considered in Example B has two arbitrary functions which depend on two variables, and possesses an infinite number of Lie point symmetries. The methods used in this chapter are not the same, although both of them are called preliminary group classification methods.

In chapter four, the equivalence and the principal Lie algebras of the system we classify are found. To find the equivalence algebra we follow Akhatov, Gazizov and Ibragimov (1991) and to find the principal Lie algebra we use the propositions stated in Ibragimov and Torrisi (1992). These algebras are utilized in chapter five. By using the preliminary group classification discussed in chapter three, we classify our system, although it becomes clear from the results we obtain that the classification is not complete.

Chapter five is concerned with the classification of the system (1.5). In classifying this system we use the Lie algebras which were introduced in chapter one. We find that this system admits a four-dimensional equivalence algebra of symmetries and a three-dimensional Principal Lie algebra of symmetries. Hence, we use the method which classifies *according to low-dimensional Lie algebras.*

The interested reader is referred to Ibragimov (1995) for more applications on the classification of partial differential equations. The method utilized in chapter six is an extension of that used in Vawda and Mahomed (1994).

Finally, in the conclusion we mention the results obtained and some of the open questions that can further be explored. In fact the main result of our work is that the arbitrary function $\psi(n)$ can only be of the form $\frac{a}{n^{1+b}}$, where a and b are constants.

1.2 Scope and objective of the study

System (1.5) has its roots in plasma physics. We are concerned here with the mathematical aspect of this problem, although the results can be analyzed physically and can be of much use in plasma physics.

The above system (1.5) is a new problem from the view-point of group classification. The method we use to classify this system is also new (see chapter five).

Our aim is to classify system (1.5) according to the group it admits and show how this new method of classifying partial differential equations works. This method classifies the system of equations according to low-dimensional Lie algebras it admits. Before embarking upon this method, we illustrate the existing methods of group classification of partial differential equations by means of examples.

1.3 Content

In chapter two we give a brief review of Lie theory of differential equations especially partial differential equations. The meaning of invariant functions and invariant differential equations are defined. Furthermore, Lie algebras are introduced in this chapter (see, e.g., Stephani 1989 and Ibragimov 1995).

Chapter three deals with the classification of partial differential equations with arbitrary functions. The only difference in the examples discussed in this chapter is that the partial differential equation considered in Example A has one arbitrary function which depends on one variable and admits a finite number of Lie point symmetries, whereas the partial differential equation

In the case of large thermal spread of the electron velocities and a negligibly weak thermal motion of ions, the collisionless plasma can be described by the combined system of the self-consistent field equations and the hydrodynamical equations for ions with the velocity $u(t, x)$ and the density $\bar{n}(x, t)$. This system in the one-dimensional case has the form (see Ibragimov 1995)

$$\begin{aligned} & f_t + v f_x + \frac{e}{m} E f_v = 0, \qquad && u_t + u u_x = \frac{\bar{e}}{\bar{m}} E, \\ & \bar{n}_t + (\bar{n} u)_x = 0, \; E_x = 4\pi\rho, && E_t + 4\pi j = 0, \\ & \rho = em \int_{-\infty}^{+\infty} dv f + \bar{e}\bar{n}, && j = em \int_{-\infty}^{+\infty} dv f + \bar{e}\bar{n} u. \end{aligned} \tag{1.3}$$

If the thermal motion of the electrons is strong enough and if their distribution is of Maxwell-Boltzman type with the density $n(t, x)$ and temperature T constant, the above system reduces to the system of differential equations

$$\begin{aligned} u_t + u u_x &= \frac{\bar{e}}{\bar{m}} E, \\ \bar{n}_t + (\bar{n} u)_x &= 0, \\ n_x &= \frac{en}{T} E, \\ E_x &= 4\pi e n + 4\pi \bar{e}\bar{n}. \end{aligned} \tag{1.4}$$

The symmetries of this system are found in Euler, Steeb and Mulser (1991, 1992). The natural generalization of system (1.4) is the following system

$$\begin{aligned} u_t + u u_x &= \frac{\bar{e}}{\bar{m}} E, \\ \bar{n}_t + (\bar{n} u)_x &= 0, \\ \psi(n)\, n_x &= \frac{e}{T} E, \\ E_x &= 4\pi e n + 4\pi \bar{e}\bar{n}, \end{aligned} \tag{1.5}$$

where (e, n, m) and $(\bar{e}, \bar{n}, \bar{m})$ are charges, densities and masses of ions and electrons, respectively; u is the flow velocity of ions, T the temperature, E the electric field and $\psi(n)$ an arbitrary function.

problem is not only of pure mathematical interest but also has practical significance. The differential equations of mathematical physics, in many cases contain parameters or functions that are determined experimentally and hence are not strictly fixed (these are called arbitrary elements).

Lie (1881) was the first to work in the problem of group classification of differential equations by classifying the linear equation

$$\begin{aligned} &R(x,y)\,z_{xx} + S(x,y)\,z_{xy} + T(x,y)\,z_{yy} \\ &+P(x,y)\,z_x + Q(x,y)\,z_y + Z(x,y)\,z = 0, \end{aligned} \tag{1.1}$$

where R, S, T, P, Q and Z are arbitrary (see Ibragimov 1994). Subsequently, many equations were classified, not only by Lie but also by others.

1.1 Statement and analysis of the problem

The one-dimensional nonrelativistic electron-ion plasma with a weak thermal motion of electrons and ions, described by the hydrodynamical model, is based on the following equations (see, e.g., Ibragimov 1995)

$$\begin{aligned} &v_t + vv_t = \frac{e}{m}E - g(n)n_x, \qquad n_t + (nv)_x = 0, \\ &u_t + uu_t = \frac{\bar{e}}{\bar{m}}E - \bar{g}(\bar{n})\bar{n}_x, \qquad \bar{n}_t + (\bar{n}u)_x = 0, \\ &E_x = 4\pi en + 4\pi\bar{e}\bar{n}, \qquad E_t + 4\pi\,(env + \bar{e}\bar{n}u) = 0. \end{aligned} \tag{1.2}$$

Here, v and u are the velocities of the electron and ion components of the plasma, the function $g(n)$ and $\bar{g}(\bar{n})$ take into account the electron and ion thermal pressure depending on the densities n and $\bar{n}$ of the electrons and ions, respectively. The temperature of each plasma component is assumed to be constant.

Chapter 1

Introduction

The great Norwergian mathematician Sophus Lie (1842–1899) initiated the study of continuous transformation groups in the second half of the previous century. Today, these groups are called Lie groups. His aim was to create a theory of integrating ordinary differential equations similar to the Abelian theory of solving algebraic equations. Lie groups became well-known, but their original field of fruitful application to a large extent remained hidden in the literature. Hence many people who could have profited from this method simply were not aware of its existence.

It is only in this century in which this theory has become widespread. Today, there are many books containing this theory. Since Lie's works were in German, most of the books are merely translations and do not contain significantly new information.

One of the main problems of group analysis of differential equations is to study the action of the group admitted by a given equation (system of equations) in a set of solutions to this equation. Another interesting and practically important problem consists of using the group analysis technique for the group classification of differential equations. The solution to this

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DEDICATION

To my parents with all my love

DECLARATION

I declare that the contents of this research report are original except where due references have been made. It has not been submited before for any degree to any other insitution.

M B Bashe

August 1995

ABSTRACT

In this work we give a brief overview of the existing group classification methods of partial differential equations by means of examples. On top of these methods we introduce another new method which classify according to low-dimensional Lie algebras. One can ask: What is the aim of introducing a new method whilst there are existing methods? This question is answered in the following paragraph.

Firstly we classify our system of non-linear partial differential equations using the preliminary group classification method (one of the existing methods). The results are not different from what Euler, Steeb and Mulser have obtained in 1991 and 1992. That is, this method does not yield new information.

This new method which classifies according to low-dimensional Lie algebras is used to classify a general system of equations from plasma physics. Finally, using this method we completely classify our system for four-dimensional algebras. For a partial differential equation to be completely classified using this method, it must admit a low-dimensional Lie algebra.

EQUIVALENCE AND SYMMETRY GROUPS OF A NONLINEAR EQUATION IN PLASMA PHYSICS

M B BASHE

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If a differential equation admits the vector fields X_1 and X_2, then it also admits their commutator $[X_1, X_2]$ (see Ovsyannikov 1982). The largest admitted Lie algebra is called the *full* Lie algebra of the equation. In this section we are concerned only with finite dimensional real Lie algebras, because in applications one mainly encounters real differential equations.

One and two-dimensional algebras both have the same structures over the real and over the complex numbers. In suitable bases, these can be taken to be $[X_1, X_2] = 0$ and $[X_1, X_2] = X_1$ for two-dimensional algebras, and $[X, X] = 0$ for any X for a one-dimensional algebra.

We next consider the three-dimensional real algebras. There are eleven Lie algebras of dimension three (decomposable and indecomposable) two of which depend on parameters.

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where the prolongation formulae, ζ_2, ζ_{11} and ζ_{22} are as given in (2.19), (2.21) and (2.23), respectively. Substituting $v_{tt} = v_x v_{xx} + v_x^2 + x$, (2.29) has the following independent variables $t, x, v, v_t, v_x, v_{tt}, v_{tx}$ and v_{xx}, and the unknown functions ξ^1, ξ^2 and η depending only on t, x and v. Equating all the coefficient of $v_x v_{xx}, v_{xx}, v_{xt}, v_x$, to zero and solving the resulting equations, one ends up with the general solution

$$\xi^1 = -C_1,\ \xi^2 = C_2,\ \eta = C_3 t + \frac{C_2}{2}t^2 + C_4, \tag{2.30}$$

where C_i, $i = 1, 2, ..., 4$, are arbitrary constants. Solutions (2.30) for the determining equation (2.29) generate a four-dimensional Lie algebra L_4 spanned by the following operators

$$X_1 = -\frac{\partial}{\partial t},\ X_2 = \frac{t^2}{2}\frac{\partial}{\partial v} + \frac{\partial}{\partial x},\ X_3 = t\frac{\partial}{\partial v},\ X_4 = \frac{\partial}{\partial v}. \tag{2.31}$$

2.2 Lie algebras

A Lie algebra is a vector space L endowed with a bilinear product $[X_1, X_2]$ such that the following holds: The Lie algebra is *skew symmetric*, i.e.,

$$[X_1, X_2] = -[X_2, X_1]$$

and satisfies the *Jacobi identity*,

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0, \text{ for all } X_1, X_2, X_3 \in L.$$

In group analysi of differential equations one deals mostly with real Lie algebras of operators and hence here we consider only vector spaces over the field of real numbers.

We define the Lie bracket [,] on a set L of vector fields of the form (2.7) as the commutator

$$[X_1, X_2] = X_1 X_2 - X_2 X_1. \tag{2.32}$$

The difference between Definition 2.2 and Definition 2.2$'$ is that 2.2$'$ does not assume the knowledge of solutions and the invariance can be tested on any given differential equation using the following

$$X_{(s)} F_k \Big|_{(2.5)} = 0, \; k = 1, \dots, p, \tag{2.27}$$

where $X_{(s)}$ is the sth prolongation of the generator X of the group G and the notation $\Big|_{(2.5)}$ means evaluated on the frame. Equations (2.27) are called the determining equations of the generator X of the symmetry group G. The determining equations are a system of linear homogeneous partial differential equations for the unknown vector X. In most practical instances, these equations can be solved by elementary methods and the general solution determines the most general infinitesimal symmetry of the system. The solutions of the determining equation generate a set of vector fields denoted by L_r, where r is the number of the elements in the set, which can be infinite, and L_r generates a Lie algebra of dimension r. We study Lie algebras in more detail in the next section.

To illustrate the above, let us consider the second-order partial differential equation

$$v_{tt} = v_x v_{xx} + v_x^2 + x. \tag{2.28}$$

The above equation is a special case of

$$v_{tt} = f(x, v_x) v_{xx} + g(x, v_x).$$

For more information about the above equation see Ibragimov, Torrisi and Valenti (1991). For the invariance conditions of (2.28) we use (2.18) where we replace u by v, which reduces the determining equation (2.27) to the form

$$\zeta_{11} - v_x \zeta_{22} - v_{xx} \zeta_2 - 2 v_x \zeta_2 - \xi^2 = 0. \tag{2.29}$$

As is well-known, this problem can be solved by constructing solutions of the system of ordinary differential equations

$$\frac{dz^1}{\vartheta^1(z)} = \frac{dz^2}{\vartheta^2(z)} = \dots = \frac{dz^s}{\vartheta^s(z)}. \tag{2.25}$$

Solutions of (2.25) take the form

$$J_1(z) = C_1,\ J_2(z) = C_2, \dots, J_{s-1}(z) = C_{s-1}, \tag{2.26}$$

in which $C_1, \dots, C_{s-1}$ are constants of integration and the $J_i(z)$ are functions independent of the C_js. One can note that if $\{J_1(z), \dots, J_{s-1}(z)\}$ is a basis of invariants of a group of transformations G, then $J(z) = F(J_1(z), \dots, J_{s-1}(z))$ is also an invariant, where F is a smooth function of its argument (see also Ovsyannikov 1962, Olver 1986). For example, $z = (x, u)$, then $F = F(x, u)$ in (2.24) and we have an operator of the form (2.8).

To illustrate the foregoing, consider the rotation group which has infinitesimal generator $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. The corresponding characteristics system is

$$\frac{dx}{x} = -\frac{dy}{y}.$$

Here $z = (x, y)$ and $n = 2$, i.e., we have only one invariant $J(z) = xy = C$, where C is an arbitrary constant.

Definition 2.2′

The system of differential equations (2.5) is said to be invariant under the group G if the frame of the system is an invariant surface with respect to the prolonged group $G_{(s)}$.

and operator (2.8) has the form

$$X = \xi^1(x,y,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u}. \tag{2.16}$$

The first and second prolongations of (2.16) are given by

$$X_{(1)} = X + \zeta_1\frac{\partial}{\partial u_x} + \zeta_2\frac{\partial}{\partial u_y}, \tag{2.17}$$

$$X_{(2)} = X_{(1)} + \zeta_{11}\frac{\partial}{\partial u_{xx}} + \zeta_{12}\frac{\partial}{\partial u_{xy}} + \zeta_{22}\frac{\zeta}{\partial u_{yy}} \tag{2.18}$$

respectively, where

$$\zeta_1 = D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2), \tag{2.19}$$

$$\zeta_2 = D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2), \tag{2.20}$$

$$\zeta_{11} = D_x(\zeta_1) - u_{xx} D_x(\xi^1) - u_{xy} D_x(\xi^2), \tag{2.21}$$

$$\zeta_{12} = D_y(\zeta_1) - u_{xx} D_y(\xi^1) - u_{xy} D_y(\xi^2), \tag{2.22}$$

$$\zeta_{22} = D_y(\zeta_2) - u_{xy} D_y(\xi^1) - u_{yy} D_y(\xi^2). \tag{2.23}$$

(see also Stephani 1989).

Theorem 1.1

The function $F(z)$ is an invariant function of the group G with symbol X if and only if it satisfies the linear partial differential equation

$$X(F) \equiv \vartheta^\kappa(z)\frac{\partial F}{\partial z^\kappa} = 0. \tag{2.24}$$

$$\xi^i = \frac{\partial f^i}{\partial a}\Big|_{a=0}\,,\ \eta^\alpha = \frac{\partial g^\alpha}{\partial a}\Big|_{a=0}\,. \tag{2.9}$$

The first prolongation of the generator (2.8) is given by

$$X_{(1)} = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha\frac{\partial}{\partial u_i^\alpha}, \tag{2.10}$$

or

$$X_{(1)} = X + \zeta_i^\alpha\frac{\partial}{\partial u_i^\alpha}, \tag{2.11}$$

where ξ^i and η^α are as given in (2.9) and

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{2.12}$$

with D_i as in (2.2). The second prolongation of the generator X is denoted by $X_{(2)}$ and is given by

$$X_{(2)} = X_{(1)} + \zeta_{i_1 i_2}^\alpha\frac{\partial}{\partial u_{i_1 i_2}^\alpha}, \tag{2.13}$$

where

$$\zeta_{i_1 i_2}^\alpha = D_{i_2}(\zeta_{i_1}^\alpha) - u_{j i_1}^\alpha D_{i_2}(\xi^j). \tag{2.14}$$

Following (2.12) and (2.14), the higher prolongation is defined as

$$\zeta_{i_1 i_2 \ldots i_s}^\alpha = D_{i_s}(\zeta_{i_1 i_2 \ldots i_{s-1}}^\alpha) - u_{j i_1 i_2 \ldots i_{s-1}}^\alpha D_{i_s}(\xi^j). \tag{2.15}$$

We now look at the situation when x, y are independent variables, and u a dependent variable. In this case the derivative operator (2.2) takes the form

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y} + \ldots \\ D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y} + \ldots, \end{aligned}$$

$$f(x, y) = x - y,$$

is an invariant function of G_c since

$$f(x + c\varepsilon,\ y + c\varepsilon) = f(x, y), \text{ for all } \varepsilon,$$

and the function

$$f(x, y) = x + y$$

is not an invariant of G_c since

$$f(x + c\varepsilon,\ y + c\varepsilon) \neq f(x, y),$$

(see Olver 1986).

Definition 2.2

The system of differential equations (2.5) is said to be invariant under the group G of (2.7) if G converts every solution of the system under consideration into a solution of the same system.

This definition assumes that the solution of the system is known a priori (cf. Definition 2.2′).

A group of transformations of the above form (2.7) is called a group G of point transformations in the space of dependent and independent variables.

Corresponding to the group G there is an infinitesimal generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{2.8}$$

where

$$\bar{u}^\alpha = g^\alpha(x, u, a),\ g^\alpha|_{a=0} = u^\alpha. \tag{2.7}$$

These transformations form a one-parameter group G if the successive action of two transformations is equivalent to the action of another of the form (2.7), i.e., if the function $f = (f^1, ..., f^n)$ and $g = (g^1, ..., g^m)$ satisfy the property:

$$f^i\left(f\left(x, u, a\right), g\left(x, u, a\right), b\right) = f^i\left(x, u, c\right),\ i = 1, ..., n,$$

$$g^\alpha\left(f\left(x, u, a\right), g\left(x, u, a\right), b\right) = g^\alpha\left(x, u, c\right),\ \alpha = 1, ..., m,$$

where

$$c = \phi\left(a, b\right)$$

is a smooth function defined for small a and b. In the case defined above, G is referred to as a local one-parameter group (a and b small), otherwise G is called a global group (a and b can take on values from a fixed interval of reals). Here, we mainly use local one-parameter groups because of their importance in the application of group analysis.

Definition 2.1

Let G be a local group of transformations acting on a manifold M. A function $F : M \to N$, where N is another manifold, is called a G invariant function if for all $z \in M$ and all $g \in G$ such that $g.z$ is defined

$$F(g.z) = F(z).$$

Let us consider an example of the above definition. Let G_c be the group of translations

$$(x, y) \mapsto (x + c\varepsilon,\ y + c\varepsilon),\ \varepsilon \in R,$$

where c is some fixed constant. Then the function

Then (2.3) is the s-order partial differential equation. Suppose $x, u, u_{(1)}, u_{(2)}, \ldots$ are functionally independent variables connected only by (2.1), then (2.3) determines a surface in the space of independent variables $x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}$. In Ibragimov (1992) this surface is called the *frame* of the differential equation under consideration. The frame of the equation we have considered is accompanied by its differential consequences:

$$D_i F = 0, \; D_i D_j F = 0, \; \ldots, \tag{2.4}$$

where

$$D_i F = \frac{\partial F}{\partial x^i} + u_i^\alpha \frac{\partial F}{\partial u^\alpha} + \ldots + u^\alpha_{i i_1 \ldots i_s} \frac{\partial F}{\partial u^\alpha_{i_1 \ldots i_s}}.$$

All the points $(x, u, u_{(1)}, u_{(2)}, \ldots)$ satisfying the equations (2.3) and (2.4) are denoted by $[F]$ and called the extended frame.

Let us see what happens when we have a system of equations. Consider the following system

$$F_k(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) = 0, \; k = 1, \ldots p, \tag{2.5}$$

where $F_k \in \mathcal{A}$ and s is the maximum order of the k differential equations $F_k = 0$. It is assumed that

$$p = \text{rank} \left\| \frac{\partial F_k}{\partial x^i}, \frac{\partial F_k}{\partial u^\alpha}, \ldots, \frac{\partial F_k}{\partial u^\alpha_{i_1 \ldots i_s}} \right\|,$$

on the frame of the equations under consideration. In this case the extended frame is given by

$$F_k = 0, \; D_i F_k = 0, \; D_i D_j F_k = 0, \; \ldots \tag{2.6}$$

Consider the following invertible transformations on the (x, u) space

$$\bar{x}^i = f^i(x, u, a), \; f^i|_{a=0} = x^i,$$

where

$$u_i^\alpha = D_i(u^\alpha),\ u_{ij}^\alpha = D_j(u_i^\alpha) = D_jD_i(u^\alpha),\ \ldots \tag{2.1}$$

D_i is called the operator of total differentiation and is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots,\ i = 1, \ldots, n. \tag{2.2}$$

In group analysis it is expedient to consider all variables $x, u, u_{(1)}, u_{(2)}, \ldots$ as functionally independent connected only by the differential relations (2.1). As a result the variables u are referred to as *differential variables*. We denote by z the sequence

$$z = (x, u, u_{(1)}, u_{(2)}, \ldots),$$

with elements

$$z^\nu,\ \nu \geq 1,$$

where

$$z^i = x^i,\ 1 \leq i \leq n;\ z^{n+\alpha} = u^\alpha,\ 1 \leq \alpha \leq m,$$

and the remaining elements represent the derivatives of u.

A locally analytic function $f(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}),\ s < \infty$, is called a differential function of order s. The set of all differential functions of all finite orders is denoted by $\mathcal{A}$. The space $\mathcal{A}$ is closed under the derivation given by (2.2). Let $F \in \mathcal{A}$ be a differential function of order s. Consider the equation

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \tag{2.3}$$

where u depends on x, i.e.,

$$u^\alpha = u^\alpha(x),\ u_i^\alpha = \frac{\partial u^\alpha(x)}{\partial x^i},\ \ldots$$

Chapter 2

Lie theory of differential equations

In this chapter we give a brief review of Lie theory as applied to the study of invariance properties of partial differential equations. The essence of this chapter is taken from Ibragimov (1995). Most of the information in this chapter is of great use in the following chapters.

2.1 Differential equations

Firstly, we consider the universal space $\mathcal{A}$ of differential functions introduced in Ibragimov (1995). Let

$$x = \left\{x^i\right\},\ i = 1, ..., n,$$

be independent variables and

$$u = \{u^\alpha\},\ \alpha = 1, ..., m,$$

the dependent variables. The successive derivatives of u are

$$u_{(1)} = \{u_i^\alpha\},\ u_{(2)} = \left\{u_{ij}^\alpha\right\},\ ...,\ i, j = 1, ...n,\ \alpha = 1, ..., m,$$

$$u_t + uu_x = \frac{\bar{e}}{\bar{m}} E \qquad \text{(equation of motion)} \tag{4.3}$$

$$n_x = \frac{en}{T} \quad \text{(balance of pressure and electric force)} \tag{4.4}$$

$$E_x = 4\pi en + 4\pi\bar{e}\bar{n} \qquad \text{(the Poisson equation)}. \tag{4.5}$$

We denote by (e, n, m) and $(\bar{e}, \bar{n}, \bar{m})$ the respective charges, densities and masses of ions and electrons; u is the flow velocity of ions, T the temperature which is constant and E is the electric field. All the quantities are dimensionless. The symmetry Lie algebra of equations (4.2), (4.3), (4.4) and (4.5) is spanned by

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$X_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2\bar{n}\frac{\partial}{\partial \bar{n}} - E\frac{\partial}{\partial E}$$

(see Euler, Steeb and Mulser 1991 and 1992).

Comparing the two systems (4.1) and (4.2)–(4.5), we find that the function $\psi(n)$ must be $\frac{1}{n}$ in (4.1) in order for it to be equivalent to (4.2–4.5). The physical reason for the inclusion of the general interaction $\psi(n)$ instead of just $\frac{1}{n}$ can be interpreted in the trapping of electrons by plasma potential waves (see e.g., Ibragimov 1995).

4.2 Equivalence and the principal algebras

Following Akhatov, Gazizov and Ibragimov (1991), we seek the operator

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial n} + \eta^3\frac{\partial}{\partial \bar{n}} + \eta^4\frac{\partial}{\partial E} + \mu\frac{\partial}{\partial \psi}$$

Chapter 4

Equivalence and the principal Lie algebras

In this chapter, we find equivalence and principal Lie algebras for the system of equations

$$
\begin{aligned}
u_t + u u_x &= \frac{\bar{e}}{\bar{m}} E, \\
\bar{n}_t + (\bar{n} u)_x &= 0, \\
\psi(n)\, n_x &= \frac{e}{T} E, \\
E_x &= 4\pi e n + 4\pi \bar{e} \bar{n},
\end{aligned}
\tag{4.1}
$$

where $\psi(n)$ is an arbitrary function. This work is new and we motivate this in the next section.

4.1 Introduction

In 1991 and 1992, Euler, Steeb and Mulser considered the system of partial differential equations for a collisionless plasma of cold ions and warm electrons, viz.,

$$\bar{n}_t + (\bar{n} u)_x = 0 \qquad \text{(equation of continuity)} \tag{4.2}$$

dimensional subalgebras, see Chupakhin (1994). We use the above method of preliminary classification in the next chapter to classify our system.

3.4 Discussion

The disadvantage of the methods discussed above is that they are not easily applicable to all partial differential equations. Especially for the method illustrated in Example A, you cannot state a priori which differential equations can or cannot be classified using it. For Example B, it is obvious that the equation considered there is not completely classified as one is uncertain whether the equivalence algebra used for its group classification is the largest subalgebra of the infinite-dimensional equivalence algebra.

$$E_1 = \frac{\partial}{\partial t},\ E_2 = \frac{\partial}{\partial v},\ E_3 = t\frac{\partial}{\partial v},$$

$$E_4 = x\frac{\partial}{\partial v},\ E_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v},$$

$$E_6 = t\frac{\partial}{\partial t} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g},$$

$$E_7 = t^2\frac{\partial}{\partial v} + 2\frac{\partial}{\partial g}, \tag{3.7}$$

$$E_\varphi = \varphi\frac{\partial}{\partial x} + 2\varphi' f\frac{\partial}{\partial f} + \varphi'' v_x f\frac{\partial}{\partial g},$$

$$E_F = F\frac{\partial}{\partial v} - F'' f\frac{\partial}{\partial g}.$$

This became a problem as it is known that preliminary group classification is simple only when the differential equation under consideration has a finite-dimensional equivalence algebra. The above problem was solved by choosing a finite-dimensional equivalence algebra L_{10} from the infinite one L_∞. From this finite-dimensional equivalence algebra, the following extensions were obtained

$$\begin{aligned}
Y_1 &= \frac{\partial}{\partial x},\ Y_2 = \frac{\partial}{\partial v_x},\\
Y_3 &= x\frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v_x},\ Y_4 = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},\\
Y_5 &= \frac{\partial}{\partial g},\ Y_6 = g\frac{\partial}{\partial g} + v_x\frac{\partial}{\partial v_x},\\
Y_7 &= x\frac{\partial}{\partial v_x} - f\frac{\partial}{\partial g}.
\end{aligned} \tag{3.8}$$

After the optimal system of subalgebras for these extensions was obtained, certain propositions were invoked to complete the group classification with respect to one-dimensional subalgebras only. For optimal system of two-respectively. All the above symmetries are the additional symmetries to the principal Lie algebra which is given by

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial t}, \; Z_2 = \frac{\partial}{\partial x}, \; Z_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2\omega\frac{\partial}{\partial \omega}, \\ Z_4 &= x\frac{\partial}{\partial \omega}, \; Z_5 = \frac{\partial}{\partial \omega}. \end{aligned} \tag{3.5}$$

Thus, (3.1) has been successfully classified using this method.

3.3 Example B

(See Ibragimov, Torrisi and Valenti 1991)

Consider the equation

$$v_{tt} = f\left(x, v_x\right) v_{xx} + g\left(x, v_x\right). \tag{3.6}$$

An equivalence transformation for (3.6) is defined as a nondegenerate change of the variables t, x and v taking any equation of the form (3.6) into an equation of the same form, with different $f\left(x, v_x\right)$ and $g\left(x, v_x\right)$.

For this equation, an infinite-dimensional equivalence algebra has been found.

The following equation has been found through the investigation of the determining equation of (3.1), viz.,

$$a + bH + cH' + d\omega_{xx}H' = 0, \tag{3.3}$$

where a, b, c and d are constants. Solving (3.3) with the use of (3.2), different forms of H have been found. To solve this type of equation is not always easy as it turns out here (see, e.g., Ibragimov 1995). That is why many differential equations cannot be classified using this method. For certain functions H, more symmetries have been found. That is, for

$$H = \exp(\omega_{xx}), \tag{3.4}$$

the following symmetry is an additional symmetry (to the principal Lie algebra of symmetries)

$$Z_6 = 2t\frac{\partial}{\partial t} - x^2\frac{\partial}{\partial \omega}.$$

For other forms of H, we just state the function H and the corresponding additional symmetry(ies). One has

$$H = \ln \omega_{xx},$$
$$Z_6 = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial \omega}$$

and

$$H = \omega_{xx}^{\sigma},$$
$$Z_6 = (1 - \sigma)\, t\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial \omega},$$

where $\sigma \neq \pm\frac{1}{3}$. For $\sigma = \frac{1}{3}$ and $\sigma = -\frac{1}{3}$, the additional symmetries are

$$Z_7 = \omega\frac{\partial}{\partial x}$$

and

$$Z_7 = x^2\frac{\partial}{\partial x} + x\omega\frac{\partial}{\partial x},$$

and Valenti 1991). To illustrate this method we take an example from this paper.

3.2 Example A

(See Akhatov, Gazizov and Ibragimov 1991)

Consider the following equation

$$\omega_t = H(\omega_{xx}). \tag{3.1}$$

For this example, an equivalence transformation is a nondegenerate change of variables t, x and ω taking any equation (3.1) with an arbitrary H into an equation (3.1) with in general a different function $H(\omega_{xx})$.

As we have mentioned before, the first step before performing the group classification is to find the equivalence algebra. The equivalence algebra of (3.1) is

$$E_1 = t\frac{\partial}{\partial t} - H\frac{\partial}{\partial H},\ E_2 = \frac{\partial}{\partial t},\ E_3 = x\frac{\partial}{\partial x},$$

$$E_4 = \frac{\partial}{\partial x},\ E_5 = \omega\frac{\partial}{\partial \omega} + H\frac{\partial}{\partial H},\ E_6 = x^2\frac{\partial}{\partial \omega},$$

$$E_7 = x\frac{\partial}{\partial \omega},\ E_8 = t\frac{\partial}{\partial \omega} + \frac{\partial}{\partial H},\ E_9 = \frac{\partial}{\partial \omega}.$$

It follows from the above that the equivalence transformation is (see paper cited)

$$\begin{aligned} &\tilde{t} = \alpha t + \gamma_1,\ \tilde{x} = \beta_1 x + \gamma_2, \\ &\tilde{\omega} = \delta_1\omega + \delta_2 x^2 + \delta_3 x + \delta_4 t + \delta_5,\ \tilde{H} = \frac{\delta_1}{\alpha}H + \delta_4. \end{aligned} \tag{3.2}$$

use experiments to find these arbitrary functions. The Lie group approach involves using a group classification method to solve this problem.

3.1 Introduction

The most important first step before carrying out group classification is to find the equivalence algebra of the equation under consideration. The method used to find equivalence algebra is similar to the one of finding the principal Lie algebra, although it is not exactly the same (see next chapter, see also Akhatov, Gazizov and Ibragimov 1991).

One of the methods of group classification of differential equations is effected by inspecting the determining equations of the equation under consideration. Using this, hand-in-hand with the equivalence transformation which is easily found from equivalence algebras by using the Lie equation (see, e.g., Ibragimov 1994), the equation investigated can be classified. This method works well when applied to certain classes of differential equations. We illustrate this method by using one of the examples given in Akhatov, Gazizov and Ibragimov (1991).

It has been observed (see Ibragimov and Torrisi 1992) that some of the principal Lie algebra extensions are subalgebras of the equivalence algebras. Some of these extensions are similar to each other. To consider only non-similar extensions, one has to find the optimal system of these extensions (see Ovsyannikov 1982). Finding the invariants of this optimal system and invoking certain propositions, solve the problem of group classification for these extensions. It is this method which is called preliminary group classification. It has been found that this method works well when the equation investigated has a finite-dimensional equivalence algebra (see Ibragimov, Torrisi

Chapter 3

On group classification of differential equations

Here we discuss the classification of non-linear partial differential equations by means of two examples. The system we classify in chapter four is a system of non-linear partial differential equations. This group classification of partial differential equations illustrated here is not new work. S Lie (1881) initiated the group classification of partial differential equations, although his focus was on linear partial differential equations. Moreover, another method for group classification of partial differential equations which works even for non-linear partial differential equations has been introduced and is known as the preliminary group classification method (see Akhatov, Gazizov and Ibragimov 1991). We illustrate how the preliminary group classification method works. We do not discuss the method used by Lie. Here our main aim is to illustrate how to classify non-linear partial differential equations. This is done by means of two examples.

Most of partial differential equations which have physical applications possess arbitrary functions or constants. Engineers and physicists in general

The labeling is the same as in Table 2.1. The above Table 2.2 is of great utility in chapter six. We do not discuss five and higher dimensional Lie algebras as they do not arise in our work.

appearing as a superscript is to separate those which depend on parameters. Thus, $L^a_{3,6}$ denotes the sixth algebra of dimension three which depends on parameter a.

In the case of four-dimensional real Lie algebras we have twenty four Lie algebras (decomposable and indecomposable). The indecomposable real Lie algebras of dimension four are classified into twelve types (see, e.g., Mahomed 1986) with five depending on parameters. We consider only six types, with four of the standard types incorporated in the two algebras which depend on parameters. The reason for this will become apparent in chapter six.

Table 2.2

Algebra	Non-zero commutation relations
$L_{4,1}$	$[X_2, X_3] = X_1$
$L_{4,2}$	$[X_1, X_4] = 2X_1,\ [X_2, X_4] = X_2,$ $[X_3, X_4] = X_2 + X_3,\ [X_2, X_3] = X_1$
$L^b_{4,3}\ (0 < \|b\| < 1),$ $(b = -1, 0, 1)$	[illegible]$, X_3] = X_1,\ [X_1, X_4] = (1 + b)X_1,$ [illegible]$X_4] = X_2,\ [X_3, X_4] = bX_3$
$L^a_{4,4}\ (a \geq 0)$	$[X_2, X_3] = X_1,\ [X_1, X_4] = 2aX_1,$ $[X_2, X_4] = aX_2 - X_3,\ [X_3, X_4] = X_2 + aX_3$
$L_{4,5}$	$[X_1, X_4] = X_1,\ [X_3, X_4] = X_2$
$L_{4,6}$	$[X_2, X_4] = X_1,\ [X_3, X_4] = X_2$

Table 2.1

Non-isomorphic structures of real three-dimensional Lie algebras (see, e.g., Mahomed 1986)

Algebra	Non-zero commutation relations
$L_{3,1}$	
$L_{3,2}$	$[X_2, X_3] = X_1$
$L_{3,3}$	$[X_1, X_3] = X_1$, $[X_2, X_3] = X_1 + X_2$
$L_{3,4}$	$[X_1, X_3] = X_1$
$L_{3,5}$	$[X_1, X_3] = X_1$, $[X_2, X_3] = X_2$
$L^a_{3,6}$ $(0 < \|a\| < 1)$, $(a \neq 0, 1)$, $(a = -1)$	$[X_1, X_3] = X_1$, $[X_2, X_3] = aX_2$
$L^b_{3,7}$ $(b \geq 0)$	$[X_1, X_3] = bX_1 - X_2$, $[X_2, X_3] = X_1 + bX_2$
$L_{3,8}$	$[X_1, X_2] = X_1$, $[X_2, X_3] = X_3$, $[X_3, X_1] = -2X_2$
$L_{3,9}$	$[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$

Labeling: Most of the time we will use more than one Lie algebra of the same dimension, as we have done in Table 2.1. We distinguish one from another by two indices, the first referring to the dimension and the second to the number of the algebra in an arbitrary chosen ordering. The alphabet where e is a constant. The system

$$
\begin{aligned}
u_t + uu_x &= \frac{\bar{e}E}{\bar{m}}, \\
\bar{n}_t + (\bar{n}u)_x &= 0, \\
en_x &= \frac{en}{T}E, \\
E_x &= 4\pi en + 4\pi\bar{e}\bar{n},
\end{aligned}
$$

admits the four-dimensional algebra L_4, generated by the operators (4.23) and

$$Y_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2n\frac{\partial}{\partial n} - 2\bar{n}\frac{\partial}{\partial \bar{n}} - E\frac{\partial}{\partial E}.$$

Here we have exactly the same solution as the special case, $e = 1$, considered by Euler, Steeb and Mulser (1991 and 1992).

4.4 Discussion

In this chapter we have illustrated how the preliminary group classification can be simple when the equivalence algebra is finite. What is surprising is that although it works in a simple manner, one does not get new information. The principal Lie algebra obtained in this chapter is used in the next chapter to obtain a complete classification of equation (4.1) in terms of low-dimensional Lie algebras.

There is no need for us to find the optimal system of subalgebras as has been suggested in the previous chapter where this method was discussed. Using the following proposition we classify (4.1).

Proposition 4.2

Let Y be an equivalence operator. The operator

$$X = \mathrm{pr}_{(t,x,u,n,\theta,E)}(Y),$$

is a symmetry operator for the system (4.1) with function

$$\psi = f(n),$$

iff $\psi(n)$ is invariant under the group generated by

$$Z = \mathrm{pr}_{(n,\psi)}(Y)$$

(see Ibragimov and Torrisi 1992).

To find the invariants of (4.24) we can neglect the constant $2C_1$, i.e., we use

$$Z = -n\frac{\partial}{\partial n} + \psi\frac{\partial}{\partial \psi}.$$

Following the well-known method, we have

$$Z(\psi - f(n))\big|_{\psi = f(n)} = 0.$$

From this we obtain the first order ordinary differential equation

$$f + nf' = 0.$$

Solution of this equation is

$$\psi = f(n) = \frac{c}{n}.$$

This proposition has been taken from Ibragimov and Torrisi (1992). From (4.20) the equivalence operator Y for system (4.1) is

$$\begin{aligned} Y &= (C_1 t + C_3)\frac{\partial}{\partial t} + (C_1 x + C_4 t + C_5)\frac{\partial}{\partial x} \\ &\quad + C_4\frac{\partial}{\partial u} - 2C_1 n\frac{\partial}{\partial n} - 2C_1\bar{n}\frac{\partial}{\partial \bar{n}} \\ &\quad - C_1 E\frac{\partial}{\partial E} + 2C_1\psi\frac{\partial}{\partial \psi} \end{aligned} \tag{4.22}$$

and

$$\mathrm{pr}_{(\psi,n)}Y = -2C_1 n\frac{\partial}{\partial n} + 2C_1\psi\frac{\partial}{\partial \psi}.$$

Equating $\mathrm{pr}_{(\psi,n)}Y$ to zero we obtain

$$C_1 = 0.$$

Substituting $C_1 = 0$ in (4.22) we have

$$Y = C_3\frac{\partial}{\partial t} + (C_4 t + C_5)\frac{\partial}{\partial x} + C_4\frac{\partial}{\partial u}.$$

Therefore our principal algebra is

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, \\ Y_2 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ Y_3 &= \frac{\partial}{\partial x}. \end{aligned} \tag{4.23}$$

4.3 Preliminary group classification

As we see from above, that there is only one non-trivial projection $\mathrm{pr}_{(n,\psi)}(Y)$ for our equivalence operator Y which is given by

$$Z = 2C_1\left(-n\frac{\partial}{\partial n} + \psi\frac{\partial}{\partial \psi}\right). \tag{4.24}$$

$$
\begin{aligned}
\xi^2 &= C_1x + C_4t + C_5,\\
\eta^1 &= C_4,\\
\eta^2 &= -2C_1n,\\
\eta^3 &= -2C_1\bar{n},\\
\eta^4 &= -C_1E,\\
\psi &= 2C_1\psi.
\end{aligned}
\tag{4.20}
$$

Hence we have the following generators

$$
\begin{gathered}
Y_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2n\frac{\partial}{\partial n} - 2\bar{n}\frac{\partial}{\partial \bar{n}} - E\frac{\partial}{\partial E} + 2\psi\frac{\partial}{\partial \psi},\\
Y_1 = \frac{\partial}{\partial t},\\
Y_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\
Y_3 = \frac{\partial}{\partial x}.
\end{gathered}
\tag{4.21}
$$

It is easy to see that the reflections

$$x \mapsto -x,\ u \mapsto -u \ \text{ and } \ E \mapsto -E$$

also yield an equivalence transformation.

To find the principal Lie algebra, we use the following proposition.

Proposition 4.1

An operator X belongs to the principal Lie algebra for the system (4.1) iff

$$X = \mathrm{pr}_{(t,x,u,n,\bar{n},E)}(Y)$$

with an equivalence generator Y, such that

$$\mathrm{pr}_{(n,\psi)}(Y) = 0.$$

After substituting the above equations in the third equation of (4.8), we obtain

$$\mu = \psi \left[\frac{\eta^4}{E} + \xi^2_x - \eta^2_n \right]. \tag{4.17}$$

Differentiating the above equation with respect to t and x we have

$$\begin{aligned} \mu_t &= \psi \left[\frac{\eta^4_t}{E} + \xi^2_{xt} - \eta^2_{nt} \right], \\ \mu_x &= \psi \left[\frac{\eta^4_x}{E} + \xi^2_{xx} - \eta^2_{nx} \right]. \end{aligned}$$

From (4.12), the above equations become

$$\begin{aligned} \eta^4_t + E\xi^2_{tx} = 0, \\ \eta^4_x + E\xi^2_{xx} = 0. \end{aligned}$$

Bearing in mind that ξ^2 is independent of E, we deduce

$$\begin{aligned} \eta^4 &= \eta^4(E), \\ \xi^2 &= C_1 x + C_2(t). \end{aligned} \tag{4.18}$$

Solving the first equation of (4.16) by using (4.18), we get

$$\xi^1 = C_1 t + C_3.$$

Thus far our coefficients are

$$\begin{aligned} \xi^1 &= C_1 t + C_3, \\ \xi^2 &= C_1 x + C_2(t), \\ \eta^1 &= \eta^1(t, x, u, \bar{n}), \\ \eta^2 &= \eta^2(n), \\ \eta^3 &= \eta^3(t, x, u, \bar{n}), \\ \eta^4 &= \eta^4(E). \end{aligned} \tag{4.19}$$

Using (4.19) to solve equations (4.14) and (4.16), we obtain the general solution

$$\xi^1 = C_1 t + C_3,$$

after the substitution of the prolongation formulae and (4.6). Decomposing the above equation with respect to $u_x, \bar{n}_x, E_t, \frac{1}{\psi}$ and n_t, the following system results

$$
\begin{gathered}
\xi^1 = \xi^1(t,x),\ \xi^2 = \xi^2(t,x,\bar{n}),\\
-\bar{n}\eta^3_{\bar{n}} + \bar{n}\xi^1_t + 2u\bar{n}\xi^1_x + \bar{n}\eta^1_u - \bar{n}\xi^2_x + \eta^3 = 0,\\
u\xi^1_t - \xi^2_t + u^2\xi^1_x - u\xi^2_x + \eta^1 + \bar{n}\eta^1_{\bar{n}} = 0,\\
\eta^3 = \eta^3(t,x,u,\bar{n}),\\
u\eta^3_n + \bar{n}\eta^1_n = 0,\\
\eta^3_t + u\eta^3_x + \bar{n}\eta^1_x + \bar{n}(4\pi e n + 4\pi e\bar{n})\eta^1_E + \frac{eE}{m}\eta^3_u - \frac{\bar{n}eE}{m}\xi^1_x = 0.
\end{gathered}
\tag{4.14}
$$

Solving (4.13) together with (4.14), we obtain

$$
\eta^4 = \eta^4(t,x,\bar{n},E),\ \xi^1 = \xi^1(t). \tag{4.15}
$$

The first equation of (4.8), viz.,

$$
\zeta^1_1 + u\zeta^1_2 + u_x\eta^1 - \frac{e}{m}\eta^4 = 0,
$$

becomes

$$
\begin{gathered}
\eta^1_t + \left(\frac{eE}{m} - uu_x\right)\eta^1_u + n_t\eta^1_n - (\bar{n}_x u + u_x\bar{n})\eta^1_{\bar{n}} + E_t\eta^1_E - \left(\frac{eE}{m} - uu_x\right)\xi^1_t\\
-u_x\left(\xi^2_t - (\bar{n}_x u + u_x\bar{n})\xi^2_{\bar{n}}\right) + u\left[\eta^1_x + u_x\eta^1_u + \frac{eE}{\psi T}\eta^1_n + \bar{n}_x\eta^1_{\bar{n}}\right]\\
u(4\pi e n + 4\pi e\bar{n})\eta^1_E - uu_x(\xi^2_x + \bar{n}_x\xi^2_{\bar{n}}) + u_x\eta^1 - \frac{e}{m}\eta^4 = 0.
\end{gathered}
$$

Equating the coefficients of u_x, u_t and E_t to zero, we obtain the system

$$
\begin{gathered}
\xi^1_t - \xi^2_x = 0,\\
-\bar{n}\eta^1_{\bar{n}} - \xi^2_t + \eta^1 = 0,\\
\xi^2 = \xi^2(t,x),\ \eta^1 = \eta^1(t,x,u,\bar{n}),\\
\eta^1_t + \frac{eE}{m}(\eta^1_u - \xi^1_t) + u\eta^1_x - \frac{e}{m}\eta^4 = 0.
\end{gathered}
\tag{4.16}
$$

Solving the third equation of (4.13) by using the third equation of (4.16), we deduce that

$$
\eta^4 = \eta^4(t,x,E).
$$

Since μ and η^2 are independent of ψ_n,

$$\begin{aligned}\mu_x &= \mu_t = \mu_u = \mu_{\bar n} = \mu_E = 0 \Rightarrow \mu = \mu(n,\psi) \qquad (4.12)\\ \eta^2_x &= \eta^2_t = \eta^2_u = \eta^2_{\bar n} = \eta^2_E = 0 \Rightarrow \eta^2 = \eta^2(n).\end{aligned}$$

After substituting the prolongation formulae and involving (4.6), the fourth equation of (4.8) yields

$$\begin{aligned}&\eta^4_x + u_x\eta^4_u + \frac{cE}{\psi T}\eta^4_n + \bar n_x\eta^4_{\bar n} + (4\pi en + 4\pi\bar e\bar n)\eta^4_E - E_t\left(\xi^1_x + u_x\xi^1_u + \frac{cE}{\psi T}\xi^1_n\right)\\ &-E_t\bar n_x\xi^1_{\bar n} - E_t(4\pi en + 4\pi\bar e\bar n)\xi^1_E - (4\pi en + 4\pi\bar e\bar n)\left(\xi^2_x + u_x\xi^2_u + \frac{cE}{\psi T}\xi^2_n\right)\\ &-(4\pi en + 4\pi\bar e\bar n)\bar n_x\xi^2_{\bar n} - (4\pi en + 4\pi\bar e\bar n)^2\xi^2_E - 4\pi e\eta^2 - 4\pi\bar e\eta^3 = 0.\end{aligned}$$

We set each of the coefficients of $u_x, \bar n_x, \frac{1}{\psi_t}$ and E_t equal to zero in turn and split the determining equation into the following system

$$\begin{aligned}&\xi^1 = \xi^1(t,x,E),\\ &\eta^4_u - (4\pi en + 4\pi\bar e\bar n)\xi^2_u = 0,\\ &\eta^4_{\bar n} - (4\pi en + 4\pi\bar e\bar n)\xi^2_{\bar n} = 0,\\ &\eta^4_n - (4\pi en + 4\pi\bar e\bar n)\xi^2_n = 0, \qquad (4.13)\\ &\xi^1_x + (4\pi en + 4\pi\bar e\bar n)\xi^1_E = 0,\\ &\eta^4_x + (4\pi en + 4\pi\bar e\bar n)\left[\eta^4_E - \xi^2_x - (4\pi en + 4\pi\bar e\bar n)\xi^2_E\right]\\ &-4\pi e\eta^2 - 4\pi\bar e\eta^3 = 0.\end{aligned}$$

The second equation of (4.8) becomes

$$\begin{aligned}&\eta^3_t + \left(\frac{\bar cE}{\bar m} - uu_x\right)\eta^3_u + n_t\eta^3_n - (\bar n_x u + \bar n u_x)\eta^3_{\bar n} + E_t\eta^3_E + (\bar n_x u + \bar n u_x)\xi^1_t\\ &+(\bar n_x u + \bar n u_x)E_t\xi^1_E - n_x\left[\xi^2_t + \left(\frac{\bar cE}{\bar m} - uu_x\right)\xi^2_u + n_t\xi^2_n - (\bar n_x u + \bar n u_x)\xi^2_{\bar n}\right]\\ &-n_xE_t\xi^2_E + u\left[\eta^3_x + u_x\eta^3_u + \frac{cE}{\psi T}\eta^3_E + \bar n_x\eta^3_{\bar n} + (4\pi en + 4\pi\bar e\bar n)\eta^3_E\right]\\ &+u(\bar n_x u + \bar n u_x)\xi^1_x + u(\bar n_x u + \bar n u_x)(4\pi en + 4\pi\bar e\bar n)\xi^1_E + \bar n_x\eta_1\\ &+\bar n\left[\eta^1_x + u_x\eta^1_u + \frac{cE}{\psi T}\eta^1_n + \bar n_x\eta^1_{\bar n}\right] + \bar n(4\pi en + 4\pi\bar e\bar n)\eta^1_E\\ &-\bar n\left(\frac{cE}{\bar m} - uu_x\right)\left[\xi^1_x + (4\pi en + 4\pi\bar e\bar n)\xi^1_E\right] - \bar n u_x(\xi^2_x + u_x\xi^2_u)\\ &-\bar n u_x\left[\frac{cE}{\psi T}\xi^2_E + \bar n_x\xi^2_{\bar n} + (4\pi en + 4\pi\bar e\bar n)\xi^2_E\right] + u_x\eta^3 = 0,\end{aligned}$$

$$
\begin{aligned}
\tilde{D}_x &= \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \ldots, \\
\tilde{D}_u &= \frac{\partial}{\partial u} + \psi_u \frac{\partial}{\partial \psi} + \ldots, \\
\tilde{D}_n &= \frac{\partial}{\partial n} + \psi_n \frac{\partial}{\partial \psi} + \ldots, \\
\tilde{D}_{\bar{n}} &= \frac{\partial}{\partial \bar{n}} + \psi_{\bar{n}} \frac{\partial}{\partial \psi} + \ldots, \\
\tilde{D}_E &= \frac{\partial}{\partial E} + \psi_E \frac{\partial}{\partial \psi} + \ldots
\end{aligned} \tag{4.10}
$$

In view of (4.6), (4.10) becomes

$$
\begin{aligned}
\tilde{D}_t &\equiv \tilde{D}_1 = \frac{\partial}{\partial t}, \\
\tilde{D}_x &\equiv \tilde{D}_2 = \frac{\partial}{\partial x}, \\
\tilde{D}_u &\equiv \tilde{D}_3 = \frac{\partial}{\partial u}, \\
\tilde{D}_n &\equiv \tilde{D}_4 = \frac{\partial}{\partial n} + \psi_n \frac{\partial}{\partial \psi}, \\
\tilde{D}_{\bar{n}} &\equiv \tilde{D}_5 = \frac{\partial}{\partial \bar{n}}, \\
\tilde{D}_E &\equiv \tilde{D}_6 = \frac{\partial}{\partial E}.
\end{aligned} \tag{4.11}
$$

The last five equations of (4.8) become

$$\mu_x - \psi_n \eta_x^2 = 0$$

$$\mu_t - \psi_n \eta_t^2 = 0$$

$$\mu_u - \psi_n \eta_u^2 = 0$$

$$\mu_{\bar{n}} - \psi_n \eta_{\bar{n}}^2 = 0$$

$$\mu_E - \psi_n \eta_E^2 = 0.$$

Furthermore

$$
\begin{aligned}
\mu_1 &= \tilde{D}_1(\mu) - \psi_t \tilde{D}_1\left(\xi^1\right) - \psi_x \tilde{D}_1\left(\xi^2\right) - \psi_u \tilde{D}_1\left(\eta^1\right) \\
&\quad - \psi_n \tilde{D}_1\left(\eta^2\right) - \psi_{\bar{n}} \tilde{D}_1\left(\eta^3\right) - \psi_E \tilde{D}_1\left(\eta^4\right) \\
&= \mu_t - \psi_n \eta^2_t,
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 &= \tilde{D}_2(\mu) - \psi_t \tilde{D}_2\left(\xi^1\right) - \psi_x \tilde{D}_2\left(\xi^2\right) - \psi_u \tilde{D}_2\left(\eta^1\right) \\
&\quad - \psi_n \tilde{D}_2\left(\eta^2\right) - \psi_{\bar{n}} \tilde{D}_2\left(\eta^3\right) - \psi_E \tilde{D}_2\left(\eta^4\right) \\
&= \mu_x - \psi_n \eta^2_x,
\end{aligned}
$$

$$
\begin{aligned}
\mu_3 &= \tilde{D}_3(\mu) - \psi_t \tilde{D}_3\left(\xi^1\right) - \psi_x \tilde{D}_3\left(\xi^2\right) - \psi_u \tilde{D}_3\left(\eta^1\right) \\
&\quad - \psi_n \tilde{D}_3\left(\eta^2\right) - \psi_{\bar{n}} \tilde{D}_3\left(\eta^3\right) - \psi_E \tilde{D}_3\left(\eta^4\right) \\
&= \mu_u - \psi_n \eta^2_u,
\end{aligned}
$$

$$
\begin{aligned}
\mu_5 &= \tilde{D}_5(\mu) - \psi_t \tilde{D}_5\left(\xi^1\right) - \psi_x \tilde{D}_5\left(\xi^2\right) - \psi_u \tilde{D}_5\left(\eta^1\right) \\
&\quad - \psi_n \tilde{D}_5\left(\eta^2\right) - \psi_{\bar{n}} \tilde{D}_5\left(\eta^3\right) - \psi_E \tilde{D}_5\left(\eta^4\right) \\
&= \mu_{\bar{n}} - \psi_n \eta^2_{\bar{n}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_6 &= \tilde{D}_6(\mu) - \psi_t \tilde{D}_6\left(\xi^1\right) - \psi_x \tilde{D}_6\left(\xi^2\right) - \psi_u \tilde{D}_6\left(\eta^1\right) \\
&\quad - \psi_n \tilde{D}_6\left(\eta^2\right) - \psi_{\bar{n}} \tilde{D}_6\left(\eta^3\right) - \psi_E \tilde{D}_6\left(\eta^4\right) \\
&= \mu_E - \psi_n \eta^2_E.
\end{aligned}
$$

In the above

$$
\tilde{D}_t = \frac{\partial}{\partial t} + \psi_t \frac{\partial}{\partial \psi} + \ldots,
$$

$$= \eta_t^4 + u_t\eta_u^4 + n_t\eta_n^4 + \bar{n}_t\eta_{\bar{n}}^4 + E_t\eta_E^4 - E_t\left(\xi_t^1 + u_t\xi_u^1 + n_t\xi_n^1 + \bar{n}_t\xi_{\bar{n}}^1 + E_t\xi_E^1\right)$$

$$-E_x\left(\xi_t^2 + u_t\xi_u^2 + n_t\xi_n^2 + \bar{n}_t\xi_{\bar{n}}^2 + E_t\xi_E^2\right),$$

$$\zeta_2^1 = D_x\left(\eta^1\right) - u_t D_x\left(\xi^1\right) - u_x D_x\left(\xi^2\right)$$

$$= \eta_x^1 + u_x\eta_u^1 + n_x\eta_n^1 + \bar{n}_x\eta_{\bar{n}}^1 + E_x\eta_E^1 - u_t\left(\xi_x^1 + u_x\xi_u^1 + n_x\xi_n^1 + \bar{n}_x\xi_{\bar{n}}^1 + E_x\xi_E^1\right)$$

$$-u_x\left(\xi_x^2 + u_x\xi_u^2 + n_x\xi_n^2 + \bar{n}_x\xi_{\bar{n}}^2 + E_x\xi_E^2\right),$$

$$\zeta_2^2 = D_x\left(\eta^2\right) - n_t D_x\left(\xi^1\right) - n_x D_x\left(\xi^2\right)$$

$$= \eta_x^2 + u_x\eta_u^2 + n_x\eta_n^2 + \bar{n}_x\eta_{\bar{n}}^2 + E_x\eta_E^2 - n_t\left(\xi_x^1 + u_x\xi_u^1 + n_x\xi_n^1 + \bar{n}_x\xi_{\bar{n}}^1 + E_x\xi_E^1\right)$$

$$-n_x\left(\xi_x^2 + u_x\xi_u^2 + n_x\xi_n^2 + \bar{n}_x\xi_{\bar{n}}^2 + E_x\xi_E^2\right),$$

$$\zeta_2^3 = D_x\left(\eta^3\right) - \bar{n}_t D_x\left(\xi^1\right) - \bar{n}_x D_x\left(\xi^2\right)$$

$$= \eta_x^3 + u_x\eta_u^3 + n_x\eta_n^3 + \bar{n}_x\eta_{\bar{n}}^3 + E_x\eta_E^3 - \bar{n}_t\left(\xi_x^1 + u_x\xi_u^1 + n_x\xi_n^1 + \bar{n}_x\xi_{\bar{n}}^1 + E_x\xi_E^1\right)$$

$$-\bar{n}_x\left(\xi_x^2 + u_x\xi_u^2 + n_x\xi_n^2 + \bar{n}_x\xi_{\bar{n}}^2 + E_x\xi_E^2\right),$$

$$\zeta_2^4 = D_x\left(\eta^4\right) - E_t D_x\left(\xi^1\right) - E_x D_x\left(\xi^2\right)$$

$$= \eta_x^4 + u_x\eta_u^4 + n_x\eta_n^4 + \bar{n}_x\eta_{\bar{n}}^4 + E_x\eta_E^4 - E_t\left(\xi_x^1 + u_x\xi_u^1 + n_x\xi_n^1 + \bar{n}_x\xi_{\bar{n}}^1 + E_x\xi_E^1\right)$$

$$-E_x\left(\xi_x^2 + u_x\xi_u^2 + n_x\xi_n^3 + \bar{n}_x\xi_{\bar{n}}^2 + E_x\xi_E^2\right).$$

The operators D_t and D_x are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + n_t\frac{\partial}{\partial n} + \bar{n}_t\frac{\partial}{\partial \bar{n}} + E_t\frac{\partial}{\partial E}, \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + n_x\frac{\partial}{\partial n} + \bar{n}_x\frac{\partial}{\partial \bar{n}} + E_x\frac{\partial}{\partial E}. \end{aligned} \tag{4.9}$$

System (4.7) can be rewritten as

$$\begin{aligned}
\zeta_1^1 + u\zeta_2^1 + u_x\eta^1 - \frac{\bar{e}}{\bar{m}}\eta^4 &= 0, \\
\zeta_1^3 + u\zeta_2^3 + \bar{n}_x\eta^1 + \bar{n}\zeta_2^1 + u_x\eta^3 &= 0, \\
\psi\zeta_2^2 + n_x\mu - \frac{e}{T}\eta^4 &= 0, \\
\zeta_2^4 - 4\pi e\eta^2 - 4\pi\bar{e}\eta^3 &= 0, \\
\mu_1 &= 0, \\
\mu_2 &= 0, \\
\mu_3 &= 0, \\
\mu_5 &= 0, \\
\mu_6 &= 0,
\end{aligned} \tag{4.8}$$

The prolongation formulae of (4.8) are:

$$\zeta_1^1 = D_t\left(\eta^1\right) - u_t D_t\left(\xi^1\right) - u_x D_t\left(\xi^2\right)$$

$$= \eta_t^1 + u_t\eta_u^1 + n_t\eta_n^1 + \bar{n}_t\eta_{\bar{n}}^1 + E_t\eta_E^1 - u_t\left(\xi_t^1 + u_t\xi_u^1 + n_t\xi_n^1 + \bar{n}_t\xi_{\bar{n}}^1 + E_t\xi_E^1\right)$$

$$-u_x\left(\xi_t^2 + u_t\xi_u^2 + n_t\xi_n^2 + \bar{n}_t\xi_{\bar{n}}^2 + E_t\xi_E^2\right),$$

$$\zeta_1^2 = D_t\left(\eta^2\right) - n_t D_t\left(\xi^1\right) - n_x D_t\left(\xi^2\right)$$

$$= \eta_t^2 + u_t\eta_u^2 + n_t\eta_n^2 + \bar{n}_t\eta_{\bar{n}}^2 + E_t\eta_E^2 - n_t\left(\xi_t^1 + u_t\xi_u^1 + n_t\xi_n^1 + \bar{n}_t\xi_{\bar{n}}^1 + E_t\xi_E^1\right)$$

$$-n_x\left(\xi_t^2 + u_t\xi_u^2 + n_t\xi_n^2 + \bar{n}_t\xi_{\bar{n}}^2 + E_t\xi_E^2\right),$$

$$\zeta_1^3 = D_t\left(\eta^3\right) - \bar{n}_t D_t\left(\xi^1\right) - \bar{n}_x D_t\left(\xi^2\right)$$

$$= \eta_t^3 + u_t\eta_u^3 + n_t\eta_n^3 + \bar{n}_t\eta_{\bar{n}}^3 + E_t\eta_E^3 - \bar{n}_t\left(\xi_t^1 + u_t\xi_u^1 + n_t\xi_n^1 + \bar{n}_t\xi_{\bar{n}}^1 + E_t\xi_E^1\right)$$

$$-\bar{n}_x\left(\xi_t^2 + u_t\xi_u^2 + n_t\xi_n^2 + \bar{n}_t\xi_{\bar{n}}^2 + E_t\xi_E^2\right),$$

$$\zeta_1^4 = D_t\left(\eta^4\right) - E_t D_t\left(\xi^1\right) - E_x D_t\left(\xi^2\right)$$

of the equivalence group from the condition of invariance of (4.1) written as the following system

$$
\begin{aligned}
u_t &= \frac{\bar{c}}{\bar{m}}E - uu_x, \\
\bar{n}_t &= -(\bar{n}_x u + \bar{n}u_x), \\
n_x &= \frac{cE}{\psi T}, \\
E_x &= 4\pi cn + 4\pi\bar{c}\bar{n}, \\
\psi_t &= \psi_x = \psi_u = \psi_n = \psi_{\bar{n}} = \psi_E = 0.
\end{aligned}
\tag{4.6}
$$

Here $u, n, \bar{n}, E$ and ψ are considered as differential variables: $u, n, \bar{n}, E$ in the space of (t, x) and ψ in the space of $(t, x, u, n, \bar{n}, E)$. The coordinates $\xi^1, \xi^2, \eta^1, ..., \eta^4$ of the operator Y are sought as functions of $t, x, u, n, \bar{n}$ and E, whereas the coordinate μ as a function of $t, x, u, n, \bar{n}, E, \psi$. The invariance conditions of system (4.6) are

$$
\begin{aligned}
\tilde{Y}\left(u_t + uu_x - \frac{\bar{c}}{\bar{m}}E\right) &= 0, \\
\tilde{Y}\left(\bar{n}_t + \bar{n}_x u + \bar{n}u_x\right) &= 0, \\
\tilde{Y}\left(\psi n_x - \frac{cE}{T}\right) &= 0, \\
\tilde{Y}\left(E_x - 4\pi cn - 4\pi\bar{c}\bar{n}\right) &= 0, \\
\tilde{Y}\left(\psi_t\right) &= 0, \\
\tilde{Y}\left(\psi_x\right) &= 0, \\
\tilde{Y}\left(\psi_u\right) &= 0, \\
\tilde{Y}\left(\psi_{\bar{n}}\right) &= 0, \\
\tilde{Y}\left(\psi_E\right) &= 0
\end{aligned}
\tag{4.7}
$$

provided (4.6) holds. $\tilde{Y}$ is the prolongation of Y and is

$$
\begin{aligned}
\tilde{Y} = Y &+ \zeta_1^1\frac{\partial}{\partial u_t} + \zeta_2^1\frac{\partial}{\partial u_x} + \zeta_1^2\frac{\partial}{\partial n_t} + \zeta_2^2\frac{\partial}{\partial n_x} + \zeta_1^3\frac{\partial}{\partial \bar{n}_t} \\
&+ \zeta_2^3\frac{\partial}{\partial \bar{n}_x} + \zeta_1^4\frac{\partial}{\partial E_t} + \zeta_2^4\frac{\partial}{\partial E_x} + \mu_1\frac{\partial}{\partial \psi_t} + \mu_2\frac{\partial}{\partial \psi_x} \\
&+ \mu_3\frac{\partial}{\partial \psi_u} + \mu_4\frac{\partial}{\partial \psi_n} + \mu_5\frac{\partial}{\partial \psi_{\bar{n}}} + \mu_6\frac{\partial}{\partial \psi_E}.
\end{aligned}
$$

$$[X_1, X_3] = 0$$

$$[X_1, X_2] = 0$$

$$[X_2, X_4] = 0$$

$$[X_2, X_3] = 0$$

$$[X_3, X_4] = X_2.$$

After manipulations we find that X_1 is an operator of the form

$$\begin{aligned} X_1 &= e^{-u}\left[C_1(n,\bar{n},E)\frac{\partial}{\partial t} + C_2(n,\bar{n},E)\frac{\partial}{\partial x}\right] \\ &+ e^{-u}\left[C_3(n,\bar{n},E)\frac{\partial}{\partial u} + C_4(n,n,E)\frac{\partial}{\partial n}\right] \\ &+ e^{-u}\left[C_5(n,\bar{n},E)\frac{\partial}{\partial \bar{n}} + C_6(n,\bar{n},E)\frac{\partial}{\partial E}\right]. \end{aligned}$$

Solving for the unknown coefficients of X_1, we end up with

$$X_1 = 0$$

and ψ arbitrary. In other words our system does not admit the algebra $L_{4,5}$.

F. Algebra $L_{4,6}$

Our operators are as follows

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial t}, \\ X_2 &= X_4 + \alpha X,\ \text{where } X = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \text{ and } \alpha \text{ a constant} \end{aligned}$$

and

$$\begin{aligned} X_4 &= \xi^1(t,x,u,n,\bar{n},E)\frac{\partial}{\partial t} + \xi^2(t,x,u,n,\bar{n},E)\frac{\partial}{\partial x} \\ &+ \eta^1(t,x,u,n,\bar{n},E)\frac{\partial}{\partial u} + \eta^2(t,x,u,n,\bar{n},E)\frac{\partial}{\partial n} \\ &+ \eta^3(t,x,u,n,\bar{n},E)\frac{\partial}{\partial \bar{n}} + \eta^4(t,x,u,n,\bar{n},E)\frac{\partial}{\partial E}. \end{aligned}$$

$$[X_2, X_3] = X_1$$

$$[X_3, X_4] = X_2 + aX_3.$$

We deduce that the operator X_4 which satisfies the above commutators is of the form

$$\begin{aligned} X_4 &= \left(at + u + C_5(n, \bar{n}, E)\right)\frac{\partial}{\partial t} + \left(2ax - t^2 + u^2 + C_5(n, \bar{n}, E)\,u\right)\frac{\partial}{\partial x} \\ &+ C_6(n, \bar{n}, E)\frac{\partial}{\partial x} + \left(-t + au + C_7(n, \bar{n}, E)\right)\frac{\partial}{\partial u} + \eta^2(n, \bar{n}, E)\frac{\partial}{\partial n} \\ &+ \eta^3(n, \bar{n}, E)\frac{\partial}{\partial \bar{n}} + \eta^4(n, \bar{n}, E)\frac{\partial}{\partial E}. \end{aligned}$$

Solving the determining equations we end up with an incompatibility problem as in Case B. Hence our system does not admit $L^a_{4,4}$ for any value of a .

E. Algebra $L_{4,5}$

Here we rename our operators as

$$\begin{aligned} X_1 &= \xi^1(t, x, u, n, \bar{n}, E)\frac{\partial}{\partial t} + \xi^2(t, x, u, n, \bar{n}, E)\frac{\partial}{\partial x} \\ &+ \eta^1(t, x, u, n, \bar{n}, E)\frac{\partial}{\partial u} + \eta^2(t, x, u, n, \bar{n}, E)\frac{\partial}{\partial n} \\ &+ \eta^3(t, x, u, n, \bar{n}, E)\frac{\partial}{\partial \bar{n}} + \eta^4(t, x, u, n, \bar{n}, E)\frac{\partial}{\partial E}, \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} X_2 &= \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial t}, \\ X_4 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \end{aligned}$$

The above operators satisfy the commutators

$$[X_1, X_4] = X_1$$

and

$$\psi = \sigma.$$

Case C2. For $b = 0$,

$$X_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2n\frac{\partial}{\partial n} - 2\bar{n}\frac{\partial}{\partial \bar{n}} - E\frac{\partial}{\partial E}$$

and

$$\psi = \frac{\sigma}{n}.$$

This is the special case which was considered by Euler, Steeb and Mulser (1991) and (1992), for $\sigma = 1$ (see chapter four).

Case C3. For $b = 1$,

$$X_4 = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - 2n\frac{\partial}{\partial n} - 2\bar{n}\frac{\partial}{\partial \bar{n}}$$

and

$$\psi = \frac{\sigma}{n^2}.$$

Hence (4.1) admits the algebra $L^b_{4,3}$.

D. Algebra $L^a_{4,4}$

Our principal algebra together with the unknown operator X_4 satisfies the following commutators

$$[X_1, X_4] = 2aX_1$$

$$[X_1, X_3] = 0$$

$$[X_1, X_2] = 0$$

$$[X_2, X_4] = aX_2 - X_3$$

$$[X_2, X_4] = X_2$$

$$[X_2, X_3] = X_1$$

$$[X_3, X_4] = bX_3.$$

We find that the unknown operator X_4, which satisfies the above commutators together with the principal algebra, is of the form

$$\begin{aligned} X_4 &= (t + C_2(n, \bar{n}, E))\frac{\partial}{\partial t} + ((1+b)x + C_2(n, \bar{n}, E)u + C_3(n, \bar{n}, E))\frac{\partial}{\partial x} \\ &\quad + (bu + C_4(n, \bar{n}, E))\frac{\partial}{\partial u} + \eta^2(n, \bar{n}, E)\frac{\partial}{\partial n} \\ &\quad + \eta^3(n, \bar{n}, E)\frac{\partial}{\partial \bar{n}} + \eta^4(n, \bar{n}, E)\frac{\partial}{\partial E}. \end{aligned}$$

After solving the determining equations we have the following general solution

$$\begin{aligned} \xi^1 &= t,\ \xi^2 = (1+b)x + c_3, \\ \eta^1 &= bu,\ \eta^2 = -2n,\ \eta^3 = -2\bar{n}, \\ \eta^4 &= (b-1)E,\ \psi = \frac{\sigma}{n^{1+b}}, \end{aligned}$$

where σ and c_3 are arbitrary constants. We ignore the operator $X = c_3\frac{\partial}{\partial x}$, as it is a linear combination of X_1. Hence

$$\begin{aligned} X_4 &= t\frac{\partial}{\partial t} + (1+b)x\frac{\partial}{\partial x} + bu\frac{\partial}{\partial u} \\ &\quad -2n\frac{\partial}{\partial n} - 2\bar{n}\frac{\partial}{\partial \bar{n}} + (b-1)E\frac{\partial}{\partial E}. \end{aligned}$$

Case C1. For $b = -1$,

$$X_4 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2n\frac{\partial}{\partial n} - 2\bar{n}\frac{\partial}{\partial \bar{n}} - 2E\frac{\partial}{\partial E}$$

$$[X_3, X_4] = X_2 + X_3, \tag{5.31}$$

which constitute the algebra $L_{4,2}$. In a similar manner to $L_{4,1}$ we find the operators that satisfy the above commutators which has the principal Lie algebra L_P as a subalgebra. The operator X_4 turns out to be of the form

$$\begin{aligned} X_4 &= (t+u)\frac{\partial}{\partial t} + \left(2x + u^2 + C_4(n,\bar{n},E)\right)\frac{\partial}{\partial x} \\ &\quad + (u + C_5(n,\bar{n},E))\frac{\partial}{\partial u} + \eta^2(n,\bar{n},E)\frac{\partial}{\partial n} \\ &\quad + \eta^3(n,\bar{n},E)\frac{\partial}{\partial \bar{n}} + \eta^4(n,\bar{n},E)\frac{\partial}{\partial E}. \end{aligned} \tag{5.32}$$

We use the method discussed in Case A to find the unknown coefficients of X_4 in (5.32). Starting with the fourth equation of (5.18) we have

$$\begin{aligned} &\frac{eE}{\psi T}\eta^4_n + \bar{n}_x\eta^4_{\bar{n}} + (4\pi en + 4\pi\bar{e}\bar{n})\,\eta^4_E - E_t u_x \\ &- (4\pi en + 4\pi\bar{e}\bar{n})\left[2 + 2uu_x + \frac{eE}{\psi T}C_{4n} + \bar{n}_x C_{4\bar{n}}\right] \\ &- (4\pi en + 4\pi\bar{e}\bar{n})^2 C_{4E} - 4\pi e\eta^2 - 4\pi\bar{e}\eta^3 = 0. \end{aligned}$$

This system gives rise to an incompatibility (see the term containing E_t). We conclude that our system does not admit the algebra $L_{4,2}$.

C. Algebra $L^b_{4,3}$

The algebra $L^b_{4,3}$ satisfies the following commutators

$$[X_1, X_4] = (1+b)\,X_1$$

$$[X_1, X_3] = 0$$

$$[X_1, X_2] = 0$$

Using (5.21), (5.20) gives us the solution

$$\eta^1 = 0. \tag{5.24}$$

From (5.21), (5.22), (5.23) and (5.24) we arrive at

$$\begin{aligned} \xi^1 &= 0,\ \xi^2 = C_1, \\ \eta^1 &= \eta^2 = \eta^3 = \eta^4 = 0, \\ \psi &= 0. \end{aligned} \tag{5.25}$$

Substituting (5.25) into (5.15) we obtain the operator

$$X_4 = C_1 \frac{\partial}{\partial x}.$$

This operator is equivalent to X_1. We conclude that our system does not admit the algebra $L_{4,1}$.

B. Algebra $L_{4,2}$

We consider the same operators in (5.1) and (5.2). The only difference is that here these operators are required to satisfy the following commutators

$$[X_1, X_4] = 2X_1 \tag{5.26}$$

$$[X_1, X_3] = 0 \tag{5.27}$$

$$[X_1, X_2] = 0 \tag{5.28}$$

$$[X_2, X_4] = X_2 \tag{5.29}$$

$$[X_2, X_3] = X_1 \tag{5.30}$$

$$
\begin{aligned}
&n_t\eta^3_{\bar n} - (\bar n_x u + \bar n u_x)\,\eta^3_{\bar n} + E_t\eta^3_E + (\bar n_x u + \bar n u_x)^2\,\xi^1_{\bar n} + (\bar n_x u + \bar n u_x)\,(n_t\xi^1_{\bar n}) \\
&+ (\bar n_x u + \bar n u_x)\,E_t\xi^1_E - \bar n_x\left[\left(\frac{\bar c}{\bar m}E - uu_x\right)\xi^1 + n_t u\xi^1_n - u\,(\bar n_x u + \bar n u_x)\,\xi^1_{\bar n}\right] \\
&-\bar n_x E_t u\xi^1_u + u\frac{cE}{\psi T}\eta^3_n + u\left[\bar n_x\eta^3_{\bar n} + (4\pi en + 4\pi\bar e\bar n)\,\eta^3_E - \bar n_t\left(\frac{cE}{\psi T}\xi^1_n + \bar n_x\xi^1_{\bar n}\right)\right] \\
&+u\,(4\pi en + 4\pi\bar e\bar n)\,\xi^1_E - u\bar n_x\left[u_x\xi^1 + \frac{ucE}{\psi T}\xi^1_n + \bar n_x u\xi^1_{\bar n} + u\,(4\pi en + 4\pi\bar e\bar n)\,\xi^1_E\right] \\
&+\bar n_x\eta^1 + \bar n\left[\frac{cE}{\psi T}\eta^1_n + \bar n_x\eta^1_n + (4\pi en + 4\pi\bar e\bar n)\,\eta^1_E - \left(\frac{\bar c}{\bar m}E - uu_x\right)\left(\frac{cE}{\psi T}\xi^1\right)\right] \\
&-\bar n\left(\frac{\bar c}{\bar m}E - uu_x\right)\bar n_x\xi^1_{\bar n} - \bar n\left(\frac{\bar c}{\bar m}E - uu_x\right)(4\pi en + 4\pi\bar e\bar n)\,\xi^1_E \\
&-\bar n u_x\left[u_x\xi^1 + \frac{ucE}{\psi T}\xi^1_n + \bar n_x u\xi^1_{\bar n}\right] - \bar n u_x u\,(4\pi en + 4\pi\bar e\bar n)\,\xi^1_E + u_x\eta^3 = 0.
\end{aligned}
$$

We set the coefficients of u_x, E_t, n_t, $\bar n_x$ and u equal to zero and split the determining equation into the following system

$$
\begin{gathered}
\bar n\xi^1_{\bar n} - \xi^1 = 0, \\
-\bar n\eta^3_{\bar n} + \bar n n_t\xi^1_{\bar n} + \bar n E_t\xi^1_E - nu\bar n_x\xi^1_{\bar n} + \eta^3 = 0, \\
\frac{cE}{\psi T}\eta^3_n + (4\pi en + 4\pi\bar e\bar n)\,\eta^3_E = 0,\ \eta^3_E = \eta^3_n = 0, \\
\eta^1 + \bar n\eta^1_{\bar n} = 0.
\end{gathered}
$$

Solving this system we have

$$
\xi^1 = 0,\ \xi^2 = C_1,\ \eta^3 = K_3\bar n,\ \eta^1 = \frac{K_5}{\bar n}, \tag{5.21}
$$

where C_1, K_3 and K_5 are arbitrary constants. Using the above solutions the third equation of (5.18) becomes

$$
\frac{cE}{\psi T}\eta^2\psi' + \psi\left[\frac{cE}{\psi T}\eta^2_n + \bar n_x\eta^2_{\bar n} + (4\pi en + 4\pi\bar e\bar n)\,\eta^2_E\right] = 0 \tag{5.22}
$$

The fourth equation of (5.18) becomes

$$
-4\pi e\eta^2 - 4\pi\bar e K_3\bar n = 0.
$$

Bearing in mind that η^2 is independent of $\bar n$, we have

$$
\eta^2 = K_3 = 0. \tag{5.23}
$$

The prolongation formulae are as given in chapter four (section 4.2). From the first equation of (5.18) we deduce

$$\begin{gathered}
n_t\eta^1_n - \left(\bar n_x u + \bar n u_x\right)\eta^1_{\bar n} + E_t\eta^1_E - \left(\frac{\bar e}{\bar m}E - uu_x\right)\left[n_t\xi^1_n - \left(\bar n_x u + \bar n_x u\right)\xi^1_{\bar n}\right] \\
- \left(\frac{\bar e}{\bar m}E - uu_x\right)E_t\xi^1_E - u_x\left[\left(\frac{\bar e}{\bar m}E - uu_x\right)\xi^1 + n_t\left(\xi^1_n u + C_{1n}\right)\right] \\
+u_x\left(\bar n_x u + \bar n_x u\right)\left(\xi^1_{\bar n}u + C_{1\bar n}\right) - u_x\left[E_t\left(\xi^1_E u + C_{1E}\right)\right] + u\left[\frac{eE}{\psi T}\eta^1_n + \bar n_x\eta^1_{\bar n}\right] \\
+u\left(4\pi e n + 4\pi\bar e\bar n\right)\eta^1_E - u\left(\frac{\bar e}{\bar m}E - uu_x\right)\left[\frac{eE}{\psi T}\xi^1_n + \bar n_x\xi^1_{\bar n} + \left(4\pi e n + 4\pi\bar e\bar n\right)\xi^1_E\right] \\
-uu_x\left[\xi^1 u_o + \frac{eE}{\psi T}\left(\xi^1_n u + C_{1n}\right) + \bar n_x\left(\xi^1_{\bar n}u + C_{1\bar n}\right)\right] \\
-uu_x\left[\left(4\pi e n + 4\pi\bar e\bar n\right)\left(\xi^1_E u + C_{1E}\right)\right] + u_x\eta^1 - \tfrac{\bar e}{\bar m}\eta^4 = 0.
\end{gathered}$$

Setting the coefficients of u_x, n_t, E_t and u equal to zero, we obtain the following system

$$C_{1\bar n} = 0,\ C_{1n} = 0,\ C_{1E} = 0,$$

$$-\bar n\eta^1_{\bar n} + \frac{\bar n\bar e E}{\bar m}\xi^1_{\bar n} - \frac{\bar e E}{\bar m}\xi^1 + \eta^1 = 0,$$

$$\frac{eE}{\psi T}\left(\eta^1_n - \frac{\bar e}{\bar m}E\xi^1_n\right) + \left(4\pi e n + 4\pi\bar e\bar n\right)\left(\eta^1_E - \frac{\bar e}{\bar m}E\xi^1_E\right) = 0,$$

$$n_t\eta^1_n + E_t\eta^1_E - \frac{\bar e E}{\bar m}\left(n_t\xi^1_n + E_t\xi^1_E\right) - \frac{\bar e}{\bar m}\eta^4 = 0.$$

Solving the above equations we find that

$$\begin{gathered}
C_1 = C_1,\ \eta^4 = 0, \\
\eta^1_n - \frac{\bar e E}{\bar m}\xi^1_n = 0,\ \eta^1_E - \frac{\bar e E}{\bar m}\xi^1_E = 0, \\
\eta^1 - \frac{\bar e E}{\bar m}\xi^1 = 0.
\end{gathered} \tag{5.20}$$

The second equation of (5.18) together with (5.19) and (5.20) give

Substituting the coefficients (5.14) into (5.2), we determine

$$
\begin{aligned}
X_4 \;=\;& \xi^1(n,\bar{n},E)\frac{\partial}{\partial t}+\left[\xi^1 u + C_1(n,\bar{n},E)\right]\frac{\partial}{\partial x} \\
&+\eta^1(n,\bar{n},E)\frac{\partial}{\partial u}+\eta^2(n,\bar{n},E)\frac{\partial}{\partial n} \\
&+\eta^3(n,\bar{n},E)\frac{\partial}{\partial \bar{n}}+\eta^4(n,\bar{n},E)\frac{\partial}{\partial E}.
\end{aligned}
\tag{5.15}
$$

Our unknown coefficients depend on $n,\bar{n},E$. Does (4.1) admit X_4? We consider this. For the determining equation we use

$$
\begin{aligned}
\tilde{X}_4 \;=\;& X_4+\zeta_1^1\frac{\partial}{\partial u_t}+\zeta_2^1\frac{\partial}{\partial u_x}+\zeta_1^2\frac{\partial}{\partial n_t}+\zeta_2^2\frac{\partial}{\partial n_x} \\
&+\zeta_1^3\frac{\partial}{\partial \bar{n}_t}+\zeta_2^3\frac{\partial}{\partial \bar{n}_x}+\zeta_1^4\frac{\partial}{\partial E_t}+\zeta_2^4\frac{\partial}{\partial E_x},
\end{aligned}
\tag{5.16}
$$

which is the prolonged operator of X_4. The invariance conditions of our system are given by

$$
\begin{aligned}
\tilde{X}_4\left(u_t+uu_x-\frac{\bar{e}}{\bar{m}}E\right)\Big|_{(4.1)} &= 0, \\
\tilde{X}_4\left(\bar{n}_t+(\bar{n}u)_x\right)\Big|_{(4.1)} &= 0, \\
\tilde{X}_4\left(\psi(n)\,n_x-\frac{e}{T}E\right)\Big|_{(4.1)} &= 0, \\
\tilde{X}_4\left(E_x-4\pi e n-4\pi\bar{e}\bar{n}\right)\Big|_{(4.1)} &= 0.
\end{aligned}
\tag{5.17}
$$

These invariance conditions yield

$$
\begin{aligned}
\zeta_1^1+u\zeta_2^1+u_x\eta^1-\frac{\bar{e}}{\bar{m}}\eta^4 &= 0, \\
\zeta_1^3+u\zeta_2^3+\bar{n}_x\eta^1+\bar{n}\zeta_2^1+u_x\eta^3 &= 0, \\
n_x\eta^2\psi'(n)+\psi(n)\,\zeta_2^2-\frac{e}{T}\eta^4 &= 0, \\
\zeta_2^4-4\pi e\eta^2-4\pi\bar{e}\eta^3 &= 0,
\end{aligned}
\tag{5.18}
$$

in which we substitute

$$
\begin{aligned}
u_t &= \frac{\bar{e}}{\bar{m}}E-uu_x,\; \bar{n}_t=-\left(\bar{n}_x u+\bar{n}u_x\right), \\
n_x &= \frac{eE}{\psi T},\; E_x=4\pi e n+4\pi\bar{e}\bar{n}.
\end{aligned}
\tag{5.19}
$$

$$[X_3, X_4] = 0. \tag{5.8}$$

From (5.1) and (5.2), (5.3) becomes

$$[X_1, X_4] = \xi^1_x \frac{\partial}{\partial t} + \xi^2_x \frac{\partial}{\partial x} + \eta^1_x \frac{\partial}{\partial u} + \eta^2_x \frac{\partial}{\partial n} + \eta^3_x \frac{\partial}{\partial \bar{n}} + \eta^4_x \frac{\partial}{\partial E} = 0. \tag{5.9}$$

The above equation (5.9) yields

$$\begin{aligned} \xi^i_x &= 0 \Rightarrow \xi^i = \xi^i\,(t, u, n, \bar{n}, E)\,,\ i = 1, 2 \\ \eta^j_x &= 0 \Rightarrow \eta^j = \eta^j\,(t, u, n, \bar{n}, E)\,,\ j = 1, ..., 4. \end{aligned} \tag{5.10}$$

Equations (5.4), (5.5) and (5.7) are identically satisfied. Substituting (5.1) and (5.2) into (5.6) and making use of (5.10), we obtain

$$[X_2, X_4] = \xi^1_t \frac{\partial}{\partial t} + \xi^2_t \frac{\partial}{\partial x} + \eta^1_t \frac{\partial}{\partial u} + \eta^2_t \frac{\partial}{\partial n} + \eta^3_t \frac{\partial}{\partial \bar{n}} + \eta^4_t \frac{\partial}{\partial E} = 0. \tag{5.11}$$

From this equation we have the following

$$\begin{aligned} \xi^i &= \xi^i\,(u, n, \bar{n}, E)\,,\ i = 1, 2 \\ \eta^j &= \eta^j\,(u, n, \bar{n}, E)\,,\ j = 1, ..., 4. \end{aligned} \tag{5.12}$$

Equation (5.8) gives

$$[X_3, X_4] = \xi^1_u \frac{\partial}{\partial t} + \xi^2_u \frac{\partial}{\partial x} + \eta^1_u \frac{\partial}{\partial u} + \eta^2_u \frac{\partial}{\partial n} + \eta^3_u \frac{\partial}{\partial \bar{n}} + \eta^4_u \frac{\partial}{\partial E} - \xi^1 \frac{\partial}{\partial x} = 0. \tag{5.13}$$

This equation implies the following

$$\begin{aligned} \xi^1 &= \xi^1\,(n, \bar{n}, E)\,, \\ \xi^2 &= \xi^1 u + C_1\,(n, \bar{n}, E)\,, \\ \eta^j &= \eta^j\,(n, \bar{n}, E)\,. \end{aligned} \tag{5.14}$$

A. Algebra $L_{4,1}$

The principal Lie algebra, $L_{\mathcal{P}}$, of system (4.1) has the following basis generators

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t}, \\
X_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.
\end{aligned}
\tag{5.1}
$$

This basis satisfies the commutation relations of $L_{3,2}$. We impose algebra $L_{4,1}$ on our system (4.1) which according to our labeling denotes the first algebra of dimension four. Consequently we assume that there is another element X_4, apart from those in (5.1) which constitute $L_{4,1}$, viz.,

$$
\begin{aligned}
X_4 &= \xi^1(t,x,u,n,\bar{n},E)\frac{\partial}{\partial t} + \xi^2(t,x,u,n,\bar{n},E)\frac{\partial}{\partial x} \\
&\quad + \eta^1(t,x,u,n,\bar{n},E)\frac{\partial}{\partial u} + \eta^2(t,x,u,n,\bar{n},E)\frac{\partial}{\partial n} \\
&\quad + \eta^3(t,x,u,n,\bar{n},E)\frac{\partial}{\partial \bar{n}} + \eta^4(t,x,u,n,\bar{n},E)\frac{\partial}{\partial E}.
\end{aligned}
\tag{5.2}
$$

According to Table 2.2, (5.1) together with (5.2) satisfy the commutators

$$[X_1, X_4] = 0 \tag{5.3}$$

$$[X_1, X_3] = 0 \tag{5.4}$$

$$[X_1, X_2] = 0 \tag{5.5}$$

$$[X_2, X_4] = 0 \tag{5.6}$$

$$[X_2, X_3] = X_1 \tag{5.7}$$

Chapter 5

Classification according to low-dimensional Lie algebras

In this chapter we completely classify system (4.1) according to the low-dimensional Lie algebra it admits. We do not use the methods of classification discussed in chapter three because of their inherent difficulties and disadvantages. The method we use here has its own advantages and disadvantages. This method shows no disadvantage in the group classification of system (4.1). Unlike other methods, we know when the system or equation is partially or totally classified using this method.

The principal Lie algebra L_P of system (4.1) satisfy $L_{3,2}$ of Table 2.1. $L_{3,2}$ is a subalgebra of each of the four-dimensional algebras listed in Table 2.2 (see e.g., Mahomed 1986). These are the only algebras of dimension four that has $L_{3,2}$ as a subalgebra (see Mahomed 1986). As a result of this, we use Table 2.2 to classify our system (4.1). Let us show now this method works. The work presented here is new.

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C. For $v(n) = 1 + n$, the program reads as follows

```
ECHO:TRUE
NDEP#:4 $
NIND#:2 $
DV#:DE#:{} $
DE#[1]:U(1,1)+U(1)*U(1,2)-(B/D)*U(4) $
DE#[2]:U(3,1)+U(3)*U(1,2)+U(3,2)*U(1) $
DE#[3]:(1+U(2))*U(2,2)-((A/T)*U(4)) $
DE#[4]:U(4,2)-4*#PI*A*U(2)-4*#PI*B*U(3) $
DV#[1]:U(1,1) $
DV#[2]:U(3,1) $
DV#[3]:U(2,2) $
DV#[4]:U(4,2) $
ECHO:FALSE $
RDS() $
```

We find the symmetrics

VEC# (1) = = -D# (X2)

VEC# (2) = = -D# (X1)

VEC# (3) = = -X1*D# (X2) - D# (U1)

The above results confirm the analytical results we have obtained.

VEC# (1) = = -D# (X2)

VEC# (2) = = -D# (X1)

VEC# (3) = = -X1*D# (X2) - D# (U1)

B. For $\psi(n) = \frac{C}{n^{1+P}}$, where C and P are arbitrary constants, we have

```
ECHO:TRUE
NDEP#:4 $
NIND#:2 $
DV#:DE#:{} $
DE#[1]:U(1,1)+U(1)*U(1,2)-(B/D)*U(4) $
DE#[2]:U(3,1)+U(3)*U(1,2)+U(3,2)*U(1) $
DE#[3]:C*U(2,2)-((A/T)*U(4)*U(2)^(P+1)) $
DE#[4]:U(4,2)-4*#PI*A*U(2)-4*#PI*B*U(3) $
DV#[1]:U(1,1) $
DV#[2]:U(3,1) $
DV#[3]:U(2,2) $
DV#[4]:U(4,2) $
ECHO:FALSE $
RDS() $
```

Here, the symmetries are

VEC# (1) = = P*D# (X1) / (-P + P^2)

VEC# (2) = = P*D# (X2) / (-P + P^2) - P^2*D# (X2) / (-P + P^2)

VEC# (3) = = X1*P*D# (X2) / (-P + P^2) - X1*P^2D# (X2) / (-P + P^2) + P*D# (U1) / (1 - P) - D# (U1) / (1 - P)

VEC# (4) = = U1*D# (U4) / P - X1*D# (X1) / (-P + P^2) - X2*P*D# (X2) / (-P + P^2) - X2*D# (X2) / (-P + P^2)

Appendix

We find the symmetries of the system

$$u_t + uu_x = \frac{\bar{e}}{\bar{m}}E,$$
$$\bar{n}_t + (\bar{n}u)_x = 0,$$
$$\psi(n)\, n_x = \frac{eE}{T},$$
$$E_x = 4\pi e n + 4\pi \bar{e}\bar{n},$$

by using the program LIE version 4.1 (c). We consider different forms of $\psi(n)$ for implementing the program.

A. For $\psi(n)$ arbitrary, the program gives

```
ECHO:TRUE
NDEP#:4 $
NIND#:2 $
DV#:DE#:{} $
DE#[1]:U(1,1)+U(1)*U(1,2)-(B/D)*U(4) $
DE#[2]:U(3,1)+U(3)*U(1,2)+U(3,2)*U(1) $
DE#[3]:H(U(2))*U(2,2)-((A/T)*U(4)) $
DE#[4]:U(4,2)-4*#PI*A*U(2)-4*#PI*B*U(3) $
DV#[1]:U(1,1) $
DV#[2]:U(3,1) $
DV#[3]:U(2,2) $
DV#[4]:U(4,2) $
ECHO:FALSE $
RDS() $
```

The symmetries are

alence symmetry is finite. For the equation utilized, the problem of group classification reduces to the construction of optimal system of Lie subalgebras. This method can be tedious, as it was in this example. As the name implies, it is also a partial group classification method.

In chapter four, the equivalence and principal Lie algebras of the system of equations (4.1) were found. These symmetries became useful in the following chapter. We used a new method to classify our system. Before using this new method, the preliminary group classification method was used to classify system (4.1) and the results showed that $\psi(n) = \frac{c}{n}$, where c is a constant. This new method classifies according to low-dimesional principal Lie algebras and our system (4.1) had a three dimensional Lie algebra. We invoked the Lie algebras of dimension four listed in chapter two (Table 2.2) to completely classify the system (4.1). Accordingly, we extended previous results on the symmetry group of (4.1) obtained by Euler, Steeb and Mulser (1991) and (1992). The method we have utilized is a variant of the method used by Vawda and Mahomed (1994) and has been used for the first time in the classification of partial differential equations. Our results show that $\psi(n) = \frac{A}{n^{1+\alpha}}$, where A and α are arbitrary constants, in order for the system (4.1) to admit a four-dimensional algebra. For ψ an arbitrary function of n, (4.1) admits the principal Lie algebra $L_{\mathcal{P}} \equiv L_{3,2}$. Unlike the other methods we have considered, it is not a partial group classification method.

The method we have introduced in chapter five can also be used to classify the equations of chapter three, so as to confirm the classification which was carried out by the preliminary group classification methods. For future works, the symmetries of the system (4.1) we have classified can be used to find invariant solutions and also optimal system of invariant solutions.

Chapter 6

Conclusion

In our work we have illustrated different methods of group classification of partial differential equations. We have illustrated each method by an example. Along with this, we have introduced a new method of group classification which classifies according to low-dimensional Lie algebras.

The partial differential equation we have used in Example A as an illustration has nine equivalence symmetries and five Lie symmetries. The method discussed in this chapter works with the knowledge of the determining equations and the equivalence group of the equation (system of equations) under consideration. This method worked perfectly for the example used. Since this method is a partial classification method, one is not sure if it completely classifies the equation. This is one of its disadvantages. The other is that you cannot state a priori which equation (system of equations) can or cannot be classified using it.

The preliminary group classification method discussed in chapter three was illustrated by a partial differential equation which possesses infinite number of equivalence symmetries and three Lie symmetries, viz., example B. This method works with equivalence symmetries and is simple when the equiv-

5.1 Discussion

The method used above is new. It is only applicable to equations or systems which possess low-dimensional Lie algebras, as the one considered above. Using this method we have completely classified system (4.1) for four-dimensional algebras. We find that ψ can only be a function of the form $\frac{\sigma}{n^{1+b}}$, where σ and b are constants. Thus, we have achieved a generalisation of the result of Euler, Steeb and Mulser (1991) and (1992).

Some computational work, using the program LIE by A K Head (1993), has been done to confirm the above analytical work for some functional forms of ψ. We show this in the Appendix.

The commutators satisfied by the above operators are

$$[X_1, X_4] = 0 \tag{5.34}$$

$$[X_1, X_3] = 0 \tag{5.35}$$

$$[X_1, X_2] = 0 \tag{5.36}$$

$$[X_2, X_4] = X_1 \tag{5.37}$$

$$[X_3, X_2] = 0 \tag{5.38}$$

$$[X_3, X_4] = X_2. \tag{5.39}$$

Substituting our operators in the above commutators, starting from (5.34) to (5.39), we obtain

$$\begin{aligned} X_4 \;=\; & \xi^1(n, \bar{n}, E)\frac{\partial}{\partial t} + \left[\frac{u}{\alpha} + \xi^1 - \alpha t + C_2(n, \bar{n}, E)\right]\frac{\partial}{\partial x} \\ & + \eta^1(n, \bar{n}, E)\frac{\partial}{\partial u} + \eta^2(n, \bar{n}, E)\frac{\partial}{\partial n} + \eta^3(n, \bar{n}, E)\frac{\partial}{\partial \bar{n}} \\ & + \eta^4(n, \bar{n}, E)\frac{\partial}{\partial E}. \end{aligned}$$

Solving the determining equations, we find

$$\xi^1(n, \bar{n}, E) + C(n, \bar{n}, E) + \frac{u}{\alpha} + \alpha = 0.$$

The above equation gives

$$\alpha \neq 0, \; u = 0,$$
$$\xi^1(n, \bar{n}, E) + C(n, \bar{n}, E) = -\alpha.$$

The equation $u = 0$ is impossible. Hence our system does not admit the algebra we have $L_{4,6}$.