COMBINATORIAL ASPECTS OF COLORINGS ON GROUPS

A thesis submitted by Shivani Singh

in fulfillment of a Doctor of Philosophy degree

Supervisor: Prof. Yuliya Zelenyuk

University of Witwatersrand

School of Mathematics

November 2019

In mathematics the art of proposing a question must be held of higher value than solving it.

– Georg Cantor

A mathematician is a blind man in a dark room looking for a black cat which isn't there.

– Charles R. Darwin

DECLARATION

I hereby declare that this dissertation is my own original work unless explicitly stated otherwise. Furthermore, all contributions to the research presented in this thesis, from external literary sources, have been duly recognized and referenced to the best of my knowledge in a complete bibliography. This research was conducted with the supervision and collaboration of Professor Yuliya Zelenyuk. All contributions to any collaborative research efforts, including co-authorships, is also appropriately acknowledged. This thesis is being submitted to the University of the Witwatersrand in Johannesburg, in fulfillment of a Doctor of Science degree. It has never been submitted before, in any capacity, for any other qualification to any other university.

Signature **Date**

ABSTRACT

An r-coloring of a finite group G is any mapping $\chi : G \longrightarrow \{1, 2, \cdots, r\}$. A coloring χ is symmetric if there exists $a \in G$ such that, for all $x \in G$, $\chi(ax^{-1}a) = \chi(x)$. A subset X of a group G is called symmetric if there is an element $g \in G$, such that $gX^{-1}g = X$. We first examine monochromatic symmetric subsets in r-colorings of finite abelian groups. The combinatorial aspect of this thesis counts the number of symmetric colorings and equivalence classes of symmetric colorings of dihedral groups. We also derive polynomials for the number of symmetric r-colorings of group $G \times \mathbb{Z}_2$ where G is abelian. For $k, n, r \in \mathbb{N}$, an r-coloring χ is said to be k-alternating if every set of k consecutive vertices have pairwise distinct colors. We calculate the smallest value of r for which a k-alternating r-coloring of $\mathbb{Z}_n$ exists. Lastly, we explicitly derive expressions that count the number of 2-alternating r-colorings and 2-alternating r-ary necklaces of $\mathbb{Z}_n$.

ACKNOWLEDGMENTS

I would like to express my immense gratitude to my supervisor Professor Yuliya Zelenyuk for her participation in this thesis, whose insight and guidance was invaluable to this undertaking. It is also necessary for me to acknowledge her research collaboration which was instrumental to the development of this thesis. My profound appreciation goes to the National Research Foundation whose financial support has made this doctoral research a reality. Most importantly, I am deeply grateful to my parents for their infinite love, patience and understanding. I wish to extend this thanks to all my other family members, including those who have sadly passed away, for offering me support and encouragement throughout my life. A special thanks goes to my best friends Arjun and Subhadra for their unconditional love and companionship. Last but not least, I would like to humbly dedicate this thesis to Lord Krishna who is the source of all knowledge.

LIST OF SYMBOLS

Conventions

$\exists$	There exists
$\forall$	For all
$\lceil x \rceil$	Smallest integer greater than or equal to x
$\lfloor x \rfloor$	Greatest integer smaller than or equal to x
$\Longrightarrow$	Implies
ϕ	Euler phi function
Π	Product
$\sim$	Equivalence relation

$\sum$	Summation
n-gon	Regular polygon of n sides
p	Prime number
$f : X \longrightarrow Y$	Function f maps X to Y
$f \circ g$	Composition of two functions f and g
$n!$	Factorial of n
$\binom{a}{b}$	a choose b
"$\Longleftarrow$"	Backward statement
"$\Longleftrightarrow$"	If and only if
"$\Longrightarrow$"	Forward statement
gcd$(a,\ b)$	Greatest common divisor of a and b
i.e.	That is
w.l.o.g.	Without loss of generality
w.r.t.	With respect to

Sets

$\mathbb{N}$	Natural numbers
$\mathbb{R}$	Real numbers
$\mathbb{Z}$	Integers
$\mathbb{C}$	Complex numbers
$\mathbb{Z}^+$	Set of all positive integers
$\mathbb{Z}_{\geq 1}$	Strictly positive integers (integers ≥ 1)
$\lvert X \rvert$	Cardinality of set X
$[n]$	$\{1, 2, \cdots, n\}$
$\bigcup$	Union
$\bigcap$	Intersection
$\emptyset$	Empty set
$\in$	Element of
$\neq$	Not equal to
$X \subseteq Y$	X is a subset of Y
t-term	Has t number of terms
$\notin$	Not an element of
$X \subset Y$	X is a proper subset of Y

$X \nsubseteq Y$	X is not a subset of Y
$X \not\subset Y$	X is a not a proper subset of Y
$\mathbb{R}^t$	t-ary Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$
$\mathbb{Z}^t$	t-ary Cartesian product $\mathbb{Z} \times \cdots \times \mathbb{Z}$
$(a_j)_{j=1}^n$	Sequence of terms a_j, starting from $j = 1$ to $j = n$
$x \in [1,\, n]$	$1 \leq x \leq n$

Posets and the Möbius function

P	Partially ordered set, abbreviated to poset
$I(P)$	Incidence algebra
$\alpha,\ \beta$	Incidence functions
δ	Kronecker delta
ζ	Zeta function
μ	The Möbius function
$\Omega,\ \Psi$	Partitions
$\leq_R$	Partial order relation
$\prec$	Refinement order relation
$\mu(H,\ F)$	Möbius function on the lattice of subgroups
Υ_{D_p}	Poset of optimal partitions of D_p, for prime $p > 2$

Group Theory

G	Group
$\|G\|$	Cardinality of G
$2G$	$\{2g \ : \ g \in G\}$
G^2	$\{g^2 \ : \ g \in G\}$
$B(G)$	Boolean group of G
$D(G)$	Generalized dihedral group of G
$L(G)$	Lattice of subgroups of a group G
D_n	Dihedral group of order $2n$
$\mathbb{Z}_n$	Group of all integers modulo n
$A \leq G$	A is a subgroup of G
$A \trianglelefteq G$	A is a normal subgroup of G
$[G : A]$	Index of A in G
G/A	Quotient group of A in G
Ag	Right coset of A in G
$A \cong G$	A is isomorphic to G
$A \times B$	Direct product of A and B
e	Identity element of G

$\overline{x}$	$x = x + n\mathbb{Z}$
$\overline{0}$	Identity element of $\mathbb{Z}_n$
$\cong$	Isomorphic to
$Aut(G)$	Automorphism group of G
V_4	Klein four-group
$A \ltimes B$ or $A \rtimes B$	Semidirect product of A and B
Q_8	Quaternion group of order 8
S_3	Symmetric group of order 6
$\langle a \rangle$	Cyclic subgroup of G, generated by $a \in G$
$(g_t)_{t \in T}$	Set of tuples
$\bigoplus\limits_{t \in T} G_t$	Direct sum of groups G_t
$\prod\limits_{t \in T} G_t$	Direct product of groups G_t
$V(n)$	Number of subgroups of D_n
$o(n)$	Number of all $d \in \mathbb{Z}^+$, such that $d \vert n$
$o'(n)$	Sum of all $d \in \mathbb{Z}^+$, such that $d \vert n$

Symmetric Colorings

χ	Coloring
$[\chi]$	Orbit of χ
$St(\chi)$	Stabilizer of χ
$Z(\chi)$	Centralizer of χ
r^G	Set of all r-colorings of group G
$\chi \in r^G$	χ is an r-coloring of G
$S_r(G)$	Number of all symmetric r-colorings of G
$s_r(G)$	Number of all equivalence classes of symmetric r-colorings of G
$f(r,\, G)$	Polynomial representing the number of r-colorings of G
$f(r)$	Polynomial representing $S_r(G)$
$N_r(n)$	Number of all r-ary necklaces of $\mathbb{Z}_n$
$N_r^*(n)$	Number of symmetric r-ary necklaces of $\mathbb{Z}_n$
$\lambda_r(G),\, \omega_r(G)$	Ramsey functions of the form $\dfrac{t}{\lvert G\rvert}$
$\chi\vert_G$	χ is restricted to group G
$P_r(n)$	Number of aperiodic r-colorings of $\mathbb{Z}_n$

$Q_r(n)$	Number of aperiodic symmetric r-colorings of $\mathbb{Z}_n$
(χ_1, χ_2)	Pair of r-colorings
$[(\chi_1, \chi_2)]$	Orbit of a pair of r-colorings
$St(\chi_1, \chi_2)$	Stabilizer of a pair of r-colorings
$E_r(G)$	Set of all equivalence classes of pairs of symmetric r-colorings of G

Alternating Colorings

C_n	Cycle graph of n vertices
$\chi(C_n)$	Chromatic number of Cycle graph
$\rho(k, n)$	Alternating number
$A_r(k, n)$ or $A_r(k, \mathbb{Z}_n)$	Number of k-alternating r-colorings of $\mathbb{Z}_n$
$A_r(n)$ or $A_r(\mathbb{Z}_n)$	Number of 2-alternating r-colorings of $\mathbb{Z}_n$
$e_r(G)$	Number of equivalence classes of r-colorings of G
X_n	Set of all k-alternating r-colorings of $\mathbb{Z}_n$
$a_r(k, n)$	Number of k-alternating r-ary necklaces of $\mathbb{Z}_n$
$a_r(n)$	Number of 2-alternating r-ary necklaces of $\mathbb{Z}_n$

LIST OF FIGURES

4.1 Generalized Subgroup Lattice of D_p, for prime $p > 2$ 42

4.2 Generalized Hasse diagram of Υ_{D_p} for prime $p > 2$. 44

5.1 A non-alternating $\chi \in 5^{\mathbb{Z}_8}$. 68

5.2 A 2-alternating $\chi \in 3^{\mathbb{Z}_8}$. 69

5.3 A 12-alternating $\chi \in 12^{\mathbb{Z}_{12}}$. 72

5.4 The k-alternating colorings in $k^{\mathbb{Z}_{15}}$ 73

5.4a A 3-alternating $\chi \in 3^{\mathbb{Z}_{15}}$. 73

5.4b A 5-alternating $\chi \in 5^{\mathbb{Z}_{15}}$. 73

5.5 A 3-alternating $\chi \in 4^{\mathbb{Z}_8}$. 74

5.6 Alternating numbers $\rho(2, 12)$ and $\rho(2, 15)$ 77

5.6a A 2-alternating $\chi \in 2^{\mathbb{Z}_{12}}$ 77

5.6b A 2-alternating $\chi \in 3^{\mathbb{Z}_{15}}$ 77

5.7 Two 2-colorings of $\mathbb{Z}_{10}$ 81

5.7a Properly symmetric 2-coloring of $\mathbb{Z}_{10}$ 81

5.7b Symmetric (not properly symmetric) 2-coloring of $\mathbb{Z}_{10}$ . . 81

6.1 A 4-gon or $\mathbb{Z}_4$. 88

6.2 Bond lattice of $\mathbb{Z}_4$ with Möbius function values. 89

7.1 Two equivalent 3-alternating 4-colorings of $\mathbb{Z}_9$ 94

7.1a $\chi \in 4^{\mathbb{Z}_9}$. 94

7.1b $\theta \in 4^{\mathbb{Z}_9}$. 94

LIST OF TABLES

3.1 Ramsey function values $\lambda_2(\mathbb{Z}_n)$, for $n \leq 8$ 37

3.2 Ramsey function values $\omega_2(\mathbb{Z}_n)$, for $n \leq 8$ 37

6.1 Values of $A_r(k, n)$ for $k \in [3, 10]$ 86

CONTENTS

Declaration **II**

Abstract **III**

Acknowledgments **IV**

List of Symbols **IV**

List of Figures **XV**

List of Tables **XVII**

1 Introduction **1**

2 Preliminaries **5**

2.1 Finite Groups 5

2.2 Vertex Colorings of Regular n-gons 11
2.3 Posets and Lattices 14
2.4 The Technique of Möbius Inversion 17
2.5 Multiplicative Functions 20

3 Monochromatic Structures in Colorings 25

3.1 Monochromatic Symmetric Subsets in Colorings 27
3.2 Ramsey Functions for 2-colorings of Finite Abelian Groups 34

4 Symmetric Colorings of the Dihedral group 38

4.1 Counting Symmetric Colorings of the group D_p 40
4.2 Counting Symmetric Colorings of the group D_n 46
4.3 Counting Symmetric Colorings of the group $G \times \mathbb{Z}_2$ 56

5 Alternating Colorings 66

5.1 Alternating Vertex Colorings of Regular n-gons 67
5.2 The Alternating Number 74
5.3 Symmetry and Alternating Colorings 79

6 Counting Alternating Colorings 83

6.1 Counting 2-alternating r-colorings of $\mathbb{Z}_n$ 87

7 Alternating Necklaces 94

7.1 Counting 2-alternating r-ary Necklaces of $\mathbb{Z}_n$ 96

8 Conclusion 101

References 106

CHAPTER 1

INTRODUCTION

An r-coloring of a finite group G is any mapping $\chi : G \longrightarrow [r]$. This thesis studies the notion of counting the number of r-colorings of a finite group. In particular, a regular polygon of n-sides (n-gon) may be expressed geometrically as the cyclic group $\mathbb{Z}_n$. Hence, r-colorings may be contextualized in a group theoretic environment. Therefore, the structural foundations of the thesis is modeled from several essential books on group theory such as [11], [22] and [36]. However, the overarching objective of this research is to autonomously evaluate a new type of group coloring called an *alternating coloring*, together with its respective features. We aim to present some original results pertaining to this prototypical group coloring. Classical *graph colorings* will be an enduring influence on our work on r-colorings of groups, since they possess an abundance of well-known literature and results. Thus, we benefit greatly from similarities between proper graph colorings and colorings of regular n-gons.

A *symmetry* on a group G, for all $x \in G$, is any mapping $x \longmapsto ax^{-1}a$ for $a \in G$. This classic concept of symmetry on groups stems from [27]. Consequently, any coloring of a group G is called *symmetric* if the coloring is invariant under any symmetry of G. A coloring χ is symmetric if there exists $a \in G$ such that, for all $x \in G$, $\chi(ax^{-1}a) = \chi(x)$. We revisit symmetric colorings before our analysis of alternating colorings begins. Symmetric colorings may be comprehensively studied from either a theoretic or computational perspective. This thesis attempts to introduce both of these components to the reader.

The theoretical attributes of colorings materializes in the form of Ramsey theory. Therefore this thesis seeks to investigate some aspects of Ramseyan mathematics. In particular, the articles [16], [17] and [43] study the monochromatic symmetric subsets that arise in group r-colorings. We depend upon these sources for direction and hence Ramsey theory will be re-conceived using the familiar and convenient language of symmetric colorings. This treatment of Ramsey theory comparatively deepens in complexity. We will progress to more involved results and try to place them in perspective according to the recent advancements made in this field. The governing principle of all Ramseyan results is to find order in chaos. Therefore, we will return to these classical and reliable sources on Ramsey theory [14], [15] and [25], for counsel.

A subset S of a group G is symmetric if there exists an element $g \in G$, such that $gS^{-1}g = S$. The seminal article [32] introduces the notion of searching for the existence and cardinality of symmetric subsets that are monochromatic. The articles [3] and [4] contain rigorous discussions of these symmetric subsets on various mathematical objects. Some *Ramsey-type functions* for symmetric subsets in *finite abelian groups* is analyzed intricately in [43]. The result outlined in [16] states that any 2-coloring of an infinite group G contains monochromatic symmetric subsets of arbitrarily large cardinality, that is less than the order of G. These references conceptually dictate our discourse on Ramsey theory.

Then we proceed to develop the computational aspects of symmetric colorings. This topic is motivated by previous research, in [39], on the application of combinatorics to symmetric colorings. The principal results that were highlighted in [39] are refined and augmented. The *dihedral group* of order $2n$, denoted by D_n, becomes the focus of our attention. We deal with two polynomials for the number of *symmetric r-colorings* and the number of *equivalence classes of symmetric r-colorings* of the group D_p, where $p > 2$ is prime denoted by $S_r(D_p)$ and $s_r(D_p)$, respectively. These two polynomials are explicitly derived in [28] as an extension of the result for the group D_3, found in [23]. Consequently, these findings may then be generalized to $S_r(D_n)$. This is done in [30]. Lastly, we uncover a further generalization of this outcome to $S_r(G \times \mathbb{Z}_2)$, where G is abelian. This last engaging result is published in [29].

Let χ denote an r-coloring of a regular n-gon. Then, χ is called *k-alternating* if any k consecutive vertices of this n-gon are colored into k different colors. This thesis aims to characterize this innovative type of coloring on the cyclic group $\mathbb{Z}_n$ and to make a modest effort towards solving the combinatorial problems associated with it. We endeavour to count the number of k-alternating r-colorings of $\mathbb{Z}_n$, denoted by $A_r(k, \mathbb{Z}_n)$. The method used to count the number of symmetric colorings of finite groups is based on constructing the poset of optimal partitions. We were well-acquainted with this procedure in [18], [42] and [46]. This method hinges on the *technique of Möbius inversion*. The Möbius function is a useful combinatorial tool that was exploited when counting symmetric colorings. We rely heavily on this effective application of Möbius inversion to combinatorics developed in the exalted treatise [6]. This thesis modifies the same strategy for counting symmetric colorings and applies them to alternating colorings.

Since this thesis is concerned with the many nuances of alternating colorings, we may naturally induce the definitions of *alternating necklaces*. Each equivalence class of the k-alternating r-colorings of $\mathbb{Z}_n$ is called a k-alternating r-ary necklace. The number of such necklaces is denoted by $a_r(k,\, \mathbb{Z}_n)$. The combinatorial aspect of this thesis deals with calculating the number of k-alternating r-colorings that exist for an arbitrary n. Primarily, our goal is to derive formulas for $A_r(k,\, \mathbb{Z}_n)$ and $a_r(k,\, n)$ by using the customary technique of *Möbius inversion* made familiar in [39].

Due to the versatile nature of colorings, the theoretical dimensions of this thesis broadens to accommodate such fields as group theory, graph theory, number theory, Ramsey theory and combinatorics. The inevitable purpose of most Ramseyan mathematics is to find monochromatic structures in arbitrary finite colorings of the natural numbers. However, the combinatorial perspective on symmetric colorings of finite groups are also an intriguing subject of study. Specifically, we aim to strengthen the central results previously discussed in the thesis [39]. Lastly, this thesis attempts to introduce, analyze and then enumerate alternating colorings of regular n-gons.

CHAPTER 2

PRELIMINARIES

This chapter revisits the notation and other foundational requirements applicable to r-colorings of groups. It consists of several standard definitions and results from combinatorics as well as group theory. The brevity of this narrative is intentional since this material has been examined in the prequel, [39]. As a consequence of this, proofs are omitted and the reader is requested to refer to the requisite literature for substantiation.

2.1 Finite Groups

It is well-known that the finite cyclic group $\mathbb{Z}_n$ may be geometrically represented as a regular n-gon, with its elements at the vertices. As a result of this phenomenon, this thesis draws upon several rudimentary definitions and results from group theory, which may be found in [11], [22] and [36]. Henceforth, all groups considered are strictly finite.

Definition 1. A group G is called *cyclic* if there exists an $a \in G$, such that $G = \{a^n : n \in \mathbb{Z}\} = \langle a \rangle$. This element a is called a *generator* of G. A cyclic group can have multiple generators.

Definition 2. The *cyclic group* of order $n \in \mathbb{Z}$, up to isomorphism, is represented by $\mathbb{Z}_n$. This group, under the operation of *addition modulo* n is described by the set $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$ with $\overline{0}$ as the *identity element*.

Theorem 1. *The group $\mathbb{Z}_n$ has $\phi(n)$ generators where ϕ is the Euler phi function.*

Proof. Refer to [11]. □

The theorem below presents the generic subgroup structure of finite cyclic groups.

Theorem 2. *Let group $G = \langle g \rangle$ be cyclic. Then,*

(1) *Every $H \leq G$ is cyclic. Furthermore, if $H \leq G$ either $H = \{\boldsymbol{e}\}$ or $H = \{h^a\}$, for the smallest $a \in \mathbb{Z}^+$ such that $h^a \in H$.*

(2) *Suppose $|G| = n \in \mathbb{Z}$. Then, $\forall\ d \in \mathbb{Z}^+$ that divides n, $\exists$ a unique $H \leq G$ such that $|H| = d$. Moreover, $H = \langle h^a \rangle$, for $a = \dfrac{n}{d}$. Lastly, $\forall\ m \in \mathbb{Z}$, $\langle g^m \rangle = \langle g^b \rangle$, where $b = gcd(n, m)$.*

Proof. Refer to [11]. □

Definition 3. Let G be a finite group and define a subset of G as follows;

$$B(G) = \{a \in G \ : \ a^{-1} = a\}.$$

Then, G is called a *Boolean group* if $G = B(G)$; furthermore if G is a Boolean group then G is also commutative.

Definition 4. For a finite index set T, let $\{G_t\}_{t\in T}$ be a sequence of groups. The *group direct product* of $\{G_t\}_{t\in T}$ is denoted by, $\prod_{i=1}^{t} G_t = G_1 \times \cdots \times G_t$. The elements in $\prod_{i=1}^{t} G_t$ are of the form $(a_1, \cdots, a_t)$, where $a_i \in G_i$ for $i = 1, \cdots, t$. Component-wise multiplication is defined on the elements of $\prod_{i=1}^{t} G_t$ as follows,

$$(a_1, \cdots, a_t)(a_1', \cdots, a_t') = (a_1 a_1', \cdots, a_t a_t').$$

Lastly, $\left|\prod_{i=1}^{t} G_t\right| = \prod_{i=1}^{t} |G_t|$.

Note 1. The *group direct sum* of $\{G_t\}_{t\in T}$ is denoted by, $\bigoplus_{t\in T} G_t$. As long as there is a restriction to a finite set of indices the direct product and direct sum of any abelian group, will be identical.

The following theorem gives us the complete classification of all finite abelian groups. This result has numerous manifestations and generalizations. The description below, is given in the form of direct sums and is also known as the *basis theorem of finite abelian groups*.

Theorem 3. *[Fundamental Theorem of Finite Abelian Groups] Let finite group G be abelian. Then, G is uniquely expressible as*

$$G \cong \bigoplus_{i=1}^{t} \mathbb{Z}_{p_i^{a_i}},$$

(1) p_i *is prime*, $\forall\, i \in \{1, \cdots, t\}$.

(2) $p_i \leq p_{i+1}$, $\forall\, i \in \{1, \cdots, t\}$.

Proof. Refer to [11]. □

Definition 5. Suppose that for a finite group G we have $A \leq G$ and $B \leq G$ such that

(1) $G = AB = \{ab \in G \ : \ a \in A$ and $b \in B\}$,

(2) A is a *normal subgroup* of G, i.e. $A \trianglelefteq G$ and

(3) $A \cap B = \{\mathbf{e}\}$.

If these conditions are true then G is called the *semidirect product* of A and B. Furthermore, the homomorphism $\varphi : B \longrightarrow Aut(A)$ is an action called conjugation, where $\varphi(b) = \varphi_b$ and

$$\varphi_b(a) = bab^{-1} \in A \text{ (since } A \trianglelefteq G).$$

The group G is determined up to isomorphism by A, B and φ. This is written symbolically as $G \cong A \rtimes_{\varphi} B$.

Definition 6. The group of all symmetries of a regular n-gon, which includes rotations and reflections, is called the *dihedral group*. It is denoted by D_n and has an order of $2n$. Moreover, the presentation of D_n is

$$D_n = \langle a, \, b \ : \ a^2 = \mathbf{e} = b^n, \ b^{n-1}a = ab \rangle.$$

The subgroups of D_n are categorized as either cyclic or dihedral. This characteristic naturally determines a formula for counting the number of subgroups of D_n, denoted by $V(n)$. The following result is an excerpt from the article [8].

Theorem 4 (Cavior)**.** *Suppose that* $n \geq 3$ *and let* $d \in \mathbb{Z}^+$*, such that* d *is a divisor of* n*. We let* $o(n)$ *denote the number of all* d *and* $o'(n)$ *denote the sum of all* d*.*

Then, the number of subgroups of D_n, denoted by $V(n)$, is given by

$$V(n) = o(n) + o'(n),$$

Proof. The n reflections and n rotations of a regular n-gon gives us the complete geometric representation of D_n. Hence, there are two types of subgroups of D_n.
Type 1: The subgroups of $\mathbb{Z}_n$; $o(n)$ is the number of these subgroups.
Type 2: The dihedral subgroups, where the number of rotations is equal to the number of reflections; call this number v. The v rotations belong to the unique $H \leq \mathbb{Z}_n$, where $|H| = v \implies v|n$. In the v reflections, the number of Type 2 subgroups are $\frac{n}{v}$, for every $v|n$. Thus, the total number of Type 2 subgroups is given by

$$\sum_{v|n} \frac{n}{v} = \sum_{v|n} v = o'(n).$$

Hence, $V(n) = o(n) + o'(n)$. □

Definition 7. The group D_n is *isomorphic* to the *semidirect product* of $\mathbb{Z}_n$ and $\mathbb{Z}_2$. Since, $\mathbb{Z}_2 = \langle a \rangle = \{\overline{0}, a\}$ and $\mathbb{Z}_n = \langle b \rangle$, we have;

(1) $D_n = \mathbb{Z}_n \mathbb{Z}_2$,

(2) $\mathbb{Z}_n$ is a *normal subgroup* of D_n, i.e. $\mathbb{Z}_n \trianglelefteq D_n$ and

(3) $\mathbb{Z}_n \cap \mathbb{Z}_2 = \{\overline{0}\}$

Furthermore, a acts on $\mathbb{Z}_n$ by *inversion*. Therefore, $\forall\ g \in \mathbb{Z}_n$, the following occurs;

(1) $[D_n : \mathbb{Z}_n] = 2$,

(2) the *quotient group* is defined to be, $D_n \; \mathbb{Z}_n = \mathbb{Z}_n a$ and

(3) $a^2 = \overline{0}$ and $ag = g^{-1}a$.

The generalization of the dihedral group is extremely influential in the sequel and is introduced here. The *inversion function* $\psi : G \longrightarrow G$ on a group G is defined by $\psi(g) = g^{-1}$. It is easily shown that ψ is an *automorphism* if and only if G is abelian. Due to the existence of this automorphism, the *generalized dihedral group* may be decomposed into a *semidirect product* in which the group of rotations is represented by an abelian group G and the group of reflections become $\mathbb{Z}_2$.

Definition 8. Let G be an abelian group. Define the action $\theta : \mathbb{Z}_2 \longrightarrow Aut(G)$ as follows

$$\theta_a(x) = x^{-1},\ \forall\, x \in G \text{ and } \mathbf{e} \neq a \in \mathbb{Z}_2.$$

Then, the *generalized dihedral group* is denoted by $D(G)$ and $D(G) \cong \mathbb{Z}_2 \ltimes_\theta G$.

Note 2. The elements in $D(G)$ are the ordered pairs $(x,\, a)$, where $x \in G$ and $a \in \mathbb{Z}_2$. Furthermore, $(x,\, a)(y,\, b) = (x + a(y),\, a + b)$.

Note 3. By the above definition $D(\mathbb{Z}_n) \cong D_n \cong \mathbb{Z}_2 \ltimes_\theta \mathbb{Z}_n$, as stated previously in Definition 7.

This section concludes with the statement of *Burnside's lemma* which plays an instrumental role in counting the orbits of a group action.

Lemma 1 (Burnside)**.** *Let G be a finite group and suppose that G acts on the set A. For $g \in G$, define the set of elements in A that are fixed by g as $A^g = \{a \in$*

$A : g \cdot a = a\}$. *If* A/G *denotes the set of all orbits of* G*, then*

$$|A/G| = \frac{1}{|G|}\sum_{g\in G}|A^g|.$$

Proof. Refer to [36]. □

2.2

Vertex Colorings of Regular n-gons

This section is a perfunctory review of the foundations of r-colorings of finite groups. This content is found in [42] and [46].

Definition 9. An r-coloring of a finite group G is a mapping $\chi : G \longrightarrow [r]$. The set of all r-colorings of G is denoted by r^G.

Definition 10. Any two r-colorings χ and θ of G are said to be *equivalent* if $\exists\, a \in G$ such that $\chi(ax^{-1}a) = \theta(x)\,, \forall\, x \in G$.

Definition 11. A coloring χ of G is called *symmetric* if $\exists\, a \in G$ such that, for all $x \in G$

$$\chi(ax^{-1}a) = \chi(x).$$

The finite group G naturally *acts* on the set of all r-colorings r^G by

$$\chi a(x) = \chi(xa^{-1}).$$

Definition 12. Let $\chi \in r^G$ the *orbit* and *stabilizer* of χ is defined respectively as

$$[\chi] = \{\chi y\ :\ y \in G\}\ \text{ and }\ St(\chi) = \{y \in G\ :\ \chi y = \chi\}.$$

Whenever this group action occurs we have

$$|[\chi]| = \frac{|G|}{|St(\chi)|} \text{ and } St(\chi y) = y^{-1}St(\chi)y.$$

Definition 13. A polygon of n sides (n-gon) is called *regular* if all of its sides and angles are equal.

A special case of an r-coloring on a group is when $G = \mathbb{Z}_n$ because this definition has a geometric interpretation. It is a coloring of the vertices of a regular n-gon, hence the above definitions are applied to $\mathbb{Z}_n$.

Definition 14. Let $n,\ r \in \mathbb{N}$. An r-coloring of $\mathbb{Z}_n$ is a mapping $\chi : \mathbb{Z}_n \longrightarrow [r]$. The set of all r-colorings of $\mathbb{Z}_n$ is denoted by $r^{\mathbb{Z}_n}$.

Note 4. We stipulate that all r colors are used when coloring $\mathbb{Z}_n$. Furthermore, we assume that all r-colorings are *surjective*, since they are more convenient to work with algebraically. In other words, each vertex has only one color but different vertices may have the same color.

Proposition 1. *Let $n,\ r \in \mathbb{N}$. Then r^n is the total number of r-colorings of $\mathbb{Z}_n$.*

Proof. See [46]. □

Definition 15. We define any two r-colorings, χ and θ of $\mathbb{Z}_n$ to be *equivalent* if $\exists\ a \in \mathbb{Z}_n$ such that $\chi(x - a) = \theta(x)\,, \forall\ x \in \mathbb{Z}_n$.

In geometric terms, any two r-colorings of a regular n-gon are equivalent if one may be obtained from the other by any rotation about the n-gon's center.

Note 5. The key difference between graph colorings and the r-colorings defined here is that these colorings are taken to be equivalent under rotation. Whereas

χ and θ, in the above definition, would be regarded as two *different* colorings in graph theory.

The definition of symmetry is adjusted accordingly for the case of $G = \mathbb{Z}_n$. Let $n,\, r \in \mathbb{N}$, then $\chi \in r^{\mathbb{Z}_n}$ is (properly) symmetric if $\exists\ y \in \mathbb{Z}_n$ such that, $\forall\ x \in \mathbb{Z}_n$ we have

$$\chi(y - x + y) = \chi(x), \text{ which is equivalent to } \chi(2y - x) = \chi(x).$$

The group action on $r^{\mathbb{Z}_n}$ is defined as

$$(y + \chi)(x) = \chi(x - y).$$

The orbit and stabilizer of $\chi \in r^{\mathbb{Z}_n}$ respectively becomes

$$[\chi] = \{y + \chi\, :\, y \in \mathbb{Z}_n\} \text{ and } St(\chi) = \{y \in \mathbb{Z}_n\, :\, y + \chi = \chi\}.$$

Lastly, the *centralizer* of χ is defined as follows

$$Z(\chi) = \{y \in \mathbb{Z}_n\, :\, \chi(y - x) = \chi(x),\, \forall\ x \in \mathbb{Z}_n\}.$$

Let the number of different r-colorings of a given group G be a function of r, denoted by $f(r,\, G)$. Then, in similar fashion to the classical *chromatic polynomial* of graph colorings $f(r,\, G)$ may also be expressed as a polynomial of the number of colors r. The following result proves this property.

Theorem 5. *Let $k,\, n,\, r \in \mathbb{N}$. Then, $f(r,\, G)$ is a polynomial of r.*

Proof. Suppose that we use exactly r colors to color a set of n vertices. Then this is the same as partitioning n into r-vertex sets. In order to find the total number of r-colorings we count all the colorings that give a particular partition and add them up, for all such partitions. Since G is a finite group it has a finite number of partitions. So, we need to show that the number of colorings for a single partition is a polynomial of r. Fix a partition with say p cells. If we assign a different color to each cell we get all the colorings that belong to the partition. The first color may be chosen in r possible ways, the second in $r-1$ ways and so on. Therefore, there are $r(r-1)\cdots(r-p+1)$ colorings in total which is a polynomial. □

2.3

Posets and Lattices

This section reviews *partially ordered set* theory which is critical to our understanding of Möbius inversion. The reader is encouraged to consult [1], [7] and [21] for a more detailed assessment of this subject matter.

Definition 16. A *partition* of a nonempty set X is a set of disjoint subsets of X whose union is equal to X. Each element of a partition is called a *cell*. Lastly, the *order* of a partition is the number of cells it contains.

Definition 17. Let P_X denote the set of all partitions of a set X and let $\Omega,\ \Psi \in P_X$. We define a *refinement ordering* of P_X to be a relation in which Ω is *finer* than Ψ if every cell in Ω is a subset of every cell of Ψ, where $\Omega \neq \Psi$. We denote this relation by $\Omega \prec \Psi$ and Ψ is said to be *coarser* than Ω.

Example 1. Let $X = \{1, 2, 3, 4\}$ and suppose that we have two partitions $\Omega = \{\{1, 2\}, \{3\}, \{4\}\}$ and $\Psi = \{\{1, 2, 3\}, \{4\}\}$. Then, clearly $\Omega \prec \Psi$ since every cell of Ω is a subset of every cell of Ψ.

Definition 18. Let X be a nonempty set. Then, a *poset* is an ordered pair $P = (X, \leq)$ where $\leq$ is a relation on X that satisfies the following conditions: for each $x, y, z \in P$;

(1) reflexive: $x \leq x$,

(2) antisymmetric: if $x \leq y$ and $y \leq x$, then $x = y$,

(3) transitive: if $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition 19. Partially ordered sets may be graphically displayed as a *Hasse diagram*. In this diagram, each element in the poset is represented as a vertex with a straight line connecting any two vertices that are related to each other. Moreover, for any x, y in the poset where $x \leq y$ then vertex x is drawn beneath y in the diagram.

Definition 20. A *bond* of a regular n-gon is a partition of its vertices such that all vertices in the same cell are connected to each other, within the n-gon.

The set of all bonds of a regular n-gon form what is called a *bond lattice*.

Definition 21. A *lattice* is a poset with the additional property that every pair of elements has a unique *greatest lower bound* (infimum) and a unique *least upper bound* (supremum).

The set of all natural numbers $\mathbb{N}$ is a lattice with the partial order on its elements being divisibility. The unique supremum is the least common multiple and the unique infimum is the greatest common divisor.

Definition 22. The *subgroup lattice* of a group G, denoted by $L(G)$, is the set of all subgroups of G. The partial order relation between the subgroups in this lattice is *set inclusion*.

We conclude our review of posets by discussing the *poset of optimal partitions*. The reader is encouraged to consult [23] and [39] for a more detailed explanation of this poset.

Definition 23. Let G be a finite group with Ω being a partition of G. The *stabilizer* and the *center* of Ω are defined respectively as

$$St(\Omega) = \{g \in G \,:\, \forall\, x \in G,\ x \text{ and } xg^{-1} \text{ belong to the same cell of } \Omega\} \quad \text{and}$$

$$Z(\Omega) = \{g \in G \,:\, \forall\, x \in G,\ x \text{ and } gx^{-1}g \text{ belong to the same cell of } \Omega\}.$$

Definition 24. A partition Ω of G is called an *optimal partition* if, for all partitions Ψ of G, it satisfies the following conditions;

(1) $\mathbf{e} \in Z(\Omega)$ and

(2) $\Omega \prec \Psi$, where $St(\Psi) = St(\Omega)$ and $Z(\Psi) = Z(\Omega)$.

Note 6. These optimal partitions of G form a poset under $\prec$, which is understood to be the poset of pairs of subsets of G, $(St(\Omega), Z(\Omega))$, for Ω a partition of G and $\mathbf{e} \in Z(\Omega)$.

Definition 25. The *poset of optimal partitions* of G is constructed as follows. We begin with the *finest partition* $\Omega = \{a, a^{-1} \,:\, a \in G\}$. Thereafter, we use Ω to calculate the remaining optimal partitions. Let Ω be an optimal partition of G and suppose $X \subseteq G$. Let Ω_1 be the finest partition of G, such that $\Omega \prec \Omega_1$ and

$X \subseteq St(\Omega_1)$. Also, let Ω_2 be another finest partition of G, such that $\Omega \prec \Omega_2$ and $X \subseteq Z(\Omega_2)$. Then, Ω_1 and Ω_2 are also optimal partitions.

2.4

The Technique of Möbius Inversion

Möbius inversion has a diverse array of applications in combinatorial analysis. A definitive survey of which is given in [6]. The Möbius inversion technique has proven to be an invaluable tool when enumerating colorings. The article [6] also discusses the consummate elegance and simplicity of the Möbius function, when applied to solving counting problems. This section revisits the theory behind this useful combinatorial concept. We present first the traditional number theoretic *Möbius function* which is intuitively understood to be a generalization of the *inclusion-exclusion* principle. Thereafter, we consider the Möbius function on a partially ordered set. This material is freely adapted from [6], [7] and [35].

Let $\mathbb{Z}_{\geq 1} = \{x \in \mathbb{Z} \ : \ x \geq 1\}$ denote the set of all *strictly positive integers*. Suppose that we have two functions $f : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}$ and $g : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}$ which satisfy

$$g(n) = \sum_{d|n} f(d),$$

where $d|n$ means that d divides n. It is possible to derive a formula for $f(n)$, from the one stated above, through a process known as *Möbius inversion*. This process entails finding a function μ such that the following theorem is satisfied.

Theorem 6. *For $n \in \mathbb{Z}_{\geq 1}$, there exists a function $\mu : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}$, such that*

$$g(n) = \sum_{d|n} f(d) \Longleftrightarrow f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d).$$

Proof. Refer to [35]. □

The function μ is found to be

$$\mu(n) = \begin{cases} 1, & \text{for } n = 1 \\ 0, & \text{if there exists prime } p \text{ such that } p^2|n \\ (-1)^t, & \text{if } n = p_1p_2\cdots p_t, \text{ where } p_1,\, p_2 \cdots,\, p_t \text{ are distinct primes.} \end{cases}$$

Henceforth, $\mu(n)$ shall always denote the Möbius function. It is important to note that $\mu(n)$ only accepts values from the set $\{-1, 0, 1\}$. Partially ordered sets give rise to a more general type of Möbius function.

Definition 26. Let P be a locally finite poset and consider the set

$$I(P) = \{\alpha : P \times P \longrightarrow \mathbb{R} \,:\, \alpha(x,\, y) = 0 \text{ if } x \not\leq y\}.$$

For $\alpha,\, \beta \in I(P)$ the two operations on $I(P)$ are defined as follows.

(1) Addition

$$(\alpha + \beta)(x,\, y) = \alpha(x,\, y) + \beta(x,\, y).$$

(2) Multiplication (also called *convolution*) which is associative

$$(\alpha * \beta)(x,\, y) = \sum_{x \leq z \leq y} \alpha(x,\, z)\beta(z,\, y).$$

$I(P)$ forms the *incidence algebra* over $\mathbb{R}$ and if $\alpha \in I(P)$ then α is referred to as an *incidence function*.

Definition 27. The multiplicative identity of $I(P)$ is the *Kronecker delta*,

$$\delta(x,\, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y. \end{cases}$$

Proposition 2. *Let $\alpha \in I(P)$, then the multiplicative inverse of α exists $\iff$ $\alpha(x,\, x) \neq 0$, $\forall\, x \in P$.*

Proof. Refer to [35]. □

Definition 28. Another significant function in $I(P)$ called the *zeta function* is defined as follows

$$\zeta(x,\, y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x \nleq y. \end{cases}$$

By the above proposition, the multiplicative inverse of the zeta function exists in $I(P)$ with respect to the convolution operation. This inverse is called the Möbius function μ of P, which is symbolically written as $\zeta^{-1} = \mu$. We define μ recursively by the conditions below.

$$\mu(x,\, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \nleq y \\ -\sum\limits_{x \leq z < y} \mu(x, z), & \text{if } x < y. \end{cases}$$

The inversion property of the Möbius function is a powerful enumeration tool.

This preliminary chapter is brought to an end with the formal statement of *Möbius inversion*.

Theorem 7 (The Möbius Inversion Theorem)**.** *Let P be a locally finite poset with Möbius function μ. Suppose we have two functions $\alpha : P \longrightarrow \mathbb{R}$ and $\beta : P \longrightarrow \mathbb{R}$. Assume that all the principal ideals of P are finite. Then,*

$$\beta(x) = \sum_{y \leq x} \alpha(y) \Longleftrightarrow \alpha(x) = \sum_{y \leq x} \mu(y, x)\beta(y).$$

Proof. Refer to [35]. □

2.5

Multiplicative Functions

Multiplicative number theory forms the basis of several results presented in this thesis. Multiplicative functions have an ubiquitous effect on the enumeration of colorings of $\mathbb{Z}_n$. This section discusses the interplay between important arithmetic functions and complies with [2].

Definition 29. The *Euler phi function* is defined to be the total number of $a \in \mathbb{Z}^+$, where $a \in [1, n]$ such that $\gcd(a, n) = 1$.

Definition 30. An *arithmetic function* is a function α, which is defined on the set of all strictly positive integers that has values in the set of complex numbers. Symbolically, $\alpha : \mathbb{Z}^+ \longrightarrow \mathbb{C}$.

Theorem 8 (Fundamental Theorem of Arithmetic)**.** *Suppose $n \in \mathbb{Z}$ where $n > 1$. Then n may be factorized into a product of distinct prime divisors*

$$n = \prod_{i=1}^{t} p_i^{x_i},$$

where $p_i \in \mathbb{N}$ is prime, $\forall\, x_i \in \mathbb{N}$ and $\forall\, i \in [1,\, t]$.

Proof. Refer to [2]. □

Definition 31. A *multiplicative function* α, is an arithmetic function that satisfies the following criteria;

(1) $\alpha(1) = 1$,

(2) $\alpha(ab) = \alpha(a)\alpha(b)$, where $\gcd(a,\, b) = 1$.

Theorem 9. *For a multiplicative function α and $n = p_1^{x_1} p_2^{x_2} \cdots p_t^{x_t}$, where $p_1,\, p_2,\, \cdots,\, p_t$ are primes. Then,*

$$\alpha(n) = \alpha(p_1^{x_1})\alpha(p_2^{x_2}) \cdots \alpha(p_t^{x_t}).$$

Proof. This may easily be shown by induction. □

Theorem 10. *The Möbius function $\mu(n)$ is a multiplicative function.*

Proof. Clearly $\mu(1) = 1$. Let $a,\, b \in \mathbb{Z}_{\geq 1}$ and suppose that either $\mu(a) = 0$ or $\mu(b) = 0$. So there exists a prime p such that either $p^2|a$ or $p^2|b \implies p^2|ab \implies \mu(ab) = 0$. Hence, $\mu(ab) = \mu(a)\mu(b)$, whenever $\mu(a) = 0$ or $\mu(b) = 0$. Now suppose that $\mu(a) \neq 0$ and $\mu(b) \neq 0$. Then let $a = p_1 \cdots p_s$ and $b = q_1 \cdots q_t$ where $p_i,\, q_j$ are all primes. Therefore, $ab = p_1 \cdots p_s q_1 \cdots q_t$. Since $\gcd(a,\, b) = 1 \implies \forall\, i,\, j,\, p_i \neq q_j$. Hence, there is no prime in ab whose power is greater than $1 \implies \mu(a,\, b) \neq 0$. Hence, $\mu(a,\, b) = (-1)^{s+t} = (-1)^s(-1)^t = \mu(a)\mu(b)$ and the result follows. □

Theorem 11. *The Euler phi function ϕ is a multiplicative function.*

Proof. See [2]. □

Definition 32. Let $x \in \mathbb{R}$, then $\lfloor x \rfloor$ denotes the *greatest integer* less than or equal to x. Whereas, $\lceil x \rceil$ denotes the *smallest integer* greater than or equal to x.

Theorem 12 (Divisor Sum of the Möbius function)**.** *Suppose that $n \in \mathbb{Z}$ and $n \geq 1$. Then $\forall\ t \in \mathbb{Z}^+$*

$$\sum_{t|n} \mu(t) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1, & \textit{for } n = 1 \\ 0, & \textit{for } n > 1. \end{cases}$$

Proof. The expression is true for $n = 1$. So assume that $n > 1$ and let $n = p_1^{x_1} \cdots p_i^{x_1}$, for primes $p_1, \cdots, p_i$ and $x_1, \cdots, x_i \in \mathbb{N}$. For all t that divides n, whenever $t = 1$ and whenever t is a product of distinct primes the terms in $\sum\limits_{t|n} \mu(t)$ are nonzero. There are no other nonzero terms in this summation except when these two conditions occur. Hence,

$$\begin{aligned} \sum_{t|n} \mu(t) &= \mu(1) + \mu(p_1) + \cdots + \mu(p_1 p_2) + \cdots + \mu(p_{i-1} p_i) + \cdots + \mu(p_1 \cdots p_i) \\ &= 1 + \sum_{j=1}^{i} (-1)^j \binom{i}{j} \\ &= (1-1)^i = 0. \end{aligned}$$

□

The Möbius function and the Euler phi function share a special relationship which is introduced in the theorem below.

Theorem 13. *Suppose that* $a \in \mathbb{Z}$*, where* $a \geq 1$*. Then*

$$\phi(n) = \sum_{a|n} \mu(a) \frac{n}{a}.$$

Proof. By the definition of the Euler phi function

$$\begin{aligned} \phi(n) &= \sum_{t=1}^{n} 1 \\ &= \sum_{t=1}^{n} \left\lfloor \frac{1}{\gcd(t,\, n)} \right\rfloor. \end{aligned}$$

In Theorem 12 we replace n by $\gcd(t,\, n) = m$. Hence

$$\begin{aligned} \phi(n) &= \sum_{t=1}^{n} \sum_{a|n} \mu(a) \\ &= \sum_{t=1}^{n} \sum_{\substack{a|n \\ a|t}} \mu(a). \end{aligned}$$

Fix $a|n$ now we sum over all $t \in [1,\, n]$ that are multiples of n. Therefore, if $t = ba$ for $b \in \mathbb{Z}$, then $t \in [1,\, n] \iff b \in \left[1,\, \frac{n}{a}\right]$. Hence, we obtain

$$\begin{aligned} \phi(n) &= \sum_{a|n} \sum_{b=1}^{\frac{n}{a}} \mu(a) \\ &= \sum_{a|n} \mu(a) \sum_{b=1}^{\frac{n}{a}} 1 \\ &= \sum_{a|n} \mu(a) \frac{n}{a}. \end{aligned}$$

□

Proposition 3. *Let $n \in \mathbb{Z}_{\geq 1}$ and suppose that $n = p_1^{t_1} \cdots p_m^{t_m}$, where $p_1, \cdots, p_m$ are primes. Then for every multiplicative function α,*

$$\sum_{x|n} \mu(x)\alpha(x) = \prod_{i=1}^{m} [1 - \alpha(p_i)].$$

Proof. Since μ and α are multiplicative functions this implies that $\mu\alpha$ is also multiplicative. Let $\beta(n) = \sum_{x|n} \mu(x)\alpha(x)$, then β is also multiplicative. Hence for p prime

$$\begin{aligned} \beta(p^t) &= \mu(1)\alpha(1) + \mu(p)\alpha(p) + \mu(p^2)\alpha(p^2) \cdots + \mu(p^t)\alpha(p^t) \\ &= \mu(1)\alpha(1) + \mu(p)\alpha(p) \\ &= 1 - \alpha(p). \end{aligned}$$

Therefore, for $n = p_1^{t_1} \cdots p_m^{t_m}$ we have

$$\begin{aligned} \beta(n) &= \beta(p_1^{t_1}) \cdots \beta(p_m^{t_m}) \\ &= [1 - \alpha(p_1)] \cdots [1 - \alpha(p_m)] \\ &= \prod_{i=1}^{m} [1 - \alpha(p_i)]. \end{aligned}$$

□

CHAPTER 3

MONOCHROMATIC STRUCTURES IN COLORINGS

A subset of a group G is called symmetric if it remains invariant under some symmetry of G. Symmetric subsets have the property of being able to be naturally defined in general for a variety of different algebraic and geometric objects. The problem of finding monochromatic symmetric subsets of different sizes in colorings of groups, originated with the seminal article [32]. This topic is explored more rigorously in the articles [3], [4] and [5]. The underlying motivation behind Ramseyan mathematics is to find order in chaos. Thus, the notion of searching for the existence of monochromatic structures in arbitrary colorings of groups is strongly linked to Ramsey theory.

This diverse subject is closely interlinked with a variety of mathematical disciplines which include algebraic topology, compact group theory, functional and harmonic analysis, measure theory and Riemannian geometry which break the confines of this thesis. So only a cursory outline of the developments made in this field is presented below. These results are solicited from the paper [4], whereas an updated version of this survey may be found in [5].

Definition 33. For $t \in \mathbb{N}$ and $X \subset \mathbb{R}^t$ subset X is called symmetric with respect to a point $a \in \mathbb{R}^t$ if and only if $X = 2a - X$.

The article [32] posed the question of whether every r-coloring of $\mathbb{Z}^r$ yields an infinite monochromatic subset say X, such that X is symmetric with respect to some point in $\mathbb{Z}^r$. The solution to this problem for $r = 2$, is elementary. However, for $r > 2$ the solution is complex and involves algebraic topology. This classical problem has many variations and generalizations and is studied widely by mathematicians. The traditional form of this problem is stated in the theorem below. A solution for non-abelian groups is still outstanding.

Theorem 14. *Let an infinite abelian group G be finitely colored. Then there exists a monochromatic symmetric subset $X \subset G$, such that X has an arbitrarily large finite size.*

Proof. Refer to [32]. □

It is proven in [16] that whenever any *infinite group* G is 2-colored, then it yields *monochromatic symmetric subsets* of arbitrarily large cardinality that is less than $|G|$. The article [17] also generalizes several results on 2-colorings of infinite commutative groups as seen in [32]. However, this thesis examines monochromatic symmetric subsets of colorings in a *finite commutative group setting*, modeled from the material in [43]. Whereas, the succeeding chapters discuss the computational aspect of colorings on groups, this chapter approaches these colorings from a Ramsey-theoretic perspective.

3.1

Monochromatic Symmetric Subsets in Colorings

This section evaluates the symmetric subsets in *finite abelian groups*. We follow the article [43] closely. The latter studies *Ramsey type functions* for symmetric subsets in finite abelian groups.

Definition 34. Let G be a finite group. A subset $Y \subseteq G$ is called *symmetric* if $\exists\, g \in G$, such that $gY^{-1}g = Y$. The element g is called the *centre of symmetry*.

The two numbers $\lambda_r(G)$ and $\omega_r(G)$ are introduced in the definitions below.

Definition 35. Let G be a finite group and $r,\, t \in \mathbb{N}$. We define $\lambda_r(G)$ to be the *maximum number* represented by $\frac{t}{|G|}$, such that every r-coloring of G yields a monochromatic symmetric subset Y, where $|Y| = t$.

Definition 36. Let G be a finite group and $r,\, t \in \mathbb{N}$. Then $\omega_r(G)$ is the *maximum number* represented by $\frac{t}{|G|}$, such that every r-coloring χ of G yields a subset $Y \subseteq G$, where $|Y| = t$ and $g \in G$ such that $\chi(y) = \chi(gy^{-1}g),\ \forall\, y \in Y$.

Note 7. The two values $\lambda_r(G)$ and $\omega_r(G)$ may be identified as *Ramsey functions*. They have a notable affiliation to *Ramsey numbers*.

Lemma 2. *Let G be a finite group and $r,\, t \in \mathbb{N}$. Then*

(1) $\lambda_r(G) \leq \frac{1}{r} + \frac{1}{|G|}$,

(2) $\omega_r(G) \leq 1$ *and*

(3) $\lambda_r(G) \geq \dfrac{\omega_r(G)}{r}$.

Proof. These inequalities are easily deducible from the definition. $\square$

Definition 37. Let G be a finite group, $\chi : G \longrightarrow [r]$ and $g \in G$. Define

$$T(\chi,\, g) = |\{y \in G \ : \ \chi(y) = \chi(gy^{-1}g)\}| \text{ and}$$

$$\omega(\chi) = \frac{1}{|G|} \max_{g \in G} T(\chi,\, g).$$

Then

$$\omega_r(G) = \min_{\chi : G \longrightarrow [r]} \omega(\chi).$$

Lastly, $\forall\ a \in G$ let

$$c(a) = |\{y \in G \ : \ y^2 = a\}|.$$

Lemma 3. *For all* $\chi : G \longrightarrow [r]$ *and* $S_n = \chi^{-1}(n)$

$$\sum_{g \in G} T(\chi,\, g) = \sum_{n=1}^{r} \sum_{(x,\, y) \in S_n^2} c(yx^{-1}).$$

Proof. There are two ways to calculate the number of elements $(g,\, x,\, g)$ that belong to the direct product $G \times G \times G$, such that $gx^{-1}g = y$. Therefore,

$$\sum_{g \in G} T(\chi,\, g) = \sum_{n=1}^{r} \sum_{(x,\, y) \in S_n^2} |\{g \in G \ : \ \ gx^{-1}g = y\}|.$$

However

$$|\{g \in G \ : \ \ gx^{-1}g = y\}| = |\{g \in G \ : \ \ gx^{-1}gx^{-1} = yx^{-1}\}| = c(yx^{-1}).$$

The result follows. □

Theorem 15. *Let G be a finite group of odd order or a finite abelian group. For $r \in \mathbb{N}$*

$$\omega_r(G) \geq \frac{1}{r} \text{ and } \lambda_r(G) \geq \frac{1}{r^2}.$$

Proof. Suppose $\chi : G \longrightarrow [r]$ and $S_n = \chi^{-1}(n)$. By Lemma 3,

$$\sum_{g \in G} T(\chi, g) = \sum_{n=1}^{r} \sum_{(x, y) \in S_n^2} c(yx^{-1}).$$

Case 1: Assume that G is a group of odd order. It follows that $c(yx^{-1}) = 1, \forall\, x, y \in G$. Now the function

$$x_1^2 + \cdots + x_r^2,$$

where $x_1 + \cdots + x_r = B$ reaches a minimum whenever

$$x_1 + \cdots + x_r = \frac{B}{r}.$$

So

$$\sum_{g \in G} T(\chi, g) = \sum_{n=1}^{r} |S_n^2| \geq \underbrace{\left(\frac{|G|}{r}\right)^2 + \cdots + \left(\frac{|G|}{r}\right)^2}_{r \text{ times}} = \frac{|G|^2}{r}.$$

Case 2: Assume that G is an abelian group. So we have that $c(yx^{-1}) > 0 \iff yx^{-1} \in G^2 = \{g^2 : g \in G\}$. Suppose that $yx^{-1} \in G^2$ then $c(yx^{-1}) = [G : G^2]$. Let H_j, for $1 \leq j \leq t$, be cosets of G modulo G^2. Then

$H_{j,n} = H_j \bigcap S_n$ and

$$\sum_{g \in G} T(\chi, g) = \sum_{n=1}^{r} \sum_{j=1}^{t} |H_{j,n}|^2 \cdot t \geq rt \left(\frac{|G|}{rt} \right)^2 \cdot t = \frac{|G|^2}{r}.$$

Hence, in both cases $\exists\, g \in G$ such that $T(\chi, g) \geq \dfrac{|G|}{r} \implies \omega_r(\chi) \geq \dfrac{1}{r}$.

□

Note 8. For a *non-abelian* group G the statement $\lambda_r(G) \geq \dfrac{1}{r^2}$ is false. This may be shown by the following *counterexample*. Consider the *Quaternion group* $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Then $\lambda_2(Q_8) = \dfrac{1}{8} \neq \dfrac{1}{2^2}$.

Next we classify $\omega_r(G) = \dfrac{1}{r}$ and $\omega_r(G) = 1$ for finite abelian groups. The following two theorems are found in [43].

Theorem 16. *Let G be a finite abelian group and $r \in \mathbb{N}$. We have $\omega_r(G) = \dfrac{1}{r} \iff r$ is a divisor of $|2G|$.*

Proof. "$\implies$" Suppose $\omega_r(G) = \dfrac{1}{r}$. Let the subgroups $2G = \{2g \ : \ g \in G\}$ and the *Boolean group* $B(G) = \{g \in G \ : \ 2g = 0\}$. Denote the order of these subgroups as $|2G| = l$ and $|B(G)| = t$. Clearly $|G| = lt$. Now suppose that r does not divide l. Choose an arbitrary r-coloring of group G say χ. For $1 \leq j \leq t$ let H_j be cosets of G modulo $2G$. Now

$$H_{j,n} = H_j \bigcap \chi^{-1}(n) \text{ and}$$

$$\sum_{n=1}^{r} |H_{j,n}|^2 > r \left(\frac{l}{r} \right)^2 = \frac{l^2}{r}.$$

Therefore,

$$\sum_{g \in G} T(\chi, g) = t \cdot \sum_{n=1}^{t} \sum_{j=1}^{r} |H_{j,n}|^2 > t^2 \cdot \frac{l^2}{r} = \frac{|G|^2}{r}.$$

So, we have shown that $\exists\, g \in G$ such that $T(\chi, g) > \dfrac{|G|}{r} \implies \omega(\chi) > \dfrac{1}{r}$. This is a contradiction therefore we conclude that r divides l.

"$\Longleftarrow$" Suppose that r divides l. By Theorem 15 we have that $\omega_r(G) \geq \dfrac{1}{r}$. Hence, it is sufficient to create an r-coloring χ of G with $\omega(\chi) = r$. Choose a subgroup C of G, such that $B(G) \subseteq C$ and $[G : C] = r$. Then clearly $[2G : 2C] = r$. Define a coset K of G modulo $2C$. There are r such cosets K. Moreover, all of these r cosets K form a coset of G modulo $2G$. We construct χ such that it satisfies the following criteria;

(1) every coset K is monochromatic and

(2) all r cosets K are colored in r different colors.

So for $g,\ h \in G$, we have

$$\begin{aligned}
\chi(h) = \chi(2g - h) &\iff h - (2g - h) \in 2C \\
&\iff 2(h - g) \in 2C \\
&\iff \exists\, c \in C,\ \text{such that } 2(h - g - c) = 0 \\
&\iff \exists\, c \in C,\ \text{such that } h - g - c \in B(G) \\
&\iff h - g \in C + B(G) = C \\
&\iff h \in g + C.
\end{aligned}$$

Therefore $T(\chi,\, g) = |C|,\ \forall\, g \in G$. Hence, $\omega(\chi) = \dfrac{|C|}{|G|} = \dfrac{1}{[G:C]} = \dfrac{1}{r}$ and the proof is complete.

□

The result below characterizes all finite abelian groups in which each r-coloring is symmetric.

Theorem 17. *Let G be a finite abelian group and $r \in \mathbb{N}$. Then we have $\omega_r(G) = 1 \iff$ one of the following cases is true:*

(1) $r = 1$*;*

(2) $r = 2$ *or G is cyclic such that $|G| = 3$ or $|G| = 5$;*

(3) *G is a Boolean group.*

Proof. "$\Longrightarrow$" The forward statement is easily verifiable.

"$\Longleftarrow$" Assume that *none* of the above cases (1) to (3) is true. We will give a proof by contradiction. The proof is broken into two cases for $|G|$ even and for $|G|$ odd.

Case 1: Assume that $|G|$ is even. Then the subgroups $2G = \{2x \ : \ x \in G\}$ and $B(G) = \{x \in G \ : \ 2x = 0\}$ are not equal to G. Choose $x,\, y \in G$, such that $x + y \notin 2G$ and $x - y \notin B(G)$. It follows that $2g \neq x + y,\ \forall\, g \in G$ and $2x - 2y \neq 0$. Now define an r-coloring $\chi : G \longrightarrow \{1,\, 2\}$ as follows

$$\chi(g) = \begin{cases} 1, & \text{if } g \in \{x,\, y\} \\ 2, & \text{otherwise.} \end{cases}$$

Let $g \in G$. Since $x + y \notin 2G \implies 2g - x \neq y$. Suppose that $2g - x = x \implies 2g - y \neq y$ since $x - y \notin B(G)$. Consequently either $\chi(x) \neq \chi(2g - x)$ or $\chi(y) \neq \chi(2g - y)$. If either statement is true we have a contradiction and the result follows.

Case 2: Assume that $|G|$ is odd. Then clearly $2G = G$. We also observe that $|G| \geq 7$. We may choose $x, y, z \in G$, where $x \neq y \neq z$, such that $\forall\ g, a \in \{x, y, z\}$ where $g \neq a$, $2g - a \notin \{x, y, z\}$. To verify this we take $x, y \in G$, where $x \neq y$. Then there exists a unique $g \in G$, such that $y = 2g - x$. Let $H = G \backslash \{x, y, g, 2x - y, 2y - x\}$ and take $z \in H$. Now define an r-coloring $\chi : G \longrightarrow \{1, 2\}$ as follows

$$\chi(g) = \begin{cases} 1, & \text{if } g \in \{x, y, z\} \\ 2, & \text{otherwise.} \end{cases}$$

Take $g \in G$. Suppose $g \notin \{x, y, z\}$ and $2g - x \neq y$ then $2g - z \notin \{x, y, z\}$. Suppose $g \in \{x, y, z\}$ and $g = x$ then $2g - y \notin \{x, y, z\}$. Therefore, it follows that $\exists\ a \in \{x, y, z\}$, such that $\chi(a) \neq \chi(2g - a)$. This is a contradiction and hence the result follows.

□

3.2

Ramsey Functions for 2-colorings of Finite Abelian Groups

This section classifies $\lambda_2(G)$ for a finite abelian group G in the theorem below. The proof of this result is delayed until after we recount several auxiliary statements. However, the proofs of these statements are omitted. We recommend that the reader consult with [43] for the complete proofs of these results.

Theorem 18. *Let $n \in \mathbb{N}$ and suppose G is a finite abelian group. The following statements are true:*

(1) $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_4$ *is a subgroup of* $G \Longrightarrow \lambda_{2^n}(G) = \dfrac{1}{4^n}$

(2) $\mathbb{Z}_4$ *is not a subgroup of* $G \Longrightarrow \lambda_2(G) > \dfrac{1}{4}$.

Lemma 4. *Let $n \in \mathbb{N}$.*

$$\lambda_{2^n}\left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_4\right) = \frac{1}{4^n}.$$

Proof. Refer to [43]. □

Lemma 5. *Suppose that G is a finite group such that there exists a surjective homomorphism $\psi : G \longrightarrow X$. Let χ be a coloring of X. Define a coloring γ of G as $\gamma = \chi \circ \psi$. Then $\lambda(\gamma) = \lambda(\chi)$.*

Proof. Refer to [43]. □

The lemma above induces the following corollary.

Corollary 1. *Let $r \in \mathbb{N}$ and suppose that G is a finite group. Let X be the homomorphic image of G. Then $\lambda_r(G) \leq \lambda_r(X)$.*

□

Now we are prepared to prove our principal result.

Proof of Theorem 18.

Case 1: We may deduce from Theorem 15 that $\lambda_{2^n}(G) \geq \dfrac{1}{4^n}$. Now suppose that $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_4 \leq G$. There exists a homomorphism say, $f : G \longrightarrow \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_4$. Therefore, by Corollary 1, $\lambda_{2^n}(G) \leq \lambda_{2^n}\left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_4\right)$. Moreover, by Lemma 4, $\lambda_{2^n}\left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_4\right) = \dfrac{1}{4^n}$. Thus $\lambda_{2^n}(G) = \dfrac{1}{4^n}$ and the proof is complete.

Case 2: Suppose that $\mathbb{Z}_4$ is not a subgroup of G and let $Y \leq G$, such that $|Y|$ is odd. Furthermore, let B denote the Boolean group of G. Then $G = Y \times B$. Suppose χ is any 2-coloring of G. Define the coloring χ_x on Y for $x \in B$ as follows:

$$\chi_x(g) = \chi(g, x), \forall \ g \in Y.$$

Recall that

$$T(\chi, g) = |\{x \in G \ : \ \chi(x) = \chi(gx^{-1}g)\}|.$$

Thus we have

$$\sum_{y\in Y} T(\chi, y) = \sum_{y\in Y} \sum_{x\in B} T(\chi_x, y) = \sum_{x\in B} \sum_{y\in Y} T(\chi_x, y).$$

Since $|Y|$ is odd we have

$$\sum_{y \in Y} T(\chi_x,\, y) > \frac{|Y|^2}{2}.$$

Thus we obtain

$$\sum_{y \in Y} T(\chi,\, y) = \sum_{y \in Y} \sum_{x \in B} T(\chi_x,\, y) > |B| \cdot \frac{|Y|^2}{2}.$$

This implies that $\exists\ y \in Y$ such that

$$T(\chi,\, y) > \frac{|B| \cdot |Y|}{2} = \frac{|G|}{2}.$$

Hence $\omega(\chi) > \dfrac{1}{2}$. It follows that $\lambda(\chi) > \dfrac{1}{4}$ and the proof is complete.

□

The above theorem gives rise to two observations listed in the corollaries below.

Corollary 2. *Let G be a finite abelian group, $\lambda_2(G) = \dfrac{1}{4} \iff \mathbb{Z}_4$ is a subgroup of G.*

□

Moreover, whenever $G = \mathbb{Z}_n$ the following equivalence arises.

Corollary 3. *Let $n \in \mathbb{N}$ $\lambda_2(\mathbb{Z}_n) = \dfrac{1}{4} \iff 4$ divides n.*

□

This section culminates with two tables that depict the Ramsey function values $\lambda_2(\mathbb{Z}_n)$ and $\omega_2(\mathbb{Z}_n)$, for $n \leq 8$. Each of these Ramsey functions yield corresponding

2-colorings of $\mathbb{Z}_n$, drawn in the article [43]. It should also be noted that there is a conspicuous dearth of known values for $\lambda_r(\mathbb{Z}_n)$ and $\omega_r(\mathbb{Z}_n)$, whenever $r > 2$.

$\boldsymbol{n}$	1	2	3	4	5	6	7	8
$\mathbf{t}$	1	1	2	1	3	2	3	2
$\lambda_\mathbf{2}(\mathbb{Z}_\mathbf{n}) = \frac{t}{n}$	1	0.5	0.666666	0.25	0.6	0.333333	0.42857142	0.25

Table 3.1: Ramsey function values $\lambda_2(\mathbb{Z}_n)$, for $n \leq 8$.

$\boldsymbol{n}$	1	2	3	4	5	6	7	8
$\mathbf{t}$	1	2	3	2	5	4	5	4
$\omega_\mathbf{2}(\mathbb{Z}_\mathbf{n}) = \frac{t}{n}$	1	1	1	0.5	1	0.66666666	0.71428571	0.5

Table 3.2: Ramsey function values $\omega_2(\mathbb{Z}_n)$, for $n \leq 8$.

The Ramsey function $\lambda_r(G)$ has been studied quite extensively. For a *compact abelian group* $\lambda_r(G)$ displays asymptotic behaviour. The study of this Ramsey function is closely related to topology and is therefore not within the scope of this dissertation. Thus, the reader is referred to the articles [24] and [45] for further detail.

CHAPTER 4

SYMMETRIC COLORINGS OF THE DIHEDRAL GROUP

An r-coloring χ of any finite group G is called symmetric if $\exists\ a \in G$, such that $\forall\ x \in G$ the condition $\chi(ax^{-1}a) = \chi(x)$ is satisfied. This notion of symmetry on groups has its origins in [27]. The article [46] explicitly derived formulas that count the number of *symmetric r-colorings* and *equivalence classes of symmetric r-colorings* of group G, denoted by $S_r(G)$ and $s_r(G)$ respectively. The established combinatorial method of computing the number of symmetric group colorings was examined in the prequel [39]. This section presents a brief summary of these combinatorial results on symmetric colorings of finite groups. These key results are collectively extricated from the articles [18], [30], [39], [42] and [46]. However, when the equations reviewed here are applied to the abundance of finite groups they become computationally infeasible. This chapter contrives to streamline the laborious approach of counting symmetric colorings whenever the group under consideration is dihedral.

Theorem 19. *For a finite abelian group G and for $r \in \mathbb{N}$*

$$S_r(G) = \sum_{F \le G} \sum_{H \le G} \frac{\mu(H,\, F)|G/H|}{|B(G/H)|} r^{(|G/F|+|B(G/F)|)/2} \textit{ and}$$

$$s_r(G) = \sum_{F \le G} \sum_{H \le G} \frac{\mu(H,\, F)}{|B(G/H)|} r^{(|G/F|+|B(G/F)|)/2}.$$

Furthermore, $B(G) = \{g \in G \,:\, g^2 = \boldsymbol{e}\}$ is the Boolean group, $\le$ is the subgroup relation and $\mu(H,\, F)$ is the Möbius function on the lattice of subgroups of G.

Proof. See [46]. □

The article [18] substitutes $G = \mathbb{Z}_n$ and thereby utilizes multiplicative functions to simplify the above expressions as per the theorem below.

Theorem 20. *Let p be prime, d be a divisor of n, m be odd and $l \ge 1$. Then*

$$S_r(\mathbb{Z}_n) = \begin{cases} \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}}, & \text{for } n \text{ odd} \\ \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1}, & \text{for } n = 2^l m, \end{cases}$$

$$s_r(\mathbb{Z}_n) = \begin{cases} r^{\frac{n+1}{2}}, & \text{for } n \text{ odd} \\ \frac{1}{2}\left(r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}\right), & \text{for } n = 2^l m. \end{cases}$$

Proof. See [18] or [42]. □

Finally Theorem 19 may be generalized to include all arbitrary finite groups.

Theorem 21. *Let P denote the poset of optimal partitions of a finite group G. If*

$\mu(\Omega, \Psi)$ *denotes the Möbius function on* P*, then*

$$|S_r(G)| = |G| \sum_{\Omega \in P} \sum_{\Psi \leq \Omega} \frac{\mu(\Psi, \Omega)}{|Z(\Psi)|} r^{|\Omega|} \text{ and}$$

$$[1em]|S_r(G)/\sim| = \sum_{\Omega \in P} \sum_{\Psi \leq \Omega} \frac{\mu(\Psi, \Omega)|St(\Psi)|}{|Z(\Psi)|} r^{|\Omega|}.$$

Proof. See [18]. □

The accompanying sections in this chapter offer a more enriched perspective on the familiar results presented in [39]. The *dihedral group* becomes the center of our attention. The results contained here are in conjunction with the articles [28], [29] and [30].

4.1

Counting Symmetric Colorings of the group D_p

Let the group D_p denote the dihedral group of order $2p$, where $p > 2$ is prime. For $r \in \mathbb{N}$ the number of symmetric r-colorings and the number of symmetric r-ary necklaces of D_p is represented by $S_r(D_p)$ and $s_r(D_p)$ respectively. This section presents two polynomial expressions for computing these values. These formulas are explicitly derived in [28] and are a generalization of the result rendered in [23], for the group D_3. The direct derivation of these expressions are an application of Theorem 21, which is based on the familiar technique of constructing the poset of optimal partitions. The reader is referred back to Definition 23, Definition 24 and Definition 25. Firstly, the following result provides us with the exact number of subgroups of D_p.

Corollary 4. *For each prime $p > 2$ the number of subgroups $V(p)$ of D_p is $p+3$.*

Proof. There are two divisors of p, namely 1 and p. Hence, the number of divisors $o(p)$ of p is 2 and the sum of divisors $o'(p)$ of p is $p+1$. By *Cavior's Theorem* (Theorem 4) we have $V(p) = o(p) + o'(p) = 2 + 1 + p = p + 3$. □

Suppose that $D_p = \langle x, y \; : \; x^p = \mathbf{e} = y^2, \, x^{p-1}y = yx \rangle$ thus, the $p+3$ subgroups of D_p are listed below.

(1) D_p of order $2p$

(2) One subgroup of order p : $\langle x \rangle$

(3) p subgroups of order 2 : $\langle y \rangle, \, \langle xy \rangle, \, \langle x^2 y \rangle, \, \cdots, \, \langle x^{p-1} y \rangle$

(4) One subgroup of order 1 : $\langle \mathbf{e} \rangle$

An analysis of the subgroups of D_p for each prime $p > 2$, reveals that they have a particular structural regularity. These systematic similarities in the subgroup structure of D_p forms a pattern. This pattern is generalized in the *subgroup lattice* pictured below.

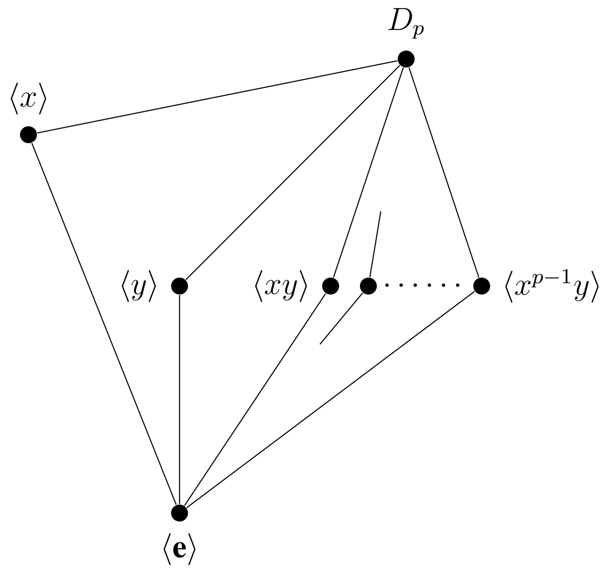


Figure 4.1: Generalized Subgroup Lattice of D_p, for prime $p > 2$

The *optimal partitions* of D_p (for every prime $p > 2$) are listed below, together with their respective stabilizers and centers. Let the poset of these optimal partitions be denoted by, Υ_{D_p}.

(1) The partition that is *finest* : $\Omega = \{\mathbf{e}\}, \{y\}, \{xy\}, \cdots, \{x^{p-1}y\}, \{x, x^{p-1}\}, \cdots$

$St(\Omega) = \{\mathbf{e}\}, Z(\Omega) = \{\mathbf{e}\}$

$|St(\Omega)| = 1, |Z(\Omega)| = 1, |\Omega| = p + 1 + \dfrac{p-1}{2} = \dfrac{3p+1}{2}$

(2) p partitions of the structure : $\Omega = \{\mathbf{e}\}, \{x, x^{p-1}\}, \cdots, \{y\}, \{xy, x^{p-1}y\}, \cdots$

$St(\Omega) = \{\mathbf{e}\}, Z(\Omega) = \{\mathbf{e}, y\}$

$|St(\Omega)| = 1, |Z(\Omega)| = 2, |\Omega| = 2\left(\dfrac{p+1}{2}\right) + 2 = p + 1$

(3) A single partition : $\Omega = \{\mathbf{e}, x, \cdots, x^{p-1}\}, \{y\}, \{xy\}, \cdots, \{x^{p-1}y\}$

$St(\Omega) = \{\mathbf{e}\}, Z(\Omega) = \{\mathbf{e}, x, \cdots, x^{p-1}\}$

$|St(\Omega)| = 1, |Z(\Omega)| = p, |\Omega| = (p-1) + 2 = p + 1$

(4) A single partition : $\Omega = \{y,\, xy,\, \cdots,\, x^{p-1}y\},\, \cdots,\, \{x,\, x^{p-1}\},\, \{\mathbf{e}\}$

$St(\Omega) = \{\mathbf{e}\},\, Z(\Omega) = \{\mathbf{e},\, x,\, \cdots,\, x^{p-1},\, y\}$

$|St(\Omega)| = 1,\, |Z(\Omega)| = p+1,\, |\Omega| = \dfrac{p-1}{2} + 2 = \dfrac{p+3}{2}$

(5) p partitions of the structure : $\Omega = \{\mathbf{e},\, x,\, \cdots,\, x^{p-1}\},\, \{y\},\, \{xy,\, x^{p-1}y\},\, \cdots$

$St(\Omega) = \{\mathbf{e}\},\, Z(\Omega) = \{\mathbf{e},\, x,\, \cdots,\, x^{p-1},\, y\}$

$|St(\Omega)| = 1,\, |Z(\Omega)| = p+1,\, |\Omega| = \dfrac{p+3}{2}$

(6) p partitions of the structure : $\Omega = \{\mathbf{e},\, y\},\, \{x\; x^{p-1},\, xy,\, x^{p-1}y\},\, \cdots$

$St(\Omega) = \{\mathbf{e},\, y\},\, Z(\Omega) = \{\mathbf{e},\, y\}$

$|St(\Omega)| = 2,\, |Z(\Omega)| = 2,\, |\Omega| = \dfrac{p-1}{2} + 1 = \dfrac{p+1}{2}$

(7) A single partition : $\Omega = \{\mathbf{e},\, x,\, \cdots,\, x^{p-1}\},\, \{y,\, xy,\, \cdots,\, x^{p-1}y\}$

$St(\Omega) = \{\mathbf{e},\, x,\, \cdots,\, x^{p-1}\},\, Z(\Omega) = D_p$

$|St(\Omega)| = p,\, |Z(\Omega)| = 2p,\, |\Omega| = 2$

(8) The partition that is *coarsest* : $\Omega = D_p$

$St(\Omega) = D_p,\, Z(\Omega) = D_p$

$|St(\Omega)| = 2p,\, |Z(\Omega)| = 2p,\, |\Omega| = 1$

The optimal partitions above conform to a particular pattern. This pattern generates a certain structural uniformity in the corresponding *Hasse diagrams* for Υ_{D_p}. The generalized Hasse diagram is illustrated below and contains the parameters $|St(\Omega)|,\, |Z(\Omega)|,\, |\Omega|$, together with the Möbius function values $\mu(x,\, \mathbf{e})$. The symmetric r-colorings of D_p is counted in Theorem 22, by generalizing the methodical pattern in which the optimal partitions are constructed.

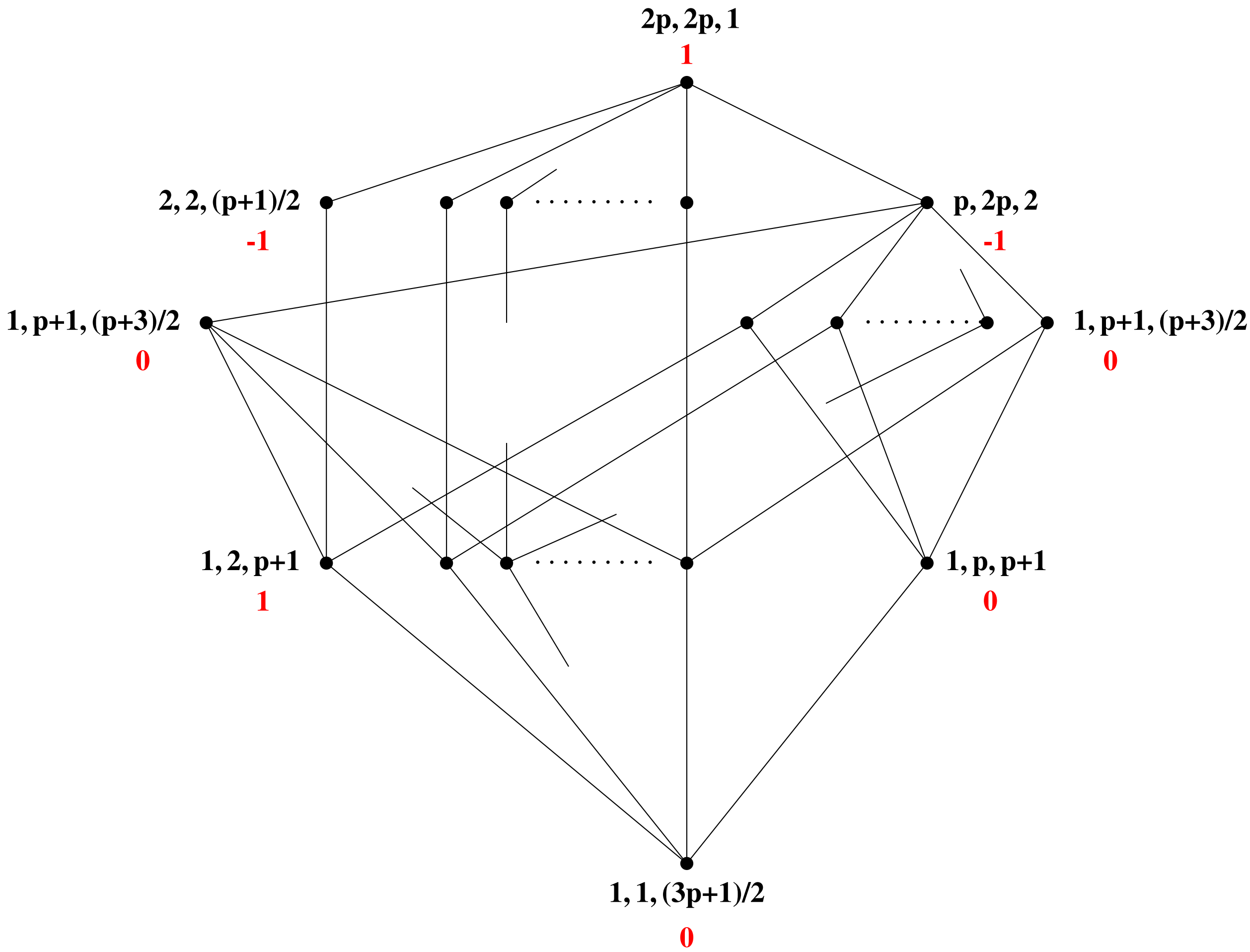


Figure 4.2: Generalized Hasse diagram of Υ_{D_p} for prime $p > 2$.

Theorem 22. *For all* $r \in \mathbb{N}$ *and primes* $p > 2$

$$S_r(D_p) = 2pr^{\frac{3p+1}{2}} - (p^2 + 2p - 2)r^{p+1} + 2pr^{\frac{p+3}{2}}(p-1) - r^2(p-1)^2 \text{ and}$$

$$s_r(D_p) = r^{\frac{3p+1}{2}} - \left(\frac{p^2+2p-2}{2p}\right) r^{p+1} + (p-1)r^{\frac{p+3}{2}} - \binom{p}{2} r^{\frac{p+1}{2}}$$

$$- \left(\frac{p^2+3p-2}{2p}\right) r^2 + \left(\frac{1-p}{2}\right) r$$

Proof. For all primes $p > 2$ let Ω and Ψ denote the optimal partitions of D_p that belong to the poset Υ_{D_p}. By Theorem 21 we have

$$S_r(D_p) = 2p \sum_{\Omega \in \Upsilon_{D_p}} \sum_{\Psi \leq \Omega} \frac{\mu(\Psi, \Omega)}{|Z(\Psi)|} r^{|\Omega|} \text{ and}$$

$$s_r(D_p) = \sum_{\Omega \in \Upsilon_{D_p}} \sum_{\Psi \leq \Omega} \frac{\mu(\Psi, \Omega)|St(\Psi)|}{|Z(\Psi)|} r^{|\Omega|}.$$

The result is obtained by rote substitution of all the parameters from Figure 4.2 into the equations above. Refer to [28] for the complete derivation. □

Note 9. Recall from [39] that the total number of all r-colorings of D_p is r^{2p}. Furthermore the total number of all equivalence classes of r-colorings of D_p is given by

$$\frac{1}{|D_p|} \sum_{g \in D_p} r^{|D_p/\langle g \rangle|} = \frac{1}{2p} \left[r^{2p} + pr^p + (p-1)r^2 \right].$$

Example 2. Consider the dihedral group of order 6 namely D_3. So $p = 3$ and by Theorem 22 we obtain

$$S_r(D_3) = 6r^5 - 13r^4 + 12r^3 - 4r^2 \text{ and}$$

$$s_r(D_3) = r^5 - \frac{13}{6}r^4 + 2r^3 + \frac{7}{6}r^2 - r.$$

This is consistent with the result found in [23].

Theorem 22 makes it far simpler to achieve polynomials for the number of symmetric colorings of D_p and equivalence classes of symmetric colorings of D_p, where $p > 2$ is prime. By this theorem the result in [23] may be accomplished more easily. However, this result is entirely dependent on the systematic predictability of the optimal partitions of D_p. In general, whenever $n \in \mathbb{N}$ is not prime the poset of optimal partitions of D_n are structurally complex in nature and do not conform to any pattern. The next section presents a superior approach to calculating the symmetric r-colorings of D_n for any $n \in \mathbb{N}$.

4.2

Counting Symmetric Colorings of the group D_n

Let the group D_n denote the dihedral group of order $2n$ where $n \in \mathbb{N}$. For $r \in \mathbb{N}$ the number of symmetric r-colorings and the number of symmetric r-ary necklaces of D_n is represented by $S_r(D_n)$ and $s_r(D_n)$, respectively. This section generalizes Theorem 22 to D_n, for all $n \in \mathbb{N}$ and derives expressions for calculating $S_r(D_n)$ and $s_r(D_n)$. These equations are explicitly derived in [30]. The computation of $S_r(D_p)$ and $s_r(D_p)$, in the previous section was dependent on the jaded method of constructing the poset of optimal partitions. However, this is a long computation especially for large values of n. This section evolves the method of counting symmetric colorings of dihedral groups. We begin by exploring some subsidiary results which are developed in [30] and [43]. The *semidirect product representation* of D_n introduced in Definition 7, holds sway over the entirety of this section.

Theorem 23. *Let coloring $\chi \in r^{\mathbb{Z}_n}$. χ is symmetric $\iff$ either $r = 1$ or $r = 2$ and $n \in \{1, 2, 3, 4, 5\}$.*

Proof. Refer to Theorem 17. □

Note 10. Recall Definition 7 and let coloring $\gamma \in r^{\mathbb{Z}_n a}$. The coloring $\gamma a \in r^{\mathbb{Z}_n}$ is defined as $\gamma a(g) = \gamma(ga)$. Furthermore, for all $\chi \in r^{D_n}$ we have $\chi|_{\mathbb{Z}_n a}a = \chi a|_{\mathbb{Z}_n}$.

Lemma 6. *Let coloring* $\chi \in r^{D_n}$. χ *is symmetric* $\Longleftrightarrow$ *at least one of the colorings* $\chi|_{\mathbb{Z}_n},\ \chi a|_{\mathbb{Z}_n} \in r^{\mathbb{Z}_n}$ *is symmetric.*

Proof. "$\Longrightarrow$" Suppose that χ is symmetric w.r.t. some $b \in \mathbb{Z}_n$ then $\chi|_{\mathbb{Z}_n}$ is symmetric. Therefore, we assume that χ is symmetric w.r.t. some $ba \in \mathbb{Z}_n a$. So

$$\chi(ba(xa)^{-1}ba) = \chi(xa)$$
$$\Longrightarrow\ \chi(bx^{-1}ba) = \chi(xa),\ \forall\ x \in \mathbb{Z}_n.$$

Hence, $\chi a|_{\mathbb{Z}_n}$ is symmetric w.r.t. $b \in \mathbb{Z}_n$.

"$\Longleftarrow$" Suppose that $\chi|_{\mathbb{Z}_n}$ is symmetric w.r.t. some $b \in \mathbb{Z}_n$. So

$$b(xa)^{-1}b = bxab = bxb^{-1}a = xa,\ \forall\ x \in \mathbb{Z}_n.$$

Therefore, χ is symmetric w.r.t. b. Next we assume that $\chi a|_{\mathbb{Z}_n}$ is symmetric w.r.t. some $b \in \mathbb{Z}_n$. This implies that

$$bax^{-1}ba = b(x^{-1}b)^{-1} = x,\ \forall\ x \in \mathbb{Z}_n \text{ and}$$
$$baxaba = bx^{-1}ba$$
$$\Longrightarrow\ \chi(ba(xa)^{-1}ba) = \chi(bax^{-1}ba) = \chi(xa),\ \forall\ x \in \mathbb{Z}_n.$$

Hence, χ is symmetric w.r.t. ba and the result follows. □

Corollary 5. *Let coloring* $\chi \in r^{D_n}$. χ *is symmetric* $\Longleftrightarrow$ *either* $r = 1$ *or* $r = 2$ *and* $n \in \{1, 2, 3, 4, 5\}$.

Proof. This statement follows from Theorem 23 and Lemma 6. □

The theorem below is our first essential result and coincides with [30].

Theorem 24. *For all* $r,\ n \in \mathbb{N}$

$$S_r(D_n) = 2r^n S_r(\mathbb{Z}_n) - [S_r(\mathbb{Z}_n)]^2.$$

Proof. Let coloring $\chi \in r^{D_n}$. There are $S_r(\mathbb{Z}_n)r^n$ colorings χ such that $\chi|_{\mathbb{Z}_n}$ is symmetric. Also there are $r^n S_r(\mathbb{Z}_n)$ colorings χ such that $\chi|_{\mathbb{Z}_n a}a$ is symmetric. Moreover, there are $[S_r(\mathbb{Z}_n)]^2$ colorings χ such that both $\chi|_{\mathbb{Z}_n}$ and $\chi|_{\mathbb{Z}_n a}a$ are symmetric. As a result of this, by the *inclusion-exclusion principle*, there are $2r^n S_r(\mathbb{Z}_n) - [S_r(\mathbb{Z}_n)]^2$ colorings χ, such that $\chi|_{\mathbb{Z}_n}$ or $\chi|_{\mathbb{Z}_n a}a$ is symmetric. By Lemma 6, these colorings gives us exactly the symmetric colorings of D_n and the statement follows. □

By the above result the equation for $S_r(D_p)$ in Theorem 22 may be achieved in a more efficient manner.

Corollary 6. *For all* r, $n \in \mathbb{N}$ *and primes* $p > 2$

$$S_r(D_p) = 2pr^{\frac{3p+1}{2}} - (p^2 + 2p - 2)r^{p+1} + 2p(p-1)r^{\frac{p+3}{2}} - (p-1)^2 r^2.$$

Proof. From Theorem 20 we have that

$$S_r(\mathbb{Z}_n) = pr^{\frac{p+1}{2}} - (p-1)r$$

and by Theorem 24

$$
\begin{aligned}
S_r(D_p) &= 2r^p \left[pr^{\frac{p+1}{2}} - r(p-1)\right] - \left[pr^{\frac{p+1}{2}} - r(p-1)\right]^2 \\
&= 2pr^{\frac{3p+1}{2}} - 2(p-1)r^{p+1} - p^2r^{p+1} + 2p(p-1)r^{\frac{p+3}{2}} - (p-1)^2r^2 \\
&= 2pr^{\frac{3p+1}{2}} - (p^2+2p-2)r^{p+1} + 2p(p-1)r^{\frac{p+3}{2}} - (p-1)^2r^2.
\end{aligned}
$$

□

Example 3. Consider the dihedral group D_3. By Theorem 20 we have $S_r(\mathbb{Z}_3) = 3r^2 - 2r$. So by Theorem 24

$$
\begin{aligned}
S_r(D_3) &= 2r^3 S_r(\mathbb{Z}_3) - [S_r(\mathbb{Z}_3)]^2 \\
&= 2r^3(3r^2 - 2r) - (3r^2 - 2r)^2 \\
&= 6r^5 - 13r^4 + 12r^3 - 4r^2.
\end{aligned}
$$

This remains consistent with Example 2.

Computing the number of symmetric necklaces of D_n, denoted by $s_r(D_n)$, is a more complex problem. Several preparatory results must be constructed first in order to derive an expression for $s_r(D_n)$.

Lemma 7. *Suppose $\chi \in r^{D_n}$ and let $\gamma = \chi|_{\mathbb{Z}_n}$ and $\varphi = \chi a|_{\mathbb{Z}_n}$. Furthermore let $X = St(\gamma)$ and $Y = St(\varphi)$. If γ and φ are not equivalent we have $St(\chi) = X \cap Y$. Lastly, if $\varphi = \gamma b$ for $b \in \mathbb{Z}_n$ then $St(\chi) = X \cup Xba$.*

Proof. Let $g \in \mathbb{Z}_n$, now $g \in St(\chi) \iff \chi g|_{\mathbb{Z}_n} = \gamma$ and $\chi g|_{\mathbb{Z}_n a} = \varphi$. Moreover,

$ga \in St(\chi) \iff \chi ga|_{\mathbb{Z}_n} = \gamma$ and $\chi ga|_{\mathbb{Z}_n a} = \varphi$. For all $x \in \mathbb{Z}_n$ we have

$$\chi g(x) = \chi(xg^{-1}) = \gamma(xg^{-1}) = \gamma g(x)$$

$$\chi g(xa) = \chi(xag^{-1}) = \chi(xga) = \varphi(xg) = \varphi g^{-1}(x),$$

$$\chi ga(x) = \chi(xga) = \varphi(xg) = \varphi g^{-1}(x) \text{ and}$$

$$\chi ga(xa) = \chi(xaga) = \chi(xg^{-1}) = \gamma(xg^{-1}) = \gamma g(x).$$

This implies that the following two equivalences occur:

(1) $g \in St(\chi) \iff \gamma g = \gamma$ and $\varphi g^{-1} = \varphi$,

(2) $ga \in St(\chi) \iff \gamma g = \varphi$.

Consequently from **(1)** we deduce that $g \in St(\chi) \iff g \in X \cap Y$. Furthermore from **(2)** we also deduce that if γ and φ are not equivalent, $ga \notin St(\chi)$ and if $\varphi = \gamma b$ then $ga \in St(\chi) \iff gb^{-1} \in X$. This completes the proof. □

Definition 38. Let $\chi \in r^G$, χ is called an *aperiodic* r-coloring if $St(\chi) = \{1\}$.

Lemma 8. *Let G be an abelian group with a subgroup Y. The number of $\chi \in r^G$ such that $St(\chi) = Y$ is equal to the number of aperiodic r-colorings of G/Y. Symbolically*

$$|\{\chi \in r^G \ : \ St(\chi) = Y\}| = |\{\chi \in r^{G/Y} \ : \ St(\chi) = \{1\}\}|$$

Proof. This is easily deducible from the definition of aperiodicity. □

The above lemma holds true for symmetric colorings as well. To this end we may define a *bijection* for $\chi/Y \in r^{G/Y}$

$$\rho : \chi \longrightarrow \chi/Y,$$

where $\chi/Y(gY) = \chi(g), \ \forall \ gY \in G/Y$.

Definition 39. Let $r, \, n \in \mathbb{N}$ and define $P_r(n)$ to be the *number of aperiodic r-colorings* of $\mathbb{Z}_n$. Furthermore, we abbreviate $P_r(n)$ to $P(n)$.

Definition 40. Let $r, \, n \in \mathbb{N}$ and define $Q_r(n)$ to be the *number of aperiodic symmetric r-colorings* of $\mathbb{Z}_n$. Furthermore, we abbreviate $Q_r(n)$ to $Q(n)$.

Theorem 25. *Let $r, \, n, \, t \in \mathbb{N}$. For Möbius function μ*

$$P_r(n) = \sum_{t|n} \mu(t) r^{\frac{n}{t}}.$$

Proof. Refer to [6]. □

Theorem 26. *Let G be a finite abelian group and $H \leq G$ and let $\chi \in r^G$ be symmetric. The number of orbits of χ such that $St(\chi) = H$ is*

$$|[\chi]| = \frac{1}{B(G/H)} \sum_{H \leq F \leq G} \mu(H, \, F) r^{\frac{|G/F| + |B(G/F)|}{2}}$$

Proof. Refer to [18]. □

The above theorem leads to the following result.

Corollary 7. *Let $r, \, n \in \mathbb{N}$. Then*

$$Q_r(n) = \frac{n}{2 - \delta(n)} \sum_{t|n} \mu(t) r^{\frac{\frac{n}{t} + 2 - \delta(\frac{n}{t})}{2}},$$

where $\delta(n)$ is defined as

$$\delta(n)=\begin{cases}1, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even.}\end{cases}$$

Proof. Refer to [18]. □

Hence, we proceed to the main result of this section.

Theorem 27. *Let $r,\ n,\ s,\ t \in \mathbb{N}$. It follows that*

$$s_r(D_n)=\sum_{\substack{s,\,t|n\\ s<t}}\frac{(s,\,t)}{n}\left[Q\left(\frac{n}{s}\right)P\left(\frac{n}{t}\right)+P\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)-Q\left(\frac{n}{s}\right)\right]$$
$$+\sum_{s|n}\frac{s}{2n}Q\left(\frac{n}{s}\right)\left[2P\left(\frac{n}{s}\right)-Q\left(\frac{n}{s}\right)+\frac{n}{s}\right].$$

Proof. Let $\chi \in r^{D_n}$ be symmetric such that the colorings $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ have different stabilizers of order s and t, respectively i.e. $|St(\chi|_{\mathbb{Z}_n})| = s$ and $|St(\chi a|_{\mathbb{Z}_n})| = t$ where $s \neq t$. The number of symmetric $\chi \in r^{D_n}$ is

$$Q\left(\frac{n}{s}\right)P\left(\frac{n}{t}\right)+P\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)-Q\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right).$$

By Lemma 7 the number of orbits of the colorings $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ is

$$\frac{(s,\,t)}{2n}\left[Q\left(\frac{n}{s}\right)P\left(\frac{n}{t}\right)+P\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)-Q\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)\right].$$

It follows that the number of symmetric $\chi \in r^{D_n}$ such that $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ have different stabilizers is

$$\sum_{\substack{s,\,t|n\\ s\neq t}} \frac{(s,\,t)}{2n}\left[Q\left(\frac{n}{s}\right)P\left(\frac{n}{t}\right)+P\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)-Q\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)\right].$$

This equation is the same as

$$\sum_{\substack{s,\,t|n\\ s<t}} \frac{(s,\,t)}{n}\left[Q\left(\frac{n}{s}\right)P\left(\frac{n}{t}\right)+P\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)-Q\left(\frac{n}{s}\right)Q\left(\frac{n}{t}\right)\right].$$

Now the number of symmetric $\chi \in r^{D_n}$ such that $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ have the same stabilizer of order s, where $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ are not equivalent is

$$\begin{aligned}
&2Q\left(\frac{n}{s}\right)\left[P\left(\frac{n}{s}-\frac{n}{s}\right)\right]-Q\left(\frac{n}{s}\right)\left[Q\left(\frac{n}{s}\right)-\frac{n}{s}\right]\\
&=Q\left(\frac{n}{s}\right)\left[2P\left(\frac{n}{s}\right)-Q\left(\frac{n}{s}\right)-\frac{n}{s}\right].
\end{aligned}$$

Hence the number of orbits of $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ is

$$\frac{s}{2n}\,Q\left(\frac{n}{s}\right)\left[2Q\left(\frac{n}{s}\right)-Q\left(\frac{n}{s}\right)-\frac{n}{s}\right].$$

Moreover, the number of symmetric $\chi \in r^{D_n}$ such that $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ have the same stabilizer of order s, where $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ are equivalent is

$$\frac{n}{s}\,Q\left(\frac{n}{s}\right).$$

Hence the number of orbits of $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ is

$$\frac{s}{n} \cdot \frac{n}{s} Q\left(\frac{n}{s}\right) = Q\left(\frac{n}{s}\right).$$

It follows that the number of orbits of symmetric $\chi \in r^{D_n}$ such that $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ have the same stabilizer of order s is

$$\frac{s}{2n} Q\left(\frac{n}{s}\right)\left[2P\left(\frac{n}{s}\right) - Q\left(\frac{n}{s}\right) - \frac{n}{s}\right] + Q\left(\frac{n}{s}\right)$$
$$= \frac{s}{2n} Q\left(\frac{n}{s}\right)\left[2P\left(\frac{n}{s}\right) - Q\left(\frac{n}{s}\right) + \frac{n}{s}\right].$$

Therefore the number of orbits of symmetric $\chi \in r^{D_n}$ such that $\chi|_{\mathbb{Z}_n}$ and $\chi a|_{\mathbb{Z}_n}$ have the same stabilizer is

$$\sum_{s|n} \frac{s}{2n} Q\left(\frac{n}{s}\right)\left[2P\left(\frac{n}{s}\right) - Q\left(\frac{n}{s}\right) + \frac{n}{s}\right].$$

So the result follows. □

Thus, Theorem 27 makes it possible to derive the equation for $s_r(D_p)$ from Theorem 22 in a more efficacious manner.

Corollary 8. *For all $r \in \mathbb{N}$ and primes $p > 2$*

$$s_r(D_p) = r^{\frac{3p+1}{2}} - \left(\frac{p^2+2p-2}{2p}\right) r^{p+1} + (p-1)r^{\frac{p+3}{2}} - \left(\frac{p}{2}\right) r^{\frac{p+1}{2}}$$
$$- \left(\frac{p^2+3p-2}{2p}\right) r^2 + \left(\frac{1-p}{2}\right) r.$$

Proof. By Theorem 27 we have

$$s_r(D_p) = \frac{1}{p}\left[Q(p)P(1) + P(p)Q(1) - Q(p)Q(1)\right] \tag{4.1}$$

$$+ \frac{1}{2p}Q(p)\left[2P(p) - Q(p) + p\right] \tag{4.2}$$

$$+ \frac{1}{2}Q(1)\left[2P(1) - Q(1) + 1\right]. \tag{4.3}$$

Now by Theorem 25 and Corollary 7

$$P(1) = Q(1) = r, \tag{4.4}$$

$$P(p) = r^p - r \text{ and} \tag{4.5}$$

$$Q(p) = pr^{\frac{p+1}{2}} - pr. \tag{4.6}$$

We substitute the formulas (4.4), (4.5) and (4.6) into (4.1), (4.2) and (4.3) respectively. Thereby (4.1) becomes

$$\frac{P(1)}{p}\left[Q(p) + P(p) - Q(p)\right] = \frac{P(1)}{p}P(p) = \frac{r}{p}(r^p - r). \tag{4.7}$$

Whereas (4.2) reduces to

$$\frac{1}{2p}\left(pr^{\frac{p+1}{1}} - pr\right)\left(2r^p - 2r - pr^{\frac{p+1}{2}} + pr + p\right) \tag{4.8}$$

$$= r^{\frac{3p+1}{2}} - \frac{p+2}{2}r^{p+1} + (p-1)r^{\frac{p+3}{2}} + \frac{p}{2}r^{\frac{p+1}{2}} - \frac{p-2}{2}r^2 - \frac{p}{2}r. \tag{4.9}$$

Lastly (4.3) reduces to

$$\frac{P(1)}{2}\left[P(1)+1\right] = \frac{r}{2}(r+1). \tag{4.10}$$

The result is achieved by combining the simplified expressions (4.7), (4.9) and (4.10). □

4.3

Counting Symmetric Colorings of the group $G \times \mathbb{Z}_2$

This section scrutinizes the number of symmetric r-colorings of $G \times \mathbb{Z}_2$ for an abelian group G, denoted by $S_r(G \times \mathbb{Z}_2)$. Suppose that $S_r(G)$ is represented by the polynomial $f(r)$. It will be shown that $f(r^2)$ becomes the polynomial that represents $S_r(G \times \mathbb{Z}_2)$. This result improves the efficacy of counting symmetric group colorings. The chapter concludes by extending these results to include *dihedral groups*. Theorem 28 discloses an expression for $S_r(G \times \mathbb{Z}_2)$ and is the highlight of this section. However, the derivation of this formula is suspended until after several prerequisite results have been examined. The central results discussed in this section are rigorously portrayed in the article [29].

Theorem 28. *Let G be a finite abelian group and $r \in \mathbb{N}$. We have*

$$S_r(G \times \mathbb{Z}_2) = \sum_{F \le G} \sum_{H \le G} \frac{\mu(H, F)|G/H|}{|B(G/H)|} r^{(|G/F|+|B(G/F)|)}. \tag{4.11}$$

Lemma 9. *Let $r \in \mathbb{N}$ and suppose that $\chi : G \times \mathbb{Z}_2 \longrightarrow [r]$. Also $\forall\, a \in \mathbb{Z}_2$ define $\chi_a : G \longrightarrow [r]$, such that $\chi_a(g) = \chi(g,\, a)$. Then χ is symmetric $\Longleftrightarrow$ $\exists\, x \in G$ such that every χ_a is symmetric w.r.t. x.*

Proof. "$\Longrightarrow$" Assume that χ is symmetric $\exists\, (x,\, b) \in G \times \mathbb{Z}_2$ such that

$$\chi((x,\, b)(g,\, a)^{-1}(x,\, b)) = \chi(g,\, a),\ \forall\, a \in G \times \mathbb{Z}_2.$$

Now we have that

$$\begin{aligned}(x,\, b)(g,\, a)^{-1}(x,\, b) &= (x,\, b)(g^{-1},\, a)(x,\, b)\\ &= (xg^{-1}x,\, bab)\\ &= (xg^{-1}x,\, a).\end{aligned}$$

Since $\chi(xg^{-1}x,\, a) = \chi(g,\, a)$ we obtain $\chi_a(xg^{-1}x) = \chi_a(g)$. Hence χ_a is symmetric w.r.t. $x \in G$.

"$\Longleftarrow$" Assume that $\exists\, x \in G$ such that every χ_a is symmetric w.r.t. x. Since

$$\begin{aligned}\chi((x,\, b)(g,\, a)^{-1}(x,\, b)) &= \chi(xg^{-1}x,\, a)\\ &= \chi_a(xg^{-1}x)\\ &= \chi_a(g)\\ &= \chi(g,\, a).\end{aligned}$$

Therefore $\forall\, b \in \mathbb{Z}_2$ χ is symmetric w.r.t. $(x,\, b)$ and the result is achieved. □

Definition 41. A *pair* of r-colorings of G, namely $(\chi_1,\, \chi_2)$ is *symmetric* if $\exists\, x \in G$ such that every χ_a is symmetric w.r.t. x. This means that $\forall\, a \in \mathbb{Z}_2$ and $\forall\, g \in G$ we have $\chi_a(xg^{-1}x) = \chi_a(g)$.

Corollary 9. *Let $r \in \mathbb{N}$. The number of symmetric r-colorings of $G \times \mathbb{Z}_2$ (denoted by $S_r(G \times \mathbb{Z}_2)$) is equal to the number of symmetric pairs of r-colorings of G.*

Proof. The mapping $\chi \longmapsto (\chi_1, \chi_2)$ defined in Lemma 9, between $r^{G \times \mathbb{Z}_2}$ and the set of *pairs of r-colorings* of G is bijective. This is trivially verifiable. Lemma 9 shows that this bijection also maps the set of *symmetric* r-colorings of $G \times \mathbb{Z}_2$ onto the set of *pairs of symmetric r-colorings* of G. So the result follows. □

Definition 42. The *group action* of G on the pairs of r-colorings of G is defined as follows $\forall\ g \in G$

$$(\chi_1, \chi_2)g = (\chi_1 g, \chi_2 g).$$

Definition 43. The *orbit* and *stabilizer* of the pair (χ_1, χ_2) is denoted respectively as $[(\chi_1, \chi_2)]$ and $St(\chi_1, \chi_2)$, where

$$[(\chi_1, \chi_2)] = \{(\chi_1, \chi_2)g \ : \ g \in G\} \text{ and}$$

$$St(\chi_1, \chi_2) = \{g \in G \ : \ (\chi_1, \chi_2)g = (\chi_1, \chi_2)\}.$$

Therefore, in the case of an existing group action we have

$$|[(\chi_1, \chi_2)]| = \frac{|G|}{|St(\chi_1, \chi_2)|}.$$

Lemma 10. *Suppose that (χ_1, χ_2) is symmetric w.r.t. $x \in G$. Then $\forall\ y \in G$, $(\chi_1, \chi_2)y$ is symmetric w.r.t. $xy \in G$.*

Proof. Assume that (χ_1, χ_2) is symmetric w.r.t. $x \in G$. Then $\forall\ g \in G$ we have

$$\begin{aligned}\chi_a y(xyg^{-1}xy) &= \chi_a(xyg^{-1}xyy^{-1}) \\ &= \chi_a(xyg^{-1}x) \\ &= \chi_a((yg^{-1})^{-1}) \\ &= \chi_a(gy^{-1}) \\ &= \chi_a y(x).\end{aligned}$$

This established the lemma. □

This lemma naturally produces the following two corollaries.

Corollary 10. *If (χ_1, χ_2) is symmetric, then every pair of r-colorings belonging to the orbit $[(\chi_1, \chi_2)]$ is also symmetric.*

□

Corollary 11. *Suppose that (χ_1, χ_2) is symmetric. Then $\forall\ x \in G$, we have $[(\chi_1, \chi_2)]_x \neq \emptyset$.*

□

Note 11. We remind the reader that for all abelian groups the additive notation is used. However **e** still denotes the identity element.

Lemma 11. *Suppose that G is an abelian group and let $y \in G$. If we let (χ_1, χ_2) be symmetric w.r.t. **e** $\in G$, then $(\chi_1, \chi_2) + y$ is symmetric w.r.t. **e** $\iff 2y \in St(\chi_1, \chi_2)$.*

Proof. Let $h \in G$ we have

$$\begin{aligned}\chi_a(-h) = \chi_a y(h) &\Longleftrightarrow \chi_a y(h+y) = \chi_a y(h-y)\\ &\Longleftrightarrow \chi_a(-h) = \chi_a(h-2y)\\ &\Longleftrightarrow \chi_a(h) = \chi_a(h-2y)\\ &\Longleftrightarrow \chi_a = \chi_a 2y.\end{aligned}$$

So the statement follows. □

This lemma leads to the corollary below.

Corollary 12. *Let G be an abelian group and suppose (χ_1, χ_2) is symmetric. If $St(\chi_1, \chi_2) = H$, where $\boldsymbol{e} \in G$ and $B(G)$ denotes the Boolean group, then*

$$|[\chi_1, \chi_2]_{\boldsymbol{e}}| = |B(G/H)|,$$

Proof. Assume that (χ_1, χ_2) is symmetric w.r.t. $g \in G$. We define $F = \{x \in G \ : \ 2x \in H\}$. Now $(\chi_1, \chi_2) - g$ is symmetric w.r.t. $\mathbf{e} \in G$. Also we have that $St((\chi_1, \chi_2) - g) = H$. Therefore

$$\{x \in G \ : \ (\chi_1, \chi_2) + x \in [(\chi_1, \chi_2)]_{\mathbf{e}}\} = F - g.$$

This follows because F is the *inverse image* of $B(G/H)$ w.r.t. the homomorphism $f : G \longrightarrow G/H$, i.e. $F = f^{-1}(B(G/H))$. Furthermore, $F - g$ contains $|B(G/H)|$ cosets of H in G.

□

Finally we are equipped to prove the main result namely Equation 4.11.

Proof of Theorem 28. By Corollary 9 we are required to count the number of pairs of symmetric r-colorings of group G. Suppose that $H \leq G$ and let $S(H)$ denote the set of all pairs of (χ_1, χ_2) symmetric w.r.t. $\mathbf{e} \in G$, such that $St(\chi_1, \chi_2) = H$. For all $(\chi_1, \chi_2) \in S(H)$ we have $|[\chi_1, \chi_2]| = |G/H|$. By Corollary 12,

$$|[\chi_1, \chi_2]_{\mathbf{e}}| = |B(G/H)|.$$

This implies that the total number of symmetric pairs of r-colorings, (χ_1, χ_2) is

$$\sum_{H \leq G} \frac{|S(H)|}{B(G/H)} |G/H|.$$

Now

$$\begin{aligned} \sum_{H \leq F \leq G} |S(H)| &= \left(r^{\frac{|G/H| - |B(G/H)|}{2} + |B(G/H)|}\right)^2 \\ &= \left(r^{\frac{|G/H| + |B(G/H)|}{2}}\right)^2 \\ &= r^{|G/H| + |B(G/H)|}. \end{aligned}$$

We may apply Möbius inversion (Theorem 7). Thus we obtain

$$|S(H)| = \sum_{H \leq F \leq G} \mu(H, F) r^{|G/F| + |B(G/F)|}.$$

Therefore the total number of symmetric pairs of r-colorings of group G is

$$\begin{aligned} S_r(G \times \mathbb{Z}_2) &= \sum_{H \leq G} \frac{|G/H|}{|B(G/H)|} \sum_{H \leq F \leq G} \mu(H, F) r^{|G/F| + |B(G/F)|} \\ &= \sum_{F \leq G} \sum_{H \leq F} \frac{\mu(H, F)|G/H|}{|B(G/H)|} r^{|G/F| + |B(G/F)|}. \end{aligned}$$

So Equation 4.11 is achieved. □

The proof above also establishes the number of equivalence classes of pairs of symmetric r-colorings.

Corollary 13. *Let $E_r(G)$ denote the set of all equivalence classes of pairs of symmetric r-colorings of group G. Then*

$$|E_r(G)| = \sum_{F \leq G} \sum_{H \leq F} \frac{\mu(H, F)}{|B(G/H)|} r^{|G/F|+|B(G/F)|}.$$

□

Note 12. The above corollary does not permit us to compute $s_r(G \times \mathbb{Z}_2)$. Contrary to Corollary 9 $s_r(G \times \mathbb{Z}_2) \neq |E_r(G)|$. This may be proven with the following counterexample.

$$s_r(\mathbb{Z}_3 \times \mathbb{Z}_2) = \frac{1}{2}(r^4 + r^2), \text{ but}$$
$$E_r(\mathbb{Z}_3) = r^4.$$

Another corollary is obtained as a result of Theorem 28.

Corollary 14. *Let G be a finite abelian group and $r \in \mathbb{N}$. Suppose that $S_r(G) = f(r)$ where $f(r)$ is a polynomial. Then $S_r(G \times \mathbb{Z}_2) = f(r^2)$ and consequently for $n \in \mathbb{N}$, $S_r(G \times \prod_n \mathbb{Z}_2) = f(r^{2^n})$.*

□

Note 13. The above corollary is true only for $S_r(G \times \mathbb{Z}_2)$. It does not hold for $s_r(G \times \mathbb{Z}_2)$. We prove this with the following counterexample. Using the values computed from Theorem 20

$$s_r(\mathbb{Z}_4) = \frac{1}{2}(r^3 + 1), \text{ but}$$

$$s_r(\mathbb{Z}_4 \times \mathbb{Z}_2) = \frac{1}{4}(r^6 + 2r^3 + r^2).$$

It is possible to extend the compelling result in Corollary 14 to include dihedral groups.

Lemma 12. *Let G be an abelian group and B be the Boolean group. Then*

$$D(G \times B) = D(G) \times B.$$

Proof. Define a mapping $f : D(G \times B) \longrightarrow D(G) \times B$ by

$$f((x, y), a) = ((x, a), y).$$

This mapping is clearly *bijective*. Next we verify that f is a *homomorphism*. So let $g = ((x_1, y_1), a)$ and $h = ((x_2, y_2), b)$. Then

$$\begin{aligned} gh &= ((x_1, y_1), a)((x_2, y_2), b) \\ &= ((x_1, y_1) + a(x_2, y_2)), a + b) \\ &= ((x_1, y_1) + (a(x_2), y_2), a + b)) \\ &= ((x_1 + a(x_2), y_1 + y_2), a + b). \end{aligned}$$

Therefore we have

$$\begin{aligned} f(gh) &= f(((x_1, y_1), a)((x_2, y_2), b))) \\ &= f((x_1 + a(x_2), y_1 + y_2), a + b) \\ &= ((x_1 + a(x_2), a + b), y_1 + y_2), \end{aligned}$$

$$\begin{aligned} f(g)f(h) &= f((x_1, y_1), a)f((x_2, y_2), b) \\ &= ((x_1, a), y_1)((x_2, b), y_2) \\ &= ((x_1, a) + (x_2, b), y_1 + y_2) \\ &= ((x_1 + a(x_2), a + b), y_1 + y_2). \end{aligned}$$

So $f(gh) = f(g)f(h)$ and the lemma is established. □

In Theorem 24 the values of $S_r(D(\mathbb{Z}_n))$ and $s_r(D(\mathbb{Z}_n))$ were determined. The following theorem strengthens this critical result by an extension to all dihedral groups $D(G)$, where G is abelian. The proof is identical to that in Theorem 24.

Theorem 29. *Let G be a finite abelian group and $r \in \mathbb{N}$. Then*

$$S_r(D(G)) = 2r^{|G|}S_r(G) - [S_r(G)]^2$$

Proof. See [30]. □

Let $S_r(G) = f(r)$ and $|G| = m$. By the above theorem we have

$$S_r(D(G)) = h(r) = 2r^m f(r) - [f(r)]^2.$$

So we obtain the following result.

Corollary 15. *For finite abelian group G*

$$S_r(D(G) \times \mathbb{Z}_2) = h(r^2).$$

Proof. By Corollary 14 $S_r(G \times \mathbb{Z}_2) = f(r^2)$. It follows from the above theorem that

$$S_r(D(G) \times \mathbb{Z}_2) = 2r^{2m} f(r^2) - [f(r^2)]^2 = h(r^2).$$

Lastly, by Lemma 12 $D(G \times \mathbb{Z}_2) = D(G) \times \mathbb{Z}_2$ and so the result follows. □

The outcomes of this section exclusively apply to finite abelian groups. We wish to extend the results to non-commutative groups.

CHAPTER 5

ALTERNATING COLORINGS

This chapter studies the concept of an alternating coloring on the vertices of a regular n-gon. As opposed to the conventional symmetric colorings which was studied previously in [39], alternating colorings exhibit more complex structural behavior. This property presents considerable difficulties when attempting to achieve our combinatorial objective of enumerating them. Since there is a notable absence of existing literature on the subject, our primary concern is to create a range of meaningful results on this dynamic new type of coloring.

We draw upon distinguished works in graph colorings such as [12], [20], [26] and [47] for inspiration. This concept bears a marginal resemblance to chromatic graph theory. For an introduction to *chromatic polynomials* the reader is referred to [34]. However we eschew graphs in favor of the cyclic group of order n, denoted by $\mathbb{Z}_n$. This chapter is analytical in nature and aims to provide precise descriptions of alternating colorings. It is seasoned with figures that illustrate the theoretical concepts. These graphic representations help us to navigate this unfamiliar territory. The results attained in this chapter are surveyed in [40].

5.1

Alternating Vertex Colorings of Regular n-gons

In this section we systematically investigate the structural properties of k-alternating r-colorings of $\mathbb{Z}_n$.

Definition 44. Let $k,\, n,\, r \in \mathbb{N}$. An r-coloring of a regular n-gon is said to be k*-alternating* if every set of k consecutive vertices has *pairwise distinct* colors.

The above definition may be translated into more mathematically precise terminology.

Definition 45. A coloring $\chi : \mathbb{Z}_n \longrightarrow [r]$ is k-alternating if $\forall\ x \in \mathbb{Z}_n$ the *restriction* of χ to the set $\{x,\, x+1,\, \cdots,\, x+k-1\}$ is *injective*.

Note 14. The above definition implies that an r-coloring $\chi : \mathbb{Z}_n \longrightarrow [r]$ is k-alternating if and only if $\forall\, a \in [1,\, k-1]$ and $x \in \mathbb{Z}_n$

$$\chi(x) \neq \chi(x+a).$$

The following proposition considers the trivial case of $k = 1$.

Proposition 4. *Let* $n,\, r \in \mathbb{N}$. *All* r*-colorings of* $\mathbb{Z}_n$ *are one-alternating.*

Proof. Let χ be any r-coloring of $\mathbb{Z}_n$. Then χ colors every vertex in $\mathbb{Z}_n$ with one color. Hence χ is a one-alternating r-coloring of $\mathbb{Z}_n$. $\square$

The above proposition clarifies the definition of k-alternating by stating that all r-colorings of $\mathbb{Z}_n$, even the monochromatic ones are trivially one-alternating.

Henceforth, we regard r-colorings to be k-alternating if and only if $k \geq 2$ since they are non-trivial.

Note 15. A 2-alternating r-coloring may be considered analogous to a *proper graph coloring*.

Definition 46. Let $n,\ r \in \mathbb{N}$. An r-coloring of $\mathbb{Z}_n$ that has at least two consecutive monochromatic vertices is referred to as *non-alternating*[1].

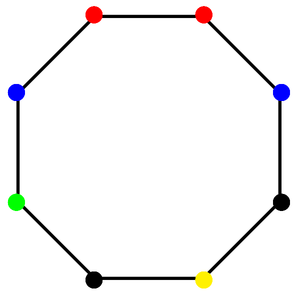


Figure 5.1: A non-alternating $\chi \in 5^{\mathbb{Z}_8}$.

Proposition 5. *Let $k,\ n,\ r \in \mathbb{N}$. For all k-alternating r-colorings of $\mathbb{Z}_n$, $k \leq r$.*

Proof. Suppose we have a k-alternating r-coloring of $\mathbb{Z}_n$ such that $k > r$. By the pigeonhole principle if k consecutive vertices are assigned r different colors, then at least one pair of the k consecutive vertices must have the same color. This is a contradiction. Therefore by the definition of a k-alternating r-coloring, $k \leq r$. □

The above proposition simply conveys that it is not possible for k to exceed the number of colors r in any k-alternating $\chi \in r^{\mathbb{Z}_n}$. Otherwise our coloring would

[1]This coloring will however be trivially one-alternating.

have monochromatic vertices and therefore be non-alternating. Henceforth $k \leq r$.

The following example offers us a visual representation of the definition of a k-alternating r-coloring.

Example 4. Let us consider a basic 3-coloring of a regular 8-gon. In Figure 5.2 we observe that this 3-colored 8-gon is 2-alternating. However, it is not 3-alternating since not every three consecutive vertices have distinct colors.

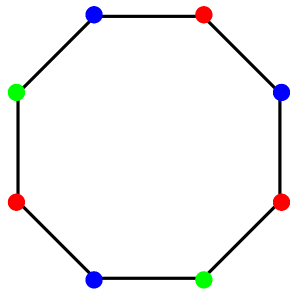


Figure 5.2: A 2-alternating $\chi \in 3^{\mathbb{Z}_8}$.

It is easily discernible from the definition of k-alternating that any k-alternating r-coloring is also $(k-1)$-alternating, $(k-2)$-alternating, $(k-3)$-alternating and so on. This is stated in the theorem below in more accurate terms.

Theorem 30. *Let $k, l, n, r \in \mathbb{N}$. For $l < k$, any k-alternating r-coloring of $\mathbb{Z}_n$ is also l-alternating.*

Proof. This is trivially deducible by the definition of k-alternating. □

The following theorems explore some salient properties of k-alternating $\chi \in r^{\mathbb{Z}_n}$.

Theorem 31. *Let $k, n, r \in \mathbb{N}$. Any r-coloring of $\mathbb{Z}_n$ that is equivalent to a k-alternating r-coloring is also k-alternating.*

Proof. Let χ and $\theta \in r^{\mathbb{Z}_n}$ such that θ is k-alternating and χ is equivalent to θ. It is required to show that χ is also k-alternating. We proceed by induction on k.

Let $k = 2$: If χ is equivalent to a 2-alternating r-coloring θ, this implies that $\exists\ g \in \mathbb{Z}_n$ such that $\forall\ x \in \mathbb{Z}_n$, $\chi(x) = \theta(x-g)$. Since θ is 2-alternating $\exists\ x,\ y \in \mathbb{Z}_n$ such that $\theta(x) \neq \theta(y)$ where $x \neq y$. We need to show that χ is also 2-alternating. So we suppose that χ is *not* 2-alternating. Then we have either one of the following two cases occurring.

Case 1: χ is l-alternating for $l \geq 2$. By Theorem 30 χ is 2-alternating and we are done.

Case 2: χ has at least two monochromatic vertices i.e. $\exists\ x,\ y \in \mathbb{Z}_n$ such that $\chi(x) = \chi(y)$ where $x \neq y$. Since χ is equivalent to θ we have

$$\begin{aligned} \chi(x) &= \chi(y) \\ \implies \theta(x-g) &= \theta(y-g) \\ \implies x - g &= y - g \\ \implies x &= y. \end{aligned}$$

This is a contradiction. Hence χ is 2-alternating.

Let $k > 2$: Assume that the statement is true for $k-1$. That is χ and $\theta \in r^{\mathbb{Z}_n}$ where χ is equivalent to the $(k-1)$-alternating r-coloring θ. This implies that χ is $(k-1)$-alternating. Now suppose that χ is equivalent to the k-alternating r-coloring θ. We want to show that χ is k-alternating. So suppose that χ is *not* k-alternating. Then we have either one of the following two cases occurring.

Case 1: χ is l-alternating where $2 \leq l \leq k-1$. By the induction hypothesis χ cannot be equivalent to θ. This is a contradiction. Hence χ is k-alternating.

Case 2: The r-coloring χ yields l consecutive monochromatic vertices such that $2 \leq l \leq n$, i.e.

$$\chi(x_1) = \chi(x_2) = \cdots = \chi(x_l).$$

Since χ is equivalent to θ, $\exists\ g \in \mathbb{Z}_n$ such that $\forall\ x \in \mathbb{Z}_n$, $\chi(x) = \theta(x - g)$. Therefore

$$\theta(x_1 - g) = \theta(x_2 - g) = \cdots = \theta(x_l - g).$$

This implies that θ yields l consecutive monochromatic vertices where $2 \leq l \leq n$. This is a contradiction since θ is k-alternating. Hence χ is k-alternating. $\square$

Now we shall consider the special case of $k = n$. This leads to an interesting observation in the theorem below.

Theorem 32. *Let $n, r \in \mathbb{N}$. An r-coloring of $\mathbb{Z}_n$ is n-alternating $\iff r = n$.*

Proof. "$\Longrightarrow$" Suppose we have a n-alternating r-coloring of $\mathbb{Z}_n$. By Proposition 5 $n \leq r$. Suppose $n < r$ then by the pigeonhole principle if r colors are assigned to the n vertices at least one vertex must have more than one color. This is a contradiction therefore $n = r$.

"$\Longleftarrow$" Suppose that we have an r-coloring of $\mathbb{Z}_n$ such that $r = n$. Clearly every vertex must be assigned a different color for this n-coloring to exist. Hence this r-coloring is n-alternating. $\square$

The following example illustrates the above argument.

Example 5. Figure 5.3 depicts a 12-alternating 12-coloring of a 12-gon.

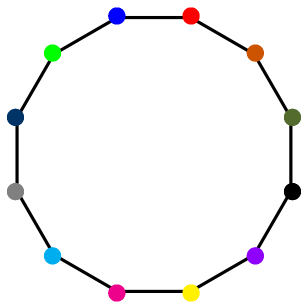


Figure 5.3: A 12-alternating $\chi \in 12^{\mathbb{Z}_{12}}$.

Now consider the case of $k = r$. Then k-alternating k-colorings exist whenever the following theorem is satisfied. This result becomes the distinguishing characteristic of all k-alternating $\chi \in k^{\mathbb{Z}_n}$.

Theorem 33. *Let $k, n, r \in \mathbb{N}$. Then for any k-alternating r-coloring of $\mathbb{Z}_n$, $k = r \iff k$ is a divisor of n.*

Proof. "$\Longleftarrow$" Assume that $k|n$ then $k = \frac{n}{t}$ for $t \in \mathbb{N}$. Define a k-alternating r-coloring of $\mathbb{Z}_n$ by $\chi : \mathbb{Z}_n \longrightarrow [r]$ such that $\forall\ x \in \mathbb{Z}_n$

$$\chi(x) = \chi(x + k),\ \forall\ k \in \mathbb{N}.$$

So the r-coloring χ partitions the n vertices into monochromatic subsets of order $\dfrac{n}{k}$. There are $\dfrac{n}{t}$ of these partitions. Hence $r = \dfrac{n}{t} = k$.

"$\Longrightarrow$" Assume that $k = r$ for any k-alternating r-coloring of $\mathbb{Z}_n$. Define a coloring $\chi : \mathbb{Z}_n \longrightarrow [r]$ such that $\forall\ x \in \mathbb{Z}_n$

$$\chi(x) \neq \chi(x + a),\ \forall\ a \in [1,\ k - 1].$$

Now suppose that k doesn't divide n then $\exists\ b,\ t \in \mathbb{Z}^+$ such that $n = bk + t$, where $t < k$. Therefore, $t \in [1,\ k-1]$ but since $k = r$, $\exists\ x \in \mathbb{Z}_n$ such that $\chi(x) = \chi(x+a)$, $\forall\ a \in [1,\ k-1]$. This implies that χ is not k-alternating. This is a contradiction. Hence k divides n. □

Example 6. Consider $\mathbb{Z}_{15}$. The divisors of 15 that do not belong to the set $\{1,\ 15\}$ will belong to the set $\{3,\ 5\}$. Therefore by Theorem 33, a 3-alternating 3-coloring and a 5-alternating 5-coloring of $\mathbb{Z}_{15}$ exists. Figure 5.4 illustrates these two colorings respectively.

(a) A 3-alternating $\chi \in 3^{\mathbb{Z}_{15}}$.

(b) A 5-alternating $\chi \in 5^{\mathbb{Z}_{15}}$.

Figure 5.4: The k-alternating colorings in $k^{\mathbb{Z}_{15}}$

Example 7. Refer to Example 4. By Theorem 33, a 3-alternating 3-coloring of $\mathbb{Z}_8$ does not exist since 3 is not a divisor of 8. However, a 3-alternating 4-coloring of $\mathbb{Z}_8$ does exist. This is drawn in Figure 5.5. More precisely by Theorem 33, a 3-alternating $\chi \in r^{\mathbb{Z}_8}$ only exists if $r > 3$.

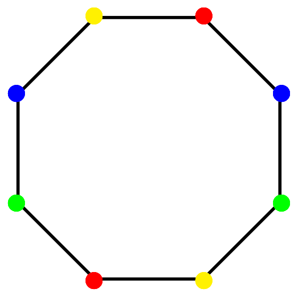


Figure 5.5: A 3-alternating $\chi \in 4^{\mathbb{Z}_8}$.

5.2

The Alternating Number

As mentioned previously this thesis is consciously influenced by graph colorings. The concept of the *chromatic number* of a graph is of particular interest to us and may be customized to suit our particular study of alternating colorings. The chromatic number of graph G is defined to be the smallest number of colors that is required to color the vertices of G so that no two adjacent vertices have the same color. More precisely the chromatic number of G, denoted by $\chi(G)$ is the smallest value of k which makes it possible to obtain a k-coloring of G.

One of the major questions that emerge from the definition of an alternating coloring is determining the smallest number of colors needed in order for a coloring of a regular n-gon to be alternating. In more mathematical terms the minimum r required for a k-alternating r-coloring of $\mathbb{Z}_n$ to exist. This notion may be interpreted as a variation of the chromatic number of a graph, although here we name it the *alternating number* of an r-colored n-gon. This concept is also guided by the notion of an *Alternation graph* explored in [19]. This expository section provides a detailed account of the fundamentals of the alternating number as seen in [40].

Definition 47. Let $k,\, n,\, r \in \mathbb{N}$ with $k \leq n$. The *minimum* value of r such that a k-alternating r-coloring of $\mathbb{Z}_n$ exists is called the *alternating number* of $\mathbb{Z}_n$. This number is denoted by $\rho(k,\, n)$.

The above definition is somewhat opaque, since it is not immediately possible to determine in general what this minimal value of r is. However, for certain values of k only specific choices for r are available. Let us consider the simplest case of $k = 1$. This produces the following proposition.

Proposition 6. *Let $n \in \mathbb{N}$. Then, $\rho(1,\, n) = 1$. That is only one color is required for a one-alternating r-coloring of $\mathbb{Z}_n$ to exist.*

Proof. Let $\chi \in \mathbb{Z}_n$ such that χ is one-alternating. By Proposition 4 we may assume that χ is monochromatic, i.e. $r = 1$. Therefore the minimum number of colors for a one-alternating r-coloring to occur is one. So $\rho(1,\, n) = 1$. □

Corollary 16. *Let $k,\, n \in \mathbb{N}$. It follows that*

$$k \leq \rho(k,\, n) \leq n.$$

Proof. This statement is clear from Definition 47. □

It is important that we describe *cycle graphs* in graph theory, since they are known to be interchangeable with regular n-gons. They have a significant impact on our investigation of alternating numbers.

Definition 48. A *cycle graph* is a graph which has only one cycle. This means that all of its vertices are connected by a closed *path*. A cycle graph of n vertices is denoted by C_n.

From our earlier explanation of the chromatic number, the alternating number of 2-alternating r-colorings $\rho(2,\, n)$ is analogous to the chromatic number of a cycle graph $\chi(C_n)$. So the following expression for $\rho(2,\, n)$ corresponds to the one for $\chi(C_n)$. This is a common and pervasive result in graph colorings and may be found in [9] and [31]. However, this result is translated into our specific terminology below.

Theorem 34. *Let $n \in \mathbb{N}$. Then*

$$\rho(2,\, n) = \begin{cases} 2, & \textit{if } n \textit{ is even} \\ 3, & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. By Proposition 5 $r \geq 2$. Suppose that n is even. The minimum number of colors needed for any 2-alternating r-coloring of $\mathbb{Z}_n$ to exist is 2. Hence for n even $\rho(2,\, n) = 2$. Suppose that n is odd and $\rho(2,\, n) = 2$. Then it is impossible *not* to obtain two vertices that are monochromatic, whenever $\mathbb{Z}_n$ is 2-colored. This is a contradiction to the r-coloring of $\mathbb{Z}_n$ being 2-alternating. Hence $\rho(2,\, n) > 2$. However a 2-alternating 2-coloring of $n - 1$ vertices exists whenever n is odd. Therefore, if we color one vertex with a different color then we obtain a 2-alternating 3-coloring of n vertices. Hence, $\rho(2,\, n) = 3$ for n odd. □

The following example illustrates the argument made in the above theorem.

Example 8. Consider the groups $\mathbb{Z}_{12}$ and $\mathbb{Z}_{15}$. By Theorem 34 we have $\rho(2,\, 12) =$

2 and $\rho(2,\,15)=3$. The colorings that correspond to these alternating numbers are drawn in Figure 5.6.

(a) A 2-alternating $\chi \in 2^{\mathbb{Z}_{12}}$

(b) A 2-alternating $\chi \in 3^{\mathbb{Z}_{15}}$

Figure 5.6: Alternating numbers $\rho(2,\,12)$ and $\rho(2,\,15)$

Our various deductions about the alternating number may be assembled into a singular decisive statement as given below.

Theorem 35. *Let $k,\,n \in \mathbb{N}$ with $k \le n$. Suppose that $n = mk + l$ and $l = k_0 m + m_0$, where $0 \le l \le k$ and $0 \le m_0 \le m$. Then*

$$\rho(k,\,n) = \left\lceil \frac{n}{m} \right\rceil = k + \left\lceil \frac{l}{m} \right\rceil = \begin{cases} k + k_0, & \textit{if } m_0 = 0 \\ k + k_0 + 1, & \textit{otherwise.} \end{cases}$$

Proof.

Case 1: Suppose that the r-coloring $\chi : \mathbb{Z}_n \longrightarrow [r]$ is k-alternating. For all $i \in [r]$ we have $|\chi^{-1}(i)| \le m$. This statement is true otherwise there exists an increasing sequence $(a_j)_{j=1}^{m+1}$ in $\{0,\,1,\,\cdots,\,n-1\}$, such that $a_{j+1} - a_j \ge k$ for all $j \le m$ and $a_1 + n - a_{m+1} \ge k$. This implies that $(m+1)k \le n$ which in turn implies $r \ge \left\lceil \frac{n}{m} \right\rceil$.

Case 2: Suppose that $r = k + \left\lceil \frac{l}{m} \right\rceil$. We may partition $\mathbb{Z}_n$ into m consecutive blocks as follows. The first m_0 blocks have $k + k_0 + 1$ elements and the next $m - m_0$ blocks have $k + k_0$ elements. This is possible because

$$m_0(k + k_0 + 1) + (m - m_0)(k + k_0) = m(k + k_0) + m_0 = n.$$

Now we define an r-coloring $\chi : \mathbb{Z}_n \longrightarrow [r]$ on each block of the partition $\{a + 1,\, a + 2,\, \cdots,\, a + i,\, \cdots\}$, by $\chi(a + i) = i$. It is easy to confirm that χ is indeed k-alternating and we are done. □

Theorem 35 yields exact values for the alternating number $\rho(k, n)$. This result generates the following two corollaries.

Corollary 17. *Let $k, n \in \mathbb{N}$, with $k \leq n$. We have the following*

(1) $\rho(k, n) = k \Longleftrightarrow k|n$,

(2) $\rho(k, n) = n \Longleftrightarrow 2k > n$ *and*

(3) $2k \leq n\rho(k, n) \leq \left\lceil \frac{n}{2} \right\rceil$.

Proof. Write $n = mk + l$ and $l = k_0 m + m_0$ where $0 \leq l \leq k$ and $0 \leq m_0 \leq m$.

(1) Then, $\rho(k, n) = k \Longleftrightarrow m_0 = 0$ and $k_0 = 0,$ that is $l = 0$.

(2) We have $2k > n \Longleftrightarrow m = 1$. So suppose that $m = 1$ then we have $\rho(k, n) = \left\lceil \frac{n}{1} \right\rceil = n$. Now we suppose that $m \geq 2$. Consequently this implies that $\rho(k, n) = \left\lceil \frac{n}{m} \right\rceil \leq \left\lceil \frac{n}{2} \right\rceil < n$. But $\left\lceil \frac{n}{2} \right\rceil < n$ since $2 \leq m \leq n$.

(3) Suppose that $2k \leq n$. This implies $m \geq 2$ so $\rho(k, n) \leq \left\lceil \frac{n}{2} \right\rceil$.

□

Corollary 18. *Let $k,\, n \in \mathbb{N}$. If $n \geq k^2$ then*

$$\rho(k,\, n) = \begin{cases} k, & \text{if } k|n \\ k+1, & \text{otherwise.} \end{cases}$$

Proof. Write $n = mk + l$ and $l = k_0 m + m_0$ where $0 \leq l \leq k$ and $0 \leq m_0 \leq m$. Now suppose that $n \geq k^2$. This implies that $m \geq k$. Therefore $k_0 = 0$ and $m_0 = l$. □

5.3

Symmetry and Alternating Colorings

An r-coloring of a regular n-gon is called *(properly) symmetric* if it is invariant with respect to some reflection about an axis. This axis must connect two vertices of the n-gon and pass through its center. There are in fact two notions of symmetry on finite abelian groups which are defined in the article [18]. In this section we discuss these two descriptions of symmetry on a regular n-gon and investigate the effect of symmetry on alternating colorings.

Definition 49. Let G be a finite abelian group. A *symmetry* of G is defined to be the mapping $x \mapsto g - x$ for $g \in G$ and $x \in G$.

Definition 50. If $\chi : G \longrightarrow \{1, \cdots, r\}$ is a *symmetric* r-coloring of a finite abelian group G, then there exists $a \in G$ such that $\chi(a - x) = \chi(x)$ for all $x \in G$.

Definition 51. Let G be a finite abelian group. A *proper symmetry* of G is defined to be the mapping $x \mapsto 2g - x$ for $g \in G$ and $x \in G$.

Definition 52. If $\chi : G \longrightarrow \{1, \cdots, r\}$ is a *properly symmetric* r-coloring of a finite abelian group G, then there exists $a \in G$ such that for all $x \in G$ $\chi(2a-x) = \chi(x)$.

There is a natural *group action* between the group G and the set of all r-colorings.

Definition 53. Let G be finite abelian group with an r-coloring χ. Then for $a \in G$ we define another r-coloring χa as follows: $\chi a(x) = \chi(x-a)$, for every $x \in G$.

Proposition 7. *Let G be finite abelian group with an r-coloring χ that is symmetric w.r.t $a \in G$. Then the r-coloring χb is symmetric w.r.t $a + b \in G$.*

Proof. Suppose that $a, b \in G$. Then we have

$$\begin{aligned}\chi b(2(a+b)-x) &= \chi(2(a+b)-x-b)\\ &= \chi(2a-(x-b))\\ &= \chi(x-b)\\ &= \chi b(x).\end{aligned}$$

□

Note 16. For the group $\mathbb{Z}_n$, whenever n is odd the proper symmetries are identical to the symmetries and whenever n is even the proper symmetries yield half the total amount of symmetries. Hence the definition of proper symmetry produces an incomplete evaluation of the symmetries of $\mathbb{Z}_n$ in contrast to the definition of symmetry. However, proper symmetries have the ability to be defined in general on any group G. This is achieved by taking proper symmetries to be the mapping $x \mapsto ax^{-1}a$ for $a, x \in G$. Whereas symmetries cannot be defined in general.

So when we refer to an r-coloring of $\mathbb{Z}_n$ as symmetric we actually mean properly symmetric. These two interpretations of symmetry on cyclic groups are graphically represented in the figure below.

(a) Properly symmetric 2-coloring of $\mathbb{Z}_{10}$

(b) Symmetric (not properly symmetric) 2-coloring of $\mathbb{Z}_{10}$

Figure 5.7: Two 2-colorings of $\mathbb{Z}_{10}$

Theorem 36. *For an even $n \in \mathbb{N}$ any 2-alternating 2-coloring of $\mathbb{Z}_n$ is symmetric.*

Proof. Let $\chi : \mathbb{Z}_n \longrightarrow \{a, b\}$ be a 2-alternating 2-coloring of $\mathbb{Z}_n$. Then $\forall\ x \in \mathbb{Z}_n$, $\chi(x+1) \neq \chi(x)$. Fix $\chi(x) = a$ then $\chi(x+1) = b$ and $\forall\ x \in \mathbb{Z}_n$ we have

$$\chi((x+1)+1) = a = \chi(x)$$
$$\Longrightarrow \chi(x+2) = \chi(x)$$
$$\Longrightarrow \chi(2-(-x)) = \chi(x).$$

Hence χ is symmetric. □

The above theorem establishes a relationship between alternating and symmetric colorings. However, this inference may not be applied to all k-alternating r-colorings as is proven below.

Theorem 37. *Let $k,\, n \in \mathbb{N}$, where $k > 2$. Then any k-alternating r-coloring of $\mathbb{Z}_n$ is not symmetric.*

Proof. Let $\chi : \mathbb{Z}_n \longrightarrow [r]$ be k-alternating and suppose that χ is symmetric w.r.t $g \in \mathbb{Z}_n$. Since χ is k-alternating we have for any $x \in \mathbb{Z}_n$,

$$\chi(x) \neq \chi(x + a),\ \forall\, a \in [1,\, k - 1].$$

Consequently there is at least one element $x \in \mathbb{Z}_n$ such that $\chi(x) = \chi(x + k)$. Otherwise the r-coloring χ will be $(k + 1)$-alternating. Since χ is symmetric w.r.t. $g \in \mathbb{Z}_n$ we have $\chi(g - x) = \chi(x),\ \forall\, x \in \mathbb{Z}_n$. We fix $x \in \mathbb{Z}_n$ such that $\chi(x) = \chi(x + k)$. Therefore $\chi(g - (x + k)) = \chi(x + k)$ and we have

$$\chi(g - x) = \chi(x) = \chi(x + k) = \chi(g - (x + k)) = \chi(g - x - k).$$

Hence χ has at least five monochromatic vertices. Furthermore $\forall\, x \neq y \in \mathbb{Z}_n$ we have $\chi(y) = \chi(g - y)$. Suppose $y = x + k - 1$ then

$$\chi(x + k - 1) = \chi(g - (x + k - 1)) = \chi((g - x - k) + 1).$$

Let $g - x - k = z \in \mathbb{Z}_n$. Therefore $\chi(z) = \chi(x + k)$ and $\chi(z + 1) = \chi(x + k - 1)$. This implies that $\chi(x + k + 1) = \chi(x + k - 1)$. Thus χ is not k-alternating and this is a contradiction. Hence χ is not symmetric w.r.t. $g \in \mathbb{Z}_n$.

□

CHAPTER 6

COUNTING ALTERNATING COLORINGS

This subsequent chapter attempts to overcome the computational challenges associated with counting the number of k-alternating r-colorings that exist for an arbitrary n. This number is denoted by $A_r(k,\, n)$ and we seek to derive a function that describes this value. This notion is closely related to the *chromatic polynomial* from graph theory, except here we take an extra variable k into account. The source [10] facilitates our work since it studies the connection between Möbius inversion and graph colorings. Whereas the text [34] makes it clear that finding the chromatic polynomial is dependent on the structure of the graph under consideration. Similarly $A_r(k,\, n)$ is subject to certain restrictions on k and r.

The goal of this chapter is to launch an inquiry into $A_r(k,\, n)$. The articles [33] and [41] discusses the application of the Möbius inversion theorem to the chromatic polynomial of a graph. Additionally [6] reflects on the various combinatorial aspects of the Möbius function. This familiar technique of enumeration via Möbius inversion, has yielded fruitful results for symmetric colorings. We rely on these auxiliary methods detailed in these articles for direction. These sources play a key role in deriving formulae for $A_r(k,\, n)$. Since our prerogative is to be as exhaustive as possible we consider each and every value of k. The solutions realized for $A_r(k,\, n)$ will be in an increasing order of maturity. After which we administer a more sophisticated method of enumerating alternating colorings in general. This material remains deferential to the results composed in [40].

Definition 54. Let $k,\, n,\, r \in \mathbb{N}$. The number of all k-alternating r-colorings of $\mathbb{Z}_n$ is denoted by $A_r(k,\, \mathbb{Z}_n)$.

If several restrictions are imposed on $k \in \mathbb{N}$, then $A_r(k,\, \mathbb{Z}_n)$ becomes fairly straightforward to calculate as outlined in the following results.

Theorem 38. *Let $n,\, r \in \mathbb{N}$. If $k = 1$ then*

$$A_r(1,\, \mathbb{Z}_n) = r^n$$

Proof. This equation follows immediately from Proposition 1 and Proposition 4. □

Theorem 39. *Let $n,\, r \in \mathbb{N}$. If $n = k$ then*

$$A_r(n,\, \mathbb{Z}_n) = r!$$

Proof. Theorem 36 implies that $r = n$. Define $\chi : \{1,\, \cdots,\, n\} \longrightarrow \{1,\, \cdots,\, r\}$ such that $\chi(x) \neq \chi(y),\ \forall\, x \neq y \in \{1,\, \cdots,\, n\}$. Then χ is a bijective r-coloring. Furthermore χ assigns one of r colors to any of the n vertices. The next vertex is assigned one of $r - 1$ colors and so on, until every vertex has a different color.

Therefore there are $r(r-1)\cdots 1 = r!$ of these colorings. Hence

$$A_r(n, \mathbb{Z}_n) = r!$$

□

Proposition 8. *Let $k, n, r \in \mathbb{N}$. Suppose $2k > n$. Then for any k-alternating r-coloring of $\mathbb{Z}_n$, $k = n$.*

Proof. By Corollary 17 for $2k > n$ we have $\rho(k, n) = n \implies r = n$. Therefore by Theorem 32 $r = n = k$ □

Corollary 19. *Let $k, n, r \in \mathbb{N}$. If $2k > n$ then*

$$A_r(k, \mathbb{Z}_n) = r!$$

Proof. By Proposition 8 we have $n = k$. It follows from Theorem 39 that $A_r(k, n) = r!$ □

Theorem 33 made the assertion that $k = r$ if and only if k divides n. This restriction on k is a desirable property for any k-alternating r-coloring to possess because it considerably simplifies the enumeration of this type of alternating coloring.

Theorem 40. *Let $k, n, r \in \mathbb{N}$, where $k = r$ and $k > 2$. Then*

$$A_r(k, n) = n(r-1)!$$

Proof. By Corollary 17 $\rho(k, n) = k = r \iff k|n$. Let us define a partition on n vertices, denoted by $\{A_1, \cdots, A_k\}$ where $|A_i| = \dfrac{n}{k}$ for $i = 1, \cdots, k$. Then define a coloring $\chi_i : A_i \longrightarrow \{1, \cdots, r\}$ as follows:

$$\chi_i(a_i) = \chi(a_i + k), \ \forall \ a_i \in A_i \text{ and } i = 1, \cdots, k.$$

Now define $\chi : \mathbb{Z}_n \longrightarrow \{1, \cdots, r\}$ as follows:

$$\chi(a) = \chi_i(a_i), \ \forall \ a \in \mathbb{Z}_n \text{ and } i = 1, \cdots, k.$$

Consequently χ results in monochromatic A_i's for all $i = 1, \cdots, k$. There are $r(r-1) \cdots (r-(k-1))$ number of r-colorings χ_i. Hence the number of colorings χ is

$$\begin{aligned} A_r(k, \mathbb{Z}_n) &= |A_i| r(r-1) \cdots (r-(k-1)) \\ &= \frac{nr}{k}(r-1) \cdots (r-(k-1)) \\ &= \frac{nr}{r}(r-1) \cdots (r-(r-1)) \\ &= n(r-1)! \end{aligned}$$

□

Example 9. The following table uses Theorem 40 to summarize the values of $A_r(k, n)$, for $k > 2$ where $k|n, \ \forall \ k, n \in \mathbb{N}$.

$\boldsymbol{k}$	3	4	5	6	7	8	9	10
$\boldsymbol{A_r(k, n)}$	$2n$	$6n$	$24n$	$120n$	$720n$	$5040n$	$40320n$	$362880n$

Table 6.1: Values of $A_r(k, n)$ for $k \in [3, 10]$.

So far we have solved for $A_r(k, \mathbb{Z}_n)$, whenever k satisfies these three restrictions; $k = n$, $k = r$ and $2k > n$. In order to fulfill our quest of enumerating alternating colorings we need to devise a practical method to compute $A_r(k, \mathbb{Z}_n)$ for any $k \in \mathbb{N}$. The ensuing section attempts to resolve this issue.

6.1

Counting 2-alternating r-colorings of $\mathbb{Z}_n$

The reader is cognizant of the fact that 2-alternating r-colorings of $\mathbb{Z}_n$ are interchangeable with *proper colorings* of the cycle graph, denoted by C_n. In this section we use the technique of *Möbius inversion* to derive a polynomial expression for $A_r(2, \mathbb{Z}_n)$, which corresponds to the chromatic polynomial of C_n. Alternatively this result is also known as the *chromatic polynomial of an n-gon* found in [13] and [34]. This strategy of utilizing the Möbius inversion theorem to attain chromatic polynomials for various types of graphs is pioneered in [33] and [41].

Consider the group $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ represented by a 4-gon in Figure 6.1. Recall from Proposition 1 that the total number of r-colorings of $\mathbb{Z}_4$ is r^4. A polynomial for the number of 2-alternating r-colorings of $\mathbb{Z}_4$ will be derived and then generalized, appropriated from material in [33] and [41].

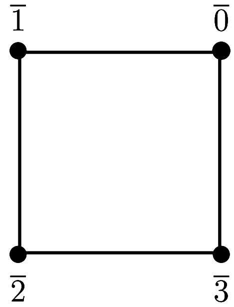


Figure 6.1: A 4-gon or $\mathbb{Z}_4$.

The bond partitions of $\mathbb{Z}_4$ together with their respective orders are evaluated below.

(1) The finest bond of order 4,

$\Omega_1 = \{\overline{0}\}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{3}\}.$

(2) Four bonds of order 3,

$\Omega_2 = \{\overline{0}, \overline{1}\}, \{\overline{2}\}, \{\overline{3}\},\ \Omega_3 = \{\overline{1}, \overline{2}\}, \{\overline{0}\}, \{\overline{3}\},\ \Omega_4 = \{\overline{2}, \overline{3}\}, \{\overline{0}\}, \{\overline{1}\},$

$\Omega_5 = \{\overline{0}, \overline{3}\}, \{\overline{1}\}, \{\overline{2}\}.$

(3) Six bonds of order 2,

$\Omega_6 = \{\overline{0}, \overline{1}, \overline{2}\}, \{\overline{3}\},\ \Omega_7 = \{\overline{1}, \overline{2}, \overline{3}\}, \{\overline{0}\},\ \Omega_8 = \{\overline{0}, \overline{2}, \overline{3}\}, \{\overline{1}\},$

$\Omega_9 = \{\overline{0}, \overline{1}, \overline{3}\}, \{\overline{2}\},\ \Omega_{10} = \{\overline{0}, \overline{1}\}, \{\overline{2}, \overline{3}\},\ \Omega_{11} = \{\overline{1}, \overline{2}\}, \{\overline{0}, \overline{3}\}.$

(4) The coarsest bond of order 1,

$\Omega_{12} = \mathbb{Z}_4.$

The *bond lattice* of $\mathbb{Z}_4$ with the corresponding *Möbius function values* attached is drawn below in Figure 6.2. For a thorough explanation on how to obtain these Möbius function values, consult with [39]. Hence $A_r(2, \mathbb{Z}_4)$ is derived from this relevant information in the theorem that follows.

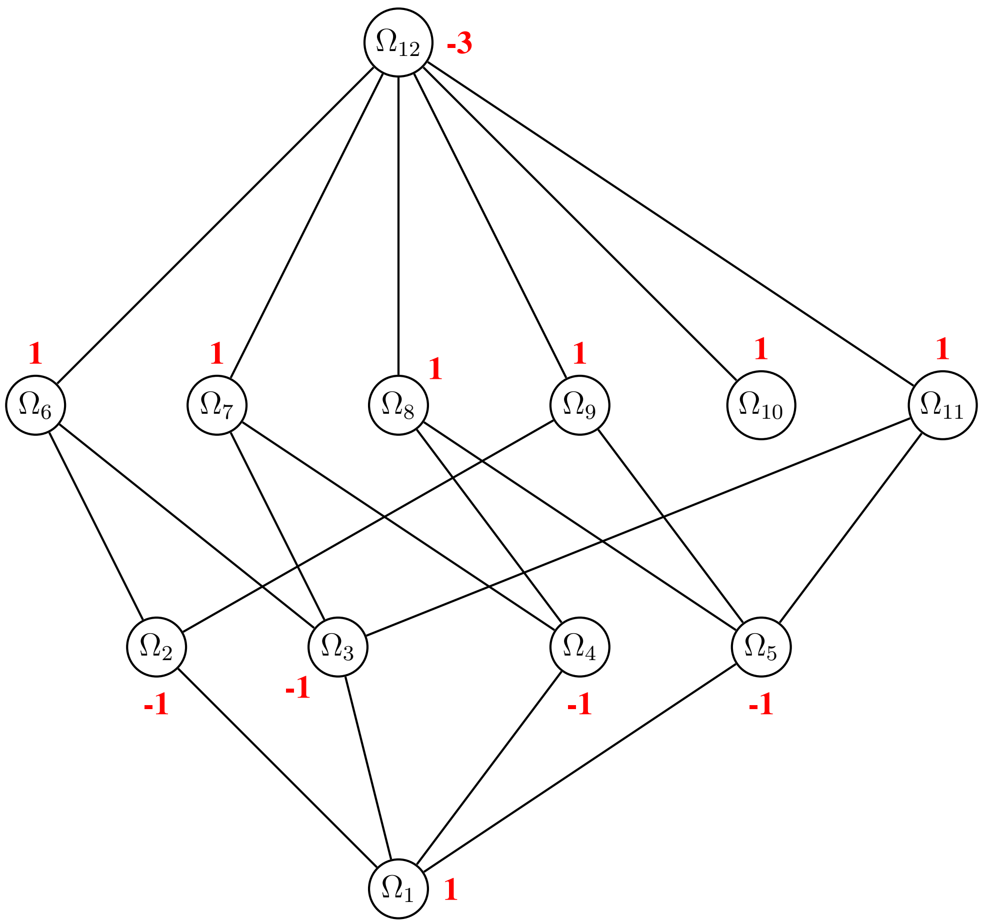


Figure 6.2: Bond lattice of $\mathbb{Z}_4$ with Möbius function values.

Theorem 41. *Let $n,\, r \in \mathbb{N}$. The total number of 2-alternating r-colorings of $\mathbb{Z}_4$ is*

$$A_r(2,\, \mathbb{Z}_4) = (r-1)[(r-1)^3 + (-1)^4].$$

Proof. Let $\Pi_b(\mathbb{Z}_4)$ denote the set of all bond partitions of $\mathbb{Z}_4$. The finest bond $x = \{\bar{0}\},\, \{\bar{1}\},\, \{\bar{2}\},\, \{\bar{3}\}$ represents all r-colorings in which no two consecutive vertices have the same color. Let $A_r(2,\, \mathbb{Z}_4) = f(y)$ represent the number of all

2-alternating r-colorings. Suppose that

$$g(x) = \sum_{x \leq y} f(y), \; \forall \; y \in \Pi_b(\mathbb{Z}_4).$$

Applying Möbius inversion (Theorem 7) we obtain

$$f(x) = \sum_{x \leq y} \mu(x, \, y) g(y), \; \forall \; y \in \Pi_b(\mathbb{Z}_4).$$

The Möbius function values $\mu(x, y)$ were computed in the lattice Figure 6.2. We notice that all $y \in \Pi_b(\mathbb{Z}_4)$ that are on the same level in the lattice have the same number of cells. Therefore $\forall \; y \in \Pi_b(\mathbb{Z}_4)$ on the same level $g(y) = r^{|y|}$. The summation of the Möbius function values for all the bonds on a particular level are the coefficients in our polynomial. Hence

$$\begin{aligned} f(x) &= \sum_{x \leq y} \mu(x, \, y) r^{|y|} \\ &= r^4 - 4r^3 + 6r^2 - 3r \\ &= (r-1)^4 + (-1)^4(r-1) \\ &= (r-1)[(r-1)^3 + (-1)^4]. \end{aligned}$$

Thus $A_r(2, \mathbb{Z}_4) = (r-1)[(r-1)^3 + (-1)^4]$. □

Theorem 41 is a natural antecedent to enumerating all 2-alternating r-colorings of $\mathbb{Z}_n$. Theorem 42 is an extension of Theorem 41 to all groups $\mathbb{Z}_n$. This result is extracted from [40]. First we abbreviate $A_r(2, \mathbb{Z}_n)$ to $A_r(n)$.

Theorem 42. *Let $n,\, r \in \mathbb{N}$. The number of 2-alternating r-colorings of $\mathbb{Z}_n$ is*

$$A_r(n) = (r-1)^n + (-1)^n(r-1). \tag{6.1}$$

Proof. Define the coloring $\chi : \{0,\, 1,\, \cdots,\, n-2\} \longrightarrow [r]$ such that $\forall\ i < n-2$ $\chi(i) \neq \chi(i+1)$. There are $r(r-1)^{n-2}$ such colorings. Moreover, the number of colorings with $\chi(0) = \chi(n-2)$ is $A_r(n-2)$. Consequently

$$\begin{aligned} A_r(n) &= (r-1)A_r(n-2) + (r-2)[r(r-1)^{n-2} - A_r(n-2)] \\ &= (r-2r)(r-1)^{n-2} + A_r(n-2). \end{aligned}$$

Now we have that

$$\begin{aligned} &A_r(2) = r(r-1) = (r^2-2r) + r \text{ and} \\ &A_r(3) = r(r-1)(r-2) = (r^2-2r)(r-1). \end{aligned}$$

This implies that whenever n is even

$$\begin{aligned} A_r(n) &= (r^2-2r)[(r-1)^{n-2} + (r-1)^{n-4} + \cdots + 1] + r \\ &= (r^2-2r)\frac{(r-1)^n - 1}{(r-1)^2 - 1} + r \\ &= (r-1)^n - 1 + r \\ &= (r-1)^n + (r-1). \end{aligned}$$

Whenever n is odd

$$\begin{aligned}
A_r(n) &= (r^2 - 2r)[(r-1)^{n-2} + (r-1)^{n-4} + \cdots + (r-1)] \\
&= (r^2 - 2r)(r-1)\frac{(r-1)^{n-1} - 1}{(r-1)^2 - 1} \\
&= (r-1)[(r-1)^{n-1} - 1] \\
&= (r-1)^n - (r-1).
\end{aligned}$$

If we combine these two expressions for $A_r(n)$ we obtain $A_r(n) = (r-1)^n + (-1)^n(r-1)$, for all $n \in \mathbb{N}$. □

An interesting case of the above result is realized whenever $k = r = 2$. Then Equation 6.1 may be used to count the number of 2-alternating 2-colorings of $\mathbb{Z}_n$, denoted by $A_2(2, \mathbb{Z}_n)$ and abbreviated to $A_2(n)$.

Corollary 20. *Let $n \in \mathbb{N}$. The number of symmetric alternating colorings of $\mathbb{Z}_n$ is*

$$A_2(n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Theorem 36 and Theorem 37 asserted that the symmetric property of k-alternating r-colorings only occurs when $k = 2 = r$. Hence $A_2(n)$ counts the total number of symmetric alternating colorings. Consequently by Theorem 42 we have

$$\begin{aligned} A_2(n) &= (2-1)[(2-1)^{n-1} + (-1)^n] \\ &= 1^{n-1} + (-1)^n \\ &= 1^n + (-1)^n \\ &= \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

□

This section concludes with an application of Equation 6.1 detailed in the following example.

Example 10. We shall enumerate all 2-alternating 4-colorings of $\mathbb{Z}_{12}$ using Equation 6.1. Recall from Proposition 1 that the total number of 4-colorings of $\mathbb{Z}_{12}$ is 4^{12}. By Theorem 42

$$A_4(2, \mathbb{Z}_{12}) = 3(3^{11} + 1) = 531\,444.$$

Furthermore, the total number of symmetric r-colorings of $\mathbb{Z}_{12}$ is

$$2r^2 - 4r^3 - 3r^4 + 6r^7,$$

which is taken from [39]. Hence the number of symmetric 4-colorings of $\mathbb{Z}_{12}$ is

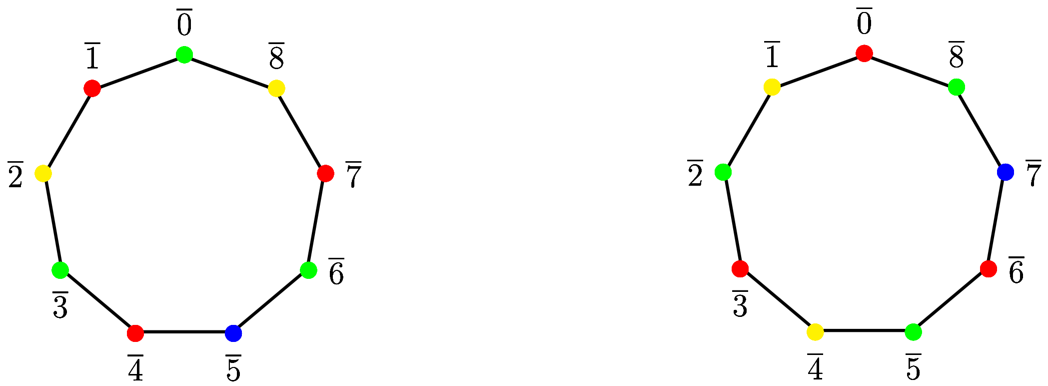
$$2(4)^2 - 4(4)^3 - 3(4)^4 + 6(4)^7 = 97\,312.$$

This example shows that there are more 2-alternating 4-colorings than symmetric 4-colorings of $\mathbb{Z}_{12}$.

CHAPTER 7

ALTERNATING NECKLACES

Any two r-colorings of $\mathbb{Z}_n$ are called *equivalent* if one may be obtained from the other by any *rotation about the n-gon's center*. The figure below supplies a graphic representation of the concept of equivalence in relation to alternating colorings. We observe in the following figure that the two 3-alternating 4-colorings, namely χ and θ are equivalent since $\chi(x+\overline{2}) = \theta(x),\ \forall\ x \in \mathbb{Z}_9$.



(a) $\chi \in 4^{\mathbb{Z}_9}$.

(b) $\theta \in 4^{\mathbb{Z}_9}$.

Figure 7.1: Two equivalent 3-alternating 4-colorings of $\mathbb{Z}_9$

Let us denote the number of equivalence classes of r-colorings of a group G by $e_r(G)$. By an application of *Burnside's Lemma* (Lemma 1)

$$e_r(G) = \frac{1}{|G|} \sum_{g \in G} r^{\frac{|G|}{|\langle g \rangle|}},$$

where $\langle g \rangle \leq G$ is the subgroup generated by the element $g \in G$. An equivalence class of r-colorings of $\mathbb{Z}_n$ taking all rotations as equivalent is called an *r-ary necklace of length n*. The number of all necklaces $\mathbb{Z}_n$ is denoted by $N_r(n)$. If we take $G = \mathbb{Z}_n$ the formula for $e_r(G)$ becomes

$$N_r(n) = \left(\frac{1}{n}\right) \sum_{d|n} \phi(d) r^{\left(\frac{n}{d}\right)}.$$

The proof of this expression may be found in [39]. In [44] a formula was derived for the number of *symmetric r-ary necklaces of length n*, denoted by $N_r^*(n)$. It was shown that the number of classes of equivalent symmetric r-colorings of $\mathbb{Z}_n$ is completely determined by the expression below,

$$N_r^*(n) = \begin{cases} r^{\frac{n+1}{2}}, & \text{for } n \text{ odd} \\ \frac{1}{2}\left(r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}\right), & \text{for } n = 2^l m,\ l \geq 1 \text{ and } m \text{ odd.} \end{cases}$$

In this chapter we contend with the issue of counting the number of equivalence classes of k-alternating r-colorings of $\mathbb{Z}_n$. These are referred to as *k-alternating r-ary necklaces* and their number is denoted by $a_r(k, \mathbb{Z}_n)$. It also is important to note that by virtue of Theorem 31 any r-coloring that is equivalent to a k-alternating r-coloring is also k-alternating. We first enumerate the alternating necklaces for several trivial cases before pursuing more refined results.

Counting alternating necklaces for the following two restrictions on k is undemanding. The next two theorems demonstrate these scenarios.

Corollary 21. *Let* $k,\, n,\, r \in \mathbb{N}$*, where* $k = 1$*. Then*

$$a_r(1,\, \mathbb{Z}_n) = N_r(n) = \left(\frac{1}{n}\right) \sum_{d|n} \phi(d) r^{\left(\frac{n}{d}\right)}.$$

Proof. This follows immediately from Proposition 4. □

Theorem 43. *Let* $k,\, n,\, r \in \mathbb{N}$*. If* $2k > n$ *then*

$$a_r(k,\, \mathbb{Z}_n) = 1.$$

Proof. Recall from Theorem 39 and Corollary 19 that $A_r(n,\, \mathbb{Z}_n) = r!$ whenever $2k > n$. These $r!$ colorings are all obtained by rotating the n consecutive vertices of the n-gon $r!$ times. Therefore they are all equivalent, i.e. all the $r!$ colorings are regarded as the same coloring. More precisely there is one equivalence class of k-alternating r-colorings of $\mathbb{Z}_n$ for $k = n = r$. Hence, they all belong to a single necklace and the statement follows. □

7.1

Counting 2-alternating r-ary Necklaces of $\mathbb{Z}_n$

The aim of this section is to advance the stratagem executed previously of counting alternating necklaces. We strive to find a function, denoted by $a_r(k,\, n)$ that expresses the number of *equivalence classes of k-alternating r-colorings*. We assay the concept of *counting orbits of symmetric colorings* and thereby depend on the sources [18], [42] and [46] for insight. In the previous chapter the chromatic polynomial of the aforementioned cycle graph was found to be identical to $A_r(2, \mathbb{Z}_n)$.

A systematic approach was used in [33] and [41] to derive this expression for $A_r(2, n)$. This approach was based on using the Möbius inversion theorem to give us the coefficients of the polynomial. We embrace the same technique of counting symmetric necklaces from [42]. These fundamental results are holistically presented in accordance with the article [40]. First, we review the following definitions. For $\chi \in r^{\mathbb{Z}_n}$ the *orbit* of χ is

$$[\chi] = \{g + \chi \,:\, g \in \mathbb{Z}_n\}.$$

Hence two colorings are equivalent if they belong to the same orbit. The *stabilizer* of χ is

$$St(\chi) = \{g \in \mathbb{Z}_n \,:\, g + \chi = \chi\}.$$

Lastly, the *centralizer* of χ is

$$Z(\chi) = \{g \in \mathbb{Z}_n \,:\, \chi(g - x) = \chi(x),\, \forall\, x \in \mathbb{Z}_n\}.$$

Furthermore it is established that

$$|[\chi]| = \frac{|G|}{|St(\chi)|}$$

and $St(\chi) = St(\varphi),\ \forall\, \varphi \in [\chi]$.

Definition 55. Let $k,\, m,\, n,\, r \in \mathbb{N}$. Then for every $m|n$ we define the set of all k-alternating r-colorings of $\mathbb{Z}_m$ to be X_m.

Lemma 13. *For every $g \in \mathbb{Z}_n$*

$$|\{\chi \in X_n \ :\ \chi + g = \chi\}| = A_r(k,\, n/|\langle g\rangle|).$$

Proof. Let $|\langle g\rangle| = d$ and take a coloring $\psi \in X_{\frac{n}{d}}$. Define a coloring $\overline{\psi} \in X_n$ as

$$\overline{\psi}\left(i + \left(\frac{n}{d}\right) j\right) = \psi(i),$$

where $i \in \{0,\, 1,\, \cdots,\, \dfrac{n}{d} - 1\}$ and $j \in \{0,\, 1,\, \cdots,\, d-1\}$. Consequently for $g \in \mathbb{Z}_n$, we have $\overline{\psi} + g = \overline{\psi}$. Now define a mapping

$$X_{\frac{n}{d}} \ni \psi \longmapsto \overline{\psi} \in \{\chi \in X_n \ :\ \chi + g = \chi\}$$

We claim that this mapping is bijective. Clearly this mapping will be injective. Let $\chi \in X_n$ and suppose that $\chi + g = \chi$ for $g \in \mathbb{Z}_n$. Define coloring $\psi \in X_{\frac{n}{d}}$ to be the restriction of χ to $\{0,\, 1,\, \cdots,\, \dfrac{n}{d} - 1\}$. This implies that $\overline{\psi} = \chi$ hence the mapping is surjective and the proof is complete. □

Theorem 44. *Let $d,\, k,\, n,\, r \in \mathbb{N}$. Then if ϕ denotes the Euler phi function*

$$a_r(k,\, n) = \frac{1}{n} \sum_{d|n} \phi(d) A_r(k,\, \frac{n}{d}), \tag{7.1}$$

Proof. We apply Burnside's lemma (Lemma 1). This shows that

$$a_r(k,\, n) = \frac{1}{n} \sum_{g \in \mathbb{Z}_n} |\{\chi \in X_n \;:\; \chi + g = \chi\}|$$

Furthermore by Lemma 13 for all $g \in \mathbb{Z}_n$ we have that

$$|\{\chi \in X_n \;:\; \chi + g = \chi\}| = A_r(k,\, n/|\langle g\rangle|).$$

By Theorem 2 for every $d|n$, there is exactly one subgroup of $\mathbb{Z}_n$ of order d. Moreover, by Theorem 1 the number of this subgroup's generators is $\phi(d)$. Therefore the number of equivalence classes of k-alternating r-colorings of $\mathbb{Z}_n$ is

$$a_r(k,\, n) = \frac{1}{n} \sum_{d|n} \phi(d) A_r(k,\, \frac{n}{d}).$$

□

Consequently the above theorem leads to the following result in which we denote the number of 2-alternating r-ary necklaces of $\mathbb{Z}_n$ by $a_r(n)$.

Corollary 22. *Let* $d,\, n,\, r \in \mathbb{N}$. *Then*

$$a_r(n) = \frac{1}{n} \sum_{d|n} \phi(d)[(r-1)^{\frac{n}{d}} + (-1)^{\frac{n}{d}}(r-1)]. \tag{7.2}$$

Proof. This equation follows immediately from Theorem 42 and Theorem 44. □

Example 11. We compute the number of 2-alternating r-ary necklaces of $\mathbb{Z}_{12}$, denoted by $a_r(\mathbb{Z}_{12})$. Therefore by Corollary 22 we obtain

$$
\begin{aligned}
a_r(\mathbb{Z}_{12}) &= \frac{1}{12}\sum_{d|12}\phi(d)[(r-1)^{\frac{12}{d}}+(-1)^{\frac{12}{d}}(r-1)] \\
&= \frac{1}{12}\{(r-1)^{12}+(r-1)+(r-1)^6+(r-1)+2[(r-1)^4+(r-1)] \\
&+2[(r-1)^3-(r-1)]+2[(r-1)^2+(r-1)]+4[(r-1)-(r-1)]\} \\
&= \frac{(r-1)}{12}\left[(r-1)^{11}+(r-1)^5+2(r-1)^3+2(r-1)^2+2(r+1)\right]
\end{aligned}
$$

CHAPTER 8

CONCLUSION

This concluding chapter synthesizes the research outcomes that were developed by the dissertation and reflects upon the relevance and limitations of this research endeavor, as a whole. Chapter 1 was the introduction and described the prevailing aims and objectives of the thesis. Thereafter, Chapter 2 consolidated the theoretical background of the thesis. This preliminary chapter codified a host of definitions and minor results, pertaining to various interconnecting subjects including group theory, partially ordered set theory and some number theoretic concepts. The content in this chapter was not exhaustive and merely served to introduce the notation and other basic results required in the successive chapters. Burnside's lemma and the technique of Möbius inversion, Lemma 1 and Theorem 7 respectively, were the highlights of this chapter.

The next installment of the thesis, Chapter 3, offered a cohesive evaluation of some theoretical aspects of symmetric colorings. This chapter was didactic in nature and imparted an analysis of monochromatic symmetric subsets in r-colorings of finite abelian groups. The numbers $\lambda_r(G)$ and $\omega_r(G)$ for $r \in \mathbb{N}$, were defined and examined. Several bounds for these numbers were proved in Lemma 2 and Theorem 15. Likewise, exact values for $\omega_r(G)$ were resolved in Theorem 16. Finite abelian groups for which $\omega_r(G) = 1$ were classified in Theorem 17. This result played an important role in the sequel. Moreover, Section 3.2 discussed the Ramsey function $\lambda_2(G)$ for a finite abelian group G. Finally, Table 3.1 and Table 3.2 produced values for $\lambda_2(G)$ and $\omega_2(\mathbb{Z}_n)$, where $n \leq 8$.

The purview of Chapter 4 examined symmetric colorings in the rarefied setting of dihedral groups. Several well-known formulas for counting symmetric colorings were restated in Theorem 19, Theorem 20 and Theorem 21. This chapter sought to advance these previously established results. Expressions for $S_r(D_p)$ and $s_r(D_p)$, where p is prime and $r \in \mathbb{N}$ were presented in Section 4.1. These formulas were detailed in Theorem 22 and are derived through the familiar technique of constructing the poset of optimal partitions. This technique was predicated on the structural predictability of the subgroup lattice and the Hasse diagram, pictured in Figure 4.1 and Figure 4.2 respectively. Thereafter Example 2 demonstrated the simplicity and ease with which these equations may be employed. This section is a prequel to Section 4.2 in which these results are generalized to include $S_r(D_n)$ and $s_r(D_n)$ for all $r,\ n \in \mathbb{N}$.

Thus, Theorem 24 proved that $S_r(D_n)$ may be obtained from $S_r(\mathbb{Z}_n)$ for all $r,\ n \in \mathbb{N}$, due to the semidirect product representation of D_n. It was then verified, in Corollary 6 that $S_r(D_p)$ may be derived in more sophisticated manner by this scheme. A formula for $s_r(D_n)$ was contrived in Theorem 27. This theorem was the focal point of this section. As a byproduct of this statement Corollary 8 more effectively rendered an expression for $s_r(D_p)$.

The thesis then proceeded to Section 4.3 which was the crux of Chapter 4. Our first significant result was Theorem 28 which counted the symmetric colorings of the group $G \times \mathbb{Z}_2$, for an abelian group G. The proof of this theorem was postponed until after we had introduced several ancillary results on pairs of symmetric colorings. These results were namely, Corollary 9, Corollary 10, Lemma 11 and Corollary 12. Theorem 28 brought about two deductions. The first of which concerned the polynomial representation of $S_r(G \times \mathbb{Z}_2)$ stated in Corollary 14. Whereas the second calculated the number of equivalence classes of pairs of symmetric colorings in Corollary 13. However, this corollary did not provide a solution for $s_r(G \times \mathbb{Z}_2)$. The chapter concluded by verifying in Theorem 29, that $S_r(D(G))$ may be expressed in terms of $S_r(G)$ for any abelian group G. This result is considered to be a generalization of the assertion made in Theorem 24.

Alternating colorings became the target of our scrutiny in Chapter 5. Firstly, several structural properties of alternating colorings were explored and the reader was provided with numerous illustrative examples. These observations were outlined in Proposition 4, Proposition 5, Theorem 30 and Theorem 32. Theorem 31 proved that every coloring equivalent to an k-alternating r-coloring is also k-alternating. Whereas, Theorem 33 demonstrated the signature characteristic of k-alternating k-colorings. The notion of an alternating number, denoted by $\rho(k, n)$, was investigated in Section 5.2. This concept vaguely alludes to the chromatic number of a graph. Indeed, Theorem 34 verified that $\rho(2, n)$ is identical to the chromatic number of the cycle graph. Theorem 35 solved for $\rho(k, n)$, for arbitrary values of k and

n. Corollary 17 and Corollary 18 emerged from this main result. Section 5.3 presented a detailed analysis of symmetry on the cyclic group $\mathbb{Z}_n$ and its relation to alternating colorings. Thereby Theorem 37 showed that for all $k > 2$ any k-alternating r-coloring will not be symmetric.

Chapter 6 proceeded to count alternating colorings. This chapter attempted to overcome the computational difficulties of enumerating these colorings. In this chapter's incipient stage, $A_r(k, n)$ was computed for several trivial values of $k, r \in \mathbb{N}$. These values are exemplified in Theorem 38 to Theorem 40. Section 6.1 exploited the technique of Möbius inversion to achieve an expression for $A_r(2, n)$ in Theorem 41. This result is generalized in Theorem 42. Alternating necklaces, which are also known as equivalence classes of alternating colorings, were discussed in Chapter 7. Section 7.1 counted the equivalence classes of 2-alternating r-colorings of $\mathbb{Z}_n$. This expression is denoted by $a_r(n)$ and is derived in Theorem 44. Our modest contribution to the body of research surrounding colorings on groups was Corollary 22 which was published in [40].

This thesis analyzed both the computational and theoretical aspects of symmetric colorings on groups. We also introduced the idea of an alternating coloring and expounded upon this conception at length. This notion proved to be a potent landscape for several interesting results. The combinatorial angle of this subject was strongly emphasized throughout this research enterprise. This standpoint allowed for the publication of two articles [29] and [40]. Furthermore, several aspects of this research were orally presented at two conferences. The first of which was the South African Mathematical Society's 60th annual congress. This event was held at the North-West University in Potchefstroom, on the 20–22 November 2017. The second was the Southern African Mathematical Sciences Association conference. This event was hosted at the Botswana International University of Science and Technology in Palyapye Botswana, on the 19–22 November 2018. An abstract of the last presentation is published in the book [37].

The findings that were published in the article [29] are compatible with the underlying theory of symmetric colorings. The computational techniques developed in this thesis are a contribution to the hierarchy of results surrounding this topic. The research outcomes for symmetric colorings were limited to abelian, cyclic and dihedral groups. Future research can potentially aim to extend these results to other finite groups. Alternating colorings were exclusively defined on a cyclic group in the article [40]. This definition may be developed further from its nascent stage in this thesis and may be used as a guide towards exploring this type of coloring, on a different variety of finite groups. Group colorings are an eclectic field which unites a variety of mathematical disciplines. A new discovery made in this field may have dramatic and far-reaching consequences.

REFERENCES

[1] M. Aigner, Combinatorial theory, Springer-Verlag Berlin Heidelberg, New York, 1979. 14

[2] T. M. Apostol, Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Berlin-Heidelberg, 1976. 20, 21, 22

[3] T. Banakh, O. Verbitsky and Y. Vorobets, *A Ramsey treatment of symmetry*, Electron. J. Comb. **7** (2000), Research paper R52, 25 p. 2, 25

[4] T. Banakh and I. Protasov, *Symmetry and Colorings: Some results and open problems*, Gomel University, Voprosy Algebry. **17** (2001), 4–15. 2, 25, 26

[5] T. Banakh, *Symmetry and Colorings: Some results and open problems II*, arXiv:11111.1015, preprint. 25, 26

[6] E. A. Bender and J. A. Goldman, *On the applications of Möbius inversion in combinatorial analysis*, The American Mathematical Monthly **82**(8) (1975), 789–803. 4, 17, 51, 83

[7] P. J. Cameron, Combinatorics: Topics, Techniques, Algorithms, Cambridge University Press, 1994. 14, 17

[8] S. R. Cavior, *The Subgroups of the Dihedral Group*, Mathematics Magazine, **48**(2) (1975), 107–107. 8

[9] G. Chartrand and P. Zhang, Chromatic Graph Theory 2nd ed, CRC Press, Taylor & Francis Group, A Chapman & Hall Book, Boca Raton London New York, 2009. 76

[10] S. M. Cioabă and M. Ram Murty, A First Course in Graph Theory and Combinatorics, Texts and Readings in Mathematics **55**, Hindustan Book Agency, New Delhi, 2009. 83

[11] D. Dummit and R. Foote, Abstract Algebra 3rd ed, John Wiley & Sons, New York, NY, 2004. 1, 5, 6, 7

[12] P. Formanowicz and K. Tanas, *A survey of Graph Coloring - Its Types, Methods and Applications*, Foundation of Computing and Decision Sciences. **37**, No. 3 (2012). 66

[13] R. S. Al-Gounmeein and J. Maan, *On the Chromatic Polynomial of the Cycle graph*, International Journal of Applied Mathematics **25**:6 (2012), 825–832. 87

[14] R. Graham, Rudiments of Ramsey theory, CBMS Regional conference series in mathematics volume 123, American Mathematical Society, 2015. 2

[15] R. Graham, B. Rothschild and J. Spencer, Ramsey Theory 2nd ed, John Wiley & Sons, 1990. 2

[16] Y. Gryshko, *Monochrome symmetric subsets in 2-colorings of groups*, Electron. J. Combin. **10** (2003), Research Paper #R28. 2, 3, 26

[17] Yu. Gryshko, A. Khelif, *Monochrome symmetric subsets of colored groups*, J. Combin. Theory Ser. A **112**:2 (2005), 212–221. 2, 26

[18] Yu. Gryshko, *Symmetric colorings of regular polygons*, Ars Combinatoria **78** (2006), 277–281. 3, 38, 39, 40, 51, 52, 79, 97

[19] M. Halldórsson, S. Kitaev and A. Pyatkin, *Alternation graphs*, Lecture Notes in Comp. Sci. **6986** (2011), 191–202. 75

[20] T. R. Jensen and B. Toft, Graph Coloring Problems, John Wiley & Sons Inc, Canada, 1995. 66

[21] C. Jongsma, Discrete Mathematics: Chapter 7, Posets, Lattices, & Boolean Algebra, Faculty Work: Comprehensive List. Paper 427, 2016. 14

[22] M. I. Kargapolov and Ju. I. Merzljakov, Fundamentals of the Theory of Groups, Springer-Verlag, New York-Berlin-Heidelberg, 1979. 1, 5

[23] I. Kashuba and Yu. Zelenyuk, *The number of symmetric colorings of the dihedral group* D_3, Quaestiones Mathematicae **39**(1), DOI:10.2989/16073606.2015.1015646 (2016), 65–71. 3, 16, 40, 45

[24] M. Korostenski and Yu. Zelenyuk, *Ramsey Functions for Symmetric Subsets in Compact Abelian Groups*, Quaestiones Mathematicae, **33**:2, DOI:10.2989/16073606.2010.490992 (2010), 161–169. 37

[25] B. M. Landman and A. Robertson, Ramsey theory on the Integers, Student Mathematical Library volume 24, American Mathematical Society, 2004. 2

[26] R. M. R. Lewis, A Guide to Graph Colouring Algorithms and Applications, Springer International Publishing, Switzerland, 2016. 66

[27] O. Loos, Symmetric Spaces I: General Theory, W. A. Benjamin, Inc. New York Amsterdam, 1969. 1, 38

[28] J. Phakathi, D. Radnell and Yu. Zelenyuk, *The Number of Symmetric Colorings of the Dihedral Group D_p*, Appl. Math. Inf. Sci. **10**, No. 6 (2016) 2373–2376. 3, 40, 45

[29] J. Phakathi, S. Singh, Ye. Zelenyuk and Yu. Zelenyuk, *Counting Symmetric Colorings of the group $G \times \mathbb{Z}_2$*, Journal of Algebra and Its Applications **18**(10) (2019), World Scientific Publishing Company, DOI:10.1142/S0219498819501998. 3, 40, 56, 104, 105

[30] J. Phakathi, W. Toko, Ye. Zelenyuk and Yu. Zelenyuk, *Symmetric colorings of the dihedral group*, Communications in Algebra **46**(4) (2018), Taylor & Francis Group DOI:10.1080/00927872.2017.1350694, 1554–1559. 3, 38, 40, 46, 47, 64

[31] G. Pica, *Generalised chromatic numbers of some graphs*, Riv. Mat. Univ. Parma **4:9** (1983), 259–264. 76

[32] I. V. Protasov, *Asymmetrically resolvable abelian groups*, Mat. Zametki. **59** (1996), 468–471. 2, 25, 26

[33] S. Rao, V. Rao and A. Kumar, *Classification of Algebraic Properties of Chromatic Polynomials*, J. Sci. Res. **5**(3) (2013), 469–477. 83, 87, 97

[34] R. C. Reed, *An Introduction to Chromatic Polynomials*, Journal of Combinatorial Theory **4** (1968), 52–71. 66, 83, 87

[35] G–C. Rota, *On the foundations of combinatorial theory I: Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie **2** (1964), 340–368. 17, 18, 19, 20

[36] J. Rotman, An Introduction to the Theory of Groups 4th ed, Graduate Texts in Mathematics, Springer-Verlag, New York, 1995. 1, 5, 11

[37] SAMSA Conference 2018, Book of Abstracts: Page 21, Botswana International University of Science and Technology, Palyapye Botswana, https://sites.google.com/a/biust.ac.bw/samsa2018/Author-Information/book-of-abstracts. 105

[38] F. Simon, P. Tittmann and M. Trinks, *Counting Connected Set Partitions of Graphs*, Electron. J. Combin., **18**(1) (2011), Paper #P14.

[39] S. Singh, Combinatorial Aspects of Symmetries on Groups, MSc dissertation, University of Witwatersrand, Johannesburg, ETD Collection (2016), https://hdl.handle.net/10539/21683. 3, 4, 5, 16, 38, 40, 45, 66, 88, 93, 95

[40] S. Singh and Yu. Zelenyuk, *Alternating Colorings of the Vertices of a Regular Polygon*, Bull. Aust. Math. Soc. 13743 (2019), DOI:10.1017/S0004972719000157. 66, 75, 84, 90, 97, 104, 105

[41] R. V. N. SinivasaRao, J. VenkateswaraRao and H. M. GebreYohannes, *Extension of Chromatic Polynomials by Utilizing Möbius Inversion Theorem*, Contemporary Mathematics and Statistics **2** (2013), DOI:10.7726/cms.2013.1005, 54–70. 83, 87, 97

[42] Yu. Zelenyuk, *Counting Symmetric colorings of the vertices of a regular polygon*, Bull. Aust. Math. Soc. **90** (2014), 1–8. 3, 11, 38, 39, 97

[43] Yu. Zelenyuk, *Monochrome Symmetric Subsets in Colorings of Finite Abelian Groups*, Symmetry **3** (2011), 126–133. 2, 26, 27, 30, 34, 37, 46

[44] Ye. Zelenyuk and Yu. Zelenyuk, *Counting symmetric bracelets*, Bull. Aust. Math. Soc. **8** (2014), 431–436. 95

[45] Ye. Zelenyuk and Yu. Zelenyuk, *Transformations and colorings of groups*, Can.Math. Bull. 50 (2007), 632—636. 37

[46] Yu. Zelenyuk, *Symmetric colorings of finite groups*, Groups St Andrews in Bath, LMS Lecture Note Series (2011), 580–590. 3, 11, 12, 38, 39, 97

[47] P. Zhang, A Kaleidoscopic View of Graph Colorings, SpringerBriefs in Mathematics, Springer International Publishing AG, Switzerland, 2010. 66