An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning

KONA SAGAREN KANAKASABY NAIDOO

An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning

KONA NAIDOO

A research report submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in partial fulfillment of the requirements for the degree of Master of Sciences.

Edenvale, March 2009

DECLARATION

I declare that this research report is my own work that has been unaided. My research report is submitted for the degree of Master of Sciences at the University of the Witwatersrand in Johannesburg, South Africa. I further declare that this research report has not been submitted previously for any degree or examination at any other university.

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Signature of candidate

6th day of March 2009

ABSTRACT

Research in early algebra is critical because a smooth transition from arithmetic to algebra will influence future algebra learning that is central to school mathematics. This study investigated learners' interpretation of letters in different levels of generalised arithmetic activities. Thirty grade nine learners from one inner city school participated in this study. All learners engaged with seventeen paper and pencil tasks encompassing six different interpretations of letters and six learners were then interviewed.

Analysis of the data showed that the overall performance of learners was very poor and most learners have not been successful in making the transition from arithmetic to algebra. Learner responses suggested a strong arithmetical influence and a poor understanding of algebraic letter and basic manipulative skills. Throughout the data a number of misconceptions surfaced which suggested that most learners in this sample were lacking 'symbol sense'.

KEYWORDS/PHRASES

Algebra

Early algebra

Generalised arithmetic

Transition from arithmetic to algebra

‘Symbol sense’

Constructivism

Misconceptions

Interpretations of letters

Levels of understanding

Children’s errors and strategies

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To my wife Vani and daughter Kyrah, thanks for your love, support and patience during my research. Without you it wouldn't have been possible. A big thanks to my parents Ganas and Logie, as well, for their immense support.

DEDICATION

AUM SAI RAM !

I offer my Pranaams at the divine lotus feet and dedicate this thesis to our beloved Bhagawan Sri Sathya Sai Baba, my guru, teacher, preceptor and God. Baba was my inspiration and inner motivator who helped me graduate through many years of study, culminating in this Research Project. In this journey we are mere instruments, actors on the stage of life, He is the director. In making this offering I do so not alone but with a deep sense of appreciation and love to the combined effort of all those who assisted me. I pray to Bhagawan to bless this humble task and to shower his infinite love on all those who read this book.

Life is a game – play it.
Life is love – share it.
Life is a dream – realize it.

JAI SAI RAM !

ABBREVIATIONS

CSMS……………………Concepts in Secondary Mathematics and Science project

Wits……………………...University of the Witwatersrand

NCS……………………...National Curriculum Statement

MSc…………………….....Master of Sciences

Dr………………………...Doctor

Etc....................................etcetera

GDE..................................Gauteng Department of Education

LIST OF TABLES

TABLE OF CONTENTS

Chapter three: <u>Methodology</u>……………………………………………………24

Chapter 4: <u>Analysis and data interpretation</u>……………………………………...35

Chapter one: <u>Introduction and Rationale</u>

1.1 Introduction

Central to secondary school mathematics curricula in South Africa and abroad is algebra. Yet many teachers and teacher educators share similar notions of the problematic nature of the algebraic realm.

> Even in the best circumstances, with an appropriately reformed curriculum and drastic changes in how we teach algebra, I suspect that the job of teaching algebra to students who have not been successful in mathematics will remain a difficult challenge for those teachers willing to take it on. (Chazan, 1996, p. 475)

The reported difficulties in implementing algebra curricula and the relatedness of algebra to other mathematical sections prompted me to embark on research in early algebra or generalised arithmetic. The CSMS[1] (1979) project was an influential study that reports on how learners engage with letters in early algebra. The major transition from arithmetic to algebra in the South African mathematics curriculum is in the senior phase (grade 7–9) which forms the foundation for learners' understanding of algebra. This transition can be overwhelming and results in many learners being unable to cope.

Letters are used in different ways in mathematics but I discuss letters as being symbols of algebra that stand for numbers. Letters are central in generalised arithmetic and Schoenfeld and Arcavi (1988) emphasise that an understanding of letters forms the foundation for the transition from arithmetic to algebra.

[1] Concepts in Secondary Mathematics and Science project [CSMS, 1979, reported in Hart, 1981, Children's Understanding of Mathematics, Chapter 8, by Küchemann (1981)].

Therefore, the centrality of letters in early algebra has influenced the purpose of my research, which is to investigate learners' interpretations of letters and misconceptions related to these in generalised arithmetic contexts.

1.2 Research problem and research questions

Competency when interpreting and manipulating letters is crucial for proficiency in algebra and mathematics. However, the conception of letters in mathematics is not single dimensioned as Usiskin (1998) explains that letters can stand for functions, points, matrices and vectors. Therefore, letters have many different definitions and interpretations but very often learners work with letters with minimal sense and logic. Therefore, this study explores how grade nine learners grapple with and interpret letters across tasks of different levels of understanding.

The research problem that is driving my research is that grade nine learners often display misconceptions when engaging with generalised arithmetic tasks. There is local and international research that reports on some of these misconceptions (See, for example, Stacey & MacGregor, 2000 and Olivier, 1989). Adoption of incorrect strategies and errors are the result of misconceptions. Are misconceptions related to prior knowledge, levels of understanding or possibly language? Therefore, I investigate the nature of misconceptions in generalised arithmetic settings that could possibly have contributions/implications to my/other teachers' teaching and to the learning of symbolic proficiency.

The following are critical research questions that guide this study:

1. How do learners interpret symbols/letters during engagement with generalised arithmetic activities?

2. Why do learners adopt certain methods, strategies and common errors when engaging with algebraic problems?

3. How and why are learner interpretations of symbols different across a range of activities reflecting different levels of algebraic understanding?

4. What are possible similarities and differences between the present sample's interpretations of letters and that of the CSMS (1979) sample?

1.3 Rationale

My study focuses on generalised arithmetic in early algebra by investigating how learners interpret letters in problem solving tasks. Interpreting letters is central to the 'core activities of algebra' (Kieran, 2004) and 'symbol sense' (Arcavi, 2005) and a deep understanding and appropriate uses of letters will contribute to competency in algebraic activities. Therefore, the importance of letters in algebra is reflected in my first three research critical questions.

However, interpretation of letters is complex and multi–faceted. Schoenfeld and Arcavi (1988) show that depending on the mathematical instance, the concept of variable can take alternate forms. The many alternative definitions of variable below illustrate the complex and abstract nature of interpreting letters in algebra. Therefore, teachers of algebra need to carefully and strategically introduce and nurture the interpretations of letters in their learners. As a mathematics teacher I became fully aware of multiple interpretations of letters only in my academic studies and hence a study of letters in algebra interests me.

A few definitions of the concept of variable are listed below:

- A quantity that may assume any one of a specified set of values.
- A variable is a named entity possessing a value that may change during execution of a program.
- Any symbol whose meaning is not determinate is called a variable.
- Variable means something that does indeed vary, or that has multiple values. (Schoenfeld & Arcavi, 1988, p. 421-422)

I embarked on this research with the assumption that learners will display misconceptions when interpreting letters because of the wide local and international research on this issue. For example, Küchemann (1981) and Arcavi (2005) report that learners display misconceptions when interpreting letters which relates to my research questions. However, my research is situated in a South African context, which is different to the above authors, and it is interesting to investigate if internationally reported misconceptions are also prevalent locally.

1.3.1 The importance of studying letters in early algebra in the current South African context: Location of algebra in the senior phase curriculum.

In this section, I discuss the location of algebra in the school mathematics curriculum that my study is concerned with. The National Curriculum Statement (NCS, 2002) provides guidelines in terms of content and depth of algebraic skills and knowledge needed to be taught in a specific grade. In the senior phase the NCS (2002) prescribes one learning outcome, out of five, that broadly deals with algebra. This learning outcome which is learning outcome 2 is named 'patterns, functions and algebra' (NCS, 2002). The learning outcome focus envisages that central tenets of 'patterns, functions and algebra' are manipulation skills and the use of symbolic expressions (NCS, 2002). These central tenets of manipulation and interpreting symbolic expressions are in line with the focus of the four research questions that inform this study.

Algebra is introduced in grade seven and the presence of letters is strong in two assessment standards relating to relationships between variables and interpreting of equations and expressions. As expected, there is prevalent content and context progression across the three grades in the senior phase within learning outcome 2. In grade eight there is one broad assessment standard that deals with the simplification of expressions in detail whereas in grade nine assessment standards include the distributive law, factorization, laws of exponents, solving equations and simplification of expressions.

The aim of my research is to investigate misconceptions related to the central aspect of letters in algebra which forms the basis of secondary school algebra. In learning outcome 2, for grade nine, all nine assessment standards require learners to work with letters in different contexts. However, in all other learning outcomes, although implicit, letters are embedded in many assessment standards. Learning outcome 1, which is ‘numbers operations and relationships’, has a strong presence of letters in the assessment standard that deals with exponential laws (NCS, 2002). Learning outcome 3, which is geometry has its last assessment standard that deals with representations of ordered pairs and the Cartesian plane that involves the use and interpretation of letters. Learning outcome 4, which is measurement, encompasses perimeter and area formulae, which includes a strong presence of letters. Learning outcome 5 which is, data handling, requires learners to draw different graphs to represent data and letters are also infused into this assessment standard. Therefore, the presence of letters is strongly embedded in the grade nine mathematics curriculum and research involving letters is crucially important.

My first three research questions aim to explicitly report on methods, strategies and common errors that learners adopt when engaging with generalised arithmetic. Küchemann (1981) reports on six interpretations of letters and how incorrect interpretation of letters leads to greater difficulty in solving tasks. Arcavi (2005) reports on ‘symbol sense’ which relates to proficiency with symbols which is “distinct” from interpretations of letters (I will delve deeper into this distinction in Chapter 2). Therefore, the centrality of letters in the studies by Küchemann (1981) and Arcavi (2005) enables me to think about symbolic competencies that learners possess or are without in relation to my first three research questions.

1.4 Summary

In this chapter, I discussed the aims, research problem, research questions and rationale of my study. I established coherent links between my research idea and research critical questions. A rationale for my study, which stems from the centrality of algebraic conceptions of letters in mathematics, was provided. I argued that a clear understanding of the concept of algebraic letters, which forms the basis of South Africa’s school algebra curriculum, is related to competency.

In Chapter 2, I provide a survey of literature relevant to this study with a focus on interpretations of letters and levels of understanding, ‘symbol sense’ and Kieran’s

(2004) three core activities of algebra. I will also elaborate on the theoretical framework that underpins and guides my study.

In Chapter 3, I elaborate on the methodology that guides my research with a focus on the research context of this study, the research instruments, the data gathering and analysis processes, ethical considerations and the piloted study.

In Chapter 4, I analyse and discuss data collected from the two research instruments resulting in the establishment of four themes.

In Chapter 5, I conclude my study by elaborating on findings of my study with reference to my research questions. I also reflect on the study and discuss its limitations.

Chapter two: <u>Theoretical framework and Literature review</u>

2.1 Introduction

In this chapter I provide a survey of literature pertinent to my study. Discussions focus on perspectives of algebra, the transition from arithmetic to algebra, generalised arithmetic and ‘symbol sense’. I discuss constructivism by focussing on core misconceptions in the transition from arithmetic to algebra. In the latter sections of this chapter the research of the CSMS (1979) based on interpretations of letters and levels of understanding are discussed in detail.

2.2 A perspective of algebra: Generalised arithmetic

Letters and variables will take on different roles depending on the view of algebra adopted. Van Amerom (2003, p. 64) explains that ‘it is useful to distinguish four basic perspectives: (1) algebra as generalised arithmetic, (2) algebra as a problem solving tool, (3) algebra as the study of relationships and (4) algebra as the study of structures’. There are other authors that also define algebra with many sharing similar notions of what algebra is [See, Sfard (1995), Lins & Kaput (2004), Lee (2001, as cited in Lins & Kaput, 2004)]. However, due to my study focussing on the transition from arithmetic to algebra (grade 7–9, NCS, 2002) and hence early algebraic understanding the perspective of algebra adopted in this study is generalised arithmetic.

Letters are not used in arithmetic where numbers and operations receive centre stage. In generalised arithmetic we often see numbers linking with letters in expressions or equations as constants and coefficients and letters are used as representations of numbers. Kieran (2004, p. 24) explains generalised arithmetic as the ‘unknown takes priority over the variable and expressions and equations tend to be viewed as representations of numerical processes rather than functional relations’. It follows that, generalised arithmetic is essentially linking letters and numbers to operations or manipulations or as Wong (1997, p. 285) explains ‘elementary algebra could be regarded as generalised arithmetic with the use of letters to represent numbers its principal characteristic’.

The core competencies of algebra such as symbolic interpretations and manipulations are crucial when engaging with generalised arithmetic. Activities encompassing the gist of algebra involving equations, expressions and using algebra as an instrument/tool will be meaningless with a poor understanding of symbols in generalised arithmetic.

2.3 Core activities of algebra

Kieran (2004) explains that school algebra consists of three core activities which she calls ‘generational, transformational, and global/meta–level’. ‘Generational activities’ encompass ‘forming of the expressions and equations that are the objects of algebra. The focus of generational activities is the representation (and interpretation) of situations, properties, patterns, and relations’ (Kieran, 2004, p. 23). ‘Transformational activities’ encompass simplification of expressions, factorization, substitution, solving equations, manipulation and equivalence, etc. ‘The manipulative process is as much a conceptual object in algebra learning as are the typical algebra objects – unknown, variable, expression and equation – and one of the main manipulative process is that which deals with equivalence of expressions and its conceptualization’ (Kieran, 2004, p. 25).

‘Global/meta–level activities’ are when algebra is used in other fields to solve problems as an instrument/tool in activities that are not always inclusive of algebra. Other authors share a similar view of the core activities of algebra (See, Kendal & Stacey, 2004 and Kilpatrick, Swafford & Findell, 2001). However, my study focuses more on ‘transformational activities’. Investigations focused on misconceptions in generalised arithmetic where numerical processes involving letters were more central than representational relationships between contexts (‘generational activities’) or using algebra as a tool (‘global-level activities’).

2.4 The problematic realm of the transition from arithmetic to algebra

Chazan (1996) criticizes the link between arithmetic and algebra with the following comments:

'The traditional algebra curriculum is regularly criticized on this score. It does not adequately explain the nature of symbolic expressions and the purpose and goal of the manipulations of symbols' Chazan (1996, p. 459).

Other authors share the same sentiments as Chazan (1996) because they also explain that the transition from arithmetic to algebra is problematic (See, Bishop, Clements, Keitel, Kilpatrick and Laborde, 1996 & Boulton–Lewis, Cooper, Atweh, Pillay and Wilss, 1998). Much of the work in algebra requires students to use their prior arithmetic skills. Arithmetic involves calculations with numbers whereas algebra 'requires reasoning about unknown or variable quantities' (Van Amerom, 2003, p. 64). According to Van Amerom, (2003, p. 65) algebra and arithmetic are interrelated and 'algebra relies heavily on arithmetical operations and arithmetical expressions are sometimes treated algebraically'. However, Stacey and MacGregor (2000, p. 150) explain that 'cognitive discontinuities' are evident in the transition from arithmetic to algebra. They discuss the missing link in the transition from working with numbers in arithmetic to unknowns in algebra as a 'cognitive gap' (Herscovics & Linchevski, 1994, as cited in Stacey and MacGregor, 2000) or 'cut-point' and 'didactic point' (Filloy & Rojano, 1989, as cited in Stacey and MacGregor, 2000). Furthermore, they explain that unless students are taught algebraic methods they will opt for arithmetic methods.

According to Linchevski (1995, as cited in Boulton–Lewis et al., 1998, p. 144) pre – algebra should be taught within the *cognitive gap* (after teaching of arithmetic) and should encompass 'substitution of numbers for letters, dealing with equivalent equations through substitution and allowing students to build cognitive schemas through spontaneous procedures'. In this way early algebra could be introduced in relation to arithmetical operations involving numbers, substitution and equivalence and learners could make some sense of the use of letters in algebra in relation to arithmetic.

2.5 ‘Symbol sense’

Proficient learners of algebra must have an understanding of letters or what Arcavi (2005) refers to as ‘symbol sense’. He argues that having ‘symbol sense’ is central to algebra and teaching should be geared towards achieving ‘symbol sense’. This is in line with Slavit (1998, p. 357) who explains that communication in mathematics is viable if symbolic systems are understood and relations between systems could be used to enhance symbolic understanding. Algebraic symbols are central to this study’s research questions and were entrenched in all the paper and pencil tasks that learners in this sample engaged with.

Empirical findings of Arcavi (2005) which are pertinent to my study included the fundamentals of ‘symbol sense’ comprising of six components.

> 1. Friendliness with symbols: this includes understanding of and an aesthetic feel for the power of symbols.
> 2. An ability to manipulate and also to ‘read through’ symbolic expressions as two complimentary aspects in solving algebraic problems.
> 3. The awareness that one can successfully engineer symbolic relationships that express verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions.
> 4. The ability to select one possible symbolic representation for a problem.
> 5. The realization of the need to check for symbol meanings.
> 6. The realizations that symbols can play different roles in different contexts. (Arcavi, 2005, pp. 42–43)

However, the six components of ‘symbol sense’ are interrelated and closely linked. In other words, if a learner has one component then she/he will probably display other components but not having one component might result in not having any of the components. In other words, if a learner has ‘friendliness with symbols’ then she/he should be able to ‘manipulate and also read through symbolic expressions as two complimentary aspects’ (Arcavi, 2005, pp. 42–43). Due to learners in the present sample being relatively low achieving learners (see Chapter 4, section 4.2) many did not display any of the components of Arcavi’s (2005) framework. Thus, the framework was less useful in the analysis of my data than I had initially expected. However, Arcavi (2005) contributes to my second research question in terms of what components are needed by learners to be proficient when working in generalised arithmetic contexts.

2.6 Theoretical Framework

The theoretical perspective adopted in this study is constructivism because an assumption of this study, with respect to learning of mathematics, is learners possess certain misconceptions when engaging with algebraic activities. Mathematical thinking is considered in this study to be largely an internal facet which could involve the internalization of prior learning and misconceptions. There are two influences for my choice of viewing learning through a lens of cognitive theories.

Firstly, in constructivism misconceptions are seen as fundamental in learning because learners can create misconceptions in the sense making process of knowledge acquisition. This is in line with Hatano (1996, p. 201) who explains that misconceptions can be seen as the ‘strongest pieces of evidence for the constructive nature of knowledge acquisition’. I investigate mathematical thinking in terms of interpretation of symbols, strategies in algebraic problem solving and common errors or misconceptions. Therefore, constructivism will help me to explain knowledge acquisition involving letters or variables.

Secondly, I draw on many different aspects of Küchemann’s (1981) study which is clearly linked to constructivism. This is evident as he attempts to create links between levels of algebra and Piagetian sub–stages and issues of misconceptions, which are major contributions of cognitive theorists. However, in my study I do not try to create links to Piagetian sub–stages but rather focus on misconceptions. The alignment to the Piagetian learning theory is partly influenced by the time of Küchemann’s (1981) study. Constructivism in the first ten years (1976–1985) ‘…swept through mathematics education… some argued that it’s quick ascension demonstrated the tendency of the field to respond too quickly to fashions’ (Confrey & Kazak, 2006, p.

310). Although Confrey and Kazak (2006) suggest that the relevance of constructivism needs careful consideration the centrality of misconceptions in relation to my topic of study influences me to view my study through a constructivist lens.

Hatano (1996) discusses knowledge acquisition, from a constructivist perspective as having the characteristics of constructing, restructuring and situating in contexts. Constructivists explain that the construction of knowledge takes place through equilibration, assimilation and accommodation. Assimilation is the ability of individuals to interpret incoming information to match prior knowledge or thinking. Accommodation is the adaptation of old ways of thinking to new situations of learning while equilibration 'encompasses both assimilation and accommodation. It refers to the overall interaction between existing ways of thinking and new experience' (Siegler, 2005, p.38). However, since my study focuses on results from learners working on a test and what students know rather than on the process of knowledge acquisition I do not draw on the construction of knowledge in the analysis of my study. Hence, I will not elaborate on these. However, I will provide a thorough discussion of misconceptions below which is more central to my study.

2.6.1 Misconceptions

The focus of constructivism is what the child brings to the activity and it is the child's active participation, the well-learned concepts and misconceptions that have a central role in learning. As alluded to above, this study focuses on misconceptions as a conceptual lens through which to view learners written and spoken words.

'The notion of misconception denotes a line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather than sporadic and non-systematic errors' (Nesher, 1987, p. 34). Therefore, a source of errors in mathematics is misconceptions although there are other sources of errors like carelessness or misleading language in tasks. Prior knowledge based on misconceptions will hinder the process of new knowledge acquisition and will cause learners to make errors during engagement with algebraic activities. In Olivier's (1989) study, 58% of grade 8 learners got a numerical solution of 12 by equating $e = f = g = 4$ and hence $e + f + g =$

$4 + 4 + 4 = 12$ (in, If $e + f = 8$ then $e + f + g =$ __). According to Olivier (1989, p. 13) the above example has only letters but ‘if a pupil’s arithmetic schema is retrieved, it will require that *numbers* be added’ and since ‘no values can be given for the letters the schema will make a default evaluation and somehow manage to produce replacement numbers’. Therefore, Olivier (1989) is suggesting that previous arithmetic knowledge can cause errors in algebraic problems. Moreover, this strengthens the argument that prior knowledge based on misconceptions will cause learners to make errors.

Nesher (1987) explains that misconceptions are difficult to identify or diagnose because at certain instances a learner might have a misconception but still manage to arrive with a correct solution. Smith, DiSessa and Roschelle (1993) and Olivier (1989) share the same sentiments as Nesher (1987) because they also argue that misconceptions give rise to patterns of errors, stem from learners’ prior knowledge and are very resistant to change. Misconceptions are therefore critical in the learning of algebra because if they are left undetected they will surface and resurface at various different stages of a learner’s mathematics development and will hamper the child’s learning process. Therefore, misconceptions have an important place in the mathematics classroom and the mathematics teacher ‘should provide opportunity to the student to manifest his misconceptions, and then relate his subsequent instruction to these misconceptions’ (Nesher, 1987, p. 39).

Nesher (1987) states that ‘a good instructional program will have to predict types of errors and purposely allow for them in the process of learning’. Olivier (1989, p. 13) also stresses that direct teaching of previous knowledge to confront misconceptions is not as viable as using other strategies such as ‘successful remediation’ or ‘cognitive conflict’. Moreover, if misconceptions are made explicit and used in learning learners could restructure and construct new richer knowledge.

2.6.2 Core misconceptions in generalised arithmetic and the transition from arithmetic to algebra

There is good local and international research on misconceptions in early algebra and in this section I cite a few of these reported misconceptions. A frequently cited error is the arithmetic instinctiveness of wanting to find a single answer solution. 'When encountered in an algebraic sentence x signals here's something to be calculated' (Novotna & Kubinova, 2001, as cited in Drouhard & Teppo, 2004, p. 241). Stacey and MacGregor (2000, p. 151) explain that this tendency for computation of a solution hinders students from using algebraic methods in problem solving. An example is when the expression $2a + 3b + a$ needs to be simplified, some learners give answers like $5ab$ or $6ab$ by combining all the terms in the expression to get a single answer solution. Boulton–Lewis et al. (1997, p. 89) suggest an alternate misconception for the instinctiveness of calculating a solution related to the equals sign as being 'a symbol indicating where the answer should be written or to do something'. They suggest that due to the equals sign students give a solution of $4m$ for $4 + m$ instead of leaving the expression as the solution.

In a similar way MacGregor and Stacey (1997) speak of *conjoining* when learners join terms during addition. This could also be seen as a tendency to find single answer solutions. Learners in MacGregor and Stacey's (1997, p. 7) sample (in response to the question: 'Con is 8 cm taller than Kim. Kim is y cm tall. What can you write for Con's height?) wrote the terms $8y$, $y8$, etc. in which they denote a combination of the number 8 and the unknown number y, their errors being due to conjoining terms for addition'. Other typical examples of *conjoining* could include $4ab$ as the solution to $4 + ab$ or $8ab$ for $3a + 2b + 3$.

MacGregor and Stacey (1997, p. 10) also found that 'some errors were due to the *letter = 1* belief. Students with this *letter = 1* misconception during addition of $3 + n + 4$ could get solutions such as $8n$ or 8 due to the n being assigned a value of 1. One likely cause is a misunderstanding of what teachers mean when they say x without a coefficient means $1x$'. MacGregor and Stacey (1997) also explain that another reason for students making the above error is due to the power of x being 1 and x to the power 0 equals to 1.

Although the context of Stacey and MacGregor's (2000, p. 150) research involved word problems and my study does not deal specifically with word problems they found that students 'used one letter to stand for many different quantities that are present even in the simple situations portrayed in the problems'. The issue and misconception is being unaware of when letters are equal, when letters have a single numerical value and when letters can have more than one numerical value. It follows that, this misconception could cause errors such as interpreting the letters x and y, in $x + y = 10$, as being equal or possibly randomly equating the letters x and y to any numbers that make a sum of 10. Therefore, this misconception, related to the incorrect interpretation of the letter, could cause errors because learners are unable to see that x and y could be any real numbers but when added must make 10.

2.7 The research of the CSMS (1979)

It is the strong presence of symbols in the senior phase (grade 7–9) and the difficulties learners' experience when working with symbols that prompted me to investigate learners' misconceptions when engaging with algebraic activities. There is a wide range of local and international research on misconceptions in generalised arithmetic but the research of the CSMS (1979) is a particularly influential large scale study that is still being drawn on in today's research and is presently re–emerging in the United Kingdom[2]. The CSMS (1979) has influenced, over the last three decades, ongoing research related to interpretation and understanding of symbols. [See Hart (1981), Kieran & Sfard (1999) and Drouhard & Teppo (2004)]. The empirical field of Küchemann's (1981) study which evaluated learners' interpretation of letters across different levels of understanding is located in the study of the CSMS (1979) which was conducted in England with a sample of about ten thousand learners. Learners' ages ranged from 11 to 15 years and the study was conducted over three years (1976-1978).

Algebraic test activities of the CSMS (1979) included generalised arithmetic tasks because letters were used as representations of numbers. (See section 2.2 above for discussion on generalised arithmetic). Küchemann (1981) reports on misconceptions in these generalised arithmetic activities for fourteen year olds in the CSMS (1979) sample which interested me and enables me to think about my first three research

[2] Personal communication, Professor Margaret Brown, Kings College, London.

questions. Therefore, it is to this end that I extensively review the literature of the CSMS (1979).

2.8 Symbolic interpretation

Küchemann (1981) found that learners had varying interpretations of letters. He found that interpretation of the letters were influenced by the type of algebraic activity and that younger children encountered greater difficulty than their older counterparts. He attributes the latter empirical finding to 'performance was dependant more on cognitive development than on specific experiences of algebra' which is a generalisation of the study (Küchemann, 1981, p. 117). Some constructivists would agree that the dependency of learning on cognitive development is a central tenet of constructivism.

However, my research is situated in a South African context where language and socio-economic contexts are different to that of England. It has been many years since the research of the CSMS (1979) and in addition since this research South Africa has reformed its curriculum. A key reform is the introduction of algebra in grade 7 as opposed to grade 8 in the former curriculum and hence younger children are introduced to algebra. Therefore, in my research I investigated, many years later, using an adaptation of the CSMS (1979) instrument, the nature of misconceptions in early algebra and the extent to which these findings relate to or are different to the CSMS (1979) study. However, I must emphasize that my research is not attempting to repeat the CSMS (1979) study. My methodological approach is very different to that of the CSMS (1979) study because my sample comprised only thirty learners and I have analysed learner interviews as my primary data source.

Küchemann's (1981) study also provides conceptual tools that are useful for my study. His six categories below are discussed in relation to how letters were interpreted across different test items by learners in the CSMS (1979) sample. At certain instances learners interpreted the *letters as variables, objects or specific unknowns,* etc. Therefore, Küchemann (1981) evaluated learners' thinking and hence makes reference to 'children's interpretations of letters'.

Letter evaluated

'This category applies to responses where the letter is assigned a numerical value from the outset' (Küchemann, 1981, p. 104). This is an example of a question where learners can make use of the *letter evaluated* interpretation: What can you say about a if $a + 5 = 8$?

The letter a can be evaluated as a number and by inspection or substitution the equation can be solved. Albeit the letter is interpreted numerically or arithmetically the letter is given some meaning and needs to be used to solve the task but the letter is not interpreted as an unknown. This differs from the next interpretation where the letter can be ignored in solving the task.

Letter not used

'Here the children ignore the letter, or at best acknowledge its existence but without giving it meaning' (Küchemann, 1981, p. 104). This is an example of a question where learners can make use of the *letter not used* interpretation: If $a + b = 43$, then $a + b + 2 =$ __

This interpretation is also arithmetical and learners don't have to use or assign any meaning to the letter. Matching and logic could be used to solve the equations. In other words, matching of $a + b = 43$ in the latter equation without interpreting the letter results in $a + b + 2 = 45$. Hence, in this way the letter is not interpreted as an unknown to solve the equations.

Letter used as an object

'The letter is regarded as shorthand for an object or as an object in its own right' (Küchemann, 1981, p. 104). This is an example of a question where learners can make use of the *letter used as an object* interpretation: $2a + 5a =$ __

In this interpretation the $2a$ and $5a$ can be interpreted as objects and added to get $7a$. The letter is not interpreted as an unknown nor is the letter *ignored* or *evaluated* as a number but in solving the task $2a$'s need to be added to $5a$'s and hence children treat the *letters as objects*.

As explained above, for the first three interpretations the letters are evaluated arithmetically and not as unknowns. However, in the latter three interpretations the letter needs to be interpreted algebraically, given meaning and operations need to be performed directly on the letter.

Letter used as a specific unknown

'Children regard a letter as a specific but unknown number, and can operate on it directly' (Küchemann, 1981, p. 104). This is an example of a question where learners can make use of the *letter used as a specific unknown* interpretation: Multiply $n + 5$ by 4.

For *letter used as a specific unknown* the letter has a particular value albeit this value is unknown. Moreover, in the example the unknown n must be multiplied by 4 to achieve the solution.

Letter used as a generalised number

'The letter is seen as representing, or at least as being able to take, several values rather than just one' (Küchemann, 1981, p. 104). This is an example of a question where learners can make use of the *letter used as a generalised number* interpretation: What can you say about c if $c + d = 10$ and c is less than d?

In this category, the letter is not an *unknown* taking one specific value but it can be seen that the letter c above has a set of values less than 5 and is therefore a generalised number. It follows that, 'it may be the case that children get an understanding of specific unknown first' before this interpretation where the letter is assigned many values is properly understood (Küchemann, 1981, p. 109).

Letter used as variable

'The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between such sets of values' (Küchemann, 1981, p. 104). This is an example of a question where learners can make use of the *letter used as variable* interpretation: Which is larger $2n$ or $n + 2$?

In this interpretation the letter can be a range of values which seems similar to the interpretation as *generalised number* but 'the concept of a variable implies an understanding of an unknown as its value changes' (Küchemann, 1981, p. 110). In the example, learners have to reason about how varying n affects the magnitude of $2n$ and $n + 2$. Therefore, n must be interpreted as belonging to a set of real numbers which suggests that this interpretation is the most abstract of the six.

Küchemann (1981) explains that the selection of a specific interpretation by learners in the CSMS (1979) sample depended on the task structure as certain interpretations were not always relevant. However, the test items are also categorised under each of the interpretations of the letter which implies that certain tasks require the letters to be interpreted in certain ways. To sum up, Küchemann's (1981) six interpretations of letters emerged from learners' responses which I use to analyse and make sense of learners' responses in the present sample.

Questions 1–3 in the paper and pencil tasks (refer to Appendix 2) were categorised as the *letter needs to be evaluated*. However, the letter can still be interpreted in a different way to the specified category. Question 1 is: What can you say about a if $a + 5 = 8$. The letter a can be interpreted as an *object, specific unknown or not used*. In this study, I discuss interpretations of the letter as the minimum meaning needed to be given to the letter to solve a particular task. It could then be argued that the interpretations of letters follow a hierarchical order with *letter evaluated* tasks needing minimum 'symbol sense' and minimum understanding of the meaning given to the letter whereas using the *letter as a variable* needs the most sophisticated use of the letter.

Therefore, if a learner uses a lower level interpretation for a particular task it could result in incorrect solutions. For example, in task 15 learners have to interpret the letter c (What can you say about c, if $c + d = 10$ and c is less than d?) as a *generalised number* to achieve the correct solution. If learners *use the letter as specific unknown* (lower interpretation) then learners could possibly yield one solution and not a range of responses, which would be incorrect. Moreover, to discuss how learners interpret letters in algebraic settings it seemed interesting and appropriate to have tasks involving various uses of letters. In this way, I was able to discuss for which interpretations of letters learners in this sample were successful and which interpretations posed problems. (I further discuss how learners performed for the different interpretations of letters in Chapter 4.)

2.9 Levels of understanding

The CSMS (1979) study classified tasks of different levels of understanding with some being easier than others. Level 1 is the lowest level of understanding and the levels follow a hierarchical nature with level 4 being the most abstract level. Learners in the CSMS (1979) sample were categorised as having a particular level of understanding if she/he achieved correct solutions for two–thirds of tasks in that level. Küchemann (1981) determined the levels of understanding by two dimensions which are the interpretation of letters and the structural complexity of tasks.

A characteristic of structural complexity could be the number of variables of the task. For example, '$3a + 4b + c =$ __' has a more complex structure than '$4a + 2a + a =$ __' because the former task has 3 variables as compared to the 1 for the latter. However, this characteristic 'is clearly not sufficient for some of the other item pairs' (Küchemann, 1981, p. 103). An example is, '$(a - b) + b =$ __' has the same number of variables as 'if $a + b = 43$, then $a + b + 2 =$ __' but the former task has a more complex structure. This more complex structure is mainly due to learners needing to perform one operation for latter task but two operations, the simplification of the bracketed term and then addition of the three terms, for the former task.

Küchemann (1981) explains that within the lower level (level 1 and 2) and higher level tasks (level 3 and 4) there are structural differences. An example is "whilst the letter in '$a + 5 = 8$' (level 1) can be evaluated immediately, in '$u = v + 3$, $v = 1$' (level 2) the child first has to cope with an ambiguous statement" (Küchemann, 1981, p. 116). Hence, this 'ambiguous statement' or structure of the level 2 task makes it more complex than the level 1 task. Another characteristic of structural complexity is the number of operations in the task. "Whilst 'add 4 onto $3n$' (level 3) only requires a single operation in 'multiply $n + 5$ by 4 (level 4) this leads to an ambiguous answer ($n + 5 \times 4$) and it becomes necessary to coordinate two operations" (Küchemann, 1981,

p. 116). In this way the level 4 task, having two operations, increases in structural complexity from the level 3 task. Therefore, structural complexity affects difficulty of the tasks and hence influences the level of understanding of tasks.

'The items at level 1 and 2 can all be solved without having to operate on letters as unknowns, whereas at levels 3 and 4 the letters have to be treated at least as specific unknowns and in some cases generalised numbers or variables' (Küchemann, 1981, p. 116). Level 1 tasks were 'extremely easy, purely numerical and had a simple structure' (Küchemann, 1981, p. 113). The level 1 tasks could be solved by *evaluating the letter, not using the letter* and using the *letter as an object*. The level 2 tasks increase in structural complexity but could also be solved by *evaluating the letter, not using the letter* and using *the letter as an object*. The equations for the level 1 and 2 tasks are arithmetically inclined and could be solved without giving meaning to the letter and performing operations directly on the letter (as in $3a + 5a = 8a$). Therefore, the level 1 and 2 tasks require a more arithmetic notion of the letter to solve the task. Learners on these levels might not be able to cope with the interpretations of *letters as unknowns, generalised numbers* and *variables* because these require a more algebraic notion of the letter.

Level 3 increases further in structural complexity from level 2 and the tasks need to be solved by interpreting the *letters as unknowns, generalised numbers or variables*. Küchemann (1981) demonstrated that using *letters as variables, unknowns or pattern generalisers* were of greater difficulty because the letter could not be avoided and greater meaning of the letter was required. Therefore, a more algebraic notion of the letter is required to solve tasks in level 3. Learners on this level are able to view solutions like $a + 3$ as having meaning 'despite the lack of closure of the answers' (Küchemann, 1981, p. 114). Level 4 items also need an algebraic notion of the letter and are the most abstract and complex level in terms of structure. Tasks on level 4 also need to be solved by interpreting the *letters as unknowns, generalised numbers or variables*. In this study, I also interpret and discuss levels of understanding involving the two dimensions of interpretation of letters and structural complexity of the tasks.

Table 1 below provides a summary of how the four levels of understanding relate to the two dimensions of interpretation of letters and structural complexity.

Table 1: Illustration of the two dimensions determining levels of understanding

Level	***Interpretation of letters***	***Structural complexity***
level 1	*letter evaluated* *letter not used* *letter as an object*	'Extremely easy, purely numerical and simple structure' (Küchemann, 1981, p. 113).
level 2	*letter evaluated* *letter not used* *letter as an object*	Increases in *complexity* from the level 1 items. Learners on this level might not cope with the interpretation of *letters as variables, unknowns or pattern generalisers*. 'A willingness to accept answers which are to some extent incomplete or ambiguous' (Küchemann, 1981, p. 113).
level 3	*letter as specific unknown* *letter as generalised number* *letter as variable*	Increases in *complexity* from level 2. 'Children at this level can use letters as specific unknowns though only when the item-structure is simple' and are able to view solutions such as $a + 3$ as having meaning 'despite the lack of closure of the answer' (Küchemann, 1981, p. 114).
level 4	*letter as specific unknown* *letter as generalised number* *letter as variable*	Children at this level 'can cope with items that require specific unknowns and which have a complex structure' (Küchemann, 1981, p. 115). Learners at this level are also able to carry out multiple operations in one item and are able to interpret letters as variables.

2.10 Summary

The three conceptual frameworks discussed by Kieran (2004), Arcavi (2005) and Küchemann (1981) are strongly related and have crucial links to my study. The 'core activities', children's interpretations of letters and 'symbol sense' are all describing or encompassing algebraic situations which are in line with my study. In other words, my study is investigating learners' interpretation of letters and 'symbol sense' in generalised arithmetic contexts which is an integral part of the three core activities of algebra. Therefore, my investigation is informed by the above three frameworks.

In this chapter, I strengthened my rationale for the use of generalised arithmetic in my study by showing how generalised arithmetic is embedded in the transition from arithmetic to algebra and by creating links to the 'three core activities of algebra'. I discussed constructivism as the theoretical framework that underpins this study by explaining two major influences for adopting a cognitive theory of learning. I elaborated on a central tenet of constructivism that of misconceptions, argued that misconceptions could be used to enhance pedagogy and provided a summary of core misconceptions. In the next chapter I unpack the methodology adopted in this study.

Chapter three: <u>Methodology</u>

3.1 Introduction

In this chapter, I explain the methodology, methodological orientation and features of case study research that have guided my research. I also discuss the selection of the school and learners in this sample. Thereafter, discussions centre on issues of data collection, research instruments and the pilot study. In the latter part of this chapter I discuss issues of rigour in research and ethical considerations.

3.2 Methodological orientation

My research is a case study of qualitative nature. The purpose of my study is to gain insight into learners' interpretations of letters and misconceptions in algebraic settings. Therefore, I focused on one test and six in-depth interviews 'seeking to maximize understanding of events and facilitating the interpretation of data' (Hitchcock & Hughes, 1995, p. 296). Moreover, analysis involved looking for patterns and themes in the data therefore a qualitative approach was seen as being useful.

3.3 Features of case study research

Denscombe (2007, p. 35) explains case studies as a 'focus on one instance of a particular phenomenon with the view to providing an in-depth account of events, relationships, experiences or processes occurring in that particular situation'. Therefore, I chose to have six in-depth interviews which were preceded by paper and pencil tests written by thirty grade nine learners (See Chapter 4 for detailed rationale for selection of learners for interviews). Analysis of the data from these instruments was critically and closely analysed (discussed later in this chapter) which enabled me to provide detailed discussions of how learners engaged with algebraic tasks.

According to Denscombe (2007, p. 35) a strength of case study research involves focussing on a few cases that enables the researcher to focus on 'the subtleties of complex situations'. Therefore, due to my study adopting a case study approach I was able to provide a thorough analysis of the data. In other words, data from the two instruments were analysed in detail by focussing on each learner's responses which enabled me to discuss common themes emerging from the data.

The sample of this study comprised only thirty learners from one school who engaged with one test for thirty minutes followed by interviews of only six learners. Therefore, the sample and research instruments might not allow me to make generalisations. This concurs with Denscombe (2007, p. 45) who explains that a disadvantage of case study research is the justification or 'credibility' of generalising results. However, generalisations across similar cases could be established.

3.4 Selection of the school

Through my relationship with a colleague from previous studies who taught at a school near the university, the school in my study was selected. In this way it was a convenience sample. The selected school is an inner city government school located in Johannesburg, Gauteng, South Africa. It has classes from grade 8 to grade 12 and has a 98% Black enrolment with a high immigrant population from many different countries in Africa. There was also an array of teachers from different cultural and ethnic backgrounds. I observed that the school afforded private garden services, a maintenance manager and security personnel which suggested that the school had reasonable financial resources. From my discussions with the deputy principal, I also gathered that the learning and teaching resources were sufficient. The latter comments are based on my professional judgement during several visits to the school. Moreover, the grade nine classroom used during my data gathering had its own overhead projector, chalkboard, set of mathematics textbooks and the walls had relevant grade nine mathematics charts.

3.5 The sample

The selected inner city school had three grade nine classes which were not streamed according to ability and were taught by the same mathematics teacher. Therefore, it did not matter which class was used and I based my choice on availability, hence convenience sampling. The sample comprised thirty grade nine learners from the one available class. All learners were Black. The majority of learners in this sample do not speak English as their home language but are accustomed to learning in English as this is the language of learning and teaching at their school.

3.6 Research instruments

3.6.1 Paper and pencil test

The central aim of my first three critical research questions was to investigate learners' interpretation of letters in algebraic contexts. Therefore, Küchemann's (1981) research was reviewed in detail because the structure and nature of his study assists me with my first three research questions. The CSMS (1979) test was useful as a first step in my investigations followed by interviews. Therefore, the research by Küchemann (1981) enables me to think about my empirical field, setting and findings because my fourth research question enables me to make a comparison to the CSMS (1979) study.

However, there are two key methodological differences that I will now foreground. Firstly, the scope of my study does not allow me to achieve a similar sample (10 000 learners) as that of the CSMS (1979). However, due to the paper and pencil tasks being selected directly from the CSMS (1979) study I did align my research to that of the CSMS (1979) in terms of the age or grade level of learners. Secondly, the CSMS (1979) study administered paper and pencil tasks as the principal data-collecting instrument whereas interviews were my principal instrument and were a more central focus.

Based on two factors, I selected seventeen algebra tasks from the CSMS (1979) study. The first factor I considered was the tasks involved generalised arithmetic in the transition from arithmetic to algebra because my study is also concerned with generalised arithmetic in early algebra. The second factor I considered in choosing the tasks was the centrality of letters in the tasks because a central tenet of my study is to investigate learners engaging with letters in algebra.

The paper and pencil tasks were categorised into six interpretations of letters and four levels of understanding. The different interpretations relate to the minimum meaning needed to be given to the letter to solve the task. Therefore, I am able to analyse and discuss how learners solved tasks that involved different uses of letters in algebra. On the other side of the same coin, I discuss levels of understanding involving the two dimensions of interpretation of letters and structural complexity of the tasks. Level 1 and 2 tasks have an arithmetical structure and could be solved by *evaluating the letter, not using the letter* and using the *letter as an object*. A more algebraic notion of the letter is required to solve tasks in level 3 and 4 by interpreting the *letters as unknowns, generalised numbers* or *variables*. Level 4 items are the most abstract and have a complex structure. (For a more thorough discussion of interpretations of letters and levels of understanding refer to section 2.8 and 2.9).

Table 2 below relates the tasks for each interpretation to the different levels of understanding. (See Appendix 2 for paper and pencil test.)

Table 2: Illustration of the composition of the paper and pencil tasks

Interpretation of letter	***Total number of tasks***	***Number of level 1 tasks***	***Number of level 2 tasks***	***Number of level 3 tasks***	***Number of level 4 tasks***
Letter evaluated (relates to questions 1–3)	3	1	2	na	na
Letter not used (relates to questions 5, 6, 12)	3	2	1	na	na
Letter as object (relates to questions 8&9)	2	1	1	na	na
Letter as specific unknown (relates to questions 4, 7, 10, 11, 13,14)	6	na	na	4	2
Letter as generalised number (relates to questions 15&16)	2	na	na	1	1
Letter as variable (relates to question 17)	1	na	na	na	1

*na: not applicable

3.6.2 Interviews

An assumption of this study was that incorrect strategies and misconceptions will surface in learners' responses to the paper and pencil tasks. Interviews allowed me to probe to see what incorrect strategies and misconceptions are adopted by learners during algebraic problem solving.

Interviews were my principal research tool and were crucial for my investigation gaining a deeper insight to the "why's?" of my research questions. (Why do learners adopt certain methods….? And, why are learners' interpretations of symbols different across a range of activities…?). Paper and pencil tasks enabled me to reflect on my first and fourth research questions but did not allow me to respond fully to my second and third research questions. Therefore, I interviewed six learners that showed common misconceptions or interesting results that yielded sufficient data to get a deeper understanding of why learners responded to the tasks in certain ways.

3.6.2.1 Preparation prior to interviews

Interviews were semi–structured because questions were developed based on learners' solutions to the paper and pencil tasks but learners' responses to my interview questions guided me to what further needed to be asked. Therefore, my interpretation of semi–structured interviews is in line with Drever (1995, no page numbers) who explains that 'some researchers use semi-structured interviews which have some pre-set questions, but allow more scope for open-ended answers'.

Each learner showed different interesting aspects at different stages of the paper and pencil tasks hence one uniform interview schedule was not seen as viable. Based on each learner's responses to the paper and pencil tasks I prepared different interview schedules which assisted me to gain a deep insight into the reasoning behind responses. Interview questions were related to research critical question/s and designed so that each research question was covered by at least one or two interview questions. I did not select all tasks in the tests to base my interview questions because my interviews were scheduled for only thirty minutes and 'one interview question might provide answers to several research questions' (Krale, 1996, p. 125).

Interview questions were prepared in relation to the theoretical framework of constructivism and the literature review of Küchemann (1981) and Arcavi (2005). In other words, interview questions focused on possible misconceptions across different interpretations and levels of understanding, which are central to this study. The focus was on how and why learners interpreted letters in certain ways. Interview questions also focused on the possible display of the components of 'symbol sense' which are also central to this study. Questioning during interviews varied and I used introducing, follow-up and probing questions. Introducing questions were used to introduce learners to the topic or aspects for discussion in the interview. Follow up questions were used to investigate key responses from learners in relation to research questions. I also made use of probing questions to probe or "push" learners to explain their thinking during engagement with the tasks but was careful not to funnel learners' responses.

3.7 Piloting the instruments

I piloted my instruments at a high school in Johannesburg. Five grade nine learners were randomly selected to write the paper and pencil tasks for thirty minutes under similar conditions as the sample would be exposed to. I then piloted one interview with one of the initial five learners who was selected based on availability during my visit to the school.

There were two central reasons for piloting my instruments. Firstly, I wanted to view the validity of the instruments in terms of eliciting data that will enable me to discuss my critical research questions. Secondly, I needed to practice for the interview situations that were to follow at a later stage. This concurs with Krale's (1996, p. 126) view that prior practising of interviews is crucial for the success of interviews and a 'substantial part of the investigation should take place before the tape recorder is turned on in the actual interview situation'.

Therefore, data gathering started by piloting the paper and pencil tasks and responses assisted me in terms of the validity of the test. In other words, if learners had difficulties with the instrument (e.g. maybe tasks were too difficult or had misleading language) or the instrument did not yield anticipated data then I would have adjusted the test to accommodate problems. This was not the case and the test was therefore not adjusted. On the other hand, piloting the interview enabled me to investigate how letters were interpreted and possible misconceptions. I also practiced interviewing techniques such as probing. From piloting my two instruments I learnt that misconceptions are prevalent in learners' interpretation of letters in algebra. Furthermore, piloting resulted in analysis of data which revealed that the instruments would assist in answering my research questions and support the aims of this research. I provide a full analysis of my pilot study in Appendix 4.

In the next section I discuss the data gathering process after the piloted study.

3.8 Data gathering process

Due to ethical reasons, I administered my two research instruments in August 2008 when the school calendar does not have examinations. At this stage of the school year grade nine learners should be fully equipped with the algebraic background to engage with the instrument. I provided the necessary stationery, observed and invigilated without taking field notes. The two instruments were administered after school hours which did not disturb teaching and learning contact time. Moreover, it seemed likely that if the time frame between the interviews and tests was too long learners could have forgotten their strategies and methods used in the tests. Therefore, the interviews were conducted ten days after the tests and during these ten days I analysed the tests, selected the six learners for the interviews and prepared interview schedules.

The paper and pencil tasks were administered in a similar style to a test where learners worked individually on the tasks and were only allowed to make use of a scientific calculator. No other resources were needed because the tasks encompassed basic arithmetic and interpretation of algebraic symbols. Learners had thirty minutes to work through the tasks and no verbal explanations about any of the tasks were provided. After my initial instrument I interviewed six learners from the sample for which a comprehensive motivation for selection is provided in Chapter 4.

All interviews were individual with only the researcher and learner present. They were tape recorded and transcribed. Interview schedules were prepared for each interview. Although similar questions were asked the questions depended on what students wrote in their responses to the paper and pencil tasks. At the start and end of every interview aspects relevant to those stages of the interview were explained to the learner. This is in line with Krale (1996) who explains that interviewees should be given introductory and concluding remarks. Each interview lasted for a maximum of thirty minutes. This time frame was informed by the piloted interview that suggested thirty minutes was sufficient and hence interview schedules were prepared accordingly.

3.9 Data analysis process

Küchemann's (1981) framework was used as a lens through which I viewed the data collected. However, categories and themes also emerged from the data. Therefore, to some extent my analysis was inductive in nature. The inductive nature of this qualitative study involved 'looking for patterns, themes, consistencies and exceptions to the rule' (Hitchcock & Hughes, 1995, p. 296).

The process of analysing data was long and comprised many steps. All tests were marked and learners were ranked according to their scores (see Table 3, Chapter 4) which enabled me to discuss the overall performance of learners in the test. This initial analysis also enabled me to determine common errors and to select learners for interviews. I discuss common errors as similar incorrect answers for particular tasks and based on frequency I categorized first, second and third most common erroneous answers. In other words, the first common erroneous answer was viewed most frequently in learners' solutions.

Learners' responses were then summarized according to the different interpretations of letters and levels of understanding. A close analysis of each task for the different interpretations of the letter was the next step of the process. Themes for the paper and pencil tasks were established if the theme was viewed frequently across many different learners' responses. Based on common errors that were evident in learners' responses two themes and two tentative themes emerged from the paper and pencil tasks. I refer to tentative themes as potential patterns that were emerging from the data. However, at that stage there was insufficient data from the responses to the written test to support naming these potential patterns themes.

The next step focused on analysing the interviews to see if they provided corroborating evidence to strengthen the themes and tentative themes that emerged from the paper and pencil tasks. Themes were established if the misconception was frequently used by most of the interviewed learners. The two tentative themes from the paper and pencil tasks as well as the two themes were strengthened and hence four themes emerged from the interviews. Two themes were evident in all six learners' interviews, while the other two themes were evident in data collected for three and five learners respectively. The steps of data analysis that I followed was in line with Potter (1996, as cited in Hatch, 2002, p. 161) who explains that an inductive analysis 'begins with an examination of the particulars within data, moves to looking for patterns across individual observations, then arguing for those patterns as having the status of general explanatory statements'.

3.10 Rigour in research

Reliability in research has to be maintained through the whole data gathering process and not only the data or the instrument. According to Bell (1999, as cited in Scaife, 2004, p. 66) 'reliability is the extent to which a test produces similar results under constant conditions on all occasions'. Therefore, although it is not the central focus of my study, for the reliability of possible comparisons to the CSMS (1979) study, tasks used by the CSMS (1979) were also used in this study. Furthermore, the CSMS (1979) study conducted interviews with thirty learners from different schools in London to check the reliability of test items. Hart (1981, p. 1) explains that tasks in the CSMS (1979) study were 'free of technical words and these were tried on interviews with children and then replaced or revised'.

'Validity refers to the degree to which a method, a test or research tool actually measures what it is supposed to measure' (Wellington, 2000, as cited in Scaife, 2004, p. 68). It seemed viable that paper and pencil tasks would be the most valid instrument to investigate how learners interpret algebraic letters which is central to my research critical questions. However, the paper and pencil tasks did not enable me to discuss why learners adopted incorrect strategies and interviewing a portion of learners enabled me to investigate this.

3.11 Ethical considerations

There was no disruption to the running of the school in terms of contact time, as my research was carried out after school hours. I am willing to share overall general results with the mathematics teacher but will maintain the confidentiality of individual learners. Participation in this study was voluntary and anonymity of learners in this sample, the mathematics teacher and school has been maintained throughout my study. I received written consent from each parent of the learners for both the interviews and the paper and pencil tasks (see Appendix 3). I did also seek written consent from the principal of the school, all learners in this study as well as the mathematics teacher for each step of the data gathering process such as the tests, interviews and tape recording of interviews. Moreover, I applied to the Gauteng Department of Education and the University of the Witwatersrand for ethical clearance and have received written ethical clearance from both entities. It follows that, in this study and my research report all names of learners, the school and the mathematics teacher are pseudonyms.

3.12 Summary

In this chapter, I discussed the methodological orientation of my research as a case study of qualitative nature. I also provided descriptions of the context of my research and the sample of people that were part of this study. A rationale for the selection of the two research tools that enabled me to gather data to respond to my research critical questions was provided. Discussions also centred on the administration of the research instruments and that interviews were my principal research tool. I explained how the data was analysed. Lastly, issues of reliability and validity were discussed and ethical considerations were made explicit.

Chapter 4: <u>Analysis and data interpretation</u>

4.1 Introduction

In this chapter I provide an analysis of the data collected. My data analysis will be based on data derived from the two instruments which are the paper and pencil tasks (30 grade nine learners) and the six interviews, as discussed in Chapter 3.

The learners wrote the paper and pencil tasks on the 15th August 2008 and the six interviews were conducted on the 25th of August 2008. The paper and pencil tasks were administered in a similar style to a test. Learners worked individually on the tasks and were only allowed to make use of a scientific calculator. However, during my invigilation I observed only two learners who opted to make use of a calculator. Most learners spent the full thirty minutes allocated working through the tasks in the instrument. Learners were not asked to study for the engagement with the instrument and I did not provide additional verbal explanations about any of the tasks.

In the discussions that follow I provide an overview of learners' solutions for the paper and pencil tasks. I then briefly recap on the influence of Küchemann's (1981) study followed by an analysis of learners' responses with respect to the different interpretations of the letter (See section 2.8 in Chapter 2 for discussion of the different interpretations). An analysis of learners' solutions across the different levels of understanding is then provided (See section 2.9 in Chapter 2 for discussion on levels of understanding). Thereafter, I provide a thorough analysis of the tests. Lastly, I supplement this with an analysis of the interviews which helps me to explain why learners experienced difficulty in interpreting letters.

4.2 Overview of learners' solutions for the paper and pencil tasks

Table 3 below provides an overview of learners' solutions for the 17 paper and pencil tasks. The numbers in the bracket, after each learner's name, indicates the number and percentage of correct solutions. The answers that are bold are correct solutions. Table 3 also highlights the learners that were selected for the interviews for which a rationale is provided later in this chapter. It can be clearly seen that learners' overall performance was very poor and they experienced great difficulty engaging with the tasks. In total the learners managed to get 113/510 solutions correct (22%). The highest score was 7/17 correct responses (41%) achieved by two learners while the lowest was 0/17 scored by one learner. The most unanswered task was task 16 where 7 learners did not write any solution. The vast majority of learners (24 learners) overall performance was below 30%, 4 learners scored between 30% and 40% and 2 learners scored 41%.

In the next section I discuss how the research by Küchemann (1981) is used to frame my data analysis.

Table 3: My sample's solutions for all tasks

Learner/ Task	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***	***13***	***14***	***15***	***16***	***17***
Ilo (0; 0%)	3a	3	4n	30	$2ab^2$	763	9	9a	7ab	3a-b	abcd	9n	7n	9n	1		2n
Geo (1, 6%)	13*a*	3*uv*	4*m*+ 5*n*	30*rs t*	**45**	763	16	$7a^2$	$2a^2 +5b$	$2a^2b$	ab^2	9n	7n	1n	yes	never	2n
Que (1; 6%)	**3**	1	4	3p0	43	762	8	$7a^2$	$7a^2b$	$3a^2b$	$ab+b^2$	9n	7n	20n	10		2n
Emma (1; 6%)	8	**4**	35	60	2ab	763	12	8a	ab	3ab	10	9n	7n	29	10	never	n+2
Many (1; 6%)	**3**	1	35	10	45ab	1000	16ef g		63ab	31ab	2ab	10n	35n	9n	4		2n
Lee (2; 12%)	**3**	**3**	4mn	30rst	2ab	**761**	12	$7a^2$	$7a^2b$	$3a^2b$	ab	9n	7n	20n	10cd	15	2n
Ky (2; 12%)	**3**				5	493	12	**7a**	8b	3ab	ab^2	10n	7n	20n	9	never	n+2
Leila (3; 18%)	**3**	3	4n	**15**	2	**761**	8	7a	3a5b	4ab	ab^2	9n	7n	20n	4	never	none
Greg (3; 18%)	**3**	2		30	**45**	763	12	**7a**	7a	3ab	ab	9n	7n	20n	10		none
Tide (3; 18%)	**3**	3	3	10	43	1008	8	**7a**	8ab	3ab	ab	4n+a	7n	**4n+20**	10		2n
Nelli (3; 18%)	**3**	v+3	6	**15**	2ab	**761**	16	$7a^2$	$7a^2b$	3ab	$ab-b^2$	9n	7n	20n	3	never	+n2
Mik (3; 18%)	**3**	**4**	4n	30rst	48ab	**761**	8g	$7a^2$	$8a^2b$	5ab	1ba	9n	7n	20n	10	never	2+n
Tibo (3; 18%)	**3**	2	10	10	**45**	**761**	2	$7a^2$	$8a^2b$	$5a^2b$	$3ab^2$	9n	7n	1n	10	never	
Depla (3; 18%)	**3**	3	4n	10	**45**	269	12	**7a**	7ab	3a-b	a-b	9n	7n	20n	10	never	
Louis (3; 18%)	**3**	**4**	35	30	5	512	12	**7a**	8ab	5ab					10	**someti mes**	2n
Kay (4; 24%)	**3**	**4**	**13**	30	**45**	515	8g	$10a^2$	$3a^2 +b$	$4a^2-b$	$ab-b^2$	10n	7n	20	10	never	n+2
Elli (4; 24%)	8-5		1	30				$7a^2$	**3a+5b**	**4a-b**	$ab-b^2$	**n+9**	7n	**4n+20**	5	always	2n
Thabo (4; 24%)	5	1	1		4	514n	g	$7a^2$	**3a+5b**	**4a-b**	$ab-b^2$	4n+5	7n	4n+2a			
Leya (5; 29%)	**3**	2	35	10	**45**	**761**	12	**7a**	8a	**4a-b**	2b-a	1017	7n	48	4	never	no
Pink (5; 29%)	5	2	6n	20	41	**761**	6	**7a**	**3a+5b**	**4a-b**	ab +b	9n	7n	**4n+20**	4	never	none
Soji (5; 29%)	**3**	2	0	10	**45**	763	8g	**7a**	3a5b	4ab	$ab-b^2$	9n	7n	20n	4	never	
Kate (5; 29%)	**3**	2	0	**15**	**45**	513	9	**7a**	**3a+5b**	2a-b	$a-b^2$	10n	7n	21n	10	never	n+2
Rami (5; 29%)	**3**	**4**	8	10	**45**	**761**	9	**7a**	7ab	3a	ab	9n	7n	20n	3+d	never	2n
Jays (5; 29%)	**3**	1	4+n	30	**45**	349	12	**7a**	**3a+b**	**4a-b**	1a-2b	9n	7n	20n	4	always	n+2
Tiel (6; 35%)	**3**	**4**	35	10	**45**	269	12	**7a**	**3a+5b**	**4a-b**	a-2b	10n	7n	24n	4	never	n+2
Gill (6; 35%)	**3**	**4**	**13**	20	**45**	**761**	9	**7a**	7ab	3a	ab^2	9n	7n	20n	6		none
Vuyo (6; 35%)	**3**	**4**	5	10	**45**	**761**	12	**7a**	**3a+5b**	2a-b	4a	9n	9n	20n		never	none
Agi (6; 35%)	**3**	**4**	35	10	46	**761**	12	**7a**	**3a+5b**	**4a-b**	**a**	9n	7n	20n	4a+b	always	n+2
Laizal (7; 41%)	**3**	**4**	**13**	1	**45**	763	8	7a	8a	**4a-b**	b-b+a	**n+9**	**4+3n**	20+n	10	never	2n
Signy (7, 41%)	**3**	**4**	**13**	5	**45**	**761**	9	**7a**	7ab	3ab	**a**	9n	7n	$20n^4$	10	never	none

*Open space indicates no response by the learner.

4.3 The influence of Küchemann's (1981) study in my data analysis

In my analysis I used Küchemann's (1981) framework of the six different interpretations of letters and the four levels of understanding to discuss how learners in this sample engaged with the generalised arithmetic tasks. As alluded to above, levels of understanding involve two dimensions, interpretation of letters and structural complexity. The interpretation of letters refers to the minimum meaning needed to be given to the letter to solve the task while structural complexity relates to the "structure" of the task. Level 1 and 2 tasks could be solved by *evaluating the letter, not using the letter* and using the *letter as an object*. Level 3 and 4 tasks increase in structural complexity and could be solved by interpreting the *letters as unknowns, generalised numbers* or *variables*. (For a more thorough discussion of interpretations of letters and levels of understanding refer to section 2.8 and 2.9). Moreover, the central aspect of my data analysis involved analysing each question but in doing this I also looked at the different levels of understanding because the levels encompass both the interpretation of letters and the structural complexity.

In the next section I discuss the average correct responses for the different interpretations of letters.

4.4 Overview of the learners' responses to the different interpretations of letters

<u>Table 4</u>: Average correct responses (average refers to the mean scores of the different questions) for the different interpretations of letters

	Interpretation of letter	***Average correct response by learners (%)***
	Letter evaluated (relates to questions 1–3)	43%
Lower levels	*Letter not used* (relates to questions 5, 6, 12)	32%
	Letter as object (relates to questions 8&9)	40%
	Letter as specific unknown (relates to questions 4, 7, 10, 11, 13,14)	9%
	Letter as generalised number (relates to questions 15&16)	2%
	Letter as variable (relates to question 17)	0%

The above table shows that the overall performance was poor for the six different uses of the letter because the highest average correct responses were only 43%. Learners were fairly successful (relative to their overall performance) with the tasks where the *letter needed to be evaluated, not used* and *used as an object* (I refer to these three categories as the first three interpretations of the letter) because they scored, on average, 38% correct solutions across these three categories. However, almost all learners struggled (the average correct responses was only 4%) across the last three categories where the letter needed to be interpreted as a *specific unknown, generalised number* and a *variable.*

The poor performance in the latter interpretations suggest that learners were lacking components of Arcavi's (2005) 'symbol sense' such as manipulations and different uses of symbols and knowledge of algebraic expressions. This is an interesting finding which I will discuss further in Chapter 5. The above comparative performance analysis relates to my first research question and shows how learners interpreted letters during engagement with generalised arithmetic activities. Moreover, this finding is similar to Küchemann's (1981) study because in his study using *letters as variables*, *unknowns* or *pattern generalisers* were also of greater difficulty for the fourteen year old learners, although overall results are much weaker for learners in my study.

In the section that follows I discuss the number of correct solutions achieved by learners across the different levels of understanding.

4.5 Overview of learners' responses to the different task levels

The table below shows the number of correct solutions achieved out of the total possible solutions that could have been achieved by the 30 learners. It can be seen that there were tasks of different levels for the six interpretations of the letter. As explained above, the level of complexity and difficulty of tasks increased from level 1 to 4.

Table 5: Learners' correct solutions across the different levels

Interpretation of letter	***Level 1 (4 tasks)***	***Level 2 (4 tasks)***	***Level 3 (5 tasks)***	***Level 4 (4 tasks)***
Letter evaluated (relates to questions 1–3)	24/30	15/60	na	na
Letter not used (relates to questions 5, 6, 12)	27/60	2/30	na	na
Letter as object (relates to questions 8&9)	16/30	8/30	na	na
Letter as specific unknown (relates to questions 4, 7, 10, 11, 13,14)	na	na	13/120	5/60
Letter as generalised number (relates to questions 15&16)	na	na	0/30	1/30
Letter as variable (relates to question 17)	na	na	na	0/30
Total correct responses	67/120	25/120	13/150	6/120

*na: not applicable

The assumption that learners will interpret letters differently across the different levels of understanding can be seen clearly in the above table. The total correct responses across the different levels suggest that learners were fairly competent with the level 1 tasks (relative to their own performance) and experienced difficulties with the level 2, 3 and 4 tasks. The correct solutions were 67 out of a possible 120 for the level 1 tasks, 25 out of a possible 120 for the level 2 tasks, 13 for the level 3 tasks and 6 for the level 4 tasks. Thus, while there were more level 3 and 4 tasks learners got fewer correct solutions in these levels.

A possible reason for learners experiencing difficulty with the higher levels could be the combination of difficulty and weakness in learners' use of letters. In question 13 (level 3) where learners needed to add 4 to $3n$ it seems likely that learners were unable to interpret the *letter as a specific unknown* because this interpretation of the letter is of greater difficulty. Moreover, this difficulty and weakness in learners' use of the letter is reflected by 87% of learners giving the incorrect solution of $7n$. In contrast, 31 % of Küchemann's (1981) sample also gave this incorrect solution.

At the lower levels the structural complexity does seem to make a difference on the performance as is suggested by the 35% decrease in the total correct responses from level 1 to level 2. However, the structural complexity seems to have no impact in the higher levels where there is a 3,7% decrease in the total correct responses from level 3 to level 4. Furthermore, it seems more likely that for the higher levels the interpretation of letters bogs down the learners in this sample because the poor performance in these levels suggests they could not interpret *letters as variables*, *unknowns* or *pattern generalisers*.

In the section that follows I provide a detailed analysis of how learners interpreted letters for the different tasks and common errors across the different levels of understanding. Common misconceptions and themes emerge from this analysis that suggest further reasons for learners performing poorly. I provide comparisons to the CSMS (1979) study in the bottom two sections of each table in section 4.6.

4.6 Analysis of paper and pencil tasks

4.6.1 Analysis of letter evaluated tasks

Level	*Level 1 (question 1)*	*Level 2 (question 2)*	*Level 2 (question 3)*
Task : Letter evaluated	What can you say about a if $a + 5 = 8$	What can you say about u if $u = v + 3$ and $v = 1$	What can you say about m if $m = 3n + 1$ and $n = 4$
% correct for this sample	80%	37%	13%
Most common erroneous answer (Percentage of learners)	5 (6%)	2 (20%)	35 (20%)
% correct for CSMS (1979) sample	92%	61%	62%
Most common erroneous answer for CSMS (1979) (percentage of learners)		2 (14%)	Other values (14%) [These values are not provided in Hart, (1981)]

*Open space indicates no common error was provided by CSMS (1979, as cited in Hart, 1981).

In each of these questions learners could evaluate the letter either by inspection or substitution. We see the level 1–2 increase in structural complexity which leads to an increase in error. A large percentage of learners (80%) were able to evaluate a in the first task. To get the $a = 3$ learners could have subtracted the 5 from the 8 or calculated by inspection that a should be 3 as $3 + 5 = 8$. The task contained one letter with a coefficient of 1 and all numbers were less than 10 which enabled this task to be easily solved using arithmetic strategies by inspection. Question 2 (level 2) has two letters and two equations as compared to the level 1 task which had only one equation with one letter. This change of structure increases the level of difficulty which contributed to a decline in the correct solutions (43% decline) from level 1 to 2.

There was a further drop in the number of correct solutions for question 3 to 13%. In question 3 the coefficient of the letter n is 3. In questions 1 and 2 the coefficients of the letters were 1. The coefficient of 3 is likely to be the reason that 20% of learners got a solution of 35 by simply equating the $3n$ to 34 as $n = 4$ and placing the 4 in the place of n. The issue is the implicit multiplication in the notation of $3n$ which was not an issue for $1a$. However, the substitution seems logical because substitute means put the number in place of the letter which was not problematic in question 2, for $u = v + 3$ and $v = 1$, where substitution involved replacing the v by 1. Moreover, four learners obtained $4n$ (2nd most common error) which suggests that these learners added the 3 and 1 (in, $m = 3n + 1$) and joined the answer to n.

4.6.2 Analysis of letter not used tasks

Level	***Level 1 (question 5)***	***Level 1 (question 6)***	***Level 2 (question 12)***
Task: Letter not used	If $a + b = 43$ then $a + b + 2 =$	If $n - 246 = 762$ then $n - 247 =$	Add 4 to $n + 5$
% correct in this sample	50%	40%	6%
Most common erroneous answer (Percentage of learners)	$2ab$ (10%)	763 (20%)	$9n$ (63%)
% correct for CSMS (1979) sample	97%	74%	68%
Most common erroneous answer for CSMS (1979) (percentage of learners)		763 (13%)	9 (20%)

In each question learners could evaluate the letter by systematic matching or through using logic to solve the equations. In question 5, 50% of learners were able to correctly match and substitute for $a + b$. It is suggested by the most common error of $2ab$ that learners ignored the first equation of the task and were unaware of the implicit multiplication in $2ab$. The most common error involved joining the three numbers $a + b + 2 = 2ab$. This is similar to learners being unaware of the implicit multiplication of $4n$, in question 3 above, resulting in the joining of the $3n$ and 1 to get $4n$.

The most common error in question 6 of 763 (20% of learners) was due to the structure of the task. Since 247 is larger than 246, learners were prompted to add 1 to 762 instead of subtracting one. However, 40% of learners were able to correctly identify that they needed to subtract 1 from 762. 63% of learners got an answer of $9n$ for question 12 (level 2 task). Learners could have got the solution of $9n$ by adding 4 and 5 and then joining the n to the 9 to get $9n$. It appears that a large percentage (63%) of the learners thought similarly in an incorrect way to get this solution.

4.6.3 Analysis of letter used as an object tasks

Level	***Level 1 (question 8)***	***Level 2 (question 9)***
Task: Letter as object	**Note: $x + 3x$ can be written as $4x$ *(Hint for questions 8 – 11)*** Simplify $2a + 5a =$	Simplify $2a + 5b + a =$
% correct in this sample	53%	27%
Most common erroneous answer (Percentage of learners)	$7a^2$ (23%)	$7ab$ (17%)
% correct for CSMS (1979) sample	86%	60%
Most common erroneous answer for CSMS (1979) (percentage of learners)		

Learners could have solved each equation above by simply treating the *letter as an object*. In other words, in the level 1 task the $2a$ and $5a$ could have been interpreted as objects ($2a$'s and $5a$'s) and added to get $7a$.

There were 53% correct responses for the level 1 task. 27% of learners got the solution of $7a^2$ by adding the $2a$ and $5a$ and then using an inappropriate rule to get the a^2. The initial level 1 task for *letter not used* has two equations and two variables. (If $a + b = 43$, then $a + b + 2 =$ __). The level 1 task for *letter evaluated* has letters and numbers. (What can you say about a if $a + 5 = 8$). However, question 8 which is the level 1 for *letter used as an object* has only one equation with one letter which suggests the difficulty is lower than the previous two level 1 tasks. Hence, there should have been more correct responses, but this was not the case. 47% of learners struggled with question 8 involving the gathering of like terms which is a basic skill taught in the introduction of algebra in the South African mathematics curriculum. This suggests significant difficulty was experienced by learners even with the lower level interpretation of *letters as objects*. This is confirmed by their responses to question 9 which had a slightly greater structural complexity. 27% of learners achieved the correct solution for question 9 which was the level 2 task while 17% of learners got $7ab$.

4.6.4 Analysis of letter used as a specific unknown task

Level	*Level 3 (question 4)*	*Level 3 (question 7)*	*Level 3 (question 10)*	*Level 4 (question 11)*	*Level 3 (question 13)*	*Level 4 (question 14)*
Task: letter as specific unknown	What can you say about r if $r = s + t$ and $r + s + t = 30$	If $e + f = 8$ then $e + f + g =$	Simplify $3a - b + a =$	Simplify $(a - b) + b =$	Add 4 to $3n$	Multiply $n + 5$ by 4
% correct in this sample	10%	0%	30%	6%	3%	10%
Most common erroneous answer (Percentage of learners)	10 (33%)	12 (37%)	$3ab$ (17%)	ab (17%)	$7n$ (87%)	$20n$ (47%)
% correct for CSMS (1979) sample	35%	41%	47%	23%	36%	17%
Most common erroneous answer for CSMS (1979) (percentage of learners)	10 (21%)	12 (26%)			$7n$ (31%)	$n + 20$ (31%)

In the above tasks learners have to interpret the *letter as a specific unknown* by performing operations directly on the letter. It is from this interpretation onwards that learners in this sample experienced great difficulty (as can be seen by the average correct solutions in the above table).

Question 4 has three letters and requires learners to evaluate the letter *r as an unknown*. Substitution of $s + t$ by r results in $2r = 30$ which requires the unknown r to be evaluated. This more complex task structure is likely to have contributed to only 10% of the learners achieving the correct solution. Moreover, 33% of learners equated $r = s = t = 10$ which suggests that the letter was evaluated numerically and the concept of algebraic *letter as unknown* and *variable* is not understood by these learners. The equating of the letters also suggests that learners made an assumption of the letters having equal magnitude, which I discuss as a theme in section 4.7.

Question 7 needed learners to match the $e + f$ in $e + f + g$ with 8 to get a solution of 8 + g. There were no learners that were able to get the correct solution in this level 3 question. Five learners gave the solution of 9 which was the second most common error. It seems that these learners were able to get the solution of 8 + g and then added the 8 to the coefficient of g to get 9. Hence, it would appear that learners were treating the g as 1 because the coefficient of g is 1. Three learners got the solution of $8g$ possibly by joining $8 + g = 8g$. The most common error of 12 (37%) is obtained by taking $e = f = 4 = g$. Therefore learners used a lower level and *evaluated the letters* numerically to get $e + f + g = 12$. The equating of letters is similar to question 4 where learners equated the letters which further illustrates a poor understanding of variable and the use of the algebraic letter in the task.

17% of learners got the solution of $3ab$ for question 10. This error which was also evident in questions 3, 5 and 9 suggests that the numbers that could be seen were added and joined to the letters. Question 11 (level 4) was correctly answered by only 6% of learners. This low achievement could be due to the structure of the task and the presence of brackets which prompted learners to multiply as 73% of responses were products, such as ab, $2ab$, ab^2, $ab - b^2$ and $ab + b$.

87% of learners got the solution of $7n$ for question 13 while only 3% managed to get the correct solution. Moreover, joining of numbers and letters has been seen throughout my analysis of the paper and pencil tasks but is incredibly consistent here because the vast majority of learners used the same joining rule to get the solution of $7n$. Although a minimal increase, 10% of learners achieved correct solutions for question 14. Question 14 was different from all the other tasks in this interpretation of the letter because the task involved multiplication. 47% of learners got the solution of $20n$ possibly by saying n (5 x 4) instead of saying $4(n + 5)$.

4.6.5 Analysis of letter used as a generalised number tasks

Level	***Level 3 (question 15)***	***Level 4 (question 16)***
Task: letter as generalised number	What can you say about c if $c + d = 10$ and c is less than d	L+ M + N = L + P + N Always Sometimes Never
% correct in this sample	0%	3%
Most common erroneous answer (Percentage of learners)	10 (37%)	Never (60%)
% correct for CSMS (1979) sample	11%	25%
Most common erroneous answer for CSMS (1979) (percentage of learners)	c = 1, 2, 3, 4 (19%)	Never (51%)

In each question learners had to interpret the letter as a generalised number. The level 3 task has the letter taking on a set or range of values less than 5. In the level 4 task the letter is a different generalised number because the letter takes on a much broader set of values as M and P could be any number provided they are equal. The two tasks for the *letter used as a generalised number* are considerably more difficult than all the previous tasks in the instrument. The two tasks involve the interpretation of the *letter being a generalised number* which is more abstract than interpreting the *letter as a specific unknown* or *an object*.

The level of difficulty of interpreting the *letter as a generalised number* is also seen by learners' poor percentage of correct solutions. Virtually no learner managed correct solutions for the two tasks. 40% of learners got the solution of 10 for the level 3 task. This solution suggests that the presence of the 10 as the only number in the question prompted learners to pick the 10 as the answer for *c*. 23% of learners got a solution of 4 which suggests that these learners were able to find one value for *c* and did not proceed to find other values or did not visualise that c could have a range of values and therefore the solution could be generalised.

60% of learners got the solution of never for the level 4 task. One reason for this solution could be taking M and P to be unequal because they are different letters. This is a contradiction of the rule that learners used in task 4 and 7 where the letters were given equal magnitudes and hence shows inconsistencies with own incorrect rules. There are, however, other reasons for learners choosing the answer never like guessing, etc. One learner selected the correct solution of sometimes but was unable to justify the cases when the identity will be true.

4.6.6 Analysis of letter used as a variable tasks

Level	***Level 4 (question 17)***
Task: letter as variable	Which is larger? $2n$ or $n + 2$? Explain!
% correct in this sample	0%
Most common erroneous answer (Percentage of learners)	$2n$ (33%)
% correct for CSMS (1979) sample	6%
Most common erroneous answer for CSMS (1979) (percentage of learners)	$2n$ (71%)

In the above task learners were required to visualise the instances when $2n$ is larger and cases where $n + 2$ is larger. No learner was able to get the correct solution because the structure of the response required makes this one of the most difficult tasks in the instrument. Moreover, learners had to realise that n is the same number in $2n$ and $n + 2$ and that the two expressions are the same when n is 2. Therefore, the complexity of the structure of the task and the response thereof seemed too complex for learners in this sample. 33% of learners wrote their solution as $2n$ while 30% selected the $n + 2$. However, no learner was able to justify the cases or the instances when the numbers are larger.

In the section that follows, I discuss the establishment of themes and tentative themes from the paper and pencil tasks based on what learners were doing by hypothesising reasons for errors.

4.7 Themes and tentative themes for paper and pencil tasks

Although it was easier to interpret, formulate and justify themes for the interviews because transcripts provided concrete evidence, two tentative themes and two key themes emerged from the analysis of the paper and pencil tasks. Themes were formed based on the frequent use of common errors across many learners' responses and could be concretely justified. On the other hand, tentative themes are patterns, such as learner responses suggesting certain misconceptions, that I felt were emerging from the data which I could not concretely justify. I've called these two themes "joining" and "inconsistencies with own rules". In the sections that follow I discuss these tentative themes and themes.

4.7.1 Tentative themes emerging from the paper and pencil tasks

There were two tentative themes that emerged from the paper and pencil tasks. The notions of the usage of the coefficient of 1 and picking/combining numbers and operations randomly with no 'symbol sense' were emerging although I could not draw any firm conclusions. In question 7, 17% of learners added $8 + g$ to get 9 which suggests that the g is taken as 1 due to the coefficient of g being 1. A similar usage of the coefficient of the letter of 1 was seen in question 12 (Add 4 to $n + 5$). Five learners got the solution of $10n$ which further suggests that the n was assigned a value of 1 due to the coefficient of n being 1. Hence, $4 + 1 + 5 = 10$. However, in this task the learners also join the letter n to the 10 which is inconsistent to the rule used in question 7 where the letter was not joined.

13% and 40% of learners chose the numbers 8 and 10 respectively as their solutions for question 7 and 15 possibly due to these being the only numbers in the given tasks. Lee (a learner in this sample) got solutions for question 3 and 4 (refer to section 4.6. above) of $4mn$ and $30rst$ by combining any given numbers in a random manner. Although Lee's solutions might suggest a similarity to the theme of joining (the theme of joining is discussed below) the difference is that Lee joins any numbers from any side of any equation or even different equations. At this stage the two tentative themes are not fully unpacked but will be thoroughly explained during the establishment of themes for the interviews.

4.7.2 Themes emerging from the paper and pencil tasks

4.7.2.1 Joining

The theme of joining is a misconception which involves the joining of numbers during addition and was seen throughout my analysis of the paper and pencil tasks. It was common to see joining such as $a + b + 2 = 2ab$, $4 + n + 5 = 9n$ and $4 + 3n = 7n$. 10% of the learners got the $2ab$, while 63 % and 87% of learners got the solutions of $9n$ and $7n$ respectively.

The rule of joining which was extremely consistent both across different learners tasks and within individual learner's solutions, was add the numbers that can be seen and join the answer to the letters. There were 8 tasks for which joining could have been used from the onset due to the task structure involving addition of numbers and letters. These were questions 3, 5, 9, 10, 11, 12, 13 and 14 which was greater than 50% of the total tasks. At least one of the three most common errors for each of the 8 tasks was calculated using the joining rule. For example, the 2nd most common error for question 3 was $4n$ and the most common error for question 4 was $2ab$ whereas the 3rd most common error for question 9 was $8a$ which are incorrect solutions gained by applying the joining rule. Joining could not be used from the start for tasks such as questions 14 and 17 because the tasks either involved multiplication or the structure of expressions/question made the joining rule difficult to use from the onset. However, even in these tasks there was evidence of joining.

MacGregor and Stacey (1997, p. 12) explain ‘conjoining’ in a similar way to what I call joining because they explain that ‘conjoining’ results in solutions of the form $7n$ for $7 + n$. In their study, students in year 7 (11–13 years old) in Australia also wrote solutions such as $8y$ and $y8$ for $8 + y$. They explain that students who made the joining error were trying to ‘denote a combination of the number 8 and the unknown number y, their errors being due to conjoining terms for addition’ which is similar to the general rule that learners in the present study used for joining (MacGregor & Stacey, 1997, p. 7). Furthermore, Liebenberg, Linchevski, Olivier and Sasman (1998, p. 3) explain that learners who make the above error are unable to see the ‘hidden structure’ of algebraic terms. They refer to the ‘hidden structure’ of a term such as $2ab$ as the implicit multiplication of 2 x a x b. Boulton et al. (1997) also found in their research that many students could not explain the term $3x$. Therefore, I would hypothesise that learners in this sample, who got solutions of $7n$ for $4 + 3n$ and 34 evaluated for $3n$ since n was 4, did not understand implicit relations such as $7n$ is 7 x n and $3n$ is 3x n.

<u>4.7.2.2 Inconsistencies with own rules</u>

I have called the second theme inconsistencies with own rules which involves learners having their own non–algebraic rules but change these rules at different instances. 37% and 33% of learners respectively took the letters in the expressions $r + s + t$ and $e + f + g$ to be equal in questions 4 and 7 (see section 4.6 above). However, for question 16, 60% of learners said that L + M + N never equals to L + P + N. It follows that; if the given identity is never true then M and P are never equal, which is a contradiction to the former rule.

Nelli’s (a learner in this sample) initial rule for adding numbers is $a + a = a^2$. For $2a + 5a$ she gets $7a^2$ and for $2a + 5b + a$ she gets $7a^2b$, which is consistent. However, for $3a – b + a$ she gets $3ab$ and no longer adds the exponents which is an inconsistency in her rule. The above cited cases suggest that certain learners, when engaging with algebraic tasks, have their own set of rules but also change these rules at different instances. Moreover, for the latter three interpretations incorrect rules became more dominant and a vast number of solutions were randomly picked.

I have established two themes above which are joining and inconsistencies with own rules which are also themes evident in the interviews. In the section that follows I explain the rationale for the selection of learners for interviews.

4.8 Selection of learners for interviews

Six learners were selected for the interviews based on three factors. Firstly, I based my selection on a spread of common errors across solutions. At least one of the learners selected made the same error as the most common error in 15 out of the 17 tasks. Secondly, I ranked learners according to the total number of correct solutions and wanted to have a spread of levels of performance. Therefore, I selected two top performing learners (Laizal and Agi), two bottom performing learners (Lee and Emma) and two average performing learners (Kate and Nelli). Thirdly, I carefully studied all 30 learners' responses and tried to establish which responses might guide me in answering my research critical questions. Furthermore, the latter motivation for selection was conducted first and I then checked that the former justifications for selection were suitably met.

Table 6 below shows the interviewed learners' performance and although not intentional there were five girls and one boy (Lee) selected. In the table, the answers underlined indicate the most common error, italics reflects the second and third most common error while the bold responses are correct solutions. The ranking shows the number of correct solutions achieved by each learner.

In the section that follows I explain the establishment of the four themes from the data collected for the interviews which strengthen and support tentative themes and themes that were evident in the paper and pencil tasks.

Table 6: Analysis of six learners to be interviewed

Learner/ Task	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***	***13***	***14***	***15***	***16***	***17***	***Score out of 17***
Laizal	**3**	**4**	**13**	1	**45**	763	*8*	**7*a***	*8a*	**4*a*-b**	*b-b+a*	***n*+9**	**4+3*n***	20+*n*	10	never	2*n*	7
Lee	**3**	*3*	4*mn*	*30rst*	2*ab*	**761**	12	$7a^2$	$7a^2b$	$3a^2b$	*ab*	9*n*	7*n*	20*n*	10*cd*	15	2*n*	2
Kate	**3**	2	0	**15**	**45**	513	*9*	**7*a***	**3*a*+5*b***	2*a-b*	$a-b^2$	*10n*	7*n*	21*n*	10	never	*n*+2	5
Emma	*8*	**4**	35	60	2*ab*	763	12	8*a*	*ab*	3*ab*	10	9*n*	7*n*	29	10	never	*n* +2	1
Nelli	**3**	*v*+3	6	**15**	2*ab*	**761**	16	$7a^2$	$7a^2b$	3*ab*	$ab-b^2$	9*n*	7*n*	20*n*	3	never	*n*+2	3
Agi	**3**	**4**	35	10	46	**761**	12	**7*a***	**3*a*+5*b***	**4*a-b***	*a*	9*n*	7*n*	20*n*	4*a*+*b*	*always*	*n*+2	6
Most common erroneous answer	5	2	35	10	2*ab*	763	12	$7a^2$	7*ab*	3*ab*	*ab*	9*n*	7*n*	20*n*	10	never	2*n*	
2nd Most common erroneous answer	8	3	4*n*	30	43;5	269	9		$7a^2b$	4*ab*;$3a^2b$; 3*a*;5*ab*	ab^2; $ab-b^2$	10*n*	35*n*;9*n*	1*n*;9*n*	4	always	*n*+2	
3rd Most common erroneous answer	8-5	1	0 &1	30*rst*			8		8*a*			4*n*+a				sometimes	None (meaning neither 2*n* nor *n*+ 2)	

* Blank spaces indicate there was no 2nd and 3rd most common error.

4.9 Process of establishing themes

Two themes and two tentative themes emerged from the analysis of the paper and pencil tasks. I will be looking for corroborating evidence in the interviews to strengthen arguments for establishment of these themes. The theme of joining which involved the joining of numbers and letters during addition was established because it was a misconception that was commonly used across several tasks by the vast majority of learners. The second theme was inconsistencies with own rules. It was seen that learners formulated their own non–mathematical rules which were inconsistent. Therefore, the second theme was established because most learners seemed to equate letters in some instances but not others. Moreover, to justify this theme I also showed (in section 4.7.2) how there were inconsistencies in Nelli's solutions.

Tentative themes were assigning a value of 1 to the letter with coefficient of 1 and picking/combining numbers and operations from the task randomly. However, these tentative themes are further evidenced in the interviews and are now established as themes. Moreover, I worked through each interview repeatedly and themes were established if the misconception was common in most of the interviewed learners' data (see Table 7 below). In the sections that follow I discuss the themes in greater detail.

4.10 Themes for interviews

4.10.1 Random Picking

Learners' responses to the paper and pencil tasks suggested some learners arrived at their answers by arbitrarily picking numbers and letters and where necessary combining them in some way. At certain instances, all learners that were interviewed showed evidence of the same strategy. This strategy can thus be considered a theme which I will refer to as random picking. Errors in this category emerged in three different forms. The first form involves selecting numbers from the task as being the solution to the task which was seen in questions 7 and 15. Task 15 was: what can you say about c if $c + d = 10$ and c is less than d. 40% of learners in this sample chose 10 as the solution which suggests that the 10 was selected because it was the only number in the task. This is in line with Collis (1975, as cited in Küchemann, 1981, p. 103) who found that learners when working with simultaneous equations 'gave numerical values to the letters before manipulating them in any way'. Stacey and MacGregor (2000, p. 160) also found that students use the letter x in three ways, one of which was assigning any unknown value to the letter.

The second form involves combining any numbers and letters or assigning any numerical value to the letter for the sake of achieving a solution. Task 4 was: what can you say about r if $r = s + t$ and $r + s + t = 30$. Three learners including Lee combined numbers and letters in a random manner to get a solution of $30rst$. Lee confirms this notion because he explains in his interview that he added the r, s and t to get rst which he places or joins to the 30 to get $30rst$. The issue is that Lee did not understand the equation which shows that the sum of the three letters is 30. This resulted in him randomly combining all the letters and the number even though they were on different sides of the equation, for the sake of achieving a solution.

The second form was also typical for many of the other learners that were interviewed. It can be seen in the excerpt below how Nelli, in her last sentence, explains that she assigns any numbers to the letters.

Kona	Can we look at number 7? For number 7 you said $e + f + g = 16$. How do you get the 16?
Nelli	Eh. I just said eh cos[3] $e + f$. No I said, I think I just added. I just put, I just gave the numbers, the alphabets[3] numbers.

The third form involves randomly performing an operation even when the task does not have that particular operation. Similar to Kate's explanation below Johanning (2004, p. 381) found in her research that learners who performed poorly 'randomly tried operations using the numbers in the problem'. Stacey and MacGregor (2000, p.

[3] Throughout the interviews learners used the word 'alphabet' as meaning a letter and 'cos' to mean because.

151) also explain that the tendency for computation of a solution hinders students from using algebraic methods in problem solving. The excerpt below which was typical for learners in this sample suggests that Kate performs non–algebraic procedures for the sake of completing the task and randomly picks operations. (Kate needs to calculate m in $m = 3n + 1$, if $n = 4$). It can be seen that she performs any operation using any numbers with no 'symbol sense'. After adding $3n$ to 1 to get 4 she performs subtraction even though there is no subtraction or hint to subtract in the given task.

Kona	Ok that's fine. For number 3 if you said $m = 0$, can you explain your thinking there? How did you get $m = 0$?
Kate	$m = 0$. Ok what I said there was, Ok so I said $3n$ plus 1 so I added 1 to 3 so that makes it 4. And then again I subtracted 4 from 4 so it gave me zero.
Kona	Ok. But why did you subtract? Why not add or multiply?
Kate	Because it says $n = 4$ and the first answer has an addition sign so I added the first one and then when I got to the second answer which was equals to 5, I subtracted 4 from 4, so I got nothing.

Lee also chooses operations randomly as can be seen in his second response in the excerpt below. Lee selects multiplication even though there is no multiplication or hint of multiplication in the question. Lee's solution of 3 suggests that he assigns a 1 to the u and hence u x 3 = 3 which relates to what all the interviewed learners referred to as 'invisible 1'. However, this task involves multiplication whereas most of the other viewed uses of the 'invisible 1', as reported in this study, involved addition. Therefore, Lee's use of this misconception in this task is different to other learners' use of this misconception. (I discuss the 'invisible 1' misconception in the next section.)

Kona	Here's your test. Can we look at number 2? It says, what can you say about u if $u = v + 3$ and $v = 1$. You said u is 3. Can you explain how you getting the 3?
Lee	Here sir, what I've done is is I thought of the number a number that isn't it a letter isn't it they always say eh a letter is always with eh eh invisible 1?
Kona	Yes
Lee	So, eh, I multiply this each letter by 3. I said 3 times, ah, u x 3. Then I got a 3. Then I also said v. I made a v as like as like a letter I said $v + 3$ and then it gave me a 3. So that's when I thought of writing a 3 down.

Therefore, I discuss random picking as having three forms. In summary, the three forms include randomly selecting numbers and operations and combining any numbers and letters or assigning any numerical value to the letter for the sake of achieving a solution. The desire to achieve a solution is similar to my reference (in Chapter 2) to Novotna and Kubinova (2001, as cited in Drouhard and Teppo, 2004, p. 241) who explain that x in an expression triggers learners to perform calculations. In line with the theme of random picking, Bell (1995, p. 46) explains that for learners who pick operations and symbols randomly the main aim of algebra is seen as symbols undergoing manipulations which is incongruent to the essence of algebra as providing generalisations and conjecturing. Moreover, all learners that were interviewed used random picking at different stages which suggests that these learners viewed algebra as operating on symbols.

4.10.2 Invisible 1

Another hypothesis stemming from the paper and pencil tasks is that learners assigned a value of 1 to a letter with a coefficient of 1. This tentative theme was strengthened by the interview analysis because it was interesting that all learners mentioned the phrase ‘invisible 1’. The ‘invisible 1’ refers to the one in the number of the form $1x$ because the one is not seen when the term is written as x and is therefore ‘invisible’. In the same way, it could then be argued that the square root of a number has an invisible 2 in front of the square root sign because the 2 is generally not written. Lee repeatedly talked about the ‘invisible 1’ and explains that ‘they always say a letter is always with eh invisible 1’ and Kate mentions that ‘Mr Fani[4] said there’s always an invisible 1 in front of a number you’re working with’. I use the learners’ phrase ‘invisible 1’ to refer to the theme of assigning a value of 1 to a letter with coefficient of 1.

[4] Mr Fani is the mathematics teacher of learners in this sample.

Five learners, including Kate, gave the answer of 9 to task 7 (in, if $e + f = 8$ then $e + f + g =$). Küchemann (1981, p. 106) explains that learners in the CSMS (1979) sample who got the answer of 9 'just added 1 because this was the simplest way of making the answer bigger'. However, according to Kate's explanation below it is clearly seen how she assigns 1 to g in $8 + g$ to get 9. My argument is that the 'invisible 1' misconception is applied by Kate to get the solution of 9 instead of her wanting a 'way of making the answer bigger' or applying the misconception that addition makes bigger (Küchemann, 1981, p. 106). Kate also used this misconception elsewhere as can be seen in the latter part of the excerpt below for question 12. Kate adds the 4 and 5 (in add 4 to $n + 5$) and then instead of leaving her answer as $n + 9$ she equates n to 1 to get a solution of 10.

Kona	Ok, that's fine. Can we check number 7? Your answer for number 7 was 9. For $e + f + g$ you got 9. How did you get the 9?
Kate	Ok, so if the $e + f = 8$ so the second one says $e + f + g$ so in front of the g there was an invisible 1 so I added the 1 to the 8 which makes it 9.
Kona	Which makes it 9?
Kate	Ja
Kona	Ok that's fine. And then number 12. Can we look at number 12? We said add 4 to $n + 5$ and you got $10n$. How did you get the $10n$?
Kate	$10n$? Well I said 4 + 5 right, is $9 + n$ which has got an invisible 1 which equals to 10.
Kona	$10n$?
Kate	Ja

The excerpt below shows that based on her understanding of 'invisible 1', Agi concludes that $n + 2$ is larger than $2n$. The n in $n + 2$ has an 'invisible 1'which results in Agi equating the n to 1 and hence $n + 2$ ($1n + 2 = 3n$) is larger than $2n$. However, it does seem like joining was used but from the joining rule, discussed earlier, of adding the numbers that can be seen and join to the letter, the answer for $n + 2$ would have been $2n$ and not $3n$. Furthermore, it seems more likely that assigning 1 to n resulted in Agi getting a solution of $3n$ for $n + 2$. According to Küchemann (1981, p. 112) learners in the CSMS (1979) sample needed 'sufficient processing capacity to consider the possible effect of n on the relative size of $2n$ and $n + 2$, whereas children without this capacity will go for something simpler and more immediate'. Therefore, Agi's solution of $3n$ for $n + 2$ suggests that she did not have 'sufficient processing capacity' and opted for an easier route in solving the task by using her prior knowledge involving the 'invisible 1' misconception.

Kona	So, look at number 17, the last one. It says: Which is larger $2n$ or $n + 2$. And you said $n + 2$.
Agi	Ja
Kona	Because there's an invisible 1[5]. Can you explain what you meant by that?
Agi	Like next to the n there's invisible 1, so if you add the n and the + 2 it will give you $3n$.
Kona	Ok

However, there were very few cases where learners understood the concept of 'invisible 1', which helped to solve the task. It can be seen in Kate's excerpt below that she was able to add the $2a$ and a which has an 'invisible 1' in $2a + 5b + a$, to get the solution of $3a + 5b$. This is very interesting because in question 7 where the *letter was an unknown* she was unable to use the 'invisible 1' with the same success. (In question 7 she got $8 + g = 9$). This suggests that the 8 prompted Kate to add or she was unable to interpret the letter g as an unknown because it is a higher level of understanding.

Kona	Ok. Can we turn over? Let's look at number 9. Can you explain your thinking for number 9? Your answer was $3a + 5b$.
Kate	Ok, well actually what I did there was I added $2a + a$ which has got an invisible 1 which makes it $3a + 5b$ and I left it like that.

[5] The excerpt suggests that I was leading Agi to respond to the issue of 'invisible 1' but her written response to question 17 in the test was '$n + 2$ invisible 1'.

4.10.3 Joining

The theme of joining was also evidenced in the majority of the paper and pencil tasks. A vast majority of learners used joining during addition for tasks such as $4 + n + 5 = 9n$ and $4 + 3n = 7n$. I have explained joining as the adding of the seen numbers and joining the letters which was evident in all interviewed learners' data except Laizal.

In section 4.7, I explained that the joining rule was incredibly consistent both across different learners' tasks and within individual learner's solutions. At least one of the three most common errors in eight tasks, where joining could have been used from the onset, suggested that the joining rule was used. It was interesting that all these common errors resulting from joining were single answers such as $3ab$ and $9n$ with no visible operation signs. This is in line with my Chapter 2 reference that very often learners have an instinctiveness of wanting to find a single answer solution. It was also very interesting that there was an overwhelming desire to have single terms or single numbers as solutions as is suggested by all 17 most common erroneous answers being single answer solutions. (At least 6 of the first 14 tasks' correct solutions were not single answer solutions. Questions 15–17 are not included because their solutions involve inequalities and words such as *sometimes*.)

There were also other types of joining for which the rules were not always clear. In question 9 (simplify $2a + 5b + a =$) there were two learners who got the solution of $8ab$ and another two learners who got the solution of $8a^2b$. In question 12 (add 4 to $n + 5$) five learners got the solution of $10n$. The solutions of $8ab$, $8a^2b$ and $10n$ suggest that the three terms in each expression were merely collapsed together during addition to form one term which differs from the joining rule that I discuss above. However, the three solutions also suggest the 'invisible 1' misconception could have been used and therefore the joining rules were not always fully clear. To further illustrate the point, by equating $a = 1$ in the first case and $n = 1$ in the second case, which is in line with the 'invisible 1' misconception, and collapsing all terms in each case the solutions of $8ab$ and $10n$ could have been determined.

The excerpt below illustrates how the misconception of joining becomes problematic in algebraic settings. Emma (in, $a + b + 2$) adds the a and b to get ab which she joins to 2 to get 2*ab*. In a similar way, Agi when asked how she gets 7*n* when adding 4 to 3*n* explains 'I added 3 and 4… and it gave me 7*n*'. According to Küchemann (1981, p. 108) the *letter was not used* because 'elements that were meaningful (the numbers 3 and 4) were 'properly' combined but the letter was simply left as it was'. However, contrary to Küchemann (1981), I hypothesise that 87% of learners got 7*n* by adding 4 and 3 and then joining the *n* to the 7. The letter was not ignored as Lee explains: 'I said 4 + 3 it was 7. I also wrote the *n*. Since, there was a 3*n* I couldn't leave it alone so I had to add to get the 7*n*'. This strengthens the argument that the numbers were added and then joined to the letter which was not ignored.

Kona	And then it says: $a + b + 2 =$ __ and you've got 2*ab*. Can you explain how you get the 2*ab*?
Emma	How I got the 2*ab* is like I added $a + b$ and the 2, so I got the 2*ab*. I added all of them together.

4.10.4 Inconsistencies with own rules

I refer to inconsistencies with own rules as having a non–mathematical rule that changes at different instances which was one of the two themes established from the paper and pencil tasks. During my discussion in section 4.7, I explained how Nelli changes her own rules at different stages. This is in line with Bowie's (2000, p. 4) explanation that learners are often unable to visualise the mathematical objects and structures which results in the creation of learners own meanings that is 'not coherent and lacks rich relationships'.

Moreover, it seemed like the rule of joining was very consistent (discussed in 4.7.2.1) while the rules involved in using the 'invisible 1' were at times inconsistent as could be seen in Kate's excerpts above. Kate initially assigns a 1 to the letter in $8 + g$ but in $2a + b + a$ she does not assign 1 to the *a* but interprets the *a* as 1*a*. Lee also had inconsistent rules which can be seen in his latter responses in the excerpt below. He has a rule for adding numbers initially, $v + 3 = 3$, which ignores the letter but he changes the rule for $4 + mn = 4mn$ by joining the number and the letters.

Kona	Here's your test. Can we look at number 2? It says, what can you say about u if $u = v + 3$ and $v = 1$. You said u is 3. Can you explain how you getting the 3?
Lee	Here sir, what I've done is is I thought of the number a number that isn't it a letter isn't it they always say eh a letter is always with eh eh invisible 1?
Kona	Yes
Lee	So, eh, I multiply this each letter by 3. I said 3 times, ah, u x 3. Then I got a 3. Then I also said v. I made a v as like as like a letter I said $v + 3$ and then it gave me a 3. So that's when I thought of writing a 3 down.
Kona	Ok, fine. Can we look at another one? Can we look at number 3? Ok, you said that the question is, what can you say about n if $m = 3n + 1$ and $n = 4$? You said its $4nm$.
Lee	Ja, here sir I said 3. I plussed, added 3 + 1 and then I added this m, $m + n = mn$, I added this 3 + 1. That gave me $4mn$.

4.10.4.1 Meaning given to letters

I discuss meaning given to letters in relation to the theme of inconsistencies with own rules. It was seen in the paper and pencil tasks that learners equated $r + s + t$ and $e + f + g$ for questions 4 and 7 but did not equate the letters M and P in question 16 which was inconsistent. The excerpt below, for Emma, sheds light on this inconsistent rule. It can be clearly seen how she equates $r = s = t = 30$ in question 4. However, for question 16 she explains that different letters cannot be equal when she says: 'let me make an example like *abc* it equals to *zmy*, you can't say that'. Therefore, Emma's solutions also reflect her inconsistencies with rules as she got 60 for question 4 by equating s and t but in question 16 she says that L + M + N is never equal to L + P + N which suggests that she feels that M and P can never be equal.

Kona	Ok, that's fine. Can you look at number 4? You got 60. Number 4 says: $r = s + t$ and then $r + s + t = 30$. You got your answer as 60. Can you explain how you got the 60?
Emma	Isn't it the z? r, s plus t it equals to 30, so I like, I add them like all of them like 30, 30, 30 and I got 60. I add $s + t$ and I got 60. It's $r + t$, I mean it's $s + t$ because you've got… So I… its like 30 + 30 equals to 60.
Kona	Ok, Ok. So you're saying $s = t$?
Emma	Equals to t.
Kona	Ok. And then number 16. You said that this is never equal to that. Can you explain why?
Emma	I think because, sir, you can see by the letters you can't just say… Let me make an example like *abc* it equals to *zmy*, you can't say that. If it was like maybe M N L, I would say sometimes or maybe always if like the letters they were the same, but then not put it like accordingly. But, ja, the letters are not the same.

In their interviews, Nelli and Lee were explicit about the order of the letters of the alphabet being related to magnitude while two other learners (Kate and Emma) felt that different letters could not have the same magnitude. The excerpt below shows how Nelli feels that due to a being the first letter of the alphabet it should be the biggest while z must be the smallest.

Kona	Ok. Good. So which is bigger, m or n?
Nelli	m. I believe it's bigger.
Kona	m, why do you say m?
Nelli	Because it comes first than n.
Kona	It comes first, so it means a will be the biggest?
Nelli	Of them all.
Kona	Of them all, so which is the smallest?
Nelli	z

4.11 Display of themes for six interviewed learners

Table 7 below shows which learners displayed the four themes during their interviews. The "Yes", in the table, indicates that the theme was evidenced in the learner's data while the "No" indicates the theme was not evident. It can be seen that all learners used random picking and 'invisible 1', only Laizal did not join during addition and three learners were inconsistent with their own rules.

Table 7: Themes for learners interviewed

Learner/Theme	***Random picking***	***Invisible 1***	***Inconsistencies with rules***	***Joining***
Nelli	Yes	Yes	Yes	Yes
Laizal	Yes	Yes	No	No
Lee	Yes	Yes	Yes	Yes
Agi	Yes	Yes	No	Yes
Kate	Yes	Yes	No	Yes
Emma	Yes	Yes	Yes	Yes

4.12 Summary

In this chapter I focused on the qualitative analysis of data collected from the two research instruments which were the paper and pencil tasks and interviews. It was seen that learners overall performance was very poor. Learners struggled with the latter interpretations of the letter and the level 2, 3 and 4 tasks. Two themes and two tentative themes emerged from the analysis of the paper and pencil tasks but corroborating evidence from the interviews strengthened arguments for the establishment of four themes. In my last chapter I will conclude my study by discussing my findings, reflecting on my study and providing concluding remarks.

Chapter 5: <u>Findings, Reflections and Conclusion</u>

5.1 Introduction

The aim of this study was to understand ways in which grade nine learners interpret letters in different levels of generalised arithmetic activities. In grade nine the NCS (2002) prescribes the interpretation of letters in all learning outcomes. This strong presence of letters and the problematic nature of the transition from arithmetic to algebra, which can impact negatively on future algebra learning, suggested that research in early algebra is useful.

In Chapter 1, the aims, research critical questions and rationale were established. In Chapter 2, a survey of literature was provided by creating coherent links to my research idea and research critical questions. Conceptual frameworks that are crucial to this study included the interpretation of letters, levels of understanding, 'symbol sense' and the 'core activities of algebra'. Misconceptions as viewed by constructivists were used as a lens through which to view learners' written and spoken words in my study. In Chapter 3, I discussed the methodology of this research by explaining the context of my investigations. I also discussed how the school and sample of people that participated in this study were selected, the administration of my research instruments and the data analysis process. I analysed the data collected, in Chapter 4, which culminated in the establishment of themes that relate to common errors and misconceptions. In this chapter, I conclude my study by discussing the findings and reflecting on a few core implications that emerge from this study.

5.2 Findings

5.2.1 Instinctiveness of finding single answer solutions

It was seen throughout the paper and pencil tests and interviews that learners had an instinctiveness to find single answer/one term solutions. The misconception of joining which also emerged in many of the interviews seemed to be an underlying cause for the vast majority of learners' single term answers in the paper and pencil tests. This tendency to find single answer solutions suggests a link to arithmetic thinking where answers are commonly single terms. This is in line with Malisani and Spagnolo (2008, no page numbers) who explain arithmetic problem solving as involving operations to gain a 'solution almost always unique'. Moreover, this drive to get a single answer was in conflict with coming to understand the meaning behind the implicit relations represented by the syntax of algebra. For example, the drive to simplify 2 + *a* + *b* to 2*ab* mitigates against internalising 2*ab* as signifying 2 x *a* x *b*. Learners' lack of internalising of implicit relations after two years of algebra teaching with a teacher who is regarded as competent is cause for concern. This suggests a strong interference of prior knowledge on current learning and perhaps whether there are ways of introducing algebra to overcome this interference needs investigation.

The excerpt below, which was typical for many interviewed learners, illustrates Agi's poor understanding of internalising 3*m* as being 3 x *m*. Due to not understanding the implicit relation represented by the syntax of algebra, Agi substitutes the value of *m* in the term 3*m* without considering the multiplication between the 3 and *m*. It can be seen that she evaluates 3*m* as 34 when *m* is 4 and 39 when *m* is 9.

Kona	35. So what is your understanding of 3*m*? What does 3*m* mean to you?
Agi	3*m* means to me that the *m* is 4 so it's 34, I understand the *n*'s 3.
Kona	Ok. So if the *m* was 9 then what would 3*m* be?
Agi	That would be 39 plus 1.
Kona	Will give you?
Agi	It will give me 40.

5.2.2 Creation of own rules

The themes of random picking and inconsistencies with own rules suggest strongly that at certain stages, when learners did not understand the algebra, they created their own rules or ways of making meaning. It seems reasonable to suggest that a lack of basic manipulations of early algebra contributed to these rules being temporary as the rules changed at different instances. There were instances when learners randomly combined any numbers and letters, selected random numbers from tasks as solutions and randomly performed operations for the sake of achieving solutions.

Laizal was the only interviewed learner that showed a basic understanding of manipulations of letters and hence a basic understanding of the ‘knowledge landscape’ (Greeno, 1991, as cited in Daniels, 2001, p. 25) of early algebra. Therefore, Laizal was able to display more components of ‘symbol sense’ and performed better. However, if more learners were aware of relations between symbolic systems and were proficient in manipulations then this could have improved their performance.

Nelli had her own set of rules where the “basics” of symbolic manipulations were limited. At certain stages she equated letters to any constant or variable and took the order of letters of alphabet as proportional to magnitude. The excerpt below shows how Nelli needs to find a “final answer” and relates her prior knowledge to any situation by modifying her understanding to suit the respective task. In task 10 ($3a - b + a =$ __) she just drops off $1a$ and has a rule of $3a^2 + b = 3ab$. In task 11 she does not drop off any letters but adds the $1b$ twice and has new rules of $b + b = b^2$ and $a + b = ab$. This suggests that she had her own logic but the logic was temporary because she made up rules as she progressed from one task to another.

Kona	That’s fine. Can we turn over? Can we look at number 10? Ok. Number 10 says simplify $3a - b + a$ and your answer was $3ab$. Can you explain how you getting your answer of $3ab$?
Nelli	Eh. What I just said, I just put the *a* and *b* the *a* and *a*, the both *a*’s aside and just said $3a - b$ is and then I thought let me just remove $1a$ instead of writing a^2 because they the same thing they the same alphabet so instead of writing $3a^2 + b$, I just say $3ab$ that is like a suitable answer.
Kona	$3ab$. Ok. So you just removed the $1a$?
Nelli	Just removed the $1a$.
Kona	Then for number 11. You see for number 10 you dropped the *a*. It seems like for number 11 you didn’t drop off the *b*.
Nelli	Didn’t.
Kona	Can you explain what was your thinking in number 11?
Nelli	Well I just added the $2b$’s and just made them b^2. Then, I also added this positive *a* plus this positive *b* and made it an *ab* so that I can strain and use this negative which is saying $- b^2$.

It was fascinating that throughout many interviews learners created their own different rules when interpreting letters. These temporary rules, which were for “now” rather than "later", suggest that learners had their own logic or desire to find an answer by relating misconceptions in prior learning to any situation. The excerpt below shows how Lee, due to his poor manipulative skills, almost instantly creates different rules of u x 3 = 3, v + 3 = 3 and $m + n = mn$. Moreover, throughout his interview Lee made new rules that changed which was also typical for many of the other learners.

Lee	So, eh, I multiply this each letter by 3. I said 3 times, ah, u x 3. Then I got a 3. Then I also said v. I made a v as like a letter I said v + 3 and then it gave me a 3. So that's when I thought of writing a 3 down.
Kona	Ok, fine. Can we look at another one? Can we look at number 3? Ok, you said that, the question is, what can you say about m if $m = 3n + 1$ and $n = 4$? You said its $4nm$.
Lee	Ja, here sir, I said 3. I plussed, added 3 + 1 and then I added this m, $m + n = mn$, I added this 3 + 1. That gave me $4mn$.

5.2.3 Understanding of algebraic letters

According to Küchemann's (1981) 'levels of understanding' most learners in the present sample were operating below level 1. This means they were able to work with some of the tasks that required a more arithmetic notion of the letter. These tasks were 'extremely easy, purely numerical and had a simple structure' where the letters needed to be *evaluated, not used* and *used as objects* (Küchemann, 1981, p. 113). However, due to an increase in structural complexity many learners did not cope with the level 2 tasks where the letter also needed to be *evaluated*, *not used* and *used as an object*. It follows that, tasks where the letter needed to be interpreted as *specific unknowns, generalised numbers* or *variables*, where a more algebraic notion of the letter was needed, were too sophisticated for the present sample.

A central focus of algebra as envisaged by the NCS (2002, p. 63) is that grade nine learners should 'investigate patterns between variables and express rules governing patterns in algebraic language or symbols'. However, most learners in the present sample were operating below level 1 according to Küchemann's (1981) levels of understanding and seem to have no algebraic notion of the letter and the syntax of algebra. Therefore, the findings of this study conflicts with what is prescribed for grade nine learners in the algebra curriculum.

Due to lacking a conceptual understanding of letters in algebra learners created their own ways to give meaning to the letter. Therefore, using misconceptions such as ‘invisible 1’ were common as all interviewed learners mentioned the phrase ‘invisible 1’ with little algebraic understanding. It seems like the phrase was learnt in the mathematics class but with a limited understanding of its reference to the syntax of algebra.

It was also seen that at some instances letters were equated but not at other stages and some learners related the order of the letters of the alphabet to magnitude. This further suggests that learners created their own ways to give meaning to the letter due to them lacking a conceptual understanding of the syntax of algebra. Lee and Nelli explain, in the excerpts below, that the order of the letters of the alphabet affects the magnitude of the letters. Lee explains that M will be bigger than P because it comes before P in the alphabet whereas Nelli explains that the letter *a* has the biggest value because it is the first letter of the alphabet. This relation of the magnitude of the letter to order of the alphabet implies that Lee and Nelli find their own ways to give meaning to letters as a result of having a poor understanding of algebraic letters.

Kona	What do you mean before?
Lee	Isn’t it M in letters we count M, it comes first and then P follows. Ja, so they’ll never be equal in that way.
Kona	So are you saying one is bigger?
Lee	Ja. One is bigger and one is lesser.
Kona	Which would be bigger and which would be lesser?
Lee	Eh, I think P would be bigger, ja, I think P would be bigger than. No, M would be bigger cos it comes before P.

Nelli	Eh. In the alphabetic way. I just thought of because *n* already had a number so instead of using another number like 7 or something I just used 6 because I wanted to find a number for *m* because *n* comes after it.
Kona	Ok. Good. So which is bigger, *m* or *n*?
Nelli	*m*, I believe it’s bigger.
Kona	*m*, why do you say *m*?
Nelli	Because it comes first than *n*.
Kona	It comes first, so it means *a* will be the biggest?
Nelli	Of them all.
Kona	Of them all, so which is the smallest?
Nelli	*z*

Malisani and Spagnolo (2008, no page numbers) emphasise that 'the introduction of the concept of variable represents a critical point in the arithmetic-algebraic transition'. The data analysis suggests that this fundamental issue of algebraic letter is not understood by learners in the present sample. This is likely to lead to errors and misconceptions in future algebraic learning where 'making sense of letters' is a critical aspect (Malisani & Spagnolo, 2008, no page numbers). This is another cause for concern because in Chapter 1, I argued that competency when interpreting and manipulating letters is crucial for proficiency in algebra and mathematics.

5.2.4 Overall performance of learners

Table 3 showed that the overall performance of learners across the 17 tasks for the six different uses of the letter was poor. No learner coped with the level 3 and 4 tasks where a more algebraic notion of the letter was needed. The interviews suggested that misconceptions such as random picking and 'invisible 1' were possible causes for this overall poor performance. It was seen throughout the interviews that learners in this grade nine class lacked the components of 'symbol sense' such as manipulations, different uses of symbols and knowledge of algebraic expressions. The interviews also suggested that the transition from arithmetic to algebra has not been made and that learners are still on the arithmetic level. The implications of this poor interpretation of letters seems cause for great concern because grade ten sections dealing with functions, simplification of algebraic expressions and solving equations, etc. require understanding and proficiency of the uses of letters as a prerequisite.

5.3 Research questions

This study was guided by the following research questions:

1. How do learners interpret symbols/letters during engagement with generalised arithmetic activities?

2. Why do learners adopt certain methods, strategies and common errors when engaging with algebraic problems?

3. How and why are learner interpretations of symbols different across a range of activities reflecting different levels of algebraic understanding?

4. What are possible similarities and differences between the present sample's interpretations of letters and that of the CSMS (1979) sample?

5.3.1 How do learners interpret symbols/letters during engagement of generalised arithmetic activities?

Learners in this sample experienced great difficulty during engagement with the generalised arithmetic tasks. Joining of numbers and letters during addition was common and problematic in algebraic settings. The three forms of random picking suggest that learners performed non–algebraic procedures for the sake of completing tasks with no 'symbol sense'. There were instances when any operation was performed to get a solution which is in line with Bell (1995) who explains that learners are merely viewing symbols as undergoing manipulations. Moreover, the themes of joining and random picking suggest that most learners in this sample engaged with the generalised arithmetic activities with little or no understanding of algebraic symbol systems and symbolic manipulations.

5.3.2 Why do learners adopt certain methods and strategies, and common errors when engaging with algebraic problems?

An assumption of this study was that misconceptions will be displayed by learners which was confirmed by my data analysis. Misconceptions in prior learning of concepts such as substitution, implicit multiplication and gathering of like terms were seen to be problematic and caused learners to make errors in many tasks. Learners had a poor understanding of the concept of algebraic letter and basic symbolic manipulations and hence did not have 'symbol sense'. This resulted in random picking, joining of numbers and letters during addition and the creation of rules that were not consistent. Moreover, this poor algebraic understanding also seems to have contributed to learners creating their own ways to give meaning to the letter such as the 'invisible 1' misconception which also caused errors.

5.3.3 How and why are learner interpretations of symbols different across a range of activities reflecting different levels of algebraic understanding?

Learners performed satisfactorily for the level 1 tasks but experienced difficulties with the level 2, 3 and 4 tasks. Due to the structural complexity of the higher level tasks and weakness in learners' use of letters the assumption that letters will be interpreted differently across the different levels of understanding was seen clearly in Table 5. Learners struggled to interpret the *letter as a specific unknown, generalised number* and *variable* because these interpretations of the letter were of greater difficulty and needed a more algebraic notion of the letter.

5.3.4 What are possible similarities and differences between the present samples' interpretations of letters and that of the CSMS (1979) sample?

Küchemann (1981) found that using *letters as variables*, *unknowns* or *pattern generalisers* were of greater difficulty for the fourteen year old learners. Although overall results are much weaker for learners in my study they also struggled with the latter three interpretations of letters. Therefore, a similarity is that both samples experienced greater difficulty with the higher levels of interpretation.

The present sample's percentage of total correct solutions are below the CSMS (1979) sample's for all of the 17 paper and pencil tasks. I have also noted that both samples had a decline in correct solutions from level 1 to level 4 and learners in both samples seemed to have equated letters in tasks 4 and 7. Another striking similarity is that there were 7 out of 10 tasks that had the same most common error for both samples.

5.4 Reflections

5.4.1 Arcavi's (2005) conceptual framework of symbol sense

I thought that the framework by Arcavi (2005) would be useful at the time of going into the empirical field but during my interview analysis it became apparent that this framework would not be as useful as I initially thought. Arcavi (2005) discusses six components of 'symbol sense' that are interrelated and dependant. In other words, if the component of 'symbols can play different roles in different contexts' is not understood then other components such as 'friendliness with symbols' and the 'ability to select one possible symbolic representation for a problem' will also be absent (Arcavi, 2005, pp. 42–43). Therefore, due to being lower achieving learners the six interviewed learners did not display the components of 'symbol sense'.

In this way, Arcavi's (2005) conceptual framework was not useful. However, the framework helped me to analyse and discuss symbolic proficiencies that learners lacked which informed my initial three research critical questions. This is an interesting reflection for me that one might begin a research project with a framework only to find that the framework might not be related to one's sample.

5.4.2 The issue of language

Language of the given text is critical in explaining reasons for learners interpreting symbols in certain ways and is an area of mathematics education I would enjoy researching in the future. For example, Agi interpreted the instruction "add" in task 13 as needing to do something. She felt that she needed to add 4 to $3n$ which is what she did to get $7n$ that was a solution reached by 87% of learners. Agi says: 'the add is like you must add the 4 to 3 which gives me 7'. Therefore, the solution of $7n$ suggests that the word "add" in the question prompted learners to "do something" and joined the 4 and $3n$. Furthermore, it would be interesting to research whether phrasing the question differently as $4 + 3n =$ __ or can you simplify $4 + 3n$, without the word "add", would yield different results.

I hypothesise that the mathematics teacher's use of mathematical language has strongly influenced the mathematical language of learners. This is seen with the issue of 'invisible 1' because all learners interviewed mentioned the phrase 'invisible 1' which was problematic. The intention of using phrases such as 'invisible 1' or FOIL (FOIL stands for 'first outer inner last' and is used in the teaching of the distributive law) is to simplify the mathematics but careful and strategic implementation is needed so that misconceptions could be avoided. A focus by mathematics teachers on common misconceptions and the roots of them are crucial in teaching and learning and should form part of the planning of instructional programs. Perhaps, teacher development and training needs to focus on understanding these and other misconceptions related to learners' interpretation of letters.

5.5 Conclusion: An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning

Grade nine learners who participated in this study seemed to have an arithmetical inclination during problem solving and have not yet made the transition from arithmetic to algebra. Central to this transition is the introduction of letters which was problematic because most learners had a poor understanding of the different interpretations of letters which could impact negatively on future algebra learning. The various misconceptions uncovered suggest that learners also lacked basic introductory manipulative skills involving letters. Therefore, this was a very worthwhile study because the findings are interesting and useful to the teaching of early algebra.

Learners in this sample were simplifying and solving equations using various misconceptions which if left undetected will resurface in future learning and hamper the learning process. Olivier (1989, p. 13) explains that direct teaching of previous knowledge to confront misconceptions is not as successful as using strategies such as 'successful remediation' or 'cognitive conflict'. Therefore, the mathematics teacher 'should provide opportunity to the student to manifest his misconceptions, and then relate his subsequent instruction to these misconceptions' (Nesher, 1987, p. 39).

My study has uncovered a few misconceptions in early algebra learning such as joining, inconsistencies with own rules, random picking and 'invisible 1' which are useful for teachers of early algebra. It is useful to have studies like this that point out possible misconceptions and where they come from. Moreover, this allows teachers to know that strange answers like evaluating $3m$ as 34 when m is 4 are not individual idiosyncrasies but manifestations of misconceptions.

Even towards the end of grade nine, learners in this sample are not algebraically prepared for grade ten where a more complex notion of letters is prescribed by the NCS (2002) in sections such as functions, simplification of algebraic expressions and solving equations. If the cited misconceptions in this study are not confronted and used in learning it would seem that 'the job of teaching algebra to students who have not been successful in mathematics will remain a difficult challenge for those teachers willing to take it on' (Chazan, 1996, p. 475).

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APPENDIX 1

Example of one Interview Schedule

Name of learner: Agi   ½ hour

Task	*Research question*	*Interview question*	*Notes*
3	1	Is that your final answer? If I put another line and =?	Quick question and move on.
4	1,2,3	What you have written here is interesting. Can you write this differently?	Important to probe because Laizal was happy with no closure.
6	1,2	Can you explain your thinking? Why did you assume that n = -516?	
7	1,2,3	Why is the 8 so close to the =? Was this your final answer? Where or what happened to g?	Critical question! Important to probe in this question.
9	1,2,3	Can you explain your thinking?	
10	1,2	What is different to 9 above?	Might not need this question if task 9 is a slip!
11	1	Can you simplify this?	Similar response to task 3 above.
14,15	1,2,3	Can you explain your thinking?	Probe for task 15, if you had to state one sentence for c what would c be?
16	1,2,3	Can you explain your thinking? Can M and P ever be the same? If M = 3 can P = 3?	
17	1,2,3	Can you explain your thinking? Your response says multiplication makes bigger. Does it work with all numbers? Can you think of any numbers that don't work?	

APPENDIX 2

Research Instrument: Paper and Pencil test

Grade nine ***30min.***

Code of learner: ________________________________

1. What can you say about a if $a + 5 = 8$ ________________

2. What can you say about u if $u = v + 3$
and $v = 1$ ________________

3. What can you say about m if $m = 3n + 1$
and $n = 4$ ________________

4. What can you say about r if $r = s + t$
and $r + s + t = 30$ ________________

5. If $a + b = 43$
then $a + b + 2 =$ ______________

6. If $n - 246 = 762$
then $n - 247 =$ ______________

7. If $e + f = 8$
then $e + f + g =$ _____________

Note: $\boldsymbol{x + 3x}$ **can be written as** $\boldsymbol{4x}$ ***(Hint for questions 8 – 11)***

8. Simplify $2a + 5a =$ _________

9. Simplify $2a + 5b + a =$ _________

10. Simplify $3a - b + a =$ __________

11. Simplify $(a - b) + b =$ __________

12. Add 4 to $n + 5$ _____________

13. Add 4 to $3n$ ______________

14. Multiply $n + 5$ by 4 ________________

15. What can you say about c if $c + d = 10$
and c is less than d ________________________

16. L + M + N = L + P + N

always ☐

sometimes (when) ☐ _______________________

never ☐

17. Which is larger?

$2n$ or $n + 2$? explain! __________________________________

Thank you for participating in this study!

APPENDIX 3: All letters of permission

Letter of permission to school principal

The Principal
High School

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am Kona Naidoo and am currently undertaking research for my MSc. Degree (in Science Education) under the supervision of Doctor Mellony Graven and Lynn Bowie at the University of the Witwatersrand. The aim of my study is to investigate grade nine students' algebraic thinking. My research instrument is a test that comprises algebraic tasks. I will also interview five learners to gain insight into the algebraic thinking during engagement with the algebraic tasks. Children will be selected to participate in this study based on his/her June exam mathematics results and your suggestions.

I applied to pilot my research instrument in May 2008 but due to time constraints I had to use another school. However, I would like to conduct my research at your school according to the attached action plan. I hope that your learners will be available to participate in my study.

Kindly note the following information with regards to your child's participation in this/my study:

- Your learners' participation in this study is voluntary and if he/she refuses to participate there will be no penalty or any loss of benefits to which he/she might be entitled.
- Your learners' may discontinue their participation in this study at any given time without any penalty or loss of benefits.
- The duration of your learners' participation will be 30 minutes to engage with the algebraic activities that involve paper and pencil tasks for which there is no prior preparation required. The length of the interviews will be 30 minutes.
- Your learners may be selected to participate in an interview after his/her involvement in the paper and pencil tasks for which he/she will be informed in due time and I will ask for your written consent.
- There are no foreseeable risks, discomforts, side effects or benefits from participating in this study.
- I have received GDE ethics approval to conduct my research.

Yours Faithfully

____

Kona Naidoo
(0837507607)

Supervisors

__________________	_____________
Dr. Mellony Graven	Lynn Bowie
Lecturer Mathematics Education	Wits Education
(011) 717 3413	(011) 717 3412
Mellony.graven@wits.ac.za	lynn.bowie@wits.ac.za

Letter of permission to parent

Dear parent

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am seeking permission to involve your child in my study. I am currently undertaking research for my MSc. Degree (in Science education) under the supervision of Doctor Mellony Graven and Lynn Bowie at the University of the Witwatersrand.

The aim of my study is to investigate grade nine students' algebraic thinking and my research instrument is an activity–based test embedding a range of algebraic tasks. The aim is to engage learners who are willing to participate in this study. I will also want to interview a few learners from this study to gain insight into the algebraic thinking during engagement with the algebraic tasks.

The focus of my study is to understand how learners interpret mathematical letters when engaging with algebraic activities. Learner engagement with the tasks and the interview would each entail a maximum of 30 minutes. The information collected from the study will be kept confidential and will only be used to inform my research. Names and personal particulars of all participants will remain confidential. Moreover, the name of the school will also be confidential and the information gathered from this study will be ultimately destroyed.

I will appreciate your child's participation in my study although participation is voluntary and there will be no negative consequences if your child does not want to participate in my study. Your child will be allowed to withdraw at any time and information gathered from your child if your child withdraws will not be used in my research analyses. If you have any queries you can communicate to my supervisor or myself. Kindly indicate if you consent your child to participate in this study.

Yours Faithfully

____

Kona Naidoo
(0837507607)

Supervisors

____

Dr. Mellony Graven
Lecturer Mathematics Education
(011) 717 3413
Mellony.graven@wits.ac.za

____

Lynn Bowie
Wits Education
(011) 717 3412
lynn.bowie@wits.ac.za

I give permission for my child to participate in the research project run by Kona Naidoo.

Child's name:

Parent's name:

YES	**NO**
☐	☐

Parent's signature:

Letter of permission to learner

Dear learner

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am seeking permission to involve you in my study. I am currently undertaking research for my MSc. Degree (in Science education) under the supervision of Doctor Mellony Graven and Lynn Bowie at the University of the Witwatersrand.

The aim of my study is to investigate grade nine students' algebraic thinking and my research instrument is an activity–based test embedding a range of algebraic tasks. The aim is to engage you if you are willing to participate in this study. I could also want to interview you in this study to gain insight into the algebraic thinking during engagement with the algebraic tasks. However, you will be notified accordingly and I will request your consent if needed.

The focus of my study is to understand how learners interpret mathematical letters when engaging with algebraic activities. Your engagement with the tasks and the interview would each entail a maximum of 30 minutes. The information collected from the study will be kept confidential and will only be used to inform my research. Your name and personal particulars will remain confidential. Moreover, the name of the school will also be confidential and the information gathered from this study will be ultimately destroyed. Results from this study will not be disclosed to anyone at your school and will not affect your mathematics assessment at school in any way.

I will appreciate your participation in my study although participation is voluntary and there will be no negative consequences if you don't want to participate in my study. You will be allowed to withdraw at any time and information gathered from you if you withdraw will not be used in my research analyses. If you have any queries you can communicate to my supervisor or myself. Kindly indicate if you consent to participate in this study.

Yours Faithfully

____
Kona Naidoo
(0837507607)

Supervisors

________________	________________
Dr. Mellony Graven	Lynn Bowie
Lecturer Mathematics Education	Wits Education
(011) 717 3413	(011) 717 3412
Mellony.graven@wits.ac.za	lynn.bowie@wits.ac.za

I give consent to participate in the research project run by Kona Naidoo.

Child's name:

Parent's name:

YES ☐ **NO** ☐

Child's signature:

Letter to mathematics teacher

Mathematics Teacher

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am Kona Naidoo and am currently undertaking research for my MSc. Degree (in Science education) under the supervision of Doctor Mellony Graven and Lynn Bowie at the University of the Witwatersrand. The aim of my study is to investigate grade nine students' algebraic thinking. My research instrument is a test that comprises algebraic tasks. I will also interview five learners to gain insight into the algebraic thinking during engagement with the algebraic tasks. Children will be selected to participate in this study based on his/her June exam mathematics results and your suggestions.

Kindly note the following information with regards to your child's participation in this/my study:

- Your child's participation in this study is voluntary and if he/she refuses to participate there will be no penalty or any loss of benefits to which he/she might be entitled.
- Your child may discontinue his/her participation in this study at any given time without any penalty or loss of benefits.
- The duration of your child's participation will be 30 minutes to engage with the algebraic activities that involve paper and pencil tasks for which there is no prior preparation required.
- Your child may be selected to participate in an interview after his/her involvement in the paper and pencil tasks for which he/she will be informed in due time and I will ask for written consent.
- There are no foreseeable risks, discomforts, side effects or benefits from participating in this study.

Yours Faithfully

____

Kona Naidoo
(0837507607)

Supervisor

____
Dr. Mellony Graven
Lecturer Mathematics Education
(011) 717 3413
Mellony.graven@wits.ac.za

____
Lynn Bowie
Wits Education
(011) 717 3412
lynn.bowie@wits.ac.za

Letter of permission to learner for interview

Dear learner

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am seeking permission to further involve you in my study. I am currently undertaking research for my MSc. Degree (in Science education) under the supervision of Doctor Melony Graven and Lynn Bowie at the University of the Witwatersrand. You have participated in the activity-based test, which was the initial data gathering process.

The aim of my study is to investigate grade nine students' algebraic thinking and my second research instrument is a clinical interview. The aim is to engage learners who are willing to participate in this study. I am seeking permission to interview you to gain insight into the algebraic thinking involved during engagement with the algebraic tasks.

The focus of my study is to understand how learners interpret mathematical letters when engaging with algebraic activities. Learner engagement with the clinical interview would entail a maximum of 30 minutes. The information collected from the study will be kept confidential and will only be used to inform my research. Names and personal particulars of all participants will remain confidential. Moreover, the name of the school will also be confidential and the information gathered from this study will be ultimately destroyed.

I will appreciate your participation in my study although participation is voluntary and there will be no negative consequences if you do not want to participate in my study. You will be allowed to withdraw at any time and information gathered from you if you withdraw will not be used in my research analyses. If you have any queries you can communicate to my supervisor or myself. Kindly indicate if you consent to participate in this study.

Yours Faithfully

____

Kona Naidoo
(0837507607)

Supervisors

___________________	________________
Dr. Mellony Graven	Lynn Bowie
Lecturer Mathematics Education	Wits Education
(011) 717 3413	(011) 717 3412
Mellony.graven@wits.ac.za	lynn.bowie@wits.ac.za

Child's name:

Parent's name:

I give consent to participate in the research project run by Kona Naidoo

YES	**NO**
☐	☐

Learner's signature:

Letter of permission to parent for interview

Dear parent

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am seeking permission to further involve your child in my study. I am currently undertaking research for my MSc. Degree (in Science education) under the supervision of Doctor Melony Graven and Lynn Bowie at the University of the Witwatersrand. Your child participated in the activity-based test, which was the initial data gathering process.

The aim of my study is to investigate grade nine students' algebraic thinking and my second research instrument is a clinical interview. The aim is to engage learners who are willing to participate in this study. I am seeking permission to interview your child to gain insight into the algebraic thinking involved during engagement with the algebraic tasks.

The focus of my study is to understand how learners interpret mathematical letters when engaging with algebraic activities. Learner engagement with the clinical interview would entail a maximum of 30 minutes. The information collected from the study will be kept confidential and will only be used to inform my research. Names and personal particulars of all participants will remain confidential. Moreover, the name of the school will also be confidential and the information gathered from this study will be ultimately destroyed.

I will appreciate your child's participation in my study although participation is voluntary and there will be no negative consequences if you child does not want to participate in my study. Your child will be allowed to withdraw at any time and information gathered will not be used in my research analyses. If you have any queries you can communicate to my supervisor or me.

Kindly indicate if you give consent for your child to participate in this study.

Yours Faithfully

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Kona Naidoo
(0837507607)

Supervisors

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Lecturer Mathematics Education	Wits Education
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Child's name:

Parent's name:

I give consent to participate in the research project run by Kona Naidoo

YES	**NO**
☐	☐

Parent's signature:

Letter of permission to parent to audio record

Dear parent

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am seeking permission to audio record the interview with your child. I am currently undertaking research for my MSc. Degree (in Science education) under the supervision of Doctor Mellony Graven and Lynn Bowie at the University of the Witwatersrand.

The aim of my study is to investigate grade nine students' algebraic thinking and my research instrument is an activity–based test embedding a range of algebraic tasks. The aim is to engage learners who are willing to participate in this study. I am seeking permission to audio record the interview with your child. The interview will be approximately 30 minutes and is based on your child's responses to the paper and pencil tasks that were administered in my study. The interview aims to gain insight into the algebraic thinking during engagement with the algebraic tasks.

The focus of my study is to understand how learners interpret mathematical letters when engaging with algebraic activities. The information collected from the audio recordings will be kept confidential and will only be used to inform my research. Names and personal particulars of your child will remain confidential. Moreover, the name of the school will also be confidential and the information gathered from this study will be ultimately destroyed.

All learners participating in this study are allowed to withdraw at any time and information gathered from such learners will not be used in my research analyses. If you have any queries you can communicate to my supervisor or myself.

Kindly indicate on the form below if you give permission for your child to participate in this study.

Yours Faithfully

____
Kona Naidoo
(0837507607)

Supervisors

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Dr. Mellony Graven	Lynn Bowie
Lecturer Mathematics Education	Wits Education
(011) 717 3413	(011) 717 3412
Mellony.graven@wits.ac.za	lynn.bowie@wits.ac.za

I give permission for my child to participate in the research project run by Kona Naidoo.

Child's name:

Parent's name:

YES	**NO**
☐	☐

Parent's signature:

Letter of permission to learner to audio record

Dear learner

Research title: **An Investigation of Learners' Symbol Sense and Interpretation of Letters in Early Algebraic Learning**

I am seeking permission to audio record my interview with you. I am currently undertaking research for my MSc. Degree (in Science education) under the supervision of Doctor Mellony Graven and Lynn Bowie at the University of the Witwatersrand.

The aim of my study is to investigate grade nine students' algebraic thinking and my research instrument is an activity–based test embedding a range of algebraic tasks. The aim is to engage learners who are willing to participate in this study. I am seeking permission to audio record the interview that I will conduct with you. The interview will be approximately 30 minutes and is based on your responses to the paper and pencil tasks that were administered in my study. The interview aims to gain insight into your algebraic thinking during engagement with the algebraic tasks.

The focus of my study is to understand how you interpreted mathematical letters when engaging with the algebraic activities. The information collected from the audio recordings will be kept confidential and will only be used to inform my research. Your name and personal particulars will remain confidential. Moreover, the name of the school will also be confidential and the information gathered from this study will be ultimately destroyed.

All learners participating in this study are allowed to withdraw at any time and information gathered from such learners will not be used in my research analyses. If you have any queries you can communicate to my supervisor or myself.

Kindly indicate on the form below if you give permission for yourself to participate in this study.

Yours Faithfully

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Kona Naidoo
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Child's name:

Parent's name:

I give consent to participate in the research project run by Kona Naidoo

YES	**NO**
☐	☐

Learner's signature:

APPENDIX 4

Pilot analysis: Paper and pencil tasks and one interview

1. Two boys and three girls wrote the paper and pencil tasks while one learner was interviewed.
2. The time allocated was 30 minutes for the tasks and 30 minutes for the interview.
3. The tasks in the pilot analysis were marginally different to the final instrument.

Pilot analysis: Paper and pencil tasks

Letter evaluated

Level	***Level 1***	***Level 2***	***Level 2***
Task : Letter evaluated	$x + 4 = 9$	$x + 6 = b$ and $b = 12$	$x = 4a + 2$ and $a = 3$
Analysis of responses	$x = 5$ 100% correct	$x = 6$ 100% correct	$x = 14$, 25 % correct. $x = 9$, 50% correct (by adding 4+3). $x = 45$, 25% correct (by making $4a = 43$).

Letter not used

Level	***Level 1***	***Level 1***	***Level 2***
Task: Letter not used	$u + v = 56$ $u + v + 5 =$ ____	If $m - 124 = 257$ $m - 125 =$ ___	Add 3 to $a + 3$
Analysis of responses	Correct solution: 61 100% correct.	Correct solution: 256 25% correct. Incorrect solution of 258 75% incorrect.	25% got correct solution of $a + 6$. 50% got $6a$. 25% got $7a$.

Letter used as an object

Level	***Level 1***	***Level 2***
Task: Letter as object	$3x + 4x =$	$3x + 4y + x =$
Analysis of response	Correct solution. of $7x$ by 75%. 25% got 3 + 4 as solution.	$4x + 4y$ by 75%. $3 + 4y$ by 25% by canceling x's.

Letter used as a specific unknown

Level	***Level 3***	***Level 4***	***Level 3***	***Level 3***	***Level 4***	***Level 4***
Task: letter as specific unknown	Add 3 to $4a$	$x = u + v$ and $x + u + v = 16$	If $u + v = 10$ $u + v + w =$ ___	$3x - 2y + x =$	$(x - y) + 2x =$	Multiply a + 3 by 5
Analysis of responses	25% got correct solution of $3 + 4a$. 75% got solution of $7a$.	$x = 8$, 25% correct. $x = 5$, 50%. (1 learner took letters to be equal). $x = 10$ (by assigning any number to the letter).	Solution of 15 given by 75% of learners by taking $u = v = w = 15$.	75% got $4x$-$2y$. 25% got 3+$4y$.	25% got the correct solution of $3x - y$. 50% got $2x^2 - 2xy$. 25% got solution $-2y$.	25% got the correct solution of $5a + 15$. 50% got $15a$. 25% got 20.

Letter used as a generalised number

Level	***Level 3***	***Level 4***
Task: letter as generalised number	What can you say about a if $a + b = 20$ and a is less than b?	When is $a + b + c = a + d + c$?
Analysis of responses	75% assigned only one numerical value for a. 25% got many numerical values for a.	100% solutions indicated the answer never.

Letter used as a variable

Level	***Level 4***
Task: letter as variable	Which is bigger, $3a$ or $a + 3$? Explain.
Analysis of responses	50% say equal. 25% say $a + 3$ and 25% say $3a$.

The relation of learner responses to my research questions

<u>1. How do learners interpret symbols/letters during engagement of generalised arithmetic activities?</u>

Learners were able to interpret letters for levels 1 and 2. Levels 3 and 4 posed problems. Learners confused magnitude in the second level 1 task for *letter not used*, which seemed like an error. Taking away 1 more results in 1 less but learners got 258. Learners were fairly successful with the *letter as an object* tasks. The mere presence of the brackets in the second level 4 task for *letter as a specific unknown* posed problems. The word "add" created the issue of "completing" or "finishing" the task. Learners are unable to use the *letter as a generalised number*. Learners are unable to distinguish between a and $a + 3$ possibly due to a poor understanding of the letter and the difference between the two terms.

<u>2. Why do learners adopt certain methods and strategies, (and resultant errors), when interpreting algebraic symbols?</u>

The interview with one learner suggested some reasons for adoption of strategies, like misconceptions. Methods show misconceptions for level 2, 3 and 4 tasks. $4a$ is not evaluated correctly but taken as $4 + a$ or $4a$ has become 43 by placing the value for $a = 3$ next to 4. There were stages where learners took letters to be equal and assumed equality. Learners were also unable to distinguish between like terms and could not apply the distributive law.

<u>3. How and why are learner interpretations of symbols different across a range of activities reflecting different levels of algebraic understanding?</u>

A decline of correct solutions is evident as levels of understanding increase. Also, no correct solution for using letters as *generalised numbers* and *variable*. There were few correct responses for the level 3 and 4 tasks. Why? Could look at task structure. Moreover, learners struggled with the level 2, 3 and 4 tasks.

<u>4. What are possible similarities and differences between the present samples' interpretations of letters and that of the CSMS (1979) sample?</u>

Interpreting letters as being equal in certain tasks was similar for both samples. A careful analysis could find more links in terms of percentage correctness and misconceptions.

Pilot analysis/notes: interview

Learner struggled to explain her thinking during the interview because the interview took place about three weeks after the test was written.

Task 3. Learner adds 4 + 3 + 2 = 9. However, she notices her error and quickly corrects her mistake and gets the correct answer of 14.
Task 5. Learner explains that the letters are variables. She also says that she has to find x which is the subject of the formula.
Task 6. Kona: Are they equal?
Learner: Yes. If they had different coefficients but all have an invisible 1.
Task 7. 4 + 4 + 2 = 10. Learner took numbers that make 10.
Kona: Can I put 7, 2, 1 or 6, 2, 2?
Learner: Yes.
Kona: How do we know which numbers to put?
Learner: Numbers that add to 10.
Task 9. Kona: Explain what you did?
Learner explained grouping of terms correctly.
Task 11. Learner multiplied using the 'distributive law'.
Kona: Is the given expression different from $(x - y) + 2x$
Learner: No. Learner explains that there is a + in front of the $2x$.
Kona: Is there any difference to $+ 2x + (x - y)$?
Learner: No even though there is no positive sign we don't have to put it.
Task 12 Kona: How do you get 7?
Learner: $a = 1$ due to the coefficient of a being 1. Therefore result of 7.
Kona: Why wasn't $4x + 4y = 8$?
Learner: Because of variables, if variables are the same then I would add but x and y above are not the same. She explains that $4x + 5x = 9x^2$.

Task 13. Kona: Are the variables the same?
Learner: No
Kona: Then why add?
Learner: Because it says add. Therefore, she adds 3 to 4*a* and gets 7*a*.
Task 14. Kona: How do you get 20?
Learner: 1 + 3 = 4 x 5 = 20.
Task 15. Kona: Are there any other possible answers?
Learner does not see the generalisation. When asked which option she would choose she responded by saying 18 and 2, as this was the first option she thought of.
Task 16. Kona: Why did you say 'never'?
Learner: The problem was *d* if it was *b* then they will be equal… Interviewer should have probed further.
Task 17. Kona: Why *a* + 3?
Learner: Because *a* + 3 contains an invisible 1 and when added you will get 4, therefore it is larger.

Notes in terms of research questions

Research question 1. Similar to the analysis of the tasks. Lower levels are manageable whereas the higher levels seem too sophisticated.

Research question 2. The interview suggests some reasons for learners adopting certain methods, strategies and misconceptions.

Research question 3. I can see the pattern as previously where learners are able to interpret the letters in the lower levels but not the higher levels.

Research question 4. Need further analyses and comparisons.

APPENDIX 5: All transcript analyses

Analysis of Transcript of interview for Agi

Transcript	***Close findings***
Kona: Can we start? Can we look at the 3rd question?	
Agi: Ok	
Kona: Ok, it said what can we say about n if $n = 3m + 1$ and $m = 4$? You wrote your answer as 35. Can you explain your thinking? What were you thinking when you wrote down 35?	
Agi: I was thinking that if the m… that if m was 34, so I just wrote the 3 and then 1 plus 4 which gave me 5.	
Kona: Ok	
Agi: So 35. I add the? to 35	
Kona: So can you explain that further? How did you get the 35 again? What did you add to get the 35?	
Agi: The 3 I just wrote it down and I just said 4 plus 1 is 5.	The rule of $3m = 34$ if $m = 4$. Therefore, $3m + 1 = 35$.
Kona: Ok. And then you get?	
Agi: 35	
Kona: 35. So what is your understanding of $3m$? What does $3m$ mean to you?	Poor understanding of the syntax of algebra.
Agi: $3m$ means to me that the m is 4 so it's 34, I understand (?) the n's 3.	
Kona: Ok. So if the m was 9 then what would $3m$ be?	
Agi: That would be 39 plus 1.	
Kona: Will give you?	
Agi: It will give me 40.	
Kona: Ok, good. And then can we look at number 4?	

Agi: Ok	
Kona: I think what you have written in number 4 is very interesting. Again, can you explain your thinking? It says: What can you say about *r*?	
Agi: I just put the answer which is 30 because I just think of a number which gave me 30 which is 10 plus 10 plus 10.	
Kona: So what is the value of *r*?	
Agi: It is 10.	
Kona: Oh, it's 10.	
Agi: Ja	
Kona: Ok. Could you add any other numbers for *r*, *s* and *t*?	
Agi: Yes. 5	
Kona: *r* could be 5?	
Agi: Ja	
Kona: And then *s* and *t*, what could those be?	
Agi: The other one would be 15 and the other one would be 10.	
Kona: So we get?	
Agi: We get 30.	
Kona: We get 30. Ok. Then how would you know which numbers to put for *r*, *s* and *t*? So if I can explain that – you're saying that it could be 10, 10 and 10 or it could be 5, 15 and 10 – but how do you know which numbers to put – whether you put *r* is 5 or *r* is 10?	
Agi: Well you can just put any number like as long as it gives you 30. Like when you add all of them, as long as it gives you 30.	Random picking for *r*, *s*, and *t*. but learner understands that $r + s + t = 30$.
Kona: As long as it gives you 30?	
Agi: Ja	

Kona: Ok. And then can we look at number 6?	
Agi: Ja	
Kona: Ok, number 6 was fine. Number 7. You got $e + f + g = 12$. (Are you with me?)	
Agi: Ja	
Kona: Can you explain your thinking there. How do you get the 12?	
Agi: I just said what one can give me f which is 4 + 4 and then I added all of them to give me 12.	
Kona: Ok, so e is 4, f is 4 and g is 4?	
Agi: Ja	
Kona: And that 4 + 4 + 4 gives you 12?	Random picking $e + f = 8$. She says e could be 4 and f could be 4 but she then says any other combination is also fine and e and f don't have to be equal.
Agi: Ja	
Kona: Ok. So tell me, do you think e is the same as f, is the same as g?	
Agi: Ja	
Kona: Now again, similar to question 4 – how do you know which values e and f and g are equal to?	
Agi: Well I just… So they're 8 which gave me like 4 + 4 is 8. I just checked the number what can give you 8, then I add everything which gives me the answer (?).	
Kona: Ok. So could e be 5 and f be 3 because that's still going to give you 8?	
Agi: Ja. Ja, it can.	
Kona: And what else could e and f be?	
Agi: e and f can also be 3 and, no, 2 and… could also	

be 2 and 6.	
Kona: Ok. Then it means they're not equal any more.	
Agi: No, they don't have to be.	
Kona: They don't have to be equal?	
Agi: No, as long as it gives you 8.	
Kona: As long as it gives you 8. Ok, good. Then can we turn over and let's look at task number 11.	
Agi: Ok	
Kona: I was quite surprised to see that your answer for task 11 was a (?) and then minus. You wrote that, am I right?	
Agi: Ja, I left the *a*, it was just minus for my answer.	
Kona: So what's your final answer?	
Agi: There's nothing, there's no answer.	
Kona: Is there no answer?	
Agi: Ja	
Kona: Can you explain a bit further what you mean by "no answer"?	
Agi: Ok. $a - b$, *a* is positive and *b* is minus so you cannot add or subtract because there's no number. It's like the invisible 1, so you cannot subtract or add because there's no bigger number or smaller number. And then class (?) *b* there's no number also because there's no number like in front of it, there's only invisible 1 and it's standing (?) for a positive (?), you cannot add or subtract.	'Invisible 1'; notion of 'invisible 1' as being the 1 in front of the letter in *a* and *b*, etc. 1*a* and 1*b*.
Kona: Ok, because of the invisible 1?	
Agi: Ja and because of the sign.	
Kona: Ok. So can you repeat that? You said you've got $a - b$.	
Agi: Ja	

Kona: And then?	
Agi: Like it's positive and negative and there's invisible 1, so you cannot subtract or like add. If there was 2 like next to the b then you were gonna take the? bigger number and subtract, but they're both equal.	Theme of 'invisible 1'
Kona: Ok, yes I do understand. So if you had $a - 2b$	
Agi: Ja, you were gonna subtract.	
Kona: And then what would your answer be?	
Agi: It would be $1ab$	
Kona: $1ab$?	
Agi: Ja, negative $1ab$	Rule of $a - 2ab = -ab$.
Kona: Negative $1ab$? And then plus the b?	
Agi: Plus the b also. It's going to be negative $1ab + b$. It will give you negative $1ab^2$	Rule of $1ab + b = 1ab^2$.
Kona: Negative $1ab^2$. Ok, and how do you get the b^2?	
Agi: Because there are like 2 b's. b x b is b^2.	
Kona: b x b is b^2. Ok. Now tell me, where do you get the times from?	
Agi: The bracket represents multiplication, ja.	
Kona: The bracket represents multiplication, Ok, good. Can we try another one? Can we look at number 12? It said: Add 4 to $n + 5$ and you wrote down $9n$. How do you get the $9n$?	
Agi: Because I add the 4 and the 5 and it gave me 9.	Rule of $4 + n + 5 = 9n$.
Kona:	
Agi: So it means n is 9.	
Kona: n is 9?	
Agi: Ja	
Kona: Ok. And then for number 13? Number 13 said:	

Add 4 to 3*n*. How did you get 7*n*?	
Agi: I added 3 and 4.	
Kona: Mmm	
Agi: And it gave me 7*n*	Rule of 4 + 3*n* = 7*n*.
Kona: 7*n*?	
Agi: Ja	
Kona: If you had to write any other answer for number 13 what would you write?	
Agi: It would still be 7 because it said add 4 to 3*n*. The add is like you must add the 4 to 3 which gives me 7.	The word *add* is telling Agi that she must do something therefore she adds 4 + 3*n* = 7*n*.
Kona: Ok. I've heard somewhere that you can only add like terms. What do you understand by like terms?	
Agi: By like terms is if the letters… no, not the letters… Ja, if the letters are like the same.	
Kona: Ok. And then are the letters the same there in number 13?	
Agi: Number 13? No.	
Kona: But you added it?	
Agi: Ja	
Kona: You did say now you can only add when the letters are the same.	
Agi: It's not multiplication. If it was multiplication then we were gonna add the… Ja, we were gonna add them.	Adding during multiplication.
Kona: Ok, it's fine. Can we check number 14? Now it's multiplication.	
Agi: Ja	
Kona: So it's: multiply *n* + 5 by 4 and you got 5*n* times 4 gives us 20*n*. Can you explain your thinking there?	For 4 (*n* + 5) then *n* + 5 = 5*n* 5*n* x 4 = 20*n*.

Agi: Well I just thought it said multiply and plus 5 by 4 but I just multiplied the… and I just add the $n + 5$ which gave me $5n$ times 4.	
Kona: Gives you?	
Agi: Gives you $20n$	
Kona: $20n$. Ok. So it's the same as number 12 and 13?	
Agi: Ja	
Kona: Where it said $n + 5$, you're saying $n + 5$ is just $5n$	
Agi: Ja	
Kona: And then you're multiplying that you ? 4. $5n$ multiplied by 4 so its $20n$.	
Agi: Ja	
Kona: Ok. Number 15. Number 15 says: What can you say about c? That's the question. If $c + d = 10$ and c is less than d and you said $4 + 6$.	
Agi: Ja. Well I just thought of a number which gave me 10, but then c must be less than d, so I said $4 + 6$ gives you 10.	Understands the question clearly, but is unable to see the letter as a generalised number.
Kona: So if you had to answer the question, what would your answer be?	
Agi: It would be $4 + 6$? 10	
Kona: Ok. But the question says: What can you say about c?	
Agi: I'd say it's gonna be 4	Agi is able to get 1 option for c.
Kona: c is going to be 4?	
Agi: Ja	
Kona: Ok, because you say $4 + 6$ must give you?	
Agi: 10	

Kona: 10. Ok. Now tell me, can *c* be any other number?	
Agi: Yes it can.	
Kona: What other number could *c* be?	
Agi: It can be 3 plus 7. It can also be 2 + 5.	
Kona: Will you get 10?	
Agi: No, but… No, it will be 3 plus 7.	
Kona: Mmm. And any other numbers?	
Agi: 1 + 9	
Kona: 1 + 9. So how do you know which number *c* is? If the question is then: What can you say about *c*? You said *c* could be 4, it could be 3, it could be 1, but how do you know which one to write down?	
Agi: You can just, or you can just see the answer 10 and then you can say which number, which numbers can go, which number like that it can go to 10.	
Kona: Ok, yes. So which answer would you write down if it was your test? Would you write *c* is 4 or 3 or 1?	
Agi: I would write any one.	Does not see *c* as a generalised number as she admits to just writing any one answer.
Kona: You would write any of those ones?	
Agi: Ja, as long as it gives me 10.	
Kona: Ok, good. Number 16. Can you explain your thinking? You said L + M + N = L + P + N. And the question is when. Is it always, sometimes or never? You said "always". Can you explain why you said always?	
Agi: Because they always like have to be in the... It depends like if it's a multiplication so you have to, let's say it was 2L + 2M + 2N then you have to say… you just add everything and you write the L, M, N.	
Kona: So why is it "always"? You ticked "always". What made you tick "always"?	

Agi: Because they are letters. Whenever they are there are you have to write them.	
Kona: Ok. So are all the letters equal?	
Agi: No, they're not.	
Kona: They're not equal?	
Agi: Ja	
Kona: They're different?	
Agi: Ja	
Kona: So look at number 17, the last one. It says: Which is larger $2n$ or $n + 2$. And you said $n + 2$.	
Agi: Ja	
Kona: Because there's an invisible one. Can you explain what you meant by that?	Theme of 'invisible 1'
Agi: Like next to the n there's invisible 1, so if you add the n and the + 2 it will give you $3n$.	'Invisible 1' again creates rule of $n + 2 = 3n$.
Kona: Ok	
Agi: And that one's the larger because…	
Kona: Because?	
Agi: Because $2n$… If you say $2n$ then there's the invisible 1, but it's not added to the 2 but I once added (?) to (?)	
Kona: Ok. So you're saying $n + 2 = 2$?	
Agi: 3	
Kona: 3. And then $3n$ is bigger than…	
Agi: $2n$	
Kona: $2n$. So what is your understanding by $2n$? If someone asks you "What does $2n$ mean to you?" what would you say?	
Agi: $2n$, it's only $2n$ and the invisible 1, but it's not	

added. If only there was a plus between both of them then it would be 3*n*.	
Kona: So where's the invisible 1 for 2*n*?	
Agi: Right here.	
Kona: Its 2*n* and then 1.	
Agi: Ja. No… Like 2 1*n*	
Kona: 2 and then 1*n*?	
Agi: Ja	
Kona: So won't that look like 21*n*?	
Agi: It will always if it was visible.	
Kona: If it was visible?	
Agi: Ja	
Kona: But now it's invisible.	
Agi Ja	
Kona: I think we're done.	
End of interview	

Analysis of Transcript of interview for Emma

Kona: Emma, for task number 3 you got your answer as 35, Ok. Can you explain how you got the 35?	
Emma: How I got the 35?	
Kona: Yes	
Emma: It's *m* across to 3n. Or it's like 3 workings about *m* when *m* is 4, so I thought if *m* is 4, and they're both because it's like 3*n*, I thought maybe it's 34 plus 1, then I got 35.	3*m* is 34 if $m = 4$
Kona: Ok, so what do you understand by 3*n*?	
Emma: 3*n* – what do I understand it?	
Kona: What does 3*n* mean?	
Emma: It's like a whole number and I write it together, mixed together.	Issue of mixed together
Kona: Ok. So does it mean maybe $3 + n$?	
Emma: $3 + n$? Well if like it's $3 + n$ and the *n* is equal to 4, like I know that Ok, it's a whole number plus *n* and the number *n* is equal to 4, so I'll add 4 to substitute the *n*.	
Kona: Ok. So is 3*n* $3 + n$?	Does not understand what is 3*n*.
Emma: Sorry?	
Kona: 3*n* – is 3*n* $3 + n$ or $3 – n$ or…. or it's just 3*n*?	
Emma: It's I think 3*n* it's like $n + 3$, I think so, you got 3*n*.	Rule of $n + 3 = 3n$.
Kona: Ok, that's fine. Can you look at number 4? You got 60. Number 4 says: $r = s + t$ and then $r + s + t = 30$. You got your answer as 60. Can you explain how you got the 60?	
Emma: Isn't it the *z*? *r*, *s* plus *t* it equals to 30, so I	random picking

like, I add them like all of them like 30, 30, 30 and I got 60. I add $s + t$ and I got 60. It's $r + t$, I mean it's $s + t$ because you've got… So I… it's like 30 + 30 equals to 60.	
Kona: Ok, Ok. So you're saying $s = t$?	
Emma: Equals to t	
Kona: Because both are 30?	
Emma: Both are 30.	
Kona: So how do you know $s = t$?	
Emma: s is not equal to t because both are 30 and there's a plus sign between them, so if there wasn't a sign maybe it was… the answer was going to be like 30 only if there wasn't a sign between them.	
Kona: Ok, that's fine. Can we look at number 5?	
Emma: Yes, Sir	
Kona: Number 5 it says: $a + b = 43$. 43	
Emma: Yes Sir	
Kona: And then it says: $a + b + 2 = 2$ and you've got $2ab$. Can you explain how you get the $2ab$?	
Emma: How I got the $2ab$ is like I added $a + b$ and the 2, so I got the $2ab$. I added all of them together.	Rule of $a + b + 2 = 2ab$.
Kona: Ok. So number 7, how come you didn't add them all to get $3fg$?	
Emma: Um… *(long pause)* It's like… Or maybe I thought like e and f are 8 so I thought like e is like an individual e, it's 4 plus it's 4. And I thought also a g well if this? I thought also g is 4 and so I got 12.	random picking
Kona: Ok, that's fine. Can we look at number 8? You've got $2a + 5a$ and then your answer you wrote down is $8a$. Can you explain how you're getting the $8a$?	

Emma: It's like I got 7, like 2 + 5 that's 7 and the letter is *a* and *a* has the invisible 1, so the answer is *a* and it has the invisible 1, so that's why I also got 1(?), it can stay(?)	'Invisible 1' creates confusion. $2a + 5a = 7a$ plus the 'invisible 1' for *a* and learner gets solution of $8a$.
Kona: Ok, that's fine. Check over here. Number 9. You've got $2a + 5b + a$ and you've got $8ab$.	
Emma: Ja, this is how I did it. It's like… The same way that I explained it in number 8 because a there is the invisible 1.	
Kona: Ok, so you said $2a$ and $5b$ gives you 7.	
Emma: Yes, $7ab$.	
Kona: $7ab$?	
Emma: Mmm	
Kona: Plus the…	
Emma: 1	
Kona: $1a$ gives you	
Emma: $8ab$.	
Kona: $8ab$. Ok. Can we see number 10. You've got $3ab$.	Rule of $2a + 5b + a = 7ab$.
Emma: Yes	
Kona: But from what you've been telling me shouldn't number 10 your answer there be $4ab$?	
Emma: Oh, ja.	
Kona: Or can you explain your thinking there?	
Emma: It's like… Here it's…isn't it, it's $3a – b$. You minus the *b* and you stay with the $3a$. There it should have been 4(?)?	
Kona: Can you explain it the way you were explaining it?	
Emma: Well the way I was thinking of it it's like I minus the 2 and I was left with $3a$ plus the *b*	random picking

because you cannot have the minus sum, the negative, that's not wanted (?), so I turned it into a positive.	
Kona: Ok. And then number 13, can you explain how you're getting $7n$? Is it the same way?	
Emma: Yes	
Kona: 4 + 3. Is it the same way?	
Emma: Yes, because here it says add.	The word *add* means do something similar to Agi.
Kona: Ok. And then number 16. You said that this is never equal to that. Can you explain why?	
Emma: I think because, Sir, you can see by the letters you can't just say… Let me make an example like *abc* it equals to *zmy*, you can't say that. If it was like maybe M, N, L, I would say sometimes or maybe always if like the letters they were the same, but then not put it like accordingly. But, ja, the letters are not the same.	Poor understanding of variable as she says that different letters cannot be the same. Her intention is *similar* to magnitude being related to order of letters of the alphabet.
Kona: Ok. So what do you understand by the letter?	
Emma: By the letter?	
Kona: Yes, let's take any letter. The letter *l* or the letter *n*. What's your understanding of it?	
Emma: I think that… I think they're just like the letters to substitute the number. Instead of numbers you're writing letters.	But, here there is a fair understanding of letters.
Kona: Ok.	
Emma: Like instead of writing 567 you write those letters and you know that, Ok, each letter substitute a 1, that there's no like number in front of the letter, the letter has 1.	
Kona: Ok. And then can 2 different letters ever have the same value? So you get an *a* being equal to *b*?	
Emma: *a* being equal to *b*? No. *a* being… No	No understanding of variable

Kona: It can't be equal, Ok, it's fine. Let's look at the last one. Which is bigger $2n$ or $n + 2$. You said $n + 2$ because it has an invisible 1. Will you explain your thinking there?	'Invisible 1' again.
Emma: That's because… isn't it I said I'd like to substitute a 1, a 1. So when I mix $n + 2$ it equals to like 3, but $2n$ it stays as $2n$, it doesn't change like it's a space where in a sum you just put $2n$ and here you can change as a 3.	'Invisible 1' causes Emma to view $n + 2$ as 3 hence larger.
Kona: Ok	
end of interview	

Analysis of Transcript of interview for Kate

Kona: Ok, as I said, be calm, be relaxed, feel free to explain yourself. So Kate, let's see. Number 1. You've got $a = 3$. Can you explain your thinking there?	
Kate: $a = 3$ I don't actually understand, so I was just trying. Like you said I was just going to write the test so I was just trying to see if maybe I understood some of the things.	
Kona: Ok. So how did you get the 3? You got your answer as 3. How did you get the 3?	
Kate: I like… I just subtracted.	
Kona: You did?	
Kate: Ja	
Kona: Which numbers did you subtract?	
Kate: It was $a + 5$ equals to 8, so I subtracted 3, ja.	
Kona: Ok. And then for number 2. For number 2 you said $u = 2$. How can you check that answer?	
Kate: Ok, like Mr. Fani said there's always an invisible 1 in front of a number you're working with, I mean there's letters, so I subtracted 1 from 3.	Random picking, even though theres a plus sign Kate subtracts.
	Issue of 'invisible 1' comes in early responses.
Kona: And then you got the 2?	
Kate: Ja	
Kona: But if you had to check your answer, how would you check your answer to see if your answer is correct or incorrect?	
Kate: I don't know, but? If Mr. Fani writes the corrections then maybe I'll try and figure out what did I do wrong.	
Kona: Ok. But in this question you say $u = 2$.	

Kate: Ja	
Kona: How can you check if that is the right answer or the wrong answer for yourself?	
Kate: For myself is because of the bases are not the same.	Incorrect link to prior knowledge as exponents are linked here with limited reasoning.
Kona: Ok	
Kate: So I subtracted.	
Kona: So you subtracted?	
Kate: Ja	
Kona: Ok that's fine. For number 3 if you said $m = 0$, can you explain your thinking there? How did you get $m = 0$?	
Kate: $m = 0$. Ok what I said there was, Ok so I said $3n$ plus 1 so I added 1 to 3 so that makes it 4. And then again I subtracted 4 from 4 so it gave me zero.	Random picking
Kona: Ok. But why did you subtract? Why not add or multiply?	
Kate: Because it says $n = 4$ and the first answer has an addition sign so I added the first one and then when I got to the second answer which was equals to 5, I subtracted 4 from 4, so I got nothing.	Reasoning does not make sense.
Kona: Ok, that's fine, can we look at number 4? In number 4 you said $r + s + t = 15$.	
Kate: Ja	
Kona: But the question says "What can you say about r if $r = s + t$ and $r + s + t = 30$". Ok, so what is r?	
Kate: I don't know what's r but what I thought was that I had to like divide, ja. So I got 15 which is half of 30.	
Kona: So are you saying that $r = 15$?	
Kate: Ja, ja.	

Kona: When you said "divide", how did you know you must divide and what must you divide by?	
Kate: I just saw the letter sir; I didn't actually understand what was going on so I just divided by 2.	Random picking of the operation.
Kona: Ok. So why divide by 2? Why not divide by 3 or 4?	Just picking any operation.
Kate: Because that's like two (?) of the easiest (?) numbers we use.	
Kona: Ok, that makes sense. Can we check number 6? For number 6 your answer was 513/530(?). Can you explain how you got the 513/530(?)?	
Kate: What I did was I, how can I call it, I swopped the… Ok, let's say 246 never belonged there like where it is right now. So I crossed it over and I added. So when I crossed it over it was going to automatically change to an addition sign so I added the two so which makes it m = 513/530(?).	Simple arithmetic not understood.
	Kate does any operation with no 'symbol sense'.
Kona: If you just add the two?	
Kate: Ja. I'm not sure what I did; Sir, but I don't remember how I got both of them, but two.	
Kona: So can I check with you? You said that the 246 does not belong there. What did you mean by that "does not belong there"?	
Kate: Like on the other side is 262 so what I actually did was that I put the numbers together and I left the letters alone.	Not wanting to work with letters.
Kona: ?	
Kate: Ja, so that's what I did. I worked out only the numbers and I left m alone.	
Kona: And that's how you got 513/530(?)?	
Kate: Ja	
Kona: Ok, that's fine. Can we check number 7? Your answer for number 7 was 9. For $e + f + g$ you got 9.	

How did you get the 9?	
Kate: Ok, so if the $e + f = 8$ so the second one says $e + f + g$ so in front of the g there was an invisible 1 so I added the 1 to the 8 which makes it 9.	Classic case of the 'invisible 1'. $8 + g = 9$.
	There are misconceptions in prior learning.
Kona: Which makes it 9?	
Kate: Ja	
Kona: Ok. Can we turn over? Let's look at number 9. Can you explain your thinking for number 9? Your answer was $3a + 5b$.	
Kate: Ok, well actually what I did there was I added $2a + a$ which has got an invisible 1 which makes it $3a + 5b$ and I left it like that.	Uses the 'invisible 1' correctly here.
Kona: Ok that's fine. And then number 12. Can we look at number 12? We said add 4 to $m + 5$ and you got $4m$. How did you get the $10m$?	
Kate: $10m$? Well I said $4 + 5$, right is $9 + m$ which has got an invisible 1 which equals to 10.	'Invisible 1'; $4 + m + 5 = 4 + 5 + 1$. the 1 is the 'invisible 1' for the m.
Kona: $10m$?	
Kate: Ja	
Kona: Can we maybe look at number 12, Ok, and number 9. Number 12 and number 9 are very similar sums.	
Kate: Ja	
Kona: But you've done them differently. It seems like you've got different rules for number 9 and number 12.	
Kate: Number 12 I actually left… It's like, how can I put it, ja, it's got different rules but it's done the same. Here it's like a is alone so they say every time when the letter is alone it's got an invisible 1 and it also has the same thing but I can't really explain it. *(long pause)* It's sort of like I did the same thing, but? The rules (?) part.	Admits that there are different rules.

Kona: You see the difference in your answers in number 9 and number 12. In number 9 you didn't join $3a + 5b$. You could have joined it to make it $8ab$, but you didn't. In number 12 you joined everything and made it $10a$. So why in one you didn't join and in the other one you joined it? Did you have any reasons for that?	
Kate: Not really, because sometimes in class when we work out sums like this we sometimes do them like this or sometimes we do continue, do not continue the sum, carry on, so I decided to leave it like this.	Prior knowledge, confusion of algebraic rules.
Kona: For number 9 you said it's fine like that?	
Kate: Ja	
Kona: But for number 12 you said you wanted to carry on?	
Kate: Ja, because I was a bit confused so I just added everything together.	Learner is confused about the rules.
Kona: Ok that's fine. We'll look at number 14. Number 14 you got $21n$. Can you explain your thinking? How did you get $21n$?	
Kate: 21. I think I multiplied.	
Kona: ?	
Kate: Which was 5 times (?) 4 which makes it 20, ja, which makes it 20. So I added n which is invisible number 1 which makes it 21.	'Invisible 1'; $5 \times 4 + n = 20 + 1n = 21$.
Kona: Ok. So why did you add the n?	
Kate: Why do I add the n? Because you say like it's got an invisible 1 so I added 20 + 1.	
Kona: To give you 21?	
Kate: Ja	
Kona: Ok, that's fine. When we look at number 16 you've got $L + M + N = L + P + N$ and you said they're never equal. Can you explain what you were thinking?	

Kate: There I was thinking that the letters are not the same so you just cannot add them, so I got (?) them out.	There is some understanding here.
Kona: ?	
Kate: Ja, that's what I was thinking.	
Kona: That the letters are not the same?	
Kate: Ja	
Kona: So you cannot add them?	
Kate: Ja	
Kona: Ok. If you knew the values of the letters, if you knew L was 2 and M was a certain number and N was a certain number, then would your answer still be "never"?	
Kate: Ja, I think so.	
Kona: Why?	
Kate: Because they said like if the bases are not the same you cannot add the powers or work out the powers because like now L is just going to be, I'm still gonna leave L like that if L is equal to ? I'm just gonna leave L + M + N just like that.	Relating to most recent learning of exponents.
	There's no 'symbol sense' through the entire transcript.
Kona: Ok. So if I told you L is 2	
Kate: Ja	
Kona: Ok. And *n*, *n* is 3 – you with me?	
Kate: Ja	
Kona: So on the right hand side of the equals to sign, L is going to be 2 and N is going to be 3.	
Kate: Ja	
Kona: Now you know what L is and you know what N is but you don't know M and P, would you still say it's "never"?	

Kate: Actually I'm going to put L in because it's L.	
Kona: For?	
Kate: For M and P.	
Kona: For M and P?	
Kate: Ja	
Kona: Ok. Then what would your answer be?	
Kate: I'm gonna add… I'm gonna add L + N, L+ L which makes it L to the 4(?) and then I'm going to make N and N which is going to be N to the 6 and then I was going to leave M and P because they both have the number?	Doing anything with no 'symbol sense' and random picking.
Kona: Ok. So would your answer be always equal, be sometimes equal or be never equal?	
Kate: It's gonna be sometimes equal.	
Kona: It'll be sometimes equal?	
Kate: Ja	
Kona: But then it says "when"?	
Kate: When like the bases are the same.	Relating prior learning of exponents incorrectly again.
Kona: The bases are the same? Ok. Can we look at number 17, the last one? It says, "Which is larger $2n$ or $n + 2$" and you said $n + 2$ because it's got an invisible 1. Can you explain that to me?	Similar reasoning by other learners. $n + 2$ is larger due to 'invisible 1'. $n + 2 = 3$ which is larger than $2n$ which is 2.
Kate: I actually write it $n + 2$ because it was n plus the invisible 1 $n + 2$ + the invisible 1 is 3, ja so that makes it larger.	
Kona: So 3 is larger than?	
Kate: $2n$	
Kona: ? . Ok. And then I just want to ask you also if we turn over, for Question 7 there was no g, Kate in your answer. You see you've just got your answer as	$8 + g = 9$

9. What happened to the *g* there?	
Kate: I think I forgot to write it like next to the 9.	
Kona: So should the *g* go down?	
Kate: Ja, I think I should write 9*g*.	
Kona: 9*g*?	
Kate: Ja	
Kona: Will that be your final answer – 9*g*?	
Kate: Ja, I'll make it my final answer.	Kate has got the ability to control and decide on her final answer.
Kona: Ok that's fine. Thank you.	
End of interview.	

Analysis of Transcript of interview for Laizal

Kona	Ok. Laizal. Can we look at task number 3? Task 3 says what can you say about *m* if *m* = 3*n* + 1 and *n* = 4. How did you get your answer? Can you explain your thinking there for task 3?	
Laizal	Firstly, I thought cos you said eh 3*n* but so here they said *n* = 4. so I decided that [coughs, sorry] that *n* is 4 cos *n* = 4 and then ja and I don't know how did I get this *m* cos they said *m* = but all I thought is that *m* = 3*n* + 1 so 3*n* + 1 is 8*m* = 4 so 3*m* that *m* I put in the bracket cos that *m* is 4 so that's how I got my answer.	Is able to substitute 3(4) + 1. Laizal understands that 3*n* is 3 x *n*.
Kona	Ok. Is that your final answer?	
Laizal	No. I have to work it more but I thought you just want to work it out there and here.	There is some 'symbol sense', some friendliness with symbols and manipulations, can also debate that other components of 'symbol sense' are present.
Kona	Ok. So what would your final answer be?	
Laizal	mm. I was gonna say eh. 3 x 4 in the bracket and then 16 right, 3 x 4, ja, 16 [pause]	
Kona	3 x 4?	
Laizal	12, 12	
Kona	Ok. 12	
Laizal	So then I was going to 12+ 1 is 13. That's my answer.	
Kona	Ok. Perfect. That's correct. That's exactly the. That's the correct answer. Can we look at number 4? Ok. Once again what you have written in number 4 I thought was very interesting. You've got *r* + *s* + *t* over 30 equals to 1. Can you explain your thinking there again? How did you get that?	
Laizal	Ok. Ok. Eh. What can you say about *r* if *r* = *s* + *t*. I decided because they said *r* + *s* and *t*. I decided to divide it because, I don't know. I just, I just. Its just out of my mind but I just thought that maybe this is gonna be a right answer if I divide all those by 30 because they didn't give us the numbers of *r, s* and *t*.	
Kona	[pause] Ok. So, if we had to go back and answer the question. The question says what can you say about *r*? What would you say about *r*?	

Laizal	[pause] em. What does this question mean? Like what could I say about *r*?	
Kona	[pause] if the question was what can I say about Wits High School? I would say it's beautiful, it's clean, and it's in the city centre. In other words, I'll be describing Wits high School.	
Laizal	Ok. But I think. What I thought is that this *r* was like. In my mind I thought that this *r* is there because maybe there is some number. It's just that you gave us a little bit of a sum.	Understands that *r* is a number but also sees the need to calculate something.
Kona	Ok. So when you say numbers. What do you mean by numbers?	
Laizal	Like there is. This sum maybe its longer. It's just that in the sum you gave us that small part of the sum.	
Kona	Ok. So when you say longer on which side of the sum will it be?	
Laizal	Which side?	
Kona	Like the equals to sign is there ok. $r = s + t$. on which side is the sum? Is it on the left, on the right, on the top?	
Laizal	Is. On the. Is on the. Is on the left.	
Kona	Is on the left hand side.	
Laizal	Yes. The r +. I mean the $s + t$.	
Kona	Ok. If you had to put numbers on your own. You did speak about numbers. What numbers could you put for *r*, *s*, and *t*?	
Laizal	Could put eh, eh 10 for *r* and then for *s* I could put 15 and 15.	
Kona	So if you'll add it you'll get?	
Laizal	30. No I mean, I wanted to say it's gonna be 30 and then I would need a 15, *s* and other 5. Eh. And then when I add it all [I get 30 and 30 divided by 30].	
Kona	Ok, so you need to get 30? [Inaudible] can we try number 5. For number 5 you wrote your answer as 45. How did you get the 45?	
Laizal	I thought because they said $a + b$ is 43 and then we had to maybe this. There is somewhere that made me understand. [Pause] ja, I thought maybe because this was maybe this was the, was the numbers that give us 43. So I took the 43 plus 2 so that's the 45.	There's is 'symbol sense' here. Most of the six components of *symbol sense* can be seen here.
Kona	Ok. That's correct. That's fine. Number 7. Can you explain in number 7 how you get the 8? What I found interesting was: you wrote the 8 on the far left. Why did you	Learner was not certain with the answer of 8.

	write it so far? The 8, and not in the middle of the line, like if you look at your answer for 7*a* you wrote your answer in the middle. What was your reason for writing your answer so far to the left?	
Laizal	I don't know. Maybe, I thought, I don't know, maybe it's because I just wrote it there the 8 or maybe I had another answer in my mind.	
Kona	Ok. Ok. Fine. Can you explain your thinking in number 7? How you getting the 8?	
Laizal	For number 7?	
Kona	Yes	
Laizal	Ok. Isn't it they said here $e+f$ is 8 so then in my thinking I thought maybe this answer is it stays the same because $e+f+g$ maybe the g was an invisible number or something cos $e+f$ is 8.	Issue of the 'invisible 1'. Laizal does not see g as a variable.
Kona	Ok.	
Laizal	Then I thought that maybe this is an invisible number so you can't just say you cannot just say so I just decided to write the 8 there.	
Kona	Ok. [Pause] so you drop of the g because you taking the g as being an invisible number? What do you mean by invisible number?	
Laizal	I mean like for instance, like here you said $e+f=8$ so here $e+f$, this $e+f=8$ now this g because it came there I just took it as an invisible, invisible alphabet there like, we don't know the answer there so we needed to find the answer so I just decided to write the 8 until maybe if this sum was another sum then I was gonna get the g what is the g.	
Kona	So if I told you [pause] that g was 4 then what would $e+f+g$ equal to?	
Laizal	12	If probed she is able to find $e+f+g$.
Kona	12, yes that's correct. So what are you saying g equals to in the answer? If you saying $e+f+g=8$ what are you saying g equals to there?	
Laizal	If my answer is written 8 and [pause] the g would be, like 0.	
Kona	The g would be 0.	
Laizal	0	
Kona	Yes. Now, can you just assume the g is 0?	

Laizal	No.	
Kona	[Pause] Because it could be any number.	Not sure if she sees this or if I funnelled the answer.
Laizal	Any number.	
Kona	But you taking the g as being 0. So if you had to re write your answer what would you write it as?	
Laizal	What do you mean rewrite my answer?	
Kona	If you had to say that 8 is no longer your answer what would your new answer be for $e + f + g$?	
Laizal	I don't know, maybe it was gonna be another answer. Maybe I would put that g as another number.	Is aware that g is any number.
Kona	Ok. Then how would you know what number that g is?	
Laizal	You know, I think maybe this g you had to guess what is the answer because you just gave us the alphabet and then you said here $e + f = 8$ and then we had to guess what is $e + f$ is and then here too maybe its $4 + 4$ and $+ 0$, that's what I thought. Now if maybe I could make up answer I was gonna say maybe $4 + 4 + 2$ or $4 + 4 + 1$.	Random picking of numbers for g although there is some 'symbol sense'.
Kona	So the question in my mind is how do you decide what number g must be? So you agree that you took that, you've taken g to be 0, then you say maybe g could be 4 or maybe g could be 2, but how do you know what number g must be. Must it be 0 or 2 or 4?	
Laizal	I don't know. I think it's only by guessing.	Laizal does not comprehend that g is a variable. She is taking g as an unknown that must be one number only.
Kona	Ok. That's fine. Can we try the next page Laizal? Let us look at example 12. Ok. In example 12 you said add 4 to $n + 5$ your answer is $n + 9$. Are you with me?	
Laizal	Yes	
Kona	Now, can we compare that task 12? Ok, you said your answer is $n + 9$. Can you compare that answer in the one we just discussed, the one with the g. Can you find the difference, or the link or the similarity with the two?	
Laizal	The, the difference is that this one they said add 4 to $n + 9$, isn't it you add when the when the like the alphabets are the same you can add n + maybe if there was another	She would see $e + f = 8$ but because she was not told to add she could not get $8 + g$.

	number so here I said $4 + 5 = 9$ now this n has no eh like another number that so you can add it together. I think difference is that eh here they gave us the numbers and there they gave us only alphabets and the answer.	
Kona	So, could you have written down your answer for the g sum as your answer is $8 + g$?	
Laizal	Yes	
Kona	See your answer here is $n + 9$. Could you have written down your answer there as $8 + g$ and leave it as that.	She's able to get $n + 9$ easily because the question says add.
Laizal	Ja, I think so [pause]	
Kona	Why?	
Laizal	Because I think, because eh $e + f$ is 8, they gave us that. And, now g, we don't have an answer like here we don't have an answer for n so I would say $8 + g$, so that I could find out my own g. I don't know how was I gonna find out my own g but I was just like here I was just gonna do it $8 + g$.	
Kona	And leave it as that?	
Laizal	Yes	
Kona	Ok that's fine. Can we look at number 13? It says add 4 to $3n$. And you got $4 + 3n$. Can you explain your thinking there?	
Laizal	Eh. I thought. I always have this thing. I dunno if this sum is right or wrong but I always have this thing that 4 you cannot add it to $3n$ cos 4 doesn't have an alphabet but now it came on my mind, maybe that when they say add 4 to $3n$ they meant like add 4 to that $3n$ so that it can be $6n$, $7n$, is fine.	There is 'symbol sense' here. Understands that $4 + 3n = 4 + 3n$ but changes her mind later.
Kona	Yes. You did write down $7n$. can you see the $7n$	
Laizal	Ja	
Kona	You struck it off. You thought maybe it could be $7n$ but then you didn't take that as your answer. Can you explain it again to me?	
Laizal	Explain eh?	
Kona	Why you didn't take $7n$? You see you wrote $7n$ down and then in your mind you thought no I don't want $7n$ and then you wrote your answer as $4 + 3n$.	
Laizal	Ja. I thought isn't it when we like when Mr Fani shows that we don't have to add like if maybe it was $4 + 3n + 6 + 4n$ so we would add like [4] 3 + that $4n$ then 4 + that 5 and	She has two options and she chose not to combine $4 + 3n$.

	then just write the answers like that. So I thought that maybe cos when you you multiply that's when you can, you can add [pause]. Ja. So I decided that ok maybe this $4n$ I could just leave like that 4. I mean, I could just leave it like that and don't add it to the $3n$. I only had two answers that maybe you can add it or you cannot add it.	
Kona	So what's your final answer? Can you add it or can you not add it?	
Laizal	I think you can add it	She changes her mind here.
Kona	You can add it [pause] to get $7n$.	
Laizal	Yes	
Kona	But when you wrote the test you thought that you couldn't add it?	
Laizal	So I just. Ja. So I just wrote my final answer as $4 + 3n$.	
Kona	Ok. Fine. Can we try another one? Can you look at number 14? Ok. Number 14 said multiply $n + 5$ by 4. And you got $20 + n$. Can you explain your thinking there? How did you get $20 + n$?	
Laizal	Oh, I said eh, they said multiply. I thought that maybe I just said 5 x 4 is 20 and the n missed out here cos I had to say 5 x 4 is 20 plus n x 4 is $4n$.	Just made an error.
Kona	Ok. So what would your final answer be?	
Laizal	Its gonna be $20 + 4n$	Correct answer here.
Kona	Ok, fine, can we look at number 15? Can you explain your thinking there for number 15? What can you say about c if $c + d = 10$ and c is less than d?	
Laizal	[mumbles] I don't know how did I get this answer. [Pause] Oh, but I thought when say $c < d$. So maybe the the alphabet c will be a less like here for example 4, I wrote 4 + 5 [pause]. Oh I wanted to write 4 + 6 because they said $c < d$ so $c < d$. The number c has to be less than the number d.	
Kona	Ok. c is 4?	
Laizal	Ja	
Kona	And then 4 + 6 will give you 10?	
Laizal	10	Is able to see that c could be many values but she wants one answer and not a generalised answer.
Kona	Ok [pause] so could c be any other value?	
Laizal	That is less than d.	
Kona	From what is written here, can c be any	

	other value?	
Laizal	Yes [but as long as its < *d*]	
Kona	So what other values could *c* be?	
Laizal	Could be 2 and 1.	
Kona	Then what must *d* be?	
Laizal	*d* if its 1 it's gonna be 9, if its 2 it's gonna be 8, or it's gonna be 7.	
Kona	Ok. Fine. So, how do you know which numbers you must take for *c*? Because you said *c* could be 4, it could be 3, it could be 2, it could be 1. How do you know which number you must take for *c*?	
Laizal	I think I should take any because as long as its less than *d* cos they said *c* has to be less than *d*. So I think its any number between those numbers that has to be less than *d*.	She sees the generalised number.
Kona	Ok. So, if you had to state one sentence for *c* what would it be? Remember the question says what can you say about *c*? If you had to say or state one sentence for *c* what would that sentence be?	
Laizal	Eh. I could say is any value less than *d* any value less than *d*, any number that is less than *d*.	Funnelling.
Kona	Ok, which is correct. But if *c* is 1 it means *d* must be?	
Laizal	9	
Kona	9, if *d* is 9 you saying *c* can be any number less than 9?	
Laizal	9	
Kona	So it means *c* could be 8?	
Laizal	No, no, ok, ja. I think when when when I said less than *d* the numbers must be added that could make 10. So if 9 if *d* is 9 the number has to be 1, so that they could make up 10.	After funnelling Laizal is able to see *c* as a variable although not for all cases/variations.
Kona	Yes. So how can you restate your sentence about *c*?	
Laizal	Ok. I could say any number no not any number. I don't know. I could say a number, a number less than *d* that could make up 10. Any number less than *d* that could make up 10 that could be added to *d* to make up 10.	
Kona	Ok, makes sense, makes sense. Can we look at number 16? Ok. Number 16 says L + M +N = L+ P + N. Is it always equal, is it sometimes equal, is it never equal? You said it's never equal. Why did you say that?	
Laizal	Because there is M and then there they said	Letter *used as unknown* is not

	it is equal to P. It's not equal because L + M + N and then there they said L + P. Now the P and M is not the same thing so it's ever gonna be equal.	fully understood.
Kona	Ok. Can M and P ever be the same?	
Laizal	No	
Kona	They can never be the same, why do you say that?	
Laizal	Because they both different alphabets that aren't the same.	
Kona	So different alphabets can never be the same?	
Laizal	Maybe they can when like eh if maybe it was a sum maybe they can because something in alphabets there's some hidden numbers there like always a 1. But here I think it's not cos they just said L + M and here L + P and it's not equal.	Issue of 'invisible 1' is in Laizal's mind without understanding.
Kona	So if I told you M = 3 and P = 3 then which one would you select always, sometimes or never?	
Laizal	Always	
Kona	Then it always will be equal? Ok. Can we look at number 17? It says which is larger $2n$ or $n + 2$ and you said $2n$ because $2n$ is when you multiply and $n + 2$ is when you add. Can you explain your thinking there?	
Laizal	Ja. I thought maybe $2n$ maybe that n it's like they gave us n as a number there. For instance I'm just making an example maybe they gave us 2 for that n and then I was gonna put that n in a bracket 2 so when you multiply its 4. But here ok. Ja. It's the same thing here [laughs] 2 + 2 is 4. Like maybe [inaudible] ja, maybe let me go to the bigger number, they gave us 6. I was gonna say 2 x 6 is 12 and then here they gave us 6 again 6 + 2 is 8. So I think $2n$ is larger than $n + 2$.	Here Laizal can see that n is any unknown. Very nice usage of letter as an unknown.
Kona	Ok. But when you put 2 it doesn't work out to be larger.	
Laizal	It's the same with 2.	
Kona	0 if you put 0?	
Laizal	0 its 0 they both. No that's when its gonna be 2. This one is gonna be larger and this one is smaller.	
Kona	So can you rethink your answer?	
Laizal	I think they same	She sees that her reasoning does not work for all cases and changes her mind.

Kona	They are the same, [pause] fine I think we'll stop here.	
	end of interview	

Analysis of Transcript of interview for Lee

Kona	Here's your test. Can we look at number 2? It says, what can you say about *u* if $u = v + 3$ and $v = 1$. You said *u* is 3. Can you explain how you getting the 3?	
Lee	Here sir, what I've done is is I thought of the number a number that isn't it a letter isn't it they always say eh a letter is always with eh eh invisible 1?	'invisible 1'
Kona	Yes	
Lee	So, eh, I multiply this each letter by 3. I said 3 times, ah, *u* x 3. Then I got a 3. Then I also said *v*. I made a *v* as like as like a letter I said $v + 3$ and then it gave me a 3. So that's when I thought of writing a 3 down.	Lack of 'symbol sense' 3 x $u = 3$ and $v + 3 = 3$. In 1 response Lee changes his rules.
Kona	Ok, fine. Can we look at another one? Can we look at number 3? Ok, you said that the question is, what can you say about *n* if $m = 3n + 1$ and $n = 4$? You said its $4nm$.	
Lee	Ja, here sir I said 3. I plussed, added 3 + 1 and then I added this *m*, $m + n = mn$, I added this 3 + 1. That gave me $4mn$.	$m + n = mn$ $3 + 1 + mn = 4mn$
Kona	Ok. And why did you do that? What was the reason for doing that?	
Lee	For adding these numbers?	
Kona	Mm.	
Lee	Ja sir, the reason was that, was that, eh what Mr Fani taught us is you add the numbers. You add the numbers to get the exponents so I added this because because I saw that here down its written *n*, *n* equals 4 which they made here which was 3 + 4, 3 + 1 that gave them $n = 4$.	random picking Has been learning exponents a x $a = a^{1+1}$ is used arbitrarily. Also, $m = 3n + 1$ is interpreted as $3 + 1 = 4$ therefore $n = 4$.
Kona	Ok. Fine. Can we look at number 4? How do you get the $30rst$?	
Lee	Almost the same as that one. I, also added this the *r*, *s*, + the *t*. I eh, I, I, wrote this 30 down then I added this all three, all 3 letters to get [that].	Random picking; from $r + s + t = 30$, answer became $30rst$.
Kona	Ok. So you saying $r + s + t = rst$?	Rule of $r + s + t = rst$.
Lee	Yes, sir.	
Kona	Ok, fine. Then number 5? It means you've done number 5 the same way.	
Lee	Yes, sir	
Kona	You said $a +$. Can you explain it?	
Lee	It's $a = b + 2 = 2ab$.	Same rule as before, $a + b + 2 = 2ab$.
Kona	Ok fine. And then number 7. How do you get the 12? For $e + f + g$?	Changes rule here, now $e + f = 8$, Lee takes all

		numbers' coefficients + 8 to get 13.
Lee	Ok, number 8, number 7. I [pause] this is as like saying I added each, each, each letter. I gave it a 1. I like eh I, I, I, ja, I added the invisible 1 to get the 12. I added the invisible 1 to to to the 8.	
Kona	Ok, but where, how many invisible ones do you have?	
Lee	Its 5, its 5 sir.	
Kona	So that's five	
Lee	Which gives us 13?	
Kona	So your answer should have been 13?	
Lee	Yes	
Kona	Ok, but can I ask you? You see for number 5 you said $a + b + 2 = 2ab$. How come for number 7 you didn't say $e + f + g = efg$?	No consistencies with rules, applies any rule anywhere just to get an answer.
Lee	[pause] Hey I think that's my mistake also. It's a mistake that I have done.	
Kona	Ok. Number 8. Can you explain how you getting the $7a^2$?	
Lee	Ja number 8. They say simplify so I added these 2 numbers together. I said 2 + 5 which gives us 7 then I, I added these 2, 2 exponents to get to get a^2.	The rule of $2a + 5a = 7a^2$.
Kona	Ok, that's fine. Let's look at another one. Can you explain your thinking for number 9? How you getting that answer?	
Lee	I almost done the same thing as, as number 8 but I, I also added 5 + 2 which gives, which gives 7 then $a + a$ which gives us a^2 and then eh 5 aah and then ab so since there's no b I just wrote the b alone and then I wrote the a^2 .	Process of simplification: $2a + 5b + a$: $2 + 5 = 7$ $a + a = a^2$ (maintains this rule here) $7a^2b$. Lee wrote the b alone because there was only $1b$.
Kona	Ok, number 10?	
Lee	Number 10. It's the same thing that I've done with number 9. I also wrote the 3 down because there's no other number that I can add the 3 with so I I said $3a$ plus the a here. It gave me $3a^2$ + + I I wrote this b down on the [last].	Rule to get $3a^2b$ same as $7a^2b$ above.
Kona	Ok. And then number 13? How do you get $7n$?	
Lee	Number 13, number 13. I said 3 3 + ah 4 + 3 it was 7 I also wrote the n. Since, since there was a $3n$ I couldn't leave it alone so I had to I had to add to get to get the $7n$.	Rule of $4 + 3n = 7n$.
Kona	Ok, fine. Can we you check number 16? For number 16 you saying that L + M + N is always = L + P + N. Why did you say always?	

lee	I think, I think, I, I, I made a mistake here. First I didn't understand it very well. So here I wrote this and I think I rubbed them off because I didn't have a tippex. So I ticked this sometimes. I ticked sometimes. So this [mumbles] L, M + N is not always LPN, its not always L + P + N, its not always, sometimes.	
Kona	Ok, when?	Lee changes his answer now to sometimes.
Lee	When the, when the, ja, when I think, I think, when this number M here is used so so we replace with P. I'm really not sure.	However, Lee has no idea why he changes his answer to sometimes.
Kona	So when will those 2 be equal? When will L + P + N be equal to L + M + N? [pause] because you saying sometimes it will be equal. So the question would be is when, when are they equal?	
Lee	These numbers? These the letters?	
Kona	Yes, ok, you see there's a left hand side and a right hand side. When will the left hand side be equal to the right hand side?	
Lee	This will never be equal cos the number it will never be equal. [mumbles] This will always stay stay smaller than these ones this 2 L and L they will always be equal and N and N will always be equal but P and M will never be equal. Because since, since M since M comes before P so there's no way they can be equal.	Contradicts himself similar to Laizal and other learners, he says P and M can never be equal.
Kona	What do you mean before?	
Lee	Isn't it M in letters we count M, it comes first and then P follows. Ja, so they'll never be equal in that way.	
Kona	So are you saying one is bigger?	
Lee	Ja. One is bigger and one is lesser.	
Kona	Which would be bigger and which would be lesser?	
Lee	Eh, I think P would be bigger, ja, I think P would be bigger than. No, M would be bigger cos it comes before P.	M is bigger because M comes before P in the order of letters of the alphabet. How is this different to Nelli?
Kona	Are you sure?	
Lee	Yes	
Kona	Ok, the last one, number 17. Which is larger $2n$ or $n + 2$? And you said $2n$? Can you explain why $2n$?	
Lee	It's $2n$ because I think it begins with a number. Ja, I think it begins with a number that's why its $2n$.	
Kona	[pause] So what is your understanding of $2n$?	

Lee	Eh, $2n$, sir is, is, I think you first put a number before a letter so my understanding is that eh $n2$ cannot be bigger than cannot be bigger than this number here.	Lee is lacking 'symbol sense' here and through the whole interview.
Kona	And why, why can't be bigger?	
Lee	Can't be bigger cos this this it begins with a b = number and this it begins with a letter.	
Kona	[pause] And what about the invisible 1?	
Lee	Invisible 1 of n?	
Kona	Yes	
Lee	They might be equal cos they both have have invisible ones.	
Kona	Where's the invisible one for $2n$?	
Lee	Its here by n after n.	
Kona	Ok, so for the last time Lee. Can you tell me which is bigger?	
Lee	Which of these 2?	
Kona	Ja, $2n$ or $n + 2$?	
Lee	[pause] Hi sir, I, I, still think its $2n$.	
Kona	$2n$, and your reason for that?	
Lee	It's because this $2n$ it has an invisible 1 and this and, hi no sir. I think they are equal. Ja, they are equal sir cos this they both have invisible ones and and they stick to the same positions but just that they've swopped the the numbers they put the letter first the other one then the other one they put the letter second. So I think they equal.	Lee has no clue as to which is larger due to him not understanding the numbers $2n$ and $n + 2$.
Kona	Ok, thanks.	
	end of interview	

Analysis of Transcript of interview for Nelli

Kona	Can we look at task number 2? It says: What can you say about *u* if $u = v + 3$ and $v = 1$. And then you wrote down $u = x$, which $= v + 3$. Can you explain your thinking there?	
Nelli	Basically we just simplifying here because eh I hardly understand, understood the exponents and whatever we doing here, so I just simplified this question [learner mumbles].	
Kona	Ok, so what's, the question says; what can you say about *u*?	
Nelli	Oh, I thought we were supposed to like find the value of *x* or something like that, so and see what makes the value of *x* or the value of *u*. So I thought of writing the value *x* as $v + 3$.	Nelli substitutes *x* for *u* and says $x = u = v + 3$. She did not understand that *u* needed to be calculated.
Kona	So u is $v + 3$. Ok. That's fine. Can we look at number 3? Number 3: can you explain your method that you used there? You got to $n = 6$. How do you get the 6?	
Nelli	Well, I wanted a number that will get the answer to 10. So I just said because there was $3a + 1$ so which makes it 4. I wanted a number because *n* is after *m* so I thought if I could find the number that is before *n*, that would becomes, that would make it 10, so I thought of 6.	Learner wants the answer to be 10. Its random picking as learner equates variables to any number, in this case 10.
Kona	Ok. Where do you get the 10 from?	Rule of $3a + 1 = 4$.
Nelli	The 10 was just that I'm writing an answer as a 10. I just added to the numbers so that's why I write it as a 10.	
Kona	Ok. So, could you say any other number/ could you say 20?	
Nelli	I could say 20. I could say 30. But I just have decided on 10. The smallest number I could get.	
Kona	You decided on 10.	
Nelli	[Yes]	
Kona	Ok. So, can you explain it a little further? The method you using?	
Nelli	Eh. Basically because *m* comes first then I said just said because *n* is already = 4, *n* already has a number, so I just said I'll take, I'll get a number for *m*. then I decided to use 6 and then I just added the numbers and made it 10.	
Kona	Ok. What do you mean when you say *m* comes before *n*?	
Nelli	Eh. In the alphabetic way. I just thought of	Order of letters of the

	because *n* already had a number so instead of using another number like 7 or something I just used 6 because I wanted to find a number for *m* because *n* comes after it.	alphabet is related to size. Similar to Lee.
Kona	Ok. Good. So which is bigger, *m* or *n*?	
Nelli	[*m*] [I believe it's bigger]	
Kona	*m*, why do you say *m*?	
Nelli	Because it comes first than *n*.	
Kona	It comes first, so it means *a* will be the biggest?	
Nelli	Of them all.	
Kona	Of them all, so which is the smallest?	
Nelli	*z*	
Kona	*z* will be the smallest, ok. And then the 10…..? Ok. That's fine. Let's look at number 4. Number 4 again. Can you explain your thinking there, Nelli?	
Nelli	Eh. Well in number 4, just because I saw the 15, I wanted to find two numbers that will make another 15; so that I can make the 30 = 15 + 15 is 30. So I just used 10 and 5. [So I worked it to get 15].	No 'symbol sense' here. Learner is assigning any number to any variable with no reasoning.
Kona	So you get 30?	
Nelli	Yes	
Kona	Ok. So it means *r* must be 15? Now, can you use any other numbers there?	Nelli gets correct answer and there is some reasoning here, this is similar to Nesher (1987).
Nelli	You, I think you can……..pause….. ja I think 15 and 10 is suitable.	
Kona	15 and 10?	
Nelli	I mean 10 and 15	
Kona	So could *r*, you said *r* = 15?	
Nelli	Yes.	
Kona	Could *r* possibly be equal to 20?	
Nelli	It could be 20 and then two will be 5. No. it wouldn't work out as you saw. I'd prefer 15. I'd prefer 10 and 15.	
Kona	Why, why won't it work out with 20?	
Nelli	Cos if eh I say maybe eh *s* = 20, right, then I'll say *t* = 5, it will make a number bigger than 30. and I don't want a number that will exceed 30. Just wanted a number that will be lesser or get it exactly.	
Kona	Ok. So that's *s*, *s* cannot be 20. What about *r*? Could *r* be 20?	
Nelli	No, it couldn't.	
Kona	Why?	
Nelli	Cos if it's if *r* is 20, and then maybe I'll say my *s* is 5 it will still be 25. But then I was looking	Again, no 'symbol sense' here. Nelli is assigning any

	for a number that will make 15. [Numbers doesn't make 15].	number to any variable.
Kona	Ok. That's good. Can we look at number 7? For number 7 you said $+f+g$ =16. How do you get the 16?	
Nelli	Eh. I just said eh cos $e+f$. No I said, I think I just added. I just put, I just gave the numbers, the alphabets numbers. [I just gave the numbers].	
Kona	Ok. When you looked at number 7, in your mind something told you something is not right there. I could sense it from your reaction. If you had to change your answer what would your new answer be?	
Nelli	Maybe 20 [I think it's a suitable answer].	
Kona	20?	
Nelli	Cos I was just simplifying.	
Kona	Ok. What do you mean by suitable answer?	
Nelli	Eh. Maybe I could add the numbers or if I had to work it out cos I just didn't work it out. It just confused me. So I just write, I just wrote done any answer.	
Kona	Ok. Can we look at number 8? Can you explain how you getting that answer? The answer you got was $7a^2$.	
Nelli	Well I just, cos its $2a + 1a$. I just added the 2+5 which makes 7 then the $2a$'s which makes a^2.	Misconception/rule of $a + a = a^2$.
Kona	Which makes a^2? So what does, what does $7a^2$ mean to you?	
Nelli	It's just the answer that I got, nothing much, just an answer.	
Kona	So, that's your answer. Ok, but what's the difference between7, 7, $7a$, $7a^2$? Is there any difference between the three numbers 7, $7a$, $7a^2$?	
Nelli	I think there is a difference cos the other 7 doesn't have an a neither an a^2 and the other one doesn't have a squared a. If there were like two numbers you had to add like the $7a^2$ to another number maybe $6a^2$ you would make an answer that would have an exponent at the top but then why if there was just a 7 and another 6 you wouldn't have exponents, it would just be the answer.	
Kona	Ok. So can I go back to my initial question? What does $7a^2$ mean to you? If you had to explain someone the number $7a^2$?	
Nelli	I'd just say it's a number that you got from that particular, if maybe there were two alphabets	No 'symbol sense' as she is viewing objects as

	involved in the sum or maybe three you just had $7a$. I believe most of the time when you add your bases you sometimes add the powers and multiply the powers so I just added them. [ok. $7a^2$].	processes.
Kona	That's fine. Can we turn over? Can we look at number 10? Ok. Number 10 says: simplify $3a - b + a$ and your answer was $3ab$. Can you explain how you getting your answer of $3ab$?	$3a + a = 3a$ Nelli breaks her rule here.
Nelli	Eh. What I just said, I just put the a and b the a and a, the both a's aside and just said $3a - b$ is and then I thought let me just remove $1a$ instead of writing a^2 because they the same thing they the same alphabet so instead of writing $3a^2 + b$ I just say $3ab$ [that is like a suitable answer].	
Kona	$3ab$. Ok. So you just removed the $1a$?	
Nelli	Just removed the $1a$.	
Kona	Do you think you can do that in maths? Just remove an a?	
Nelli	[not sure, some people just remove anything]	
Kona	Ok. Then why, why did you remove it?	
Nelli	Sometimes simplifying is just harder. I prefer working the sum so when you also simplify you just write anything that comes close to the line.	
Kona	Then for number 11. How did you? You see for number 10 you dropped the a. It seems like for number 11 you didn't drop off the b.	
Nelli	Didn't	
Kona	Can you explain what was your thinking in number 11?	
Nelli	Well I just added the $2b$'s and just made them b^2 then I also added this positive a plus this positive b and made it an ab so that I can [strain] and use this negative which is saying $-b^2$.	$b + b = b^2$ $a + b = ab$. Nelli adds the $1b$ twice.
Kona	So you added?	
Nelli	I added eh the $2b$'s, the one inside the bracket and the one outside then I also added the positive a and the positive b and just put them together and then I said $-b^2$.	
Kona	Ok. So why did you select the minus sign and not the plus sign?	
Nelli	[inaudible] I just took the sign of the big of the bases.	There's a confusion of rules learned in algebra. Work learned in exponents is applied for adding numbers.
Kona	Ok. That's fine. Can we look at number 15? Ok. For number 15 it says: what can you say	

	about c if $c + d = 10$ and c is lees than d? You said that $c = 3$ and $d = 4$. Can you explain what was your thinking there?	
Nelli	Well they said eh if c is less than d. So I just looked for a number that will be lesser than the number that I gave to d. So I just chose 3 but then I didn't concentrate in the real number that was supposed to be 10.	Again, assigning any number to the letter.
Kona	Ok. So if you had to change your answer what would your new answer be?	
Nelli	I'd probably say something like – 5 d ok. d = -5 and then c will be 1 or something.	
Kona	Ok. And then will you get $c + d = 10$? If you say -5 and 1?	
Nelli	Eh. I will not.	
Kona	Can you maybe rethink it?	
Nelli	Or maybe I'd say eh minus. I'd say $c = -5$ and then $d = + 5$ and then I'd just add the powers, the positive and the negative which is a positive and then I'll get a positive 10.	Learner uses rules across with no understanding of context.
Kona	Ok. Can we look at number 17? I thought it was very interesting what you wrote for your final answer. Can I ask you, what does the word final answer mean to you?	
Nelli	Like the final answer that you get at the end of the sum that you positive about. That you won't have to break again until you find another answer just the final one, the end of the answer [the] sum.	Learner does senseless calculations to get answers in her mind, hence random picking.
Kona	Because you see you wrote: $n + 2$ is larger because you still adding it's not final yet. So what did you say final means again? Your final answer?	
Nelli	Just your final answer, yes.	
Kona	Ok. So are you saying if you adding its going to get bigger?	
Nelli	Yes. It happens that you can still break the maybe you'd say what got gave you n or what gave you 2 something like that but when you just write it as $2n$ its just written as your final answer so I believe eh $n + 2$ cos you still gonna maybe make a few steps before you reach the final answer.	Hinting towards adding makes larger, although not explicit.
Kona	Ok. While you making these few steps could your answer get smaller?	
Nelli	Yes, it can get smaller because you. [mumbles]. The more steps maybe the less the answer will be as you making the more steps.	

Kona	Ok. I think that’s fine. Thank you.	
	End of interview	