Symmetric colorings of finite groups

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A Thesis presented for the degree of
Doctor of Philosophy

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DEDICATED TO

My future self

Abstract

Given a finite group G and $r \in \mathbb{N}$, an r-coloring(or coloring) of G is a mapping $\chi : G \longrightarrow \{1, 2, 3, ..., r\}$. The group G naturally acts on its colorings by $\chi(xg^{-1}) = \chi(x)$. Colorings χ and ψ are equivalent if there is $g \in G$ such that $\chi(xg^{-1}) = \psi(x)$ for all $x \in G$. A coloring χ of G is called symmetric if there is $g \in G$ such that $\chi(gx^{-1}g) = \chi(x)$ for all $x \in G$. Let $|S_r(G)|$ denote the number of symmetric r-colorings of G and $|S_r(G)/\sim|$ the number of equivalence classes of symmetric r-colorings of G. We present methods for computing $|S_r(G)/\sim|$ and $|S_r(G)|$ and derive explicit formulas in some cases, in particular cyclic group $\mathbb{Z}_n$ and the dihedral group D_n.

Declaration

The work in this thesis is based on research carried out at the University of Witwatersrand, School of Mathematics in South Africa. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

This thesis would not have been possible without an input and support from many sides. Special thanks are due to Yuliya Zelenyuk and all my colleagues at the Department of Mathematics, University of Witwatersrand.

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Chapter 1

Introduction

In this chapter, we give a brief review of basic concepts in group theory and incidence algebra that will be used throughout the thesis. We also review existing results in the literature that are relevant to the problem solved in the thesis.

1.1 Preliminaries

1.1.1 Groups

In the following definition, the set G is assumed to be non-empty.

Definition 1.1 The set G is said to be a **group** under $\cdot$ if the following conditions hold:

(1) $a \cdot b \in G$.

(2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(3) There is $1 \in G$ such that $a \cdot 1 = 1 \cdot a = a$.

(4) There is $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

for all $a, b, c \in G$. The notation $\cdot$ denotes a map that assigns each ordered pair of elements of G to an an element of G.

The element 1 is called an **identity** of G and a^{-1} is called an **inverse** of a. Moreover, we say G is **commutative** if $a \cdot b = b \cdot a$ for all $a, b \in G$, otherwise G is a non-commutative group. If G is finite set, we call G a finite group, otherwise G is an infinite group. The number of elements or the cardinality of G is denoted by

$|G|$ and called the **order** of G.

Remark: For the sake of simplicity, we shall write ab instead of $a \cdot b$ for all $a, b \in G$.

Example 1.1 The following are the examples of groups:

(1) The set of all integers $\mathbb{Z}$ under the standard addition of integers. 0 is the identity and the inverse of $m \in \mathbb{Z}$ is $-m$.

(2) The set $D_n := \{1, a, ..., a^{n-1}, b, ab, ..., a^{n-1}b\}$ where: $b^2 = 1, a^n = 1$ and $ab = ba^{-1}$ forms a group under the standard operation. The group is denoted by $D_n = \langle a, b : \ a^n = b^2 = 1, ab = ba^{-1} \rangle$ and called a **dihedral group of order 2n**. The subset $\{1, a, ..., a^{n-1}\}$ of D_n is also a group under the same operation. It is called a **cyclic group of order n** generated by a. It is denoted by $\langle a \rangle$ or $\mathbb{Z}_n$. Usually the set $\{1, a, ..., a^{n-1}\}$ is represented by the set of the exponents: $\{0, 1, 2, ..., n-1\}$. The group $\mathbb{Z}_n$ is also called a group of integers modulo n. All the elements of $\mathbb{Z}_n$ which are coprime to n also form a group denoted by $U(\mathbb{Z}_n)$. $U(\mathbb{Z}_n)$ is called the multiplicative group of integers modulo n. The order of the group $U(\mathbb{Z}_n)$ is denoted by the **Euler's totient function** $\phi(n)$.

(3) The **unitary circle** $\{z \in \mathbb{C} : |z| = 1\}$ is a group under standard complex multiplication. $| \cdot |$ denotes complex modulus. The unitary circle is usual denoted by S^1.

(4) Let G be a commutative group. The subset $B(G) := \{x \in G : x^2 = 1\}$ of G is also a group the same operation. If it happens that G is a cyclic group of order n, we have:

$$B(G) = \begin{cases} \{0\} & \text{if} \quad n \text{ is odd} \\ \{0, \frac{n}{2}\} & \text{if} \quad n \text{ is even} \end{cases}$$

If the condition $B(G) = G$ holds, G is called a **Boolean group**.

(5) Let G be a group. The set denoted by $Z(G) := \{a \in G : ab = ba \text{ for all } b \in G\}$ is called a **center** of G. The set $Z(G)$ is also a group contained in a larger group G. Another important subset of G is the **normalizer** of a non-empty subset K of G defined as the set of all elements in $g \in G$ such $gKg^{-1} = K$ where $gKg^{-1} := \{gkg^{-1} : k \in K\}$.

The logical consequences of the above definition are the following (see [5]):

Proposition 1.1 Let G be a group. Then

(1) The identity of G is unique.

(2) For all a in G, a^{-1} is unique.

(3) $(a^{-1})^{-1} = a$ for all a in G.

(4) $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$ for all a, b in G.

Some of the groups in Example 1.1 above demonstrate the possibility of having a group contained in another larger group. Such groups are called subgroups, i.e., given a group G, a non-empty subset of group G that is also a group under the same operation is said to be a **subgroup** of G. Showing that a non-empty subset of group G is a subgroup of G is a very tedious process but fortunately there exist a simple test given below using only one condition (see [5]).

Proposition 1.2 Let K be a non-empty subset of the group G. If $ab^{-1} \in K$ for all $a, b \in K$, then K is a subgroup of G and we write $K \leq G$.

Let G be a group. If G is commutative, the Boolean group $B(G)$ is a subgroup of G. The center $Z(G)$ and the normalizer of G are also subgroups of G.

Definition 1.2 Let G be a group and A be a non-empty sets. For all $g, h \in G$ and for all $a \in A$, If the following conditions hold:

(1) $1 \cdot a = a$.

(2) $g \cdot (h \cdot a) = (gh) \cdot a$.

we "G acts on A". The notation $\cdot$ denotes a map that assigns each ordered pair of elements of G and A to an an element of A .

Another concept we need is the "equivalence relation $\sim$ on a non-empty set A" (see [1] for definition). The set $[a] := \{b : b \sim a\}$ is called an **equivalent class** of $a \in A$ and each element of $[a]$ is said to be **equivalent** to $a \in A$. Two equivalence classes of elements of A are either equal or disjoint, hence the set of equivalence classes partition A. $A/\sim$ denotes the set of all equivalence classes and $|A/\sim|$ denotes the cardinality.

Suppose that a group G acts on a non-empty set A and that for any two elements $a, b \in A$ we have $a = g \cdot b$ for some $g \in G$. If we use the notation $a \sim b$ instead, the relation becomes an equivalence relation on A. Moreover, the **equivalence class** or the **orbit** of $a \in A$ is given by $[a] = \{g \cdot a : g \in G\}$. The subgroup $St(a) := \{g \in G : g \cdot a = a\}$ is called a **stabilizer** of $a \in A$. For finite groups, $|[a]|$ the **cardinality**(number of elements) of [a] is related to $|St(a)|$.

Theorem 1.3 Let the finite group G act on a finite non-empty set A. Then $|[a]||St(a)| = |G|$ for all $a \in A$.

The result is known as the **orbit-stabilizer theorem** (see [5]). The immediate consequence of the orbit-stabilizer theorem is the **Burnside's lemma** (see [6]).

Theorem 1.4 Let the finite group G act on a finite non-empty set A. Define $\mathrm{Fix}(g) := \{a \in A : g \cdot a = a\}$. Then

$$\frac{1}{|G|}\sum_{g \in G} |\mathrm{Fix}(g)|$$

denotes the number of distinct orbits of G.

Proof: Using the last theorem, we have

$$\begin{aligned}\sum_{g \in G} |\mathrm{Fix}(g)| &= \sum_{a \in A} |St(a)| \\ &= |G| \sum_{a \in A} \frac{1}{|[a]|} \\ &= |G| \sum_{[a] \in A/\sim} 1\end{aligned}$$

where $A/\sim$ denotes the set of distinct orbits. Hence

$$\frac{1}{|G|}\sum_{g \in G} |\mathrm{Fix}(g)|$$

is the number of distinct orbits of G. $\square$

Corollary 1.5 Let G be a finite group. The map $\chi : G \longrightarrow \{1, 2, 3, ..., r\}$ where $r \in \mathbb{N}$ is called an r-coloring or coloring of G. Define the group action of G on the set of all colorings of G by $\chi g(x) := \chi(xg^{-1})$ for all $x \in G$. The number of orbits is given by:

$$\frac{1}{|G|}\sum_{g\in G} r^{|G:\langle g\rangle|}$$

where $|G : \langle g \rangle | := |G|/| \langle g \rangle |$ (the index of $\langle g \rangle$ in G).

Two distinctly defined groups can have "similar" group structures and such groups are said to be isomorphic: Mathematically, the groups G and H are said to be isomorphic if there is an invertible map $\varphi : G \longrightarrow H$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$. The notation $G \cong H$ is used in case the groups G and H are isomorphic. Furthermore, the map φ is called a homomorphism and an isomorphism if invertible.

Theorem 1.6 Let K be the subgroup of a finite group G. Then $|K|$ divides $|G|$.

The result is known as the **Lagrange's theorem** (see [5]).

Proof of Lagrange's theorem : Consider the map $\varphi : K \longrightarrow gK$ defined by $\varphi(k) := gk$. The set gK is derived from page 2 where $\overline{g}$ is an identity element. The set gK is called a **coset** or **left-coset** of K. The **right-coset** of K is defined in the similar manner. The map φ is invertible. Hence all the cosets of K have the same cardinality or size. In general, two cosets of K are equal if and only if they are disjoint, then $|G| = |G : K||K|$ where $|G : K|$ denotes the number of cosets of K(possibly infinite). $\square$

The notation $|G : K|$ is called the **index** of K in G.

If the normalizer of the subgroup K of G is equal to G, K is said to be a **normal subgroup** of G and denoted by $K \lhd G$. As a consequence, we have the following simple result.

Proposition 1.3 Let G be a group and K a subgroup of $Z(G)$. Then K is a normal subgroup of G.

The sets of all cosets of the normal subgroup K of G is denoted by G/K and it is a group under "coset-multiplication" (see [5]).

Proposition 1.4 The set G/K is a group under coset-multiplication.

We define a "coset-multiplication" as $(gK) \cdot (\overline{g}K) = (g\overline{g})K$ all $g, \overline{g} \in G$ and the group G/K is called a **quotient group** of G by K.

Corollary 1.7 If $\varphi : G \longrightarrow K$ is a homomorphism of groups G and K, then $\ker\varphi \lhd G$ and $G/\ker\varphi \cong \varphi(G)$ where $\varphi(G) := \{\varphi(g) : g \in G\}$ and $\ker\varphi := \{g \in G : \varphi(g) = 1\}$.

The result is known as the **First Isomorphism Theorem** (see [5]).

Using the First Isomorphism Theorem: If G is a commutative group, then $G/B(G) \cong G^2$ where $G^2 := \{x^2 : x \in G\}$:

Proof of orbit-stabilizer: Let $a \in A$ and define S to be a set of all cosets of $St(a)$. For each $b \in [a]$, define the map $\varphi : [a] \longrightarrow S$ to be $\varphi(b) := gSt(a)$ where $b = g \cdot a$ for some $g \in G$. Then

$$\begin{aligned} hSt(a) = gSt(a) &\Leftrightarrow g^{-1}h \in St(a) \\ &\Leftrightarrow g^{-1}h \cdot a = a \\ &\Leftrightarrow h \cdot a = g \cdot a \end{aligned}$$

where $h \in G$. Hence φ is invertible and by using the Lagrange's theorem, we get $|[a]||St(a)| = |G|$. $\square$

Another important concept to focus on is the **automorphism of a group**. Given a group G, we define the automorphism of G as the set of all isomorphisms from G to itself. The automorphism of G is denoted by $\mathrm{Aut}(G)$ and is a group under map composition.

Let H and K be subgroups of a group G. Under the condition:

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1\theta_{k_1}(h_2), k_1k_2)$$

where $h_1, h_2 \in H$ and $k_1, k_2 \in K$ and the map $\theta : K \longrightarrow \mathrm{Aut}(H)$ is assumed be a homomorphism, $H \times K$ turns into a subgroup of G. The subgroup is denoted by $H \rtimes_\theta K$ whenever θ is non-trivial, meaning $\theta \neq 1$.

Proposition 1.5 (see [5]) Let K and H be subgroups of G and the map $\theta : K \longrightarrow \mathrm{Aut}(H)$ be a homomorphism such that

(1) $G = HK$

(2) $H \trianglelefteq G$

(3) $K \cap H = 1$

Then $G \cong H \rtimes_\theta K$ and $\theta_k(h) = khk^{-1}$ for all $h \in H$ and $k \in K$.

Define $HK := \{hk : h \in H \text{ and } k \in K\}$

Proof: The second statement follows from the fact that $(h,k)^{-1} = (\theta_{k^{-1}}(h^{-1}), k^{-1})$ for each $(h,k) \in H \rtimes_\theta K$:

$$\begin{aligned}(1,k)(h,1)(1,k)^{-1} &= (1,k)(h,1)(1,k^{-1})\\ &= (\theta_k(h),k)(1,k^{-1})\\ &= (\theta_k(h),1)\end{aligned}$$

Define the map $\varphi : HK \longrightarrow H \rtimes_\theta K$ to be $\varphi(hk) = (h,k)$. Since $H \cap K = 1$, then every element of G can be uniquely written as product of $h \in H$ and $k \in K$. For all $h_1, h_2 \in H$ and $k_1, k_2 \in H$:

$$\begin{aligned}\varphi(h_1k_1h_2k_2) &= \varphi(h_1k_1h_2(k_1^{-1}k_1)k_2)\\ &= \varphi(h_1(k_1h_2k_1^{-1})k_1k_2)\\ &= (h_1\theta_{k_1}(h_2), k_1k_2)\\ &= (h_1,k_1)(h_2,k_2)\\ &= \varphi(h_1k_1)\varphi(h_2k_2)\end{aligned}$$

Hence φ is an isomorphism and $(1,1)$ is an identity of G. $\square$

The group G is called an **internal semi-direct product** of H and K.

In the following result, we assume the hypothesis of Proposition 1.5.

Corollary 1.8 The homomorphism θ is trivial if and only if K is a normal subgroup of G.

Proof: Let θ be a trivial map. Then $kh = hk$ for all $h \in H$ and $k \in K$. The normalizer of K in G is equal to G, hence K is a normal subgroup of G. Now, let K be a normal subgroup of G. Since H is also a subgroup of G, then $hk = kh$ for all $h \in H$ and $k \in K$. Hence θ is trivial. $\square$

The group $D_n = \{1, a, ..., a^{n-1}, b, ba, ..., ba^{n-1}\}$ can be "generated" using only the products of $S := \{a, b\}$ called a **generated set** of D_n and the equations $a^n = 1$ and $b^2 = 1$ called **relations** of D_n. Usually we write D_n as $D_n := \langle a, b \ : a^n = 1, b^2 = 1, ab = ba^{-1}\rangle$ and it is called a **presentation** of D_n. In general, every group G has a presentation [2]. Presentation of G is written as $G = \langle S : R\rangle$ or $G = \langle S\rangle$ where S and R are generator and relation sets respectively. The group G is called a cyclic group is S has one element. In fact there are two types of cyclic groups, namely $\mathbb{Z} \cong \langle 1\rangle$ and $\mathbb{Z}_n$.

Example 1.2 Let $D_n := \langle a, b \ : a^n = 1, b^2 = 1, ab = ba^{-1}\rangle$. It is obvious that $\langle a\rangle$ and $\langle b\rangle$ are subgroups of D_n and $D_n = \langle a\rangle \langle b\rangle$. Furthermore the subgroup $\langle a\rangle$ is the normal subgroup of D_n and $\langle a\rangle \cap \langle b\rangle = 1$. Define the homomorphism θ : $\langle b\rangle \longrightarrow \text{Aut}(\langle a\rangle)$ as :

$$\theta_k(h) := \begin{cases} h & \text{if} \quad k = 1 \\ h^{-1} & \text{if} \quad k = b \end{cases}$$

for all $h \in \langle a\rangle$. Hence $D_n \cong \langle a\rangle \rtimes_\theta \langle b\rangle \cong \mathbb{Z}_n \rtimes_\theta \mathbb{Z}_2$.

Let n be a positive odd integer. The sets $\langle a^2, b\rangle \cong D_n$ and $\langle a^n\rangle \cong \mathbb{Z}_2$ are subgroups of D_{2n} where $\langle a^2, b\rangle$ is normal subgroup of $D_{2n} = \langle a^2, b\rangle \langle a^n\rangle$ (the index of $\langle a^2, b\rangle$ in D_{2n} is 2). Since the subgroup $\langle a^n\rangle$ is a subgroup of $Z(D_{2n})$, then $\langle a^n\rangle$ is a normal subgroup of D_{2n} and $D_{2n} \cong D_n \times \mathbb{Z}_2$.

Final remarks: Even though the literature on group theory is vast, most of the definitions and the results of the notes discussed above come from a book by Dummit and Foote (see [5]). The thesis titled : "The symmetric colorings of finite groups" (see [11]) also discuss some of the material. The proof of the Burnside lemma is given in the book by Erickson (see [6]).

1.1.2 Incidence algebra

We now discuss partial ordered sets.

Definition 1.9 Let P be a non-empty set and $\preceq$ be a relation on P. The set P is called a **partially ordered set** under $\preceq$ if the following conditions hold:

(1) For each $a \in P$, $a = a$.

(2) If $a \preceq b$ and $b \preceq a$, then $a = b$.

(3) If $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Unless it is necessary, the "under $\preceq$" is usually ignored.

Remark We always assume $a \preceq b$ and $b \succeq a$ have the same meaning. If $a \preceq b$ but $a \neq b$, we write $a \prec b$. Hence $a \succ b$ mean $b \prec a$.

Definition 1.10 Let P be a partial ordered set. An **interval** between a and b is defined by

$$[a, b] := \{u \in P : a \preceq u \preceq b\}$$

When the interval between two elements of P is finite P is said to be **locally finite**.

Remark For the rest of the thesis unless stated every partial ordered set is assumed to be locally finite.

Just like many other mathematical objects, every partially ordered set has a graphical representation called a **Hasse-diagram**. To draw one: Let the elements of a partially ordered set P represent the vertices. Draw an edge or a line segment between elements $a, b \in P$ whenever there is no element $u \in P$ such that $a \preceq u \preceq b$ (the element b is always place above the element a).

We now define an **incidence algebra** of a partial ordered set P.

Definition 1.11 Let P be a partially ordered set. Define the **incidence algebra** of P as :

$$\mathbb{A}(P) := \{f : P \times P \longrightarrow \mathbb{R} : f(a, b) = 0 \text{ if } a \npreceq b\}$$

The set $\mathbb{A}(P)$ is rich in mathematical structures, in fact it is real vector space under standard addition.

Definition 1.12 Let P be a partially ordered set. Define the **convolution** of f and g as :

$$(f * g)(a, b) := \sum_{a \preceq u \preceq b} f(a, u) g(u, b)$$

for all $a, b \in P$.

Obviously f and g are assumed to be in $\mathbb{A}(P)$.

Definition 1.13 Let P be a partially ordered set. The map

$$\delta(a, b) := \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{if} \quad a \neq b \end{cases}$$

is called a **delta function**.

If the **convolution** of f and g (or the converse) is equal to δ, we say f has a multiplicative inverse g or g has a multiplicative inverse f.

For each $f, g, h \in \mathbb{A}(P)$:

$$\begin{aligned} f * (g * h)(a, b) &= \sum_{a \preceq u \preceq b} f(a, u) g * h(u, b) \\ &= \sum_{a \preceq u \preceq b} f(a, u) \big(\sum_{u \preceq v \preceq b} g(u, v) h(v, b) \big) \\ &= \sum_{a \preceq v \preceq b} \big(\sum_{a \preceq u \preceq v} f(a, u) g(u, v) \big) h(v, b) \\ &= \sum_{a \preceq v \preceq b} (f * g)(a, v) h(v, b) \\ &= (f * g) * h(a, b) \end{aligned}$$

It then follows that $f * (g * h) = (f * g) * h$.

Proposition 1.6 Let P be a partially ordered set. Each $f \in \mathbb{A}(P)$ has a multiplicative inverse if and only if $f(a, a)$ is non-zero for all $a \in P$.

Definition 1.14 Let P be a partially ordered set. The map

$$\zeta(a, b) := \begin{cases} 1 & \text{if} \quad a \preceq b \\ 0 & \text{if} \quad a \npreceq b \end{cases}$$

is called a **zeta-function**.

By Proposition 1.6, there exist a multiplicative inverse denoted by μ of ζ called a **Möbius function** of P and given by:

$$\mu(a,b) := \begin{cases} 1 & \text{if} \quad a = b \\ -\sum_{a \preceq u \prec b} \mu(a,u) & \text{if} \quad a \prec b \\ 0 & \text{otherwise} \end{cases}$$

Now:

Theorem 1.15 Let P be a partially ordered set. For each $f, g \in \mathbb{A}(P)$

$$f(x) = \sum_{x \preceq y} g(y) \quad \Leftrightarrow \quad g(x) = \sum_{x \preceq y} f(y)\mu(y,x)$$

for all $x \in P$.

Proof: Let $f, g \in \mathbb{A}(P)$ such that $g := f * \mu$. Then

$$\begin{aligned} g &= f * \mu \\ g * \zeta &= f * (\mu * \zeta) \\ &= f * \delta \\ &= f \end{aligned}$$

□

The result is called **Möbius inversion theorem**.

Example 1.3 The following are examples of partially ordered sets:

(1) The set of all subgroups of a finite group G under set inclusion. The partially ordered set is called a **lattice of subgroups**.

(2) The set of all natural numbers under divisibility. For each $a, b \in \mathbb{N}$

$$\mu(a,b) = \begin{cases} 1 & \text{if} \quad a = b \\ (-1)^r & \text{if} \quad \frac{a}{b} \text{ is a product of } r \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

Notice that if $a|b$, meaning a divides b, we have $\mu(a,b) = \mu(1, \frac{b}{a}) := \mu(n)$ where $n := \frac{a}{b}$. The "classical" Möbius function of $\mathbb{N}$ under divisibility is oldest of all such functions. It is due to August Ferdinand Möbius.

Final remark: For more detailed analysis on incidence algebra, consult the book(see [16]).

1.2 Literature review

In this section, we give a survey of some results related to the main problem of the thesis. These results fall under a field of mathematics known as Ramsey theory. In this field, one considers the partition of some mathematical object and ask the following question.

Question: How large must the mathematical object be to guarantee that one part of the partition has a desired property ?

The mathematical object of interest in this section is a group partitioned by coloring its elements using finite number of colors and the desired property involves the existence and sizes of monochromatic symmetric subsets.

Definition 1.16 Let G be a group. The map $\chi : G \longrightarrow \{1, 2, 3, ..., r\}$ where $r \in \mathbb{N}$ is called a r-coloring of G or coloring of G.

Consider the set of all maps $s : G \longrightarrow G$ defined by $s(x) := gx^{-1}g$ for some $g \in G$. These maps are called **symmetries**. Moreover, a set $B \subseteq G$ is said to be **symmetric** if $s(B) := B$ for some $g \in G$ and the **color class** is the set of $g \in G$ such that $\chi(g) = j$ for fixed $j \leq r$. If the set is measurable (see [1] for a definition), it is said to be a **monochromatic** subset of G.

The search for the existence of monochromatic symmetric subsets of various sizes (possibly infinite) has it's roots in the 1995 seminal paper (see [15]) by I.V Protasov, where he asked the following question.

Question: Is it true that for every n-coloring of an infinite group G there exist an infinite monochromatic symmetric subset of G, in particular the group $\mathbb{Z}^n$.

The general solution of the question asked by I.V Protasov regrading the group $(\mathbb{Z})^n$ is non-elementary, it relies on the on some concept from algebraic topology (see [15]). Fortunately, for $n = 2$, the solution is elementary.

Lemma 1.17 (see [8]) Let χ be a 2-coloring of $\mathbb{Z}^2$. Suppose χ does not contain an infinite monochromatic symmetric subset, then for each $a \in \mathbb{Z}^2$, the set $\{x \in \mathbb{Z}^2 : \chi(x) \neq \chi(axa)\}$ is finite .

Proof: The sets

$$\{x \in \mathbb{Z}^2 : \chi(x) = \chi(x^{-1}) = 0\} \text{ and } \{x \in \mathbb{Z}^2 : \chi(x) = \chi(x^{-1}) = 1\}$$

are finite. Moreover, the sets

$$\{x \in \mathbb{Z}^2 : \chi(axa) = \chi(x^{-1}) = 0\} \text{ and } \{x \in \mathbb{Z}^2 : \chi(axa) = \chi(x^{-1}) = 1\}$$

are also finite. The union of these four finite sets include the set in statement. $\square$

Lemma 1.18 (see [5]) Let K be a subgroup of $\mathbb{Z}^2$. Then $K \cong \mathbb{Z}^2$ if and only if the index of K in $\mathbb{Z}^2$ is finite.

Theorem 1.19 Every 2-coloring of $\mathbb{Z}^2$ contain an infinite monochromatic symmetric subset.

Proof: Let χ be a coloring of $(\mathbb{Z})^2$ and assume the coloring does not contain an infinite monochromatic symmetric subset. The set $\{x \in G : \chi(x) \neq \chi(axa)\}$ is finite. Consider $\langle a^2 \rangle \cong \mathbb{Z}$, then $x \langle a^2 \rangle$ where $x \in (\mathbb{Z})^2$ are monochromatic except for finitely many cosets, hence $|(\mathbb{Z})^2 : \langle a^2 \rangle|$ is finite. $\square$

There are other modifications of the I.V Protasov question.

Question: Is it true that for every n-coloring of an infinite commutative group G there exist a monochromatic symmetric subset of G of arbitrary finite size.

For a proof by I.V Protasov (see [15]). The problem is still unsolved for non-commutative infinite groups. For more related problems posed by other mathematicians (see [3],[8]).

For a finite group G, the most preferred modification of I.V Protasov problem involves $\overline{ms(G,r)}$: The number $\overline{ms(G,r)}$ affirms that for a given r-coloring of G, we can guarantee the existence of the monochromatic symmetric subset of size $\geq \overline{ms(G,r)}$. Define by

$$ms(G,r) := \frac{\overline{ms(G,r)}}{|G|}$$

the **maximal symmetry number** of G.

Theorem 1.20 (see [3] and [7]) Let Let G be a finite commutative group. For every $r \in \mathbb{N}$, we have

(1) $ms(G,r) \geq \frac{1}{r^2}$.

(2) $ms(G) \leq \frac{1}{r^2} + \left(\frac{1}{r} - \frac{1}{r^2}\right)\left(\frac{B(G)}{|G|}\right) + 3\sqrt{2\ln((2r|G|)/|G|)}$.

(3) $ms((G,r) = \frac{1}{r^2}$ for $r = 2^n$ where $n \in \mathbb{N}$ if G contains a subgroup isomorphic to $(\mathbb{Z}_4)^n$.

Proof: Define the coloring $\chi : (\mathbb{Z}_4)^n \longrightarrow (\mathbb{Z}_2)^n$ by

$$(\chi(x))_i = \begin{cases} 0 & \text{if} \quad (x)_i \in \{0,1\} \\ 1 & \text{if} \quad (x)_i \in \{2,3\} \end{cases}$$

For $g \in (\mathbb{Z}_4)^n$. If $\chi(x) = \chi(gx^{-1}g)$ then $(\chi(x))_i = (\chi(gx^{-1}g))_i$, thus $ms(\mathbb{Z}_4, r) = \frac{1}{4}$.

Given χ the coloring of G. Let $s(\chi)$ denote the cardinality of the largest monochrome symmetric subset G divided by $|G|$.

Let χ be a coloring of a finite commutative group H. Let $f : G \longrightarrow H$ be a surjective homomorphism. Define the coloring φ of G by $\varphi := \chi \circ f$. Then $s(\varphi) = s(\chi)$. Let S be a monochrome subset of G symmetric with respect to $g \in G$. By definition of φ it follows that $\varphi(x) = \varphi(gx^{-1}g)$ if and only if $\chi(f(x)) = \chi(f(g)f(x)^{-1}f(g))$. So, $f(S)$ is a monochrome subset of H symmetric with respect to $f(g)$. Since $|S| \leq |\text{ker} f||f(S)|$, then

$$\frac{|S|}{|G|} \leq \frac{|\text{ker} f||f(S)|}{|G|} = \frac{f(S)}{|H|}$$

Thus $s(\varphi) \leq s(\chi)$.

Conversely, let S be a monochrome subset of H symmetric with respect $h \in H$. Then $f^{-1}(S)$ is a monochrome subset of G symmetric with respect to any $g \in f^{-1}(h)$. Since $|f^{-1}(S)| = |\text{ker} f||f(S)|$

$$\frac{|f^{-1}(S)|}{|G|} = \frac{|S|}{|H|}$$

Thus $s(\varphi) \geq s(\chi)$. Since G contains $(\mathbb{Z}_4)^n$, then there exist a homomorphism from G to $(\mathbb{Z}_4)^n$. Then $ms(G,r) = \frac{1}{r^2}$ since $ms(G,r) \leq \frac{1}{r^2}$. $\square$

The theorem implies the following results.

Corollary 1.21 $ms(\mathbb{Z}_n, 2) = \frac{1}{4}$ if and only if $4|n$.

Corollary 1.22

$$\lim_{n\to\infty} ms(\mathbb{Z}_n, r) = \frac{1}{r^2}$$

.

Another question related to the I.V Protasov problem is the following.

General question: How many colorings χ of a finite group G satisfy the following condition

$$\chi(gx^{-1}g) := \chi(x)$$

for all $x \in G$ and for some $g \in G$.

Let χ be a coloring of G. The group action $\chi g(x) := \chi(xg^{-1})$ for all $x \in G$ where $g \in G$ partitions the finite set of colorings of G into finite number of distinct equivalence classes or orbits. A coloring χ of G is said to be **symmetric** with respect to g if there is $g \in G$ such that $\chi(gx^{-1}g) = \chi(x)$ for all $x \in G$. If χh where $h \in G$ is symmetric with respect to $g \in G$, then χ is symmetric with respect to $gh^{-1} \in G$. Denote by $|S_r(G)|$ the number of symmetric r-colorings of G and $|S_r(G)/\sim|$ the number of equivalence classes (or symmetric orbits) of symmetric r-colorings of G.

The method for computing $|S_r(G)|$ and $|S_r(G)/\sim|$ was formulated by Yuliya Zelenyuk (2010) (see [18]). Using the formulas, she managed to derive formulas for computing $|S_r(G)|$ and $|S_r(G)/\sim|$ where $G \cong Q_8$ and A_4 (see [18]). For the majority of finite groups, the method is impossible to use, particularly when dealing with infinite families of finite groups like D_n, etc. In the next chapter, we formulate methods for computing $|S_r(D_n)|$ and $|S_r(D_n)/\sim|$.

Thesis question: Compute $|S_r(D_n)/\sim|$ and $|S_r(D_n)|$.

Chapter 2

Symmetric colorings of D_n

This chapter is devoted to the formulation of methods for computing $|S_r(D_n)/\sim|$ and $|S_r(D_n)|$. One method is formulated in Section 2.1 and the other in Section 2.2.

The results in these sections arise in two papers with the following titles :

- The number of symmetric colorings of D_p (see [12]).
- The number of symmetric colorings of the dihedral group (see [13]).

2.1 Special case

Let be χ be a coloring of the finite group G.

$$Z(\chi) := \{g \in G : \chi(gx^{-1}g) = \chi(x) \quad \text{for all } x \in G \}$$

Hence χ is symmetric if and only if $Z(\chi)$ is non-empty.

Lemma 2.1 For every $h \in G$

$$Z(\chi h) = Z(\chi)h.$$

Proof: Given $g \in G$:

$$\begin{aligned} g \in Z(\chi h) &\Leftrightarrow \chi h(gx^{-1}g) = \chi h(x) \\ &\Leftrightarrow \chi(gx^{-1}gh^{-1}) = \chi(xh^{-1}) \\ &\Leftrightarrow \chi(gh^{-1}(xh^{-1})^{-1}gh^{-1}) = \chi(xh^{-1}) \\ &\Leftrightarrow \chi(gh^{-1}x^{-1}gh^{-1}) = \chi(x) \\ &\Leftrightarrow gh^{-1} \in Z(\chi) \\ &\Leftrightarrow g \in Z(\chi)h \end{aligned}$$

□

For every coloring χ of G and $g \in G$, the set of all symmetric colorings with respect g in the orbit of χ is denoted by $[\chi]_g$.

Lemma 2.2 For every $g, h \in G$, $[\chi]_g h = [\chi]_{gh}$.

Proof: Let $\varphi \in [\chi]$, then

$$\begin{aligned} \varphi \in [\chi]_g h &\Leftrightarrow \varphi h^{-1} \in [\chi]_g \\ &\Leftrightarrow g \in Z(\varphi h^{-1}) \\ &\Leftrightarrow g \in Z(\varphi)h^{-1} \\ &\Leftrightarrow gh \in Z(\varphi) \\ &\Leftrightarrow \varphi \in [\chi]_{gh} \end{aligned}$$

□

Lemma 2.3 For every $g \in G$, $|[\chi]_g| = |[\chi]_1|$.

Proof: $[\chi]_1 g = [\chi]_g$ and $[\chi]_g g^{-1} = [\chi]_1$. It follows that $|[\chi]_g| = |[\chi]_1|$ and $|[\chi]_g| = |[\chi]_1|$. Hence $|[\chi]_g| = |[\chi]_1|$ □

Lemma 2.4 Let χ be the coloring of G.

(1) $Z(\chi) = \{g \in G : \quad 1 \in Z(\chi g^{-1})\}$

(2) $Z(\chi) \cdot St(\chi) = Z(\chi)$

Proof: Given $g \in G$:

$$\begin{aligned} g \in Z(\chi) &\Leftrightarrow \chi(gx^{-1}g) = \chi(x) \\ &\Leftrightarrow \chi(g(xg)^{-1}g) = \chi(xg) \\ &\Leftrightarrow \chi(x^{-1}g) = \chi(xg) \\ &\Leftrightarrow \chi g^{-1}(x^{-1}) = \chi g^{-1}(x) \\ &\Leftrightarrow 1 \in Z(\chi g^{-1}) \end{aligned}$$

Consider the function $\phi : G \longrightarrow [\chi]$ given by $\phi(g) = \chi g^{-1}$

$$\begin{aligned} \chi g^{-1} = \chi h^{-1} &\Leftrightarrow \chi g^{-1} h = \chi \\ &\Leftrightarrow g^{-1} h \in St(\chi) \\ &\Leftrightarrow h \in g \cdot St(\chi) \end{aligned}$$

□

Lemma 2.5 Let χ be the coloring of G and $1 \in Z(\chi)$, then:

(1) $Z(\chi) = \{g \in G : \chi(gxg) = \chi(x) \quad \text{for all } x \in G\ \}$

(2) $St(\chi) \cdot Z(\chi) = Z(\chi)$

(3) For every $a \in Z(\chi)$, $\langle a \rangle \subseteq Z(\chi)$

Proof: Let $g \in G$, then:

$$\begin{aligned} g \in Z(\chi) &\Leftrightarrow \chi(gx^{-1}g) = \chi(x) \\ &\Leftrightarrow \chi(g(gxg)^{-1}g) = \chi(gxg) \\ &\Leftrightarrow \chi(x^{-1}) = \chi(gxg) \\ &\Leftrightarrow \chi(x) = \chi(gxg) \end{aligned}$$

Let $h \in St(\chi)$ and $g \in Z(\chi)$. Now, for every x in G:

$$\begin{aligned} \chi(ghx^{-1}gh) &= \chi((ghx^{-1}gh)^{-1}) \\ &= \chi(g^{-1}h^{-1}xg^{-1}) \\ &= \chi(gx^{-1}hg) \\ &= \chi(h^{-1}x) \\ &= \chi(x) \end{aligned}$$

hence, $hg \in Z(\chi)$. Clearly $Z(\chi)$ is contained in $St(\chi) \cdot Z(\chi)$. $\square$

Let P be the set of all pairs $x := (St(\chi), Z(\chi))$ for some coloring χ of G with $1 \in Z(\chi)$. Define the relation $\preceq$ on P by $x \preceq y$ if and only if each co-ordinate of x is contained in the corresponding co-ordinate of y. Note that P becomes a partially ordered set.

Let π be the partition of G. The **stabilizer** and **center** of π are defined by

$$St(\pi) := \{g \in G : x \text{ and } xg^{-1} \text{ belong to the same cell of } \pi\}$$
$$Z(\pi) := \{g \in G : x \text{ and } gx^{-1}g \text{ belong to the same cell of } \pi\}$$

We say that a partition π of G is **optimal** if $1 \in Z(\pi)$ and for every partition π' of G with $St(\pi) = St(\pi')$ and $Z(\pi) = Z(\pi')$, one has $\pi \preceq \pi'$. The relation $\preceq$ means that every cell of π is contained in some cell of x'. Clearly the set of all optimal partition of G is a partially ordered set under the relation $\preceq$. Since $St(\pi) = St(\chi)$ and $Z(\pi) = Z(\chi)$ for some coloring χ of G, it immediately follows that the set P is "identical" to the partially ordered set of all optimal partitions of G under the relation $\preceq$.

Theorem 2.6 [18, *p*.580] Let P be the set of optimal partitions of G. Then

$$|S_r(G)/\sim| = \sum_{x \in P} \sum_{y \leqslant x} \frac{\mu(y,x)|St(y)|}{|Z(y)|} r^{|x|}$$

and

$$|S_r(G)| = \sum_{x \in P} \sum_{y \leqslant x} \frac{\mu(y,x)}{|Z(y)|} r^{|x|}$$

Proof: Let $C := \{\chi \in S_r(G) : \ 1 \in Z(\chi)\ \}$ and for every $x \in P$, let

$$C(x) := \{\chi \in C : St(\chi) = St(x) \text{ and } Z(\chi) = Z(x)\}$$

Then $\{C(x) : \ x \in P\ \}$ is a partition of C and for every $x \in P$ and $\chi \in C(x)$, one has $|[\chi]_1| = \dfrac{|Z(x)|}{|St(x)|}$ and $|[\chi]| = \dfrac{|G|}{|St(x)|}$ but in general, $[\chi]_1 \subsetneq C(x)$. To correct this situation, define the equivalence $\equiv$ on P by

$$x \equiv y \Leftrightarrow St(y) = aSt(x)a^{-1} \text{ and } Z(y) = Z(x)a^{-1} \text{ for some } a \in Z(x)$$

Clearly $\equiv$ is induced by $\sim$. For every $x \in P$, let $\bar{x}$ denote the $\equiv$ class containing x and let $C(\bar{x}) = \cup_{y \in \bar{x}} C(y)$. Then whenever $y \in \bar{x}$ and $\chi \in C(y)$,

$$[\chi]_1 \subset C(\bar{x}), |[\chi]_1| = \frac{|Z(x)|}{|St(x)|} \text{ and } |[\chi]| = \frac{|G|}{|St(x)|}$$

then

$$
\begin{aligned}
|C(\bar{x})/\sim| &= \frac{|Z(x)|C(\bar{x})}{|St(x)|} \\
&= \sum_{y\in\bar{x}} \frac{|St(x)||C(y)|}{|Z(y)|}
\end{aligned}
$$

and the number of all colorings equivalent to colorings from $C(\bar{x})$ is

$$
|C(\bar{x})/\sim| \cdot \frac{|G|}{|St(x)|} = |G| \sum_{y\in\bar{x}} \frac{|C(y)|}{|Z(y)|}
$$

Hence

$$
\begin{aligned}
|C(G)/\sim| &= |S_r(G)/\sim| \\
&= \sum_{y\in P} \frac{|St(x)||C(y)|}{|Z(y)|}
\end{aligned}
$$

$$
|S_r(G)| = |G| \sum_{y\in P} \frac{|C(y)|}{|Z(y)|}
$$

Now to compute $|C(y)|$, note that

$$
r^{|y|} = \sum_{y\in x} |C(y)|
$$

Hence,

$$
|C(y)| = \sum_{y\leqslant x} \mu(y,x)|r^{|x|}
$$

and

$$
|S_r(G)/\sim| = \sum_{x\in P} \sum_{y\leqslant x} \frac{\mu(y,x)|St(y)|}{|Z(y)|} r^{|x|}
$$

$$
|S_r(G)| = \sum_{x\in P} \sum_{y\leqslant x} \frac{\mu(y,x)}{|Z(y)|} r^{|x|}
$$

$\square$

The basic optimal partition of G is given by $\pi := \{\{x, x^{-1}\} : x \in G\}$. Here, the stabilizer and center of π are $St(\pi) = \langle 1 \rangle$ and $Z(\pi) = \langle 1 \rangle$ respectively. All the other optimal partitions contain the optimal partition π.

Now, we use the above theorem to compute $|S_r(D_p)/\sim|$ and $|S_r(D_p)|$ where p is positive odd prime number.

Lemma 2.7 Let G be a subgroup of D_n. Then:

(1) $G \cong \langle a^d \rangle$ or $G \cong \langle a^d, a^r b \rangle$ where $d|n$ and $0 \leq r < d$.

(2) The number of positive divisors of n counts the total number of subgroups of the form $\langle a^d \rangle$. The sum of positive divisors of n counts the total number of subgroups of the form $\langle a^d, a^r b \rangle$.

Remark: The subgroups in (1) are $\langle a^d \rangle \cong \mathbb{Z}_{n/d}$ and $\langle a^d, a^r b \rangle \cong D_{n/d}$.

Using Lemma 2.7, there are five cyclic subgroups of D_3, namely $\langle a \rangle \cong \mathbb{Z}_3$, $\langle a^3 \rangle \cong \mathbb{Z}_1$, $\langle a^3, b \rangle \cong D_1 \cong \mathbb{Z}_2 \cong \langle b \rangle$, $\langle a^3, ab \rangle \cong D_1 \cong \mathbb{Z}_2 \cong \langle ab \rangle$ and $\langle a^3, a^2 b \rangle \cong D_1 \cong \mathbb{Z}_2 \cong \langle a^2 b \rangle$. The only commutative subgroup of D_3 is $\langle a, b \rangle \cong D_3$.

We now list all the representative of the optimal partitions of D_3.
One optimal partitions of the form

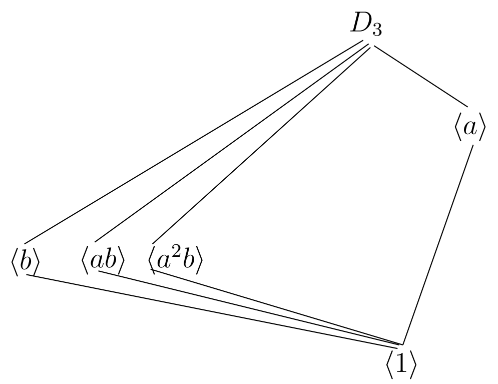

Figure 2.1: Lattice of subgroups of D_3

$$\pi : \{1\}, \{a, a^2\}, \{ab\}, \{a^2 b\}, \{b\}$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \langle 1 \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = 5.$$

Three optimal partitions of the form

$$\pi : \{1\}, \{a, a^2\}, \{b\}, \{ab, a^2 b\}$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \langle b \rangle,$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 4.$$

One optimal partition of the form

$$\begin{aligned}
&\pi : \{1, a, a^2\}, \{b\}, \{ab\}, \{a^2b\} \\
&St(\pi) = \langle 1 \rangle, Z(\pi) = \langle a \rangle, \\
&|St(\pi)| = 1, |Z(\pi)| = 3, |\pi| = 4.
\end{aligned}$$

One optimal partition of the form

$$\begin{aligned}
&\pi : \{1\}, \{a, a^2\}, \{b, ab, a^2b\} \\
&St(\pi) = \langle 1 \rangle, Z(\pi) = \{1, b, ab, a^2b\}, \\
&|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 3.
\end{aligned}$$

Three optimal partitions of the form

$$\begin{aligned}
&\pi : \{1, a, a^2\}, \{b\}, \{ab, a^2b\} \\
&St(\pi) = \langle 1 \rangle, Z(\pi) = \{1, a, b, a^2\}, \\
&|St(\pi)| = 1, |Z(\pi)| = 4, |\pi| = 3.
\end{aligned}$$

Three optimal partitions of the form

$$\begin{aligned}
&\pi : \{1, b\}, \{a, a^2, ab, a^2b\} \\
&St(\pi) = \langle b \rangle, Z(\pi) = \langle b \rangle, \\
&|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = 2.
\end{aligned}$$

One optimal partition of the form

$$\begin{aligned}
&\pi : \{1, a, a^2\}, \{b, ab, a^2b\} \\
&St(\pi) = \langle a \rangle, Z(\pi) = D_3, \\
&|St(\pi)| = 3, |Z(\pi)| = 6, |\pi| = 2.
\end{aligned}$$

One optimal partition of the form

$$\begin{aligned}
&\pi : D_3 \\
&St(\pi) = D_3, Z(\pi) = D_3, \\
&|St(\pi)| = 6, |Z(\pi)| = 6, |\pi| = 1.
\end{aligned}$$

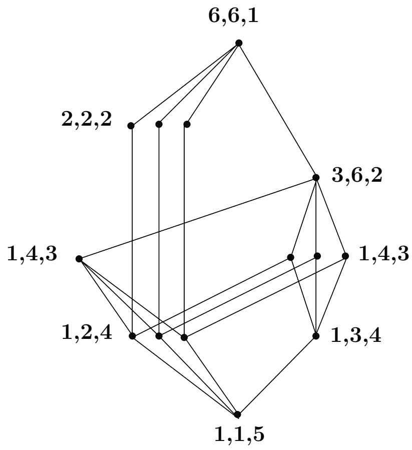

Figure 2.2: Optimal partitions of D_3

By Theorem 2.6

$$
\begin{aligned}
|S_r(D_3)| &= 6\Big(r^5 + 3\Big(\frac{1}{2} - 1\Big)r^4 + \Big(\frac{1}{3} - 1\Big)r^4 + 3\Big(\frac{1}{4} - \frac{1}{2} - \frac{1}{3} + 1\Big)r^3 + \Big(\frac{1}{4} - \frac{3}{2} + 2\Big)\\
&\quad + 3\Big(\frac{2}{2} - \frac{1}{2}\Big)r^2 + \Big(\frac{1}{6} - \frac{1}{4} - \frac{3}{4} + \frac{3}{2} + \frac{2}{3} - 2\Big)r^2 + \Big(\frac{1}{6} - \frac{1}{6} - \frac{3}{2} + \frac{3}{2}\Big)r\Big)\\
&= 6\Big(r^5 - \frac{3}{2}r^4 - \frac{2}{3}r^4 + \frac{5}{4}r^3 + \frac{3}{4}r^3 - \frac{2}{3}r^2\Big)\\
&= 6r^5 - 13r^4 + 12r^3 - 4r^2
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(D_3)/\sim| &= r^5 + 3\Big(\frac{1}{2} - 1\Big)r^4 + \Big(\frac{1}{3} - 1\Big)r^4 + 3\Big(\frac{1}{4} - \frac{1}{2} - \frac{1}{3} + 1\Big)r^3 + \Big(\frac{1}{4} - \frac{3}{2} + 2\Big)r^3\\
&\quad + 3\Big(\frac{2}{2} - \frac{1}{2}\Big)r^2 + \Big(\frac{3}{6} - \frac{1}{4} - \frac{3}{4} + \frac{3}{2} + \frac{2}{3} - 2\Big)r^2 + \Big(\frac{6}{6} - \frac{3}{6} - \frac{6}{2} + \frac{3}{2}\Big)r\\
&= r^5 - \frac{13}{6}r^4 + 2r^3 + \frac{7}{6}r^2 - r.
\end{aligned}
$$

Theorem 2.8 (see [10]) For every $r \in \mathbb{N}$, $|S_r(D_3)| = 6r^5 - 13r^4 + 12r^3 - 4r^2$ and $|S_r(D_3)/\sim| = r^5 - \frac{13}{6}r^4 + 2r^3 + \frac{7}{6}r^2 - r$.

Now we analyze D_5. By Lemma 2.7, there are seven cyclic subgroups of D_5. We have $\langle a\rangle \cong \mathbb{Z}_5$ and $\langle a^5\rangle = \langle 1\rangle \cong \mathbb{Z}_1$. Furthermore $\langle a^5, ab\rangle \cong D_1 \cong \mathbb{Z}_2 \cong \langle ab\rangle$. The other cyclic subgroups are shown in Figure 2.3. $D_5 \cong \langle a, b\rangle$ is the only non-cyclic subgroup.

The optimal partitions are the following.

One optimal partition of the form

$$\begin{aligned}
&\pi : \{1\}, \{a, a^4\}, \{a^2, a^3\}, \{b\}, \{ab\}, \{a^2b\}, \{a^3b\}, \{a^4b\} \\
&St(\pi) = \langle 1 \rangle, Z(\pi) = \langle 1 \rangle, \\
&|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = 8.
\end{aligned}$$

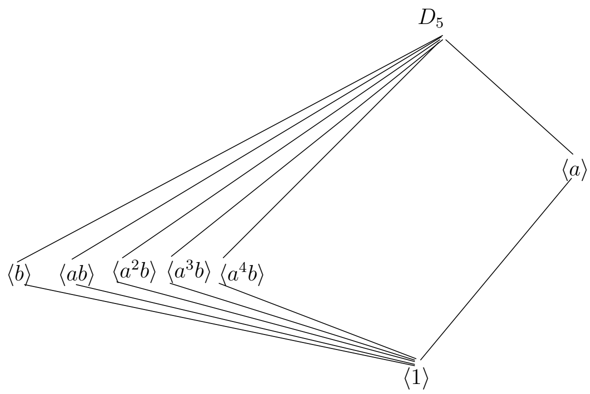

Figure 2.3: Lattice of subgroups of D_5

Five optimal partitions of the form

$$\begin{aligned}
&\pi : \{1\}, \{a, a^4\}, \{a^2, a^3\}, \{b\}, \{ab, a^4b\}, \{a^2b, a^3b\}, \\
&St(\pi) = \langle 1 \rangle, Z(\pi) = \langle b \rangle, \\
&|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = 6.
\end{aligned}$$

One optimal partition of the form

$$\begin{aligned}
&\pi : \{1, a, a^2, a^3, a^4\}, \{b\}, \{ba\}, \{ba^2\}, \{ba^3\}, \{ba^4\} \\
&St(\pi) = \langle 1 \rangle, Z(\pi) = \langle a \rangle, \\
&|St(\pi)| = 1, |Z(\pi)| = 5, |\pi| = 6.
\end{aligned}$$

One optimal partition of the form

$$\pi : \{1\}, \{a, a^4\}, \{a^2, a^3\}, \{b, ab, a^2b, a^3b, a^4b\}$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \{1, a, a^2, a^3, a^4, b\}$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 4.$$

Five optimal partitions of the form

$$\pi : \{1, a, a^2, a^3, a^4\}, \{b\}, \{ab, a^4b\}, \{a^2b, a^3b\}$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \{1, a, a^2, a^3, a^4, b\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 6, |\pi| = 4.$$

Five optimal partitions of the form

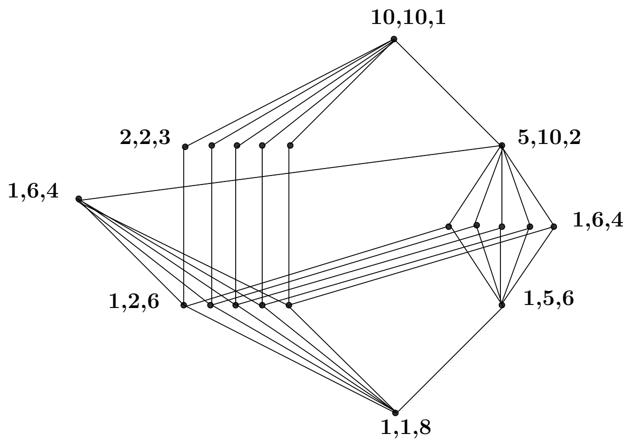

Figure 2.4: Optimal partitions of D_5

$$\pi : \{1, b\}, \{a, a^4, ab, a^4b\}, \{a^2, a^4, a^2b, a^3b\}$$
$$St(\pi) = \langle b \rangle, Z(\pi) = \langle b \rangle,$$
$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = 3.$$

One optimal partition of the form

$$\pi : \{1, a, a^2, a^3, a^4\}, \{b, ab, a^2b, a^3b, a^4b\}$$
$$St(\pi) = \langle a \rangle, Z(\pi) = D_5$$
$$|St(\pi)| = 5, |Z(\pi)| = 10, |\pi| = 2.$$

One optimal partition of the form

$$
\begin{aligned}
&\pi : D_5\\
&St(\pi) = D_5, Z(\pi) = D_5,\\
&|St(\pi)| = 10, |Z(\pi)| = 10, |\pi| = 1.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
|S_r(D_5)| = 10\Big(& r^8 + 5\Big(\frac{1}{2} - 1\Big)r^6 + \Big(\frac{1}{5} - 1\Big)r^6 + 5\Big(\frac{1}{6} - \frac{1}{2} - \frac{1}{5} + 1\Big)r^4 + \Big(\frac{1}{6} - \frac{5}{2} + 4\Big)r^4\\
&+ 5\Big(\frac{1}{2} - \frac{1}{2}\Big)r^3 + \Big(\frac{1}{10} - \frac{1}{6} - \frac{5}{6} + \frac{5}{2} + \frac{4}{5} - 4\Big)r^2 + \Big(\frac{1}{10} - \frac{1}{10} - \frac{5}{2} + \frac{5}{2}\Big)r\Big)\\
&= 10r^8 - 33r^6 + 40r^4 - 16r^2
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(D_5)/\sim| = {}& r^8 + 5\Big(\frac{1}{2} - 1\Big)r^6 + \Big(\frac{1}{5} - 1\Big)r^6 + 5\Big(\frac{1}{6} - \frac{1}{2} - \frac{1}{5} + 1\Big)r^4 + \Big(\frac{1}{6} - \frac{5}{2} + 4\Big)r^4\\
&+ 5\Big(\frac{2}{2} - \frac{1}{2}\Big)r^3 + \Big(\frac{5}{10} - \frac{1}{6} - \frac{5}{6} + \frac{5}{2} + \frac{4}{5} - 4\Big)r^2 + \Big(\frac{10}{10} - \frac{5}{10} - \frac{10}{2} + \frac{5}{2}\Big)r\\
&= r^8 - \frac{33}{10}r^6 + 4r^4 + \frac{5}{2}r^3 - \frac{6}{5}r^2 - 2r.
\end{aligned}
$$

Theorem 2.9 (see [11]) For every $r \in \mathbb{N}$, $|S_r(D_5)| = 10r^8 - 33r^6 + 40r^4 - 16r^2$ and $|S_r(D_5)/\sim| = r^8 - \frac{33}{10}r^6 + 4r^4 + \frac{5}{2}r^3 - \frac{6}{5}r^2 - 2r$.

Clearly, the analysis demonstrate that symmetric colorings of D_3 and D_5 are linked structurally. The Hasse-diagrams of their optimal partitions follow a predictable pattern. Similarly, the optimal partitions also follow a predictable pattern. Generalization of these similarities lead to the computation $|S_r(D_p)|$ and $|S_r(D_p)/\sim|$ for odd prime number p :

The optimal partition of D_p are listed below.

One optimal partition of the form

$$\pi : \{1\}, \{b\}, \{ab\}, ..., \{a^{p-1}b\}, \{a, a^{p-1}\}, ...$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \langle 1 \rangle$$
$$|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = \frac{3p+1}{2}$$

p optimal partitions of the form

$$\pi : \{1\}, \{a, a^{p-1}\}, ..., \{b\}, \{ab, a^{p-1}b\}, ...$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \langle b \rangle$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = p+1$$

One optimal partition of the form

$$\pi : \{1, a, ..., a^{p-1}\}, \{b\}, \{ab\}, ..., \{a^{p-1}b\}$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \langle a \rangle$$
$$|St(\pi)| = 1, |Z(\pi)| = p, |\pi| = p+1$$

One optimal partition of the form

$$\pi : \{b, ab, ..., a^{p-1}b\}, ..., \{a, a^{p-1}\}, \{1\}$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \{1, a, ..., a^{p-1}, b\}$$
$$|St(\pi)| = 1, |Z(\pi)| = p+1, |\pi| = \frac{p+3}{2}$$

p optimal partitions of the form

$$\pi : \{1, a, ..., a^{p-1}\}, \{b\}, \{ab, a^{p-1}b\}, ...$$
$$St(\pi) = \langle 1 \rangle, Z(\pi) = \{1, a, ..., a^{p-1}, b\}$$
$$|St(\pi)| = 1, |Z(\pi)| = p+1, |\pi| = \frac{p+3}{2}$$

p optimal partitions of the form

$$\pi : \{1, b\}, \{a, a^{p-1}, ab, a^{p-1}b\}, ...$$
$$St(\pi) = \langle b \rangle, Z(\pi) = \langle b \rangle$$
$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = \frac{p+1}{2}$$

One optimal partition of the form

$$
\begin{aligned}
&\pi : \{1, a, ..., a^{p-1}\}, \{b, ab, ..., a^{p-1}b\} \\
&St(\pi) = \langle a \rangle, Z(\pi) = D_p \\
&|St(\pi)| = p, |Z(\pi)| = 2p, |\pi| = 2
\end{aligned}
$$

One optimal partition of the form

$$
\begin{aligned}
&\pi : \{D_p\} \\
&St(\pi) = D_p, Z(\pi) = D_p \\
&|St(\pi)| = 2p, |Z(\pi)| = 2p, |\pi| = 1
\end{aligned}
$$

Hence (see [12])

$$
\begin{aligned}
|S_r(D_p)| &= |D_p| \sum_{x \in P} \sum_{y \leqslant x} \frac{\mu(y,x)}{|Z(y)|} r^{|x|} \\
&= 2p(r^{\frac{3p+1}{2}} - \frac{pr^{p+1}}{2} + r^{p+1}(\frac{1}{p} - 1) + pr^{\frac{p+3}{2}}(1 + \frac{1}{p+1} - \frac{1}{2} - \frac{1}{p}) + \\
&r^{\frac{p+3}{2}}(\frac{1}{p+1} - \frac{p}{2} + p - 1) + pr^{\frac{p+1}{2}}(\frac{1}{2} - \frac{1}{2}) + \\
&r^2(\frac{1}{2p} - \frac{1}{p+1} - \frac{p}{p+1} + \frac{p}{2} + \frac{p-1}{p} - p + 1) + r(\frac{1}{2p} - \frac{1}{2p} - \frac{p}{2} + \frac{p}{2})) \\
&= 2pr^{\frac{3p+1}{2}} + (-p^2 - 2p + 2)r^{p+1} + 2p(p-1)r^{\frac{p+3}{2}} - (p-1)^2 r^2.
\end{aligned}
$$

and

$$
\begin{aligned}
|S_r(D_p)/\sim| &= |D_p| \sum_{x \in P} \sum_{y \leqslant x} \frac{\mu(y,x)|St(y)|}{|Z(y)|} r^{|x|} \\
&= r^{\frac{3p+1}{2}} - \frac{pr^{p+1}}{2} + r^{p+1}(\frac{1}{p} - 1) + pr^{\frac{p+3}{2}}(1 + \frac{1}{p+1} - \frac{1}{2} - \frac{1}{p}) + \\
&r^{\frac{p+3}{2}}(\frac{1}{p+1} - \frac{p}{2} + p - 1) + pr^{\frac{p+1}{2}}(\frac{2}{2} - \frac{1}{2}) + \\
&r^2(\frac{p}{2p} - \frac{1}{p+1} - \frac{p}{p+1} + \frac{p}{2} + \frac{p-1}{p} - p + 1) + r(\frac{2p}{2p} - \frac{p}{2p} - \frac{2p}{2} + \frac{p}{2}) \\
&= r^{\frac{3p+1}{2}} + (\frac{-p^2 - 2p + 2}{2p})r^{p+1} + (p-1)r^{\frac{p+3}{2}} + \frac{p}{2}r^{\frac{p+1}{2}} \\
&+ \frac{-p^2 + 3p - 2}{2p} r^2 + \frac{1-p}{2} r
\end{aligned}
$$

As shown above, Theorem 2.6 is easy to use when n is an odd prime. However, for general case the complexity of Hasse-diagrams of optimal partitions make it near impossible to evaluate $|S_r(D_n)/\sim|$ and $|S_r(D_n)|$, hence a need for a new approach (see [11]. It demonstrates how D_4 deviates from the above pattern).

2.2 General case

Now, we present a general method for computing $|S_r(D_n)/\sim|$ and $|S_r(D_n)|$.

Firstly:

Lemma 2.10 (see [9]) Let G be a finite commutative group. Then

$$|S_r(G)| = \sum_{Y\leqslant G}\sum_{X\leqslant Y}\frac{\mu(X,Y)|G/X|}{|B(G/X)|}r^{\frac{|G/Y|+|B(G/Y)|}{2}}$$

and

$$|S_r(G)/\sim| = \sum_{Y\leqslant G}\sum_{X\leqslant Y}\frac{\mu(X,Y)}{|B(G/X)|}r^{\frac{|G/Y|+|B(G/Y)|}{2}}$$

where $\mu(X,Y)$ is the Möbius function on a lattice of subgroups of G.

Proof: Let X be a subgroup of G and let $C(X)$ be the set of all colorings of G with the stabilizer X symmetric with respect to the identity. Now, given $\chi\in C(X)$ we claim that the number of colorings of G equivalent to χ and symmetric with respect the identity is equals to $B(G/X)$. Remember that all the colorings of G equivalent to χ are stabilized by X.

$$\begin{aligned}
\chi g(x^{-1}) = \chi g(x) &\Leftrightarrow \chi g((xg^{-1})^{-1}) = \chi g(xg^{-1})\\
&\Leftrightarrow \chi g(gx^{-1}) = \chi g(xg^{-1})\\
&\Leftrightarrow \chi(gx^{-1}g^{-1}) = \chi(xg^{-2})\\
&\Leftrightarrow \chi(x^{-1}) = \chi g^2(x)\\
&\Leftrightarrow \chi(x) = \chi g^2(x)\\
&\Leftrightarrow g^2\in X
\end{aligned}$$

So, the number of symmetric orbits of G with the stabilizer X is equals to

$$\frac{|C(X)|}{|B(G/X)|}$$

Hence

$$|S_r(G)/\sim| = \sum_{Y \leqslant G} \frac{C(Y)}{|B(G/Y)|}$$

and

$$|S_r(G)| = \sum_{Y \leqslant G} \frac{|G/Y||C(Y)|}{|B(G/Y)|}$$

notice that

$$\sum_{Y \leqslant X \leqslant G} |C(X)| = r^{\frac{|G/X|+|B(G/X)|}{2}}$$

By Möbius inversion function on a lattice of subgroups of G, it follows that

$$|C(Y)| = \sum_{Y \leqslant X \leqslant G} \mu(Y,X) r^{\frac{|G/X|+|B(G/X)|}{2}}$$

□

Corollary 2.11 (see [8]) Let p be a prime. If n is odd then

$$|S_r(\mathbb{Z}_n)| = \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}}$$
$$|S_r(\mathbb{Z}_n)/\sim| = r^{\frac{n+1}{2}}$$

if $n = 2^l m$, where $l \geq 1$ and m is odd then

$$|S_r(\mathbb{Z}_n)| = \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1}$$
$$|S_r(\mathbb{Z}_n)/\sim| = \frac{r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}}{2}$$

Proof: By Lemma 2.10 :

$$|S_r(\mathbb{Z}_n)/\sim| = \sum_{d|n} \sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} r^{\frac{\frac{n}{d}+2-\delta(\frac{n}{k})}{2}}$$

and

$$|S_r(\mathbb{Z}_n)| = \sum_{d|n} \sum_{k|d} \frac{\mu(\frac{d}{k})\frac{d}{k}}{2-\delta(\frac{n}{k})} r^{\frac{\frac{n}{d}+2-\delta(\frac{n}{k})}{2}}$$

where

$$\delta(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Let $\dfrac{n}{d}$ be even, then $\delta(\dfrac{n}{k}) = 0$.

$$\begin{aligned} \sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{n}{k})} &= \sum_{k|d} \frac{\mu(\frac{d}{k})}{2} \\ &= \begin{cases} \frac{1}{2} & \text{if} \quad d = 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \end{aligned}$$

again

$$\begin{aligned} \sum_{k|d} \frac{\mu(\frac{d}{k})\frac{d}{k}}{2-\delta(\frac{n}{k})} &= \sum_{k|d} \frac{\mu(\frac{d}{k})\frac{d}{k}}{2} \\ &= \frac{1}{2}\prod_{p|d}(1-p) \end{aligned}$$

Let $\dfrac{n}{d}$ be odd, then $\delta(\dfrac{n}{k}) = \delta(\dfrac{d}{k})$. Hence,

$$\begin{aligned} \sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{d}{k})} &= \prod_{p|d}(1 - \frac{1}{2-\delta(p)}) \\ &= \begin{cases} 1 & \text{if } \ d = 1 \\ \dfrac{1}{2} & \text{if } \ d = 2^i \\ 0 & \quad \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \sum_{k|d} \frac{\mu(\frac{d}{k})\frac{d}{k}}{2-\delta(\frac{d}{k})} &= \prod_{p|d}(1 - \frac{p}{2-\delta(p)}) \\ &= \begin{cases} 0 & \qquad\qquad \text{otherwise} \\ \prod_{p|d}(1-p) & \text{if } \ d \text{ is even} \end{cases} \end{aligned}$$

Now, if n is odd, then for $d|n$ one has

$$\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{d}{k})} = \begin{cases} 1 & \text{if } \ d = 1 \\ 0 & \quad \text{otherwise} \end{cases}$$

and

$$\sum_{k|d} \frac{\mu(\frac{d}{k})\frac{d}{k}}{2-\delta(\frac{d}{k})} = \prod_{p|d}(1-p)$$

if n is even, then

$$\sum_{k|d} \frac{\mu(\frac{d}{k})}{2-\delta(\frac{d}{k})} = \begin{cases} \dfrac{1}{2} & \text{if } d = 2^i \text{ or } d = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{k|d} \frac{\mu(\frac{d}{k})\frac{d}{k}}{2-\delta(\frac{d}{k})} = \begin{cases} 0 & \text{otherwise} \\ \frac{1}{2}\prod_{p|d}(1-p) & \text{if } d|\frac{n}{2} \end{cases}$$

and consequently

If n is odd then

$$|S_r(\mathbb{Z}_n)| = \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}}$$

$$|S_r(\mathbb{Z}_n)/\sim| = r^{\frac{n+1}{2}}$$

if $n = 2^l m$, where $l \geq 1$ and m is odd then

$$|S_r(\mathbb{Z}_n)| = \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{\frac{d+1}{2}}$$

$$|S_r(\mathbb{Z}_n)/\sim| = \frac{r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}}{2}$$

$\square$

Geometrically, the vertex set of an n-polygon can be used to represent the elements of the cyclic group $\mathbb{Z}_n$ and r-colored n-polygons represent the r-colorings of $\mathbb{Z}_n$. In this framework, the symmetric r-colorings of $\mathbb{Z}_n$ are represent r-colored n-polygons that are invariant with respect to some axial symmetry with an axis crossing the center of the n-polygon and one of its vertices.

Finally:

Theorem 2.12 (see [13]) For any $r \in \mathbb{N}$

$$|S_r(D_n)| = 2r^n|S_r(\mathbb{Z}_n)| - (|S_r(\mathbb{Z}_n)|)^2$$

and

$$\begin{aligned}|S_r(D_n)/\sim| = &\sum_{\substack{d,m|n\\ d<m}} \frac{(d,m)}{n}(B(\frac{n}{d})A(\frac{n}{m}) + A(\frac{n}{d})B(\frac{n}{m}) - B(\frac{n}{d})B(\frac{n}{m}))\\ &+ \sum_{d|n} \frac{d}{2n}B(\frac{n}{d})(2A(\frac{n}{d}) - B(\frac{n}{d}) - \frac{n}{d})\end{aligned}$$

where $B := B_r$ and $A := A_r$ are given by:

$$A_r(n) := \sum_{d|n} \mu(d) r^{\frac{n}{d}}$$

and

$$B_r(n) := \frac{n}{2-\delta(n)} \sum_{d|n} \mu(d) r^{\frac{\frac{n}{d}2-\delta(\frac{n}{d})}{2}}$$

where

$$\delta(n) := \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Example 2.1 Using Theorem 2.12

$$\begin{aligned}|S_r(D_p)| &= 2r^p|S_r(\mathbb{Z}_p)| - (|S_r(\mathbb{Z}_p)|)^2\\ &= 2r^p(pr^{\frac{p+1}{2}} - (p-1)r) - (pr^{\frac{p+1}{2}} - (p-1)r)^2\\ &= 2pr^{\frac{3p+1}{2}} - (p^2+2p-2)r^{p+1} + 2p(p-1)r^{\frac{p+3}{2}} - (p-1)^2r^2\end{aligned}$$

Since $A(1) = B(1) = r$ and $A(p) = r^p - r$ and $B(p) = pr^{\frac{p+1}{2}} - pr$, then

$$\begin{aligned}|S_r(D_p)/\sim| = &r^{\frac{3p+1}{2}} - (\frac{p^2+2p-2}{2p})r^{p+1} + (p-1)r^{\frac{p+1}{2}}\\ &+ \frac{p}{2}r^{\frac{p+1}{2}} + (\frac{p^2-3p+2}{2p})r^2 - \frac{p-1}{2}r\end{aligned}$$

Now, for D_4

$$\begin{aligned}|S_r(D_4)| &= 2r^4|S_r(\mathbb{Z}_4)| - (|S_r(\mathbb{Z}_4)|)^2\\ &= 2r^4(2r^3 - r^2) - (2r^3 - r^2)^2\\ &= 4r^7 - 6r^6 + 4r^5 - r^4\end{aligned}$$

We have $B(4) = 2(r^3 - r^2)$ and $A(4) = r^4 - r^2$. Hence

$$|S_r(D_4)/\sim| = \frac{1}{2}r^7 - \frac{3}{4}r^6 + \frac{1}{2}r^5 + r^3 - \frac{1}{4}r^2$$

Now given D_6, we have

$$\begin{aligned}|S_r(D_6)| &= 2r^6|S_r(\mathbb{Z}_6)| - (|S_r(\mathbb{Z}_6)|)^2 \\ &= 2r^6(3r^4 - 2r^2) - (3r^4 - 2r^2)^2 \\ &= 6r^{10} - 13r^8 + 12r^6 - 4r^4\end{aligned}$$

In this case $B(6) = 3(r^4-2r^2+r)$ and $A(6) = r^6-r^3-r^2+r$. The first sum is given by $\frac{2}{3}r^8-\frac{3}{2}r^6-\frac{1}{2}r^5+\frac{7}{2}r^4-2r^3-\frac{1}{6}r^2$. The second sum is $\frac{1}{2}r^{10}-\frac{7}{4}r^8+\frac{5}{2}r^6+r^5-\frac{13}{4}r^4+3r^3-r^2$ Hence

$$|S_r(D_6)/\sim| = \frac{1}{2}r^{10} - \frac{13}{12}r^8 + r^6 + \frac{1}{2}r^5 + \frac{1}{4}r^4 + r^3 - \frac{7}{6}r$$

Since $D_n \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$, denote the coloring χ of D_n restricted to $\mathbb{Z}_n$ and $\mathbb{Z}_n h$ by $\chi_{|\mathbb{Z}_n}$ and $\chi h_{|\mathbb{Z}_n}$ respectively.

Lemma 2.13 Let χ be the r-coloring of D_n. The following statements are equivalent:

(1) χ is a symmetric r-coloring of D_n.

(2) $\chi_{|\mathbb{Z}_n}$ or $\chi h_{|\mathbb{Z}_n}$ is symmetric.

Proof: Let χ be a symmetric r-coloring of D_n with respect to some c in $\mathbb{Z}_n$, then $\chi_{|\mathbb{Z}_n}$ is also symmetric. Now assume that χ is symmetric with respect to some ch in $\mathbb{Z}_n h$, then

$$\begin{aligned}\chi(xh) &= \chi(ch(xh)^{-1}ch) \\ &= \chi(c(x)^{-1}ch) \\ &= \chi h(cx^{-1}c)\end{aligned}$$

for all x in $\mathbb{Z}_n$. Hence $\chi h_{|\mathbb{Z}_n}$ is symmetric with respect to c.

Now let $\chi_{|\mathbb{Z}_n}$ be symmetric with respect to c in $\mathbb{Z}_n$. Since $c(xh)^{-1}c = xh$, then χ is symmetric. Again assume that $\chi h_{|\mathbb{Z}_n}$ be symmetric with respect to c in $\mathbb{Z}_n$. $chx^{-1}ch = x$ and $ch(xh)^{-1}ch = cx^{-1}ch$, then χ is symmetric since

$$\begin{aligned}\chi(ch(xh)^{-1}ch) &= \chi(cx^{-1}ch)\\ &= \chi(xh)\end{aligned}$$

$\square$

Corollary 2.14 (see [17]) The r-coloring χ of D_n is symmetric if and only if one of the following cases hold:

(1) $r = 1$

(2) $r = 2$ and $n = 3$ or 5.

(3) $D_n \cong \mathbb{Z}_2$.

(4) $D_n \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof: Let the order of the group $G \cong \mathbb{Z}_n$ be even. We want to show that there exist a coloring of G that is not symmetric, hence the existence of a coloring of D_n that is not symmetric. Define the coloring $\chi : G \longrightarrow \{1, 2\}$ by:

$$\chi(x) := \begin{cases} 1 & \text{if } x \in \{a, b\} \\ 2 & \text{otherwise} \end{cases}$$

where $a, b \in G$ satisfy the conditions: $ab \notin G$ and $ab^{-1} \notin B(G)$. The condition $ab \notin G$ implies that $ga^{-1}g = b$ for all $g \in G$. Since $ab \notin G^2$, $ga^{-1}g \neq b$. If $ga^{-1}g = a$ then $gb^{-1}g \neq b$ because $ab^{-1} \notin B(G)$. It follows that either $\chi(a) \neq \chi(ga^{-1}g)$ or $\chi(b) \neq \chi(gb^{-1}g)$, hence a contradiction.

Now let the order of G be odd. Then $B(G) = 1$ and the order of G is at least 7. Choose three distinct $a, b, c \in G$ such that for any distinct $g, x \in \{a, b, c\}$ we have $gx^{-1}g \notin \{a, b, c\}$. Now pick any distinct $a, b \in G$. There is a unique $g \in G$ such that $b = ga^{-1}g$. Pick $c \in G - \{a, b, g, a^2b^{-1}, b^2a^{-1}\}$. Define $\chi : G \longrightarrow \{1, 2\}$ by:

$$\chi(x) := \begin{cases} 1 & \text{if } x \in \{a, b, c\} \\ 2 & \text{otherwise} \end{cases}$$

Let $g \in G$. If $g \notin \{a,b,c\}$ and $ga^{-1}g = b$. then $gc^{-1}g \notin \{a,b,c\}$. If $g \in \{a,b,c\}$, say $g = a$ then $gb^{-1}g \notin \{a,b,c\}$. It follows that there is $x \in \{a,b,c\}$ such that $\chi(x) \neq \chi(gx^{-1}g)$, hence a contradiction. $\square$

Proof of the first part Theorem 2.12: There are $r^n|S_r(\mathbb{Z}_n)|$ colorings of $\mathbb{Z}_n$ such that $\chi_{|\mathbb{Z}_n}$ is symmetric and there are also $r^n|S_r(\mathbb{Z}_n)|$ colorings of D_n such that $\chi h_{|\mathbb{Z}_n}$ is symmetric. $(|S_r(\mathbb{Z}_n)|)^2$ colorings of D_n such that $\chi h_{|\mathbb{Z}_n}$ and $\chi_{|\mathbb{Z}_n}$ are symmetric. Hence, there are $2r^n|S_r(\mathbb{Z}_n)| - (|S_r(\mathbb{Z}_n)|)^2$ colorings of D_n such that $\chi h_{|\mathbb{Z}_n}$ or $\chi_{|\mathbb{Z}_n}$ is symmetric. Hence $2r^n|S_r(\mathbb{Z}_n)| - (|S_r(\mathbb{Z}_n)|)^2$ is the number of symmetric colorings of D_n. $\square$

To evaluate the number of symmetric orbits of D_n, the following lemma is necessary.

Lemma 2.15 Let χ be a coloring of D_n. Let $\phi := \chi_{|\mathbb{Z}_n}$ and $\varphi := \chi h_{|\mathbb{Z}_n}$.

(1) If $\phi \nsim \varphi$, then $St(\chi) = St(\phi) \cap St(\varphi)$.

(2) If $\phi \sim \varphi$ with respect to $b \in \mathbb{Z}_n$, then $St(\chi) = St(\phi) \cup St(\phi)bh$.

Proof: Let $a \in \mathbb{Z}_n$, then $a \in St(\chi)$ if and only if $\chi a_{|\mathbb{Z}_n} = \phi$ and $\chi ah_{|\mathbb{Z}_n} = \varphi h$. $ah \in St(\chi)$ if and only if $\chi ah_{|\mathbb{Z}_n} = \phi$ and $\chi a_{|\mathbb{Z}_n} = \varphi h$. For every $x \in \mathbb{Z}_n$,

$$\begin{aligned}
\chi a(x) &= \chi(xa^{-1}) = \phi(xa^{-1}) = \phi a(x)\\
\chi a(xh) &= \chi(xha^{-1}) = \chi(xah) = \varphi(xa) = \varphi a^{-1}(x)\\
\chi ah(x) &= \chi(xah) = \varphi(xa) = \varphi a^{-1}(x)\\
\chi ah(xh) &= \chi(xhah) = \chi(xa^{-1}) = \phi(xa^{-1}) = \phi a(x)
\end{aligned}$$

It follows that

$$a \in St(\chi) \quad \text{if and only if} \quad \phi a = \phi \quad \text{if and only if} \quad \varphi a^{-1} = \varphi$$
$$ah \in St(\chi) \quad \text{if and only if} \quad \phi a = \varphi$$

From the first equivalence we obtain that

$$a \in St(\chi) \quad \text{if and only if} \quad a \in St(\phi) \cap St(\varphi)$$

From the second equivalence we obtain that

$$ah \in St(\chi) \quad \text{if and only if} \quad ab^{-1} \in St(\phi)$$

$\square$

A coloring χ of a finite group G is **aperiodic** if $St(\chi) = 1$. If G is abelian and H is a subgroup of G, then the number of colorings χ of G with $St(\chi) = H$ is equal to the number of aperiodic colorings of G/H. The corresponding bijection is given by $\chi \longmapsto \chi/H$ where χ/H is the coloring of G/H defined by $\chi/H(xH) = \chi(x)$.
Let $H \cong \mathbb{Z}_n$ where $n \in \mathbb{N}$. Then $A_r(n)$ or just $A(n)$ is the number of aperiodic colorings of H and $B_r(n)$ or just $B(n)$ is the number of aperiodic symmetric colorings of H.

Proof of the second part of Theorem 2.12: The number of symmetric colorings of D_n such that the colorings of $\chi_{|\mathbb{Z}_n}$ and $\chi h_{|\mathbb{Z}_n}$ have different stabilizers of orders d and m is

$$B\Big(\frac{n}{d}\Big)A\Big(\frac{n}{m}\Big) + A\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big) - B\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big)$$

and the number of their orbits is

$$\frac{(d,m)}{2n}B\Big(\frac{n}{d}\Big)A\Big(\frac{n}{m}\Big) + A\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big) - B\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big)$$

Hence, the number of orbits of all symmetric colorings of D_n such that the colorings $\chi_{|\mathbb{Z}_n}$ and $\chi h_{|\mathbb{Z}_n}$ have different stabilizers is

$$\sum_{\substack{d,m|n\\ d\neq m}} \frac{(d,m)}{2n}\Big(B\Big(\frac{n}{d}\Big)A\Big(\frac{n}{m}\Big) + A\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big) - B\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big)\Big)$$

which is equals to

$$\sum_{\substack{d,m|n\\ d<m}} \frac{(d,m)}{n}\Big(B\Big(\frac{n}{d}\Big)A\Big(\frac{n}{m}\Big) + A\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big) - B\Big(\frac{n}{d}\Big)B\Big(\frac{n}{m}\Big)\Big)$$

The number of symmetric colorings χ of D_n such that the colorings $\chi_{|\mathbb{Z}_n}$ and $\chi h_{|\mathbb{Z}_n}$ have the same stabilizer of d and are not equivalent is

$$2B\Big(\frac{n}{d}\Big)\Big(A\Big(\frac{n}{d}\Big) - \frac{n}{d} - B\Big(\frac{n}{d}\Big)\Big)\Big(B\Big(\frac{n}{d}\Big) - \frac{n}{d}\Big) = B\Big(\frac{n}{d}\Big)\Big(2A\Big(\frac{n}{d}\Big) - B\Big(\frac{n}{d}\Big) - \frac{n}{d}\Big)$$

So the number of their orbits is

$$\frac{d}{2n}B\Big(\frac{n}{d}\Big)\Big(2A\Big(\frac{n}{d}\Big) - B\Big(\frac{n}{d}\Big) - \frac{n}{d}\Big)$$

The number of symmetric colorings χ of D_n such that the colorings $\chi_{|\mathbb{Z}_n}$ and $\chi h_{|\mathbb{Z}_n}$ have the same stabilizer of d and are equivalent is

$$\frac{n}{d}B\Big(\frac{n}{d}\Big)$$

So the number of their orbits is

$$\frac{d}{n}\frac{n}{d}B\Big(\frac{n}{d}\Big) = B\Big(\frac{n}{d}\Big)$$

Hence, the number of orbits of symmetric colorings χ of D_n such that the colorings $\chi_{|\mathbb{Z}_n}$ and $\chi h_{|\mathbb{Z}_n}$ have the same stabilizer of order d

$$\frac{d}{2n}B\Big(\frac{n}{d}\Big)\Big(2A\Big(\frac{n}{d}\Big) - B\Big(\frac{n}{d}\Big) - \frac{n}{d}\Big) + B\Big(\frac{n}{d}\Big) = \frac{d}{2n}B\Big(\frac{n}{d}\Big)\Big(2A\Big(\frac{n}{d}\Big) - B\Big(\frac{n}{d}\Big) + \frac{n}{d}\Big)$$

Hence, the number of orbits of all symmetric colorings χ of D_n such that the colorings $\chi_{|\mathbb{Z}_n}$ and $\chi h_{|\mathbb{Z}_n}$ have the same stabilizer is

$$\sum_{d|n}\frac{d}{2n}B\Big(\frac{n}{d}\Big)\Big(2A\Big(\frac{n}{d}\Big) - B\Big(\frac{n}{d}\Big) + \frac{n}{d}\Big)$$

$\square$

Chapter 3

Conclusion

3.1 Main results

Recall that the basis of the thesis was the computation $|S_r(D_n)|$ and $|S_r(D_n)/\sim|$. The results from page 29 (paper [12]) and Theorem 2.12 (paper [13]) from the preceding chapter presented two different approaches for computing $|S_r(D_n)|$ and $|S_r(D_n)/\sim|$. The first method relies on the construction of optimal partitions which is a more difficult process. Hence the limited use and need to formulate a new method.

Summary:

First main result: Let p be a prime number. Then

$$|S_r(D_p)| = 2pr^{\frac{3p+1}{2}} + (-p^2 - 2p + 2)r^{p+1} + 2p(p-1)r^{\frac{p+3}{2}} - (p-1)^2r^2.$$

and

$$|S_r(D_p)/\sim| = r^{\frac{3p+1}{2}} + (\frac{-p^2 - 2p + 2}{2p})r^{p+1} + (p-1)r^{\frac{p+3}{2}} + \frac{p}{2}r^{\frac{p+1}{2}} + \frac{-p^2 + 3p - 2}{2p}r^2 + \frac{1-p}{2}r$$

The generalization

Second main result: For any $r \in \mathbb{N}$.

$$|S_r(D_n)| = 2r^n|S_r(\mathbb{Z}_n)| - (|S_r(\mathbb{Z}_n)|)^2$$

and

$$\begin{aligned}|S_r(D_n)/\sim| &= \sum_{\substack{d,m|n \\ d<m}} \frac{(d,m)}{n}(B(\frac{n}{d})A(\frac{n}{m}) + A(\frac{n}{d})B(\frac{n}{m}) - B(\frac{n}{d})B(\frac{n}{m})) \\ &+ \sum_{d|n} \frac{d}{2n}B(\frac{n}{d})(2A(\frac{n}{d}) - B(\frac{n}{d}) - \frac{n}{d})\end{aligned}$$

where

$$A(n) = \sum_{d|n} \mu(d) r^{\frac{n}{d}}$$

and

$$B(n) = \frac{n}{2-\delta(n)} \sum_{d|n} \mu(d) r^{\frac{\frac{n}{d}2-\delta(\frac{n}{d}}{2}}$$

also

$$\delta(n) := \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

3.2 Suggestions for future work

There are numerous unsolved problems related to the problem addressed in the thesis. For example, recall in Chapter 1 it was shown that for any positive odd integer n, $D_{2n} \cong D_n \times \mathbb{Z}_2$. Now, let $|S_r(D_n)| = f(r)$ and $|S_r(\mathbb{Z}_n)| = g(r)$ where f and g are polynomials with coefficients in $\mathbb{Q}$. By Corollary 2.11:

$$\begin{aligned}|S_r(\mathbb{Z}_{2n})| &= \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{d+1} \\ &= \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p)(r^2)^{\frac{d+1}{2}} \\ &= g(r^2)\end{aligned}$$

Therefore, using Theorem 2.12 the condition $|S_r(D_{2n})| = f(r^2)$ holds. Numerous other finite groups have been tried, the condition seems to hold for all the finite groups:

Open question: Let K be a subgroup of a finite group G such that $G \cong K \times \mathbb{Z}_2$. If $|S_r(K)| = f(r)$ where f is polynomial with coefficients in $\mathbb{Q}$, is it always true that $|S_r(G)| = f(r^2)$?

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