SOME STOCHASTIC CONVERGENCE THEOREMS ON VECTOR LATICES

David Francis Rodda

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Declaration

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Doctor of Philosophy to the University of the Witwatersrand contains my own, independent work and has not been submitted for any degree at any other university.

Name: David Francis Rodda

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Name David Rodda
Supervisors Dr W-C Kuo, Prof B A Watson
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Abstract

This thesis presents new generalisations of some classical convergence theorems, thus developing the theory of stochastic processes in vector lattices as set out by Kuo, Labuschagne and Watson [58–60]. This setting has been receiving considerable attention since its introduction, and has already proven to be fruitful by providing generalisations of many classical stochastic process results, which have the classical results as special cases. Such results include, amongst others, generalisations of Brownian motion and Itô integration [44], martingale theory [59, 60], Markov processes [88, 89], mixingales [68], modes of stochastic convergence [6], and various important inequalities, see for example [7, 9, 38]. Hence it has been shown how these classical theories depend greatly on the order structure and not on measure theory. In so doing, some new results have been brought even to the classical measure space setting, for example in [65].

This thesis builds on previous work and gives a number of new results. The first main result obtained is the strong sequential completeness of the space $\mathcal{L}^1(T)$, the natural domain of the conditional expectation operator T. Strong completeness of $\mathcal{L}^\infty(T)$ is also proved. A maximal inequality is generalised which in the classical setting is due to Hájek-Rényi and Chow in [16, Theorem 1], see [30, Proposition (6.1.4)]. The final main result is a measure-free version of Chow's martingale law of large numbers [16, 17] having Kolmogorov's and Lévy's laws of large numbers as special cases. This is quite a general result having many areas of application, including $\mathcal{L}^p(T)$ martingales for arbitrary $p > 1$, submartingales, and independent random variables. In the course of proving the above we develop the understanding of functional calculus on Riesz spaces, prove Riesz space versions of Kronecker's Lemma and Hölder's inequality for sums, give a submartingale limit law and gain other results.

To the Triune God, whom I believe, and whose word promises a sure convergence:
all things work together to conform those who love Him to the image of Christ.
This convergence is by the strong law of the Spirit of life.
(Romans 8:2, 28-30)

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Contents

Chapter 1

Introductory Chapter

1.1 Introduction

In this thesis, the completeness of the natural domain of a conditional expectation operator on a vector lattice, i.e. Riesz space, is proved, as well as other results related to convergence of stochastic processes in vector lattices. The final result of the thesis is a generalisation of Chow's strong law of large numbers to Riesz spaces.

1.2 Outline of the thesis

This introductory chapter describes the landscape into which the work of the current thesis fits. We mention some applications of the study of Riesz spaces and give a literature survey of existing research on stochastic processes in Riesz spaces.

Chapter 2 introduces concepts that are used later in the thesis.

In Chapter 3 strong sequential completeness of the natural domain of a conditional expectation operator on a Riesz space is proved.

In Chapter 4 a generalisation of the Hájek-Rényi-Chow maximal inequality for submartingales in Riesz spaces is given.

In Chapter 5 Kronecker's Lemma is proved in the Riesz space setting, as well as a theorem related to the convergence of Cesàro sums. Making use of these results and the Hájek-Rényi-Chow maximal inequality of the previous chapter, a limit law for submartingales is proved.

Chapter 6 culminates with a generalisation to Riesz spaces of Chow's L^p laws of large numbers for martingale difference sequences.

1.3 Survey of Riesz space research

Speaking at the 1928 International Mathematical Congress in Bologna, Italy, F. Riesz initiated the exploration into the theory of Riesz spaces [77]. Riesz spaces are vector spaces endowed with a partial ordering compatible with the vector operations such that every set of two elements has a supremum. These spaces are also known as lattice ordered vector spaces or vector lattices. As surveyed in [3, Preface to the first edition], the theory of Riesz spaces was first axiomatically formulated and developed in the 1930s independently by H. Freudenthal [33] and L. V. Kantrovich [52], [51]. Most of the early developments were devoted to the algebraic side; however, notable analytical developments were made by A. Zaanen and W. Luxemburg [71] as well as H. H. Schaefer [78]. In particular, books covering topological Riesz spaces have been written by D. H. Fremlin [31] and by C. D. Aliprantis [3].

One significant use of Riesz spaces comes from the fact that the classical measure spaces $L^p(\Omega)$ are special cases of a vector lattice. Hence many results from measure theory are particular cases of theorems holding on Riesz spaces. As an example, the classical Radon-Nikodym Theorem can be deduced quite simply from the Freudenthal's Spectral Theorem on Riesz spaces.

In addition, some useful applications of Riesz spaces have been found in optimisation theory and economic modelling. The first use of Riesz spaces in economic theory was by Charalambos Aliprantis and Donald Brown in the 1980s in [4], as mentioned in [3, p. 215]; who were inspired by the work of mathematical economist Hukukane Nikaido in his classic book *Convex Structures and Economic Theory* [72].

Riesz spaces have also been found useful in the study of generalised functions and partial differential equations, see for example [2, 85] and references therein.

Another area of application, the subject of this thesis, is the study of stochastic processes in Riesz spaces. The existing literature on this topic can be roughly divided into the following groups: initial generalisations of martingales to vector lattices; discrete-time stochastic processes in the setting where the conditional expectation operators map weak order units to weak order units; Banach space valued stochastic processes; other processes including martingale generalisations and Markov processes; stochastic integration and continuous processes in Riesz spaces; and generalisations of convergence and L^p space concepts.

1.3.1 Initial generalisations of martingales to vector lattices

In the classical setting, stochastic processes are based on measure theory. Traditionally, discrete-time stochastic processes are defined as sequences of measurable functions on a probability space, and continuous stochastic processes are defined analogously where the index set is an interval of real numbers. The 1966 paper *A martingale convergence theorem in vector lattices* by R. DeMarr [27] is, to our knowledge, the first time that stochastic processes were generalised to the setting of Riesz spaces. DeMarr's definition of a conditional expectation requires only that it be a positive linear projection, and it is therefore a generalisation of classical conditional expectations. Examples in his paper show that martingales can be defined in this generalised sense that are not martingales in the classical sense. In DeMarr's own words, examples in that paper "illustrate that the martingale concept extends beyond its original confines in probability (i.e., measure) theory." DeMarr proved a submartingale decomposition theorem and Doob's martingale convergence theorem for bounded submartingales meeting certain conditions, which are met in the classical setting of almost everywhere bounded random variables $L^\infty(\Omega)$. Though his generalisation did not have Doob's $L^1(\Omega)$ martingale convergence theorem as a special case, his result allows a new proof of it without using upcrossings.

In the 1970's and 1980's, order theoretic approaches were also applied to the theory of Markov processes, see for example Doob [28] and Schaefer [78].

The work initiated by DeMarr was continued by G. Stoica in the 1990's, see [80–83]. Stoica's setting required additionally that the range of a conditional expectation be a "component" of the vector lattice. This is a special case of DeMarr's setting which no longer contains the classical setting as a special case (see subsection 2.2.2 below). Stoica's work applies to some of DeMarr's example processes which are not martingales in the classical setting.

Troitsky's paper [84] presented and studied a different but related special case of DeMarr's setting. In Troitsky's paper a conditional expectation is defined as a positive contractive projection on a Banach space. As the paper points out, this definition is a generalisation of the classical conditional expectation since, in the classical case, conditional expectations are exactly the positive contractive projections preserving constant functions, as proved by Douglas in [29].

In the 1970's, M. Rao and de Jonge noted how some properties of conditional expectations, and hence stochastic processes, are related to the order structure of $L^0(\Omega)$ in which they are defined, see [75] and [24]. Conditional expectations have also been studied as operators on the Banach lattice $L^p(\Omega)$ by de Pagter, Dodds, Grobler, Huijsmans and Rao in [26, 41, 74]. De Jonge provided four sufficient conditions for linear operators on $L^\infty(\Omega)$ to be representable as conditional expectation operators on $L^\infty(\Omega)$.

1.3.2 Discrete-time stochastic processes in the new setting

Building on the above insights, Kuo, Labuschagne and Watson in [58] define conditional expectation operators as positive order-continuous projections mapping weak order units to weak order units and having Dedekind complete range. With this definition, conditional expectation operators are shown to commute with certain band projections. The averaging properties of these operators are then shown, which leads to the extension of the domains of such operators to what is called their maximal domain. In [60] and [55] the newly defined conditional expectation was used to generalise martingales, submartingales, stopping times and optional stopping theorems to vector lattices. In [62], Kuo, Labuschagne and Watson presented a generalisation of the classical upcrossing theorem and showed how it could be applied to give a fuzzy upcrossing theorem for fuzzy processes.

In [63] the concept of independence was generalised, as well as the Borel-

Cantelli Lemma and Kolmogorov's Zero-One Law. These results were also shown to have applications in classical and fuzzy probability theory. In [59] and [56] a generalised version of Doob's martingale convergence theorem was proved for bounded martingales using the Riesz space upcrossings theorem, and the space of convergent martingales was characterised with the help of a Riesz-Krickeberg-like decomposition. In [61] some ergodic theorems were generalised, and these were used in conjunction with Kolmogorov's Zero-One Law to prove a strong law of large numbers for sequences of independent random variables.

Weak laws of large numbers were given for more general processes in [67,68]. In Stoica's initial Riesz space setting, some stronger limit laws and laws of large numbers were proved in [79]. In the setting of Kuo, Labuschagne and Watson, Chow's laws of large numbers had not been attempted, this forms the topic of the latter half of this thesis.

1.3.3 Banach space valued stochastic processes

In a related field, Labuschagne, Cullender and others studied Banach space valued stochastic processes in the above-mentioned setting of Troitsky. In [20] a summary of Banach space valued stochastic processes and stochastic processes on Riesz spaces was given. Results there were applied to give results in the classical setting. In [21–23], where the stochastic elements were operators, results relating to decompositions, stopping times, convergence and the Radon-Nikodym property in Banach spaces were obtained. The study of Banach space valued stochastic processes has been continued by Gao, Labuschagne, Marraffa, Troitsky and Xanthos, for example in [69] and [35].

1.3.4 Other processes on Riesz spaces

In addition to martingales, various stochastic processes have been generalised to the Riesz space setting of Kuo, Labuschagne and Watson. A generalisation of amarts to Riesz spaces was given in [57]. In this paper the martingale convergence theorem of [59] was adapted to prove convergence of amarts in a Riesz space. Reverse martingales were generalised along with a convergence theorem for reverse martingales in [54].

Kuo, Vardy and Watson generalised Markov processes [88,89]; mixingales, with a related weak law of large numbers [68]; Quasi-martingales, with related decomposition theorems [90]; and Bernoulli processes, with a related law of large numbers, the Bienaymé inequality, and Poisson's theorem [67] to Riesz spaces.

Rogans, Kuo, and Watson generalised mixing processes and obtained Riesz space mixing inequalities [65]. Their results were applied in the classical setting to get new results for σ-finite processes. The concept of near-epoch dependence was generalised in [66] and it was proved that if a process is near-epoch dependent, then it is a mixingale .

1.3.5 Integration and continuous stochastic processes on Riesz spaces

Additional developments were made which allowed the study of integration and continuous stochastic processes on Riesz spaces. Watson gave a Hahn-Jordan decomposition as well as an Andô-Douglas-Radon-Nikodym Theorem in the Riesz space setting in [91]. Related results are given by L. Hong in [50]. Labuschagne and Watson generalised discrete stochastic integration to Riesz spaces in [70]. Here the f-algebra structure of the universal completion of a Riesz space was utilised to consider generalisations of L^2 spaces of martingales, and use was also made of the Radon-Nikodym Theorem to define stochastic integration with respect to discrete-time Brownian motion.

Grobler [36] obtained results for continuous stochastic processes, which he defined in a slightly different setting. In order to make use of duality theory, requirements were added to the order-dual space of the Riesz space in question. Indeed, the main result of the paper, a decomposition theorem for continuous submartingales, requires a perfect Riesz space. In Grobler's setting the Riesz space is not required to have a weak order unit, but instead the more general requirement is that the band generated by the range of a conditional operator be the Riesz space. His definitions helped define continuous stochastic processes on Riesz spaces.

For continuous time processes, Grobler in [37] considered general stopping times, and adapted T-universal completeness and a Radon-Nikodym Theorem (from [58] and [91] respectively) to get a generalisation of Doob's optional stopping Theorem to continuous stochastic processes in Riesz spaces.

In [38] Grobler discusses in detail how to define a functional calculus in a Riesz space setting so that functions like x^p make sense for a Riesz space element x and real number p. Grobler does this by defining a Daniell integral, and making use the sup-completion of a Riesz space. Using this functional calculus, he then proves Jensen's inequality in the Riesz space setting, which together with an upcrossing theorem give a number of martingale inequalities for both discrete and continuous processes.

In [39, 40] Grobler generalises a Kolmogorov-Čentsov Theorem and Hölder-continuity to Riesz spaces. Conditional independence is discussed and continuous Brownian motion is generalised to the Riesz space setting, extending the discrete Brownian motion given by Labuschagne and Watson in [91].

Grobler and Labuschagne in [43–47] discuss Itô integration with respect to Brownian motion, martingales or local martingales, and extend classical integrals such as the Bochner integral. Results relating to continuity and quadratic variation of a continuous process are given in [47] where a "localization" technique is used to deal with not necessarily bounded martingales, where a truncation technique would have been insufficient; Itô's rule is studied in [43] where Lévy's characterisation of Brownian motion is proved, and in [42] generalisations of Girsanov's Theorem, a Kunita-Watanabe inequality, exponential processes, cross-variation processes, Itô's formula for multidimensional processes and the integration by parts formula for martingales are given.

1.3.6 Generalisations of convergence and L^p spaces

Various inequalities, modes of convergence, and other concepts related to L^p spaces spaces have been generalised to the Riesz space setting. In [48] Grobler, Labuschagne and Marraffa studied the quadratic variation of discrete-time martingales and obtained a number of related inequalities. In [6] Azouzi, Kuo, Ramdane, and Watson extended concepts of convergence in mean, i.e. strong convergence, and convergence in probability to the Riesz space setting. This paper also included a study of how these modes of convergence relate to the generalised concepts of T-uniform integrability (first introduced in [68]) and T-boundedness.

Azouzi and Trabelsi in [9] introduced and studied generalised $\mathcal{L}^p(T)$ spaces for $p > 1$ by using the groundwork laid by the functional calculus in [38]. The paper [9] also gives a comparison of two different ways of establishing a functional calculus on a Riesz space, one with the Daniell integral used by Grobler, and another with the homomorphism formulation of Buskes, Pagter and van Rooij as in [14]. These two formulations are shown to be equivalent on the ideal generated by the weak order unit, and therefore functional calculus results of the homomorphism formulation can be applied in settings involving the Daniell integral formulation. A study of convex functions on $\mathcal{L}(T)^p$-spaces gives an $\mathcal{L}^1(T)$-valued norm on $\mathcal{L}^p(T)$, as well as the inequalities of Minkowski, Young, Hölder, Lyapunov, and Chebychev. The paper also characterises the T-uniform nets as being equal to T-bounded nets for nets in $\mathcal{L}^p(T)$ spaces with $p > 1$, and it is shown that convergence in $\mathcal{L}^p(T)$ is equivalent to having a T-uniform tail and converging in T-conditional probability.

In [7] Azouzi and Ramdane generalised Burkholder's inequality to Riesz spaces. (This inequality is used the proof of Chow's strong law of large numbers for $p > 2$ in Chapter 6 below.) The paper [7] also introduces Riemann integration in a Riesz space setting and obtains an integral representation of p-powers, related to what is done in [38, Theorem 6.5]. In addition, a number of technical results are obtained relating to $\mathcal{L}^p(T)$ norms and stopping times, as well as a Riesz space Cauchy-Schwarz type inequality.

Azouzi and Ramdane generalised distribution functions to Riesz spaces in [8]. In this paper a result is given analogous to the classical result of getting an expectation by taking an integral with respect to a density function. It is also shown that every function meeting the three defining properties of a T-conditional distribution is the distribution function of some Riesz space element.

Azouzi in [5] showed that a Riesz space is universally complete if and only if it is unbounded order complete, *uo*-complete. This gives an analogue to the fact that Dedekind complete spaces are order-complete and has applications in the Banach space valued stochastic processes studied by Gao, Troitsky, Xanthos and others, see for example [35] and references in [5]. Another application is that the range $\mathcal{R}(T)$ of a conditional expectation operator T is *uo*-complete if T has been extended to its natural domain, since in that case $\mathcal{R}(T)$ is universally complete, see [65, Theorem 2.2] and [48, Proposition 2.1 (4)]. This paper [5] also includes a study of the sup-completion used in [38, 48] and it is shown that the two definitions of the natural domain $dom(T)$ of a conditional expectation operator on a Riesz space are equivalent - the one using the sup-completion as in [48], and the other using a universal completion as in [58].

Our recent publication, [64], considers a type of completeness specifically related to settings with a conditional expectation operator. Here the strong sequential completeness of $\mathcal{L}^p(T)$ spaces is shown, meaning that strongly Cauchy sequences, with respect to the $\mathcal{R}(T)$ valued norm induced by T, will converge in the space. This publication forms the content of Chapter 3 below. This was proved without the need of duality theory or a perfect Riesz space and is a first step in an attempt to remove the perfect Riesz space requirement of Grobler and Labuschagne.

Chapter 2

Preliminaries

In this chapter we discuss classical convergence concepts, laws of large numbers, and some other limit theorems.

In addition, preliminary Riesz space theory is introduced, both in general and concerning stochastic processes, which will be used later in the thesis.

2.1 Classical stochastic processes

Unexplained measure theory can be found, for example, in [10], [12] and [15]. We start by defining some modes of stochastic convergence.

Definition 1 (Almost sure convergence) *A sequence of real-valued functions X_n defined on a measure space $(\Omega, \mathcal{F}, \mu)$ is said to converge almost surely to X if there exists a set $\mathcal{N}$ in $\mathcal{F}$ such that $\mu(\mathcal{N}) = 0$ and $X_n(\omega) \to X(\omega)$ for all ω in $\Omega \setminus \mathcal{N}$.*

A different mode of convergence is convergence in measure, also known as convergence in probability when the measure space is in fact a probability space. A sequence of real-valued functions is sometimes called a sequence of "random variables" when the underlying measure space is a probability space.

Definition 2 (Convergence in probability) *A sequence of random variables X_n is said to converge to X in probability if for all $\epsilon > 0$, we have the convergence as $n \to \infty$ that $\mu\left(\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}\right) \to 0$.*

It can be shown that almost sure convergence implies convergence in probability. Another mode is convergence in r-th mean, also known as strong convergence, referring to convergence in the L^r-norm.

Definition 3 (Strong convergence) *A sequence of random variables X_n is said to converge strongly (or in the L^r-norm) to a random variable X if*

$$\mathbb{E}(|X_n - X|^r) \to 0 \text{ as } n \to \infty.$$

Strong convergence for any $r \geq 1$ implies convergence in probability, but strong convergence and almost sure convergence are independent concepts.

Different modes of convergence have their own corresponding concepts of completeness, based on the corresponding meaning of being Cauchy. Here we define strong completeness.

Definition 4 *A sequence X_n in a normed space L is called a strong Cauchy sequence if for any $\epsilon > 0$, there exists an N such that for any $m, n \geq N$ we have $\|X_m - X_n\| < \epsilon$.*

Hence we can define strong completeness as follows:

Definition 5 *A normed vector space L is said to be strongly-complete if every strong Cauchy sequence X_n converges strongly (i.e. in norm) to a limit in L.*

Applications of completeness include the ability to determine the existence of solutions to partial differential equations, and the ability to define integrals as the limit of a Cauchy sequence. The first novel result in this thesis gives strong sequential completeness of the natural domain of a Riesz space conditional expectation operator, see Chapter 3.

Another contribution of this thesis is the generalisation to Riesz spaces of certain laws of large numbers. A law of large numbers gives conditions under which a stochastic process, usually representing sample means, will converge. In the classical theory, if the convergence is almost sure convergence then the law is called a strong law of large numbers (SLLN); and if the convergence is in probability then it is called a weak law of large numbers. Since almost sure convergence implies convergence in probability, a strong law implies a weak law.

Chebyshev's law of large numbers states that given a sequence of independent identically distributed random variables from a distribution with finite mean and variance, the sample average of the first n results will converge in probability to the mean as $n \to \infty$. This is a weak law of large numbers which has been improved, for example to the following strong law of Kolmogorov.

Theorem 1 (Kolmogorov's SLLN) *If X_n is a sequence of independent (not necessarily identically distributed) random variables with mean zero and*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}(X_n^2) < \infty$$

then we have the almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0.$$

There are a number of versions of Kolmogorov's strong law, including for martingale-difference sequences (see, for example, [32, Sections 273 and 276]). A martingale stochastic process can be defined as a sequence of random variables where a random variable in the sequence is equal to the previous variable plus a random variable having zero expectation. A martingale difference sequence consists of the jumps $Y_n = X_n - X_{n-1}$ of a martingale X_n.

Another law of large numbers, containing Kolmogorov's laws as a special case, is due to Y. S. Chow (see [16, 17] and [30, Section 6.1]). There are two results, one for $1 < p \leq 2$ and the other for $p > 2$. We have generalised Chow's strong laws of large numbers for martingale difference sequences to the Riesz space setting, see Chapter 6 below. The following formulation of Chow's laws of large numbers can be found in [30, pp. 259-260].

Theorem 2 (Chow's SLLN for $1 < p \leq 2$**)** *Let p be a real number satisfying $1 < p \leq 2$, let $Y_n = \sum_{i=1}^{n} X_i$ be a martingale, and let a_i be a positive increasing sequence of real numbers. If*

$$\sum_{i=1}^{\infty} \frac{1}{a_i^p} \mathbb{E}\left(|X_i|^p\right) < \infty,$$

then we have the almost sure convergence:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0.$$

Theorem 3 (Chow's SLLN for $p > 2$**)** *Let p be a real number satisfying $2 < p \leq \infty$, and let $Y_n = \sum_{i=1}^{n} X_i$ be a martingale. Suppose*

$$\sum_{i=1}^{\infty} \frac{1}{i^{1+p/2}} \mathbb{E}\left(|X_i|^p\right) < \infty.$$

Then we have the almost sure convergence:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0.$$

More details and applications of laws of large numbers can be found in [76] and [18].

Apart from laws of large numbers, there are other limit theorems, such as the Arzelà-Ascoli Theorem, which ensures convergence of a subsequence of certain sequences; and Komlós' Lemma, which guarantees the existence of Cesàro summable subsequences (see [32, p. 396] and [53]). Since Komlós' Lemma can be proved using a martingale law of large numbers, it is hoped that the generalisation of Chow's laws will contribute towards generalising Komlós' Lemma to Riesz spaces, similarly to what was done for Banach lattices in [35]. The classical proof of Komlós' Lemma also uses concepts such as weak convergence and relative weak compactness, as well as the Arzelà-Ascoli Theorem. These may therefore need to be generalised to the Riesz space setting in order to prove Komlós' Lemma.

By using Komlós' Lemma, one can obtain a short proof of the Doob-Meyer decomposition Theorem for continuous-time martingales (see [11]). Since the Doob-Meyer decomposition is crucial for stochastic integration (see [36]), a Riesz space version of Komlós' Lemma might assist in removing the perfect Riesz space requirement for continuous stochastic processes on Riesz spaces.

2.2 Preliminaries for Riesz spaces and generalised stochastic processes

2.2.1 Riesz spaces

Here we give some basic definitions and concepts for work in Riesz spaces. Some of these concepts will be developed further in the sections of the thesis where they are needed. For additional or unexplained details one can consult [13], [25], and [92].

A partial ordering "$\leq$" on a set A is a collection of statements "$a \leq b$" about pairs of elements in A, such as (a, b) satisfying, 1) if $a \in A$ then $a \leq a$, 2) if $a \leq b$ and $b \leq a$ then $a = b$, and 3) if $a \leq b$ and $b \leq a$ then $a = b$. The statement $a \leq b$ can also be written as $b \geq a$, and if for two elements we have that either $a \leq b$ or $b \leq a$ then we say that a and b are "comparable." The difference between a partial ordering and a linear ordering is that for a partial ordering not all elements need to be comparable, but with a linear ordering, any two elements need to be comparable.

An upper bound of a subset C of a partially ordered space A is an element $a \in A$ such that $a \geq c$ for every $c \in C$. Similarly a lower bound of C is an element a of A such that $a \leq c$ for every $c \in C$. A subset C of a partially ordered space is said to be bounded above if there exists at least one upper bound for C in A. The supremum of C, if it exists, is the lowest upper bound of C, that is, the supremum of C is an element $b \in A$ such that $b \leq a$ for every upper bound a of C. Similarly the infimum of a set C is defined as the largest lower bound. If the set C contains only two elements, c_1 and c_2, then the supremum is denoted $c_1 \vee c_2$, and the infimum is denoted $c_1 \wedge c_2$. The supremum of a set C is denoted $\sup(C)$, and the infimum is denoted $\inf(C)$. If the supremum of C is an element contained in C then it can be denoted $max(C)$, the maximum element in C, and similarly for the minimum.

A ordered vector space is defined as a vector space L on which a partial ordering is defined, such that the partial order is compatible with the scalar multiplication and addition of the vector space. That is, for a scalar $\lambda \in \mathbb{R}$ and vectors $x, y, z \in L$ such that $x \leq y$, then the following two conditions must hold, $\lambda x \leq \lambda y$ and $x + z \leq y + z$.

A Riesz space (also called a vector lattice) is an ordered vector space E such that for any two vectors $a, b \in E$, their supremum and infimum exist in E. That is, $a \vee b \in E$ and $a \wedge b \in E$. It can be shown that if $a \vee b \in E$ for every $a, b \in E$ then also $a \wedge b \in E$ for every $a, b \in E$, hence sometimes only the supremum requirement is used when defining a Riesz space.

An example of a linear ordering is the relationship $\leq$ on $\mathbb{R}$ defined as the usual less than or equal relationship. An example of a partial ordering on a Riesz space is the pointwise less that or equal to almost everywhere relationship on $L^0(\Omega, \mathcal{F}, \mu)$, where $f \leq g$ if $f(x) \leq g(x)$ for almost every $x \in \Omega$.

On a Riesz space E, for any element $x \in E$ we can define the positive part $x^+ := x \vee 0$ and the negative part $x^- := (-x) \vee 0 = -(x \wedge 0)$. The modulus of x is defined as $|x| := x^+ + x^- = -x \vee x$. Each of x^+, x^-, and $|x|$ are in the positive cone E^+ of E, i.e. $x^+ \geq 0$, $x^- \geq 0$, and $|x| \geq 0$.

A Riesz subspace B of E is a vector subspace B of E which inherits the same partial ordering as that in E and such that B is a Riesz space in its own right, i.e. for any $x, y \in B$ we have $x \vee y \in B$.

An ideal in a Riesz space E is a Riesz subspace A of E which is solid, meaning that if $x \in A$ and $y \in E$ are such that $|y| \leq |x|$ in E then $y \in A$. A band B in a Riesz space E is an ideal in E with the property that if the supremum of any subset of B exists in E, then it exists in B as well. The smallest band containing an element $x \in E$ is called the principle band generated by x and is denoted by B_x. Principal ideals are defined analogously, the principal ideal generated by an element $u \in E$ is the ideal

$$E_u := \{x \in E \text{ such that } |x| \leq n|u| \text{ for some } n \in \mathbb{N}\}.$$

If an element $e \in E$ is such that the ideal generated by e is the entire space, i.e. if $E_e = E$, then e is called a strong order unit for E. Similarly, if $w \in E$ is such that the band B_w equals the whole space E then w is called a weak order unit for E.

Two Riesz space elements $x, y \in E$ are said to be disjoint if $|x| \wedge |y| = 0$, and we write $x \perp y$. The disjoint complement of a non-empty subset D is the set of elements disjoint to every element in D and is denoted D^d. That is, $D^d = \{x \in E \text{ such that } x \perp y \text{ for all } y \in D\}$.

A projection band in a Riesz space E is a band B in E such that any element u of E can be written uniquely as $u = x + y$ where $x \in B$ and $y \in B^d$, i.e. a projection band $B \subset E$ is a band satisfying $E = B \oplus B^d$. If B is a projection band in E, then the operator P_B which maps $u \in E$ to the corresponding $x \in B$ is called the band projection onto B. The identity operator on a Riesz space E will be denoted by I, meaning $Iu = u$ for every $u \in E$. For a band projection P onto a projection band B, we define $P^d := I - P$ to be the band projection onto B^d, the disjoint complement of B.

An operator T on E is said to be positive if $Tu \geq 0$ whenever $u \in E^+$.

An Archimedean Riesz space is one satisfying the property that if $u, x \in E^+$ and $x \leq \frac{1}{n}u$ for every $n \in \mathbb{N}$ then $x = 0$. A Riesz space E is said to have the principal projection property if every principle band B_x in E is a projection Band, and E is said to have the projection property if every band B in E is a projection band.

A Dedekind complete Riesz space E (also called order complete by some authors) is one in which every non-empty bounded above subset of E has a supremum in E. It follows from Riesz space properties that if E is a Dedekind complete Riesz space then every subset of E that is bounded below has an infimum. A Riesz space E is called Dedekind σ-complete (also called order σ-complete) if every non-empty countable subset of E which is bounded above has a supremum in E. Dedekind complete Riesz spaces are clearly also Dedekind σ-complete, but Dedekind σ-complete spaces are not necessarily Dedekind complete. An important result in the study of Riesz spaces is the main inclusion theorem which can be found, for example, in [71, Theorem 25.1], which connects certain properties concerning Dedekind completeness and band projections. For our study, we mention the following implications for a Riesz space E: if E is Dedekind complete, then it has the projection property; if E has the projection property then it has the principal projection property; and if E has the principal projection property, then E is Archimedean. Hence Dedekind complete spaces have a number of useful properties, and will form the base of our generalised study of stochastic processes. This is justifiable since, in the classical setting of a probability space, the set of measurable functions $L^0(\Omega, \mathcal{F}, \mu)$ is a Dedekind complete Riesz space, see [92, Example 12.5 (iii)] for proof.

A Riesz space E is called laterally complete if every subset consisting of mutually disjoint elements has a supremum in E. If E is both Dedekind complete and laterally complete then it is said to be universally complete.

If we have a Riesz space E, and V is a Riesz subspace of E, then we say V is order dense in E if, for every non-zero $u \in E^+$, there is a $v \in V$ such that $v \leq u$. Such a subspace V is said to be strongly order dense in E if for any $u \in E^+$ we have that $u = sup\{v \in V \text{ such that } 0 \leq v \leq u\}$. For an Archimedean Riesz space E, a Riesz subspace is strongly order dense in E if and only if it is order dense in E [92, Theorem 23.4].

The universal completion E^u of a Riesz space E is the unique space (up to Riesz isomorphism) such that E^u is universally complete and contains E as an order dense subspace, see [71, Definition 50.4]. Every Archimedean Riesz space E can be extended uniquely (up to Riesz isomorphism) to a universally complete space E^u containing E as an order dense subspace, see [71, Theorem 50.8] and [94].

Another completion used in the literature is the sup-completion of a Dedekind complete Riesz space E, denoted E_s, which plays a similar role for E as $\mathbb{R} \cup \{\infty\}$ does for $\mathbb{R}$. More details can be found in [38], [9] and [5]. One property of E_s is that it is a tight cone for E, which means (among other properties) that if $u \in E_s$ and $u \leq v$ for some $v \in E$ then $u \in E$.

A non-empty subset D of a Riesz space E is said to be downwards directed if for every two elements $u, v \in D$, there exists an element in D that is smaller or equal to both, i.e. $\exists h \in D$ such that $h \leq u \wedge v$. Upwards directed subsets of a Riesz space are similarly defined. If a downwards directed D set has infimum d we write $D \downarrow d$ (similarly for a decreasing sequence v_n with infimum v, we write $v_n \downarrow v$).

In this thesis, a sequence x_n in a Riesz space is said to converge in order to x if there exists a sequence $v_n \downarrow 0$ in the same Riesz space such that $|x_n - x| \leq v_n$ for all n. In this case we write $x_n \xrightarrow{o} x$. We note that this type of order convergence is sometimes called 1-convergence or O_1-convergence in the literature (see [1] and references there). For sequences this definition is correct despite the version for nets requiring a less restrictive definition, namely that $x_\alpha \xrightarrow{o} x$ if and only if there is a net $(v_\alpha) \downarrow 0$ in the same Riesz space and an index α_0 such that $|x_\alpha - x| \leq v_\alpha$ for all $\alpha \geq \alpha_0$. A different version of order convergence sometimes used, called 2-convergence, states that a net x_α 2-converges to x if there is a net $(v_\beta)_{\beta \in B} \downarrow 0$ such that for each $\beta \in B$ there is an index $\alpha_0 \in A$ such that $|x_\alpha - x| \leq v_\beta$ for all $\alpha \geq \alpha_0$. The second type of convergence, having two potentially different index sets, holds whenever the first convergence holds. But this second type of order convergence is slightly more general and does not necessarily imply the first, see [1, Example 1.4]. On Dedekind complete Riesz spaces, the two definitions are equivalent, and on Dedekind σ-complete spaces, the sequential versions of the two are equivalent (see [1] and [35, Remark 2.2]).

An operator T on a Riesz space E is said to be order-continuous if for every downwards directed subset D of E having infimum zero (i.e. $D \downarrow 0$ in E), we have $inf\{|Tu| : u \in D\} = 0$. If T is positive, a consequence is that for every decreasing sequence $u_n \downarrow 0$ we have that $Tu_n \downarrow 0$.

2.2.2 Stochastic processes on Riesz spaces

Having laid out some foundational definitions for Riesz spaces, we turn now to preliminaries specific to the study of generalised stochastic processes, starting with generalised conditional expectations.

As defined by Kuo, Labuschagne and Watson in [58, 60], a positive operator T on a Riesz space E having a weak order unit is called a conditional expectation operator if the following properties are present:
1) T is order-continuous,
2) T is a projection (i.e. T is idempotent, $TTu = Tu$ for any $u \in E$),
3) T maps weak order units to weak order units, and
4) The range $\mathcal{R}(T)$ of T is a Dedekind complete Riesz subspace of E.

We say a conditional expectation operator T is strictly positive if $T|f| = 0$ implies that $f = 0$. Some authors include strict positivity of T in the definition of a conditional expectation operator on a Riesz space, such as in [38] and [9]. The preference in this thesis is to leave the strict condition out of the definition, and assume strict positivity only when required, mentioning it explicitly in such cases. In any case, as remarked in [59] and [91], the assumption of strict positivity does not necessarily pose a restriction, since in such cases the quotient space $E/\mathrm{Ker}\{T\}$ can be considered, on which the map induced by T is then a strictly positive conditional expectation.

If a conditional expectation operator is defined on a Riesz space E, then we say that E is T-universally complete if every increasing net (x_α) in E^+ with (Tx_α) bounded above in E^u is order convergent in E. This is the definition used in the context of $L^p(T)$ spaces for example in [9, 65, 70]. In some literature a slightly more inclusive definition is used, with "bounded above in E" replacing "bounded above in E^u" (for example [59, 91]). A brief note on the differences between these notions can be found in [46].

It was shown in [58] that the domain of a conditional expectation operator T on E can always be extended in a natural way to its maximal domain in the universal completion E^u of E. This domain is called the natural domain and is denoted as $\mathcal{L}^1(T)$, or sometimes as $dom(T)$. In particular, $\mathcal{L}^1(T) = D(\tau) - D(\tau)$ where $f \in D(\tau)$ if $0 \leq f \in E^u$ and there exists an upwards directed net f_α in E^+ with order limit f and with the net Tf_α order bounded in E^u; and in this case the value assigned to Tf is the order limit in E^u of the net Tf_α. The natural domain $\mathcal{L}^1(T)$ is T-universally complete, is Dedekind complete having same weak order unit as E, and is an order dense subspace of E^u. Other properties from E carry over to the extended domain in such a way that it does not add any benefit to define T on anything less than its natural domain (see introductory paragraphs of [9]). With the "in E^u" definition of T-universal completeness used in this thesis, it can be shown that a set E is T-universally complete if and only if it is equal the natural domain $\mathcal{L}^1(T)$ (see [65]).

Now we come to concepts specific to stochastic processes on Riesz spaces. A stochastic process on a Riesz space can be defined as a net of elements on a Riesz space. In this thesis, dealing only with countable processes, a stochastic process on a Riesz space is simply a sequence of elements in that space. When the Riesz space is $L^0(\Omega, \mathcal{F}, \mu)$ then stochastic processes are sequences of measurable functions and the theory is reduced to the classical setting.

A filtration on a Riesz space E having a weak order unit is a family of conditional expectation operators $(T_i)_{i\in\mathbb{N}}$ defined on E such that $T_iT_j = T_jT_i = T_i$ for all $i < j$. This implies that the ranges are increasing, i.e. $\mathcal{R}(T_i) \subseteq \mathcal{R}(T_j)$. We say that a sequence of elements $(f_i)_{i\in\mathbb{N}}$ on a Riesz space is adapted to a filtration $(T_i)_{i\in\mathbb{N}}$ if for every $i \in \mathbb{N}$ we have that $f_i \in \mathcal{R}(T_i)$. A conditional expectation operator T on a Riesz space E having weak order unit is said to be compatible with a filtration $(T_i)_{i\in\mathbb{N}}$ on E if $T_iT = TT_i = T$ for all $i \in \mathbb{N}$.

A Riesz space (sub-, super-) martingale is a double sequence $(f_i, T_i)_{i\in\mathbb{N}}$ such that $(f_i)_{i\in\mathbb{N}}$ is adapted to the filtration $(T_i)_{i\in\mathbb{N}}$ and $T_if_j(\geq, \leq) = f_i$ whenever $i < j$.

We mention here a comparison to the setting used by Stoica (for example in [79]). In that setting, taking the place of the above conditional expectation operators are linear order bounded projections from E onto subsets (called "components") $\mathcal{E}$ of E satisfying $E = \mathcal{E} \bigoplus \mathcal{E}^d$. This requirement on the components is equivalent to requiring that they be projection bands. This can be seen, for example, from the fact that a projection, say F, onto a component $\mathcal{E}$ satisfies $0 \leq Fx \leq x$ for any $x \in E^+$, and is therefore a band projection by Theorem 11.4 (iii) in [92]. Hence these projections, if they are not the identity operator, do not map weak order units to weak order units of the space E like the requirement for a conditional expectation in the definition above (if they did, the corresponding projection band would contain a weak order unit for E, and would hence equal the whole space E, meaning that the projection was just the identity operator). In the classical setting of $L^0(\Omega, \mathcal{F}, \mu)$, conditional expectations map constant functions to themselves, and non-zero constant functions are weak order units of $L^0(\Omega, \mathcal{F}, \mu)$. Hence conditional expectations in the classical setting are not band projections, and cannot be seen as a special case of the linear ordered bounded projections use by Stoica.

Definitions of T-strong convergence and convergence in T-conditional probability can be found in [6]. Note that order convergence theorems implies T-strong convergence by the continuity of the conditional expectation operator T, and thereby also implies convergence in T-conditional probability [6, Lemma 5.3]. Hence the strong laws of large numbers in Riesz spaces imply their weak counterparts, and both have the classical laws as special cases, noting that order convergence is equivalent to almost sure convergence in the space of measurable functions, see [6, Remark 2.5] .

Note that on a Dedekind complete Riesz spaces with weak order unit e, we can define a Riesz space multiplication with e as the multiplicative unit. This multiplication shares many of the properties of multiplication in the real-numbers, see [58].

Lastly for the preliminaries, it is noted that a functional calculus can be developed on the Riesz spaces we are looking at. The reader is referred to [9] for the details, whence we can define functions such as $f(x) = x^p$ for arbitrary $p > 0$ on Riesz spaces. This also led naturally to the defining of $\mathcal{L}^p(T)$ spaces in [9] as

$$\mathcal{L}^p(T) = \{x \in \mathcal{L}^1(T) : |x|^p \in \mathcal{L}^1(T)\}.$$

Chapter 3

Sequential Completeness

3.1 Introductory notes

The first novel result in this thesis, a stochastic convergence theorem, is the strong sequential completeness of the natural domain of a conditional expectation operator in Riesz spaces. This is important in the theory of stochastic integrals on Riesz spaces, and in the study of stochastic processes in Riesz spaces as a whole.

This rest of this chapter, coming from our publication [64], starts with its own introduction and preliminaries, which set the question in context, point to relevant research, and state an application to measure spaces with infinite measure. The contribution of the paper starts with Lemma 4 which is actually a version of the Beppo-Levi Theorem (see [15, Theorem 4.30]) for T-universally complete Riesz spaces. This result is known in the Riesz space setting, and has been used by other authors (for example in [87, Theorem 6.2.7]), but we are not aware of it being published in this form before.

The initial Lemma is used to prove strong sequential completeness of $\mathcal{L}^p(T)$. After this, in an easier way, strong sequential completeness of $\mathcal{L}^\infty(T)$ is also proved.

3.2 Introduction

Strong convergence and convergence in probability were generalised in [6] to Dedekind complete Riesz spaces with a conditional expectation operator as T-strong convergence and T-conditional convergence in conditional probability, respectively. Generalised L^p spaces for $p = 1, 2, \infty$ were discussed in the setting of Riesz spaces as $\mathcal{L}^p(T)$ spaces in [70]. More recently, in [9], the $\mathcal{L}^p(T)$ spaces for $p \in (1, \infty)$ were considered. It was discovered in [9] that $R(T)$ is an f-algebra and that $\mathcal{L}^1(T)$ is an $R(T)$-module. An $R(T)$ valued norm, for the cases of $p = 1, \infty$, was introduced on the $\mathcal{L}^p(T)$ spaces in [65] where it was also shown that $R(T)$ is a universally complete f-algebra and that these spaces are $R(T)$-modules. We refer the reader to [86] for an interesting study of sequential order convergence in vector lattices using convergence structures and filters, and to [10] for the well known proof of the strong sequential completeness of $L^1(\Omega, \mathcal{F}, \mu)$.

In this paper we prove the strong sequential completeness of the natural domain, $\mathcal{L}^1(T)$, of the Riesz space conditional expectation operator T, i.e. that each strong Cauchy sequence in $\mathcal{L}^1(T)$ converges strongly in $\mathcal{L}^1(T)$. The term strong here means with respect to the vector valued norm induced by the conditional expectation operator T in the given space. These results can be extended to the convergence of strong Cauchy nets which contain a sequence as a subnet. We conclude by showing the strong completeness of $\mathcal{L}^\infty(T)$, i.e. that every strong Cauchy net in $\mathcal{L}^\infty(T)$ is strongly convergent.

Interest in the completeness studied in this paper came at least in part from [44]. There the Riesz space $\mathcal{L}^1(T)$ is endowed with a locally solid, locally convex linear topology where, for every positive order-continuous linear functional φ on $\mathcal{L}^1(T)$, a semi-norm p_φ, is defined as $p_\varphi(f) := \varphi(T|f|)$. Under the condition that $\mathcal{L}^1(T)$ is a perfect Riesz space, in [44] it was proved $\mathcal{L}^1(T)$, endowed with the topology generated by these semi-norms, is complete. The results of the current work supply a partial answer to whether one can avoid duality theory and the assumption of the Riesz space being perfect to obtain that $\mathcal{L}^1(T)$ is complete with suitable definitions of completeness and convergence.

The issue of completeness of $\mathcal{L}^1(T)$ is important in the theory of stochastic integrals in Riesz spaces, since these integrals are defined to be limits of Cauchy nets in $\mathcal{L}^1(T)$. The results also impact on the study of martingales in Riesz spaces, see [80, 84].

3.3 Preliminaries

For general background on Riesz spaces and order convergence we refer the reader to [1, 71, 93].

A conditional expectation operator, T, on a Dedekind complete Riesz space, E, with weak order unit, say e, is a positive order-continuous projection which maps weak order units to weak order units and has $R(T)$ a Dedekind complete Riesz subspace of E, see [58]. In addition we assume in this paper that T is strictly positive, in that if $v \in E_+$ with $v \neq 0$ then $Tv \neq 0$ ($Tv \geq 0$ as T

is positive). This last condition is required for both the construction of the T-universal completion of E, i.e. the natural domain, $\mathcal{L}^1(T)$, of T and so that the mapping $f \mapsto T|f|$ defines an $R(T)$ valued norm on $\mathcal{L}^1(T)$.

The Riesz space $\mathcal{L}^1(T)$ is defined to be the T-universal completion of E or natural domain of T, see [26] and [58]. We recall that T has a unique extension to $\mathcal{L}^1(T)$ as a conditional expectation operator. In particular $\mathcal{L}^1(T)$ is characterized by the property that if (x_α) is an upward directed net in $\mathcal{L}^1(T)$ with (Tx_α) bounded in E^u (the universal completion), then (Tx_α) is order convergent in $\mathcal{L}^1(T)$.

We recall from [65] that in $\mathcal{L}^1(T)$, $R(T)$ is a universally complete f-algebra and that $\mathcal{L}^1(T)$ is an $R(T)$-module. It thus makes sense, as was done in [65], to define an $R(T)$-valued norm on $\mathcal{L}^1(T)$ by $\|f\|_{T,1} := T|f|$. This norm takes its values in $R(T)^+$, is homogeneous with respect to multiplication by elements of $R(T)^+$, is strictly positive and obeys the triangle inequality. For more details on this norm we refer the reader to [65]. Convergence with respect to this norm was called T-strongly convergence in [6] where various of its properties were studied in relation to other modes of convergence.

The other space that will be of interest in this work is

$$\mathcal{L}^\infty(T) := \{f \in \mathcal{L}^1(T) \,:\, |f| \leq g \text{ for some } g \in R(T)\}$$

with $R(T)$-valued norm

$$\|f\|_{T,\infty} := \inf\{g \in R(T) \,:\, |f| \leq g\}.$$

We refer the reader to [65] for more details and for the readers convenience we give an abbreviated version of the example presented there.

Example: Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, which to be interesting should have $\mu(\Omega) = \infty$ and suppose that there is $(\Omega_n)_{n\in\mathbb{N}}$, an $\mathcal{A}$-measurable partition of Ω into sets of finite positive measure. Let Σ be the sub-σ-algebra of $\mathcal{A}$ generated by $(\Omega_n)_{n\in\mathbb{N}}$. We take as the starting Riesz space $E = L^\infty(\Omega, \mathcal{A}, \mu)$ and the conditional expectation operator $T = \mathbb{E}[\,\cdot\,|\Sigma]$.

For $f \in E$ we have

$$Tf(\omega) = \frac{\int_{\Omega_n} f\, d\mu}{\mu(\Omega_n)}, \quad \text{for} \quad \omega \in \Omega_n.$$

The universal completion, E^u, of E is the space of equivalence classes of $\mathcal{A}$-measurable functions. Here the $\mathcal{A}$-measurable functions f and g are equivalent if $f = g$ a.e. with respect to the measure μ.

The T-universal completion of E is the space

$$\mathcal{L}^1(T) = \left\{ f \in E^u \;\middle|\; \int_{\Omega_n} |f|\, d\mu < \infty \text{ for all } n \in \mathbb{N} \right\},$$

which is characterized by $f|_{\Omega_n} \in L^1(\Omega, \mathcal{A}, \mu)$, for each $n \in \mathbb{N}$.

We note that E has weak order unit $e = 1$, the equivalence class of functions a.e. identically 1 on Ω, which again is a weak order unit for $\mathcal{L}^1(T)$. The range of the generalised conditional expectation operator T (as extended to $\mathcal{L}^1(T)$) is

$$R(T) = \{f \in E^u \mid f \text{ a.e. constant on } \Omega_n, n \in \mathbb{N}\},$$

which is an f-algebra.

Finally

$$\mathcal{L}^\infty(T) = \{f \in E^u \mid f \text{ essentially bounded on } \Omega_n \text{ for each } n \in \mathbb{N}\}.$$

The vector norms on $\mathcal{L}^1(T)$ and $\mathcal{L}^\infty(T)$ are

$$\begin{aligned} \|f\|_{T,1}(\omega) &= T|f|(\omega) = \frac{\int_{\Omega_n} |f|\,d\mu}{\mu(\Omega_n)}, \quad \text{for} \quad \omega \in \Omega_n, f \in \mathcal{L}^1(T), \\ \|f\|_{T,\infty}(\omega) &= \text{ess sup}_{\Omega_n}|f|, \quad \text{for} \quad \omega \in \Omega_n, f \in \mathcal{L}^\infty(T). \end{aligned}$$

The following lemma will assist in the proof of strong sequential completeness.

Lemma 4 *Let (h_n) be a sequence in $\mathcal{L}^1(T)$ with $s := \sum_{n=1}^{\infty} T|h_n|$ order convergent in the universal completion of $\mathcal{L}^1(T)$, then the summation $\sum_{n=1}^{\infty} h_n$ is order convergent in $\mathcal{L}^1(T)$.*

Proof. Let $s_n^\pm = \sum_{i=1}^{n} h_i^\pm$, then the partial sums s_n of $\sum_{n=1}^{\infty} h_n$ are given by $s_n = s_n^+ - s_n^-$. Here $(s_n^\pm)$ are increasing sequences with

$$Ts_n^\pm = \sum_{i=1}^{n} Th_i^\pm \le \sum_{i=1}^{n} T|h_i| \le s.$$

The T-universal completeness of $\mathcal{L}^1(T)$ now allows us to conclude that the sequences $(s_n^\pm)$ are convergent in $\mathcal{L}^1(T)$ to limits, say $h^\pm$. Setting $h = h^+ - h^-$ we have that

$$s_n = s_n^+ - s_n^- \to h^+ - h^- = h \in \mathcal{L}^1(T)$$

in order as $n \to \infty$. ■

Definition 5 *We say that a net (f_α) in $\mathcal{L}^p(T)$, $p = 1, \infty$, is a strong Cauchy net if*

$$v_\alpha := \sup_{\beta,\gamma \ge \alpha} \|f_\beta - f_\gamma\|_{T,p}$$

is eventually defined and has order limit zero.

3.4 Strong sequential completeness of $\mathcal{L}^1(T)$

We now show that $\mathcal{L}^1(T)$ is strongly sequentially complete - i.e. that for every sequence (f_n) in $\mathcal{L}^1(T)$ with $\sup_{i,j\geq n} T|f_i - f_j| \downarrow 0$ there is $f \in \mathcal{L}^1(T)$ so that $T|f_n - f| \to 0$ in order as $n \to \infty$.

Theorem 6 $\mathcal{L}^1(T)$ *is strongly sequentially complete.*

Proof. Let (f_n) be a strong T-Cauchy sequence in $\mathcal{L}^1(T)$. From the definition of a strong Cauchy sequence, we can define

$$v_n := \sup_{r,s\geq n} T|f_r - f_s|$$

where the sequence $(v_n) \subset R(T)$ decreases with infimum zero. As $e, v_n \in R(T)$, it follows that $(\frac{1}{2^j}e - v_n)^+ \in R(T)$ and hence the band projections $P_{j,n} := P_{(\frac{1}{2^j}e - v_n)^+}, j, n \in \mathbb{N}$, commute with T, see [58]. For $n = 0$ define $P_{j,n} = 0$. We observe that $P_{j,n}$ is increasing in n and decreasing in j. In particular, $\lim_{n\to\infty} P_{j,n} = I$, since $v_n \downarrow 0$. Hence $\sum_{n=0}^{\infty}(P_{j,n+1} - P_{j,n}) = I$ for each $j \in \mathbb{N}$.

We now construct a sequence $(g_j) \in \mathcal{L}^1(T)$ that is both asymptotically close to (f_n) and is strongly convergent in $\mathcal{L}^1(T)$. As band projections commute with Riesz space absolute value, we have

$$T|(P_{j,n} - P_{j,n-1})f_{\max\{j,n\}}| = (P_{j,n} - P_{j,n-1})T|f_{\max\{j,n\}}|, \quad n, j \in \mathbb{N}.$$

Here, for $m \neq n$, $(P_{j,n} - P_{j,n-1}) \wedge (P_{j,m} - P_{j,m-1}) = 0$ so

$$\begin{aligned}\sum_{n=1}^{\infty} T|(P_{j,n} - P_{j,n-1})f_{\max\{j,n\}}| &= \sum_{n=1}^{\infty}(P_{j,n} - P_{j,n-1})T|f_{\max\{j,n\}}| \\ &= \sup_{n\in\mathbb{N}}(P_{j,n} - P_{j,n-1})T|f_{\max\{j,n\}}| \\ &=: K \in E^u\end{aligned}$$

exists in the universal completion E^u. Lemma 4 can now be applied to give that the summation

$$g_j = \sum_{n=1}^{\infty}(P_{j,n} - P_{j,n-1})f_{\max\{j,n\}}, \quad j \in \mathbb{N},$$

converges in order in $\mathcal{L}^1(T)$.

We now show that the sequence (g_j) converges in $\mathcal{L}^1(T)$. Consider $T|g_j - g_{j+1}|$. Because $\sum_{n=0}^{\infty}(P_{j,n+1} - P_{j,n}) = I$ for each $j \in \mathbb{N}$, we have that

$$T|g_j - g_{j+1}| \;=\; T\left|\sum_{n=1}^{\infty}(P_{j,n} - P_{j,n-1})f_{\max\{j,n\}} - \sum_{m=1}^{\infty}(P_{j+1,m} - P_{j+1,m-1})f_{\max\{j+1,m\}}\right|$$

$$= T\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(P_{j+1,m}-P_{j+1,m-1})(P_{j,n}-P_{j,n-1})(f_{\max\{j,n\}}-f_{\max\{j+1,m\}})\right|$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(P_{j+1,m}-P_{j+1,m-1})(P_{j,n}-P_{j,n-1})T|f_{\max\{j,n\}}-f_{\max\{j+1,m\}}|.$$

Here we have used that

$$(P_{j+1,m}-P_{j+1,m-1})(P_{j,n}-P_{j,n-1})\wedge(P_{j+1,x}-P_{j+1,x-1})(P_{j,y}-P_{j,y-1})=0$$

for $(m,n)\neq(x,y)$.

For $m\geq n$ we have

$$(P_{j,n}-P_{j,n-1})T|f_{\max\{j,n\}}-f_{\max\{j+1,m\}}| \leq (P_{j,n}-P_{j,n-1})v_n \leq \frac{1}{2^j}(P_{j,n}-P_{j,n-1})e$$

while for $m<n$ we have

$$(P_{j+1,m}-P_{j+1,m-1})T|f_{\max\{j,n\}}-f_{\max\{j+1,m\}}| \leq (P_{j+1,m}-P_{j+1,m-1})v_m \leq \frac{1}{2^{j+1}}(P_{j+1,m}-P_{j+1,m-1})e.$$

Thus

$$T|g_j-g_{j+1}|\leq\frac{1}{2^j}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(P_{j+1,m}-P_{j+1,m-1})(P_{j,n}-P_{j,n-1})e=\frac{1}{2^j}e$$

and the summation $\sum_{j=1}^{\infty}T|g_j-g_{j+1}|$ is e-uniformly (and hence order) convergent. Application of Lemma 4 gives that the summation $\sum_{j=1}^{\infty}(g_j-g_{j+1})$ is order convergent, which is equivalent to the order limit $\lim_{j\to\infty}(g_1-g_{j+1})$ existing. We can thus define g to be the order limit of the sequence (g_j) in $\mathcal{L}^1(T)$. Order-continuity of T now gives that $\lim_{n\to\infty}T|g_n-g|=0$ and that (g_n) converges strongly to g in $\mathcal{L}^1(T)$.

From the order-continuity of T and the order convergence of (g_n) to g we have that $T|g_n-g|\to 0$ in order. Hence to show that g is the strong limit of the (f_n) it suffices to prove that $T|g_n-f_n|\to 0$ in order. As $\sum_{n=0}^{\infty}(P_{j,n+1}-P_{j,n})=I$ we have

$$g_j-f_j = \sum_{n=1}^{\infty}(P_{j,n}-P_{j,n-1})(f_{\max\{j,n\}}-f_j)$$
$$= \sum_{n=1}^{j}(P_{j,n}-P_{j,n-1})(f_j-f_j)+\sum_{n=j+1}^{\infty}(P_{j,n}-P_{j,n-1})(f_n-f_j)$$

$$= \sum_{n=j+1}^{\infty} (P_{j,n} - P_{j,n-1})(f_n - f_j).$$

The order-continuity of T gives

$$\begin{aligned} T|g_j - f_j| &\leq \sum_{n=j+1}^{\infty} (P_{j,n} - P_{j,n-1})T|f_n - f_j| \\ &\leq \sum_{n=j+1}^{\infty} (P_{j,n} - P_{j,n-1})v_j \\ &\leq v_j \end{aligned}$$

and $v_j \downarrow 0$ as $j \to \infty$. Thus $T|f_j - g| \to 0$ in order as $n \to \infty$. ∎

These results extended to the convergence of strong Cauchy nets which contain a sequence as a subnet. More can be said in the case of $p = \infty$, as we see in the following section.

3.5 Strong completeness of $\mathcal{L}^\infty(T)$

For the case of $\mathcal{L}^\infty(T)$ we can prove a stronger result, i.e. that $\mathcal{L}^\infty(T)$ is strongly complete. The proof, unfortunately, cannot be extended to $\mathcal{L}^p(T)$ for $p \in [1, \infty)$.

Theorem 7 *Each strong Cauchy net in $\mathcal{L}^\infty(T)$ is strongly convergent in $\mathcal{L}^\infty(T)$.*

Proof. Let (f_α) be a strong Cauchy net in $\mathcal{L}^\infty(T)$, then eventually

$$v_\alpha := \sup_{\beta,\gamma \geq \alpha} \|f_\beta - f_\gamma\|_{T,\infty} = \inf\{g \in R(T) : |f_\beta - f_\gamma| \leq g \text{ for all } \beta, \gamma \geq \alpha\}$$

exists as an element of $R(T)$ and $v_\alpha \downarrow 0$. Hence eventually $|f_\beta - f_\gamma| \leq v_\alpha$, for $\beta, \gamma \geq \alpha$, and the Cauchy net (f_α) is eventually bounded. We can thus define $\underline{f} := \liminf_\alpha f_\alpha$, $\overline{f} := \limsup_\alpha f_\alpha$ in $\mathcal{L}^\infty(T)$. Now

$$0 \leq \overline{f} - \underline{f} = \lim_\alpha (\sup_{\beta \geq \alpha} f_\beta - \inf_{\gamma \geq \alpha} f_\gamma) = \lim_\alpha \sup_{\beta,\gamma \geq \alpha} (f_\beta - f_\gamma) \leq \lim_\alpha v_\alpha = 0.$$

So $\overline{f} = \underline{f}$ and we can set $f := \overline{f} = \underline{f}$ with (f_α) converging in order to f, see [6]. However $|f_\beta - f_\gamma| \leq v_\alpha$ for all $\beta, \gamma \geq \alpha$, so taking the order limit in the index γ we have $|f_\alpha - f| \leq v_\alpha$ and hence $\|f_\alpha - f\|_{T,\infty} \leq v_\alpha \downarrow 0$. ∎

3.6 Summary

The results of this chapter are important stepping stones in the generalisation of stochastic processes on Riesz spaces. The strong sequential completeness of $\mathcal{L}^1(T)$, and also that of $\mathcal{L}^\infty(T)$ proved here is likely to spark and be followed by proofs for more general $p > 1$. It is hoped that such developments would then contribute to the generalisation of certain theories related to the duality theory of $L^2(\Omega, \mathcal{F}, \mu)$ spaces, weak convergence, relative weak compactness, Komlós' Lemma, and stochastic integration for non-perfect Riesz spaces. Hence the work of this chapter has been a significant contribution to the theory of stochastic processes in vector lattices.

Chapter 4

The Hájek-Rényi-Chow Maximal Inequality

4.1 Introduction

The second main development of the theory of stochastic processes in Riesz spaces produced in this thesis is a generalisation of the Hájek-Rényi-Chow inequality[1], a maximal inequality for submartingales. While this does not depend on the previous result of strong sequential completeness, it does contribute results in the same line, because this maximal inequality is crucial for the development of other convergence results, such as Chow's L^p laws of large numbers for martingale difference sequences. Chow's law of large numbers is proved below in Chapter 6. Proving this law of large numbers is likely to contribute towards generalising Komlós' Lemma and to help formulate and develop the theory of stochastic integration in Riesz spaces.

In the classical sense the Hájek-Rényi-Chow inequality in [16] is a generalisation to semi-martingales (including submartingales) of the well known Hájek-Rényi maximal inequality in [49] for independent random variables. Both have Kolmogorov's maximal inequality and Doob's maximal inequality as special cases (see for example [16, 49] and [30, at (1.4.18)]). The Hájek-Rényi maximal inequality for independent random variables was generalised to the Riesz space setting in Vardy's thesis [87, Chapter 6]. There it was obtained as a special case of Rao's quasi-martingale maximal inequality [73, Lemma 2.1] (which was also generalised to Riesz spaces in [87, Chapter 6]). Discrete and continuous time versions of Doob's maximal inequality were proved in the Riesz space setting by Grobler in [38].

Remark: The classical Hájek-Rényi maximal inequality has two equivalent formulations. Both start with $x_1, ..., x_n$ an independent sequence of random variables with zero mean, $\mathbb{E}(x_k) = 0$ for all k, and finite variance, $\mathbb{E}(x_k^2) < \infty$ for all k. Also, both involve a non-decreasing sequence (a_k) of positive real number weights. The original paper [49] as well as [87, Chapter 6] formulate the inequality as follows:

[1] The terminology *Hájek-Rényi-Chow inequality* can be found in [30, p. 256].

$$\epsilon^2 P\left(\sup_{n\le k\le m}|x_1+x_2+...+x_k|\ge a_k\epsilon\right)\le \frac{1}{a_n^2}\sum_{i=1}^{n}\mathbb{E}(x_i^2)+\sum_{i=n+1}^{m}\mathbb{E}\left(\frac{x_i^2}{a_i^2}\right),\quad (4.1)$$

for all $\epsilon > 0$, whereas Chow's paper states the Hájek-Rényi maximal inequality only with the version where $n = 1$:

$$\epsilon^2 P\left(\sup_{1\le k\le m}\frac{1}{a_k}|x_1+x_2+...+x_k|\ge \epsilon\right)\le \sum_{i=1}^{m}\mathbb{E}\left(\frac{x_i^2}{a_i^2}\right).\quad (4.2)$$

To see that (4.2) implies (4.1), consider the sequence of random variables where the first n terms of the $x_i's$ are grouped together, and take the $n-1$ tail of the real sequence (a_i). I.e. set $y_1 = \sum_{i=1}^{n} x_i$ and $y_k = x_{n+k-1}$ for $k \ge 2$, and set up the real sequence $(b_i) := (a_{n-1+i})$. Note that due to independence and zero mean, $\mathbb{E}(y_1^2) = \sum_{i=1}^{n}\mathbb{E}(x_i^2)$. By applying (4.2), which can be done since the $y_k's$ satisfy the independence, zero mean and finite variance conditions, we have:

$$\epsilon^2 P\left(\sup_{1\le k\le m-n+1}\frac{1}{b_k}|y_1+y_2+...+y_k|\ge \epsilon\right)\le \sum_{i=1}^{m}\frac{1}{b_i^2}\mathbb{E}(y_i^2),$$

which can be rewritten as

$$\begin{aligned}\epsilon^2 P\left(\sup_{n\le k\le m}\frac{1}{a_k}|x_1+x_2+...+x_k|\ge \epsilon\right)&\le \frac{1}{b_1^2}\mathbb{E}(y_1^2)+\sum_{i=2}^{m}\frac{1}{b_i^2}\mathbb{E}(y_i^2)\\ &=\frac{1}{a_n^2}\sum_{i=1}^{n}\mathbb{E}(x_i^2)+\sum_{i=n+1}^{m}\frac{1}{a_i^2}\mathbb{E}(x_i^2)\end{aligned}$$

which is the same formulation as (4.1).

In contrast, the Hájek-Rényi-Chow maximal inequality in the classical setting for submartingales is the following improvement [16]: let $(y_k)_k \in \{1, 2, 3, ...\}$ be a submartingale, let a_i be a non-decreasing real sequence, and let ϵ be any positive real number, then we have

$$\epsilon P\left(\sup_{1\le k\le m}\frac{1}{a_k}y_k\ge \epsilon\right)\le \frac{1}{a_1}\mathbb{E}(y_1^+)+\sum_{i=1}^{m-1}\frac{1}{a_{i+1}}\mathbb{E}(y_{i+1}^+-y_i^+).$$

The Hájek-Rényi maximal inequality is deduced from the Hájek-Rényi-Chow inequality by letting $z_n = \sum_{i=1}^{n} x_i$. Note that z_n^2 is a positive submartingale, and that $\mathbb{E}(z_n^2) = \sum_{i=1}^{n}\mathbb{E}(x_n^2)$ by independence and as the $x_i's$ have zero mean, so $\mathbb{E}(z_{n+1}^2 - z_n^2) = \mathbb{E}(x_{n+1}^2)$. Hence we have

$$\begin{aligned}\epsilon^2 P\left(\sup_{1\le k\le m}\frac{1}{a_k}|x_1+x_2+...+x_k|\ge \epsilon\right)&=\epsilon^2 P\left(\sup_{1\le k\le m}\frac{1}{a_k^2}z_k^2\ge \epsilon^2\right)\\ &\le \frac{1}{a_1^2}\mathbb{E}(z_1^2)+\sum_{i=1}^{m-1}\frac{1}{a_{i+1}^2}\mathbb{E}(z_{i+1}^2-z_i^2)\\ &=\sum_{i=1}^{m}\frac{1}{a_i^2}\mathbb{E}(x_i^2),\end{aligned}$$

which is the Hájek-Rényi inequality of (4.2).

These deductions and implications regarding the relationship between the above inequalities also hold true with the Riesz space versions of these inequalities.

The sequence and ideas for the proofs in this chapter were inspired by subsections (6.1.1) to (6.1.4) in [30].

4.2 A stopping time inequality

Our first result is a generalisation of [30, Lemma (6.1.1)] to vector lattices.

Lemma 8 *Let $(T_i)_{i\in\mathbb{N}}$ be a filtration on a Dedekind complete Riesz space E with weak order unit e. Let $(X_i) \subset E$ be adapted to (T_i) and let $g \in \mathcal{R}(T_1)^+$. For $i \in \mathbb{N}$ define the band projections $P_i := P_{(g-X_i)^+}$ and $Q_i := \prod_{j=1}^{i} P_j$.*

Then for all $n \geq 1$

$$Q_n^d g \leq X_1 + \sum_{i=1}^{n-1} [Q_i(X_{i+1} - X_i)] - Q_n X_n$$

Remark: $Q := (Q_i^d)_{i\in\mathbb{N}}$ is a Riesz Space stopping time (see [60] for the definition of stopping times and stopped processes), and can be thought of as stopping the first time the process (X_i) goes above g. The result of this Lemma is one form of the intuitively obvious statement that if (X_i) does go above g by time n, then at the first time it does so, it is greater or equal to g; i.e. $Q_n^d(X_{Q\wedge n} - g) \geq 0$. In fact the Lemma and proof do not require the filtration, and (X_i) can be any sequence of Riesz space elements, but the filtration is included in the statement to allow the stopping time interpretation. Also note that if $X_n \geq 0$ then this Lemma implies

$$Q_n^d g \leq X_1 + \sum_{i=1}^{n-1} Q_i(X_{i+1} - X_i)$$

Proof. If we define $Q_0 := I$ then

$$Q_n^d = \sum_{i=1}^{n} [Q_i^d - Q_{i-1}^d] = \sum_{i=1}^{n} Q_i^d Q_{i-1} = \sum_{i=1}^{n} P_i^d Q_{i-1}.$$

Also $P_i^d g \leq P_i^d X_i$ for all $i \in \mathbb{N}$ as is evident from

$$P_i(g - X_i) = P_{(g-X_i)^+}(g - X_i) = (g - X_i)^+ \geq 0$$

together with the definition of P_i^d and linearity of the operators.

Hence

$$Q_n^d g = \sum_{i=1}^{n} P_i^d Q_{i-1} g$$

$$\begin{aligned}
&\le \sum_{i=1}^{n} P_i^d Q_{i-1} X_i \\
&= \sum_{i=1}^{n} [Q_i^d - Q_{i-1}^d] X_i \leftarrow (\text{this is } = Q_n^d X_{Q\wedge n}) \\
&= \sum_{i=1}^{n} [Q_{i-1} - Q_i] X_i \\
&= \sum_{i=1}^{n} Q_{i-1} X_i - \sum_{i=1}^{n} Q_i X_i \\
&= X_1 + \sum_{i=2}^{n} Q_{i-1} X_i - \sum_{i=1}^{n} Q_i X_i \\
&= X_1 + \sum_{i=1}^{n-1} Q_i X_{i+1} - \sum_{i=1}^{n-1} Q_i X_i - Q_n X_n \\
&= X_1 + \sum_{i=1}^{n-1} [Q_i (X_{i+1} - X_i)] - Q_n X_n. \ \blacksquare
\end{aligned}$$

By applying T_1 to both sides and taking $g = \lambda e$ for real $\lambda > 0$, the just-proved lemma can also be used to prove the following maximal inequality for a positive supermartingale $(X_i, T_i)_{i\in\mathbb{N}}$, proved also in [48, Lemma 3.1]:

$$\lambda T_1 Q_n^d e \le X_1.$$

Now we can state and prove a vector lattice version of Chow's maximal inequality.

4.3 Chow's maximal inequality

Theorem 9 *Let $(T_i)_{i\in\mathbb{N}}$ be a filtration on a Dedekind complete Riesz space E with weak order unit e and $(Y_i, T_i)_{i\in\mathbb{N}}$ be a submartingale. Let $(a_i)_{i\in\mathbb{N}}$ be a non-decreasing sequence of positive real numbers, and $g \in \mathcal{R}(T_1)^+$, then for each $n \in \mathbb{N}$ we have*

$$T_1 U_n^d g \le \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1 \left[\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}} \right] \tag{4.3}$$

where $U_n := \prod_{i=1}^{n} P_{\left(g - \frac{Y_i}{a_i}\right)^+} = P_{\bigwedge_{i=1}^{n} \left(g - \frac{Y_i}{a_i}\right)^+}.$

Proof. Let $Q = P_g$. Consequently $Qg = g$ and $Q^d g = 0$. As $g \in \mathcal{R}(T_1)^+$ it follows that T_1 and Q commute, see [58, Theorem 3.2]. By linearity, T_1 commutes with Q^d also,

$$T_1 Q^d = T_1(I - Q) = T_1 - T_1 Q = T_1 - Q T_1 = (I - Q) T_1 = Q^d T_1.$$

The right-hand side of (4.3) is greater or equal zero because (Y_i^+, T_i) is a submartingale (see [38, Corollary 4.5]) and hence $T_1(Y_{i+1}^+ - Y_i^+) \geq 0$. Therefore

$$Q^d(T_1U_n^dg) = T_1U_n^dQ^dg = 0 \leq Q^d\left(\frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1\left[\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right]\right). \tag{4.4}$$

Applying Lemma 8 where $X_i := \dfrac{Y_i^+}{a_i}$ for all $i \in \mathbb{N}$ gives, for all $n \in \mathbb{N}$,

$$Q_n^dg \leq \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} Q_i\left[\frac{Y_{i+1}^+}{a_{i+1}} - \frac{Y_i^+}{a_i}\right] \leq \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} Q_i\left[\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right], \tag{4.5}$$

where $Q_i := \prod_{j=1}^{i} P_{\left(g - \frac{Y_j^+}{a_j}\right)^+} = P_{\bigwedge_{j=1}^{i}\left(g - \frac{Y_j^+}{a_j}\right)^+}$.

For $i \leq n$, the generating elements $\left(g - \frac{Y_i^+}{a_i}\right)^+$ are in $\mathcal{R}(T_n)^+$ because $\mathcal{R}(T_n)$ is a Riesz space. Hence for $i \leq n$, $Q_iT_n = T_nQ_i$. Also, $T_1 = T_1T_n$ for $n \geq 1$, hence for all $i \geq 1$

$$T_1Q_i\left(Y_{i+1}^+ - Y_i^+\right) = T_1T_iQ_i\left(Y_{i+1}^+ - Y_i^+\right) = T_1Q_iT_i\left(Y_{i+1}^+ - Y_i^+\right)$$

and since $0 \leq Q_i \leq I$ and $T_i\left(Y_{i+1}^+ - Y_i^+\right) \geq 0$ it follows that

$$T_1Q_i\left(Y_{i+1}^+ - Y_i^+\right) \leq T_1T_i\left(Y_{i+1}^+ - Y_i^+\right) = T_1\left(Y_{i+1}^+ - Y_i^+\right). \tag{4.6}$$

From (4.5) and (4.6) we have

$$T_1Q_n^dg \leq \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1\left[\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right]. \tag{4.7}$$

Since $g \geq 0$, we have for each i that

$$g \wedge \left(\left(g - \frac{Y_i}{a_i}\right) \vee 0\right) = \left(g \wedge \left(g - \frac{Y_i}{a_i}\right)\right) \vee (g \wedge 0) = \left(g - \left(0 \vee \frac{Y_i}{a_i}\right)\right) \vee 0,$$

giving $g \wedge \left(g - \frac{Y_i}{a_i}\right)^+ = \left(g - \frac{Y_i^+}{a_i}\right)^+$, thus $QQ_i = U_i$, and $QQ_i = QQQ_i = QU_i$. Hence for all i we have

$$QQ_i^d = Q(I - Q_i) = Q - QQ_i = Q - QU_i = Q(I - U_i) = QU_i^d.$$

Now, applying Q to (4.7) and recalling that $QT_1 = T_1Q$ we have

$$Q(T_1U_n^dg) \leq Q\left(\frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1\left[\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right]\right). \tag{4.8}$$

Combining (4.4) with (4.8) gives (4.3). ■

We note here, as in [30, p. 257], that if the $a_i = 1$ for all $i \in \mathbb{N}$, and $g = \lambda e$ for a positive real number $\lambda > 0$, then Theorem 9 implies Doob's martingale inequality

$$\lambda T_1 Q_n^d e \leq T_1 \left[X_n^+\right]$$

where $Q_n := \prod_{i=1}^{n} P_{(\lambda e - X_i)^+}$. A continuous time version of Doob's martingale inequality is proved in [38], and a version for quasi-martingales in Riesz spaces can be found in [87, Theorem 6.2.10].

4.4 Summary

We have generalised a submartingale maximal inequality which was developed in the classical setting by Hájek and Rényi, and improved by Chow. The papers of these authors, including the original results, can be found in [16, 49]. As in the classical setting, this inequality is applicable in proving certain laws of large numbers such as Chow's martingale laws of large numbers which are generalised to the Riesz space setting in Chapter 6.

Chapter 5

A Limit Law for Submartingales

5.1 Introduction

In this chapter, a limit law for submartingales is generalised to the Riesz space setting, which is crucial for proving Chow's law of large numbers in the next chapter. In order to prove this submartingale limit law, a few extra results are required. These include a Cesàro type convergence result (not limited in applications to stochastic processes). This Cesàro convergence result is used to prove Kronecker's Lemma, the next result. The proof here takes its inspiration from the classical formulation and proof in Fremlin's Measure Theory, volume 2 [32, 273C on p. 349]. A Band version of a monotone convergence property of measures is also given. Here we deduce that Riesz space Band projections are left-continuous with respect to their generating elements.

As with the Hájek-Rényi-Chow inequality in the previous chapter, the proof of the submartingale limit law in this chapter is based on Section 6.1 of Edgar and Sucheston [30], in particular proposition (6.1.7).

5.2 Archimedean Riesz space convergence results

The first step in proving our submartingale limit law is to generalise Kronecker's Lemma to Riesz spaces, a useful result that holds for all Archimedean Riesz spaces. This generalisation will be done with the help of the following Cesaȓo convergence result.

Lemma 10 *If s_n converges in order to s in an Archimedean Riesz space, E, and if b_n is a non-decreasing sequence of positive real numbers diverging to $+\infty$, then $z_n := \frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1}-b_i)|s_i - s|$ converges to zero.*

Proof. By the order convergence of $(s_n)_{n\in\mathbb{N}}$, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in E such that $|s_n - s| \leq v_n \downarrow 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose $m_n \in \mathbb{N}$ large enough such that $m_n > n$ and $m_n > m_{n-1}$ if $n > 1$, so that $\frac{b_n}{b_{m_n}} < \frac{1}{n}$. We now show that whenever $\mathbb{N} \ni m' \geq m_n$, we have $0 \leq z_{m'} \leq \frac{1}{n}v_1 + v_n$, the right-hand terms here forming a decreasing sequence in n with infimum zero.

Let $m' \geq m_n$, then

$$\begin{aligned} z_{m'} &= \frac{1}{b_{m'}}\sum_{i=1}^{n-1}(b_{i+1}-b_i)|s_i - s| + \frac{1}{b_{m'}}\sum_{i=n}^{m'-1}(b_{i+1}-b_i)|s_i - s| \\ &\leq \frac{1}{b_{m'}}\sum_{i=1}^{n-1}(b_{i+1}-b_i)v_1 + \frac{1}{b_{m'}}\sum_{i=n}^{m'-1}(b_{i+1}-b_i)v_n \\ &= \frac{b_n - b_1}{b_{m'}}v_1 + \frac{b_{m'}-b_n}{b_{m'}}v_n \\ &\leq \frac{b_n}{b_{m'}}v_1 + v_n \\ &\leq \frac{b_n}{b_{m_n}}v_1 + v_n \\ &\leq \frac{1}{n}v_1 + v_n. \end{aligned}$$

Let $(y_n)_{n\in\mathbb{N}} \subset E^+$ with

$$y_i = \begin{cases} 2v_1 & \text{if } 1 \leq i < m_1 \\ \frac{1}{n}v_1 + v_n & \text{if } m_n \leq i < m_{n+1},\ n \in \mathbb{N} \end{cases}$$

then by (5.1) $0 \leq z_n \leq y_n$ for all $n \in \mathbb{N}$. Moreover, $(y_n)_{n\in\mathbb{N}}$ is a decreasing sequence with infimum zero, as E is Archimedean and $v_n \downarrow 0$. Hence $z_n \xrightarrow{o} 0$. ■

Remark: If $(s_n) \xrightarrow{o} s$ in an Archimedean Riesz space E, then the sequence is also Cesàro summable, with Cesàro sum s, i.e.

$$\frac{1}{n}\sum_{i=1}^{n} s_i \xrightarrow{o} s \text{ as } n \to \infty.$$

This can be seen by taking $b_i := i - 1, 1 < i \in \mathbb{N}$, in Lemma 10, which gives the order limit

$$0 = \lim_{n\to\infty} \frac{1}{n-1}\sum_{i=1}^{n-1}|s_i - s| \geq \limsup_{n\to\infty}\left|\frac{1}{n-1}\sum_{i=1}^{n-1} s_i - s\right|.$$

This result, that order convergence is preserved under taking Cesàro sums, has been shown in [35, Lemma 3.14] to hold for 2-convergence in a Dedekind σ-complete Riesz space. The proof there obtains the 2-convergence result by essentially proving the 1-convergence result after using the fact that the two order convergence definitions coincide on a Dedekind σ-complete Riesz space.

Lemma 11 (Kronecker's Lemma for Archimedean Riesz spaces) *Let (x_n) be a summable sequence in an Archimedean Riesz space E. Let $(b_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of positive real numbers diverging to $+\infty$. Then*

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} b_i x_i = 0 \textit{ in order.}$$

Proof. Define $s_n = \sum_{i=1}^n x_i$ and $s = \lim_{n\to\infty} s_n$, which exists by hypothesis. Then

$$\begin{aligned}
\frac{1}{b_n}\left[\sum_{i=1}^n b_i x_i\right] &= \frac{1}{b_n}\left[b_1 s_1 + b_2(s_2 - s_1) + ... + b_n(s_n - s_{n-1})\right] \\
&= \frac{1}{b_n}\left[b_n s_n - (b_2 - b_1)s_1 - (b_3 - b_2)s_2 - ... - (b_n - b_{n-1})s_{n-1}\right] \\
&= \frac{1}{b_n}[b_n s_n - \sum_{i=1}^{n-1}(b_{i+1} - b_i)s_i] \\
&= s_n - \frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1} - b_i)s_i
\end{aligned}$$

(The manipulation in the square brackets above is known as summation by parts and it works whenever the x_n's are from a vector space and the b_n's are from the corresponding field of scalars.) Taking absolute values we have

$$\begin{aligned}
\left|\frac{1}{b_n}\sum_{i=1}^n b_i x_i\right| &= \left|s_n - \frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1} - b_i)s - \frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1} - b_i)(s_i - s)\right| \\
&= \left|s_n - \frac{b_n - b_1}{b_n}s - \frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1} - b_i)(s_i - s)\right| \\
&\leq |s_n - s| + \frac{b_1}{b_n}|s| + \frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1} - b_i)\,|s_i - s|
\end{aligned}$$

The term $|s_n - s|$ converges in order to zero by hypothesis, and the term $\frac{b_1}{b_n}|s|$ converges to zero because E is Archimedean. The third term
$z_n := \frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1} - b_i)|s_i - s|$ converges to zero by Lemma 10. ■

5.3 Two familiar results

In the next theorem we give a submartingale limit law which will be used in the proof of Chow's strong law. We first require two lemmas.

Lemma 12 *[Band projections are left-continuous] Let x_n be a sequence of positive elements in a Riesz Space E with the principal projection property such that $\sup x_n$ exists in E. Then*

$$\bigvee_{i=1}^{\infty} P_{x_i} u = P_{\bigvee_{i=1}^{\infty} x_i} u \tag{5.1}$$

holds for every $u \in E$.

Proof. Due to linearity of the band projection, it is sufficient to show the equality holds when the band projections are applied to positive elements

$z \in E^+$. For any $z \in E^+$, and any principle band projection P_w, it is well known that

$$P_w z = \sup(z \wedge (nw) : n = 1, 2, ...),$$

see for example [92, Theorem 11.5]. Hence

$$\begin{aligned} P_{\bigvee_{i=1}^{\infty} x_i} z &= \sup_n \left(z \wedge n \left(\bigvee_{i=1}^{\infty} x_i \right) \right) \\ &= \sup_n \left(z \wedge \bigvee_{i=1}^{\infty} n x_i \right) \\ &= \sup_n \left(\bigvee_{i=1}^{\infty} (z \wedge n x_i) \right) \quad \text{[92, Theorem 6.1]} \\ &= \bigvee_{i=1}^{\infty} \sup_n (z \wedge n x_i) \\ &= \bigvee_{i=1}^{\infty} P_{x_i} z. \ \blacksquare \end{aligned}$$

Remark: The result of this lemma is known, see [8, Proposition 2.3] and [5, Proposition 5], but the proof is included for completeness.

This lemma is related to right-continuity of the distribution function $F_X(x) := Pr(X \leq x)$. Right-continuity of the distribution function has been shown to hold for a generalised distribution function in [8, Proposition 2.3], where the result of Lemma 12 above is implicitly assumed.

We cannot replace the suprema in (5.1) with infima, but we can get an inequality for infima. If $0 \leq g_n \downarrow g$ then $g_n \geq g \geq 0$ for all $n \in \mathbb{N}$, so $P_{g_n} \geq P_g \geq 0$ for all $n \in \mathbb{N}$, and hence

$$\bigwedge_{n=1}^{\infty} P_{g_n} z \geq P_g z = P_{\bigwedge_{n=1}^{\infty} g_n} z \text{ for all } z \text{ in } E^+. \tag{5.2}$$

The next lemma is a band projection inequality. If an expectation operator is applied, the result can be compared to the inequality

$$Pr\left(\sup_{m \leq i \leq n} Y_i > \nu \right) \leq Pr\left(\sup_{m \leq i \leq n} Y_i \geq \nu \right).$$

Lemma 13 *If Y_i is a positive sequence in E^+ and likewise $\nu \in E^+$, then*

$$P_{((\bigvee_{i=m}^n Y_i) - \nu)^+} = P_{\bigvee_{i=m}^n (Y_i - \nu)^+} \leq I - P_{\bigwedge_{i=m}^n (\nu - Y_i)^+} = P^d_{\bigwedge_{i=m}^n (\nu - Y_i)^+}. \tag{5.3}$$

Proof. To see the equality on the left of (5.3), note that

$$\left(\left(\bigvee_{i=m}^n Y_i \right) - \nu \right)^+ = \left(\bigvee_{i=m}^n (Y_i - \nu) \right) \vee 0 = \bigvee_{i=m}^n ((Y_i - \nu) \vee 0) = \bigvee_{i=m}^n (Y_i - \nu)^+.$$

The inequality in (5.3) can be seen by the fact that the two generating elements $\bigvee_{i=m}^{n}(Y_i-\nu)^+$ and $\bigwedge_{i=m}^{n}(\nu-Y_i)^+$ are disjoint, since

$$\begin{aligned}0 \le \left[\bigvee_{i=m}^{n}(Y_i-\nu)^+\right]\bigwedge\left[\bigwedge_{i=m}^{n}(\nu-Y_i)^+\right] &= \left[\bigvee_{i=m}^{n}(\nu-Y_i)^-\right]\bigwedge\left[\bigwedge_{j=m}^{n}(\nu-Y_j)^+\right]\\ &= \bigvee_{i=m}^{n}\left[(\nu-Y_i)^-\bigwedge\left(\bigwedge_{j=m}^{n}(\nu-Y_j)^+\right)\right]\\ &\le \bigvee_{i=m}^{n}\left[(\nu-Y_i)^-\wedge(\nu-Y_i)^+\right]\\ &= \bigvee_{i=m}^{n}0=0. \blacksquare\end{aligned}$$

5.4 A submartingale limit law

We now have all the tools needed to generalise the desired limit law for submartingales.

Theorem 14 *Let $p \ge 1$ and let $(X_i, T_i)_{i\in\mathbb{N}}$ be a non-negative submartingale, with T_1 strictly positive, in the Dedekind complete Riesz space E with e a weak order unit such that $T_1e=e$. Let $(a_i)_{i\in\mathbb{N}}$ be a positive, non-decreasing, sequence of real numbers diverging to ∞. If*

$$\sum_{i=1}^{\infty}T_1\left(\frac{X_{i+1}^p-X_i^p}{a_{i+1}^p}\right) \text{ exists,} \tag{5.4}$$

then

$$\frac{X_n}{a_n}\xrightarrow{o}0.$$

Proof. Note that the process (X_n^p) is also a non-negative submartingale [38, Corollary 4.5]. Applying Theorem 9 to the non-negative submartingale $(X_m^p, X_{m+1}^p, ...)$ with the corresponding filtration $(T_m, T_{m+1}, ...)$ also starting at index m, we have for any real $\lambda>0$ and index $n>m$ that $\lambda^pe\in\mathcal{R}(T_m)^+$ so

$$\lambda^pT_mP^d_{\bigwedge_{i=m}^{n}(\lambda^pe-\frac{X_i^p}{a_i^p})^+}e = T_mP^d_{\bigwedge_{i=m}^{n}(\lambda^pe-\frac{X_i^p}{a_i^p})^+}\lambda^pe \le \frac{X_m^p}{a_m^p}+\sum_{i=m}^{n-1}T_m\left[\frac{X_{i+1}^p-X_i^p}{a_{i+1}^p}\right]. \tag{5.5}$$

From Lemma 13 with $\nu:=\lambda^pe$ and $Y_i:=\frac{X_i^p}{a_i^p}$ we have

$$P_{\bigvee_{i=m}^{n}(\frac{X_i^p}{a_i^p}-\lambda^pe)^+}\le P^d_{\bigwedge_{i=m}^{n}(\lambda^pe-\frac{X_i^p}{a_i^p})^+}\le I.$$

Hence, by the positivity of the operator T_m and since $\lambda^p > 0$, from (5.5) we get

$$\lambda^p T_m P_{\bigvee_{i=m}^{n}(\frac{X_i^p}{a_i^p}-\lambda^p e)^+} e \leq \frac{X_m^p}{a_m^p} + \sum_{i=m}^{n-1} T_m \left[\frac{X_{i+1}^p - X_i^p}{a_{i+1}^p}\right]. \tag{5.6}$$

Applying the conditional expectation T_1 to both sides of (5.6), noting that $T_1 T_m = T_1$, and taking the order limit as $n \to \infty$ gives

$$\lambda^p T_1 P_{\bigvee_{i=m}^{\infty}(\frac{X_i^p}{a_i^p}-\lambda^p e)^+} e \leq T_1\left[\frac{X_m^p}{a_m^p}\right] + \sum_{i=m}^{\infty} T_1 \left[\frac{X_{i+1}^p - X_i^p}{a_{i+1}^p}\right]. \tag{5.7}$$

The limit could be taken through the conditional expectation on the left because conditional expectations are order-continuous, and the limit could move from outside the band projection to its generating element by Lemma 12.

Taking the limit as $m \to \infty$ on both sides of (5.7) makes the right-hand side zero. The limit of the second term on the right-hand side is zero as seen from the hypothesis (5.4). Applying Lemma 11 to the summable sequence in (5.4) and the increasing sequence of scalars (a_{i+1}^p), we have

$$\frac{1}{a_{m+1}^p} \sum_{i=1}^{m} a_{i+1}^p T_1 \left(\frac{X_{i+1}^p - X_i^p}{a_{i+1}^p}\right) \xrightarrow{o} 0 \text{ as } m \to \infty.$$

This simplifies to

$$T_1\left[\frac{X_{m+1}^p}{a_{m+1}^p}\right] - T_1\left[\frac{X_1^p}{a_{m+1}^p}\right] \xrightarrow{o} 0 \text{ as } m \to \infty,$$

which implies that

$$T_1\left[\frac{X_m^p}{a_m^p}\right] \xrightarrow{o} 0.$$

Since the left-hand side of (5.7) is greater or equal to zero, when we take the limit $m \to \infty$ on both sides of (5.7) we are left with

$$\lambda^p T_1 \left[\lim_{m\to\infty} P_{\bigvee_{i=m}^{\infty}(\frac{X_i^p}{a_i^p}-\lambda^p e)^+} e\right] = 0.$$

Taking $\lambda = \frac{1}{t}$ gives for any real $t > 0$ that

$$T_1 \left[\lim_{m\to\infty} P_{\bigvee_{i=m}^{\infty}(\frac{X_i^p}{a_i^p}-(\frac{1}{t})^p e)^+} e\right] = 0.$$

The strict positivity of T_1 now gives

$$\lim_{m\to\infty} P_{\bigvee_{i=m}^{\infty}(\frac{X_i^p}{a_i^p}-(\frac{1}{t})^p e)^+} e = 0 \text{ for all } t > 0.$$

Since the generating elements are decreasing in m, and bounded below by zero, their infimum exists in E and it follows from equation (5.2) in the remark of Lemma 12 that

$$0 \leq P_{\bigwedge_m \bigvee_{i=m}^{\infty}(\frac{X_i^p}{a_i^p}-(\frac{1}{t})^p e)^+} e \leq \lim_{m\to\infty} P_{\bigvee_{i=m}^{\infty}(\frac{X_i^p}{a_i^p}-(\frac{1}{t})^p e)^+} e = 0 \text{ for all } t > 0,$$

that is,

$$P_{\limsup_i(\frac{X_i^p}{a_i^p}-(\frac{1}{t})^p e)^+} e = 0 \text{ for all } t > 0.$$

Hence the generating element is disjoint from the weak order unit e, and must therefore be zero, so

$$\left(\limsup_i \frac{X_i^p}{a_i^p} - (\frac{1}{t})^p e\right)^+ = \limsup_i \left(\frac{X_i^p}{a_i^p} - (\frac{1}{t})^p e\right)^+ = 0 \text{ for all } t > 0.$$

Taking the limit now as $t \to \infty$ we have

$$\left(\limsup_i \frac{X_i^p}{a_i^p}\right)^+ = 0.$$

Both the submartingale and the sequence a_i are positive, hence

$$\limsup_i \frac{X_i^p}{a_i^p} = 0,$$

and

$$0 \leq \liminf_i \frac{X_i^p}{a_i^p} \leq \limsup_i \frac{X_i^p}{a_i^p} = 0,$$

thus

$$\frac{X_i^p}{a_i^p} \xrightarrow{o} 0 \text{ as } i \to \infty.$$

This implies the existence of a sequence $(v_i) \in E^+$ with $0 \leq \frac{X_i^p}{a_i^p} \leq v_i \downarrow 0$. Define the increasing concave function $f := (\cdot)^{\frac{1}{p}}$ on E^+. Note that $f(v_i) \in E$ for all i, as is well known, and shown in Appendix B for completeness. We then have

$$v_i \downarrow 0 \implies f(v_i) \downarrow 0$$

since the limit of a positive decreasing sequence can commute with an increasing concave function by [9, p. 8] or a simple modification of [38, Theorem 4.6.2]. Then by [38, Theorem 4.6.1] we have

$$0 \leq f(\frac{X_i^p}{a_i^p}) \leq f(v_i) \downarrow 0.$$

Putting this together with the fact from [38, Theorem 3.9] that $f \circ g(\cdot) = f(g(\cdot))$ gives the desired conclusion where g is set to be f's inverse $g(\cdot) := (\cdot)^p$,

$$\frac{X_i}{a_i} = I(\frac{1}{a_i}X_i) = f(g(\frac{1}{a_i}X_i)) = f(\frac{1}{a_i^p}X_i^p) = f(\frac{X_i^p}{a_i^p}) \xrightarrow{o} 0. \blacksquare$$

5.5 Conclusion

In this chapter a limit law for submartingales was given. It will be used in the proof of the strong laws of large numbers for $\mathcal{L}^p(T)$ martingales in the next chapter. Along the way a Cesàro convergence result, Kronecker's Lemma and the left-continuity of band projections were proved.

Chapter 6

Chow's Laws of Large Numbers

6.1 Introduction

In this chapter we will prove Chow's strong laws of large numbers in the Riesz space setting. See [30, Theorems 6.1.8 and 6.1.9] for one classical formulation or [16] and [17] for Chow's original papers.

As noted in the Chapter 2, there are a number of different laws of large numbers, in particular, weak ones and strong ones, ones requiring independence of the random variables, and those requiring a martingale or other special process. Examples of strong laws of large numbers in the classical setting can be found in Fremlin's measure theory, Sections 273 and 276 of [32] as well as Chow's strong laws cited in the previous paragraph. Some progress has been made on the generalisation of such laws to the Riesz space setting, for example, the weak law of large numbers for mixingales in [68], the strong law of large numbers for independent variables in [61], and Bernoulli's law of large numbers in [67]. Then there are also the limit laws proved by Stoica in [79], albeit in a different Riesz space setting to ours.

Chow's laws of large numbers are very general. Indeed, as mentioned in the introductory paragraphs of [16], Chow's laws include, as special cases, extensions of Kolmogorov's law of large numbers made by Brunk, Chung, Kawata and Udagawa, and Prohorov for independent random variables as well as extensions made by Lévy and Loève for dependent random variables. The particular case of $p = 2$ gives a generalisation of a strong law of large numbers originally due to Paul Lévy, see [30, p. 259]. When the martingale is obtained as a sum of independent random variables with zero expectation, the cases $p = 2$ and $p = 1$ give Kolmogorov's SLLN, see [30, p. 259] and [32, Theorem 273D]. Indeed, it was shown in [88, Corollary 5] that the partial sums of a sequence of T-conditionally independent Riesz space elements with zero expectation necessarily give a Riesz space martingale. Hence the results in this chapter for martingale differences can be applied to sequences of T-conditionally independent elements.

In this chapter, all conditional expectations are assumed to be strictly positive. In theorems where $\mathcal{L}^p(T)$ (for some p) is introduced together with a martingale $(X_n)_{n\in\mathbb{N}}$, it should be understood that T is a strictly positive conditional expectation operator that is compatible with the filtration $(T_i)_{i\in\mathbb{N}}$ of the martingale. Strict positivity is required for the construction of the natural domain $\mathcal{L}^1(T)$ and also ensures that Theorem 14 can be applied. In fact, strict positivity of T implies strict positivity of each of the operators in the filtration $(T_i)_{i\in\mathbb{N}}$. Indeed, for any i, if $T_i a = 0$ for some $a \geq 0$ in the Riesz space, then $Ta = TT_i a = T0 = 0$ and by strict positivity of T we have that $a = 0$, establishing strict positivity of T_i.

6.2 The martingale SLLN for $p = 2$

Theorem 15 *Let $(X_n, T_n)_{n\in\mathbb{N}}$ be a martingale with $\mathcal{L}^2(T) \ni X_n =: \sum_{i=1}^n Y_i$ for all $n \in \mathbb{N}$, and $(a_i)_{i\in\mathbb{N}}$ be a positive, non-decreasing sequence of real numbers diverging to ∞. If*

$$\sum_{i=1}^{\infty} T_1\left(\frac{|Y_i|^2}{a_i^2}\right) \text{ converges in } \mathcal{L}^2(T)$$

then

$$\frac{X_n}{a_n} \xrightarrow{o} 0.$$

Proof. From Jensen's inequality, $(|X_i|, T_i)_{i\in\mathbb{N}}$ is a non-negative submartingale [38, Corollary 4.5]. Hence we just need to apply Theorem 14 to the below, keeping in mind that $T_i Y_{i+1} = 0$ because $(X_n, T_n)_{n\in\mathbb{N}}$ is a martingale, and T_i is an averaging operator so elements in $\mathcal{R}(T_i)$ can be taken out from the conditional expectation

$$\begin{aligned}
T_1\left[\frac{X_{i+1}^2 - X_i^2}{a_{i+1}^2}\right] &= T_1\left[\frac{(X_{i+1} + X_i)(X_{i+1} - X_i)}{a_{i+1}^2}\right] \\
&= T_1\left[\frac{(Y_{i+1} + 2X_i)Y_{i+1}}{a_{i+1}^2}\right] \\
&= T_1 T_i\left[\frac{Y_{i+1}^2 + 2X_i Y_{i+1}}{a_{i+1}^2}\right] \\
&= T_1\left[\frac{T_i(Y_{i+1}^2) + 2X_i T_i(Y_{i+1})}{a_{i+1}^2}\right] \\
&= T_1\left[\frac{Y_{i+1}^2}{a_{i+1}^2}\right].
\end{aligned}$$

Hence by the hypothesis

$$\sum_{i=1}^{\infty} T_1\left[\frac{X_{i+1}^2 - X_i^2}{a_{i+1}^2}\right]$$

exists. ■

Note the sufficient condition of the weak law of large numbers in [68, Lemma 4.1] is also a sufficient condition for a strong law. To see this, notice that the process (g_i, T_i) in [68, Lemma 4.1] is an e-uniformly bounded martingale difference sequence, hence $\sum_{i=1}^{\infty} T_1\left(\frac{|g_i|^2}{i^2}\right)$ will converge, thus by Theorem 15 we have order convergence of $\overline{g}_n := \frac{1}{n}\sum_{i=1}^{n} g_i$ as well as T-strong convergence.

6.3 Chow's strong law for $1 < p < 2$

In Theorem 29 of Appendix A, the following inequality is proved for Riesz spaces.

Theorem 16 *For $x, y \in E^u$, we have that*

$$|x+y|^p + |x-y|^p \leq 2(|x|^p + |y|^p) \text{ for all } 1 < p < 2.$$

Using this inequality we can prove the law of large numbers for $1 < p < 2$.

Theorem 17 *Let p be a fixed number, $1 < p < 2$, and let $(X_n, T_n)_{n\in\mathbb{N}}$ be a martingale with $\mathcal{L}^p(T) \ni X_n =: \sum_{i=1}^{n} Y_i$ for all $n \in \mathbb{N}$, and let $(a_i)_{i\in\mathbb{N}}$ be a positive, non-decreasing sequence of real numbers diverging to ∞. If*

$$\sum_{i=1}^{\infty} T_1\left(\frac{|Y_i|^p}{a_i^p}\right) \text{ converges in } \mathcal{L}^1(T) \tag{6.1}$$

then

$$\frac{X_n}{a_n} \xrightarrow{o} 0.$$

Proof. Note first that $|X_i|$ and $|X_i|^p$ are submartingales adapted to $(T_n)_{n\in\mathbb{N}}$ by Corollary 4.5 in [38]. Using the definition of a martingale and Jensen's inequality, Theorem 4.4 of [38], we get

$$\begin{aligned} T_i(|X_i|^p) &= |X_i|^p = |T_i X_i|^p \\ &= |T_i(2X_i - X_{i+1})|^p \\ &\leq T_i(|2X_i - X_{i+1}|^p). \end{aligned}$$

It follows from this, Theorem 16, and the positivity of the conditional expectation operator that

$$\begin{aligned} T_i(|X_{i+1}|^p - |X_i|^p) &\leq T_i(|X_{i+1}|^p - |X_i|^p) + T_i(|2X_i - X_{i+1}|^p) - T_i(|X_i|^p) \\ &= T_i(|X_{i+1}|^p + |2X_i - X_{i+1}|^p) - 2T_i(|X_i|^p) \\ &= T_i(|X_i + Y_{i+1}|^p + |X_i - Y_{+1}|^p) - 2T_i(|X_i|^p) \\ &\leq T_i(2(|X_i|^p + |Y_{i+1}|^p)) - 2T_i(|X_i|^p) \\ &= 2T_i(|Y_{i+1}|^p). \end{aligned}$$

Applying T_1 to both sides and dividing by a_{i+1}^p, it is evident that

$$T_1\left(\frac{|X_{i+1}|^p - |X_i|^p}{a_{i+1}^p}\right) \leq 2T_1\left(\frac{|Y_{i+1}|^p}{a_{i+1}^p}\right). \tag{6.2}$$

From the hypothesis (6.1) it now follows that we can sum both sides of (6.2) for $i = 1$ to ∞ to give that

$$\sum_{i=1}^{\infty} T_1\left(\frac{|X_{i+1}|^p - |X_i|^p}{a_{i+1}^p}\right)$$

converges. Since $|X_i|$ is a non-negative submartingale the proof is completed by application of Theorem 14. ■

6.4 Preliminaries for $p > 2$, including Hölder's inequality for sums

Chow's SLLN for $p > 2$ will make use of Riesz space versions of Burkholder's inequality, see [7, Theorem 16], and Hölder's inequality for sums.

Using notation as in [7], the Riesz space p-norm is defined as $N_p(x) := T(|x|^p)^{\frac{1}{p}}$, and if $(X_n)_{n\in\mathbb{N}}$ is a martingale with $X_n = \sum_{i=1}^n Y_i$ for all $n \in \mathbb{N}$, then its quadratic variation process S_n is defined by $S_n := \sum_{i=1}^n Y_i^2$.

Theorem 18 (Burkholder's inequality) *For every $p \in (1, \infty)$, there exist constants c_p and C_p such that*

$$C_p N_p(X_n) \leq N_p(S_n^{\frac{1}{2}}) \leq c_p N_p(X_n)$$

for all positive martingales $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{L}^p(T)$ with quadratic variation $(S_k)_{k\in\mathbb{N}}$.

In [38] and [9], the following version of Hölder's inequality for conditional expectation operators is given

Theorem 19 (Hölder's inequality for conditional expectations) *Let $\mathbb{F}$ be a conditional expectation with natural domain $\mathcal{L}^1(\mathbb{F})$ and $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in \mathcal{L}^p(\mathbb{F})$ and $y \in \mathcal{L}^q(\mathbb{F})$, then $xy \in \mathcal{L}^1(\mathbb{F})$ and*

$$\mathbb{F}|xy| \leq (\mathbb{F}|x|^p)^{\frac{1}{p}} (\mathbb{F}|y|^q)^{\frac{1}{q}}.$$

An application of this result gives Hölder's inequality for sums.

Theorem 20 (Hölder's inequality for sums) *Let T be a conditional expectation with natural domain $\mathcal{L}^1(T)$ and $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $n \in \mathbb{N}$. If $x_i \in \mathcal{L}^p(T)$ and $y_i \in \mathcal{L}^q(T)$ for all $i \in \{1, 2, ..., n\}$, then $x_i y_i \in \mathcal{L}^1(T)$ for each i, and*

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}}.$$

Proof. The existence of these terms in $\mathcal{L}^1(T)$ is shown in Appendix B.

Consider $\underline{x} := (x_1, ..., x_n)$ in the n-fold Cartesian product $(\mathcal{L}^p(T))^n$ and $\underline{y} := (y_1, ..., y_n) \in (\mathcal{L}^q(T))^n$. Both of these Cartesian products are contained in the Dedekind complete Riesz space Cartesian product $(\mathcal{L}^1(T))^n$ where multiplication and ordering are defined component-wise. The application functions will also be done component wise. Then by Theorem 19 with $\mathbb{F} = T$ (or Appendix B) we know $\underline{xy} \in (\mathcal{L}^1(T))^n$. Then define the operator

$$\mathbb{F} : \left(\mathcal{L}^1(T)\right)^n \longrightarrow \left(\mathcal{L}^1(T)\right)^n \text{ by } \mathbb{F}\underline{a} := \left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)_{i=1}^{n}$$

for $\underline{a} = (a_1, ..., a_n) \in (\mathcal{L}^1(T))^n$, i.e. $\mathbb{F}$ maps an n-tuple $\underline{a}$ to the n-tuple where each of the components are equal the average of the a_i's. It is evident that $\mathbb{F}$ is a conditional expectation operator on $(\mathcal{L}^1(T))^n$, being a positive linear order-continuous projection mapping weak order units to weak order units, and having Dedekind complete range

$$\{(y_1, ..., y_n) \in \left(\mathcal{L}^1(T)\right)^n | y_1 = ... = y_n \in \mathcal{L}^1(T)\}$$

isomorphic to $\mathcal{L}^1(T)$. Hence we can apply Theorem 19 which gives

$$\mathbb{F}|x_i y_i| \le (\mathbb{F}|x|^p)^{\frac{1}{p}} (\mathbb{F}|y|^q)^{\frac{1}{q}},$$

and considering any of the identical components of the result on both sides of the inequality gives

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} |x_i y_i| &\le \left(\frac{1}{n}\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\frac{1}{n}\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{n}\right)^{\frac{1}{p}+\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \end{aligned}$$

and since $\frac{1}{p} + \frac{1}{q} = 1$, multiplying both sides by n completes the proof. ■

6.5 Chow's strong law for $p > 2$

For $p > 2$, we will prove the simpler version of Chow's strong law by taking $a_i = i$ for all $i \in \mathbb{N}$. A result for more general choice of a_i, as in Chow's original papers [16, 17], can probably also be proved in the Riesz space setting.

Theorem 21 *Let $p > 2$ be a fixed number and let $(X_n, T_n)_{n\in\mathbb{N}}$ be a martingale with $\mathcal{L}^p(T) \ni X_n =: \sum_{i=1}^{n} Y_i$ for all $n \in \mathbb{N}$. If*

$$\sum_{i=1}^{\infty} T_1\left(\frac{|Y_i|^p}{i^{1+\frac{p}{2}}}\right) \text{ converges in } \mathcal{L}^1(T) \tag{6.3}$$

then

$$\frac{X_n}{n} \xrightarrow{o} 0.$$

Proof. By Theorem 14 it will suffice to prove the convergence as $n \to \infty$ of

$$Z_n := \sum_{i=2}^{n} T_1\left(\frac{|X_i|^p - |X_{i-1}|^p}{i^p}\right) = \sum_{i=2}^{n} \frac{T_1(|\sum_{k=1}^{i} Y_k|^p) - T_1(|\sum_{k=1}^{i-1} Y_k|^p)}{i^p}. \tag{6.4}$$

A rearrangement gives

$$Z_n = \frac{-T_1(|X_1|^p)}{2^p} + \sum_{i=2}^{n-1}\left(\frac{1}{i^p} - \frac{1}{(i+1)^p}\right) T_1(|X_i|^p) + \frac{T_1(|X_n|^p)}{n^p}. \tag{6.5}$$

Hence it is sufficient to prove that the middle term converges and that the rightmost term converges to zero. To do this, we will use the Hölder and Burkholder inequalities to get an upper bound for $T_1(|X_i|^p) = T_1(|\sum_{k=1}^{i} Y_k|^p)$.

Applying Hölder's inequality, Theorem 20, with $x_i = e$ and $y_i = Y_i^2$ for all $i \in \{1, 2, ..., n\}$, and with exponents of x_i and y_i on the right-hand side as $\frac{p}{p-2}$ and $\frac{p}{2}$ respectively, we get

$$S_n = \sum_{i=1}^{n} Y_i^2 \leq \left(\sum_{i=1}^{n} e^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \left(\sum_{i=1}^{n} (Y_i^2)^{\frac{p}{2}}\right)^{\frac{2}{p}} = n^{1-\frac{2}{p}} \left(\sum_{i=1}^{n} |Y_i|^p\right)^{\frac{2}{p}}.$$

Then by taking both sides to the power $\frac{1}{2}$ and applying N_p, the left-hand side of Burkholder's inequality, Theorem 18, gives, for some constant $C_p \in \mathbb{R}^+$, that

$$C_p N_p(X_n) \leq N_p(S_n^{\frac{1}{2}}) \leq N_p\left(n^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{i=1}^{n} |Y_i|^p\right)^{\frac{1}{p}}\right).$$

Raising to the power p, dividing by C_p^p, setting $K = C_p^{-p}$, and rewriting gives the bound

$$T_1(|X_n|^p) \leq K T_1\left(n^{\frac{p}{2}-1}\left(\sum_{i=1}^{n} |Y_i|^p\right)\right). \tag{6.6}$$

Applying this to the rightmost term of (6.5) we have

$$\frac{T_1(|X_n|^p)}{n^p} \leq K n^{\frac{-p}{2}-1} T_1\left(\sum_{i=1}^{n} |Y_i|^p\right). \tag{6.7}$$

From condition (6.3) of this Theorem, and Kronecker's Lemma, Theorem 11, taking $b_i := i^{1+\frac{p}{2}}$, we have that

$$\lim_{n\to\infty} \frac{1}{n^{\frac{p}{2}+1}} T_1\left(\sum_{i=1}^{n} |Y_i|^p\right) = 0$$

and hence from (6.7) the rightmost term of (6.5) converges to 0.

The proof will be completed by showing the middle term of (6.5) converges. For this we need two inequalities for real numbers, the first is

$$\frac{1}{n^p} - \frac{1}{(n+1)^p} \leq \frac{p}{n^{p+1}}. \tag{6.8}$$

To prove it, note that $f(x) := x^p$ is convex, and hence the gradient of the tangent of f at $(n+1, f(n+1))$ will be greater than the gradient of the chord joining $(n, f(n))$ to $(n+1, f(n+1))$. Hence $f'(n+1) \geq f(n+1) - f(n)$ for any convex function f, and in this case $p(n+1)^{p-1} \geq (n+1)^p - n^p$. Dividing both sides by $n^p(n+1)^p$ gives

$$\frac{p}{n^p(n+1)} \geq \frac{1}{n^p} - \frac{1}{(n+1)^p}$$

and (6.8) is proved by noting that $\dfrac{p}{n^{p+1}} \geq \dfrac{p}{n^p(n+1)}$.

The second inequality is

$$\sum_{n=i}^{\infty} n^{-\frac{p}{2}-2} \leq \left(\frac{p2^{\frac{p}{2}+2}}{p+2}\right) i^{\frac{-p}{2}-1} = Ai^{\frac{-p}{2}-1} \tag{6.9}$$

where A is understood to be some constant depending on p. To prove it, let $i \geq 1$ in $\mathbb{N}$, then

$$\begin{aligned}
\sum_{n=i}^{\infty} n^{-\frac{p}{2}-2} &= \sum_{n=i}^{\infty} \left(\frac{1}{n}\right)^{\frac{p}{2}+2} \\
&\leq \sum_{n=i}^{\infty} \left(\frac{2}{n+1}\right)^{\frac{p}{2}+2} \quad \text{since } \left(\frac{2}{n+1}\right) \Big/ \left(\frac{1}{n}\right) = \frac{2n}{n+1} \geq 1 \text{ for } n \geq 1 \\
&= 2^{\frac{p}{2}+2} \sum_{n=i+1}^{\infty} \left(\frac{1}{n}\right)^{\frac{p}{2}+2} \\
&\leq 2^{\frac{p}{2}+2} \int_i^{\infty} \left(\frac{1}{x}\right)^{\frac{p}{2}+2} dx \\
&= 2^{\frac{p}{2}+2} \left[\frac{1}{-\frac{p}{2}-1} x^{-\frac{p}{2}-1}\right]_i^{\infty} \\
&= 2^{\frac{p}{2}+2} \left(\frac{2}{p+2}\right) i^{-\frac{p}{2}-1}.
\end{aligned}$$

Using inequalities (6.6), (6.8) and (6.9) we get

$$
\begin{aligned}
\sum_{i=2}^{n-1}\left(\frac{1}{i^p}-\frac{1}{(i+1)^p}\right)T_1(|X_i|^p) &\le \sum_{i=2}^{n-1}\left(\frac{1}{i^p}-\frac{1}{(i+1)^p}\right)Ki^{\frac{p}{2}-1}T_1\left(\sum_{k=1}^{i}|Y_k|^p\right)\\
&\le \sum_{i=2}^{n-1}\left(\frac{p}{i^{p+1}}\right)Ki^{\frac{p}{2}-1}T_1\left(\sum_{k=1}^{i}|Y_k|^p\right)\\
&\le KpT_1\left(\sum_{i=1}^{n-1}i^{-\frac{p}{2}-2}\sum_{k=1}^{i}|Y_k|^p\right)\\
&= KpT_1\left(\sum_{k=1}^{n-1}|Y_k|^p\sum_{i=k}^{n-1}i^{-\frac{p}{2}-2}\right)\\
&\le KApT_1\left(\sum_{k=1}^{n-1}|Y_k|^p k^{\frac{-p}{2}-1}\right)\\
&\le KAp\sum_{k=1}^{\infty}T_1\left(\frac{|Y_k|^p}{k^{1+\frac{p}{2}}}\right).
\end{aligned}
\tag{6.10}
$$

The final term in (6.10) exists by (6.3) of the Theorem, and therefore $\sum_{i=2}^{n-1}\left(\frac{1}{i^p}-\frac{1}{(i+1)^p}\right)T_1(|X_i|^p)$ converges as $n\to\infty$. Hence (Z_n) in (6.4) converges. Application of Theorem 14 then gives absolute convergence, implying convergence. ■

Chapter 7

Concluding Chapter

A number of results have been contributed to the literature of stochastic processes on vector lattices as introduced by Kuo, Labuschagne and Watson in [58, 60].

The completeness result in Chapter 3 is the first step towards the development of concepts such as weak convergence and relative weak compactness in the Riesz space setting, without requiring a perfect Riesz space. These would be important to the study of stochastic integration, as in [44], where there is currently a requirement of a perfect Riesz space. It is hoped that this work forms a foundation for the development of an Arzelà-Ascoli Theorem, which may be useful in generalising Komlós' Lemma. Chapters 4, 5 and 6 culminate in Chow's strong laws of large numbers for martingale difference sequences in Riesz spaces. Kolmogorov's and Lévy's laws of large numbers are special cases. The convergence in order implies both T-strong convergence and convergence in T-conditional probability.

On the topic of completeness, the following questions have not yet been answered in literature: is $\mathcal{L}^2(T)$ strongly sequentially complete? Does the strong sequential completeness hold also for arbitrary $p \geq 1$? What are the implications of such results to the theory of stochastic integration on Riesz spaces?

Appendix A

Functional Calculus and Clarkson's Inequality

To prove Chow's strong law of large numbers for $1 < p < 2$ we need the following inequality to be lifted from the real numbers to the Riesz space setting.

$$|a+b|^p + |a-b|^p \leq 2(|a|^p + |b|^p) \text{ for all } 1 < p < 2. \tag{A.1}$$

A proof of this inequality for real numbers can be found in [30, p. 260]. This inequality is one of Clarkson's inequalities, see [19, Theorem 2] for the original paper and see [34] for a simpler proof.

To do this we will make use of functional calculus. There are two approaches to the functional calculus in which terms like those in this inequality can be defined, they are the Daniell integral version and the Riesz homomorphism version. A function such as $f(\cdot) := |\cdot|^p$ will be written as f^D for the Daniell integral version, and as f^H for the Riesz homomorphism version, following the notation in [9]. The Daniell integral is what was defined and used in [38] for proving Jensen's inequality (and other results), but is only defined for functions of a single variable. The Riesz homomorphism version is applicable in lifting real-valued functions on $\mathbb{R}^N$ to Riesz spaces. To define and apply the Riesz homomorphism version of the functional calculus, we will need to apply theorems in [14] which involve certain concepts detailed below. A real-valued Riesz homomorphism ω is a linear operator from a Riesz space E into $\mathbb{R}$ such that for all $a_1, a_2 \in E$ we have

$$\omega(a_1 \vee a_2) = \omega(a_1) \vee \omega(a_2) \text{ and } \omega(a_1 \wedge a_2) = \omega(a_1) \wedge \omega(a_2).$$

If the Riesz space E is also an algebra, we can define a multiplicative Riesz homomorphism as a Riesz homomorphism which additionally preserves the algebra's multiplication, i.e.

$$\omega(a_1 a_2) = \omega(a_1)\omega(a_2).$$

An f-algebra (see for example [93, definition 140.8]) is defined as a Riesz space E in which there exists a multiplication which is associative and distributive and where $a_1 a_2 \geq 0$ for all $a_1, a_2 \in E^+$ and having the property that

$$a_1 \wedge a_2 = 0 \implies (a_1 w) \wedge a_2 = (w a_1) \wedge a_2 = 0 \text{ for all } w \in E^+.$$

For an f-algebra E to be semiprime means that $w^2 = 0$ in E implies that $w = 0$, and it can be shown that any Archimedean f-algebra which has a multiplicative unit is semiprime [93, p. 670 and Theorem 142.5]. As mentioned in [58], both the universal completion $\mathcal{L}^1(T)^u$ and the ideal $\mathcal{L}^1(T)_e$ are f-algebras (where T is any strictly positive conditional expectation operator), and since they are Archimedean with multiplicative unit e, these spaces are also semiprime. Another property applicable to theorems in [14] is uniform completeness. For our current purposes is it sufficient to note that any Dedekind σ-complete Riesz space is uniformly complete [71, observation after Theorem 42.2] . The following functional calculus definition is given in [14, Definition 4.2].

Definition 22 *Let $a_1, ...a_N$ be elements in an Archimedean semiprime f-algebra E, and let g be a real-valued continuous function on $\mathbb{R}^N$ with $g(\mathbf{0}) = 0$. If $b \in E$ is such that $g(\omega(a_1), ..., \omega(a_N)) = \omega(b)$ for all multiplicative Riesz homomorphisms ω on the f-subalgebra of E generated by $\{a_1, ...a_N, b\}$, then we can write $g^H(a_1, ..., a_N) = b$.*

In [14, Lemma 4.4(iii)], it was shown that $g^H(a_1, ..., a_N) = b$ implies $g(\omega(a_1), ..., \omega(a_N)) = \omega(b)$ for all multiplicative Riesz homomorphisms ω on any f-subalgebra of E containing $a_1, ...a_N, b$. Moreover, a sufficient condition for $g^H(a_1, ..., a_N) = b$ is if $g(\omega(a_1), ..., \omega(a_N)) = \omega(b)$ for all multiplicative Riesz homomorphisms ω on the f-subalgebra of E generated by any countable subset $L \supseteq \{a_1, ...a_N, b\}$ (this follows from [14, Lemma 4.4(ii)] together with [14, corollary 2.7]).

The Riesz homomorphism version in Definition 22 and the Daniell integral version used in [38] have been shown in [9] to be equivalent for functions of a single variable on the ideal E_e generated by e, where E is any Dedekind complete Riesz space with weak order unit $e > 0$. To lift inequality (A.1) to Riesz spaces we will start from the Riesz homomorphism version and by that show the Daniell integral version of (A.1) also holds on $\mathcal{L}^1(T)_e$. To finish the job we will prove using limit arguments the validity of the Daniell integral version on the whole universal completion $\mathcal{L}^1(T)^u$ of $\mathcal{L}^1(T)$.

Proposition 23 *Inequality (A.1) holds in the Riesz homomorphism functional calculus for all a, b in the universal completion $\mathcal{L}^1(T)^u$.*

Proof. For a given p with $1 < p < 2$, consider the two real-valued functions on R^2:

$$g(x, y) := |x + y|^p + |x - y|^p, \text{ and } h(x, y) := 2(|x|^p + |y|^p). \tag{A.2}$$

Both of these functions are continuous functions of polynomial growth, both being dominated above by $2(|x| + |y|)^p$. Furthermore the universal completion $\mathcal{L}^1(T)^u$ is an f-algebra with unit e, and is Dedekind complete and hence uniformly complete. Therefore the conditions of Theorem 4.12 in [14] are met, so that for any $a, b \in \mathcal{L}^1(T)^u$ the functional calculus

$$\Psi(\cdot) : f \mapsto f(a, b) \in \mathcal{L}^1(T)^u$$

is a Riesz homomorphism, and hence positive. The inequality $g(x,y) \le h(x,y)$ holds for $x, y \in \mathbb{R}$ ([30, p. 260]) and the positiveness of Ψ implies that for all $a, b \in \mathcal{L}^1(T)^u$:

$$\begin{aligned} &\Psi(g) \le \Psi(h) \\ \implies &g^H(a,b) \le h^H(a,b) \end{aligned} \tag{A.3}$$

completing the proof. ■

Now, since the Daniell integral is only defined for functions of a single variable, we will use the following two simple lemmas to process the above before getting a Daniell integral version of inequality (A.1).

Lemma 24 *Let E be a Dedekind complete Riesz space that is also an f-algebra. If $f_1 = f_2 + f_3$ for f_1, f_2, f_3 real valued functions on $\mathbb{R}^2$, then if $f_2^H(u,v)$ and $f_3^H(u,v)$ exist in the Riesz space for some $u, v \in E$, we have that $f_1^H(u,v) = f_2^H(u,v) + f_3^H(u,v)$.*

Proof. Let $b_i = f_i^H(u,v)$ for $i = 2, 3$, and let ω be a multiplicative real-valued Riesz homomorphism on the f-subalgebra generated by the set $\{u, v, b_2, b_3, b_2 + b_3\}$. Then

$$\begin{aligned} f_1(\omega(u), \omega(v)) &= f_2(\omega(u), \omega(v)) + f_3(\omega(u), \omega(v)) \\ &= \omega(b_2) + \omega(b_3) \\ &= \omega(b_2 + b_3) \end{aligned}$$

so

$$\begin{aligned} f_1^H(u,v) &= (b_2 + b_3) \\ &= f_2^H(u,v) + f_3^H(u,v). \ \blacksquare \end{aligned}$$

Lemma 25 *Let E be a Dedekind complete Riesz space that is also an f-algebra, and consider real-valued functions f_1, f_3 defined on $\mathbb{R}^2$ and real-valued function f_2 defined on $\mathbb{R}$. If $f_1 = f_2 \circ f_3$, then when $f_3^H(u,v)$ and $f_2^H(f_3^H(u,v))$ exist in E for some $u, v \in E$, we have $f_1^H(u,v) = f_2^H(f_3^H(u,v))$.*

Proof. Let $b_3 = f_3^H(u,v)$ and let $b_2 = f_2^H(f_3^H(u,v)) = f_2^H(b_3)$. Then let ω be any multiplicative real-valued Riesz homomorphism on the f-subalgebra generated by the set $\{u, v, b_2, b_3\}$. Then we have for all u, v in the Riesz space that:

$$\begin{aligned} f_1(\omega(u), \omega(v)) &= f_2(f_3(\omega(u), \omega(v))) \\ &= f_2(\omega(f_3^H(u,v))) \\ &= f_2(\omega(b_3)) \\ &= \omega(f_2^H(b_3)) \\ &= \omega(b_2). \end{aligned}$$

This implies that

$$f_1^H(u,v) = b_2 = f_2^H(f_3^H(u,v))$$

as required. ■

Remark: Lemma 25 implies that for the homomorphism version of the functional calculus, powers of a Riesz space element w can be combined in the same way as for real numbers. For example for $x \in \mathbb{R}$, let $f_a(x) = x^a$, $f_b(x) = x^b$, $h(x) = f_b(f_a(x))$ and $\times(x,y) = xy$. Then $h(x) = (x^a)^b = x^{ab}$ so in a Riesz space,

$$w^{ab} = h^H(w) = f_b^H(f_a^H(w)) = (w^a)^b.$$

Similarly, for real numbers, $\times(f_a(x), f_b(x)) = x^a x^b = x^{a+b}$, so also in a Riesz space,

$$x^a x^b = \times^H(f_a^H(x), f_b^H(x)) = x^{a+b}.$$

If the exponent is 1, the result is simply the identity function, $f_1(x) = x^1 = x$ so in a Riesz space $x^1 = f_1^H(x) = x$. For an exponent of zero, we have for any strictly positive or negative number x that $x^0 = 1$, the multiplicative unit, so in a Riesz space for a strictly positive or negative element y we have $y^0 = e$. Also, as with real numbers, in a Riesz space, $(xy)^a = x^a y^a$.

Now we can process the right- and left-hand sides of (A.3) to be in terms of simple operations and functions of a single variable. Let g be as in (A.2), then defining $g_1(x,y) = |x+y|^p$ and $g_2(x,y) = |x-y|^p$, we have

$$g(x,y) = g_1(x,y) + g_2(x,y)$$

and hence, by Lemma 24,

$$g^H(x,y) = g_1^H(x,y) + g_2^H(x,y).$$

Now, define $+(x,y) = x+y$ and $-(x,y) = x-y$ (keeping in mind that the Riesz space function $+^H(x,y) = x+y$ as well, and likewise $-^H(x,y) = x-y$). Then define

$$f(x) = |x|^p.$$

We have that $g_1(x,y) = f(+(x,y))$ and $g_2(x,y) = f(-(x,y))$ which implies by Lemma 25 that

$$g_1^H(x,y) = f^H(+^H(x,y)) = f^H(x+y)$$

and

$$g_2^H(x,y) = f^H(-^H(x,y)) = f^H(x-y)$$

with the result that we can say, whenever all the terms are defined, that

$$g^H(x,y) = f^H(x+y) + f^H(x-y). \tag{A.4}$$

Similarly, considering the function $d(x)$ defined by $d(x) = 2x$ (realizing that $d^H(x) = 2x$) and using Lemmas 24 and 25, we have for the function $h(x,y)$ in (A.2) that

$$h^H(x,y) = 2(f^H(x) + f^H(y)). \tag{A.5}$$

Hence, combining inequality (A.3) with (A.4) and (A.5) we have, for $f(\cdot) = |\cdot|^p$, whenever the below terms exist, that

$$f^H(x+y) + f^H(x-y) \leq 2(f^H(x) + f^H(y))$$

for all $x, y \in \mathcal{L}^1(T)^u$, and for any real p in the range $1 < p < 2$.

We are now in a position to state a Daniell integral version of the above inequality. By Lemma 2.3 of [9] and the comments after [9, Lemma 2.7], we know for all elements in $\mathcal{L}^1(T)_e$ (the ideal generated by the unit e) that $f^D(\cdot)$ and $f^H(\cdot)$ coincide. Hence for all $x, y \in \mathcal{L}^1(T)_e$, keeping in mind that $x+y$ and $x-y$ will then also be in $\mathcal{L}^1(T)_e$, we have the inequality:

$$f^D(x+y) + f^D(x-y) \leq 2(f^D(x) + f^D(y)).$$

So with the Daniell integral version of the functional calculus we can say for all $x, y \in E_e$ that whenever the terms defined, we have

$$|x+y|^p + |x-y|^p \leq 2(|x|^p + |y|^p) \text{ for all } 1 < p < 2. \tag{A.6}$$

For the rest of this paper all use of functions lifted from $\mathbb{R}$ will be considered to be by the Daniell integral version of the functional calculus.

The next job is to prove that inequality (A.6) holds for arbitrary x, y in $\mathcal{L}^1(T)^u$, and hence in $\mathcal{L}^p(T)$ also. This can be done by the below six steps. Step 1 and step 5 will be shown in more detail after the list. Here and below we let $E := \mathcal{L}^1(T)$ for brevity of writing:

1. Show the inequality holds for $x \in E^{u+}, y \in E_e^+$.

2. By swapping x and y show it holds for $x \in E_e^+, y \in E^{u+}$. This is so because the quantities on the left and right are unchanged by swapping the inputs, i.e. by defining $LHS = |x+y|^p + |x-y|^p =: f(x,y)$ and $RHS = 2(|x|^p + |y|^p) =: g(x,y)$ then $f(x,y) = f(y,x)$ and $g(x,y) = g(y,x)$.

3. Applying the result of the first two steps and using the same argument as in step 1 but letting $0 \leq y \in E^{u+}$, it can be shown that the inequality holds for $0 \leq x, y \in E^{u+}$.

4. It is clear then that the inequality holds whenever each of the inputs are either negative or positive (i.e. for $x, y \in E^{u+} \cup E^{u-}$). This is so because neither side of the inequality depends on the sign of either of the inputs, i.e. $LHS = f(x,y) = f(x,-y) = f(-x,-y) = f(-x,y)$ and likewise for the right-hand side.

5. It can then be shown that the inequality holds for arbitrary $x \in E^u$ and $0 \leq y \in E^{u+}$.

6. Finally, using the same logic as in step 5, and remembering that the order of the inputs does not change either side of the inequality (step 2), the inequality is proved for arbitrary $x, y \in E^u$.

Elaboration of step 1: first we prove a Lemma, which is an analogue of [38, Theorem 4.6.2] for downward directed sets (this analogue was used, though not explicitly proved or stated, in the proof of [38, Theorem 6.5]).

Lemma 26 *If* $0 \leq x_n \downarrow x$ *in a Riesz space, and* f *is a strictly increasing convex or concave function on* $[0, \infty)$ *and if for all* $n \in \mathbb{N}$ *we have that the Riesz space elements* $f(x_n)$ *and* $f(x)$ *are defined, then* $f(x_n) \downarrow f(x)$*.*

Proof. Theorem 4.6.2 of [38] has the result for the case $0 \leq x_n \uparrow x$ which will be used to prove this result. Define the real-valued function g on $\mathbb{R}$ by $g(r) = -(f(-r))$ for $r \in (-\infty, 0]$, then g is strictly increasing and convex or concave (whichever f is not) and $g(-x)$ is defined in the Riesz space whenever $f(x)$ is defined. Since $0 \leq x_n \downarrow x$, it follows that $-x_n \uparrow -x$, and by Theorem 4.6.2 in [38] we now have that

$$g(-x_n) \uparrow g(-x) \iff -f(-(-x_n)) \uparrow -f(-(-x)) \iff -f(x_n) \uparrow -f(x)$$

and the result $f(x_n) \downarrow f(x)$ is therefore proved. ■

Now to prove step 1 of the six steps above:

Theorem 27 $|x+y|^p + |x-y|^p \leq 2(|x|^p + |y|^p)$ *for all* $1 < p < 2$ *and for all* $x \in E^{u+}$ *and* $y \in E_e^+$*.*

Proof. Since $x \in E^{u+}$ and $y \in E_e^+$ are given, consider $x_n := x \wedge ne$, then each x_n is in E_e^+ and $x_n \uparrow x$ because e is a weak order unit for E^u. Now because $y \geq 0$, it follows that $0 \leq x_n + y \uparrow x + y$. The function $|\cdot|^p$ for $1 < p < 2$ is strictly increasing and convex on $[0, \infty)$ and hence by Theorem 4.6.2 of [38] we have that

$$|x_n + y|^p \uparrow |x+y|^p. \tag{A.7}$$

For the second term on the left-hand side of (A.6) things are slightly more complicated as $x_n - y$ is not necessarily greater than or equal to zero. For this we note that

$$\begin{aligned} |x-y|^p &= [(x-y)^+ + (x-y)^-]^p \\ &= ((x-y)^+)^p + ((x-y)^-)^p \end{aligned} \tag{A.8}$$

because of the definition of $|\cdot|$ and by proposition 2.6 of [9] since $(x-y)^+$ and $(x-y)^-$ are orthogonal, $0^p = 0$, and $(\cdot)^p$ is increasing and continuous on E^{u+}. Because $x_n \uparrow x$, it follows that $x_n - y \uparrow x - y$ and so $0 \leq (x_n - y)^+ \uparrow (x-y)^+$; and in an opposite direction $-(x_n - y) \downarrow -(x-y)$ so

$$0 \geq (x_n - y)^- = (-(x_n - y))^+ \downarrow (-(x-y))^+ = (x-y)^-.$$

It follows now from Theorem 4.6.2 in [38] and Lemma 26 above that $((x_n - y)^+)^p \uparrow ((x-y)^+)^p$ and $((x_n - y)^-)^p \downarrow ((x-y)^-)^p$. From (A.8) this implies that

$$|x_n - y|^p \to |x-y|^p.$$

Adding this result to that of (A.7) it is now evident that

$$|x_n + y|^p + |x_n - y|^p \to |x+y|^p + |x-y|^p. \tag{A.9}$$

For each $n \in \mathbb{N}$, the left-hand side of (A.9) is bounded above by $2(|x_n|^p+|y|^p)$ as in (A.6) because the x_n are in E_e^+. But since $x_n \uparrow x$ it follows that $2(|x_n|^p+|y|^p)$ is bounded above by $2(|x|^p + |y|^p)$ for all $n \in \mathbb{N}$. Since for all $n \in \mathbb{N}$ the left-hand side of (A.9) is bounded above by $2(|x|^p + |y|^p)$, it follows that the limit $|x+y|^p + |x-y|^p$ must also be bounded above by $2(|x|^p + |y|^p)$, and this proof is complete. ■

Following steps 2 to 4 above it can be shown that the inequality holds as long as x and y are either positive or negative. The next step, step 5, is to show that the inequality holds where one of them is neither altogether in the positive cone, nor altogether negative.

Theorem 28 *Inequality (A.6) holds for arbitrary $x \in E^u$ and positive $y \in E^{u+}$.*

Proof. Let $P := P_{x^+}$ be the band projection onto the band generated by the positive part of x, namely, x^+. By P^d we mean $I - P$, so that $I = P + P^d$. Then P is a positive linear operator that maps x to the positive element $x^+ \in E^{u+}$. Similarly, $P^d x = (I - P)x = x - Px = x - x^+ = -x^-$. Band projections also map positive elements to positive elements, so Py and $P^d y$ are both in E^{u+}. Note also that because the ranges of P and P^d are orthogonal it follows by proposition 2.6 of [9] that, for arbitrary z in the Riesz space, $|P(z) + P^d(z)|^p = |P(z)|^p + |P^d(z)|^p$. By applying P and P^d, we can break the left-hand side of the inequality (A.6) into two sets, where in these new sets the elements involved are either positive or negative, and there the inequality has already been proven to apply:

$$\begin{aligned}
|x+y|^p + |x-y|^p &= |I(x+y)|^p + |I(x-y)|^p \\
&= |P(x+y) + P^d(x+y)|^p + |P(x-y) + P^d(x-y)|^p \\
&= |P(x+y)|^p + |P^d(x+y)|^p + |P(x-y)|^p + |P^d(x-y)|^p \\
&= |Px+Py|^p + |P^d x + P^d y|^p + |Px - Py|^p + |P^d x - P^d y|^p \\
&= |Px+Py|^p + |Px-Py|^p \\
&\quad + |P^d x + P^d y|^p + |P^d x - P^d y|^p \\
&= |x^+ + Py|^p + |x^+ - Py|^p \\
&\quad + |-x^- + P^d y|^p + |-x^- - P^d y|^p \\
&\leq 2(|x^+|^p + |Py|^p) \\
&\quad + 2(|-x^-|^p + |P^d y|^p) \\
&= 2(|x^+ - x^-|^p + |Py + P^d y|^p) \\
&= 2(|x|^p + |y|^p)
\end{aligned}$$

And hence step 5 is proved. ■

By step 6 above, the inequality is fully generalised to E^u:

Theorem 29 *For $x, y \in E^u$, we have that*

$$|x+y|^p + |x-y|^p \leq 2(|x|^p + |y|^p) \text{ for all } 1 < p < 2.$$

Appendix B

Existence of p-Powers and Related Products in a Riesz Space

We remark how the p-powers in Theorem 14 and the terms used in Theorems 19 and 20 exist in the relevant Riesz space. Let E be a Dedekind complete Riesz space with weak order unit $e > 0$. From page 281 of [38] we know for $x \in E$ and $p > 0$ that $|x|^{\frac{1}{p}}$ is defined in the sup-completion E_s because $|\cdot|^{\frac{1}{p}}$ is positive and continuous. From [9, Lemma 3.5 (ii)] we have for $x \in E$ and $p \geq 1$ that $\frac{1}{p}$ and $1 - \frac{1}{p}$ are both in $[0, \infty)$ and hence

$$|x|^{\frac{1}{p}} = |x|^{\frac{1}{p}} e^{1-\frac{1}{p}} \leq \frac{1}{p} x + \left(1 - \frac{1}{p}\right) e \in E,$$

and since E_s is a tight cone for E it follows that $|x|^{\frac{1}{p}} \in E$. For the case $E = \mathcal{L}^1(T)$ it is clear then that $|x|^{\frac{1}{p}} \in \mathcal{L}^p(T)$ since $(|x|^{\frac{1}{p}})^p = |x| \in \mathcal{L}^1(T)$.

If $x \in L^p(T)$ and $y \in L^q(T)$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $xy \in \mathcal{L}^1(T)$; this is part of [9, Theorem 3.7] and can be proved by using Young's inequality (see [9, proposition3.6]). Indeed, $xy \in \mathcal{L}^1(T)^u$ because $\mathcal{L}^1(T)^u$ is an f-algebra; then Young's inequality gives

$$|xy| \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q \in \mathcal{L}^1(T),$$

and it follows that $|xy| \in \mathcal{L}^1(T)$ because $\mathcal{L}^1(T)$ is an ideal in $\mathcal{L}^1(T)^u$ (see [58, Lemma 5.2]). Hence the right and left-hand sides of Hölder's inequality for sums, Theorem 19, do exist in $\mathcal{L}^1(T)$; to see this for the right-hand side, note that

$$|x_i|^p \in \mathcal{L}^1(T)\ \forall i \implies \sum_{i=1}^{n} |x_i|^p \in \mathcal{L}^1(T) \implies \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \in \mathcal{L}^p(T)$$

and similarly $(\sum_{i=1}^{n} |y_i|^q)^{\frac{1}{q}} \in L^q(T)$, and their product must therefore exist in $\mathcal{L}^1(T)$.

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