# An Analysis of the Invariance and Conservation Laws of some Classes of Nonlinear Wave Equations

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in requirement for the degree Master of Science.

Johannesburg, 2011

## DECLARATION

I declare that the contents of this dissertation are original, except where due references have been made. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other institution.

S. Jamal

## Acknowledgements

Praise, Glory, and thanks be to Allah, the most Gracious, the most Merciful. He who enabled me to successfully complete this research project, and peace be upon his Prophet.

I would like to express my sincere appreciation and thanks to Professor A. H. Kara, my supervisor, for his great assistance in making this project possible. His vast knowledge, guidance, suggestions and technical assistance has been much appreciated.

This work has been made possible, in part by the financial aid received from the University of the Witwatersrand and the National Research Foundation (NRF) of South Africa (Grant number FA - 2007041200006).

Finally, my deepest and greatest gratitude goes to my family, for their encouragement and prayers, and my mum for her patience and encouragement.

Sameerah Jamal
June 2011.

*"This, therefore, is Mathematics:*
*She reminds you of the invisible forms of the soul;*
*she gives life to her own discoveries;*
*she awakens the mind and purifies the intellect;*
*she brings to light our intrinsic ideas;*
*she abolishes oblivion and ignorance which are ours by birth ..."*

Diadochus Proclus (410-485)

## Abstract

We analyse nonlinear partial differential equations arising from the modelling of wave phenomena. A large class of wave equations with dissipation and source terms are studied using a symmetry approach and the construction of conservation laws. Some previously unknown conservation laws and symmetries are obtained. We then proceed to use the multiplier (and homotopy) approach to construct conservation laws from which we obtain some surprisingly interesting higher-order variational symmetries. We also find the corresponding conserved quantities for a large class of Gordon-type equations similar to those of the sine-Gordon equation and the relativistic Klein-Gordon equation. In particular, we direct our research and analysis towards a wave equation with non-constant coefficient terms, that is, coefficients dependent on time and space. Finally, we study a class of multi-dimensional wave equations.

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# Introduction

Nonlinear partial differential equations (PDEs) that admit conservation laws arise in many disciplines of the applied sciences including physical chemistry, fluid mechanics, particle and quantum physics and nonlinear optics, to name a few. Conservation laws have many important applications, both physical and mathematical. These laws state that certain physical quantities, for example, momentum or energy, will not change with time during physical processes.

In the analysis of differential equations and symmetries, two names come to mind, Sophus Lie and Emmy Noether. Sophus Lie's profound contribution to mathematics can be seen through his discovery that the special techniques required in solving various equations are in fact just special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. In 1918, Emmy Noether proved two extraordinary theorems relating symmetry groups of a variational integral to properties of its associated Euler-Lagrange equations. In the first of her theorems, Noether shows how each one-parameter variational symmetry group gives rise to a conservation law, for example, energy conservation comes from the invariance of the problem under a group of time translations [18].

The modelling of wave phenomena using simple linear and more complex, nonlinear partial differential equations is well known and the methods used to analyse and solve the resulting equations are varied and well established, both exactly and numerically. Complex versions of the models necessitates equations with greater nonlinearities (see [22, 21]). Symmetry methods have been used to solve/analyse such PDEs and also, for determining conservation laws (see [13, 19, 4], inter alia).

A symmetry of a given PDE is a transformation (mapping) that maps any solution of the PDE into another solution of the PDE. Once a symmetry group of a system of differential equations has been determined, one can firstly use the defining property of such a group and construct new solutions to the system from known solutions, thus allowing us to classify different symmetry classes of solutions. Secondly, the symmetry group can be used to classify families of differential functions depending on arbitrary parameters or functions, where equations with a high degree of symmetry are useful in mathematics and physics. Also the existence of infinitely many generalized symmetries is closely connected with the possibility of linearizing a system, [19]. In our study, we investigate the existence of higher-order symmetries and conserved flows of various nonlinear wave equations.

We come across the Klein-Gordon equation, which was the first relativistic wave equation. The equation was named after the physicists Oskar Klein and Walter Gordon in 1927. The nonlinear Klein-Gordon equation has widespread applications in physics and wave propagation.

In Chapter one, the mathematics which we require for our research along with notation and theory are presented. We describe two different approaches used in the investigation, namely, the multiplier method and the use of Noether's theorem.

Chapter two discusses the dissipative wave equations which have Lagrangians; one of the equations needs to be multiplied by a *variational integration factor*. Consequently, we can obtain Noether symmetries for these equations using Noether's theorem. We also determine a bigger class of conservation laws for these equations which were previously unknown. In this chapter, we also introduce and analyse a generalized Gordon-type equation and present some special cases of this equation.

In Chapter three, we use the multiplier method and find *new* higher-order symmetries for the canonical form of the Gordon-type equations and thereafter we prove that these symmetries are variational. In our study we also present the associated conservation laws for these symmetries. We conclude with some special and interesting results.

In the fourth Chapter, we pursue a study of some multi-dimensional Gordon-type equations in classical form, and via the multiplier approach, we obtain previously unknown higher-order symmetries. We hope to gain insight into equations with higher dimension and thus greater complexity.

# Chapter 1

# Preliminaries

## 1.1 Introduction

A systematic way of determining conservation laws for systems of Euler-Lagrange equations once their Noether symmetries are known is via the celebrated Noether's theorem ([18], [3], see also the books by [20], [14], [4], [10] and [19]). This theorem relies on the availability of a Lagrangian and many works have been devoted to the inverse problem in the calculus of variation, i.e., to determine when a differential equations system has a Lagrangian formulation for a suitable Lagrangian function (see e.g. [2]). The Euler and Lie-Bäcklund or generalised operators also play a fundamental role in the investigation of algebraic properties in variational calculus and differential equations (see, e.g. [20], [14], [4] and [19] and references therein). The other approach used in finding conserved quantities, called the multiplier approach, is described below. We also present some of the definitions and notations used. Note that the summation convention is adopted throughout.

## 1.2 Differential Functions

Intrinsic to a Lie algebraic treatment of differential equations is the universal space $\mathcal{A}$ (see, e.g. [19] or [14]). A locally analytic function $f(x,\, u,\, u_{(1)},\dots,u_{(k)})$ of a finite number of variables is called a *differential function of order* $k$. The variables $u_{(1)}, u_{(2)},\dots,u_{(k)}$ denote the collections of all first, second, ..., $k$th-order partial derivatives, that is

$$u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$$

respectively, with the total differentiation operator with respect to $x^i$ given by,

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1,\dots,n. \tag{1}$$

The space $\mathcal{A}$ is the vector space of all differential functions of all finite orders and forms an algebra. A total derivative converts any differential function of order $k$ to a differential function of order $k+1$. Hence, the space $\mathcal{A}$ is closed under total derivations $D_i$. There are also other operators on $\mathcal{A}$ and some of the important ones which we utilize in the sequel are presented in the definitions below. These are well-known and can be found in the books of [20], [14], [19] and [4] (see also [15]).

## 1.3 The Multiplier Approach

Consider an $r$th-order system of PDEs of $n$ independent variables $x = (x^1, x^2, \dots, x^n)$ and $m$ dependent variables $u = (u^1, u^2, \dots, u^m)$

$$G^\mu(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \mu = 1, \dots, \tilde{m}, \tag{2}$$

A current $\Phi = (\Phi^1, \dots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{3}$$

along the solutions of (2). It can be shown (see [9]) that every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{4}$$

holds identically (i.e., off the solution space) for some current $\Phi^i$. The conserved current may then be obtained by the homotopy operator which is discussed in section (1.6). See also [1, 11].

## 1.4 Fundamental Operators

**Definition 1.** The *Euler* operator, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\,\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\;, \qquad \alpha = 1,\ldots,m\;. \tag{5}$$

The terms Euler operator and variational derivative are used interchangeably.

**Definition 2.** If there exists a function $L = L(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) \in \mathcal{A}; \quad s \leq r$, $r$ being the order of (2), such that

$$\frac{\delta L}{\delta u^\alpha} = 0, \qquad \alpha = 1,\ldots,\tilde{m}$$

then $L$ is called a Lagrangian of (2) and $\dfrac{\delta}{\delta u^\alpha}$ is the corresponding Euler operator in (5). $\dfrac{\delta L}{\delta u^\alpha} = 0$ are known as the Euler-Lagrange equations.

**Note.** In most literature, a variational problem consists of finding the extrema (maxima or minima) of a *functional*

$$\mathcal{L}[u] = \int_\Omega L(x, u_{(n)})dx$$

in some class of functions $u = f(x)$ defined over $\Omega$, where $\Omega \subset X$ is an open, connected subset with smooth boundary $\partial\Omega$, (we consider the Euclidean space with $X = R^n$). The integrand $L(x, u_{(n)})$, called the Lagrangian of the variational problem $\mathcal{L}$, is a smooth function of $x, u$ and various derivatives of $u$ [19].

**Definition 3.** The Lie-Bäcklund or generalised operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} , \qquad \xi^i,\ \eta^\alpha \in \mathcal{A} . \tag{6}$$

This operator is an abbreviated form of the following infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1\ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \tag{7}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} , \\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s > 1. \end{aligned} \tag{8}$$

In (8), $W^\alpha$ is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j . \tag{9}$$

One can write the Lie-Bäcklund or generalised operator (7) in the characteristic form

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}} \tag{10}$$

**Definition 4.** We call $X$ a Lie point symmetry if $\xi^i$ and $\eta^\alpha$ in (6) are functions of $x^i$ and $u^\alpha$ only. A Lie (point) symmetry is characterized by an infinitesimal transformation which leaves the given differential equation invariant under the transformation of all independent variables $x^i$ and dependent variables $u^\alpha$.

**Definition 5**. Lie-Bäcklund or generalised operators $\tilde{X}$ and $X$ are said to be *equivalent* if

$$X - \tilde{X} = \lambda^i D_i, \quad \lambda^i \in \mathcal{A}\,.$$

In particular, a generalized operator of the form $\tilde{X} = Q^\alpha \partial/\partial u^\alpha + \cdots$, where $Q^\alpha \in \mathcal{A}$, is called a *canonical* or *evolutionary* representation of $X$, and $Q^\alpha$ is called its *characteristic.*

**Definition 6.** The Noether operator associated with a Lie-Bäcklund operator $X$ is given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{i i_1 \cdots i_s}}\,, \quad i = 1, \ldots, n, \tag{11}$$

where the Euler-Lagrange operators with respect to derivatives of $u^\alpha$ are obtained from (5) by replacing $u^\alpha$ by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{i j_1 \cdots j_s}} \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m\,. \tag{12}$$

The Euler, Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i\,. \tag{13}$$

**Definition 7.** A Lie-Bäcklund operator $X$ of the form (6) is called a Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$, if there exists a vector $B^i = (B^1, \dots, B^n), B^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{14}$$

If $B^i = 0, (i = 1, \dots, n)$, then $X$ is referred to as a strict Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$.

**Noether's Theorem.** For any Noether symmetry $X$ corresponding to a given Lagrangian $L \in \mathcal{A}$, there exists a current $\Phi^i = (\Phi^1, \dots, \Phi^n), \quad \Phi^i \in \mathcal{A}$, defined by

$$\Phi^i = B^i - N^i(L) , \qquad i = 1, \dots, n , \tag{15}$$

which is a conserved current of the Euler-Lagrange equations $\dfrac{\delta L}{\delta u^\alpha} = 0$ , where $N^i$ and $B^i$ are defined above. See also [18, 19, 16, 17].

**Definition 8.** Let $G$, $G \in \mathcal{A}$, be a system of differential equations. A recursion operator for $G$ is a linear operator $\Re : \mathcal{A}^q \to \mathcal{A}^q$ in the space of $q-$tuples of differential functions, with the property that whenever $X = Q\partial_u$ is an evolutionary vector field of $G$, so is $X = \tilde{Q}\partial_u$ with $\tilde{Q} = \Re Q$.

**Proposition 1.** Let $\Gamma$, $\Gamma \in \mathcal{A}$, be a linear system of differential functions, with $\Gamma$ denoting a linear differential operator. A second linear differential operator $\Re : \mathcal{A}^q \to \mathcal{A}^q$ not depending on $u$ or its derivatives is a recursion operator for $\Gamma$, if and only if $Q = \Re[u]$ is the characteristic of a 'linear' generalized symmetry to the system.

## 1.5 Fréchet Derivatives and Variational Symmetries

**Definition 9.** Consider the differential functions $G$ in (2). The *Fréchet derivative* of $G$ is the differential operator $D_G : \mathcal{A}^q \to \mathcal{A}^r$ defined so that

$$D_G(Q) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} G[u + \epsilon Q[u]]$$

for any $Q \in \mathcal{A}^q$. More simply, to evaluate $D_G(Q)$, we replace $u$ and the derivatives of $u$ in $G[u]$ by $u + \epsilon Q$ and then differentiate the resulting expression with respect to $\epsilon$.

**Example:** If $G[u] = u_x u_{xx}$ then,

$$D_G(Q) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (u_x + \epsilon D_x Q)(u_{xx} + \epsilon D_x^2 Q) = u_x D_x^2 Q + u_{xx} D_x Q.$$

**Definition 10.** If

$$\mathcal{D} = \sum_J P_J[u] D_J, \quad P_J \in \mathcal{A},$$

is a differential operator, then the adjoint of $\mathcal{D}$ is the differential operator $\mathcal{D}^*$ which satisfies

$$\int_\Omega P \cdot \mathcal{D} Q dx = \int_\Omega Q \cdot \mathcal{D}^* P dx$$

for every pair of differential functions $P, Q \in \mathcal{A}$, which vanish when $u = 0$, every domain $\Omega \subset R^n$ and every function $u = f(x)$ of compact support in $\Omega$. It can also be shown by integration by parts that

$$\mathcal{D}^* = \sum_J (-D)_J \cdot P_J,$$

meaning that for any $Q \in \mathcal{A}$,

$$\mathcal{D}^* Q = \sum_J (-D)_J [P_J[Q]] \qquad \text{(see[19])}.$$

**Example:** If $\mathcal{D} = D_x^2 + uD_x$, then $\mathcal{D}$'s adjoint is,

$$\mathcal{D}^* = (-D_x)^2 + (-D_x) \cdot u = D_x^2 - uD_x - u_x.$$

**Definition 11.** A Lie-Bäcklund operator $X$ of the form (6) is called a variational symmetry if it leaves invariant the functional $\mathcal{L}[u] = \int L(x, u_{(n)})dx$.

**Note.** Our focus is to investigate the existence of *higher-order* symmetries. When we use the multiplier method (which is not a variational technique like Noether's theorem) in search of higher-order symmetries, we need to verify that these symmetries are variational as we may encounter symmetries that are non-variational. Variational symmetries are important if they are 'associated' with the components of the conservation law and therefore may be used for double reductions and the integrability of the differential equation (see [16] for more on the relationship between symmetries and conservation laws).

The following theorem defines the conditions under which a symmetry is variational.

**Theorem 1.** For variational PDEs, $E = 0$, $E \in \mathcal{A}$, an evolutionary vector field $X$ is a variational symmetry if and only if $XE + \mathcal{A}\mathcal{F}_Q E = 0$, where $\mathcal{A}\mathcal{F}$ is the adjoint *Fréchet derivative* [19].

## 1.6 The Homotopy Operator

In a dynamical problem, we may encounter one of the independent variables as the time $t$, the remaining variables may be the spatial variables, $X = (x, y, z)$. Now, the conservation law takes the form

$$D_t\Phi^t + D_x\Phi^x + D_y\Phi^y + D_z\Phi^z = 0.$$

$\Phi^t$ is known as the conserved density and $(\Phi^x, \Phi^y, \Phi^z)$ are the conserved fluxes, which are functions of $x, y, z, t, u$ and the derivatives of $u$.

The continuous homotopy operator is a powerful tool that can be used to compute densities and fluxes explicitly. It allows one to invert the total divergence operator $D_i$, by computing higher variational derivatives followed by a one-dimensional integration with respect to a single auxiliary parameter. This operator can be applied to problems in which integration by parts of arbitrary functions in multi-variables is essential. A literature search done by the authors of [12] revealed that homotopy operators are used in integrability testing and inversion problems involving PDEs, differential-difference equations (DDEs), lattices and beyond.

In defining the Homotopy operator, note that Definition 13 is a calculus-based formula for the homotopy operators in one dimension (the formulae for higher dimensions and higher Euler operators are analogous and can be found in [12]). Other definitions which involve differential forms do exist (see [19]). The calculus based formulae are easier to implement in software packages such as *Mathematica*. In our calculations, we made extensive use of the computer software involving the calculus-based formulae.

**Definition 12.** The higher Euler operators (Lie-Euler operators) in one dimension (with variable $x$) are

$$\mathcal{E}_{u(x)}^{(i)} = \sum_{k=i}^{\infty} \binom{k}{i} (-D_x)^{k-i} \frac{\partial}{\partial u_{kx}}.$$

Note that the higher Euler operator for $i = 0$ would give us the Euler operator (in one dimension with variable $x$) in Definition 1 .

**Definition 13.** The homotopy operator in one dimension with variable $x$ is

$$\mathcal{H}_{u(x)}(f) = \int_0^1 \sum_{j=1}^{m} I_{u^j}(f)[\lambda u] \frac{d\lambda}{\lambda},$$

where $f \in \mathcal{A}$, $u^j$ is the $j$th component of the dependent variable $u$ and the integrand $I_{u^j}(f)$ is given by

$$I_{u^j}(f) = \sum_{i=0}^{\infty} D_x^i \left( u^j \mathcal{E}_{u^j(x)}^{(i+1)}(f) \right),$$

where $\mathcal{E}_{u^j(x)}^{(i+1)}$ denotes the $(i+1)$th higher Euler operator, $m$ is the number of dependent variables, $\lambda$ is an auxiliary variable and $I_{u^j}(f)[\lambda u]$ means that in $I_{u^j}(f)$, we replace $u \to \lambda u, u_x \to \lambda u_x$, etc., [11].

# Chapter 2

# Dissipative Wave Equations

## 2.1 Introduction

Firstly, we note that the basic linear (1+1) wave equation with dissipation is given by

$$u_{tt} + u_t - u_{xx} = 0 \tag{16}$$

and the general nonlinear version is given by $u_{tt} + \phi(u)u_t - (g(u)u_x)_x = 0$. The symmetries and conservation laws of (16) are listed in [13] with the conserved vectors obtained via the Lagrangian

$$L = \frac{1}{2}e^t(u_t^2 - u_x^2). \tag{17}$$

In the first of our analysis below, we revisit these equations for illustrative purposes - to determine whether a bigger class of conservation laws are available for (16).

Furthermore, when the dissipative wave equation above is subject to a source term

$k(u)$, we may obtain some classes of (dissipative) Gordon-type equations, viz.,

$$u_{tt} + u_t - u_{xx} + k(u) = 0 \tag{18}$$

in which, for e.g., $k(u) = \sin u$ is the corresponding sine-Gordon equation and for $k(u)$ being a linear polynomial in $u$, we have the equivalent of the Klein-Gordon equation.

We consider a variety of PDEs of interest. This section will not only serve to illustrate the method of Noether's theorem, but also to construct previously unknown conserved flows of the PDEs in question. Suppose $(t, x)$ and $u$ are the independent and dependent variables, respectively. For simplicity we have looked at point type symmetry operators and we have restricted the gauge terms to be independent of derivatives. One can equally well try to find Lie-Bäcklund type symmetry operators and the method still applies. In this case however, the calculations are quite tedious and are best done by a computer algebra package.

## 2.2 Illustrative Example 1

Consider the wave equation with dissipation (16) and let

$$X = \xi(t, x, u)\partial_x + \tau(t, x, u)\partial_t + \phi(t, x, u)\partial_u$$

be a Noether point operator that satisfies (14) with gauge vector $(f, g)$. This becomes, for the Lagrangian given by (17),

$$\begin{aligned} &\phi^t(e^t u_t) - \phi^x(e^t u_x) + \tau(\tfrac{1}{2}e^t u_t^2 - \tfrac{1}{2}e^t u_x^2) + \tfrac{1}{2}e^t(u_t^2 - u_x^2)[D_t\tau + D_x\xi] \\ &= D_t f + D_x g. \end{aligned} \tag{19}$$

Separation by derivatives of $u$ yields the following overdetermined system

$$\begin{aligned}
u_t^3 &: \tau_u = 0 \\
u_x^3 &: \xi_u = 0 \\
u_x^2 &: \phi_u = -\tfrac{1}{2}\tau + \tfrac{1}{2}\tau_t - \tfrac{1}{2}\xi_x \\
u_t u_x &: \tau_x = \xi_t \\
u_t &: e^t \phi_t = f_u \\
u_x &: -e^t \phi_x = g_u \\
1 &: f_t + g_x = 0
\end{aligned}$$

which leads to $-\tau_t - 2\tau_{txx} + \tau_{ttt} + \xi_{xxx} = 0$ and $\phi = \frac{1}{2}u(-\tau + \tau_t - \xi_x) + \alpha(x,t)$. The gauge terms are,

$$f = \frac{u^2}{4}e^t(-\tau_t + \tau_{tt} - \xi_{xt}) + ue^t\alpha_t + \gamma(x,t)$$

and

$$g = -\frac{u^2}{4}e^t(-\tau_x + \tau_{xt} - \xi_{xx}) - ue^t\alpha_x + \delta(x,t),$$

where $\delta_x + \gamma_t = 0$.

Some particular choices of $\tau$ and $\xi$ that satisfy these equations lead to the following symmetries

$$\begin{aligned}
X_1 &= \partial_x \\
X_2 &= \partial_t - \tfrac{u}{2}\partial_u \\
X_3 &= t\partial_x + x\partial_t - \tfrac{xu}{2}\partial_u \\
X_\infty &= \alpha\partial_u
\end{aligned}$$

where $\alpha$ satisfies the equation $\alpha_t + \alpha_{tt} - \alpha_{xx} = 0$. With the use of (15), we now list the corresponding conserved flows.

(i) $X_1 = \partial_x, \quad f = \gamma(x,t)$ and $g = \delta(x,t)$,

$$\Phi_1^t = -u_x u_t e^t - \gamma, \quad \Phi_1^x = \tfrac{1}{2} e^t (u_t^2 - u_x^2) + u_x^2 e^t - \delta,$$

Thus,

$$\begin{aligned} D_x\Phi_1^x + D_t\Phi_1^t &= -u_{tt}u_x e^t - u_x u_t e^t - u_{tx}u_t e^t + u_{tx}u_t e^t + u_{xx}(-u_x e^t + 2u_x e^t) \\ &\quad -(\delta_x + \gamma_t) \\ &= -u_x e^t\,(u_{tt} - u_{xx} + u_t) - (\gamma_t + \delta_x) \\ &= 0, \quad \text{along the solutions of (16).} \end{aligned}$$

(ii) $X_2 = \partial_t - \frac{u}{2}\partial_u, \quad f = \gamma(x,t)$ and $g = \delta(x,t)$,

$$\Phi_2^t = \tfrac{1}{2} e^t (u_t^2 - u_x^2) - (\tfrac{u}{2} + u_t) e^t u_t - \gamma, \quad \Phi_2^x = (\tfrac{u}{2} + u_t) e^t u_x - \delta,$$

Thus,

$$\begin{aligned} D_t\Phi_2^t + D_x\Phi_2^x &= \tfrac{1}{2} e^t (u_t^2 - u_x^2) - e^t u_t(\tfrac{u}{2} + u_t) + u_{tt}(u_t e^t - 2u_t e^t - \tfrac{u}{2} e^t) - u_{tx}u_x e^t \\ &\quad -\tfrac{1}{2}u_t^2 e^t + u_{tx}u_x e^t + u_{xx}e^t(\tfrac{u}{2} + u_t) + \tfrac{1}{2}u_x^2 e^t - (\gamma_t + \delta_x) \\ &= -\left(u_t + \tfrac{u}{2}\right) e^t\,(u_{tt} - u_{xx} + u_t) - (\gamma_t + \delta_x) \\ &= 0, \quad \text{along the solutions of (16).} \end{aligned}$$

(iii) $X_3 = t\partial_x + x\partial_t - \frac{xu}{2}\partial_u, \quad f = \gamma(x,t)$ and $g = \frac{u^2}{4} e^t + \delta(x,t)$,

$$\begin{aligned} \Phi_3^t &= \tfrac{1}{2} x e^t (u_t^2 - u_x^2) - (\tfrac{xu}{2} + xu_t + tu_x) e^t u_t - \gamma, \\ \Phi_3^x &= \tfrac{1}{2} t e^t (u_t^2 - u_x^2) + (\tfrac{xu}{2} + xu_t + tu_x) e^t u_x - \tfrac{u^2}{4} e^t - \delta. \end{aligned}$$

Thus,

$$\begin{aligned} D_t\Phi_3^t + D_x\Phi_3^x &= \tfrac{1}{2} x e^t (u_t^2 - u_x^2) + \tfrac{1}{2} x e^t (2u_t u_{tt} - 2u_x u_{xt}) + e^t u_t(-\tfrac{xu}{2} - xu_t - tu_x) \\ &\quad + e^t u_{tt}(-\tfrac{xu}{2} - xu_t - tu_x) + e^t u_t(-\tfrac{x}{2}u_t - u_x - tu_{tx} - xu_{tt}) \\ &\quad + \tfrac{1}{2} t e^t (2u_t u_{tx} - 2u_x u_{xx}) - e^t u_{xx}(-\tfrac{xu}{2} - xu_t - tu_x) \\ &\quad - e^t u_x(-\tfrac{u}{2} - \tfrac{1}{2}xu_x - tu_{xx} - u_t - xu_{tx}) - \tfrac{1}{2} e^t u u_x - (\gamma_t + \delta_x) \\ &= \left(-\tfrac{1}{2}xu - xu_t - tu_x\right) e^t\,(u_{tt} - u_{xx} + u_t) - (\gamma_t + \delta_x) \\ &= 0, \quad \text{along the solutions of (16).} \end{aligned}$$

**Note**

In [13], the conserved vector $(\Phi_2^t, \Phi_2^x)$ is listed as being associated with $\partial_t$ which is, in fact, NOT a Noether symmetry even though it is a Lie symmetry of (16) (the notation of the conserved vector in [13] for the symmetry $\partial_t$ is $(C^t, C^x)$). The symmetry $\partial_t$ is 'lost' as the standard Lagrangian (17) was obtained by multiplying (16) by a *variational integration factor* $e^t$. The Lie symmetry, $\partial_t$ arises if we consider $X_2 + \frac{1}{2}X_1$. Also, the conserved vector $(\Phi_3^t, \Phi_3^x)$ is not listed in [13].

## 2.3 Example 2 - The Gordon Class of Equations

We now conduct an analysis of the Gordon classes of equations (18), that is,

$$u_{tt} + u_t - u_{xx} + k(u) = 0.$$

As before, let

$$X = \xi(t,x,u)\partial_x + \tau(t,x,u)\partial_t + \phi(t,x,u)\partial_u$$

be a Noether point operator that satisfies (14) with gauge vector $(f, g)$. Invoking the Lagrangian,

$$L = e^t \left( -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + k_1(u) \right),$$

where $k_1 = \int k du$, we find

$$\tau e^t \left[ -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + k_1(u) \right] + \phi k(u)e^t - u_t e^t \phi^t + u_x e^t \phi^x +$$

$$e^t \left( -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + k_1(u) \right) [D_t\tau + D_x\xi] = D_t f + D_x g$$

Separation of the derivatives of $u$ yields the following system

$$\begin{aligned}
u_t^3 &: \tau_u = 0 \\
u_x^3 &: \xi_u = 0 \\
u_x^2 &: \phi_u = -\tfrac{1}{2}\tau \\
u_t u_x &: \tau_x = \xi_t \\
u_t &: -e^t\phi_t = f_u \\
u_x &: e^t\phi_x = g_u \\
1 &: f_t + g_x = \tau k_1(u)e^t + \phi k(u)e^t
\end{aligned}$$

The last of these calculations yields the equation,

$$k_1(u)\,[Ax + C] + k(u)\left[-\frac{u}{2}\,(Ax + C) + \gamma(x,t)\right] = u\,(-\gamma_t - \gamma_{tt} + \gamma_{xx}) + \alpha_t + \delta_x. \quad (20)$$

where $\gamma = \gamma(x,t)$, $\delta = \delta(x,t)$, $\alpha = \alpha(x,t)$ and where $A, B$ and $C$ are arbitrary constants.

We obtain the following general symmetry,

$$X = (At + B)\,\partial_x + (Ax + C)\,\partial_t + \left(-\frac{u}{2}\,(Ax + C) + \gamma(x,t)\right)\partial_u, \qquad (21)$$

The corresponding gauge vector is $(f, g) = \left(-ue^t\gamma_t + \alpha(x,t), ue^t\left(-\frac{u}{2}A + \gamma_x\right) + \delta(x,t)\right)$.

We now look at two special cases of $k(u)$ in detail.

Case (a):

$$k(u) = u$$

Case (b):

$$k(u) = u^n, \text{where n} \neq 1, \qquad k(u) = \sin u, \qquad k(u) = e^u \quad \text{or} \quad k(u) = constant.$$

**Case (a):** $k(u) = u$

This case yields the most interesting results. From (20), we obtain

$$\frac{u^2}{2}\left[Ax + C\right] + u\left[-\frac{u}{2}\left(Ax + C\right) + \gamma(x,t)\right] = u\left(-\gamma_t - \gamma_{tt} + \gamma_{xx}\right) + \alpha_t + \delta_x. \qquad (22)$$

Separating by powers of $u$, we find that $\alpha_t + \delta_x = 0$ and $-\gamma_t - \gamma_{tt} + \gamma_{xx} - \gamma = 0$.

We also obtain four symmetries, namely,

$$\begin{aligned} X_1 &= \partial_x \\ X_2 &= \partial_t - \tfrac{u}{2}\partial_u \\ X_3 &= t\partial_x + x\partial_t - \tfrac{xu}{2}\partial_u \\ X_\infty &= \gamma\partial_u \end{aligned}$$

We obtain the gauge vector $(f, g) = (-e^t\gamma_t + \alpha, ue^t(-\frac{u}{2}A + \gamma_x) + \delta)$.

The corresponding conserved quantities are;

(i) $X_1 = \partial_x, \quad f = \alpha(x,t)$ and $g = \delta(x,t)$,

$$\Phi_1^t = u_x u_t e^t - \alpha,$$
$$\Phi_1^x = e^t\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + k_1(u)\right) - e^t u_x^2 - \delta.$$

Thus,

$$\begin{aligned} D_t\Phi_1^t + D_x\Phi_1^x &= u_{tt}u_xe^t + u_xu_te^t + u_{tx}u_te^t - u_{tx}u_te^t + u_{xx}(u_xe^t - 2u_xe^t) + u_xe^tu \\ &\quad -(\alpha_t + \delta_x) \\ &= u_xe^t\left(u_{tt} - u_{xx} + u_t + u\right) - (\alpha_t + \delta_x) \\ &= 0, \quad \text{along the solutions of (18).} \end{aligned}$$

(ii) $X_2 = \partial_t - \frac{u}{2}\partial_u, \quad f = \alpha(x,t)$ and $g = \delta(x,t)$,

$$\Phi_2^x = e^t\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + k_1(u)\right) - e^t u_x^2 - \delta,$$
$$\Phi_2^t = e^t(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + k_1(u)) + e^t u_t(\tfrac{u}{2} + u_t) - \alpha.$$

Thus,

$$\begin{aligned} D_x\Phi_2^x + D_t\Phi_2^t &= -\tfrac{1}{2}e^t u_x^2 - e^t u_{xx}(\tfrac{u}{2} + u_t) \\ &\quad + e^t(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{u^2}{2}) + e^t(u_{tx}u_x + u_t u_{tt} + uu_t) + u_t e^t(\tfrac{u}{2} + u_t) \\ &\quad - u_{tx}u_x e^t + u_{tt}e^t(\tfrac{u}{2}) + 2e^t u_t u_{tt} + \tfrac{1}{2}u_t^2 e^t - (\alpha_t + \delta_x) \\ &= e^t\left(u_t + \tfrac{u}{2}\right)(u_{tt} - u_{xx} + u_t + u) - (\alpha_t + \delta_x) \\ &= 0, \quad \text{along the solutions of (18).} \end{aligned}$$

(iii) $X_3 = t\partial_x + x\partial_t - \frac{xu}{2}\partial_u, \quad f = \alpha(x,t)$ and $g = -e^t\frac{u^2}{2} + \delta(x,t)$,

$$\Phi_3^t = xe^t(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{u^2}{2}) + u_t e^t(\tfrac{ux}{2} + xu_t + tu_x) - \alpha,$$
$$\Phi_3^x = te^t(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{u^2}{2}) - u_x e^t(\tfrac{ux}{2} + xu_t + tu_x) + \tfrac{u^2}{2}e^t - \delta.$$

Thus,

$$\begin{aligned} D_x\Phi_3^x + D_t\Phi_3^t &= te^t(-u_{tx}u_t + u_x u_{xx} + uu_x) + e^t u_{xx}(-\tfrac{xu}{2} - xu_t - tu_x) \\ &\quad + e^t u_x(-tu_{xx} - xu_{tx} - u_t - \tfrac{u}{2} - \tfrac{x}{2}u_x) + \tfrac{1}{2}e^t uu_x \\ &\quad + xe^t(-u_t u_{tt} + u_x u_{tx} + u_t u) - e^t u_t(-\tfrac{ux}{2} - tu_x - xu_t) \\ &\quad - e^t u_{tt}(-\tfrac{xu}{2} - xu_t - tu_x) - e^t u_t(-\tfrac{x}{2}u_t - u_x - tu_{tx} - xu_{tt}) \\ &\quad + xe^t(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{u^2}{2}) - (\alpha_t + \delta_x) \\ &= e^t\left(\tfrac{1}{2}xu + xu_t + tu_x\right)(u_{tt} - u_{xx} + u_t + u) - (\alpha_t + \delta_x) \\ &= 0, \quad \text{along the solutions of (18).} \end{aligned}$$

**Case (b):** $k(u) = u^n$, where $n \neq 1$, $k(u) = \sin u$, $k(u) = e^u$ or $k(u) = constant$.

The above choices for $k(u)$ all yield the same results, that is, we find one symmetry, $X = \partial_x$ with conserved vector,

$$(\Phi^t, \Phi^x) = (u_t u_x e^t - \alpha, e^t(-\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + k_1(u)) - u_x^2 e^t - \delta).$$

We also find that $\alpha_t + \delta_x = 0$. Thus, the total divergence is,

$$\begin{aligned} D_x\Phi^x + D_t\Phi^t &= u_{tt}u_x e^t + u_x u_t e^t + u_{tx}u_t e^t - 2u_{xx}u_x e^t + e^t(-u_t u_{tx} + u_x u_{xx} + k(u)u_x) \\ &\quad -(\alpha_t + \delta_x) \\ &= u_x e^t\,(u_{tt} - u_{xx} + u_t + k(u)) - (\alpha_t + \delta_x) \\ &= 0, \quad \text{along the solutions of (18).} \end{aligned}$$

## 2.4 Discussion and Conclusion

In addition to the obvious symmetry $\partial_x$, we have found two more Noether symmetries of the wave equation with dissipation and therefore, we are able to present a bigger class of conserved forms for this particular wave equation. We have also ruled out $\partial_t$ as a Noether symmetry of the equation. The generalized dissipative Gordon-type equation, when analyzed, admits a special case with four symmetries and its associated four conserved vectors. The other cases considered offer only one symmetry, namely $\partial_x$ and hence only one conserved vector. At this stage we must point out that the use of Noether's theorem in constructing symmetries of a given PDE results in finding low-order symmetries, that is, first order symmetries.

# Chapter 3

# Gordon-type Equations and Higher-Order Variational Symmetries and Conservation Laws

## 3.1 Introduction

It can be proved that with the well-known transformations,

$$X = \frac{1}{2}(x - t) \quad \text{and} \quad T = \frac{1}{2}(x + t),$$

the classical wave equation $u_{tt} - u_{xx} - k(u) = 0$ can be transformed to

$$u_{XT} - k(u) = 0. \tag{23}$$

These equations have been extensively studied in terms of their symmetries and variational properties [19]. In particular, the sine-Gordon equation $u_{XT} - \sin u = 0$ has been shown to have higher-order variational symmetries, $X = Q\partial_u$, giving rise to interesting, nontrivial conservation laws, e.g., the symmetries,

$$\begin{aligned}
&X_1 = (u_{XXX} + \tfrac{1}{2}u_X^3)\partial_u,\\
&X_2 = (u_{XXXXX} + \tfrac{5}{2}u_X^2 u_{XXX} + \tfrac{5}{2}u_X u_{XX}^2 + \tfrac{3}{8}u_X^5)\partial_u,\\
&X_3 = (u_{TTT} + \tfrac{1}{2}u_T^3)\partial_u,
\end{aligned} \tag{24}$$

are variational as they satisfy Theorem 1, [19].

The first two symmetries in (24) lead to the corresponding densities

$$\Phi_1^T = -\frac{1}{2}u_{XX}^2 + \frac{1}{8}u_X^4, \quad \Phi_2^T = \frac{1}{2}u_{XXX}^2 - \frac{5}{4}u_X^2 u_{XX}^2 + \frac{1}{16}u_X^6. \tag{25}$$

We carry out the symmetry/conservation law analysis to the interesting wave and Gordon related class of equations (23) in canonical form. We investigate the existence of higher-order symmetries and possible conserved densities via the multiplier method. We then make extensive use of Theorem 1 in proving that the higher-order symmetries found are variational. We obtain some interesting results with some far reaching consequences.

Finally, we focus our analysis towards our main example of the chapter. A general homogeneous second-order linear wave equation on a two-dimensional space-time is exactly solvable when its general solution can be expressed as the sum of (two) progressing waves (see [5] for details). The simplest case being the wave equation $u_{XT} = 0$. It is implicit in [7] and explicit in [6, 8] that a generalization leads to the family of equations

$$u_{XT} - \frac{m(m+1)}{(T-X)^2}u = 0, \tag{26}$$

where $m$ is a non-negative integer.

## 3.2 The Canonical Wave Equation

We investigate *higher-order variational symmetries* for the class of equations (23). In [19], the case $k(u) = \sin u$ is discussed, and we have summarized the results above. We illustrate the method by highlighting another higher-order variational symmetry, not discussed in detail in [19]. The analysis is carried out via the multiplier approach. That is, we solve for the multiplier $Q$ in

$$\frac{\delta}{\delta u}[Q(u_{xt} - k(u))] = 0 \tag{27}$$

(for convenience, we use lower case $(x,t)$ instead of $(X,T)$).

Ultimately, we find that the only other cases of $Q$ that are derivative dependent are, when $k = u$ and $k = e^u$. That is,

$$u_{xt} - u = 0, \tag{28}$$

$$u_{xt} - e^u = 0. \tag{29}$$

Firstly, if $k(u) = \sin u$, [19] lists

$$\begin{aligned} X &= Q\,\partial_u \\ &= \left(u_{xxx} + \tfrac{1}{2}u_x^3\right)\partial_u \end{aligned}$$

as a higher-order variational symmetry. We now prove using Theorem 1 that

$$\begin{aligned} X^* &= Q^*\,\partial_u \\ &= \left(u_{ttt} + \tfrac{1}{2}u_t^3\right)\partial_u \end{aligned}$$

is also a higher-order variational symmetry of (23) with $k(u) = \sin u$, i.e.,

$$\begin{aligned} (X^*)^{[p]}E + \mathcal{A}\mathcal{F}_Q E &= \phi^{xt} - \phi\cos u + \left(-\tfrac{3}{2}u_t^2 D_t - 3u_t u_{tt} - D_{ttt}\right)(u_{xt} - \sin u) \\ &= u_{ttttx} + 3u_t u_{tt} u_{tx} + \tfrac{3}{2}u_t^2 u_{ttx} - (u_{ttt} + \tfrac{1}{2}u_t^3)\cos u \\ &\quad -\tfrac{3}{2}u_t^2 u_{xtt} + \tfrac{3}{2}u_t^3\cos u - 3u_t u_{xt} u_{tt} + 3u_t u_{tt}\sin u - u_{xtttt} \\ &\quad +u_{ttt}\cos u - u_t u_{tt}\sin u - 2u_t u_{tt}\sin u - u_t^3\cos u \\ &= 0, \end{aligned}$$

where $p$ denotes that the symmetry is prolonged.

**Case (a):** $u_{xt} - u = 0$.

For this case, the tedious calculations of the multiplier method reveal that, inter alia, the possibilities for $Q$ depending on third-order derivatives, are

$$Q_1 = u_{xxx} + ne^{x+t}, \qquad n \text{ is an arbitrary constant,} \tag{30}$$

$$Q_2 = u_{ttt}. \tag{31}$$

Therefore we have the two higher-order symmetries,

$$X_1 = (u_{xxx} + ne^{x+t})\partial_u \quad \text{and} \quad X_2 = (u_{ttt})\partial_u$$

for which the respective conserved quantities are

$$\begin{aligned}
\Phi_1^x &= \tfrac{1}{4}(2({u_x}^2 + u_{xt}u_{xx} - u_x u_{xxt}) + u_t(2e^{t+x}n + u_{xxx}) + u(-2e^{t+x}n - 4u_{xx} + u_{xxxt})),\\
\Phi_1^t &= \tfrac{1}{4}(u_x(2e^{t+x}n + u_{xxx}) - u(2e^{t+x}n + u_{xxxx})),
\end{aligned}$$

$$\begin{aligned}
\Phi_2^x &= \tfrac{1}{4}(u_t u_{ttt} - u u_{tttt}),\\
\Phi_2^t &= \tfrac{1}{4}(2{u_t}^2 + u_{ttt}u_x + 2u_{tt}u_{xt} - 2u_t u_{xtt} + u(-4u_{tt} + u_{xttt})).
\end{aligned}$$

We now show that the generalized symmetries $X_1$ and $X_2$ are variational using Theorem 1.

(i) $X_1 = (u_{xxx} + ne^{x+t})\partial_u$

$$\begin{aligned}
X_1^{[p]}E + \mathcal{A}\mathcal{F}_{Q_1}E &= \phi^{xt} - \phi + (-D_{xxx})(u_{xt} - u)\\
&= u_{xxxxt} + ne^{x+t} - (u_{xxx} + ne^{x+t}) - u_{xxxxt} + u_{xxx}\\
&= 0.
\end{aligned}$$

(ii) $X_2 = (u_{ttt})\partial_u$

$$\begin{aligned} X_2^{[p]}E + \mathcal{A}\mathcal{F}_{Q_2}E &= \phi^{xt} - \phi + (-D_{ttt})\,(u_{xt} - u) \\ &= u_{ttttx} - u_{ttt} - u_{ttttx} + u_{ttt} \\ &= 0. \end{aligned}$$

**Case (b)**: $u_{xt} - e^u = 0$.

Here, we obtain the third-order multiplier

$$Q = u_{xxx} - \frac{1}{2}u_x^3, \tag{32}$$

and hence the symmetry $X = Q\partial_u$ with conserved vector $(\Phi^t, \Phi^x)$, where

$$\begin{aligned} \Phi^t &= \tfrac{1}{16}(-u_t {u_x}^3 + {u_x}^2(8e^u - 3uu_{xt}) - 8u_x u_{xxt} + 4((-4e^u + 2u_{xt})u_{xx} + u_t u_{xxx} \\ &\quad + uu_{xxxt})), \\ \Phi^x &= \tfrac{1}{16}(-{u_x}^4 + 3u{u_x}^2 u_{xx} + 4u_x u_{xxx} - 4uu_{xxxx})). \end{aligned}$$

We prove that the generalized symmetry $X = (u_{xxx} - \dfrac{1}{2}u_x^3)\partial_u$ is variational.

$$\begin{aligned} X^{[p]}E + \mathcal{A}\mathcal{F}_Q E &= \phi^{xt} - \phi e^u + \left(\tfrac{3}{2}u_x^2 D_x + 3u_x u_{xx} - D_{xxx}\right)(u_{xt} - e^u) \\ &= u_{xxxxt} - 3u_x u_{xx} u_{xt} - \tfrac{3}{2}u_x^2 u_{xxt} - e^u(u_{xxx} - \tfrac{1}{2}u_x^3) \\ &\quad + \tfrac{3}{2}u_x^2(u_{txx} - u_x e^u) + 3u_x u_{xx} u_{xt} - 3u_x u_{xx} e^u - u_{txxxx} \\ &\quad + u_{xxx}e^u + u_x u_{xx} e^u + 2u_x u_{xx} e^u + u_x^3 e^u \\ &= 0. \end{aligned}$$

The condition for Theorem 1 is satisfied, and therefore, $X = \left(u_{xxx} - \dfrac{1}{2}u_x^3\right)\partial_u$ is a higher-order variational symmetry of $u_{xt} - e^u = 0$.

## 3.3 Main Example

We now determine the higher-order conserved vectors of equation (26), i.e.,

$$u_{XT} - \frac{m(m+1)}{(T-X)^2}u = 0,$$

which has a Lagrangian; the Noether symmetries and Noether's theorem would provide only the first-order multipliers. If we consider

$$\frac{\delta}{\delta u}\left[Q(u_{xt} - \frac{m(m+1)}{(t-x)^2}u)\right] = 0, \tag{33}$$

with

$$Q = f(x, t, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{xxx}, u_{ttt}),$$

the tedious calculations after expansion and separation by monomials reveal that $Q$ is of order three in derivatives with respect to $u$. The multiplier method yields the third-order multiplier

$$Q = \frac{u_x + u_t}{(t-x)^2} + \frac{u_{xxx} + u_{ttt}}{3m(m+1)}$$

and hence we have the symmetry $X = Q\partial_u$, whose corresponding conserved vector has components

$$\begin{aligned}
\Phi^x = \quad & -\tfrac{1}{12m(1+m)(t-x)^4}(6m(2+3m+2m^2+m^3)u^2 \\
& -(t-x)^2(3m(1+m){u_t}^2 + 2(m(1+m){u_x}^2 + (t-x)^2u_{xt}u_{xx} \\
& -(t-x)^2u_xu_{xxt}) + u_t((t-x)^2u_{ttt} + 3m(1+m)u_x + (t-x)^2u_{xxx})) \\
& +(t-x)u(-6m(1+m)u_t + 3m(1+m)(t-x)u_{tt} + t^3u_{tttt} - 3t^2xu_{tttt} \\
& +3tx^2u_{tttt} - x^3u_{tttt} - 2mu_x - 2m^2u_x - 3mtu_{xt} - 3m^2tu_{xt} \\
& +3mxu_{xt} + 3m^2xu_{xt} + 4mtu_{xx} + 4m^2tu_{xx} - 4mxu_{xx} - 4m^2xu_{xx} \\
& -t^3u_{xxxt} + 3t^2xu_{xxxt} - 3tx^2u_{xxxt} + x^3u_{xxxt})),
\end{aligned}$$

$$
\begin{aligned}
\Phi^t = \quad & -\tfrac{1}{12m(1+m)(t-x)^4}(6m(2+3m+2m^2+m^3)u^2 \\
& -(t-x)^2(2m(1+m){u_t}^2+(t-x)^2u_{ttt}u_x+3m{u_x}^2+3m^2{u_x}^2 \\
& +2t^2u_{tt}u_{xt}-4txu_{tt}u_{xt}+2x^2u_{tt}u_{xt}+u_t(3m(1+m)u_x-2(t-x)^2u_{xtt}) \\
& +t^2u_xu_{xxx}-2txu_xu_{xxx}+x^2u_xu_{xxx}) \\
& +(t-x)u(2m(1+m)u_t+4m(1+m)(t-x)u_{tt}+6mu_x+6m^2u_x \\
& -3mtu_{xt}-3m^2tu_{xt}+3mxu_{xt}+3m^2xu_{xt}-t^3u_{xttt} \\
& +3t^2xu_{xttt}-3tx^2u_{xttt}+x^3u_{xttt}+3mtu_{xx}+3m^2tu_{xx}-3mxu_{xx}-3m^2xu_{xx} \\
& +t^3u_{xxxx}-3t^2xu_{xxxx}+3tx^2u_{xxxx}-x^3u_{xxxx})).
\end{aligned}
$$

As before, we now establish that the higher-order symmetry $X = Q\partial_u$ is variational,

$$
\begin{aligned}
X^{[p]}E + \mathcal{A}\mathcal{F}_Q E \quad &= \quad \phi^{xt} - \phi\tfrac{m(m+1)}{(t-x)^2}u \\
& \quad +(-\tfrac{D_t+D_x}{(t-x)^2} - \tfrac{1}{3}(D_{ttt}+D_{xxx}))(u_{xt} - \tfrac{m(m+1)}{(t-x)^2}u) \\
&= \quad \tfrac{1}{3}(u_{xxxxt}+u_{ttttx}) + m(m+1)(t-x)^{-2}(u_{xxt}+u_{ttx}) + \\
& \quad 2m(m+1)(t-x)^{-3}(u_{xx}-u_{tt}) - 6m(m+1)(t-x)^{-4}(u_x+u_t) \\
& \quad -m(m+1)(t-x)^{-2}(\tfrac{1}{3}(u_{xxx}+u_{ttt}) + \tfrac{u_x}{(t-x)^2} + \tfrac{u_t}{(t-x)^2}) \\
& \quad -(t-x)^{-2}(u_{ttx} - m(m+1)(t-x)^{-2}u_t + u_{txx} \\
& \quad -m(m+1)(t-x)^{-2}u_x) + \tfrac{m(m+1)}{3}((t-x)^{-2}u_{ttt} \\
& \quad +6u_{tt}(t-x)^{-3} + 18u_t(t-x)^{-4}) + \tfrac{m(m+1)}{3}((t-x)^{-2}u_{xxx} \\
& \quad +6u_{xx}(t-x)^{-3} + 18u_x(t-x)^{-4}) - \tfrac{1}{3}(u_{ttttx}+u_{txxxx}) \\
&= \quad 0.
\end{aligned}
$$

The evolutionary higher-order symmetry $X = \left(\dfrac{u_x+u_t}{(t-x)^2} + \dfrac{u_{xxx}+u_{ttt}}{3m(m+1)}\right)\partial_u$ is variational as Theorem 1 is satisfied.

The Lie point symmetry generators of equation (26) are

$$\partial_t + \partial_x, \quad u\partial_u, \quad x\partial_x + t\partial_t, \quad x^2\partial_x + t^2\partial_t, \quad F^1(x,t)\partial_u$$

where

$$mF^1 + m^2F^1 - t^2F^1_{xt} + 2txF^1_{xt} - x^2F^1_{xt} = 0.$$

The action of the symmetries on $Q$ are given below.

(i). $(\partial_x + \partial_t)Q = 0,$

(ii). $(u\partial_u)Q = Q,$

(iii). $(x\partial_x + t\partial_t)Q = -Q + \dfrac{2(u_x + u_t)}{(t-x)^2},$

(iv).

$$\begin{aligned}(x^2\partial_x + t^2\partial_t)Q &= 2(x+t)Q - \frac{6}{3m(m+1)}(u_{xx} + u_{tt}) - 2t\left(\frac{u_t}{(t-x)^2} + \frac{u_{xxx}}{3m(m+1)}\right) \\ &\quad -2x\left(\frac{u_x}{(t-x)^2} + \frac{u_{ttt}}{3m(m+1)}\right),\end{aligned}$$

(v). $(F^1(x,t)\partial_u)\, Q = 0.$

## 3.4 Discussion and Conclusion

The use of the multiplier approach en route to finding conservation laws resulted in obtaining new and interesting higher-order symmetries for the Gordon-type equations. Symmetry methods and the use of Noether's Theorem restrict our findings to symmetries of low order. In an effort to prove that the higher-order symmetries obtained are variational, we employed the use of *Adjoint Fréchet Derivatives*. Our results were then extended to the wave equation with non-constant coefficient terms and we were able to construct higher-order variational symmetries for this particular equation. The higher-order symmetries found may be used for double reductions and integrability of the equations considered.

# Chapter 4

# Multi-dimensional Gordon-Type Equations

## 4.1 Introduction

The multipliers and conservation laws in the higher-order cases of the Gordon equations $u_{XT} - k(u) = 0$ can be recast in the well known classical form

$$u_{tt} - u_{xx} - k(u) = 0. \tag{34}$$

The purpose for doing this is to gain an insight into extending the results to the multi-dimensional Gordon equations which are somewhat cumbersome if one pursues a canonical form. That is, the multi-dimensional Gordon equations are best considered as

$$u_{tt} - \triangle u - k(u) = 0, \tag{35}$$

where $\triangle$ denotes the Laplacian. This equation is of the classical form rather than 'canonical' form. We investigate the existence of higher-order symmetries (variational) for classical forms of Gordon equations with dimension one and two, and we find possible conserved densities via the multiplier method. We obtain some special results.

## 4.2 (1+1) Gordon-type Equations Revisited

Consider the following special cases of equation (34) in the form
(a). $u_{xx} - u_{tt} - \sin u = 0$
(b). $u_{xx} - u_{tt} - u = 0$
(c). $u_{xx} - u_{tt} - e^u = 0$

In each case, we list one of the higher-order multipliers and conserved densities which arise as a consequence of the transformation of the canonical forms. Case (a) is done in detail.

$$(a) \qquad u_{xx} - u_{tt} - \sin u = 0$$

In converting the equation $u_{xx} - u_{tt} - \sin(u) = 0$ into $u_{XT} - \sin u$, we used the transformations,

$$X = \frac{1}{2}(x - t), \qquad T = \frac{1}{2}(x + t)$$

and obtained the higher-order symmetries (24).

In the reverse direction, we use transformations,

$$x = X + T, \qquad t = T - X$$

to obtain higher-order symmetries for the classical equation, which we call $\bar{\mathcal{X}}$. We will now illustrate the method using $X_1 = (u_{XXX} + \frac{1}{2}u_X^3)\partial_u$ where $X_1$ comes from (24). Since,

$$
\begin{aligned}
u_X &= u_x x_X + u_t t_X \\
&= u_x - u_t, \\
u_{XX} &= u_{xx} x_X + u_{xt} t_X - u_{tx} x_X - u_{tt} t_X \\
&= u_{xx} - 2u_{xt} + u_{tt}, \\
u_{XXX} &= u_{xxx} x_X + u_{xxt} t_X - 2(u_{xtx} x_X + u_{xtt} t_X) + u_{ttx} x_X + u_{ttt} t_X \\
&= u_{xxx} - 3u_{xxt} + 3u_{xtt} - u_{ttt},
\end{aligned}
$$

we have that the equivalent of $X_1$ is

$$
\begin{aligned}
\bar{\mathcal{X}}_1 &= Q_1 \partial_u \\
&= \left(u_{xxx} - 3u_{xxt} + 3u_{xtt} - u_{ttt} + \tfrac{1}{2}(u_x - u_t)^3\right)\partial_u,
\end{aligned}
$$

so that $\bar{\mathcal{X}}_1 = Q_1\partial_u$ is the evolutionary generator leading to a conserved density

$$
\begin{aligned}
\Phi_1^t = \; & \tfrac{1}{8}({u_t}^4 + 4{u_{tt}}^2 - 3{u_t}^3 u_x + 4u_{ttt}u_x - 4\cos(u){u_x}^2 + 8u_{tt}(\sin(u) - u_{xt}) \\
& -16\sin(u)u_{xt} + 3u{u_x}^2 u_{xt} - 4u_x u_{xtt} + 4uu_{xttt} + {u_t}^2(-4\cos(u) + 3{u_x}^2 \\
& +3u(u_{xt} - u_{xx})) + 8\sin(u)u_{xx} - 3u{u_x}^2 u_{xx} + 8u_{xt}u_{xx} - 4{u_{xx}}^2 \\
& -4u_x u_{xxt} - 12uu_{xxtt} + 4u_x u_{xxx} + u_t(-{u_x}^3 + u_x(8\cos(u) \\
& -6u(u_{xt} - u_{xx})) - 8(u_{xtt} - 2u_{xxt} + u_{xxx})) + 12uu_{xxxt} - 4uu_{xxxx}).
\end{aligned}
$$

$$(b) \qquad u_{xx} - u_{tt} - u = 0$$

$\mathcal{Q}_2 = u_{xxx} - u_{ttt} - 3u_{txx} + 3u_{ttx} + e^x,$

where $\mathcal{Q}_2$ corresponds to $Q_1$ in (30) where $n = 1$. The corresponding conserved density for $X_2 = \mathcal{Q}_2\partial_u$ is

$$\begin{aligned}\Phi_2^t = \ & \tfrac{1}{2}(-{u_t}^2 + {u_{tt}}^2 + u_{ttt}u_x - {u_x}^2 - 2u_{tt}u_{xt} - u_x u_{xtt} + 2u_{xt}u_{xx} - {u_{xx}}^2 \\ & -u_x u_{xxt} + u_x u_{xxx} - 2u_t(e^x - u_x + u_{xtt} - 2u_{xxt} + u_{xxx}) + u(2u_{tt} - 4u_{xt} \\ & +u_{xttt} + 2u_{xx} - 3u_{xxtt} + 3u_{xxxt} - u_{xxxx})).\end{aligned}$$

$$(c) \qquad u_{xx} - u_{tt} - e^u = 0$$

$\mathcal{Q}_3 = u_{xxx} - u_{ttt} - 3u_{txx} + 3u_{ttx} - \frac{1}{2}(u_x - u_t)^3,$

where $\mathcal{Q}_3$ corresponds to $Q$ in (32). The corresponding conserved density for $X_3 = \mathcal{Q}_3\partial_u$ is

$$\begin{aligned}\Phi_3^t = \ & \tfrac{1}{8}(-{u_t}^4 + 4{u_{tt}}^2 + 3{u_t}^3 u_x + 4u_{ttt}u_x - 4e^u{u_x}^2 + 8u_{tt}(e^u - u_{xt}) \\ & -16e^u u_{xt} - 3u{u_x}^2 u_{xt} - 4u_x u_{xtt} + 4uu_{xttt} - {u_t}^2(4e^u + 3{u_x}^2 \\ & +3u(u_{xt} - u_{xx})) + 8e^u u_{xx} + 3u{u_x}^2 u_{xx} + 8u_{xt}u_{xx} - 4{u_{xx}}^2 - 4u_x u_{xxt} \\ & -12uu_{xxtt} + 4u_x u_{xxx} + u_t({u_x}^3 + u_x(8e^u + 6u(u_{xt} - u_{xx})) \\ & -8(u_{xtt} - 2u_{xxt} + u_{xxx})) + 12uu_{xxxt} - 4uu_{xxxx}).\end{aligned}$$

## 4.3 (1+2) Gordon-type Equations

After some lengthy calculations, it appears that for the multi-dimensional Gordon-type equations (35), higher-order symmetries/multipliers (and the corresponding conserved quantities) may be determined for the case $k(u) = u$ only. The underlying calculations produce negative results despite spending a huge amount of time on it. This seems to be a consequence of the underlying differential operator being linear only if $k(u) = u$ (see Proposition 1). In what follows, we first assume a form of a multiplier for equation (36) and secondly we take a formal approach (the multiplier method) for finding multipliers of the equation.

We assumed the form of the multiplier $Q$ for the two-dimensional (Klein-) Gordon equation

$$u_{xx} + u_{yy} - u_{tt} - u = 0 \qquad (36)$$

by an extrapolation of a study of the one-dimensional classical form in case (b) above. We obtained the corresponding higher-order symmetries $X_A = \mathcal{Q}_A \partial_u$ and $X_B = \mathcal{Q}_B \partial_u$ for the two dimensional Klein-Gordon equation (36), namely

(i). $X_A = \left(-u_{xxx} - u_{yyy} + u_{ttt} + 3u_{txx} + 3u_{tyy} - 3u_{ttx} - 3u_{tty}\right)\partial_u$

with conserved density

$$\mathcal{T}_A^t = \tfrac{1}{2}({u_t}^2 - {u_{tt}}^2 - u_{ttt}u_y + {u_y}^2 + 2u_{tt}u_{yt} + u_y u_{ytt} - 2u_{yt}u_{yy} + {u_{yy}}^2 + u_y u_{yyt} - u_y u_{yyy} - u_{ttt}u_x + u_{yyt}u_x + {u_x}^2 + 2u_{tt}u_{xt} - 2u_{yy}u_{xt} + u_x u_{xtt} - u_x u_{xyy} - 2u_{yt}u_{xx} + 2u_{yy}u_{xx} - 2u_{xt}u_{xx} + {u_{xx}}^2 + u_y u_{xxt} + u_x u_{xxt} - u_y u_{xxy} - u_x u_{xxx} + u_t(-2u_y + 2u_{ytt} - 4u_{yyt} + 2u_{yyy} - 2u_x + 2u_{xtt} + u_{xyy} - 4u_{xxt} + u_{xxy} + 2u_{xxx}) - u(2u_{tt} - 4u_{yt} + u_{yttt} + 2u_{yy} - 3u_{yytt} + 3u_{yyyt} - u_{yyyy} - 4u_{xt} + u_{xttt} + 2u_{xyyt} + 2u_{xx} - 3u_{xxtt} + 2u_{xxyt} - 2u_{xxyy} + 3u_{xxxt} - u_{xxxx})).$$

and

(ii). $X_B = (u_{xxx} + u_{yyy} - u_{ttt} - 3u_{txx} - 3u_{tyy} + 3u_{ttx} + 3u_{tty} + e^x + e^y)\,\partial_u$

$\mathcal{T}_B^t = \frac{1}{2}(-{u_t}^2 + {u_{tt}}^2 + u_{ttt}u_y - {u_y}^2 - 2u_{tt}u_{yt} - u_y u_{ytt} + 2u_{yt}u_{yy} - {u_{yy}}^2 - u_y u_{yyt} + u_y u_{yyy} + u_{ttt}u_x - u_{yyt}u_x - {u_x}^2 - 2u_{tt}u_{xt} + 2u_{yy}u_{xt} - u_x u_{xtt} + u_x u_{xyy} + 2u_{yt}u_{xx} - 2u_{yy}u_{xx} + 2u_{xt}u_{xx} - {u_{xx}}^2 - u_y u_{xxt} - u_x u_{xxt} + u_y u_{xxy} + u_x u_{xxx} - u_t(2e^x + 2e^y - 2u_y + 2u_{ytt} - 4u_{yyt} + 2u_{yyy} - 2u_x + 2u_{xtt} + u_{xyy} - 4u_{xxt} + u_{xxy} + 2u_{xxx}) + u(2u_{tt} - 4u_{yt} + u_{yttt} + 2u_{yy} - 3u_{yytt} + 3u_{yyyt} - u_{yyyy} - 4u_{xt} + u_{xttt} + 2u_{xyyt} + 2u_{xx} - 3u_{xxtt} + 2u_{xxyt} - 2u_{xxyy} + 3u_{xxxt} - u_{xxxx}))$.

More formally, we consider

$$\frac{\delta}{\delta u}[\mathcal{Q}(u_{xx} + u_{yy} - u_{tt} - u)] = 0 \tag{37}$$

where $\mathcal{Q} = \mathcal{Q}(x, y, t, u_x, u_x, u_{xx}, u_{xy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy})$. Although not pursued here, the calculations may include derivatives of $u$ with respect to $t$. Then

$$\mathcal{Q}[(u_{xx} + u_{yy} - u_{tt} - u)] = D_t\mathcal{T} + D_x\mathcal{X} + D_y\mathcal{Y},$$

where $(\mathcal{T}, \mathcal{X}, \mathcal{Y})$ is the conserved flow ($\mathcal{T}$ being the conserved density).

We obtained the following for multipliers $\mathcal{Q}$,

$$\begin{aligned}\mathcal{Q} \;=\;& \tfrac{1}{6}\{-3(-\tfrac{1}{3}qC_4x^3 + (aC_4 + pyC_4 + pC_3 + qC_1)x^2 \\ &+(-ny^2C_4 + ((-b + c)C_4 - 2C_1p - 2nC_3)y - 4pC_5 - 2aC_1 + (-b + c)C_3 \\ &-2nC_6 + (-2C_6 + 2C_2)q - 2\beta C_{11})x + \tfrac{1}{3}my^3C_4 + (nC_1 - aC_4 + mC_3)y^2 \\ &+((2C_6 - 2C_2)p + (b - c)C_1 - 2aC_3 + 4nC_5 + 2mC_6 + 2C_{11}\alpha)y - 2pC_8 \\ &-2nC_7 - 2aC_2 - 2qC_{10} + (-2c + 2b)C_5 - 2\alpha C_{13} - 2\beta C_{12} - 2mC_9)\}\end{aligned} \tag{38}$$

where the $C_i's$ $\quad (i = 1, 2, 3, \ldots 13)$ are arbitrary constants and where

$$\alpha = u_x, \qquad \beta = u_y, \qquad a = u_{xy}, \qquad b = u_{xx}, \qquad c = u_{yy},$$

$$m = u_{xxx}, \qquad n = u_{xxy}, \qquad p = u_{xyy}, \qquad q = u_{yyy}.$$

The multipliers $\mathcal{Q}$ form a vector space and the set of $C_i's$ are a basis for the vector space. When we separate (38) (i.e. we let $C_1 = 1$ and set the rest of the $C_i's = 0$ ($i \neq 1$), etc.), we obtain the set of multipliers $\mathcal{Q}_i$ together with their conserved densities $\mathcal{T}_i^t$, namely,

$\mathcal{Q}_1 = yu_{xx} + x^2u_{yyy} - yu_{yy} + y^2u_{yxx} - 2xyu_{yyx} - 2xu_{xy}$,
$\mathcal{T}_1^t = \frac{1}{2}(-u_t(-yu_{yy} + x^2u_{yyy} - 2xu_{xy} - 2xyu_{xyy} + yu_{xx} + y^2u_{xxy}) + u(-yu_{yyt} + x^2u_{yyyt} - 2xu_{xyt} - 2xyu_{xyyt} + yu_{xxt} + y^2u_{xxyt}))$,

$\mathcal{Q}_2 = -u_{yx} + xu_{yyy} - yu_{xyy}$,
$\mathcal{T}_2^t = \frac{1}{2}(u_t(-xu_{yyy} + u_{xy} + yu_{xyy}) + u(xu_{yyyt} - u_{xyt} - yu_{xyyt}))$,

$\mathcal{Q}_3 = -xu_{xx} + x^2u_{xyy} + xu_{yy} + y^2u_{xxx} - 2xyu_{yxx} - 2yu_{xy}$,
$\mathcal{T}_3^t = \frac{1}{2}(-u_t(xu_{yy} - 2yu_{xy} + x^2u_{xyy} - xu_{xx} - 2xyu_{xxy} + y^2u_{xxx}) + u(xu_{yyt} - 2yu_{xyt} + x^2u_{xyyt} - xu_{xxt} - 2xyu_{xxyt} + y^2u_{xxxt}))$,

$\mathcal{Q}_4 = -xyu_{xx} - \frac{1}{3}x^3u_{yyy} + \frac{1}{3}y^3u_{xxx} + xyu_{yy} - y^2u_{xy} - xy^2u_{xxy} + yx^2u_{xyy} + x^2u_{xy}$,
$\mathcal{T}_4^t = \frac{1}{6}(u_t(-3xyu_{yy} + x^3u_{yyy} - 3x^2u_{xy} + 3y^2u_{xy} - 3x^2yu_{xyy} + 3xyu_{xx} + 3xy^2u_{xxy} - y^3u_{xxx}) + u(3xyu_{yyt} - x^3u_{yyyt} + 3x^2u_{xyt} - 3y^2u_{xyt} + 3x^2yu_{xyyt} - 3xyu_{xxt} - 3xy^2u_{xxyt} + y^3u_{xxxt}))$,

$\mathcal{Q}_5 = u_{xx} - 2xu_{xyy} + 2yu_{xxy} - u_{yy}$,
$\mathcal{T}_5^t = \frac{1}{2}(u_t(u_{yy} + 2xu_{xyy} - u_{xx} - 2yu_{xxy}) + u(-u_{yyt} - 2xu_{xyyt} + u_{xxt} + 2yu_{xxyt}))$,

$\mathcal{Q}_6 = -xu_{xxy} - xu_{yyy} + yu_{xyy} + yu_{xxx}$,
$\mathcal{T}_6^t = \frac{1}{2}(u_t(xu_{yyy} - yu_{xyy} + xu_{xxy} - yu_{xxx}) + u(-xu_{yyyt} + yu_{xyyt} - xu_{xxyt} + yu_{xxxt}))$,

$\mathcal{Q}_7 = u_{xxy}$,
$\mathcal{T}_7^t = \frac{1}{2}(-u_tu_{xxy} + uu_{xxyt})$,

$\mathcal{Q}_8 = u_{xyy}$,
$\mathcal{T}_8^t = \frac{1}{2}(-u_t u_{xyy} + u u_{xyyt})$,

$\mathcal{Q}_9 = u_{xxx}$,
$\mathcal{T}_9^t = \frac{1}{2}(-u_t u_{xxx} + u u_{xxxt})$,

$\mathcal{Q}_{10} = u_{yyy}$,
$\mathcal{T}_{10}^t = \frac{1}{2}(-u_t u_{yyy} + u u_{yyyt})$,

$\mathcal{Q}_{11} = x u_y - y u_x$,
$\mathcal{T}_{11}^t = \frac{1}{2}(u_t(-x u_y + y u_x) + u(x u_{yt} - y u_{xt}))$,

$\mathcal{Q}_{12} = u_y$,
$\mathcal{T}_{12}^t = \frac{1}{2}(-u_t u_y + u u_{yt})$,

$\mathcal{Q}_{13} = u_x$,
$\mathcal{T}_{13}^t = \frac{1}{2}(-u_t u_x + u u_{xt})$.

We now select a few of the multipliers above and prove that $X_2 = \mathcal{Q}_2 \partial_u$, $X_3 = \mathcal{Q}_3 \partial_u$ and $X_9 = \mathcal{Q}_9 \partial_u$ are variational using Theorem 1 (the rest of the symmetries $X_i = \mathcal{Q}_i \partial_u$ ($i = 1, 4, 5, 6, 7, 8, 10, 11, 12, 13$) prove to be variational as well but the details are omitted here). That is,

$$\begin{aligned}
X_2^{[p]} E + \mathcal{A}\mathcal{F}_{\mathcal{Q}_2} E &= \phi^{xx} + \phi^{yy} - \phi^{tt} - \phi \\
&\quad + (D_{xy} + y D_{xyy} - x D_{yyy})(u_{xx} + u_{yy} - u_{tt} - u) \\
&= u_{xxxy} + u_{xyyy} - u_{xytt} - u_{xy} + y(u_{xxxyy} + u_{xyyyy} - u_{ttxyy} - u_{xyy}) \\
&\quad - x(u_{xxyyy} + u_{yyyyy} - u_{ttyyy} - u_{yyy}) + 2u_{xyyy} \\
&\quad + x u_{xxyyy} - y u_{xxxyy} - u_{xxxy} + u_{yyyyy} - 2u_{xyyy} - y u_{xyyyy} \\
&\quad - u_{xyyy} - x u_{yyytt} + y u_{xyytt} + u_{xytt} - x u_{yyy} + y u_{xyy} + u_{xy} \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
X_3^{[p]}E + \mathcal{A}\mathcal{F}_{\mathcal{Q}_3}E &= \phi^{xx} + \phi^{yy} - \phi^{tt} - \phi \\
&\quad +(xD_{xx} + 2yD_{xy} - xD_{yy} - y^2 D_{xxx} + 2xyD_{xxy} \\
&\quad -x^2 D_{xyy})(u_{xx} + u_{yy} - u_{tt} - u) \\
&= x(u_{xxxx} + u_{xxyy} - u_{xxtt} - u_{xx}) + 2y(u_{xxxy} + u_{xyyy} - u_{ttxyy} - u_{xy}) \\
&\quad -x(u_{xxyy} + u_{yyyyy} - u_{ttyy} - u_{yy}) \\
&\quad -y^2(u_{xxxxx} + u_{xxxyy} - u_{xxxtt} - u_{xxx}) + 2xy(u_{xxxxy} + u_{xxyyy} \\
&\quad -u_{ttxxy} - u_{xxy}) - x^2(u_{xxxyy} + u_{yyyyx} - u_{xyytt} - u_{xyy}) \\
&\quad +4u_{xyy} + 4xu_{xxyy} + x^2 u_{xxxyy} - 4yu_{xxxyy} - 2xyu_{xxxxy} + xu_{xxyy} \\
&\quad -xu_{xxxx} + y^2 u_{xxxxx} - 2yu_{xxxy} + x^2 u_{xyyyy} - 4xu_{xxyy} \\
&\quad -2xyu_{xxyyy} - xu_{xxyy} + 2yu_{xxxy} + 2u_{xxx} + 2yu_{xxxy} + y^2 u_{xxxyy} \\
&\quad -4u_{xyy} - 2yu_{xyyy} - x^2 u_{xyytt} + 2xyu_{xxytt} - xu_{yytt} + xu_{xxtt} \\
&\quad -y^2 u_{xxxtt} + 2yu_{xytt} - 2u_{xxx} - x^2 u_{xyy} + 2xyu_{xxy} - xu_{yy} \\
&\quad +xu_{xx} - y^2 u_{xxx} + 2yu_{xy} + xu_{yyyy} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
X_9^{[p]}E + \mathcal{A}\mathcal{F}_{\mathcal{Q}_9}E &= \phi^{xx} + \phi^{yy} - \phi^{tt} - \phi + (-D_{xxx})\,(u_{xx} + u_{yy} - u_{tt} - u) \\
&= u_{xxxxx} + u_{xxxyy} - u_{xxxtt} - u_{xxx} \\
&\quad -u_{xxxxx} - u_{xxxyy} + u_{xxxtt} + u_{xxx} \\
&= 0.
\end{aligned}
$$

Therefore, we conclude that the evolutionary higher-order symmetries $X_i = \mathcal{Q}_i \partial_u$ are variational.

## 4.4 Discussion and Conclusion

In the first of our analysis of multi-dimensional Gordon equations, we recalled results from Chapter 3 that found new higher-order symmetries with their conserved densities by an analysis of equations in canonical form. We then focussed on Gordon-type equations in classical form and we established interesting results of higher-order symmetries and conserved flows. Finally we investigated multi-dimensional Gordon-type equations. We were able to deduce higher-order symmetries and conserved densities for a two dimensional linear Klein-Gordon equation which was based on an extrapolation of a study of the one-dimensional linear Klein-Gordon equation in classical form. In applying the multiplier approach to the two dimensional Klein-Gordon equation, we determined a set of thirteen higher-order variational symmetries and conserved densities.

# Conclusion

We began our study by analyzing a dissipative wave equation and we identified a bigger class of conserved forms for this particular (1+1) wave equation. We discovered that the generalized dissipative Gordon-type equation has a special case with four Noether symmetries and its associated four conserved vectors. Symmetry methods and the use of Noether's Theorem restrict our findings to symmetries of low order, that is, first order symmetries.

We then employed the multiplier approach on Gordon-type equations, where the multipliers led to a large class of interesting and higher-order conserved flows that would not be obtained by variational techniques such as Noether's theorem. In particular we obtained higher-order symmetries. Noether's theorem will naturally provide variational symmetries, but when we use the multiplier method which is not a variational technique, we may obtain non-variational symmetries. It was necessary to verify that the higher-order symmetries found are in fact variational (since the nature of the symmetry has a bearing on the integrability of the equation). We then made extensive use of a theorem involving *Adjoint Fréchet Derivatives*. We extended our results to the wave equation with non-constant coefficient terms and we obtained higher-order variational symmetries. The higher-order symmetries constructed from the multipliers prove to be variational in every example. This may be attributed to the fact that each of the PDEs considered is known to have a Lagrangian.

We researched multi-dimensional Gordon-type equations; like most multi-dimensional equations it can be cumbersome and tedious to solve. In an effort to simplify our analysis, we studied one-dimensional Gordon-type equations and we successfully deduced the symmetries of the two dimensional linear Klein-Gordon equation. In addition we obtained thirteen new higher-order variational symmetries with their conserved densities for the higher-dimensional equations.

Symmetry analysis is a special tool used to understand and construct solutions of differential equations. The existence of infinitely many generalized symmetries previously found using the recursion operator, is of great importance since equations with a high degree of symmetry are useful in mathematics and physics [19]. In particular the symmetries found in this investigation may be used for double reductions and integrability of the equations. In the analysis of differential equations, conservation laws have many significant uses, particularly with regard to integrability and linearization, constants of motion, analysis of solutions and numerical solution methods. Consequently, numerous studies are dedicated to finding the conservation laws for given differential equations. In this study we identified numerous higher-order variational symmetries and conserved flows of some nonlinear wave equations.

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