THE DILATATION OPERATOR OF CONFORMAL $\mathcal{N} = 4$ SUPER YANG-MILLS THEORY USING SCHUR POLYNOMIALS

by

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ABSTRACT

In this dissertation we study the anomalous dimension of operators with R-charge of order N in the 't Hooft limit. This is a highly non-trivial task as the planar approximation, which is usually used to simplify the problem, is not accurate. One is forced to sum all Feynman diagrams. The operators of interest are dual to a system of two excited giant gravitons. We build the Yang Mills operators using restricted Schur polynomials that are labeled by Young diagrams that have at most two columns. We learn from our analysis that the giant gravitons behave as a two coupled harmonic oscillators.

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DECLARATION

I, the under signed, hereby certify that sources directly quoted and ideas used including figures, tables, sketches, drawings and photos, have been correctly denoted. Those not otherwise indicated belong to the author.
This dissertation is submitted to the University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Master of Science. It has not been submitted before for any degree or examination in any other university.

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Contents

List of Figures

Chapter 1

Introduction

One of the most important and exciting research topics in theoretical high energy physics is the AdS/CFT correspondence. Presently the correspondence is a conjectured equivalence between gauge theory and gravity. There are four fundamental interactions (forces) in nature: electromagnetism, strong nuclear, weak nuclear and gravity. Electromagnetism and both strong and weak nuclear dominate at the microscopic scale and can all be written as Quantum Field Theories (QFTs). At the macroscopic level, gravity is dominant and it can be written as a geometrical theory (General Relativity or simply GR) on spacetime. The AdS/CFT correspondence claims that a theory of gravity on a 10-dimensional background involving Anti-de Sitter (AdS) space is equivalent to a 4-dimensional Conformal Field Theory (CFT-a type of QFT).

QFT's model elementary particles as point-like excitations of some fields on a spacetime background. This description has been extremely successful in unifying electromagnetism and the weak force. The strong force can also be written as a QFT, namely Quantum Chromodynamics (QCD), which is a non-Abelian gauge theory based on the gauge group $SU(3)$ where the three represents the different types of charge, called colour. The Standard Model is a Yang-Mills theory based on the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ (the subscript c denotes colour, L means "left handed" and Y is hypercharge) and it attempts to unify the electro-weak interaction with QCD. The Standard Model has thus far passed every empirical test put to it [1]. However, there are strong indications that new physics will be involved at very short distances (high energies), comparable to the Planck scale[2]. At these distances quantum gravity effects are important. A quantum theory of gravity has proven very difficult to write down. However, string theory– which purports that elementary particles are not point like objects but vibrations on a one-dimensional string– appears to allow for a consistent quantum theory of both elementary particles and gravity. This is because all string theories include a massless spin 2 particle which can only be associated with the graviton.

The effective coupling of QCD is very weak at high energies (very short distances), this is called asymptotic freedom, whereas at low energies the coupling is very strong hence it is very hard to make predictions. 't Hooft suggested that a gauge theory based on $SU(N)$, for large N, might be simpler than QCD[3]. In which case, then one could solve the $N = \infty$ theory exactly and then do an expansion in $1/N \sim 1/3$. Working with the large N limit gauge theories led to the realization of the close resemblance of the theory to a free string theory with coupling constant $1/N$. The AdS/CFT correspondence (also known as the Maldecena conjecture, gauge/gravity duality or gauge string duality), was first mentioned as a result of studies of D-branes and black holes in string theory [4; 5; 6] and is a concrete realization of 't Hooft's suggestion.

The AdS/CFT correspondence maps strong coupling dynamics in Super Yang Mills theory onto a gravitational theory defined on a weakly curved spacetime, it is a strong/weak coupling duality in the 't Hooft coupling of the field. The duality is a non perturbative equivalence, therefore the quantities used to begin to probe the duality must be exact expressions or receive small corrections, thereby enabling us to extrapolate weak coupling results to strong coupling [7]. Quantities protected by supersymmetry such as those associated with BPS states serve this purpose. Giant gravitons are half-BPS solutions that represent stable, spherical D3-branes on a $\mathrm{Ad}S_5 \times S^5$ background [8]. The sphere giants, which are giant gravitons propagating in S^5 of $\mathrm{Ad}S_5 \times S^5$, have energies protected by supersymmetry [8] and thus have been extensively utilized to calculate useful quantities on both sides of the correspondence. Operators on the gauge theory side that are dual to giant gravitons were proposed to be Schur polynomials in [9; 10]. Schur polynomials are labeled by Young diagrams. A Schur polynomial with n columns of length $\mathcal{O}(N)$ is dual to a bound state of n sphere giants, while n rows of length $\mathcal{O}(N)$ means that the operator is dual to a bound state of giant gravitons which have expanded in $\mathrm{Ad}S_5$ of $\mathrm{Ad}S_5 \times S_5$. There is compelling evidence to support this proposition [11; 12; 13; 7; 14; 15; 16; 17], however much still needs to be understood about this particular duality. This is part of our motivation in this project.

Operators in the Yang Mills theory that are dual to giant gravitons are to have R-charge of $\mathcal{O}(N)$. These operators are of the form

$$Tr(YZZZYZZZZYZZZY).$$

One can think of the operator as a lattice of Z's on which the impurities (Y) hop. Our aim in this project is to compute the anomalous dimension of an operator dual to an excited two giant graviton state. We study an operator of R-charge of order N with two impurities, in the form of a restricted Schur polynomial labeled by a Young diagram with $n+2$ boxes, a Young diagram with n boxes and a Young diagram with 2 boxes (where $n \sim \mathcal{O}(N)$). This task is highly non-trivial as the planar approximation, which is usually used to simplify such problems, is not accurate. Instead one is forced to sum all (planar and non-planar) Feynman diagrams. We carry out our analysis by diagonalizing the dilatation operator in the restricted Schur polynomial basis and numerically solve for its eigenvalues (the anomalous dimensions).

The layout of the dissertation is as follows. In Chapter 2 we give an overview of the AdS/CFT correspondence which includes a review of the key fields of study it seeks to unify. In Chapter 3 we discuss giant gravitons. The main results of the dissertation are presented in Chapter 4. The numerical test and overall discussion are presented in Chapter 5. The appendices contain useful tools we employ in the calculations of Chapter 4. Appendix A describes the coset expansion and hop off identities. Appendix B shows a small calculation used to remove twisted states. In Appendix C we review correlation functions and compute the two point functions of all the operators that appear in the computation of the main result.

The results presented in this dissertation have been published in the Journal of American Physical Society (DOI:10.1103/PhysRevD.81.106009 or see [18]).

Chapter 2

The AdS-CFT Correspondence

In order to understand and truly appreciate the AdS/CFT correspondence, it is important that the reader first be exposed to the multitude of different theories that this correspondence seeks to unify. In this chapter we will present a brief overview of the basic concepts and ideas of the theories that constitute conformal field theory and gravity. The first five sections set the backdrop for the main subject, which is motivated and defined in the last section.

2.1 Gauge theory: Quantum field theory basics

Nature likes gauge theories. One might even go as far as claiming that any pure (fundamental physical) theory that describes nature has to be a gauge theory. Quantum Chromo-dynamics (QCD), Quantum Electrodynamics (QED), etc. are all gauge theories. It seems as if it is a prerequisite. The study of gauge theories is a broad field with far reaching applications. We will, however, concern ourselves with the basics of gauge theory in Quantum Field Theory (QFT). QFT incorporates Special Relativity into Quantum Theory. In the transition from Quantum Mechanics (QM) to QFT we have to let go of concepts such as position operators and adopt new ideas of a dynamic vacuum in which particles can "pop" into and out of existence.

In introducing gauge theory, one can start off with the action of a free gauge theory and then proceed to discuss its symmetries and the algebra of the fields. We will start by discussing a scalar field theory and then build gauge invariance into it, this will allow us to first touch on QFT in general.

Quantum Field Theory: The basics

In order to define a QFT one needs the following key elements

- The action S, which is related to the Lagrangian density $\mathcal{L}$ by

$$S = \int d^d x \mathcal{L}(\ldots, \ldots)$$

  d is the dimension of spacetime. The arguments of Lagrangian can be the fields and derivatives thereof. For a free complex scalar theory we have

$$S = \int d^d x \mathcal{L}(\phi, \partial_\mu \phi) = \int d^d x \partial_\mu \phi \partial^\mu \phi^*$$

- The canonical momentum of the field ϕ is denoted by π and it is defined as

$$\pi(\bar{x}, t) = \frac{\partial \mathcal{L}}{\partial(\dot{\phi}(\bar{x}, t))}$$

  where $\dot{\phi} = \partial_0 \phi(x)$.

- The canonical commutation relation (equal time),

$$[\phi(\bar{x}, t), \pi(\bar{y}, t)] = i\hbar \delta(\bar{x} - \bar{y}).$$

- The Hamiltonian

$$\mathcal{H}(x) = \pi(x)\dot{\phi}(x) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)).$$

Using these four key elements one should be able to compute any observable in the theory. We know from QM that observables are computed from expectation values of hermitian operators. In QFT we compute vacuum expectation values (VEVs) of operators $\langle \hat{O}(\bar{x},t)\rangle$. The procedure described above is the *canonical quantization*. There is an alternative quantization procedure of studying QFT using "path integrals". The map between these two prescriptions is easily expressed via VEVs

$$\langle 0|\hat{O}(\bar{x},t)|0\rangle = \int D_\phi e^{-iS[\phi]}\hat{O}(\bar{x},t).$$

The measure $\int D_\phi$ represents a discretization of spacetime, which the field is integrated over [19]

$$\int D_\phi = \lim_{\substack{\Delta x_i \to 0 \\ N\to\infty}} \frac{1}{B(\Delta x_i)} \cdot \prod_{i=1}^{N} \int_{-\infty}^{\infty} \frac{d\phi(x_i)}{B(\Delta x_i)},$$

where $B(\Delta x_i)$ is a weighting factor. A generalization of VEVs is a correlation function or an n-point function G_n

$$G_n(x_1,x_2,\ldots,x_n) = \langle 0|T\{\hat{O}(x_1)\hat{O}(x_2)\ldots\hat{O}(x_n)\}|0\rangle.$$

The generating function of the correlation function is called the partition function

$$Z[J] = \int D_\phi e^{-iS[\phi]+i\int d^dx J(x)\phi(x)}$$

and the following holds:

$$G_n(x_1,x_2,\ldots,x_n) = \frac{1}{i^n}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\Big|_{J=0}.$$

T is a time ordering instruction. It commands that the operators be arranged in order of decreasing time from left to right. The functional integrals described above are notoriously hard to evaluate and often involve large combinatorial factors. The best way to deal with these functions is perturbatively with the aid of pictorial representation of the terms appearing in the Lagrangian, the so-called Feynman diagrams.

Symmetries

A symmetry is a transformation of the fields that leaves the action invariant. Consider a transformation T_ϵ of the field ϕ, where ϵ is an infinitesimal, spacetime dependent parameter. Let us consider the free complex scalar field,

$$\phi \to \phi' = T_\epsilon \phi.$$

Suppose $T_\epsilon = e^{-i\epsilon Q}$ is a symmetry, we say that

- T_ϵ is a global symmetry if ϵ is constant.
- T_ϵ is a local symmetry if $\epsilon \equiv \epsilon(\bar{x}, t)$.

With an infinitesimal transformation parameter the variation in the action will be of the form

$$\begin{aligned} \delta S &= \int d^4x j^\mu \partial_\mu \epsilon \\ &= \int d^4x \partial_\mu (j^\mu \epsilon) - \int d^4x \epsilon (\partial_\mu j^\mu) \end{aligned} \tag{2.1.1}$$

we call the vector field $j = (j^0, \vec{j})$ the *Noether current*. We choose ϵ to vanish at the boundary of spacetime, this reduces the first term in eqn(2.1.1) to zero. We are then left with

$$\delta S = -\int d^4x \epsilon (\partial_\mu j^\mu).$$

Noether's theorem states that there exists a conserved current j^μ for every global symmetry. Thus

$$\begin{aligned} \delta S = \epsilon \textstyle\int d^4x \partial_\mu j^\mu = 0 \\ \Rightarrow \partial_\mu j^\mu = 0. \end{aligned} \tag{2.1.2}$$

If we think of $\vec{j}$ as an electric current density and j^0 as electric charge density then the eqn(2.1.2) is the continuity equation

$$\partial_0 j^0 - \vec{\nabla} \cdot \vec{j} = 0$$

which tells us about the conservation of charge. At the quantum level a global symmetry is a transformation that must leave both the action and the measure invariant i.e. $S[\phi] = S[\phi']$ and $D_\phi = D_{\phi'}$. The quantum Noether theorem requires

$$\langle \partial_\mu j^\mu \rangle = \int [D_\phi] \, e^{-iS[\phi]} \partial_\mu j^\mu = 0$$

A classical symmetry which cannot be preserved when the theory is quantized, because $D_\phi \neq D_{\phi'}$, is said to have an "anomaly". Now in order for us to make our global symmetry local we let the transformation parameter depend on x. In the straightforward application of this rule, we encounter a problem: $\partial_\mu \phi$ does not transform covariantly, i.e.

$$\partial_\mu \phi \longrightarrow \partial_\mu \phi'(x) \neq e^{i\alpha(x)Q} \partial_\mu \phi(x).$$

We can remedy the problem by introducing a gauge field A_μ and replacing ∂_μ with a covariant derivative $D_\mu \equiv \partial_\mu - iQA_\mu$. Then

$$D_\mu \phi \longrightarrow D'_\mu \phi' = e^{i\epsilon(x)Q} D_\mu \phi,$$

on condition that the gauge transformation of A_ν is $A'_\mu = A_\mu + \frac{i}{Q} e^{-i\epsilon Q} \partial_\mu e^{i\epsilon Q}$. The free action of the gauge field A_μ is

$$S_{gauge} = -\frac{1}{4} \int d^d x F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu} = \frac{1}{Q}[D_\mu, D_\nu]$ is the field strength of the gauge field. $F_{\mu\nu}$ is invariant under the gauge transformation. The complete scalar field action that is invariant under the local symmetry, in this case a $U(1)$ gauge symmetry, is

$$S = \int d^4 x \left[(D_\mu \phi)(D^\mu \phi^*) - \frac{1}{4} F_{\nu\mu} F^{\nu\mu} \right].$$

Non Abelian Gauge theories

The generalization of the $U(1)$ local transformation, dealt with above, to a $U(N)$, $SU(N)$ or $SO(N)$ $(N > 1)$ transformation yields a non Abelian Gauge theory, also known as a Yang-Mills (YM) theory [20]. Let $U \in U(N)$, then we can write

$$U(x) = e^{-ig\epsilon^a(x)T^a} \qquad a = 1, \ldots, N^2.$$

The $(T^a)_{ij}$ $(i, j = 1, \ldots, N)$ are the gauge group generators in representation R, and they satisfy the following relations:

$$[T_a, T_b] = f_{ab}{}^c T_c \qquad \text{(Lie algebra)}$$

$$tr(T_a T_b) = \delta_{ab} \qquad \text{(normalization)}$$

$$\sum_a (T_a)_{ij}(T_a^*)_{kl} = \delta_{ik}\delta_{jl} \qquad \text{(Completeness)}$$

The Yang Mills fields $(A_\mu)_{ij}$ are again used to define the covariant derivative

$$(D_\mu)_{ij} \equiv \delta_{ij}\partial_\mu - ig(A_\mu)_{ij},$$

which ensures that the derivative terms transform covariantly, again on condition that

$$A'_\mu = UA_\mu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger.$$

The field strength of the gauge field is again

$$F_{\mu\nu} \equiv \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\nu, A_\mu],$$

previously in $U(1)$ gauge theory the last term was zero (Abelian). We can decompose our gauge field A_μ into a collection of scalar fields A_μ^a and generators T^a [20],

$$A_\mu = A_\mu^a T^a$$

such that

$$A'^a_\mu = A_\mu^a - gA_\mu^c \epsilon^b f^{bca} - \partial_\mu \epsilon^a.$$

This allows us to re-express our relations in terms of the generators of the gauge group. The adjoint representation is defined by $(T^a)_{bc} = f^a{}_{bc}$. The gauge fields lives in the adjoint representations and the field strength is [21]

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^a{}_{bc} A^b_\mu A^c_\nu.$$

The gauge invariant, non-Abelian action is

$$\begin{aligned} S &= -\int d^4x \frac{1}{4} tr F_{\mu\nu} F^{\mu\nu} \\ &= -\int d^4x \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \\ &= \int d^4x \left[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2} g (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f^{abc} A^{b\mu} A^{c\nu} \right. \\ &\quad \left. + \frac{1}{2} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{e\nu} A^{d\mu} \right]. \end{aligned} \tag{2.1.3}$$

The presence of the second and third terms in eqn(2.1.3) show that the gauge fields of YM theory themselves carry charge and hence interact with each other. Pure YM gauge interactions are studied by treating the second and third terms in eqn(2.1.3) as small perturbations to the first term. These additional interaction terms are important in understanding phenomenon such as asymptotic freedom.

2.2 General Relativity, AdS space and p-branes

General Relativity

The theory of special relativity was discovered by Einstein during a time of conflict between electro-magnetism and Newtonian mechanics. Electromagnetic waves have a constant speed, c, when propagating through a vacuum, regardless of the motion of an inertial observer. This phenomenon was inconsistent with Galilean (or Newtonian) relativity which predicts that an inertial observer should measure a faster/slower speed for light when moving towards/away the wave. Special relativity begins with two simple assumptions

- Physics is the same in all inertial reference frames,
- The speed of light, c, is constant for all inertial observers.

As a consequence of these assumptions we have that the line element ds (also often called the metric) defined by

$$ds^2 = -dt^2 + d\bar{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \qquad \eta_{\mu\nu} = diag(-,+,+,\dots,+) \tag{2.2.1}$$

is a physical constant. That is, all inertial observers will agree on lengths defined via eqn(2.2.1). The set of transformations which leave this line element invariant, form a group SO(1,d-1) called the Lorentz group. The striking resemblance between eqn(2.2.1) and the geometric theorem of Pythagoras

$$ds^2 = dx_1^2 + dx_2^2 + \dots + dx_d^2 = \delta_{ij} dx^i dx^j,$$

naturally compels one to think of the 1 dimension of time and the $d-1$-spacial dimensions as describing some geometry. We call this geometry *spacetime*. Some important characteristics of a general geometry are coded in the *metric tensor*, g_{ij}, an arbitrary function that maps two vector space elements to a scalar, $g : V \otimes V \to \mathbb{R}$. The metric tensor is used to define lengths on a geometry as follows

$$ds^2 = g_{ij}(x)dx^i dx^j.$$

We will have to modify some of our Euclidean notions and introduce new ones, in order for us to do calculus on general geometries. We define the *Christoffel symbol* $\Gamma^\mu{}_{\sigma\rho}$ which helps us compare vectors at different positions in the space (also known as parallel transport). It is useful in defining the generalized directional derivative, the so called covariant derivative D_μ,

$$D_\mu \bar{v} = (\partial_\mu v^\alpha)\bar{e}_\alpha - v^\rho \, \Gamma^\sigma{}_{\mu\rho} \, \bar{e}_\sigma$$

where vectors are expanded as follows: $\bar{v} = v^\alpha \bar{e}_\alpha$ and $\bar{e}_\alpha$ are basis vectors in some arbitrary coordinate system. In terms of the metric tensor the Christoffel symbol is

$$\Gamma^\mu{}_{\sigma\rho} = \frac{1}{2} g^{\mu\nu} \left(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\sigma\rho} \right),$$

where $g^{\mu\nu} = (g^{-1})_{\mu\nu}$. Next we define the Riemann tensor

$$\left(R^\mu{}_\nu \right)_{\rho\sigma} (\Gamma) = \partial_\rho (\Gamma^\mu{}_\nu)_\sigma - \partial_\sigma (\Gamma^\mu{}_\nu)_\rho + (\Gamma^\mu{}_\lambda)_\rho (\Gamma^\lambda{}_\nu)_\sigma - (\Gamma^\mu{}_\lambda)_\sigma (\Gamma^\lambda{}_\nu)_\rho.$$

Written in this manner it appears that the Riemann tensor is a "field strength" of the "gauge field" $(\Gamma^\mu{}_\nu)_\sigma$, when contrasted with the field strength of a Yang Mills theory (in the adjoint representation)

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^{ac}_\mu A^{cb}_\nu - A^{ac}_\nu A^{cb}_\mu.$$

The Ricci tensor $R_{\mu\nu}$ is the Riemann tensor with the contraction

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$$

and the Ricci scalar is

$$R = R_{\mu\nu} g^{\mu\nu}.$$

The Ricci scalar is an invariant quantity that has encoded in it features that completely characterize a geometry.

Some comments: Tensors are indexed objects that transform according to the rule

$$T_{\mu\nu}{}^{\sigma}(x) \to T'_{\mu\nu}{}^{\sigma} = \frac{\partial x'^{\sigma}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} T_{\gamma\beta}{}^{\alpha} \, .$$

So far the tensors we have come across are $g_{\mu\nu}$, $(R^{\mu}{}_{\nu})_{\rho\sigma}$ and $R_{\mu\nu}$, the Christoffel symbol is not a tensor as it does not satisfy the above transformation rule. The covariant derivative D_{μ} is a tensor. The action of the covariant derivative on a tensor is as follows

$$D_{\mu} T_{\rho\nu}{}^{\sigma} \equiv \partial_{\mu} T_{\rho\nu}{}^{\sigma} + \Gamma^{\sigma}{}_{\alpha\mu} T_{\rho\nu}{}^{\alpha} - \Gamma^{\alpha}{}_{\rho\mu} T_{\alpha\nu}{}^{\sigma} - \Gamma^{\alpha}{}_{\nu\mu} T_{\rho\alpha}{}^{\alpha}$$

therefore on a scalar $D_{\mu}\phi \equiv \partial_{\mu}\phi$.

Einstein's theory of **general relativity** resolves the conflict between special relativity and Newtons gravitation. Newtonian gravity allowed for gravitational interactions that propagate at speeds faster than that of light, however according to SR nothing can travel faster than light. The assumptions of general relativity are:

1. Physics is invariant under general (local) coordinate transformation $x \to x'(x)$. As an example: *we must have*

$$\begin{aligned} ds^2 &= g_{ij}(x) dx^i dx^j \\ &= g'_{ij}(x) dx'^i dx'^j \end{aligned}$$

2. Locally there is no measurable difference between acceleration and gravity. For example: *an observer in a lift will not be able to distinguish whether the lift is*

in a weak gravitational field or it is being accelerated by an engine in free space.

Hence

$$m_i = m_g \qquad \text{where } \bar{F} = m_i \bar{a} \;\text{ and } \bar{F}_G = m_G \frac{GM}{r^2}\hat{r}.$$

In special relativity all bodies trace out a spacetime trajectory, called a *world line*. Free bodies(bodies with no forces acting on them) have geodesic (a generalization of a straight line) world lines which is consistent with Newtons first law of motion, while accelerating bodies have non-geodesic world lines. Gravity accelerates matter, i.e. it forces matter to have curved (non-geodesic) world lines on what we assume to be a flat spacetime background. We can reconcile SR and gravity if we think of gravity as a spacetime curvature, then the equivalence between inertial and gravitational mass implies that one cannot distinguish between a curved world line on a flat spacetime background and geodesic on a curved spacetime. So bodies continue to move on geodesics if no resultant forces act upon them and the effect of gravity is actually the result of spacetime curvature.

To construct a theory of gravity (curved spacetime) we need an action that is invariant (in line with assumption 1), and at the same time that captures information about the curvature of space time. One such action is the Einstein Hilbert action

$$S_G = \frac{1}{16\pi G}\int d^d x\sqrt{-g}R$$

where G is Newtons constant, $\int d^d x\sqrt{-g}$ is a coordinate invariant measure and R is the Ricci scalar discussed above. The equation of motion for this action is

$$\frac{\delta S_G}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \qquad (2.2.2)$$

If we add matter to the theory we get that the right hand side of eqn(2.2.2) is no longer zero, but

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \qquad (2.2.3)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}$. Equation (2.2.3) is often referred to as the Einstein equation. We learn from eqn (2.2.3) that matter is the source of spacetime curvature whilst at the same time recalling that curvature dictates the world lines of matter through spacetime.

Solutions to eqn(2.2.3) with non-zero mass and an event horizon[1] are called **black holes**. One such solution is the Reissner-Nordst$\ddot{o}$rm metric; which is a static, spherically symmetric, non rotating and charged black hole,

$$ds^2 = -(1 - \frac{2MG}{r} + \frac{Q^2G}{r^2})dt^2 + \frac{dr^2}{1 - \frac{2MG}{r} + \frac{Q^2G}{r^2}} + r^2 d\Omega_2^2. \tag{2.2.4}$$

where t is the time, r is a radial coordinate and Ω is the solid angle on a 2-sphere. The electric field is a point charge electric field, $A_t = -\frac{Q}{r} \Rightarrow F_{r,t} = \frac{Q}{r^2}$. Notice that the metric appears to be singular ($g^{rr} \to \infty$) at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}.$$

Where we have set $G = 1$. For convenience let us consider the $Q = 0$ limit, this is called the Schwarzschild metric. There seems to be two singularities, at $r = 0$ and at $r = M$ in eqn(2.2.4). The former is coordinate independent, the Ricci scalar diverges, hence it is a real singularity and it labels the position of infinite energy density. The latter is a coordinate singularity because in an appropriately chosen coordinate system, we learn that the geometry near the $r = 2M$ bound is non-singular and almost flat for big enough M [22]. The apparent singularity is called the *event horizon* and represents a causal boundary, in that no signal can escape the gravitational pull once it traverses the event horizon. The case where $M = |Q|$, is called the *extremal black hole* and it has a metric given by

$$ds^2 = -(1 - \frac{M}{r})^2 dt^2 + (\frac{dr}{1 - \frac{M}{r}})^2 + r^2 d\Omega_2^2.$$

[1]This will be defined in the next paragraph when we discuss an example of a black hole soution.

Extremal black holes are a rich probe in the study of black hole thermodynamics and in the low energy limit of superstrings. They are commonly written in the isotropic coordinates systems, which take on the the form

$$ds^2 = -H^{-2}(\bar{x})dt^2 + H^2(\bar{x})d\bar{x}^2$$

where $d\bar{x}^2 = \eta_{\mu\nu}dx^\nu dx^\mu$ and $H(\bar{x})$ is a harmonic function [22]. The Reissner-Nordst$\ddot{o}$rm metric is written with the harmonic function

$$H(\bar{x}) = \left(1 + \frac{Q}{\rho}\right)$$

where ρ is the isotropic coordinate that places the horizon at $\rho = 0$.

A generalization of black holes, in a higher dimensional space, is called a ***p*-brane**. Where p is the number of spatial dimensions the horizon extends in. We will be interested in "extremal p-branes" in the low energy limit of $d = 10$ superstring theory. The solution of the $d = 10$ superstring theory that reduces to supergravity in the low energy limit is [21]

$$\begin{aligned} ds^2_{string} &= H_p^{-\frac{1}{2}}(-dt^2 + d\bar{x}_p^2) + H_p^{\frac{1}{2}}(dr^2 + r^2 d\Omega^2_{8-p}) \\ &= H_p^{-\frac{1}{2}}(-dt^2 + d\bar{x}_p^2) + H_p^{\frac{1}{2}} d\bar{x}^2_{9-p}, \end{aligned}$$

where H_p is a harmonic function of $\bar{x}_{9-p}$ and it has the form $H_p = 1 + \frac{(\dots)Q}{r^{7-p}}$.

AdS space

The abbreviation AdS stands for anti de Sitter. Therefore in trying to understand AdS we first have to come to terms with de Sitter space. A space of Lorentzian signature is a space where the diagonal part of the metric has the form $(-, +, \cdots, +)$, the time component of the lines element ds will have factor of -1. Now a de Sitter space is the Lorentzian analogue of a Euclidean sphere with positive curvature. In d

dimensions we define it by embedding it in a $d+1$ dimensional space, we write: the metric of the higher dimensional space

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 + dx_d^2$$

and the equation of the Lorentzian sphere

$$\Rightarrow -x_0^2 + \sum_{i=1}^{d-1} x_i^2 + x_d^2 = R^2.$$

Now consider a d dimensional Euclidean surface of constant negative curvature, called the *Lobachevski* space. We model this geometry by embedding it in a $d+1$ Minkowski space[2]. We write

$$x_1^2 + x_2^2 + \cdots + x_{d-1}^2 - x_d^2 = -R^2$$

Anti de-Sitter space is the Lorentzian analog of the Lobachevski space, in that it is Lorentzian space of constant negative curvature. To describe it in d dimensions we embed it in a larger non Lorentzian $d+1$ dimensional space, the metric of the embedding space and equation describing the AdS space are

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 - dx_d^2$$

$$-x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_d^2 = -R^2. \tag{2.2.5}$$

AdS space is invariant under the $SO(2, d-1)$ group.

The boundary of AdS_d is $\mathbb{R} \times S_{d-2}$, a time coordinate times a $(d-2)$ sphere. As we will see later, this space is conformally equivalent to $\mathbb{R}^{1,d-2}$ Minkowski space. AdS space is an interesting geometry in GR, it is a maximally symmetric, vacuum solution to the Einstein equation (eqn(2.2.3)) with a constant energy-momentum tensor $T_{\mu\nu} = \Lambda g_{\mu\nu}$, $\Lambda < 0$. Λ is known as the cosmological constant.

[2]Notice that we cannot embed this surface in a Euclidean space

2.3 String theory and D-branes

The "strings" in string theory are not the everyday strings that we are used to, i.e. strings that stretch and can possibly snap. The strings we are accustomed to are composed of tiny little particles, that move closer and further apart from each other as one compresses or stretches the string. This is not the case in string theory, the strings of string theory are fundamental strings which means that they are not made up of any smaller constituents. So when one stretches a fundamental string it increases in length but the tension remains fixed. As a result there can only be transverse waves propagating on a fundamental string.

Dp-branes are p dimensional hyper-surfaces where strings are allowed to end. D-branes are not inert objects that make up the backdrop of the string's background, they are dynamical entities that have an existence independent of strings.
In this section we will briefly review some important string actions and then end off with a brief discussion on D-branes. The structure and content in this section is a fusion between [21] and [23].

String theory actions

A free point particle has an action given by the proper length of its world line and it evolves in a manner that will extremize its action. Extending this analogy to a one dimensional object, say a string, this object sweeps out a surface in space-time (a world sheet) instead of a world line. Therefore naturally one would expect that the string will evolve in a manner that extremizes the area of its world sheet. In other words, we are guessing (and correctly so) that the action is the area of the world sheet. Consider a string in an arbitrary spacetime with metric $g_{\mu\nu}$. Let $X^{\mu}(\sigma,\tau)$ be the coordinates of the position of the string, where σ is the world sheet length and τ

is the world sheet time. The simplest string action is the Nambu-Goto action,

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-(\dot{X})^2(X')^2 + (\dot{X}_\nu X'^\nu)^2}$$

where $\frac{1}{2\pi\alpha'} = T$ is the string tension, $\dot{X} = \frac{\partial}{\partial\tau}X$ and $X' = \frac{\partial}{\partial\sigma}X$. The induced metric h_{ab} (*pullback* of the spacetime metric) on the world sheet is

$$h_{ab}(\sigma,\tau) = g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu.$$

The Nambu-Goto action simplifies to

$$S_{NG} = -T\int d\sigma d\tau\sqrt{-det(h_{ab})}.$$

In order to have a tractable expression for the equations of motion, we first define the conjugate momenta

$$\Pi^\tau_\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} = -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}},$$
$$\Pi^\sigma_\mu = \frac{\partial\mathcal{L}}{\partial X'^\mu} = -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}}.$$

The equations of motion in flat space are given by

$$\frac{\partial\Pi^\tau_\mu}{\partial\tau} + \frac{\partial\Pi^\sigma_\mu}{\partial\sigma} = 0.$$

There is an action equivalent to the Nambu-Goto action called the Polyakov action, in flat spacetime it is

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu\eta_{\nu\mu}$$

where $\gamma(\sigma,\tau)$ is an independent wolrdsheet metric and $\gamma \equiv \det\gamma$. To see that the Polyakov action is equivalent to Nambu-Goto action we compute the equation of motion for the independent world sheet metric (use $\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} =$

$\frac{1}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta})$

$$h_{ab} - \frac{1}{2}\gamma_{ab}(\gamma^{cd}h_{cd}) = 0 \quad (2.3.1)$$

$$\Rightarrow \sqrt{-\gamma} = \frac{2}{\gamma^{cd}h_{cd}}\sqrt{-h}. \quad (2.3.2)$$

Plugging the result in eqn(2.3.2) into the Polyakov action reduces it to the Nambu-Goto action. This equivalence was demonstrated only at the classical level.

Symmetries

The Nambu-Goto and Polyakov action have the following symmetries:

- Spacetime Poincaré invariance.
- World sheet reparametrization (diffeomorphism) invariance defined by two transformations $(\tau, \sigma) \to (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$ that give $X'^{\mu}(\sigma', \tau') = X^{\mu}(\sigma, \tau)$.

The Polyakov action has an additional symmetry called Weyl symmetry.

- Weyl invariance is a symmetry that results from the transformations of the form $X^{\mu}(\tau, \sigma) \to X^{\mu}(\tau, \sigma)$, while the metric becomes

  $$\gamma_{\alpha\beta}(\tau, \sigma) \to e^{2\phi(\tau,\sigma)}\gamma_{\alpha\beta}(\tau, \sigma).$$

  The Weyl transformation is a conformal transformation which means that it preserves angles and not distances. The independence of the action on the factor of $\Omega^2 = e^{2\phi(\tau,\sigma)}$, which is very much like the factor $\gamma^{cd}h_{cd}$, is a gauge symmetry of the theory since Ω is a function of the world sheet coordinates (τ, σ). This means that two metrics that are related by a Weyl transformation are to be considered as the same physical state, just as gauge related vector potentials in electrodynamics produce the same electromagnetic field.

Fixing a gauge

The Polyakov action has three world sheet invariances (2 diffeomorphisms and 1 Weyl) therefore we have a degree of freedom in choosing three components of the world sheet metric $\gamma_{\alpha\beta}$ as we please. Naturally we will select to work in a gauge that we are very familiar with,

$$\gamma_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$$

called the conformal gauge. Weyl invariance allows us to set ϕ to zero, thus the world sheet metric reduces to the Minkowski metric. Then the Polyakov action in flat spacetime becomes

$$S = -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\nu\mu}.$$

The X^μ equation of motion gives the 2 dimensional wave equation

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu = -4\partial_+\partial_- X^\mu = 0$$

we have made use of the light-cone coordinates after the first equality. Light-cone coordinates are defined as follows: $\sigma^\pm = \tau \pm \sigma \Rightarrow \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. The general solution of the 2-dimensional wave equation is

$$X^\mu(\sigma,\tau) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+). \tag{2.3.3}$$

This is where the closed and open strings part ways: we will defer the discussion of open strings to a later subsection and continue with the discussion of closed strings. Closed strings do not require the imposition of boundary conditions, instead the range of the world sheet space coordinate becomes periodic $\sigma \in [0\ , 2\pi)$

$$X^\nu(\sigma + 2\pi) = X^\nu(\sigma).$$

The mode expansion of the solution to the 2 dimensional wave equation is

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{l^2}{4}p^\mu\sigma^- + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}$$
$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{l^2}{4}p^\mu\sigma^+ + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}$$

where $l^2/2 = \alpha'$ and the subscripts R and L stand for right-moving and left-moving waves respectively. In addition to fixing a gauge we have to make sure that the equations of motion of the world sheet metric is satisfied. The equations of motion of the Polyakov action for the metric γ is

$$\frac{2}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}} \equiv T_{\alpha\beta} = 0 \tag{2.3.4}$$

Imposing our gauge condition $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ into the equations of motion

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\lambda}\partial_\rho X \partial_\lambda X = 0.$$

The equations of motion of γ imposes the following constraints on X

$$T_{01} = \dot{X}\cdot X' = 0$$
$$T_{00} = T_{11} = \tfrac{1}{2}(\dot{X}^2 + X'^2) = 0.$$

In appropriately chosen coordinates, we can interpret the constraints physically. The first constraint tells us that the physical modes of the string are only transverse oscillations. The second constraint equation relates the length of the string to the instantaneous velocity of the string, as the string fluctuates in size (length).
In the light cone coordinates it is sufficient to ensure

$$(\partial_+ X)^2 = 0 \quad \text{and} \quad (\partial_- X)^2 = 0,$$

in order to impose the constraints of eqn(2.3.4). Start with

$$\begin{aligned}\partial_- X^\mu &= \partial_- X^\mu_R(\sigma^-)\\ &= \frac{\alpha'}{2}p^\mu + \sqrt{\frac{\alpha'}{2}}\sum_{n>0}\alpha^\mu_n e^{-in\sigma^-}\\ &= \sqrt{\frac{\alpha'}{2}}\sum_{n=0}\alpha^\mu_n e^{-in\sigma^-},\end{aligned}$$

where we have defined $\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu$. Then

$$\begin{aligned}(\partial_- X)^2 &= \frac{\alpha'}{2}\sum_{m,p}\alpha_m\cdot\alpha_p e^{-i(m+p)\sigma^-}\\ &= \frac{\alpha'}{2}\sum_{m,n}\alpha_m\cdot\alpha_{n-m}e^{-in\sigma^-}\\ &= \alpha'\sum_n L_n e^{-in\sigma^-},\end{aligned}$$

where we have defined the sum of oscillator modes $L_n = \frac{1}{2}\sum_m \alpha_m\cdot\alpha_{n-m}$. The same procedure can be repeated for $(\partial_+ X)^2$, we again define the sum of left moving oscillators $\tilde{L}_n = \frac{1}{2}\sum_m \tilde{\alpha}_m\cdot\tilde{\alpha}_{n-m}$ and the zero mode $\tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu$. There are infinitely many constraints to impose

$$L_{n\geqslant 0} = \tilde{L}_{n\geqslant 0} = 0.$$

The zero modes are special in that they tell us about the mass spectrum in terms of excited oscillator modes,

$$\begin{gathered}L_0 = \alpha_0\cdot\alpha_0 + \textstyle\sum_{m>0}\alpha_m\cdot\alpha_{-m} = 0\\ \Rightarrow p\cdot p \equiv -M^2 = -\frac{4}{\alpha'}\textstyle\sum_{m>0}\alpha_m\cdot\alpha_{-m}\end{gathered}$$

and similarly

$$p\cdot p \equiv -M^2 = -\frac{4}{\alpha'}\sum_{m>0}\tilde{\alpha}_m\cdot\tilde{\alpha}_{-m} = 0$$

Quantization

In quantizing the string we follow the standard procedure of promoting the mode weights α_m^μ and $\tilde{\alpha}_m^\mu$ to operators and define their algebra. We treat α_m^μ and $\tilde{\alpha}_m$ as annihilation operators whereas α_{-m}^μ and $\tilde{\alpha}_{-m}^\mu$ are creation operators for $m > 0$,

$$\alpha_m^\mu = \sqrt{m} a_m^\mu, \quad \alpha_{-m}^\mu = \sqrt{m}\; {a^\dagger}^\mu_m\,, \quad \tilde{\alpha}_m^\mu = \sqrt{m}\tilde{a}_m^\mu, \quad \tilde{\alpha}_{-m}^\mu = \sqrt{m}\; {\tilde{a}^\dagger}{}^\mu_m\,.$$

The creation and annihilation operators obey the algebra

$$[a_m^\mu, {a^\dagger}^\nu_n] = \delta_{mn}\eta^{\mu\nu} \quad \text{ and } \quad [\tilde{a}_m^\mu, {\tilde{a}^\dagger}{}^\nu_n] = \delta_{mn}\eta^{\mu\nu}.$$

We also have to take care of the constraint operators, this is done by imposing the condition that the expectation of these operators vanish, it is sufficient to require[3]

$$L_m|phys\rangle = 0, \quad \tilde{L}_m|phys\rangle = 0$$
$$(L_0 - b)|phys\rangle = (\tilde{L}_0 - b)|phys\rangle = 0,$$

for some normal ordering constant b that is related to the dimension of spacetime. The closed string spectrum is now given by

$$\alpha'\, M^2 = 4\left(-b + \sum_{m\geqslant 1} mN_m\right) = 4\left(-b + \sum_{m\geqslant 1} m\tilde{N}_m\right),$$

where we have defined level number operators $N_m = {a^\dagger}_m \cdot a_m^\mu$ and $\tilde{N}_m = {\tilde{a}^\dagger}_m \cdot \tilde{a}_m^\mu$. The relativistic string on Minkowski background can only be quantized consistently in a spacetime with 26 dimensions (required for the Lorentz group $SO(1, d-1)$ invariance) and $b(26) = 1$[21]. To get back to a 4-dimensional theory we could make use of the Kaluza-Klein compactification ideas.

The closed string mass spectrum has a ground state with mass $\alpha' M^2 = -4$, this is a *tachyon*. Denote this state by $|0; k\rangle$ and it obeys $a_1|0; k\rangle = \tilde{a}_1|0; k\rangle = 0$. The

[3]Since $L_n^\dagger = L_{-n}$.

first excited states are on the level $m = 1$ and they have an equal mass of $\alpha' M^2 = 4(-1+1) = 0$, i.e. massless states. Denote the states by $a_1^{\dagger\,\mu}\,\tilde{a}_1^{\dagger\,\nu}\,|0,k\rangle$, which is effectively a tensor state $A^{\mu\nu}$. The tensor state decomposes into three components: a symmetric traceless tensor part $g_{\mu\nu}$ (the graviton), an antisymmetric tensor part $B_{\mu\nu}$ (B field) and a trace part ϕ (the dilaton).

What we have been considering so far has been the simplest bosonic string action and we have seen one of its biggest short coming is the tachyonic ground state. The tachyon ground state indicates that we are expanding about a local maximum in the potential of the tachyon field as opposed to a minimum. This implies that what we were referring to as a "ground state" is an unstable state and the system can decay into possibly[4] another(or the true) ground state.

Opens strings and D-branes

We would like the generic points on a string to be governed by local physics: this means that a generic point on a string will not be able to distinguish if it is part of an open or closed string. This was implied, though in a subtle manner, when we discussed string actions by the fact that we did not distinguish between open and closed string actions. An open string system can be described by the Polyakov action. Let us work in the conformal gauge with $\phi = 0$,

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X.$$

The variation of the action is given by

$$\begin{aligned}\delta S &= -\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \partial_\alpha X \cdot \partial^\alpha \delta X \\ &= -\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[\frac{\partial}{\partial\tau}(\delta X\cdot\dot{X}) + \frac{\partial}{\partial\sigma}(\delta X \cdot X') - \delta X\cdot(\partial^\alpha\partial_\alpha X)\right].\end{aligned}$$

[4]It has not yet been established that there is indeed a stable ground state .i.e. if a stable minimum exists

In the second equality we have made use of integration by parts. The first term, is a total derivative and evaluates to zero since we require that $\delta X^\mu = 0$ at $\tau = \tau_i$ and $\tau = \tau_f$. The second term can be made to vanish independently in two ways

- Neumann boundary conditions (string ends are free to move): $\partial^\sigma X_\mu = 0$ at $\sigma = 0$ and $\sigma = \pi$. This condition implies that the the end points will move at the speed of light.

- Dirichlet boundary conditions (string ends are fixed): $\delta X^\mu = 0$ at $\sigma = 0$ and $\sigma = \pi$. This condition implies that the end points are fixed at a constant position, $\bar{X} = \bar{c}$.

We can choose $p+1$ Neumann boundary conditions for p spatial and one time dimension, and $d-p-1$ Dirichlet boundary conditions. This means that the endpoints live on $p+1$ dimensional hyper-surfaces in spacetime. We call these hyper-surfaces *D-branes* or $Dp-branes$ when we want to specify their dimension. The D stands for Dirichlet. In this language, a $D0$-brane is a particle, a $D1$-brane is a string and a $D2$-brane is a membrane. D-branes are dynamical and hence can interact with the strings. The mode expansion of the solution to the 2 dimensional wave equation is

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{l^2}{2}p^\mu\sigma^- + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}$$
$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{l^2}{2}p^\mu\sigma^+ + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}.$$

Imposing the boundary conditions sets the following relations between the modes:

- Neumann boundary conditions, $\partial_\sigma X^a = 0$ at the end points implies

$$\alpha_n^a = \tilde{\alpha}_n^a \qquad a = 0, \ldots, p$$

- Dirichlet boundary conditions, $X^I = c^I$ at the end points implies

$$x^I = c^I, \quad p^I = 0, \quad \alpha_n^I = -\tilde{\alpha}_n^I \qquad I = p+1, \ldots, d-1.$$

Notice that in both boundary conditions the left and right moving modes are not independent. The mass spectrum is given by

$$\alpha' M^2 = \sum_{i=0}^{p}\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{i=p+1}^{d-1}\sum_{n>0}\alpha^i_{-n}\alpha^i_n - b.$$

The first sum is over modes parallel to the brane, the second over the modes perpendicular to the brane and again b is the normal ordering constant. Lorentz symmetry under the group $SO(1,p)\times SO(d-p-1)$ at the quantum scale, is preserved only in a 26 dimensional spacetime and with $b=1$. These are exactly the same restrictions on closed strings. This an indication that open and closed strings can coexist, a stronger statement would be that they are different states of the same theory.

The mass spectra has a ground state with mass $\alpha' M^2 = -1$, a tachyon, as was the case with closed strings. It is defined by $|0;p\rangle$ and obeys $a_n|0;p\rangle = \tilde{a}_n|0;p\rangle = 0, \quad n>0$. The first excited states are massless. There are two classes:

- Oscillators longitudinal to the brane

$$a^{\dagger\, r}_{\ 1}\,|0;p\rangle, \qquad r = 0,\ldots,p.$$

  These are vector states on the brane that transform under the $SO(1,p)$ Lorentz group. We represent the state with a spin 1 gauge field A^r, a photon.

- Oscillators transverse to the brane

$$a^{\dagger\, I}_{\ 1}\,|0;p\rangle \qquad I = p+1,\ldots,d-1.$$

  These are vector states transverse to the brane that transform under the $SO(d-p-1)$ rotation group. It is also useful to think of these states as scalars under the $SO(1,p)$ Lorentz group of the brane. We can think of these states as excitations of a scalar field ϕ^I living on the brane. We interpret these states as fluctuations of the brane in the transverse direction, a sign that the D-brane is dynamic.

D-branes

In the language of D-branes we understand strings as one dimensional branes. Therefore we may guess that the dynamics of a Dp-brane are captured by a p-dimensional Nambu-Goto action

$$S_{\mathrm{D}_p} = -T_p \int d^{p+1}\xi \sqrt{-\det \gamma},$$

where T_p is the tension of the Dp-brane and $\xi^a, a = 0, \ldots, p$ are the worldvolume coordinates of the brane. Again γ_{ab} is the pullback of the spacetime metric. The transverse excitations of the brane we spoke of above, can be recovered explicitly in Minkowski space and in the static gauge with the reparametrization $X^a = \xi^a$,

$$X^I(\xi) = 2\pi\alpha'\phi^I(\xi), \qquad I = p+1, \ldots, d-1.$$

The quantum theory of D-branes is, as yet, not well understood [23]. Beside it proving to be very difficult to quantize a brane (Weyl invariance does not hold for branes), there are some conceptual difficulties involved [23]. Quantizing a membrane or higher dimensional object results in a continuous spectrum of energy states! Therefore we lose the interpretation of discrete states corresponding to a particle. The expectation is that the quantum membrane should describe an ensemble of particles, which would explain the continuous spectrum. We have already established that open strings end on D-branes. And there is nothing preventing a string to have its end points on two different branes. We can keep track of the end points by the labels $|i\rangle, |j\rangle$ called *Chan-Patton factors*, for a string stretching between D-brane i and D-brane j. For N D-branes the possible open string configurations $|i\rangle|j\rangle\lambda^a_{ij}$ forms a $N \times N$ matrix. We can then write the massless fields

$$\left(\phi^I\right)^m_n \qquad \text{and} \qquad \left(A_a\right)^m_n. \tag{2.3.5}$$

The gauge fields A_a on a single brane is responsible for a local $U(1)$ symmetry. It can be shown that if there are N coincident branes then the $U(1)^{\otimes N}$ gauge symmetry of the brane is enhanced to a $U(N)$ gauge symmetry. The fields transform in the adjoint representation of $U(N)$ in eq(2.3.5). In fact in can be shown that the lower energy limit $\alpha' \to 0$ of N coincident D-branes is $\mathcal{N} = 4$ Super Yang Mills theory with gauge group $U(N)$.

There are a plethora of different D-brane interactions, however a special one worth mentioning is the RR charge[5] which D-branes carry. The RR charge is responsible for the repulsive force between D-branes. At the same time the D branes interact via an attractive potential which is mediated by an exchange of gravitons and dilatons. A favourable property of D-branes is that their tension is equal to the RR charge of the brane, this results in the RR repulsion being matched by the gravitational and dilaton attraction [25]. D-branes possess a number of favourable properties because they are BPS states. As we shall mention in the next subsection, type II string theory has $\mathcal{N} = 2$ supersymmetry in 9+1 dimensions however in the presence of a D-brane the symmetry is reduced to $\mathcal{N} = 1$, *half* the symmetry is preserved. We will see later on how these properties and interactions make the D-branes interesting probes for the AdS-CFT correspondence.

Superstrings

The RR modes we spoke of above exist only in superstring theories. Adding fermion modes to a string theory results in a supersymmetric string theory, often referred to as *superstring* theory. Superstrings are compelling for three primary reasons:

- their quantum consistency requires that spacetime be 10 dimensional for both open and closed strings.

[5] Joeseph Polchinski argued in 1995, D-branes are charged objects that act as sources for RR fields [24]

- there is no tachyon
- The ground state is composed of the massless states $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ and some other supersymmetric fields [21].

There is a freedom in adding the fermion fields onto the bosonic string. We decomposed the string modes into right and left moving. This gives rise to two different classes of string theory.

- Type II strings have both left and right-moving world sheet fermions. Quantum consistency in this theory requires a spacetime of 10 dimensions with $\mathcal{N} = 2$ supersymmetry, which means 32 supercharges.
- The Heterotic strings only have right-moving fermions, with $\mathcal{N} = 1$ supersymmetry, or 16 supercharges.

2.4 Supersymmetry and Supergravity

The Pioncaré symmetry consists of two types of generators, the Lorentz group generators J_{ab} which generate rotations and boost, and the translational symmetry generators P_a. We have also encountered internal symmetries such as the local gauge symmetries: $U(1)$ for Q.E.D., $SU(3)$ for QCD etc. The generators, T_i, for the internal symmetries satisfy a Lie algebra

$$[T_i, T_j] = f_{ij}{}^k T_k.$$

It was not too long after the discovery of Yang Mills theories that physicists pondered the existence of a larger symmetry incorporating both internal symmetries and the Pioncaré symmetry. In particular the scientists were looking for a larger encompassing Lie algebra. The answer came in the form of the Coleman-Mandula theorem, which says that if the Pioncaré and internal symmetries were to satisfy a combined Lie Algebra, the S matrices for all process would be zero, i.e. observables would be trivial. However the theorem can be circumvented if the overall algebra is a *graded Lie algebra*. A graded Lie algebra is an algebra that has some generators that satisfy an anti-commuting law

$$\{Q^i_\alpha, Q^j_\beta\} = \textit{ other generators}.$$

The graded algebra has the following general structure: call the operators that commute with each other "even" generators and those that anti-commute with each other "odd" generators, then

$$[even, even] = \ even; \quad \{odd, odd\} = \ even; \quad [even, odd] = \ odd. \tag{2.4.1}$$

The anti-commuting nature of Q^i_α allows us to choose the spinor representation for the odd generators. And since the result of the product of a spinor and bosonic field is a spinor field, we must have that when Q_α acts on a bosonic field the result should be a spinor field. Therefore Q_α gives a symmetry between bosons and fermions. We call this symmetry *supersymmetry* and the algebra, of the type in eq(2.4.1), is called the *supersymmetry algebra*. Any Z_2 graded Lie algebra is called a *superalgebra*. For a super algebra to close, the supersymmetric theories must have an equal number of bosonic and fermionic degrees of freedom. This also extends to requirements such as equal masses. A favourable consequence of this larger symmetry is that it rids QFT of some divergences [26]. Supersymmetry was actually first discovered in string theory. Herein we will also briefly discuss the basic ideas behind supergravity, which is a local supersymmetric theory in which gravity manifests. We will follow the presentations of [21] and [26]. For illustration purposes we will work through the Wess-Zumino supersymmetry and supergravity models. However before we get into the models let us briefly revise spinors using [27].

Spinors

The algebra of the Lorentz group $SO(3,1)$ is

$$[J_l, J_m] = i\epsilon_{lmn}J_n$$

$$[J_p, K_q] = i\epsilon_{pqr}K_r$$

$$[K_a, K_b] = -i\epsilon_{abc}J_c.$$

Now define the following objects $J_{\pm i} \equiv (J_i \pm iK_i)$. these operators satisfy the algebra

$$[J_{+i}, J_{+j}] = i\epsilon_{ijk}J_{+k}$$

$$[J_{-i}, J_{-j}] = i\epsilon_{ijk}J_{-k}$$

$$[J_{+i}, J_{-j}] = 0.$$

This tells us that J_+ and J_- form two separate $SU(2)$ algebras. We say that $SO(3,1)$ is isomorphic to $SU(2) \times SU(2)$. The representations of $SU(2)$ are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, therefore in order to label the representations of $SO(3,1)$ we need two labels (j^+, j^-), one for each $SU(2)$ subgroup. The four dimensional representation $(\frac{1}{2}, \frac{1}{2})$ is a Lorentz vector representation, i.e. the objects that transform under this particular $SU(2) \otimes SU(2)$ representation are Lorentz vectors. The 4-dimensional representation $(\frac{1}{2}, \frac{1}{2})$ is reducible in terms of the two dimensional representations $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The two component "vectors", called *Weyl spinors*, transforming in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation are denoted by ψ_α and $\bar{\chi}^{\dot{\alpha}}$ respectively. The Dirac spinor consists of two Weyl spinors

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

A Majorana spinor is a Dirac spinor that satisfies the condition $\Psi = \Psi^C$ (The C represents charge conjugation), that is it is made up of one Weyl spinor

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}.$$

Supersymmetry: The Wess-Zumino model

Herein we work with the simplest possible supersymmetry model [21], so that we can have a feel of how supersymmetry works.

The 2-dimensional model

The action of a free boson and a free fermion in two dimensional Minkowski space is

$$S = \frac{1}{2} \int d^2x [(\partial_\mu \phi)^2 + \bar{\psi} \partial\!\!\!/ \psi].$$

Because the action is dimensionless in natural units, we have that $[\phi] = 0$ and $[\psi] = \frac{1}{2}$. The infinitesimal supersymmetry transformation, with parameter ϵ, between the boson and the fermion is given by: first on the boson,

$$\delta\phi = \bar{\epsilon}\psi. \tag{2.4.2}$$

Dimensional analysis tells us that $[\epsilon] = -\frac{1}{2}$. The fermion case

$$\delta\psi = \partial\!\!\!/\phi\epsilon, \tag{2.4.3}$$

again we can use dimensional reasoning to understand the presence of $\partial\!\!\!/$. We can check that the above transformation leaves the action invariant. We will need the following Majorana spinor identities

$$1)\ \bar{\epsilon}\chi = \bar{\chi}\epsilon \qquad 2)\ \bar{\epsilon}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\epsilon$$

The variation of the action due to a supersymmetry transformation is

$$\begin{aligned}\delta S &= \int d^2x \left[-\phi\partial^2\delta\phi + \frac{1}{2}\delta\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\bar{\psi}\partial\!\!\!/\delta\psi\right] \\ &= \int d^2x \left[-\phi\partial^2\delta\phi + \bar{\psi}\partial\!\!\!/\delta\psi\right] \\ &= \int d^2x \left[-\phi\partial^2\bar{\epsilon}\psi + \bar{\psi}\partial\!\!\!/^2\phi\epsilon\right] \\ &= 0.\end{aligned}$$

In the first line we made use of integration by parts. In the second line we made use of integration by parts and identity (2). To get the last equality we integrate by parts twice, use identity (1) and the identity $\partial\!\!\!/^2 = \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu = \partial_\mu\partial_\nu\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = \partial_\mu\partial_\nu g^{\mu\nu}$. Indeed we have found a global symmetry. The next thing we have to verify is the Anti–commutator law. The generator of the supersymmetry Q_α which is also known as the supercharge, is a Grassman object and hence obeys an anti-commutation relation of the form

$$\{Q^i_\alpha, Q^j_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ij}.$$

In our simple 2-dimensional model, since $\phi \to \delta\phi = \bar{\epsilon}Q\phi$ is a bosonic operation, the anti commutation relation of the super charges is obtained via the commutation relation

$$[\epsilon_1 Q_1, \epsilon_2 Q_2]\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\begin{pmatrix}\phi\\ \psi\end{pmatrix},$$

(the relation holds on condition that the symmetry is on shell i.e. $\not{\partial}\psi = 0$ and $\partial^2\phi = 0$). Notice that

$$\begin{aligned}[\epsilon_1 Q_1, \epsilon_2 Q_2] &= \epsilon_{1\alpha}Q_1^\alpha\epsilon_{2\beta}Q_2^\beta - \epsilon_{2\beta}Q_2^\beta\epsilon_{1\alpha}Q_1^\alpha\\ &= -\epsilon_{1\alpha}\epsilon_{2\beta}Q_1^\alpha Q_2^\beta - \epsilon_{1\alpha}\epsilon_{2\beta}Q_2^\beta Q_1^\alpha\\ &= -\epsilon_{1\alpha}\epsilon_{2\beta}\left\{Q_1^\alpha, Q_2^\beta\right\}\end{aligned}$$

Supersymmetric models

The 2 dimensional Wess-Zumino model is an example of $\mathcal{N} = 1$ supersymmetry since there was only one supercharge Q^i_α. Therefore $\mathcal{N}$ represents the number of different types of super charges in the theory i.e $i = 1, .., \mathcal{N}$. Herein we will list some spin $\leqslant 1$ supersymmetric multiplets in 3+1 dimensions as described in [21].

- The $\mathcal{N} = 1$ chiral multiplet with (ϕ, ψ)
- The $\mathcal{N} = 2$ vector multiplet as made of one $\mathcal{N} = 1$ vector multiplet (A_μ, λ) and one $\mathcal{N} = 1$ chiral multiplet (ψ, ϕ).
- The $\mathcal{N} = 2$ hypermultiplet, is made of two $\mathcal{N} = 1$ chiral multiplets (ψ_1, ϕ_1) and (ψ_2, ϕ_2).
- The $\mathcal{N} = 1$ vector multiplet (ψ^a, A^a), the action is

$$S_{\mathcal{N}=1\ SYM} = -2\int d^4x tr\left[-\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}\bar{\psi}\not{D}\psi\right]$$

where $D_\mu = \partial_\mu + igA_\mu$ and $\psi = \psi^a T_a$ is an adjoint fermion ($trT_aT_b = \frac{-1}{2}\delta_{ab}$). The transformation rules are

$$\begin{aligned}
\delta A_\mu^a &= \bar{\epsilon}\gamma_\mu\psi^a \\
\delta\psi^a &= (\frac{-1}{2}\gamma^{\mu\nu}F_{\mu\nu}^a + i\gamma_5 D^a)\epsilon \\
\delta D^a &= i\bar{\epsilon}\gamma_5\partial\!\!\!/\psi^a
\end{aligned}$$

- The $\mathcal{N}=4$ Super Yang Mills[6] theory contains a $\mathcal{N}=1$ vector multiplet (A_μ, ψ_4) and 3 $\mathcal{N}=1$ hyper multiplets (ϕ_i, ψ_i), $i = 1,2,3$. They can be arranged into $(A_\mu^a, \psi^{ai}, \phi_{[ij]})$, where $i = 1,\ldots,4$ is a $SU(4)$ index and $[i,j]$ is the 6 dimensional antisymmetric representation of $SU(4)$. The action is

$$\begin{aligned}
S_{\mathcal{N}=4\ SYM} &= -2\int d^4x\ tr\left[-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\phi}_i D\!\!\!\!/\psi^i - \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij}\right. \\
&\left. +ig\bar{\psi}^i[\phi_{ij},\psi^j] - g^2\left[\phi_{ij},\phi_{kl}\right]\left[\phi^{ij},\phi^{kl}\right]\right]
\end{aligned}$$

where the $D_\mu = \partial_\mu + g\,[A_\mu,\ ]$ [21]. The supersymmetry rules are

$$\begin{aligned}
\delta A_\mu^a &= \bar{\epsilon}_i\gamma_\mu\psi^{ai} \\
\delta\phi_a^{[ij]} &= \frac{i}{2}\bar{\epsilon}^{[i}\psi_a^{j]} \\
\delta\psi^{ai} &= -\frac{\gamma^{\mu\nu}}{2}F_{\mu\nu}^a\epsilon^i + 2i\gamma^\mu D_\mu\phi^{a,[ij]}\epsilon_j - 2g\ f^a{}_{bc}\,(\phi^b\phi^c)^{[ij]}\epsilon^j
\end{aligned}$$

where $(\phi^a\phi^b)^i{}_j \equiv \phi^{a,i}{}_k\ \phi^{b,k}{}_j$ [21].

Vielbeins and spin connections

Most of our discussions of physics, thus far, have been done in a flat spacetime background. We chose to work in a flat spacetime for convenience as our discussions

[6] $\mathcal{N}=4$ Super Yang Mills can also be formulated as an $\mathcal{N}=2$ vector multiplet and $\mathcal{N}=2$ hypermultiplet

are merely for illustrative purposes. However, some subtleties can creep in due to our gross simplifications. For one, objects such as spinors are naturally defined on flat backgrounds. We can then ask ourselves how do we deal with such objects when we try to generalize to curved geometries, such the ones studied in general relativity? We will use [28] to answer this question. Consider a general coordinate transformation $x \longrightarrow x'$, a vector V^μ will transform as follows

$$V^\mu \to V'^\mu = Z^\mu_\nu V^\nu,$$

where $Z^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ is an invertible matrix and is in the group of invertible, real, $n \times n$ matrices $GL(n, R)$. General relativity requires that matter fields that couple with gravity form representations of $GL(n, R)$ [28]. But spinors form a projective representation of $SO(n)$ that is not part of any representation of $GL(n, R)$ [28]. Therefore in order to consistently define spinors in a curved space time we need to find transformations Z that are elements of $SO(1, n-1) \subset GL(n, R)$. Any smooth(analytic) curved space is locally flat [21]. That means on these special types of spacetime geometries we can always find a small neighbourhood around any arbitrary point that is flat (tangent space). The orthonormal basis on this locally flat region is called the *vielbein*[7], $e^a_\mu(x)$, (German: "viel"=many and "bein"=legs). The index μ is a "curved"(global) index that transforms by a general coordinate transformation and a is a new "flat" (local) index that transforms by a local Lorentz gauge. We therefore have

$$g_{\mu\nu}(x) = e^a_\mu e^b_\nu \eta_{ab},$$

where $g_{\mu\nu}$ is the curved spacetime metric and η_{ab} is the Minkowski metric. We can also rewrite any vector field V^μ in the vielbein basis $V^\mu = e^\mu_a V^a$. A local Lorentz transformation implies

$$e^a_\mu \to \tilde{e}^a_\mu(x) = \lambda^a{}_b(x) e^b_\mu(x).$$

[7] Vielbein are also known as tetrad.

To ensure that the physics is independent of the choice of vielbein we introduce a gauge field, called the *spin connection*, $\omega_{\mu}{}^{a}_{b}$ of Lorentz group $SO(1, n-1)$. The spin connection satisfies a gauge like transformation $\omega_\mu \rightarrow \lambda\omega_\mu\lambda^{-1} - \partial_\mu\lambda \cdot \lambda^{-1}$. The covariant derivative on V^a will be

$$D_\mu V^a = \partial_\mu V^a + \omega_{\mu}{}^{a}_{b} V^b.$$

However V^μ and V^a are the same vector and hence for consistency we must have that

$$D_\mu V^\nu = e^\nu_a D_\mu V^a$$
$$\Rightarrow D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\sigma{}_{\mu\nu}\, e^a_\sigma + \omega_{\mu}{}^{a}_{b}\, e^b_\nu = 0. \tag{2.4.4}$$

The Christoffel symbol and the spin connections can both be defined by eqn(2.4.4), the so called "vielbein postulate". We can define the analog of the Riemann curvature tensor or a gauge field strength

$$(R_{\mu\nu})^a_b(\omega) = \partial_\mu\, \omega_{\nu}{}^{a}_{b} - \partial_\nu\, \omega_{\mu}{}^{a}_{b} + [\omega_\mu, \omega_\nu]^a_b.$$

We have now sufficient technology to describe the coupling of spinors to gravity. The role of our transformation matrix Z is now played by the local Lorentz transformation λ^a_b and the covariant derivative on a spinor $\psi(x)$ is defined by

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\,\omega_\mu{}^{ab}\,\Sigma_{ab}\psi$$

where Σ_{ab} is the generator of the local Lorentz transformation.

Supergravity: The Wess-Zumino model

Einstein's general relativity models gravity as an effect of a curved spacetime, and since this curvature is brought about by matter, it follows that we would be interested in understanding how to write down QFT's in curved backgrounds. Up to now we have only considered field theories in Minkowski space, for example the scalar theory

$$S = \int d^4x \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{2.4.5}$$

In a general spacetime the action would be

$$S = \int d^4x \sqrt{-\det g_{\mu\nu}} \left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{2.4.6}$$

in this form the action is invariant under a general coordinate transformation:

$$x^\mu \to x'^\mu(x) = \Lambda^\mu_\nu(x)x^\nu + a^\mu(x)$$

$$\phi \to \phi'(x') = \phi(x).$$

We have written the general coordinate transformation as a local Pioncaré invariance. From a particle physicist point of view it seems in order to make the Minkowski scalar action invariant under a local Poincaré invariance one had to add a gauge field $g_{\mu\nu}$. The particle associated with this field is called a *graviton*, a massless spin 2 particle. The graviton action is given by the Einstein-Hilbert action

$$S_g = -\frac{1}{16\pi G_N}\int d^4x\sqrt{-\det g_{\mu\nu}}R.$$

Therefore a complete theory of matter that includes gravity should be described by the composite action $S = S_g + S_M$, where S_M could be an action like eqn(2.4.6). However GR is notoriously known for being non linear. We can get around this by expanding the metric about the Minkowski metric

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + k\ h_{\mu\nu}$$

where $k = \sqrt{8\pi G_N} = \frac{\sqrt{8\pi}}{M_{Planck}} \equiv \frac{1}{M_p}$ is a constant put in for dimensional consistency. Consider perturbation of the free scalar action

$$\begin{aligned}\textstyle\int d^4x\sqrt{-\det g_{\mu\nu}}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi &= \int d^4x\frac{1}{2}\eta^{\nu\mu}\partial_\mu\phi\partial_\nu\phi \\ &-k\int d^4x\frac{1}{2}h^{\mu\nu}\overline{\partial_\mu\phi\partial_\nu\phi} - k^2\ldots.\end{aligned}$$

Dimensional analysis gives us $[k] = \left[M_P^{-1}\right] = -1$ i.e. negative mass dimension [26]. Now from power counting in QFT we know that any interaction in the action with a negative mass dimension is non renormalizable. Even the graviton action is plagued by couplings of the form k^{2n}, which implies that gravity is non renormalizable.

We have seen how gravity is the gauge theory of global Poincaré transformation. Next we will extend the analogy to show that supergravity is the gauge theory of the global supersymmetry transformation. Consider again the simple Wess-Zumino model, a scalar field ϕ with its super partner the spin 1/2 fermion ψ. The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 + \frac{1}{2}\bar{\psi}\not{\partial}\psi$$

is invariant under the global supersymmetry transformation (see eqn(2.4.2,2.4.3)). Now to make the transformation local requires $\epsilon = \epsilon(x)$ such that $\delta\phi = \bar{\epsilon}(x)\psi$ and $\delta\psi = -i(\not{\partial}\phi)\epsilon(x)$, the Lagrangian variation is [26]

$$\delta\mathcal{L} = \partial_\mu \epsilon^\alpha K^\mu_\alpha + h.c.$$

with

$$K^\alpha_\mu \equiv -(\partial_\mu\phi)\psi^\alpha - \frac{i}{2}\gamma_\mu\psi^\alpha\not{\partial}\phi.$$

Making use of the "Noether method" we will add terms to the Lagrangian and continue to do so until the action is locally invariant. The first term we add is the Noether coupling

$$\mathcal{L}_N = k\ K^\alpha_\mu \Psi^\mu_\alpha$$

where k is a constant introduced for dimensional consistency and Ψ is a Majorana vector spinor field with spin 3/2, called the gravitino. The Lagrangian is still not invariant,

$$\delta(\mathcal{L} + \mathcal{L}_N) = \frac{1}{2}k\bar{\Psi}_\mu\gamma_\nu\epsilon T^{\mu\nu}$$

given that the gravitino transforms as follows:

$$\delta \Psi_\mu^\alpha = \frac{1}{k} D_\mu \epsilon, \qquad \text{recall that} \;\; D_\mu \epsilon^\alpha = \partial_\mu \epsilon^\alpha + \frac{1}{2}\, \omega_\mu{}^{mn}\, \Sigma_{mn} \epsilon^\alpha .$$

So we add a new term

$$\mathcal{L}_g = -\frac{1}{2} g_{\mu\nu} T^{\mu\nu}$$

and require that under local supersymmetry [29]

$$\delta g_{\mu\nu} = -\frac{k}{2} \Psi_\mu \gamma_\mu \epsilon - \frac{k}{2} \bar{\Psi}_\mu \gamma_\mu \epsilon$$

or more formally, in terms of vielbeins [29]

$$\delta e_\mu^a = \frac{k}{2} \bar{\epsilon} \gamma_\mu \psi^a .$$

The iterative process terminates and $\mathcal{L} + \mathcal{L}_N + \mathcal{L}_g$ is invariant under a local supersymmetry transformation.

Supergravity models

The Wess-Zumino supergravity model we discussed in the previous subsection is an $\mathcal{N} = 1$ supergravity model with two multiplets (e_μ^a, Ψ_μ^a) and (ϕ, ψ). Again $\mathcal{N}$ is the number of super generators, it necessarily has to be the number of gravitini too since the gravitini appear to play the role of a gauge field to the supergravity generators. In what follows we will denote the multiplets by their spin bracket, i.e. chiral multiplets (ϕ, ψ) of spin $(0, 1/2)$, gauge multiplets (A_μ, ψ) of spin $(1, 1/2)$, gravitino multiplets (Ψ_μ^a, A_μ) of spin $(3/2, 1)$ and the supergravity multiplet (e_μ^a, Ψ_μ^a) of spin $(2, 3/2)$. Here are some examples of supergravity models [21]

- $\mathcal{N} = 3$ supergravity: supergravity multiplets $(2, 3/2)$, two gravitino multiplets $(3/2, 1)$ and one vector multiplets $(1, 1/2)$. The fields are $\{e_\mu^a, \Psi_\mu^i, A_\mu^i, \lambda\}$, where $i = 1, 2, 3$.

- $\mathcal{N}=4$ supergravity: supergravity multiplets $(2,3/2)$, three gravitino multiplets $(3/2,1)$, three vector multiplets $(1,1/2)$ and one chiral multiplets $(1/2,0)$. The fields are $\{e^a_\mu, \Psi^i_\mu, A^k_\mu, B^k_\mu, \lambda^i, \phi, B\}$ where $i=1,2,3,4$; $k=1,2,3$. A is a vector, B is an axial vector, ϕ is a scalar and B is a pseudoscalar.

2.5 Conformal field theories

In the Wilsonian approach to QFT [30], we learn that physics should be organized according to scale. In performing the renormalization group (RG) flow we see that coupling constants are in fact dynamic and vary according to scale, this realization led to the important Callan-Symanzik equations which allow us to track how couplings flow. The running of the coupling constant with scale is captured in the β function,

$$\beta(\lambda, \epsilon) = \mu \frac{d\lambda}{d\mu}|_{m_0, \lambda_0, \epsilon}$$

where μ is the renormalization scale, λ is the renormalized coupling, ϵ is the cutoff and λ_0 the bare coupling. It took some time to fully appreciate that most field theories exhibit a RG flow from some scale invariant (often free) UV RG fixed point to some scale invariant IR RG fixed point [2]. Conformal field theories were born out of trying to understand physics at these illusive limits, where the β function vanishes.
Our discussion on conformal field theories closely follows [2].

The conformal group and algebra

The conformal group is the group of transformations which preserve the form of the metric up to an arbitrary scale factor $g_{\mu\nu}(x) \longrightarrow \Omega^2(x) g_{\mu\nu}(x)$ $(\mu, \nu = 0, 1, ..., d-1)$. The conformal group of Minkowski space ($\mathbb{R}^{1,d-1}$,$d > 2$)[8] is generated by the Poincaré transformations, the scale transformations

$$x^\mu \longrightarrow \lambda x^\mu$$

and the special conformal transformations

$$x^\mu \longrightarrow \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}.$$

[8] $d = 2$ is a special case, in that the conformal group has an infinite set of generators.

Denote the generators of these transformations by $M_{\mu\nu}$, P_μ, D, K_μ corresponding respectively to Lorentz transformation, translation, scaling transformation and special conformal transformation. The vacuum of a conformal theory is annihilated by all of these generators. They obey the conformal algebra

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu); \qquad & [M_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\rho\nu}M_{\mu\sigma} \pm \ \textit{permutations} & [M_{\mu\nu}, D] &= 0; \quad [D, K_\mu] = iK_\mu; \\
[D, P_\mu] &= -iP_\mu; & [P_\mu, K_\nu] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D. \qquad (2.5.1)
\end{aligned}$$

We can arrange the generators of flat space conformal symmetry to form a group defined by an antisymmetric $(d+2) \times (d+2)$ matrix:

$$\bar{J}_{MN} = \begin{pmatrix} J_{\mu\nu} & \bar{J}_{\mu,d+1} & \bar{J}_{\mu,d+2} \\ -\bar{J}_{\nu,d+1} & 0 & \bar{J}_{d+1,d+2} \\ -\bar{J}_{\nu,d+2} & -\bar{J}_{d+1,d+2} & 0 \end{pmatrix}$$

where we have defined

$$J_{\mu\nu} = M_{\mu\nu}, \quad \bar{J}_{\mu,d+1} = \frac{1}{2}(K_\mu - P_\mu), \quad \bar{J}_{\mu,d+1} = \frac{1}{2}(K_\mu + P_\mu), \quad \bar{J}_{d+1,d+2} = D.$$

In this form the isomorphism between conformal symmetry and the symmetry group $SO(2,d)$ is made clearer as the Lie algebra of $\bar{J}_{MN}$ shows that the metric in the $d+2$ direction is negative [21]. This is the first clue that alludes to the AdS/CFT correspondence since the symmetry group $SO(2,d)$ is the symmetry of $d+1$-dimensional AdS space. It is sometime useful to study conformal field theory in Euclidean space since $\mathbb{R}^d$ is conformally equivalent to S^d, so a field theory on $\mathbb{R}^d$(with appropriate boundary conditions) is isomorphic to a field theory on S^d[2].

Primary fields and Correlation functions

There is an interesting representation of the conformal group that involves operators (or fields) which are eigenfunctions of the scaling operator D, also known as the dilatation operator. Under a scaling transformation

$$\phi(x) \longrightarrow \phi'(x) = \lambda^{\Delta}\phi(\lambda x),$$

if $\phi(x)$ is an eigenfunction of D then

$$D\phi(x) = -i\Delta\phi(x),$$

where Δ is the scaling(conformal) dimension of the field. The conformal algebra, in eqn(2.5.1) implies that the operators P_μ raise the dimension of a field while the operator K_μ lowers it, akin to the creation/annihilation operators $a^\dagger/a$.

$$[D, P_\mu] = -iP_\mu \Rightarrow D(P_\mu\phi) = P_\mu(D\phi) - iP_\mu\phi = -i(\Delta+1)(P_\mu\phi)$$

Operators of lowest dimension Φ_0, those that are annihilated by K_μ, i.e. $K_\mu\Phi_0 = 0$, are called *primary operators* as they form a basis from which all other operators may be constructed

It has been shown by Luscher and Mack [31] that Euclidean Greens functions of CFT may be analytically continued to Minkowski space and that the resulting Hilbert space carries a unitary representation of the Lorentzian group [2]. A consequence of their result is that correlation functions in Euclidean and Minkowski space are equivalent (up to some normalization factors). It can be shown that the correlation functions of fields of different dimensions vanish while for a single scalar field of dimension Δ we get:

$$\langle\phi(x)\phi(y)\rangle = \frac{M}{|x-y|^{2\Delta}},$$

where M is a normalization constant. The 3-point function has the form

$$\langle\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = \frac{Mc_{ijk}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_3+\Delta_2-\Delta_1}}$$

again M is a normalization constant. The 2-point and 3-point functions serve to illustrate the importance of the conformal dimensions, in that they largely determine the correlation functions.

Superconformal algebras and field theories

The next natural point of inquiry would be to ask if we could marry supersymmetry with conformal symmetry to get an even larger symmetry that includes the Poincaré group. Indeed this is possible, but only for $d \leqslant 6$, the result was illustrated by Nahm[2]. In order to construct the superconformal group and algebra, one needs to introduce new type of generators: the fermionic generators S (one for each supersymmetry generator) and R-symmetry generators. Due to the complexity introduced by supersymmetry, in that in different dimensions the spinor representations of conformal group behave differently and for different R-symmetry groups, one cannot write the general structure for the exact commutation relations.

Many interacting theories are classically scale invariant however this scale invariance does not always extend to the quantum theory like Yang Mills theories (Wilsonian cutoff breaks amongst other symmetries scale invariance). In four dimensions $\mathcal{N} = 4$ Super Yang Mills theory (SYM) is a special case of a superconformal theory at both the classical and quantum level. $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group has the following fields: $\{A^a_\mu, \psi^{a\ i}_\alpha, \phi^a_{[ij]}\}$ spin 1 vector field, fermion field and scalar fields respectively. Here the R-symmetry is $SU(4)$ and the fermoinic generators are in $(4,4) + (\bar{4},\bar{4})$ of $SO(4,2) \times SU(4)$ [2].

2.6 AdS/CFT correspondence

Herein we will bring together all the preceding sections in this chapter to outline the ideas behind the Maldecena conjecture and some motivations that make it plausible. We leverage heavily on presentations of [21] and [2].

The AdS/CFT correspondence suggests that a d+1 dimensional theory of gravity is equivalent to a conformal field theory (CFT) in d-dimensions. Kinematics, dynamics and all! The first step in trying to explain the correspondence is in showing that D-branes are p-branes.

D-branes are p-branes

The fact that the dynamical hyper-planes on which strings end, D-branes, are the extremal solutions of supergravity, p-branes, was shown by Polchinski in [24]. In proving this equivalence, it is necessary to argue that D-branes are charged objects that curve spacetime, akin to p-branes. It is understood that N $D3$-branes correspond to the supergravity solution [21]

$$\begin{aligned} ds^2 &= H^{-1/2}(r)(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{1/2}(r)(dr^2 + r^2 d\Omega_5^2) \\ F_5 &= (1+\star)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge (dH^{-1}) \\ H(r) &= 1 + \tfrac{R^4}{r^4}; \quad R = 4\pi g_s N\, {\alpha'}^2; \quad Q = g_s N. \end{aligned} \tag{2.6.1}$$

We argued in subsection 2.3 that N coincident $D3$-branes will contain a $\mathcal{N} = 4$ Super Yang Mills with gauge group $U(N)$. Indeed if D-branes are p-branes we should be able to see this maximally symmetric CFT manifest in the supergravity solution.

D-branes Recall, open strings can only end on dynamical walls we call D-branes and we keep track of the end points of the string with Chan-Paton factors. We represent a string state by $\lambda^a_{ij}|i\rangle \otimes |j\rangle$, with λ^a_{ij} the generator of the adjoint representation of $U(N)$, for the case of N $D3$-branes. We showed that the first excited states (ground state for the Supestring case) of an open string are massless gauge fields A^a_μ that live on the brane. In the $\alpha' \to 0$ limit, only the massless states remain and it can be shown that the $U(1)^{\otimes N}$ is enhanced to a SYM theory with $U(N)$ gauge group. The number of supercharges, $\mathcal{N} = 4$, can be explained as follows. String theory has 32 supercharges, which form (11-dimensional spinors or) 8 4-dimensional spinors, this is $\mathcal{N} = 8$ supersymmetry in 4 dimension. In a similar manner that the $SO(1, d)$ symmetry of closed strings is broken when we move to open strings, the presence of D-branes reduces the supersymmetry to $\mathcal{N} = 4$.

***p*-branes** The low-energy limit, $\alpha' \to 0$, of superstrings is supergravity, of which p-branes are solutions. We learned previously that p-branes have $Q = M$, and are constrained $|Q| \leqslant M$. This constraint is natural both in gravity and supersymmetry. In gravity, it is consistent with "no naked singularity"(black holes without horizon) theorems. In supersymmetry the bound results from the super algebra and in this bound the maximum amount of symmetry, 1/2 the supersymmetry , is preserved. Therefore the $\mathcal{N} = 8$ supersymmetry in 4 dimension of superstrings is constrained to $\mathcal{N} = 4$ supersymmetry in 4 dimension.

Thus both p-branes and D-branes admit a $\mathcal{N} = 4$ supersymmetry in 4 dimensions and both are solutions of supergravity.

Motivation

The dual nature of D-branes[9] enables us to have two different descriptions of the same physical systems. It is through the comparison of these two pictures that AdS/CFT

[9] String end point and solutions to supergravity.

correspondence was conjectured.

Point of view no. 1

Consider type IIB string theory in a 10 dimensional Minkowski space. Suppose there are N coincident $D3$-branes in this background. There are three dynamical objects in this theory, namely: the closed strings in the bulk, the open strings attached to the branes and the branes themselves. The open string excitations also include excited modes of the brane therefore we effectively have two perturbative excitations at play. Note: *In what follows low-energy limit refers to energies of the scale* $E << 1/\sqrt{\alpha'}$*, where* $\alpha' = \frac{1}{2\pi T} = l_s^2$.

- The open strings live on the $D3$-brane. In the low-energy limit only the massless states are attainable and they are the gauge fields that reduce to $\mathcal{N} = 4$ super-Yang Mills with gauge group $U(N)$.
- The closed strings live in the bulk of spacetime. They describe a supergravity theory coupled to the massive modes of the strings. In the low-energy limit the massive modes decouple and only the supergravity theory is left.

The open and closed strings can interact. Thus the effective action of the massless modes is

$$S = S_{bulk} + S_{brane} + S_{interaction}.$$

This is an effective Lagrangian as it only involves massless fields, the massive fields have been integrated out in the Wilsonian sense. S_{int} describes the interactions between the bulk modes and the brane modes. In the low-energy limit, the massive string modes drop out: the closed strings in the bulk reduce to supergravity and the gauge fields on the branes will reduce to $\mathcal{N} = 4$ SYM. The interaction strength is a function of the string parameter α'

$$S_{int} \propto \sqrt{G_N} \sim g_s(\alpha')^2$$

thus in the low-energy limit, with the string coupling fixed and with $\alpha' \to 0$, we see that the interaction coupling $\sqrt{G_N}$ is switched off and the bulk and open string modes do not interact with each other. Therefore, in the low-energy limit, we get two decoupled systems: gravity in the bulk and 4 dimensional $\mathcal{N} = 4$ supersymmetric gauge theory on $D3$-branes.

Point of view no. 2

Now lets treat the N $D3$-branes as supergravity solutions, p-branes. Near the p brane, spacetime is curved (asymptotically Ad$S_5 \times S^5$ space) and far away from the brane the spacetime is asymptotically flat. We see this by taking $r \to 0$ in eqn(2.6.1), the harmonic function becomes $H = 1 + \frac{R^4}{r^4} \sim \frac{R^4}{r^4}$ and the supergravity background reduces to

$$ds^2 \approx \frac{r^2}{R^2}(-dt^2 + d\bar{x}_3^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2,$$

perform the change of coordinates $\frac{r}{R} \equiv \frac{R}{x_0} \Rightarrow dr = \frac{-R^2}{(x_0)^2}dx_0$. Then

$$ds^2 \approx \frac{R^2}{x_0^2}\left(-dt^2 + d\bar{x}_3^2 + dx_0^2\right) + R^2 d\Omega_5^2$$

the metric of Ad$S_5 \times S^5$, the AdS_5 is written in Poincaré coordinates.

The energy E_r measured at a point r and the energy E measured at infinity are related by

$$E_r \sim \frac{d}{d\tau} = \frac{1}{\sqrt{-g_{00}}}\frac{d}{dt} \sim \frac{1}{\sqrt{-g_{00}}}E \Rightarrow E = H^{-1/4}E_r.$$

For fixed E_r and very small r, $H = 1 + \frac{R^4}{r^4} \sim \frac{R^4}{r^4}$ and hence the energy is $E \sim rE_r$, so as $r \to 0$ the energy observed at infinity is redshifted, E tends to zero. In this description, the low-energy system consists of :

- At large distances, far away from the p-branes,we get gravity in flat space. This is supported by the fact that the effective dimensionless coupling $G_N E \to 0$ in the low-energy regime. We consider low-energy excitations of closed strings in this sector.

- Near the horizon (within the vicinity of the warped spacetime due to the p-branes) closed strings in this sector cannot escape the p-brane potential. The finite excitations in this sector are redshifted and are seen as low-energy excitations by observers at infinity.

The two low-energy systems are decoupled because the lo- energy absorption cross section of waves near the horizon and in the bulk is proportional to the energy, $\sigma \sim \omega^3 R^8$ [32; 4] where ω is the energy. Once again we have two decoupled lo- energy systems: gravity in the bulk and type IIB closed strings in a Ad$S_5 \times S^5$ background.

The two descriptions in the appropriate low-energy limit, both have free gravity in the bulk. The natural implication is that *the 4 dimensional gauge theory that lives on the D3-branes, $\mathcal{N} = 4$ SYM with gauge group $U(N)$ is dual to the type IIB superstring theory on $AdS_5 \times S^5$ [5].* The fact that we worked with perturbative excitation means that we can only conjecture the equivalence between the two theories, this argument cannot be used as a proof [2].

Some comments: on the $\alpha' \to 0$ limit in the near horizon region. We would like to keep the energies of the objects in the near horizon region fixed, in string units, even as we take the limit $r \to 0$. This implies that $\sqrt{\alpha'} E_r$ must be fixed. Since $H \approx R^4/r^4 \propto \alpha'^2/r^4$, we have that the energy measured at infinity is

$$E \sim E_r \frac{r}{\sqrt{\alpha'}}.$$

We should also keep the energies measured at infinity fixed, to be consistent with field theory[2], as we take $r \to 0$. We can ensure this by concurrently taking $\alpha' \to 0$

, keeping r/α' fixed, i.e. $E \sim E_r\sqrt{\alpha'}\frac{r}{\alpha'}$. Defining a new variable $U \equiv \frac{r}{\alpha'}$, the supergravity solution, becomes

$$ds^2 = \alpha\left[\frac{U^2}{\sqrt{4\pi g_s N}}\left(-dt^2 + d\bar{x}_3^2\right) + \sqrt{4\pi g_s N}\left(\frac{dU^2}{U^2} + d\Omega_5^2\right)\right].$$

Symmetry discussion

The validity of the correspondence is supported by matching global symmetries in both theories. The $\mathcal{N} = 4$ SYM is a conformal theory and the conformal group in 3+1 dimensions has a symmetry of $SO(4,2)$. We can make sense of the low-energy theory on the D-branes possessing this symmetry by the fact that the near horizon geometry of the branes is Ad$S_5 \times S^5$ and AdS_5 has $SO(4,2)$ symmetry. In addition, the conformal theory is invariant under a $SU(4)_R$ R-symmetry which rotates the six scalar fields and the four fermions. Now $SU(4) \cong SO(6)$ global symmetry, where $SO(6)$ is the rotation symmetry of the five sphere, S^5.

Parameters and limits

Mapping the degrees of freedom between the gravitational theory and field theory is non-trival because of the different dimensions in which the two theories live. The starting point is with the couplings on either side.

$$g_{YM}^2 = g_s. \tag{2.6.2}$$

The coupling of the massless gauge fields living on the brane and described by the open string excitations is g_{YM}. The closed strings in type IIB have a coupling g_s. We can make sense of the relation in eqn(2.6.2) by considering some string Feynman diagrams. Two of the allowed string interactions are that a single string (open or

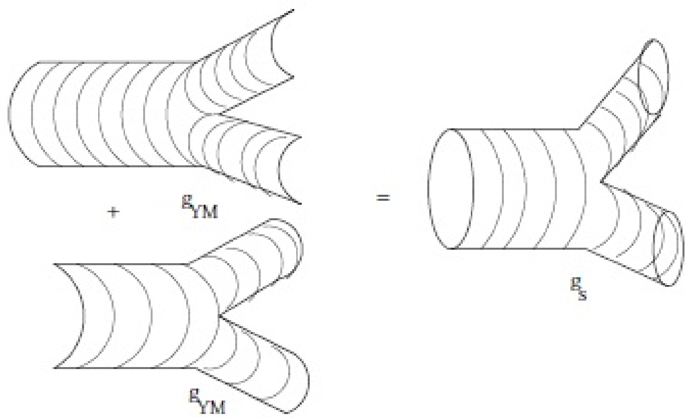


Figure 2.1: Two open string splitting interactions can be joined to make the most basic a closed string interaction,represented by the so call "pants diagram". Thus explaining the relation $g^2_{YM} = g_s$.[source of picture [21]]

closed) can split into two strings and that two open strings can join to make a closed string; therefore out of two open string splitting interactions one can make a closed string splitting interaction [21], see Fig 2.1. The radius of the Ad$S_5 \times S^5$ space is related to the gauge field coupling by

$$R^2 = \sqrt{g^2_{YM} N} \alpha'.$$

Most field theories cannot be solved exactly, hence we make computations perturbativley. In Yang-Mills theories this means that we expand in powers of the 't Hooft coupling $\lambda \equiv g^2_{YM} N$ in the so called 't Hooft limit $\lambda << 1$. On the gravity side, working in the 't Hooft limit means $\frac{R^4}{\alpha'} << 1$, high curvature is very difficult to work in. On the other hand classical gravity description becomes reliable when the radius of curvature R is large $\frac{R^4}{(\alpha')^2} >> 1$. In this limit λ is large, this is the strongly coupled regime of QFT which is very difficult to work in. It is for this reason that the correspondence is called the strong/weak duality. If true, as it so far promises to be, the correspondence will allow us to easily access the difficult ends of both theories by working in easier regimes. However at the same time this makes it very hard to prove.

It is hoped (in the scientific community) that the correspondence holds for all values of g_s and N, this is the strongest form of the correspondence. A weaker form would be, if the conjecture is valid for finite $g_s N$ but only in the $N \to \infty$ limit, so that the α' corrections would agree with the field theory but not the g_s [2].

The field theory is defined on $\mathbb{R}^{3,1}$ while the gravitation theory is defined on $\mathrm{Ad}S_5 \times S^5$. We can think of the field theory living on the boundary of $\mathrm{Ad}S_5$, which is $\mathbb{R} \times S^3$ and it is conformally equivalent to $\mathbb{R}^{3,1}$. This idea is in line with the *holographic principle* of quantum gravity. The holographic principle is motivated by consistency between the laws of black hole mechanics and the laws of thermodynamics. In a quantum gravity system the horizon area is the analog of the entropy for a thermodynamic system. And in the same fashion that the entropy completely determines a thermodynamic system, the complete dynamics of a quantum gravity system are encoded in the boundary of the system. Moreover the correspondence, in the strongest form, claims a complete equivalence between the field theory on the boundary and the quantum gravity in the bulk.

It is one of the objectives of this dissertation to perform a calculation that can be used to verify and better understand the dictionary of the correspondence.

Chapter 3

Giant Gravitons

Giant Gravitons are BPS particles that have expanded into a membrane. They have the peculiar characteristic, common in non-commutative field theories, that their angular momentum L is proportional to their size. When a graviton ("point particle") moves on the m-sphere of $\mathrm{Ad}S_n \times S^m$ with a m-form flux through it, the BPS states undergo an expansion into a $m-2$ dimensional spherical brane which is analogous to the theory of electric dipoles moving in a magnetic field. As the particle moves through the space with increasing angular momentum it grows in size until it reaches the radius of S^m, at which stage the growth can no longer continue and the tower of Kaluza-Klein states terminate[8].

In our review of giant gravitons we begin by discussing an electric dipole moving in a magnetic field, this sets the stage for section two where we consider giant gravitons in $\mathrm{Ad}S_5 \times S^5$. The discussion in the first two sections leverages primarily off [8]. In the third section we will briefly review the Schur polynomial giant graviton duality.

3.1 Dipoles in Magnetic Fields

Consider a pair of unit charges of opposite sign moving on a plane with a constant magnetic flux through the plane. The Lagrangian is

$$\mathcal{L} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{B}{2}\epsilon_{ij}(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{K}{2}(x_1 - x_2)^2 \tag{3.1.1}$$

Let us suppose that that the mass is so small that the mass term in eq(3.1.1) can be ignored. Introduce the centre of mass and relative coordinates

$$X = \frac{1}{2}(x_1 + x_2)$$
$$\Delta = \frac{1}{2}(x_1 - x_2).$$

The Lagrangian becomes

$$\mathcal{L} = 2B\epsilon_{ij}\dot{X}^i\Delta^j - 2K\Delta^2.$$

The commutator of X and Δ is

$$[X^i, \Delta^j] = i\frac{\epsilon^{ij}}{B}.$$

The center of mass momentum conjugate to X is

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{X}^i} = B\epsilon_{ij}\Delta^j,$$

which implies that

$$|\Delta| = |P|/B,$$

the size of the dipole is proportional to the momentum.

Now suppose the dipole is moving on a sphere of radius R. To get a uniform magnetic flux normal to the plane we will need a magnetic monopole at the centre of the sphere, with strength

$$2\pi N = \Omega_2 B R^2.$$

Using heuristic reasoning we can already see how some of the bounds arise: say the size of the dipole Δ is about R then

$$P \sim BR \Rightarrow L \sim PR \sim BR^2 \sim N,$$

the angular momentum is of the order of the total magnetic flux. Let us show this more concretely and make a stronger statement.

Parametrizing the sphere using spherical coordinates, we label a point on the sphere by (ϕ, θ). The angle ϕ measures the distance from the equator, it is $\pm\pi/2$ at the poles. The azimuthal angle θ goes from 0 to 2π. We pick a gauge in which the θ component of the vector potential is non-zero,

$$A_\theta = N \frac{1 - \sin\phi}{2R\cos\phi}.$$

For a unit charged point particle moving on the sphere the term coupling the velocity to the vector potential is

$$\mathcal{L}_A = A_\theta R \cos\phi\dot{\theta} = NR\frac{1 - \sin\phi}{2R}\dot{\theta}.$$

Orient the axis such that the dipole centre of mass moves along the equator, in particular the positive charge is at position (θ, ϕ) and the negative charge is at $(\theta, -\phi)$. For the motion we consider ϕ is time independent and hence

$$\mathcal{L}_A = -N\sin\phi\dot{\theta}.$$

The chordal size of the dipole is $\Delta = 2R\sin\phi$. Therefore the coupling is

$$\mathcal{L}_S = -\frac{k}{2}R^2 \sin^2\phi.$$

Again we drop the kinetic term, since we want a slow moving dipole with a very small mass relative to the magnetic field i.e. $mR << N$. The effective Lagragian is

$$\mathcal{L} = -\frac{k}{2}R^2\sin^2\phi - N\sin\phi\dot{\theta}$$

and the angular momentum is $L = -N \sin\phi$. The angular momentum will reach its maximum

$$|L_{max}| = N$$

when $\phi = \pi/2$ at which point the chordal size of the dipole is equal to $|\Delta| = 2R$, the diameter of the sphere.

3.2 Ad$S_5 \times S^5$

We extend the analogy of the dipole on a sphere with a magnetic flux: the dipole is replaced by a BPS particle, the background now is the S^5 part of Ad$S_5 \times S^5$ and the magnetic field is replaced by the 5-form field strength on the sphere.

Consider now an Ad$S_5 \times S^5$ background with radius

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s \tag{3.2.1}$$

where g_s is the string coupling and l_s string length. We work in the large N limit while $g_s N$ is fixed and large. Denote the 5-form flux by B, quantization of the flux requires

$$\Omega_5 B R^5 = 2\pi N$$

The graviton has zero net charge, however it couples to the background field strength in a similar fashion as the dipole in a magnetic field.

Let us work with the S^5 space by embedding it inside a 6 dimensional flat space, the coordinate transformation between Cartesian and spherical coordinates is

$$\begin{aligned}
X_1 &= R\cos\theta_1 \\
X_2 &= R\sin\theta_1\cos\theta_2 \\
X_3 &= R\sin\theta_1\sin\theta_2\cos\theta_3 \\
X_4 &= R\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\
X_5 &= R\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\cos\theta_5 \\
X_6 &= R\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5.
\end{aligned}$$

The angles $\theta_1, \ldots, \theta_4$ range from 0 to π. The angle θ_5 is azimuthal and it ranges from 0 to 2π. Then the 5-sphere is

$$(X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 + (X_5)^2 + (X_6)^2 = R^2.$$

We want to know what happens to a graviton[1] when it moves on a 5-sphere in the presence of the 5-form field strength. Let us now consider a $D3$brane wrapped on an S^3 in S^5, and parametrize it with $\theta_3, \theta_4, \theta_5$. The brane is allowed to move in the X_1, X_2 plane and its size is dependent on where it is on the plane according to

$$r = R\sin\theta_1\sin\theta_2.$$

The brane is at its maximum size, at $r = R$, this is when $\theta_1 = \theta_2 = \pi/2$ that is, the brane is at the origin $X_1 = X_2 = 0$. The trajectory of the brane is given by

$$(X_1)^2 + (X_2)^2 = R^2 - r^2.$$

In terms of polar coordinates on the plane, we have

$$X_1 = \sqrt{R^2 - r^2}\cos\phi, \qquad X_2 = \sqrt{R^2 - r^2}\sin\phi.$$

[1] In general, this analysis can be applied to any massless graviton moving on S^m in the presence of a m-from flux [8].

The S^5 is now parametrized by $r, \phi, \theta_3, \theta_4$ and θ_5 and the metric becomes

$$\begin{aligned} ds^2 &= dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2 + dX_6^2 \\ &= \frac{R^2}{R^2 - r^2} dr^2 + (R^2 - r^2) d\phi^2 + r^2 d\Omega_3^2 \end{aligned}$$

with $d\Omega_3^2$ the metric on the unit 3-sphere. The infinitesimal volume element is

$$d\omega = Rr^3 dr d\phi d\Omega_3.$$

Let us now built the Lagrangian. We are studying the dynamics of a large and fixed size brane (r is fixed and close to R) that moves in the X_1, X_2 plane of radius $\sqrt{R^2 - r^2}$ with angular velocity $\dot{\phi}$. The kinetic energy term is given by the Dirac-Born-Infeld Lagrangian

$$\begin{aligned} \mathcal{L}_{\mathcal{DBI}} &= -\frac{T_{D3}}{g_s} \sqrt{|\det G_{ab}|} \\ &= -T_{D3} \Omega_3 r^3 \sqrt{1 - (R^2 - r^2)\dot{\phi}^2} \end{aligned}$$

where G_{ab} is the induced metric on the $D3$-brane and

$$T_{D3} = \frac{1}{(2\pi)^3 l_s^4 g_s}$$

is the tension of the $D3$-brane. Next we add the Chern-Simons coupling, which describes the coupling of the brane to the background field. The contribution of the five-form field potential to the action of the brane per orbit around the S^5 is

$$S_{CS} = \int_{world\ volume} C = \int_{\Sigma} F$$

where $F = dC$ is the five-form flux and Σ is a five-manifold in S^5 whose boundary is the 4-dimensional surface which the $D3$-brane sweeps out during one-orbit. The background flux is given by $F = Bd\omega$, with B the constant flux density. We therefore have

$$S_{CS} = B\omega$$

$$\Rightarrow \quad \mathcal{L}_{CS} = \frac{S_{CS}}{T_{D3}} = B\omega \frac{\dot{\phi}}{2\pi} = \dot{\phi} N \frac{r^4}{R^4}.$$

The total bosonic Lagrangian is

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{CS} = -T_{D3}\Omega_3 r^3 \sqrt{1-(R^2-r^2)\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4}.$$

We will use the relation

$$T_{D3}\Omega_3 = \frac{N}{R^4}.$$

The angular momentum in terms of $\dot{\phi}$ is

$$L = \frac{m\dot{\phi}(R^2-r^2)}{\sqrt{1-(R^2-r^2)\dot{\phi}^2}} + N\frac{r^4}{R^4},$$

where $m = T_{D3}\omega_3 r^3 = (N/R^4)r^3$. Again we see that the angular momentum is bounded by N since $0 \leqslant r \leqslant R$ and $0 \leqslant \dot{\phi}R \leqslant 1$. It is clear that as r tends to R the upper bound of the angular momentum is N. The energy is

$$E = \sqrt{m^2 + \frac{(L - Nr^4/R^4)^2}{R^2 - r^2}}.$$

Varying the energy with respect to r at fixed L, we find in this case a stable minimum when

$$r^2 = \frac{L}{N}R^2.$$

At this stable minimum the energy matches the BPS bound when L is large for $N >> 1$[8]:

$$E = \frac{L}{R}.$$

This equivalence is, at least at the (semi-) classical level, good evidence that Kaluza-Klein gravitons are best described in terms of branes as opposed to fundamental strings [8].

3.3 Schur/Giants duality

The AdS/CFT correspondence suggests an equivalence between SYM theory and quantum gravity on Ad$S_5 \times S^5$ spacetime. One way of verifying the conjecture is to perform a change of variables, in this case the Operator-State correspondence tells us that operators (fields) of $\mathcal{N} = 4$ SYM can be transformed into states in Type II B string theory. In the $\frac{1}{2}$-BPS sector the correspondence relates the $\mathcal{R}-$charge of an operator with the angular momentum (J) of a state. The natural observable in both theories are correlation functions of gauge invariant local operators. Preparing a gauge invariant operator in SYM, with gauge group $U(N)$, involves taking a trace over $U(N)$ indices. Thus we need to find an orthonormal basis which spans all the gauge invariant operators.Schur polynomials offer a candidate for such a basis, since one can construct a $u(N)$ representation in an orthonormal basis.

A Schur polynomial labeled by a Young diagram R with n boxes is given by

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_n}_{i_{\sigma(n)}}$$

where $\chi_R(\sigma) = Tr(\Gamma_R(\sigma))$ is called the character of the element σ in representation R. The matrices Z are $u(N)$ adjoint representations of scalar Higgs fields (for a detailed analysis see [7; 12; 13]). The operators build from Schur polynomial basis are suggested as duals of giant gravitons states on the string theory side. The seminal work of [9] suggested the duality by describing the duality in a Ad$S_5 \times S^5$ background, in the large N limit:

- A Schur polynomial labeled by a Young diagram which has a single column of length L ($L \sim \mathcal{O}(N)$), is dual to $D3$-brane which has expanded in S^5 (a sphere giant). The giant graviton size, and hence angular momentum, is bounded above by the stringy exclusion principle. In the dual description the length of column of a Young diagram is bounded above by N, where the Young diagram labels a representation of SU(N).

- A Schur polynomial labeled by a Young diagram which has a single row of length L ($L \sim \mathcal{O}(N)$), has no upper bound and the series has angular momentum increasing in units of 1 [9]. These properties suggest that symmetric representation are dual to AdS giants, since AdS giants have no upper bound on their angular momentum [33].

We may extend the analogy and make a general statement. A Schur polynomial labeled by a Young diagram with $\mathcal{O}(1)$ columns and $\mathcal{O}(N)$ rows is dual to a bound state of sphere giants and that a Schur polynomial labeled by a Young diagram with $\mathcal{O}(N)$ columns and $\mathcal{O}(1)$ rows is dual to a bound state of AdS giants [7].

We have seen in the previous section that the size of the giant graviton is related to its angular momentum, J. And by Myers effect, an increase in J results in the graviton expanding, due to the coupling to the background RR five-form flux. Let R be the radius of the $\frac{1}{2}$-BPS state and R_{AdS} be the radius of Ad$S_5 \times S^5$ (given in eqn(3.2.1)), the relation is

$$R = \sqrt{\frac{J}{N}} R_{AdS} \tag{3.3.1}$$

in the large N limit while $g_s N$ is fixed and large. This relation motivates our choice of size of the Young diagram, since $J \sim \mathcal{O}(N)$ describes an object of size order R_{AdS}, the giant gravitons of [8].

A giant graviton can be excited by attaching an open string to it. In the dual description we represent this by attaching a string word W, comprising of $\mathcal{O}(\sqrt{N})$ letters(fields), to the Schur polynomial ($J \sim \mathcal{O}(\sqrt{N})$ the operator will be dual to an object of order l_s). The letters will generally be Higgs fields in the limit we'll be working in, but in principle they can be fermions or covariant derivatives of these fields. The dual of an excited giant with strings attached is called a restricted Schur polynomial,

$$\chi_{R,R_1}(Z,W^{(1)},W^{(2)}\dots,W^{(k)})=\tfrac{1}{(n-k)!}\sum_{\sigma\in S_n}Tr_{R_1}(\Gamma_R(\sigma))tr(\sigma Z^{\otimes(n-k)}W^{(1)}W^{(2)}\dots W^{(k)})$$

with

$$tr\left(\sigma Z^{\otimes(n-k)}W^{(1)}W^{(2)}\dots W^{(k)}\right)=Z^{i_1}_{i_{\sigma 1}}\dots Z^{i_{n-k}}_{i_{\sigma n-k}}(W^{(1)})^{i_{n-k+1}}_{i_{\sigma(n-k+1)}}(W^{(2)})^{i_{n-k+2}}_{i_{\sigma(n-k+2)}}\dots(W^{(n)})^{i_n}_{i_{\sigma(n)}}.$$

The representation R is a Young diagram with n boxes, while R_1 is a Young diagram with $n-k$ boxes. Therefore R and R_1 are representations of S_n and S_{n-k}. By subducing[2] R one can automatically build an irreducible representations of $S_{n-k}\otimes(S_1)^{\otimes k}$ subgroup if the string words are distinguishable and a reducible representation of $S_{n-k}\otimes S_k$ subgroup if all the string words are indistinguishable. Imagine that we are dealing with distinguishable string words then one of the subduced $S_{n-k}\otimes(S_1)^{\otimes k}$ irreducible representations is R_1. The operation $Tr_{R_1}(\Gamma_R(\cdot))$ is an instruction to take the trace over the R_1 subspace of R i.e.$Tr_{R_1}(\Gamma_R(\cdot))=Tr(P_{R\to R_1}\Gamma_R(\sigma)P_{R\to R_1})$, where $P_{R\to R_1}$ is a projection operator. Let us illustrate with an example:

Example 1. *Suppose R has the following possible subductions: R_1,R_2 and R_3, all representations of $S_{n-2}\otimes S_1\otimes S_1$. Then in an appropriately chosen basis we can write*

$$\Gamma_R(\sigma)=\begin{bmatrix}\Gamma_{R_1}(\sigma)&0&0\\0&\Gamma_{R_2}(\sigma)&0\\0&0&\Gamma_{R_3}(\sigma)\end{bmatrix}\qquad,\sigma\in S_{n-2}\otimes S_1\otimes S_1$$

$$\Gamma_R(\sigma)=\begin{bmatrix}A^{(1,1)}_{i_1j_1}&A^{(1,2)}_{i_1j_2}&A^{(1,3)}_{i_1j_3}\\A^{(2,1)}_{i_2j_1}&A^{(2,2)}_{i_2j_2}&A^{(2,3)}_{i_2j_3}\\A^{(3,1)}_{i_3j_1}&A^{(3,2)}_{i_3j_2}&A^{(3,3)}_{i_3j_3}\end{bmatrix}\qquad,\sigma\notin S_{n-2}\otimes S_1\otimes S_1$$

[2]Subduction is the process of legally removing boxes from a Young diagram. See [13] for more details.

Thus $Tr_{R_2}(\Gamma_R(\sigma)) = Tr(\Gamma_{R_2}(\sigma))$ *for* $\sigma \in S_{n-2} \otimes S_1 \otimes S_1$ *and* $Tr_{R_2}(\Gamma_R(\sigma)) = Tr(A^{(2,2)})$ *for* $\sigma \notin S_{n-2} \otimes S_1 \otimes S_1$.

For the case where the strings are indistinguishable we can compute the restricted trace $Tr_{r,s}(\Gamma_R(\sigma))$ by rewriting it in terms of restricted traces in the the $S_{n-k} \otimes (S_1)^{\otimes k}$ basis (to see an example of this change of basis see Chapter 4.1). For a more detailed treatment of restricted Schur polynomials see [7; 12; 11]. We interpret the "on diagonal" blocks, $Tr(A^{(1,1)}), Tr(A^{(2,2)}), Tr(A^{(3,3)})$, as states in which the two open strings are each on a specific giant and the "off diagonal" blocks, $A^{i,j}$ $(i \neq j)$, as states in which the two strings are stretched between the giants (twisted states).

Chapter 4

The Anomalous Dilatation Operator

The gauge/gravity duality predicts that the spectrum of scaling dimension Δ of a conformal gauge theory should coincide with the energy spectrum of some string states [34]. Thus, by independently studying the string theory energy spectrum and the conformal dimension of the corresponding gauge theory operator, we can test the correspondence. We will study the Lorentzian $\mathcal{N} = 4$ SYM theory on $R \times S^3$ with the action

$$S = \frac{N}{4\pi\lambda} \int dt \int_{S^3} \frac{d\Omega_3}{2\pi^2} \left(\frac{1}{2}(D\phi^i)(D\phi^i) + \frac{1}{4}([\phi^i, \phi^j])^2 - \frac{1}{2}\phi^i\phi^i + \cdots \right),$$

where λ is the t' Hooft coupling and $i, j = 1, \ldots, 6$. In the above action, we have omitted the fermion and the gauge kinetic terms. The six Higgs fields in our theory can be grouped into the following complex combinations

$$Z = \phi^1 + i\phi^2, \quad Y = \phi^3 + i\phi^4, \quad X = \phi^5 + i\phi^6$$

The scalar fields transform in the adjoint representation of $u(N)$, the complex combinations Z, X and Y are $N \times N$ complex matrices. The free field theory propagators we use are

$$\langle Z^\dagger_{ij}(t)Z_{kl}(t)\rangle = \langle X^\dagger_{ij}(t)X_{kl}(t)\rangle = \frac{4\pi\lambda}{N}\delta_{il}\delta_{jk}.$$

The state operator correspondence of conformal field theory tells us that the Hamiltonian in the conformal $\mathcal{N}=4$ SYM theory defined on $R\times S^3$ is dual to the generator of dilations in $\mathcal{N}=4$ SYM theory defined on R^4. In CFT, the conformal dimension, Δ, of an operator is defined by the two point correlation function

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(y)\rangle = \frac{\delta_{\alpha\beta}}{|x-y|^{2\Delta}}.$$

The full conformal dimension is a combination of the classical scaling dimension Δ_0 and the anomalous dimension[1] $\gamma(g_{YM})$, a quantum correction,

$$\Delta = \Delta_0 + \gamma.$$

The scaling dimension of the operator $\mathcal{O}_\alpha(x)$ correspond to the eigenvalues of a matrix of anomalous dimension. The matrix of anomalous dimensions can be mapped into the Hamiltonian of a closed spin chain [35]. Beisert, Kristjansen and Staudacher [36] developed a more efficient means of calculating the matrix of anamolous dimensions for the vacuum $\mathcal{N}=4$ SYM case which is dual to a string theory on Ad$S_5\times S^5$ by introducing a dilatation operator D:

$$D\mathcal{O}_\alpha = \Delta_\alpha\mathcal{O}_\alpha.$$

The operator admits a perturbative expansion:

$$D = \sum_{k=0}^{\infty}(\frac{g^2_{YM}}{16\pi^2})^k D_{2k}$$

[1] $\gamma = \frac{dlnZ}{dln\Lambda}$, where Z is the wave function renormalization and Λ is the regularization scale parameter.

where D_{2k} is the k-th loop anomalous dilatation operator. The one-loop anomalous dilatation operator we will use [37]

$$D_2 = - : Tr\left[\phi_i, \phi_j\right]\left[\frac{d}{d\phi_i}, \frac{d}{d\phi_j}\right] : -\frac{1}{2} : Tr\left[\phi_i, \frac{d}{d\phi_j}\right]\left[\phi_i, \frac{d}{d\phi_j}\right] :$$

We study an operator of R-charge of order N with two impurities, in the form of a restricted Schur polynomial labeled by a Young diagram with $n+2$ boxes, a Young diagram with n boxes and a Young diagram with 2 boxes (where $n \sim \mathcal{O}(N)$). This task is highly non-trivial as the planar approximation, which is usually used to simplify such problems, is not accurate. Instead one is forced to sum all (planar and non-planar) Feynman diagrams. In addition we want to diagonalize the anomalous dilatation operator D_2, to one loop, in the following basis $\chi^{(2)}_{R,(R'',r)}(Z,Y)$, where R is a Young diagram representing the irreducible representation (irrep.) of the symmetric group S_{n+2}, R'' is the Young diagram R with two boxes removed and $r = \square\!\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}$. In particular we want to consider Young diagrams with two columns; this is because we are interested in studying the interaction between two sphere giants that are close to one another. For our operators of interest, we get the effective anomalous dilatation operator to be:

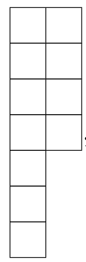
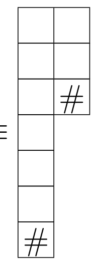
$$D_2 = 2 : Tr[Y, Z][\frac{d}{dZ}, \frac{d}{dY}] : .$$

4.1 Labeling convention and change of basis

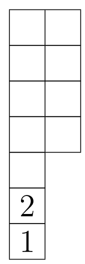
Consider a Young diagram R that consist of two columns. Our labeling convention is as follows: the number of empty boxes in second column from the left (col. 2) is b_0 and the number of empty boxes in the left most column (col. 1) is $b_0 + b_1$, therefore defining b_1 as the difference in the number of empty boxes between col. 2 and col. 1.

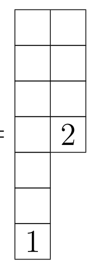
The structure of 2 column Young diagrams imposes default constraints on b_0 and b_1: $b_0 \geqslant 1$ and $b_1 \geqslant 0$.

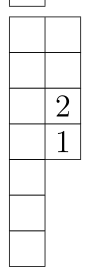
Example 2.

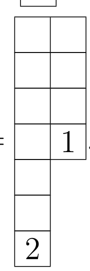
$R(b_0 = 4, b_1 = 3) \equiv$ , $\quad R(b_0 = 2, b_1 = 4) \equiv$ 

There are four possible ways of removing 2 boxes from R. We label the box which is removed first with 1 and the box removed second with 2.

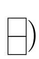
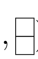
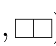
$R_1'' =$ 

$R_2'' =$  (4.1.1)

$S_1'' =$ 

$S_2'' =$ . (4.1.2)

After removing the two boxes the Young diagram that is left behind we denote it by either R_1'', R_2'', S_1'' and S_2''. The removed boxes can then be used to make an irrep. of S_2. Suppose R is composed of $n+2$ boxes then we can build an irrep of $S_n \times S_2$, as described above. The irreps are:

$(R_1'',$ $) \quad (S_1'',$ $) \quad (R'',$ $) \quad (R'',$ $).$

The basis of the carrier space (the space on which the group elements act on) of

$S_n \times S_2$ can be written in terms of the basis of $S_n \times S_1 \times S_1$:

$$
\begin{aligned}
|R, (R_1'', \begin{smallmatrix}\square\\\square\end{smallmatrix})\rangle_i &= |R_1''\rangle_i \\
|R, (S_1'', \begin{smallmatrix}\square\\\square\end{smallmatrix})\rangle_i &= |S_1''\rangle_i \\
|R, (R'', \begin{smallmatrix}\square\\\square\end{smallmatrix})\rangle_i &= \alpha|R_2''\rangle_i + \beta|S_2''\rangle_i \\
|R, (R'', \square\square)\rangle_i &= \bar{\alpha}|R_2''\rangle_i + \bar{\beta}|S_2''\rangle_i
\end{aligned}
$$

Where $i = 1, ..., d_{R''}$. The constants $\alpha, \beta, \bar{\alpha}$ and $\bar{\beta}$ satisfy

$$((n+2, n+1) - 1)\,|R, (R'', \square\square)\rangle_i = 0$$

and

$$((n+2, n+1) + 1)\,|R, (R'', \begin{smallmatrix}\square\\\square\end{smallmatrix})\rangle_i = 0.$$

The action of the group elements of the form $\Gamma_R(i, i+1)$ is reviewed in Appendix A.1 Aside 1. We simply present the result

$$(\bar{\alpha}, \bar{\beta}) = \left(\pm\sqrt{\frac{b_1}{2(b_1+1)}}, \pm\sqrt{\frac{b_1+2}{2(b_1+1)}}\right) = (-\beta, \alpha).$$

The freedom in choosing a sign $\pm$ will not have any bearing on the computation of the operator, therefore we can choose that $\bar{\alpha}$ and $\bar{\beta}$ both be positive. The change of basis is performed solely for convenience, as all the technology available to us of working with Schurs has been developed in the $S_n \times S_1 \times S_1$ basis [7; 13; 12]. However, in the end we will want to revert back to the $S_n \times S_2$ basis because it orthogonal unlike the $S_n \times S_1 \times S_1$ basis.

We get the following relations between the characters[2]:

$$\chi_{R,(R_1'',\begin{smallmatrix}\square\\\square\end{smallmatrix})} = \chi_{R,R_1''}$$

$$\chi_{R,(S_1'',\begin{smallmatrix}\square\\\square\end{smallmatrix})} = \chi_{R,S_1''}$$

$$\chi_{R,(R'',\begin{smallmatrix}\square\\\square\end{smallmatrix})} = \tfrac{b_1+2}{2(b_1+1)}\chi_{R,R_2''} + \tfrac{b_1}{2(b_1+1)}\chi_{R,S_2''} - \sqrt{\tfrac{b_1(b_1+2)}{4(b_1+1)^2}}(\chi_{R\to R_2'',S_2''} + \chi_{R\to S_2'',R_2''})$$

$$\chi_{R,(S'',\square\square)} = \tfrac{b_1+2}{2(b_1+1)}\chi_{R,S_2''} + \tfrac{b_1}{2(b_1+1)}\chi_{R,R_2''} + \sqrt{\tfrac{b_1(b_1+2)}{4(b_1+1)^2}}(\chi_{R\to R_2'',S_2''} + \chi_{R\to S_2'',R_2''})$$

The restricted characters $\chi_{R\to R_2'',S_2''}$ can be rewritten in terms of $\chi_{R,R_2''}$ and $\chi_{R,S_2''}$ for our special case- we call this removing twisted states- see Appendix B for details.

$$\chi_{R\to R_2'',S_2''} + \chi_{R\to S_2'',R_2''} = \sqrt{b_1(b_1+2)}(\chi_{R,S_2''} - \chi_{R,R_2''}).$$

The character relations clean up to

$$\chi_{R,(R_1'',\begin{smallmatrix}\square\\\square\end{smallmatrix})}(\sigma) = \chi_{R,R_1''}(\sigma) \tag{4.1.3}$$

$$\chi_{R,(S_1'',\begin{smallmatrix}\square\\\square\end{smallmatrix})}(\sigma) = \chi_{R,S_1''}(\sigma) \tag{4.1.4}$$

$$\chi_{R,(R'',\begin{smallmatrix}\square\\\square\end{smallmatrix})}(\sigma) = \tfrac{b_1+2}{2}\chi_{R,R_2''}(\sigma) - \tfrac{b_1}{2}\chi_{R,S_2''}(\sigma) \tag{4.1.5}$$

$$\chi_{R,(R'',\square\square)}(\sigma) = \tfrac{b_1+2}{2}\chi_{R,S_2''}(\sigma) - \tfrac{b_1}{2}\chi_{R,R_2''}(\sigma) \tag{4.1.6}$$

[2] I have used the short hand $\chi_{R,(R'',\begin{smallmatrix}\square\\\square\end{smallmatrix})}$ for $\chi_{R,(R'',\begin{smallmatrix}\square\\\square\end{smallmatrix})}(\sigma)$ in order to have tractable expressions.

4.2 The action of the anomalous dilatation operator

In this section we calculate the action of the anomalous dilatation[3] operator on the general state:

$$\chi^{(2)}_{R,(s,r)}(Z^{\otimes n},Y^{\otimes 2}) = \frac{1}{n!2!}\sum_{\psi\in S_{n+2}} Tr_{(s,r)}(\Gamma_R(\psi))tr\left(\psi Z^{\otimes n}Y^{\otimes 2}\right).$$

The Dilatation operator is $D_2 \equiv g^2_{YM}tr\left([Y,Z][\frac{d}{dZ},\frac{d}{dY}]\right)$, for convenience we define $W^i_j \equiv [Y,Z]^i_j$. The matrix derivative operator $(D_Z)^i_j = \frac{d}{dZ^j_i}$, works as follows:

$$(D_Z)^i_j Z^k_l = \delta^k_j \delta^i_l.$$

The action of the dilatation operator:

$$\begin{aligned}
D_2\chi^{(2)}_{R,(s,r)}(Z^{\otimes n},Y^{\otimes 2}) &= W^k_l[\frac{d}{dZ},\frac{d}{dY}]^l_k \times \\
&\quad \frac{1}{2!n!}\sum_{\psi\in S_{n+2}} Tr_{(s,r)}(\Gamma_R(\psi))Z^{i_1}_{i_{\psi(1)}}Z^{i_2}_{i_{\psi(2)}}\cdots Z^{i_n}_{i_{\psi(n)}}Y^{i_{n+1}}_{i_{\psi(n+1)}}Y^{i_{n+2}}_{i_{\psi(n+2)}} \\
&= \frac{W^k_l}{2!n!}\sum_{\psi\in S_{n+2}} Tr_{(s,r)}(\Gamma_R(\psi))Z^{i_1}_{i_{\psi(1)}}Z^{i_2}_{i_{\psi(2)}}\cdots Z^{i_{n-1}}_{i_{\psi(n-1)}}Y^{i_{n+1}}_{i_{\psi(n+1)}} \\
&\quad \times\left(2n\delta^{i_{n+2}}_{i_{\psi(n)}}\delta^{i_n}_k\delta^l_{i_{\psi(n+2)}} - 2n\delta^{i_n}_{i_{\psi(n+2)}}\delta^{i_{n+2}}_k\delta^l_{i_{\psi(n)}}\right) \\
&= \frac{1}{(n-1)!}\sum_{\psi\in S_{n+2}} Tr_{(s,r)}(\Gamma_R(\psi))Z^{i_1}_{i_{\psi(1)}}Z^{i_2}_{i_{\psi(2)}}\cdots Z^{i_{n-1}}_{i_{\psi(n-1)}}Y^{i_{n+1}}_{i_{\psi(n+1)}} \\
&\quad \times\left(\delta^{i_{n+2}}_{i_{\psi(n)}}W^{i_n}_{\psi(n+2)} - \delta^{i_n}_{i_{\psi(n+2)}}W^{i_{n+2}}_{\psi(n)}\right) \\
&\text{Implement the change of variable } \phi\equiv\psi(n,n+2)\Rightarrow\psi=\phi(n,n+2) \\
&= \frac{1}{(n-1)!}\sum_{\phi\in S_{n+2}} Tr_{(s,r)}(\Gamma_R(\phi(n,n+2)))Z^{i_1}_{i_{\phi(1)}}Z^{i_2}_{i_{\phi(2)}}\cdots Z^{i_{n-1}}_{i_{\phi(n-1)}}Y^{i_{n+1}}_{i_{\phi(n+1)}} \\
&\quad \times\left(\delta^{i_{n+2}}_{i_{\phi(n+2)}}W^{i_n}_{\phi(n)} - \delta^{i_n}_{i_{\phi(n)}}W^{i_{n+2}}_{\phi(n+2)}\right).
\end{aligned}$$

[3] We will from here on use the term dilatation operator when we referring the anomalous dilatation operator

It would be very convenient if we had all the deltas in the $n+2$ position for both terms. We can apply the subgroup swap rule (for details see [7; 12]) on the second term to swap the labels on W and δ.

$$
\begin{aligned}
&D_2 \chi^{(2)}_{R,(s,r)}(Z^{\otimes n}, Y^{\otimes 2}) \\
&= \tfrac{1}{(n-1)!} \textstyle\sum_{\phi \in S_{n+2}} Tr_{(s,r)} \left(\Gamma_R(\phi(n,n+2) - (n,n+2)\phi(n,n+2)(n,n+2)) \right) \\
&\qquad \times Z^{i_1}_{i_{\phi(1)}} Z^{i_2}_{i_{\phi(2)}} \cdots Z^{i_{n-1}}_{i_{\phi(n-1)}} W^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}} \delta^{i_{n+2}}_{i_{\phi(n+2)}} \\
&= \tfrac{1}{(n-1)!} \textstyle\sum_{\phi \in S_{n+2}} Tr_{R,(s,r)} \left([\phi, (n, n+2)] \right) Z^{i_1}_{i_{\phi(1)}} Z^{i_2}_{i_{\phi(2)}} \cdots Z^{i_{n-1}}_{i_{\phi(n-1)}} W^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}} \delta^{i_{n+2}}_{i_{\phi(n+2)}}.
\end{aligned}
$$

The presence of the delta looks like a reduction[4] has taken place on a string word at position $n+2$ so we can rewrite our expression as follows:

$$
\begin{aligned}
&D_2 \chi^{(2)}_{R,(s,r)}(Z^{\otimes n}, Y^{\otimes 2}) \\
&= \tfrac{tr(D_V)}{(n-1)!} \textstyle\sum_{\phi \in S_{n+2}} Tr_{R,(s,r)} \left([\phi, (n, n+2)] \right) Z^{i_1}_{i_{\phi(1)}} Z^{i_2}_{i_{\phi(2)}} \cdots Z^{i_{n-1}}_{i_{\phi(n-1)}} W^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}} V^{i_{n+2}}_{i_{\phi(n+2)}}.
\end{aligned}
$$

We cannot continue the computation beyond this point using the general irrep $R,(s,r)$, we need to consider each basis state individually. From here on we will only compute the action of D_2 on $\chi_{R,(R_1'', \begin{smallmatrix}\square\\\square\end{smallmatrix})}$ in detail, the steps that are taken are generic and can be applied to the rest of the basis elements. At the end of each major step we will state the corresponding result for the other basis elements too.

Before we proceed, let us introduce some convenient short hand notation:

$$\Psi_a(b_0, b_1) \equiv \chi^{(2)}_{R,(R_1'', \begin{smallmatrix}\square\\\square\end{smallmatrix})}(Z^{\otimes n}, Y^{\otimes 2}) \tag{4.2.1}$$

$$\Psi_b(b_0, b_1) \equiv \chi^{(2)}_{R,(S_1'', \begin{smallmatrix}\square\\\square\end{smallmatrix})}(Z^{\otimes n}, Y^{\otimes 2}) \tag{4.2.2}$$

$$\Psi_d(b_0, b_1) \equiv \chi^{(2)}_{R,(R'', \square\square)}(Z^{\otimes n}, Y^{\otimes 2}) \tag{4.2.3}$$

$$\Psi_e(b_0, b_1) \equiv \chi^{(2)}_{R,(R'', \begin{smallmatrix}\square\\\square\end{smallmatrix})}(Z^{\otimes n}, Y^{\otimes 2}). \tag{4.2.4}$$

[4] For the definition and overview of reduction rule see [7] Appenidix B and C.

We can apply the reduction rules developed in [7] to our interim expression, even though our polynomial traces the product $\phi(n, n+2)$.

$$D_2\Psi_a(b_0, b_1) = (N - b_0 - b_1)\frac{1}{(n-1)!}\sum_{\phi \in S_{n+1}} Tr_{R,R_1''}([\phi,(n,n+2)]) \times Z^{i_1}_{i_{\phi(1)}} \cdots Z^{i_{n-1}}_{i_{\phi(n-1)}} W^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}}.$$

Where $N - b_0 - b_1$ is the weight of the first box to be removed from the irrep. R_1''. In the Appendix A we compute the characters above, we can draw on that result.

$$\mathrm{Tr}_{R''}\left(\Gamma_R(\sigma(n, n+2))\right) = \sum_\alpha \tau_\alpha \mathrm{Tr}_{R_\alpha'''}\left(\Gamma_{R'}(\sigma)\right) + \sum_\beta \lambda_\beta \mathrm{Tr}_{R_\beta''',T_\beta'''}\left(\Gamma_{R'}(\sigma)\right)$$

where τ and λ are constants defined in Appendix A.1. In the second sum T_β''' must be the same shape as R_β'''. Also note that λ_β is transpose invariant, i.e. $\lambda_{T''',R'''} = \lambda_{R''',T'''}$. Therefore

$$\begin{aligned}
\mathrm{Tr}_{R''}\left(\Gamma_R([\sigma,(n,n+2)])\right) &= \sum_\alpha \tau_\alpha \mathrm{Tr}_{R_\alpha'''}\left(\Gamma_{R'}(\sigma)\right) + \sum_\beta \lambda_\beta \mathrm{Tr}_{R_\beta''',T_\beta'''}\left(\Gamma_{R'}(\sigma)\right) \\
&\quad - \sum_\alpha \tau_\alpha \mathrm{Tr}_{R_\alpha'''}\left(\Gamma_{R'}(\sigma)\right) - \sum_\beta \lambda_\beta \mathrm{Tr}_{T_\beta''',R_\beta'''}\left(\Gamma_{R'}(\sigma)\right) \\
&= \sum_\beta \lambda_\beta \left[\mathrm{Tr}_{R_\beta''',T_\beta'''}\left(\Gamma_{R'}(\sigma)\right) - \mathrm{Tr}_{T_\beta''',R_\beta'''}\left(\Gamma_{R'}(\sigma)\right)\right]
\end{aligned}$$

Applying this result to our calculation, we get:

$$D_2\Psi_a(b_0, b_1) = C_{(1,R_1'')}\lambda_{R_1'''}\left(\chi^{(2)}_{(b_0-1,b_1+1)\to R_1''',R_2'''} - \chi^{(2)}_{(b_0-1,b_1+1)\to R_2''',R_1'''}\right)(Z,Y,W) \tag{4.2.5}$$

$$D_2\Psi_b(b_0, b_1) = C_{(1,S_1'')}\lambda_{S_1'''}\left(\chi^{(2)}_{(b_0,b_1-1)\to S_1''',S_2'''} - \chi^{(2)}_{(b_0,b_1-1)\to S_2''',S_1'''}\right)(Z,Y,W) \tag{4.2.6}$$

$$D_2\Psi_d(b_0, b_1) = \frac{b_1}{2}D^{(1)}\Psi_a(b_0+1, b_1-2) - \frac{b_1+2}{2}D^{(1)}\Psi_b(b_0-1, b_1+2) \tag{4.2.7}$$

$$D_2\Psi_e(b_0, b_1) = \frac{b_1}{2}D^{(1)}\Psi_b(b_0-1, b_1+2) - \frac{b_1+2}{2}D^{(1)}\Psi_a(b_0+1, b_1-2) \tag{4.2.8}$$

Where

$$C_{(1,R_1'')} = (N - b_0 - b_1 - 1), \qquad \lambda_{R_1'''} = \frac{1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}$$

and

$$C_{(1,S_1'')} = (N - b_0), \qquad \lambda_{S_1'''} = \frac{-1}{b_1}\sqrt{\frac{b_1+1}{b_1-1}}.$$

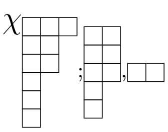
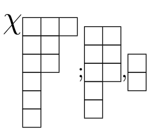
After substituting the full expression for W, it is evident that the terms in each basis state have been encountered before. We come across such terms when we coset expand restricted Schurs such as $\chi^{(2)}_{R,R''}(Z^{\otimes n-2}, Y, Y)$, $\chi^{(2)}_{R,S''}(Z^{\otimes n-2}, Y, Y)$, $\chi^{(2)}_{R\to R'',S''}(Z^{\otimes n-2}, Y, Y)$ and $\chi^{(2)}_{R\to S'',R''}(Z^{\otimes n-2}, Y, Y)$ (see Appendix A.1). The hop off identities of Appendix A.2 help us express the restricted Schurs in eqns(4.2.5-4.2.8) above in terms of the $S_n \times S_1 \times S_1$ basis. The coset expansion gives us expansions of the form

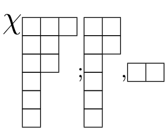
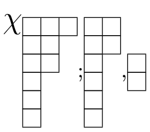
$$\chi^{(2)}_L(Z, Y, Y) = \sum_Q C_Q \chi^{(2)}_Q(Z, Y, YZ).$$

The hop off identities will enable us to invert these expressions as follows:

$$\chi^{(2)}_Q(Z, Y, YZ) = \sum_L \tilde{C}_L \chi^{(2)}_L(Z, Y, Y).$$

At which point we will have the action of the Dilatation operator in the $S_n \times S_1 \times S_1$ basis. The hop off identities introduce new operators

$\Psi_f(b_0, b_1) = \chi$  $\qquad \Psi_g(b_0, b_1) = \chi$  (4.2.9)

$\Psi_h(b_0, b_1) = \chi$  $\qquad \Psi_i(b_0, b_1) = \chi$  (4.2.10)

To get the action in the $S_n \times S_2$ basis of interest, we can change basis by inverting eqns(4.1.3-4.1.6).

4.2.1 Final Result

Continuing with the notation introduced in eqns(4.2.1-4.2.4) and eqns(4.2.9, 4.2.10), the action of the Dilatation operator on the basis states of $S_n \times S_2$ is:

$$\begin{aligned}
D_2\Psi_a(b_0,b_1) \quad = \quad & \frac{4b_0\,(\mathrm{N}-b_0-b_1-1)}{(2+b_1)^{\,2}\,(3+b_0+b_1)}\Psi_a(b_0,b_1) \\
& -\frac{2b_0\,(\mathrm{N}-b_0-b_1-1)}{(1+b_0)\,(2+b_1)}\Psi_d(b_0,b_1) \\
& +\frac{2b_0b_1\,(\mathrm{N}-b_0-b_1-1)}{(1+b_0)\,(2+b_1)^{\,2}}\Psi_e(b_0,b_1) \\
& +\frac{4\,(1+b_1)\,(2+b_0+b_1)\,(\mathrm{N}-b_0-b_1-1)}{(1+b_0)\,(2+b_1)^{\,2}\,(3+b_1)}\Psi_b(b_0-1,b_1+2) \\
& +\frac{2\,(\mathrm{N}-b_0-b_1-1)\,(1+b_1)\,(2+b_0+b_1)}{(3+b_0+b_1)\,(6+5b_1+b_1^2)}\Psi_d(b_0-1,b_1+2) \\
& -\frac{2\,(\mathrm{N}-b_0-b_1-1)\,(2+b_0+b_1)\,(4+5b_1+b_1^2)}{(2+b_1)^{\,2}\,(3+b_1)\,(3+b_0+b_1)}\Psi_e(b_0-1,b_1+2) \\
& +\frac{2b_0\,(\mathrm{N}-b_0-b_1-1)}{(1+b_0)\,(2+b_1)}\Psi_f(b_0,b_1) \\
& -\frac{2b_0\,(b_0+b_1)\,(\mathrm{N}-b_0-b_1-1)}{(1+b_0)\,(2+b_1)\,(3+b_0+b_1)}\Psi_g(b_0,b_1) \\
& -\frac{2\,(\mathrm{N}-b_0-b_1-1)\,(1+b_1)\,(2+b_0+b_1)}{(3+b_0+b_1)\,(6+5b_1+b_1^2)}\Psi_h(b_0-1,b_1+1) \\
& +\frac{2\,(-1+b_0)\,(1+b_1)\,(2+b_0+b_1)\,(\mathrm{N}-b_0-b_1-1)}{(1+b_0)\,(3+b_0+b_1)\,(6+5b_1+b_1^2)}\Psi_i(b_0-1,b_1+2)
\end{aligned}$$

$$\begin{aligned}
D_2\Psi_b(b_0,b_1) \; = \; & \frac{4\,(\mathrm{N}-b_0)\,(1+b_0)\,(1+b_1)}{(-1+b_1)\,b_1^2\,(2+b_0+b_1)}\Psi_a(b_0+1,b_1-2) \\
& -\frac{2\,(\mathrm{N}-b_0)\,(1+b_0)\,(1+b_1)}{(2+b_0)\,(-1+b_1)\,b_1}\Psi_d(b_0+1,b_1-2) \\
& +\frac{2\,(\mathrm{N}-b_0)\,(1+b_0)\,(-2-b_1+b_1^2)}{(2+b_0)\,(-1+b_1)\,b_1^2}\Psi_e(b_0+1,b_1-2) \\
& +\frac{4\,(\mathrm{N}-b_0)\,(1+b_0+b_1)}{(2+b_0)\,b_1^2}\Psi_b(b_0,b_1) \\
& +\frac{2\,(\mathrm{N}-b_0)\,(1+b_0+b_1)}{b_1\,(2+b_0+b_1)}\Psi_d(b_0,b_1) \\
& -\frac{2\,(\mathrm{N}-b_0)\,(2+b_1)\,(1+b_0+b_1)}{b_1^2\,(2+b_0+b_1)}\Psi_e(b_0,b_1) \\
& +\frac{2\,(\mathrm{N}-b_0)\,(1+b_0)\,(1+b_1)}{(2+b_0)\,(-1+b_1)\,b_1}\Psi_f(b_0+1,b_1-2) \\
& -\frac{2\,(\mathrm{N}-b_0)\,(1+b_0)\,(1+b_1)\,(-1+b_0+b_1)}{(2+b_0)\,(-1+b_1)\,b_1\,(2+b_0+b_1)}\Psi_g(b_0+1,b_1-2) \\
& -\frac{2\,(\mathrm{N}-b_0)\,(1+b_0+b_1)}{b_1\,(2+b_0+b_1)}\Psi_h(b_0,b_1) \\
& +\frac{2\,(\mathrm{N}-b_0)\,b_0\,(1+b_0+b_1)}{(2+b_0)\,b_1\,(2+b_0+b_1)}\Psi_i(b_0,b_1)
\end{aligned}$$

$$\begin{aligned}
D_2\Psi_d(b_0,b_1) &= \tfrac{b_1}{2}D_2\Psi_a(b_0+1,b_1-2)-\tfrac{b_1+2}{2}D_2\Psi_b(b_0-1,b_1+2) \\
D_2\Psi_e(b_0,b_1) &= -\tfrac{b_1+2}{2}D_2\Psi_a(b_0+1,b_1-2)+\tfrac{b_1}{2}D_2\Psi_b(b_0-1,b_1+2)
\end{aligned}$$

Comments

The factor of $N-b_0-b_1-1$ in the action of the anomalous dilatation operator on Ψ_a, ensures that the first column of the Young diagram does not grow longer than N boxes. For the Ψ_b its the $N-b_0$ factors that contain the maximum size of the first column and the factors $\frac{1}{b_1}$, $\frac{1}{b_1-1}$ reinforce the constraints that $b_1>1$ which guarantees that the first column is always longer than the second. These are the built in boundary conditions.

Normalization

The states are normalized to have a unit two-point function. The dictionary between

Schur polynomial operators and normalized states is as follows:

$$\chi^{(2)}_{R,R''}(Z,W^{(2)},W^{(1)}) \longleftrightarrow \mathbb{N}_{R,R''}|R,R'',W^{(2)},W^{(1)}\rangle$$

such that

$$\langle\chi^{(2)}_{R,R''}(Z,W^{(2)},W^{(1)})\chi^{(2)}_{R,R''}(Z,W^{(2)},W^{(1)})^{\dagger}\rangle = (\mathbb{N}_{R,R''})^2$$

In the next section we will normalize our states and express (the action of) the dilatation operator in a normalized basis

4.2.2 The Dilatation operator in a normalized basis

Here in we express the dilatation operator in the normalized basis. The two point functions of all the relevant operators have been computed in Appendix C.3, using the technology developed in [7; 12]. In what follows we have dropped the three column states $\{f,g,h,i\}$ contributions because their two point functions are sub-leading in b_0 relative to the two column states $\{a,b,d,e\}$. We are allowed to do this since we are concerned with $b_0 \sim \mathcal{O}(N)$ in the large N limit while g_sN is fixed and large. This is a favourable outcome as it ensures the action of the dilatation operator closes on the two membrane states. We insert back the g_{YM} factors for completeness.

$$\begin{aligned}
D_2|a,b_0,b_1\rangle \;&= g^2_{YM}\frac{4\,(N-b_0-b_1+1)}{(2+b_1)^{\,2}}|a,b_0,b_1\rangle \\
&-g^2_{YM}2\frac{\sqrt{(N-b_0-b_1-1)(N-b_0+1)}}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}|d,b_0,b_1\rangle \\
&+g^2_{YM}2b_1\frac{\sqrt{(N-b_0-b_1-1)(N-b_0+1)}}{(b_1+2)^2}\sqrt{\frac{b_1+3}{b_1+1}}|e,b_0,b_1\rangle \\
&+g^2_{YM}\frac{4\sqrt{(N+1-b_0)(N-1-b_0-b_1)}}{(2+b_1)^{\,2}}|b,b_0-1,b_1+2\rangle \\
&+g^2_{YM}2\frac{N-b_0-b_1-1}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}}|d,b_0-1,b_1+2\rangle \\
&-g^2_{YM}2\frac{(N-b_0-b_1-1)(b_1+4)}{(b_1+2)^2}\sqrt{\frac{b_1+1}{b_1+3}}|e,b_0-1,b_1+2\rangle
\end{aligned}$$

$$
\begin{aligned}
D_2|b,b_0,b_1\rangle &= g_{YM}^2 4\frac{\sqrt{(N-b_0)(N-b_0-b_1)}}{(b_1)^2}|a,b_0+1,b_1-2\rangle\\
&-g_{YM}^2 2\frac{N-b_0}{b_1}\sqrt{\frac{b_1+1}{b_1-1}}|d,b_0+1,b_1-2\rangle\\
&+g_{YM}^2 2\frac{(N-b_0)(b_1-2)}{(b_1)^2}\sqrt{\frac{b_1+1}{b_1-1}}|e,b_0+1,b_1-2\rangle\\
&+g_{YM}^2\frac{4\,(N-b_0)}{(b_1)^2}|b,b_0,b_1\rangle\\
&+g_{YM}^2 2\frac{\sqrt{(N-b_0)(N-b_0-b_1)}}{(b_1)^2}\sqrt{\frac{b_1-1}{b_1+1}}|d,b_0,b_1\rangle\\
&-g_{YM}^2 2\frac{(b_1+2)\sqrt{(N-b_0)(N-b_0-b_1)}}{(b_1)^2}\sqrt{\frac{b_1-1}{b_1+1}}|e,b_0,b_1\rangle
\end{aligned}
$$

$$
\begin{aligned}
D_2|d,b_0,b_1\rangle &= \frac{b_1}{2}\sqrt{\frac{b_0(b_1+3)}{b_0+b_1+2}}D_2|a,b_0+1,b_1-2\rangle\\
&-\frac{b_1+2}{2}\sqrt{\frac{(b_0+b_1+1)(b_1-1)}{b_0+b_1}}D_2|b,b_0-1,b_1+2\rangle
\end{aligned}
$$

$$
\begin{aligned}
D_2|e,b_0,b_1\rangle &= -\frac{b_1+2}{2}\sqrt{\frac{(b_0+b_1+1)(b_1-1)}{b_0+b_1}}D_2|a,b_0+1,b_1-2\rangle\\
&+\frac{b_1}{2}\sqrt{\frac{b_0(b_1+3)}{b_0+b_1+2}}D_2|b,b_0-1,b_1+2\rangle
\end{aligned}
$$

Chapter 5

Numerical results and Discussion

Part of the duality dictionary tells us that we can map the energy of a string state to the conformal dimension of the corresponding operator. This necessarily requires us to think of the anomalous dilatation operator as the Hamiltonian of some string states. In our case, the operators represented by Schur polynomials are known to be dual to Giant gravitons, therefore the string states are in fact the giant graviton $D3$-branes. The dilatation operator allows us to access the energy spectrum of the giant gravitons.

The anomalous dilatation operator D_2 is one of the generators of conformal symmetry, its eigen-values also tells us about the the anomalous dimension γ of an operator $\hat{O}$ in conformal field theory. Therefore [36]

$$\textit{Hamiltonian} \quad \mathcal{H} = g_{YM} D_2$$

$$\textit{Energies} \quad \mathcal{E} = g_{YM} \gamma.$$

Here in we diagonalize the anomalous dilatation operator and determine its eigen-values. We conclude with a discussion.

5.1 Numerical results

To determine the spectrum of the one loop anomalous dimension, one would have to analytically solve for the eigenvalues (and eigen-vectors) of the dilatation operator as expressed in terms of normalized states (see Chapter 4.2.2). In this project we focused on solving for the spectrum numerically. The dual description of the dilatation operator is that it is the Hamiltonian for our two three-brane system of interest. We want to study two sphere giants that are separated by a distance of order string length. We excite the gravitons by inserting two impurities and observe their interaction. The physics we are interested in, constrains the parameters N, b_0 and b_1 as follows

- N must be large, since we are working in the large N regime.
- b_0 must be $\mathcal{O}(N)$ in order to describe sphere giants with radius comparable to the radius of $\mathrm{Ad}S_5 \times S^5$
- b_1 controls the separation distance between the branes, hence it must fall in the range $0 \leqslant b_1 \leqslant \mathcal{O}(\sqrt{N})$.

To perform the numerical calculations we chose $N = 100000$ and considered an operator built from 199800 Zs and 2 Ys. In this configuration there are a total of 398 excited two three-brane states. The energy spectrum of our system is shown in Fig 5.1. The most striking feature is that almost[1] three quarters of the states have zero energy. The nonzero (massive) states have a linear energy spectrum, this alludes that the two three-brane system behaves as a harmonic oscillator. We have not demonstrated this assertion analytically in this project. To get further insight into the system we tracked the separation parameter b_1 to see how it varies with energy. Fig 5.2 shows the spectrum of non zero anomalous dimensions together with the

[1] To be precise two less than three quarters.

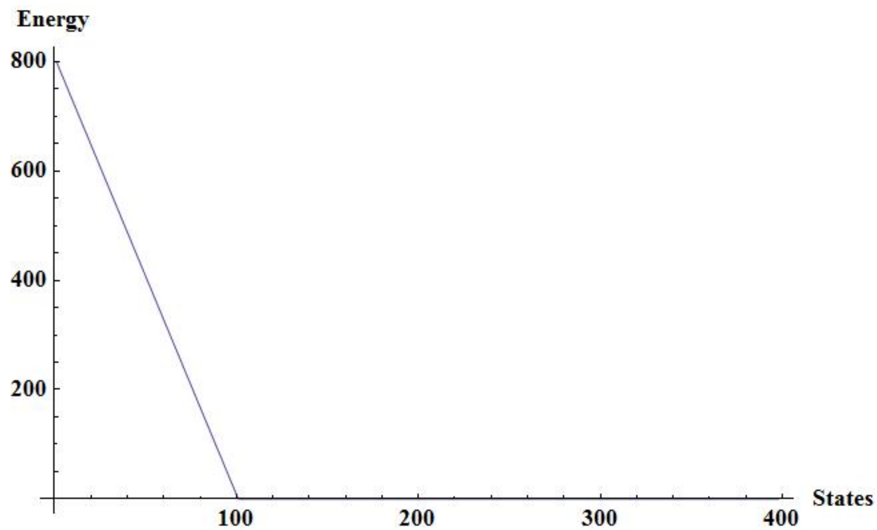


Figure 5.1: This is the energy level per state of the two sphere giant graviton system. The plot was produced with the $N = 100000$ and an operator built from 199800 Zs and two Ys. There are a total of 398 possible, excited states for the given parameters.

expected value of b_1. The plot shows that the average separation of the brane grows with energy until it saturates and then it diminishes as the energy continues to grow. It is also clear that the average separation of the three-branes does not influence the equal increment between consecutive states. Again there are hints that reinforce the idea of harmonic behaviour. It is well known that coupled harmonic oscillators have two normal modes. The region where the branes are diverging can be associated with the branes oscillating in phase, in this regime the open strings stretching between the branes are hardly excited. The second region where the branes are converging can be associated with the branes oscillating out of phase by π, in this regime the open strings stretching between the branes will be excited, hence the higher energy levels [18].

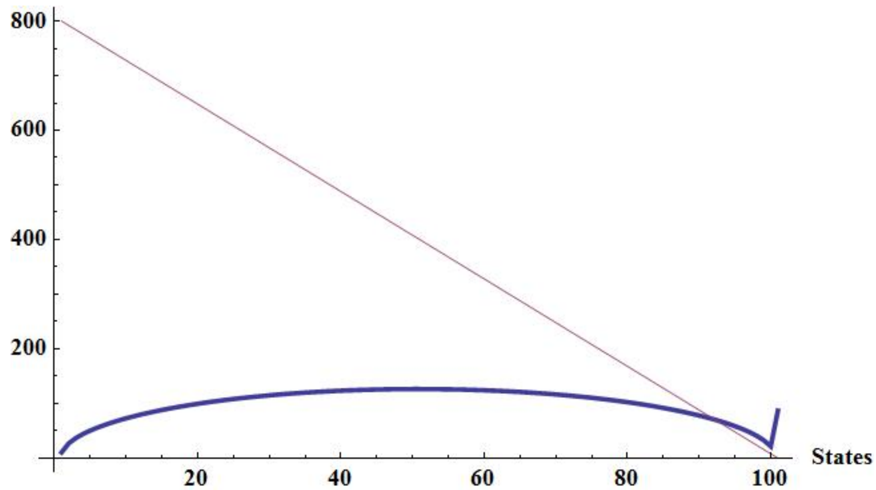


Figure 5.2: This plot shows the energy level (thin line) and the average separation $\langle b_1 \rangle$ (thick line) per state of the two sphere giant graviton system. The plot was produced with the $N = 100000$ and an operator built from 199800 Zs and two Ys.

5.2 Discussion

In this project we set out to determine the anomalous dimension of operators with R-charge of $\mathcal{O}(N)$ and constructed with the aid of Schur polynomials. The operators have a dual interpretation of two excited three-brane states. Our strategy was to diagonalize the anomalous dilatation operator and numerically solve for its eigenvalues. We worked in the large N limit, while keeping $g_s N$ fixed. In this limit we saw the mixing between two column Schurs and Schurs of different number of columns being suppressed. The number of columns in a Schur is identified with the number of three-branes, hence in the dual theory this meant that at weak string coupling the three-brane number is conserved. The anomalous dimensions gave us the energy spectrum of the two three-brane system. The numerical analysis showed that the three-brane system behaves like two coupled harmonic oscillators.

There are a number of coherent extension of our study. One can study the effects of increasing the number of Y fields in the impurities and/or use a wider variety of fields, such as other Higgs fields and fermions, to compose the impurity. It would also be interesting to check whether the excited three-brane dynamics are integrable.

Appendix A

Coset Expansions and Hop off identities

Herein we review the coset expansion and hop off identities using [13].

A.1 Coset Expansion

Let R denote an irreducible representation of S_{n+2}, with the following possible subductions shown in Fig A.1.

Consider the restricted Schur polynomial

$$\chi^{(2)}_{R,R''}(Z^{\otimes n}, Y^{\otimes 2}) = \frac{1}{n!2!} \sum_{\sigma \in S_{n+2}} Tr_{R''}(\Gamma_R(\sigma)) Z^{i_1}_{i_{\sigma(1)}} \dots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} Y^{i_{n+2}}_{i_{\sigma(n+2)}}$$

The coset expansion enables us to express a restricted Schur polynomial of irrep. R in terms of the subduced irrep.s R', S', etc.. The technique exploits the fact that any element, σ, of S_n can be factored into the product of ψ an element of S_{n-1} and (i,n), where $i \leqslant n$. We illustrate the procedure by way of an example.

Example 3. *Let us build the elements of S_3 starting from S_1. To get the elements of S_2, multiply all the elements of S_1 by $\{1,(1,2)\}$ and the result is trivial. Now to*

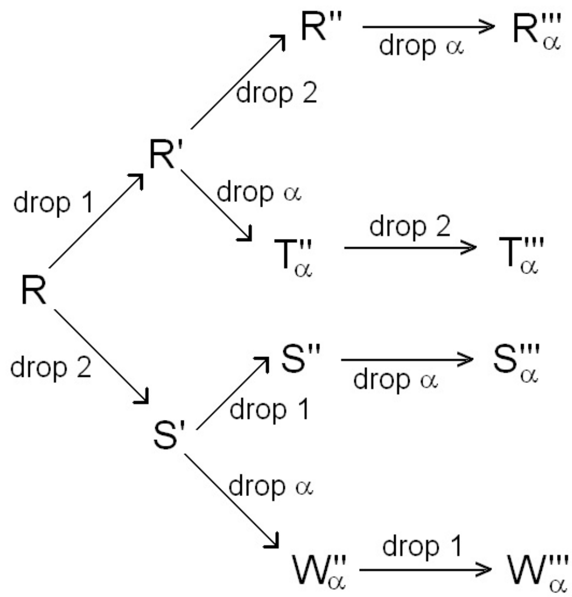


Figure A.1: This diagram shows the different subduction routes we follow in Appendix A.1. (source [13])

recover S_3 *multiply all the elements of* S_2 *by the set* $\{(i,3) : i \leqslant 3\}$.

$$\begin{aligned} S_2 \times 1 &= S_2 \\ S_2 \times (2,3) &= (23),(123) \\ S_2 \times (1,3) &= (13),(132). \end{aligned}$$

The right coset expansion of $\chi^{(2)}_{R,R''}(Z^{\otimes n}, Y^{\otimes 2})$ is as follows:

$$\begin{aligned} \chi^{(2)}_{R,R''}(Z^{\otimes n}, Y^{\otimes 2}) &= \frac{1}{n!2!} \sum_{\phi \in S_{n+1}} \Big[Tr_{R''}\left(\Gamma_{R'}(\phi)\right) Z^{i_1}_{i_{\phi(1)}} \dots Z^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}} tr(Y) \\ &+ \sum_{a<n+1} Tr_{R''}\left(\Gamma_R(\phi(a,n+2))\right) Z^{i_1}_{i_{\phi(1)}} \cdots (ZY)^{i_a}_{i_{\phi(a)}} \cdots Z^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}} \\ &+ Tr_{R''}\left(\Gamma_R(\phi(n+1,n+2))\right) Z^{i_1}_{i_{\phi(1)}} \cdots Z^{i_n}_{i_{\phi(n)}} (YY)^{i_{n+1}}_{i_{\phi(n+1)}} \Big]. \end{aligned}$$

The left coset expansion is exactly the same except $\phi(a, n+2)$ is replaced with $(a, n+2)\phi$ and ZY with YZ.

Aside 1

Before we try simplify the coset expansion we will need to review computing traces of group elements. A good and thorough treatment of computing traces of groups elements using strand diagrams is presented in Appendix B.3 of [13]. Herein we will only present the summarized result by way of two examples drawn from our current computation. An arbitrary element of S_n can be factorized into a product of cycles of the form $(i, i+1)$. For example $(n+2, n) = (n, n+1)(n+1, n+2)(n, n+1)$. We also have

$$Tr_{R^{***}}(\Gamma_R(\sigma)) = \ _j\langle R^{***}|\sigma|R^{***}\rangle_j$$

where R^{***} represents an arbitrary subduction of R. Therefore in order to compute the trace it is sufficient to compute the action of $\Gamma_R(i, i+1)$ on $|R^{***}\rangle_j$.

Example 4.

$$\begin{aligned}
\Gamma_R(n+2, n+1)|R''\rangle &= \frac{1}{c_1 - c_2}|R''\rangle + \sqrt{1 - \frac{1}{(c_1 - c_2)^2}}|S''\rangle \\
\Gamma_R(n+1, n)|R'''_\alpha\rangle &= \frac{1}{c_2 - c_\alpha}|R'''_\alpha\rangle + \sqrt{1 - \frac{1}{(c_2 - c_\alpha)^2}}|T'''_\alpha\rangle \\
\Gamma_R(n+2, n+1)|T''_\alpha\rangle &= \frac{1}{c_1 - c_\alpha}|T''_\alpha\rangle
\end{aligned}$$

where c_m is the weight of the box with label m on it.

Notice that a group element can permute the order of subduction, where possible, and/or leave it inert. A group element permutes the order of subduction with a weighting of $f_{\text{swap}} = \sqrt{1 - \frac{1}{(c_m - c_{m-1})^2}}$ and it leaves the order of subduction inert with a weighting of $f_{\text{no swap}} = \frac{1}{c_m - c_{m-1}}$.

Now back to the coset expansion. The first term in the coset expansion is easily identified to be $\frac{1}{2}tr(Y)\chi^{(1)}_{R',R''}(Z,Y)$. The last term we compute below

$$\begin{aligned} Tr_{R''}\left(\Gamma_R(\sigma(n+1,n+2))\right) &= {}_i\langle R''|\sigma(n+1,n+2)|R''\rangle_i \\ &= {}_i\langle R''|\sigma|R''\rangle_i \cdot \frac{1}{c_1-c_2} + {}_i\langle R''|\sigma|S''\rangle_i \cdot \sqrt{1-\frac{1}{(c_1-c_2)^2}} \\ &= \frac{1}{c_1-c_2} \cdot {}_i\langle R''|\sigma|R''\rangle_i. \end{aligned}$$

To get the last equality we used $\sigma \in S_{n+1} \Rightarrow \sigma(n+2) = n+2$, therefore σ cannot map from R'' subspace to S'' subspace i.e. $Tr_{R'',S''}(\sigma) = 0$. The constants c_1 and c_2 are the weights of the first and second box to be removed in rep. R'', respectively. Hence the last term in the coset expansion is $\frac{1}{2(c_1-c_2)}\chi^{(1)}_{R''}(Z,YY)$.

The remaining n middle terms will each have the same contribution, this is because the Z's are indistinguishable and all are in the carrier subspace of S_n. Therefore we can interchange index labels without changing the expression.

$$\begin{aligned} &\frac{1}{2!n!}\sum_{a<n+1}\sum_{\phi\in S_{n+1}} Tr_{R''}\left(\Gamma_R(\phi(a,n+2))\right) Z^{i_1}_{i_{\phi(1)}}\cdots (ZY)^{i_a}_{i_{\phi(a)}}\cdots Z^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}} \\ &\quad = \frac{1}{2!(n-1)!}\sum_{\phi\in S_{n+1}} Tr_{R''}\left(\Gamma_R(\phi(n,n+2))\right) Z^{i_1}_{i_{\phi(1)}}\cdots (ZY)^{i_n}_{i_{\phi(n)}} Y^{i_{n+1}}_{i_{\phi(n+1)}}. \end{aligned}$$

What we do next is to factor out the contribution of $(n, n+2)$ from the trace.

$$\begin{aligned} Tr_{R''}\left(\Gamma_R(\sigma(n,n+2))\right) &= \sum_i^{dim_{R''}} {}_i\langle n+2,n+1|\Gamma_R(\sigma(n,n+2))|n+2,n+1\rangle_i \\ &= \sum_i^{dim_{R'''_\alpha}}\sum_\alpha {}_i\langle n+2,n+1,\alpha|\Gamma_R(\sigma(n,n+2))|n+2,n+1,\alpha\rangle_i, \end{aligned}$$

where α labels the next box that can be legally removed from R'' and has index label n associated with it. Insert a completeness relation in the carrier space of R', of the form

$$\begin{aligned}
1_{R'} &= \sum_i |n+2\rangle_i \, {}_i\langle n+2| \\
&= \sum_j^{dim_{R''_\beta}} \sum_\beta |n+2,\beta\rangle_j \, {}_j\langle n+2,\beta| \\
&= \sum_j^{dim_{R'''_{\beta,\alpha}}} \sum_{\alpha\beta} |n+2,\beta,\alpha\rangle_j \, {}_j\langle n+2,\beta,\alpha|
\end{aligned}$$

in between $\Gamma_{R'}(\sigma)$ and $\Gamma_R((n,n+2))$.

$$\begin{aligned}
Tr_{R''}\left(\Gamma_R(\sigma(n,n+2))\right) &= \sum_j^{dim_{R'''_\alpha}} \sum_i^{dim_{R'''_\beta}} \sum_{\alpha,\beta,\gamma} {}_j\langle n+2,n+1,\alpha|\Gamma_R(\sigma)|n+2,\gamma,\beta\rangle_i \\
&\quad \underbrace{\times \, {}_i\langle n+2,\gamma,\beta|\Gamma_R((n,n+2))|n+2,n+1,\alpha\rangle_j}_{\bowtie}
\end{aligned}$$

Aside 2

In this aside we will go through a heuristic argument that will help us factor out the action of $(n,n+2)$ in our expression.

$$\begin{aligned}
\bowtie &= {}_i\langle n+2,\gamma,\beta|\Gamma_R((n,n+2))|n+2,n+1,\alpha\rangle_j \\
&= {}_i\langle n+2,\gamma,\beta|\Gamma_R((n,n+1)(n+1,n+2)(n,n+1))|n+2,n+1,\alpha\rangle_j.
\end{aligned}$$

From inspection we can expect γ and β to be either $n+1$ or α only. By considering the action of $(n,n+2)$ on the ket only, it is clear that the operator permutes the labels n, $n+1$ and $n+2$. Therefore, since we are using an orthonormal basis, the contributing terms in the series will be the ones in the carrier space of R', in particular $R'''_{\gamma,\beta}$ must have the same shape as $R'''_{n+1,\alpha}$. *For a more rigorous (and compelling) proof of this* *assertion please see [13]*

$$\begin{aligned}\bowtie &= {}_i\langle n+2, n+1, \alpha|\Gamma_R((n,n+1)(n+1,n+2)(n,n+1))|n+2,n+1,\alpha\rangle_j \delta_{\gamma n+1}\delta_{\beta\alpha} \\ &+ {}_i\langle n+2, \alpha, n+1|\Gamma_R((n,n+1)(n+1,n+2)(n,n+1))|n+2,n+1,\alpha\rangle_j \delta_{\beta n+1}\delta_{\gamma\alpha} \\ &= \tau_\alpha \delta_{ij}\delta_{\gamma n+1}\delta_{\beta\alpha} + \lambda_\alpha \delta_{ij}\delta_{\beta n+1}\delta_{\gamma\alpha}\end{aligned}$$

The factors τ_α and λ_α can be readily calculated using strand diagrams of [13]. Strand diagram analysis yields

$$\tau_\alpha = \frac{1}{c_1 - c_\alpha}\left(1 + \frac{1}{(c_2 - c_\alpha)(c_1 - c_2)}\right)$$
$$\lambda_\alpha = \frac{1}{c_1 - c_\alpha}\frac{1}{c_1 - c_2}\sqrt{1 - \frac{1}{(c_2 - c_3)^2}}$$

where c_1, c_2 and c_α are the weights of the boxes with label 1, 2 and α in Young diagram R respectively. This concludes the aside.

Therefore

$$Tr_{R''}\left(\Gamma_R(\sigma(n,n+2))\right) = \sum_\alpha \tau_\alpha Tr_{R'''_\alpha}\left(\Gamma_{R'}(\sigma)\right) + \sum_\alpha \lambda_\alpha Tr_{R'''_\alpha, T'''_\alpha}\left(\Gamma_{R'}(\sigma)\right).$$

The n middle terms reduce to

$$\frac{1}{2}\sum_\alpha \tau_\alpha \chi^{(2)}_{R',R'''_\alpha}(Z,Y,ZY) + \frac{1}{2}\sum_\alpha \lambda_\alpha \chi^{(2)}_{R'\to R'''_\alpha,T'''_\alpha}(Z,Y,ZY)$$

Finally putting all these results together we obtain the coset expansion identities:

Right Coset expansion

$$\begin{aligned}\chi^{(2)}_{R,R''}(Z,Y,Y) &= \frac{1}{2}tr(Y)\chi^{(1)}_{R',R''}(Z,Y) + \frac{1}{2(c_1-c_2)}\chi^{(1)}_{R',R''}(Z,YY) \\ &+ \frac{1}{2}\sum_\alpha \tau_\alpha \chi_{R',R'''_\alpha}(Z,Y,ZY) + \frac{1}{2}\sum_\alpha \lambda_\alpha \chi^{(2)}_{R'\to R'''_\alpha,T'''_\alpha}(Z,Y,ZY)\end{aligned}$$

Left Coset expansion

$$\begin{aligned}\chi^{(2)}_{R,R''}(Z,Y,Y) &= \frac{1}{2}tr(Y)\chi^{(1)}_{R',R''}(Z,Y) + \frac{1}{2(c_1-c_2)}\chi^{(1)}_{R',R''}(Z,YY) \\ &+ \frac{1}{2}\sum_\alpha \tau_\alpha \chi^{(2)}_{R',R'''_\alpha}(Z,Y,YZ) + \frac{1}{2}\sum_\alpha \lambda_\alpha \chi^{(2)}_{R'\to T'''_\alpha,R'''_\alpha}(Z,Y,YZ)\end{aligned}$$

Another restricted Schur worth considering is

$$\begin{aligned}\chi^{(2)}_{R\to R'',S''}(Z^{\otimes n},Y^{\otimes 2}) &= \frac{1}{n!2!}\sum_{\phi\in S_{n+1}}\Big[Tr_{R'',S''}\left(\Gamma_{R'}(\phi)\right)Z^{i_1}_{i_{\phi(1)}}\dots Z^{i_n}_{i_{\phi(n)}}Y^{i_{n+1}}_{i_{\phi(n+1)}}tr(Y)\\ &+\sum_{a<n+1}Tr_{R'',S''}\left(\Gamma_R(\phi(a,n+2))\right)Z^{i_1}_{i_{\phi(1)}}\cdots(ZY)^{i_a}_{i_{\phi(a)}}\cdots Z^{i_n}_{i_{\phi(n)}}Y^{i_{n+1}}_{i_{\phi(n+1)}}\\ &+Tr_{R'',S''}\left(\Gamma_R(\phi(n+1,n+2))\right)Z^{i_1}_{i_{\phi(1)}}\cdots Z^{i_n}_{i_{\phi(n)}}(YY)^{i_{n+1}}_{i_{\phi(n+1)}}\Big].\end{aligned}$$

The first term is equal to 0 as argued before. The last term becomes $\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{R',R''}$. The middle terms can be evaluated in a similar fashion as highlighted above, the $(n,n+2)$ contribution is

$$\begin{aligned}Tr_{R'',S''}(\Gamma_R(\sigma(n,n+2))) &= \sum_\alpha \zeta_\alpha\ {}_j\langle n+2,n+1,\alpha|\Gamma_{R'}(\sigma)|n+2,n+1,\alpha\rangle_j\\ &+\sum_\alpha \epsilon_\alpha\ {}_j\langle n+2,n+1,\alpha|\Gamma_{R'}(\sigma)|n+2,\alpha,n+1\rangle_j\end{aligned}$$

Once again strand diagrams help us to compute the coefficients

$$\begin{aligned}\zeta_\alpha &= \frac{1}{c_1-c_\alpha}\frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\\ \epsilon_\alpha &= \frac{1}{c_2-c_\alpha}\sqrt{1-\frac{1}{(c_2-c_1)^2}}\sqrt{1-\frac{1}{(c_2-c_1)^2}}.\end{aligned}$$

The final results:

Right Coset expansion

$$\begin{aligned}\chi^{(2)}_{R\to R'',S''}(Z,Y,Y) &= \frac{1}{2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{R',R''}(Z,YY)+\frac{1}{2}\sum_\alpha\zeta_\alpha\chi^{(2)}_{R',R'''_\alpha}(Z,Y,ZY)\\ &+\frac{1}{2}\sum_\alpha\epsilon_\alpha\chi^{(2)}_{R'\to R'''_\alpha,T'''_\alpha}(Z,Y,ZY)\end{aligned}$$

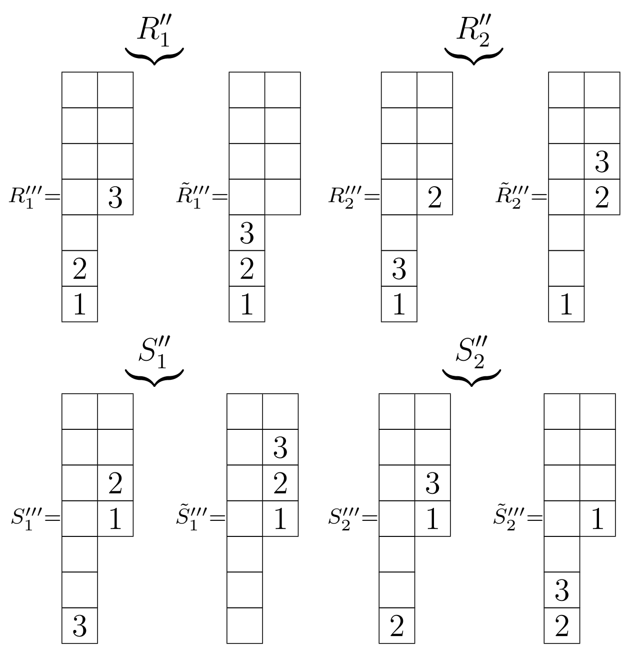
Left Coset expansion

$$\begin{aligned}\chi^{(2)}_{R\to R'',S''}(Z,Y,Y) &= \frac{1}{2}\sqrt{1-\frac{1}{(c_1-c_2)^2}}\chi^{(1)}_{S',S''}(Z,YY)+\frac{1}{2}\sum_\alpha\zeta_\alpha\chi^{(2)}_{S',S'''_\alpha}(Z,Y,YZ)\\ &+\frac{1}{2}\sum_\alpha\epsilon_\alpha\chi^{(2)}_{R'\to W'''_\alpha,S'''_\alpha}(Z,Y,YZ)\end{aligned}$$

A.2 Hop off identities

The $S_n \times S_1 \times S_1$ basis represent two giants that have two distinguishable impurities attached. When we coset expand these states we learn that we can rewrite these states as a "string joining" term $\chi^{(1)}(Z, YY)$, a "closed string emission"(radiation) term $tr(X)\chi^{(1)}(Z, Y)$ and a linear sum of *hop off interaction* terms $\chi^{(2)}(Z, Y, ZY)$ where a Z field has hopped off the membrane and onto one of the impurities. The hop off identities invert the coset expansion to rewrite a single hop off interaction as a linear combination of two branes with two impurities attached, a "string joining" term and a "closed string emission" term[1].

Before we proceed, we introduce some new notation. Building on to the notation introduced in section 4.1, eqn(4.1.1,4.1.2):



[1]The inverted commas are to emphasize that these are merely labels. The field Y is not, by itself, a string word (see Chapter 3.3).

Please note: *the Y fields in our problem are indistinguishable, however in what follows we consider Y fields that are distinguishable. The difference is an overall factor of* $\frac{1}{2}$.

In what follows we do not write the argument of the Schurs but they can be easily kept track of: for the Schurs with two words attached the argument is $(Z, Y^{(2)}, Y^{(1)})$, the Schurs with pre-factor of $tr(X)$ have the argument (Z, X) and the Schurs with one string word attached have the argument (Z, YY).

We define the vectors $\bar{y}_{R'}|_2|_1$ and $\bar{y}_{S'}|_2|_1$:

$$\bar{y}_{R'}|_2|_1 = \frac{1}{2}\begin{bmatrix} \chi^{(2)}_{R(b_0+1,b_1-2),R''_1} - tr(Y^{(2)})\chi^{(1)}_{R'(b_0+1,b_1-2),R''_1} + \chi^{(1)}_{R'(b_0+1,b_1-2),R''_1} \\ \chi^{(2)}_{R(b_0+1,b_1-2),\tilde{S}''_2} - tr(Y^{(2)})\chi^{(1)}_{R'(b_0+1,b_1-2),R''_1} - \frac{1}{b_1-1}\chi^{(1)}_{R'(b_0+1,b_1-2),R''_1} \\ \chi^{(2)}_{R(b_0+1,b_1-2)\to\tilde{S}''_2,\tilde{R}'''_2} - (b_1)_{-1}\chi^{(1)}_{R'(b_0+1,b_1-2),R''_1} \\ \chi^{(2)}_{R(b_0,b_1),R''_2} - tr(Y^{(2)})\chi^{(1)}_{R'(b_0,b_1),R''_2} + \frac{1}{b_1+1}\chi^{(1)}_{R'(b_0,b_1),R''_2} \\ \chi^{(2)}_{R(b_0+1,b_1-2),\tilde{S}''_1} - tr(Y^{(2)})\chi^{(1)}_{R'(b_0+1,b_1-2),R''_2} + \chi^{(1)}_{R'(b_0+1,b_1-2),R''_2} \\ \chi^{(2)}_{R(b_0,b_1)\to R''_2,S''_2} - (b_1)_1\chi^{(1)}_{R'(b_0,b_1),R''_2} \end{bmatrix}$$

$$\bar{y}_{S'}|_2|_1 = \frac{1}{2}\begin{bmatrix} \chi^{(2)}_{R(b_0-1,b_1+2),S''_1} - tr(Y^{(2)})\chi^{(1)}_{S'(b_0-1,b_1+2),S''_1} + \chi^{(1)}_{S'(b_0-1,b_1+2),S''_1} \\ \chi^{(2)}_{R(b_0-1,b_1+2),\tilde{R}''_2} - tr(Y^{(2)})\chi^{(1)}_{S'(b_0-1,b_1+2),S''_1} + \frac{1}{b_1+3}\chi^{(1)}_{S'(b_0-1,b_1+2),S''_1} \\ \chi^{(2)}_{R(b_0-1,b_1+2)\to\tilde{R}''_2,\tilde{S}'''_2} - (b_1)_3\chi^{(1)}_{S'(b_0-1,b_1+2),S''_1} \\ \chi^{(2)}_{R(b_0,b_1),S''_2} - tr(Y^{(2)})\chi^{(1)}_{S'(b_0,b_1),S''_2} - \frac{1}{b_1+1}\chi^{(1)}_{S'(b_0,b_1),S''_2} \\ \chi^{(2)}_{R(b_0-1,b_1+2),\tilde{R}''_1} - tr(Y^{(2)})\chi^{(1)}_{S'(b_0-1,b_1+2),S''_2} + \chi^{(1)}_{S'(b_0-1,b_1+2),S''_2} \\ \chi^{(2)}_{R(b_0,b_1)\to S''_2,R''_2} - (b_1)_1\chi^{(1)}_{S'(b_0,b_1),S''_1} \end{bmatrix}$$

where

$$(b_1)_n = \frac{\sqrt{b_1+n-1}\sqrt{b_1+n+1}}{b_1+n}.$$

Define the vectors $\bar{x}_{R'}(ZY)$, $\bar{x}_{R'}(YZ)$, $\bar{x}_{S'}(ZY)$ and $\bar{x}_{S'}(YZ)$:

$$\bar{x}_{R'}(ZY)|_1|_2 = \begin{bmatrix} \chi^{(2)}_{R'(b_0,b_1-1),R_1'''} \\ \chi^{(2)}_{R'(b_0+1,b_1-2),\tilde{R}_1'''} \\ \chi^{(2)}_{R'(b_0,b_1-1)\to R_1''',R_2'''} \\ \chi^{(2)}_{R'(b_0,b_1-1),R_2'''} \\ \chi^{(2)}_{R'(b_0-1,b_1+1),\tilde{R}_2'''} \\ \chi^{(2)}_{R'(b_0,b_1-1)\to R_2''',R_1'''} \end{bmatrix} (Z^{\otimes n-1}, Y^{(1)}, ZY^{(2)})$$

$$\bar{x}_{R'}(YZ)|_1|_2 = \begin{bmatrix} \chi^{(2)}_{R'(b_0,b_1-1),R_1'''} \\ \chi^{(2)}_{R'(b_0+1,b_1-2),\tilde{R}_1'''} \\ \chi^{(2)}_{R'(b_0,b_1-1)\to R_2''',R_1'''} \\ \chi^{(2)}_{R'(b_0,b_1-1),R_2'''} \\ \chi^{(2)}_{R'(b_0-1,b_1+1),\tilde{R}_2'''} \\ \chi^{(2)}_{R'(b_0,b_1-1)\to R_1''',R_2'''} \end{bmatrix} (Z^{\otimes n-1}, Y^{(1)}, Y^{(2)}Z)$$

$$\bar{x}_{S'}(ZY)|_1|_2 = \begin{bmatrix} \chi^{(2)}_{S'(b_0-1,b_1+1),S_1'''} \\ \chi^{(2)}_{S'(b_0-2,b_1+3),\tilde{S}_1'''} \\ \chi^{(2)}_{S'(b_0-1,b_1+1)\to S_1''',S_2'''} \\ \chi^{(2)}_{S'(b_0-1,b_1+1),S_2'''} \\ \chi^{(2)}_{S'(b_0,b_1-1),\tilde{S}_2'''} \\ \chi^{(2)}_{S'(b_0-1,b_1+1)\to S_2''',S_1'''} \end{bmatrix} (Z^{\otimes n-1}, Y^{(1)}, ZY^{(2)})$$

$$\bar{x}_{S'}(YZ)|_1|_2 = \begin{bmatrix} \chi^{(2)}_{S'(b_0-1,b_1+1),S_1'''} \\ \chi^{(2)}_{S'(b_0-2,b_1+3),\tilde{S}_1'''} \\ \chi^{(2)}_{S'(b_0-1,b_1+1)\to S_2''',S_1'''} \\ \chi^{(2)}_{S'(b_0-1,b_1+1),S_2'''} \\ \chi^{(2)}_{S'(b_0,b_1-1),\tilde{S}_2'''} \\ \chi^{(2)}_{S'(b_0-1,b_1+1)\to S_1''',S_2'''} \end{bmatrix} (Z^{\otimes n-1}, Y^{(1)}, Y^{(2)}Z)$$

Isolating the hop off interactions derived from the coset expansion we get

$$2\bar{y}_{R'}|_2|_1 = A_{R'}\bar{x}_{R'}(ZY)|_1|_2$$

$$2\bar{y}_{R'}|_2|_1 = A_{R'}\bar{x}_{R'}(YZ)|_1|_2$$

$$2\bar{y}_{S'}|_2|_1 = A_{S'}\bar{x}_{S'}(ZY)|_1|_2$$

$$2\bar{y}_{S'}|_2|_1 = A_{S'}\bar{x}_{S'}(YZ)|_1|_2$$

where $|_m|_n$ is an instruction about the order of subduction. It is read from left to right, say for $|_1|_2$, it instructs to remove the box with string label (1) first and then the box with string label (2) second. These relations show that $\bar{x}_{R'}(ZY) = \bar{x}_{R'}(YZ)$ and $\bar{x}_{S'}(ZY) = \bar{x}_{S'}(YZ)$.

The matrices $A_{R'}$ and $A_{S'}$ are as follows

$$A_{R'} = \begin{bmatrix} -\frac{1}{b_1} & -1 & \frac{(b_1)_0}{b_1+1} & 0 & 0 & 0 \\ -1+\frac{1}{b_1(b_1-1)} & \frac{1}{b_1-1} & \frac{-(b_1)_0}{b_1-1} & 0 & 0 & 0 \\ \frac{(b_1)_0}{b_1} & \frac{-(b_1)_{-1}}{b_1-2} & -(b_1)_{-1}(b_1)_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+\frac{1}{b_1(b_1+1)} & \frac{-1}{b_1+1} & \frac{(b_1)_0}{b_1+1} \\ 0 & 0 & 0 & \frac{1}{b_1} & -1 & \frac{-(b_1)_0}{b_1-1} \\ 0 & 0 & 0 & \frac{-(b_1)_1}{b_1+1} & \frac{(b_0)_1}{b_1+2} & -(b_1)_0(b_1)_1 \end{bmatrix}$$

$$A_{S'} = \begin{bmatrix} \frac{1}{b_1+2} & -1 & \frac{-(b_1)_2}{b_1+1} & 0 & 0 & 0 \\ -1+\frac{1}{(b_1+2)(b_1+3)} & \frac{-1}{b_1+3} & \frac{(b_1)_2}{b_1+3} & 0 & 0 & 0 \\ \frac{-(b_1)_3}{(b_1+2)} & \frac{(b_1)_3}{b_1+4} & -(b_1)_3(b_1)_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+\frac{1}{(b_1+2)(b_1+1)} & \frac{1}{b_1+1} & \frac{-(b_1)_2}{b_1+1} \\ 0 & 0 & 0 & \frac{-1}{b_1+2} & -1 & \frac{(b_1)_2}{b_1+3} \\ 0 & 0 & 0 & \frac{(b_1)_1}{(b_1+2)} & \frac{-(b_1)_1}{b_1} & -(b_1)_1(b_1)_2 \end{bmatrix}$$

These matrices are a direct sum of two 3×3 matrices that can be easily inverted. The hop off identities read

$$\bar{x}_{R'}(YZ)|_1|_2 = 2(A_{R'})^{-1}\bar{y}_{R'}|_2|_1$$
$$\bar{x}_{S'}(ZY)|_1|_2 = 2(A_{S'})^{-1}\bar{y}_{S'}|_2|_1$$

For our main task we would like to express our results in the $S_n \times S_2$ basis, we know how to change basis between our desired basis and the $S_n \times S_1 \times S_1$ basis, via eqns(4.1.3-4.1.6) and removing twisted states. As it stands the hop off identities have "string joining", $\chi^{(1)}_{R',R''}(Z,YY)$, and "closed string emission", $tr(Y)\chi^{(1)}_{R',R''}(Z,Y)$ terms. We need to rewrite these terms in the $S_n \times S_2$ basis. To do this we will use dual characters, developed in [38].

$$\begin{aligned}
\chi^{(1)}_{R',R''}(Z,YY) &= \frac{1}{n!}\sum_{\sigma\in S_{n+1}} \chi_{R',R''}(\sigma) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}(YY)^{i_{n+1}}_{i_{\sigma(n+1)}} \\
&= \frac{1}{n!}\sum_{\sigma\in S_{n+1}} \chi_{R',R''}(\sigma) Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{n+2}} Y^{i_{n+2}}_{i_{\sigma(n+1)}} \\
&= \frac{1}{n!}\sum_{\sigma\in S_{n+1}} \chi_{R',R''}(\sigma) \sum_{R,(r_1,r_2)} \chi^{R,(r_1,r_2)}(\sigma(n+1,n+2))\chi_{R,(r_1,r_2)}(Z,Y) \\
&= \sum_{R,(r_1,r_2)} \alpha_{R,(r_1,r_2)}\chi_{R,(r_1,r_2)}(Z,Y)
\end{aligned}$$

Where $\chi^{R,(r_1,r_2)}(\cdot)$ is the dual character and it is related to the usual character as follows [38]

$$\chi^{R,(r_1,r_2)}(\sigma(n+1,n+2)) = \frac{d_R n! 2!}{d_{r_1} d_{r_2}(n+2)!}\chi_{R,(r_1,r_2)}(\sigma(n+1,n+2))$$

For a Young diagram T labeling an irrep. of S_n, the dimension of the representation is denoted by d_T. An interesting feature is that dual characters give us an expression that is already in the $S_n \times S_2$ basis. The coefficients $\alpha_{R,(r_1,r_2)}$ are [38]

$$\begin{aligned}
\alpha_{R,(r_1,r_2)} &= \frac{1}{n!}\sum_{\sigma\in S_{n+1}} \chi_{R',R''}(\sigma)\chi^{R,(r_1,r_2)}(\sigma(n+1,n+2)) \\
&= \frac{d_R 2!}{d_{r_1}d_{r_2}(n+2)!}\sum_{\sigma\in S_{n+1}} \chi_{R',R''}(\sigma)\chi_{R,(r_1,r_2)}(\sigma(n+1,n+2)) \\
&= \frac{d_R 2!}{d_{r_1}d_{r_2}(n+2)!} Tr_{R,(r_1,r_2)}\left(\sum_{\sigma\in S_{n+1}} \chi_{R',R''(\sigma)}\sigma(n+1,n+2)\right).
\end{aligned}$$

The sum in the big brackets is the projection operator we will encounter in Appendix C.1

$$\sum_{\sigma\in S_{n+1}} \chi_{R',R''(\sigma)}\sigma = \frac{(n+1)!}{d_{R'}}P_{R',R''}.$$

For our problem $r_2 = \yng(2), \yng(1,1)$ which implies $d_{r_2} = 1$ and $(n+1,n+2)|R,(r_1,r_2)\rangle = (-1)^{r_2}|R,(r_1,r_2)\rangle$, thus

$$\alpha_{R,(r_1,r_2)} = (-1)^{r_2}\frac{2\cdot h_{R'}}{h_R d_{r_1}}\sum_{i,j}\left(\,_i\langle R,(r_1,r_2)|R',R''\rangle_j\right)^2$$

where $(-1)^{r_2} = 1$ if $r_2 = \yng(2)$ and $(-1)^{r_2} = -1$ if $r_2 = \yng(1,1)$.

The "closed string emission" term can be rewritten by following the same prescription

$$\chi_{R',R''}(Z,Y)Tr(Y) = \sum_{R,(r_1,r_2)} \beta_{R,(r_1,r_2)}\chi_{R,(r_1,r_2)}(Z,Y)$$

where $\beta_{R,(r_1,r_2)} = \frac{2\cdot h_{R'}}{h_R d_{r_1}}\sum_{i,j}\left(\,_i\langle R,(r_1,r_2)|R',R''\rangle_j\right)^2$.

Before we go onto the results, we need to introduce new operators

$$\Psi_f(b_0,b_1) = \chi_{\yng(3,2,2,1,1,1);\yng(2,2,2,1,1),\yng(2)} \qquad \Psi_g(b_0,b_1) = \chi_{\yng(3,2,2,1,1,1);\yng(2,2,2,1,1),\yng(1,1)}$$

$$\Psi_h(b_0,b_1) = \chi_{\yng(3,2,2,1,1,1);\yng(2,2,1,1,1,1),\yng(2)} \qquad \Psi_i(b_0,b_1) = \chi_{\yng(3,2,2,1,1,1);\yng(2,2,1,1,1,1),\yng(1,1)}$$

Together with the notation of eqns(4.2.1-4.2.4), the results:

"String joining" terms (please note the argument of all the right hand side terms, in what follows, is (b_0, b_1)).

$$\begin{aligned}\chi^{(1)}_{R'_2(b_0,b_1),R''_2}(Z,YY) &= \frac{1}{b_0+b_1+2}\Psi_d - \frac{b_1+2}{b_1(b_0+b_1+2)}\Psi_e - \frac{2(b_1-1)}{b_1(b_0+2)}\Psi_b \\ &+ \frac{b_1+b_0+1}{b_0+b_1+2}\Psi_h - \frac{b_0(b_1+b_0+1)}{(b_0+2)(b_0+b_1+2)}\Psi_i\end{aligned}$$

$$\begin{aligned}\chi^{(1)}_{S'(b_0,b_1),S''_2}(Z,YY) &= \frac{1}{b_0+1}\Psi_d - \frac{b_1}{(b_1+2)(b_0+1)}\Psi_e - \frac{2(b_1+3)}{(b_1+2)(b_0+b_1+2)}\Psi_a \\ &+ \frac{b_0}{b_1+1}\Psi_f - \frac{b_0(b_1+b_0+1)}{(b_0+1)(b_0+b_1+3)}\Psi_g\end{aligned}$$

"Closed string emission" terms (please note the argument of all the right hand side terms, in what follows, is (b_0, b_1)).

$$\begin{aligned}tr(Y)\chi^{(1)}_{R'_2(b_0,b_1),R''_2}(Z,Y) &= \frac{1}{b_0+b_1+2}\Psi_d + \frac{b_1+2}{b_1(b_0+b_1+2)}\Psi_e + \frac{2(b_1-1)}{b_1(b_0+2)}\Psi_b \\ &+ \frac{b_1+b_0+1}{b_0+b_1+2}\Psi_h + \frac{b_0(b_1+b_0+1)}{(b_0+2)(b_0+b_1+2)}\Psi_i\end{aligned}$$

$$\begin{aligned}tr(Y)\chi^{(1)}_{S'_2(b_0,b_1),S''_2}(Z,YY) &= \frac{1}{b_0+1}\Psi_d + \frac{b_1}{(b_1+2)(b_0+1)}\Psi_e + \frac{2(b_1+3)}{(b_1+2)(b_0+b_1+2)}\Psi_a \\ &+ \frac{b_0}{b_1+1}\Psi_f + \frac{b_0(b_1+b_0+1)}{(b_0+1)(b_0+b_1+3)}\Psi_g\end{aligned}$$

Appendix B

Removing twisted states

Consider

$$\chi^{(2)}_{R,S''_2}(Z^{\otimes n},Y^{\otimes 2}) = \frac{1}{n!2!}\sum_{\sigma\in S_{n+2}}\chi_{R,S''_2}(\sigma)Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}Y^{i_{n+2}}_{i_{\sigma(n+2)}}.$$

The Y's are indistinguishable therefore the Schur will be invariant if we swap them.

With the aid of the subgroup swap rule (see [7] Appendix D for details):

$$\begin{aligned}\chi^{(2)}_{R,S''_2}(Z^{\otimes n},Y^{\otimes 2})|_1|_2 \;&= \frac{1}{(b_1+1)^2}\chi^{(2)}_{R,S''_2}(Z^{\otimes n},Y^{\otimes 2})|_2|_1 + (1-\frac{1}{(b_1+1)^2})\chi^{(2)}_{R,R''_2}(Z^{\otimes n},Y^{\otimes 2})|_2|_1\\ &+\frac{1}{b_1+1}\sqrt{1-\frac{1}{(b_1+1)^2}}\left(\chi^{(2)}_{R\to S''_2,R''_2}+\chi^{(2)}_{R\to R''_2S''_2}\right)(Z^{\otimes n},Y^{\otimes 2})|_2|_1\end{aligned}$$

The above relation must also hold for the characters i.e.

$$\chi_{R,S''_2} = \frac{1}{(b_1+1)^2}\chi_{R,S''_2} + (1-\frac{1}{(b_1+1)^2})\chi_{R,R''_2} + \frac{1}{b_1+1}\sqrt{1-\frac{1}{(b_1+1)^2}}\left(\chi_{R\to S''_2,R''_2}+\chi_{R\to R''_2S''_2}\right)$$

Make the twisted characters the subject of the equation

$$\chi_{R\to S''_2,R''_2}+\chi_{R\to R''_2,S''_1} = (b_1+1)\sqrt{1-\frac{1}{(b_1+1)^2}}\left(\chi_{R,S''_2}-\chi_{R,R''_2}\right) \qquad \text{(i)}$$

Now consider

$$\chi^{(2)}_{R\to S''_2,R''_2}(Z^{\otimes n},Y^{\otimes 2}) = \frac{1}{n!2!}\sum_{\sigma\in S_{n+2}}\chi_{R\to S''_2,R''_2}(\sigma)Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}Y^{i_{n+2}}_{i_{\sigma(n+2)}},$$

follow the same procedure, subgroup swap rule:

$$\begin{aligned}\chi_{R\to S_2'',R_2''} &= \tfrac{-1}{(b_1+1)^2}\chi_{R\to S_2'',R_2''} + (1-\tfrac{1}{(b_1+1)^2})\chi_{R\to R_2'',S_2''} \\ &\quad + \tfrac{1}{b_1+1}\sqrt{1-\tfrac{1}{(b_1+1)^2}}\left(\chi_{R,S_2''}-\chi_{R,R_2''}\right)\end{aligned}$$

rearrange terms and we have

$$\begin{aligned}&\left(\tfrac{(b_1+1)^2+1}{(b_1+1)^2}\right)\chi_{R\to S_2'',R_2''} - \left(\tfrac{(b_1+1)^2-1}{(b_1+1)^2}\right)\chi_{R\to R_2'',S_2''} \\ &= \tfrac{1}{b_1+1}\sqrt{1-\tfrac{1}{(b_1+1)^2}}\left(\chi_{R,S_2''}-\chi_{R,R_2''}\right)\end{aligned} \tag{ii}$$

Solved simultaneously, equations (i) and (ii) yield the solutions

$$\chi_{R\to S_2'',R_2''} = \chi_{R\to R_2'',S_1''} = \frac{(b_1+1)}{2}\sqrt{1-\frac{1}{(b_1+1)^2}}\left(\chi_{R,S_2''}-\chi_{R,R_2''}\right).$$

Appendix C

Correlation functions

We have proposed previously that Schur polynomials are operators dual to giant gravitons. In trying to work out the dynamics of the Schur/giant duality, we will need to compute two point functions of the Schur polynomials. Working out two point functions will be the focus in this section.

There are six Higgs fields in our $\mathcal{N} = 4$ super Yang-Mills theory ϕ^i, $i = 1, \ldots, 6$, these can be grouped into the following complex combinations

$$Z = \phi^1 + i\phi^2, \quad Y = \phi^3 + i\phi^4, \quad X = \phi^5 + i\phi^6$$

- The giant gravitons are built out of the Z fields
- Our impurities are build out of Y fields.

C.1 Schur polynomial two point functions

Like in QFT the two point function involves contracting fields, we will start with two point functions of ordinary Schur polynomials and then later discuss restricted Schurs.

The conventions for contracting the fields are

$$\langle Z^\dagger_{ij}(t) Z_{kl}(t)\rangle = \frac{4\pi\lambda}{N}\delta_{il}\delta_{jk}$$
$$\langle Y^\dagger_{ij}(t) Y_{kl}(t)\rangle = \frac{4\pi\lambda}{N}\delta_{il}\delta_{jk}$$
$$\langle X^\dagger_{ij}(t) X_{kl}(t)\rangle = \frac{4\pi\lambda}{N}\delta_{il}\delta_{jk}$$

All indicies range from $1, \ldots, N$

For convenience we will often omit the $\frac{4\pi\lambda}{N}$ factor in examples but we will include it for important (general) results. Here are a few basic examples to illustrate the two point function calculations. Consider

$$\begin{aligned}
\chi_{\square\square}(Z) &= \frac{1}{2!}\sum_{\sigma\in S_2} Tr_{\square\square}(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \\
&= \frac{1}{2}\left(tr(Z)^2 + tr(Z^2)\right) \\
\chi_{\substack{\square\\\square}}(Z) &= \frac{1}{2!}\sum_{\sigma\in S_2} Tr_{\substack{\square\\\square}}(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \\
&= \frac{1}{2}\left(tr(Z)^2 - tr(Z^2)\right)
\end{aligned}$$

Example 5.

$$\begin{aligned}
\langle \chi_{\square\square}(Z)\chi^\dagger_{\substack{\square\\\square}}(Z)\rangle &= \frac{1}{4}\langle\left(tr(Z)^2 + tr(Z^2)\right)\left(tr(Z)^2 - tr(Z^2)\right)\rangle \\
&= \langle\left[tr(Z)^2tr(Z^\dagger)^2 - tr(Z)^2tr(Z^\dagger) + tr(Z^2)tr(Z^\dagger)^2 - tr(Z^2)tr((Z^\dagger)^2)\right]\rangle \\
&= \frac{1}{4}(2N^2 - 2N + 2N - 2N^2) \\
&= 0
\end{aligned}$$

Example 6.

$$\begin{aligned}
\langle \chi_{\square\square}(Z)\chi^\dagger_{\square\square}(Z)\rangle &= \frac{1}{4}(2N^2 + 2N + 2N + 2N^2) \\
&= N(N+1)
\end{aligned}$$

Notice that $N(N+1)$ is the product of the weights of Young diagram $\square\square$.

Example 7.

$$\begin{aligned}\langle \chi_{\yng(1,1)}(Z)\chi^{\dagger}_{\yng(1,1)}(Z)\rangle &= \frac{1}{4}(2N^2 - 2N - 2N + 2N^2) \\ &= N(N-1)\end{aligned}$$

Notice again that $N(N-1)$ *is the product of the weights of Young diagram* $\yng(1,1)$*.*

The examples allude to the assertion that Schur polynomials of different shapes are orthogonal and that the two point function of Schurs with the the same shapes is (up to a factor) equal to the product of the weights. The assertion is indeed true, we prove it next. Consider a general Schur polynomial $\chi_R(Z)$ with R a young diagram consisting of n boxes.

$$\begin{aligned}\chi_R(Z) &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)tr(\sigma Z^{\otimes n}) \\ &= tr\left(\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma Z^{\otimes n}\right) \\ &= \frac{1}{d_R}tr\left(P_R Z^{\otimes n}\right)\end{aligned}$$

In the above $(\sigma)^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_n} = \delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\dots\delta^{i_n}_{j_{\sigma(n)}}$ and we define the operator

$$P_R \equiv \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma.$$

Claim 1. P_R *is a projection operator of the subspace of* R *and its in the centre of* S_n*.*

Proof.

$$\begin{aligned}
tr\,(P_R P_S) &= tr\left(d_R d_S(\frac{1}{n!})^2 \sum_{\phi\in S_n}\sum_{\psi\in S_n}\chi_R(\phi)\chi_S(\psi)\phi\psi\right)\\
&= tr\left(d_R d_S(\frac{1}{n!})^2 \sum_{\phi\in S_n}\sum_{\tau\in S_n}\chi_R(\phi)\chi_S(\tau\phi^{-1})\phi\tau\phi^{-1}\right) \qquad ,\tau=\psi\phi\\
&= tr\left(d_R d_S(\frac{1}{n!})^2 \sum_{\tau\in S_n}\tau\Gamma_S(\tau)_{jk}\sum_{\phi\in S_n}\Gamma_R(\phi)_{ii}\Gamma_S(\phi^{-1})_{kj}\right)\\
&= tr\left(d_R d_S(\frac{1}{n!})^2 \sum_{\tau\in S_n}\tau\Gamma_S(\tau)_{jk}\delta_{kj}\frac{n!}{d_R}\delta^{RS}\right)\\
&= tr\left(\frac{d_S}{n!}\sum_{\tau\in S_n}\tau\chi_S(\tau)\delta^{RS}\right)\\
&= tr\left(\delta^{RS}P_S\right)
\end{aligned}$$

In the third last line we made use of the Fundamental Orthogonality Relation of representation theory.

Let $\psi \in S_n$ then

$$\begin{aligned}
(P_R\psi) &= \left(\frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma\psi\right)\\
&= \left(\frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi\tau\psi^{-1})\psi\tau\right) \qquad ,\sigma=\psi\tau\psi^{-1}\\
&= \left(\psi\frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\tau)\tau\right)\\
&= (\psi P_R)
\end{aligned}$$

□

Theorem 1 (Schur polynomial two point function)**.**

$$\langle\chi_R(Z)\chi_S^{\dagger}(Z)\rangle = \delta_{R,S}f_R(\frac{4\pi\lambda}{N})^n$$

Where f_R is the product of the weights of representation R.

Proof. We follow a proof by [39]

$$
\begin{aligned}
\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle &= \langle \frac{1}{d_R} tr\left(P_R Z^{\otimes n}\right)\frac{1}{d_S} tr\left(P_S Z^{\otimes n}\right)\rangle \\
&= \frac{1}{d_S}\frac{1}{d_R}(P_R)^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}(P_S)^{k_1 k_2 \ldots k_n}_{l_1 l_2 \ldots l_n}\langle Z^{j_1}_{i_1} Z^{j_2}_{i_2}\ldots Z^{jn}_{in}(Z^\dagger)^{l_1}_{k_1}(Z^\dagger)^{l_2}_{k_2}\ldots(Z^\dagger)^{l_n}_{k_n}\rangle \\
&= \frac{1}{d_S}\frac{1}{d_R}(P_R)^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}(P_S)^{k_1 k_2 \ldots k_n}_{l_1 l_2 \ldots l_n}\left[(\delta^{j_1}_{k_2}\delta^{j_2}_{k_3}\delta^{j_3}_{k_1}\delta^{j_4}_{k_4}\ldots\delta^{j_n}_{k_n})(\delta^{l_1}_{i_3}\delta^{l_2}_{i_1}\delta^{l_3}_{i_2}\delta^{l_4}_{i_4}\ldots\delta^{l_n}_{i_n})\right. \\
&\qquad \left. +\text{all other permutations}\right](\frac{4\pi\lambda}{N})^n \\
&= \frac{1}{d_S}\frac{1}{d_R}(P_R)^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}(P_S)^{k_1 k_2 \ldots k_n}_{l_1 l_2 \ldots l_n}\left[(123)^{j_1 j_2 \ldots j_n}_{k_1 k_2 \ldots k_n}(132)^{l_1 l_2 \ldots l_n}_{i_1 i_2 \ldots i_n}+\cdots\right](\frac{4\pi\lambda}{N})^n \\
&= \frac{1}{d_S}\frac{1}{d_R}\sum_{\sigma\in S_n} tr\left(P_R\sigma P_S\sigma^{-1}\right)(\frac{4\pi\lambda}{N})^n \\
&= \frac{1}{d_S}\frac{1}{d_R}\sum_{\sigma\in S_n} tr\left(\sigma P_R P_S\sigma^{-1}\right)(\frac{4\pi\lambda}{N})^n \qquad P_R\in Com(S_n) \\
&= \frac{1}{d_S}\frac{1}{d_R}\sum_{\sigma\in S_n} tr\left(P_R P_S\sigma^{-1}\sigma\right)(\frac{4\pi\lambda}{N})^n \qquad \text{, Trace is Cyclic} \\
&= n!\frac{1}{d_S}\frac{1}{d_R} tr\left(P_R P_S\right)(\frac{4\pi\lambda}{N})^n \\
&= n!\frac{1}{(d_R)^2} tr(P_R)\delta^{RS}(\frac{4\pi\lambda}{N})^n
\end{aligned}
$$

Recall from representation theory that $\chi_R(1) = Dim_R = \frac{f_R}{h_R}$, where 1 is the identity in $U(N)$, and also $d_R = \frac{n!}{h_R}$, therefore

$$
\begin{aligned}
\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle &= n!\frac{1}{(d_R)^2} d_R\chi_R(1)\delta^{RS}(\frac{4\pi\lambda}{N})^n \\
&= f_R\delta^{RS}(\frac{4\pi\lambda}{N})^n
\end{aligned}
$$

□

C.2 Restricted Schur polynomial two point functions

In this section we will present the rules for computing the two point functions of the most general restricted Schur polynomials. We start off by noting that there are no contractions between the Higgs fields (Ys and Zs) that make up the open strings and the Higgs fields (Zs) that make up the giant graviton. This is because these contributions are subleading at large N. As a result, the two point function factorizes as [7]

$$\begin{aligned}
\langle\chi^{(k)}_{R,R'}(\chi^{(k')}_{S,S'})^\dagger\rangle &= \frac{1}{(n-k)!(n'-k')!}\sum_{\phi\in S_n}\sum_{\psi\in S_{n'}} Tr_{R,R'}(\phi)Tr_{S,S'}(\psi) \\
&\times\langle Tr(\phi Z^{\otimes(n-k)}W^{(1)}\dots W^{(k)})Tr(\psi(Z^\dagger)^{\otimes(n'-k')}(W^\dagger)^{(1)}\dots(W^\dagger)^{(k')})\rangle \\
&= \frac{1}{(n-k)!(n'-k')!}\sum_{\phi\in S_n}\sum_{\psi\in S_{n'}} Tr_{R,R'}(\phi)Tr_{S,S'}(\psi) \\
&\times\langle Z^{i_1}_{i_{\phi(1)}}Z^{i_2}_{i_{\phi(2)}}\dots Z^{i_{n-k}}_{i_{\phi(n-k)}}(Z^\dagger)^{j_1}_{j_{\psi(1)}}(Z^\dagger)^{j_2}_{j_{\psi(2)}}\dots(Z^\dagger)^{j_{n'-k'}}_{j_{\psi(n'-k')}}\rangle \\
&\times\langle(W^{(1)})^{i_{n-k+1}}_{i_{\phi(n-k+1)}}\dots(W^{(n)})^{i_n}_{i_{\phi(n)}}((W^{(1)})^\dagger)^{j_{n'-k'+1}}_{j_{\psi(n'-k'+1)}}\dots((W^{(k)})^\dagger)^{j_{n'}}_{j_{\psi(n')}}\rangle
\end{aligned}$$

Let h_i be the total number of Higgs fields in an open string word, W^i. The contraction of the string words has a general form

$$\langle(W^{(a)})^i_j(W^{(a)\dagger})^k_l\rangle = (\frac{4\pi\lambda}{N})^{h_a}(F_0\delta^i_l\delta^k_j + F_1\delta^i_j\delta^k_l)$$

The coefficients F_0 and F_1 are, in general, a function of N and angular momentum J. The exact expressions of the coefficient are directly dependent on the composition of the string words, for further analysis of these coefficients please see [7]. We will also assume that

$$\langle(W^{(r)})^i_j((W^{(s)})^\dagger)^k_l\rangle \propto \delta^{rs}$$

in the large N limit. In the large N limit non-planar diagrams contributing to the open string correlator above are suppressed [13]. With out loss of generality and motivated by tractability, from here on we will work with restricted Schurs that only have one string word attached and are labeled by representations of S_n .

$$\begin{aligned}\langle\chi^{(1)}_{R,R'}(\chi^{(1)}_{S,S'})^\dagger\rangle &= \frac{1}{((n-1)!)^2}\sum_{\sigma,\phi}Tr_{R'}(\Gamma_R(\sigma))Tr_{S'}(\Gamma_S(\phi))\\ &\quad\times\langle Tr\left(\sigma Z^{\otimes n-1}W^{(1)}\right)Tr\left(\phi(Z^\dagger)^{\otimes n-1}(W^{(1)})^\dagger\right)\rangle\\ &= \frac{1}{((n-1)!)^2}\sum_{\sigma,\phi}Tr_{R'}(\Gamma_R(\sigma))Tr_{S'}(\Gamma_S(\phi))\\ &\quad\times\langle Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_{n-1}}_{i_{\sigma(n-1)}}(Z^\dagger)^{j_1}_{j_{\phi(1)}}\dots(Z^\dagger)^{j_{n-1}}_{j_{\phi(n-1)}}\rangle(F_1\delta^{i_n}_{i_{\sigma(n)}}\delta^{j_n}_{j_{\phi(n)}}+F_0\delta^{i_n}_{j_{\phi(n)}}\delta^{j_n}_{i_{\sigma(n)}})\end{aligned}$$

At this stage we have two terms, one with coefficient F_0 and one with F_1, which we will deal with separately. We start with the F_1 term:

$$\begin{aligned}&\tfrac{1}{((n-1)!)^2}\textstyle\sum_{\sigma,\phi}Tr_{R'}(\Gamma_R(\sigma))Tr_{S'}(\Gamma_S(\phi))\langle Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_{n-1}}_{i_{\sigma(n-1)}}(Z^\dagger)^{j_1}_{j_{\phi(1)}}\dots(Z^\dagger)^{j_{n-1}}_{j_{\phi(n-1)}}\rangle F_1\delta^{i_n}_{i_{\sigma(n)}}\delta^{j_n}_{j_{\phi(n)}}\\ &\qquad= F_1\langle D_{W^{(1)}}\chi^{(1)}_{R,R'}(D_{W^{(1)}}\chi^{(1)}_{S,S'})^\dagger\rangle\\ &\qquad= F_1(C_{n,R'})^2\langle\chi^{(1)}_{R,R'}(\chi^{(1)}_{S,S'})^\dagger\rangle\end{aligned}$$

where $D_{W^{(1)}}\equiv\frac{d}{dW^{(1)}{}^i_i}$ and $C_{n,R'}$ is the weight of the box with string word $W^{(1)}$. The two point function here can be evaluated using Theorem 1. Next we introduce the notation $|_n$ which is an instruction to sum over all Wick contractions that contract $Z^{i_1}_{i_{\sigma(n)}}$ with $(Z^\dagger)^{j_1}_{j_{\phi(n)}}$. The F_0 term:

$$\begin{aligned}&\tfrac{1}{((n-1)!)^2}\textstyle\sum_{\sigma,\phi}Tr_{R'}(\Gamma_R(\sigma))Tr_{S'}(\Gamma_S(\phi))\langle Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_{n-1}}_{i_{\sigma(n-1)}}(Z^\dagger)^{j_1}_{j_{\phi(1)}}\dots(Z^\dagger)^{j_{n-1}}_{j_{\phi(n-1)}}\rangle F_0\delta^{i_n}_{j_{\phi(n)}}\delta^{j_n}_{i_{\sigma(n)}}\\ &\qquad= F_0(\tfrac{n!}{(n-1)!})^2\langle\chi_{R,R'}(Z)(\chi_{S,S'}(Z))^\dagger\rangle|_n\end{aligned}$$

The factor of $(\frac{n!}{(n-1)!})^2$ compensates for replacing the string word in the restricted Schur with a Z field. The restricted correlator rule states [7]

$$\langle\chi_{R,R'}(Z)(\chi_{S,S'}(Z))^\dagger\rangle|_{n,n-1,\dots,n-k}=\frac{(n-k-1)!d_{R'}f_R}{n!d_R}\delta_{R\to R',S\to S'},$$

for a trace running over an "on the diagonal block" and

$$\langle\chi_{R,R'T'}(Z)(\chi_{S,U'S'}(Z))^\dagger\rangle|_{n,n-1,\dots,n-k}=\frac{(n-k-1)!d_{R'}f_R}{n!d_R}\delta_{R\to(R'T'),S\to(U'S')}\delta_{U'T'}\delta_{R'S'},$$

for a trace running over an “off the diagonal block”. The delta $\delta_{R\to R',S\to S'}$ indicates that all representations appearing in the intermediate steps of restricting R to R' must match all the representations appearing in the intermediate steps of restricting S to S'. The parameter k is the number of string words attached minus one.

Lets work through an example to familiarize ourselves with the restricted Schur correlator rules:

Example 8.

$$
\begin{aligned}
\langle \chi^{(2)}_{\young(\ 1,2)} (\chi^{(2)}_{\young(\ 1,2)})^\dagger \rangle &= \sum_{\phi,\psi} Tr_{R''}(\phi) Tr_{R''}(\psi) \langle Z^{i_1}_{i_{\phi(1)}} (Z^{j_1}_{j_{\psi(1)}})^\dagger \rangle \\
&\quad \times \left(F^1_0 \delta^{i_2}_{j_{\psi(2)}} \delta^{j_2}_{i_{\phi(2)}} + F^1_1 \delta^{i_2}_{i_{\phi(2)}} \delta^{j_2}_{j_{\psi(2)}} \right) \left(F^2_0 \delta^{i_3}_{j_{\psi(3)}} \delta^{j_3}_{i_{\phi(3)}} + F^2_1 \delta^{i_3}_{i_{\phi(3)}} \delta^{j_3}_{j_{\psi(3)}} \right) \\
&= F^1_0 F^2_0 \langle \chi_{\young(\ 1,2)}(Z) \chi^\dagger_{\young(\ 1,2)}(Z) \rangle|_{3,2} \cdot 3! + F^1_1 F^2_1 (N-1)^2 (N+1)^2 \langle \chi_{\square}(Z) \chi^\dagger_{\square}(Z) \rangle \\
&\quad + F^1_1 F^2_0 (N+1)^2 \langle \chi_{\young(\ ,2)}(Z) \chi^\dagger_{\young(\ ,2)}(Z) \rangle|_2 \cdot 2! + F^1_0 F^2_1 (N-1)^2 \langle \chi_{\young(\ 1)}(Z) \chi^\dagger_{\young(\ 1)}(Z) \rangle|_2 \cdot 2! \\
&= F^1_0 F^2_0 \frac{1}{2} (N-1)(N+1)N + F^1_1 F^2_1 (N-1)^2 (N+1)^2 N \\
&\quad + F^1_1 F^2_0 (N+1)^2 (N-1) N + F^1_0 F^2_1 (N-1)^2 (N+1) N
\end{aligned}
$$

C.3 Correlators that appear

Herein we compute the correlators of our bases, for both $S_n \times S_1 \times S_1$ and $S_n \times S_2$ spaces.

C.3.1 The basis of $S_n \times S_1 \times S_1$

We start with :

$\chi^{(2)}_{R(b_0,b_1),R''_1}(Z, Y^{\otimes 2})$

$$\langle \chi^{(2)}_{R,R''_1}(b_0,b_1)(\chi^{(2)}_{R,R''_1}(b_0,b_1))^\dagger \rangle$$

$$= (\tfrac{1}{n!2!})^2 \textstyle\sum_{\sigma,\phi\in S_{n+2}} Tr_{R,R''_1}(\sigma) Tr_{R,R''_1}(\phi) \langle Z^{i_1}_{i_{\sigma(1)}} \dots Z^{i_n}_{i_{\sigma(n)}} (Z^\dagger)^{i_1}_{i_{\phi(1)}} \dots (Z^\dagger)^{i_n}_{i_{\phi(n)}} \rangle$$
$$\times \underbrace{\langle Y^{i_{n+1}}_{i_{\sigma(n+1)}} Y^{i_{n+2}}_{i_{\sigma(n+2)}} (Y^\dagger)^{i_{n+1}}_{i_{\phi(n+1)}} (Y^\dagger)^{i_{n+2}}_{i_{\phi(n+2)}} \rangle}_{\delta^{i_{n+1}}_{j_{\phi(n+1)}} \delta^{j_{n+1}}_{i_{\sigma(n+1)}} \delta^{i_{n+2}}_{j_{\phi(n+2)}} \delta^{j_{n+2}}_{i_{\sigma(n+2)}} + \delta^{i_{n+1}}_{j_{\phi(n+2)}} \delta^{j_{n+1}}_{i_{\sigma(n+2)}} \delta^{i_{n+2}}_{j_{\phi(n+1)}} \delta^{j_{n+2}}_{i_{\sigma(n+1)}}}$$

Following the procedure from restricted Schur polynomial two point function, we see that we have two terms. The first term can be dealt with using restricted correlator rules

$$(\tfrac{1}{n!2!})^2 \textstyle\sum_{\sigma,\phi\in S_{n+2}} Tr_{R,R''_1}(\sigma) Tr_{R,R''_1}(\phi) \langle Z^{i_1}_{i_{\sigma(1)}} \dots Z^{i_n}_{i_{\sigma(n)}} (Z^\dagger)^{i_1}_{i_{\phi(1)}} \dots (Z^\dagger)^{i_n}_{i_{\phi(n)}} \rangle$$
$$\times \delta^{i_{n+1}}_{j_{phi(n+1)}} \delta^{j_{n+1}}_{i_{\sigma(n+1)}} \delta^{i_{n+2}}_{j_{\phi(n+2)}} \delta^{j_{n+2}}_{i_{\sigma(n+2)}}$$
$$= \langle \chi_{R(b_0,b_1),R''}(Z) \chi^\dagger_{R(b_0,b_1),R''}(Z) \rangle |_{n+2,n+1} (\tfrac{(n+2)!}{n!2!})^2$$
$$= \tfrac{1}{4} \tfrac{(b_0+b_1+2)(b_0+b_1+3)(b_1+1)}{b_1+3} f_R,$$

again f_R is the product of the weights of Young diagram R. A closer look at the indices on the deltas of the second term one notices that we could use the restricted correlator rules again if only the lower indices where rearranged as follows $\sigma(n+2) \leftrightarrow \sigma(n+1)$ and similarly $i_{n+2} \leftrightarrow i_{n+1}$. This can be achieved by employing the subgroup swap rule on the Y field in $\chi^{(2)}_{R,R''_1}(Z,Y,Y)$ i.e.

$$(\tfrac{1}{n!2!})^2 \textstyle\sum_{\sigma,\phi\in S_{n+2}} Tr_{R,R''_1}(\sigma) Tr_{R,R''_1}(\phi) \langle Z^{i_1}_{i_{\sigma(1)}} \dots Z^{i_n}_{i_{\sigma(n)}} (Z^\dagger)^{i_1}_{i_{\phi(1)}} \dots (Z^\dagger)^{i_n}_{i_{\phi(n)}} \rangle$$
$$\times \delta^{i_{n+1}}_{j_{\phi(n+2)}} \delta^{j_{n+1}}_{i_{\sigma(n+2)}} \delta^{i_{n+2}}_{j_{\phi(n+1)}} \delta^{j_{n+2}}_{i_{\sigma(n+1)}}$$
$$= (\tfrac{1}{n!2!})^2 \textstyle\sum_{\sigma,\phi\in S_{n+2}} Tr_{R,R''_1}((n+1,n+2)\sigma(n+1,n+2)) Tr_{R,R''_1}(\phi)$$
$$\times \langle Z^{i_1}_{i_{\sigma(1)}} \dots Z^{i_n}_{i_{\sigma(n)}} (Z^\dagger)^{i_1}_{i_{\phi(1)}} \dots (Z^\dagger)^{i_n}_{i_{\phi(n)}} \rangle \delta^{i_{n+1}}_{j_{\phi(n+1)}} \delta^{j_{n+1}}_{i_{\sigma(n+1)}} \delta^{i_{n+2}}_{j_{\phi(n+2)}} \delta^{j_{n+2}}_{i_{\sigma(n+2)}}$$
$$= (-1)^2 \langle \chi_{R(b_0,b_1),R''}(Z) \chi^\dagger_{R(b_0,b_1),R''}(Z) \rangle |_{n+2,n+1} (\tfrac{(n+2)!}{n!2!})^2$$

In the last line the factor of $(-1)^2$ appears as a result of the subgroup swap rule. Since the Y fields in our problem are indistinguishable we will readily use this technique to compute correlation functions.

Our final result

$$\langle\chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle = \frac{1}{2}\frac{(b_0+b_1+2)(b_0+b_1+3)(b_1+1)}{b_1+3}f_R$$

where $f_R = \frac{N!}{(N-b_1-b_0-2)!}\frac{(N+1)!}{(N-b_0+1)!}$ for this particular subduction.

$\underline{\chi^{(2)}_{R(b_0,b_1),S_1''}(Z,Y^{\otimes 2})}$

$$\langle\chi^{(2)}_{R,S_1''}(b_0,b_1)(\chi^{(2)}_{R,S_1''}(b_0,b_1))^\dagger\rangle = \frac{1}{2}\frac{(b_0+1)(b_0+2)(b_1+1)}{b_1-1}f_R$$

where $f_R = \frac{N!}{(N-b_1-b_0)!}\frac{(N+1)!}{(N-b_0-1)!}$ for this particular subduction.

$\underline{\chi^{(2)}_{R(b_0,b_1),R_2''}(Z,Y^{\otimes 2}) \text{ and } \chi^{(2)}_{R(b_0,b_1),S_2''}(Z,Y^{\otimes 2})}$

$$\langle\chi^{(2)}_{R,R_2''}(b_0,b_1)(\chi^{(2)}_{R,R_2''}(b_0,b_1))^\dagger\rangle = \frac{1}{4}(1+\frac{1}{(b_1+1)^2})(b_0+1)(b_0+b_1+2)f_R$$
$$\langle\chi^{(2)}_{R,S_2''}(b_0,b_1)(\chi^{(2)}_{R,S_2''}(b_0,b_1))^\dagger\rangle = \frac{1}{4}(1+\frac{1}{(b_1+1)^2})(b_0+1)(b_0+b_1+2)f_R$$
$$\langle\chi^{(2)}_{R,S_2''}(b_0,b_1)(\chi^{(2)}_{R,R_2''}(b_0,b_1))^\dagger\rangle = \frac{1}{4}(1-\frac{1}{(b_1+1)^2})(b_0+1)(b_0+b_1+2)f_R$$
$$\langle\chi^{(2)}_{R,R_2''}(b_0,b_1)(\chi^{(2)}_{R,S_2''}(b_0,b_1))^\dagger\rangle = \frac{1}{4}(1-\frac{1}{(b_1+1)^2})(b_0+1)(b_0+b_1+2)f_R$$

where $f_R = \frac{N!}{(N-b_1-b_0-1)!}\frac{(N+1)!}{(N-b_0)!}$ for these particular subductions.

C.3.2 The basis of $S_n \times S_2$

We will use the notation introduced in the main matter, eqns(4.2.1-4.2.4) and eqns(4.2.9, 4.2.10). In addition, we will apply the results obtained in the previous subsection.

The technology used here was built in [40]

Note: *All other correlators not shown are zero. This illustrates that indeed our $S_n \times S_2$ basis is orthogonal.*

$\underline{\Psi_a(b_0, b_1)}$

$$\begin{aligned}\langle \Psi_a(b_0,b_1)\Psi_a^\dagger(b_0,b_1)\rangle &= \langle \chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle \\ &= \left(\frac{4\pi\lambda}{N}\right)^{n+2}\frac{1}{2}\frac{(b_0+b_1+2)(b_0+b_1+3)(b_1+1)}{b_1+3}f_R^{(a)}\end{aligned}$$

where $f_R^{(a)} = \frac{N!}{(N-b_1-b_0-2)!}\frac{(N+1)!}{(N-b_0+1)!}$.

$\underline{\Psi_b(b_0, b_1)}$

$$\begin{aligned}\langle \Psi_b(b_0,b_1)\Psi_b^\dagger(b_0,b_1)\rangle &= \langle \chi^{(2)}_{R,S_1''}(b_0,b_1)(\chi^{(2)}_{R,S_1''}(b_0,b_1))^\dagger\rangle \\ &= \left(\frac{4\pi\lambda}{N}\right)^{n+2}\frac{1}{2}\frac{(b_0+2)(b_0+1)(b_1+1)}{b_1-1}f_R^{(b)}\end{aligned}$$

where $f_R^{(b)} = \frac{N!}{(N-b_1-b_0)!}\frac{(N+1)!}{(N-b_0-1)!}$.

$\underline{\Psi_d(b_0, b_1)}$

$$\begin{aligned}&\langle \Psi_d(b_0,b_1)\Psi_d^\dagger(b_0,b_1)\rangle \\ &= (\tfrac{b_1+2}{2})^2\langle \chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle + (\tfrac{b_1}{2})^2\langle \chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle \\ &\quad -\tfrac{b_1(b_1+2)}{4}\left[\langle \chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,S_1''}(b_0,b_1))^\dagger\rangle + \langle \chi^{(2)}_{R,S_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle\right] \\ &= \left(\tfrac{4\pi\lambda}{N}\right)^{n+2}\tfrac{1}{2}(b_0+b_1+2)(b_0+1)f_R^{(d)}\end{aligned}$$

where $f_R^{(d)} = \frac{N!}{(N-b_1-b_0-1)!}\frac{(N+1)!}{(N-b_0)!}$.

$\underline{\Psi_e(b_0, b_1)}$

$$\langle \Psi_e(b_0,b_1)\Psi_e^\dagger(b_0,b_1)\rangle$$
$$= (\tfrac{b_1}{2})^2 \langle \chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle + (\tfrac{b_1+2}{2})^2 \langle \chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle$$
$$-\tfrac{b_1(b_1+2)}{4}\left[\langle \chi^{(2)}_{R,R_1''}(b_0,b_1)(\chi^{(2)}_{R,S_1''}(b_0,b_1))^\dagger\rangle + \langle \chi^{(2)}_{R,S_1''}(b_0,b_1)(\chi^{(2)}_{R,R_1''}(b_0,b_1))^\dagger\rangle\right]$$
$$= \left(\tfrac{4\pi\lambda}{N}\right)^{n+2}\tfrac{1}{2}(b_0+b_1+2)(b_0+1)f_R^{(e)}$$

where $f_R^{(e)} = \frac{N!}{(N-b_1-b_0-1)!}\frac{(N+1)!}{(N-b_0)!}$.

Remark 1. *Notice that the $S_n \times S_2$ Schur polynomials have two point functions of the form*

$$\langle \chi^{(2)}_{R,(r,s)}(\chi^{(2)}_{R,(r,s)})^\dagger\rangle \propto \frac{h_R}{h_r h_2} f_R$$

C.3.3 "String joining" and "Closed string emission"

"String joining"

$$\langle \chi^{(1)}_{R',R''}(Z,YY)(\chi^{(1)}_{R',R''}(Z,YY))^\dagger\rangle$$
$$= (\tfrac{1}{n!})^2 \sum_{\psi,\sigma\in S_{n+1}} Tr_{R',R''}(\sigma)Tr_{R',R''}(\psi)\langle Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_n}_{i_{\sigma(n)}}(Z^\dagger)^{i_1}_{i_{\phi(1)}}\dots(Z^\dagger)^{i_n}_{i_{\phi(n)}}\rangle$$
$$\times \underbrace{\langle Y_k^{i_{n+1}} Y^k_{i_{\sigma(n+1)}}(Y^\dagger)^{j_{n+1}}_l (Y^\dagger)^l_{j_{\psi(n+1)}}\rangle}_{F_1\delta^{i_{n+1}}_{i_{\sigma(n+1)}}\delta^{j_{n+1}}_{j_{\psi(n+1)}}+F_0\delta^{i_{n+1}}_{j_{\psi(n+1)}}\delta^{j_{n+1}}_{i_{\sigma(n+1)}}}$$

here $F_0 = N$ and $F_1 = 1$. Applying the techniques for restricted correlators we get

$$\langle \chi^{(1)}_{R',R''}(Z,YY)(\chi^{(1)}_{R',R''}(Z,YY))^\dagger\rangle$$
$$= \langle D_V\chi^{(1)}_{R,R''}(Z,V)(D_V\chi^{(1)}_{R,R''}(Z,V))^\dagger\rangle + (\tfrac{(n+1)!}{n!})^2\langle \chi_{R,R''}(Z)\chi^\dagger_{R,R''}(Z)\rangle|_{n+1}\cdot N$$
$$= (N-b_0-b_1)f_{R''}(N-b_0-b_1+N\tfrac{(b_0+b_1+2)(b_1+1)}{(b_1+2)})$$

$$\langle\chi^{(1)}_{R',S''}(Z,YY)(\chi^{(1)}_{R',S''}(Z,YY))^\dagger\rangle$$

$$= \langle D_V\chi^{(1)}_{R,R''}(Z,V)(D_V\chi^{(1)}_{R,S''}(Z,V))^\dagger\rangle + (\tfrac{(n+1)!}{n!})^2\langle\chi_{R,S''}(Z)\chi^\dagger_{R,R''}(Z)\rangle|_{n+1}\cdot N$$

$$= (N-b_0+1)f_{R''}(N-b_0+1+N\tfrac{(b_0+1)b_1}{(b_1-1)})$$

"Closed String emission"

$$\langle tr(Y)\chi^{(1)}_{R',R''}(Z,Y)(tr(Y)\chi^{(1)}_{R',R''}(Z,Y))^\dagger\rangle$$

$$= (\tfrac{1}{n!})^2\sum_{\psi,\sigma\in S_{n+1}} Tr_{R',R''}(\sigma)Tr_{R',R''}(\psi)\langle Z^{i_1}_{i_{\sigma(1)}}\dots Z^{i_n}_{i_{\sigma(n)}}(Z^\dagger)^{i_1}_{i_{\phi(1)}}\dots(Z^\dagger)^{i_n}_{i_{\phi(n)}}\rangle$$

$$\times \underbrace{\langle Y^{i_{n+1}}_{i_{\sigma(n+1)}}Y^k_k(Y^\dagger)^{j_{n+1}}_{j_{\psi(n+1)}}(Y^\dagger)^l_l\rangle}_{F_1\delta^{i_{n+1}}_{i_{\sigma(n+1)}}\delta^{j_{n+1}}_{j_{\psi(n+1)}}+F_0\delta^{i_{n+1}}_{j_{\psi(n+1)}}\delta^{j_{n+1}}_{i_{\sigma(n+1)}}}$$

This expression is identical to the "string joining" correlator of the same representation. Therefore

$$\begin{aligned}\langle tr(Y)\chi^{(1)}_{R',R''}(Z,Y)(tr(Y)\chi^{(1)}_{R',R''}(Z,Y))^\dagger\rangle &= \langle\chi^{(1)}_{R',R''}(Z,YY)(\chi^{(1)}_{R',R''}(Z,YY))^\dagger\rangle\\ &= (N-b_0-b_1)f_{R''}\\ &\quad\times\left(N-b_0-b_1+N\frac{(b_0+b_1+2)(b_1+1)}{(b_1+2)}\right)\end{aligned}$$

$$\begin{aligned}\langle tr(Y)\chi^{(1)}_{R',R''}(Z,Y)(tr(Y)\chi^{(1)}_{R',R''}(Z,Y))^\dagger\rangle &= \langle\chi^{(1)}_{R',S''}(Z,YY)(\chi^{(1)}_{R',S''}(Z,YY))^\dagger\rangle\\ &= (N-b_0+1)f_{R''}\left(N-b_0+1+N\frac{(b_0+1)b_1}{(b_1-1)}\right)\end{aligned}$$

C.3.4 Three column Schurs

When dealing with hop off identities A.2 we included three column Schur polynomials, herein we present their two point functions (calculated based on remark 1)

$$\langle\Psi_h(b_0,b_1)\Psi^\dagger_h(b_0,b_1)\rangle = \left(\frac{4\pi\lambda}{N}\right)^{n+2}\frac{1}{2}\frac{(b_0+b_1+2)(b_1+1)(b_0+2)}{(b_1-1)(b_0+b_1+1)}\frac{N+2}{N-b_0-b_1}f^{(h)}_R$$

$$\langle \Psi_i(b_0, b_1)\Psi_i^\dagger(b_0, b_1)\rangle = \langle \Psi_h(b_0, b_1)\Psi_h^\dagger(b_0, b_1)\rangle$$

$$\langle \Psi_f(b_0, b_1)\Psi_f^\dagger(b_0, b_1)\rangle = \left(\frac{4\pi\lambda}{N}\right)^{n+2} \frac{1}{2}\frac{(b_0+b_1+3)(b_1+1)(b_0+1)}{(b_1+2)b_0}\frac{N+2}{N-b_0+1} f_R^{(f)}$$

$$\langle \Psi_g(b_0, b_1)\Psi_g^\dagger(b_0, b_1)\rangle = \langle \Psi_f(b_0, b_1)\Psi_f^\dagger(b_0, b_1)\rangle$$

Bibliography

[1] Dine, M.: *"Supersymmetry and string theory"*. Cambridge, 2007.

[2] Aharony, O., Gubser, S.S., Maldacena, J.M., Ooguri, H. and Oz, Y.: "Large N field theories, string theory and gravity". *Phys. Rept.*, vol. 323, pp. 183–386, 2000. hep-th/9905111.

[3] 't Hooft, G.: "A planar diagram theory for strong interactions". *Nucl. Phys.*, vol. B72, p. 461, 1974.

[4] Gubser, S.S., Klebanov, I.R. and Tseytlin, A.A.: "String theory and classical absorption by three-branes". *Nucl. Phys.*, vol. B499, pp. 217–240, 1997. hep-th/9703040.

[5] Maldacena, J.M.: The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, vol. 2, pp. 231–252, 1998. hep-th/9711200.

[6] Witten, E.: "Anti-de Sitter space and holography". *Adv. Theor. Math. Phys.*, vol. 2, pp. 253–291, 1998. hep-th/9802150.

[7] de Mello Koch, R., Smolic, J. and Smolic, M.: "Giant Gravitons - with Strings Attached (I)". *JHEP*, vol. 06, p. 074, 2007. hep-th/0701066.

[8] McGreevy, J., Susskind, L. and Toumbas, N.: "Invasion of the giant gravitons from anti-de Sitter space". *JHEP*, vol. 06, p. 008, 2000. hep-th/0003075.

[9] Corley, S., Jevicki, A. and Ramgoolam, S.: "Exact correlators of giant gravitons from dual N = 4 SYM theory". *Adv. Theor. Math. Phys.*, vol. 5, pp. 809–839, 2002. hep-th/0111222.

[10] Berenstein, D.: "A toy model for the AdS/CFT correspondence". *JHEP*, vol. 07, p. 018, 2004. hep-th/0403110.

[11] Balasubramanian, V., Berenstein, D., Feng, B. and Huang, M.-x.: "D-branes in Yang-Mills theory and emergent gauge symmetry". *JHEP*, vol. 03, p. 006, 2005. hep-th/0411205.

[12] de Mello Koch, R., Smolic, J. and Smolic, M.: "Giant Gravitons - with Strings Attached (II)". *JHEP*, vol. 09, p. 049, 2007. hep-th/0701067.

[13] Bekker, D., de Mello Koch, R. and Stephanou, M.: "Giant Gravitons - with Strings Attached (III)". *JHEP*, vol. 02, p. 029, 2008. 0710.5372.

[14] Kimura, Y. and Ramgoolam, S.: "Branes, Anti-Branes and Brauer Algebras in Gauge-Gravity duality". *JHEP*, vol. 11, p. 078, 2007. 0709.2158.

[15] Brown, T.W., Heslop, P.J. and Ramgoolam, S.: "Diagonal multi-matrix correlators and BPS operators in N=4 SYM". *JHEP*, vol. 02, p. 030, 2008. 0711.0176.

[16] Brown, T.W., Heslop, P.J. and Ramgoolam, S.: "Diagonal free field matrix correlators, global symmetries and giant gravitons". *JHEP*, vol. 04, p. 089, 2009. 0806.1911.

[17] Kimura, Y.: "Non-holomorphic multi-matrix gauge invariant operators based on Brauer algebra". *JHEP*, vol. 12, p. 044, 2009. 0910.2170.

[18] de Mello Koch, R., Mashile, G. and Park, N.: "emergent three-brane lattices". *Phys. Rev. D*, vol. 81, no. 10, p. 106009, May 2010.

[19] Michael E. Peskin and Daniel V. Schroeder: *"An Introduction to Quantum Field Theory "*. Perseus Books, 2008. ISBN 0201503972.

[20] Srednicki, M.: *"Quantum Field Theory"*. Cambridge, UK: Cambridge University Press, 2007.

[21] Nastase, H.: "Introduction to AdS-CFT". 2007. 0712.0689.

[22] Dabholkar, A.: "Black Hole Entropy and Attractors". Shanghai Summer School on Recent Trends in M/String Theory, 2005.

[23] Tong, D.: Lectures on String Theory. 2009. 0908.0333.

[24] Polchinski, J.: "Dirichlet Branes and Ramond-Ramond Charges". *Phys. Rev. Lett.*, vol. 75, no. 26, pp. 4724–4727, Dec 1995.

[25] Szab, R.J.: *"An Introduction to String Theory and D-Brane Dynamics"*. Imperial Coll., London, 2004.

[26] Cerdeño, D. G. and Muñoz, C.: "An introduction to supergravity". In: *Corfu Summer Institute on Elementary Particle Physics*. Proceedings of Science PoS (corfu98), 1998.

[27] Zee, A.: *"Quantum field theory in a nutshell"*. Princeton, UK: Princeton Univ. Pr. (2003).

[28] M.B.Green, J.H.Schwarz, E.: *"Superstring Theory"*. Cambridge, London, 1988.

[29] Van Nieuwenhuizen, P.: "Supergravity". *Phys. Rept.*, vol. 68, pp. 198–205, 1981.

[30] Wilson, K.G.: "Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior". *Phys. Rev. B*, vol. 4, no. 9, pp. 3184–3205, Nov 1971.

[31] Luscher, M. and Mack, G.: "Global conformal invariance and quantum field theory,". *Comm. Math. Phys.*, vol. 41, p. 203, (1975).

[32] Klebanov, I.R.: World-volume approach to absorption by non-dilatonic branes. *Nucl. Phys.*, vol. B496, pp. 231–242, 1997. hep-th/9702076.

[33] Das, S.R., Jevicki, A. and Mathur, S.D.: "Giant gravitons, BPS bounds and noncommutativity". *Phys. Rev.*, vol. D63, p. 044001, 2001. hep-th/0008088.

[34] Beisert, N.: "The dilatation operator of N = 4 super Yang-Mills theory and integrability". *Phys. Rept.*, vol. 405, pp. 1–202, 2005. hep-th/0407277.

[35] Minahan, J.A. and Zarembo, K.: The Bethe-ansatz for N = 4 super Yang-Mills. *JHEP*, vol. 03, p. 013, 2003. hep-th/0212208.

[36] Beisert, N., Kristjansen, C. and Staudacher, M.: The dilatation operator of N = 4 super Yang-Mills theory. *Nucl. Phys.*, vol. B664, pp. 131–184, 2003. hep-th/0303060.

[37] Beisert, N., Kristjansen, C., Plefka, J., Semenoff, G.W. and Staudacher, M.: BMN correlators and operator mixing in N = 4 super Yang- Mills theory. *Nucl. Phys.*, vol. B650, pp. 125–161, 2003. hep-th/0208178.

[38] Bhattacharyya, R., de Mello Koch, R. and Stephanou, M.: Exact Multi-Restricted Schur Polynomial Correlators. *JHEP*, vol. 06, p. 101, 2008. 0805.3025.

[39] de Mello Koch, R.: "Schur polynomial two point function". Discussion, 2009.

[40] Bhattacharyya, R., Collins, S. and Koch, R.d.M.: "Exact Multi-Matrix Correlators". *JHEP*, vol. 03, p. 044, 2008. 0801.2061.