty)$ for some complex number λ_0, then, for arbitrary complex λ, every solution of $Ly = \lambda y$ is of class $\mathcal{L}^2(0, \infty)$.*

No more than one of the linearly independent solutions to $Ly = \lambda y$ is square integrable on $(0, \infty)$ in the limit-point case [14, p.226]. The proof of the theorem given in [14, p.228] demonstrates that there is precisely one such solution, regardless of the choice of λ. However, it is required that $\Im(\lambda)$, that is the imaginary part of λ, be non-zero.

2.3.4 Levinson's Limit-Point Criterion

From [20] we have *Levinson's limit-point criterion*, which provides a means for classifying operators as being either in the limit-circle or in the limit-point. If a function Q can be found, which is positive and differentiable, with constants k_1 and k_2 both positive, such that for some positive number a the following conditions hold

$$q(t) \geq -k_1 Q(t), \qquad p(t) Q'^2(t) Q^{-3}(t) \leq k_2, \qquad t \geq a,$$
$$\int_0^\infty \{p(t)Q(t)\}^{-1/2} dt = \infty,$$

then the operator L, as given in (2.3.12), is in the limit-point case. In particular, (2.3.12) is in the limit-point case if

$$p(t) = 1, \qquad q(t) \geq -k_1 t^\alpha, \qquad k_1 > 0, \quad t \geq a, \quad \alpha = 2.$$

Here $\alpha = 2$ is the maximal value of α for which Levinson's condition is satisfied with $Q(t) = t^\alpha$. Classification criteria other than the one given here can be found in the literature.

2.3.5 Sturm-type Theorems for Orthogonal Polynomials

In [35, pp.19-21], Szegö discusses the interactions between and properties of the orthogonal polynomial solutions of Sturm-Liouville-type problems. These results will be stated formally and proved in Section 5.2, though it may be worthwhile to high-light here some of the main concepts.

The first result is concerning the interlacing property. Consider two Sturm-Liouville differential equations on (a, b), given by

$$y'' + q(x)y = 0 \qquad \text{and} \qquad Y'' + Q(x)Y = 0.$$

Let these be satisfied by functions $y(x)$ and $Y(x)$ respectively, neither of which vanish identically. For functions $q(x)$ and $Q(X)$ continuous such that $q(x) \leq Q(x)$, the function $Y(x)$ has at least one zero in between two consecutive zeros, x_1 and x_2, of $y(x)$, conditional on $q(x) \neq Q(x)$ in $[x_1, x_2]$. This notion also remains valid at the boundary points when $x_1 = a$ or $x_2 = b$ if it is true that

$$\lim_{x \to x_0 + 0} [y'(x)Y(x) - y(x)Y'(x)] = 0. \tag{2.3.13}$$

Furthermore, the Wronskian $W = y'(x)Y(x) - y(x)Y'(x)$ is increasing in the interval $[x_1, \xi]$, where ξ is the first zero of $Y(x)$ in $[x_1, x_2]$. Also, W vanishes at $x = x_1$ if $Y(x_1) = 0$ and is thus positive in (x_1, ξ), whence we have that $y(x)/Y(x)$ is an increasing function here. If condition (2.3.13) is met then the statement is also valid for $x_1 = a$.

Chapter 3

The Darboux-Crum Transformation

3.1 Crum's Result Introduced

The Darboux-Crum transformation, as presented by Crum in [15], is derived by considering a regular Sturm-Liouville problem

$$-y'' + qy = \lambda y, \tag{3.1.1}$$

with $q(x)$ a potential function and $0 < x < 1$. The boundary conditions are given by

$$y'(0) = h_{(0)}y(0), \qquad y'(1) = h_{(1)}y(1).$$

For $s, n \in \mathbb{N}$, the set of eigenvalues is denoted λ_s, with ordering $\lambda_0 < \lambda_1 < \ldots$ and associated eigenfunctions ϕ_s. As a consequence of letting $q(x)$ be repeatedly differentiable on the interval $(0, 1)$, we have that the ϕ_s are repeatedly differentiable as well. For $n \geq 1$, in [15] the transformation function

$$\phi_{ns} = \frac{W_{ns}}{W_n}, \tag{3.1.2}$$

where W_{ns}, with $s \geq n$, represents the Wronskian (defined in Section 2.2) of the set $\phi_0, \phi_1, \ldots, \phi_{n-1}, \phi_s$ of $n + 1$ functions and W_n the Wronskian of the set

$\phi_0, \phi_1, \ldots, \phi_{n-1}$ of n functions, is defined. Hence, the transformation (3.1.2) transforms the original Sturm-Liouville equation (3.1.1) into

$$-y'' + q_n(x)y = \lambda y \qquad 0 < x < 1 \tag{3.1.3}$$

with (in general) singular boundary conditions of the form

$$\lim_{x\to 0} y(x) = 0, \qquad \lim_{x\to 1} y(x) = 0. \tag{3.1.4}$$

The potential is transformed into

$$q_n(x) = q(x) - 2\frac{d^2}{dx^2}\ln W_n$$

and ϕ_{ns} is an eigenfunction corresponding to the eigenvalue λ_s. The transformed problem is only regular for $n = 1$, but not for $n > 1$, where instead

$$q_n(x) \approx \begin{cases} n(n-1)x^{-2} & \text{if} \quad x \to 0, \\ n(n-1)(1-x)^{-2} & \text{if} \quad x \to 1. \end{cases}$$

Furthermore, for $s < n$, ϕ_{ns} is identically zero but when $s > n$, it has $s - n$ zeros on the interval. The Wronskian W_n is non-zero on the interval and q_n is a continuous function. Lastly, Crum [15] points out that the family of functions ϕ_{ns} for $s \geq n$, is $\mathcal{L}^2$-closed and complete over $(0, 1)$, as will be demonstrated later.

3.1.1 Transformation via Commutation

The reader is now referred back to the discussion provided after equation (2.2.5). Note that the Darboux-Crum transformation is both a factor of the original equation and a solution to the new transformed equation. This can be seen by considering the simplest case, i.e. the first transformation. Consider the linear second-order differential equation

$$y'' + [q(x) - \lambda]y = 0, \tag{3.1.5}$$

which has a factorisation expressed as

$$(-D - u(x))(D - v(x))y = \lambda y.$$

Setting

$$\tilde{y} = (D - v(x))y = y' - v(x)y, \tag{3.1.6}$$

(3.1.5) becomes

$$(-D - u(x))\tilde{y} = \lambda y.$$

When acted upon by $(D - v)$, we have

$$(D - v)(-D - u)\tilde{y} = \lambda(D - v)y = \lambda\tilde{y}.$$

So, $\tilde{y}$ satisfies an equation of the same form as (3.1.5) but the factors have been permuted. The equation can be transformed back into its original form by using the other factor to generate a transformation equation.

3.1.2 Commutation via Transformation

Suppose that z is an eigenfunction (3.1.5), with Dirichlet boundary conditions, corresponding to the least eigenvalue, which we have assumed to be $\lambda = 0$, i.e. $z''+qz = 0$. The Darboux-Crum transformation of a solution y to (3.1.5) as in (1.0.2) can be differentiated to produce, after some substitutions and manipulation,

$$\tilde{y}' = \lambda y - \frac{z'}{z}\tilde{y}.$$

The second derivative is then

$$-\tilde{y}'' + \tilde{q}\tilde{y} = -\lambda\tilde{y},$$

which is a differential equation in the original form but with q replaced by $\tilde{q}$, where

$$\tilde{q} = 2\left(\frac{z'}{z}\right)^2 - q.$$

In other words, the equation factorises into

$$-y'' + qy = \left(-D - \frac{z'}{z}\right)\left(D - \frac{z'}{z}\right)y = \lambda y.$$

Clearly, permuting the factors on the right-hand side allows for transformation between equations, as motivated by [17].

3.1.3 A More General Differential Equation

The Darboux-Crum transformation can be modified slightly to allow for the transformation of weighted problems. Consider an equation of the form

$$-(p(x)y')' + q(x)y = r(x)\lambda y. \tag{3.1.7}$$

The Darboux-Crum transformation for this equation is given by

$$\tilde{y} = py' - \frac{pz'}{z}y, \tag{3.1.8}$$

where z is a solution of (3.1.7) with no zeros and $\lambda = \mu$. Differentiating the $\tilde{y}$ produces, after some manipulation

$$p\tilde{y}' = pr(\mu - \lambda)y - \frac{pz'}{z}\tilde{y}.$$

Then dividing through by pr gives

$$\frac{1}{r}\tilde{y}' = (\mu - \lambda)y - \frac{pz'}{prz}\tilde{y}.$$

Differentiating the above equation, we find that

$$\left(\frac{1}{r}\tilde{y}'\right)' = (\mu - \lambda)\frac{\tilde{y}}{p} + \left[\frac{\mu}{p} - \frac{q}{pr} + \left(\frac{z'}{z}\right)^2 \frac{1}{r} + \frac{pz'}{(prz)^2}(prz)'\right]\tilde{y}.$$

Thus, the transformed equation is of the form

$$-\left(\frac{1}{r}\tilde{y}'\right)' + \tilde{q}(x)\tilde{y} = \lambda\frac{1}{p}\tilde{y},$$

where

$$\tilde{q} = \frac{\mu}{p} - \frac{q}{pr} + \left(\frac{z'}{z}\right)^2 \frac{1}{r} + \frac{pz'}{(prz)^2}(prz)',$$

and

$$\tilde{\lambda} = \lambda - \mu.$$

3.2 The Derivation of Crum's Results

3.2.1 A First Application of the Darboux-Crum Transformation

The first transformation corresponds to the case $n = 1$ in (3.1.3), where the Wronskian corresponding to the lowest eigenvalue is $W_1 = \phi_0$ and does not change sign in the interval $0 \leq x \leq 1$. Then the first transformation is given by

$$\phi_{1s} := \phi_s' - \frac{\phi_0'}{\phi_0}\phi_s = \phi_s' - v\phi_s \tag{3.2.9}$$

where

$$v := \frac{\phi_0'}{\phi_0}.$$

Choosing the zero-th eigenfunction ensures that this transformation does not behave singularly at any point inside of the interval under consideration. Equation (3.2.9) is the equivalent of (3.1.6). Manipulation of the equation $\phi_0'' + (\lambda - q)\phi_0 = 0$ leads to

$$v' + v^2 = q - \lambda. \tag{3.2.10}$$

Later, when calculating the first derivative of ϕ_{1s}, it is useful to know that

$$\begin{aligned} \frac{d}{dx}(\phi_0\phi_{1s}) &= \frac{d}{dx}(\phi_0\phi_s' - \phi_0'\phi_s) \\ &= \phi_0\phi_s'' - \phi_0''\phi_s \\ &= (\lambda_0 - \lambda_s)\phi_0\phi_s \end{aligned} \tag{3.2.11}$$

where the last step is obtained by referring back to equation (1.0.1). From the boundary conditions,

$$\phi_{1s}(0) = 0 = \phi_{1s}(1) \tag{3.2.12}$$

we have

$$\begin{aligned} \phi_0\phi_{1s} &= (\lambda_0 - \lambda_s)\int_0^x \phi_0(\xi)\phi_s(\xi)d\xi \\ &= -(\lambda_0 - \lambda_s)\int_x^1 \phi_0\phi_s d\xi \end{aligned} \tag{3.2.13}$$

which also implies the eigenfunction orthogonality relation

$$\int_0^1 \phi_0\phi_s d\xi = 0 \qquad \forall s \neq 0.$$

Therefore, by putting (3.2.11) and (3.2.13) together, the first derivative of ϕ_{1s} is

$$\phi'_{1s} = (\lambda_0 - \lambda_s)\phi_s - v\phi_{1s}. \tag{3.2.14}$$

Differentiating again yields

$$\begin{aligned}\phi''_{1s} &= (\lambda_0 - \lambda_s)\phi'_s - v'\phi_{1s} - v[(\lambda_0 - \lambda_s)\phi_s - v\phi_{1s}] \\ &= (\lambda_0 - \lambda_s - v' + v^2)\phi_{1s} \\ &= (q_1 - \lambda_s)\phi_{1s}\end{aligned} \tag{3.2.15}$$

where

$$q_1 = \tilde{q} = \lambda_0 - v' + v^2 = q - 2v' = q - 2\frac{d^2}{dx^2}(\ln W_1). \tag{3.2.16}$$

From (3.2.9) it may be deduced that

$$\frac{\phi_{1s}}{\phi_0} = \frac{1}{\phi_0}\left(\phi'_s - \frac{\phi'_0}{\phi_0}\phi_s\right) = \frac{\phi'_s\phi_0 - \phi_s\phi'_0}{\phi_0^2} = \frac{d}{dx}\left(\frac{\phi_s}{\phi_0}\right)$$

and since, by Sturm's oscillation theorem, ϕ_s has precisely s zeros inside $(0,1)$, it can be inferred that ϕ_{1s} has $s-1$ or more zeros in the same interval. However, equations (3.2.11) and (3.2.12), gives that $\frac{d}{dx}(\phi_{1s}\phi_0)$ has exactly s zeros in $(0,1)$. But $\phi_{1s}\phi_0$ has at least $s-1$ zeros and thus has exactly $s-1$ zeros in $(0,1)$.

For $s \geq 1$, it follows that the ϕ_{1s} give all the eigenfunctions corresponding to eigenvalues λ_s, for the regular system (transformed problem)

$$y'' + [\lambda - \tilde{q}(x)]y = 0 \qquad 0 < x < 1 \tag{3.2.17}$$

$$\lim_{x\to 0} y(x) = 0, \qquad \lim_{x\to 1} y(x) = 0. \tag{3.2.18}$$

The general solution for (3.2.17), when $\lambda \neq \lambda_0$ (the least eigenvalue is removed), is given by

$$X_1 = \frac{W(\phi_0, \chi)}{W_1},$$

where χ is the general solution of the original differential equation (1.0.1). In the case where $\lambda = \lambda_0$, we have that $W(\phi_0, \chi)$ is constant and (3.2.17) has solution given by

$$X_1 = \frac{1}{W_1} = \frac{1}{\phi_0}, \tag{3.2.19}$$

with two independent solutions of (3.2.19) given by

$$\frac{1}{\phi_0}\int_0^x \phi_0^2(\xi)d\xi, \qquad \frac{1}{\phi_0}\int_x^1 \phi_0^2(\xi)d\xi. \tag{3.2.20}$$

Note that any two of the solutions in (3.2.19) and (3.2.20) form an independent pair of solutions, that can be verified by substituting into (1.0.1). Finally, the fact that the ϕ_{1s} ($s \geq 1$) are the only solutions of (3.2.17) that satisfy (3.2.18) follows from ϕ_{1s} being a fundamental set of solutions as shown in Subsection 3.2.2.

3.2.2 Sequential Transformations

For $n > 1$ (that is, a second transformation followed by a third and so on) the determinant W_{ns} is considered, where $s \geq n$. Jacobi's theorem is applied to this determinant to generate

$$W_{ns}W_{n-1} = W_n\frac{d}{dx}W_{n-1,s} - W_{n-1,s}\frac{d}{dx}W_n.$$

So

$$\begin{aligned}\phi_{ns} = \frac{W_{ns}}{W_n} &= \frac{1}{W_{n-1}}\frac{d}{dx}(W_{n-1}\phi_{n-1,s}) - \phi_{n-1,s}\frac{1}{W_n}\frac{d}{dx}W_n \\ &= \phi'_{n-1,s} - v_{n-1}\phi_{n-1,s} \\ &= \frac{1}{\phi_{n-1,n-1}}W(\phi_{n-1,n-1}, \phi_{n-1,s}) \end{aligned} \tag{3.2.21}$$

where

$$v_n = \frac{\phi'_{nn}}{\phi_{nn}}, \qquad v_{n-1} = \frac{W'_n}{W_n} - \frac{W'_{n-1}}{W_{n-1}}. \tag{3.2.22}$$

Therefore, by induction on n and by conducting a procedure much like the one in section (3.2.1), the following relations may be derived:

$$v'_n + v_n^2 = q_n - \lambda_n, \tag{3.2.23}$$

$$\frac{d}{dx}(\phi_{n-1,n-1}\phi_{ns}) = (\lambda_{n-1} - \lambda_s)\phi_{n-1,n-1}\phi_{n-1,s}, \tag{3.2.24}$$

$$\phi''_{ns} = (q_n - \lambda_s)\phi_{ns}, \qquad q_n = q_{n-1} - 2v'_{n-1},$$

$$q_n + 2\frac{d}{dx}\left(\frac{W'_n}{W_n}\right) = q_{n-1} + 2\frac{d}{dx}\left(\frac{W'_{n-1}}{W_{n-1}}\right) = q. \tag{3.2.25}$$

It will now be shown, by induction on n, that the following relations hold:

$$\phi_{ns} = C_{ns}\prod_{t=0}^{n-1}(\lambda_t - \lambda_s)x^n[1 + O(x^2)] \qquad (C_{ns} \neq 0), \tag{3.2.26}$$

$$\phi'_{ns} = nx^{-1}\phi_{ns}[1 + O(x^2)], \tag{3.2.27}$$

$$v_n = nx^{-1}[1 + O(x^2)], \tag{3.2.28}$$

all as $x \to 0$, with similar relations as $x \to 1$. In addition, it must be shown that

$$\phi_{ns} \quad \text{has} \quad s - n \quad \text{zeros inside} \quad (0, 1). \tag{3.2.29}$$

Now, assuming that the preceding statement is true for n, it follows that ϕ_{nn} is non-zero in $(0, 1)$, in which case W_{n+1} is non-zero inside $(0, 1)$ as well. This is shown by recalling (3.2.22) and considering

$$\phi_{nn} = \phi'_{nn}\frac{W_{n+1}W_n}{W'_{n+1}W_n - W'_nW_{n+1}}.$$

Consequently, q_{n+1} and $\phi_{n+1,s}$ are continuous in the interval by (3.2.25) and (3.2.21) respectively.

As is shown in [15], any of the eigenfunctions from the fundamental set of solutions may be used in setting up the transformation relation, though only the use of the eigenfunction ϕ_0 corresponding to the least eigenvalue λ_0 yields a regular transformed problem.

Proof. We begin by showing that the equations (3.2.26) through (3.2.29) hold for the case $n = 1$. Now, expanding ϕ_{1s} by Taylor's Theorem yields

$$\phi_{1s}(x) = \phi_{1s}(0) + \phi'_{1s}(0)x + \frac{1}{2!}\phi''_{1s}(0)x^2 + \cdots + \frac{1}{n!}\phi^{(n)}_{1s}(0)x^n + R_{n+1}(x),$$

since ϕ_{1s} has $n+1$ continuous derivatives on the interval $(0, x)$. The remainder $R_{n+1}(x)$ is given by

$$R_{n+1}(x) = \frac{1}{n!}\int_0^x \phi_{1s}^{(n+1)}(t)(x-t)^n dt,$$

which is estimated by

$$|R_{n+1}(x)| \leq \left(\max_{t\in(0,1)} |\phi_{1s}^{(n+1)}(t)|\right) \frac{|x|^{n+1}}{(n+1)!}.$$

Recalling boundary condition (3.2.12) and the first derivative of ϕ_{1s} in (3.2.14), leads to

$$\phi_{1s}'(0) = (\lambda_0 - \lambda_s)\phi_s(0)$$

which satisfies the aforementioned conditions for $x \to 0$. In addition, by (3.2.24) and again (3.2.12)

$$\phi_{1s}''(0) = (q_1 - \lambda_s)\phi_{1s}(0) = 0,$$

which, together with the preceding approximation, implies (3.2.26) for $n = 1$ as

$$\phi_{1s} = C_{1s}(\lambda_0 - \lambda_s)x[1 + O(x^2)] \qquad (C_{1s} \neq 0). \tag{3.2.30}$$

Now, keeping in mind that ϕ_s satisfies the boundary condition $y'(0) = h_{(0)}y(0)$ and expanding ϕ_s by Taylor's Theorem, we arrive at

$$\phi_s = \phi_s(0)[1 + h_{(0)}x + O(x^2)].$$

This, together with (3.2.30) and

$$\frac{d}{dx}(\phi_0\phi_{1s}) = (\lambda_0 - \lambda_s)\phi_0\phi_s,$$

from (3.2.11), results in

$$\phi_{1s}' = x^{-1}\phi_{1s}[1 + O(x^2)],$$

which is the analogue of (3.2.27) for $n = 1$. This, in turn, yields

$$v_1 = x^{-1}[1 + O(x^2)]$$

as in (3.2.28).

We now proceed with proving (3.2.26)-(3.2.29) for the case $n+1$. First, assume (3.2.26)-(3.2.29) true for the case n. From (3.2.21) we have

$$\phi_{n+1,s} = \frac{1}{\phi_{nn}} W(\phi_{nn}, \phi_{ns}),$$

which together with (3.2.27) and (3.2.28), gives

$$\phi_{n+1,s} = \phi_{ns} \left[\frac{n}{x} + O(x) - \frac{n}{x} + O(x) \right] = o(1),$$

as $x \to 0$. Now, from (3.2.24) and the orthogonality of ϕ_{nn} and ϕ_{ns}, it follows that

$$\phi_{nn}\phi_{n+1,s} = (\lambda_n - \lambda_s) \int_0^x \phi_{nn}\phi_{ns} d\xi. \tag{3.2.31}$$

The integral in (3.2.31) is calculated by considering (3.2.26) so that

$$\phi_{nn}\phi_{n+1,s} = (\lambda_n - \lambda_s) \int_0^x \phi_{nn}\phi_{ns} d\xi = \phi_{nn} \frac{C_{ns}}{2n+1} \prod_{t=0}^{n} (\lambda_t - \lambda_s) x^{n+1} [1 + O(x^2)].$$

We thus obtain (3.2.26) for $n+1$ as

$$\phi_{n+1,s} = C_{n+1,s} \prod_{t=0}^{n} (\lambda_t - \lambda_s) x^{n+1} [1 + O(x^2)],$$

where

$$C_{n+1,s} = \frac{1}{2n+1} C_{ns} \neq 0.$$

From an analogue of (3.2.14) for $\phi_{n+1,s}$, we have

$$\phi'_{n+1,s} = (n+1) x^{-1} \phi_{n+1,s} [1 + O(x^2)],$$

which is equation (3.2.27) with $n+1$. Now, (3.2.28) with $n+1$, is

$$v_{n+1} = (n+1) x^{-1} [1 + O(x^2)],$$

which follows easily from the expression for $\phi'_{n+1,s}$ given above.

Inside $(0,1)$, the function $\phi_{n+1,s}$ has at least $s-n-1$ zeros, as inferred by (3.2.21) with $n+1$ and statement (3.2.29). However, from (3.2.24) for $n+1$ and the inductive hypotheses (3.2.29) and (3.2.26), it can be deduced that $\phi_{n+1,s}$ has at most $s-n-1$

zeros in the interval $(0,1)$. Thus, $\phi_{n+1,s}$ has $s+n-1$ zeros exactly in $(0,1)$, giving (3.2.29).

Finally, we show that the expression

$$q_n(x) = n(n-1)x^{-2} + O(1) \tag{3.2.32}$$

for the potential function holds as $x \to 0$. A similar equation exists for $x \to 1$. So, by (3.2.32) and the assumption (3.2.28) we have that

$$q_{n+1} = q_n - 2v_n' = 2\lambda_n + 2v_n^2 - q_n = O(1) + n(n+1)x^{-2}$$

so that the relation (3.2.32) with $n+1$ is attained. □

The general solution for the transformed problem (3.1.3) for $\lambda \neq \lambda_s$ with $s < n$ is given by

$$y = \chi_n = \frac{W(\phi_0, \phi_1, \ldots, \phi_{n-1}, \chi)}{W_n},$$

where χ is the general solution of the original problem (1.0.1). A solution for the case when $\lambda = \lambda_{n-1}$ is given by

$$\begin{aligned} y &= \frac{1}{\phi_{n-1,n-1}} W(\phi_{n-1,n-1}, \chi_{n-1,n-1}) \\ &= \frac{C}{\phi_{n-1,n-1}} \\ &= C\frac{W(\phi_0, \phi_1, \ldots, \phi_{n-2})}{W(\phi_0, \phi_1, \ldots, \phi_{n-1})}. \end{aligned} \tag{3.2.33}$$

For $\lambda = \lambda_s$, with $s \leq n-1$, a solution for (3.1.3) is

$$y = \psi_{ns} = \frac{W_n^{(s)}}{W_n}, \tag{3.2.34}$$

where $W_n^{(s)}$ is the Wronskian of the set of $n-1$ functions $\{\phi_0, \ldots, \phi_{n-1}\}$ for $s < n-1$.

As is pointed out in Deift [17], Crum's result demonstrates how, by modification of the potential $q(x)$ into $\tilde{q}(x)$, one may add in or remove eigenvalues of the Sturm-Liouville operators $-\frac{d^2}{dx^2} + q(x)$.

3.2.3 Completeness of the family ϕ_{ns}

Given that the system (3.1.3) is not regular at the end-points, for $n > 1$, it is necessary to show that for $s \geq n$, the family of solutions ϕ_{ns} is $\mathcal{L}^2$-complete over $(0,1)$. A consequence of this is that the ϕ_{ns} are the only solutions of (3.1.3) which are bounded. It is sufficient to verify that the completeness of the family $\phi_{n+1,s}$ is implied by that of the family ϕ_{ns}, given that (3.1.3) for $n = 1$ is known to be regular.

Theorem 3.2.1. *The family of solutions ϕ_{ns}, satisfying the problem* (3.1.3)*, is $\mathcal{L}^2$-complete over* $(0,1)$.

Proof. Let $f(x)$ be of $\mathcal{L}^2(0,1)$. Then, given $\epsilon > 0$, there exists $g(x)$ such that

(i) $\lim_{x\to 0} g(x) = \lim_{x\to 1} g(x) = 0$,

(ii) $g'(x)$ is continuous in $(0,1)$,

(iii) $\int_0^1 |f-g|^2 d\xi < \epsilon$.

Consider the derivative

$$\begin{aligned} \frac{d}{dx}(\phi_{nn}g) &= \phi'_{nn}g + \phi_{nn}g' \\ &= \phi_{nn}\left[g' + \frac{\phi'_{nn}}{\phi_{nn}}g\right] \\ &= \phi_{nn}[g' + v_n g]. \end{aligned} \tag{3.2.35}$$

Let $h = g' + v_n g$. Then h is square integrable on $(0,1)$. Moreover, h satisfies the orthogonality relation

$$\int_0^1 h\phi_{nn}\, d\xi = [g\phi_{nn}]_0^1 = 0.$$

By assuming that the family of functions ϕ_{ns} is complete, it follows that they form an orthonormal basis in $\mathcal{H}$ so that, for $s \geq n$, we may write

$$h = \sum_{s=n+1}^{N} c_s\phi_{ns} + \eta, \tag{3.2.36}$$

where

$$\int_0^1 |\eta^2| dx < \epsilon.$$

Now, from (3.2.35) and the series expansion of h given above,

$$\begin{aligned}\phi_{nn} g &= \int_0^x \phi_{nn} h \, d\xi \\ &= \sum_{s=n+1}^{N} c_s \int_0^x \phi_{nn}\phi_{ns} \, d\xi + \int_0^x \phi_{nn}\eta \, d\xi \\ &= \phi_{nn} \sum_{s=n+1}^{N} C_s \phi_{n+1,s} + \phi_{nn}\zeta.\end{aligned}$$

The last line is obtained by using (3.2.31), so that

$$C_s = c_s(\lambda_n - \lambda_s)^{-1}$$

and

$$\zeta = \frac{1}{\phi_{nn}} \int_0^x \phi_{nn}\eta \, d\xi.$$

Again we refer back to (3.2.31) where, for $x \to 0$ with a similar equation for $x \to 1$, we can deduce that

$$\int_0^x \phi_{nn}^2 \, dx = O(\phi_{nn}^2) \qquad \text{and} \qquad \int_x^1 \phi_{nn}^2 \, dx = O(\phi_{nn}^2),$$

for $x \to 0$ and $x \to 1$ respectively. By the Cauchy-Schwartz inequality

$$|\zeta^2| \leq \frac{1}{|\phi_{nn}^2|} \left(\int_0^1 |\phi_{nn}|^2 dx \right) \left(\int_0^1 |\eta|^2 dx \right) < M_n \int_0^1 |\eta^2| dx < M_n \epsilon,$$

where

$$M_n = \sup \frac{1}{|\phi_{nn}^2|} \int_0^1 |\phi_{nn}^2| \, dx.$$

Therefore, by integrating over $(0, 1)$ we have

$$\int_x^1 \phi_{nn}^2 |\zeta^2| dx < M_n \epsilon,$$

giving that h can be approximated by a series expansion in $\phi_{n+1,s}$. This final result infers the completeness of the family $\phi_{n+1,s}$, as desired. □

Chapter 4

Darboux-Crum Related Results

4.1 A Modification for Successive Transformations

Successive applications of the Darboux-Crum transformation, together with its adaptation, is discussed by Adler in [2]. It is pointed out by Binding, Browne and Watson in [7] that it is possible to solve differential equations via successive transformations. In fact, it is Adler's result (Lemma 4.1.2) which enables the creation of the transformation relation proposed in [8], which captures Crum's iterative Darboux transformation procedure in a single application. Now,

$$\tilde{\phi}_j = \frac{W_n(\phi_j)}{W_n}, \tag{4.1.1}$$

satisfies the Sturm-Liouville differential equation

$$-\tilde{\phi}_n'' + \tilde{q}(x)\tilde{\phi}_n = \lambda_j \tilde{\phi}_n.$$

This $\tilde{\phi}_j$ is equivalent to the transformation relation (3.1.2) in Subsection 3.1. The subscript n denotes the n^{th} successive application of the Darboux transformation, while j indexes the solutions of the original and transformed equations. We note that $\tilde{\phi}_j = 0$ for $j = 0, 1, 2, \ldots, n-1$. Adler requires that the initial potential exhibit the following asymptotic behavior:

$$q(x) \sim \frac{\alpha}{(x-a)^2}, \quad \text{as} \quad x \to a, \qquad q(x) \sim \frac{\beta}{(x-b)^2}, \quad \text{as} \quad x \to b \tag{4.1.2}$$

where α, β are strictly positive. In [15] the unit interval is used, which is here generalized by Adler to an arbitrary open interval (a, b). The Sturm-Liouville problem to be investigated is set up by the equation

$$L\phi = -\phi'' + q\phi = \lambda\phi \tag{4.1.3}$$

with eigenvalues $\lambda_0 < \lambda_1 < \dots$. We are dealing with a problem that has singularities at the end point of (a, b) so that the applicable boundary conditions are

$$\lim_{x \to a^+} \phi(x) = 0 = \lim_{x \to b^-} \phi(x). \tag{4.1.4}$$

We denote by ϕ_m an eigenfunction corresponding to the m^{th} eigenvalue λ_m and make the assumption $\psi_j = \phi_{m_j}$. Taking a selection of these eigenfunctions ϕ_{m_j} for $j = 0, 1, \dots, n-1$, we have an equivalent formulation of Crum's potential $\tilde{q}$ in (3.2.25) in the form of

$$\tilde{q} = q - 2\frac{d^2}{dx^2}[\ln W(\phi_{m_0}, \dots, \phi_{m_{n-1}})] \tag{4.1.5}$$

in (a, b), where, in general, it may possess singularities. Adler seeks to address the problem of selecting the m_j's so that the potential $\tilde{q}$ is regular. The main result, given in [2], provides the following.

Theorem 4.1.1. *Let the numbers $m_0, \dots, m_{n-1}$, arranged in ascending order, form several fragments of the natural series*

$$0, \dots, M_0'; \quad M_1, \dots, M_1'; \quad \dots \quad ; \quad M_s, \dots, M_s'; \qquad M_j' < M_{j+1} - 1$$

(the first fragment may be absent). A necessary and sufficient condition for the potential (4.1.5) to be regular is that all the fragments $M_j, \dots, M_j'$, except for $0, \dots, M_0'$, consist of an even number of terms. Then the spectrum of $\tilde{q}$ is identical to the spectrum of q, with the eigenvalues $\lambda_{m_0}, \dots, \lambda_{m_{n-1}}$ deleted.

Constructing the eigenfunction $\tilde{\phi}_{n+s}$ as in (4.1.1) results in $\tilde{\phi}_{n+s}$ being the eigenfunction corresponding to the s^{th} eigenvalue of $-\tilde{y}'' + \tilde{q}\tilde{y} = \lambda\tilde{y}$ with boundary conditions $y(a^+) = 0 = y(b^-)$. Removing the first n eigenvalues from the original operator involving q, gives the spectrum of the operator with potential $\tilde{q}$. Adler points out that more than one option is available when selecting an appropriate sequence of eigenvalues for removal, as suggested in Theorem 4.1.1.

4.1.1 Preliminary Results

What makes Adler's result so useful is that he was able to construct a single regular formula to replace the iterated application of the Darboux-Crum transformation. For the sequential procedure one would have the j^{th} application given by

$$v_j = \frac{\psi'_{j,j}}{\psi_{j,j}}, \qquad q_{j+1} = q_j - 2v'_j, \tag{4.1.6}$$

$$\psi_{j+1,i} = \left(\frac{d}{dx} - v_j\right)\psi_{j,i}, \qquad i > j, \tag{4.1.7}$$

where $\psi_{0,i} = \phi_i$. (These are precisely those formulae given in Subsection 3.2.2, with Crum's indices n and s being replaced here by j and i respectively.) Comparing with (3.2.25) and (4.1.1) in the introduction, it is evident that

$$q_0 = q, \quad q_n = \tilde{q}, \quad \psi_{0,j} = \psi_j = \phi_{m_j}, \quad \psi_{n,j} = \tilde{\psi}_j.$$

The eigenfunctions of the operator $L_j = -D^2 + q_j$ are transformed via the operator $A_j = D - v_j$ into eigenfunctions of the operator $L_{j+1} = -D^2 + q_{j-1}$, which is essentially what was demonstrated in Subsection 3.1.1. Similarly, it can be shown that the formal adjoint operator, written $A_j^+ = -D - v_j$ transforms eigenfunctions of L_{j+1} to eigenfunctions of L_j.

The asymptotic behaviour displayed by each function v_j is found, by induction, to be

$$v(x) \sim \frac{\gamma}{x-a} \quad \text{as} \quad x \to a^+, \qquad v(x) \sim \frac{\delta}{x-b} \quad \text{as} \quad x \to b^-. \tag{4.1.8}$$

As was mentioned in the introduction, (4.1.2) holds for potentials q_j. The eigenfunctions (4.1.1) satisfy the boundary conditions given in (4.1.4), together with

$$\psi'(a^+) = \psi'(b^-) = 0. \tag{4.1.9}$$

The potential obtained after the first application of the Darboux-Crum transformation is

$$q_1 = q - 2\frac{d^2}{dx^2}[\ln W(\phi_0, \ldots, \phi_{n-1})], \tag{4.1.10}$$

on the interior of (a, b), having chosen $\psi_0 = \phi_0$ and the eigenfunction corresponding to the lowest eigenvalue (i.e. ground-state function) ϕ_0 does not change sign in (a, b). This point would suggest that conditions for the regularity of potentials q_n in (a, b) need to be formulated as well. However, we know that the lowest eigenvalue in the spectrum is removed to render the spectrum of $-y'' + qy = \lambda y$ with q replaced by q_1 and the applicable boundary conditions given by (4.1.9), whence we have the interior regularity of all potentials taking on the form

$$q_n = q - 2\frac{d^2}{dx^2}[\ln W(\phi_0, \ldots, \phi_{n-1})].$$

This outcome will be substantiated by the next result from [2] which suggests that the Wronskian will not have any zeros if two successive eigenfunctions are used in its construction.

Lemma 4.1.2. *The function $W = \phi_m\phi_n' - \phi_m'\phi_n$, $m < n$, maintains a constant sign on (a, b) if, and only if, $n = m + 1$.*

Proof. By differentiating the Wronskian given in the statement, we have

$$\frac{d}{dx}W = (\lambda_m - \lambda_n)\phi_m\phi_n, \tag{4.1.11}$$

with second derivative

$$\frac{d^2}{dx^2}W = (\lambda_m - \lambda_n)(\phi_m\phi_n' + \phi_m'\phi_n).$$

Representing the zeros of ϕ_m in (a, b) by $x_1 < \cdots < x_m$ and those of ϕ_n by $z_1, \ldots, z_n$, it follows immediately that

$$\frac{d}{dx}W(x_i) = \frac{d}{dx}W(z_i) = 0, \tag{4.1.12}$$

and since

$$W(x_i) = -\phi_m'\phi_n \quad \text{and} \quad W(z_i) = \phi_m\phi_n', \tag{4.1.13}$$

it follows that

$$\frac{d^2}{dx^2}W(x_i) = -(\lambda_m - \lambda_n)W(x_i), \qquad \frac{d^2}{dx^2}W(z_i) = (\lambda_m - \lambda_n)W(z_i). \tag{4.1.14}$$

Initially assume that the eigenfunctions are not successive, i.e. $n > m + 1$, and partition the interval (a, b) into $m + 1$ sets

$$(a, x_1), [x_1, x_2), \dots, [x_m, b).$$

For some $j = 1, \dots, m$, there is an interval $[x_j, x_{j+1})$ in which at least two of the elements of the set $\{z_1, \dots, z_n\}$ are located. Say z_k and z_{k+1} are two such elements, then

$$x_j \leq z_k < z_{k+1} < x_{j+1}.$$

Without loss of generality, we take $\phi_m > 0$ in (x_j, x_{j+1}) and $\phi_n > 0$ in (z_k, z_{k+1}), with $\phi_n < 0$ in $(z_{k-1}, z_k) \cup (z_{k+1}, z_{k+2})$. This gives $\phi'(z_k) > 0$ and $\phi'(z_{k+1}) < 0$. Then by (4.1.13), $W(z_k) > 0$ and $W(z_{k+1}) < 0$, giving that W has a zero in (z_k, z_{k+1}). The intervals (a, x_1) and $[x_m, b)$ can be treated in a similar manner.

We now assume $n = m + 1$. The zeros of ϕ_m and ϕ_n interlace, giving

$$z_1 < x_1 < z_2 < \dots < x_m < z_{m+1},$$

which are precisely the zeros of W' in (a, b) by (4.1.11). Without loss of generality, we will assume that $\phi_m > 0$ on (a, x_1) and $\phi_m < 0$ on (x_1, x_2), in which case $\phi'_m(a) > 0$, $\phi'_m(x_1) < 0$ and $\phi'_m(x_2) > 0$. Further we take, without loss of generality, $\phi_n > 0$ on (a, z_1), $\phi_n < 0$ on (z_1, z_2) and $\phi_n > 0$ on (z_2, z_3), so that

$$\phi'_n(a) > 0 \qquad \phi'_n(z_1) < 0 \qquad \phi'_n(z_2) > 0. \tag{4.1.15}$$

Since $z_1 \in (a, x_1)$, it follows that $\phi_m(z_1) > 0$ and by taking (4.1.15) into account, we find that by (4.1.13) $W(z_1) < 0$. Moreover, $z_2 \in (x_1, x_2)$ so that $\phi_m(z_2) < 0$, which together with (4.1.15) implies $W(z_2) < 0$. This reasoning is repeated at each of the zeros of ϕ_n, whereby it can be concluded that

$$W(z_k) < 0 \qquad \forall k = 1, \dots, n.$$

Similarly, $x_1 \in (a, z_1)$, where $\phi_n(x_1) > 0$ and $\phi'_m(x_1) < 0$, so that $W(x_1) < 0$. Reproducing this argument for each of the zeros of ϕ_m, it is deduced that

$$W(x_i) < 0 \qquad \forall i = 1, \ldots, m+1.$$

Given that the respective zeros of ϕ_m and ϕ_n are the only zeros of W, it follows that W maintains a constant sign on $[z_1, z_{m+1}]$. From (4.1.11) we know that W' does not have zeros situated outside of $[z_1, z_{m+1}]$, motivating the monotonicity of W and by (4.1.4), it is clear that $W(a) = W(b) = 0$, so that W does not vanish on (a, b). □

Consequently, the potential

$$q_2 = q - 2\frac{d^2}{dx^2}[\ln W(\phi_m, \phi_{m+1})], \tag{4.1.16}$$

is regular on the interior of (a, b) when choosing $\psi_0 = \phi_m, \psi_1 = \phi_{m+1}$, even though a singularity may occur in the intermediate potential, that is, the potential that would have come about for sequential applications of the transformation via q_1.

The next result from [2] formalises a consequence of the transformation which Crum fails to mention, that being, that the operator obtained via the Darboux-Crum transformation is almost isospectral to the original differential operator, since the eigenvalue corresponding to the eigenfunction used in setting up the transformation is removed from the spectrum of the second operator. Adler extends this result to account for the use of functions other than the ground-state function in formulating the n-fold transformations.

Lemma 4.1.3. *The respective spectra corresponding to the potentials* (4.1.10) *and* (4.1.16) *are identical to the spectrum corresponding to the potential q but with the terms λ_0 and λ_m, λ_{m+1}, respectively, deleted.*

Proof. We begin by recalling the expressions for the behaviour of v_j given in (4.1.8) and (4.1.9). Now, the operator $A_0^+ = -D - v_0$ applied to the eigenfunctions ϕ_{1s} of the transformed problem, with

$$q_1 = q - 2\frac{d^2}{dx^2}(\ln \phi_0),$$

gives back the original (untransformed) problem. Similarly, by considering $A_0^+ A_1^+ = (-D - v_0)(-D - v_1)$ applied to the eigenfunctions ϕ_{2s} associated with the transformed potential

$$q_2 = q - 2\frac{d^2}{dx^2}[\ln W(\phi_m, \phi_{m+1})]$$

returns the original problem. In both cases, transforming back to the original eigenvalue problem

$$Ly = -y'' + qy = \lambda y$$

does not result in the addition of any new eigenvalues to the spectrum of operator L. In fact, λ_0 has been re-introduced as an eigenvalue (i.e. supporting the notion that it is possible to add eigenvalues to spectra by means of transformations) and with this the associated eigenfunction as well.

In a similar manner, the operator $A_0 = D - v_0$ can be applied to an eigenfunction ϕ corresponding to the original potential q and, with the exception of the least eigenvalue λ_0, the eigenvalues of both operators, L and its transformed variant, are identical.

To show that λ_0 is not present as an eigenvalue in the spectrum of the operator associated with the potential q_1, we recall that by the transformation relation (4.1.1), the eigenfunction corresponding to the eigenvalue λ_1, given by

$$\tilde{\psi}_j = \psi_{1,1} = \frac{W(\phi_0, \phi_1)}{\phi_0},$$

is associated with the potential

$$q_1 = q - 2\frac{d^2}{dx^2}(\ln \phi_0).$$

As a result of using ϕ_0 and ϕ_1, by Lemma 4.1.2, the transformed eigenfunction $\psi_{1,1}$, does not have any zeros in (a, b), whereby it follows that λ_1 is the least eigenvalue in the spectrum of the operator associated with q_1.

Moreover, except for λ_m and λ_{m+1}, all of the eigenvalues associated with the original operator L are eigenvalues of the transformed operator. Here the transformation $A_1A_0 = (D - v_1)(D - v_0)$ takes eigenfunctions of L to eigenfunctions of the transformed operator. To substantiate this claim we must show that eigenvalues λ_m and λ_{m+1} are absent from the spectrum of the transformed operator, which has potential

$$q_2 = q - 2\frac{d^2}{dx^2}[\ln W(\phi_m, \phi_{m+1})].$$

If we begin by using eigenfunctions ϕ_{m-1} and ϕ_m to construct the transformation then the resulting potential is

$$\tilde{q}_2 = q - 2\frac{d^2}{dx^2}[\ln W(\phi_{m-1}, \phi_m)].$$

Given that this potential is of the same form as q_2 in (4.1.16), $\tilde{q}_2$ inherits internal regularity by virtue of Lemma 4.1.2.

Now, another potential, say q_4, may be constructed in a similar way from eigenfunctions associated with transformed potential $\tilde{q}_2$. Suppose we choose to use the functions $\tilde{\phi}_{2,m+1}$ and $\tilde{\phi}_{2,m+2}$ to set up q_4 as

$$q_4 = q - 2\frac{d^2}{dx^2}[\ln W(\tilde{\phi}_{2,m+1}, \tilde{\phi}_{2,m+2})],$$

then for the same reason as the one given above, q_4 has no internal zeros. However, it is also possible to build q_4 using eigenfunctions $\phi_{2,m-1}$ and $\phi_{2,m+2}$ associated with the problem characterised by potential q_2. We recall that

$$q_2 = q - 2\frac{d^2}{dx^2}[\ln W(\phi_m, \phi_{m+1})].$$

Both $\tilde{q}_2$ and q_2 are on the same transformation level (i.e. the first level), though each uses two different consecutive ϕ's taken from the original set of eigenfunctions. Since q_4 is internally regular, it must be (again by Lemma 4.1.2) that the functions $\phi_{2,m-1}$ and $\phi_{2,m+2}$ used in the construction of q_4 correspond to two consecutive eigenvalues associated with the operator L_2, which is characterised by potential q_2.

It may thus be concluded that the two eigenvalues λ_m and λ_{m+1} are missing from the spectrum of L_2. □

Adler's result makes it possible to generate an n-fold application of the Darboux-Crum transformation in one step, as opposed to performing each one of the n transformations in succession.

4.1.2 Proof of Fundamental Theorem

Proof of Theorem 4.1.1.

Proof. To prove sufficiency, we suppose that the m_j satisfy the conditions stated in the theorem, that is, all fragments $M_j, \dots, M_j'$ except for $0, \dots, M_0'$ consist of an even number of terms. Then, the potential

$$\tilde{q} = q - 2\frac{d^2}{dx^2}[\ln W(\phi_{m_0}, \dots, \phi_{m_{n-1}}), \qquad (4.1.17)$$

is obtained by several iterations of the transformation that is implemented to generate either of the internally regular potentials q_1, given in (4.1.10), or q_2, as in (4.1.16), by the equivalence of the two approaches. It follows then that $\tilde{q}$ does not possess any internal zeros either. Consequently, the spectra of operators $L = -D^2 + q$ and $\tilde{L} = -D^2 + \tilde{q}$, aside from the sequence $\lambda_{m_0}, \dots, \lambda_{m_{n-1}}$, are identical.

To show necessity, we begin by assuming that the potential $\tilde{q}$ is regular and that one of the m_j is a fragment consisting of an odd number of terms. Suppose this to be the fragment $M_l, \dots, M_l'$ on the extreme right-hand side of the natural series. Starting with $\tilde{q}$, new potentials are created such that they conform to the formulation equivalent to (4.1.10), given by

$$q_n = q - 2\frac{d^2}{dx^2}[\ln W(\phi_0, \dots, \phi_{n-1})],$$

whereby the regularity of the newly created potentials are ensured. After each application of the potential construction procedure, one of the fragments to the left of $M_l, \dots, M_l'$ will have been deleted. Eventually it will be possible to consider the remaining fragments as one fragment.

The very next potential, say $\tilde{q}^*$, to be created will utilise those eigenfunctions ϕ_m indexed by

$$0, \ldots, M'_{l-1}; \quad M_l, \ldots, M'_l; \quad M_{l+1}, \ldots, M'_{l+1}; \ldots .$$

A potential, say $\tilde{q}^{**}$, having no internal zeros, is achieved when constructed from the same functions as those used to build $\tilde{q}^*$ without the inclusion of the functions $\phi_{m'_l}$. As the eigenvalues corresponding to $\phi_{M'_l}$ are not the least eigenvalues of the operator associated with the potential

$$\tilde{q}^* = \tilde{q}^{**} - 2\frac{d^2}{dx^2}[\ln \phi_{M'_l}],$$

potential $\tilde{q}^*$ will contain internal zeros. This deduction contradicts the initial assumption stipulating that every $\tilde{q}$ generated be regular on (a, b). □

4.2 Factorisation and Commutation

The factorisation method was first introduced in [34], by Schrödinger in 1940. The use of creation and annihilation operations allows for (1.0.1) to be solved for successive eigenvalues. Infeld and Hull put together a large catalogue [25] consisting of equations of mathematical physics that can be transformed by means of this technique. In 1956, Ince published [24], in which the commutation of operators is discussed. Almost four decades after Schrödinger's publication, Deift put the two concepts together in [17]. Deift approaches Crum's method from an operator theory perspective. In [17] Crum's result is generated by shifting then factorising the operators, after which the factors are permuted. Consideration of the Friedrichs extension enables the transformation of the boundary conditions. Deift too obtains an "almost" isospectral transformed operator with the least eigenvalue having been removed.

4.2.1 Transformation via Factorisation in Detail

Deift applies a commutation formula to ordinary differential equations characterised by the Sturm-Liouville type operator

$$-\frac{d^2}{dx^2} + q(x)$$

where $q = q(x)$ represents the potential function. He proposes a function $b = b(x)$ as a solution to the equation

$$-b'' + qb = 0$$

in which case, it would follow that $q = b^{-1}(b'')$. He computes the following

$$\left(b\frac{d}{dx}b^{-1}\right)^* \left(b\frac{d}{dx}b^{-1}\right) = -b^{-1}\frac{d}{dx}b^2\frac{d}{dx}b^{-1} = -\frac{d^2}{dx^2} + q(x), \tag{4.2.18}$$

where * denotes adjoint. Deift goes on to demonstrate that, by commutation,

$$\left(b\frac{d}{dx}b^{-1}\right) \left(b\frac{d}{dx}b^{-1}\right)^* = -b\frac{d}{dx}b^{-2}\frac{d}{dx}b = -\frac{d^2}{dx^2} + \tilde{q}(x) \tag{4.2.19}$$

where $\tilde{q}(x) = b(b^{-1})''$, is isospectral to (4.2.18) (i.e. both operators possess the same spectrum) with zero being a possible exception. Consequently, given a potential $q(x)$, another potential $\tilde{q}(x)$ exists such that the operators

$$-\frac{d^2}{dx^2} + q(x) \qquad \text{and} \qquad -\frac{d^2}{dx^2} + \tilde{q}(x)$$

share the same spectrum. Again, the value zero is not common to both. As pointed out in [15], the two potentials are related by

$$\tilde{q} = q - 2\frac{d^2}{dx^2}\ln b.$$

It is, of course, necessary to establish those boundary conditions for which these statements would be valid.

In [17], Deift presents the *commutation formula* by assuming two bounded operators in a Hilbert space, namely A and B. The formula is then written as

$$\lambda(AB + \lambda)^{-1} + A(BA + \lambda)^{-1}B = I \tag{4.2.20}$$

where $-\lambda \in \rho(AB)$ if $0 \neq -\lambda \in \rho(BA)$. Here $\rho(AB)$ denotes the resolvent set of the operator AB. Further, the resolvent of AB, given by $(AB+\lambda)^{-1}$, is obtained by manipulating (4.2.20) and generating $\lambda^{-1}(I - A(BA+\lambda)^{-1}B)$. By interchanging operators A and B in the preceding discussion, it is evident that

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}. \tag{4.2.21}$$

4.2.2 Proof of the Commutation Formula

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with linear operators $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_1$. Consider the product operator BA having domain

$$D(BA) \equiv \{f \in \mathcal{H}_1 : f \in D(A), \quad Af \in D(B)\} \tag{4.2.22}$$

with the action

$$(BA)f \equiv [B(Af)], \qquad f \in D(BA).$$

Operator BA is said to be *naturally defined* in $\mathcal{H}_1$. The first theorem, taken from [17], is stated next.

Theorem 4.2.1. *Let A be a (densely defined) closed linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$. Then A^*A, defined naturally, is a (densely defined) positive, self-adjoint operator in $\mathcal{H}_1$. Moreover*

$$D(A) = Q(A^*A) = D(\sqrt{A^*A})$$

where $Q(A^*A)$ is the form domain of A^*A.

Deift goes on to prove the following theorem, titled *Commutation.* He considers two cases of which only one will be given here as it pertains to this investigation.

Theorem 4.2.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Suppose that A is a (densely defined) closed linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and that $B = A^*$. Let $S = AA^*$ and $T = A^*A$, defined naturally (as self-adjoint operators). Then the spectra of S and T are identical away from zero. Moreover, if $-\lambda \neq 0$ is an eigenvalue of S*

(respectively T), then $-\lambda$ is an eigenvalue of T (respectively S) and B (respectively A) is a surjection of $N(S+\lambda)$ (respectively $N(T+\lambda)$) onto $N(T+\lambda)$ (respectively $N(S+\lambda)$). In particular $N(T+\lambda)$ and $N(S+\lambda)$ have the same dimension.
Finally we have the commutation formulae

$$B(AA^*+\lambda)^{-1} = (A^*A+\lambda)^{-1}A^* \tag{4.2.23}$$

$$A(A^*A+\lambda)^{-1} = (AA^*+\lambda)^{-1}A^* \tag{4.2.24}$$

and

$$\lambda(AA^*+\lambda)^{-1} + A(A^*A+\lambda)^{-1}A^* = I \tag{4.2.25}$$

$$\lambda(A^*A+\lambda)^{-1} + A^*(AA^*+\lambda)^{-1}A = I \tag{4.2.26}$$

for $0 \neq -\lambda \in [\rho(AA^*) \setminus \{0\}] = [\rho(A^*A) \setminus \{0\}]$.

Proof. By assumption we have $S = AA^*$ and $T = A^*A$. First, we'll prove (4.2.25) and (4.2.26), after which (4.2.23) and (4.2.24) will be established.

Assuming $0 \neq -\lambda \in \rho(T)$ (i.e. not in the spectrum of T), we show that $-\lambda \in \rho(S)$ as well. Firstly, operator $(S+\lambda)$ is shown to be a surjection. Let $f \in \mathcal{D}(A^*)$. Then by (4.2.22)

$$[I - A(A^*A+\lambda)^{-1}A^*]f = f \in \mathcal{D}(A^*)$$

and by permutation of the operators we have

$$\begin{aligned} A^*[I - A(A^*A+\lambda)^{-1}A^*]f &= A^*f - [(A^*A+\lambda)-\lambda](A^*A+\lambda)^{-1}A^*f \\ &= A^*f - A^*f + \lambda(A^*A+\lambda)^{-1}A^*f \\ &= \lambda(A^*A+\lambda)^{-1}A^*f \in \mathcal{D}(T) \subset \mathcal{D}(A). \end{aligned} \tag{4.2.27}$$

Thus

$$[I - A(A^*A+\lambda)^{-1}A^*]f \in \mathcal{D}(S)$$

and, again by permuting factors, we see that from (4.2.27) that

$$(S+\lambda)[\lambda^{-1}[I - A(A^*A+\lambda)^{-1}A^*]f]$$

$$= \lambda^{-1}\{A(A^*[I - A(A^*A + \lambda)^{-1}A^*]) + \lambda[I - A(A^*A + \lambda)^{-1}A^*]\}f$$
$$= \lambda^{-1}\lambda A[(A^*A + \lambda)^{-1}]A^*f + [I - A(A^*A + \lambda)^{-1}A^*]f$$
$$= A[(A^*A + \lambda)^{-1}A^*]f + [I - A(A^*A + \lambda)^{-1}A^*]f$$
$$= f. \tag{4.2.28}$$

Now, it is shown that $A(A^*A + \lambda)^{-1}A^*$ is a bounded operator. By manipulation

$$\begin{aligned}[A(A^*A + \lambda)^{-1}]A^* &= A[(A^*A + I)^{-1/2}(A^*A + I)(A^*A + \lambda)^{-1}(A^*A + I)^{-1/2}]A^* \\ &\subset [A(A^*A + I)^{-1/2}][(A^*A + I)(A^*A + \lambda)^{-1}][A(A^*A + I)^{-1/2}]^*,\end{aligned}$$

as $\mathcal{D}((A^*A + I)^{-1/2}A^*) \subset \mathcal{D}((A(A^*A + I)^{-1/2})^*$. By the Closed Graph Theorem 2.3.5 and Theorem 4.2.1, $A(A^*A + I)^{-1/2}$ is bounded. This is evidently also true of $(A^*A + I)(A^*A + \lambda)^{-1}$. Therefore, $A(A^*A + \lambda)^{-1}A^*$ is bounded as well.

Now, let f be any vector in $\mathcal{H}_2$. $\mathcal{D}(A^*)$ is dense in $\mathcal{H}_2$, that is, the closure of $\mathcal{D}(A^*)$ is equal to the whole of $\mathcal{H}_2$. A sequence is then chosen in the form of

$$\{f_n\}_{n=1}^{\infty} \subset \mathcal{D}(A^*)$$

so that f_n converges to $f \in \mathcal{H}_2$. Then the sequence defined by

$$g_n := \lambda^{-1}[I - A(A^*A + \lambda)^{-1}A^*]f_n$$

is Cauchy. Now, since S is a self-adjoint closed operator, from (4.2.28),

$$(S + \lambda)[\lambda^{-1}[I - A(A^*A + \lambda)^{-1}A^*]f] = f \tag{4.2.29}$$

and the operator $(S + \lambda)$ is surjective. Moreover, if $\lambda \notin \mathbb{R}$ then that $S + \lambda$ is an injective mapping is a trivial consequence of the fact that S is self-adjoint. If $\lambda \in \mathbb{R}$ then

$$N(S + \lambda) = [\mathrm{Ran}(S + \lambda)]^{\perp} = 0.$$

Therefore, $-\lambda \in \rho(S)$ and the operator $(S + \lambda)^{-1}$ exists and is bounded. Formula (4.2.25) is then given by (4.2.29), while (4.2.26) if obtained in a similar manner, though by considering the operator $(T + \lambda)$ instead.

In order to prove (4.2.23), one would have to show that

$$(A^*A + \lambda)^{-1}A^*f = A^*(AA^* + \lambda)^{-1}f \tag{4.2.30}$$

for $f \in \mathcal{D}(A^*)$. It is sufficient to demonstrate that

$$(A^*A + \lambda)A^*(AA^* + \lambda)^{-1}f = A^*f. \tag{4.2.31}$$

Suppose $f \in \mathcal{D}(A^*)$. Then

$$A^*(AA^* + \lambda)^{-1}f \in \mathcal{D}(A)$$

and

$$A[A^*(AA^* + \lambda)^{-1}f] = A(A^*A + \lambda)^{-1}A^*f = [I - \lambda(AA^* + \lambda)^{-1}]f \in \mathcal{D}(A^*). \tag{4.2.32}$$

Hence, to show (4.2.31) we apply the operator $(A^*A + \lambda)$ to the right-hand side of (4.2.30), which results in

$$(A^*A + \lambda)A^*(AA^* + \lambda)^{-1}f = A^*f.$$

The proof of (4.2.23) is hereby complete.

Next we show that the dimensions of $N(A^*A + \lambda)$ and $N(AA^* + \lambda)$ are equal for $\lambda \neq 0$, since

$$R(A^*A + \lambda) = R[(A^*A + \lambda)^*] = N(A^*A + \lambda)^{\perp}.$$

First, we assume $f \in N(A^*A + \lambda) \setminus \{0\}$, then by the definition of $N(A^*A + \lambda)$,

$$(A^*A + \lambda)f = 0 \Rightarrow A^*Af = -\lambda f, \tag{4.2.33}$$

from which we have that $A^*Af \in \mathcal{D}(A)$. By applying A to (4.2.33), it is seen that

$$A(A^*A)f = (AA^*)Af = -\lambda(Af).$$

Since $\lambda \neq 0$ and $f \neq 0$, by (4.2.33) it follows that $Af \neq 0$. So $f \mapsto Af$ defines a map from $N(A^*A+\lambda)$ into $N(AA^*+\lambda)$. This map is injective by virtue of its linearity. Now, suppose $h \in N(AA^*+\lambda)$, then

$$A^*h \in N(A^*A+\lambda).$$

Since

$$(AA^*+\lambda)h = 0 \Rightarrow (-\lambda)^{-1}AA^*h - h = 0 \Rightarrow h = (-\lambda)^{-1}AA^*h,$$

we have that

$$A[(-\lambda)^{-1}A^*h] = h.$$

That is, $f \mapsto Af$ is surjective from $N(A^*A+\lambda)$ onto $N(AA^*+\lambda)$. Consequently, $f \mapsto Af$ is a bijection. A similar argument substantiates the bijectivity of A^* from $N(AA^*+\lambda)$ onto $N(A^*A+\lambda)$. □

Essentially, what the theorem aims to demonstrate is that the operators AB and BA are (formally) similar, in which case we may write

$$AB = A(BA)A^{-1}$$

so that the similarity transform is provided by A. The next theorem in Deift's paper is now presented.

Theorem 4.2.3. [17] *Let A and $B = A^*$ be as in Theorem 4.2.2. Let $K_1 = N(A)^\perp, K_2 = N(A^*)^\perp$, then K_1 and K_2 reduce A^*A and AA^* respectively and $(A^*A \restriction K_1)$ is unitarily equivalent to $(AA^* \restriction K_2)$.*

Two closed operators A_i in Hilbert spaces $\mathcal{H}_i$ respectively, $i = 1, 2$, are called *essentially isospectral* if

$$\sigma(A_1) \setminus \{0\} = \sigma(A_2) \setminus \{0\}.$$

As it happens, in Theorem 4.2.2, operators AB and BA are essentially isospectral.

4.2.3 Applications to Ordinary Differential Equations

In the introduction we began by suggesting that two operators

$$-\frac{d^2}{dx^2}+q(x)=\left(b\frac{d}{dx}b^{-1}\right)^*\left(b\frac{d}{dx}b^{-1}\right) \quad \text{and} \quad -\frac{d^2}{dx^2}+\tilde{q}(x)=\left(b\frac{d}{dx}b^{-1}\right)\left(b\frac{d}{dx}b^{-1}\right)^*, \tag{4.2.34}$$

where

$$q=b^{-1}b'' \quad \text{and} \quad \tilde{q}=q-2\frac{d^2}{dx^2}(\ln b),$$

are essentially isospectral, subject to suitable boundary conditions. Deift considers three kinds of problems: those on $\mathcal{L}^2(0,1)$, $\mathcal{L}^2(-\infty,\infty)$ and $\mathcal{L}^2(0,\infty)$. Several preliminary results are discussed before the main results are given, but first is stated some definitions. Let

$$\mathcal{AC}_0(0,1)=\{f\in\mathcal{AC}(0,1):f(0)=f(1)=0\}.$$

Define a quadratic form on $\mathcal{L}^2(0,1)$ by

$$t_D(f)\equiv\int_0^1(|f'(x)|^2+q|f(x)|^2)dx$$

with domain $D(t_D)\equiv\mathcal{AC}_0(0,1)$. The subscript D denotes "Dirichlet". We are interested in the Dirichlet case in particular.

Theorem 4.2.4. [17, Theorem7] *t_D as defined above, is a closed form on its domain and bounded from below. $\mathcal{C}_0^\infty(0,1)\equiv\{f\in\mathcal{C}^\infty[0,1]:f(0)=f(1)\}$ is a core for t_D.*

Let H_D be the self-adjoint operator in $\mathcal{L}^2(0,1)$ associated with t_D and let

$$\mathcal{AC}^2(0,1)=\{f\in\mathcal{AC}(0,1):f' \text{ is absolutely continuous on } [0,1]\}.$$

Theorem 4.2.5. [17, Theorem8]

$$D(H_D)=\{f\in\mathcal{AC}^2(0,1):-f''+qf\in\mathcal{L}^2(0,1)\}$$

and

$$H_Df=-f''+qf, \qquad f\in D(H_D)\subset D(t_D).$$

The next result is a particularly useful one as it describes the spectrum of operator H.

Theorem 4.2.6. [17, Theorem9] *The operator H_D has only discrete spectrum. Each eigenvalue of H_D has multiplicity one. The least eigenvalue of H_D can be chosen to be non-negative and its corresponding eigenfunction vanishes only at 0 and 1.*

Now, Deift defines an operator P_D by restricting the closed operator $i^{-1}\frac{d}{dx}$ to $\mathcal{AC}_0(0,1)$. A core of P_D is $\mathcal{C}_0^\infty(0,1)$.

Theorem 4.2.7. [17, Theorem10] *Let $b_D(x)$ be an eigenfunction to the least eigenvalue of H_D. Let $b_D P_D' b_D^{-1}$ denote the (densely defined) closure of $b_D\left(i^{-1}\frac{d}{dx}\right)b_D^{-1} \upharpoonright \mathcal{C}_0^\infty(0,1)$, then*

$$D(b_D P_D' b_D^{-1}) = D(t_d) = \mathcal{AC}_0(0,1)$$

and

$$H_D - \lambda_D = (b_D P_D' b_D^{-1})^*(b_D P_D' b_D^{-1}). \tag{4.2.35}$$

Here the operators on the right are defined naturally and λ_D is the least eigenvalue of H_D.

The proof can be found in [17, p.279]. Crum's result for removing eigenvalues is presented in the subsequent theorem.

Theorem 4.2.8. [17, Theorem12] *Let $\{\lambda_i\}_{i=0}^\infty, \{\phi_i\}_{i=0}^\infty, W_{ns}, W_n$ be as above. For each $n \geq 1$, define*

$$q_n(x) \equiv q(x) - 2\frac{d^2}{dx^2}\ln W_{n-1}. \tag{4.2.36}$$

Then $q_n \in \mathcal{L}^1_{loc}(0,1)$ and $\left(-\frac{d^2}{dx^2} + q_n(x)\right) \upharpoonright \mathcal{C}_0^\infty(0,1)$ is semi-bounded; let H_n denote its Friedrichs extension. We have the basic result

$$\sigma(H_n) = \{\lambda_s\}_{s\geq n}$$

with associated eigenfunctions

$$\phi_{ns} = \frac{W_{ns}}{W_{n-1}}, \qquad s \geq n \geq 1.$$

With regards to the Friedrichs extension, the reader is referred to [18, p.98] and [23, p.85].

4.3 Unitary Equivalence of Darboux-Crum Transformations

In the paper by Schmincke [33], the Schrödinger operator

$$A = -\frac{d^2}{dx^2} + q$$

exists on a densely defined domain, in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. With suitable boundary conditions, this operator is self-adjoint and is further characterized by a gap in its spectrum denoted by G. Now, Schmincke suggests that it is possible to create another Schrödinger operator in $\mathcal{H}$, which is unitarily equivalent to operator A, represented as

$$\tilde{A} = -\frac{d^2}{dx^2} + \tilde{q}.$$

Definition 4.3.1. [19, p.116] *An operator $U : \mathcal{H} \mapsto \mathcal{H}$ is called an* isometry *if $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$. If an operator U is an isometry and $\Im(U) = \mathcal{H}$, then U is called a* unitary operator.

The authors of [19] go on to say that two operators Q and $\tilde{Q}$ are termed *unitary equivalent* if a unitary operator U can be found such that

$$Q = U^{-1}\tilde{Q}U.$$

In addition, for two such operators, Q would in addition be *isospectral* to $\tilde{Q}$.

Now, Schmincke's argument uses two numbers μ and ν existing in the gap G , so that commutation results in

$$(A - \mu)(A - \nu) = B^*B \tag{4.3.37}$$

where B is a second order differential operator derived from eigensolutions corresponding to A. Furthermore, B happens to be the transformation between A and $\tilde{A}$ such that $BA = \tilde{A}B$. And then, since B is an invertible operator, one may construct a unitary operator $U = B|B|^{-1}$, thereby yielding a means for transforming A, as seen here

$$\tilde{A} = UAU^*,$$

where U^* denotes the adjoint of U. This formula is easily verified by

$$\tilde{A} = BAB^{-1} = BA|B|^{-1}|B|B^{-1} = B|B|^{-1}A'(B|B|)^{-1} = UAU^*$$

as A is self-adjoint.

Necessary and sufficient conditions, to enable the construction of B as above, emerge from the following considerations. For the $\mu, \nu \in G$, and as such, not in the spectrum of A, we have a non-trivial real-valued solution f for

$$-y'' + qy = \mu y$$

such that f is square integrable in $(0, \infty)$. Moreover, there is another real-valued function g, also non-trivial, and square integrable in $(-\infty, 0)$, which is a solution for

$$-y'' + qy = \nu y.$$

Linear independence of eigenfunctions f and g means that the Wronskian $W := fg' - f'g$ does not have any zeros. It is, however, possible to set up the problem with $\mu = \nu$. The new operator $\tilde{A}$ has potential function

$$\tilde{q} := q - 2(\ln|W|)''$$

as is given by Crum's result.

Whereas Deift is primarily concerned with how commutation of factors of operator A generates operator $\tilde{A}$, in [33] Schmincke deals with the problem of finding such a closed differential operator B satisfying (4.3.37), so that, as in [17],

$$|B| := \sqrt{B^*B}.$$

Chapter 5

Orthogonal Polynomials

5.1 Introduction

According to [3, p.240], orthogonal functions were first introduced in 1835 by Murphy. However, it was in the work done by Chebyshev that the significance of these functions stimulated the development of a self-standing and important branch of mathematics.

The connection between orthogonal functions and continued fractions is motivated by the fact that all orthogonal polynomials satisfy a three-term recurrence relation. The nature of the zeros of the orthogonal polynomials has implications for a range of different problems. The Christoffel-Darboux formula provides a means for investigating these zeros. Gauss's expansion of the expression

$$\frac{\log(1+x)}{1-x}$$

as a continued fraction with successive convergents gave rise to polynomials that he then used in his quadrature approximation theory. It was in 1826 that Jacobi recognised these polynomials as Legendre polynomials and observed the far-reaching implications resulting from their orthogonality.

The concept of special orthogonal polynomials (that in recent years became established as classical orthogonal polynomials) only became formalised some time after they first started to appear in mathematics. Legendre polynomials were used by both Legendre and Laplace in their studies of celestial mechanics, while Laplace's work on probability made use of Hermite polynomials. The polynomials of Jacobi, Laguerre and Hermite are those which have been most extensively studied and are the best established.

For a definition of the orthogonal polynomials, we appeal to [3, p.244] and [35, p.8].

Definition 5.1.1. *A sequence of polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *where each* p_n *has exact degree* n *is called* orthogonal *with respect to the weight function* $w(x) > 0$ *on the interval* (a, b) *if*

$$\int_a^b w(x)p_m(x)p_n(x)dx = h_n\delta_{mn} \qquad \delta_{mn} := \begin{cases} 0 & \text{if} \quad m \neq n, \\ 1 & \text{if} \quad m = n. \end{cases} \tag{5.1.1}$$

Information regarding the moments of orthogonal polynomials is found in [35, p.25] and [3, p.244].

Definition 5.1.2. *It is required that the weight function* $w(x)$ *be positive and continuous on the interval* (a, b) *so that the moments given by*

$$\mu_n := \int_a^b w(x)x^n dx \qquad n = 0, 1, 2, ...$$

exist. The inner product *of the polynomials* f *and* g *is denoted by the integral*

$$\langle f, g\rangle := \int_a^b w(x)f(x)g(x)dx.$$

The interval (a, b) *is called the* interval of orthogonality*, which need not be finite.*

If $h_n = 1$ for each $n \in 0, 1, 2, ...$ the sequence of polynomials is called *orthonormal*, and if the leading coefficient k_n is unity then the polynomials are called *monic*.

A note on the presence of the weight function seems appropriate here. From [3, p.244], we are informed that function $w(x)$ is non-decreasing with an infinite number of points of increase on interval $[a, b]$ (one may have boundary points at infinity). As such, moments of all orders must exist. In other words, $\int_a^b x^n dw(x)$ exists for $n = 0, 1, 2, \ldots$. From [28, p.238] we source a means for finding the weight function associated with each of the classical orthogonal polynomials. We consider a sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ which satisfies a differential equation expressed as

$$lp_n = (l_{22}x^2 + l_{21}x + l_{20})p_n'' + (l_{11}x + l_{10})p_n' = \lambda_n p_n,$$

where

$$\lambda_n = l_{11}n + l_{22}n(n-1), \qquad n = 0, 1, 2, \ldots.$$

The value l_{ij} is an element taken from the matrix representation of operator l. All four of the polynomials studied in this investigation, so those of Jacobi, Legendre, Hermite and the generalised Laguerre, are of the form given above.

The weight function must be such that the differential expression wl be symmetric over $\{p_n\}_{n=0}^{\infty}$. That is to say, $wl = (wl)^*$, which is expanded as

$$(wa_2)y'' + (wa_1)y' = (wa_2y)'' - (wa_1y)',$$

where

$$a_2 = l_{22}x^2 + l_{21}x + l_{20} \qquad \text{and} \qquad a_1 = l_{11}x + l_{10}.$$

An equivalent constraint is given by

$$S_0 = (wa_2)' - (wa_1) = 0, \tag{5.1.2}$$

which is solved by

$$w = \frac{C}{a_2} exp\Big\{ \int \frac{a_1}{a_2} dx \Big\},$$

where C is arbitrary. Most often, C is chosen to be 1.

A theorem regarding the orthogonalization procedure is taken from [35, p.23], and presented here next.

Theorem 5.1.3. *Let the real-valued functions*

$$f_0(x), f_1(x), f_2(x), \dots, f_l(x), \tag{5.1.3}$$

with l being finite or infinite, be of the class $\mathcal{L}^2_\alpha(a,b)$ and linearly independent. Then an orthonormal set

$$\phi_0(x), \phi_1(x), \phi_2(x), \dots, \phi_l(x) \tag{5.1.4}$$

exists such that, for $n = 0, 1, 2, \dots, l$,

$$\phi_n(x) = \lambda_{n0} f_0(x) + \lambda_{n1} f_1(x) + \cdots + \lambda_{nn} f_n(x), \qquad \lambda_{nn} > 0. \tag{5.1.5}$$

The set (5.1.4) *is uniquely determined.*

The *orthogonalization* procedure refers to the process whereby (5.1.4) is derived from (5.1.3). More commonly this procedure is referred to as *Gram-Schmidt orthogonalization* (see [36, p.134].

The following theorem implies that all sequences of orthogonal polynomials satisfy a three-term recurrence relation. It may be found in [3, p.244] or [35, p.42]. The converse of this theorem also holds and is known as *Favard's Theorem.*

Theorem 5.1.4. *A sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies*

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) + C_n p_{n-1}(x) \qquad n = 1, 2, 3, ...$$

where

$$A_n = \frac{k_{n+1}}{k_n} \quad n = 0, 1, 2, ... \qquad \textit{and} \qquad C_n = -\frac{A_n}{A_{n-1}} \frac{h_n}{h_{n-1}} \quad n = 1, 2, 3, \dots.$$

Proof. For each $n \in 0, 1, 2, \dots$ we have that $p_n(x)$ is of degree n, from which we deduce that the sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ exhibits linear independence. Now, by choosing $A_n = \frac{k_{n+1}}{k_n}$, the expression $p_{n+1}(x) - A_n x p_n(x)$ represents a polynomial of degree at most n. Hence

$$p_{n+1}(x) - A_n x p_n(x) = \sum_{k=0}^{n} c_k p_k(x).$$

The orthogonality property suggests that

$$\langle p_{n+1}(x) - A_n x p_n(x), p_k(x)\rangle = \sum_{m=0}^{n} c_m \langle p_m(x), p_k(x)\rangle = c_k \langle p_k(x), p_k(x)\rangle = c_k h_k$$

which, in turn, implies that for $k = 1, 2, \ldots, n$

$$\begin{aligned} h_k c_k &= \langle p_{n+1}(x) - A_n x p_n(x), p_k(x)\rangle \\ &= \langle p_{n+1}(x), p_k(x)\rangle - A_n \langle x p_n(x), p_k(x)\rangle \\ &= -A_n \langle p_n(x), x p_k(x)\rangle. \end{aligned}$$

For $k < n-1$ we have that the degree of $xp_k(x)$ is strictly less than n, which implies that $\langle p_n(x), xp_k(x)\rangle = 0$. Hence, $c_k = 0$ for $k < n-1$. This proves that the polynomials satisfy the three-term recurrence relation

$$p_{n+1}(x) - A_n x p_n(x) = c_n p_n(x) + c_{n-1} p_{n-1}(x) \qquad n = 1, 2, 3, \ldots.$$

Furthermore

$$h_{n-1} c_{n-1} = -A_n \langle p_n(x), x p_{n-1}(x)\rangle = -A_n \frac{k_{n-1}}{k_n} h_n$$

which implies that

$$c_{n-1} = -\frac{A_n}{A_{n-1}} \frac{h_n}{h_{n-1}}.$$

□

It should be noted that for a sequence of monic orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$, the three-term recurrence relation has the form

$$p_{n+1}(x) = x p_n(x) + B_n p_n(x) + C_n p_{n-1}(x) \quad \text{with} \quad C_n = -\frac{h_n}{h_{n-1}}, \quad n = 1, 2, 3, \ldots. \tag{5.1.6}$$

Also, we have from [3, p.246] that the $\mathcal{L}^2$ norm of $p_n(x)$ may be expressed as

$$h_n = h_0 \frac{c_1 c_2 \ldots c_n}{a_0 a_1 \ldots a_{n-1}},$$

and is derived from the three-term recurrence as suggested by the following result, taken from [3, p.245].

Corollary 5.1.5.

$$h_n = \frac{A_0}{A_n} C_1 C_2 \dots C_n h_0.$$

A consequence of the three-term recurrence relation is the following formula, found in [35, p.43], with a proof given there as well.

Theorem 5.1.6. *A sequence of orthogonal polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *satisfies the* Christoffel-Darboux formula *given by*

$$\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{k_n}{h_n k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y}, \tag{5.1.7}$$

as well as its confluent form represented by

$$\sum_{k=0}^{n} \frac{[p_k(x)]^2}{h_k} = \frac{k_n}{h_n k_{n+1}} \left[p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)\right], \tag{5.1.8}$$

for $n = 0, 1, 2, \dots$.

Proof. Multiplying (5.1.6) through by $p_n(y)$ produces

$$p_{n+1}(x)p_n(y) = (A_n x + B_n)p_n(x)p_n(y) + C_n p_{n-1}(x)p_n(y)$$

and, similarly,

$$p_{n+1}(y)p_n(x) = (A_n y + B_n)p_n(y)p_n(x) + C_n p_{n-1}(y)p_n(x)$$

which, when subtracted from one another, results in

$$p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x) = A_n(x-y)p_n(x)p_n(y) + C_n[p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)].$$

Now, using the fact that $C_n = -\frac{A_n}{A_n - 1}\frac{h_n}{h_{n-1}}$ as defined in (5.1.4), dividing the equation above through by A_n and taking the sum from 1 to n, generates the telescoping series

$$\begin{aligned}(x-y)\sum_{k=1}^{n} \frac{p_k(x)p_k(y)}{h_k} &= \sum_{k=1}^{n} \frac{p_{k+1}(x)p_k(y) - p_{k+1}(y)p_k(x)}{A_k h_k} - \sum_{k=1}^{n} \frac{p_k(x)p_{k-1}(y) - p_k(y)p_{k-1}(x)}{A_{k-1}h_{k-1}} \\ &= \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{A_n h_n} - \frac{k_0^2(x-y)}{h_0}.\end{aligned}$$

This implies that

$$(x-y)\sum_{k=1}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{A_n h_n} - \frac{k_0^2(x-y)}{h_0}$$

$$= \frac{k_n}{h_n k_{n+1}} \left[p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x) \right]$$

and so (5.1.7) is proved. The confluent form is easily obtained by taking the limit $y \to x$ so that we have

$$\begin{aligned}
&\lim_{y \to x} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x - y} \\
&\qquad = \lim_{y \to x} \frac{p_n(x)(p_{n+1}(x) - p_{n+1}(y)) - p_{n+1}(x)(p_n(x) - p_n(y))}{x - y} \\
&\qquad = p_n(x)p'_{n+1}(x) - p_{n+1}(x)p'_n(x).
\end{aligned}$$

□

5.2 The Zeros of Orthogonal Polynomials

The locations of the zeros of orthogonal polynomials are of particular importance as this information provides a means of finding the eigenvalues associated with each particular eigenfunction, thereby enabling us to derive an explicit description of the spectral characteristics.

5.2.1 Results from Kellogg

Orthogonal polynomials appear frequently in the solutions to problems in mathematical physics. It is commonly seen that a particular polynomial from a collection of orthogonal functions experiences one more sign change than the preceding polynomial of one lesser degree, when considered in the same interval.

For example, consider the interval $(0, 1)$. Let $\phi_0(x) = 1$ and let $\phi_1(x)$ be defined on $(0, 1)$ by

$$\phi_1(x) = \begin{cases} 27x - 8 & \text{for} \quad 0 \leq x \leq \frac{1}{3}, \\ 18x - 5 & \text{for} \quad \frac{1}{3} \leq x \leq \frac{1}{2}, \\ -18x + 13 & \text{for} \quad \frac{1}{2} \leq x \leq \frac{2}{3}, \\ 1 & \text{for} \quad \frac{2}{3} \leq x \leq 1. \end{cases}$$

Furthermore, let $\phi_2(x) = \phi_1(1-x)$. All three functions are orthogonal to one another on $(0,1)$. The first function is a constant, and as such does not vanish. The second function changes sign once, as expected. However, the third function changes sign only once as well, and not twice as intuition would have suggested.

Kellogg begins to explain this apparent anomaly by letting the particular point at which $\phi_2(x)$ has a zero be called $x = a$. Given that the three functions are orthogonal, it is not possible for either of $\phi_0(x)$ or $\phi_1(x)$ to have a zero at point $x = a$ as well. This would mean that the function $\phi_0(a)\phi_1(x) - \phi_1(a)\phi_0(x)$ (i.e. a linear combination, with constants $\phi_0(a)$ and $\phi_1(a)$) is orthogonal to $\phi_2(x)$ and does not vanish at $x = a$ either. Now, choosing any two points $x_o < x_1$ lying inside of $(0,1)$, and assuming that the function $\phi_0(x_0)\phi_1(x_1) - \phi_1(x_1)\phi_0(x_0) > 0$ (that is, $\phi_1(x)$ is increasing between x_0 and x_1). It is then required that $\phi_2(x)$ have two zeros in $(0,1)$, by construction of $\phi_2(x)$.

Kellogg notes that for the linear combination $c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x)$, coefficients $c_0, c_1, \ldots, c_n$ can be found such that this function will intersect another function $f(x)$ in the interval $n+1$ times, as long as both functions do not vanish simultaneously.

5.2.2 Oscillation Properties

A set of functions, each of which is real and continuous in $(0,1)$, is denoted by $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$. Further, this set is orthonormal. Now, for any $x_0 < x_1 < \cdots < x_n$ in $(0,1)$, the determinants that are formulated by

$$D(x_0, \ldots, x_n) = \begin{vmatrix} \phi_0(x_0) & \phi_1(x_0) & \ldots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \ldots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \ldots & \phi_n(x_n) \end{vmatrix} \tag{5.2.9}$$

are positive for $n = 0, 1, 2, \ldots$. We note, however, that interchanging two elements of the sequence $x_0, \ldots, x_n$ multiplies (5.2.9) by a factor of (-1). Moreover, it is clear that $D_0(x_0) = \phi_0(x_0)$. An elementary requirement on (5.2.9) is that it be non-zero for orthogonal functions. We will make use of the function

$$\Phi_{m,n}(x) = c_m\phi_m(x) + c_{m+1}\phi_{m+1}(x) + \cdots + c_n\phi_n(x) \qquad m \leq n. \tag{5.2.10}$$

Let $n+1$ distinct points inside of $(0, 1)$ be represented by $x_0, x_1, \ldots, x_n$, then $\Phi_{0,n}(x)$ can be constructed to take on particular values at these points by choice of coefficients $c_0, c_1, \ldots, c_n$. This result is inferred by the continuity of polynomials. Suppose then that $\Phi_{0,n}(x)$ is zero at each of n distinct points, then the sign of $\Phi_{0,n}(x)$ will change at each of these points. This result follows since $\Phi_{0,n}(x)$ is equal to $D(x_0, x_1, \ldots, x_{n-1})$ up to a non-zero factor. Also, the function $\phi_n(x)$ cannot have more than n zeros, else this could result in the determinant $D(x_0, x_1, \ldots, x_n)$ vanishing as well.

Theorem 5.2.1. [27] *It is not possible for $\Phi_{0,n}(x)$ to vanish at $n+1$ distinct points in the interior of $(0, 1)$ without vanishing identically.*

If $\Phi_{0,n}(x)$ did indeed vanish at $n+1$ distinct points in $(0, 1)$ then $\phi_0(x)$ would have to be zero, in which case more than one of the orthogonal functions comprising $\Phi_{0,n}(x)$ would have a zero at the same point in $(0, 1)$. We know that this cannot happen for orthogonal polynomials.

Theorem 5.2.2. [27] *Every continuous function $\psi(x)$ orthogonal to $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ on the interval $(0, 1)$ changes sign at least $n + 1$ times.*

Proof. A function $\psi(x)$ is assumed to have zeros $x_0, x_1, \ldots, x_k$ only, where $k < n$, then a function $\Phi_{0,k+1}(x)$ could be found having these same zeros, i.e. both functions having $k + 1 \leq n$ zeros. In addition, at another point x_{k+1}, the function $\Phi_{0,k+1}(x)$ would exhibit the same sign as that of $\psi(x)$ (i.e. $\phi_{k+1}(x_{k+1}) = 0$). Given that $k+1 \leq n$, each term comprising $\Phi_{0,k+1}(x)$ would be orthogonal to $\psi(x)$ (by the assumption in the statement) which in turn would imply that $\Phi_{0,k+1}(x)$ is orthogonal to $\psi(x)$.

However, this cannot be, since $\psi(x)\Phi_{0,k+1}(x)$ is continuous and does not vanish identically. Also, functions $\psi(x)$ and $\Phi_{0,k+1}(x)$ have the same signs in $(0,1)$. Thus, $\psi(x)$ cannot have n zeros or less if it is orthogonal to each of $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$.

□

The fact that $\phi_n(x)$ cannot vanish more than n times together with the result from Theorem 5.2.2 imply that $\phi_n(x)$ changes sign at each of exactly n zeros. The system comprised of functions $\phi_0(x), \phi_1(x), \ldots, \phi_n(x), \ldots$ forms a closed set with regards to all continuous functions that change sign a finite number of times [27]. For any constants $c_m, c_{m+1}, \ldots, c_n$, not all zero, the function (5.2.10) has no more than n and no less than m sign changes for $m \leq n$ [27].

5.2.3 Separation Theorems

The results given next describe the interaction between two polynomials of the same family. The first given here is seen in [3, p.253] or [35, p.44].

Theorem 5.2.3. *If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval (a,b) with respect to the weight function $w(x)$, then the polynomial $p_n(x)$ has exactly n simple, real roots on the interval (a,b).*

Proof. The polynomial $p_n(x)$ has at most n roots since the degree of $p_n(x)$ is n. Suppose that $p_n(x)$ has $m \leq n$ distinct real roots $x_1, x_2, \ldots, x_m$ in the interval (a,b) of odd order each. The polynomial

$$p_n(x)(x - x_1)(x - x_2)\ldots(x - x_m)$$

is constructed such that all of the factors are of even multiplicity. This means that the polynomial does not change sign on (a,b). Consequently,

$$\int_a^b w(x)p_n(x)(x - x_1)(x - x_2)\ldots(x - x_m)dx \neq 0.$$

If $m < n$ then, by the property of orthogonality, the above integral equals zero. Therefore, we must have that $m = n$, so that $p_n(x)$ has n distinct real roots of odd order in (a,b). This proves that all n zeros are distinct and simple. □

The next result, regarding the interlacing property of orthogonal polynomials may be found in [3, p.253] or [35, p.46].

Theorem 5.2.4. *If* $\{p_n(x)\}_{n=0}^{\infty}$ *is a sequence of orthogonal polynomials on the interval* (a, b) *with respect to the weight function* $w(x)$*, then the zeros of* $p_n(x)$ *and* $p_{n-1}(x)$ *separate each other.*

Proof. By the definition of orthogonal polynomials

$$h_n = \int_a^b w(x)[p_n(x)]^2 dx > 0 \qquad n = 0, 1, 2, \dots.$$

This suggests that we must have $[p_n(x)]^2 > 0$, since the weight function $w(x)$ is known to be continuous and positive on (a, b). Then by the (5.1.8)

$$\frac{k_n}{k_{n+1}} \cdot \big(p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)\big) > 0.$$

Denote two consecutive zeros of $p_n(x)$ by $x_{n,k} < x_{n,k+1}$, so that

$$p_n(x_{n,k}) = p_n(x_{n,k+1}) = 0.$$

Then by Theorem 5.2.3, $p_n(x)$ has n real simple zeros exactly, in (a, b). It follows that one of $p'_n(x_{n,k})$ and $p'_n(x_{n,k+1})$ is positive, while the other is negative, so that

$$p'_n(x_{n,k}) \cdot p'_n(x_{n,k+1}) < 0.$$

Consequently, the inequality

$$p_{n-1}(x_{n,k}) \cdot p_{n-1}(x_{n,k+1}) < 0$$

is evident given that $p_{n-1}(x)$ and $p'_n(x)$ are of the same degree. As n is arbitrary, it is also true that

$$p_{n+1}(x_{n,k}) \cdot p_{n+1}(x_{n,k+1}) < 0.$$

At least one zero of $p_{n+1}(x)$ can be found in between $x_{n,k}$ and $x_{n,k+1}$, since $p_{n+1}(x)$ is defined for all $x \in (a, b)$. This argument can be extended to any two of the zeros of $p_n(x)$ that follow immediately one after the other. $\square$

Represent the consecutive zeros of $p_n(x)$ and $p_{n+1}(x)$ by $\{x_{n,k}\}_{k=1}^{n}$ and $\{x_{n+1,k}\}_{k=1}^{n+1}$ respectively, then the interlacing property of orthogonal polynomials can be depicted as

$$a < x_{n+1,1} < x_{n,1} < x_{n+1,2} < x_{n,2} < \cdots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1}, b.$$

Kellogg also states and proves separation results in [27].

5.3 Gauss-Jacobi Mechanical Quadrature

At times it may not be possible to determine the precise value of an integral. The interpolation method was introduced to deal with this difficulty, whereby an interpolating polynomial is created as an approximation to the original function, and then integrated to provide an approximation to the sought after integral. From [3, p.249] we have a definition.

Definition 5.3.1. *The Lagrange interpolation polynomial is a polynomial of degree $n-1$ that takes the value $f(x_i)$ at x_i for $i = 1, \ldots, n$. This polynomial is given by*

$$P_n(x) = \sum_{i=1}^{n} f(x_i) \frac{p(x)}{p'(x_i)(x - x_i)}, \tag{5.3.11}$$

where

$$p(x) = (x - x_1) \ldots (x - x_n).$$

As given in [3, p.248] and [35, p.47], we now state the principal result in this section.

Theorem 5.3.2. *The* Gauss Quadrature Formula *is expressed as*

$$\int_a^b w(x) f(x) dx = \sum_{i=1}^{n} \lambda_{n,i} f(x_i),$$

with

$$\lambda_{n,i} := \int_a^b \frac{w(x) p_n(x)}{(x - x_i) p_n'(x_i)} dx$$

for $i = 1, 2, \ldots, n$.

Proof. If f is a continuous function on (a, b) and $x_1 < x_2 < \cdots < x_n$ are n distinct points in (a, b), then there exists exactly one polynomial P with degree $\leq n-1$ such that $P(x_i) = f(x_i)$ for all $i = 1, 2, \ldots, n$. Lagrange interpolation (Definition 5.3.1) is used to find this polynomial P. But the derivative of $p(x)$ (also as in Definition 5.3.1) is of a degree one less than that of $p(x)$ and would thus have one less factor. Say, the i^{th} factor. So then

$$P(x) = \sum_{i=1}^{n} f(x_i) \frac{(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)}{(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)}.$$

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials on the interval (a, b) with respect to the weight function $w(x)$. Then $x_1 < x_2 < \cdots < x_n$ are taken to be the n distinct real zeros of the polynomial $p_n(x)$. If f is a polynomial of degree $\leq 2n - 1$, then $f(x) - P(x)$ is a polynomial of degree $\leq 2n - 1$ with at least the zeros $x_1 < x_2 < \cdots < x_n$, i.e. the zeros common to both polynomials $f(x)$ and $P(x)$. Next, let

$$f(x) = P(x) + r(x)p_n(x),$$

where $r(x)$ is a polynomial of degree $\leq n - 1$. This may also be represented as

$$f(x) = \sum_{i=1}^{n} f(x_i) \frac{p_n(x)}{(x - x_i)p_n'(x_i)} + r(x)p_n(x).$$

Multiplying through by some weight function $w(x)$ and integrating over (a, b) gives

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} f(x_i) \int_a^b \frac{w(x)p_n(x)}{(x - x_i)p_n'(x_i)} dx + \int_a^b w(x)r(x)p_n(x)dx.$$

Recall that the degree of $r(x)$ is $n - 1$ and thus, by the property of orthogonality, the second integral on the right-hand side vanishes. What remains is the desired result. □

If f is a polynomial of degree less than or equal to $2n - 1$ and the value of $f(x_i)$ is known for the zeros $x_1 < x_2 < \cdots < x_n$ of the polynomial $p_n(x)$, then the Gauss quadrature formula can be used to calculate the value of the integral. Else, an approximation to the integral may be found instead. The same formula can be used in finding this approximation. The coefficients $\{\lambda_{n,i}\}_{i=1}^{n}$ are known as *Christoffel* *numbers* and do not depend on the function f. The next theorem is taken from [35, p.48].

Theorem 5.3.3. *The Christoffel numbers* λ_n *are all positive, and*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \int_a^b d\alpha(x) = \alpha(b) - \alpha(a).$$

The following representations hold:

$$\begin{aligned}
\lambda_v &= \int_a^b \left(\frac{p_n(x)}{p_n'(x_v)(x - x_v)} \right)^2 d\alpha(x), \\
\lambda_v &= \frac{k_{n+1}}{k_n} \frac{-1}{p_{n+1}(x_v)p_n'(x_v)} = \frac{k_n}{k_{n-1}} \frac{1}{p_{n-1}(x_v)p_n'(x_v)}, \\
\lambda_v^{-1} &= [p_0(x_v)]^2 + [p_1(x_v)]^2 + \cdots + [p_n(x_v)]^2 = K_n(x_v, x_v).
\end{aligned}$$

The following discussion demonstrates the theorem.

Proof. Let

$$l_{n,i}(x) := \frac{p_n(x)}{(x - x_i)p_n'(x_i)}, \qquad i = 1, 2, \ldots, n,$$

then

$$\lambda_{n,i} = \int_a^b w(x) l_{n,i}(x) dx, \qquad i = 1, 2, \ldots, n.$$

By the assumption above, the polynomial $l_{n,i}^2 - l_{n,i}$ is of degree at most $2n-2$. This polynomial vanishes when $p_n(x) = 0$. Let $q(x)$ represent some polynomial of degree at most $n-2$ and write

$$l_{n,i}^2 - l_{n,i} = p_n(x)q(x).$$

The orthogonality relation suggests that

$$\int_a^b w(x)p_n(x)q(x)dx = 0,$$

implying that

$$\int_a^b w(x) \left[l_{n,i}^2 - l_{n,i} \right] dx = 0.$$

Thus, by substituting back and again by the orthogonality condition, we have that

$$\lambda_{n,i} = \int_a^b w(x) l_{n,i}(x) dx = \int_a^b w(x)[l_{n,i}(x)]^2 dx > 0.$$

□

This next result is with regards to the zeros of orthogonal polynomials and is found in [3, p.253]). The proof invokes Gauss quadrature.

Theorem 5.3.4. *Let* $\{p_n(x)\}_{n=0}^{\infty}$ *be a sequence of orthogonal polynomials on the interval* (a, b) *with respect to the weight function* $w(x)$ *and let* $m < n$*. Then between any two zeros of* $p_m(x)$ *there is at least one zero of* $p_n(x)$*.*

Proof. The result is proved by contradiction. Assume that no zero of $p_n(x)$ is found in the interval bounded by two consecutive zeros of $p_m(x)$, i.e. $(x_{m,k}, x_{m,k+1})$. Suppose

$$g(x) = \frac{p_m(x)}{(x - x_{m,k})(x - x_{m,k+1})}.$$

Obviously, $g(x)$ is a polynomial that does not possess either of the zeros $x_{n,k}$ or $x_{n,k+1}$. Further, suppose $x \notin (x_{m,k}, x_{m,k+1})$ so that $x > x_{m,k}, x_{m,k+1}$ or $x < x_{m,k}, x_{m,k+1}$, implying that $(x - x_{m,k})(x - x_{m,k+1}) > 0$ in both instances. Hence

$$g(x)p_m(x) = \frac{[p_m(x)]^2}{(x - x_{m,k})(x - x_{m,k+1})} \geq 0.$$

Integrating over (a, b) yields

$$\int_a^b w(x)g(x)p_m(x)dx = \sum_{i=1}^{n} \lambda_{n,i} g(x_{n,i})p_m(x_{n,i}),$$

by the Gauss quadrature formula. But $\{x_{n,i}\}_{i=1}^{n}$ are the zeros of $p_n(x)$. The absence of zeros of $p_n(x)$ in $(x_{n,k}, x_{n,k+1})$ is true by our initial assumption, so that for all $i = 1, \ldots, n$ it follows that $g(x_{n,i})$ and $p_m(x_{n,i})$ are of the same sign. Also, the Christoffel numbers $\lambda_{n,i}$ are strictly positive for all $i = 1, \ldots, n$, as was shown before. Hence, the right-hand side sum cannot vanish (see [3, p.253]). However, the orthogonality condition dictates that the left-hand side integral must vanish as $g(x)$ and $p_m(x)$ are not of the same degree. A contradiction is achieved rendering our initial assumption false. It may be concluded that there exists, between any two consecutive zeros of $p_m(x)$, at least one zero of $p_n(x)$. □

5.4 Completeness of Orthogonal Polynomials

We now present two results, sourced from [3, pp.306-308], that are relevant for demonstrating the completeness of orthogonal polynomials. Proving the completeness of polynomial families such as the Hermite or Laguerre class of polynomials depends primarily on the uniqueness of the Fourier transforms of integrable functions. The first theorem is from [3, p.306] and is regarding the uniqueness of Fourier transforms.

Theorem 5.4.1. *If f is integrable on $(-\infty, \infty)$ and if*

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{ixt}dt \equiv 0 \tag{5.4.12}$$

then $f = 0$ almost everywhere.

Proof. Assuming (5.4.12) is given, then it is also true that

$$\int_{-\infty}^{\infty} f(t)e^{ix(t-a)}dt \equiv 0.$$

Certainly, if $a \in \mathbb{R}$, then

$$\int_{-\infty}^{a} f(t)e^{ix(t-a)}dt = -\int_{a}^{\infty} f(t)e^{ix(t-a)}dt. \tag{5.4.13}$$

Two new complex valued functions are defined by

$$L(z) = \int_{-\infty}^{a} f(t)e^{ix(t-a)}dt \qquad R(z) = -\int_{a}^{\infty} f(t)e^{ix(t-a)}dt.$$

Evidently, $L(z)$ exists for $\Im(z) \leq 0$ and is analytic in $\Im(z) < 0$, while $R(z)$ exists for $\Im(z) \geq 0$ and is analytic in $\Im(z) > 0$. Moreover, as a consequence of the Dominated Convergence Theorem 5.5.1 we have that

$$\lim_{y\to\infty} F(iy) = -\lim_{y\to\infty} \int_{a}^{\infty} f(t)e^{-y(t-a)}dt = 0.$$

Ergo, $F(z) \equiv 0$ and, in particular, $F(0) = 0$, which implies - as a is non-specific - that

$$\int_{-\infty}^{a} f(t)dt = 0,$$

for all $a \in \mathbb{R}$. Hence, $f = 0$ almost everywhere (a.e.). □

A function $p(t)$ is constructed, as recommended by [3, p.307], which is square integrable and disappears at an exponential rate when approaching infinity. Define

$$p(t) = O(e^{-\alpha|t|}) \quad \text{for some } \alpha > 0 \text{ as } |t| \to \infty. \tag{5.4.14}$$

This leads into the next theorem from [3, p.307] which is a result on the uniqueness of integrable functions.

Theorem 5.4.2. *Let $-\infty \leq a < b \leq \infty$. Let $p(t) \in \mathcal{L}^2(a,b)$, with $p(t)$ different from zero a.e., and let $p(t)$ satisfy* (5.4.14)*, if $a = -\infty$ or $b = \infty$. If $f \in \mathcal{L}^2(a,b)$ and*

$$\int_a^b t^n f(t)p(t)dt = 0 \qquad n = 0, 1, 2, \ldots, \tag{5.4.15}$$

then $f = 0$ a.e..

Proof. With $z = x + iy$ the following complex valued function is defined

$$F(z) = \int_a^b e^{izt} P(t) f(t)dt.$$

The function F is entire (i.e. holomorphic over the whole of the complex plane) when $-\infty < a < b < \infty$. Else, F is merely analytic in $-\alpha < y < \alpha$. The n^{th} derivative is thus

$$F^{(n)}(z) = i^n \int_a^b e^{izt} t^n p(t) f(t)dt.$$

By assumption (5.4.15) is true, whence we have that $F^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$. Hence $F(z) = 0$ in $-\alpha < y < \alpha$. In particular,

$$F(x) = \int_a^b e^{ixt} p(t) f(t)dt = 0.$$

It follows, by the uniqueness of the Fourier transform and the integrability of $p(t)f(t)$ on interval (a, b), that $p(t)f(t) = 0$. But, by construction, $p(t)$ is non-zero a.e., so that, necessarily, $f(t) = 0$ a.e. □

5.5 Useful Results

The Dominated Convergence Theorem is sourced from [26, p.19].

Theorem 5.5.1. *Let (X, A, μ) be a measure space. Let $\{f_n\}$ be a sequence of complex measurable functions on X that converge pointwise to the function f. Suppose there exists $g \in \mathcal{L}'(\mu)$ (with values in $[0, \infty)$) such that*

$$|f_n| \leq g \qquad n = 1, 2, \ldots .$$

Then $f, f_n \in \mathcal{L}'(\mu)$ for all n, and $f_n \to f$ in the $\mathcal{L}'(\mu)$-metric.

Leibniz's identity for the derivative will be employed in subsequent derivations and is found in [26, p.371].

Lemma 5.5.2.

$$D^n[f(x)g(x)] = \sum_{k=0}^{n} \binom{n}{k} D^k f(x) D^{n-k} g(x) \qquad n = 0, 1, \ldots . \tag{5.5.16}$$

Proof. An inductive proof is used. The case $n = 0$ is trivial. The case $n = 1$ is easily shown to be

$$\begin{aligned}\sum_{k=0}^{1} \binom{1}{k} D^k f(x) D^{1-k} g(x) &= \binom{1}{0} D^0 f(x) D^1 g(x) + \binom{1}{1} D^1(x) D^0 g(x) \\ &= f(x)g'(x) + f'(x)g(x).\end{aligned}$$

Assuming (5.5.16) true for $n = m$, it is required to prove that it also holds for $n = m + 1$. So

$$\begin{aligned}D^{m+1}[f(x)g(x)] &= D\{D^m[f(x)g(x)]\} \\ &= D\left\{\sum_{k=0}^{m} \binom{m}{k} D^k f(x) D^{m-k} g(x)\right\} \\ &= D\left\{\binom{m}{0} D^0 f(x) D^m g(x) + \cdots + \binom{m}{m} D^m f(x) D^0 g(x)\right\} \\ &= \binom{m}{0} f(x) D^{m+1} g(x) + \left[\binom{m}{0} + \binom{m}{1}\right] Df(x) D^m g(x) + \ldots \\ &\quad \cdots + \left[\binom{m}{m-1} + \binom{m}{m}\right] D^m f(x) Dg(x) + \binom{m}{m} D^{m+1} f(x) g(x).\end{aligned} \tag{5.5.17}$$

Pascal's triangle identity,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \qquad k = 1, 2, \ldots, n,$$

and keeping in mind that $\binom{m}{0} = \binom{m}{m} = 1 \quad \forall m \in \mathbb{Z}$, suggests that (5.5.17) be reformulated as

$$\begin{aligned} D^{m+1}[f(x)g(x)] &= \binom{m+1}{0} f(x)D^{m+1}g(x) + \binom{m+1}{1} Df(x)D^m g(x) + \dots \\ &\quad \dots + \binom{m+1}{m} D^m f(x)Dg(x) + \binom{m+1}{m+1} D^{m+1}f(x)g(x) \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} D^k f(x)D^{m-k+1}g(x), \end{aligned} \tag{5.5.18}$$

achieving the desired result. □

Definition 5.5.3. [3, p.64] *The* hypergeometric function ${}_2F_1(a,b;c;x)$ *is defined by the series*

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

for $|x| < 1$*, and by continuation elsewhere.*

Euler's integral representation of the hypergeometric function (1769) can be found in [3, p.65]. The Pochhammer symbol, found in [3, p.2], is defined as

$$(x)_n = \begin{cases} 1 & \text{if} \quad n = 0, \\ x(x+1)\dots(x+n-1) & \text{if} \quad n > 0. \end{cases} \tag{5.5.19}$$

Theorem 5.5.4. [19, p.175] *Let* $x(\lambda)$ *be an analytic function for all* $\lambda \in \mathbb{C}$ *(such functions are called entire functions) which is assumed to be uniformly bounded, that is, there exists* $C > 0$ *so that* $\|x(\lambda)\| \leq C$ *for all* $\lambda \in \mathbb{C}$*. Then* $x(\lambda)$ *is constant.*

In [3, p.6] the gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \qquad \text{for} \quad \Re(x) > 0. \tag{5.5.20}$$

Note, the following useful property of this function given in [3, p.3] as

$$\Gamma(n+1) = n! \qquad n \geq 0, \quad n \in \mathbb{Z}. \tag{5.5.21}$$

Chapter 6

The Jacobi Polynomials

6.1 Introduction

The Jacobi polynomials are denoted by $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ and satisfy the ordinary differential equation

$$ly = -\frac{[(1-x)^{\alpha+1}(1+x)^{\beta+1}y']'}{(1-x)^{\alpha}(1+x)^{\beta}} = n(n+\alpha+\beta+1)y \tag{6.1.1}$$

as can be seen in [3], [6], [28, p.239] and [35].

The weight function pertaining to $P_n^{(\alpha,\beta)}(x)$ follows a beta distribution (see [3, p.300]) and is expressed as $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, which is proved in [3, p.299]. The weight function suggests the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}] \qquad n = 0, 1, \ldots. \tag{6.1.2}$$

The polynomials can be expanded and written explicitly using the following series representations (see [32, p.255])

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^k$$

which can quite easily be manipulated into other representations, which can be found in [28, p.240]. The various series representations for $P_n^{(\alpha,\beta)}(x)$ are found by taking the n^{th} derivative of $w(x)$ and then implementing Leibniz's rule. So

$$\begin{aligned}
&\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] \\
&\qquad = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(1-x)^{n+\alpha}\frac{d^{n-k}}{dx^{n-k}}(1+x)^{n+\beta} \\
&\qquad = \sum_{k=0}^{n}\binom{n}{k}(-1)^k(n+\alpha)(n+\alpha-1)\dots(n+\alpha-k+1)(1-x)^{n+\alpha-k} \\
&\qquad\qquad \times(n+\beta)(n+\beta-1)\dots(\beta+k+1)(1+x)^{\beta+k} \\
&\qquad = n!\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-x)^{n+\alpha+k}(1+x)^{\beta+k},
\end{aligned}$$

for $n=0,1,\dots$. This last expression is substituted into (6.1.2) to give (as per [32, p.255])

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-x)^{n-k}(1+x)^k \qquad n=0,1,\dots. \tag{6.1.3}$$

It is evident from this expression that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n. An interesting feature of $P_n^{(\alpha,\beta)}(x)$ is that it exhibits symmetry (see [21, p.30]) in the form of

$$P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(x) \qquad n=0,1,\dots. \tag{6.1.4}$$

Furthermore, it is obvious that [32, p.254]

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n} \quad \text{and} \quad P_n^{(\alpha,\beta)}(-1)=(-1)^n\binom{n+\beta}{n} \qquad n=0,1,\dots.$$

A particularly noteworthy characteristic of the classical orthogonal polynomials is that they can each be expressed in terms of the hypergeometric function. A derivation using Gram determinants may be found in [3, p.295]. We, however, refer back to (6.1.3), and noting that $(-1)^n(1-x)^n=(x-1)^n$, write

$$P_n^{(\alpha,\beta)}(x)=\left(\frac{x-1}{2}\right)^n\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x+1}{x-1}\right)^k \qquad n=0,1,\dots, \tag{6.1.5}$$

on condition that $x\neq 1$. This restriction follows through for the duration of the first part of this derivation. Now

$$\left(\frac{x+1}{x-1}\right)^k=\left(1+\frac{2}{x-1}\right)^k=\sum_{i=0}^{k}\binom{k}{i}\left(\frac{2}{x-1}\right)^i \qquad k=0,1,\dots,$$

which, when substituted back into (6.1.5) and rearranging, gives

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{i=0}^{n}\sum_{k=i}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\binom{k}{i}\left(\frac{2}{x-1}\right)^i \quad n = 0, 1, \ldots.$$

Replace k by $k+i$ to obtain

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{i=0}^{n}\sum_{k=0}^{n-i} \binom{n+\alpha}{i+k}\binom{n+\beta}{n-i-k}\binom{i+k}{i}\left(\frac{2}{x-1}\right)^i \quad n = 0, 1, \ldots,$$

and i by $n-i$ to arrive at

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{i=0}^{n}\sum_{k=0}^{n} \binom{n+\alpha}{n-i+k}\binom{n+\beta}{i-k}\binom{n-i+k}{n-i}\left(\frac{2}{x-1}\right)^{n-i} \quad n = 0, 1, \ldots.$$

Since

$$\left[\left(\frac{x-1}{2}\right)\left(\frac{2}{x-1}\right)\right] = 1$$

the preceding equation becomes

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \sum_{i=0}^{n}\sum_{k=0}^{n} \binom{n+\alpha}{n-i+k}\binom{n+\beta}{i-k}\binom{n-i+k}{n-i}\left(\frac{x-1}{2}\right)^i \\
&= \sum_{i=0}^{n}\sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{(n-i+k)!\Gamma(i-k+\alpha+1)} \\
&\quad \times \frac{\Gamma(n+\beta+1)}{(i-k)!\Gamma(n-i+k+\beta+1)}\frac{(n-i+k)!}{(n-i)!k!}\left(\frac{x-1}{2}\right)^i \\
&= \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{i=0}^{n}\frac{(-n)_i}{\Gamma(i+\alpha+1)i!\Gamma(n-i+\beta+1)}\left(\frac{1-x}{2}\right)^i \\
&\quad \sum_{k=0}^{n}\frac{(-i)_k(-i-\alpha)_k}{(n-i+\beta+1)_k k!}. \qquad (6.1.6)
\end{aligned}$$

Note that this representation is now valid for $x = 1$. From [3, p.67], we source the *Chu-Vandermonde* identity

$${}_2F_1(-n, a; c; 1) = \frac{(c-a)_n}{(c)_n}. \qquad (6.1.7)$$

Then, for $i \in \{0, 1, \ldots, n\}$,

$$\sum_{k=0}^{n}\frac{(-i)_k(-i-\alpha)_k}{(n-i+\beta+1)_k k!} = {}_2F_1(-i, -i-\alpha; n-i+\beta+1; 1)$$

$$= \frac{(n+\alpha+\beta+1)_i}{(n-i+\beta+1)_i} \tag{6.1.8}$$

By making use of the simplification

$$\Gamma(n-i+\beta+1)(n-i+\beta+1)_i = \Gamma(n+\beta+1)$$

we have that, for $n = 0, 1, \dots,$

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{\Gamma(n+\alpha+1)}{n!} \sum_{i=0}^{n} \frac{(-n)_i(n+\alpha+\beta+1)_i}{\Gamma(i+\alpha+1)i!} \left(\frac{1-x}{2}\right)^i \\ &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \sum_{i=0}^{n} \frac{(-n)_i(n+\alpha+\beta+1)_i}{i!(\alpha+1)_i} \left(\frac{1-x}{2}\right)^i. \end{aligned} \tag{6.1.9}$$

The conclusion is a hypergeometric function representation for the Jacobi polynomials [32, p.254]

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \qquad n = 0, 1, \dots \tag{6.1.10}$$

true for $x = 1$ as well.

Hypergeometric representation (6.1.10) enables the extraction of the derivative of $P_n^{(\alpha,\beta)}(x)$ in terms of "itself". As an aside, recall the symmetry property (6.1.4), and write

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n \binom{n+\beta}{n} {}_2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right), \qquad n = 0, 1. \dots$$

This formula is also given in [3, p.297]. Now, differentiating $P_n^{(\alpha,\beta)}(x)$ once and using (6.1.10) enables

$$\begin{aligned} \frac{d}{dx} P_n^{(\alpha,\beta)}(x) &= \binom{n+\alpha}{n} (-1) \frac{1}{2} \frac{(-n)(n+\alpha+\beta+1)}{\alpha+1} \\ &\quad \times {}_2F_1\left(-n+1, n+\alpha+\beta+2; \alpha+2; \frac{1+x}{2}\right) \\ &= \frac{n+\alpha+\beta+1}{2} \binom{n+\alpha}{n-1} {}_2F_1\left(-n+1, n+\alpha+\beta+2; \alpha+2; \frac{1-x}{2}\right) \\ &= \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x), \end{aligned} \tag{6.1.11}$$

for $n = 1, 2, \ldots$.

As will be seen later, we require an explicit representation for the leading coefficient of $P_n^{(\alpha,\beta)}(x)$. We again refer to (6.1.10) and write

$$k_n = \binom{n+\alpha}{n} \frac{(-n)_n(n+\alpha+\beta+1)_n}{(\alpha+1)_n} \frac{(-1)^n}{2^n}$$
$$= \frac{(n+\alpha+\beta+1)_n}{2^n n!}, \tag{6.1.12}$$

for non-negative $n \in \mathbb{Z}$. See [3, p.297] or [21, p.30].

The next theorem is with regards to the generating function for the Jacobi polynomials and is taken from [3, p.298]. The proof given there makes use of the Lagrange inversion procedure (see [37, p.138]). But before we proceed, we require the following Lemma (see [3, p.298]).

Lemma 6.1.1. *Suppose that $\phi(y)$ is analytic in a neighbourhood of $y = x$,*

$$t = \frac{y-x}{\phi(y)} = \sum_{n=1}^{\infty} a_n (y-x)^n \qquad a_1 \neq 0 \tag{6.1.13}$$

and f is analytic in a neighbourhood of $y = x$. Then $f(y)$ can be expanded in powers of t:

$$f(y) = f(x) + \sum_{n=1}^{\infty} \frac{d^{n-1}}{dx^{n-1}} \Big\{ f'(x)[\phi(x)]^n \Big\} \frac{t^n}{n!}. \tag{6.1.14}$$

Theorem 6.1.2. *The generating function for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is given by*

$$F(x,t) = 2^{\alpha+\beta} \frac{1}{R}(1-t+R)^{-\alpha}(1+t+R)^{-\beta}$$

when

$$R = \sqrt{1-2xt+t^2}.$$

Proof. Refer to (6.1.13) and begin by taking $\phi(y) = \frac{1}{2}(y^2-1)$, which would mean that

$$t = \frac{2(y-x)}{y^2-1} \quad \Longrightarrow \quad ty^2 - 2 = 2y - 2x,$$

which in turn implies that

$$y = \frac{1}{t} - \frac{1}{t}\sqrt{1-2xt+t^2} = \frac{1}{t} - \frac{R}{t}.$$

Differentiating (6.1.14) with respect to x yields

$$f'(y)\frac{dy}{dx} = f'(x) + \sum_{n=1}^{\infty} \frac{d^n}{dx^n}\left\{f'(x)[\phi(x)]^n\right\}\frac{t^n}{n!}.$$

Now, setting $f'(x) = (1-x)^\alpha(1+x)^\beta$ suggests that

$$\frac{1}{R}(1-y)^\alpha(1+y)^\beta = (1-x)^\alpha(1+x)^\beta + \sum_{n=1}^{\infty}\frac{d^n}{dx^n}\left[2^{-n}(1-x)^\alpha(1+x)^\beta(x^2-1)^n\right]\frac{t^n}{n!}.$$

We divide this equation through by $(1-x)^\alpha(1+x)^\beta$ to arrive at

$$\frac{1}{R}\left(\frac{1-y}{1-x}\right)^\alpha\left(\frac{1+y}{1+x}\right)^\beta = 1+\sum_{n=1}^{\infty}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[2^{-n}(1-x)^\alpha(1+x)^\beta(x^2-1)^n\right]\frac{t^n}{n!}. \tag{6.1.15}$$

Noting that $(x^2-1)^n = (-1)^n(1-x^2)^n$ and recalling the Rodrigues formula for $P_n^{(\alpha,\beta)}(x)$, i.e. (6.1.2), the right-hand side of (6.1.15) becomes

$$1+\sum_{n=1}^{\infty}(-1)^n P_n^{(\alpha,\beta)}(x)\frac{t^n}{n!}.$$

Furthermore, $R^2 = 1-2xt+t^2$. Rearranging this expression, we are presented with

$$\frac{1}{1-x}\left[\frac{R-(1-t)}{t}\right] = \frac{2}{R+(1-t)} \implies \frac{1-y}{1-x} = \frac{2}{1-t+R}.$$

Similarly,

$$\frac{1+y}{1+x} = \frac{2}{1+t+R}.$$

Thus, the left-hand side of (6.1.15) is now

$$\frac{1}{R}\left(\frac{2}{1-t+R}\right)^\alpha\left(\frac{2}{1+t+R}\right)^\beta.$$

Finally, the generating function is attained, i.e.

$$\sum_{n=0}^{\infty}(-1)^n P_n^{(\alpha,\beta)}(x)\frac{t^n}{n!} = 2^{\alpha+\beta}\frac{1}{R}(1-t+R)^{-\alpha}(1+t+R)^{-\beta}. \qquad \square$$

6.2 The Orthogonality Relation

The orthogonality condition for the Jacobi polynomials, in [28, p.241], is

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = 0 \qquad n \neq m \tag{6.2.16}$$

can be derived from (6.1.1). Furthermore, we have the norm square given by [28, p.241], as

$$\int_{-1}^{1} [P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)n!\Gamma(\alpha+\beta+n+1)} \tag{6.2.17}$$

which is obtained by manipulation of (6.3.21), for $m, n \in \{0, 1, \dots\}$ and on condition that $\alpha > -1$ and $\beta > -1$.

The norm square (6.2.17) is verified in the following manner. The expression

$$\int_{-1}^{1} [P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]dx,$$

comes about by introducing the Rodrigues definition (6.1.2). Integration by parts n times and (6.1.12), lends itself to

$$[P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta dx = \frac{(n+\alpha+\beta+1)_n}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx,$$

for $n = 0, 1, \dots$. But

$$\frac{(n+\alpha+\beta+1)_n}{2^{2n}(n!)^2} = \frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)2^{2n}n!}.$$

From [3, p.4] is taken the definition of the beta function.

Definition 6.2.1. *The* Beta Integral *is defined for* $\Re(x) > 0$, $\Re(y) > 0$ *by*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{6.2.18}$$

Thus, letting $1 - x = 2t$ suggests that

$$\begin{aligned}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx &= \int_0^1 (2t)^{n+\alpha}(2-2t)^{n+\beta}2dt \\ &= 2^{2n+\alpha+\beta+1}\int_0^1 t^{n+\alpha}(1-t)^{n+\beta}dt\end{aligned}$$

$$= 2^{2n+\alpha+\beta+1}B(n+\alpha+1, n+\beta+1). \tag{6.2.19}$$

From [3, p.5] we source a gamma function representation for the beta function.

Theorem 6.2.2.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{6.2.20}$$

Therefore

$$\begin{aligned}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx &= 2^{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}\\ &= 2^{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)}\end{aligned}$$

yields the desired relation for $n = 0, 1, 2, \ldots$.

6.3 The Three-Term Recurrence Relation

As was mentioned in the introduction, all orthogonal polynomials satisfy a three-term recurrence relation. The particular relation associated with the Jacobi polynomials, which we have taken from [28, p.241], is

$$\begin{aligned}&2n(\alpha+\beta+n)(\alpha+\beta+2n-2)P_n^{(\alpha,\beta)}(x)\\ &\quad= (\alpha+\beta+2n-1)[\alpha^2-\beta^2+x(\alpha+\beta+2n)(\alpha+\beta+2n-2)]P_{n-1}^{(\alpha,\beta)}(x)\\ &\qquad-2(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n)P_{n-2}^{(\alpha,\beta)}(x).\end{aligned} \tag{6.3.21}$$

6.4 The Associated Boundary Conditions

The boundary conditions are of particular importance as we are interested to know where in $\mathcal{L}^2(-1,1;(1-x)^\alpha(1+x)^\beta)$ the differential operator l is defined. The expression $ly = 0$ is satisfied by $y_1 = 1$ and $y_2 = Q_0^{(\alpha,\beta)}(x)$, with the Jacobi function of the second kind is defined, as in [6], by

$$Q_n^{(\alpha,\beta)}(x) = 2^{-n-1}(x-1)^{-\alpha}(x+1)^{-\beta}\int_{-1}^{1}(1-t)^{n+\beta}(x-t)^{-n-1}dt.$$

Wronskians of y_1 and y_2 give the values at the boundaries $+1$ and -1. So, in the domain of the maximal operator, the boundary values for y are, as seen in [28, p.241],

$$\begin{aligned}
B_{1,1}(y) &= \lim_{x\to 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}[-y'(x)],\\
B_{1,2}(y) &= \lim_{x\to 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}[(Q_0^{(\alpha,\beta)})'y(x) - Q_0^{(\alpha,\beta)}y'(x)],\\
B_{-1,1}(y) &= \lim_{x\to -1}(1-x)^{\alpha+1}(1+x)^{\beta+1}[-y'(x)],\\
B_{-1,2}(y) &= \lim_{x\to -1}(1-x)^{\alpha+1}(1+x)^{\beta+1}[(Q_0^{(\alpha,\beta)})'y(x) - Q_0^{(\alpha,\beta)}y'(x)].
\end{aligned}$$

For all $\alpha > -1$ and $\beta > -1$, the Jacobi polynomials satisfy $B_{1,1}(y) = 0$ and $B_{-1,1}(y) = 0$.

Consider once again the Jacobi operator l. [28] suggests that when $\alpha \in (-1, 1)$, then $x = 1$ is in the limit circle, in which case boundary values $B_{1,1}$ and $B_{1,2}$ both exist. The operator l is in the limit point case when $\alpha \geq 1$. Then $B_{1,1}$ is always zero, termed an *annihilator* boundary value, and $B_{1,2}$ may not exist. A similar scenario arises for the case where $x = -1$.

From [28, p.241], we have a definition for the self-adjoint Jacobi operator in $\mathcal{L}^2(-1, 1; (1-x)^\alpha(1+x)^\beta)$.

Definition 6.4.1. *Denoted by D_L will be those elements y in $\mathcal{L}^2(-1, 1; (1-x)^\alpha(1+x)^\beta)$ that satisfy the following conditions*

(i) y is a.e. differentiable on $(-1, 1)$,

(ii) $(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)$ is a.e. differentiable on $(-1, 1)$ and ly is in $\mathcal{L}^2(-1, 1; (1-x)^\alpha(1+x)^\beta)$ and

(iii) $B_{1,1}(y) = 0$ and $B_{-1,1}(y) = 0$.

Now, by letting $Ly = ly$ for all y in D_L, the Jacobi operator L is defined and it then follows that the eigenfunctions associated with operator L are the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ with eigenvalues $\{\lambda_n = n(n+\alpha+\beta+1)\}_{n=0}^\infty$.

In the introductory chapter, a theorem was given stating the completeness of orthogonal polynomials. The following theorem from [28, p.242] extends this notion to the Jacobi polynomials.

Theorem 6.4.2. *The Jacobi polynomials* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ *form a complete orthogonal set in* $\mathcal{L}^2(-1, 1; (1-x)^{\alpha}(1+x)^{\beta})$.

Proof. Suppose that in $\mathcal{L}^2(-1, 1; (1-x)^{\alpha}(1+x)^{\beta})$ there is to be found an element f which is orthogonal to the sequence $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$. In this case f is orthogonal to $\{x^n\}_{n=0}^{\infty}$ and $e^{i\lambda x}$, for some arbitrary λ, as well. These sequences converge uniformly. So, for all λ,

$$\int_{-1}^{1} f(x)e^{i\lambda x}(1-x)^{\alpha}(1+x)^{\beta}dx = 0.$$

This integral is zero as it is the Fourier transform of a function in $\mathcal{L}^1(-\infty, \infty)$. Therefore

$$f(x)(1-x)^{\alpha}(1+x)^{\beta} = 0$$

a.e. on $(-1, 1)$, which in turm implies that $f = 0$ a.e. on $(-1, 1)$. □

Chapter 7

The Legendre Polynomials

7.1 Introduction

The Legendre polynomials are a particular case of the Jacobi polynomials, when $\alpha = \beta = 0$. Consequently, most of what was stated regarding the Jacobi polynomials applies for the Legendre polynomials as well. We denote the set of Legendre polynomials by $\{P_n(x)\}_{n=0}^{\infty}$ in $\mathcal{L}^2(-1,1)$ and we have that these satisfy the Legendre differential equation

$$ly = -((1-x^2)y')' = \lambda_n y \tag{7.1.1}$$

where

$$\lambda_n = n(n+1), \qquad n = 0, 1, \ldots .$$

as taken from [28, p.243]. A means for defining these polynomials exists in the form of their Rodrigues formula

$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x^2)^n], \qquad n = 0, 1, \ldots . \tag{7.1.2}$$

A weight function of $w(x) = 1$ is applicable here.

The symmetry relation for $P_n(x)$, in accordance with (6.1.4) is

$$P_n(-x) = (-1)^n P_n(x).$$

Other obvious relations are $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ for $n = 0, 1, \ldots$. By (6.1.10), the Hypergeometric representation for the Legendre polynomials is

$$P_n(x) = P_n^{(0,0)}(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right), \qquad n = 0, 1, \ldots. \tag{7.1.3}$$

Polynomial $P_n(x)$ has a leading coefficient in the form of

$$k_n = \frac{(-1)_n (n+1)_n}{(1)_n n!} \frac{(-1)^n}{2^n} = \frac{(2n)!}{2^n (n!)^2}, \qquad n = 0, 1, \ldots.$$

as per (6.1.12) and [21, p.27].

Theorem 7.1.1. *The generating function for the Legendre polynomials $P_n(x)$ is given by*

$$\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}. \tag{7.1.4}$$

Proof. We start with

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(1)_k k!} \left(\frac{1-x}{2}\right)^k t^n.$$

Now, interchanging the order of summation, we have

$$\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)_k (n+1)_k}{(k!)^2} \left(\frac{1-x}{2}\right)^k t^n.$$

Let n be replaced by $n + k$, so that the right-hand side becomes

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-n-k)_k (n+k+1)_k}{(k!)^2} \left(\frac{1-x}{2}\right)^k t^{n+k} = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} \left(\frac{1-x}{2}\right)^k t^k \left[\sum_{n=0}^{\infty} \frac{(2k+1)_n}{n!} t^n\right].$$

The expression inside of the square brackets reduces to $(1-t)^{-2k-1}$, by the binomial theorem. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(x) t^n &= \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} [2(x-1)t]^k (1-t)^{-2k-1} \\
&= (1-t)^{-1} \sqrt{1 - \frac{2(x-1)t}{(1-t)^2}} \\
&= [(1-t)^2 - 2(x-1)t]^{-1/2} \\
&= [1 - 2xt + t^2]^{-1/2}.
\end{aligned}$$

Thus (7.1.4) is shown to be valid. □

7.2 The Three-Term Recurrence Relation

The three-term recurrence satisfied by the Legendre polynomials is

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \qquad n = 0, 1, 2, \ldots, \tag{7.2.5}$$

also from [28, p.244]. This relation may be elicited via generating function (7.1.4). We begin by defining

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}.$$

Differentiating with respect to t yields

$$\begin{aligned}\frac{\partial}{\partial t}G(x,t) &= -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t)\\ &= \frac{x-t}{(1-2xt+t^2)^{3/2}}.\end{aligned}$$

from which we infer the equation

$$(1-2xt+t^2)\frac{\partial}{\partial t}G(x,t) - (x-t)G(x,t) = 0.$$

The generating function (7.1.4) suggests that a Fourier series solution be implemented. We proceed as is outlined in [12]. Substituting for $G(x,t)$ renders

$$(1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - (x-t)\sum_{n=0}^{\infty} P_n(x)t^n = 0.$$

After multiplying out the brackets and regrouping like terms, the equation above becomes

$$\sum_{n=0}^{\infty} \left[(n+1)P_{n+1}(x) - x(2n+1)P_n(x) - nP_{n-1}(x)\right] t^n = 0.$$

The expression inside of the square parenthesis is equal to zero, thereby presenting us with the three-term recurrence for $P_n(x)$.

7.3 The Orthogonality Relation

The orthogonality relation is elicited from the differential equation and found to be

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, \qquad m \neq n. \tag{7.3.6}$$

By manipulating the three-term recurrence, we obtain the norm square for the Legendre case, here shown to be

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}, \qquad n = 0, 1, \ldots . \tag{7.3.7}$$

We verify these relations now. By using (7.1.2) we get

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} P_m(x) \frac{d^n}{dx^n}[(1-x^2)^n] dx,$$

to which integration by parts is applied n times such that the right-hand side is reduced to

$$\frac{1}{2^n n!} \int_{-1}^{1} \frac{d^n}{dx^n} P_m(x) (1-x^2)^n dx.$$

For $m < n$, the integral in this expression vanishes. However, for $m = n$ it becomes

$$k_n n! \int_{-1}^{1} (1-x^2)^n dx$$

and is further simplified to

$$\frac{(2n)!}{2^n n!} \int_{-1}^{1} (1-x^2)^n dx.$$

As before, when dealing with the Jacobi polynomials, we seek a beta function expression for the integral in the previous expression. Naturally, we rely once again on the substitution $1 - x = 2t$ to initiate

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^n dx &= \int_{-1}^{1} (1-x)^n (1+x)^n dx \\ &= \int_{-1}^{1} (2t)^n (2-2t)^n 2dx \\ &= 2^{2n+1} B(n+1, n+1) \\ &= 2^{2n+1} \frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+2)} \\ &= \frac{2^{2n+1}(n!)^2}{(2n+1)!} \end{aligned}$$

for $n = 0, 1, \ldots$. Putting everything together results in

$$\begin{aligned} \int_{-1}^{1} [P_n(x)]^2 dx &= \frac{(2n)!}{2^{2n}(n!)^2} \frac{2^{2n+1}(n!)^2}{(2n+1)!} \\ &= \frac{2}{2n+1}, \qquad n = 0, 1, \ldots . \end{aligned} \tag{7.3.8}$$

7.4 The Associated Boundary Conditions

The differential operator is investigated in $\mathcal{L}^2(-1,1)$, for which we require the solutions to $ly = 0$. Now, ly vanishes for $y_1 = 1$ and $y_2 = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ and, since both of these are in $\mathcal{L}^2(-1,1)$, it follows that at both $x = -1$ and $x = 1$ the limit circle case is valid. As such, the boundary conditions from [28, p.244], for elements in the maximal operator's domain, are then

$$\begin{aligned}
B_{1,1}(y) &= \lim_{x\to 1}(1-x^2)[-y'(x)],\\
B_{1,2}(y) &= \lim_{x\to 1}(1-x^2)\left[\frac{1}{1-x^2}y(x) - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)y'(x)\right],\\
B_{-1,1}(y) &= \lim_{x\to -1}(1-x^2)[-y'(x)],\\
B_{-1,2}(y) &= \lim_{x\to -1}(1-x^2)\left[\frac{1}{1-x^2}y(x) - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)y'(x)\right].
\end{aligned}$$

The boundary values satisfied by the Legendre polynomials are $B_{1,1}(y) = 0$ and $B_{-1,1}(y) = 0$, leading to the following definition sourced from [28, p.245].

Definition 7.4.1. *Denote by D_L those elements y in $\mathcal{L}^2(-1,1)$ satisfying*

(i) y is a.e. differentiable on $(-1,1)$,

(ii) $(1-x^2)y'(x)$ is a.e. differentiable on $(-1,1)$ and $ly = -[(1-x^2)y']'$ is in $\mathcal{L}^2(-1,1)$, and

(iii) $B_{1,1}(y) = 0$ and $B_{-1,1}(y) = 0$.

The Legendre operator is defined by setting $Ly = ly$ for all y in D_L. The next theorem, to be found in [28, p.245], states that the Legendre polynomials are complete in $\mathcal{L}^2(-1,1)$.

Theorem 7.4.2. *The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ form a complete orthogonal set in $\mathcal{L}^2(-1,1)$.*

The proof is similar to the one pertaining to the Jacobi functions, with α and β being zero instead. As such, the proof is omitted here.

Chapter 8

The Hermite Polynomials

8.1 Introduction

Once again, we refer to [28, p.249]. The differential equation satisfied by the Hermite polynomials is given by

$$ly = -\frac{(e^{-x^2}y')'}{e^{-x^2}} = 2ny. \tag{8.1.1}$$

A Rodrigues formula is known which can be used for defining Hermite polynomials. Here, taken from [35, p.250], it is given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n = 0, 1, 2, \ldots, \tag{8.1.2}$$

where $w(x) = e^{-x^2}$ is the applicable weight function. This polynomial follows a normal distribution and is its own Fourier transform [3, p.278]. $H_n(x)$ is a polynomial of degree n, which can be proved by induction, where the Rodrigues formula implies that $H_0(x) = 1$. Evidently, $H_{2n}(x)$ is even while H_{2n+1} is odd.

The Hermite polynomials have a series representation

$$H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}, \qquad n = 0, 1, \ldots, \tag{8.1.3}$$

as taken from [28, p.250]. Here the integer part of $\frac{1}{2}n$ is represented by $\left[\frac{n}{2}\right]$.

Once again, (8.1.2) is employed to find the value of the leading coefficient of $H_n(x)$. We presently replace the notation $\frac{d^n}{dx^n}$ by D^n.

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} D^n e^{-x^2} \\ &= (-1)^n e^{x^2} D^{n-1}(-2xe^{-x^2}) \\ &= (-1)^n e^{x^2} D^{n-2}(-2e^{-x^2} + 2^2x^2e^{-x^2}). \end{aligned}$$

Continuing in this manner, it becomes apparent that

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} D[\dots(-1)^{n-1}2^{n-1}x^{n-1}e^{-x^2}] \\ &= (-1)^n e^{x^2}[\dots(-1)^{n-1}(n-1)2^{n-1}x^{n-2}e^{-x^2} + (-1)^n 2^n x^n e^{-x^2}] \\ &= (-1)^{2n}2^n x^n + (-1)^{2n-1}(n-1)2^{n-1}x^{n-2} + \dots \\ &= 2^n x^n - (n-1)2^{n-1}x^{n-2} + \dots . \end{aligned}$$

Clearly, the leading coefficient is $k_n = 2^n$ (see [21, p.31]).

A formula for the derivative of $H_n(x)$ is derived as follows.

$$\frac{d^{n+1}}{dx^{n+1}}e^{-x^2} = \frac{d}{dx}\frac{d^n}{dx^n}e^{-x^2} = (-1)^n \frac{d}{dx}[e^{-x^2}H_n(x)], \qquad n = 0, 1, \dots$$

by the Rodrigues formula. Differentiating the right-hand side yields

$$\begin{aligned} \frac{d^{n+1}}{dx^{n+1}}e^{-x^2} &= (-1)^n[-2xe^{-x^2}H_n(x) + e^{-x^2}H_n'(x)] \\ &= (-1)^{n+1}e^{-x^2}[2xH_n(x) - H_n'(x)], \qquad n = 0, 1, \dots, \end{aligned}$$

and finally

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x), \qquad n = 0, 1, \dots . \tag{8.1.4}$$

This expression is also to be found in [35, p.106], together with another useful relation

$$H_m'(x) = 2nH_{n-1}(x).$$

The following result for the generating function for the Hermite polynomials may be found in [3, p.279].

Theorem 8.1.1. *The generating function for the Hermite polynomials is given by*

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}. \tag{8.1.5}$$

Proof. Expanding the function

$$f(t) = e^{-(x-t)^2} = e^{-x^2}e^{2xt-t^2} \tag{8.1.6}$$

about the origin in terms of a Fourier series gives

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!} \tag{8.1.7}$$

where

$$f^{(n)}(0) = \left[\frac{d^n}{dx^n}e^{-(x-t)^2}\right]_{t=0}.$$

Let $x - t = u$. Then for $n = 0, 1, \ldots,$ we have

$$\begin{aligned} f^{(n)}(0) &= (-1)^n \left[\frac{d^n}{du^n}e^{-u^2}\right]_{u=x} \\ &= e^{-x^2}e^{x^2}(-1)^n\frac{d^n}{dx^n}e^{-x^2} \\ &= e^{-x^2}H_n(x). \end{aligned} \tag{8.1.8}$$

Combining (8.1.6),(8.1.7) and (8.1.8) as

$$f(t) = e^{-x^2}e^{2xt-t^2} = \sum_{n=0}^{\infty} e^{-x^2}H_n(x)\frac{t^n}{n!}$$

concludes the derivation. □

8.2 The Orthogonality Relation

The orthogonality relation and norm square can be found in [28, p.250], and is given by

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)dx = 2^n n!\delta_{mn}, \qquad m, n \in 0, 1, 2, \ldots. \tag{8.2.9}$$

This relation is derived by employing the Rodrigues formula (8.1.2) once again to obtain

$$\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)dx = (-1)^n\int_{-\infty}^{\infty} H_m(x)D^n e^{-x^2}dx.$$

The right-hand side integral is evaluated using integration by parts, i.e.

$$\int_{-\infty}^{\infty} H_m(x) D^n e^{-x^2} dx = \left[H_m(x) D^{n-1} e^{-x^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H_m'(x) D^{n-1} e^{-x^2} dx.$$

The first term vanishes due to the presence of the exponential factor. Integration by parts is again applied to the second term with similar results as before. One may continue in this manner until

$$\int_{-\infty}^{\infty} H_m^{(n-1)}(x) D e^{-x^2} dx = H_m^{(n-1)}(x) e^{-x^2} - \int_{-\infty}^{\infty} H_m^{(n)}(x) e^{-x^2} dx.$$

But if $m < n$ then $H_m^{(n)}(x) = 0$. Hence,

$$\int_{-\infty}^{\infty} H_m(x) D^n e^{-x^2} dx = 0.$$

By similar reasoning, if $n < m$,

$$\int_{\infty}^{\infty} H_n(x) D^m e^{-x^2} dx = 0$$

so that for $m \neq n$

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) dx = 0.$$

Thus the orthogonality relation is obtained. For $m = n$, integration by parts results in

$$\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = \int_{-\infty}^{\infty} D^n H_n(x) e^{-x^2} dx.$$

Now, consider the operation of D^n on the leading term of $H_n(x)$.

$$\begin{aligned} D^n[k_n x^n] &= D^{n-1}[n k_n x^{n-1}] \\ &= D^{n-2}[n(n-1) k_n x^{n-2}] \\ &\vdots \\ &= n! k_n. \end{aligned}$$

Monomials of degree $< n$ vanish after differentiating n times. Therefore,

$$D^n H_n(x) = n! k_n,$$

so that

$$\int_{-\infty}^{\infty} D^n H_n(x) e^{-x^2} dx = k_n n! \int_{-\infty}^{\infty} e^{-x^2} dx.$$

It is well-known from the Laplace transform (see [3, p.6]) that

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$$

and it has already been demonstrated that $k_n = 2^n$, rendering the result

$$\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = 2^n n! \sqrt{\pi}.$$

8.3 The Three-Term Recurrence Relation

The three-term recurrence, given in [28, p.250], satisfied by the same polynomials is

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \qquad n = 0, 1, \ldots. \tag{8.3.10}$$

With Leibniz's result in hand we turn to the derivation of the three-term recurrence. Thus, by (5.5.16),

$$\begin{aligned} D^{n+1}[w(x)] &= D^n[-2xw(x)] \\ &= -2 \sum_{k=0}^{n} \binom{n}{k} D^k[x] D^{n-k}[w(x)] \\ &= -2\binom{n}{0} x D^n[w(x)] - 2\binom{n}{1} D[x] D^{n-1}[w(x)] \\ &\quad - 2\binom{n}{2} D^2[x] D^{n-2}[w(x)] + \ldots. \end{aligned}$$

All terms starting from the third onwards vanish so that

$$D^{n+1}[w(x)] = -2xD^n w(x) - 2nD^{n-1} w(x),$$

which when multiplied by $(-1)^{n+1} e^{x^2}$ and recalling that $w(x) = e^{-x^2}$ admits

$$(-1)^{n+1} e^{x^2} D^{n+1} w(x) = 2x(-1)^n e^{x^2} D^n w(x) + 2n(-1)(-1)^{n-1} e^{x^2} D^{n-1} w(x),$$

known to be $H_{n=1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ for $n = 1, 2, \ldots$ by virtue of (8.1.2).

It is possible to generate (8.1.1) by substituting (8.1.4) into the three-term recurrence above, which brings us to

$$H_n(x) = H'_{n-1}(x) + H_n(x) - 2(n-1)H_{n-2}(x).$$

By rearranging the terms, we obtain

$$H'_{n-1}(x) = 2(n-1)H_{n-2}(x). \tag{8.3.11}$$

Differentiating (8.3.11) yields

$$H''_{n-1}(x) = 2(n-1)H'_{n-2}(x) = 2(n-1)[2xH_{n-2}(x) - H_{n-1}(x)].$$

By a shifting of the index and again the use of (8.3.11) the differential equation

$$H''_n(x) = 2xH'_n(x) - 2nH_n(x)$$

is obtained. This equation is equation is an equivalent form of (8.1.1).

8.4 The Associated Boundary Conditions

Now, [28, p.250] provides the following discussion regarding the boundary conditions of the Hermite problem. To show that the boundaries at both positive and negative infinity are limit points, one would apply Levinson's criterion. We set $ly = 0$ and find that the only solution to consider is $H_0 = 1$, giving rise to the following values at the boundaries:

$$B_\infty(y) = -\lim_{x\to\infty} e^{-x^2}y'(x)$$

and

$$B_{-\infty}(y) = -\lim_{x\to-\infty} e^{-x^2}y'(x)$$

as two annihilator conditions satisfied by all elements y in the domain of the maximal operator. The preceeding discussion leads to the next definition, found in [28, p.250], delimiting the domain on which the Hermite operator is defined.

Definition 8.4.1. *Denote by D_L those elements y in $\mathcal{L}^2(-\infty,\infty;e^{-x^2})$ satisfying* *(i) y is differentiable a.e. on $(-\infty, \infty)$, and*

(ii) $e^{-x^2}y'(x)$ is differentiable a.e. on $(-\infty, \infty)$ and ly is in $\mathcal{L}^2(-\infty, \infty; e^{-x^2})$.

Furthermore, for all y in D_L, the Hermite operator L may be defined by letting $Ly = ly$, so that the eigenfunctions of L are then $\{H_n(x)\}_{n=0}^{\infty}$, the Hermite polynomials. In addition, the eigenvalues are given by $\{\lambda_n = 2n\}_{n=0}^{\infty}$. The previous statements and the upcoming theorem concerning the completeness of the Hermite polynomials, are found in [28, p.251].

Theorem 8.4.2. *The Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ form a complete orthogonal set in $\mathcal{L}^2(-\infty, \infty; e^{-x^2})$.*

Proof. It is possible to find a function f in $\mathcal{L}^2(-\infty, \infty; e^{-x^2})$ which is orthogonal to the sequence $\{x^n\}_{n=0}^{\infty}$ if this function is chosen such that it is orthogonal to the Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$. Then by the orthogonality condition, we have the relation

$$\int_{-\infty}^{\infty} x^n f(x) e^{-x^2} dx = 0, \qquad n = 0, 1, \ldots.$$

This suggests that we consider the function

$$H(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x) e^{-x^2} dx,$$

where $\{e_j^{i\lambda x}\}_{j=1}^{\infty}$ forms a complete orthonormal system in $\mathcal{L}^2(-\infty, \infty; e^{-x^2})$. But $H(\lambda)$ is the $\mathcal{L}^1$ Fourier transform of $f(x)e^{-x^2}$ and as such it is a unitary operator. By taking the absolute value and using Schwarz's inequality, we have

$$\begin{aligned} |H(\lambda)| &\leq \int_{-\infty}^{\infty} e^{-\Im(\lambda)x} |f(x)| e^{-x^2} dx \\ &\leq \left(\int_{-\infty}^{\infty} e^{-2\Im(\lambda)x} e^{-x^2} dx \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx \right)^{\frac{1}{2}}. \end{aligned} \tag{8.4.12}$$

By Liouville's Theorem (5.5.4), $H(\lambda)$ is an entire analytic function of λ, which implies that it has a power series expansion

$$H(\lambda) = \sum_{n=0}^{\infty} \frac{H^{(n)}(0)\lambda^n}{n!}.$$

But now

$$\frac{d^n H(0)}{d\lambda^n} = \int_{-\infty}^{\infty} (ix)^n e^{i\lambda x} f(x) e^{-x^2} dx \bigg|_{\lambda=0} = 0$$

when the orthogonality relation is taken into account with $\lambda = 0$. Therefore, $H(\lambda)$ is identically zero. But then $f(x)$ vanishes a.e. in $\mathcal{L}^2(-\infty, \infty; e^{-x^2})$, which implies that $\{H_n(x)\}_{n=0}^{\infty}$ is a complete orthogonal set. $\square$

Chapter 9

The Generalised Laguerre Polynomials

9.1 Introduction

Let us denote the set of Generalised Laguerre Polynomials by $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$. The differential equation satisfied by these polynomials (see [28, p.245]) is

$$ly = -\frac{(x^{\alpha+1}e^{-x}y')'}{x^{\alpha}e^{-x}} = ny. \tag{9.1.1}$$

A useful representation of these polynomials exists in the form of the Rodrigues formula

$$L_n^{(\alpha)} = \frac{1}{n!}e^{x}x^{-\alpha}\frac{d^n}{dx^n}[e^{-x}x^{n+\alpha}], \qquad n = 0, 1, \ldots \tag{9.1.2}$$

where $w(x) = e^{-x}x^{\alpha}$ is the weight function. These polynomials follow a normal distribution (see [3, p.282]). The ordinary Laguerre polynomials are found by setting $\alpha = 0$.

The Laguerre polynomials may be represented by their series formulation given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k(1+\alpha)_n x^k}{k!(n-k)!(1+\alpha)_k} \tag{9.1.3}$$

and found in [28, p.246]. In fact, the series representation is acquired via (9.1.2). By Leibniz's rule, we have

$$\begin{aligned} D^n[e^{-x}x^{n+\alpha}] &= \sum_{k=0}^{n} \binom{n}{k} D^k e^{-x} D^{n-k} x^{n+\alpha} \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{-x} D^{n-k} x^{n+\alpha} \end{aligned}$$

where

$$\begin{aligned} D^{n-k}x^{n+\alpha} &= D^{n-k-1}[(n+\alpha)x^{n+\alpha-1}] \\ &= D^{n-k-2}[(n+\alpha)(n+\alpha-1)x^{n+\alpha-2}] \\ &\vdots \\ &= D^{n-k-(n-k-1)}[(n+\alpha)(n+\alpha-1)\dots(n+\alpha-n+k+2)x^{n+\alpha-(n-k)+1}] \\ &= D[(n+\alpha)(n+\alpha-1)\dots(\alpha+k+2)x^{n+k+1}] \\ &= (n+\alpha)(n+\alpha-1)\dots(\alpha+k+2)(\alpha+k+1)x^{\alpha+k}. \end{aligned}$$

Therefore,

$$D^n[e^{-x}x^{n+\alpha}] = \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{-x}(n+\alpha)(n+\alpha-1)\dots(\alpha+k+2)(\alpha+k+1)x^{\alpha+k}.$$

We recall that the gamma function has a factorial representation in the form of $\Gamma(n) = (n-1)!$ (see [3, p.3]) so that

$$D^n[e^{-x}x^{n+\alpha}] = e^{-x}x^{\alpha} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1)} x^k.$$

Hence, the series representation for the Laguerre polynomials

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{n!} e^x x^{-\alpha} \left\{ e^{-x}x^{\alpha} \sum_{k=0} (-1)^k \binom{n}{k} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1)} x^k \right\} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \end{aligned} \qquad (9.1.4)$$

for $n = 0, 1, \dots$. Thus, since $(k+\alpha+1)_{n-k} = (k+\alpha+1)\dots(n+\alpha)$, it is clear that

$$\frac{\Gamma(n+\alpha+1)}{(n-k)!\Gamma(k+\alpha+1)} = \frac{(k+\alpha+1)_{n-k}}{(n-k)!}.$$

Furthermore,

$$(-1)\binom{n+\alpha}{n-k} = \frac{(\alpha+1)_n}{n!}\frac{(-n)_k}{(\alpha+1)_k}, \qquad k=0,1,\ldots,n,$$

so that, for $n=0,1,\ldots,$

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}. \tag{9.1.5}$$

As with the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, there exist a relation between $L_n(x)$ and the hypergeometric function. We now find this relation by returning to the previous expression for $L_n(x)$. Now,

$$\frac{(\alpha+1)_n}{n!} = \binom{\alpha+n}{n}$$

together with

$$\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!} = {}_1F_1(-n;\alpha+1;x),$$

admits

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}{}_1F_1(-n;\alpha+1;x).$$

By expanding $L_n^{(\alpha)}(x)$ via the series expression (9.1.5), we obtain

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\frac{(-n)_n}{(\alpha+1)_n}\frac{x^n}{n!} + \text{terms of lesser degree.}$$

Noting that $(-n)_n = (-1)^n n!$, we find that the leading coefficient for each of the Laguerre polynomials is $\frac{(-1)^n}{n!}x^n$ for $n=0,1,\ldots,$ as in [21, p.31]. An expression for the first derivative of $L_n^{(\alpha)}(x)$ is easily calculated. Recall (9.1.4), then

$$\begin{aligned}\frac{d}{dx}L_n^{(\alpha)}(x) &= \frac{d}{dx}\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}\\ &= \frac{(n+\alpha)!}{(n-1)!(\alpha+1)!} + \cdots + (-1)^n\frac{nx^{n-1}}{n!}\\ &= \sum_{k=1}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^{k-1}}{(k-1)!}.\end{aligned}$$

By shifting the index, we arrive at

$$\frac{d}{dx}L_n^{(\alpha)}(x) = \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n+\alpha}{n-k-1}\frac{x^k}{k!}.$$

Therefore, as per [35, p.102], the first derivative is found to be

$$\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x) \qquad n = 1, 2, \ldots. \tag{9.1.6}$$

Theorem 9.1.1. [3, p.283] *The Laguerre polynomials* $L_n^{(\alpha)}(x)$ *are also found via their generating function*

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1-t)^{-\alpha-1} exp\left(-\frac{xt}{1-t}\right) \tag{9.1.7}$$

Proof. We begin by inserting (9.1.5) into the left-hand side of (9.1.7) to get

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{n!} t^n \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}.$$

The "Snake Oil" method[1], as described in [37, p.108], is employed so that we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(\alpha+1)_n}{(\alpha+1)_k} \frac{(-1)^k x^k t^n}{k!(n-k)!} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha+1)_{n+k}}{(\alpha+1)_k} \frac{(-1)^k x^k t^{n+k}}{k! n!}\end{aligned}$$

by replacing n with $n + k$. Finally

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n &= \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!} \sum_{n=0}^{\infty} \frac{(\alpha+k+1)_n}{n!} t^n \\ &= \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!} (1-t)^{-\alpha-k-1} \\ &= (1-t)^{-\alpha-1} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{xt}{1-t}\right)^k.\end{aligned} \tag{9.1.8}$$

But the series in (9.1.8) is the Taylor expansion for $\exp\left(-\frac{xt}{1-t}\right)$, thereby rendering the derivation complete. □

We now derive an addition formula for $L_n^{(\alpha)}(x)$. A generalisation of this formula is given in [11, p.552].

[1] It was speculated that this old-wive's remedy could cure all sorts of ailments. Wilf thought it a fitting name for a method that makes it possible to solve difficult summation problems.

Theorem 9.1.2.

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha)}(x) L_{n-k}^{(\beta)}(y), \qquad n = 0, 1, \dots. \tag{9.1.9}$$

Proof. Generating function (9.1.7) infers that

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n^{(\alpha+\beta+1)}(x+y) t^n &= (1-t)^{-\alpha-\beta-2} \exp\left(-\frac{(x+y)t}{1-t}\right) \\
&= (1-t)^{-\alpha-1} \exp\left(-\frac{xt}{1-t}\right) \cdot (1-t)^{-\beta-1} \exp\left(-\frac{yt}{1-t}\right) \\
&= \sum_{k=0}^{\infty} L_k^{(\alpha)}(x) t^k \cdot \sum_{m=0}^{\infty} L_m^{(\beta)}(y) t^m.
\end{aligned} \tag{9.1.10}$$

By what we know about the convolution of Fourier series, the last line of (9.1.10) can be reformulated as

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} L_k^{(\alpha)}(x) L_{n-k}^{(\beta)}(y) \right) t^n.$$

Thus, the addition formula (9.1.9) is attained. □

9.2 The Orthogonality Relation

The orthogonality relation is given by

$$\int_0^{\infty} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^{\alpha} e^{-x} dx = 0, \quad n \neq m, \tag{9.2.11}$$

with the norm square as

$$\int_0^{\infty} \left[L_n^{(\alpha)}(x)\right]^2 x^{\alpha} e^{-x} dx = \frac{\Gamma(\alpha+1+n)}{n!}. \tag{9.2.12}$$

Both of these can be found in [28, p.247]. In order to show that $\mu_n = \Gamma(n+\alpha+1)$ we begin by integrating by parts

$$\int_0^{\infty} e^{-x} x^{n+\alpha} dx = -\left[e^{-x} x^{n+\alpha}\right]_0^{\infty} + \int_0^{\infty} e^{-x} (n+\alpha) x^{n+\alpha-1} dx.$$

As $x \to \infty$, we have that $e^{-x} \to 0$, and $x^{n+\alpha} \to 0$ as $x \to 0$. Integration by parts renders

$$\int_0^{\infty} e^{-x} x^{n+\alpha} dx = (n+\alpha)! \int_0^{\infty} e^{-x} dx$$

$$= \Gamma(n + \alpha + 1)$$

and then (9.1.2) is made use of to procure

$$\int_0^\infty e^{-x} x^\alpha L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \int_0^\infty e^{-x} x^\alpha L_m^{(\alpha)}(x) \frac{1}{n!} e^x x^{-\alpha} D^n [e^{-x} x^{n+\alpha}]$$

$$= \frac{1}{n!} \int_0^\infty L_m^{(\alpha)}(x) D^n [e^{-x} x^{n+\alpha}]. \qquad (9.2.13)$$

Evaluating the integral by parts yields

$$\int_0^\infty L_m^{(\alpha)}(x) D^n [e^{-x} x^{n+\alpha}] dx = L_m^{(\alpha)}(x) D^{n-1} [e^{-x} x^{n+\alpha}]_0^\alpha - \int_0^\infty D L_m^{(\alpha)}(x) D^{n-1} [e^{-x} x^{n+\alpha}] dx.$$

The first term on the right-hand side vanishes and a second integration by parts is needed in order to evaluate the second integral. We find that

$$\int_0^\infty L_m^{(\alpha)}(x) D^n [e^{-x} x^{n+\alpha}] dx = (-1)^n \int_0^\infty D^n L_m^{(\alpha)}(x) [e^{-x} x^{n+\alpha}] dx$$

For $m < n$, it is obvious that $D^n L_m^{(\alpha)} = 0$ (since $L_m^{(\alpha)}$ is a polynomial of degree at most m) in which case

$$\int_0^\infty L_m^{(\alpha)}(x) D^n [e^{-x} x^{n+\alpha}] dx = 0.$$

A similar scenario arises if the roles of m and n are interchanged in (9.2.13). The conclusion is that the orthogonality relation for when $m \neq n$ is given by (9.2.11).

The norm square (9.2.12) is also easily attained by using (9.1.2) once again. Starting with

$$D^n L_n^{(\alpha)}(x) = D^n \Big\{ \frac{1}{n!} e^x x^{-\alpha} D^n [e^{-x} x^{n+\alpha}] \Big\}$$

$$= D^n \Big\{ \frac{1}{n!} e^x x^{-\alpha} e^{-x} x^\alpha \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} x^k \Big\}.$$

It was shown previously that the coefficient of x^n is given by

$$(-1)^n \binom{n}{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+1)} = \frac{(-1)^n}{n!}.$$

Therefore,

$$D^n L_n^{(\alpha)}(x) = D^n \left[\frac{(-1)^n}{n!} x^n + \dots \right] = D^{n-1} [n k_n x^{n-1} + \dots] = \dots = n! k_n,$$

so that

$$\int_0^\infty D^n L_n^{(\alpha)}(x)e^{-x}x^{n+\alpha}dx = k_n n! \int_0^\infty e^{-x}x^{n+\alpha}dx = (-1)^n\Gamma(n+\alpha+1)$$

achieves the desired result.

9.3 The Three-Term Recurrence Relation

The three-term recurrence relation associated with the Laguerre polynomials can be found in [28, p.247], and is given by

$$nL_n^{(\alpha)} = (2n-1+\alpha-x)L_{n-1}^{(\alpha)}(x) - (n-1+\alpha)L_{n-2}^{(\alpha)}(x). \tag{9.3.14}$$

The generating function is implemented to derive the three-term recurrence relation specific to the Laguerre polynomials by defining the function

$$G(x,t) := (1-t)^{-\alpha-1}\text{exp}\left(-\frac{xt}{1-t}\right)$$

and computing its partial derivatives. With respect to x, the derivative is

$$\frac{\partial}{\partial x}G(x,t) = -t(1-t)^{-\alpha-2}\text{exp}\left(-\frac{xt}{1-t}\right),$$

resulting in the equation

$$(1-t)\frac{\partial}{\partial x}G(x,t) + tG(x,t) = 0. \tag{9.3.15}$$

The derivative with respect to t is

$$\begin{aligned}\frac{\partial}{\partial t}G(x,t) = (-\alpha-1)(1-t)^{-\alpha-2}(-1)\text{exp}\left(-\frac{xt}{1-t}\right)\\ + \left[-\frac{x(1-t)+(-1)xt}{(1-t)^2}\right](1-t)^2\text{exp}\left(-\frac{xt}{1-t}\right),\end{aligned}$$

which simplifies to

$$\frac{\partial}{\partial t}G(x,t) = \left[\alpha+1-\frac{x}{1-t}\right](1-t)^{-\alpha-2}\text{exp}\left(-\frac{xt}{1-t}\right),$$

whence the equation

$$(1-t)^2\frac{\partial}{\partial t}G(x,t) + [x-(\alpha+1)(1-t)]G(x,t) = 0. \tag{9.3.16}$$

Equation (9.3.15), together with (9.1.7), leads to

$$(1-t)\sum_{n=0}^{\infty}\frac{d}{dx}L_n^{(\alpha)}(x)t^n + t\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n = 0$$

or equivalently

$$\sum_{n=0}^{\infty}\left[\frac{d}{dx}L_{n+1}^{(\alpha)}(x) - \frac{d}{dx}L_n^{(\alpha)}(x) + L_n^{(\alpha)}(x)\right]t^n = 0 \qquad n = 0, 1, \ldots .$$

From what has been shown regarding the derivative of $L_n^{(\alpha)}(x)$ in (9.1.6), the preceding equation becomes

$$\frac{d}{dx}L_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) - L_n^{(\alpha+1)}(x) \qquad n = 0, 1, \ldots . \tag{9.3.17}$$

From (9.3.16) and (9.1.7) it follows that

$$(1-t)^2\sum_{n=0}^{\infty}\frac{d}{dt}L_n^{(\alpha)}(x)t^n + [x-(\alpha+1)(1-t)]\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n = 0,$$

which, when multiplied out and by a shifting of indices, leads to

$$\sum_{n=1}^{\infty}[(n+1)L_{n+1}^{(\alpha)}(x) - 2nL_n^{(\alpha)}(x) + (n-1)L_{n-1}^{(\alpha)}(x) + \ldots$$
$$\cdots + xL_n^{(\alpha)}(x) - (\alpha+1)L_n^{(\alpha)}(x) + (\alpha+1)L_{n-1}^{(\alpha)}(x)]t^n = 0 \qquad n = 1, 2, \ldots .$$

This simplifies to the three-term recurrence (9.3.14). From the re-arrangement of the terms in (9.3.14) emerges the following useful form of the recurrence relation

$$xL_n^{(\alpha)}(x)+(n+1)[L_{n+1}^{(\alpha)}(x)-L_n^{(\alpha)}(x)]-(n+\alpha)[L_n^{(\alpha)}(x)-L_{n-1}^{(\alpha)}(x)] = 0, \qquad n = 1, 2, \ldots \tag{9.3.18}$$

Given that (9.3.14) defines $L_n^{(\alpha)}(x)$, it is only natural that this very recurrence could be used to determine the differential equation satisfied by $L_n^{(\alpha)}(x)$. The alternative expression for the recurrence, i.e. (9.3.18) is differentiated and (9.3.17) is employed, to get

$$x\frac{d}{dx}L_{n+1}^{(\alpha)}(x) + L_n^{(\alpha)}(x) - (n+1)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0 \qquad n = 1, 2, \ldots .$$

Rearranging this equation leads to

$$x\frac{d}{dx}L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \qquad n = 1, 2, \ldots , \tag{9.3.19}$$

which is differentiated to yield

$$x\frac{d^2}{dx^2}L_n^{(\alpha)}(x) + \frac{d}{dx}L_n^{(\alpha)}(x) = (n+\alpha)\left[\frac{d}{dx}L_n^{(\alpha)}(x) - \frac{d}{dx}L_{n-1}^{(\alpha)}(x)\right] - \alpha\frac{d}{dx}L_n^{(\alpha)}(x).$$

Since (9.3.17) applies to the expression inside of the square brackets, using (9.3.19), gives the second order differential equation satisfied by $L_n^{(\alpha)}(x)$ as

$$x\frac{d^2}{dx^2}L_n^{(\alpha)}(x) + (\alpha+1-x)\frac{d}{dx}L_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0,$$

where n takes a non-negative integer value.

9.4 The Associated Boundary Conditions

We would like to look at what happens to the differential operator in $\mathcal{L}^2(0,\infty; x^\alpha e^{-x})$ and as such it is necessary to set up appropriate boundary conditions. The Laguerre differential equation ly vanishes for $y_1 = 1$ and

$$y_2 = \int_1^x \frac{c^\xi}{\xi^{\alpha+1}}d\xi,$$

as in [28, p.247]. At the lower boundary where x vanishes, for elements y in the domain of the maximal operator, we have that

$$B_1(y) = -\lim_{x\to 0} x^{\alpha+1}e^{-x}y'(x),$$

and

$$B_2(y) = -\lim_{x\to 0} x^{\alpha+1}e^{-x}\left[y(x)\left(\frac{e^x}{x^{\alpha+1}}\right) - y'(x)\int_1^x \frac{e^\xi}{\xi^{\alpha+1}}d\xi\right].$$

When α lies between -1 and 1, we have that $x = 0$ is in the limit circle and both $B_1(y)$ and $B_2(y)$ are applicable boundary values. $B_1(y) = 0$ is an annihilator condition for $\alpha \leq 1$ and is the only existing boundary value. In the limit point case, when $x \to \infty$, the automatic boundary condition

$$B_\infty(y) = -\lim_{x\to\infty} x^{\alpha+1}e^{-x}y'(x) = 0$$

applies. From [28, p.247], we have the following definition concerning the differential operator for the Laguerre differential equation.

Definition 9.4.1. *We denote by D_L, those elements y in $\mathcal{L}^2(0,\infty;x^\alpha e^{-x})$ satisfying*

(i) y is differentiable a.e. on $(0,\infty)$,

(ii) $x^{\alpha+1}e^{-x}y'(x)$ is differentiable a.e. on $(0,\infty)$ and ly is in $\mathcal{L}^2(0,\infty;x^\alpha e^{-x})$ and

(iii) $\lim_{x\to+0} -x^{\alpha+1}e^{-x}y'(x) = 0$.

By letting $Ly = ly$ for all y in D_L, we are able to define the Laguerre operator L. It then follows that the eigenfunctions associated with the Laguerre operator are precisely those polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ satisfying (9.1.1), with eigenvalues $\{\lambda_n = n\}_{n=0}^\infty$.

Theorem 9.4.2. [28, p.248] *The Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_0^\infty$ form a complete orthogonal set in $\mathcal{L}^2(0,\infty;x^\alpha e^{-x})$.*

Proof. Suppose f is a function that is orthogonal to the Laguerre polynomials, $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$, and which exists in $\mathcal{L}^2(0,\infty;x^\alpha e^{-x})$. Consequently, f is also orthogonal to $\{x^n\}_{n=0}^\infty$. We thus have the orthogonality condition

$$\int_0^\infty e^{-x}x^{\alpha+n}f(x)dx = 0, \qquad n = 0, 1, ... \tag{9.4.20}$$

Using the $\mathcal{L}^1$ Fourier transform of $e^{-x}x^\alpha f(x)$, we construct the following function

$$G(\lambda) = \int_0^\infty e^{i\lambda x}e^{-x}x^\alpha f(x)dx.$$

Taking the absolute value gives

$$\begin{aligned} |G(\lambda)| &= \left| \int_0^\infty e^{i\lambda x}e^{-x}x^\alpha f(x)dx \right| \\ &\leq \int_0^\infty e^{\Im(\lambda)x}e^{-x}x^\alpha |f(x)|dx. \end{aligned} \tag{9.4.21}$$

Schwarz's inequality applied to the right-hand side yields

$$|G(\lambda)| \leq \left(\int_0^\infty e^{-2(\Im(\lambda)+\frac{1}{2})x}x^\alpha dx\right)^{\frac{1}{2}} \left(\int_0^\infty e^{-x}x^\alpha f(x)^2 dx\right)^{\frac{1}{2}}.$$

The function $G(\lambda)$ is well-defined and it is analytic in the half-plane $\Im(\lambda) > -\frac{1}{2}$. We then have that the Fourier expansion of $G(\lambda)$, about the point $x = 0$, exists and write

$$G(\lambda) = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)\lambda^n}{n!}. \tag{9.4.22}$$

This corresponds to the lowest eigenvalue, $\lambda = 0$, which implies that $G(0) = 0$. Furthermore, taking the n^{th} derivative, we have

$$\frac{d^n G(0)}{d\lambda^n} = \int_0^{\infty} (ix)^n e^{i\lambda x} e^{-x} x^{\alpha} f(x) dx \bigg|_{\lambda=0} = 0$$

and when substituted into the power series expansion (9.4.22) implies that the function $G(\lambda)$ vanishes identically. Therefore, $f(x)$ is also zero a.e., and specifically, in $\mathcal{L}^2(0, \infty; x^{\alpha}e^{-x})$. The set of Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is thus orthogonal and complete. $\square$

Chapter 10

Applications of Darboux-Crum Transformations

By motivation given in [28], it known that the operators corresponding to each of the classical orthogonal polynomial types given in Chapters 6, 7, 8 and 9 are unbounded linear self-adjoint operators. The differential equations associated with each of these polynomials will now undergo transformation via the Darboux-Crum transformation. We expect the new differential equations to be characterised by self-adjoint operators as well. Furthermore, we anticipate spectral alterations.

10.1 The Jacobi Polynomials

10.1.1 The Darboux-Crum Transformation

The Jacobi polynomials satisfy the differential equation

$$-\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}y'\right]' = (1-x)^{\alpha}(1+x)^{\beta}n(n+\alpha+\beta+1)y \qquad n=0,1,\dots, \tag{10.1.1}$$

where, in equation (3.1.7), $p=(1-x)^{\alpha+1}(1+x)^{\beta+1}$, $q=0$ and $r=(1-x)^{\alpha}(1+x)^{\beta}$. Now the transformation as suggested by Crum and given in (3.1.8) is written as

$$\tilde{y} = (1-x)^{\alpha+1}(1+x)^{\beta+1}\left[y' - \frac{z'}{z}y\right].$$

We will say that z is a solution to (10.1.1) for $n = 0$, in which case it is known that the corresponding Jabobi polynomial is $z = 1$. Therefore

$$\tilde{y} = (1-x)^{\alpha+1}(1+x)^{\beta+1}y'. \tag{10.1.2}$$

In order to generate the transformed equation, we begin by differentiating (10.1.2) to get

$$\begin{aligned}\tilde{y}' &= [(1-x)^{\alpha+1}(1+x)^{\beta+1}y']' \\ &= -(1-x)^{\alpha}(1+x)^{\beta}n(n+\alpha+\beta+1)y. \end{aligned}\tag{10.1.3}$$

Then

$$\begin{aligned}\left[(1-x)^{-\alpha}(1+x)^{-\beta}\tilde{y}'\right]' &= -n(n+\alpha+\beta+1)y' \\ &= -n(n+\alpha+\beta+1)(1-x)^{-\alpha-1}(1+x)^{-\beta-1}\tilde{y}, \end{aligned}\tag{10.1.4}$$

for $n = 1, 2, \ldots$. Hence, the transformation produces an equation of the desired form. The boundary conditions transform into

$$\lim_{x\to\pm1}\tilde{y} = 0,$$

$$\lim_{x\to\pm1}\left\{-\frac{(1-x)(1+x)}{n(n+\alpha+\beta+1)}Q_0^{\prime(\alpha,\beta)}\tilde{y}' - Q_0^{(\alpha,\beta)}\tilde{y}\right\} = 0.$$

10.1.2 An Alternative Darboux-Crum Transformation

If, instead, we carry on by using $z' = (1-x)^{-\alpha-1}(1+x)^{-\beta-1}k$, where k is some constant, then we have that $z = k\int_{-1}^{1}(1-x)^{-\alpha-1}(1+x)^{-\beta-1}dx$. The Crum transformation is now

$$\tilde{y} = (1-x)^{\alpha+1}(1+x)^{\beta+1}y' - \frac{k}{z}y,$$

for which the first derivative is

$$\begin{aligned}\tilde{y}' &= \left[(1-x)^{\alpha+1}(1+x)^{\beta+1}y'\right]' + \frac{k^2}{z^2}\frac{1}{(1-x)^{\alpha+1}(1+x)^{\beta+1}}y - \frac{k}{z}y' \\ &= -(1-x)^{\alpha}(1+x)^{\beta}n(n+\alpha+\beta+1)y - \frac{k}{z}(1-x)^{-\alpha-1}(1+x)^{-\beta-1}\tilde{y}. \end{aligned}\tag{10.1.5}$$

The first derivative is divided by $(1-x)^{\alpha}(1+x)^{\beta}$ to get

$$(1-x)^{-\alpha}(1+x)^{-\beta}\tilde{y}' = -n(n+\alpha+\beta+1)y - \frac{k}{z}(1-x)^{-2\alpha-1}(1+x)^{-2\beta-1}\tilde{y}.$$

Let $\left[(1-x)^{-\alpha}(1+x)^{-\beta}\tilde{y}'\right]' = \tilde{\varphi}'$. The transformed equation is then, for $n = 1, 2, \ldots,$

$$\begin{aligned}\tilde{\varphi}' &= n(n+\alpha+\beta+1)y' - \left[\frac{k}{z}(1-x)^{-2\alpha-1}(1+x)^{-2\beta-1}\right]'\tilde{y} \\ &\quad - \left[\frac{k}{z}(1-x)^{-2\alpha-1}(1+x)^{-2\beta-1}\right] \\ &\qquad \left[(1-x)^{\alpha}(1+x)^{\beta}n(n+\alpha+\beta+1)y + \frac{k^2}{z^2}(1-x)^{-\alpha-1}(1+x)^{-1-\beta}y - \frac{k}{z}y'\right] \\ &= n(n+\alpha+\beta+1)(1-x)^{-\alpha-1}(1+x)^{-\beta-1}\tilde{y} \\ &\quad - \left[\frac{k}{z}(1-x)^{-2\alpha-1}(1+x)^{-2\beta-1}\right]'\tilde{y} + \frac{k^2}{z^2}(1-x)^{-3\alpha-2}(1+x)^{-3\beta-2}\tilde{y}.\end{aligned}$$

10.1.3 Solution Corresponding to the First Eigensolution

From the alternative formulation of the transformation of the Jacobi equation we have

$$z = k\int_{-\infty}^{\infty}(1-x)^{-\alpha-1}(1+x)^{-\beta-1}dx. \tag{10.1.6}$$

Now, say

$$F(x) = \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}dx \tag{10.1.7}$$

and let $t = \frac{1+x}{2}$ so that $x = 2t-1$ and $dx = 2dt$. Therefore,

$$\begin{aligned}F(t) &= \int_{0}^{1}(1-2t+1)^{\alpha}(2t)^{\beta}2dt \\ &= 2^{\alpha+\beta+1}\int_{0}^{1}(1-t)^{\alpha}t^{\beta}.\end{aligned} \tag{10.1.8}$$

Therefore, (10.1.6) is transformed into

$$\begin{aligned}z &= k2^{-\alpha-1-\alpha-1+1}\int_{0}^{1}(1-t)^{-\alpha-1}t^{-\beta-1}dt \\ &= 2^{-(\alpha+\beta+1)}k\int_{0}^{1}(1-t)^{-\alpha-1}t^{-\beta-1}dt \\ &= \frac{k}{2^{\alpha+\beta+1}}B(-\beta,-\alpha)\end{aligned} \tag{10.1.9}$$

keeping in mind that it is required that $\Re(-\alpha), \Re(-\beta) > 0$.

The transformed Jacobi equation is then

$$\begin{aligned}\tilde{\varphi}' &= -n(n+\alpha+\beta+1)(1-x)^{-\alpha-1}(1+x)^{-\beta-1}\tilde{y} \\ &\quad + \frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\frac{1}{(1-x^2)^2}\left\{\frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\frac{1}{(1-x)^{\alpha}(1+x)^{\beta}}\right. \\ &\quad \left. - 2x - \frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\frac{1}{(1-x)^{\alpha+1}(1+x)^{\beta+1}}\right\}\tilde{y} \\ &= -n(n+\alpha+\beta+1)(1-x)-\alpha-1(1+x)^{-\beta-1}\tilde{y} + \frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\times \\ &\quad \left[\frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\frac{1}{(1-x)^{\alpha}(1+x)^{\beta}}\left[\frac{2-x^2}{1-x^2}\right] - 2x\right]\tilde{y}.\end{aligned}$$

The boundary conditions transformation for this problem is

$$\lim_{x\to\pm1}\left\{\left\{1-\left[\frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\right]^2\frac{(1-x)^{-2\alpha-1}(1+x)^{-2\beta-1}}{n(n+\alpha+\beta+1)}\right\}\tilde{y} - \frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{n(n+\alpha+\beta+1)}\tilde{y}'\right\} = 0,$$

$$\begin{aligned}\lim_{x\to\pm1}\Bigg\{&\left\{1-\left[\frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\right]^2\frac{(1-x)^{-2\alpha-1}(1+x)^{-2\beta-1}}{n(n+\alpha+\beta+1)}\right. \\ &\left. - \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{n(n+\alpha+\beta+1)}\frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\right\}Q_0^{(\alpha,\beta)}\tilde{y} \\ &- \left\{-\frac{(1-x)(1+x)}{n(n+\alpha+\beta+1)}\frac{2^{\alpha+\beta+1}}{B(-\beta,-\alpha)}\frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{n(n+\alpha+\beta+1)}\right\}Q_0^{(\alpha,\beta)}\tilde{y}'\Bigg\} = 0.\end{aligned}$$

10.2 The Legendre Polynomials

10.2.1 The Darboux-Crum Transformation

Legendre polynomials satisfy the differential equation

$$-[(1-x^2)y']' = \lambda_n y \qquad \lambda_n = n(n+1) \qquad n = 0, 1, \ldots, \tag{10.2.10}$$

for which we have that $p = 1 - x^2$, $q = 0$ and $r = 1$ in (3.1.7). Suppose that z is a solution to (10.2.10) for the case $n = 0$, i.e. $-[(1 - x^2)z']' = 0$. Then

$$(1 - x^2)z' = k, \tag{10.2.11}$$

where k is some constant. But it is known that $z = 1 \Rightarrow z' = 0$. Therefore, the Crum transformation is given by

$$\tilde{y} = (1 - x^2)\left[y' - \frac{z'}{z}y\right] = (1 - x^2)y'. \tag{10.2.12}$$

This is our transformation relation, which holds for $n = 1, 2, 3, \ldots$ since the first derivative of the original Legendre function is present in the new solution. As expected, the least eigenvalue falls away. Now, (10.2.12) is differentiated to produce

$$\tilde{y}' = -2xy' + (1 - x^2)y'' = -\lambda_n y. \tag{10.2.13}$$

Differentiating once more results in

$$\tilde{y}'' = -\lambda_n y' = -\lambda_n \frac{1}{1 - x^2}\tilde{y} \qquad n = 1, 2, \ldots, \tag{10.2.14}$$

10.2.2 An Alternative Darboux-Crum Transformation

If we were to carry on with finding the solution to (10.2.11) then we would have

$$z' = \frac{k}{1 - x^2}$$

which can be solved as

$$z = -k\int_{-1}^{1} \frac{1}{x^2 - 1}dx = k\,\mathrm{arctanh}(x). \tag{10.2.15}$$

The Crum transformation is

$$\tilde{y} = (1 - x^2)\left[y' - \frac{z'}{z}y\right],$$

which, after substituting for z and z', becomes

$$\tilde{y} = (1 - x^2)\left[y' - \frac{1}{1 - x^2}\frac{1}{\mathrm{arctanh}(x)}y\right]$$

$$= (1-x^2)y' - \text{arcoth}(x)y. \tag{10.2.16}$$

Differentiating (10.2.16) results in

$$\begin{aligned}
\tilde{y}' &= \left[(1-x^2)y'\right]' + \frac{k}{1-x^2}\frac{1}{k}\text{arcoth}^2(x)y - \text{arcoth}(x)y' \qquad (10.2.17)\\
&= -\lambda_n y - \frac{1}{1-x^2}\text{arcoth}(x)\left[(1-x^2)y' - \text{arcoth}(x)y\right]\\
&= -\lambda_n y - \frac{1}{1-x^2}\text{arcoth}(x)\tilde{y},
\end{aligned}$$

after which, differentiating again produces

$$\begin{aligned}
[\tilde{y}']' &= -\lambda_n y' - \left[\frac{1}{1-x^2}\text{arcoth}(x)\right]'\tilde{y} - \left[\frac{1}{1-x^2}\text{arcoth}^2(x)\right]\tilde{y}'\\
&= -\lambda_n \tilde{y} - \frac{2\text{arcoth}(x)}{(1-x^2)^2}\left[x - \text{arcoth}(x)\right]\tilde{y}, \qquad (10.2.18)
\end{aligned}$$

for $n = 1, 2, \ldots$.

10.2.3 Transformation of Boundary Conditions

The boundary conditions associated with the Legendre polynomials are given as

$$B_{\pm1,1}(y) = \lim_{x\to\pm1}(1-x^2)[-y'(x)] = 0 \tag{10.2.19}$$

$$B_{\pm1,2}(y) = \lim_{x\to\pm1}(1-x^2)\left[\frac{1}{1-x^2}y(x) - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)y'(x)\right] \tag{10.2.20}$$

Now, $B_{\pm1,1}(y)$ transforms into

$$B_{\pm1,1}(\tilde{y}) = \lim_{x\to\pm1}\left[-\tilde{y}(x)\right] = 0,$$

So that

$$\lim_{x\to\pm1}\tilde{y}(x) = 0.$$

and $B_{\pm1,2}(y)$ becomes

$$\begin{aligned}
B_{\pm1,2}(\tilde{y}) &= \lim_{x\to\pm1}(1-x^2)\Big\{\frac{1}{1-x^2}\left[-\frac{1}{\lambda_n}\tilde{y}'\right] - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\left[\frac{1}{1-x^2}\tilde{y}(x)\right]\Big\}\\
&= \lim_{x\to\pm1}\left[-\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\tilde{y}(x) - \frac{1}{\lambda_n}\tilde{y}'\right]\\
&= 0. \qquad (10.2.21)
\end{aligned}$$

For the alternative case we have that

$$\lim_{x\to\pm1}\left[\frac{1}{\lambda_n}\mathrm{arcos}(x)\tilde{y}'+\left(\frac{1}{\lambda_n}\frac{\mathrm{arcos}^2(x)}{1-x^2}-1\right)\tilde{y}\right]=0.$$

and

$$\lim_{x\to\pm1}\left\{\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\frac{1}{\lambda_n}\mathrm{arcos}(x)-\frac{1}{\lambda_n}\right]\tilde{y}'\right.$$
$$\left.-\left[\frac{1}{\lambda_n}\frac{\mathrm{arcos}(x)}{1-x^2}+\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\left(1-\frac{1}{\lambda_n}\frac{\mathrm{arcos}^2(x)}{1-x^2}\right)\right]\tilde{y}\right\}=0.$$

10.2.4 Spectral Changes

For $n=0$ we have that $\tilde{y}''=0$ so that $\tilde{y}'=k_1$ and therefore,

$$\tilde{y}=k_1x+k_2.$$

Imposing boundary condition (10.2.19) results in

$$\lim_{x\to\pm1}\tilde{y}(x)=\lim_{x\to\pm1}[k_1x+k_2]=0.$$

So, then $k_1+k_2=0\implies k_1=-k_2$ and $k_1-k_2=0\implies k_1=k_2$ which together imply that $k_1=k_2=0$. Hence, the eigenvalue corresponding to $n=0$ falls away.

10.2.5 Orthogonality Relation for Transformed Problem

Given that the Legendre polynomials are a particular case of the Jacobi polynomials, we will start by finding an orthogonality relation for the transformed Jacobi functions. The Darboux-Crum transformation for the Jacobi functions is

$$\tilde{y}=(1-x)^{\alpha+1}(1+x)^{\beta+1}y' \quad \text{and} \quad \tilde{y}'=-(1-x)^{-\alpha}(1+x)^{-\beta}n(n+\alpha+\beta+1)y.$$

The first derivative relation is

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=\frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

which transforms into

$$\tilde{P}_n'^{(\alpha,\beta)}=-\frac{2n(n+\alpha+\beta+1)}{n+\alpha+\beta}\tilde{P}_{n+1}^{(\alpha-1,\beta-1)}.$$

And so, the original orthogonality relation transforms into

$$\int_{-1}^{1}(1+x)^{-(\alpha+1)}(1+x)^{-(\beta+1)}\tilde{P}_m^{(\alpha,\beta)}(x)\tilde{P}_n^{(\alpha,\beta)}(x)dx = 0.$$

The relation associated with the Legendre functions is then

$$\int_{-1}^{1}(1-x^2)^{-1}\tilde{P}_m(x)\tilde{P}_n(x)dx = 0.$$

The Jacobi norm square transforms into

$$\int_{-1}^{1}(1-x)^{-(\alpha+1)}(1+x)^{-(\beta+1)}[\tilde{P}_n^{(\alpha,\beta)}(x)]^2dx$$
$$= \frac{2^{\alpha+\beta+1}(n+\alpha+\beta+2)^2\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+2)(n-1)!}.$$

From this, we infer that the norm square associated with the transformed Legendre functions is

$$\int_{-1}^{1}(1-x^2)^{-1}[\tilde{P}_n(x)]^2dx = 2\frac{(n+2)[\Gamma(n+1)]^2}{(2n+1)(n-1)!\Gamma(n+2)}.$$

10.3 The Hermite Polynomials

10.3.1 The Darboux-Crum Transformation

The Hermite polynomials satisfy the differential equation

$$-[e^{-x^2}y']' = 2e^{-x^2}ny \qquad n = 0, 1, \ldots, \tag{10.3.22}$$

where $p = e^{-x^2}$, $q = 0$ and $r = 2e^{-x2}$ in (3.1.7). The Crum transformation is then given by

$$\tilde{y} = e^{-x^2}\left[y' - \frac{z'}{z}y\right]. \tag{10.3.23}$$

Suppose that z is a solution to (10.3.22) for $n = 0$, i.e. $[e^{-x^2}z']' = 0$. Then $e^{-x^2}z' = k$, where k is some constant. But it is known that for $n = 0$ the Hermite polynomial is $z = 1$ so (10.3.23) becomes

$$\tilde{y} = e^{-x^2}y',$$

which, when differentiated, results in

$$\tilde{y}' = -2xe^{-x^2}y' + e{-}x^2y'' = [e^{-x^2}y']'.$$

So, after multiplying through by $-e^{x^2}$, we have

$$-e^{x^2}\tilde{y}' = 2ny.$$

Differentiating again results in

$$-[e^{x^2}\tilde{y}']' = 2ny' = 2ne^{x^2}\tilde{y} \qquad n = 1, 2, \ldots.$$

This transformed equation would suggest a new weight function $\tilde{w}(x) = e^{x^2}$.

10.3.2 Transformation of Boundary Conditions

The boundary conditions

$$\lim_{x\to\pm\infty} e^{-x^2}y'(x) = 0$$

transforms into

$$\lim_{x\to\pm\infty} \tilde{y}(x) = 0.$$

Under this transformation, the eigenvalue corresponding to $n = 0$ falls away, since

$$-[e^{x^2}\tilde{y}'] = 2ne^{x^2}\tilde{y}.$$

So for $n = 0$ we have that $e^{-x^2}\tilde{y}'(x) = k_0$, where k_0 is some constant. Then $\tilde{y}'(x) = k_0e^{x^2}$ and

$$\tilde{y}(x) = k_1 + k_0\int_0^x e^{t^2}dt.$$

Given that we are required to have $\tilde{y} \to 0$ as $x \to \infty$, and since $e^{t^2} \to \infty$ as $x \to \infty$, $k_1 = 0$ and $k_0 = 0$.

10.3.3 An Alternative Darboux-Crum Transformation

Else, if we carry on with solving $z' = ke^{x^2}$, then we obtain

$$z = k\int_{-\infty}^{\infty} e^{x^2}dx,$$

which can then be substituted into (10.3.23) to get

$$\tilde{y} = e^{-x^2}\left[y' - \frac{ke^{x^2}}{z}y\right] = e^{-x^2}y' - \frac{k}{z}y.$$

The above result is differentiated and simplified into

$$\begin{aligned}
\tilde{y}' &= \left[e^{-x^2}y'\right]' + \frac{kz'}{z^2}y - \frac{k}{z}y' \\
&= -2e^{-x^2}ny + \frac{k^2e^{x^2}}{z^2}y - \frac{k}{z}y' \\
&= -2e^{-x^2}ny - \frac{ke^{x^2}}{z}\left[e^{-x^2}y' - \frac{k}{z}y\right] \\
&= -2e^{-x^2}ny - \frac{ke^{x^2}}{z}\tilde{y}. \qquad (10.3.24)
\end{aligned}$$

After multiplying the equation through by e^{x^2}, we arrive at

$$\left[e^{x^2}\tilde{y}'\right] = -2ny - \frac{ke^{2x^2}}{z}\tilde{y}.$$

We differentiate once more, thereby obtaining an equation in the desired form, as

$$\left[e^{x^2}\tilde{y}'\right]' = -2ny' - \left(\frac{ke^{2x^2}}{z}\tilde{y}\right)'.$$

Finally,

$$\left[e^{x^2}\tilde{y}'\right]' = -2ne^{x^2}\tilde{y} - \left[\frac{ke^{2x^2}}{z}\left(4x - \frac{z'}{z}\right) - \frac{k^2e^{3x^2}}{z^2}\right]\tilde{y} \qquad n = 1, 2\ldots.$$

10.3.4 Hermite Solution Corresponding to First Eigenvalue

From the alternate Crum transformation we obtained the integral solution

$$z = k\int_0^x e^{t^2}dt. \qquad (10.3.25)$$

Now let $u = t^2$ which would imply that $t = \pm\sqrt{u}$, from which the positive branch will be selected, and then $dt = \frac{1}{2}\frac{1}{\sqrt{u}}du$. Further, the limits of integration change from $t = x$ to $u = x^2$ and $t = 0$ to $u = 0$ so that

$$z = \frac{1}{2}k\int_0^{x^2} e^u u^{-1/2}du$$

in which case setting $\alpha = \frac{1}{2}$ in (5.5.20) produces

$$z = \frac{1}{2} k \Gamma \left(\frac{1}{2} \right) = \frac{1}{2} k \sqrt{\pi}.$$

The transformed equation can now be fully expressed as

$$\begin{aligned} \left[e^{-x^2} \tilde{y}' \right] &= -2ne^{x^2} \tilde{y} - \frac{2}{k\sqrt{\pi}} e^{2x^2} \left[4x - \frac{2}{\sqrt{\pi}} e^{x^2} \right] \tilde{y} + \frac{4}{\pi} e^{3x^2} \tilde{y} \\ &= -2ne^{x^2} \tilde{y} - \left[\frac{8}{k\sqrt{\pi}} x e^{2x^2} - \left(\frac{1}{k} + 1 \right) \frac{4}{\pi} e^{3x^2} \right] \tilde{y}. \end{aligned}$$

Since it was only required that we have "a solution" corresponding to the least eigenvalue, we are able to restrict our interval of integration, thereby inhibiting the growth rate of this particular integral. The form of this solution, however, suggests that it is a solution which behaves erratically at the boundary points of the interval $(-\infty, \infty)$. This consideration motivates the presence of the forcing factor e^{-x^2} seen in the boundary conditions.

10.3.5 Alternative Transformation of Boundary Conditions

The transformed boundary conditions for the case $z' = ke^{x^2}$ are

$$\lim_{x \to \pm\infty} \left[\left(1 - \frac{2e^{2x^2}}{n\pi} \right) \tilde{y} - \frac{e^{x^2}}{n\sqrt{\pi}} \tilde{y}' \right] = 0.$$

10.3.6 Orthogonality Relation for Transformed Problem

The original Hermite polynomials are known to satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 0.$$

Also, the Hermite polynomials have the first derivative expression

$$H_n'(x) = 2nH_{n-1}(x) \qquad n = 1, 2, \ldots. \tag{10.3.26}$$

We have the Darboux-Crum transformation relations

$$y = -\frac{1}{2n} e^{x^2} \tilde{y}' \qquad y' = e^{x^2} \tilde{y}, \tag{10.3.27}$$

which, when inserted into (10.3.26), results in a relation for the first derivative of the transformed polynomials

$$\tilde{H}_n' = \frac{-n}{n+1}\tilde{H}_{n+1}. \tag{10.3.28}$$

Using the transformation relation (10.3.27), we now get

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2} y_n y_m dx &= \int_{-\infty}^{\infty} e^{-x^2}\left(-\frac{}{2n}e^{x^2}\tilde{y}_n'\right)\left(-\frac{1}{2m}e^{x^2}\tilde{y}_m'\right)dx \\ &= \int_{-\infty}^{\infty} e^{x^2}\frac{1}{4(n+1)(m+1)}\tilde{y}_{n+1}\tilde{y}_{m+1}dx\end{aligned}$$

where the final line follows from (10.3.28). Whence, we have the orthogonality relation for the transformed Hermite polynomials as

$$\int_{-\infty}^{\infty} e^{x^2}\tilde{y}_n\tilde{y}_m dx = 0. \tag{10.3.29}$$

By following a similar procedure as the one used to find the orthogonality relation, the norm square is derived as

$$\int_{-\infty}^{\infty} e^{x^2}[\tilde{y}_n]^2 dx = n^2(n-1)!2^{n+1}\sqrt{\pi}. \tag{10.3.30}$$

10.4 The Generalized Laguerre Polynomials

10.4.1 The Darboux-Crum Transformation

The generalized Laguerre polynomials satisfy the differential equation

$$-\left[x^{\alpha+1}e^{-x}y'\right]' = x^{\alpha}e^{-x}ny \qquad n = 0, 1, \ldots, \tag{10.4.31}$$

in which case we have that $p = x^{\alpha+1}e^{-x}$, $q = 0$ and $r = x^{\alpha}e^{-x}$ in (3.1.7). Now, the Crum transformation is consequently given by

$$\tilde{y} = x^{\alpha+1}e^{-x}\left[y' - \frac{z'}{z}y\right].$$

Assume z to be the solution to (10.4.31) for the case where $n = 0$. In other words, $[x^{\alpha+1}e^{-x}z']' = 0$ so that $x^{\alpha+1}e^{-x}z' = k$, where k is some constant. However, it is known that $z = 1$, whereby $z' = 0$ and hence

$$\tilde{y} = x^{\alpha+1}e^{-x}y'. \tag{10.4.32}$$

We now generate the first derivative of (10.4.32)

$$\tilde{y}' = \left[x^{\alpha+1}e^{-x}y'\right]' = -x^{\alpha}e^{-x}ny.$$

Then multiplying through by $x^{-\alpha}e^{x}$ yields

$$x^{-\alpha}e^{x}\tilde{y}' = -ny$$

which, when differentiated again, gives rise to a transformed equation in the expected form. So

$$\left[x^{-\alpha}e^{x}\tilde{y}'\right]' = -ny' = -nx^{-\alpha-1}e^{x}\tilde{y} \qquad n = 1, 2, \ldots.$$

10.4.2 Transformation of Boundary Conditions

These are

$$B_1(y) = -\lim_{x\to 0} x^{\alpha+1}e^{-x}y'(x) = 0 \tag{10.4.33}$$

$$B_2(y) = -\lim_{x\to 0} x^{\alpha+1}e^{-x}\left[y(x)\left(\frac{e^{x}}{x^{\alpha+1}}\right) - y'(x)\int_1^x \frac{e^{\xi}}{\xi^{\alpha+1}}d\xi\right]. \tag{10.4.34}$$

Note that the expression in (10.4.34) equals 0 only when $\alpha \geq 1$. Furthermore, we are given the automatic boundary condition

$$B_{\infty}(y) = -\lim_{x\to\infty} x^{\alpha+1}e^{-x}y'(x) = 0. \tag{10.4.35}$$

So, these three boundary conditions are transformed into

$$\lim_{x\to 0} -x^{\alpha+1}e^{-x}\left[x^{-\alpha-1}e^{x}\tilde{y}(x)\right] = \lim_{x\to 0}\left[-\tilde{y}(x)\right] = 0$$

with

$$\begin{aligned} &-\lim_{x\to 0} x^{\alpha+1}e^{-x}\left\{-\frac{1}{n}x^{\alpha}e^{-x}\tilde{y}'(x)\left(\frac{e^{x}}{x^{\alpha+1}}\right) - x^{-\alpha-1}e^{x}\tilde{y}(x)\int_1^x \frac{e^{\xi}}{\xi^{\alpha+1}}d\xi\right\} \\ &= \lim_{x\to 0}\left\{\frac{1}{n}x^{\alpha}e^{-x}\tilde{y}'(x) + \tilde{y}(x)\int_1^x \frac{e^{\xi}}{\xi^{2}}d\xi\right\} \\ &= 0 \qquad \text{only when } \alpha \geq 1 \end{aligned} \tag{10.4.36}$$

and

$$-\lim_{x\to\infty} x^{\alpha+1}e^{-x}\left\{x^{-\alpha-1}e^{x}\tilde{y}(x)\right\} = -\lim_{x\to\infty}\tilde{y}(x) = 0.$$

10.4.3 Spectral Changes

Whether any eigenvalues are lost during the transformation of the differential equation is of interest. Consider the case where $n = 0$, that is $[x^{-\alpha}e^{-x}\tilde{y}']' = 0$ so that $\tilde{y}' = k_0 x^{\alpha} e^{x}$, where k_0 is some constant value. Then

$$\tilde{y} = k_0 \int_0^x t^{\alpha} e^{t} dt + k_1.$$

In order for $B_1(\tilde{y}) = 0$ to be satisfied, it is required that $k_1 = 0$. Furthermore, for $B_\infty(\tilde{y}) = 0$ to hold, we must have $k_0 = 0$. Thus, the eigenvalue corresponding to $n = 0$ falls away for the transformed equation.

10.4.4 Orthogonality Relations and Norm Square for Transformed Problem

From the Darboux-Crum transformation we have

$$y' = x^{-(\alpha+1)} e^{x} \tilde{y} \quad \text{and} \quad y = -x^{\alpha} e^{-x} \frac{1}{n} \tilde{y}'.$$

Applying these to the first derivative expression for the Laguerre polynomials yields the analogue for the transformed variant

$$\tilde{L}_n'^{\alpha}(x) = n \tilde{L}_{n+1}^{(\alpha-1)}(x).$$

Using these relations in transforming the original orthogonality relation, the new relations are then

$$\int_0^\infty x^{-(\alpha+1)} e^{x} \tilde{L}_n^{(\alpha)}(x) \tilde{L}_m^{(\alpha)}(x) dx = 0.$$

By similar means as above, the norm square transforms into

$$\int_0^\infty x^{-(\alpha+1)} e^{x} [\tilde{L}_n^{(\alpha)}(x)]^2 dx = \frac{\Gamma(n+\alpha+1)}{(n-1)!} \qquad \alpha > -1.$$

10.4.5 An Alternative Transformation

If instead, we were to continue by solving $z' = x^{-\alpha-1} e^{x} k$, another solution would be

$$z = k \int_0^x x^{-\alpha-1} e^{x} dx.$$

This implies that we now have the transformation

$$\tilde{y} = x^{\alpha+1}e^{-x}y' - \frac{k}{z}y.$$

Differentiating this expression gives

$$\begin{aligned}
\tilde{y}' &= \left[x^{\alpha+1}e^{-x}y'\right]' + \frac{kz'}{z^2}y - \frac{k}{z}y' \\
&= -x^{\alpha}e^{-x}ny + \frac{k^2}{z^2}x^{-\alpha-1}e^{x}y - \frac{k}{z}y' \\
&= -x^{\alpha}e^{-x}ny - \frac{k}{z}x^{-\alpha-1}e^{x}\left[x^{\alpha+1}e^{-x}y' - \frac{k}{z}y\right] \\
&= -x^{\alpha}e^{-x}ny - \frac{k}{z}x^{-\alpha-1}e^{x}\tilde{y}. \qquad (10.4.37)
\end{aligned}$$

We multiply through by $x^{-\alpha}e^{x}$, then differentiate, and after some manipulation, we arrive at

$$\left[x^{-\alpha}e^{x}\tilde{y}'\right]' = -nx^{-\alpha-1}e^{x}\tilde{y} - \frac{k}{z}x^{-2\alpha-1}e^{2x}\left[2 - \frac{2k}{z}x^{-\alpha-1}e^{x} - \frac{2\alpha-1}{x}\right]\tilde{y}.$$

The function z is expressed in terms of the gamma function (5.5.20). Set $\chi = \int_1^x \frac{e^{\xi}}{\xi^{\alpha+1}}d\xi$ and $\varphi = \frac{(-1)^{\alpha}}{\Gamma(-\alpha)}$. The transformed equation is then expressed as

$$(x^{-\alpha}e^{x}\tilde{y}')' = -nx^{-\alpha-1}e^{x}\tilde{y} - \varphi x^{-2\alpha-1}e^{2x}\left[2 - 2\varphi x^{-\alpha-1}e^{x} - \frac{2\alpha-1}{x}\right]\tilde{y}.$$

The boundary conditions associated with the transformed equation are

$$\lim_{x\to 0}\left\{\varphi x^{-\alpha}e^{x}\frac{1}{n}\tilde{y}' + \left[\varphi^2 x^{-2\alpha-1}e^{x}\frac{1}{n} - 1\right]\tilde{y}\right\} = 0,$$

$$\lim_{x\to \infty}\left\{\varphi x^{-\alpha}e^{x}\frac{1}{n}\tilde{y}' + \left[\varphi^2 x^{-2\alpha-1}e^{x}\frac{1}{n} - 1\right]\tilde{y}\right\} = 0,$$

$$\lim_{x\to 0}\left\{\left[1 - \frac{\varphi\chi}{n}\right]\tilde{y}' + \left[\chi + \frac{\varphi}{n}x^{-\alpha-1}e^{2x} - \frac{\varphi^2\chi}{n}x^{-2\alpha-1}e^{2x}\right]\tilde{y}\right\} = 0.$$

10.5 Transformation by a Formula for Successive Applications of the Darboux-Crum Procedure

The differential equations under consideration are of the form

$$-(py')' = \lambda y.$$

Thus, the second Darboux-Crum transformation, as suggested in [2], is given by

$$\begin{aligned}\tilde{y} &= \frac{W(y_0, y_1, y)}{W(y_0, y_1)} \\ &= \frac{y_0[y_1'(py')' - y'(py_1')'] - y_1[y_0'(py')' - y'(py_0')'] + y[y_0'(py_1')' - y_1'(py_0')']}{y_0 y_1' - y_0' y_1}.\end{aligned}$$

The presence of the second derivative of the original Hermite solution and the fact that $\tilde{y}$ for $n = 2, 3, \ldots,$ is evidence of Adler's result concerning the removal of eigenvalues from the spectrum of transformed operator.

10.5.1 A Transformation of the Legendre Polynomials via Adler's Method

The differential equation satisfied by the Legendre polynomials is

$$-[(1 - x^2)y']' = \lambda_n y,$$

where $\lambda_n = n(n+1)$ for $n = 0, 1, 2, \ldots.$ We know that for the Legendre polynomials $y_0 = 1$ and $y_1 = x$ so that the Darboux-Crum transformation is thus

$$\tilde{y} = 2xy' - \lambda_n y.$$

This equation is differentiated and multiplied by $(1 - x^2)$ to arrive at

$$(1 - x^2)\tilde{y}' = (2 - \lambda_n)(1 - x^2)y' + 2x\tilde{y},$$

which is differentiated again and simplified to produce a transformed differential equation of the expected form

$$[(1 - x^2)\tilde{y}']' = (2 - \lambda_n)\tilde{y} + \frac{(2x)^2}{1 - x^2}\tilde{y} \qquad n = 2, 3, \ldots.$$

A transformation of the boundary conditions associated with this polynomial yields

$$\lim_{x\to\pm1}[(1-x^2)\tilde{y}'-2x\tilde{y}]=0$$

and

$$\lim_{x\to\pm1}\left\{\left[\frac{2x}{\lambda_n}-\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)(1-x^2)\right]\tilde{y}'+\left[\frac{2x^3}{\lambda_n}\frac{1}{1-x^2}+\ln\left(\frac{1+x}{1-x}\right)x\right]\tilde{y}\right\}=0.$$

10.5.2 A Transformation of the Hermite Polynomials via Adler's Method

Hermite polynomials satisfy the differential equation

$$-[e^{-x^2}y']'=2ne^{-x^2}y \qquad n=0,1,\ldots. \tag{10.5.38}$$

From what is known of the Hermite polynomials $y_0=1$ and $y_1=x$. The Darboux-Crum transformation is given by

$$\tilde{y}=-2e^{x^2}[xy'-ny] \qquad n=2,3,\ldots. \tag{10.5.39}$$

The first derivative of (10.5.39) is

$$\tilde{y}'=-2x(x+1)\tilde{y}+2e^{-x^2}(1-n)y'.$$

This equation is then multiplied by $e^{\frac{2}{3}x^3+2x^2}$ and is then differentiated to produce

$$[e^{\frac{2}{3}x^3+2x^2}\tilde{y}']'=-2[\frac{2}{3}x^3+2x^2+2x+1-2x^2(x^2+1)^2+1-n]e^{\frac{2}{3}x^3+2x^2}\tilde{y} \qquad n=2,3,\ldots.$$

Transforming the associated boundary conditions, we find that

$$\lim_{x\to\pm\infty}\frac{1}{2}\frac{1}{1-n}[\tilde{y}'+2x(x+1)\tilde{y}]=0.$$

10.5.3 A Transformation of the Generalized Laguerre Polynomials via Adler's Method

The Darboux-Crum transformation, in accordance with Adler's formulation given in [2], is

$$\tilde{y}=-x^{\alpha}e^{-x}[ny-(x-1)y'].$$

By differentiating, we have that

$$\tilde{y}' = \frac{\alpha}{x}\tilde{y} + x^\alpha e^{-x}\left[ny' - y' + \frac{(x-1)(\alpha+1)}{x}y' - \frac{n}{x}y\right].$$

We introduce the weight function

$$e^{\frac{1}{2}x^2-2x}[(n-1+\alpha)x-\alpha]^{1/\alpha}x^{3-2\alpha-1/\alpha}$$

by multiplying it into the equation for $\tilde{y}'$. Let

$$m(x) = n - 2 + \frac{(x-1)(\alpha+1}{x} + \frac{1}{x}.$$

After differentiating weighted equation for $\tilde{y}'$, the transformed equation is found to be

$$\begin{aligned}&\left[e^{\frac{1}{2}x^2-2x}[(n-1+\alpha)x-\alpha]^{1/\alpha}x^{3-2\alpha-1/\alpha}\tilde{y}'\right]\\ &\quad= \left\{\frac{\alpha^2-3\alpha+2}{x^2} - m + \frac{1-\alpha}{x}\left[\frac{2(\alpha-1)}{x} + \frac{\alpha}{mx^2} + 2 - \frac{\alpha+1}{x} - x\right]\right\}\times\dots\\ &\qquad\dots\times e^{\frac{1}{2}x^2-2x}[(n-1+\alpha)x-\alpha]^{1/\alpha}x^{3-2\alpha-1/\alpha}\tilde{y},\end{aligned}$$

for $n = 2, 3, 4, \dots$. Under this transformation, the boundary conditions become

$$B_1(\tilde{y}) = -\lim_{x\to 0}\frac{x}{n}\left[\left(\frac{\alpha}{x}-1\right)\tilde{y} - \tilde{y}'\right] = 0$$
$$B_\infty(\tilde{y}) = -\lim_{x\to \infty}\frac{x}{n}\left[\left(\frac{\alpha}{x}-1\right)\tilde{y} - \tilde{y}'\right] = 0$$

when $\alpha \geq 1$.

Chapter 11

Concluding Remarks

Having applied the Darboux-Crum transformation to the four types of classical orthogonal polynomials, we return to the questions posed in the introduction to this investigation to decide upon the extent to which these questions have been answered.

The boundary conditions associated with each orthogonal polynomial type remain singular after the first application of the transformation – for the simple case with $z' = 0$ as well as for the alternative where explicit expressions for z are known. For the two-fold application via Adler, the Legendre boundary conditions reduce to zero as $x \to \pm 1$, while the boundary conditions for the Hermite and Laguerre polynomials remain singular.

It does not appear as though the Darboux-Crum transformation enables transformation between different classes of polynomials. Orthogonality is preserved, as expected, with respect to the weight function implied by the transformed equation.

The original and transformed problems are almost isospectral, with the least eigenvalue absent from the spectrum of the transformed problem. Adler's result for the removal of successive eigenvalues is also apparent in the applications. It is also possible to add eigenvalues to the spectrum associated with the original problem by means of a similar sort of transformation.

Having taken a look at the continuous case, one might be tempted to ask what would occur for the discrete case. It is possible to construct a transformed solution, similar to that of (3.1.2), as a difference equation and to investigate the changes undergone by the three term recurrence relation [16].

Another aspect related to the analysis of eigenvalue problems is the inverse problem, whereby spectral properties of the operator are used to investigate other characteristics of the operator, such as the corresponding potential function $q(x)$. Transformations of Sturm-Liouville problems that are commonly considered for determining eigenfunctions are transformation to an integral equation or to a system in polar form [29], of which the Prüfer-type angle is an example (see [10], [4] and [5]). Both approaches rely on the special relationship that exists between the zeros of the eigenfunctions and the eigenvalue to which the eigenfunction corresponds. Solving the inverse problem, amongst other things, asserts that a differential operator is uniquely characterized by its spectral properties. The Darboux-Crum transformation in particular is studied in [13], where the analyticity of the transformation is proved. In [22], a specific type of Bessel problem is considered. Their method suggests an approach to the inverse problem which can be implemented for dealing with other singular problems.

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