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Lie group analysis of Prandtl‘s two-dimensional laminar boundary layer equation: analytical and numerical solutions for scaling and non-scaling symmetries

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Declaration

I Moloko Boloka declare that this dissertation is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signature

Abstract

The Lie point symmetries of Prandtl‘s two-dimensional boundary layer equation expressed in terms of the stream function are derived. The general form of the invariant solutions and boundary conditions, which include slip, suction and blowing at the boundary, are obtained. The analytical solutions for boundary layer flow in convergent and divergent channels generated by Lie point symmetries, which are not scaling symmetries, are investigated. When an ordinary differential equation and some associated boundary conditions are invariant under a scaling transformation, the boundary value problem for the ordinary differential equation can be transformed to an initial value problem which is then solved. This is known as the non-iterative transformation method. The Blasius equation is invariant under a scaling transformation while the Falkner-Skan equation is not invariant. The Blasius and Falkner-Skan equations are ordinary differential equations derived from Prandtl‘s boundary layer partial differential equation for the stream function and describe boundary layer flow over a flat plate and wedge respectively. In the case of the Falkner-Skan equation, which is non-invariant under a scaling transformation method, a modified boundary value problem is derived which is invariant under an extended scaling group. The modified problem is then transformed to an initial value problem.

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Chapter 1

Introduction

The concept of a boundary layer was introduced over a century ago by Ludwig Prandtl [1], who made a significant contribution to aerodynamics. It explains the discrepancies between the theory of inviscid flow and experiments. Prandtl observed that near solid boundaries viscous forces are the same in magnitude as inertial forces. This region is usually confined to a thin layer adjacent to a solid boundary, which is referred to as the boundary layer. The application of the boundary layer equations was demonstrated by Blasius [2] for flow past a flat plate at zero incidence.

We will calculate the Lie point symmetries and the group invariant solution of the two-dimensional Prandtl boundary layer equation for a mainstream flow which depends on the distance x along the boundary and which is expressed in terms of the stream function. We will consider a linear combination of the Lie point symmetries of the third order partial differential equation for the stream function as was done by Mason [3] for the two-dimensional laminar jet.

Lie symmetry analysis gives a unified approach for the derivation of invariant solutions. This includes solutions which can be derived by a scaling transformation and solutions which are not obtained from a scaling transformation. Solutions which can be derived by a scaling transformation, that is similarity solutions, can generally be derived by more elementary methods. We will investigate solutions which are not scale-invariant. We will consider the special solution stated in Rosenhead [4] that describes a convergent channel flow with blowing of fluid at the walls with a given strength of blowing $\lambda = -\frac{5}{\sqrt{3}}$ by using Lie group analysis. We provide a symmetry-based justification for the choice of this value of λ. We will also investigate other values of λ and solutions in a convergent and divergent channel with blowing and suction of fluid at the walls

Further work done includes that by Matthews and Hill [5].[6, 7] who considered a nano boundary layer formed in a convergent channel with a sink and a boundary condition that contains an arbitrary index parameter, denoted by $n > 0$. The resulting ordinary differential equation for different values of a parameter n was solved analytically. They suggested that at the micro and nano scales the standard no slip boundary condition of classical fluid mechanics does not apply and must be replaced by a boundary condition that allows some degree of tangential slip.

In this study we investigate Prandtl‘s two-dimensional boundary layer equation in terms of the stream function with a slip boundary condition and blowing boundary condition at the walls. In the blowing boundary condition fluid is injected into the flow normal to the boundary. The injected fluid is the same as in the boundary layer flow. The blowing and slip boundary conditions are generally independent but they must be compatible with the existence of a Lie point symmetry. From the invariant solution generated by the Lie point symmetry of Prandtl‘s two-dimensional boundary layer equation, the Blasius equation and Falkner-Skan equation arise, both of which do not have an analytical solution. The Blasius equation, however, admits a scaling transformation. By using this scaling transformation the boundary value problem for boundary layer flow over a flat plate can be re-formulated as an Initial Value Problem (IVP) [8, 9]. The non-iterative transformation method will be employed to reduce the Blasius boundary value problems to initial value problems which are easier to solve.

The Falkner-Skan equation does not admit a scaling transformation. A modified numerical method will be deployed to transform the BVP for the Falkner-Skan equation to an IVP [10, 11]. This method is called the iterative transformation method. MATLAB ode45 will be used to solve the initial value problems. Details on how the differential equations are solved using ode45 will be discussed in Chapter 2. MATLAB bvp4c solver will be used to compare how the modified numerical method works in Chapter 5. The code used for the numerical calculations in Chapter 5 is commented on and presented in Appendix A.

Chapter 2

Analytical and Numerical Methods

2.1 Introduction

In this chapter a brief overview of the mathematical methods used in the dissertation is given. The derivation of the Lie point symmetries of the Partial Differential Equations (PDEs) and Ordinary Differential Equations (ODEs) are briefly discussed. The numerical method of transforming a Boundary Value Problem (BVP) into an Initial Value Problem (IVP) is explained.

2.2 Lie group analysis of differential equations

Sophus Lie introduced a method called Lie group analysis, which unifies all the seemingly ad-hoc techniques used to solve first order ODEs where the symmetries of a differential equation could be found and exploited [12].

2.2.1 Local one-parameter group of continuous transformation

A symmetry is a transformation that leaves an object unchanged or invariant when specified transformation is applied. In general, we consider transformations

$$\bar{x}_i = f_i(x_j, \epsilon), \qquad i, j = 1, 2, 3, \cdots, n. \tag{2.2.1}$$

The transformations form a Lie group with parameter ϵ if they satisfy the following axioms, where G is the group and $\Phi(a, b)$ the law of composition.

1. ***Closure property***. If $a, b \in G$, then $\Phi(a, b) \in G$.

2. ***Associative property***. If $a, b, c \in G$, then $\Phi(a, \Phi(b, c)) = \Phi(\Phi(a, b), c)$.

3. ***Identity element***. If $a \in G$, then there exists an $e \in G$ such that $\Phi(a, e) = \Phi(e, a) = a$.

4. ***Inverse element***. If $a \in G$, then there exists a unique element $a^{-1} \in G$ such that $\Phi(a, a^{-1}) = \Phi(a^{-1}, a) = e$.

A Lie group further satisfies the following properties:

1. f_i is a smooth function of the variable x_j.

2. f_i is an analytic function in the parameter ϵ, that is, a function with a convergent Taylor series in ϵ.

3. $\epsilon = 0$ can always be chosen to correspond with the identity element e.

4. The law of composition can be taken as $\Phi(a, b) = a + b$.

2.2.2 Infinitesimal transformations

Consider

$$\bar{x} = f(x, y, \epsilon), \quad \bar{y} = g(x, y, \epsilon), \tag{2.2.2}$$

with

$$f(x, y, 0) = x \quad \text{and} \quad g(x, y, 0) = y. \tag{2.2.3}$$

If we assume ϵ is small, then a Taylor series of (2.2.2) about $\epsilon = 0$ becomes

$$\bar{x} = f(x, y, 0) + \left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + O(\epsilon^2), \tag{2.2.4}$$

$$\bar{y} = g(x, y, 0) + \left.\frac{\partial g}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + O(\epsilon^2). \tag{2.2.5}$$

Define

$$\left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} = \xi(x, y), \quad \left.\frac{\partial g}{\partial \epsilon}\right|_{\epsilon=0} = \eta(x, y), \tag{2.2.6}$$

and substitute (2.2.3) and (2.2.6) into (2.2.4) and (2.2.5) . This gives

$$\bar{x} = x + \xi(x, y)\epsilon + O(\epsilon^2), \quad \bar{y} = y + \eta(x, y)\epsilon + O(\epsilon^2). \tag{2.2.7}$$

Equations 2.2.7 for $\bar{x}$ and $\bar{y}$ are referred to as infinitesimal transformations and ξ and η simply as infinitesimals.

2.2.3 Lie's invariant condition

Consider the ODE

$$\frac{dy}{dx} = F(x, y). \tag{2.2.8}$$

Under the infinitesimal transformation

$$\bar{x} = x + \xi(x, y)\epsilon + O(\epsilon^2), \quad \bar{y} = y + \eta(x, y)\epsilon + O(\epsilon^2), \tag{2.2.9}$$

we obtain

$$\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d}{dx}(y + \eta(x, y)\epsilon + O(\epsilon^2))}{\frac{d}{dx}(x + \xi(x, y)\epsilon + O(\epsilon^2))} \\
&= \frac{\frac{dy}{dx} + [\eta_x + \eta_y y']\epsilon + O(\epsilon^2)}{1 + [\xi_x + \xi_y y']\epsilon + O(\epsilon^2)} \\
&= \left(\frac{dy}{dx} + [\eta_x + \eta_y y']\epsilon + O(\epsilon^2)\right)\left(1 - [\xi_x + \xi_y y']\epsilon + O(\epsilon^2)\right) \\
&= \frac{dy}{dx} + (\eta_x + [\eta_y - \xi_x]y' - \xi_y y'^2)\epsilon + O(\epsilon^2),
\end{aligned} \tag{2.2.10}$$

where the subscript denotes partial differentiation. Equation (2.2.10) shows how the derivative $\frac{dy}{dx}$ transforms under the infinitesimal transformation.

We now consider the ODE

$$\frac{d\bar{y}}{d\bar{x}} = F(\bar{x}, \bar{y}). \tag{2.2.11}$$

Substituting (2.2.9) and the first order derivative transformation (2.2.10) into (2.2.11) gives

$$\frac{dy}{dx} + (\eta_x + [\eta_y - \xi_x]y' - \xi_y y'^2)\epsilon + O(\epsilon^2) = F(x + \xi(x, y)\epsilon + O(\epsilon^2), y + \eta(x, y)\epsilon + O(\epsilon^2)). \tag{2.2.12}$$

Expanding to order $O(\epsilon^2)$ gives

$$\frac{dy}{dx} + (\eta_x + [\eta_y - \xi_x]y' - \xi_y y'^2)\epsilon + O(\epsilon^2) = F(x, y) + (\xi F_x + \eta F_y)\epsilon + O(\epsilon^2), \tag{2.2.13}$$

and imposing

$$\frac{dy}{dx} = F(x, y) \tag{2.2.14}$$

shows that equation (2.2.13) is satisfied to $O(\epsilon^2)$ if

$$\eta_x + (\eta_y - \xi_x)F - \xi_y F^2 = \xi F_x + \eta F_y. \tag{2.2.15}$$

This is known as Lie's Invariance Condition. For a given $F(x, y)$, any functions $\xi(x, y)$ and $\eta(x, y)$ that solve equation (2.2.15) are the infinitesimals we seek.

2.2.4 The infinitesimal operator

We define the infinitesimal operator for higher order equation as

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \tag{2.2.16}$$

Consider $F(\bar{x}, \bar{y})$, then we can write

$$\begin{aligned} F(\bar{x}, \bar{y}) &= F(x + \xi(x, y)\epsilon + O(\epsilon^2), y + \eta(x, y)\epsilon + O(\epsilon^2)) \\ &= F(x, y) + (\xi F_x + \eta F_y)\epsilon + O(\epsilon^2) \\ &= F(x, y) + XF\epsilon + O(\epsilon^2), \end{aligned} \tag{2.2.17}$$

where

$$XF = \Big(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\Big)F. \tag{2.2.18}$$

2.2.5 The extended operator

The invariance of $\dfrac{dy}{dx} = F(x, y)$ gives from (2.2.15)

$$\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 = \xi F_x + \eta F_y. \tag{2.2.19}$$

We introduce the extended operator

$$X^{[1]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}. \tag{2.2.20}$$

If we define Δ such that

$$\Delta = \frac{dy}{dx} - F(x, y) = 0, \tag{2.2.21}$$

then

$$X^{[1]}\Delta = \zeta_x - \xi F_x - \eta F_y = 0. \tag{2.2.22}$$

Comparing (2.2.19) and (2.2.22) shows that we can define ζ_x as

$$\zeta_x = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2. \tag{2.2.23}$$

When we substitute $y' = F$ into

$$X^{[1]}\Delta = 0, \tag{2.2.24}$$

we obtain Lie's invariance condition. (2.2.15) Thus,

$$X^{[1]}\Delta\Big|_{\Delta=0} = 0. \tag{2.2.25}$$

2.2.6 Extension to higher orders

From equation (2.2.10) and (2.2.23), we can write

$$\frac{d\bar{y}}{d\bar{x}} = \frac{dy}{dx} + \zeta_x\epsilon + O(\epsilon^2), \tag{2.2.26}$$

where

$$\zeta_x = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2. \tag{2.2.27}$$

We can write higher derivatives as

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{d^2y}{dx^2} + \zeta_{xx}\epsilon + O(\epsilon^2), \tag{2.2.28}$$

where

$$\zeta_{xx} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'' \tag{2.2.29}$$

and

$$\frac{d^3\bar{y}}{d\bar{x}^3} = \frac{d^3y}{dx^3} + \zeta_{xxx}\epsilon + O(\epsilon^2), \tag{2.2.30}$$

where

$$\begin{aligned}\zeta_{xxx} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy}y'^4 \\ &\quad + 3[\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2]y'' - 3\xi_y y''^2 + (\eta_y - 3\xi_x - 4\xi_y y')y'''\end{aligned} \tag{2.2.31}$$

and so on.

2.2.7 Total differential operator

Consider

$$\begin{aligned}\zeta_x &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 \\ &= \eta_x + \eta_y y' - (\xi_x + \xi_y y')y'.\end{aligned} \tag{2.2.32}$$

We define the total differential operator as

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y}, \tag{2.2.33}$$

then

$$\zeta_x = D_x(\eta) - D_x(\xi) y'. \tag{2.2.34}$$

We call ζ_x an extended infinitesimal.

Consider $\frac{d\bar{y}}{d\bar{x}}$ using the total differential operator D_x. Therefore,

$$\begin{aligned} \frac{d\bar{y}}{d\bar{x}} &= \frac{\frac{d}{dx}(y + \eta\epsilon + O(\epsilon^2))}{\frac{d}{dx}(x + \xi\epsilon + O(\epsilon^2))} \\ &= \frac{\frac{dy}{dx} + D_x(\eta)\epsilon + O(\epsilon^2)}{1 + D_x(\xi_x)\epsilon + O(\epsilon^2)} \\ &= \Big(\frac{dy}{dx} + D_x(\eta)\epsilon + O(\epsilon^2)\Big)\Big(1 - D_x(\xi)\epsilon + O(\epsilon^2)\Big) \\ &= \frac{dy}{dx} + (D_x(\eta) - D_x(\xi)y') + O(\epsilon^2) \\ &= \frac{dy}{dx} + \zeta_x\epsilon + O(\epsilon^2). \end{aligned} \tag{2.2.35}$$

We now consider second-order extended infinitesimal

$$\begin{aligned} \frac{d^2\bar{y}}{d\bar{x}^2} &= \frac{d}{d\bar{x}}\Big(\frac{d\bar{y}}{d\bar{x}}\Big) \\ &= \frac{d}{dx}\Big(\frac{d\bar{y}}{d\bar{x}}\Big) \Big/ \frac{d\bar{x}}{dx} \\ &= \frac{\frac{d}{dx}\Big[\frac{dy}{dx} + [D_x(\eta) - D_x(\xi)y']\epsilon + O(\epsilon^2)\Big]}{1 + D_x(\xi)\epsilon + O(\epsilon^2)} \\ &= \Big(\frac{d^2y}{dx^2} + D_x(\zeta_x)\epsilon + O(\epsilon^2)\Big)\Big(1 - D_x(\xi)\epsilon + O(\epsilon^2)\Big) \\ &= \frac{d^2y}{dx^2} + (D_x(\zeta_x) - D_x(\xi)y^{''})\epsilon + O(\epsilon^2), \end{aligned} \tag{2.2.36}$$

so

$$\zeta_{xx} = D_x(\zeta_x) - y^{''} D_x(\xi). \tag{2.2.37}$$

As ζ_x contains x, y and y', we need to extend the definition of D_x, so

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y^{''} \frac{\partial}{\partial y'} + y^{'''} \frac{\partial}{\partial y''} + \cdots \tag{2.2.38}$$

and expanding ζ_{xx} gives

$$
\begin{aligned}
\zeta_{xx} &= D_x(\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2) - y'' D_x(\xi), \\
&= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'' \qquad (2.2.39)
\end{aligned}
$$

and expanding ζ_{xxx} gives

$$
\begin{aligned}
\zeta_{xxx} &= D_x(\zeta_{xx}) - y''' D_x(\xi), \\
&= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy} y'^4 \\
&\quad + 3[\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy} y'^2]y'' - 3\xi_y y''^2 + (\eta_y - 3\xi_x - 4\xi_y y')y''' \\
&\qquad (2.2.40)
\end{aligned}
$$

and so on.

2.2.8 Third order partial differential equations

We now outline the derivation of the Lie point symmetries of 3^{rd} order PDEs

$$\Delta(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{xxx}, u_{yyy}, u_{xxy}, u_{xyy}) = 0, \qquad (2.2.41)$$

where x and y are the independent variables, u denotes the dependent variable.

The PDE for the stream function that will be studied in this dissertation is of the form (2.2.41). The infinitesimal generator of the one parameter Lie group of point transformations of the PDE (2.2.41) is of the form

$$
\begin{aligned}
X^{[3]} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}} \\
+ \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}} + \zeta_{yyy} \frac{\partial}{\partial u_{yyy}} + \zeta_{xxy} \frac{\partial}{\partial u_{xxy}} + \zeta_{xyy} \frac{\partial}{\partial u_{xyy}}, \qquad (2.2.42)
\end{aligned}
$$

where

$$\xi^1 = \xi^1(x, y, u), \quad \xi^2 = \xi^2(x, y, u), \quad \eta = \eta(x, y, u). \qquad (2.2.43)$$

The extended infinitesimals ζ satisfy the recursive relations

$$\zeta_i = D_i(\eta) - u_j D_i(\xi^j), \qquad (2.2.44)$$

$$\zeta_{i_1 i_2 \ldots i_k} = D_{i_k}(\zeta_{i_1 i_2 \ldots i_{k-1}}) - u_{i_1 i_2 \ldots i_{k-1} j} D_{i_k}(\xi^j), \qquad (2.2.45)$$

where $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ and with $k \geq 2$. and summation over repeated indices is implied. Expanding the recursive relations, we obtain

$$\zeta_x = D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2), \tag{2.2.46}$$

$$\zeta_y = D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2), \tag{2.2.47}$$

$$\zeta_{xy} = D_y(\zeta_x) - u_{xx} D_y(\xi^1) - u_{xy} D_y(\xi^2), \tag{2.2.48}$$

$$\zeta_{yy} = D_y(\zeta_y) - u_{xy} D_y(\xi^1) - u_{yy} D_y(\xi^2), \tag{2.2.49}$$

$$\vdots$$

$$\zeta_{yyy} = D_y(\zeta_{yy}) - u_{xyy} D_y(\xi^1) - u_{yyy} D_y(\xi^2), \tag{2.2.50}$$

with the total derivative with respect to the independent variables x and y being

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + u_{yyx} \frac{\partial}{\partial u_{yy}} + u_{xyx} \frac{\partial}{\partial u_{xy}} + u_{xxx} \frac{\partial}{\partial u_{xx}}, \tag{2.2.51}$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + u_{xxy} \frac{\partial}{\partial u_{xx}} + u_{xyy} \frac{\partial}{\partial u_{xy}} + u_{yyy} \frac{\partial}{\partial u_{yy}}. \tag{2.2.52}$$

Then a Lie point symmetry is admitted by the 3^{rd} order PDE (2.2.41) if and only if

$$X^{[3]}(\Delta) = 0 \quad \text{when} \quad \Delta = 0. \tag{2.2.53}$$

Since the coefficients of X in (2.2.53) do not involve partial derivatives, one may separate (2.2.53) with respect to the derivatives of u and solve the resulting over-determined system of linear homogeneous partial differential equations known as the determining equations.

2.2.9 Ordinary differential equations

We now outline the derivation of the Lie point symmetries of the 3^{rd} order ODE

$$y''' = f(x, y, y', y''). \tag{2.2.54}$$

We define Δ as

$$\Delta = y''' - f(x, y, y', y'') = 0 \tag{2.2.55}$$

and the infinitesimal operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{2.2.56}$$

where $\xi(x,y)$ and $\eta(x,y)$ are the infinitesimals, to be determined.

The extended operator for the 3^{rd} order ODE (2.2.54) is

$$\begin{aligned} X^{[3]} &= \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta_1(x,y,y')\frac{\partial}{\partial y'} \\ &+ \zeta_2(x,y,y',y'')\frac{\partial}{\partial y''} + \zeta_3(x,y,y',y'',y''')\frac{\partial}{\partial y'''} \end{aligned} \tag{2.2.57}$$

where the extended infinitesimals are defined by

$$\zeta_1 = D_x(\eta) - y' D_x(\xi), \tag{2.2.58}$$

$$\zeta_2 = D_x(\zeta_1) - y'' D_x(\xi), \tag{2.2.59}$$

$$\zeta_3 = D_x(\zeta_2) - y''' D_x(\xi), \tag{2.2.60}$$

and the total derivative with respect to x is

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''}. \tag{2.2.61}$$

Lie's invariance criterion is

$$X^{[3]}\Delta\Big|_{\Delta=0} = 0 \quad \text{when} \quad y''' = f. \tag{2.2.62}$$

As in the case of equation (2.2.53), the coefficients of X in (2.2.62) do not involve derivatives and one may separate this equation with respect to the derivatives of y and solve the resulting system of linear homogeneous partial differential equations known as the determining equations.

2.3 Numerical method of solution

In general, numerical results are easily obtained when BVPs are transformed. The transform into IVPs methods for converting BVPs to IVPs are covered by Klamkin and Fazio [8, 9, 10, 13, 14].

The scaling groups are of paramount importance. This transformation group has been used to obtain similarity solutions to a lot of PDE models of parabolic type arising in fluid mechanics [14]. The scaling group are also used in the conversion of a BVP to an IVP.

In this dissertation, the Lie group method is employed to convert the PDE for the stream function, derived from Prandtl‘s boundary layer equation, to a BVP for an ODE. The BVP is of the form:

$$\frac{d^3F}{d\eta^3} = H\left(\eta, F, \frac{dF}{d\eta}, \frac{d^2F}{d\eta^2}\right), \tag{2.3.1}$$

$$F(0) = A, \qquad \frac{dF}{d\eta}(0) = B, \qquad \frac{dF}{d\eta}(\infty) = 1, \tag{2.3.2}$$

where $A, B \in \mathbb{R}$. When $B = 0$ and $A = 0$, there is no slip and no fluid suction at the boundary plate.

In (2.3.1), if we have

$$H\left(\eta, F, \frac{dF}{d\eta}, \frac{d^2F}{d\eta^2}\right) = -F\frac{d^2F}{d\eta^2}, \tag{2.3.3}$$

then the Blasius equation is obtained.

If we have

$$H\left(\eta, F, \frac{dF}{d\eta}, \frac{d^2F}{d\eta^2}\right) = -\left(F\frac{d^2F}{d\eta^2} + \frac{2m}{1+m}\left(1 - \left(\frac{dF}{d\eta}\right)^2\right)\right), \tag{2.3.4}$$

then the Falkner-Skan equation is obtained. The invariance property of these two ordinary differential equations differ, which informs the transformation methods employed in numerically solving them in Chapter 5.

If the ODE (2.3.1) and boundary conditions (2.3.2) are invariant under a scaling transformation given by

$$\overline{\eta} = \lambda^a \eta, \qquad \overline{F}(\overline{\eta}) = \lambda^b F(\eta), \tag{2.3.5}$$

then (2.3.1) transforms to

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} = H\left(\bar{\eta}, \bar{F}, \frac{d\bar{F}}{d\bar{\eta}}, \frac{d^2\bar{F}}{d\bar{\eta}^2}\right), \tag{2.3.6}$$

provided a and b satisfy a linear relation of the form $b = \sigma a, \ \sigma \in \mathbb{R}$. Using (2.3.5),

$$F(0) = A \Rightarrow \bar{F}(0) = \lambda^a F(0) = \lambda^a A = \bar{A} \tag{2.3.7}$$

and

$$\frac{dF}{d\eta}(\eta) = \lambda^{a-b}\frac{d\bar{F}}{d\bar{\eta}}, \tag{2.3.8}$$

therefore

$$\frac{dF}{d\eta}(0) = B \Rightarrow \lambda^{a-b}\frac{d\bar{F}}{d\bar{\eta}}(0) = B \Rightarrow \frac{d\bar{F}}{d\bar{\eta}}(0) = B\lambda^{b-a} = \bar{B}. \tag{2.3.9}$$

Differentiating (2.3.8) yields

$$\frac{d^2F}{d\eta^2}(\eta) = \lambda^{2a-b}\frac{d^2\bar{F}}{d\bar{\eta}^2}. \tag{2.3.10}$$

Let

$$\frac{d^2\bar{F}}{d\bar{\eta}^2}(0) = 1, \tag{2.3.11}$$

therefore

$$\frac{d^2F}{d\eta^2}(0) = \lambda^{2a-b}. \tag{2.3.12}$$

The third boundary condition (2.3.2) is not invariant under the transformation (2.3.5). The non-invariance of this boundary condition in (2.3.2) is necessary in order to compute the group parameter λ, as will be shown later in this section. By a method known as the non-iterative transformation method, equation (2.3.1) and (2.3.2) transform to

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} = H\left(\bar{\eta}, \bar{F}, \frac{d\bar{F}}{d\bar{\eta}}, \frac{d^2\bar{F}}{d\bar{\eta}^2}\right), \tag{2.3.13}$$

$$\bar{F}(0) = \bar{A}, \qquad \frac{d\bar{F}}{d\bar{\eta}}(0) = \bar{B}, \qquad \frac{d^2\bar{F}}{d\bar{\eta}^2}(0) = 1, \tag{2.3.14}$$

where

$$\bar{A} = \lambda^a A, \quad \bar{B} = \lambda^{b-a} B. \tag{2.3.15}$$

In order to solve numerically, the IVP for a third order ODE given by (2.3.13) and (2.3.14) is transformed into an IVP for a system of first order ODEs given by

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{\eta}} = \bar{F}_2, \tag{2.3.16}$$

$$\frac{\mathrm{d}\bar{F}_2}{\mathrm{d}\bar{\eta}} = \bar{F}_3, \tag{2.3.17}$$

$$\frac{\mathrm{d}\bar{F}_3}{\mathrm{d}\bar{\eta}} = H(\bar{\eta}, \bar{F}, \bar{F}_2, \bar{F}_3), \tag{2.3.18}$$

$$\bar{F}(0) = \bar{A}, \qquad \bar{F}_2(0) = \bar{B}, \qquad \bar{F}_3(0) = 1, \tag{2.3.19}$$

which is easily solved for $\bar{F}(\bar{\eta})$ and the group parameter λ for a given value of $\bar{A}$ and $\bar{B}$, using the IVP solver ode45 in MATLAB. Once $\bar{F}(\bar{\eta})$ is obtained, the solution $F(\eta)$ is given by the transformation in (2.3.5), provided that we have an approximation for $\dfrac{d\bar{F}}{d\bar{\eta}}(\infty)$ such that the group parameter $\lambda > 0$ is obtained from

$$\lambda = \Big(\frac{d^2F}{d\eta^2}(0)\Big)^{1/(2a-b)} = \Big(\frac{d\bar{F}}{d\bar{\eta}}(\infty)\Big)^{1/(b-a)}. \tag{2.3.20}$$

This is known as the non-iterative transformation method.

If equation (2.3.1) is not invariant under (2.3.5), a modified BVP, which is invariant under an extended group can be set up and solved by a method known as the Iterative Transformation method. This approach is employed to numerically solve the Falkner-Skan equation in Chapter 5.

In order to show the accuracy of these transformation methods, we consider solving the Blasius boundary value problem in which there is no slip and no suction at the rigid boundary, given by

$$\frac{\mathrm{d}^3F}{\mathrm{d}\eta^3} + F\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2} = 0, \tag{2.3.21}$$

$$F(0) = 0, \quad \frac{\mathrm{d}F}{\mathrm{d}\eta}(0) = 0, \qquad \frac{\mathrm{d}F}{\mathrm{d}\eta}(\infty) = 1. \tag{2.3.22}$$

The BVP (2.3.21) and (2.3.22) is discussed in Chapter 5 in detail. It is presented here to show the accuracy of the non-iterative transformation method. Using (2.3.5), the associated IVP is

$$\frac{\mathrm{d}^3\bar{F}}{\mathrm{d}\bar{\eta}^3} + \bar{F}\frac{\mathrm{d}^2\bar{F}}{\mathrm{d}\bar{\eta}^2} = 0, \tag{2.3.23}$$

$$\bar{F}(0) = 0, \quad \frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{\eta}}(0) = 0, \qquad \frac{\mathrm{d}^2\bar{F}}{\mathrm{d}\bar{\eta}^2}(0) = 1. \tag{2.3.24}$$

provided that $b = -a$, and subsequently, the value for a is chosen to be unity.

The IVP (2.3.23) and (2.3.24) are solved for $\bar{F}(\bar{\eta})$. It is important to note that the value of $\bar{F}''(0)$ in the third boundary condition (2.3.24) need not be unity. This condition can be replaced with $\bar{F}''(0) = \alpha, \ \alpha \in \mathbb{R}$. While solution $\bar{F}(\bar{\eta})$, and more importantly $\bar{F}'(\bar{\eta})$, will be different for each α value, the solution obtained for the velocity profile $F'(\eta)$ using (2.3.5) is unique and independent of α, as shown in Figure 2.3.1 and 2.3.2. In Figure 2.3.1, $\bar{F}'(\bar{\eta})$ is plotted against $\bar{\eta}$ for varying values of λ, and it can be seen that the asymptotic value of $\bar{F}'(\bar{\eta})$ plotted against $\bar{\eta}$ decreases as λ increases. This inverse relationship between $\bar{F}'(\bar{\eta})$ and λ shown in Figure 2.3.1 is in fact necessary in order to preserve solution $F'(\bar{\eta})$ given by the product $\lambda^2\bar{F}(\bar{\eta})$, for any choice of $F''(0)$, as shown in Figure 2.3.2.

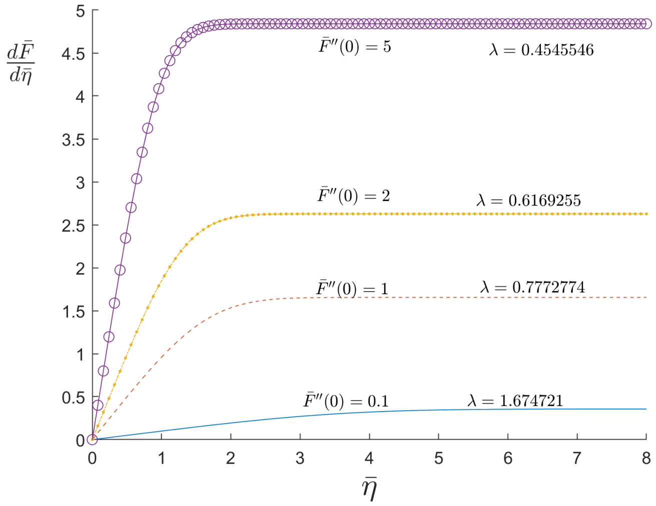

Figure 2.3.1: Graph of $\bar{F}'(\bar{\eta})$ plotted against $\bar{\eta}$ for varying values of $\bar{F}''(0)$. Solution is obtained using the non-iterative transformation method.

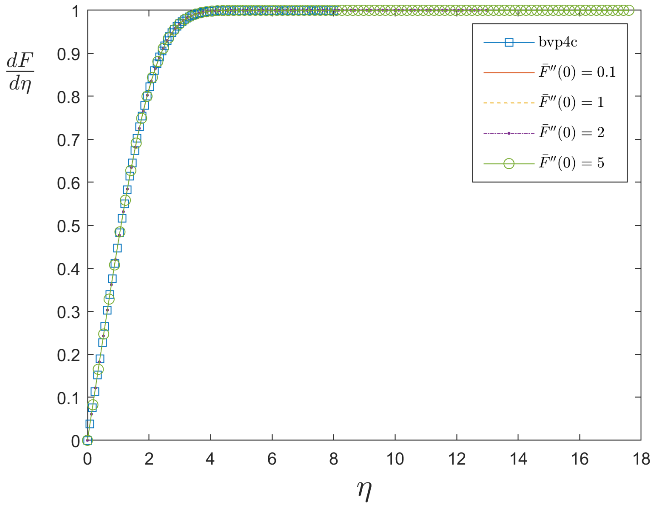

Figure 2.3.2: Graph of $F'(\eta)$ plotted against η for varying values of $\bar{F}''(0)$. Solution for $F'(\eta)$ obtained from (2.3.5b) is compared with that obtained using BVP4C in MATLAB.

2.4 Conclusions

The analytical and numerical methods briefly described in this chapter will be applied to Prandtl‘s two-dimensional boundary layer equation for the stream function and the reduced ordinary differential equations in Chapter 3, 4 and 5. In brief, the algorithm for Lie point symmetries of a PDE and ODE are essentially the same.

1. We first define the function $\Delta = 0$.

2. Define the infinitesimal operator X for the independent and dependent variables, in terms of ξ and η.

3. Define the extended operator or prolongation $X^{[3]}$.

4. Define the total derivatives D_x and D_y extended infinitesimals ζ.

5. Apply all defined variables to derive the invariance condition and separate the invariance condition according to the powers and products of independent partial derivatives. We then use the resulting over-determined system of equations to solve for the infinitesimals ξ and η.

Chapter 3

Derivation of the Lie point symmetries of Prandtl‘s two-dimensional boundary layer equations and group invariant solutions

3.1 Introduction

Model

A boundary layer is a layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant. Figure 3.1.1 depicts the boundary layer with mainstream velocity $U(x)$. In this dissertation we will investigate the laminar boundary layer and model it using Prandtl‘s two-dimensional steady state laminar boundary layer equations for the x and y components of the fluid velocity, $v_x(x,y)$ and $v_y(x,y)$ and the fluid pressure $p(x,y)$,

$$v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 v_x}{\partial y^2}, \tag{3.1.1}$$

$$\frac{\partial p}{\partial y}=0 \tag{3.1.2}$$

and the continuity equation

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0. \tag{3.1.3}$$

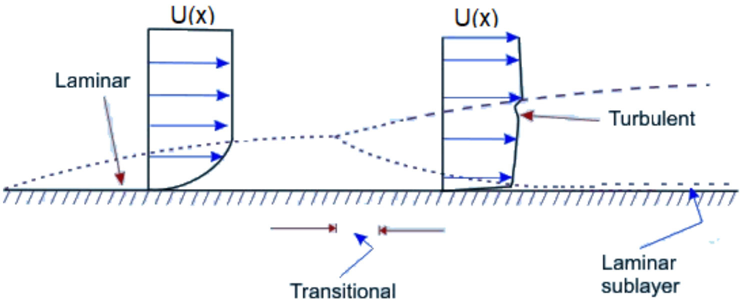

Figure 3.1.1: Viscous boundary layer flow past a flat plate with mainstream velocity $U(x)$.

Expressed in terms of the stream function $\psi(x, y)$ defined by

$$v_x = \frac{\partial \psi}{\partial y}, \; v_y = -\frac{\partial \psi}{\partial x}, \tag{3.1.4}$$

equation (3.1.1) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3}. \tag{3.1.5}$$

The continuity equation (3.1.3) is identically satisfied. From equation (3.1.2),

$$p = p(x). \tag{3.1.6}$$

The pressure is therefore constant across the boundary layer. Its value at any point is therefore known because it is the same as in the mainstream flow. The inviscid flow in the mainstream just outside the boundary layer satisfies the x-component of Euler's equation

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx}. \tag{3.1.7}$$

Equation (3.1.5) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U(x)\frac{dU(x)}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3}. \tag{3.1.8}$$

The boundary conditions are

$$v_x(x, 0) = v_t, \tag{3.1.9}$$

$$v_y(x, 0) = -v_s(x) \tag{3.1.10}$$

where $v_t(x)$ is the slip velocity, $v_s(x)$ is the suction velocity and for mainstream matching

$$v_x(x, \infty) = U(x). \tag{3.1.11}$$

3.2 Lie point symmetries

The PDE for the stream function $\psi(x,y)$ is

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U(x)\frac{dU(x)}{dx}+\nu\frac{\partial^3\psi}{\partial y^3}. \tag{3.2.1}$$

Let

$$W(x)=U(x)\frac{dU(x)}{dx}=\frac{d}{dx}\Big(\frac{1}{2}U^2(x)\Big). \tag{3.2.2}$$

The PDE becomes

$$\psi_y\psi_{xy}-\psi_x\psi_{yy}-\nu\psi_{yyy}-W(x)=0, \tag{3.2.3}$$

where $\psi_x$and ψ_y denotes partial differentiation with respect to x and y. When deriving the Lie point symmetries, x, y and all the partial derivatives of ψ with respect to x and y are regarded as independent variables. When x and y are regarded as the only independent variables then partial differentiation with respect to x and y is denoted by $\dfrac{\partial\psi}{\partial x}$ and $\dfrac{\partial\psi}{\partial y}$.

Equation (3.2.3) is of the form

$$F\Big(x,\psi_x,\psi_y,\psi_{xy},\psi_{yy},\psi_{yyy}\Big)=0, \tag{3.2.4}$$

where

$$F=\psi_y\psi_{xy}-\psi_x\psi_{yy}-\nu\psi_{yyy}-W(x). \tag{3.2.5}$$

The Lie point symmetry is of the form

$$X=\xi^1(x,y,\psi)\frac{\partial}{\partial x}+\xi^2(x,y,\psi)\frac{\partial}{\partial y}+\eta(x,y,\psi)\frac{\partial}{\partial\psi}. \tag{3.2.6}$$

The invariance condition is

$$X^{[3]}(F)=0 \qquad \text{when } F=0. \tag{3.2.7}$$

We therefore require the third prolongation

$$\begin{aligned}X^{[3]}=\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial\psi}+\zeta_x\frac{\partial}{\partial\psi_x}+\zeta_y\frac{\partial}{\partial\psi_y}&\\ +\zeta_{xy}\frac{\partial}{\partial\psi_{xy}}+\zeta_{yy}\frac{\partial}{\partial\psi_{yy}}+\zeta_{yyy}\frac{\partial}{\partial\psi_{yyy}}&\end{aligned} \tag{3.2.8}$$

where

$$\zeta_i = D_i(\eta) - \psi_s D_i(\xi^s), \tag{3.2.9}$$

$$\zeta_{ij} = D_j(\zeta_i) - \psi_{is} D_j(\xi^s), \tag{3.2.10}$$

$$\zeta_{ijk} = D_k(\zeta_{ij}) - \psi_{ijs} D_k(\xi^s), \tag{3.2.11}$$

with $x^1 = x$, $x^2 = y$ and

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + \cdots, \tag{3.2.12}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \cdots. \tag{3.2.13}$$

Hence

$$\zeta_1 = D_1(\eta) - \psi_1 D_1(\xi^1) - \psi_2 D_1(\xi^2), \tag{3.2.14}$$

$$\zeta_2 = D_2(\eta) - \psi_1 D_2(\xi^1) - \psi_2 D_2(\xi^2), \tag{3.2.15}$$

$$\zeta_{12} = D_2(\zeta_1) - \psi_{11} D_2(\xi^1) - \psi_{21} D_2(\xi^2), \tag{3.2.16}$$

$$\zeta_{22} = D_2(\zeta_2) - \psi_{12} D_2(\xi^1) - \psi_{22} D_2(\xi^2), \tag{3.2.17}$$

$$\zeta_{222} = D_2(\zeta_{22}) - \psi_{122} D_2(\xi^1) - \psi_{222} D_2(\xi^2), \tag{3.2.18}$$

which when expanded give

$$\zeta_1 = \zeta_x = \frac{\partial \eta}{\partial x} + \psi_x \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial x} - (\psi_x)^2 \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial x} - \psi_x \psi_y \frac{\partial \xi^2}{\partial \psi}, \tag{3.2.19}$$

$$\zeta_2 = \zeta_y = \frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_x \psi_y \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial y} - (\psi_x)^2 \frac{\partial \xi^2}{\partial \psi}, \tag{3.2.20}$$

$$\begin{aligned} \zeta_{12} = \zeta_{xy} = {} & \Big(\frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y}\Big)\zeta_x - \psi_{xx}\Big(\frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial \psi}\Big)\xi^1 \\ & - \psi_{yy}\Big(\frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial \psi}\Big)\xi^2, \end{aligned} \tag{3.2.21}$$

$$\begin{aligned} \zeta_{22} = \zeta_{yy} = {} & \Big(\frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y}\Big)\zeta_y - \psi_{xy}\Big(\frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial \psi}\Big)\xi^1 \\ & - \psi_{yx}\Big(\frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial \psi}\Big)\xi^2, \end{aligned} \tag{3.2.22}$$

$$\begin{aligned} \zeta_{222} = \zeta_{yyy} = {} & \Big(\frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \psi_{xxy} \frac{\partial}{\partial \psi_{xx}} + \psi_{xyy} \frac{\partial}{\partial \psi_{xy}} \\ & + \psi_{yyy} \frac{\partial}{\partial \psi_{yyy}}\Big)\zeta_{yy} - \psi_{xyy}\Big(\frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial \psi}\Big)\xi^1 - \psi_{yyy}\Big(\frac{\partial}{\partial y} - \psi_y \frac{\partial}{\partial \psi}\Big)\xi^2. \end{aligned}$$

We evaluate equation (3.2.7). Expanding (3.2.7) gives

$$-\frac{dW}{dx}\xi^1 - \psi_{yy}\,\zeta_x + \psi_{xy}\,\zeta_y + \psi_y\,\zeta_{xy} - \psi_x\,\zeta_{yy} - \nu\,\zeta_{yyy}\Big|_{F=0} = 0. \tag{3.2.23}$$

Now the invariance condition has to be evaluated at $F = 0$, that is for

$$\nu\psi_{yyy} = \psi_y\psi_{xy} - \psi_x\psi_{yy} - W(x). \tag{3.2.24}$$

We replace ψ_{yyy} in the invariance criterion by (3.2.24). We separate the resulting equation by the independent partial derivatives of ψ since $\xi^1(x, y, \psi)$, $\xi^2(x, y, \psi)$, $\eta(x, y, \psi)$ and $W(x)$ do not depend on the partial derivatives of ψ. This gives the following system of equations.

$$\psi_y\psi_{xyy}: \quad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.2.25}$$

$$\psi_{xyy}: \quad \frac{\partial \xi^1}{\partial y} = 0, \tag{3.2.26}$$

$$\psi_{xy}\psi_{xy}: \quad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.2.27}$$

$$(\psi_{yy})^2: \quad \frac{\partial \xi^2}{\partial \psi} = 0, \tag{3.2.28}$$

$$(\psi_y)^2\psi_{yy}: \quad \frac{\partial^2 \xi^2}{\partial \psi^2} = 0, \tag{3.2.29}$$

$$\psi_x\psi_y\psi_{yy}: \quad -3\nu\frac{\partial^2 \xi^1}{\partial \psi^2} + \frac{\partial \xi^2}{\partial \psi} = 0, \tag{3.2.30}$$

$$(\psi_x)^2\psi_{yy}: \quad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.2.31}$$

$$\psi_y\psi_{yy}: \quad 3\frac{\partial^2 \xi^2}{\partial y\partial \psi} - \frac{\partial^2 \eta}{\partial \psi^2} = 0, \tag{3.2.32}$$

$$\psi_x\psi_{yy}: \quad \frac{\partial \xi^1}{\partial x} - \frac{\partial \eta}{\partial \psi} - \frac{\partial \xi^2}{\partial y} + 3\nu\frac{\partial^2 \xi^1}{\partial y\partial \psi} = 0, \tag{3.2.33}$$

$$\psi_{yy}: \quad -\frac{\partial \eta}{\partial x} - 3\nu\frac{\partial^2 \eta}{\partial y\partial \psi} + 3\nu\frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{3.2.34}$$

$$(\psi_y)^2\psi_{xx}: \quad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.2.35}$$

$$\psi_y\psi_{xx}: \quad \frac{\partial \xi^1}{\partial y} = 0, \tag{3.2.36}$$

$$(\psi_y)^2\psi_{xy}: \quad 3\nu\frac{\partial^2 \xi^1}{\partial \psi^2} + \frac{\partial \xi^2}{\partial \psi} = 0, \tag{3.2.37}$$

$$\psi_x\psi_y\psi_{xy}: \quad 0 = 0, \tag{3.2.38}$$

$$\psi_y\psi_{xy}: \quad \frac{\partial \eta}{\partial \psi} + \frac{\partial \xi^2}{\partial y} - \frac{\partial \xi^1}{\partial x} + 6\nu\frac{\partial^2 \xi^1}{\partial y\partial \psi} = 0, \tag{3.2.39}$$

$$\psi_x\psi_{xy}: \quad \frac{\partial \xi^1}{\partial y} = 0, \tag{3.2.40}$$

$$\psi_{xy}: \quad \frac{\partial \eta}{\partial y} + 3\nu\frac{\partial^2 \xi^1}{\partial y^2} = 0, \tag{3.2.41}$$

$$(\psi_y)^4: \quad \frac{\partial^3 \xi^2}{\partial \psi^3} = 0, \tag{3.2.42}$$

$$\psi_x(\psi_y)^3: \quad \frac{\partial^3\xi^1}{\partial\psi^3} = 0, \tag{3.2.43}$$

$$(\psi_y)^3: \quad -\frac{\partial^2\xi^2}{\partial\psi\partial x} + 3\nu\frac{\partial^3\xi^2}{\partial y\partial\psi^2} - \nu\frac{\partial^3\eta}{\partial\psi^3} = 0, \tag{3.2.44}$$

$$(\psi_x)^2(\psi_y)^2: \quad 0 = 0, \tag{3.2.45}$$

$$(\psi_y)^2: \quad -\frac{\partial^2\xi^2}{\partial x\partial y} + \frac{\partial^2\eta}{\partial\psi\partial x} + 3\nu\frac{\partial^3\xi^2}{\partial\psi\partial y^2} - 3\nu\frac{\partial^3\eta}{\partial y\partial\psi^2} = 0, \tag{3.2.46}$$

$$(\psi_x)^2: \quad \frac{\partial^2\xi^1}{\partial y^2} = 0, \tag{3.2.47}$$

$$(\psi_x)^2\psi_y: \quad \frac{\partial^2\xi^1}{\partial y\partial\psi} = 0, \tag{3.2.48}$$

$$(\psi_y)^2\psi_x: \quad -\frac{\partial^2\xi^1}{\partial x\partial\psi} + \frac{\partial^2\xi^2}{\partial y\partial\psi} + 3\nu\frac{\partial^3\xi^1}{\partial y\partial\psi^2} = 0, \tag{3.2.49}$$

$$\psi_x\psi_y: \quad -\frac{\partial^2\eta}{\partial y\partial\psi} - \frac{\partial^2\xi^1}{\partial x\partial y} + \frac{\partial^2\xi^2}{\partial y^2} + 3\nu\frac{\partial^3\xi^1}{\partial\psi\partial y^2} = 0, \tag{3.2.50}$$

$$\psi_y: \quad \frac{\partial^2\eta}{\partial x\partial y} - 3\nu\frac{\partial^3\eta}{\partial\psi\partial y^2} + \nu\frac{\partial^3\xi^2}{\partial y^3} - 4\frac{\partial\xi^2}{\partial\psi}W(x) = 0, \tag{3.2.51}$$

$$\psi_x: \quad -\frac{\partial^2\eta}{\partial y^2} + \nu\frac{\partial^3\xi^1}{\partial y^3} - \frac{\partial\xi^1}{\partial\psi}W(x) = 0, \tag{3.2.52}$$

$$remainder: \quad \xi^1\frac{dW}{dx} + \Big(3\frac{\partial\xi^2}{\partial y} - \frac{\partial\eta}{\partial\psi}\Big)W(x) = -\nu\frac{\partial^3\eta}{\partial y^3}. \tag{3.2.53}$$

Consider first $\xi^1(x, y, \psi)$.

From (3.2.25) and (3.2.26),

$$\frac{\partial\xi^1}{\partial\psi} = 0, \ \frac{\partial\xi^1}{\partial y} = 0 \tag{3.2.54}$$

and hence

$$\xi^1(x, y, \psi) = \alpha(x). \tag{3.2.55}$$

Consider next $\xi^2(x, y, \psi)$.

From (3.2.28),

$$\frac{\partial\xi^2}{\partial\psi} = 0 \tag{3.2.56}$$

and therefore

$$\xi^2(x, y, \psi) = \beta(x, y). \tag{3.2.57}$$

Consider next $\eta(x, y, \psi)$.

From (3.2.41)

$$\frac{\partial \eta}{\partial y} = 0. \tag{3.2.58}$$

Equations (3.2.34) and (3.2.50) reduce to

$$-\frac{\partial \eta}{\partial x} + 3\nu \frac{\partial^2 \xi^2}{\partial y^2} = 0 \tag{3.2.59}$$

and

$$\frac{\partial^2 \xi^2}{\partial y^2} = 0. \tag{3.2.60}$$

Hence

$$\frac{\partial \eta}{\partial x} = 0 \tag{3.2.61}$$

and therefore

$$\eta(x, y, \psi) = h(\psi). \tag{3.2.62}$$

The remaining equations not identically satisfied are

$$\psi_y \psi_{yy} : \quad \frac{d^2 \eta}{d\psi^2} = 0, \tag{3.2.63}$$

$$\psi_x \psi_{yy} : \quad \frac{d\xi^1}{dx} - \frac{d\eta}{d\psi} - \frac{\partial \xi^2}{\partial y} = 0, \tag{3.2.64}$$

$$\psi_{yy} : \quad \frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{3.2.65}$$

$$\psi_y \psi_{xy} : \quad \frac{d\eta}{d\psi} + \frac{\partial \xi^2}{\partial y} - \frac{d\xi^1}{dx} = 0, \tag{3.2.66}$$

$$(\psi_y)^3 : \quad \frac{d^3 \eta}{d\psi^3} = 0, \tag{3.2.67}$$

$$(\psi_y)^2 : \quad \frac{\partial^2 \xi^2}{\partial x \partial y} = 0, \tag{3.2.68}$$

$$\psi_x \psi_y : \quad \frac{\partial^2 \xi^2}{\partial y^2} = 0, \tag{3.2.69}$$

$$\psi_y : \quad \frac{\partial^3 \xi^2}{\partial y^3} = 0, \tag{3.2.70}$$

$$remainder : \quad \xi^1 \frac{dW}{dx} + \Big(3\frac{\partial \xi^2}{\partial y} - \frac{d\eta}{d\psi}\Big) W(x) = 0. \tag{3.2.71}$$

The function $W(x)$ occurs only in equation (3.2.71). Expressed in terms of $\alpha(x)$, $\beta(x, y)$ and $h(\psi)$, defined by (3.2.55), (3.2.57) and (3.2.62), the above equations (3.2.63) to (3.2.71) become

$$\frac{d^2 h}{d\psi^2} = 0, \tag{3.2.72}$$

$$\frac{d\alpha(x)}{dx} - \frac{dh(\psi)}{d\psi} - \frac{\partial \beta(x, y)}{\partial y} = 0, \tag{3.2.73}$$

$$\frac{\partial^2 \beta(x, y)}{\partial y^2} = 0, \tag{3.2.74}$$

$$\frac{\partial^2 \beta(x, y)}{\partial x \partial y} = 0, \tag{3.2.75}$$

$$\alpha(x)\frac{dW}{dx} + \Big(3\frac{\partial}{\partial y}\beta(x, y) - \frac{dh(\psi)}{d\psi}\Big)W(x) = 0. \tag{3.2.76}$$

Consider now $\beta(x, y)$.

From (3.2.74),

$$\beta(x, y) = yA(x) + B(x). \tag{3.2.77}$$

Substituting (3.2.77) into (3.2.75) gives

$$\frac{dA}{dx} = 0 \tag{3.2.78}$$

and therefore

$$A(x) = A_1, \tag{3.2.79}$$

where A_1 is a constant. Hence

$$\beta(x, y) = A_1 y + B(x). \tag{3.2.80}$$

Consider next $h(\psi)$.

From (3.2.72),

$$h(\psi) = A_2\psi + A_3 \tag{3.2.81}$$

where A_2 and A_3 are constants.

In summary

$$\xi^1 = \alpha(x), \tag{3.2.82}$$

$$\xi^2 = \beta(x,y) = A_1 y + B(x), \tag{3.2.83}$$

$$\eta = h(\psi) = A_2\psi + A_3. \tag{3.2.84}$$

Substituting (3.2.82) to (3.2.84) into (3.2.73) gives

$$\frac{d\alpha(x)}{dx} - A_2 - A_1 = 0 \tag{3.2.85}$$

and therefore

$$\alpha(x) = (A_1 + A_2)x + A_4, \tag{3.2.86}$$

where A_4 is a constant.

In summary

$$\xi^1 = \alpha(x) = (A_1 + A_2)x + A_4, \tag{3.2.87}$$

$$\xi^2 = \beta(x,y) = A_1 y + B(x), \tag{3.2.88}$$

$$\eta = h(\psi) = A_2\psi + A_3 \tag{3.2.89}$$

and

$$\Big[(A_1 + A_2)x + A_4\Big]\frac{dW}{dx} + \Big[3A_1 - A_2\Big]W = 0. \tag{3.2.90}$$

Renaming the constants $A_1 = c_1$, $A_2 = c_2$, $A_3 = c_4$ and $A_4 = c_3$ and the arbitrary function $B(x) = g(x)$ we obtain

$$\xi^1 = \alpha(x) = (c_1 + c_2)x + c_3, \tag{3.2.91}$$

$$\xi^2 = \beta(x,y) = c_1 y + g(x), \tag{3.2.92}$$

$$\eta = h(\psi) = c_2\psi + c_4 \tag{3.2.93}$$

and

$$\Big[(c_1 + c_2)x + c_3\Big]\frac{dW}{dx} + \Big[3c_1 - c_2\Big]W = 0. \tag{3.2.94}$$

The Lie point symmetry is of the form

$$X = ((c_1+c_2)x+c_3)\frac{\partial}{\partial x} + (c_1 y + g(x))\frac{\partial}{\partial y} + (c_2\psi + c_4)\frac{\partial}{\partial \psi} \tag{3.2.95}$$

$$= c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + X_g, \tag{3.2.96}$$

where

$$c_1 \neq 0 : X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{3.2.97}$$

$$c_2 \neq 0 : X_2 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}, \tag{3.2.98}$$

$$c_3 \neq 0 : X_3 = \frac{\partial}{\partial x}, \tag{3.2.99}$$

$$c_4 \neq 0 : X_4 = \frac{\partial}{\partial \psi}, \tag{3.2.100}$$

$$g(x) \neq 0 : X_g = g(x)\frac{\partial}{\partial y} \tag{3.2.101}$$

where $g(x)$ is an arbitrary function.

Consider now $W(x)$ and U(x) where

$$\frac{dW}{dx} + \frac{(3c_1 - c_2)}{\Big[(c_1+c_2)x + c_3\Big]}W = 0 \tag{3.2.102}$$

and from (3.2.2)

$$\frac{d}{dx}\Big(\frac{1}{2}U^2(x)\Big) = W(x). \tag{3.2.103}$$

There are several cases depending on the values of the constants c_1, c_2 and c_3.

(i) $c_1 = -c_2, c_2 \neq 0, c_3 \neq 0, c_1 \neq c_2$

$$U(x) = \Big[\ \frac{c_3}{2c_2}W_0 \exp\Big(4\frac{c_2}{c_3}x\Big) + k\ \Big]^{\frac{1}{2}}. \tag{3.2.104}$$

(ii) $c_1 = c_2, c_2 \neq 0$

$$U(x) = \Big[2W_0 \ln\Big(x + \frac{c_3}{2c_2}\Big) + k\Big]^{\frac{1}{2}}. \tag{3.2.105}$$

(iii) $c_1 \neq -c_2, c_1 \neq c_2$

$$U(x) = \Big[\Big(\frac{c_1+c_2}{c_2-c_1}\Big)W_0\Big(x + \frac{c_3}{c_1+c_2}\Big)^{2(c_2-c_1)/(c_1+c_2)} + k\Big]^{\frac{1}{2}}, \tag{3.2.106}$$

where W_0 and k are constants.

3.3 Group invariant solution

Consider the Lie point symmetry (3.2.95). Now $\psi = \Phi(x, y)$ is a group invariant solution of the PDE (3.2.1) provided

$$X(\psi - \Phi(x, y))\Big|_{\psi=\Phi} = 0, \tag{3.3.1}$$

that is, provided

$$\Big[(c_1 + c_2)x + c_3\Big]\frac{\partial\Phi}{\partial x} + \Big[c_1 y + g(x)\Big]\frac{\partial\Phi}{\partial y} = c_2\Phi + c_4. \tag{3.3.2}$$

The differential equations of the characteristic curves of the first order linear PDE (3.2.4) are

$$\frac{dx}{(c_1 + c_2)x + c_3} = \frac{dy}{c_1 y + g(x)} = \frac{d\Phi}{c_2\Phi + c_4}. \tag{3.3.3}$$

3.3.1 Case 1: Falkner-Skan equation

For this case we assume that

$$c_1 + c_2 \neq 0, \quad c_2 \neq 0, \quad c_1 \neq c_2. \tag{3.3.4}$$

Consider the first pair of terms in (3.3.3), then

$$\frac{dy}{dx} - \frac{c_1}{(c_1 + c_2)\Big(x + \dfrac{c_3}{c_1 + c_3}\Big)}y = \frac{g(x)}{(c_1 + c_2)\Big(x + \dfrac{c_3}{c_1 + c_3}\Big)}. \tag{3.3.5}$$

The integrating factor is

$$\frac{1}{\Big(x + \dfrac{c_3}{c_1 + c_2}\Big)^{c_1/(c_1+c_2)}} \tag{3.3.6}$$

and the solution of (3.3.5) is

$$y = G(x) + \Big(x + \frac{c_3}{c_1 + c_2}\Big)^{c_1/(c_1+c_2)} A_1, \tag{3.3.7}$$

where A_1 is a constant and

$$G(x) = \frac{1}{(c_1 + c_2)}\Big(x + \frac{c_3}{c_1 + c_2}\Big)^{c_1/(c_1+c_2)} \int^x \frac{g(x)dx}{\Big(x + \dfrac{c_3}{c_1 + c_2}\Big)^{(2c_1+c_2)/(c_1+c_2)}}. \tag{3.3.8}$$

Hence

$$\frac{y - G(x)}{\Big(x + \dfrac{c_3}{c_1 + c_2}\Big)^{c_1/(c_1+c_2)}} = A_1. \tag{3.3.9}$$

Consider next the first and the last terms in (3.3.3),

$$\frac{dx}{(c_1 + c_2)\Big(x + \dfrac{c_3}{c_1 + c_2}\Big)} = \frac{1}{c_2}\frac{d\Phi}{\Big(\Phi + \dfrac{c_4}{c_2}\Big)}, \tag{3.3.10}$$

which gives

$$\frac{\Phi + \dfrac{c_4}{c_2}}{\Big(x + \dfrac{c_3}{c_1 + c_2}\Big)^{c_2/(c_1+c_2)}} = A_2, \tag{3.3.11}$$

where A_2 is a constant.

Now $\psi = \Phi(x, y)$. The group invariant solution is of the form

$$A_2 = f(A_1)$$

where f is an arbitrary function and therefore since $\psi = \Phi(x, y)$ we obtain

$$\psi(x, y) + \frac{c_4}{c_2} = \Big(x + \frac{c_3}{c_1 + c_2}\Big)^{c_1/(c_1+c_2)} f(\xi) \tag{3.3.12}$$

where

$$\xi = \frac{y - G(x)}{\Big(x + \dfrac{c_3}{c_1 + c_2}\Big)^{c_1/(c_1+c_2)}}. \tag{3.3.13}$$

The function $G(x)$ is arbitrary because $g(x)$ is arbitrary.

Substituting (3.3.12) into the PDE (3.2.1) gives

$$\nu\frac{d^3 f}{d\xi^3} + \frac{c_2}{c_1 + c_2} f(\xi)\frac{d^2 f}{d\xi^2} + \Big(\frac{c_1 - c_2}{c_1 + c_2}\Big)\Big(\frac{df}{d\xi}\Big)^2 \\ + W(x)\Big(x + \frac{c_3}{c_1 + c_2}\Big)^{(3c_1 - c_2)/(c_1+c_2)} = 0. \tag{3.3.14}$$

But from equation (3.2.102) we obtain

$$\frac{dW}{W} = (c_2 - 3c_1)\frac{dx}{\Big[(c_1 + c_2)x + c_3\Big]} \tag{3.3.15}$$

and therefore

$$W(x) = W_0\Big(x + \frac{c_3}{c_1 + c_2}\Big)^{(c_2-3c_1)/(c_1+c_2)}, \tag{3.3.16}$$

where W_0 is a constant. Substituting (3.3.16) into (3.3.14), we find that

$$\nu\frac{d^3 f}{d\xi^3} + \frac{c_2}{c_1 + c_2}f(\xi)\frac{d^2 f}{d\xi^2} + \Big(\frac{c_1 - c_2}{c_1 + c_2}\Big)\Big(\frac{df}{d\xi}\Big)^2 + W_0 = 0. \tag{3.3.17}$$

The ODE (3.3.17) is independent of $G(x)$ and depends only on W_0 and the ratio $\dfrac{c_1}{c_2}$.

In Case 3 we also assume that $c_1 \neq c_2$ and therefore the coefficient of $\Big(\dfrac{df}{d\xi}\Big)^2$ is non-zero. To obtain $\dfrac{c_1}{c_2}$, consider the boundary condition at the mainstream

$$v_x(x, \infty) = U(x). \tag{3.3.18}$$

Now

$$v_x(x, y) = \frac{\partial\psi}{\partial y} = \Big(x + \frac{c_3}{c_1 + c_2}\Big)^{(c_2-c_1)/(c_1+c_2)}\frac{df}{d\xi}. \tag{3.3.19}$$

But

$$\xi = \frac{y - G(x)}{\Big(x + \dfrac{c_3}{c_1 + c_2}\Big)^{c_1/(c_1+c_2)}} \tag{3.3.20}$$

and therefore $\xi \to \infty$ as $y \to \infty$. Hence

$$v_x(x, \infty) = \Big(x + \frac{c_3}{c_1 + c_2}\Big)^{(c_2-c_1)/(c_1+c_2)}\frac{df}{d\xi}(\infty) \tag{3.3.21}$$

and therefore

$$U(x) = \Big(x + \frac{c_3}{c_1 + c_2}\Big)^{(c_2-c_1)/(c_1+c_2)}\frac{df}{d\xi}(\infty). \tag{3.3.22}$$

Let

$$m = \frac{c_2 - c_1}{c_2 + c_1}. \tag{3.3.23}$$

Solving for $\dfrac{c_1}{c_2}$ and assuming $m \neq -1$, that is, $c_2 \neq 0$, we obtain

$$\frac{c_1}{c_2} = \frac{1 - m}{1 + m}. \tag{3.3.24}$$

Also let

$$\frac{c_3}{c_1+c_2} = -x_0. \tag{3.3.25}$$

The choice of c_3 corresponds to the choice of origin for x. Equation (3.3.22) becomes

$$U(x) = (x-x_0)^m \frac{df}{d\xi}(\infty). \tag{3.3.26}$$

The mainstream velocity must therefore be of the form

$$U(x) = U_0(x-x_0)^m \tag{3.3.27}$$

where

$$U_0 = \frac{df}{d\xi}(\infty). \tag{3.3.28}$$

For a group invariant solution we also have from (3.2.106),

$$U(x) = \Big[\Big(\frac{c_1+c_2}{c_2-c_1}\Big)W_0\Big(x+\frac{c_3}{c_1+c_2}\Big)^{2(c_2-c_1)/(c_1+c_2)} + k\Big]^{\frac{1}{2}} \tag{3.3.29}$$

provided $c_2 \neq c_1$, that is, $m \neq 0$. Equation (3.3.29) expressed in terms of m is

$$U(x) = \Big[\frac{W_0}{m}(x-x_0)^{2m} + k\Big]^{\frac{1}{2}}. \tag{3.3.30}$$

Therefore from (3.3.27) and (3.3.30)

$$U_0(x-x_0)^m = \Big[\frac{W_0}{m}(x-x_0)^{2m} + k\Big]^{\frac{1}{2}}. \tag{3.3.31}$$

Hence $k = 0$ and

$$\Big(\frac{W_0}{m}\Big)^{\frac{1}{2}} = U_0. \tag{3.3.32}$$

Equation (3.3.17) becomes

$$\nu\frac{d^3 f}{d\xi^3} + \frac{1}{2}(1+m)f(\xi)\frac{d^2 f}{d\xi^2} - m\Big(\frac{df}{d\xi}\Big)^2 + mU_0^2 = 0. \tag{3.3.33}$$

We make the transformation

$$f(\xi) = AF(\eta), \tag{3.3.34}$$

$$\xi = B\eta, \tag{3.3.35}$$

where A and B are constants to be determined. Equation (3.3.33) becomes

$$\frac{2\nu}{(1+m)AB}\frac{d^3F}{d\eta^3}+F\frac{d^2F}{d\eta^2}+\frac{2m}{(1+m)}\Big(\frac{B^2}{A^2}U_0^2-\Big(\frac{dF}{d\eta}\Big)^2\Big)=0. \tag{3.3.36}$$

We choose A and B so that

$$\frac{2\nu}{(1+m)AB}=1, \qquad \frac{B^2}{A^2}U_0^2=1. \tag{3.3.37}$$

Thus

$$A=\Big(\frac{2U_0\nu}{1+m}\Big)^{\frac{1}{2}}, \qquad B=\Big(\frac{2\nu}{(1+m)U_0}\Big)^{\frac{1}{2}} \tag{3.3.38}$$

and (3.3.33) transforms to

$$\frac{d^3F}{d\eta^3}+F\frac{d^2F}{d\eta^2}+\frac{2m}{(1+m)}\Big[1-\Big(\frac{dF}{d\eta}\Big)^2\Big]=0. \tag{3.3.39}$$

Equation (3.3.39) is the Falkner-Skan equation.

Consider now the stream function (3.3.12),

$$\psi(x,y)+\frac{c_4}{c_2}=\Big(x+\frac{c_3}{c_1+c_2}\Big)^{c_1/(c_1+c_2)}f(\xi). \tag{3.3.40}$$

After the transformation (3.3.34) and (3.3.35), equation (3.3.40) becomes

$$\psi(x,y)+\frac{c_4}{c_2}=\Big(\frac{2U_0\nu}{1+m}\Big)^{\frac{1}{2}}(x-x_0)^{\frac{1}{2}(1+m)}F(\eta). \tag{3.3.41}$$

Also ξ given by (3.3.13) transforms to

$$\eta=\Big(\frac{(1+m)U_0}{2\nu}\Big)^{\frac{1}{2}}\frac{(y-G(x))}{(x-x_0)^{\frac{1}{2}(1-m)}}. \tag{3.3.42}$$

Consider now the Lie point symmetry which generates the solution. We can take

$$\begin{aligned} c_3&=0 && \text{(fixes the origin in } x\text{)},\\ c_4&=0 && \text{(arbitrary constant in } \psi\text{)},\\ G(x)&=0 && \text{(fixes the origin in } y\text{)}. \end{aligned} \tag{3.3.43}$$

Hence

$$X=c_1X_1+c_2X_2 \tag{3.3.44}$$

$$=(c_1+c_2)x\frac{\partial}{\partial x}+c_1y\frac{\partial}{\partial y}+c_2\psi\frac{\partial}{\partial\psi}. \tag{3.3.45}$$

Next we derive $v_x(x, y)$ and $v_y(x, y)$. These velocity components will be needed in the boundary conditions later. We have

$$v_x = \frac{\partial \psi}{\partial y} = U_0(x - x_0)^m \frac{dF}{d\eta}. \tag{3.3.46}$$

We choose $G(x) = 0$ so that $y = 0$ if and only if $\xi = 0$. Thus

$$\begin{aligned} v_y = -\frac{\partial \psi}{\partial x} = &- \Big(\frac{(1+m)U_0\nu}{2}\Big)^{\frac{1}{2}} (x - x_0)^{\frac{1}{2}(m-1)} F(\eta) \\ &- (m-1)\Big(\frac{U_0\nu}{2(1+m)}\Big)^{\frac{1}{2}} (x - x_0)^{\frac{1}{2}(m-1)} \eta \frac{dF}{d\eta}. \end{aligned} \tag{3.3.47}$$

Consider now the boundary conditions. The boundary conditions do not determine the constants further. They determine v_x and v_y at $y = 0$ for a group invariant solution.

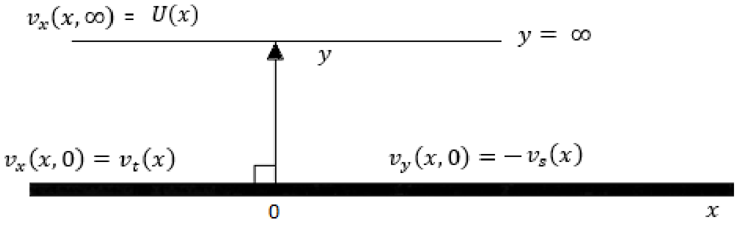

Figure 3.3.1: Viscous boundary layer flow past a flat plate with slip, suction and blowing boundary conditions

$y = \infty$: $\eta = \infty$ when $y = \infty$

$$\begin{aligned} v_x(x, \infty) &= U(x), \\ U_0(x - x_0)^m \frac{dF(\infty)}{d\eta} &= U_0(x - x_0)^m, \\ \frac{dF(\infty)}{d\eta} &= 1. \end{aligned} \tag{3.3.48}$$

$y = 0$: $\eta = 0$ when $y = 0$

$$v_x(x, 0) = v_t(x), \tag{3.3.49}$$

$$U_0(x - x_0)^m \frac{dF(0)}{d\eta} = v_t(x). \tag{3.3.50}$$

Hence for a group invariant solution, $v_t(x)$ must be of the form

$$v_t(x) = v_{t0}(x - x_0)^m \tag{3.3.51}$$

where

$$\frac{dF(0)}{d\eta} = \frac{v_{t0}}{U_0}. \tag{3.3.52}$$

Finally for v_y at $y = 0$

$$v_y(x, 0) = -v_s(x). \tag{3.3.53}$$

We have that $\dfrac{dF}{d\eta}(0)$ is finite because $v_x(x, 0)$ is finite. Hence from equation (3.3.47) at $\xi = 0$ we obtain

$$\Big(\frac{(1+m)U_0\nu}{2}\Big)^{\frac{1}{2}}(x - x_0)^{\frac{1}{2}(m-1)}F(0) = v_s(x). \tag{3.3.54}$$

Thus for a group invariant solution, $v_s(x)$ must be of the form

$$v_s(x) = v_{s0}(x - x_0)^{\frac{1}{2}(m-1)} \tag{3.3.55}$$

where

$$F(0) = \Big(\frac{2}{(1+m)U_0\nu}\Big)^{\frac{1}{2}} v_{s0}. \tag{3.3.56}$$

Case 1 : **General summary of the Falkner-Skan equation**

$$c_1 + c_2 \neq 0, \quad c_2 \neq 0, \quad c_1 \neq c_2, \tag{3.3.57}$$

$$m = \frac{c_2 - c_1}{c_2 + c_1}, \quad m \neq 0, \quad m \neq 1, \tag{3.3.58}$$

$$\psi(x, y) + \frac{c_4}{c_2} = \Big(\frac{2U_0\nu}{1+m}\Big)^{\frac{1}{2}}(x - x_0)^{\frac{1}{2}(1+m)}F(\eta), \tag{3.3.59}$$

$$\eta = \Big(\frac{(1+m)U_0}{2\nu}\Big)^{\frac{1}{2}} \frac{(y - G(x))}{(x - x_0)^{\frac{1}{2}(1+m)}}, \tag{3.3.60}$$

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} + \frac{2m}{(1+m)}\Big(1 - \Big(\frac{dF}{d\eta}\Big)^2\Big) = 0, \tag{3.3.61}$$

$$X = \frac{2}{(1+m)}x\frac{\partial}{\partial x} + \frac{(1-m)}{(1+m)}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}, \tag{3.3.62}$$

$$v_x(x,y) = U_0(x-x_0)^m \frac{dF}{d\eta}, \tag{3.3.63}$$

$$v_y(x,y) = -\Big[\frac{(1+m)U_0\nu}{2}\Big]^{\frac{1}{2}}(x-x_0)^{\frac{1}{2}(m-1)}F(\eta)$$

$$-(m-1)\Big[\frac{U_0\nu}{2(1+m)}\Big]^{\frac{1}{2}}(x-x_0)^{\frac{1}{2}(m-1)}\eta\frac{dF}{d\eta}, \tag{3.3.64}$$

$$U(x) = U_0(x-x_0)^m, \tag{3.3.65}$$

$$v_t(x) = v_{t0}(x-x_0)^m, \quad v_s(x) = v_{s0}(x-x_0)^{\frac{1}{2}(m-1)}, \tag{3.3.66}$$

$$\frac{dF}{d\eta}(\infty) = 1, \quad \frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}, \quad F(0) = \Big(\frac{2}{(1+m)U_0\nu}\Big)^{\frac{1}{2}}V_{s0}. \tag{3.3.67}$$

3.3.2 Case 2: Flow in a divergent channel and convergent channel

For this case we assume that

$$c_1 + c_2 \neq 0, \quad c_2 = 0, \quad c_1 \neq 0 \tag{3.3.68}$$

and therefore

$$m = \frac{c_2 - c_1}{c_2 + c_1} = -1. \tag{3.3.69}$$

Consider again the linear combination of the Lie point symmetries which generate the solution,

$$X = [(c_1+c_2)x + c_3]\frac{\partial}{\partial x} + (c_1 y + g(x))\frac{\partial}{\partial y} + (c_2\psi + c_4)\frac{\partial}{\partial \psi}. \tag{3.3.70}$$

When $m = -1$, we have $c_2 = 0$ and we assume that $c_1 \neq 0$. The Lie point symmetry (3.3.70) reduces to

$$X = (c_1 x + c_3)\frac{\partial}{\partial x} + (c_1 y + g(x))\frac{\partial}{\partial y} + c_4\frac{\partial}{\partial \psi}. \tag{3.3.71}$$

The Lie point symmetry is no longer a scaling symmetry.

For a group invariant solution, $\psi = \Phi(x,y)$,

$$X(\psi - \Phi(x,y))\Big|_{\psi=\Phi} = 0, \tag{3.3.72}$$

that is, provided

$$(c_1 x + c_3)\frac{\partial \Phi}{\partial x} + (c_1 y + g(x))\frac{\partial \Phi}{\partial y} = c_4. \tag{3.3.73}$$

The differential equations of the characteristic curves of the first order PDE (3.3.73) are

$$\frac{dx}{c_1 x + c_3} = \frac{dy}{c_1 y + g(x)} = \frac{d\Phi}{c_4}. \tag{3.3.74}$$

The first pair of terms give

$$\frac{y - G(x)}{\left(x + \dfrac{c_3}{c_1}\right)} = A_1 \tag{3.3.75}$$

where A_1 is a constant and

$$G(x) = \frac{1}{c_1}\left(x + \frac{c_3}{c_1}\right) \int \frac{g(x)dx}{\left(x + \dfrac{c_3}{c_1}\right)^2}.$$

The function $G(x)$ is arbitrary because g(x) is arbitrary.

Consider next the first and last term in equation (3.3.74). It gives

$$\Phi - \frac{c_4}{c_1} \ln\left(x + \frac{c_3}{c_1}\right) = A_2, \tag{3.3.76}$$

where A_2 is a constant.

Now $\psi = \Phi(x, y)$. The group invariant solution is of the form $A_2 = f(A_1)$, that is,

$$\psi(x, y) = \frac{c_4}{c_1} \ln\left(x + \frac{c_3}{c_1}\right) + f(\xi) \tag{3.3.77}$$

where $f(\xi)$ is an arbitrary function of ξ and

$$\xi = \frac{y - G(x)}{\left(x + \dfrac{c_3}{c_1}\right)}. \tag{3.3.78}$$

Note that the stream function is not simply obtained by putting $c_2 = 0$ in equation (3.3.40).

Now substitute the stream function (3.3.77) into the PDE

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = W(x) + \nu \psi_{yyy} \tag{3.3.79}$$

which gives the ODE for $f(\eta)$,

$$\nu \frac{d^3 f}{d\xi^3} + \frac{c_4}{c_1} \frac{d^2 f}{d\xi^2} + \left(\frac{df}{d\xi}\right)^2 + \left(x + \frac{c_3}{c_1}\right)^3 W(x) = 0. \tag{3.3.80}$$

But $W(x)$ satisfies the first order ODE (3.2.102) which when $c_2 = 0$ becomes

$$\frac{dW}{dx} + \frac{3c_1}{(c_1 x + c_3)} W = 0. \tag{3.3.81}$$

Thus

$$W(x)=\frac{W_0}{\left(x+\dfrac{c_3}{c_1}\right)^3}. \tag{3.3.82}$$

The differential equation (3.3.80) becomes

$$\nu\frac{d^3f}{d\xi^3}+\frac{c_4}{c_1}\frac{d^2f}{d\xi^2}+\left(\frac{df}{d\xi}\right)^2+W_0=0. \tag{3.3.83}$$

Before we can consider the boundary conditions, we first evaluate $U(x)$. We have

$$\frac{d}{dx}\left(\frac{1}{2}U^2(x)\right)=W(x) \tag{3.3.84}$$

and using (3.3.82) for $W(x)$ and integrating we obtain

$$U(x)=\left(-W_0\left(x+\frac{c_3}{c_1}\right)^{-2}+k\right)^{\frac{1}{2}}, \tag{3.3.85}$$

where k is a constant.

Consider the boundary condition at $y=\infty$ in order to obtain W_0. Now

$$v_x(x,y)=\frac{\partial\psi}{\partial y}=\frac{1}{\left(x+\dfrac{c_3}{c_1}\right)}\frac{df}{d\xi} \tag{3.3.86}$$

where ξ is defined by (3.3.78). Thus $\xi\to\infty$ as $y\to\infty$. Hence letting $y\to\infty$ in (3.3.86), we obtain

$$v_x(x,\infty)=\frac{1}{\left(x+\dfrac{c_3}{c_1}\right)}\frac{df}{d\xi}(\infty). \tag{3.3.87}$$

But $v_x(x,\infty)=U(x)$ and therefore

$$U(x)=\frac{1}{\left(x+\dfrac{c_3}{c_1}\right)}\frac{df}{d\xi}(\infty). \tag{3.3.88}$$

Hence $U(x)$ must be of the form

$$U(x)=\frac{U_0}{\left(x+\dfrac{c_3}{c_1}\right)}, \tag{3.3.89}$$

where U_0 is a constant. From (3.3.85) and (3.3.89) we obtain

$$\frac{U_0}{\left(x+\dfrac{c_3}{c_1}\right)}=\left(-W_0\left(x+\frac{c_3}{c_1}\right)^{-2}+k\right)^{\frac{1}{2}} \tag{3.3.90}$$

and therefore $k = 0$ and

$$W_0 = -U_0^2. \tag{3.3.91}$$

The ODE (3.3.83) becomes

$$\nu \frac{d^3 f}{d\xi^3} + \frac{c_4}{c_1} \frac{d^2 f}{d\xi^2} + \Big(\frac{df}{d\xi}\Big)^2 - U_0^2 = 0. \tag{3.3.92}$$

We now make a transformation of f and ξ in order to simplify the ODE (3.3.92). Let

$$f(\xi) = AF(\eta), \tag{3.3.93}$$

$$\xi = B\eta, \tag{3.3.94}$$

where A and B are constants to be determined. Equation (3.3.92) becomes

$$\frac{\nu}{AB} \frac{d^3 F}{d\eta^3} + \frac{c_4}{c_1 A} \frac{d^2 F}{d\eta^2} + \Big(\frac{dF}{d\eta}\Big)^2 - \frac{B^2}{A^2} U_0^2 = 0. \tag{3.3.95}$$

We choose

$$A = BU_0 \tag{3.3.96}$$

and equation (3.3.95) becomes

$$\frac{\nu}{B^2 U_0} \frac{d^3 F}{d\eta^3} + \frac{c_4}{BU_0 c_1} \frac{d^2 F}{d\eta^2} + \Big(\frac{dF}{d\eta}\Big)^2 - 1 = 0. \tag{3.3.97}$$

There are two cases to consider, a divergent channel with $U_0 > 0$ and a convergent channel with $U_0 < 0$.

(i) Divergent channel $U_0 > 0$.

Choose

$$\frac{\nu}{B^2 U_0} = +1 \tag{3.3.98}$$

and therefore

$$B = \pm\Big(\frac{\nu}{U_0}\Big)^{\frac{1}{2}}. \tag{3.3.99}$$

Choose

$$B = +\left(\frac{\nu}{U_0}\right)^{\frac{1}{2}}, \qquad A = BU_0 = (U_0\nu)^{\frac{1}{2}}. \tag{3.3.100}$$

Equation (3.3.97) becomes

$$\frac{d^3F}{d\eta^3} + \frac{c_4}{c_1(U_0\nu)^{\frac{1}{2}}}\frac{d^2F}{d\eta^2} + \left(\frac{dF}{d\eta}\right)^2 - 1 = 0. \tag{3.3.101}$$

Thus from (3.3.77)

$$\psi(x,y) = \frac{c_4}{c_1}\ln\left(x + \frac{c_3}{c_1}\right) + (U_0\nu)^{\frac{1}{2}}F(\eta) \tag{3.3.102}$$

and from (3.3.78)

$$\eta = \left(\frac{U_0}{\nu}\right)^{\frac{1}{2}}\frac{(y - G(x))}{\left(x + \frac{c_3}{c_1}\right)}. \tag{3.3.103}$$

Now $G(x)$ is arbitrary. We choose $G(x) = 0$ so that $\eta = 0$ when $y = 0$. Also let

$$\frac{c_3}{c_1} = -x_0. \tag{3.3.104}$$

Equations (3.3.102) and (3.3.103) become

$$\psi(x,y) = \frac{c_4}{c_1}\ln\left(x - x_0\right) + (U_0\nu)^{\frac{1}{2}}F(\eta), \tag{3.3.105}$$

$$\eta = \left(\frac{U_0}{\nu}\right)^{\frac{1}{2}}\frac{y}{\left(x - x_0\right)}. \tag{3.3.106}$$

The fluid velocity components will be used in the boundary conditions. We have

$$v_x(x,y) = \frac{\partial\psi}{\partial y} = \frac{U_0}{(x - x_0)}\frac{dF}{d\eta}, \tag{3.3.107}$$

$$v_y(x,y) = -\frac{\partial\psi}{\partial x} = -\frac{\frac{c_4}{c_1}}{(x - x_0)} + (U_0\nu)^{\frac{1}{2}}\frac{\eta\frac{dF}{d\eta}}{(x - x_0)} \tag{3.3.108}$$

and from (3.3.89),

$$U(x) = \frac{U_0}{x - x_0}. \tag{3.3.109}$$

Consider now the boundary conditions. They determine the form of v_x and v_y at $y = 0$ for a group invariant solution to exist and they also determine $\frac{c_4}{c_1}$. Consider first $v_x(x,y)$.

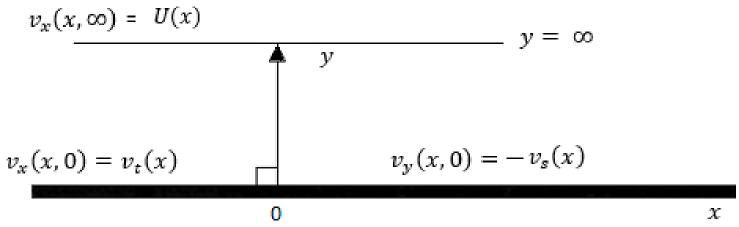

Figure 3.3.2: Viscous boundary layer flow past a flat plate with slip, suction and blowing boundary conditions.

(a) $y = \infty$: Then $\eta = \infty$ and

$$v_x(x, \infty) = U(x). \tag{3.3.110}$$

Thus from (3.3.107) and (3.3.109)

$$\frac{U_0}{(x - x_0)} \frac{dF(\infty)}{d\eta} = \frac{U_0}{(x - x_0)} \tag{3.3.111}$$

and therefore

$$\frac{dF(\infty)}{d\eta} = 1. \tag{3.3.112}$$

(b) $y = 0$: Then $\eta = 0$ when $y = 0$ and

$$v_x(x, 0) = v_t(x) \tag{3.3.113}$$

where $v_t(x)$ is the velocity of slip at the boundary. Thus from (3.3.107)

$$\frac{U_0}{(x - x_0)} \frac{dF(0)}{d\eta} = v_t(x) \tag{3.3.114}$$

and therefore for an invariant solution, $v_t(x)$ must be of the form

$$v_t(x) = \frac{v_{t0}}{(x - x_0)}. \tag{3.3.115}$$

Hence

$$\frac{U_0}{(x - x_0)} \frac{dF(0)}{d\eta} = \frac{v_{t0}}{(x - x_0)} \tag{3.3.116}$$

and therefore

$$\frac{dF(0)}{d\eta} = \frac{v_{t0}}{U_0}. \tag{3.3.117}$$

Consider next $v_y(x,y)$.

(a) $y=0$: Now

$$v_y(x,0) = -v_s(x) \tag{3.3.118}$$

where $v_s(x)$ is the suction velocity at the boundary. Thus using (3.3.108)

$$-\frac{\dfrac{c_4}{c_1}}{(x-x_0)} + (U_0\nu)^{\frac{1}{2}} \frac{\eta\dfrac{dF}{d\eta}}{(x-x_0)}\Big|_{\eta=0} = -v_s(x). \tag{3.3.119}$$

But $\dfrac{dF}{d\eta}(0)$ is finite because $v_t(x)$ is finite. Thus

$$\eta\frac{dF}{d\eta}\Big|_{\eta=0} = 0 \tag{3.3.120}$$

and therefore

$$\frac{\dfrac{c_4}{c_1}}{(x-x_0)} = v_s(x). \tag{3.3.121}$$

Hence for a group invariant solution, $v_s(x)$ must be of the form

$$v_s(x) = \frac{v_{s0}}{(x-x_0)} \tag{3.3.122}$$

and therefore

$$\frac{c_4}{c_1} = v_{s0}. \tag{3.3.123}$$

We note that this does not lead to a boundary condition on $F(0)$ as when $m \neq -1$.

The Lie point symmetry which generates the invariant solution is

$$X = ((c_1+c_2)x + c_3)\frac{\partial}{\partial x} + (c_1 y + g(x))\frac{\partial}{\partial y} + (c_2\psi + c_4)\frac{\partial}{\partial \psi}. \tag{3.3.124}$$

Now

$$c_2 = 0, \quad \frac{c_3}{c_1} = -x_0, \quad g(x) = 0 \quad \text{and} \quad \frac{c_4}{c_1} = v_{s0} \tag{3.3.125}$$

and dividing (3.3.124) by c_1, we obtain for the Lie point symmetry which generates the invariant solution for $m=-1$ and $U_0 > 0$,

$$X = (x-x_0)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v_{s0}\frac{\partial}{\partial \psi}. \tag{3.3.126}$$

Case $\mathbf{2, U_0 > 0}$ **: General summary of the divergent channel equation**

$$c_1 + c_2 \neq 0, \quad c_2 = 0, \quad c_1 \neq 0, \quad U_0 > 0, \tag{3.3.127}$$

$$m = \frac{c_2 - c_1}{c_2 + c_1} = -1, \tag{3.3.128}$$

$$\psi(x,y) = v_{s0} \ln\Big(x + \frac{c_3}{c_1}\Big) + \Big(U_0\nu\Big)^{\frac{1}{2}} F(\eta), \tag{3.3.129}$$

$$\eta = \Big(\frac{U_0}{\nu}\Big)^{\frac{1}{2}} \frac{y}{(x - x_0)}, \tag{3.3.130}$$

$$\frac{d^3F}{d\eta^3} + \frac{v_{s0}}{(U_0\nu)^{\frac{1}{2}}} \frac{d^2F}{d\eta^2} + \Big(\frac{dF}{d\eta}\Big)^2 - 1 = 0, \tag{3.3.131}$$

$$v_x(x,y) = \frac{U_0}{(x - x_0)} \frac{dF}{d\eta}, \tag{3.3.132}$$

$$v_y(x,y) = -\frac{v_{s0}}{(x - x_0)} + \frac{(U_0\nu)^{\frac{1}{2}}}{(x - x_0)} \eta \frac{dF}{d\eta}, \tag{3.3.133}$$

$$v_x(x,0) = v_t(x), \quad v_y(x,0) = -v_s(x), \tag{3.3.134}$$

$$U(x) = \frac{U_0}{(x - x_0)}, \quad v_t(x) = \frac{v_{t0}}{(x - x_0)}, \quad v_s(x) = \frac{v_{s0}}{(x - x_0)}, \tag{3.3.135}$$

$$\frac{dF}{d\eta}(\infty) = 1, \quad \frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}, \tag{3.3.136}$$

$$c_2 = 0, \quad \frac{c_3}{c_1} = -x_0, \quad \frac{c_4}{c_1} = v_{s0}, \quad g(x) = 0, \tag{3.3.137}$$

$$X = (x - x_0)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v_{s0}\frac{\partial}{\partial \psi}. \tag{3.3.138}$$

(ii) Convergent channel $U_0 < 0$.

In the differential equation (3.3.97), since $U_0 < 0$, we choose

$$\frac{\nu}{B^2 U_0} = -1 \tag{3.3.139}$$

and therefore

$$B = \pm\Big(\frac{\nu}{-U_0}\Big)^{\frac{1}{2}}. \tag{3.3.140}$$

Choose

$$B = +\Big(\frac{\nu}{-U_0}\Big)^{\frac{1}{2}} \tag{3.3.141}$$

and hence

$$A = BU_0 = -(-U_0\nu)^{\frac{1}{2}}. \tag{3.3.142}$$

Equation (3.3.97) becomes

$$\frac{d^3F}{d\eta^3} + \frac{c_4}{c_1(-U_0\nu)^{\frac{1}{2}}}\frac{d^2F}{d\eta^2} - \Big(\frac{dF}{d\eta}\Big)^2 + 1 = 0. \tag{3.3.143}$$

Thus from (3.3.77) for the stream function

$$\psi(x,y) = \frac{c_4}{c_1}\ln\Big(x + \frac{c_3}{c_1}\Big) - (-U_0\nu)^{\frac{1}{2}}F(\eta) \tag{3.3.144}$$

and from (3.3.78) for ξ

$$\eta = \Big(\frac{-U_0}{\nu}\Big)^{\frac{1}{2}}\frac{(y - G(x))}{\Big(x + \dfrac{c_3}{c_1}\Big)}. \tag{3.3.145}$$

We again choose $G(x) = 0$ so that $\eta = 0$ when $y = 0$ and let

$$\frac{c_3}{c_1} = -x_0. \tag{3.3.146}$$

Thus

$$\eta = \Big(-\frac{U_0}{\nu}\Big)^{\frac{1}{2}}\frac{y}{(x - x_0)}. \tag{3.3.147}$$

The fluid velocity components are

$$v_x(x,y) = \frac{\partial\psi}{\partial y} = \frac{U_0}{(x - x_0)}\frac{dF}{d\eta}, \tag{3.3.148}$$

$$v_y(x,y) = -\frac{\partial\psi}{\partial x} = -\frac{\dfrac{c_4}{c_1}}{(x - x_0)} - (-U_0\nu)^{\frac{1}{2}}\frac{\eta\dfrac{dF}{d\eta}}{(x - x_0)} \tag{3.3.149}$$

and from (3.3.89) we have again

$$U(x) = \frac{U_0}{x - x_0}, \tag{3.3.150}$$

where $U_0 < 0$. Consider now the boundary conditions. They again determine the form of v_x and v_y at $y = 0$ for a group invariant solution to exist and they also determine $\dfrac{c_4}{c_1}$.

Consider first $v_x(x,y)$.

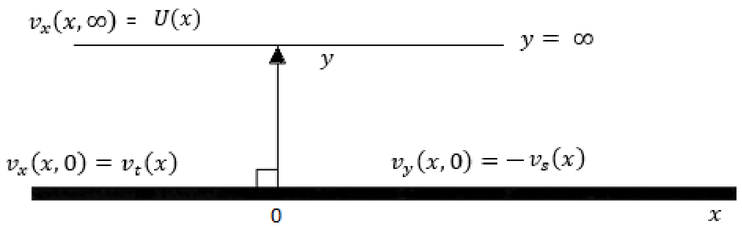

Figure 3.3.3: Viscous boundary layer flow past a flat plate with slip, suction and blowing boundary conditions.

(a) $y = \infty$. Then $\eta = \infty$ and

$$v_x(x,\infty) = U(x) \tag{3.3.151}$$

and substituting (3.3.148) and (3.3.150) into (3.3.151) gives

$$\frac{U_0}{(x-x_0)}\frac{dF(\infty)}{d\eta} = \frac{U_0}{(x-x_0)}. \tag{3.3.152}$$

We again obtain the boundary condition

$$\frac{dF(\infty)}{d\eta} = 1. \tag{3.3.153}$$

(b) $y = 0$: Now $\eta = 0$ when $y = 0$ and

$$v_x(x,0) = v_t(x), \tag{3.3.154}$$

where $v_t(x)$ is the slip velocity at the boundary. Thus using (3.3.148),

$$\frac{U_0}{(x-x_0)}\frac{dF(0)}{d\eta} = v_t(x). \tag{3.3.155}$$

Hence for a group invariant solution to exist, $v_t(x)$ must be of the form

$$v_t(x) = \frac{v_{t0}}{(x-x_0)} \tag{3.3.156}$$

where

$$\frac{dF(0)}{d\eta} = \frac{v_{t0}}{U_0}. \tag{3.3.157}$$

Consider next $v_y(x, y)$.

(a) $y = 0$: Let $v_s(x)$ be the suction velocity at the boundary. Then

$$v_y(x, 0) = -v_s(x) \tag{3.3.158}$$

and substituting (3.3.149) gives

$$-\frac{\frac{c_4}{c_1}}{(x - x_0)} - (-U_0\nu)^{\frac{1}{2}} \frac{\eta\frac{dF}{d\eta}}{(x - x_0)}\bigg|_{\eta=0} = -v_s(x). \tag{3.3.159}$$

But $\frac{dF}{d\eta}(0)$ is finite because $v_t(x)$ is finite, thus

$$\eta\frac{dF}{d\eta}\bigg|_{\eta=0} = 0 \tag{3.3.160}$$

and therefore

$$\frac{\frac{c_4}{c_1}}{(x - x_0)} = v_s(x). \tag{3.3.161}$$

For a group invariant solution to exist, $v_s(x)$ must therefore be of the form

$$v_s(x) = \frac{v_{s0}}{(x - x_0)} \tag{3.3.162}$$

where

$$\frac{c_4}{c_1} = v_{s0}. \tag{3.3.163}$$

There is no boundary condition on $F(0)$ when $m = -1$ which existed when $m \neq -1$.

The Lie point symmetry which generates the invariant solution is

$$X = ((c_1 + c_2)x + c_3)\frac{\partial}{\partial x} + (c_1 y + g(x))\frac{\partial}{\partial y} + (c_2\psi + c_4)\frac{\partial}{\partial \psi}. \tag{3.3.164}$$

Since

$$c_2 = 0, \quad \frac{c_3}{c_1} = -x_0, \quad g(x) = 0, \quad \frac{c_4}{c_1} = v_{s0}, \tag{3.3.165}$$

the Lie point symmetry reduces to

$$X = (x - x_0)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v_{s0}\frac{\partial}{\partial \psi} \tag{3.3.166}$$

which is the same as for $U_0 > 0$.

Case $2, \mathbf{U_0 < 0}$: General summary of the convergent channel equation

$$c_1 + c_2 \neq 0, \quad c_2 = 0, \quad c_1 \neq 0, \quad U_0 < 0, \tag{3.3.167}$$

$$m = \frac{c_2 - c_1}{c_2 + c_1} = -1, \tag{3.3.168}$$

$$\psi(x,y) = v_{s0} \ln(x - x_0) - (-U_0 \nu)^{\frac{1}{2}} F(\eta), \tag{3.3.169}$$

$$\eta = \Big(-\frac{U_0}{\nu} \Big)^{\frac{1}{2}} \frac{y}{(x - x_0)}, \tag{3.3.170}$$

$$\frac{d^3F}{d\eta^3} + \frac{v_{s0}}{(-U_0\nu)^{\frac{1}{2}}} \frac{d^2F}{d\eta^2} - \Big(\frac{dF}{d\eta}\Big)^2 + 1 = 0, \tag{3.3.171}$$

$$v_x(x,y) = \frac{U_0}{(x - x_0)} \frac{dF}{d\eta}, \tag{3.3.172}$$

$$v_y(x,y) = -\frac{v_{s0}}{(x - x_0)} - \frac{(-U_0\nu)^{\frac{1}{2}}}{(x - x_0)} \eta \frac{dF}{d\eta}, \tag{3.3.173}$$

$$v_x(x,0) = v_t(x), \quad v_y(x,0) = -v_s(x), \tag{3.3.174}$$

$$U(x) = \frac{U_0}{(x - x_0)}, \quad v_t(x) = \frac{v_{t0}}{(x - x_0)}, \quad v_s(x) = \frac{v_{s0}}{(x - x_0)}, \tag{3.3.175}$$

$$\frac{dF}{d\eta}(\infty) = 1, \quad \frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}, \tag{3.3.176}$$

$$c_2 = 0, \quad \frac{c_3}{c_1} = x_0, \quad \frac{c_4}{c_1} = v_{s0}, \quad g(x) = 0, \tag{3.3.177}$$

$$X = (x - x_0)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v_{s0}\frac{\partial}{\partial \psi}. \tag{3.3.178}$$

3.3.3 Case 3: Blasius equation

In this case it is assumed that $c_1 + c_2 \neq 0$, $c_1 = c_2$. Then $m = 0$. From equation (3.3.17) the ODE is

$$\nu \frac{d^3 f}{d\xi^3} + \frac{c_2}{(c_1 + c_2)} f \frac{d^2 f}{d\xi^2} + \Big(\frac{c_1 - c_2}{c_1 + c_2}\Big)\Big(\frac{df}{d\xi}\Big)^2 + W_0 = 0 \tag{3.3.179}$$

and since $c_1 = c_2$, it reduces to

$$\nu \frac{d^3 f}{d\xi^3} + \frac{1}{2} f \frac{d^2 f}{d\xi^2} + W_0 = 0. \tag{3.3.180}$$

Consider the boundary condition at $y = \infty$,

$$v_x(x, \infty) = U(x). \tag{3.3.181}$$

But from (3.3.19)

$$v_x(x, y) = \frac{\partial \psi}{\partial y} = \left(x + \frac{c_3}{c_1 + c_2}\right)^{(c_2 - c_1)/(c_1 + c_2)} \frac{df}{d\xi} = \frac{df}{d\xi}. \tag{3.3.182}$$

Also from (3.3.20)

$$\xi = \frac{y - G(x)}{\left(x + \dfrac{c_3}{c_1 + c_2}\right)^{c_1/(c_1 + c_2)}} = \frac{y - G(x)}{\left(x + \dfrac{c_3}{2c_2}\right)^{\frac{1}{2}}}. \tag{3.3.183}$$

Hence $\xi \to \infty$ as $y \to \infty$ and therefore by (3.3.182)

$$v_x(x, \infty) = \frac{df}{d\xi}(\infty). \tag{3.3.184}$$

The mainstream velocity must therefore be of the form

$$U(x) = U_0 \tag{3.3.185}$$

where

$$\frac{df}{d\xi}(\infty) = U_0. \tag{3.3.186}$$

Also, from equation (3.2.105), for a group invariant solution

$$U(x) = \left[2W_0 \ln\left(x + \frac{c_3}{2c_2}\right) + k\right]^{\frac{1}{2}}. \tag{3.3.187}$$

From equation (3.3.186) and (3.3.187)

$$U_0 = \left[2W_0 \ln\left(x + \frac{c_3}{2c_2}\right) + k\right]^{\frac{1}{2}}. \tag{3.3.188}$$

Hence

$$W_0 = 0. \tag{3.3.189}$$

The ODE (3.3.179) becomes

$$2\nu \frac{d^3 f}{d\xi^3} + f \frac{d^2 f}{d\xi^2} = 0. \tag{3.3.190}$$

We now make the transformation

$$f(\xi) = AF(\eta), \quad \xi = B\eta \tag{3.3.191}$$

and choose the constants A and B so that

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0, \quad \frac{dF}{d\eta}(\infty) = 1. \tag{3.3.192}$$

Now making the transformation (3.3.191), the ODE (3.3.190) and boundary condition (3.3.192) become

$$\frac{2\nu}{AB}\frac{d^3F}{d\eta^3} + F(\eta)\frac{d^2F}{d\eta^2} = 0 \tag{3.3.193}$$

and

$$\frac{dF}{d\eta}(\infty) = \frac{B}{A}U_0. \tag{3.3.194}$$

We choose A and B so that

$$\frac{2\nu}{AB} = 1, \tag{3.3.195}$$

$$\frac{B}{A}U_0 = 1 \tag{3.3.196}$$

and therefore

$$B = \left(\frac{2\nu}{U_0}\right)^{\frac{1}{2}}, \quad A = (2U_0\nu)^{\frac{1}{2}}. \tag{3.3.197}$$

Consider now the stream function. From (3.3.12)

$$\psi(x,y) + \frac{c_4}{c_2} = \left(x + \frac{c_3}{c_1+c_2}\right)^{c_2/(c_1+c_2)} f(\xi) = \left(x + \frac{c_3}{2c_2}\right)^{\frac{1}{2}} f(\xi). \tag{3.3.198}$$

Let

$$x_0 = -\frac{c_3}{2c_2}. \tag{3.3.199}$$

Hence

$$\psi(x,y) = (2U_0\nu)^{\frac{1}{2}}(x-x_0)^{\frac{1}{2}}F(\eta) + \frac{c_4}{c_2}. \tag{3.3.200}$$

Since an additive constant $\frac{c_4}{c_2}$ in the stream function does not contribute to the velocity components, v_x and v_y, we take $\frac{c_4}{c_2} = 0$. Consider next η. From equation (3.3.183)

$$\xi = \frac{y - G(x)}{(x - x_0)^{\frac{1}{2}}} \tag{3.3.201}$$

and hence

$$\eta = \Big(\frac{U_0}{2\nu}\Big)^{\frac{1}{2}} \frac{y - G(x)}{(x - x_0)^{\frac{1}{2}}}. \tag{3.3.202}$$

Choose $G(x) = 0$ so that $\eta = 0$ when $y = 0$. Thus

$$\eta = \Big(\frac{U_0}{2\nu}\Big)^{\frac{1}{2}} \frac{y}{(x - x_0)^{\frac{1}{2}}}. \tag{3.3.203}$$

Equation (3.3.192), which is the Blasius equation, equation (3.3.200) for $\psi(x, y)$ and (3.3.203) for η could be obtained by setting $m = 0$ in the general results for Case 1, the Falkner-Skan equation. The results from now on in this subsection are therefore the same as Case 1 with $m = 0$ given at the end of Subsection 3.3.2. We present the details of the calculation as a check.

Consider first the Lie point symmetry

$$X = ((c_1 + c_2)x + c_3)\frac{\partial}{\partial x} + (c_1 y + g(x))\frac{\partial}{\partial y} + (c_2\psi + c_4)\frac{\partial}{\partial \psi}. \tag{3.3.204}$$

Since $c_1 + c_2 \neq 0$, we divide by $c_1 + c_2$ and obtain

$$X = \Big(x + \frac{c_3}{(c_1 + c_2)}\Big)\frac{\partial}{\partial x} + \Big(\frac{c_1}{(c_1 + c_2)}y + \frac{g(x)}{(c_1 + c_2)}\Big)\frac{\partial}{\partial y} + \Big(\frac{c_2}{(c_1 + c_2)}\psi + \frac{c_4}{(c_1 + c_2)}\Big)\frac{\partial}{\partial \psi}. \tag{3.3.205}$$

Now

$$c_1 = c_2, \qquad \frac{c_3}{2c_2} = -x_0, \qquad g(x) = 0, \qquad \frac{c_4}{c_2} = 0, \tag{3.3.207}$$

and therefore

$$X = (x - x_0)\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + \frac{1}{2}\psi\frac{\partial}{\partial \psi}. \tag{3.3.208}$$

Consider next the fluid velocity components $v_x(x, y)$ and $v_y(x, y)$. Using (3.3.200) for $\psi(x, y)$ and (3.3.203) for η we obtain

$$v_x = \frac{\partial \psi}{\partial y} = U_0 \frac{dF}{d\eta} \tag{3.3.209}$$

and

$$\begin{aligned} v_y &= -\frac{\partial \psi}{\partial x} \\ &= -\Big(\frac{U_0 \nu}{2}\Big)^{\frac{1}{2}} \frac{F(\eta)}{(x-x_0)^{\frac{1}{2}}} + \Big(\frac{U_0 \nu}{2}\Big)^{\frac{1}{2}} \frac{\eta \dfrac{dF}{d\eta}}{(x-x_0)^{\frac{1}{2}}}. \end{aligned} \tag{3.3.210}$$

Consider now the boundary conditions. They determine v_x and v_y at $y = 0$.

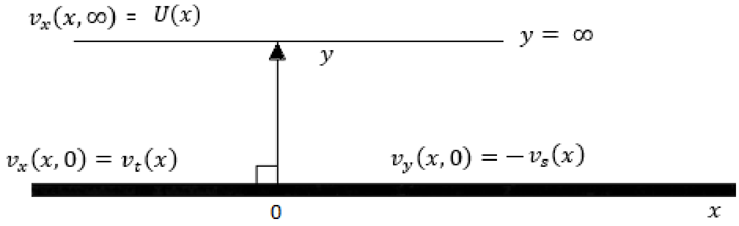

Figure 3.3.4: Viscous boundary layer flow past a flat plate with slip, suction and blowing boundary conditions.

(a) $y = \infty : \eta = \infty$ when $y = \infty$.

$$\begin{aligned} v_x(x,\infty) &= U(x), \\ U_0 \frac{dF}{d\eta}(\infty) &= U_0, \end{aligned} \tag{3.3.211}$$

which gives the boundary condition

$$\frac{dF}{d\eta}(\infty) = 1. \tag{3.3.212}$$

(b) $y = 0 : \eta = 0$ when $y = 0$.

$$\begin{aligned} v_x(x,0) &= v_t(x), \\ U_0 \frac{dF}{d\eta}(0) &= v_t(x). \end{aligned} \tag{3.3.213}$$

Hence for a group invariant solution to exist, $v_t(x)$ must be of the form

$$v_t(x) = v_{t0}, \tag{3.3.214}$$

where v_{t0} is a constant and

$$\frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}. \tag{3.3.215}$$

(c) $y = 0$:

$$v_y(x,0) = -v_s(x),$$
$$v_s(x) = \Big(\frac{U_0\nu}{2}\Big)^{\frac{1}{2}} \frac{F(0)}{(x-x_0)^{\frac{1}{2}}}. \tag{3.3.216}$$

Hence for a group invariant solution to exist, $v_s(x)$ must be of the form

$$v_s(x) = \frac{v_{s0}}{(x-x_0)^{\frac{1}{2}}} \tag{3.3.217}$$

which gives the boundary condition

$$F(0) = \Big(\frac{2}{U_0\nu}\Big)^{\frac{1}{2}} v_{s0}. \tag{3.3.218}$$

Case 3 : **General summary of the Blasuis equation**

$$c_1 + c_2 \neq 0, \quad c_1 = c_2, \quad m = \frac{c_2 - c_1}{c_2 + c_1} = 0, \qquad \frac{c_4}{c_2} = 0, \tag{3.3.219}$$

$$\psi(x,y) = \Big(2U_0\nu)^{\frac{1}{2}}(x-x_0)^{\frac{1}{2}} F(\eta), \tag{3.3.220}$$

$$\eta = \Big(\frac{U_0}{2\nu}\Big)^{\frac{1}{2}} \frac{y}{(x-x_0)^{\frac{1}{2}}}, \tag{3.3.221}$$

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0, \tag{3.3.222}$$

$$X = (x-x_0)\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + \frac{1}{2}\psi\frac{\partial}{\partial \psi}, \tag{3.3.223}$$

$$v_x(x,y) = U_0\frac{dF}{d\eta}, \tag{3.3.224}$$

$$v_y(x,y) = -\Big(\frac{U_0\nu}{2}\Big)^{\frac{1}{2}} \frac{F(\eta)}{(x-x_0)^{\frac{1}{2}}} + \Big(\frac{U_0\nu}{2}\Big)^{\frac{1}{2}} \frac{\eta\frac{dF}{d\eta}}{(x-x_0)^{\frac{1}{2}}}, \tag{3.3.225}$$

$$U(x) = U_0, \quad v_t(x) = v_{t0}, \quad v_s(x) = \frac{v_{s0}}{(x-x_0)^{\frac{1}{2}}}, \tag{3.3.226}$$

$$\frac{dF}{d\eta}(\infty) = 1, \quad \frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}, \quad F(0) = \Big(\frac{2}{U_0\nu}\Big)^{\frac{1}{2}} v_{s0}. \tag{3.3.227}$$

The results are obtained from the General summary of Case 1, equation (3.3.57) to (3.3.67), by setting $m = 0$

3.3.4 Case 4: Exponential solution

For this case $c_1+c_2 = 0$, $c_1 \neq 0$, $c_2 \neq 0$. From (3.3.2) the group invariant solution $\psi = \Phi(x, y)$ satisfies the first order linear PDE

$$\Big[(c_1 + c_2)x + c_3\Big]\frac{\partial \Phi}{\partial x} + (c_1 y + g(x))\frac{\partial \Phi}{\partial y} = c_2\Phi + c_4. \tag{3.3.228}$$

Since $c_1 + c_2 = 0$, (3.3.228) reduces to

$$c_3\frac{\partial \Phi}{\partial x} + (-c_2 y + g(x))\frac{\partial \Phi}{\partial y} = c_2\Phi + c_4. \tag{3.3.229}$$

The differential equations of the characteristic curves are

$$\frac{dx}{c_3} = \frac{dy}{-c_2 y + g(x)} = \frac{d\phi}{c_2\Phi + c_4}. \tag{3.3.230}$$

The first pair of terms give the first order ODE

$$\frac{dy}{dx} + \frac{c_2}{c_3}y = \frac{1}{c_3}g(x), \tag{3.3.231}$$

which has integrating factor $\exp\Big(\frac{c_2}{c_3}x\Big)$. Thus

$$\frac{d}{dx}\Big(y\exp\Big(\frac{c_2}{c_3}x\Big)\Big) = \frac{1}{c_3}g(x)\exp\Big(\frac{c_2}{c_3}x\Big) \tag{3.3.232}$$

and therefore

$$y = G(x) + A_1\exp\Big(\frac{-c_2}{c_3}x\Big) \tag{3.3.233}$$

where A_1 is a constant and

$$G(x) = \frac{1}{c_3}\exp\Big(\frac{-c_2}{c_3}x\Big)\int^x g(x)\exp\Big(\frac{c_2}{c_3}x\Big)dx. \tag{3.3.234}$$

The function $G(x)$ is an arbitrary function. Equation (3.3.233) can be written as

$$A_1 = (y - G(x))\exp\Big(\frac{c_2}{c_3}x\Big). \tag{3.3.235}$$

The first and last terms in (3.3.230) give

$$\frac{dx}{c_3} = \frac{d\Phi}{c_2\left(\Phi + \dfrac{c_4}{c_2}\right)}. \tag{3.3.236}$$

Thus

$$\ln\left(\Phi + \frac{c_4}{c_2}\right) = \frac{c_2}{c_3}x + constant \tag{3.3.237}$$

and therefore

$$\frac{\Phi + \dfrac{c_4}{c_2}}{\exp\left(\dfrac{c_2}{c_3}x\right)} = A_2, \tag{3.3.238}$$

where A_2 is a constant. The general solution is

$$A_2 = f(A_1), \tag{3.3.239}$$

where f is an arbitrary function. Thus

$$\frac{\Phi + \dfrac{c_4}{c_2}}{\exp\left(\dfrac{c_2}{c_3}x\right)} = f(\xi) \tag{3.3.240}$$

where

$$\xi = (y - G(x))\exp\left(\frac{c_2}{c_3}x\right). \tag{3.3.241}$$

Since $\Phi = \psi(x,y)$, equation (3.3.240) can be written as

$$\psi + \frac{c_4}{c_2} = \exp\left(\frac{c_2}{c_3}x\right)f(\xi). \tag{3.3.242}$$

Since the additive constant $\dfrac{c_4}{c_2}$ in the stream function $\psi(x,y)$ does not contribute to the velocity components $v_x(x,y)$ and $v_y(x,y)$, we take

$$\frac{c_4}{c_2} = 0 \tag{3.3.243}$$

and therefore

$$\psi(x,y) = \exp\left(\frac{c_2}{c_3}x\right)f(\xi). \tag{3.3.244}$$

The PDE for the stream function is

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = W(x) + \nu \psi_{yyy}. \tag{3.3.245}$$

Substituting (3.3.244) into the PDE (3.3.245) we obtain

$$\nu \frac{d^3 f}{d\xi^3} + \frac{c_2}{c_3} f \frac{d^2 f}{d\xi^2} - 2\frac{c_2}{c_3}\Big(\frac{df}{d\xi}\Big)^2 + W(x)\exp\Big(-4\frac{c_2}{c_3}x\Big) = 0. \tag{3.3.246}$$

But from equation (3.2.102) we obtain for a group invariant solution to exist,

$$\frac{dW}{W} = (c_2 - 3c_1)\frac{dx}{((c_1 + c_2)x + c_3)}. \tag{3.3.247}$$

But we are assuming that $c_1 + c_2 = 0$. Hence

$$\frac{dW}{W} = 4\frac{c_2}{c_3}dx \tag{3.3.248}$$

and therefore

$$W(x) = W_0 \exp\Big(4\frac{c_2}{c_3}x\Big), \tag{3.3.249}$$

where W_0 is a constant. Substituting (3.3.249) into (3.3.246), we obtain

$$\nu \frac{d^3 f}{d\xi^3} + \frac{c_2}{c_3} f \frac{d^2 f}{d\xi^2} - 2\frac{c_2}{c_3}\Big(\frac{df}{d\xi}\Big)^2 + W_0 = 0. \tag{3.3.250}$$

The ODE (3.3.250) depends only on $\frac{c_2}{c_3}$ and W_0.

To obtain $\frac{c_2}{c_3}$ and W_0, consider the boundary condition at the mainstream

$$v_x(x, \infty) = U(x). \tag{3.3.251}$$

But

$$v_x(x, y) = \frac{\partial \psi}{\partial y} = \exp\Big(2\frac{c_2}{c_3}x\Big)\frac{df}{d\xi} \tag{3.3.252}$$

and

$$\xi = (y - G(x))\exp\Big(\frac{c_2}{c_3}x\Big). \tag{3.3.253}$$

Hence $\xi \to \infty$ as $y \to \infty$. Thus

$$v_x(x, \infty) = \exp\Big(2\frac{c_2}{c_3}x\Big)\frac{df}{d\xi}(\infty) \tag{3.3.254}$$

and therefore for a group invariant solution to exist, $U(x)$ must be of the form

$$U(x) = \frac{df}{d\xi}(\infty) \exp\Big(2\frac{c_2}{c_3}x\Big). \tag{3.3.255}$$

Let

$$\frac{c_2}{c_3} = \alpha, \quad \frac{df}{d\xi}(\infty) = U_0. \tag{3.3.256}$$

Then

$$U(x) = U_0 \exp(2\alpha x). \tag{3.3.257}$$

Also, for a group invariant solution to exist, equation (3.2.104) gives

$$U(x) = \Big[\, \frac{c_3}{2c_2} W_0 \exp\Big(4\frac{c_2}{c_3}x\Big) + k\, \Big]^{\frac{1}{2}}. \tag{3.3.258}$$

Hence

$$U_0 \exp(2\alpha x) = \Big[\, \frac{1}{2\alpha} W_0 \exp\Big(4\alpha x\Big) + k\, \Big]^{\frac{1}{2}} \tag{3.3.259}$$

and therefore

$$k = 0, \quad W_0 = 2\alpha U_0^2. \tag{3.3.260}$$

The ODE (3.3.250) becomes

$$\nu \frac{d^3 f}{d\xi^3} + \alpha f \frac{d^2 f}{d\xi^2} - 2\alpha\Big(\frac{df}{d\xi}\Big)^2 + 2\alpha U_0^2 = 0. \tag{3.3.261}$$

We now make a transformation in f and ξ to simplify the ODE (3.3.261). Let

$$f(\xi) = AF(\eta), \tag{3.3.262}$$

$$\xi = B\eta, \tag{3.3.263}$$

where the constants A and B have still to be chosen. The ODE (3.3.261) becomes

$$\frac{\nu}{\alpha AB} \frac{d^3 F}{d\eta^3} + F \frac{d^2 F}{d\eta^2} - 2\Big(\frac{dF}{d\eta}\Big)^2 + 2U_0^2 \frac{B^2}{A^2} = 0. \tag{3.3.264}$$

We choose A and B to put the ODE (3.3.264) in the form

$$\frac{d^3 F}{d\eta^3} + F \frac{d^2 F}{d\eta^2} + 2\Big(1 - \Big(\frac{dF}{d\eta}\Big)^2\Big) = 0. \tag{3.3.265}$$

We choose this form because when $c_1 + c_2 = 0$, $\mid m \mid = \infty$ and the Falkner-Skan equation (3.3.61) reduces to (3.3.265) as $\mid m \mid \rightarrow \infty$. Comparing (3.3.264) with (3.3.265) we choose A and B so that

$$\frac{\nu}{\alpha AB} = 1, \tag{3.3.266}$$

$$U_0^2 \frac{B^2}{A^2} = 1. \tag{3.3.267}$$

From equation (3.3.267)

$$A = \pm BU_0. \tag{3.3.268}$$

We choose the + sign in (3.3.268). Using (3.3.268), equation (3.3.266) becomes

$$B = \pm\Big(\frac{\nu}{\alpha U_0}\Big)^{\frac{1}{2}} \tag{3.3.269}$$

provided $\alpha U_0 > 0$. We choose the +sign in (3.3.269). Using equations (3.3.268) and (3.3.269), we obtain

$$A = \Big(\frac{U_0 \nu}{\alpha}\Big)^{\frac{1}{2}}, \quad \alpha U_0 > 0. \tag{3.3.270}$$

Thus for $\alpha U_0 > 0$,

$$f(\xi) = \Big(\frac{\nu U_0}{\alpha}\Big)^{\frac{1}{2}} F(\eta), \quad \xi = \Big(\frac{\nu}{\alpha U_0}\Big)^{\frac{1}{2}} \eta. \tag{3.3.271}$$

Consider now the stream function given by (3.3.244) ,

$$\psi(x, y) = \exp\Big(\alpha x\Big) f(\xi). \tag{3.3.272}$$

After the transformation (3.3.262)

$$\psi(x, y) = \Big(\frac{U_0 \nu}{\alpha}\Big)^{\frac{1}{2}} \exp{(\alpha x)} F(\eta). \tag{3.3.273}$$

Consider now the transformation (3.3.263) for η where ξ is given by (3.3.241) and B given by (3.3.269),

$$\eta = \Big(\frac{\alpha U_0}{\nu}\Big)^{\frac{1}{2}} (y - G(x)) \exp{(\alpha x)}. \tag{3.3.274}$$

We choose $G(x) = 0$ so that $\eta = 0$ corresponds to $y = 0$. Hence

$$\eta = \left(\frac{\alpha U_0}{\nu}\right)^{\frac{1}{2}} y \exp{(\alpha x)}. \tag{3.3.275}$$

Consider $v_x(x, y)$ and $v_y(x, y)$. The velocity components will be needed in the boundary conditions later. Now

$$\frac{\partial \eta}{\partial x} = \alpha \left(\frac{\alpha U_0}{\nu}\right)^{\frac{1}{2}} y \exp{(\alpha x)} = \alpha \eta, \tag{3.3.276}$$

$$\frac{\partial \eta}{\partial y} = \left(\frac{\alpha U_0}{\nu}\right)^{\frac{1}{2}} \exp{(\alpha x)} \tag{3.3.277}$$

and therefore

$$v_x = \frac{\partial \psi}{\partial y} = U_0 \exp{(2\alpha x)} \frac{dF}{d\eta}, \tag{3.3.278}$$

$$v_y = -\frac{\partial \psi}{\partial x} = -(\alpha U_0 \nu)^{\frac{1}{2}} \exp{(\alpha x)} \Big(F(\eta) + \eta \frac{dF}{d\eta}\Big). \tag{3.3.279}$$

Consider now the boundary conditions. The boundary conditions do not determine the constants further. They determine v_x and v_y at $y = 0$ and $y = \infty$ for a group invariant solution to exist.

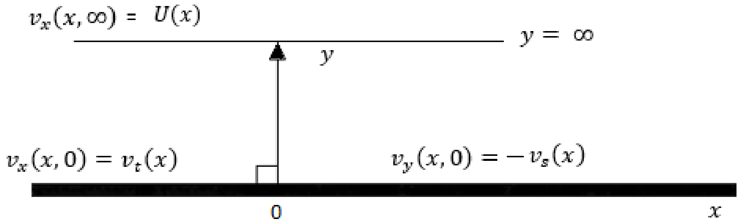

Figure 3.3.5: Viscous boundary layer flow past a flat plate with slip, suction and blowing boundary conditions.

(a) $y = \infty : \eta = \infty$ when $y = \infty$.

$$v_x(x, \infty) = U(x), \tag{3.3.280}$$

$$U_0 \exp{(2\alpha x)} \frac{dF}{d\eta}(\infty) = U_0 \exp{(2\alpha x)}, \tag{3.3.281}$$

which gives the boundary condition

$$\frac{dF(\infty)}{d\eta} = 1. \tag{3.3.282}$$

(b) $y = 0 : \eta = 0$ when $y = 0$

$$v_x(x, 0) = v_t(x), \tag{3.3.283}$$

$$U_0 \exp{(2\alpha x)} \frac{dF}{d\eta}(0) = v_t(x). \tag{3.3.284}$$

Thus for a group invariant solution to exist, $v_t(x)$ must be of the form

$$v_t(x) = v_{t0} \exp{(2\alpha x)}, \tag{3.3.285}$$

which gives the boundary condition

$$\frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}. \tag{3.3.286}$$

(c) $y = 0 : \eta = 0$ when $y = 0$

$$v_y(x, 0) = -v_s(x). \tag{3.3.287}$$

But

$$-v_s(x) = -(\alpha U_0 \nu)^{\frac{1}{2}} \exp{(\alpha x)} \Big(F(\eta) + \eta \frac{dF}{d\eta} \Big) \Big|_{\eta=0} \tag{3.3.288}$$

and therefore

$$v_s(x) = (\alpha U_0 \nu)^{\frac{1}{2}} \exp{(\alpha x)} F(0). \tag{3.3.289}$$

Thus for a group invariant solution to exist, $v_s(x)$ must be of the form

$$v_s(x) = v_{s0} \exp{(\alpha x)}.$$

Hence we obtain the boundary condition

$$F(0) = \frac{v_{s0}}{(\alpha U_0 \nu)^{\frac{1}{2}}}. \tag{3.3.290}$$

Case 4 : **General summary of the exponential solution.**

$$c_1 + c_2 = 0, \quad c_1 \neq 0, \quad c_2 \neq 0, \tag{3.3.291}$$

$$\mid m \mid = \left| \frac{c_2 - c1}{c_2 + c_1} \right| = \infty, \quad \alpha = \frac{c_2}{c_3}, \quad \frac{c_4}{c_2} = 0, \quad \alpha U_0 > 0, \tag{3.3.292}$$

$$\psi(x, y) = \left(\frac{\nu U_0}{\alpha}\right)^{\frac{1}{2}} \exp{(\alpha x)} F(\eta), \tag{3.3.293}$$

$$\eta = \left(\frac{\alpha U_0}{\nu}\right)^{\frac{1}{2}} y \exp{(\alpha x)}, \tag{3.3.294}$$

$$\frac{d^3 F}{d\eta^3} + F\frac{d^2 F}{d\eta^2} + 2\Big[1 - \Big(\frac{dF}{d\eta}\Big)^2\Big] = 0, \tag{3.3.295}$$

$$X = \frac{1}{\alpha}\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}, \tag{3.3.296}$$

$$v_x(x, y) = U_0 \exp{(2\alpha x)}\frac{dF}{d\eta}, \tag{3.3.297}$$

$$v_y(x, y) = -(\alpha \nu U_0)^{\frac{1}{2}} \exp{(\alpha x)} \Big[F(\eta) + \eta\frac{dF}{d\eta}\Big], \tag{3.3.298}$$

$$U(x) = U_0 \exp{(2\alpha x)}, \tag{3.3.299}$$

$$v_t(x) = v_{t0} \exp{(2\alpha x)}, \quad v_s(x) = v_{s0} \exp{(\alpha x)}, \tag{3.3.300}$$

$$\frac{dF}{d\eta}(\infty) = 1, \quad \frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}, \quad F(0) = \frac{v_{s0}}{(\alpha \nu U_0)^{\frac{1}{2}}}. \tag{3.3.301}$$

3.4 Conclusions

The Lie point symmetry was derived of Prandtl‘s two-dimensional boundary layer equations with mainstream velocity depending on distance x along the boundary. The problem was formulated in terms of a stream function. The Lie point symmetry depends on four arbitrary constants, $c_1,\ c_2,\ c_3$ and c_4 and on an arbitrary function $g(x)$. It is therefore a linear combination of four Lie point symmetries and an infinite number of Lie point symmetries of the form

$$X_g = g(x)\frac{\partial}{\partial x}.$$

The general form of the invariant solution for four cases was derived. The four cases were:

Case 1: Falkner-Skan equation.

Case 2: Boundary layer flow in a divergent and convergent channel.

Case 3: Blasius equation.

Case 4: Exponential solution.

The invariant solution in Cases 1 and 3 is generated by a scaling symmetry and in Cases 2 and 4 by a non-scaling symmetry. For each case a laminar boundary layer at a flat plate was considered. At the plate there is a non-zero slip velocity, fluid leak-off and fluid injection as well as matching at the free stream. The mainstream velocity is a function of distance x along the plate. It was found that for an invariant solution to exist the functional form of the fluid leak-off and fluid injection velocities, the slip velocity and the mainstream velocity could not be arbitrary but had to take a special form for each case. At the end of each case a summary of the results was given. This allowed the cases to be compared.

In the next chapter, analytical solutions for boundary layer flow in divergent and convergent channels will be considered. In Chapter 5 numerical solutions for Cases 1 and 3 will be investigated.

Chapter 4

Boundary layer flow in convergent and divergent channels

4.1 Introduction

In this chapter, we will investigate boundary layer flow in convergent and divergent channels. We will first review a model to show that this boundary layer flow is described by the case $m = -1$ of Chapter 3. The flow is therefore generated by a Lie point symmetry of Prandtl‘s boundary layer equations that is not a scaling symmetry. This is different from most boundary layer flows, including Blasius flow past a flat plate, which are generated by scaling Lie point symmetries.

We will first consider boundary layer flow in convergent and divergent channels with impermeable boundaries at which there is slip but no suction or blowing. This problem is readily solved without using Lie group analysis of ordinary differential equations.

We will then investigate the effects of suction or blowing on boundary layer flow in convergent and divergent channels with no slip at the boundaries. We will apply Lie group analysis to the ordinary differential equation for $m = -1$ with non-zero suction or blowing. The derivation of a result for flow in a convergent channel with blowing at the boundary, stated in the literature by Jones and Watson [15] without derivation, is given.

4.2 High Reynolds number flow in convergent and divergent channels

It is first necessary to show that the mainstream velocity

$$U(x) = \frac{U_0}{x} \tag{4.2.1}$$

describes flow in a convergent channel when $U_0 < 0$ and in a divergent channel when $U_0 > 0$. To do that we follow the approach of Acheson [16].

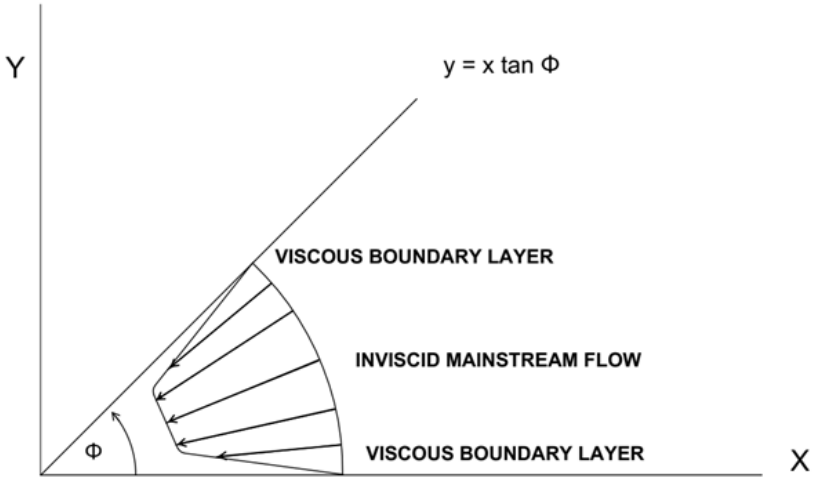

Figure 4.2.1: High Reynolds number flow in a convergent channel

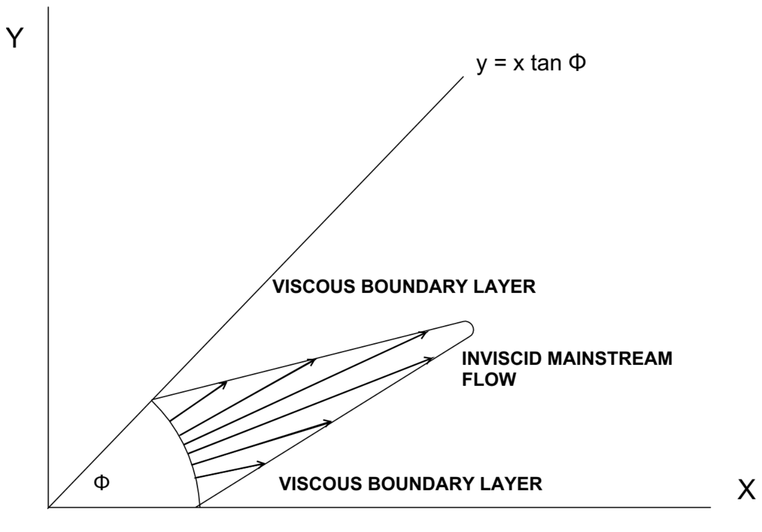

Figure 4.2.2: High Reynolds number flow in a divergent channel

Consider a steady high Reynolds number flow of an incompressible fluid between two intersecting plates, $y = 0$ and $y = x\tan\phi$, which meet at angle ϕ. Figure 4.2.1 illustrates flow in a convergent channel and Figure 4.2.2 flow in a divergent channel. For flow in a convergent channel, there is a line sink along the line of intersection of the plates through which fluid is extracted. For flow in a divergent channel, there is a line source through which fluid is injected.

It is assumed that the flow consists of an inviscid mainstream flow and thin boundary layers on the two plates in which viscosity is not neglected. The aim is to determine the mainstream velocity and show that it is given by equation (4.2.1).

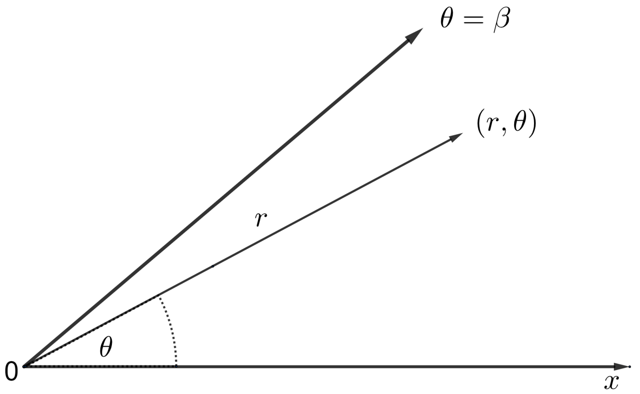

Figure 4.2.3: Cylindrical polar coordinates with the z-axis along the line of intersection of the plates.

Cylindrical polar coordinates (r, θ, z) in the channel are chosen with the z -axis along the line of intersection of the plates as shown in Figure 4.2.3. To model the mainstream flow we look for an inviscid flow in the channel that is purely radial:

$$v_r = v_r(r,\theta), \quad v_\theta = 0, \quad v_z = 0, \quad p = p(r,\theta) \tag{4.2.2}$$

and investigate the conditions put on $v_r(r,\theta)$ and $p(r,\theta)$ by the conservation of mass equation and by the Navier-Stokes equation for inviscid flow, the Euler equation.

The conservation of mass equation in cylindrical polar coordinates is

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{4.2.3}$$

and therefore for (4.2.2),

$$\frac{\partial}{\partial r}(rv_r(r,\theta)) = 0. \tag{4.2.4}$$

Hence

$$v_r(r,\theta) = \frac{U(\theta)}{r} \tag{4.2.5}$$

where $U(\theta)$ is an arbitrary function of θ.

The r, θ and z components of the steady state Euler equation for an inviscid fluid are

$$r - \text{component}: \quad v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{4.2.6}$$

$$\theta - \text{component}: \quad v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}, \tag{4.2.7}$$

$$z - \text{component}: \quad v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{4.2.8}$$

Substituting (4.2.5) into (4.2.6) and (4.2.7) gives

$$\frac{U^2(\theta)}{r^3} = \frac{1}{\rho}\frac{\partial p}{\partial r}(r,\theta), \tag{4.2.9}$$

$$0 = \frac{\partial p}{\partial \theta}(r,\theta) \tag{4.2.10}$$

and (4.2.8) is identically satisfied. From (4.2.10), $p = p(r)$ and (4.2.9) becomes

$$U^2(\theta) = \frac{r^3}{\rho}\frac{dp}{dr}(r). \tag{4.2.11}$$

By the technique of separation of variables each side of (4.2.11) must be a constant. The constant is positive because $U^2(\theta)$ is positive. Denote the constant by U_0^2. Thus

$$U^2(\theta) = U_0^2 \tag{4.2.12}$$

and

$$U(\theta) = \pm|U_0|. \tag{4.2.13}$$

Hence by (4.2.5)

$$v_r(r) = \begin{cases} -\dfrac{|U_0|}{r} & \text{convergent channel,} \\ \dfrac{|U_0|}{r} & \text{divergent channel.} \end{cases} \tag{4.2.14}$$

which can be written as

$$v_r = \frac{U_0}{r} = \begin{cases} \text{velocity in a convergent channel} \quad \text{if } U_0 < 0, \\ \text{velocity in a divergent channel} \quad \text{if } U_0 > 0. \end{cases} \tag{4.2.15}$$

In the boundary layer at $\theta = 0$, r can be replaced by x. The mainstream flow for the boundary layer is therefore given by (4.2.1) for a convergent channel if $U_0 < 0$ and a divergent channel if $U_0 > 0$.

In this chapter we will be concerned with the following two problems. The results were derived in Chapter 3.

(i) Boundary layer flow in a convergent channel

$m = -1, U_0 < 0$.

$$\frac{d^3F}{d\eta^3} + \frac{v_{s0}}{(-U_0\nu)^{\frac{1}{2}}}\frac{d^2F}{d\eta^2} + 1 - \left(\frac{dF}{d\eta}\right)^2 = 0, \tag{4.2.16}$$

$$\frac{dF}{d\eta}(\infty) = 1, \quad \frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}, \tag{4.2.17}$$

$$v_x(x,y) = \frac{U_0}{(x-x_0)}\frac{dF}{d\eta}, \tag{4.2.18}$$

$$v_y(x,y) = -\frac{v_{s0}}{(x-x_0)} - \frac{(-U_0\nu)^{\frac{1}{2}}}{(x-x_0)}\eta\frac{dF}{d\eta}, \tag{4.2.19}$$

$$\eta = \left(-\frac{U_0}{\nu}\right)^{\frac{1}{2}}\frac{y}{(x-x_0)}. \tag{4.2.20}$$

(ii) Boundary layer flow in a divergent channel

$m = -1, U_0 > 0$.

$$\frac{d^3F}{d\eta^3} + \frac{v_{s0}}{(U_0\nu)^{\frac{1}{2}}}\frac{d^2F}{d\eta^2} - \left(1 - \left(\frac{dF}{d\eta}\right)^2\right) = 0, \tag{4.2.21}$$

$$\frac{dF}{d\eta}(\infty) = 1, \quad \frac{dF}{d\eta}(0) = \frac{v_{t0}}{U_0}, \tag{4.2.22}$$

$$v_x(x,y) = \frac{U_0}{(x-x_0)}\frac{dF}{d\eta}, \tag{4.2.23}$$

$$v_y(x,y) = -\frac{v_{s0}}{(x-x_0)} + \frac{(U_0\nu)^{\frac{1}{2}}}{(x-x_0)}\eta\frac{dF}{d\eta}, \tag{4.2.24}$$

$$\eta = \left(\frac{U_0}{\nu}\right)^{\frac{1}{2}}\frac{y}{(x-x_0)}. \tag{4.2.25}$$

There are two boundary conditions given by (4.2.17) and (4.2.22) for the third ordinary differential equations (ODEs). We will find that a third boundary condition is required in the solution. The third boundary condition is

$$\frac{\partial v_x}{\partial y}(x, \infty) = 0. \tag{4.2.26}$$

Expressed in terms of $F(\eta)$ the third boundary condition is

$$\frac{d^2 F}{d\eta^2}(\infty) = 0. \tag{4.2.27}$$

In both problems only the derivatives of $F(\eta)$ occur in the ODE, boundary conditions and $v_x(x, y)$ and $v_y(x, y)$. The function $F(\eta)$ itself does not occur. The problems can therefore be formulated in terms of

$$H(\eta) = \frac{dF}{d\eta}. \tag{4.2.28}$$

The two problems can be solved at the same time by introducing the parameters λ and α defined below.

The problem can be stated as follows

$$\frac{d^2 H}{d\eta^2} + \lambda \frac{dH}{d\eta} + \varepsilon(1 - H^2) = 0, \tag{4.2.29}$$

$$H(\infty) = 1, \quad H(0) = \frac{v_{t0}}{U_0}, \quad \frac{dH}{d\eta}(\infty) = 0, \tag{4.2.30}$$

where

$$\text{convergent channel} \quad U_0 < 0 : \varepsilon = +1, \ \lambda = \frac{v_{s0}}{(-U_0 \nu)^{\frac{1}{2}}}, \tag{4.2.31}$$

$$\text{divergent channel} \quad U_0 > 0 : \varepsilon = -1, \ \lambda = \frac{v_{s0}}{(U_0 \nu)^{\frac{1}{2}}}. \tag{4.2.32}$$

Also

$$v_x(x, y) = \frac{U_0}{(x - x_0)} H(\eta), \tag{4.2.33}$$

$$v_y(x, y) = \begin{cases} -\dfrac{v_{so}}{(x - x_0)} - \dfrac{(-U_0 \nu)^{\frac{1}{2}}}{(x - x_0)} \eta H(\eta), & U_0 < 0, \qquad (4.2.34\text{a}) \\[2ex] -\dfrac{v_{so}}{(x - x_0)} + \dfrac{(U_0 \nu)^{\frac{1}{2}}}{(x - x_0)} \eta H(\eta), & U_0 > 0, \qquad (4.2.34\text{b}) \end{cases}$$

and

$$\eta = \begin{cases} \left(\dfrac{-U_0}{\nu}\right)^{\frac{1}{2}} \dfrac{y}{(x-x_0)}, \quad U_0 < 0, & (4.2.35a) \\ \left(\dfrac{U_0}{\nu}\right)^{\frac{1}{2}} \dfrac{y}{(x-x_0)}, \quad U_0 > 0. & (4.2.35b) \end{cases}$$

The coordinate x_0 is determined by the choice of the origin of the coordinates. We will choose the origin to be at the point of intersection of the two plates. We expect that if there is a singularity in the flow, it will be at the point of intersection of the plates. Thus $x_0 = 0$.

We will include slip at the boundary. Fluid can slip at a rough surface because it flows over the trapped fluid at the surface. This is illustrated in Figure 4.2.4.

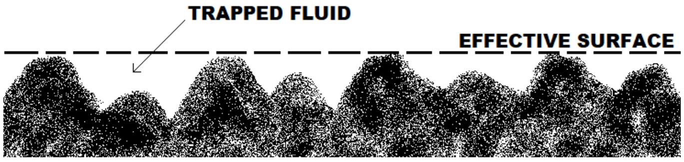

Figure 4.2.4: Illustration of a rough surface with trapped fluid. The fluid slips at the effective surface because it flows over the trapped fluid.

4.3 Boundary layer flow in convergent and divergent channels with slip but impermeable boundaries

We first investigate the effect of slip at the boundary. The fluid can slip at the boundary but there is no suction or blowing at the boundary.

Since the plates are impermeable, $v_{s0} = 0$ and (4.2.29) reduces to

$$\frac{d^2H}{d\eta^2} + \varepsilon(1 - H^2) = 0, \tag{4.3.1}$$

subject to the boundary conditions

$$H(\infty) = 1, \quad H(0) = \frac{v_{t0}}{U_0}, \quad \frac{dH}{d\eta}(\infty) = 0. \tag{4.3.2}$$

The differential equation (4.3.1) is readily solved and does not require Lie group analysis to obtain the solution [16].

4.3.1 Flow in a convergent channel, $U_0 < 0$.

For flow in a convergent channel $\varepsilon = +1$. The boundary layer flow is illustrated in Figure 4.3.1. Now

$$\frac{d^2H}{d\eta^2} = \frac{d}{dH}\Big(\frac{dH}{d\eta}\Big)\frac{dH}{d\eta} = \frac{d}{dH}\Big(\frac{1}{2}\Big(\frac{dH}{d\eta}\Big)^2\Big) \tag{4.3.3}$$

and (4.3.1) becomes

$$\frac{d}{dH}\Big(\frac{1}{2}\Big(\frac{dH}{d\eta}\Big)^2\Big) = H^2 - 1. \tag{4.3.4}$$

Hence

$$\Big(\frac{dH}{d\eta}\Big)^2 = \frac{2}{3}(H^3 - 3H + k) \tag{4.3.5}$$

where k is a constant. Imposing the boundary condition (4.3.2) at $\eta = \infty$ gives $k = 2$ and therefore

$$\Big(\frac{dH}{d\eta}\Big)^2 = \frac{2}{3}(1 - H)^2(H + 2). \tag{4.3.6}$$

The graph of $\Big(\frac{dH}{d\eta}\Big)^2$ against H is drawn in Figure 4.3.2. Since $\Big(\frac{dH}{d\eta}\Big)^2 \geq 0$, the range of H is $-2 \leq H \leq 1$. Now

$$\frac{dH}{d\eta} = \pm\Big(\frac{2}{3}\Big)^{\frac{1}{2}}(1 - H)(H + 2)^{\frac{1}{2}}. \tag{4.3.7}$$

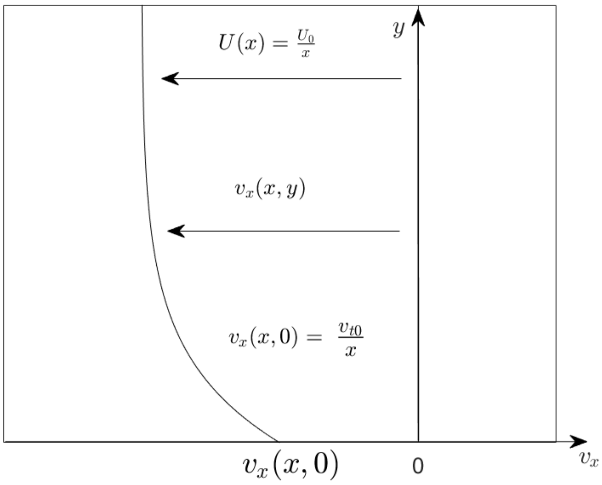

Figure 4.3.1: Boundary layer flow with slip in a convergent channel.

The differential equation is variables separable,

$$\frac{dH}{(1-H)(H+2)^{\frac{1}{2}}} = \pm\Big(\frac{2}{3}\Big)^{\frac{1}{2}} d\eta. \tag{4.3.8}$$

Let $H+2 = K^2$. Then

$$\frac{dK}{3-K^2} = \pm\Big(\frac{1}{6}\Big)^{\frac{1}{2}} d\eta. \tag{4.3.9}$$

Let $K = \sqrt{3}\tanh\theta$. Equation (4.3.9) becomes

$$d\theta = \pm\frac{1}{\sqrt{2}} d\eta \tag{4.3.10}$$

and therefore

$$\theta = \pm\frac{\eta}{\sqrt{2}} + c \tag{4.3.11}$$

where c is a constant. Thus

$$K = \sqrt{3}\tanh{(\pm\frac{\eta}{\sqrt{2}} + c)} = \pm\sqrt{3}\tanh{(\frac{\eta}{\sqrt{2}} + c)} \tag{4.3.12}$$

and therefore

$$H = 3\tanh^2{(\frac{\eta}{\sqrt{2}} + c)} - 2. \tag{4.3.13}$$

The boundary condition $H(\infty) = 1$ is satisfied. Imposing the boundary condition (4.3.2) at $\eta = 0$ gives

$$\frac{v_{t0}}{U_0} = 3\tanh^2 c - 2 \tag{4.3.14}$$

and therefore

$$c = \pm\tanh^{-1}\Big[\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1 + \frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big]. \tag{4.3.15}$$

Hence

$$H(\eta) = -2 + 3\tanh^2\Big[\frac{\eta}{\sqrt{2}} \pm \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1 + \frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big]. \tag{4.3.16}$$

The solution for the flow is not unique. This is an example of non-unique solution of a boundary layer flow. From (4.2.33) and (4.2.34b)

$$v_x(x,y) = \frac{U_0}{x}\Big[-2+3\tanh^2\Big[\Big(\frac{-U_0}{2\nu}\Big)^{\frac{1}{2}}\frac{y}{x} \pm \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big]\Big], \quad (4.3.17)$$

$$v_y(x,y) = U_0\frac{y}{x^2}\Big[-2+3\tanh^2\Big[\Big(\frac{-U_0}{2\nu}\Big)^{\frac{1}{2}}\frac{y}{x} \pm \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big]\Big]. \quad (4.3.18)$$

For the solution to be real

$$-\infty < v_{t0} \le -2U_0. \quad (4.3.19)$$

The velocity $v_x(x,y)$ approaches the mainstream velocity $\dfrac{U_0}{x}$ when

$$\Big(-\frac{U_0}{2\nu}\Big)^{\frac{1}{2}}\frac{y}{x} = O(1), \quad (4.3.20)$$

that is, when

$$y = \Big(\frac{2\nu}{-U_0}\Big)^{\frac{1}{2}}x. \quad (4.3.21)$$

The thickness of the boundary layer is therefore proportional to x and decreases as the mainstream velocity $\dfrac{U_0}{x}$ increases. Also, for large values of y,

$$v_y(x,y) = U_0\frac{y}{x^2}. \quad (4.3.22)$$

Thus $v_y(x,y)$ does not tend to zero as $y \to \infty$. It is a feature of the boundary layer approximation that the transverse velocity v_y does not tend to zero as $y \to \infty$.

We now draw the graphs of v_x against η . We will first take $v_{t0} = 0$ and then investigate the effects of slip on the solution for flow in a convergent channel, $U_0 < 0$.

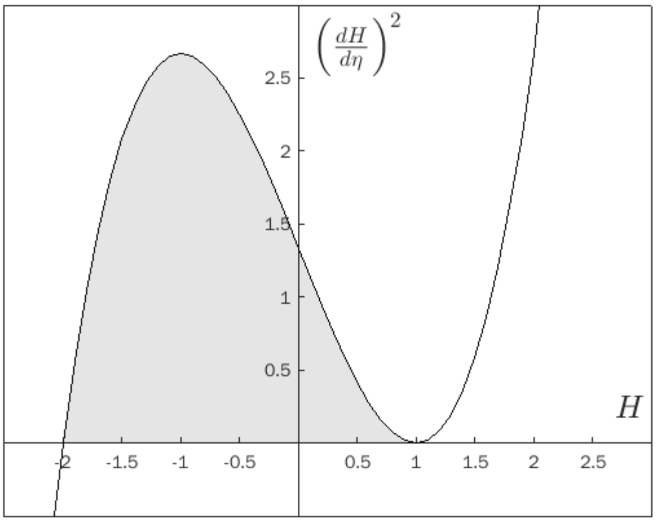

Figure 4.3.2: Flow in a convergent channel. Graph of $\left(\dfrac{dH}{d\eta}\right)^2$ plotted against H. The shaded region, $-2 \leq H \leq 1$, is the region of physical significance.

(i) **Minus sign in** (4.3.17) **and $\mathbf{v_{t0} = 0}$.**

Taking the $-$sign in the (4.3.17) and setting $v_{t0} = 0$ gives

$$v_x(x, y) = \frac{U_0}{x}\Big[3\tanh^2\Big(\frac{\eta}{\sqrt{2}} - \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big) - 2\Big]. \tag{4.3.23}$$

Now

$$v_x(x, \eta) = 0 \tag{4.3.24}$$

when

$$\eta = 2\sqrt{2}\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big), \quad \eta = 0. \tag{4.3.25}$$

Also

$$\eta = \sqrt{2}\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big): \qquad v_x = -\frac{2U_0}{x}, \tag{4.3.26}$$

$$\eta = \infty: \qquad v_x = \frac{U_0}{x}. \tag{4.3.27}$$

The graph of v_x against η is plotted in Figure 4.3.3. There is reverse flow close to the plate in the boundary layer. The region of the reverse flow is

$$0 < \eta < 2\sqrt{2}\tanh^{-1}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}\right). \tag{4.3.28}$$

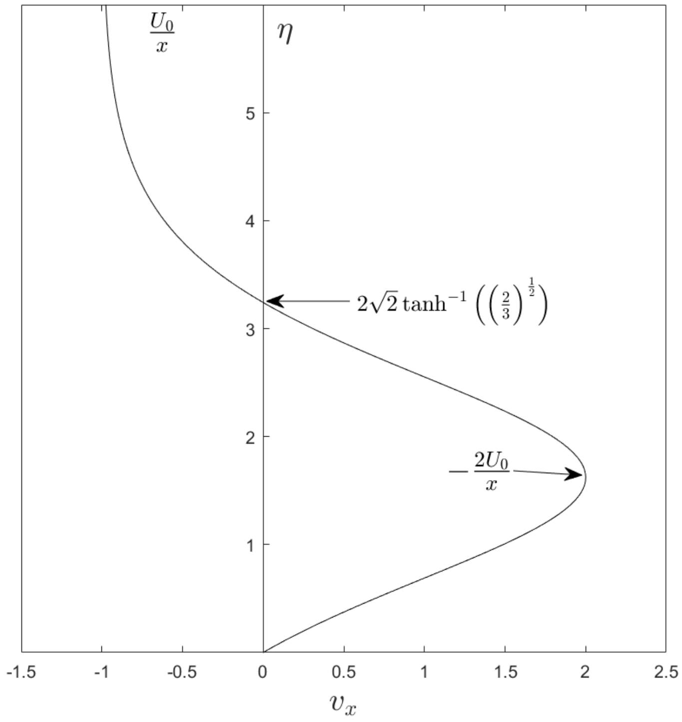

Figure 4.3.3: The velocity v_x given by (4.3.23) plotted against η for $\dfrac{U_0}{x} = -1$.

(ii) **Plus sign in** (4.3.17) **and** $v_{t0} = 0$**.**

Taking the $+$sign in (4.3.17) and setting $v_{t0} = 0$ we obtain

$$v_x(x,\eta) = \frac{U_0}{x}\left[3\tanh^2\left(\frac{\eta}{\sqrt{2}} + \tanh^{-1}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)\right) - 2\right]. \tag{4.3.29}$$

Now

$$v_x(x,\eta) = 0 \tag{4.3.30}$$

when

$$\eta = 0, \quad \eta = -2\sqrt{2}\tanh^{-1}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}\right). \tag{4.3.31}$$

The negative root is outside the physical range, $0 \leq \eta \leq \infty$. Also

$$\eta = \infty : \quad v_x = \frac{U_0}{x} < 0. \tag{4.3.32}$$

Thus $v_x(x, y) < 0$ for $0 < \eta \leq \infty$. The velocity $v_x(x, y)$ is plotted against η in Figure 4.3.4. There is no reverse flow in the boundary layer. This is the velocity profile usually observed in experiments [4].

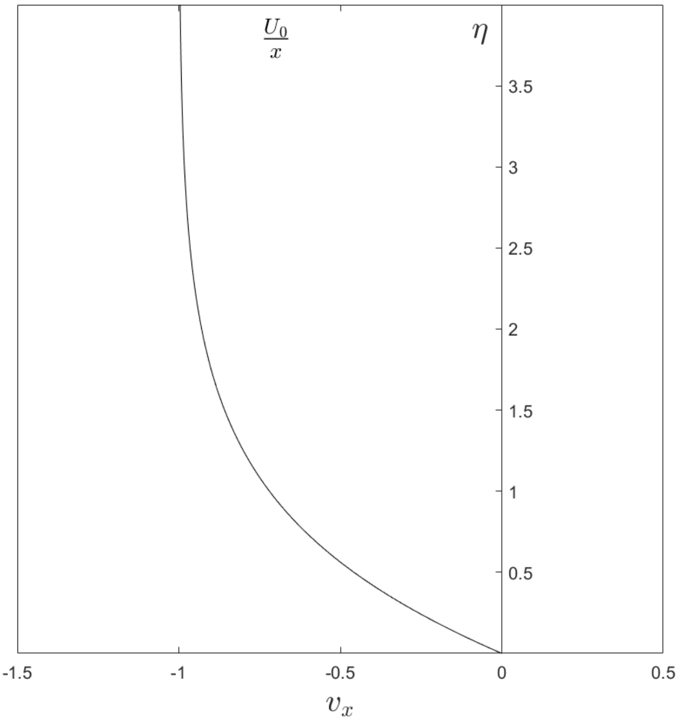

Figure 4.3.4: The velocity v_x given by (4.3.29) plotted against η for $\dfrac{U_0}{x} = -1$.

(iii) **Minus sign in** (4.3.17)**,** $-\infty < \mathbf{v_{t0}} \leq -\mathbf{2U_0}$**,** $v_{t0} \neq 0$**.**

For a real solution, $-\infty < v_{t0} \leq -2U_0$. Suppose first that $-\infty < v_{t0} < -2U_0$ with $v_{t0} \neq 0$. The limiting case $v_{t0} = -2U_0$ will be analysed later. Taking the $-$sign in (4.3.17) with $v_{t0} \neq 0$ gives

$$v_x = \frac{U_0}{x}\Big[3\tanh^2\Big(\frac{\eta}{\sqrt{2}} - \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1 + \frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big) - 2\Big]. \tag{4.3.33}$$

Now

$$v_x(x, \eta) = 0 \tag{4.3.34}$$

when

$$\eta = \eta_1 = \sqrt{2}\Big[\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big) + \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big)\Big], \tag{4.3.35}$$

and

$$\eta = \eta_2 = \sqrt{2}\Big[\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big) - \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big)\Big]. \tag{4.3.36}$$

If $0 < v_{t0} < -2U_0$, then since $U_0 < 0$,

$$\tanh^{-1}\Big[\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big] < \tanh^{-1}\Big[\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big] \tag{4.3.37}$$

and $\eta_1 > 0$ and $\eta_2 < 0$. There is only one positive root. Also

$$\eta_1 < 2\sqrt{2}\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big). \tag{4.3.38}$$

If $-\infty < v_{t0} < 0$, then

$$\tanh^{-1}\Big[\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big] > \tanh^{-1}\Big[\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big] \tag{4.3.39}$$

and $\eta_1 > 0$ and $\eta_2 > 0$ where now

$$\eta_1 > 2\sqrt{2}\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big). \tag{4.3.40}$$

There are two positive roots.

There are also the following special values

$$\eta = 0 : \quad v_x = \frac{v_{t0}}{x}, \tag{4.3.41}$$

$$\eta = \sqrt{2}\tanh^{-1}\Big[\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big] : \quad v_x = \frac{-2U_0}{x}, \tag{4.3.42}$$

$$\eta = 2\sqrt{2}\tanh^{-1}\Big[\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big] : \quad v_x = \frac{v_{t0}}{x}, \tag{4.3.43}$$

$$\eta = \infty : \quad v_x = \frac{U_0}{x}. \tag{4.3.44}$$

There are three cases to consider, $0 < v_{t0} < -2U_0$, $U_0 \leq v_{t0} < 0$ and $-\infty < v_{t0} < U_0$.

Consider first $0 < v_{t0} < -2U_0$ and therefore $\eta_1 > 0$ and $\eta_2 < 0$. In Figure 4.3.5, $v_x(x,\eta)$ given by (4.3.33) is plotted against η. There is reverse flow in the boundary layer. The range of the reverse flow is $0 < \eta < \eta_1$ where η_1 is given by (4.3.35). From the inequality (4.3.38) we see that this range is less than the range (4.3.28) for no slip. The slip boundary condition therefore reduces the depth of penetration of the reverse flow into the boundary layer.

Suppose next that $U_0 \leq v_{t0} < 0$. Then both $\eta_1 > 0$ and $\eta_2 > 0$. In Figure 4.3.6, $v_x(x, y)$ given by (4.3.33) is plotted against η. There is reverse flow in the interior of the boundary layer, not at the boundary. The range of the reverse flow is $\eta_2 < \eta < \eta_1$. From (4.3.35) and (4.3.36),

$$\eta_1 - \eta_2 = 2\sqrt{2} \tanh^{-1} \left(\left(\frac{2}{3} \right)^{\frac{1}{2}} \right). \tag{4.3.45}$$

The width of the region of reverse flow is the same as (4.3.28) for no slip.

For the third case $-\infty < v_{t0} < U_0$ where $U_0 < 0$. Then

$$\frac{v_{t0}}{U_0} > 1. \tag{4.3.46}$$

Since $-1 \leq \tanh x \leq 1$ for $-\infty \leq x \leq \infty$ it follows from (4.3.33) that for a solution to exist, it is necessary that

$$\left(\frac{2}{3} \right)^{\frac{1}{2}} \left(1 + \frac{v_{t0}}{2U_0} \right)^{\frac{1}{2}} \leq 1, \tag{4.3.47}$$

that is,

$$\frac{v_{t0}}{U_0} \leq 1. \tag{4.3.48}$$

Thus for the third case a solution does not exist.

Consider now the limiting positive value $v_{t0} = -2U_0$. Equation (4.3.33) becomes

$$v_x(x, y) = \frac{U_0}{x} \left[3 \tanh^2 \left(\frac{\eta}{\sqrt{2}} \right) - 2 \right]. \tag{4.3.49}$$

Now

$$v_x(x, y) = 0 \tag{4.3.50}$$

when

$$\eta = \eta_1 = \sqrt{2} \tanh^{-1} \left(\left(\frac{2}{3} \right)^{\frac{1}{2}} \right) \tag{4.3.51}$$

and $\eta_2 = -\eta_1 < 0$. Also,

$$\eta = 0: \quad v_x = -\frac{2U_0}{x}, \tag{4.3.52}$$

$$\eta = \infty: \quad v_x = \frac{U_0}{x}. \tag{4.3.53}$$

In Figure 4.3.7 $v_x(x, \eta)$ given by (4.3.49) is plotted against η. There is reverse flow at the boundary. The range of the reverse flow is

$$0 < \eta < \sqrt{2}\tanh^{-1}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}\right) \tag{4.3.54}$$

which is half the range, (4.3.28), when $v_{t0} = 0$.

In summary, as v_{t0} increases from 0 to $-2U_0$, the width of the region of reverse flow decreases from $2\sqrt{2}\tanh^{-1}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)$ to $\sqrt{2}\tanh^{-1}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)$. The region of reverse flow is always at the boundary. For $-\infty < v_{t0} < 0$, the region of reverse flow is in the interior of the boundary layer but its width is always $2\sqrt{2}\tanh^{-1}\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)$ which is the same as for no slip, $v_{t0} = 0$.

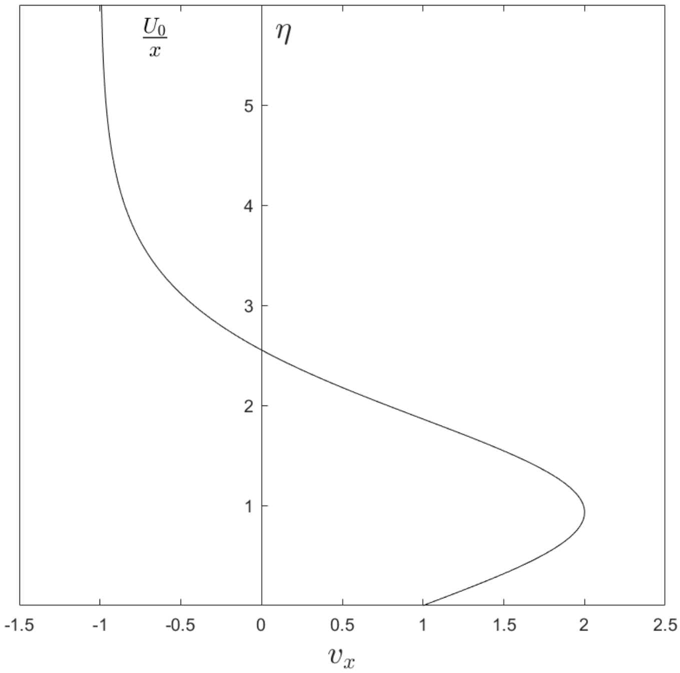

Figure 4.3.5: Boundary layer with slip, $0 < v_{t0} < -2U_0$. Velocity v_x given by (4.3.33) plotted against η for $\frac{U_0}{x} = -1$ and $\frac{v_{t0}}{U_0} = -1$.

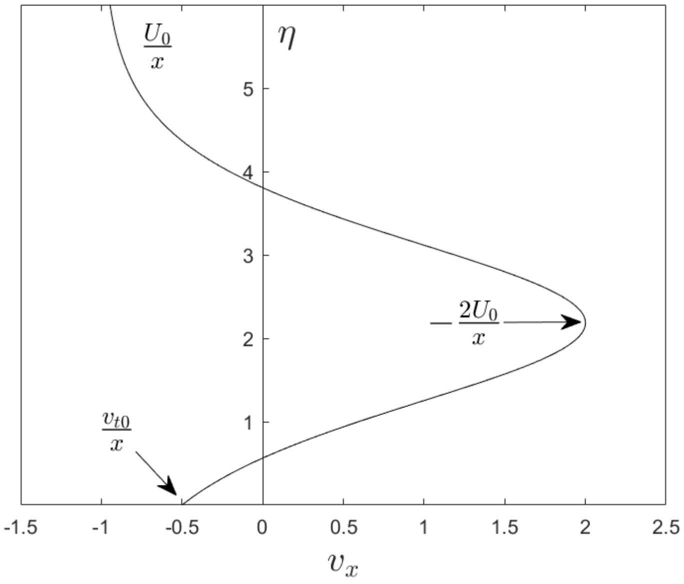

Figure 4.3.6: Boundary layer with slip, $-\infty < v_{t0} < U_0$. Velocity v_x given by (4.3.33) plotted against η for $\frac{U_0}{x} = -1$ and $\frac{v_{t0}}{U_0} = \frac{1}{2}$.

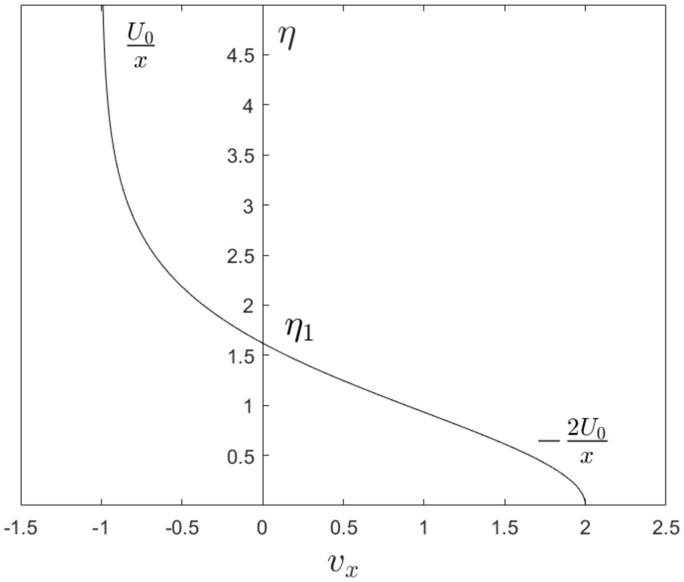

Figure 4.3.7: Boundary layer with slip, $v_{t0} = -2U_0$. Velocity v_x given by (4.3.49) plotted against η for $\frac{U_0}{x} = -1$.

(iv) **Plus sign in** (4.3.17) **with** $\mathbf{v_{t0} \neq 0}$

Taking the $+$sign in (4.3.17) with $v_{t0} \neq 0$ gives

$$v_x(x,\eta) = \frac{U_0}{x}\Big[3\tanh^2\Big(\frac{\eta}{\sqrt{2}} + \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big) - 2\Big]. \tag{4.3.55}$$

Now

$$v_x(x,\eta) = 0 \tag{4.3.56}$$

when

$$\eta = \eta_1 = \sqrt{2}\Big[\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big) - \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big], \tag{4.3.57}$$

and

$$\eta = \eta_2 = \sqrt{2}\Big[-\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big) - \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big]. \tag{4.3.58}$$

When $0 < v_{t0} \leq -2U_0$, $\eta_1 > 0$ while when $-\infty < v_{t0} \leq 0, \eta_1 < 0$. For the whole range $-\infty < v_{t0} \leq -2U_0$, we have $\eta_2 < 0$. Also

$$\eta = \infty : \quad v_x = \frac{U_0}{x} < 0. \tag{4.3.59}$$

Thus if $0 < v_{t0} \leq -2U_0$ there is one zero of v_x in the range $0 < \eta < \infty$. The graph of $v_x(x,y)$ plotted against η is shown in Figure 4.3.8. There is reverse flow in the velocity profile at the boundary for

$$0 < \eta < \eta_1 = \sqrt{2}\Big[\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big) - \tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big(1+\frac{v_{t0}}{2U_0}\Big)^{\frac{1}{2}}\Big)\Big]. \tag{4.3.60}$$

The range of the reverse flow increases from 0 for $v_{t0} = 0$ to

$$0 < \eta < \sqrt{2}\tanh^{-1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{2}}\Big) \tag{4.3.61}$$

for $v_{t0} = -2U_0$.

For $-\infty < v_{t0} < 0$, there is no zero of $v_x(x,y)$ in the range $0 < \eta < \infty$. The graph of $v_x(x,\eta)$ plotted against η is shown in Figure 4.3.9 for $U_0 \leq v_{t0} < 0$ and in Figure 4.3.10 for $-\infty < v_{t0} \leq U_0$. There is no reverse flow in the boundary layer.

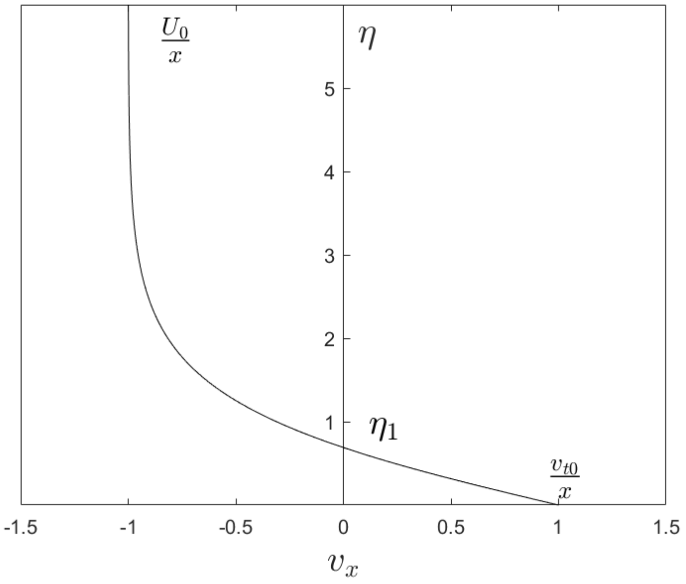

Figure 4.3.8: Boundary layer with slip, $0 < v_{t0} < -2U_0$. Velocity v_x given by (4.3.55) plotted against η for $\dfrac{U_0}{x} = -1$ and for $\dfrac{v_{t0}}{U_0} = -1$.

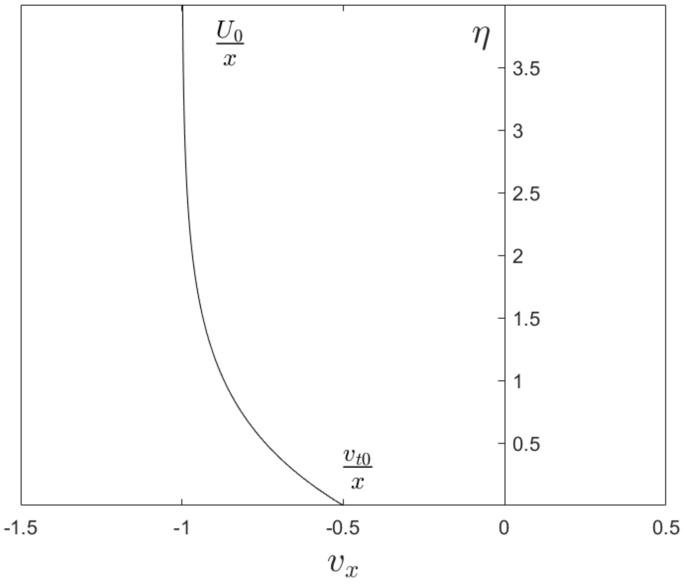

Figure 4.3.9: Boundary layer with slip, $U_0 < v_{t0} < 0$. Velocity v_x given by (4.3.55) plotted against η for $\dfrac{U_0}{x} = -1$ and for $\dfrac{v_{t0}}{U_0} = \dfrac{1}{2}$.

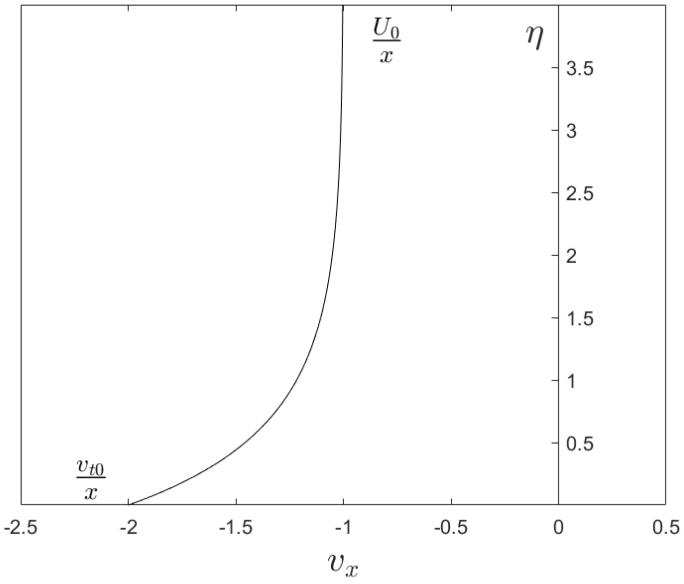

Figure 4.3.10: Boundary layer with slip, $-\infty < v_{t0} < U_0$. Velocity v_x given by (4.3.55) plotted against η for $\dfrac{U_0}{x} = -1$ and for $\dfrac{v_{t0}}{U_0} = 2$.

4.3.2 Flow in a divergent channel, $\mathbf{U_0 > 0}$.

For a divergent channel, $\varepsilon = -1$ and $U_0 > 0$. The boundary layer flow is illustrated in Figure 4.2.2. Equation (4.3.1) becomes

$$\frac{d^2H}{d\eta^2} = 1 - H^2 \tag{4.3.62}$$

and using (4.3.3) we obtain

$$\frac{d}{dH}\Big(\frac{1}{2}\Big(\frac{dH}{d\eta}\Big)^2\Big) = 1 - H^2. \tag{4.3.63}$$

Hence

$$\Big(\frac{dH}{d\eta}\Big)^2 = -\frac{2}{3}\Big(H^3 - 3H + k\Big) \tag{4.3.64}$$

where k is a constant. Imposing the boundary conditions (4.3.2) at $\eta = \infty$ gives $k = 2$. Hence

$$\Big(\frac{dH}{d\eta}\Big)^2 = -\frac{2}{3}\Big(1 - H\Big)^2\Big(H + 2\Big). \tag{4.3.65}$$

The difference between (4.3.6) for a convergent channel and (4.3.65) for a divergent channel is the minus sign.

A graph of $\left(\frac{dH}{d\eta}\right)^2$ plotted against H is shown in Figure 4.3.11. For a real solution it is necessary that

$$\left(\frac{dH}{d\eta}\right)^2 \geq 0. \tag{4.3.66}$$

Thus for a real solution

$$\text{either} \quad \left(\frac{dH}{d\eta}\right)^2 = 1 \quad \text{or} \quad -\infty < H \leq -2. \tag{4.3.67}$$

The first solution in (4.3.67) does not satisfy the boundary condition $\frac{dH}{d\eta}(\infty) = 0$ and the second solution does not satisfy the boundary condition $H(\infty) = 1$ which is not in the range $-\infty < H \leq -2$. A solution for boundary layer flow in a divergent channel with impermeable walls does not exist.

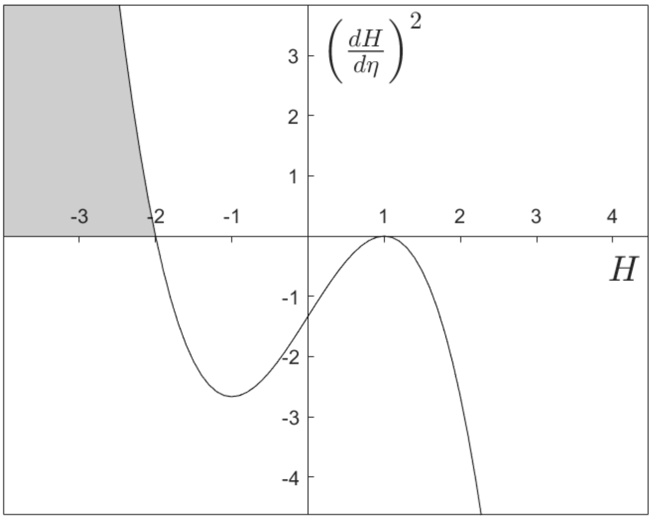

Figure 4.3.11: Flow in a divergent channel. Graph of $\left(\frac{dH}{d\eta}\right)^2$ plotted against H. The shaded region, $-\infty < H \leq -2$, is the region of physical significance.

Further confirmation that the solution satisfying the boundary conditions does not exist can be seen by imposing the remaining boundary condition

$$H(0) = \frac{v_{t0}}{U_0}. \tag{4.3.68}$$

Equation (4.3.65) becomes

$$\Big(\frac{dH}{d\eta}(0)\Big)^2 = -\frac{2}{3}\Big(1-\frac{v_{t0}}{U_0}\Big)^2\Big(2+\frac{v_{t0}}{U_0}\Big). \tag{4.3.69}$$

The graph of $\Big(\frac{dH}{d\eta}(0)\Big)^2$ plotted against $\frac{v_{t0}}{U_o}$ is given in Figure 4.3.12. The boundary condition will only be satisfied for

$$-\infty < \frac{v_{t0}}{U_0} \le -2 \quad \text{and} \quad \frac{v_{t0}}{U_0} = 1. \tag{4.3.70}$$

For the no slip boundary condition, equation (4.3.69) reduces to

$$\Big(\frac{dH}{d\eta}(0)\Big)^2 = -\frac{4}{3} \tag{4.3.71}$$

which clearly is not satisfied.

For flow in a divergent channel with impermeable walls the assumption that the flow consists of mainstream with zero viscosity and a viscous boundary layer is not correct, even if there is slip at the boundary. In the remainder of this Chapter the effects of non-zero suction or blowing on boundary layer flow in convergent and divergent channels with no slip will be investigated.

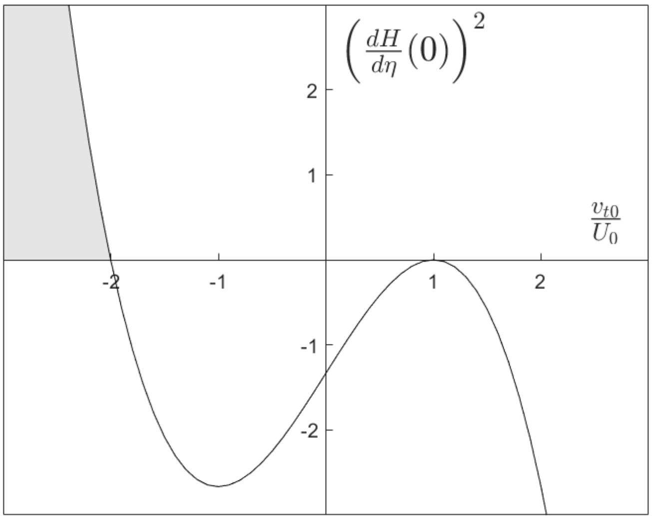

Figure 4.3.12: Flow in a divergent channel. Graph of $\Big(\frac{dH}{d\eta}(0)\Big)^2$ plotted against $\frac{v_{t0}}{U_0}$. The shaded region, $-\infty < \frac{v_{t0}}{U_0} \le -2$, is the region of physical significance.

4.4 Lie point symmerties and reduction of order

We now derive the Lie point symmetries of the differential equation (4.2.29) which we write again here,

$$\frac{d^2H}{d\eta^2} + \lambda\frac{dH}{d\eta} + \varepsilon(1 - H^2) = 0 \tag{4.4.1}$$

and use the Lie point symmetries to reduce it to a first order differential equation. The procedure described by Ibragimov [17] will be followed.

The differential equation (4.4.1) can be written as

$$G(H, H^{'}, H^{''}) = 0 \tag{4.4.2}$$

where

$$G(H, H^{'}, H^{''}) = H^{''} + \lambda H^{'} + \varepsilon(1 - H^2). \tag{4.4.3}$$

The Lie point symmetries of (4.4.1) are of the form

$$X = \alpha(\eta, H)\frac{\partial}{\partial\eta} + \beta(\eta, H)\frac{\partial}{\partial H}. \tag{4.4.4}$$

The determining equation is

$$X^{[2]}G\Big|_{G=0} = 0 \tag{4.4.5}$$

where the second prolongation $X^{[2]}$ of X is

$$X^{[2]} = X + \zeta_1(\eta, H, H^{'})\frac{\partial}{\partial H^{'}} + \zeta_2(\eta, H, H^{'}, H^{''})\frac{\partial}{\partial H^{''}} \tag{4.4.6}$$

and

$$\zeta_1 = D(\beta) - H^{'}D(\alpha), \tag{4.4.7}$$

$$\zeta_2 = D(\zeta_1) - H^{''}D(\alpha), \tag{4.4.8}$$

with

$$D = \frac{\partial}{\partial\eta} + H^{'}\frac{\partial}{\partial H} + H^{''}\frac{\partial}{\partial H^{'}} + \dots \tag{4.4.9}$$

Now

$$\zeta_1 = \frac{\partial\beta}{\partial\eta} + \Big(\frac{\partial\beta}{\partial H} - \frac{\partial\alpha}{\partial\eta}\Big)H' - \frac{\partial\alpha}{\partial H}H'^2, \tag{4.4.10}$$

$$\begin{aligned}\zeta_2 = {} & \frac{\partial^2\beta}{\partial\eta^2} + \Big(2\frac{\partial^2\beta}{\partial\eta\partial H} - \frac{\partial^2\alpha}{\partial\eta^2}\Big)H' + \Big(\frac{\partial^2\beta}{\partial H^2} - 2\frac{\partial^2\alpha}{\partial\eta\partial H}\Big)H'^2 \\ & - \frac{\partial^2\alpha}{\partial H^2}H'^3 + \Big(\frac{\partial\beta}{\partial H} - 2\frac{\partial\alpha}{\partial\eta}\Big)H'' - 3\frac{\partial\alpha}{\partial H}H'H''.\end{aligned} \tag{4.4.11}$$

The determining equation (4.4.5) applied to (4.4.3) is

$$\zeta_2 + \lambda\zeta_1 - 2\varepsilon H\beta\Big|_{G=0} = 0 \tag{4.4.12}$$

and substituting ζ_2 and ζ_1 we obtain

$$\begin{aligned}&\frac{\partial^2\beta}{\partial\eta^2} + \Big(2\frac{\partial^2\beta}{\partial\eta\partial H} - \frac{\partial^2\alpha}{\partial\eta^2}\Big)H' + \Big(\frac{\partial^2\beta}{\partial H^2} - 2\frac{\partial^2\alpha}{\partial\eta\partial H}\Big)H'^2 - \frac{\partial^2\alpha}{\partial H^2}H^{'3} + \Big(\frac{\partial\beta}{\partial H} - 2\frac{\partial\alpha}{\partial\eta}\Big)H'' \\ &\qquad -3\frac{\partial\alpha}{\partial H}H'H'' + \lambda\frac{\partial\beta}{\partial\eta} + \lambda\Big(\frac{\partial\beta}{\partial H} - \frac{\partial\alpha}{\partial\eta}\Big)H^{'} - \lambda\frac{\partial\alpha}{\partial H}H'^2 - 2\varepsilon H\beta\Big|_{G=0} = 0.\end{aligned} \tag{4.4.13}$$

Now (4.4.13) must be evaluated on $G = 0$, that is, for

$$H^{''} = -\lambda H^{'} + \varepsilon(H^2 - 1). \tag{4.4.14}$$

Replacing $H^{''}$ in (4.4.13) using (4.4.14) gives

$$\begin{aligned}&\frac{\partial^2\beta}{\partial\eta^2} + \Big(2\frac{\partial^2\beta}{\partial\eta\partial H} - \frac{\partial^2\alpha}{\partial\eta^2}\Big)H' + \Big(\frac{\partial^2\beta}{\partial H^2} - 2\frac{\partial^2\alpha}{\partial\eta\partial H}\Big)H'^2 - \frac{\partial^2\alpha}{\partial H^2}H'^3 \\ &-\lambda H^{'}\frac{\partial\beta}{\partial H} + \varepsilon H^2\frac{\partial\beta}{\partial H} - \varepsilon\frac{\partial\beta}{\partial H} + 2\lambda\frac{\partial\alpha}{\partial\eta}H' - 2\varepsilon\frac{\partial\alpha}{\partial\eta}H^2 \\ &+2\varepsilon\frac{\partial\alpha}{\partial\eta} + 3\lambda\frac{\partial\alpha}{\partial H}H'^2 - 3\alpha H^2\frac{\partial\alpha}{\partial H}H' + 3\varepsilon\frac{\partial\alpha}{\partial H}H' \\ &+\lambda\frac{\partial\beta}{\partial\eta} + \lambda\Big(\frac{\partial\beta}{\partial H} - \frac{\partial\alpha}{\partial\eta}\Big)H' - \lambda\frac{\partial\alpha}{\partial H}H'^2 - 2\varepsilon H\beta = 0.\end{aligned} \tag{4.4.15}$$

Since $\alpha(\eta, H)$ and $\beta(\eta, H)$ do not depend on H', we separate equation (4.4.15) according to powers of $H^{'}$:

$$H^{'3}: \quad \frac{\partial^2\alpha}{\partial H^2} = 0, \tag{4.4.16}$$

$$H^{'2}: \quad \frac{\partial^2\beta}{\partial H^2} + 2\lambda\frac{\partial\alpha}{\partial H} - 2\frac{\partial^2\alpha}{\partial\eta\partial H} = 0, \tag{4.4.17}$$

$$H^{'}: \quad 2\frac{\partial^2\beta}{\partial\eta\partial H} + \lambda\frac{\partial\alpha}{\partial\eta} + 3\varepsilon\frac{\partial\alpha}{\partial H} - 3\varepsilon H^2\frac{\partial\alpha}{\partial H} - \frac{\partial^2\alpha}{\partial\eta^2} = 0, \tag{4.4.18}$$

$$\begin{aligned} remainder: \quad & \frac{\partial^2\beta}{\partial\eta^2} + \lambda\frac{\partial\beta}{\partial\eta} + \varepsilon H^2\frac{\partial\beta}{\partial H} - \varepsilon\frac{\partial\beta}{\partial H} - 2\varepsilon H\beta \\ & - 2\varepsilon\frac{\partial\alpha}{\partial\eta}H^2 + 2\varepsilon\frac{\partial\alpha}{\partial\eta} = 0. \end{aligned} \tag{4.4.19}$$

From equation (4.4.16),

$$\alpha(\eta, H) = HA(\eta) + B(\eta), \tag{4.4.20}$$

where $A(\eta)$ and $B(\eta)$ are functions of η only. Substituting (4.4.20) into (4.4.17) gives

$$\frac{\partial^2\beta}{\partial H^2}(\eta, H) = 2\Big(\frac{dA(\eta)}{d\eta} - \lambda A(\eta)\Big) \tag{4.4.21}$$

and integrating twice by H, we obtain

$$\beta(\eta, H) = \Big(\frac{dA(\eta)}{d\eta} - \lambda A(\eta)\Big)H^2 + HC(\eta) + D(\eta) \tag{4.4.22}$$

where $C(\eta)$ and $D(\eta)$ are functions only of η. Substituting (4.4.20) for $\alpha(\eta, H)$ and (4.4.22) for $\beta(\eta, H)$ into (4.4.18) gives

$$4\Big(\frac{d^2A}{d\eta^2} - \lambda\frac{dA}{d\eta}\Big)H + 2\frac{dC}{d\eta} + \lambda\Big(H\frac{dA}{d\eta} + \frac{dB}{d\eta}\Big) + 3\varepsilon A(\eta) - 3\varepsilon H^2A(\eta) - H\frac{d^2A}{d\eta^2} - \frac{d^2B}{d\eta^2} = 0. \tag{4.4.23}$$

Separate (4.4.23) according to powers of H:

$$H^2: \quad \varepsilon A(\eta) = 0. \tag{4.4.24}$$

Since $\varepsilon = \pm1$, it follows that $\varepsilon \neq 0$ and therefore $A(\eta) = 0$. The coefficient of H in (4.4.23) is now zero. The remainder gives

$$remainder: \quad 2\frac{dC}{d\eta} + \lambda\frac{dB}{d\eta} - \frac{d^2B}{d\eta^2} = 0 \tag{4.4.25}$$

and integrating once with respect to η we obtain

$$C(\eta) = \frac{1}{2}\Big(\frac{dB}{d\eta} - \lambda B(\eta)\Big) + E_0, \tag{4.4.26}$$

where E_0 is a constant. Thus we have found so far that

$$\alpha(\eta, H) = B(\eta), \tag{4.4.27}$$

$$\beta(\eta, H) = \frac{1}{2}\Big(\frac{dB}{d\eta} - \lambda B(\eta)\Big)H + E_0 H + D(\eta). \tag{4.4.28}$$

We now substitute (4.4.27) and (4.4.28) into the remaining condition (4.4.19). We obtain

$$\begin{gathered}\frac{1}{2}\Big(\frac{d^3B}{d\eta^3} - \lambda\frac{d^2B}{d\eta^2}\Big)H + \frac{d^2D}{d\eta^2} + \lambda\Big[\frac{1}{2}\Big(\frac{d^2B}{d\eta^2} - \lambda\frac{dB}{d\eta}\Big)H + \frac{dD}{d\eta}\Big] \\ +\varepsilon\Big[\frac{1}{2}\Big(\frac{dB}{d\eta} - \lambda B\Big) + E_0\Big]H^2 - \varepsilon\Big[\frac{1}{2}\Big(\frac{dB}{d\eta} - \lambda B\Big) + E_0\Big] \\ -2\varepsilon\Big[\frac{1}{2}\Big(\frac{dB}{d\eta} - \lambda B\Big)H + E_0 H + D(\eta)\Big]H - 2\varepsilon\frac{dB}{d\eta}H^2 + 2\varepsilon\frac{dB}{d\eta} = 0.\end{gathered} \tag{4.4.29}$$

Separate (4.4.29) according to powers of H.

$$H^2: \quad \frac{dB}{d\eta} - \frac{\lambda}{5}B(\eta) = -\frac{2}{5}E_0, \tag{4.4.30}$$

$$H: \quad \frac{d^3B}{d\eta^3} - \lambda^2\frac{dB}{d\eta} = 4\varepsilon D(\eta), \tag{4.4.31}$$

$$remainder: \quad \frac{d^2D}{d\eta^2} + \lambda\frac{dD}{d\eta} + \frac{3}{2}\varepsilon\frac{dB}{d\eta} + \frac{1}{2}\varepsilon\lambda B(\eta) - \varepsilon E_0 = 0. \tag{4.4.32}$$

The function $B(\eta)$ is obtained by solving the first order ODE (4.4.30). The integrating factor is $\exp\Big(-\frac{\lambda}{5}\eta\Big)$. The solution is

$$B(\eta) = \frac{2E_0}{\lambda} + F_0\exp\Big(\frac{\lambda}{5}\eta\Big), \tag{4.4.33}$$

where F_0 is a constant. The function $D(\eta)$ is now obtained by substituting $B(\eta)$ into (4.4.31). This gives

$$D(\eta) = -\frac{6}{125}\varepsilon\lambda^3 F_0\exp\Big(\frac{\lambda}{5}\eta\Big). \tag{4.4.34}$$

Finally, substituting (4.4.33) and (4.4.34) into the remaining condition (4.4.32) gives a condition on λ for the Lie point symmetry to exist:

$$\lambda = \pm\frac{5}{\sqrt{3}}. \tag{4.4.35}$$

The Lie point symmetry is given by (4.4.27) and (4.4.28):

$$\alpha(\eta, H) = \frac{2}{\lambda}E_0 + F_0 \exp\Big(\frac{\lambda}{5}\Big), \tag{4.4.36}$$

$$\beta(\eta, H) = -\frac{2}{5}\lambda F_0\Big(H + \frac{3}{25}\varepsilon\lambda^2\Big)\exp\Big(\frac{\lambda}{5}\eta\Big) \tag{4.4.37}$$

and therefore

$$X = \Big[\frac{2}{\lambda}E_0 + F_0 \exp\Big(\frac{\lambda}{5}\eta\Big)\Big]\frac{\partial}{\partial\eta} - \frac{2}{5}\lambda F_0\Big(H + \frac{3}{25}\varepsilon\lambda^2\Big)\exp\Big(\frac{\lambda}{5}\eta\Big)\frac{\partial}{\partial H}. \tag{4.4.38}$$

There are two Lie point symmetries :

$$F_0 = 1, E_0 = 0 : \quad X_1 = \exp\Big(\frac{\lambda}{5}\eta\Big)\frac{\partial}{\partial\eta} - \frac{2}{5}\lambda\Big(H + \frac{3}{25}\varepsilon\lambda^2\Big)\exp\Big(\frac{\lambda}{5}\eta\Big)\frac{\partial}{\partial H}, \tag{4.4.39}$$

$$F_0 = 0, E_0 = 1 : \quad X_2 = \frac{\partial}{\partial\eta}, \tag{4.4.40}$$

where $\lambda = \pm\frac{5}{\sqrt{3}}$.

We now reduce the second order ODE (4.4.1) to a first order ODE following the procedure described by Ibragimov [17]. There are two Lie point symmetries. To determine which Lie point symmetry is used, we calculate

$$[X_1, X_2] = X_1X_2 - X_2X_1 = -\frac{\lambda}{5}X_1. \tag{4.4.41}$$

Hence X_1 is chosen to reduce the order of the ODE.

Now

$$\lambda = \frac{v_{s0}}{(|U_0|\nu)^{\frac{1}{2}}} = \pm\frac{5}{\sqrt{3}}. \tag{4.4.42}$$

We define δ by

$$\lambda = \delta\frac{5}{\sqrt{3}} \tag{4.4.43}$$

where $\delta = \pm 1$ and

$$\delta = +1 \quad \text{suction}, \qquad \delta = -1 \quad \text{blowing} . \tag{4.4.44}$$

Substituting (4.4.43) into X_1 gives

$$X_1 = \exp\Big(\frac{\delta}{\sqrt{3}}\eta\Big)\frac{\partial}{\partial\eta} - \frac{2}{\sqrt{3}}\delta(H + \varepsilon)\exp\Big(\frac{\delta}{\sqrt{3}}\eta\Big)\frac{\partial}{\partial H}. \tag{4.4.45}$$

Next we derive an invariant $U(\eta, H)$ of X_1 and a first order differential invariant $v(\eta, H, H')$ of the first prolongation $X^{[1]}$. They can both be derived from the invariance condition

$$X^{[1]}I(\eta, H, H') = 0, \tag{4.4.46}$$

where

$$X^{[1]} = X_1 + \zeta_1 \frac{\partial}{\partial H'}. \tag{4.4.47}$$

Now ζ_1 is given by (4.4.10) and using (4.4.45) it is found that

$$\zeta_1 = \left[\frac{2}{3}(H+\varepsilon) + \delta\sqrt{3}H'\right] \exp\left(\frac{\delta}{\sqrt{3}}\eta\right). \tag{4.4.48}$$

Thus

$$X^{[1]} = \exp\left(\frac{\delta}{\sqrt{3}}\eta\right)\left[\frac{\partial}{\partial \eta} - \frac{2}{\sqrt{3}}\delta(H+\varepsilon)\frac{\partial}{\partial H} - \left(\frac{2}{3}(H+\varepsilon) + \delta\sqrt{3}H'\right)\frac{\partial}{\partial H'}\right] \tag{4.4.49}$$

and the invariance condition (4.4.46) becomes

$$\frac{\partial I}{\partial \eta} - \frac{2}{\sqrt{3}}\delta(H+\varepsilon)\frac{\partial I}{\partial H} - \left(\frac{2}{3}(H+\varepsilon) + \delta\sqrt{3}H'\right)\frac{\partial I}{\partial H'} = 0. \tag{4.4.50}$$

The differential equations of the characteristic curves are

$$\frac{d\eta}{1} = -\frac{dH}{\frac{2}{\sqrt{3}}\delta(H+\varepsilon)} = -\frac{dH'}{\frac{2}{3}(H+\varepsilon) + \delta\sqrt{3}H'}. \tag{4.4.51}$$

The first pair of terms gives

$$\ln(H+\varepsilon) + \frac{2}{\sqrt{3}}\delta\eta = c_1, \tag{4.4.52}$$

where c_1 is a constant. Hence

$$u(\eta, H) = \ln(H+\varepsilon) + \frac{2}{\sqrt{3}}\delta\eta. \tag{4.4.53}$$

The last pair of terms in (4.4.51) gives the ODE for H',

$$\frac{dH'}{dH} - \frac{3}{2(H+\varepsilon)}H' = \frac{\delta}{\sqrt{3}}. \tag{4.4.54}$$

The integrating factor is $(H+\varepsilon)^{-3/2}$. The solution is

$$\frac{H'}{(H+\varepsilon)^{\frac{3}{2}}} + \frac{2\delta}{\sqrt{3}(H+\varepsilon)^{\frac{1}{2}}} = c_2, \tag{4.4.55}$$

where c_2 is a constant. The differential invariant is therefore

$$v(\eta, H, H') = \frac{H'}{(H+\varepsilon)^{\frac{3}{2}}} + \frac{2\delta}{\sqrt{3}(H+\varepsilon)^{\frac{1}{2}}}. \tag{4.4.56}$$

We now apply the following Theorem [17].

Let $u(\eta, H)$ be an invariant and $v(\eta, H, H')$ be a first order differential invariant. Then

$$w = \frac{dv}{du} = \frac{D_\eta(v)}{D_\eta(u)} = \frac{\Big(\frac{\partial}{\partial \eta} + H'\frac{\partial}{\partial H} + H''\frac{\partial}{\partial H'}\Big)v}{\Big(\frac{\partial}{\partial \eta} + H'\frac{\partial}{\partial H}\Big)u} \tag{4.4.57}$$

is a second order differential invariant. By using the second order ODE to eliminate H'', (4.4.57) *can be written as the first order ODE*

$$\frac{dv}{du} = P(u, v) \tag{4.4.58}$$

in u and v.

Now

$$D_\eta(v) = \frac{-3H'^2}{2(H+\varepsilon)^{\frac{5}{2}}} - \frac{\delta H'}{\sqrt{3}(H+\varepsilon)^{\frac{3}{2}}} + \frac{H''}{(H+\varepsilon)^{\frac{3}{2}}} \tag{4.4.59}$$

and

$$D_\eta(u) = \frac{2\delta}{\sqrt{3}} + \frac{H'}{H+\varepsilon} = (H+\varepsilon)^{\frac{1}{2}}v. \tag{4.4.60}$$

Thus substituting (4.4.59) and (4.4.60) into (4.4.57) we obtain

$$v\frac{dv}{du} = -\frac{3}{2}\frac{H'^2}{(H+\varepsilon)^3} + \frac{1}{(H+\varepsilon)^2}\Big(H'' - \frac{\delta}{\sqrt{3}}H'\Big). \tag{4.4.61}$$

But from (4.4.56),

$$\frac{H'^2}{(H+\varepsilon)^3} = v^2 - \frac{4}{3(H+\varepsilon)} - \frac{4\delta}{\sqrt{3}}\frac{H'}{(H+\varepsilon)^2} \tag{4.4.62}$$

and (4.4.61) becomes

$$v\frac{dv}{du} = -\frac{3}{2}v^2 + \frac{2}{H+\varepsilon} + \frac{1}{(H+\varepsilon)^2}\Big(H'' + \frac{5}{\sqrt{3}}\delta H\Big). \tag{4.4.63}$$

But using (4.4.43) for λ the differential equation (4.4.1) is

$$H'' = \frac{5}{\sqrt{3}}\delta H' - \varepsilon(1 - H^2). \tag{4.4.64}$$

By eliminating H'' from (4.4.63) using (4.4.64) we obtain

$$v\frac{dv}{du} = -\frac{3}{2}v^2 + \frac{2}{H+\varepsilon} - \varepsilon\frac{(1-H^2)}{(H+\varepsilon)^2}. \tag{4.4.65}$$

But by setting $\varepsilon = 1$ and $\varepsilon = -1$ in turn it can be shown that

$$\frac{2}{(H+\varepsilon)} - \frac{\varepsilon(1-H^2)}{(H+\varepsilon)^2} = \varepsilon. \tag{4.4.66}$$

Equation (4.4.65) becomes

$$v\frac{dv}{du} = -\frac{3}{2}v^2 + \varepsilon. \tag{4.4.67}$$

The second order ODE (4.4.1) has been reduced to the first order ODE (4.4.67).

Equation (4.4.67) is a first order ODE for v^2 which can be written as

$$\frac{d}{du}(v^2) + 3v^2 = 2\varepsilon. \tag{4.4.68}$$

The integrating factor is $\exp{(3u)}$. Thus

$$\frac{d}{du}\Big(v^2\exp{(3u)}\Big) = 2\varepsilon\exp{(3u)} \tag{4.4.69}$$

and therefore

$$v = \pm\Big(\frac{2}{3}\varepsilon + k\exp{(-3u)}\Big)^{\frac{1}{2}}, \tag{4.4.70}$$

where k is a constant.

Equation (4.4.70) is now expressed in terms of H and η using (4.4.53) for u and (4.4.56) for v. But

$$\exp{(-3u)} = \frac{1}{(H+\varepsilon)^3}\exp{(-2\sqrt{3}\delta\eta)} \tag{4.4.71}$$

and therefore

$$v = \pm\frac{1}{(H+\varepsilon)^{\frac{3}{2}}}\Big[\frac{2}{3}\varepsilon(H+\varepsilon)^3 + k\exp{(-2\sqrt{3}\delta\eta)}\Big]^{\frac{1}{2}}. \tag{4.4.72}$$

By substituting (4.4.56) for v a first order ODE for H is obtained :

$$H' = -\frac{2\delta}{\sqrt{3}}(H+\varepsilon) \pm \Big[\frac{2}{3}\varepsilon(H+\varepsilon)^3 + k\exp{(-2\sqrt{3}\delta\eta)}\Big]^{\frac{1}{2}}. \tag{4.4.73}$$

Integration of the second order ODE (4.4.1) for H has been transformed to the integration of two first order ODEs, (4.4.67) for v and (4.4.73) for H.

Further analysis depends on the values of ε and δ:

$\varepsilon = +1$, flow in a convergent channel

$\varepsilon = -1$, flow in a divergent channel

$\delta = +1$, suction (fluid leak-off) at the solid boundary

$\delta = -1$, blowing (fluid injection) at the solid boundary.

The boundary conditions on $H(\eta)$ are from (4.3.2) with $v_{t0} = 0$,

$$H(\infty) = 1, \quad H(0) = 0, \quad \frac{dH}{d\eta}(\infty) = 0. \tag{4.4.74}$$

The results for flow in convergent and divergent channels has been summarized by Jones and Watson [15]. We now state the properties.

For flow in a convergent channel ($\varepsilon = +1$) the solution exists for all values of λ and therefore for all values of suction and blowing. Holstein [5] derived numerical solutions for suction and blowing and also derived a solution in terms of elliptic functions for suction with $\lambda = \frac{5}{\sqrt{3}}$. For blowing with $\lambda = -\frac{5}{\sqrt{3}}$ Jones and Watson [15] state without derivation an elementary solution. We will give a derivation of this solution from (4.4.73). We will also investigate if there is an elementary solution for suction with $\lambda = +\frac{5}{\sqrt{3}}$.

For flow in a divergent channel ($\varepsilon = -1$) the solution exists only for $\lambda \geq 2\sqrt{2} = 2.828$, that is, provided the suction is sufficiently strong. Holstein [5] and Thwaites [18, 19] have derived numerical solutions. For suction with $\lambda = \frac{5}{\sqrt{3}} = 2.886$, Holstein [5] again showed that the solution could be expressed as elliptic functions. For $\lambda = \frac{5}{\sqrt{3}}$ we will investigate if an elementary solution can be derived using (4.4.73). We will also show that for $\lambda = -\frac{5}{\sqrt{3}}$ a solution of (4.4.73) and (4.4.74) does not exist in agreement with the general non-existence Theorem.

4.5 Boundary layer flow in a convergent channel with blowing ($\varepsilon = +1, \delta = -1, \lambda = -\frac{5}{\sqrt{3}}$)

For boundary layer flow in a convergent in channel with blowing (fluid injection) at the boundary, $\varepsilon = +1$ and $\delta = -1$. Equation (4.4.73) becomes

$$\frac{dH}{d\eta} = \frac{2}{\sqrt{3}}(1+H) \pm \Big[\frac{2}{3}(1+H)^3 + k\exp\left(2\sqrt{3}\eta\right)\Big]^{\frac{1}{2}} \tag{4.5.1}$$

which is subject to the boundary conditions (4.4.74). Since

$$\frac{dH}{d\eta}(\infty) = 0 \quad \text{and} \quad H(\infty) = 1, \tag{4.5.2}$$

it follows that $k = 0$. Also to satisfy (4.5.2) the $-$sign must be taken. The ODE (4.5.1) becomes

$$\frac{dH}{d\eta} = \Big(\frac{2}{3}\Big)^{\frac{1}{2}}(1+H)\Big(\sqrt{2} - (1+H)^{\frac{1}{2}}\Big). \tag{4.5.3}$$

Let

$$1 + H = G^2. \tag{4.5.4}$$

Equation (4.5.3) transforms to

$$\frac{dG}{d\eta} = \frac{1}{\sqrt{6}}G(\sqrt{2} - G) \tag{4.5.5}$$

and therefore

$$\Big(\frac{1}{G} + \frac{1}{\sqrt{2} - G}\Big)dG = \frac{1}{\sqrt{3}}d\eta. \tag{4.5.6}$$

Hence

$$G(\eta) = \frac{\sqrt{2}}{\Big(1 + B\exp\Big(-\frac{\eta}{\sqrt{3}}\Big)\Big)}, \tag{4.5.7}$$

where B is a constant and from (4.5.4)

$$H(\eta) = \frac{2}{\Big(1 + B\exp\Big(-\frac{\eta}{\sqrt{3}}\Big)\Big)^2} - 1. \tag{4.5.8}$$

But imposing the no-slip boundary condition

$$H(0) = 0 \tag{4.5.9}$$

gives

$$B = \sqrt{2} - 1 \quad \text{and} \quad B = -(\sqrt{2} + 1) \tag{4.5.10}$$

and therefore

$$H(\eta) = \frac{2}{\left[1 + (\sqrt{2} - 1)\exp\Big(-\dfrac{\eta}{\sqrt{3}}\Big)\right]^2} - 1 \tag{4.5.11}$$

and

$$H(\eta) = \frac{2}{\left[1 - (\sqrt{2} + 1)\exp\Big(-\dfrac{\eta}{\sqrt{3}}\Big)\right]^2} - 1. \tag{4.5.12}$$

But $H(\eta)$ given by (4.5.12) is infinite at

$$\eta = \sqrt{3}\ln(1 + \sqrt{2}) \tag{4.5.13}$$

and is not an acceptable solution. Equation (4.5.11) satisfies the mainstream matching boundary condition, $H(\infty) = 1$, and is the elementary solution stated by Jones and Waston [15] without derivation.

For flow in a convergent channel, $U_0 < 0$, from (4.2.33), (4.2.34a) and (4.2.35a),

$$v_x(x,y) = \frac{U_0}{x}\left[\frac{2}{\left[1 + (\sqrt{2} - 1)\exp\Big(-\Big(-\dfrac{U_0}{3\nu}\Big)^{\frac{1}{2}}\dfrac{y}{x}\Big)\right]^2} - 1\right], \tag{4.5.14}$$

$$v_y(x,y) = -\frac{v_{s0}}{x} + \frac{U_0 y}{x^2}\left[\frac{2}{\left[1 + (\sqrt{2} - 1)\exp\Big(-\Big(-\dfrac{U_0}{3\nu}\Big)^{\frac{1}{2}}\dfrac{y}{x}\Big)\right]^2} - 1\right]. \tag{4.5.15}$$

The velocity $v_x(x,y)$ approaches the mainstream velocity $\dfrac{U_0}{x}$ when

$$\Big(-\frac{U_0}{3\nu}\Big)^{\frac{1}{2}}\frac{y}{x} = O(1), \tag{4.5.16}$$

that is, when

$$y = O\Big(\Big(\frac{3\nu}{-U_0}\Big)^{\frac{1}{2}} x\Big), \tag{4.5.17}$$

which is an estimate of the thickness of the boundary layer. The thickness of the boundary layer is proportional to x and decreases as the magnitude of the mainstream velocity, $\dfrac{|U_0|}{x}$ increases as $x \to 0$ in the convergent channel.

For large values of y,

$$v_y(x,y) = -\frac{v_{s0}}{x} + U_0\frac{y}{x^2} \tag{4.5.18}$$

and $v_y(x,y)$ does not tend to zero as $y \to \infty$. In the boundary layer approximation the transverse velocity $v_y(x,y)$ does not generally vanish as $y \to \infty$.

In Figure 4.5.1, v_x is plotted against η where, using (4.5.11),

$$\begin{aligned} v_x(x,\eta) &= \frac{U_0}{x}H(\eta) \\ &= \frac{U_0}{x}\left[\frac{2}{\left[1+(\sqrt{2}-1)\exp\left(-\frac{\eta}{\sqrt{3}}\right)\right]^2} - 1\right]. \end{aligned} \tag{4.5.19}$$

There is no reverse flow in the boundary layer.

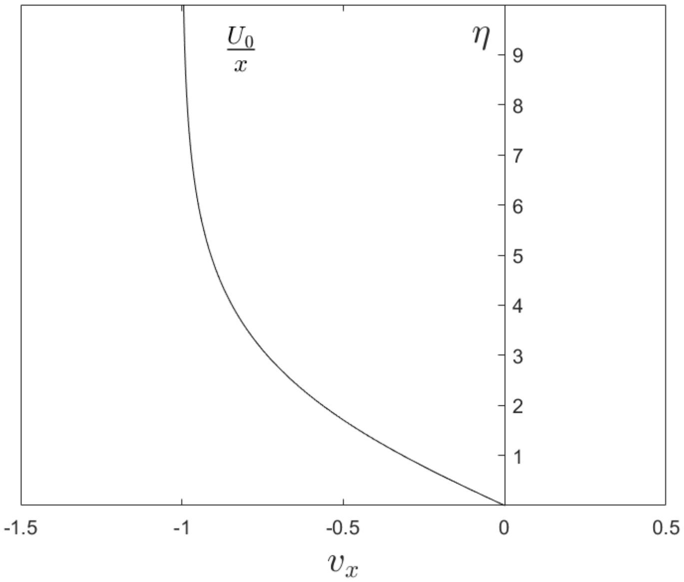

Figure 4.5.1: Boundary layer flow in a convergent channel with blowing. The velocity v_x given by (4.5.19) is plotted against η for $\frac{U_0}{x} = -1$ and $\lambda = -\frac{5}{\sqrt{3}}$.

4.6 Boundary layer flow in a convergent channel with suction ($\varepsilon = +1, \delta = +1, \lambda = +\frac{5}{\sqrt{3}}$)

Holstein [5] derived an analytical solution in terms of elliptic functions for boundary layer flow in a convergent channel with suction and $\lambda = \frac{5}{\sqrt{3}}$. We will investigate if there exists an elementary solution for $\lambda = \frac{5}{\sqrt{3}}$ similar to (4.5.11) which could be a special case of the elliptic function solution of Holstein.

When $\varepsilon = +1$ and $\delta = +1$, equation (4.4.73) becomes

$$\frac{dH}{d\eta} = -\frac{2}{\sqrt{3}}(1+H) \pm \left[\frac{2}{3}(1+H)^3 + k\exp\left(-2\sqrt{3}\eta\right)\right]^{\frac{1}{2}} \tag{4.6.1}$$

subject to the boundary conditions (4.4.74). The boundary conditions

$$H(\infty) = 1, \quad \frac{dH}{d\eta}(\infty) = 0, \tag{4.6.2}$$

are satisfied for arbitrary values of the constant k and provided the $+$sign is taken. We choose $k = 0$. This compares with the previous case in which $k = 0$ is required to satisfy the boundary conditions at $\eta = \infty$. Equation (4.6.1) becomes

$$\frac{dH}{d\eta} = -\left(\frac{2}{3}\right)^{\frac{1}{2}}(1+H)\left[\sqrt{2} - (1+H)^{\frac{1}{2}}\right]. \tag{4.6.3}$$

Making again the transformation (4.5.4), equation (4.6.3) becomes

$$\frac{dG}{d\eta} = -\frac{1}{\sqrt{6}}G(\sqrt{2} - G) \tag{4.6.4}$$

and therefore

$$\left(\frac{1}{G} + \frac{1}{\sqrt{2} - G}\right)dG = -\frac{1}{\sqrt{3}}d\eta. \tag{4.6.5}$$

Hence

$$G(\eta) = \frac{\sqrt{2}}{\left(1 + B\exp\left(\frac{\eta}{\sqrt{3}}\right)\right)} \tag{4.6.6}$$

where B is a constant. Thus from (4.5.4)

$$H(\eta) = \frac{2}{\left[1 + B\exp\left(\frac{\eta}{\sqrt{3}}\right)\right]^2} - 1. \tag{4.6.7}$$

Imposing the boundary condition $H(0) = 0$ gives $B = -(1 + \sqrt{2})$ and $B = \sqrt{2} - 1$. Hence

$$H(\eta) = \frac{2}{\left[1 - (1 + \sqrt{2}) \exp\left(\frac{\eta}{\sqrt{3}}\right)\right]^2} - 1 \tag{4.6.8}$$

and

$$H(\eta) = \frac{2}{\left[1 + (\sqrt{2} - 1) \exp\left(\frac{\eta}{\sqrt{3}}\right)\right]^2} - 1. \tag{4.6.9}$$

But for equations (4.6.8) and (4.6.9),

$$H(\infty) = -1 \tag{4.6.10}$$

which does not satisfy the mainstream matching boundary condition $H(\infty) = +1$. A solution with constant $k = 0$ satisfying the ODE (4.6.1) and the boundary conditions (4.4.74) therefore does not exist.

In order to obtain a special case of the solution of Holstein [5], it would be necessary to take $k \neq 0$. This is beyond the scope of this dissertation.

4.7 Boundary layer flow in a divergent channel with blowing $(\varepsilon = -1, \delta = -1, \lambda = -\frac{5}{\sqrt{3}})$

For boundary layer flow in a divergent channel the solution exists only for suction with $\lambda \geq 2\sqrt{2}$. It does not exists for blowing. We will prove a special case of this non-existence Theorem for blowing with $\lambda = -\frac{5}{\sqrt{3}}$.

For $\varepsilon = -1$ and $\delta = -1$, the ODE (4.4.74) becomes

$$\frac{dH}{d\eta} = \frac{2}{\sqrt{3}}(H - 1) \pm \left[-\frac{2}{3}(H - 1)^3 + k \exp\left(2\sqrt{3}\eta\right)\right]^{\frac{1}{2}} \tag{4.7.1}$$

which is subject to the boundary conditions (4.4.74). The boundary conditions

$$H(\infty) = 1, \quad \frac{dH}{d\eta}(\infty) = 0, \tag{4.7.2}$$

require $k = 0$ but are satisfied for both $+$ and $-$ signs. The ODE (4.7.1) becomes

$$\frac{dH}{d\eta} = -\left(\frac{2}{3}\right)^{\frac{1}{2}}(1 - H)[\sqrt{2} \mp (1 - H)^{\frac{1}{2}}]. \tag{4.7.3}$$

Making the transformation

$$1 - H = G^2 \tag{4.7.4}$$

we obtain the two differential equations

$$\frac{dG}{d\eta} = \frac{1}{\sqrt{6}} G(\sqrt{2} - G), \tag{4.7.5}$$

$$\frac{dG}{d\eta} = \frac{1}{\sqrt{6}} G(\sqrt{2} + G). \tag{4.7.6}$$

Consider first (4.7.6) which can be written as

$$\Big(\frac{1}{G} - \frac{1}{\sqrt{2} + G}\Big) dG = \frac{d\eta}{\sqrt{3}}. \tag{4.7.7}$$

Hence

$$G(\eta) = \frac{\sqrt{2}}{B \exp\Big(-\frac{\eta}{\sqrt{3}}\Big) - 1} \tag{4.7.8}$$

where B is a constant and therefore from (4.7.4)

$$H(\eta) = 1 - \frac{2}{\Big[B \exp\Big(-\frac{\eta}{\sqrt{3}}\Big) - 1\Big]^2}. \tag{4.7.9}$$

Imposing the boundary condition $H(0) = 0$ we obtain $B = \sqrt{2} + 1$ and $B = -(\sqrt{2} - 1)$. Hence

$$H(\eta) = 1 - \frac{2}{\Big[(\sqrt{2} + 1) \exp\Big(-\frac{\eta}{\sqrt{3}}\Big) - 1\Big]^2} \tag{4.7.10}$$

and

$$H(\eta) = 1 - \frac{2}{\Big[(\sqrt{2} - 1) \exp\Big(-\frac{\eta}{\sqrt{3}}\Big) + 1\Big]^2}. \tag{4.7.11}$$

But (4.7.10) is singular at

$$\eta = \sqrt{3} \ln(1 + \sqrt{2}) \tag{4.7.12}$$

and for both (4.7.10) and (4.7.11), $H(\infty) = -1$ and the mainstream matching boundary condition $H(\infty) = +1$ is not satisfied. Equations (4.7.10) and (4.7.11) are therefore not acceptable solutions.

Also, equation (4.7.5) is the same as the ODE (4.5.5). The solution for $G(\eta)$ is therefore given by (4.5.7) but since now $H(\eta) = 1 - G^2$ the solution for $H(\eta)$ is

$$H(\eta) = 1 - \frac{2}{\left[1 + B\exp\left(-\frac{\eta}{\sqrt{3}}\right)\right]^2}, \tag{4.7.13}$$

where B is a constant. Imposing the boundary condition $H(0) = 0$ gives $B = -(\sqrt{2}+1)$ and $B = \sqrt{2} - 1$ and the solutions (4.7.10) and (4.7.11) are again obtained.

A solution satisfying the ODE (4.7.1) and subject to the boundary conditions (4.4.74) therefore does not exist, in agreement with the existence Theorem that boundary layer flow in a divergent channel exists only for suction with $\lambda \geq 2\sqrt{2}$.

4.8 Boundary layer flow in a divergent channel with suction $(\varepsilon = -1, \delta = +1, \lambda = +\frac{5}{\sqrt{3}})$

Boundary layer flow exists in a divergent channel with suction provided the suction is sufficiently strong, such that $\lambda \geq 2\sqrt{2} = 2.828$. For $\lambda = \frac{5}{\sqrt{3}} = 2.886$, Holstein [5] showed that the boundary layer solution can be expressed in terms of elliptic functions. We will investigate if an elementary invariant solution similar to (4.5.11) for a convergent channel with $\lambda = -\frac{5}{\sqrt{3}}$ exists for a divergent channel with $\lambda = \frac{5}{\sqrt{3}}$.

When $\varepsilon = -1$ and $\delta = +1$, equation (4.4.73) becomes

$$\frac{dH}{d\eta} = -\frac{2}{\sqrt{3}}(H-1) \pm \left[-\frac{2}{3}(H-1)^3 + k\exp\left(-2\sqrt{3}\eta\right)\right]^{\frac{1}{2}}. \tag{4.8.1}$$

The boundary conditions

$$H(\infty) = 1, \quad \frac{dH}{d\eta}(\infty) = 0, \tag{4.8.2}$$

place no restriction on k and are satisfied for both $+$ and $-$ signs. We look for a solution with $k = 0$. Equation (4.8.1) becomes

$$\frac{dH}{d\eta} = \left(\frac{2}{3}\right)^{\frac{1}{2}}(1-H)[\sqrt{2} \pm (1-H)^{\frac{1}{2}}]. \tag{4.8.3}$$

By again making the transformation

$$1 - H = G^2 \tag{4.8.4}$$

we obtain the two differential equations

$$\frac{dG}{d\eta} = -\frac{1}{\sqrt{6}}G(\sqrt{2}+G), \tag{4.8.5}$$

$$\frac{dG}{d\eta} = -\frac{1}{\sqrt{6}}G(\sqrt{2}-G). \tag{4.8.6}$$

Consider first (4.8.5) which can be written as

$$\Big(\frac{1}{G} - \frac{1}{\sqrt{2}+G}\Big)dG = -\frac{1}{\sqrt{3}}dG \tag{4.8.7}$$

and hence

$$G(\eta) = \frac{\sqrt{2}}{\left[B\exp\left(\frac{\eta}{\sqrt{3}}\right) - 1\right]}. \tag{4.8.8}$$

Thus from (4.8.4),

$$H(\eta) = 1 - \frac{2}{\left[B\exp\left(\frac{\eta}{\sqrt{3}}\right) - 1\right]^2}. \tag{4.8.9}$$

Imposing the no slip boundary condition $H(0) = 0$ gives $B = \sqrt{2}+1$ and $B = -(\sqrt{2}-1)$. Thus

$$H(\eta) = 1 - \frac{2}{\left[(\sqrt{2}+1)\exp\left(\frac{\eta}{\sqrt{3}}\right) - 1\right]^2}, \tag{4.8.10}$$

and

$$H(\eta) = 1 - \frac{2}{\left[(\sqrt{2}-1)\exp\left(\frac{\eta}{\sqrt{3}}\right) - 1\right]^2}. \tag{4.8.11}$$

The mainstream matching boundary condition $H(\infty) = 1$ is satisfied by both solutions and both solutions have no singularities for $0 \leq \eta \leq \infty$.

The differential equation (4.8.6) is the same as (4.6.4) and from (4.6.6) the solution is

$$G(\eta) = \frac{\sqrt{2}}{\left[1 + B\exp\left(\frac{\eta}{\sqrt{3}}\right)\right]}, \tag{4.8.12}$$

where B is a constant. Thus from (4.8.4)

$$H(\eta) = 1 - \frac{2}{\left[1 + B\exp\left(\frac{\eta}{\sqrt{3}}\right)\right]^2}. \tag{4.8.13}$$

Imposing the no slip boundary condition we obtain $B = -(\sqrt{2}+1)$ and $B = \sqrt{2}-1$. The solutions (4.8.10) and (4.8.11) are again obtained.

There are two solutions, (4.8.10) and (4.8.11), of the differential equation (4.8.1) for the special case $k = 0$ which satisfy the boundary conditions (4.4.74). It has not been possible to compare these solutions with the elliptic function solution of Holstein [5] because this paper by Holstein cannot be traced. It was published in Germany in 1943 during the Second World War.

For flow in a divergent channel, $U_0 > 0$ and from (4.2.33), (4.2.34b) and (4.2.35b) the solutions (4.8.10) and (4.8.11) give

$$v_x(x,y) = \frac{U_0}{x}\left(1 - \frac{2}{\left[(\sqrt{2} \pm 1)\exp\left(\left(\frac{U_0}{3\nu}\right)^{\frac{1}{2}}\frac{y}{x}\right) \mp 1\right]^2}\right), \tag{4.8.14}$$

$$v_y(x,y) = -\frac{v_{s0}}{x} + \frac{U_0 y}{x^2}\left(1 - \frac{2}{\left[(\sqrt{2} \pm 1)\exp\left(\left(\frac{U_0}{3\nu}\right)^{\frac{1}{2}}\frac{y}{x}\right) \mp 1\right]^2}\right). \tag{4.8.15}$$

For large values of y, (4.8.14) becomes approximately

$$v_x(x,y) \approx \frac{U_0}{x}\left(1 - \frac{2}{(\sqrt{2} \pm 1)^2}\exp\left(-2\left(\frac{U_0}{3\nu}\right)^{\frac{1}{2}}\frac{y}{x}\right)\right) \tag{4.8.16}$$

and $v_x(x,y)$ approaches the mainstream velocity $\dfrac{U_0}{x}$ when

$$2\left(\frac{U_0}{3\nu}\right)^{\frac{1}{2}}\frac{y}{x} = O(1), \tag{4.8.17}$$

that is, when

$$y = O\left[\frac{1}{2}\left(\frac{3\nu}{U_0}\right)^{\frac{1}{2}}x\right], \tag{4.8.18}$$

which is an estimate of the thickness of the boundary layer. Although (4.5.17) for boundary layer flow in a convergent channel with blowing and (4.8.18) are order of magnitude estimates the results indicate that for given $\nu/|U_0|$ the boundary layer in the divergent channel with suction may be marginally thinner than the boundary layer in the convergent channel with blowing.

For large values of y, $v_x(x,y)$ again behaves like (4.5.18) and $v_y(x,y)$ does not tend to zero as $y \to \infty$.

In Figure (4.8.1),

$$v_x^{(1)}(x,\eta) = \frac{U_0}{x}\left[1 - \frac{2}{\left[(\sqrt{2}+1)\exp\left(\frac{\eta}{\sqrt{3}}\right) - 1\right]^2}\right] \tag{4.8.19}$$

and

$$v_x^{(2)}(x,\eta) = \frac{U_0}{x}\left[1 - \frac{2}{\left[(\sqrt{2}-1)\exp\left(\frac{\eta}{\sqrt{3}}\right) + 1\right]^2}\right], \tag{4.8.20}$$

are plotted against η. We see that the boundary layer described by $v_x^{(1)}(x,\eta)$ is broader and tends to the mainstream velocity, $\frac{U_0}{x}$, faster than the boundary layer described by $v_x^{(2)}(x,\eta)$. There is no reverse flow in the boundary layer.

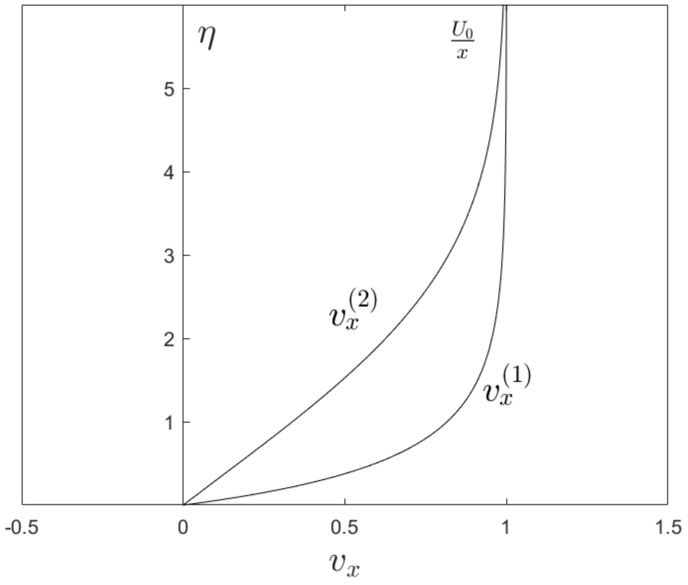

Figure 4.8.1: Boundary layer flow in a divergent channel with suction. Velocities $v_x^{(1)}(x,\eta)$ and $v_x^{(2)}(x,\eta)$, given by (4.8.19) and (4.8.20), are plotted against η for $\frac{U_0}{x} = +1$ and $\lambda = +\frac{5}{\sqrt{3}}$.

4.9 Conclusions

When there is either no slip or slip but no suction or blowing for flow in a convergent channel, the second order differential equation can be integrated without the aid of Lie point symmetries and the boundary conditions can be satisfied. There are two solutions and the solution is therefore not unique. For one of the solutions, there is reverse flow either at the plate or in the interior of the flow. For flow in a divergent channel with either no slip or slip but without suction or blowing, the solution does not exist and the assumption that the flow consists of a thin boundary layer in a mainstream flow is not satisfied.

For all the results derived for suction or blowing but no slip, the suction velocity v_{s0} was not arbitrary but satisfied

$$\lambda = \frac{v_{s0}}{(|U_0|\nu)^{\frac{1}{2}}} = \pm\frac{5}{\sqrt{3}},$$

where the $+$sign describes suction and the $-$sign describes blowing.

For suction, the constant k in the ODE (4.4.73) could not be determined from the boundary conditions. To make progress the choice $k = 0$ was made. For boundary layer flow in a convergent channel with suction it did not give a solution which satisfied the mainstream matching condition at $\eta = \infty$. For flow in a divergent channel with suction the solution was not unique. Two solutions were derived which satisfied the ODE and all the boundary conditions. Holstein [5] derived solutions in terms of elliptic functions for boundary layer flow with suction in convergent and divergent channels. This paper of Holstein was published in 1943 and could not be traced. Our solutions could therefore not be compared with the solutions of Holstein.

For blowing the mainstream matching boundary conditions gave $k = 0$ in the ODE (4.4.73). The results derived for blowing were therefore stronger than for suction because the additional condition that $k = 0$ did not need to be made. For boundary layer flow in a convergent channel with blowing the elementary invariant solution stated by Jones and Watson [15] was derived and for flow in a divergent channel with blowing, a special case of the Theorem for the non-existence of solution was established.

For a convergent channel with no suction or blowing the solution is not unique and there is reverse flow near the plate in one of the solutions. In comparison, in a convergent channel with blowing the elementary solution that was derived is unique and there is no reverse flow.

For a divergent channel with no suction or blowing, boundary layer flow does not exist. In comparison, boundary layer flow exists in a divergent channel provided the suction is sufficiently strong. The solution derived for a divergent channel with suction was not unique but there was no reverse flow in the two solutions.

Chapter 5

Numerical solutions

5.1 Introduction

In this chapter, we will be providing numerical solutions to some of the Boundary Value Problems (BVP) for an Ordinary Differential Equations (ODE) derived in Chapter 3. Solutions to flows in divergent and convergent channels were obtained analytically in Chapter 4. We will therefore be seeking numerical solutions for the Blasius and the Falkner-Skan equations.

Following from section 2.3, it is known that the one-parameter scaling group and the one-parameter spiral group have been used to transform boundary value problems to Initial Value Problems, provided that the invariance condition is met, by a method known as the non-Iterative Transformation Method [8, 9]. There is therefore a class of BVPs for which such a conversion under a one-parameter group applies.

It has been shown that when a BVP is not invariant under a one-parameter group, a modified BVP can be derived, which is invariant under an extended transformation group [10, 11, 13, 14, 20, 21] and as a result, conversion from the modified BVP to an IVP can be achieved by a method known as the iterative transformation method. This method will be used to convert the Falkner-Skan BVP to an IVP.

5.2 Numerical solutions for Blasius boundary value problems

The Blasius equation for the boundary layer flow over a flat plate inclined at zero angle of incidence is known to admit a scaling transformation. The associated boundary conditions will depend on the problem being studied. Solutions to the Blasius BVP already exists in literature for various conditions placed on the normal velocity and the horizontal velocity at the plate.

In what follows, we will be using the non-iterative transformation method to investigate the solutions for the Blasius equation.

5.2.1 Numerical solution for Blasius equation with no slip and no suction boundary condition

Consider the Blasius equation in boundary layer theory given by

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0, \tag{5.2.1}$$

$$F(0) = 0, \quad \frac{dF}{d\eta}(0) = 0, \quad \frac{dF}{d\eta}(\infty) = 1. \tag{5.2.2}$$

Using scaling transformations, the Blasius BVP can be reformulated as an IVP [8, 9].

Under the scaling transformation

$$\bar{\eta} = \lambda^a \eta, \quad \bar{F}(\bar{\eta}) = \lambda^b F(\eta) \tag{5.2.3}$$

equation (5.2.1) is invariant, provided $b = -a$. The homogeneous boundary conditions in (5.2.2) are also invariant. However, the non-homogeneous boundary condition, which is the third boundary condition in (5.2.2) is not invariant under (5.2.3). This non-invariance is necessary in order to compute the scaling group parameter λ and for the non-iterative transformation method to work [14]. Therefore, the IVP that results when the non-iterative transformation method is used to convert (5.2.1) and (5.2.2) to IVP is

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} + \bar{F}\frac{d^2\bar{F}}{d\bar{\eta}^2} = 0, \tag{5.2.4}$$

$$\bar{F}(0) = 0, \quad \frac{d\bar{F}}{d\bar{\eta}}(0) = 0, \quad \frac{d^2\bar{F}}{d\bar{\eta}^2}(0) = 1 \tag{5.2.5}$$

Once $\bar{F}(u)$ is obtained, the solution to $F(u)$ is given by (5.2.3) provided that we have an approximation for $\frac{dF}{d\eta}(\infty)$ such that λ is calculated from

$$\lambda = \left(\frac{d\bar{F}}{d\bar{\eta}}(\infty)\right)^{-\frac{1}{2a}} = \left(\frac{d\bar{F}}{d\bar{\eta}}(\infty)\right)^{-\frac{1}{2}}, \tag{5.2.6}$$

by choosing $a = 1$. The numerical solution for the boundary layer velocity F' with no slip and no suction is shown in Figure 5.2.1.

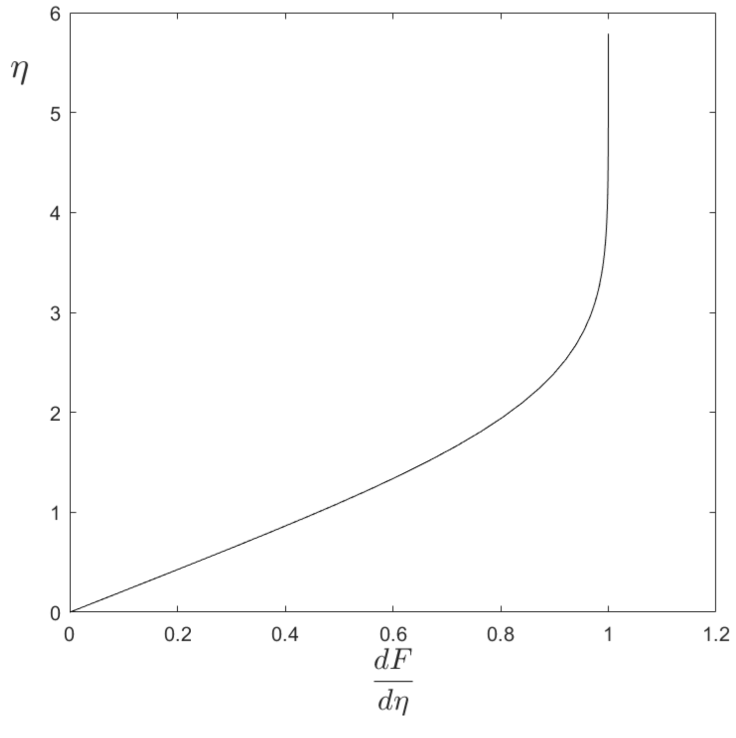

Figure 5.2.1: Numerical result of the Blasius problem for no slip and no suction.

5.2.2 Numerical solution for Blasius equation with slip boundary condition

When there is fluid slip and no-suction at the boundary plate, the Blasius BVP becomes

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0, \tag{5.2.7}$$

$$F(0) = 0, \quad \frac{dF}{d\eta}(0) = B, \quad \frac{dF}{d\eta}(\infty) = 1, \tag{5.2.8}$$

where $B = \dfrac{v_{t0}}{U_0}$. The BVP can be solved by the non-iterative transformation method and the iterative transformation method.

The invariance of equation (5.2.7) was established in Section 5.2.1. Under the transformation (5.2.3), the boundary condition (5.2.8) transforms to

$$\bar{F}(0) = 0, \quad \frac{d\bar{F}}{d\bar{\eta}}(0) = \lambda^{-2a}B, \quad \lambda^{2a}\frac{d\bar{F}}{d\bar{\eta}}(\infty) = 1. \tag{5.2.9}$$

The non-invariance of the boundary condition at infinity in (5.2.9) is vital in order to determine the group parameter λ.

To solve by the non-iterative transformation method, B must be allowed to transform according to

$$\bar{B} = \lambda^{-2a}B, \tag{5.2.10}$$

so that the BVP (5.2.7) and (5.2.8) is transformed to the IVP

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} + \bar{F}\frac{d^2\bar{F}}{d\bar{\eta}^2} = 0, \tag{5.2.11}$$

$$\bar{F}(0) = 0, \quad \frac{d\bar{F}}{d\bar{\eta}}(0) = \bar{B}, \quad \frac{d^2\bar{F}}{d\bar{\eta}^2}(0) = 1. \tag{5.2.12}$$

The procedure for solving (5.2.11) and (5.2.12) are outlined below:

1. Non-iterative transformation method

   To solve for $\bar{F}(\bar{\eta})$, first choose a value for $\bar{B}$ in (5.2.11). Once solution is obtained, the group parameter $\lambda(\bar{B})$ is obtained from

   $$\lambda = \left(\frac{d\bar{F}}{d\bar{\eta}}(\infty)\right)^{-\frac{1}{2a}} = \left(\frac{d\bar{F}}{d\bar{\eta}}(\infty)\right)^{-\frac{1}{2}} \tag{5.2.13}$$

   where a is chosen to be unity. Once $\bar{F}$ and λ are obtained, solution $F(\eta)$ is obtained from (5.2.3) and the value of B in the BVP is

   $$B = \lambda^{2a}\bar{B}. \tag{5.2.14}$$

   In Table 5.2.1, some results from using the non-iterative method to solve the Blasius problem with slip, choosing $a = 1$, are presented.

$\bar{B}$	λ	$\dfrac{\mathrm{d}F}{\mathrm{d}\bar{\eta}}(\infty)$	$\dfrac{\mathrm{d}^2F}{\mathrm{d}\eta^2}(\infty)$	B
0.0000	0.7773	1.6552	0.6041	0.0000
0.5000	0.7489	1.7829	0.5609	0.2804
1.0000	0.6957	2.0663	0.4840	0.4840
5.0000	0.4245	5.5496	0.1802	0.9010
10.0000	0.3102	10.3935	0.0962	0.9621
20.0000	0.2221	20.2795	0.0493	0.9862
50.0000	0.1412	50.1771	0.0199	0.9965
100.0000	0.0999	100.1253	0.0100	0.9987
200.0000	0.0707	200.0886	0.0050	0.9996

Table 5.2.1: Non-iterative numerical result for Blasius boundary layer with slip and no suction

The problem with this approach is that the value of $\bar{B}$ is prescribed instead of B. It is desirable to be able to specify the value of B, since the BVP requires B to be known, and not $\bar{B}$. To achieve solving the IVP (5.2.11) and (5.2.12), with a given B value, the iterative transformation method is employed.

2. Iterative transformation method for no suction and slip

   Consider the parameter transformation given by

$$\bar{B} = \lambda^{-2} B. \tag{5.2.15}$$

If we want to solve the BVP (5.2.7) and (5.2.8) for a particular value of B, say $B = \tilde{B}$, we therefore set $B = \tilde{B}$ in (5.2.15) and define a function

$$\Gamma(\bar{B}) = \bar{B} - \lambda^{-2}\tilde{B}, \tag{5.2.16}$$

where $\lambda = \lambda(\tilde{B})$. For each value of $\bar{B}$, the IVP (5.2.11)-(5.2.12) is solved and the corresponding value of λ is obtained from (5.2.13). Integration of the IVP with varying values of $\bar{B}$ is done until an interval $[\bar{B}_1, \bar{B}_2]$ on which Γ is real and continuous, such that $\Gamma(\bar{B}_1)\Gamma(\bar{B}_2) < 0$ is found.

This guarantees that indeed a root $\bar{B} = \bar{B}^*$ exists on $[\bar{B}_1, \bar{B}_2]$. Any root finding method can now be used to determine $\bar{B}^*$ such that

$$|\bar{B}^* - {\lambda^*}^{-2}\tilde{B}| < \epsilon \tag{5.2.17}$$

holds, where λ^* is the value of λ at $\bar{B} = \bar{B}^*$ and $\epsilon << 1$ is the tolerance.

For example, if we choose $B = 0.4840$, it was found that $\Gamma(0.5) < 0$ and $\Gamma(1.5) > 0$ so that the interval of $\bar{B}$ where Γ changes sign is $[\bar{B}_1, \bar{B}_2]$ = $[0.5, 1.5]$. That is, Γ satisfies the inequality $\Gamma(0.5)\Gamma(1.5) < 0$ on $[0.5, 1.5]$. Hence, there is a root such that $\bar{B} = \bar{B}^*$ in $[0.5, 1.5]$. By using the bisection method, IVP (5.2.11) and (5.2.12) is integrated with varying values of $\bar{B}$ obtained from

$$\bar{B} = \frac{\bar{B}_1 + \bar{B}_2}{2}, \tag{5.2.18}$$

where, if $\Gamma(\bar{B}) < 0$, we set $\bar{B}_1 = \bar{B}$, and if $\Gamma(\bar{B}) > 0$, we set $\bar{B}_2 = \bar{B}$. For each solution corresponding to a given $\bar{B}$, the corresponding λ value is obtained from (5.2.13). With $B = 0.4840$, the value of $\bar{B}$ obtained is $\bar{B} = 1.0001275$. The result agrees with that obtained using the non-iterative transformation method.

The velocity profile F' plotted against η for varying values of the slip velocity at $\eta = 0$ is shown in Figure 5.2.2.

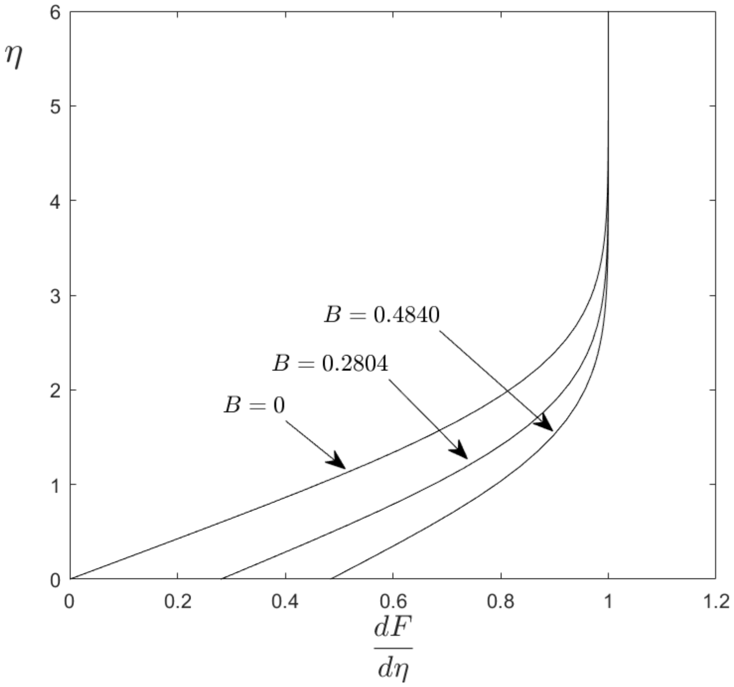

Figure 5.2.2: Numerical results for the Blasius problem for slip and no suction.

5.2.3 Numerical solution for Blasius equation with suction boundary condition

When there is fluid suction and no-slip at the boundary plate, the Blasius BVP is given by

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0 \tag{5.2.19}$$

$$F(0) = A, \quad \frac{dF}{d\eta}(0) = 0, \quad \frac{dF}{d\eta}(\infty) = 1, \tag{5.2.20}$$

where $A = \left(\frac{2}{U_0\nu}\right)^{\frac{1}{2}} v_{s0}$.

With the aid of transformation (5.2.3), the associated IVP becomes

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} + \bar{F}\frac{d^2\bar{F}}{d\bar{\eta}^2} = 0, \tag{5.2.21}$$

$$\bar{F}(0) = \bar{A}, \quad \frac{d\bar{F}}{d\bar{\eta}}(0) = 0, \quad \frac{d^2\bar{F}}{d\bar{\eta}^2}(0) = 1, \tag{5.2.22}$$

where the parameter A transforms according to

$$\bar{A} = \lambda^{-a} A. \tag{5.2.23}$$

1. Non-iterative method

   To solve for $\bar{F}(\bar{\eta})$, first choose a value for $\bar{A}$ in (5.2.22), and then solve (5.2.21) and (5.2.22). The group parameter λ is then obtained from

   $$\lambda = \Big(\frac{d\bar{F}}{d\bar{\eta}}(\infty)\Big)^{\frac{-1}{2a}} \tag{5.2.24}$$

   With $\bar{A}$ chosen and λ obtained, the value of A can be obtained from (5.2.2). In Table 5.2.2, solutions to the Blasius BVP with suction and no slip is presented, choosing $a = 1$. The non-iterative transformation method was used. The last column in Table (5.2.2) shows the values of the parameter A, calculated from $\bar{A}$ and λ using (5.2.23).

$\bar{A}$	λ	$\frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{\eta}}(\infty)$	$\frac{\mathrm{d}^2 F}{\mathrm{d}\eta^2}(\infty)$	A
0.0000	0.7773	1.6552	0.6041	0.0000
0.5000	0.9412	1.1288	0.8859	0.4706
1.0000	1.1146	0.8050	1.2423	1.1146
5.0000	2.2405	0.1992	5.0196	11.2023
10.0000	3.1631	0.1000	10.0050	31.6307
20.0000	4.4723	0.0500	20.0013	89.4455
50.0000	7.0711	0.0200	50.0002	353.5541
100.0000	10.0000	0.0100	100.0001	1000.0006
200.0000	14.1421	0.0050	200.0001	2828.4277

Table 5.2.2: Numerical results for Blasius boundary value problem with suction and no slip. The non-ITM method was used.

2. Iterative transformation method for suction and no-slip

   The approach here is similar to that implemented in Section 5.2.2. For a chosen value A, say $\tilde{A}$, define a real and continuous function

   $$\Gamma(\bar{A}) = \bar{A} - \lambda^{-1}\tilde{A}, \tag{5.2.25}$$

where $\lambda = \lambda(\bar{A})$.

For each value of $\bar{A}$ chosen by trial and error, the IVP (5.2.21) -(5.2.22) is solved until an interval is found, say, $[\bar{A}_1, \bar{A}_2]$ where $\Gamma(\bar{A}_1)\Gamma(\bar{A}_2) < 0$. This guarantees a root $\bar{A}^* \in [\bar{A}_1, \bar{A}_2]$.

Using the bisection algorithm, we then iterate with varying values of $\bar{A}$ (obtained from the midpoints of intervals). The λ value for each $\bar{A}$ is obtained from (5.2.24). Convergence is reached when

$$\Gamma(\bar{A}) = |\bar{A}^* - \lambda^{*-1}\tilde{A}| < \varepsilon. \tag{5.2.26}$$

For example, let $A = 0.4706$. An interval for $\bar{A}$ wherein the root lies is $[0.1, 1]$. Since $\Gamma(\bar{A}_1)\Gamma(\bar{A}_2) = \Gamma(0.1)\Gamma(1.0) < 0$, to proceed with iteration, we set

$$\bar{A} = \frac{\bar{A}_1 + \bar{A}_2}{2}, \tag{5.2.27}$$

and solve the IVP. The corresponding value of λ is obtained from (5.2.24). The stopping criteria for the iteration, given by (5.2.26) is invoked. If satisfied, iteration stops. If not, we check the sign of Γ.

If $\Gamma(\bar{A}) < 0$, we set $\bar{A}_1 = \bar{A}$, and if $\Gamma(\bar{A}) > 0$, we set $\bar{A}_2 = \bar{A}$. With $A = 0.4706$, the value of $\bar{A}$ obtained to 6 decimal places is 0.499983. The result corresponds with that obtained using the non-iterative transformation method. In Figure 5.2.3, the graph of the velocity profile is plotted for various values of the suction parameter A. The results show that as suction increases, the boundary layer thickness decreases.

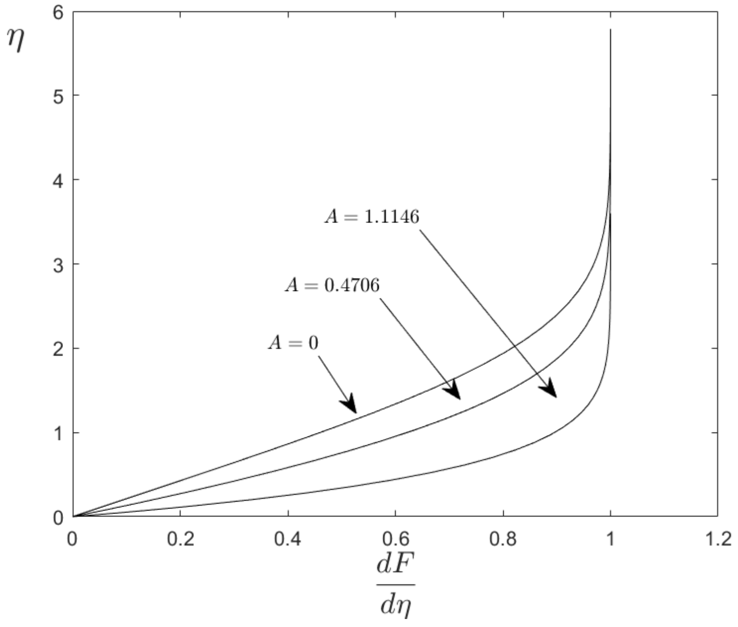

Figure 5.2.3: Numerical results of the Blasius problem for suction and no slip

5.2.4 Numerical solution for Blasius equation with slip and suction boundary condition

When there is suction and slip condition, the Blasius BVP becomes

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} = 0 \tag{5.2.28}$$

$$F(0) = A, \quad \frac{dF}{d\eta}(0) = B, \quad \frac{dF}{d\eta}(\infty) = 1, \tag{5.2.29}$$

where $A = \left(\frac{2}{U_0\nu}\right)^{\frac{1}{2}} v_{s0}$ and $B = \frac{v_{t0}}{U_0}$. The BVP is converted with the aid of the scaling transformation to the IVP

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} + \bar{F}\frac{d^2\bar{F}}{d\bar{\eta}^2} = 0, \tag{5.2.30}$$

$$\bar{F}(0) = \bar{A}, \quad \frac{d\bar{F}}{d\bar{\eta}}(0) = \bar{B}, \quad \frac{d^2\bar{F}}{d\bar{\eta}^2}(0) = 1. \tag{5.2.31}$$

A similar approach as was done in Sections 5.2.2 and 5.2.3 was followed in order to numerically solve the IVP. The non-iterative transformation method was employed. In the graphs shown in Figure 5.2.4-Figure5.2.7, the boundary layer velocity is plotted for various values of the slip parameter A and suction parameter B. The graphs show that as the suction parameter is varied, keeping the slip parameter constant, the boundary layer thickness decreases.

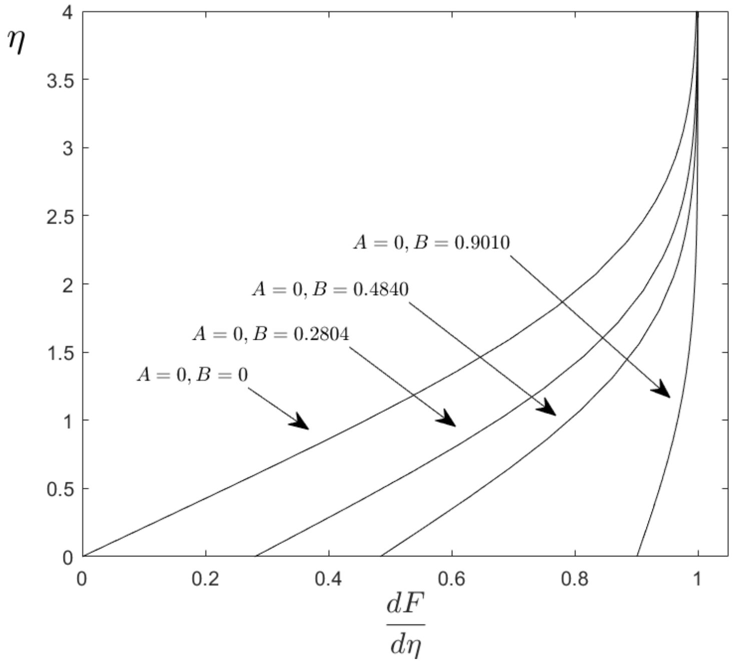

Figure 5.2.4: The graphical solution for the velocity profile for $\bar{A} = 0$ and $\bar{B} = 0, 0.5, 1, 5$

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
0.0000	0.0000	0.7773	1.6553	0.0000	0.0000
0.0000	0.5000	0.7489	1.7829	0.0000	0.2804
0.0000	1.0000	0.6957	2.0663	0.0000	0.4840
0.0000	5.0000	0.4245	5.5497	0.0000	0.9010

Table 5.2.3: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.4

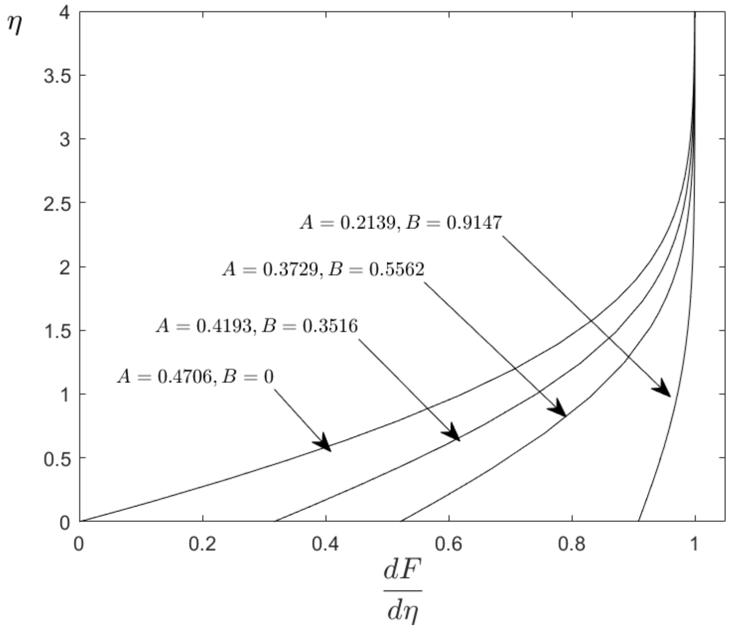

Figure 5.2.5: The graphical solution for the velocity profile for $\bar{A} = 0.5$ and $\bar{B} = 0, 0.5, 1, 5$

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}F}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
0.5000	0.0000	0.9180	1.1865	0.4706	0.0000
0.5000	0.5000	0.7951	1.5819	0.4193	0.3516
0.5000	1.0000	0.7220	1.9181	0.3729	0.5562
0.5000	5.0000	0.4262	5.5052	0.2139	0.9147

Table 5.2.4: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.5

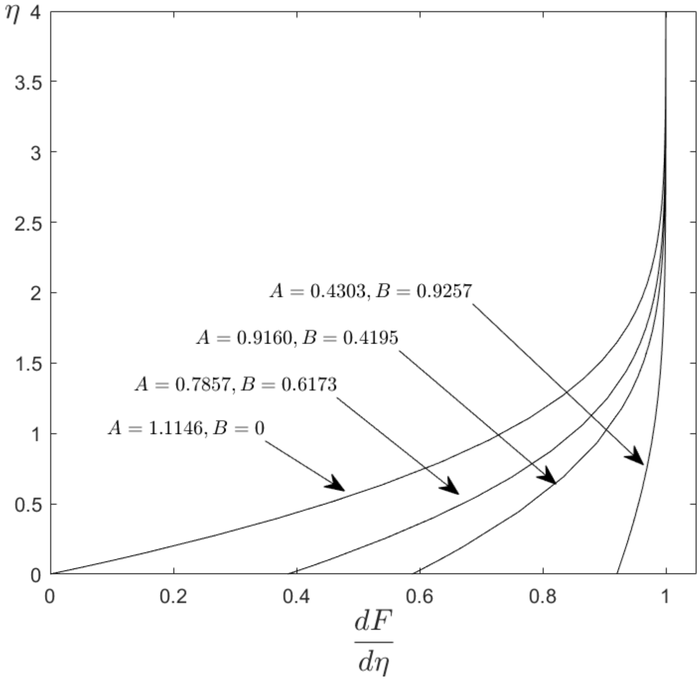

Figure 5.2.6: The graphical solution for the velocity profile for $\bar{A} = 0$ and $\bar{B} = 0, 0.5, 1, 5$

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}F}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
1.0000	0.0000	1.0275	0.9473	1.1146	0.0000
1.0000	0.5000	0.8789	1.2944	0.9160	0.4195
1.0000	1.0000	0.7669	1.7004	0.7857	0.6173
1.0000	5.0000	0.4291	5.4317	0.4303	0.9257

Table 5.2.5: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.6

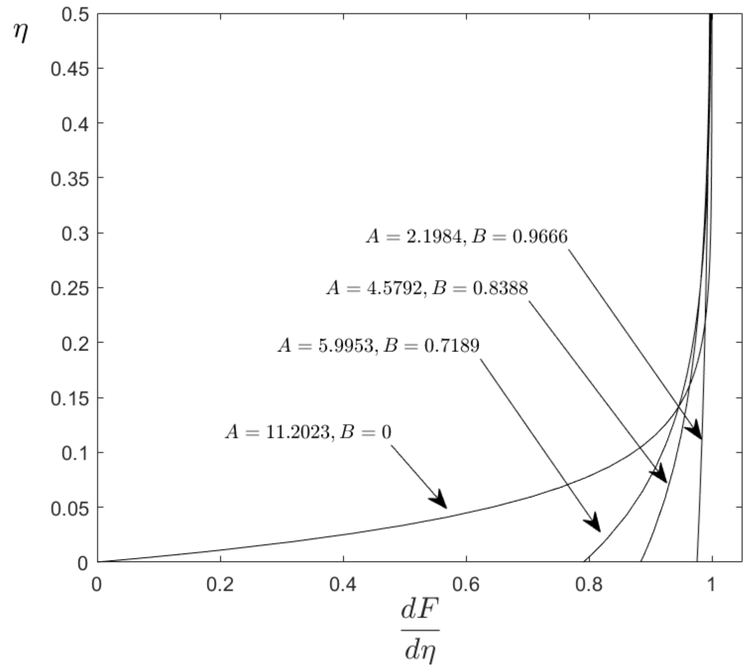

Figure 5.2.7: The graphical solution for the velocity profile for $\bar{A} = 5$ and $\bar{B} = 0, 0.5, 1, 5$

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
5.0000	0.0000	2.7402	0.1332	11.2023	0.0000
5.0000	0.5000	1.2579	0.6320	5.9953	0.7189
5.0000	1.0000	0.9403	1.1309	4.5792	0.8388
5.0000	5.0000	0.4418	5.1237	2.1984	0.9666

Table 5.2.6: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.7

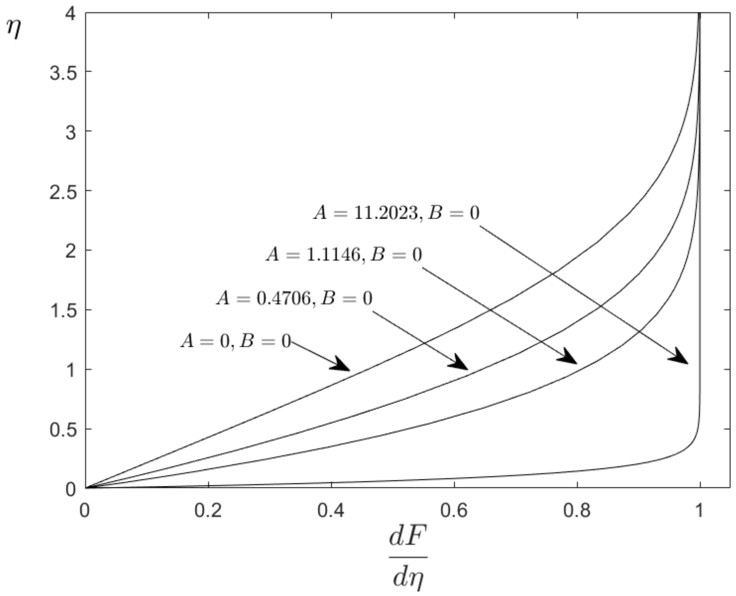

Figure 5.2.8: The graphical solution for the velocity profile for $\bar{B} = 0$ and $\bar{A} = 0, 0.5, 1, 5$

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}F}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
0.0000	0.0000	0.7773	1.6553	0.0000	0.0000
0.5000	0.0000	0.9412	1.1289	0.4706	0.0000
1.0000	0.0000	1.1145	0.8050	1.1146	0.0000
5.0000	0.0000	2.2405	0.1992	11.2023	0.0000

Table 5.2.7: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.8

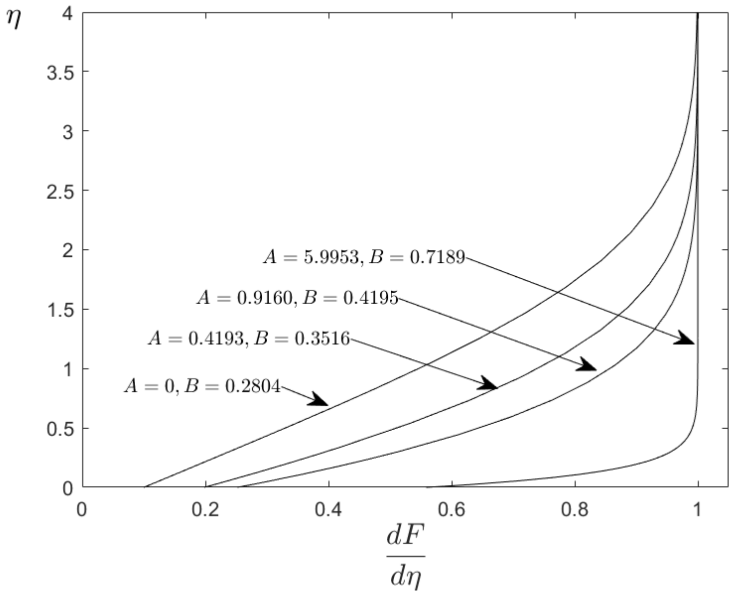

Figure 5.2.9: The graphical solution for the velocity profile for $\bar{B} = 0.5$ and $\bar{A} = 0, 0.5, 1, 5$

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}F}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
0.0000	0.5000	0.7733	1.6722	0.0000	0.2804
0.5000	0.5000	0.8907	1.2605	0.4193	0.3516
1.0000	0.5000	1.0045	0.9911	0.9160	0.4195
5.0000	0.5000	1.4951	0.4473	5.9953	0.7189

Table 5.2.8: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.9

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}F}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
0.0000	5.0000	0.3546	7.9527	0.0000	0.9010
0.5000	5.0000	0.3559	7.8945	0.2139	0.9147
1.0000	5.0000	0.3570	7.8474	0.4303	0.9257
5.0000	5.0000	0.3612	7.6643	2.1984	0.9666

Table 5.2.10: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.11

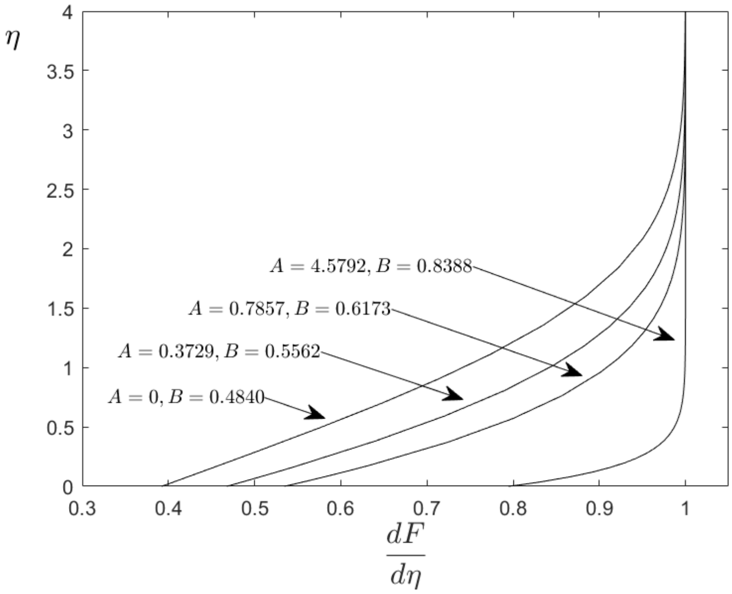

Figure 5.2.10: The graphical solution for the velocity profile for $\bar{B} = 1$ and $\bar{A} = 0, 0.5, 1, 5$

$\bar{A}$	$\bar{B}$	λ	$\frac{\mathrm{d}F}{\mathrm{d}\bar{\eta}}(\infty)$	A	B
0.0000	1.0000	0.7233	1.9117	0.0000	0.4840
0.5000	1.0000	0.7898	1.6033	0.3729	0.5562
1.0000	1.0000	0.8443	1.4028	0.7857	0.6173
5.0000	1.0000	1.0293	0.9438	4.5792	0.8388

Table 5.2.9: Numerical result of the Blasius problem with slip and suction corresponding to Figure 5.2.10

5.3 Numerical solution for Falkner-Skan problem

5.3.1 Falkner-Skan equation with suction and no slip

The Falkner-Skan BVP is given by

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} + \frac{2m}{1+m}\left(1 - \left(\frac{df}{d\eta}\right)^2\right) = 0 \tag{5.3.1}$$

$$F(0) = A, \quad \frac{dF}{d\eta}(0) = 0, \quad \frac{dF}{d\eta}(\infty) = 1, \tag{5.3.2}$$

where $A = \Big(\dfrac{2}{(1+m)U_0\nu}\Big)^{\frac{1}{2}} V_{s0}$.

The BVP (5.3.1) and (5.3.2) is of the form of the Blasius BVP. The Falkner-Skan model cannot be solved using the non-iterative transformation method method because (5.3.1) is not invariant under a scaling group, therefore the iterative transformation method would be used. To do that, a modified form of the Falkner-Skan model which is invariant under an extended scaling group will be used [14, 20].

We first introduce the scaling group of transformations

$$\bar{\eta} = h^a \eta, \quad \bar{F} = h^b F, \tag{5.3.3}$$

where h is the group parameter. This scaling group transforms (5.3.1) and (5.3.2) into a modified Falkner-Skan equation given by

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} + h^{-a-b}\bar{F}\frac{d^2\bar{F}}{d\bar{\eta}^2} + \frac{2m}{1+m}\Big(h^{-3a+b} - h^{-a-b}\Big(\frac{d\bar{F}}{d\bar{\eta}}\Big)^2\Big) = 0, \tag{5.3.4}$$

$$\bar{F}(0) = \bar{A}, \quad \frac{d\bar{F}}{d\bar{\eta}}(0) = 0, \quad h^{a-b}\frac{d\bar{F}}{d\bar{\eta}}(\infty) = 1, \tag{5.3.5}$$

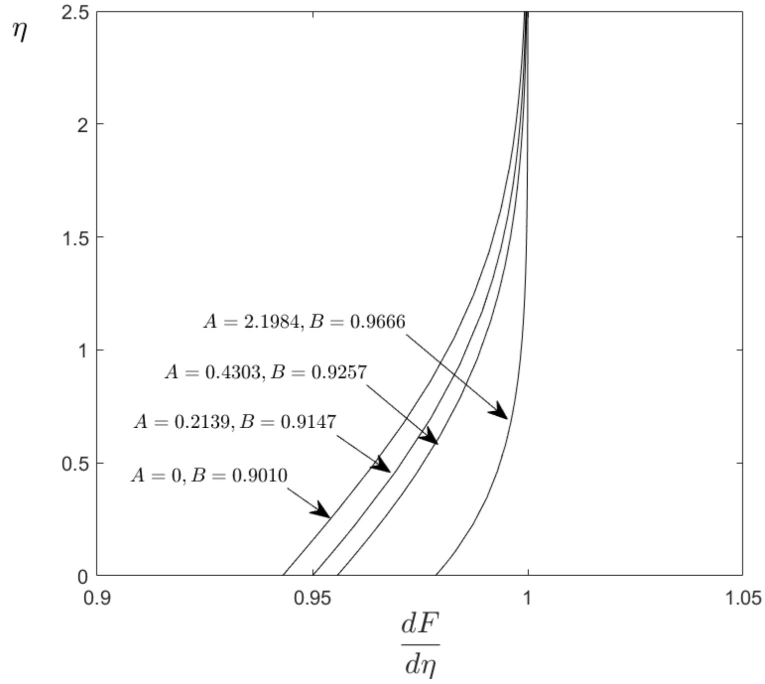

Figure 5.2.11: The graphical solution for the velocity profile for $\bar{B} = 5$ and $\bar{A} = 0, 0.5, 1, 5$

where $\bar{A} = h^b A$.

Clearly, (5.3.1) and (5.3.2) are not invariant under the scaling group of transformation (5.3.3), since (5.3.4) and the third boundary condition (5.3.5) contain the group parameter h. Equation $\bar{F}(0) = \bar{A}$ is form invariant, even though it contains the group parameter h. The non-invariance of $F'(\infty)$ in (5.3.5) under transformation (5.3.3) is necessary in order to determine the group parameter h, however, since (5.3.1) is not invariant, the boundary condition (5.3.2) could not be replaced by an appropriate initial condition of the form $\bar{F}(0) = 1$.

Consider now the modified BVP (5.3.4) and (5.3.5). If the scaling parameter h is allowed to transform, then, under the extended one-parameter scaling group

$$\eta^* = \lambda^{\alpha}\bar{\eta}, \quad F^* = \lambda^{\beta}\bar{F}, \quad h^* = \lambda h, \tag{5.3.6}$$

the modified BVP transforms to

$$\frac{d^3F^*}{d\eta^{3*}} + h^{*^{-a-b}} F^* \frac{d^2F^*}{d\eta^{*2}} + \frac{2m}{1+m}\Big(h^{*^{-3a+b}} - h^{*^{-a-b}}\Big(\frac{dF^*}{d\eta^*}\Big)^2\Big) = 0. \tag{5.3.7}$$

$$F^*(0) = A^*, \quad \frac{dF^*}{d\eta^*}(0) = 0, \quad h^{*^{a-b}}\frac{dF^*}{d\eta^*}(\infty) = 1. \tag{5.3.8}$$

where $A^* = h^{*^b}\bar{A}$ and provided $\alpha = a$ and $\beta = b$. Thus the modified problem (5.3.4)-(5.3.6) is invariant under (5.3.6) and is therefore to be solved instead of the Falkner-Skan BVP.

The IVP to solve becomes

$$\frac{d^3F^*}{d\eta^{3*}} + h^{*^{-a-b}} F^* \frac{d^2F^*}{d\eta^{*2}} + \frac{2m}{1+m}\Big(h^{*^{-3a+b}} - h^{*^{-a-b}}\Big(\frac{dF^*}{d\eta^*}\Big)^2\Big) = 0. \tag{5.3.9}$$

$$F^*(0) = A^*, \quad \frac{dF^*}{d\eta^*}(0) = 0, \quad h^{*^{a-b}}\frac{dF^*}{d\eta^*}(\infty) = 1. \tag{5.3.10}$$

The parameter h must be set to unity in (5.3.4) and (5.3.5) in order to recover the Falkner-Skan equation, and therefore, the invariance of (5.3.4) and (5.3.5) under (5.3.6) does not require a or b to be known or a relationship between a and b. Thus, provided $a = b$, thereby ensuring partial invariance of the third boundary condition in equation (5.3.6), the group parameter λ given by

$$\lambda = \Big(\frac{dF^*}{d\eta^*}(\infty)\Big)^{\frac{-1}{a-b}} \tag{5.3.11}$$

and the solution $F(\eta)$ can be obtained.

Once F^* is obtained from (5.3.9) and (5.3.10), the solution $F(\eta)$ is obtained from the second condition in (5.3.3) and the second condition in (5.3.6) as

$$F(\eta) = \lambda^{-b}F^*(\eta^*) \quad \text{and} \quad A = \frac{A^*}{\lambda^b}. \tag{5.3.12}$$

Suppose we choose $b = -a = \frac{1}{8}$, equation (5.3.9) and (5.3.10) become

$$\frac{d^3F^*}{d\eta^{*^3}} + F^*\frac{d^2F^*}{d\eta^{*^2}} + \frac{2m}{1+m}\Big(h^{*^{\frac{1}{2}}} - \Big(\frac{dF^*}{d\eta^*}\Big)^2\Big) = 0 \tag{5.3.13}$$

$$F^*(0) = A^*, \quad \frac{dF^*}{d\eta^*}(0) = 0, \quad \frac{dF^*}{d\eta^*}(\infty) = 1. \tag{5.3.14}$$

The extended transformation group, when $b = -a = \frac{1}{8}$ is

$$\eta^* = \mu^{-1}\bar{\eta}, \quad F^* = \mu\bar{F}, \quad h^* = \mu^8 h, \quad \text{where}\lambda = \mu^8 \tag{5.3.15}$$

5.3.2 Numerical solution for Falkner-Skan equation with slip boundary condition

Consider again the Falkner-Skan model for slip boundary condition with no suction

$$\frac{d^3F}{d\eta^3} + F\frac{d^2F}{d\eta^2} + \frac{2m}{1+m}\Big(1 - \Big(\frac{df}{d\eta}\Big)^2\Big) = 0 \tag{5.3.16}$$

subject to the boundary conditions

$$F(0) = 0, \quad \frac{dF}{d\eta}(0) = B, \quad \frac{dF}{d\eta}(\infty) = 1, \tag{5.3.17}$$

where $A = 0$ and $B = \dfrac{v_{t0}}{U_0}$.

We first introduce the scaling group of transformations

$$\bar{\eta} = \lambda^a\eta, \quad \bar{F} = \lambda^b F \tag{5.3.18}$$

where h is the group parameter. This group transforms (5.3.16) and (5.3.17) into a modified Falkner-Skan equation given by

$$\frac{d^3\bar{F}}{d\bar{\eta}^3} + h^{-a-b}\bar{F}\frac{d^2\bar{F}}{d\bar{\eta}^2} + \frac{2m}{1+m}\Big(h^{-3a+b} - h^{-a-b}\Big(\frac{d\bar{F}}{d\bar{\eta}}\Big)^2\Big) = 0, \tag{5.3.19}$$

$$\bar{F}(0) = 0, \quad \frac{d\bar{F}}{d\bar{\eta}}(0) = \bar{B}, \quad h^{a-b}\frac{d\bar{F}}{d\bar{\eta}}(\infty) = 1, \tag{5.3.20}$$

where $\bar{B} = \dfrac{h^b}{B}$.

Clearly, (5.3.16) and condtion $\dfrac{d\bar{F}}{d\bar{\eta}}(\infty)$ in (5.3.17) are not invariant under the scaling group of transformation (5.3.18), since (5.3.19) and condition 3 in (5.3.20) contain the group parameter h. Condition one in equation (5.3.20) is form invariant. The non-invariance of condition three in (5.3.20) under transformation (5.3.18) is necessary in order to determine the group parameter h, however, since (5.3.16) is not invariant, the boundary condition (5.3.17) could not be replaced by an appropriate initial condition of the form $\bar{F}(0) = 1$.

Consider now the modified BVP (5.3.19) and (5.3.20). If the scaling parameter h is allowed to transform, then, under the extended one-parameter scaling group

$$\eta^* = \lambda^\alpha \bar{\eta}, \quad F^* = \lambda^\beta \bar{F}, \quad h^* = \lambda h \tag{5.3.21}$$

the modified BVP transforms to

$$\frac{d^3F^*}{d\eta^{3*}} + {h^*}^{-a-b} F^* \frac{d^2F^*}{d\eta^{*2}} + \frac{2m}{1+m}\Big({h^*}^{-3a+b} - {h^*}^{-a-b}\Big(\frac{dF^*}{d\eta^*}\Big)^2\Big) = 0, \tag{5.3.22}$$

$$F^*(0) = 0, \quad \frac{dF^*}{d\eta^*}(0) = B^*, \quad {h^*}^{a-b}\frac{dF^*}{d\eta^*}(\infty) = 1, \tag{5.3.23}$$

where $B^* = \dfrac{{h^*}^b}{\bar{B}}$ and provided $\alpha = a$ and $\beta = b$. Thus the modified problem (5.3.19)-(5.3.20) is invariant under (5.3.21) and therefore, it will be solved instead of the Falkner-Skan BVP.

The IVP to solve becomes

$$\frac{d^3F^*}{d\eta^{3*}} + {h^*}^{-a-b} F^* \frac{d^2F^*}{d\eta^{*2}} + \frac{2m}{1+m}\Big({h^*}^{-3a+b} - {h^*}^{-a-b}\Big(\frac{dF^*}{d\eta^*}\Big)^2\Big) = 0, \tag{5.3.24}$$

$$F^*(0) = 0, \quad \frac{dF^*}{d\eta^*}(0) = B^*, \quad {h^*}^{a-b}\frac{dF^*}{d\eta^*}(\infty) = 1, \tag{5.3.25}$$

The parameter h must be set to unity in (5.3.4) and (5.3.5) in order to recover the Falkner-Skan equation, and therefore, the invariance of (5.3.4) and (5.3.5) under (5.3.6) does not require a or b to be known or a relationship between a and b. Thus, provided $a = b$, thereby ensuring partial invariance of the third boundary condition in equation (5.3.6), the group parameter λ given by

$$\lambda = \Big(\frac{dF^*}{d\eta^*}(\infty)\Big)^{\frac{-1}{a-b}} \tag{5.3.26}$$

and the solution $F(\eta)$ can be obtained.

Once F^* is obtained from (5.3.9) and (5.3.10), the solution $F(\eta)$ is obtained from the second condition in (5.3.3) and the second condition in (5.3.6) as

$$F(\eta) = \lambda^{-b} F^*(\eta^*) \quad \text{and} \quad B = \frac{B^*}{\lambda^{b-a}}. \tag{5.3.27}$$

Suppose we choose $b = -a = \dfrac{1}{8}$, equation (5.3.9) and (5.3.10) become

$$\frac{d^3F^*}{d\eta^{*^3}} + F^* \frac{d^2F^*}{d\eta^{*^2}} + \frac{2m}{1+m}\left(h^{*^{\frac{1}{2}}} - \left(\frac{dF^*}{d\eta^*}\right)^2\right) = 0, \tag{5.3.28}$$

$$F^*(0) = 0, \quad \frac{dF^*}{d\eta^*}(0) = B^*, \quad \frac{dF^*}{d\eta^*}(\infty) = 1. \tag{5.3.29}$$

The extended transformation group, when $b = -a = \dfrac{1}{8}$ is

$$\eta^* = \mu^{-1}\bar{\eta}, \quad F^* = \mu\bar{F}, \quad h^* = \mu^8 h, \quad \text{where} \quad \lambda = \mu^8 \tag{5.3.30}$$

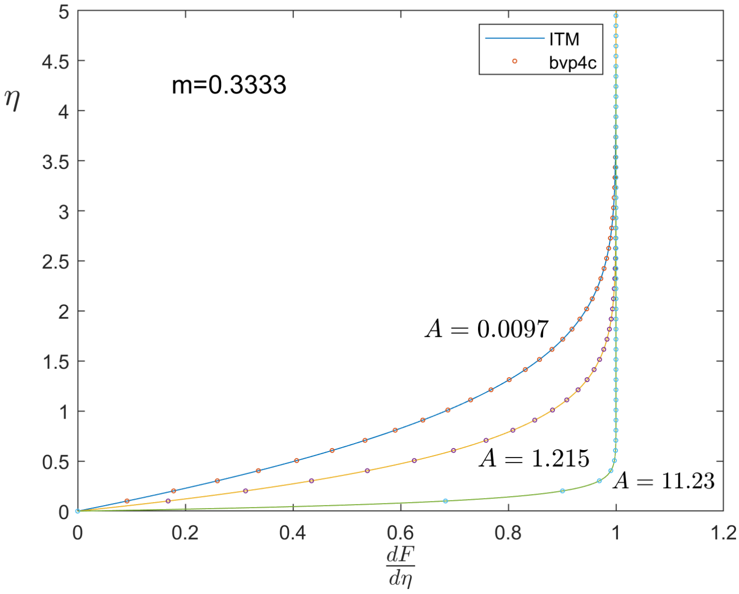

Figure 5.3.1: Graphical solution of the Falkner-Skan model with slip. Slip parameter $A = 0.0097, 1.215, 11.23$

In Figures 5.3.1, the Falkner-Skan model with slip conditions was solved, using the iterative transformation method, as well as MATLAB bvp4c for $m = 0.3333$. The graph shows a good agreement between the two results. Also, when $m = 0.052$, the graphs shows that as the slip parameter increases, the boundary layer thickness reduces.

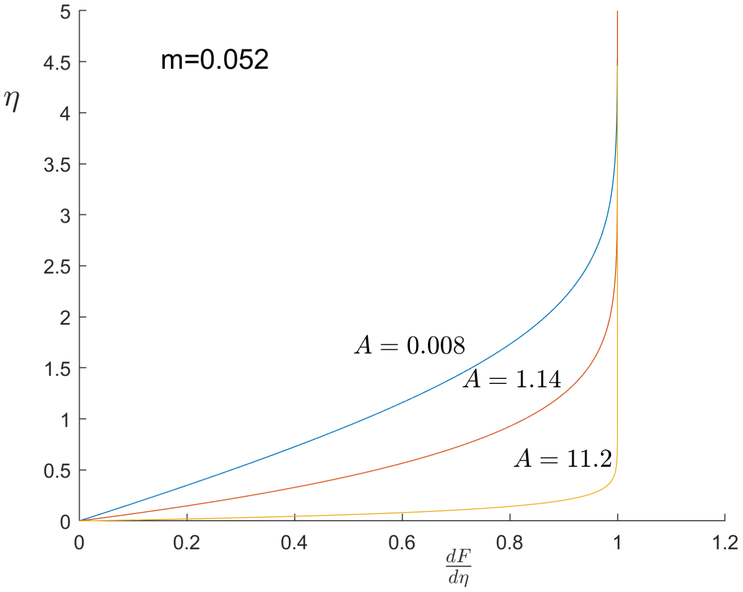

Figure 5.3.2: Graphical solution of the Falkner-Skan model with slip. Slip parameter $A = 0.008, 1.14, 11.2$

Chapter 6

Conclusions

The Lie point symmetry of Prandtl‘s two-dimensional boundary layer equation for the stream function was derived. The mainstream velocity $U(x)$ depends on distance x along the boundary layer. The Lie point symmetry depends on four arbitrary constants and an arbitrary function $g(x)$ and is therefore a linear combination of four Lie point symmetries and an infinite number of Lie symmetries of the form

$$X_g = g(x)\frac{\partial}{\partial x}$$

where $g(x)$ is an arbitrary function. The general form of the invariant solution, which includes the invariant form for the slip velocity and suction or blowing velocity at the base and the mainstream velocity $U(x)$, was derived for four cases. Two cases, the Blasius equation and the Falkner-Skan equation, where derived from scaling Lie point symmetries while two other cases, the exponential equation and flow in convergent and divergent channels, were derived from non-scaling Lie point symmetries. The Lie point symmetry analysis gave a unified and systematic approach to the derivation of the invariant solutions.

The boundary layer flow in convergent and divergent channels was studied in detail. For slip at the boundary and also for no slip, the solution for flow in a convergent channel is not unique. Two solutions were obtained. For one flow, the velocity profile is the usual form for a boundary layer. For the other solution, there is reverse flow either at the plate or in the interior of the boundary layer. When the reverse flow is in the interior of the boundary layer, its width is the same as the width of the reverse flow when there is no slip, while if its at the boundary, its width is less than the width for no slip. For flow in a divergent channel with or without slip, the solution does not exist if there is no suction.

Boundary layer flow with suction or blowing but no slip in convergent and divergent channels was also investigated. Unlike the slip boundary condition, the suction or blowing boundary condition adds a term to the ODE. The ODE admits a Lie point symmetry provided the strength of the suction λ satisfies $\lambda = \pm\dfrac{5}{\sqrt{3}}$. This explains why $\lambda = -\dfrac{5}{\sqrt{3}}$ was used by Jones and Watson [15] in the derivation of their elementary solution for flow in a convergent channel with blowing. We re-derived the elementary solution of Jones and Watson using Lie group analysis. Unlike the solution for no blowing in a convergent channel, the solution is unique and has no reverse flow. We derived an elementary solution for flow in a divergent channel with suction. This solution is not unique, but there is no reverse flow in either of the two solutions.

The numerical solution of the Blasius and Falkner-Skan model was derived by transforming the Boundary Value Problem (BVP) to an Initial Value Problem (IVP). The Blasius equation is invariant under a scaling transformation. A non-iterative method could be used for no slip and no suction boundary conditions. The Falkner-Skan equation does not admit a scaling transformation. However, a modified form of the Falkner-Skan model, which is invariant under a scaling group was used to obtain a solution. This required a root finding algorithm such as the bisection method, along with the ode45 MATLAB function. The effect of the slip and suction or blowing on the solution of the Blasius equation and Falkner-Skan equation was investigated.

Appendix A

MATLAB code

A.1 Commented MATLAB code

```
% % %Blasius equation : IVP Solution

% Define interval for root finding method
b_low = 0;
b_up = 0.6;

% Initiate variables
real_a = 0.4706;
gamma = 2;

% Loop
while(abs(gamma) > 10e-6)
%Step 1: Solve IVP
    b_bar = (b_low + b_up)/2;
    [x,y] = ode45(@Blasius,[0 4.5],[b_bar 0 1],100);

%Step 2: Find Lambda from boundary condition at f'(inf) = lambda^-2
    lambda = 1/sqrt(y(end,2));

%Step Bisection
```

```
    gamma = b_bar - (lambda)*real_a;
    if gamma < 0
        b_low = b_bar;
    else
        b_up = b_bar;
    end
end

%Step 3: solve for real_f = lambda f_bar(eta/lambda)
    eta = x/lambda;
    real_F = (lambda^2)*y(:,2);

%Step 4: Plot eta against df = y(2,:)
  plot(real_F,eta,'b-');

% Define system of first order ODEs
function dfdx = Blasius(¬,y)
dfdx = zeros(3,1);
dfdx(1) = y(2); %df/du
dfdx(2) = y(3); %d2f/du2
dfdx(3) = -1*y(1)*y(3);%d3f/du3
end
```

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