# Perturbed ODEs in Cosmology and Invariance Analysis of Ordinary Difference Equations

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A dissertation submitted to the faculty of science, University of the Witwatersrand, in partial fulfilment of the requirements for the degree of Master of Science.

Johannesburg, December 2017

# Declaration

I hereby declare that the work contained in this dissertation is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

# Abstract

This dissertation contains two parts ,the first part involves the novel study of the use of symmetries to find exact solutions of ordinary difference equations. Non-trivial symmetries of third-order difference equations of the form (1) are obtained. These symmetries are used to investigate their solutions for some random sequences $(A_n)$ and $(B_n)$. We extend the results obtained in [15] and conditions for well-defined solutions are obtained. Furthermore, a full Lie point symmetry analysis of the fifth-order rational difference equations of the form (5), where $\lambda$ and $\mu$ are real numbers, is considered and exact solutions are found. In the second part we find new geometric conditions that can be specialized to obtain the approximate Noether symmetries of a large class of variational equations. We will also look at the ordinary differential equations that model cosmological and relativistic phenomena, which we will analyse using an approximate symmetry method.

# Acknowledgments

I would like to thank Standard bank in partnership with Studytrust for sponsoring my masters programme and making it possible for me to complete my studies. I also want to extend my gratitude to Dr M. Folly-Gbetoula and Dr S. Jamal for the supervision of my dissertation and the support they gave me for my research. Last but not least, I would like to give special thanks to my family and friends for the support they have provided me throughout the year.

# Contents

# Introduction

Over a century ago, there were a number of techniques used to solve ordinary differential equations (ODEs). Marius Sophus Lie (1842-1899) studied these techniques and extended them after realizing that they were special cases of a general integration procedure based on the invariance of ODEs under a continuous group of symmetries [1]. In 1987, Shigeru Maeda showed that the Lie's method extension can also be used to solve ordinary difference equations (O$\Delta$Es). He also showed that the set of functional equations amounted from the linearized symmetry condition of the O$\Delta$Es [2]. Later, several authors studied O$\Delta$Es and some interesting results have been obtained - see [3, 4, 5, 6, 7, 8, 9] and references herein.

The Lie symmetry method for ODEs can be extended to the method used to find approximate solutions for perturbed ODEs. Among several computational techniques for approximate generators, there are two main formalisms, one proposed by Baikov, Gazizov and Ibragimov [10] and the other was presented by Shtelen and Fushchich [11]. Thereafter, the concept of approximate Noether symmetries and conservation laws emerged [12, 13]. Note that approximate symmetries form an “approximate Lie algebra” [14].

In chapter 1 in the text, we will firstly look at definitions and some notations that we will be using in this chapter. We will then perform a full Lie analysis of a third order O$\Delta$E and a fifth order O$\Delta$E. The third order O$\Delta$E that we will be studying is inspired by the work in [15], where T.F Ibrahim and N. Touafek studied the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n\left(a_n + b_n x_{n-1}x_{n-2}\right)}, \tag{1}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are real two-periodic sequences and the initial values $x_{-2},\ x_{-1},\ x_0$ are non-zero real numbers. We generalize their results to random sequences, whether $(a_n)$ and $(b_n)$ are periodic or not. The fifth order O$\Delta$E inspired by the work in [16], is given by

$$x_{n+1} = \frac{x_{n-4}x_{n-3}x_{n-2}}{x_{n-1}x_n(\lambda + \mu x_{n-4}x_{n-3}x_{n-2})}, \tag{2}$$

where $\lambda, \mu \in \mathbb{R}$. In [16], the author studied the case where $\lambda = \pm 1$ and $\mu = \pm 1$. This work extends the results in existing literature.

In chapter 2 in the text, we will stipulate the generalized approximate conditions in the case of a class of perturbed Lagrangians, up to third-order,

$$\mathcal{L}(u, u', u'', \epsilon) = \frac{1}{2}\left(-u'^2 + u^2\right) + \epsilon^i G_i(u, u', u''). \tag{3}$$

Here $i$ takes on values $1, 2\,\text{and}\,3$ and $u$ is a function of $\phi$. Our next purpose is to use the generalized conditions to find approximate divergence symmetries for several critical cases of interest. Thirdly, the latter will be used to establish the associated approximate conservation laws by invoking Noether's theorem.

Chapter 1 of the dissertation has been published [17] and chapter 2 was submitted for review in ISI journals [18].

# Chapter 1

# Invariance Analysis of Ordinary Difference Equations

In this chapter, we will perform a full Lie analysis of (1). We will use a symmetry based method to solve (1) for real sequences, $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$, and we will put less restrictions on initial values $x_{-2}$, $x_{-1}$ and $x_0$; we expect our solutions to be in 'single form'. Naturally, some ansatz will be made on the sequences in our results in order to compare them to the corresponding cases to that of [15]. For definiteness, we will consider the equation

$$u_{n+3} = \frac{u_n u_{n+1}}{u_{n+2}\left(A_n + B_n u_n u_{n+1}\right)}. \tag{4}$$

The implication is that one can only compare $x_n$ with $u_{n+2}$.

Furthermore, we use this symmetry based method to obtain exact solutions of (2) and again we study

$$u_{n+5} = \frac{u_n u_{n+1} u_{n+2}}{u_{n+3} u_{n+4}(\lambda + \mu u_n u_{n+1} u_{n+2})}, \tag{5}$$

instead. Note that solutions of (2) were found in [9]; however, their method is completely different from ours.

## 1.1 Definitions and Notations

Most of the definitions and notation used in this chapter follow those adopted by Hydon in [2].

Consider the general form of the OΔE of order $k$

$$u_{n+k} = \omega \left(n, u_n, u_{n+1}, ...., u_{n+k-1}\right), \tag{6}$$

for some function $\omega$ and $k$ a set of natural numbers.

**Definition 1.** *The shift operator is denoted by $S$ and acts on $n$ as follows*

$$S : n \rightarrow n + 1. \tag{7}$$

Now consider the point transformations

$$\Gamma_\epsilon : (n, u_n) \rightarrow (n, u_n + \epsilon Q (n, u_n)), \tag{8}$$

for some characteristic functions $Q$.

**Definition 2.** *A symmetry generator, denoted by $X$, is given by*

$$\begin{aligned} X = &Q (n, u_n) \frac{\partial}{\partial u_n} + Q (n + 1, u_{n+1}) \frac{\partial}{\partial u_{n+1}} + \cdots \\ &+ Q (n + k - 1, u_{n+k-1}) \frac{\partial}{\partial u_{n+k-1}}. \end{aligned} \tag{9}$$

To solve for the characteristic, we will need the linearized symmetry condition from [2],

$$Q(n+k, u_{n+k}) - X\omega = 0, \tag{10}$$

provided (6) holds. Given a symmetry generator for the $k^{th}$ order O$\Delta$E (6), we have a $(k-1)$th-order invariant

$$v_n = v\left(n, u_n, \ldots, u_{n+k-1}\right),$$

which satisfies

$$Xv_n = 0.$$

Assume that the characteristic $Q$ is known, then we can solve for the invariant $v_n$ using the characteristics equation

$$\frac{du_n}{Q(n,u_n)} = \frac{du_{n+1}}{Q(n+1,u_{n+1})} = \cdots = \frac{du_{n+k-1}}{Q(n+k-1,u_{n+k-1})} \left(= \frac{dv_n}{0}\right). \tag{11}$$

We use the conventions

$$\prod_{j=k}^{k-1} \alpha_j = 1, \tag{12}$$

$$\sum_{j=k}^{l} \alpha_j = 0 \text{ when } k > l \tag{13}$$

and we define the function $\theta$ as follows:

$$\theta(k, s, \alpha_{2j}) = \prod_{j=k}^{s} \alpha_{2j}. \tag{14}$$

In [2], the procedure of finding symmetries is explained at length, especially for second-order ordinary difference equations.

## 1.2 Invariance Analysis of some third-order difference equations

Consider the third-order difference equations of the form (4), i.e.,

$$u_{n+3} = \frac{u_n u_{n+1}}{u_{n+2}\left(A_n + B_n u_n u_{n+1}\right)}. \tag{15}$$

### 1.2.1 Symmetries

We impose the linearized symmetry condition (10) to (4) to get

$$\begin{aligned} & Q(n+3, u_{n+3}) + \frac{u_n u_{n+1}}{u_{n+2}^2 (A_n + B_n u_n u_{n+1})} Q(n+2, u_{n+2}) - \\ & \frac{A_n u_n}{u_{n+2}(A_n + B_n u_n u_{n+1})^2} Q(n+1, u_{n+1}) - \\ & \frac{A_n u_{n+1}}{u_{n+2}(A_n + B_n u_n u_{n+1})^2} Q(n, u_n) = 0. \end{aligned} \tag{16}$$

We then differentiate implicitly with respect to $u_n$ (keeping $\omega$ fixed and regarding $u_{n+1}$ as a function of $u_n$, $u_{n+2}$ and $\omega$). After clearing fractions in the resulting equation, we get:

$$\begin{aligned} & u_{n+1}[Q(n, u_n) - u_n Q'(n, u_n)] \\ & - u_n \left[Q(n+1, u_{n+1}) + u_{n+1} Q'(n+1, u_{n+1})\right] = 0. \end{aligned} \tag{17}$$

We differentiate (17) with respect to $u_n$ twice. This leads to

$$\frac{d^2}{du_n^2}\left[Q(n, u_n) - u_n Q'(n, u_n)\right] = 0. \tag{18}$$

The general solution of (18) is given by

$$Q(n, u_n) = k_1(n) u_n \ln(u_n) + k_2(n) u_n + k_3(n), \tag{19}$$

where $k_1$, $k_2$ and $k_3$ are functions of $n$. To find these functions $k_i,\ i \in \{1,2,3\}$, we substitute (19) into (16). We get

$$\begin{aligned}
&- k_1(n+3)u_nu_{n+1}(A_n + B_nu_nu_{n+1})\ln(A_n + B_nu_nu_{n+1}) + \\
&[k_1(n+3)(A_n + B_nu_nu_{n+1}) - (k_1(n) + k_1(n+1))A_n]\, u_nu_{n+1}\ln(u_nu_{n+1}) \\
&+ [k_1(n+2)u_{n+2} - k_1(n+3)(A_n + B_nu_nu_{n+1})]u_nu_{n+1}\ln(u_{n+2}) + \\
&k_2(n+3)u_nu_{n+1}(A_n + B_nu_nu_{n+1}) + k_3(n+3)u_{n+2}(A_n + B_nu_nu_{n+1})^2 \\
&+ u_nu_{n+1}[k_2(n+2)u_{n+2} + k_3(n+2)] - A_nu_n[k_2(n+1)u_{n+1} + k_3(n+1)] \\
&- A_nu_{n+1}[k_2(n)u_n + k_3(n)] = 0.
\end{aligned} \tag{20}$$

To ease our computation, we let $k_1 = 0$. We then solve for $k_2$ and $k_3$, and we find

$$k_2(n) = (-1)^n,$$

$$k_3(n) = 0.$$

Hence,

$$Q(n, u_n) = (-1)^n u_n$$

and the symmetry of (4) is given by

$$X = (-1)^n u_n\partial_{u_n} + (-1)^{n+1}u_{n+1}\partial_{u_{n+1}} + (-1)^{n+2}u_{n+2}\partial_{u_{n+2}}. \tag{21}$$

### 1.2.2 Reduction and exact solutions

Given that

$$Q(n, u_n) = (-1)^n u_n$$

is a characteristic, the following condition must be satisfied:

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1}u_{n+1}} = \frac{du_{n+2}}{(-1)^{n+2}u_{n+2}} \left( = \frac{dv_n}{0} \right). \tag{22}$$

We need only two combinations from (22). The first one is

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+2}}{(-1)^{n+2} u_{n+2}}. \tag{23}$$

Integrating both sides of equation (23) we get

$$c_1 = \frac{u_{n+2}}{u_n}, \tag{24}$$

where $c_1$ is a constant. We then take another combination given by

$$\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}} = \frac{du_{n+2}}{(-1)^{n+2} u_{n+2}} \tag{25}$$

which has

$$c_2 = u_{n+1} u_{n+2} \tag{26}$$

as solution for some constant $c_2$. The invariant $v_n$ is our independent variable and is represented as a function of $c_1$ and $c_2$ as shown below:

$$v_n = f(c_1, c_2) = \frac{c_1}{c_2}.$$

This means,

$$v_n = \frac{1}{u_n u_{n+1}}. \tag{27}$$

Shifting equation (27) twice yields

$$v_{n+2} = \frac{1}{u_{n+2} u_{n+3}}. \tag{28}$$

Substituting $u_{n+3}$ in (28) using equation (4) yields

$$v_{n+2} = \frac{A_n}{u_n u_{n+1}} + B_n. \tag{29}$$

Using equation (27) we can deduce that equation (29) is given by

$$v_{n+2} = A_n v_n + B_n. \tag{30}$$

To solve (30), we first split it into two categories (depending on the parity of $n$) as follows:

$$v_{2n+i} = A_{2n+i}v_{2n+i} + B_{2n+i}, \quad i = 0, 1. \tag{31}$$

Thus, the sequences $(v_{2n+i})_{i=0,1}$ satisfy the first-order linear difference equation

$$w_{n+1} = A_{2n+i}w_n + B_{2n+i}, \quad i = 0, 1, \tag{32}$$

whose solutions in closed form are given by

$$w_n = v_{2n+i} = v_i \left( \prod_{k_1=0}^{n-1} A_{2k_1+i} \right) + \sum_{l=0}^{n-1} \left( B_{2l+i} \prod_{k_2=l+1}^{n-1} A_{2k_2+i} \right), \quad i = 0, 1. \tag{33}$$

**Note**. (33) gives the solutions of (30) for all $n \in \mathbb{N}_0$.

Invoking (27), we can write $u_n$ in terms of $v_n$ as follows:

$$|u_n| = \exp\left[ (-1)^{n-1} \ln\left|\frac{1}{u_0}\right| + (-1)^{n-1} \sum_{k=0}^{n-1} -(-1)^{-k} \ln|v_k| \right], \tag{34}$$

where $v_n$ is given in (33).

**Note.** Equation (34) gives the solution of (4) in a unified manner.

The above solution can be further split for odd and even $n$. For instance,

$$|u_{2n}| = \exp\left( (-1)^{2n-1} \ln\left|\frac{1}{u_0}\right| + (-1)^{2n-1} \sum_{k=0}^{2n-1} - (-1)^{-k} \ln|v_k| \right)$$

and after expanding the summation we obtain the following:

$$\begin{aligned} |u_{2n}| &= \exp\left( \ln|u_0| + \ln|v_0| - \ln|v_1| + \ln|v_2| - ... + \ln|v_{2n-2}| - \ln|v_{2n-1}| \right) \\ &= \exp\left( \ln|u_0| + \ln\left|\frac{v_0}{v_1}\right| + \ln\left|\frac{v_2}{v_3}\right| + ... + \ln\left|\frac{v_{2n-2}}{v_{2n-1}}\right| \right) \\ &= \exp\left( \ln|u_0| + \ln\left|\frac{v_0}{v_1}\frac{v_2}{v_3}...\frac{v_{2n-2}}{v_{2n-1}}\right| \right) \\ u_{2n} &= u_0 \left( \frac{v_0}{v_1}\frac{v_2}{v_3}...\frac{v_{2n-2}}{v_{2n-1}} \right) \end{aligned}$$

and this can be represented by

$$u_{2n} = u_0 \prod_{s=0}^{n-1} \left( \frac{v_{2s}}{v_{2s+1}} \right). \tag{35a}$$

Similarly, we have

$$|u_{2n+1}| = \exp\left[ (-1)^{2n} \ln\left|\frac{1}{u_0}\right| + (-1)^{2n} \sum_{k=0}^{2n} - (-1)^{-k} \ln|v_k| \right],$$

after expanding the summation inside we obtain the following

$$\begin{aligned}
|u_{2n+1}| &= \exp\left[ \ln\left|\frac{1}{u_0}\right| - \ln|v_0| + \ln|v_1| - \ln|v_2| + ... + \ln|v_{2n-1}| - \ln|v_{2n}| \right] \\
&= \exp\left[ \ln\left|\frac{1}{u_0}\right) | - \ln|v_0| + \ln\left|\frac{v_1}{v_2}\right| + \ln\left|\frac{v_3}{v_4}\right| + ... + \ln\left|\frac{v_{2n-1}}{v_{2n}}\right| \Bigg| \right] \\
&= \exp\left[ \ln\left|\frac{1}{u_0}\right| + \ln|v_0| + \ln\left|\frac{v_1}{v_2}\frac{v_3}{v_4}...\frac{v_{2n-1}}{v_{2n}}\right| \Bigg| \right] \\
u_{2n+1} &= u_0 v_0 \left( \frac{v_1}{v_2}\frac{v_3}{v_4}...\frac{v_{2n-1}}{v_{2n}} \right),
\end{aligned}$$

$$u_{2n+1} = u_1 \prod_{s=0}^{n-1} \left( \frac{v_{2s+1}}{v_{2s+2}} \right). \tag{35b}$$

**Note.** The obtention of (35), using (27), is somewhat straightforward and it does not require the use of absolute values. However, we recall that one of the objectives of this work is to write the solution of (4) in a unified manner as in (34).

We combine (33), (35a) and (35b) to get the solutions in closed form of (4):

$$u_{2n} = \frac{{u_2}^n}{{u_0}^{n-1}} \prod_{s=1}^{n-1} \left( \frac{\left[\prod_{k_1=0}^{s-1} A_{2k_1}\right] + u_0 u_1 \left[\sum_{l=0}^{s-1} \left( B_{2l} \prod_{k_2=l+1}^{s-1} A_{2k_2} \right)\right]}{\left[\prod_{k_1=0}^{s-1} A_{2k_1+1}\right] + u_1 u_2 \left[\sum_{l=0}^{s-1} \left( B_{2l+1} \prod_{k_2=l+1}^{s-1} A_{2k_2+1} \right)\right]} \right) \tag{36a}$$

$$u_{2n+1} = \frac{u_1 u_0^n}{u_2^n} \frac{\prod_{s=1}^{n-1} \left( \left[\prod_{k_1=0}^{s-1} A_{2k_1+1}\right] + u_1 u_2 \left[\sum_{l=0}^{s-1} \left( B_{2l+1} \prod_{k_2=l+1}^{s-1} A_{2k_2+1} \right)\right] \right)}{\prod_{s=0}^{n-1} \left( \left[\prod_{k_1=0}^{s} A_{2k_1}\right] + u_0 u_1 \left[\sum_{l=0}^{s} \left( B_{2l} \prod_{k_2=l+1}^{s} A_{2k_2} \right)\right] \right)}, \tag{36b}$$

where $u_0, u_1, u_2$ are non-zero real numbers such that the denominators are different from zero, i.e.,

$$u_0 u_1 \sum_{l=0}^{s} B_{2l}\theta(l+1, s, A_{2k_2}) \neq -\theta(0, s, A_{2k_1}) \tag{36c}$$

and

$$u_1 u_2 \sum_{l=0}^{s-1} B_{2l+1}\theta(l+1, s-1, A_{2k_2+1}) \neq -\theta(0, s-1, A_{2k_1+1}). \tag{36d}$$

We may then write solutions of (1) as follows:

$$\begin{aligned} x_{2n} =& u_{2(n+1)} \\ =& \frac{x_0^{n+1}}{x_{-2}^{n}} \prod_{s=1}^{n} \left( \frac{\left[\prod\limits_{k_1=0}^{s-1} a_{2k_1}\right] + x_{-2}x_{-1}\left[\sum\limits_{l=0}^{s-1}\left(b_{2l}\prod\limits_{k_2=l+1}^{s-1} a_{2k_2}\right)\right]}{\left[\prod\limits_{k_1=0}^{s-1} a_{2k_1+1}\right] + x_{-1}x_0\left[\sum\limits_{l=0}^{s-1}\left(b_{2l+1}\prod\limits_{k_2=l+1}^{s-1} a_{2k_2+1}\right)\right]} \right) \end{aligned} \tag{37a}$$

and

$$\begin{aligned} x_{2n+1} =& u_{2(n+1)+1} \\ =& \frac{x_{-1}{x_{-2}}^{n+1}}{{x_0}^{n+1}} \frac{\prod\limits_{s=1}^{n}\left(\left[\prod\limits_{k_1=0}^{s-1} a_{2k_1+1}\right] + x_{-1}x_0\left[\sum\limits_{l=0}^{s-1}\left(b_{2l+1}\prod\limits_{k_2=l+1}^{s-1} a_{2k_2+1}\right)\right]\right)}{\prod\limits_{s=0}^{n}\left(\left[\prod\limits_{k_1=0}^{s} a_{2k_1}\right] + x_{-2}x_{-1}\left[\sum\limits_{l=0}^{s}\left(b_{2l}\prod\limits_{k_2=l+1}^{s} a_{2k_2}\right)\right]\right)}, \end{aligned} \tag{37b}$$

where $x_{-2}$, $x_{-1}$ and $x_0$ are non-zero real numbers such that the denominators are different from zero, i.e.,

$$x_{-2}x_{-1} \sum_{l=0}^{s} b_{2l}\theta(l+1, s, a_{2k_2}) \neq -\theta(0, s, a_{2k_1}) \tag{37c}$$

and

$$x_{-1}x_0 \sum_{l=0}^{s-1} b_{2l+1}\theta(l+1, s-1, a_{2k_2+1}) \neq -\theta(0, s-1, a_{2k_1+1}). \tag{37d}$$

The aim of the next subsections is to show that, in fact, the results in paper [15] are special cases of (37).

*The case where $(a_n)$ and $(b_n)$ are real two-periodic sequences*

If we let $(a_n)_{n\in\mathbb{N}_0} = \{\alpha,\, \beta,\, \dots\}$, $(b_n)_{n\in\mathbb{N}_0} = \{\zeta, \eta,\, \zeta,\, \dots\}$, $m = x_0$, $l = x_{-1}$, $h = x_{-2}$, $\Phi = \zeta lh$, $\Psi = \eta ml$, equations in (37) simplify to

$$x_{2n} = \frac{m^{n+1}}{h^n} \prod_{s=1}^{n} \left( \frac{\alpha^s + \Phi \sum\limits_{l=0}^{s-1} \alpha^l}{\beta^s + \Psi \sum\limits_{l=0}^{s-1} \beta^l} \right) \tag{38a}$$

and

$$x_{2n+1} = \frac{lh^{n+1}}{m^{n+1}} \frac{\prod\limits_{s=1}^{n} \left( \beta^s + \Psi \sum\limits_{l=0}^{s-1} \beta^l \right)}{\prod\limits_{s=0}^{n} \left( \alpha^{s+1} + \Phi \sum\limits_{l=0}^{s} \alpha^l \right)}, \tag{38b}$$

where $m$, $h$ and $l$ are non-zero real numbers such that

$$\Psi \sum_{l=0}^{s-1} \beta^l \neq -\beta^s \tag{38c}$$

and

$$\Phi \sum_{l=0}^{s} \alpha^l \neq -\alpha^{s+1}, \qquad s \leq n. \tag{38d}$$

This corresponds to the results obtained in Theorem 1 in [15] and their restriction ($\alpha,\, \beta, \Phi,\, \Psi \in (0, +\infty)$ ) is a special case of our restriction given in (38c).

*The case where $(a_n)$ and $(b_n)$ are real constants*

If we let $(a_n)_{n\in\mathbb{N}_0} = \{a, a,\, \dots\}$, $(b_n)_{n\in\mathbb{N}_0} = \{b, b,\, b,\, \dots\}$, $m = x_0$, $l = x_{-1}$, $h = x_{-2}$, $\Phi = blh$, $\Psi = bml$, equations in (37) simplify to

$$x_{2n} = \frac{m^{n+1}}{h^n} \prod_{s=1}^{n} \left( \frac{a^s + \Phi \sum\limits_{l=0}^{s-1} a^l}{a^s + \Psi \sum\limits_{l=0}^{s-1} a^l} \right) \tag{39a}$$

and

$$x_{2n+1} = \frac{lh^{n+1}}{m^{n+1}} \frac{\prod_{s=1}^{n}\left(a^s + \Psi \sum_{l=0}^{s-1} a^l\right)}{\prod_{s=0}^{n}\left(a^{s+1} + \Phi \sum_{l=0}^{s} a^l\right)}, \tag{39b}$$

where $m$, $h$ and $l$ are non-zero real numbers such that

$$\Psi \sum_{l=0}^{s-1} a^l \neq -a^s \tag{39c}$$

and

$$\Phi \sum_{l=0}^{s} a^l \neq -a^{s+1}, \qquad s \leq n. \tag{39d}$$

This corresponds to the results obtained in Theorem 4 in [15] and their restriction ($a$, $\Phi$, $\Psi \in (0, +\infty)$ ) is a special case of our restriction given in (39c).

- **The case where $a = 0$ and $b \neq 0$.**

  Equations in (39) simplify to

$$x_{2n} = \frac{m^{n+1}}{h^n} \prod_{s=1}^{n}\left(\frac{\Phi}{\Psi}\right) \tag{40a}$$

$$= m \tag{40b}$$

  and

$$x_{2n+1} = \frac{lh^{n+1}}{m^{n+1}} \frac{\prod_{s=1}^{n} \Psi}{\prod_{s=0}^{n} \Phi} \tag{40c}$$

$$= \frac{1}{mb}, \tag{40d}$$

  where $m$, $h$ and $l$ are non-zero real numbers such that

$$\Psi \neq 0 \tag{40e}$$

and

$$\Phi \neq 0. \tag{40f}$$

Note that here, $m = h$ and $b \neq 0$ and therefore condition (40e) reduces to $ml \neq 0$. This corresponds to the results obtained in Theorem 4 in [15] and their restriction coincides with our restriction in this case.

- **The case where $a \neq 0$ and $b = 0$.**

  Equations in (39) simplify to

$$x_{2n} = \frac{m^{n+1}}{h^n} \prod_{s=1}^{n} \left( \frac{a^s}{a^s} \right) \tag{41a}$$

$$= \frac{m^{n+1}}{h^n} \tag{41b}$$

and

$$x_{2n+1} = \frac{lh^{n+1}}{m^{n+1}} \frac{\prod\limits_{s=1}^{n} (a^s)}{\prod\limits_{s=0}^{n} (a^{s+1})} \tag{41c}$$

$$= \frac{lh^{n+1}}{a^{n+1} m^{n+1}}, \tag{41d}$$

where $m$, $h$ and $l$ are non-zero real numbers such that

$$0 \neq -a. \tag{41e}$$

Note that here, $a \neq 0$ and therefore condition (41e) reduces to $mlh \neq 0$. This corresponds to the results obtained in Theorem 4 in [15] and their restriction coincides with our restriction given in (41e).

- **The case where $a \neq 1$.**

  In this case, we can write equations in (39) in a less complicated way as follows:

$$x_{2n} = \frac{m^{n+1}}{h^n} \prod_{s=1}^{n} \left( \frac{a^s + \Phi \sum\limits_{l=0}^{s-1} a^l}{a^s + \Psi \sum\limits_{l=0}^{s-1} a^l} \right) \tag{42a}$$

$$= \frac{m^{n+1}}{h^n} \prod_{s=1}^{n} \left[ \frac{a^s + \Phi \left( \frac{1-a^s}{1-a} \right)}{a^s + \Psi \left( \frac{1-a^s}{1-a} \right)} \right] \tag{42b}$$

and

$$x_{2n+1} = \frac{l h^{n+1}}{m^{n+1}} \frac{\prod\limits_{s=1}^{n} \left( a^s + \Psi \sum\limits_{l=0}^{s-1} a^l \right)}{\prod\limits_{s=0}^{n} \left( a^{s+1} + \Phi \sum\limits_{l=0}^{s} a^l \right)} \tag{42c}$$

$$= \frac{l h^{n+1}}{m^{n+1}} \frac{\prod\limits_{s=1}^{n} \left[ a^s + \Psi \left( \frac{1-a^s}{1-a} \right) \right]}{\prod\limits_{s=0}^{n} \left[ a^{s+1} + \Phi \left( \frac{1-a^{s+1}}{1-a} \right) \right]}, \tag{42d}$$

where $m$, $h$ and $l$ are non-zero real numbers such that

$$\begin{aligned} &\Psi \neq -a^s(1-a)/(1-a^s), \quad 1 \leq s \leq n \text{ and} \\ &\Phi \neq -a^s(1-a)/(1-a^s), \qquad 1 \leq s \leq n+1. \end{aligned} \tag{42e}$$

## 1.3 Invariance Analysis of some fifth-order difference equations

Consider the difference equations of the form (5), i.e.,

$$u_{n+5} = \frac{u_n u_{n+1} u_{n+2}}{u_{n+3} u_{n+4} (\lambda + \mu u_n u_{n+1} u_{n+2})}.$$

### 1.3.1 Symmetries

We impose the symmetry condition (10) and we simplify the resulting equation to get

$$\begin{aligned}
&Q(n+5,u_{n+5})+\frac{u_n u_{n+1} u_{n+2}}{u_{n+3}{u_{n+4}}^2(\mu u_n u_{n+1} u_{n+2}+\lambda)}Q(n+4,u_{n+4})\\
&+\frac{u_n u_{n+1} u_{n+2}}{{u_{n+3}}^2 u_{n+4}(\mu u_n u_{n+1} u_{n+2}+\lambda)}Q(n+3,u_{n+3})\\
&-\frac{\lambda u_n u_{n+1}}{u_{n+3} u_{n+4}(\mu u_n u_{n+1} u_{n+2}+\lambda)^2}Q(n+2,u_{n+2})\\
&-\frac{\lambda u_n u_{n+2}}{u_{n+3} u_{n+4}(\mu u_n u_{n+1} u_{n+2}+\lambda)^2}Q(n+1,u_{n+1})\\
&-\frac{\lambda u_{n+1} u_{n+2}}{u_{n+3} u_{n+4}(\mu u_n u_{n+1} u_{n+2}+\lambda)^2}Q(n,u_n)=0.
\end{aligned} \tag{43}$$

We differentiate (43) with respect to $u_n$ ( keeping $\Omega$ fixed and viewing $u_{n+3}$ as a function of $u_n, u_{n+1},\ u_{n+2}$ and $\Omega$) to get

$$\begin{aligned}
&\frac{ab u_{n+3}{u_n}^2{u_{n+1}}^2{u_{n+2}}^2+\lambda u_{n+1}u_{n+2}}{u_{n+3}u_{n+4}}Q'(n+3,u_{n+3})-\frac{\lambda u_{n+1}u_{n+2}}{{u_{n+3}}^2u_{n+4}}Q(n+3,u_{n+3})\\
&+\frac{\lambda\mu u_n {u_{n+1}}^2 u_{n+2}}{u_{n+3}u_{n+4}(\mu u_n u_{n+1}u_{n+2}+\lambda)}Q(n+2,u_{n+2})\\
&+\frac{\lambda\mu u_n u_{n+1} {u_{n+2}}^2}{u_{n+3}u_{n+4}(\mu u_n u_{n+1}u_{n+2}+\lambda)}Q(n+1,u_{n+1})\\
&+\frac{(2\,\lambda\mu u_n {u_{n+1}}^2{u_{n+2}}^2+\lambda^2 u_{n+1}u_{n+2})}{u_n u_{n+3}u_{n+4}(\mu u_n u_{n+1}u_{n+2}+\lambda)}Q(n,u_n)-\frac{\lambda u_{n+1}u_{n+2}}{u_{n+3}u_{n+4}}Q'(n,u_n)=0.
\end{aligned} \tag{44}$$

We then differentiate (44) with respect to $u_n$ twice (keeping $u_{n+3}$ fixed). This gives

$$\begin{aligned}
&-\mu u_n u_{n+1}u_{n+2}Q^{(3)}(n,u_n)-\lambda Q^{(3)}(n,u_n)+\frac{\lambda}{u_n}Q^{(2)}(n,u_n)\\
&-\frac{2\,\lambda}{{u_n}^2}Q'(n,u_n)+\frac{2\,\lambda}{{u_n}^3}Q(n,u_n)=0.
\end{aligned} \tag{45}$$

The characteristic $Q$ in (45) is independent of $u_{n+1}$ and $u_{n+2}$. Therefore, we can separate with respect to these variables. This gives a system below

$$\begin{aligned} u_{n+1}u_{n+2} &: -\mu u_n Q^{(3)}(n,u_n) = 0 \\ 1 &: Q^{(3)}(n,u_n) - \frac{1}{u_n}Q^{(2)}(n,u_n) + \frac{2}{{u_n}^2}Q'(n,u_n) - \frac{2}{{u_n}^3}Q(n,u_n) = 0 \end{aligned} \tag{46}$$

whose solution is given by

$$Q_1\left(n,u_n\right) = \beta^n u_n \tag{47}$$

and

$$Q_2(n,u_n) = \bar{\beta}^n u_n, \tag{48}$$

where $\beta = \exp\left(-2\pi i/3\right)$. Thus, we obtain two characteristics with corresponding generators given by

$$\begin{aligned} X_1 =& \bar{\beta}^n u_n \partial u_n + \bar{\beta}^{n+1}u_{n+1}\partial_{u_{n+1}} + \beta\bar{\beta}^n u_{n+2}\partial_{u_{n+2}} + \bar{\beta}^n u_{n+3}\partial_{u_{n+3}} \\ &+ \bar{\beta}^{n+1}u_{n+4}\partial_{u_{n+4}} \end{aligned} \tag{49a}$$

$$\begin{aligned} X_2 =& \beta^n u_n \partial u_n + \beta^{n+1}u_{n+1}\partial_{u_{n+1}} + \bar{\beta}\beta^n u_{n+2}\partial_{u_{n+2}} + \beta^n u_{n+3}\partial_{u_{n+3}} \\ &+ \beta^{n+1}u_{n+4}\partial_{u_{n+4}}. \end{aligned} \tag{49b}$$

### 1.3.2 Reduction and exact solutions

From the characteristic equations

$$\frac{du_n}{\bar{\beta}^n u_n} = \frac{du_{n+1}}{\bar{\beta}^{n+1}u_{n+1}} = \frac{du_{n+2}}{\bar{\beta}^{n+2}u_{n+2}} = \frac{du_{n+3}}{\bar{\beta}^{n+3}u_{n+3}} = \frac{du_{n+4}}{\bar{\beta}^{n+4}u_{n+4}} \left(= \frac{V_n}{0}\right), \tag{50}$$

we obtain the invariants $c_1 = u_{n+1}^{\beta}/u_n, c_2 = u_{n+2}^{\bar{\beta}}/u_n, c_3 = u_n u_{n+3}, c_4 = u_{n+4}^{\beta}/u_n$ and $c_5 = V_n$. We readily notice that

$$S^3\left(\frac{du_n}{\bar{\beta}^n u_n} = \frac{du_{n+1}}{\bar{\beta}^{n+1}u_{n+1}}\right) = \frac{du_{n+3}}{\bar{\beta}^{n+3}u_{n+3}} = \frac{du_{n+4}}{\bar{\beta}^{n+4}u_{n+4}} \tag{51}$$

and we choose $V_n = c_1^{\bar{\beta}} c_2^{\beta}$, i.e.,

$$V_n = u_n u_{n+1} u_{n+2}. \tag{52}$$

By shifting (52) thrice, we get

$$V_{n+3} = \frac{V_n}{\lambda + \mu V_n} \tag{53}$$

whose solution is given by

$$V_n = \begin{cases} \left( \frac{\mu\left[((-1)^{2/3}-1)n-(-1)^{1/3}+1+\beta\right]}{3\left((-1)^{2/3}-1\right)} + c_6 + \bar{\beta}^n c_7 + \beta^n c_8 \right)^{-1} & \text{if} \quad \lambda = 1, \\ \left( c_{10}\lambda^{n/3} + \left[(-1)^{2/3}\lambda^{1/3}\right]^n c_8 + \left[-(-1)^{1/3}\lambda^{1/3}\right]^n c_9 + \frac{\mu}{1-\lambda} \right)^{-1} & \text{if } \lambda \neq 1. \end{cases} \tag{54}$$

The constants $c_i,\ i = 6, \ldots, 10$ can be obtained from the following equations:

$$c_6 + c_7 + c_8 = \frac{1}{u_0 u_1 u_2} - \frac{\mu\left[-(-1)^{1/3} + 1 + \beta\right]}{3\left((-1)^{2/3} - 1\right)}, \tag{55a}$$

$$c_6 + \bar{\beta} c_7 + \beta c_8 = \frac{1}{u_1 u_2 u_3} - \frac{\mu\left[\beta - (-1)^{1/3} + (-1)^{2/3}\right]}{3\left((-1)^{2/3} - 1\right)}, \tag{55b}$$

$$c_6 + \beta c_7 + \bar{\beta} c_8 = \frac{1}{u_2 u_3 u_4} - \frac{\mu\left[-1 + 2(-1)^{2/3} - (-1)^{1/3} + \beta\right]}{3\left((-1)^{2/3} - 1\right)}, \tag{55c}$$

$$c_{10} + c_8 + c_9 = \frac{1}{u_0 u_1 u_2} - \frac{\mu}{1-\lambda}, \tag{55d}$$

$$\lambda^{1/3} c_{10} + [(-1)^{2/3}\lambda^{1/3}] c_8 + [-(-1)^{1/3}\lambda^{1/3}] c_9 = \frac{1}{u_1 u_2 u_3} - \frac{\mu}{1-\lambda}, \tag{55e}$$

$$\lambda^{2/3} c_{10} + [(-1)^{2/3}\lambda^{1/3}]^2 c_8 + [-(-1)^{1/3}\lambda^{1/3}]^2 c_9 = \frac{1}{u_2 u_3 u_4} - \frac{\mu}{1-\lambda}. \tag{55f}$$

Thanks to (52), we can express $u_n$ in terms of $V_n$ as follows:

$$u_n = \exp\left( \beta^n c_{11} + \bar{\beta}^n c_{12} - \frac{2}{\sqrt{3}} \left[ \sum_{k=0}^{n-1} \operatorname{Im}(\gamma(n,k)) \ln V_k \right] \right), \tag{56a}$$

where $V_k$ is given in (54) with $\gamma(n,k) = \beta^n \bar{\beta}^{k+1}$. The constants $c_{11}$ and $c_{12}$ must satisfy

$$c_{11} + c_{12} = \ln u_0, \tag{56b}$$

$$\beta c_{11} + \bar{\beta} c_{12} = \ln u_1. \tag{56c}$$

Equations in (56) give the solutions of (5) in a unified manner.

For the sake of clarification, we now want to split solutions (56a) to realise the solutions in existing literature. Using (56a) and (56b) , we have

$$u_{6n} = u_0 \prod_{s=1}^{2n} \frac{V_{3s-2}}{V_{3s-3}}. \tag{57}$$

Using the same approach, we have shown that

$$u_{6n+i} = u_i \prod_{s=1}^{2n} \frac{V_{3(s-1)+i+1}}{V_{3(s-1)+i}}, \quad i = 0, \ldots, 5. \tag{58}$$

*The case of $\lambda = 1$*

Equation (5) becomes

$$u_{n+5} = \frac{u_n u_{n+1} u_{n+2}}{u_{n+3} u_{n+4}(1 + \mu u_n u_{n+1} u_{n+2})}. \tag{59}$$

and we said earlier that the solution of (53), in this case, is (54), i.e.,

$$V_n = \left( \frac{\mu \left[ ((-1)^{2/3} - 1)n - (-1)^{1/3} + 1 + \beta \right]}{3\left((-1)^{2/3} - 1\right)} + c_6 + \bar{\beta}^n c_7 + \beta^n c_8 \right)^{-1}. \tag{60}$$

We have that

$$V_{3s} = \left( \frac{\mu \left[ ((-1)^{2/3} - 1)(3s) - (-1)^{1/3} + 1 + \beta \right]}{3\left((-1)^{2/3} - 1\right)} + c_6 + c_7 + c_8 \right)^{-1} \tag{61}$$

and using (55a) in (61), we get

$$V_{3s} = \frac{u_0 u_1 u_2}{1 + \mu s u_0 u_1 u_2}. \tag{62}$$

Using the same approach, we have shown that

$$V_{3s} = \frac{u_0 u_1 u_2}{1 + \mu s u_0 u_1 u_2}; \tag{63a}$$

$$V_{3s+1} = \frac{u_1 u_2 u_3}{1 + \mu s u_1 u_2 u_3}; \tag{63b}$$

$$V_{3s+2} = \frac{u_2 u_3 u_4}{1 + \mu s u_2 u_3 u_4}. \tag{63c}$$

Let $a = x_{-4},\ b = x_{-3},\ c = x_{-2},\ d = x_{-1}$ and $A = u_0,\ B = u_1$, $C = u_2$, $D = u_3$ and $E = u_4$. Using (63) in (58), we obtain the solution of (59) as follows:

$$
\begin{aligned}
u_{6n} &= \frac{D^{2n}}{A^{2n-1}} \prod_{s=1}^{2n-1} \frac{1+\mu s ABC}{1+\mu s BCD},\\
u_{6n+1} &= \frac{E^{2n}}{B^{2n-1}} \prod_{s=1}^{2n-1} \frac{1+\mu s BCD}{1+\mu s CDE},\\
u_{6n+2} &= \frac{CA^{2n}B^{2n}}{D^{2n}E^{2n}} \prod_{s=0}^{2n-1} \frac{1+\mu s CDE}{1+\mu(s+1)ABC},\\
u_{6n+3} &= \frac{D^{2n+1}}{A^{2n}} \prod_{s=1}^{2n} \frac{1+\mu s ABC}{1+\mu s BCD},\\
u_{6n+4} &= \frac{E^{2n+1}}{B^{2n}} \prod_{s=0}^{2n} \frac{1+\mu s BCD}{1+\mu s CDE},\\
u_{6n+5} &= \frac{C(AB)^{2n+1}}{(DE)^{2n+1}} \prod_{s=0}^{2n} \frac{1+\mu s CDE}{1+\mu(s+1)ABC}
\end{aligned}
\tag{64}
$$

whenever the denominators do not vanish.

*The case of $\lambda \neq 1$*

In this case, as we found earlier, the solution of (53) is given by (54), i.e.,

$$
V_n = \left( c_{10}\lambda^{n/3} + [(-1)^{2/3}\lambda^{1/3}]^n c_8 + [-(-1)^{1/3}\lambda^{1/3}]^n c_9 + \frac{\mu}{1-\lambda} \right)^{-1}. \tag{65}
$$

Using this, we get

$$
V_{3s} = \left( c_{10}\lambda^s + c_8\lambda^s + c_9\lambda^s + \frac{\mu}{1-\lambda} \right)^{-1} \tag{66}
$$

and using (55d) in (66), we find

$$
V_{3s} = \frac{ABC}{\lambda^s + \mu ABC\left(\frac{1-\lambda^s}{1-\lambda}\right)}. \tag{67}
$$

Using the same approach, we have shown that

$$V_{3s+i} = \frac{u_i u_{i+1} u_{i+2}}{\lambda^s + \mu u_i u_{i+1} u_{i+2} \left(\frac{1-\lambda^s}{1-\lambda}\right)}, \quad i = 0, 1, 2. \tag{68}$$

Using (68) in (58), we obtain the solution of (5) as follows:

$$u_{6n} = \frac{D^{2n}}{A^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu\Delta_0 \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu\Delta_1 \sum\limits_{j=0}^{s-2} \lambda^j},$$

$$u_{6n+1} = \frac{E^{2n}}{B^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu\Delta_1 \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu\Delta_2 \sum\limits_{j=0}^{s-2} \lambda^j}$$

$$u_{6n+2} = C\frac{\Delta_0^{2n}}{\Delta_1^{2n}} \frac{\prod\limits_{s=2}^{2n} \left(\lambda^{s-1} + \mu\Delta_2 \sum\limits_{j=0}^{s-2} \lambda^j\right)}{\prod\limits_{s=1}^{2n} \left(\lambda^{s} + \mu\Delta_0 \sum\limits_{j=0}^{s-1} \lambda^j\right)},$$

$$u_{6n+3} = \frac{D^{2n+1}}{A^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu\Delta_0 \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu\Delta_1 \sum\limits_{j=0}^{s-2} \lambda^j}$$

$$u_{6n+4} = E\frac{\Delta_2^{2n}}{\Delta_1^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu\Delta_1 \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu\Delta_2 \sum\limits_{j=0}^{s-2} \lambda^j},$$

$$u_{6n+5} = C\frac{\Delta_0^{2n+1}}{\Delta_1^{2n+1}} \frac{\prod\limits_{s=2}^{2n+1} \left(\lambda^{s-1} + \mu\Delta_2 \sum\limits_{j=0}^{s-2} \lambda^j\right)}{\prod\limits_{s=1}^{2n+1} \left(\lambda^{s} + \mu\Delta_0 \sum\limits_{j=0}^{s-1} \lambda^j\right)}, \tag{69}$$

where $\Delta_i = u_i u_{i+1} u_{i+2}$. Equations in (69) give the exact solution of (5) for any real values of $\lambda$ and $\mu$ provided that the denominators do not vanish.

Recall that we acted the shift operator on (2) to get (5). Hence, the solutions of (2) are obtained, using (69), as follows:

$$
\begin{aligned}
x_{6n-4} &= \frac{d^{2n}}{a^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu abc \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu bcd \sum\limits_{j=0}^{s-2} \lambda^j}, \\
x_{6n-3} &= \frac{e^{2n}}{b^{2n-1}} \prod_{s=2}^{2n} \frac{\lambda^{s-1} + \mu bcd \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu cde \sum\limits_{j=0}^{s-2} \lambda^j} \\
x_{6n-2} &= \frac{c(ab)^{2n}}{(de)^{2n}} \frac{\prod\limits_{s=2}^{2n} \left( \lambda^{s-1} + \mu cde \sum\limits_{j=0}^{s-2} \lambda^j \right)}{\prod\limits_{s=1}^{2n} \left( \lambda^{s} + \mu abc \sum\limits_{j=0}^{s-1} \lambda^j \right)}, \\
x_{6n-1} &= \frac{d^{2n+1}}{a^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu abc \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu bcd \sum\limits_{j=0}^{s-2} \lambda^j}, \\
x_{6n} &= \frac{e^{2n+1}}{b^{2n}} \prod_{s=2}^{2n+1} \frac{\lambda^{s-1} + \mu bcd \sum\limits_{j=0}^{s-2} \lambda^j}{\lambda^{s-1} + \mu cde \sum\limits_{j=0}^{s-2} \lambda^j}, \\
x_{6n+1} &= \frac{c(ab)^{2n+1}}{(de)^{2n+1}} \frac{\prod\limits_{s=2}^{2n+1} \left[ \lambda^{s-1} + \mu cde \sum\limits_{j=0}^{s-2} \lambda^j \right]}{\prod_{s=1}^{2n+1} \left( \lambda^{s} + \mu abc \sum\limits_{j=0}^{s-1} \lambda^j \right)}
\end{aligned}
\tag{70}
$$

for any real values of $\lambda$ and $\mu$ as long as the denominators do not vanish.

- When $\lambda = 1$ and $\mu = 1$, equations in (70) yield the results obtained by Yasin Yazlik in Theorem 5 in [16] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \ldots \tag{71}$$

where $a,\ b,\ c,\ c,\ d$ and $e$ are positive real numbers.

- When $\lambda = 1$ and $\mu = -1$, equations in (70) yield the results obtained by Yasin Yazlik in Theorem 9 in [16] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \ldots \tag{72}$$

where $a,\ b,\ c,\ c,\ d$ and $e$ are positive real numbers with $abc \neq 1$ and $cde \neq 1$ .

**Note.** There should not be a minus sign right after the expression of $x_{3n-2}$ in Theorem 9 in [16].

- When $\lambda = -1$ and $\mu = 1$, equations in (70) yield the results obtained by Yasin Yazlik in Theorem 7 in [16] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(-1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \ldots \tag{73}$$

where $a,\ b,\ c,\ d$ and $e$ are non zero real numbers with $abc \neq 1,\ bcd \neq 1$ and $cde \neq 1$.

- When $\lambda = -1$ and $\mu = -1$, equations in (70) yield the results obtained by Yasin Yazlik in Theorem 11 in [16] for

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(-1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, 2, \ldots \tag{74}$$

where $a,\ b,\ c,\ c,\ d$ and $e$ are non zero real numbers with $abc \neq -1$ , $bcd \neq -1$ and $cde \neq -1$ .

**Note.** There should be a minus sign right after the expression of $x_{6n+1}$ in Theorem 11 in [16].

# Chapter 2

# Perturbed ODEs in Cosmology

Eliminating all the perturbed terms in the Lagrangian (3), leads to the derivation of the oscillation equation

$$u'' + u = 0. \tag{75}$$

It is easily seen that this unperturbed equation is maximally symmetric and admits the 8-dimensional Lie algebra of exact symmetries $sl(3, R)$ given by

$$\begin{aligned}
X_0^1 &= \partial_\phi,\\
X_0^2 &= \sin(2\phi)\partial_\phi + \cos(2\phi)u\partial_u,\\
X_0^3 &= \cos(2\phi)\partial_\phi - \sin(2\phi)u\partial_u,\\
X_0^4 &= \sin(\phi)\partial_u,\\
X_0^5 &= \cos(\phi)\partial_u,\\
X_0^6 &= u\partial_u,\\
X_0^7 &= u\cos(\phi)\partial_\phi - u^2\sin(\phi)\partial_u,\\
X_0^8 &= u\sin(\phi)\partial_\phi + u^2\cos(\phi)\partial_u.
\end{aligned}$$

To illustrate our main results or derived conditions, some examples are presented in the chapter. One case explores the approximate symmetries of an orbital equation that arises when a Reissner-Nordström black hole is embedded into a Friedman-Robertson-Walker (FRW) space [19]. As a second case, we investigate the modified Klein-Gordon equation of a spin-0 particle in the Generalized Uncertainty Principle (GUP) [20, 21, 22, 23, 24].

## 2.1 Point transformations

Our interest lies in point transformations, and for the convenience of the reader we insert the necessary theory pertaining to this analysis. The presentation here is for ODEs, however in the references cited, most of the theory has been generalized to partial differential equations (PDEs). For the sake of brevity, the summation convention is adopted in this text, in which there is summation over all repeated indices. For a $k$-th order perturbed system of ODEs

$$E = E_0 + \epsilon E_1 + \epsilon^2 E_2 + \ldots + \epsilon^k E_k + O(\epsilon^{k+1}), \tag{76}$$

corresponding to a Lagrangian, which is perturbed in $\epsilon$,

$$\mathcal{L}(t, x, x'^j, \epsilon) = \mathcal{L}_0(t, x, x'^j) + \epsilon \mathcal{L}_1(t, x, x'^j) + \ldots + O(\epsilon^{k+1}), \tag{77}$$

with $\int \mathcal{L} dt$ being a function that is invariant under the one-parameter group of transformations with approximate Lie symmetry generator

$$X = X_0 + \epsilon X_1 + \ldots + \epsilon^k X_k, \tag{78}$$

up to gauge

$$A = A_0 + \epsilon A_1 + \ldots + \epsilon^k A_k, \tag{79}$$

if

$$X\mathcal{L} + \left( D_t \frac{\partial \bar{t}}{\partial \varepsilon}\bigg|_{\varepsilon=0} \right) \mathcal{L} = D_t A, \tag{80}$$

where $D_t$ is the total derivative operator. In this notation, $X_0$ denotes the exact symmetry generator originating from the unperturbed Lagrangian and $X_1$ denotes the first-order approximate symmetry generator. A perturbed equation always admits the trivial approximate symmetry generator $\epsilon X_0$. A nontrivial symmetry is obtained when, $X = X_0 + \epsilon X_1$ exists with $X_0 \neq 0$ and $X_1 \neq kX_0$ (k an arbitrary constant)[25]. The first-order approximate first integrals are defined by $I = I_0 + \epsilon I_1$, with $I_0$ denoting the exact part and $I_1$ denoting the first-order approximate part of the first-order approximate first integrals

$$\begin{gathered} I_0 = \xi_0 L_0 + \left(\eta_0^\mu - x'^\mu \xi_0\right) \tfrac{\partial L_0}{\partial x'^\mu} - A_0, \\ I_1 = \xi_0 L_1 + \xi_1 L_0 + \left(\eta_0^\mu - x'^\mu \xi_0\right) \tfrac{\partial L_1}{\partial x'^\mu} + \left(\eta_1^\mu - x'^\mu \xi_1\right) \tfrac{\partial L_0}{\partial x'^\mu} - A_1. \end{gathered}$$

## 2.2 Third-Order Conditions

As mentioned above, Noether symmetries are just a specialization of Lie symmetries, and thus the $sl(3,R)$ algebra given above contains the Noether point symmetry generators. The latter comprises of a 5-dimensional Lie algebra $X_0^{1-5}$ with the corresponding gauge term (the $c_j$ are constants)

$$A_0 = u^2 \cos(2\,\phi)\, c_3 + u^2 \sin(2\,\phi)\, c_2 + \sin(\phi)\, c_5\, u - \cos(\phi)\, c_4\, u + c_6.$$

The conservation laws or Noether first integrals corresponding to each $X_0^h, h = 1, \ldots, 5$ are

$$\begin{gathered} I_0^1 = \tfrac{1}{2}\left(u^2 + u'^2\right), \\ I_0^2 = \tfrac{1}{2}\left(u'^2 - u^2\right)\sin(2\phi) - uu'\cos(2\phi), \\ I_0^3 = \tfrac{1}{2}\left(u'^2 - u^2\right)\cos(2\phi) + uu'\sin(2\phi), \\ I_0^4 = -u'\sin(\phi) + u\cos(\phi), \\ I_0^5 = -u'\cos(\phi) - u\sin(\phi). \end{gathered}$$

If we include a perturbation up to first-order in $\epsilon$, that is, the Lagrangian (3) omits the terms in $G_2(u,u',u'')$ and $G_3(u,u',u'')$, the determination of approximate symmetries takes a particular form. That is, for each term of the Noether condition (80) for the Lagrangian (3) we have the geometric condition

$$\begin{aligned}
X\mathcal{L} = {} & \left(\eta_{1,\phi} + u'\eta_{1,u} - u'\xi_{1,\phi} - (u')^2\,\xi_{1,u}\right)(-u') + \eta_1 u \\
& + \left(-2\,\sin(\phi)\cos(\phi)\,c_3\,u + 2\,c_2\,u\,(\cos(\phi))^2 + c_4\sin(\phi) + c_5\,\cos(\phi) - c_2 u\right) G_{1,u} \\
& + \left(-4\,c_2\,u\cos(\phi)\sin(\phi) + 2\,\sin(\phi)\cos(\phi)\,c_3\,u' - 2\,(\cos(\phi))^2\,c_2\,u'\right) G_{1,u'} \\
& \left(-4\,(\cos(\phi))^2\,c_3\,u - c_5\sin(\phi) + c_4\,\cos(\phi) + c_2 u' + 2\,c_3\,u\right) G_{1,u'} \\
& + \left(8\,\sin(\phi)\cos(\phi)\,c_3\,u + 6\,\sin(\phi)\cos(\phi)\,c_3\,u'' - 8c_2 u\,(\cos(\phi))^2\right) G_{1,u''} \\
& - \left(6\,(\cos(\phi))^2\,c_2\,u'' - c_4\sin(\phi) - c_5\,\cos(\phi) + 4c_2 u + 3\,c_2\,u''\right) G_{1,u''},
\end{aligned} \tag{81}$$

$$\left(D_\phi \left.\frac{\partial\bar{\phi}}{\partial\varepsilon}\right|_{\varepsilon=0}\right)\mathcal{L} = (2c_2\cos(2\phi) - 2c_3\sin(2\phi))\,G_1 + (\xi_{1,\phi} + u'\xi_{1,u})\left(-\frac{(u')^2}{2} + \frac{u^2}{2}\right), \tag{82}$$

$$D_\phi A = A_{1,\phi} + u'A_{1,u}. \tag{83}$$

On the other hand, if the perturbation is up to second-order in $\epsilon$, the Lagrangian (3) omits $G_3(u,u',u'')$, and in this case the second condition is:

$$\begin{aligned}
X\mathcal{L} = {} & + \left(\eta_{2,\phi} + u'\eta_{2,u} - u'\xi_{2,\phi} - (u')^2\,\xi_{2,u}\right)(-u') + \eta_2 u + \eta_1 G_{1,u} \\
& + \left(-u'^2\xi_{1,u} + u'\eta_{1,u} - u'\xi_{1,\phi} + \eta_{1,\phi}\right) G_{1,u'} + \left(-u'^3\eta_{1,uu} - 2\,u'^2\xi_{1,u\phi}\right) G_{1,u''} \\
& + \left(\eta_{1,uu}u'^2 - 3\,\xi_{1,u}u'u'' + 2\,\eta_{1,u\phi}u'\right) G_{1,u''} - \left(\xi_{1,\phi\phi}u' + \eta_{1,u}u'' - 2\,\xi_{1,\phi}u'' + \eta_{1,\phi\phi}\right) G_{1,u''}, \\
& + \left(-2\,\sin(\phi)\cos(\phi)\,c_3\,u + 2\,c_2\,u\,(\cos(\phi))^2 + c_4\sin(\phi) + c_5\,\cos(\phi) - c_2 u\right) G_{2,u} \\
& + \left(-4\,c_2\,u\cos(\phi)\sin(\phi) + 2\,\sin(\phi)\cos(\phi)\,c_3\,u' - 2\,(\cos(\phi))^2\,c_2\,u'\right) G_{2,u'} \\
& - \left(4\,(\cos(\phi))^2\,c_3\,u - c_5\sin(\phi) + c_4\,\cos(\phi) + c_2 u' + 2\,c_3\,u\right) G_{2,u'} \\
& + \left(8\,\sin(\phi)\cos(\phi)\,c_3\,u + 6\,\sin(\phi)\cos(\phi)\,c_3\,u'' - 8c_2 u\,(\cos(\phi))^2\right) G_{2,u''} \\
& - \left(6\,(\cos(\phi))^2\,c_2\,u'' - c_4\sin(\phi) - c_5\,\cos(\phi) + 4c_2 u + 3\,c_2\,u''\right) G_{2,u''}
\end{aligned} \tag{84}$$

$$\begin{aligned}\left(D_\phi \frac{\partial\bar{\phi}}{\partial\varepsilon}\bigg|_{\varepsilon=0}\right)\mathcal{L} = \quad & \left(\xi_{1,\phi}+u'\xi_{1,u}\right)G_1+\left(\xi_{2,\phi}+u'\xi_{2,u}\right)\left(-\tfrac{(u')^2}{2}+\tfrac{u^2}{2}\right)\\ & +\left(2c_2\cos(2\phi)-2c_3\sin(2\phi)\right)G_2,\\ & D_\phi A = A_{2,\phi}+u'A_{2,u}. \end{aligned} \tag{85}$$

Last but not least, a third-order perturbation in $\epsilon$ results in the third condition

$$\begin{gathered} X\mathcal{L} = \left(\eta_{3,\phi}+u'\eta_{3,u}-u'\xi_{3,\phi}-(u')^2\,\xi_{3,u}\right)(-u')+\eta_3 u\\ +\left(-2\,\sin(\phi)\cos(\phi)\,c_3\,u+2\,c_2\,u\,(\cos(\phi))^2+c_4\sin(\phi)+c_5\,\cos(\phi)-c_2u\right)G_{3,u}\\ +\left(-4\,c_2\,u\cos(\phi)\sin(\phi)+2\,\sin(\phi)\cos(\phi)\,c_3\,u'-2\,(\cos(\phi))^2\,c_2\,u'\right)G_{3,u'}\\ -4\left((\cos(\phi))^2\,c_3\,u-c_5\sin(\phi)+c_4\,\cos(\phi)+c_2u'+2\,c_3\,u\right)G_{3,u'}\\ +\left(8\,\sin(\phi)\cos(\phi)\,c_3\,u+6\,\sin(\phi)\cos(\phi)\,c_3\,u''-8c_2u\,(\cos(\phi))^2\right)G_{3,u''}\\ -6\left((\cos(\phi))^2\,c_2\,u''-c_4\sin(\phi)-c_5\,\cos(\phi)+4c_2u+3\,c_2\,u''\right)G_{3,u''}\\ +\eta_1 G_{2,u}+\left(-u'^2\xi_{1,u}+u'\eta_{1,u}-u'\xi_{1,\phi}+\eta_{1,\phi}\right)G_{2,u'}\\ +\left(-u'^3\eta_{1,uu}-2\,u'^2\xi_{1,u\phi}\right)G_{2,u''}\\ +\left(\eta_{1,uu}u'^2-3\,\xi_{1,u}u'u''+2\,\eta_{1,u\phi}u'\right)G_{2,u''}-\\ \left(\xi_{1,\phi\phi}u'+\eta_{1,u}u''-2\,\xi_{1,\phi}u''+\eta_{1,\phi\phi}\right)G_{2,u''}\\ +\eta_2 G_{1,u}+\left(-u'^2\xi_{2,u}+u'\eta_{2,u}-u'\xi_{2,\phi}+\eta_{2,\phi}\right)G_{1,u'}\\ +\left(-u'^3\eta_{2,uu}-2\,u'^2\xi_{2,u\phi}\right)G_{1,u''}\\ +\left(\eta_{2,uu}u'^2-3\,\xi_{2,u}u'u''+2\,\eta_{2,u\phi}u'\right)G_{1,u''}\\ -\left(\xi_{2,\phi\phi}u'+\eta_{2,u}u''-2\,\xi_{2,\phi}u''+\eta_{2,\phi\phi}\right)G_{1,u''}, \end{gathered} \tag{86}$$

$$\begin{aligned}\left(D_\phi \frac{\partial\bar{\phi}}{\partial\varepsilon}\bigg|_{\varepsilon=0}\right)\mathcal{L} = \quad & \left(\xi_{2,\phi}+u'\xi_{2,u}\right)G_1+\left(2c_2\cos(2\phi)-2c_3\sin(2\phi)\right)G_3+\\ & \left(\xi_{1,\phi}+u'\xi_{1,u}\right)G_2+\left(\xi_{3,\phi}+u'\xi_{3,u}\right)\left(-\tfrac{(u')^2}{2}+\tfrac{u^2}{2}\right), \end{aligned} \tag{87}$$

$$D_\phi A = A_{3,\phi}+u'A_{3,u}. \tag{88}$$

The separation and solution of the conditions (81)-(88) gives the approximate coefficients of the Noether point symmetry vectors. In the following sections we proceed with the applications of conditions (81)-(88) in cases of special interest.

## 2.3 The modified Klein-Gordon equation

The Klein-Gordon has great importance in the literature [26]. The modified structural form of GUP is

$$\Delta X_i \Delta P_j \geq \frac{h}{2}\left[\delta_{\alpha\beta}\left(1+\beta P^2\right)+2\beta P_\alpha P_\beta\right] \tag{89}$$

where the deformed Heisenberg algebra from (89) is

$$[X_i, P_j] = ih\left[\delta_{\alpha\beta}\left(1+\beta P^2\right)+2\beta P_\alpha P_\beta\right]. \tag{90}$$

The parameter of deformation $\beta$ is defined by $\beta = \frac{\beta_0}{M_{pl}^2 c^2} = \frac{\beta_0 l_{pl}^2}{h^2}$. The $X_\alpha = x_\beta$ is kept undeformed, the coordinate representation of the momentum operator is $P_\alpha = p_\alpha(1+\beta p^2)$ and $(x, p)$ is the canonical representation satisfying $[x_\alpha, p_\beta] = ih\delta_{\alpha\beta}$. In the relativistic four vector form, the commutation (90) can be written as

$$[X_\mu, P_\nu] = -ih\left[\left(1-\beta\left(\eta^{\mu\nu}P_\mu P_\nu\right)\right)\eta_{\mu\nu} - 2\beta P_\mu P_\nu\right] \tag{91}$$

where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$. The corresponding deformed operators in this case are

$$P_\mu = p_\mu\left(1-\beta\left(\eta^{\mu\gamma}p_\alpha p_\gamma\right)\right), \qquad X_\nu = x_\nu, \tag{92}$$

where $p^\mu = ih\frac{\partial}{\partial x_\mu}$, and $[x_\mu, p_\nu] = -ih\eta_{\mu\nu}$.

Considering the spin$-0$ particle with rest mass $m$ gives

$$\left[\eta^{\mu\nu}P_\mu P_\nu - (mc)^2\right]\Psi = 0 \tag{93}$$

where $c$ is the speed of light. Substituting $P_\mu$ from (92), yields the modified Klein-Gordon equation, which is a fourth-order PDE

$$\Delta\Psi - 2\beta h^2\Delta(\Delta\Psi) + V_0\Psi = 0 \tag{94}$$

where $V_0 = \left(\frac{mc}{h}\right)^2$, $\Delta$ is the Laplace operator and the terms $O\left(\beta^2\right)$ have been ignored. The action of the modified Klein-Gordon equation (94) is

$$S = \int dx^4 \sqrt{-g}\mathcal{L}_A(\Psi, D_\sigma\Psi),$$

where the Lagrangian $\mathcal{L}_A(\Psi, D_\sigma\Psi)$ is given by

$$\mathcal{L}_A = \frac{1}{2}\left(\sqrt{-g}g^{\mu\nu}D_\mu\Psi D_\nu\Psi - \sqrt{-g}V_0\Psi^2\right). \tag{95}$$

Changing variables $\Psi \equiv u$ and reducing Eq. (31), we obtain the reduced Klein-Gordon equation with $V_0 = 1, \epsilon = -2\beta h^2$, that is a fourth-order ODE, which then possesses the Lagrangian Eq. (95) rewritten in the form (3), with

$$G_1(u, u', u'') = -\frac{1}{2}(u'')^2 \quad \text{and} \quad G_2(u, u', u'') = G_3(u, u', u'') = 0.$$

After the application of the conditions (81)-(83) we find a system of five equations after separation of monomials. The resultant symmetries are $X_0^{1-5}$ and thus the modified Klein-Gordon equation under GUP contains no first-order nontrivial approximate symmetries.

## 2.4 The Radial Orbital equation

The orbital equation or motion equation of a planet is given by

$$u^{''} + u = \frac{M}{L^2} - \frac{Q^2 u}{L^2} + 3Mu^2 - 2Q^2u^3 - \frac{H^2}{L^2u^3}, \tag{96}$$

where $u \equiv \dfrac{1}{r}$, the prime denotes differentiation with respect to $\phi$ and the angular momentum of the planet is denoted by $L$. The terms $3Mu^2$ and $\frac{H^2}{L^2u^3}$ come from the general relativity and cosmic expansion effect, respectively. Furthermore, the term

$\frac{Q^2u}{L^2}$ and $-2Q^2u^2$ are related to charge. The ratio between the term in $H$ and $\frac{M}{L^2}$ is $8\times10^{-34}$ for Mecury and $3.6\times10^{-28}$ for Neptune. If we choose

$$\epsilon = 2M \quad \text{and} \quad \kappa\epsilon^2 = Q^2 \quad \text{and} \quad \rho\epsilon^2 = H^2,$$

the Lagrangian corresponding to Eq.(96) is given by the general Lagrangian (3) with

$$G_1(u,u',u'') = \left(-\frac{u}{2L^2} - \frac{u^3}{2}\right) \quad \text{and} \quad G_2(u,u',u'') = \left(\frac{\kappa u^2}{2L^2} + \frac{2\kappa u^4}{4} - \frac{\rho}{2L^2}u^{-2}\right). \tag{97}$$

The first step is to retain the term in $G_1(u,u',u'')$ from Eq.(96). Consequently, the conditions (81)-(83) provide a system of four equations that solve to give the first-order approximate Noether symmetry generators given by

$$\begin{aligned} X_1^\epsilon &= X_0^4 + \epsilon\left(2\sin(\phi)\partial_\phi + u\cos(\phi)\partial_u\right), \\ X_2^\epsilon &= X_0^5 - \epsilon\left(2\cos(\phi)\partial_\phi - u\sin(\phi)\partial_u\right). \end{aligned}$$

Correspondingly, the first-order approximate gauge term in this case is

$$\begin{aligned} A_1 = \; & \tfrac{1}{2L^2}\left(-\cos(2\,\phi)\,c_7\,u^2L^2 + \sin(2\,\phi)\,c_6\,u^2L^2 + \left(\left(-c_4\,u^2 - 2\,c_9\,u\right)L^2 + c_4\right)\cos(\phi)\right) \\ & + \tfrac{1}{2L^2}\left(\left(\left(c_5\,u^2 + 2\,c_{10}\,u\right)L^2 - c_5\right)\sin(\phi) + 2\,c_{11}\,L^2\right) \end{aligned}$$

The first-order approximate conservation laws related to $X_{1-2}^\epsilon$ are given by

$$\begin{aligned} I_1^1 = \; & I_0^4 + \epsilon\left(\sin(\phi)u'^2 - \cos(\phi)uu' + \tfrac{1}{2}\tfrac{(L^2u^2+1)}{L^2}\sin(\phi)\right), \\ I_1^2 = \; & I_0^5 - \epsilon\left(u'^2\cos(\phi) + uu'\sin(\phi) + \tfrac{1}{2}\tfrac{L^2u^2+1}{L^2}\cos(\phi)\right). \end{aligned}$$

In the second approximation, we retain the quadratic $\epsilon$ terms, that is the $G_1(u,u',u'')$ and $G_2(u,u',u'')$ defined for Eq.(96). We proceed with the consideration of the conditions (84)-(85) and observe that Eq.(96) possesses no nontrivial second-order approximate symmetry generators, but the first-order approximate symmetry generators are preserved.

# Conclusion

In the first part, we have performed a full Lie analysis of third-order and fifth-order difference equations. We performed the group reductions of the equations using these symmetries and solutions were given in a unified manner. We also give the condition for well-defined solutions of the equations under investigation. For conformity, we split our 'single' solutions into categories to realize some results obtained in existing literature. More importantly, we have extended the results obtained in [16, 15].

In the second part, we studied the approximate Noetherian point symmetries and conservation laws of the class of ODEs which follow from a Lagrangian perturbed up-to third-order in $\epsilon$. We presented new examples where the application of our conditions can be seen. The knowledge of approximate symmetries was used to obtain the approximate first integrals of the corresponding approximate equations. We believe that this work can be very useful in the study of various differential problems. Indeed numerous equations originate from the generalized Lagrangian (3), such as the orbital equations of perturbed spaces. Conditions (81)-(88), applied to the problems studied in [27, 28, 29], immediately gives the results on approximate symmetries, obtained in these works, for orbital equations.

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