Master of Science Dissertation

The Genus of a Nilpotent R-Powered Group

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Declaration

I declare that this dissertation is my own work. It is being submitted for the degree of Master of Science in Mathematics at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signed: ______________________________

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Abstract

In [14], Mislin define the genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N to be the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. In [6], Hilton and Mislin define an abelian group structure on the genus set $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroups. Let χ_0 be the class of finitely generated groups with finite commutator subgroup. For a χ_0-group G, the non cancellation set of G, denoted by $\chi(G)$, is the set of isomorphism classes of groups H such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. Warfield, in [19], proved that, if N is a nilpotent χ_o-group, then $\mathcal{G}(N) = \chi(N)$. In [20], the author showed that, for a χ_o-group G the non- cancellation set $\chi(G)$ has a group structure similar to the group structure on the Mislin genus of a nilpotent χ_o-group. Let R be a binomial ring. The nilpotent R-powered group, first introduced by P. Hall in [5], is a nilpotent group G extended by a binomial ring R. Many results that are found in the theory of nilpotent groups carry over to the class of nilpotent R-powered groups. In particular, Majewicz and Zyman in [13], showed that the P-localization of a nilpotent R-powered group G, for a set of primes P in R can be obtained. We study the genus of a finitely R-generated nilpotent R-powered group. We show that the for any two finitely R-generated nilpotent R-powered groups G and H and some finitely R-powered abelian group A, if $\mathcal{G}(G \times A) = \mathcal{G}(H \times A)$ then we have $\mathcal{G}(G) = \mathcal{G}(H)$.

Contents

Chapter 1

Introduction

In this study, we will investigate and expand on the studies of a genus of a group. In particular, we study the genus of a nilpotent R-powered group. We first give a brief history:

Let P be a set of primes and let P' be the set of all primes not in P. The authors, in [7], studied the P-localization of a nilpotent group. A group G is called P-local if for every product $n = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$ of primes $p_1, p_2 \dots p_k \in P'$, the mapping $\phi : G \to G$ defined by $\phi(g) = g^n$ is a bijective mapping. A mapping $e : G \to G_P$ is called a P-localization mapping if G_P is P-local and e is universal, in the sense that, for any P-local group K and a homomorphism $\phi : G \to K$, there exists a unique homomorphism $\alpha : G_P \to K$ such that $\phi = \alpha \circ e$. We call G_P the P-localization of G. If $P = \{p\}$, we write G_P as G_p and call it the p-localization of G. The main result from [7], is the Fundamental Theorem on the P-Localization of nilpotent groups, which states that, for every nilpotent group G and for any set of primes P, the P-localization G_P exists.

For a finitely generated nilpotent group N, the Mislin Genus $\mathcal{G}(N)$ is defined in [14] to be the set of all isomorphism classes of finitely generated nilpotent groups M such that they have the same p-localization, that is, $N_p \cong M_p$ for every prime p. Let M and N be finitely generated nilpotent groups. The author, also in [14], showed that if $\mathcal{G}(N \times A) = \mathcal{G}(M \times A)$ for some finitely generated abelian group A, then $\mathcal{G}(M) = \mathcal{G}(N)$. In addition, if M and N have finite commutator subgroups then $\mathcal{G}(M) = \mathcal{G}(N)$ if and only if there exists a finitely generated abelian group A such that $M \times A \cong N \times A$.

Let χ_0 be the class of finitely generated groups with finite commutator subgroups. In [6], Hilton and Mislin describe the structure of $\mathcal{G}(N)$ for a χ_0 group N. The non cancellation set of any group G, denoted by $\chi(G)$, is the set of isomorphism classes of groups H such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. Warfield showed in [19], that, if N is a Nilpotent χ_0 group, then $\mathcal{G}(N) = \chi(N)$.

Let R be a binomial ring. The nilpotent R-powered group, first introduced by P. Hall in [5], is a nilpotent group G extended by a binomial ring R, to define a unique R-exponentiation g^α for each $g \in G$ and $\alpha \in R$. Many results on nilpotent groups carry over to nilpotent R-powered groups.

In [9], Majewicz defined what is meant by a nilpotent R-powered group of finite type. In particular, the author explored a nilpotent R-powered group for the case when R is the ring $\mathbb{Q}[x]$, the set of all polynomials with rational coefficients. The author showed that these nilpotent $\mathbb{Q}[x]$-powered groups have similar properties to the usual nilpotent groups.

In [11] and [12], Majewicz and Zyman explore the root extraction of a nilpotent R-powered group. In particular, the authors investigate the case when a nilpotent R-powered group contains unique roots, or when each element of a nilpotent R-powered group has at least one root. In addition, they also considered root extraction on the set of elements created by a set of primes P in R, called P-members.

Let G be a finitely R-generated nilpotent R-powered group. In [13], Majewicz and Zyman explored the localization of a nilpotent R-powered group G. The authors, show that the Fundamental Theorem on the P-Localization holds for nilpotent R-powered groups if the binomial ring is a unique factorization domain. In addition, the authors showed that, if there is an extra requirement on R, that R is a principal ideal domain with a subring isomorphic to the rational numbers, then every P-localization map is a P-isomorphism.

Since, for a nilpotent R-powered group G we can get the p-localization of G, similarly, we can define the genus of a finitely R-generated nilpotent R-powered group and investigate its properties.

This work is subdivided as follows:

- In Chapter 2, we give preliminaries on groups and rings.
- In Chapter 3, we introduce and explore some basic results on nilpotent R-powered groups.
- In Chapter 4, we investigate P-localization of a nilpotent R-powered group.
- In Chapter 5, we define and give results on the genus of a nilpotent R-powered group.

Chapter 2

Preliminaries

In this chapter, we give some preliminaries on groups and rings. We begin by exploring groups by giving definitions and results on groups and subgroups. In Section 2.2, we recall when a subgroup is said to be normal and explore the quotient group and in Section 2.3, we explore homomorphism and the Isomorphism Theorems. In Section 2.4, we introduce the commutator subgroup and from this, we investigate nilpotent groups in Section 2.5.

In the second half of this chapter we recall notions on rings. In Section 2.6, we start by giving the definition of a ring and some basic results. Then, we recall principal ideal domains and unique factorization domains. In Section 2.8, we recall the definition of prime elements of a ring and show that the greatest common divisor and the lowest common multiple can be defined in a principal ideal domain. Lastly, we give a quick overview of R-modules and in conclusion we introduce the binomial ring.

The definitions and results for groups can be found in [3], [4], [15] and [17], while the definitions and results for rings can be found in [3], [4] and [8].

2.1 Groups and Subgroups

In this section, we define what is meant by a group and a subgroup. We prove some results that follow immediately from them. Some of the basic proofs can be found in [3], [4], [15], and [17].

Firstly, we state the definition of a group and subgroup.

Definition 2.1.1 (Group)
A non-empty set G is called a group under the binary operation $* : G \times G \to G$ if for any $x, y, z \in G$ the following properties hold:

(i) $x * y \in G$, *(Closure)*

(ii) $x * (y * z) = (x * y) * z$, *(Association)*

(iii) *there exists a specific element called identity denoted by e with property $x * e = e * x = x$, (Identity element)*

(iv) *for each $x \in G$, there exists a unique element $x^{-1} \in G$ such that $x^{-1} * x = x * x^{-1} = e$. (Inverse)*

(v) *Further, if $x * y = y * x$ for any $x, y \in G$, then G is called an abelian group.*

Definition 2.1.2 (Subgroup)
Let G be a group under a binary operation $*$ and let H be a non-empty subset of G. Then H is called a subgroup of G, denoted by $H \leq G$, if H is a group under $*$.

Notation 2.1.3
- *If multiple groups are involved, we take caution in specifying which group the identity element is from by using subscript e_G.*
- *Special attention is given to a group G if its binary operation is addition. In particular, we call G an additive group, and denote $e_G = 0$ and $x^{-1} = -x$ for any $x \in G$.*
- *However, if group G is not an additive group, we will write the product $x * y$ as plainly xy for ease of notation.*

Next, we prove that the direct product of groups is also a group.

Proposition 2.1.4
Let $n \in \mathbb{N}$ and let G_i be a group with binary operation $*_i$ for all $i = 1, 2, \dots, n$. The direct product $G_1 \times G_2 \times \cdots \times G_n$ is a group under the binary operation $*$ defined as component-wise: $(x_1, x_2, \dots, x_n) * (y_1, y_2, \dots, y_n) = (x_1 *_1 y_1, x_2 *_2 y_2, \dots, x_n *_n y_n)$, with

(a) *the identity element given by $e = (e_{G_1}, e_{G_2}, \dots, e_{G_n})$ and*

(b) *the inverse of $(x_1, x_2, \dots, x_n)$ is given by $(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1})$.*

Proof. We will show that the four axioms of Definition 2.1.1 hold.
Let $(x_1, x_2, \dots x_n), (y_1, y_2, \dots, y_n), (z_1, z_2, \dots, z_n) \in G_1 \times G_2 \times \cdots \times G_n$.

(i) We have that $(x_1, x_2, \dots, x_n) * (y_1, y_2, \dots, y_n) = (x_1 *_1 y_1, x_2 *_2 y_2, \dots, x_n *_n y_n)$. Since each G_i is a group, we have $x_i *_i y_i \in G_i$ for $i = 1, 2, \dots, n$ and any $x_i, y_i \in G_i$. Thus, $(x_1, x_2, \dots, x_n) * (y_1, y_2, \dots, y_n) = (x_1 *_1 y_1, x_2 *_2 y_2, \dots, x_n *_n y_n) \in G_1 \times G_2 \times \cdots \times G_n$.

(ii) Since each G_i is associative, we have the following:

$$\begin{aligned}
[(x_1, \dots x_n) * (y_1, \dots, y_n)] * (z_1, \dots, z_n) &= ((x_1 *_1 y_1), \dots, (x_n *_n y_n)) * (z_1, \dots, z_n) \\
&= ((x_1 *_1 y_1) *_1 z_1, \dots, (x_n *_n y_n) *_n z_n) \\
&= (x_1 *_1 (y_1 *_1 z_1), \dots, x_n *_n (y_n *_n z_n)) \\
&= (x_1, \dots x_n) * [(y_1 *_1 z_1, \dots, y_n *_n z_n)] \\
&= (x_1, \dots x_n) * [(y_1, \dots, y_n) * (z_1, \dots, z_n)].
\end{aligned}$$

Thus, $[(x_1, \dots x_n) * (y_1, \dots, y_n)] * (z_1, \dots, z_n) = (x_1, \dots x_n) * [(y_1, \dots, y_n) * (z_1, \dots, z_n)]$.

(iii) We have that $(e_{G_1}, e_{G_2}, \dots, e_{G_n}) * (x_1, x_2, \dots x_n) = (e_{G_1} *_1 x_1, e_{G_2} *_2 x_2, \dots, e_{G_n} *_n x_n) = (x_1, x_2, \dots x_n)$. Furthermore, $(x_1, x_2, \dots x_n) * (e_{G_1}, e_{G_2}, \dots, e_{G_n}) = (x_1 *_1 e_{G_1}, x_2 *_2 e_{G_2}, \dots, x_n *_n e_{G_n}) = (x_1, x_2, \dots x_n)$. Thus, $(e_{G_1}, e_{G_2}, \dots, e_{G_n})$ is the identity.

(iv) We observe that $(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}) * (x_1, x_2, \dots x_n) = (x_1^{-1} *_1 x_1, x_2^{-1} *_2 x_2, \dots, x_n^{-1} *_n x_n) = (e_{G_1}, e_{G_2}, \dots, e_{G_n})$. Also, $(x_1, x_2, \dots x_n) * (x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}) = (x_1 *_1 x_1^{-1}, x_2 *_2 x_2^{-1}, \dots, x_n *_n x_n^{-1}) = (e_{G_1}, e_{G_2}, \dots, e_{G_n})$.
Thus, $(x_1, x_2, \dots, x_n)^{-1} = (x_1^{-1}, x_2^{-1}, \dots, x_n^{-1})$.

Therefore, $G_1 \times G_2 \times \cdots \times G_n$ is a group under the binary operation $*$. □

Notation 2.1.5

For sake of notation we will omit the operations in the direct product $G_1 \times G_2 \times \dots G_n$ and just write:

$$(x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n) = (x_1 y_1, x_2 y_2, \dots, x_n y_n).$$

Theorem 2.1.6

A subset H of G is a subgroup of G if and only if

(a) *H is non-empty and*

(b) *$xy^{-1} \in H$ for all $x, y \in H$.*

Proof. For the forward implication, assume H is a subgroup of G.
By Definition 2.1.2, we have H is non-empty group. Thus, for any element $y \in H$, there exists $y^{-1} \in H$. In addition, by closure, we have $xy^{-1} \in H$, for any other element $x \in H$. Therefore, $xy^{-1} \in H$ for any $x, y \in H$.

For the converse, assume H is a non-empty subset of a group G such that $xy^{-1} \in H$ for all $x, y \in H$. We show that H is a group under the same operation as G.

- Firstly, since H is non-empty, there is at least one element $h \in H$. Furthermore, since $xy^{-1} \in H$ for all $x, y \in H$, we have $e = hh^{-1} \in H$. Hence, we have the identity element $e \in H$.
- Thus, we have $h, e \in H$ and $h^{-1} = eh^{-1} \in H$. The inverse element exists.
- Further, for any $x, y \in H$, $y^{-1} \in H$ and $x(y^{-1})^{-1} \in H$. Therefore, $xy \in H$ as $x(y^{-1})^{-1} = xy$. Hence, we have closure.
- Lastly, since G is a group, the elements of G are associative. Thus, all elements in H are associative, since H is a subset of G.

Therefore, H is a group and H is a subgroup of G. □

Next, we give the conditions for when the intersection and union of subgroups of a group is also a subgroup.

Proposition 2.1.7

Let G be a group and let $\{H_i : i \in I\}$ be a family of subgroups of G for some index set I. Then, the intersection $H = \bigcap_{i \in I} H_i$ is also a subgroup of G.

Proof. Let G be a group and let $\{H_i : i \in I\}$ be a family of subgroups of G, for some index set I.
Let $H = \bigcap_{i \in I} H_i$. Since, $H_i \leq G$ for all $i \in I$, then $e \in H_i$ for all $i \in I$ and $e \in H = \bigcap_{i \in I} H_i$. Thus, H is non-empty.

Let $g, h \in H = \bigcap_{i \in I} H_i$. Then, $g, h \in H_i$, for all $i \in I$ and $H_i \leq G$ for all $i \in H$. By Theorem 2.1.6, $gh^{-1} \in H_i$ for all $i \in I$.
Therefore, $gh^{-1} \in H$, for any $g, h \in H$ and $H = \bigcap_{i \in I} H_i$ is a subgroup of G, by Theorem 2.1.6. □

Proposition 2.1.8

Let G be a group and let $H_0 \leq H_1 \leq H_2 \leq H_3 \leq \cdots$ be a sequence of non-empty subgroups of G. We have $\bigcup_{i=0}^{\infty} H_i$ is a subgroup of G.

Proof. Let G be group and let $H_0 \leq H_1 \leq H_2 \leq H_3 \leq \cdots$ be a sequence of subgroups of G.
Let $H = \bigcup_{i=0}^{\infty} H_i$.

Since H_0 is a subgroup of G, we have $e \in H_0$. Therefore, $e \in H$ and H is non-empty.

Let $x, y \in H$. We have $x \in H_j$ and $y \in H_k$ for some $j, k \in I = \{0, 1, 2, \dots\}$.
Let $m = \max\{j, k\}$. Then, $H_j \leq H_m$, $H_k \leq H_m$ and $x, y \in H_m$. Since H_m is a subgroup of G, by Theorem 2.1.6, we have $xy^{-1} \in H_m$ and $xy^{-1} \in \bigcup_{i=0}^{\infty} H_i = H$, for any $x, y \in H$. By Theorem 2.1.6, we have $H = \bigcup_{i=0}^{\infty} H_i$ is a subgroup of G. □

Lastly, we recall when a group is said to be finitely generated and explore the generated subgroup of a subset S of a group G.

Definition 2.1.9 (Finitely Generated Group)

Let G be a group. The group G is said to be finitely generated, if there is a finite subset S of G such that $G = \langle S \rangle$. The set $\langle S \rangle$ is the intersection of all subgroups of G containing S.

Proposition 2.1.10 (Generated Subgroup)

Let G be a group and let $S \subseteq G$. Then the intersection of all subgroups containing S, denoted by $\langle S \rangle$, is a subgroup of G. Moreover, $\langle S \rangle$ is the smallest subgroup of G containing S.
We call S the set of generators of $\langle S \rangle$.

Proof. Let G be a group and let $S \subseteq G$.
Let $\mathbb{H} = \{H \mid H \leq G \text{ such that } S \subseteq H\}$ be the set of all subgroups of G containing S. By Proposition 2.1.7, we have that $\langle S \rangle = \bigcap_{H \in \mathbb{H}} H$ is a subgroup of G and by Definition of $\langle S \rangle$, we have $\langle S \rangle$ contains S.
Assume $\langle S \rangle$ is not the smallest subgroup of G containing S.
Let Q be the smallest subgroup of G containing S. Then, $Q \in \mathbb{H}$ and $\langle S \rangle = \bigcap_{H \in \mathbb{H}} H \bigcap Q$. Therefore, $\langle S \rangle \subseteq Q$, a contradiction.

Thus, $\langle S \rangle$ is the smallest subgroup of G containing S. □

Proposition 2.1.11

Let G be a group and let $S \subseteq G$.
Then we have $\langle S \rangle = \{s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \mid s_i \in S$ where $\alpha_i \in \{1, -1\}$ and $i = 1, 2, \dots, k\}$.

Proof. Let G be a group, $S \subseteq G$ and let $C = \{s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \mid s_i \in S \text{ ; } \alpha_i \in \{1, -1\} \text{ and } i = 1, 2, \dots, k\}$.
Let $\mathbb{H} = \{H \mid H \leq G \text{ ; } S \subseteq H\}$ be the set of all subgroups of G containing S. Then $\langle S \rangle = \bigcap_{H \in \mathbb{H}} H$.
For any $H \in \mathbb{H}$, we have $S \subseteq H$ and H is a subgroup. Let $x \in C$ such that $x = s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k}$ for some $s_i \in S$ where $\alpha_i \in \{1, -1\}$ for $i = 1, 2, \dots, k$. Since $S \subseteq H$, we have that each $s_i \in H$ for $i = 1, 2, \dots k$.

Since H is a subgroup, H is closed and $x = s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \in H$. Therefore, $C \subseteq H$.
Since $H \in \mathbb{H}$ was arbitrarily chosen, we have $C \subseteq H$ for all $H \in \mathbb{H}$ and $C \subseteq \bigcap_{H \in \mathbb{H}} H = \langle S \rangle$.

For the reverse containment, we will show that C is a subgroup of G that contains S.
Firstly, note that $e = s^{-1}s \in C$ and C is non-empty.
Let $x, y \in C$ where $x = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_k^{\alpha_k}$, $y = b_1^{\beta_1} b_2^{\beta_2} \dots b_m^{\beta_k}$ for some $a_i, b_j \in S$ and $\alpha_i, \beta_j \in \{1, -1\}$ for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, m$.
We have that $xy^{-1} = (a_1^{\alpha_1} a_2^{\alpha_2} \dots a_k^{\alpha_k})(b_1^{\beta_1} b_2^{\beta_2} \dots b_m^{\beta_k})^{-1} = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_k^{\alpha_k} b_m^{-\beta_k} \cdots_2^{-\beta_2} b_1^{-\beta_1}$. However, $\beta_j \in \{1, -1\}$, so $-\beta_j \in \{1, -1\}$ for $j = 1, 2, \dots, m$. Consequently, $xy^{-1} = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_k^{\alpha_k} b_m^{-\beta_k} \cdots_2^{-\beta_2} b_1^{-\beta_1} \in C$.
Therefore, by Theorem 2.1.6, C is a subgroup of G.
Furthermore, for any $s \in S$, $s = s^1 \in C$ and $S \subseteq C$. Hence, C is a subgroup of G containing S and $C \in \mathbb{H}$.
Since $\langle S \rangle = \bigcap_{H \in \mathbb{H}} H$, we have $\langle S \rangle \subseteq C$ and $\langle S \rangle = \{s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \mid s_i \in S \text{ where } \alpha_i \in \{1, -1\} \text{ and } i = 1, 2, \dots, k\}$. □

2.2 Normal Subgroup

In this section, we recall when a subgroup is said to be normal and we recall the quotient group. The section was adapted from [3], [15] and [17].

Firstly, we need to define the left and right cosets of a subgroup and when a subgroup is said to be normal.

Definition 2.2.1 (Cosets)
Let G be a group, H a subgroup of G and let $g \in G$.

(*i*) *The set $gH = \{gh \,|\, h \in H\}$ is called the left coset of H generated by g.*

(*ii*) *The set $Hg = \{hg \,|\, h \in H\}$ is called the right coset of H generated by g.*

Definition 2.2.2 (Normal Subgroup)
A subgroup H of a group G is called a normal subgroup, denoted by $H \trianglelefteq G$, if $gH = Hg$ for all $g \in G$.

Now, we define the centre of a group G and take note that, by [15, 1.5.3], the centre of G is always a normal subgroup of G.

Definition 2.2.3 (Centre)
Let G be a group. The centre of G, denoted by $Z(G)$, is defined as $Z(G) = \{g \in G \,|\, gh = hg$ for all $h \in G\}$.

The next theorem, called the Test for Normality, gives equivalent condition for a subgroup to be normal.

Theorem 2.2.4 (Test for Normality)
Let G be a group and let N be a subgroup of G. Then, N is a normal subgroup of G if and only if $gNg^{-1} \subseteq N$ for all $g \in G$.

Proof. Assume, N is a normal subgroup of G. By Definition 2.2.2, we have $gN = Ng$ for all $g \in G$. However, $gNg^{-1} = (gN)g^{-1} = (Ng)g^{-1} = Ngg^{-1} = Ne = N$ and by definition of equality of sets we have $gNg^{-1} \subseteq N$.

For the converse, assume $gNg^{-1} \subseteq N$ for all $g \in G$. Let $x \in G$. By assumption, $xNx^{-1} \subseteq N$ and $xN = xNx^{-1}x = (xNx^{-1})x \subseteq Nx$. Therefore, $xN \subseteq Nx$. In addition, since $x^{-1} \in G$, we have $x^{-1}N(x^{-1})^{-1} \subseteq N \Rightarrow x^{-1}Nx \subseteq N$. Then, $Nx = xx^{-1}Nx = x(x^{-1}Nx) \subseteq xN$ and $Nx \subseteq xN$. Consequently, $Nx = xN$. Since, x was arbitrarily chosen, by Definition 2.2.2 we have N is a normal subgroup of G. □

Given a normal subgroup N of G, we can consider the set of all cosets of N generated by elements of G. The next proposition tells us this set is actually a group and is called the quotient group.

Proposition 2.2.5 (Quotient Group)
Let G be a group and let N be a normal subgroup of G. The set of all cosets of N in G, $\{xN \,|\, x \in G\}$, is a group under the binary operation $(xN)(yN) = (xy)N$ with

(*a*) *the identity element being N and*

(*b*) $(xN)^{-1} = x^{-1}N$.

In particular, this group is called the quotient group of G modulo N and is denoted by $G/N = \{xN \mid x \in G\}$.

Proof. We will show that the four axioms of definition of a group hold.

(*i*) Let $aN, bN \in G/N$. Then, $ab \in G$ and $(ab)N \in G/N$.
Therefore, $(aN)(bN) = (ab)N \in G/N$ and G/N is closed.

(*ii*) For any $aN, bN, cN \in G/N$, since G is associative, we have $(aNbN)(cN) = ((ab)N)(cN) = (ab)cN = a(bc)N = (aN)((bc)N) = (aN)(bNcN)$. Therefore, $(aNbN)(cN) = (aN)(bNcN)$ and G/N is associative.

(*iii*) For any $aN \in G/N$, we have $NaN = eNaN = (ea)N = aN$ and $aNN = aNeN = (ae)N = aN$.
Also, $(aN)N = aN$.
Therefore, $NaN = aNN = aN$, and N is the identity element of G/N.

(*iv*) For any $aN \in G/N$ we have $aNa^{-1}N = (aa^{-1})N = eN = N$ and $a^{-1}NaN = (a^{-1}a)N = eN = N$.
Therefore, $aNa^{-1}N = a^{-1}NaN = N$ and $(aN)^{-1} = a^{-1}N$. Thus, any $aN \in G/N$ has an inverse.

Therefore, G/N is a group under the binary operation $(xN)(yN) = (xy)N$. □

Lastly, we give a relationship of how equal cosets relate to each other.

Proposition 2.2.6
Let G be a group, N a normal subgroup of G and let $gN, hN \in G/N$. Then $gN = hN$ if and only if there exists $n \in N$ such that $g = hn$.

Proof. Let G be a group, N a normal subgroup of G and let $gN, hN \in G/N$.
Assume $gN = hN$. Then, $gN = hN \Rightarrow (hN)^{-1}(gN) = N \Rightarrow (h^{-1}N)(gN) = N \Rightarrow (h^{-1}gN) = N \Rightarrow h^{-1}g \in N \Rightarrow h^{-1}g = n$ for some $n \in N$ and $g = hn$.

For the converse, assume there exists $n \in N$ such that $g = hn$. Let $gn_1 \in gN$. Then, $gn_1 = (hn)n_1 = h(nn_1) \in hN$ and $gn_1 \in hN$. Therefore, $gN \subseteq hN$.

In addition, $h = gn^{-1}$ since $g = hn$. Therefore, for any $hn_2 \in hN$, we have $hn_2 = (gn^{-1})n_2 = g(n^{-1}n_2) \in gN$ and $hN \subseteq gN$. Thus, $gN = hN$. □

2.3 Isomorphism Theorems

In this section we recall group homomorphisms and group isomorphisms. We will state the three Isomorphism Theorems.

The definitions and results for this section can be found in [3], [15] and [17].

We start by stating the definition of a group homomorphism and giving some characteristics that follow immediately from the definition.

Definition 2.3.1 (Group Homomorphism)
Let G and H be groups. A mapping $\theta : G \to H$ is called a group homomorphism if $\theta(gh) = \theta(g)\theta(h)$ for any $g, h \in G$. Further, we have:

(*i*) *If θ is injective then, θ is called a group monomorphism.*

(*ii*) *If θ is surjective then, θ is called a group epimorphism.*

(*iii*) *If θ is a bijection then, θ is called a group isomorphism and we denoted $G \cong H$.*

Note 2.3.2
We take note of the fact that the homomorphism $\theta : G \to G/N$, defined by $\theta(g) = gN$ is usually called the canonical epimorphism.

Proposition 2.3.3
Let G and H be groups and let $\varphi : G \to H$ be a group homomorphism. Then the following properties hold:

(*a*) $\varphi(e_G) = e_H$,

(*b*) $\varphi(g^{-1}) = \varphi(g)^{-1}$, *for any $g \in G$.*

Next, we introduce the kernel and image of a group homomorphism.

Definition 2.3.4 (Kernel and Image)
Let G and H be groups and let $\varphi : G \to H$ be a group homomorphism.

(*i*) *The kernel of φ, denoted by $Ker\varphi$, is defined as $Ker\varphi = \{x \in G \mid \varphi(x) = e_H\}$.*

(*ii*) *The image of φ, denoted by $Im\varphi$, is defined as $Im\varphi = \{\varphi(x) \mid x \in G\}$.*

We state the three Isomorphism Theorems, of which the proofs of them can be found in [17, Chapter 2] and [3, Section 34].

Theorem 2.3.5 (1^{st} Isomorphism Theorem)
[17, Theorem 2.24] Let G and H be groups. If $\theta : G \to H$ is a group homomorphism, then we have:

(*a*) *$Ker\theta$ is a normal subgroup of G,*

(*b*) *$G/Ker\theta \cong Im\theta$ and*

(*c*) *if θ is a group epimorphism, then $G/Ker\theta \cong H$.*

Theorem 2.3.6 (2^{nd} Isomorphism Theorem)
[17, Theorem 2.26] Let G be a group, H a subgroup of G and let N be a normal subgroup of G. Then we have the following:

(a) HN *is a subgroup of* G,

(b) N *is a normal subgroup of* HN,

(c) $H \cap N$ *is a normal subgroup of* H *and*

(d) $HN/N \cong H/(H \cap N)$.

Theorem 2.3.7 (3^{rd} Isomorphism Theorem)
[17, Theorem 2.27] Let G *be a group,* N *a normal subgroup of* G *and let* K *be a normal subgroup of* N*. Then, we have the following:*

(a) N/K *is a normal subgroup of* G/K *and*

(b) $(G/K)/(N/K) \cong G/N$.

Now we recall the definition of an equivalence relation as given in [3].

Definition 2.3.8 (Equivalence Relation)
[3, Definition 0.18] A relation $\approx$ *on a set* S *is called an equivalence relation if the following properties hold for any* $x, y, z \in S$:

(i) $x \approx x$ *(Reflexive),*

(ii) $x \approx y$*, if and only if* $y \approx x$ *(Symmetry) and*

(iii) *if* $x \approx y$ *and* $y \approx z$*, then* $x \approx z$ *(Transitive).*

We have that group isomorphisms on the set of all groups is an equivalence relation (see [15, Chapter 1.1]). We can define equivalence classes and by [3, Theorem 0.22] we have that the set of all groups can be partitioned into equivalence classes.

Definition 2.3.9 (Equivalence Classes For Group Isomorphism)
Let $\mathbb{G}$ *be the set of all groups. The equivalence class for* $G \in \mathbb{G}$*, denoted by* $[G]$*, is defined as* $[G] = \{H \in \mathbb{G} : G \cong H\}$.

2.4 Commutators and Commutators Subgroups

In this section, we recall the concept of the commutators in a group. We will prove some identities of a commutators. Next, we recall the concept of the commutator subgroup and the lower central series.

The definitions and results for this section can be found in [1].

We start by introducing the commutator of two elements and prove some useful identities.

Definition 2.4.1 (Commutator)
Let G be a group and let $x, y \in G$. The commutator of x and y, denoted by $[x, y]$, is defined as $[x, y] = xyx^{-1}y^{-1}$.

Lemma 2.4.2
Let G be a group. For any $x, y, z \in G$, the following identities hold:

(a) $[x, y] = [y, x]^{-1}$,

(b) $[xy, z] = x[y, z]x^{-1}[x, z]$,

(c) $[x, yz] = [x, y]y[x, z]y^{-1}$,

(d) $[x^{-1}, y] = x^{-1}[x, y]^{-1}x$,

(e) $[x, y^{-1}] = y^{-1}[x, y]^{-1}y$ *and*

(f) $z[x, y]z^{-1} = [zxz^{-1}, zyz^{-1}]$.

Proof. Let G be a group and let $x, y, z \in G$.

(a) $[x, y] = xyx^{-1}y^{-1} = ((y^{-1})^{-1}(x^{-1})^{-1}y^{-1}x^{-1})^{-1} = (yxy^{-1}x^{-1})^{-1} = [y, x]^{-1}$. Thus, $[x, y] = [y, x]^{-1}$.

(b) $[xy, z] = (xy)z(xy)^{-1}z^{-1} = xyzy^{-1}x^{-1}z^{-1} = xyzy^{-1}(e)x^{-1}z^{-1} = xyzy^{-1}(z^{-1}x^{-1}xz)x^{-1}z^{-1} = x(yzy^{-1}z^{-1})x^{-1}(xzx^{-1}z^{-1}) = x[y, z]x^{-1}[x, z]$. Thus, $[xy, z] = x[y, z]x^{-1}[x, z]$.

(c) $[x, yz] = x(yz)x^{-1}(yz)^{-1} = xyzx^{-1}z^{-1}y^{-1} = xy(e)zx^{-1}z^{-1}y^{-1} = xy(x^{-1}y^{-1}yx)zx^{-1}z^{-1}y^{-1} = (xyx^{-1}y^{-1})y(xzx^{-1}z^{-1})y^{-1} = [x, y]y[x, z]y^{-1}$. Thus, $[x, yz] = [x, y]y[x, z]y^{-1}$.

(d) $[x^{-1}, y] = x^{-1}y(x^{-1})^{-1}y^{-1} = x^{-1}yxy^{-1}(e) = x^{-1}yxy^{-1}(x^{-1}x) = x^{-1}(yxy^{-1}x^{-1})x = x^{-1}[y, x] = x^{-1}[x, y]^{-1}x$ by (a). Thus, $[x^{-1}, y] = x^{-1}[x, y]^{-1}x$.

(e) $[x, y^{-1}] = xy^{-1}x(y^{-1})^{-1} = xy^{-1}xy = (e)xy^{-1}xy = (y^{-1}y)xy^{-1}xy = y^{-1}(yxy^{-1}x)y = y^{-1}[y, x]y = y^{-1}[x, y]^{-1}y$ by (a). Thus, $[x, y^{-1}] = y^{-1}[x, y]^{-1}y$.

(f) $z[x, y]z^{-1} = z(xyx^{-1}y^{-1})z^{-1} = zx(e)y(e)x^{-1}(e)y^{-1}z^{-1} = zx(z^{-1}z)y(z^{-1}z)x^{-1}(z^{-1}z)y^{-1}z^{-1} = (zxz^{-1})(zyz^{-1})(zx^{-1}z^{-1})(zy^{-1}z^{-1}) = (zxz^{-1})(zyz^{-1})(zxz^{-1})^{-1}(zyz^{-1})^{-1} = [zxz^{-1}, zyz^{-1}]$. Thus, $z[x, y]z^{-1} = [zxz^{-1}, zyz^{-1}]$.

□

The next lemma gives a relationship between commutators and exponentiation.

Lemma 2.4.3
[1, Lemma 1.13] Let G be a group and let $x, y \in G$ such that $[x, y] \in Z(G)$. Then, $[x^\alpha, y] = [x, y^\alpha] = [x, y]^\alpha$ for any $\alpha \in \mathbb{Z}$.

We recall the commutator subgroup and state a proposition that follows immediately.

Definition 2.4.4 (Commutator Subgroup)
Let G be a group and let M and N be subgroups of G. The commutator subgroup generated by M and N, denoted by $[M, N]$, is defined as $[M, N] = \langle [m, n] \mid m \in M, n \in N \rangle$.
If $G = M = N$, then $[G, G]$ is called the commutator or derived subgroup of G, denoted by G'.

Proposition 2.4.5
Let G and H be groups, G_1 a subgroup of G and let H_1 be a subgroup of H. Then, $[G_1 \times H_1, G \times H] = [G_1, G] \times [H_1, H]$.

Proof. Let $u \in [G_1 \times H_1, G \times H]$. Then, $u = [(x_1, y_1), (x, y)]$ for some $x_1 \in G_1$, $y_1 \in H_1$, $x \in G$ and $y \in H$. However, $u = [(x_1, y_1), (x, y)] = (x_1, y_1)(x, y)(x_1, y_1)^{-1}(x, y)^{-1} = (x_1, y_1)(x, y)(x_1^{-1}, y_1^{-1})(x^{-1}, y^{-1}) = (x_1 x x_1^{-1} x^{-1}, y_1 y y_1^{-1} y^{-1}) = ([x_1, x], [y_1, y])$.

Hence, $u \in [G_1, G] \times [H_1, H]$ and $[G_1 \times H_1, G \times H] \subseteq [G_1, G] \times [H_1, H]$.

Next, we will show the reverse containment.
Let $w \in [G_1, G] \times [H_1, H]$. Then, $w = ([v_1, v], [z_1, z])$ for some $v_1 \in G_1$, $v \in G$, $z_1 \in H_1$ and $z \in H$. We can rewrite w as follows, $w = ([v_1, v], [z_1, z]) = (v_1 v v_1^{-1} v^{-1}, z_1 z z_1^{-1} z^{-1}) = (v_1, z_1)(v, z)(v_1^{-1}, z_1^{-1})(v^{-1}, z^{-1}) = (v_1, z_1)(v, z)(v_1, z_1)^{-1}(v, z)^{-1} = [(v_1, z_1), (v, z)]$.

Hence, $w \in [G_1 \times H_1, G \times H]$ and, $[G_1, G] \times [H_1, H] \subseteq [G_1 \times H_1, G \times H]$.

Thus, $[G_1 \times H_1, G \times H] = [G_1, G] \times [H_1, H]$. □

Lastly, we introduce the lower central series.

Definition 2.4.6 (Lower Central Series)
Let G be a group. The decreasing sequence $\{\gamma_i(G)\}$ of subgroups of G of the form:

(i) $\gamma_0(G) = G$,

(ii) $\gamma_1(G) = [G, G]$ *and*

(iii) $\gamma_{i+1}(G) = [\gamma_i(G), G]$,

is called the lower central series of G.

2.5 Nilpotent Groups

In this section, we state the definition of a nilpotent group. There are many different definitions of a nilpotent group which are found in the literature, however our study will depend on the definition defined using the lower central series. The information of this section can be found in [15, Chapter 5] and [1, Chapter 2].

Definition 2.5.1 (Nilpotent Group)
Let G be a group. The group G is called a nilpotent group if $\gamma_c(G) = \{e\}$, for some $c \in \mathbb{Z}^+$. The smallest such c is called the nilpotency class of G.

We state a theorem, which can also be found in [1]. The theorem relates the nilpotency class of a nilpotent group to its lower central series.

Theorem 2.5.2
[1, Theorem, 2.1] Let G be a group. The following conditions are equivalent:

(*a*) *G is nilpotent of nilpotency class c,*

(*b*) *$\gamma_c(G) = \{e_G\}$ but $\gamma_{c-1}(G) \neq \{e_G\}$.*

Note 2.5.3
Every abelian group G is nilpotent of nilpotency class 1, since its commutator subgroup is trivial.

Next, we recall that every subgroup and every homomorphic image of a nilpotent group is also a nilpotent group. We will show the quotient group of a nilpotent group is also a nilpotent group. Finally, we will show that the direct product of nilpotent groups is again a nilpotent group.

Proposition 2.5.4
Let G be a nilpotent group of nilpotency class c. Every subgroup of G is nilpotent with nilpotency class of at most c.

Proof. Let G be a nilpotent group of nilpotency class c and let H be a subgroup of G. Since G is nilpotent, we have, by Definition 2.5.1, that $\gamma_c(G) = \{e_G\}$. We will show $\gamma_c(H) \leq \gamma_c(G)$ by induction on c.

For $c = 0$, we have, by Definition 2.4.6, $\gamma_0(H) = H$ and $\gamma_0(G) = G$ and by assumption $H \leq G$.
Thus, $\gamma_0(H) = H \leq G = \gamma_0(G)$.
Assume that the statement is true for $c = j$, that is, $\gamma_j(H) \leq \gamma_j(G)$.
We prove that the statement is true for $j + 1$. Firstly, by Definition 2.4.6, we have that $\gamma_{j+1}(H) = [\gamma_j(H), H] = \langle [x, y] : x \in \gamma_j(H), y \in H\rangle$. However, by the assumption that $H \leq G$ and by inductive hypothesis $\gamma_j(H) \leq \gamma_j(G)$. Then, $\langle [x, y] : x \in \gamma_j(H), y \in H\rangle \leq \langle [x, y] : x \in \gamma_j(G), y \in G\rangle$ and $\gamma_{j+1}(H) = [\gamma_j(H), H] = \langle [x, y] : x \in \gamma_j(H), y \in H\rangle \leq \langle [x, y] : x \in \gamma_j(G), y \in G\rangle = [\gamma_j(G), G] = \gamma_{j+1}(G)$.
Thus, $\gamma_{j+1}(H) \leq \gamma_{j+1}(G)$ and the inductive step holds.

By the principle of mathematical induction, we conclude that $\gamma_c(H) \leq \gamma_c(G)$ for any $c \in \mathbb{N}$. Thus, $\gamma_c(H) \leq \gamma_c(G) = \{e_G\}$ and $\gamma_c(H) = \{e_G\} = \{e_H\}$. By Definition 2.5.1, H is nilpotent. Since $\gamma_c(H) = \{e_H\}$, H can only have a nilpotency class that is at most c. □

Proposition 2.5.5
Let G be a nilpotent group of nilpotency class c and let H be a group. If there exists a group epimorphism

$\theta : G \to H$, then H is also nilpotent with nilpotency class of at most c.

Proof. Let $\theta : G \to H$ be a group epimorphism. Since G is nilpotent, then, by Definition 2.5.1, $\gamma_c(G) = \{e_G\}$.
We will show $\theta(\gamma_c(G)) = \gamma_c(H)$ using induction on c.

For $c = 0$, we have by Definition 2.4.6, that $\gamma_0(G) = G$. Since θ is a group epimorphism, then $\theta(G) = H$ and $\theta(\gamma_0(G)) = \theta(G) = H = \gamma_0(H)$.
Assume that the statement is true for $c = j$, that is, $\theta(\gamma_j(G)) = \gamma_j(H)$.
We prove that the statement is true for $j + 1$. Firstly, by Definition 2.4.6, we have that $\gamma_{j+1}(G) = [\gamma_j(G), G]$. In addition, $\theta(\gamma_{j+1}(G)) = \theta[\gamma_j(G), G] = [\theta(\gamma_j(G)), \theta(G)] = [\gamma_j(H), H] = \gamma_{j+1}(H)$. Therefore, $\theta(\gamma_{j+1}(G)) = \gamma_{j+1}(H)$ and we conclude that $\theta(\gamma_c(G)) = \gamma_c(H)$, for any $c \in \mathbb{N}$.

Consequently, $\theta(\gamma_c(G)) = \theta(\{e_G\}) = \{e_H\}$ and $\gamma_c(H) = \{e_H\}$. Now, $\gamma_c(H) = \{e_H\}$, thus H can only have a nilpotency class that is at most c. □

Corollary 2.5.6
Let G be a nilpotent group and let N be a normal subgroup of G. Then, the quotient group G/N is a nilpotent group.

Proof. Let G be a nilpotent group and let N be a normal subgroup of G.
Consider the mapping $\varphi : G \to G/N$ given by $\varphi(g) = gN$. For any $g, h \in G$, we have $\varphi(gh) = (gh)N = (gN)(hN) = \varphi(g)\varphi(h)$. Thus, $\varphi : G \to G/N$ is a surjective group homomorphism. By Proposition 2.5.5, we have G/N is a nilpotent group. □

Proposition 2.5.7
Let G and H be nilpotent groups. Then $G \times H$ is also a nilpotent group.

Proof. Since G and H are nilpotent, then, by Theorem 2.5.2, there exist $k, j \in \mathbb{Z}$ such that $\gamma_k(G) = \{e_G\}$ and $\gamma_j(H) = \{e_H\}$.

Let $c = \max\{k, j\}$. We will show $\gamma_c(G \times H) = \gamma_c(G) \times \gamma_c(H)$ using induction on c.

For $c = 0$, we have, by Definition 2.4.6 that, $\gamma_0(G) = G$ and $\gamma_0(H) = H$.
Then, by Definition 2.4.6, $\gamma_0(G \times H) = G \times H = \gamma_0(G) \times \gamma_0(H)$.
Assume that the statement is true for $c = j$, that is, $\gamma_j(G \times H) = \gamma_j(G) \times \gamma_j(H)$.
We prove that the statement is true for $c = j+1$. Firstly, by Definition 2.4.6, we have that $\gamma_{j+1}(G\times H) = [\gamma_j(G \times H), G \times H]$. By induction hypothesis, $\gamma_j(G \times H) = \gamma_j(G) \times \gamma_j(H)$ and hence $\gamma_{j+1}(G \times H) = [\gamma_j(G) \times \gamma_j(H), G \times H]$.
In addition, $\gamma_j(G) \leq G$ and $\gamma_j(H) \leq H$. Then, by Proposition 2.4.5, we have $[\gamma_j(G) \times \gamma_j(H), G \times H] = [\gamma_j(G), G] \times [\gamma_j(H), H]$.
However, $[\gamma_j(G), G] = \gamma_{j+1}(G)$ and $[\gamma_j(H), H] = \gamma_{j+1}(H)$. Then, $\gamma_{j+1}(G \times H) = \gamma_{j+1}(G) \times \gamma_{j+1}(H)$ and we conclude that $\gamma_c(G \times H) = \gamma_c(G) \times \gamma_i(H)$, for any $c \in \mathbb{N}$.

Then, $\gamma_c(G \times H) = \gamma_c(G) \times \gamma_c(H) = \{e_G\} \times \{e_H\} = \{(e_G, e_H)\} = \{e_{G\times H}\}$.
Therefore, by Theorem 2.5.2, $G \times H$ is nilpotent. □

Corollary 2.5.8

Let $n \in \mathbb{N}$ such that $n \geq 2$ and let $G_1, G_2, \dots G_n$ be nilpotent groups. Then the direct product $G_1 \times G_2 \times \dots \times G_n$ is also nilpotent.

Proof. Proof by induction on n.
For $n = 2$, we have, by assumption, that G_1 and G_2 are nilpotent, hence, by Proposition 2.5.7, $G_1 \times G_2$ is nilpotent.
Assume the statement is true for $n = k$, that is if $G_1, G_2, \dots G_k$ are nilpotent groups, then $G_1 \times G_2 \times \dots \times G_k$ is also nilpotent. We prove that the statement is true for $n = k+1$. Let $G_1, G_2, \dots G_k, G_{k+1}$ be nilpotent groups. By induction hypothesis, we have $G_1 \times G_2 \times \dots \times G_k$ is a nilpotent group. Therefore, $G_1 \times G_2 \times \dots \times G_k$ and G_{k+1} are nilpotent groups. By Proposition 2.5.7 we have $G_1 \times G_2 \times \dots \times G_k \times G_{k+1}$ is nilpotent and thus $G_1 \times G_2 \times \dots \times G_n$ is also nilpotent, for any $n \geq 2$. □

Following [1], we show that for any nilpotent group of nilpotency class c, we can construct a subgroup which has nilpotency class that is strictly less than c. This, will prove useful later on for induction proofs that rely on the nilpotency class of a group.

Lemma 2.5.9

[1, Lemma 2.9] Let G be a nilpotent group of nilpotency class $c \geq 1$ and let $g \in G$. The group $H = \langle \gamma_1(G), g \rangle$ is a nilpotent group of nilpotency class that is strictly less than c.

Proof. Let G be a nilpotent group of nilpotency class $c \geq 1$, $g \in G$ and let $H = \langle \gamma_1(G), g \rangle$.
Firstly, $\{g, \gamma_1(G)\} \subseteq G$ since $\gamma_1(G) \leq G$. Consequently, by Proposition 2.1.10, we have that $H = \langle \gamma_1(G), g \rangle$ is a subgroup of G.
Since every subgroup of nilpotent group is nilpotent too, we have that H is a nilpotent group.
Now, we show that H has nilpotent class that is less than c. We will do this by showing that $\gamma_i(H) \leq \gamma_{i+1}(G)$ for every $i \geq 1$, using induction on $i \in \mathbb{N}$.
Consider the case for $i = 1$. Firstly, we note that, for any $[x, y] \in \gamma_1(G)$, we have (using Lemma 2.4.2 (f)) that $g[x, y] = g[x, y]g^{-1}g = (g[x, y]g^{-1})g = [gxg^{-1}, gyg^{-1}]g = h_*g$ for some $h_* \in \gamma_1(G)$.

Thus, every element $h \in H$, can be rearranged to a form $h = h_*g^m$ for some $h_* \in \gamma_1(G)$ and $m \in \mathbb{Z}$.
Therefore, $\gamma_1(H) = [H, H] = \langle [x, y] : x, y \in H \rangle = \langle [hg^m, kg^n] : h, k \in \gamma_1(G)$ and $m, n \in \mathbb{Z} \rangle$.
However, for any $h, k \in \gamma_1(G)$ and $m, n \in \mathbb{Z}$, we can rearrange $[hg^m, kg^n]$ in the following way:

$$\begin{aligned}
[hg^m, kg^n] &= h[g^m, kg^n]h^{-1}[h, kg^n] \text{ by Lemma 2.4.2 (b)} \\
&= h([g^m, kg^n])h^{-1}[h, kg^n] \\
&= h([g^m, k]k[g^m, g^n]k^{-1})h^{-1}[h, kg^n] \text{ by Lemma 2.4.2 (c)} \\
&= h[g^m, k](kg^m g^n g^{-m} g^{-n} k^{-1})h^{-1}[h, kg^n] \\
&= h[g^m, k](kg^{m+n-m-n}k^{-1})h^{-1}[h, kg^n] \\
&= h[g^m, k](kg^0 k^{-1})h^{-1}[h, kg^n] \\
&= h[g^m, k](kek^{-1})h^{-1}[h, kg^n] \\
&= h[g^m, k](e)h^{-1}[h, kg^n] \\
&= h[g^m, k]h^{-1}[h, kg^n] \\
&= [hg^m h^{-1}, hkh^{-1}][h, kg^n] \text{ by Lemma 2.4.2 (f).}
\end{aligned}$$

Since $h \in \gamma_1(G)$ and $kg^n \in G$, we have $[h, kg^n] \in [\gamma_1(G), G] = \gamma_2(G)$ and $hkh^{-1} \in G$. By construction of $\gamma_1(H)$, we have $hg^m h^{-1} \in \gamma_1(G)$ and $[hg^m h^{-1}, hkh^{-1}] \in \gamma_2(G)$.

Therefore, $[hg^m h^{-1}, hkh^{-1}], [h, kg^n] \in \gamma_2(G)$, for every $h, k \in \gamma_1(G)$ and $m, n \in \mathbb{Z}$. Thus, $\gamma_1(H) \leq \gamma_2(G)$.
Assume the statement is true for $i = j \geq 2$, that is $\gamma_j(H) \leq \gamma_{j+1}(G)$.
Consider the case when $i = j + 1$. By definition of the lower central series, we have $\gamma_{j+1}(H) = [\gamma_j(H), H]$. By induction hypothesis, $\gamma_j(H)$ is a subgroup $\gamma_{j+1}(G)$ and H is a subgroup. Therefore, $\gamma_{j+1}(H) = [\gamma_j(H), H] \leq [\gamma_{j+1}(G), G] = \gamma_{j+2}(G)$ and we conclude that $\gamma_i(H) \leq \gamma_{i+1}(G)$ for every $i \geq 1$.

Since, G has nilpotent class c, by Theorem 2.5.2, we have $\gamma_c(G) = \{e_G\}$ and $\gamma_{c-1}(G) \neq \{e_G\}$.
Therefore, $\gamma_{c-2}(H) \leq \gamma_{c-1}(G) \neq \{e_G\}$ and $\gamma_{c-2}(H) \neq \{e\}$. However, $\gamma_{c-1}(H) \leq \gamma_c(G) = \{e_G\}$. Therefore, $\gamma_{c-1}(H) = \{e\}$ and $\gamma_{c-2}(H) \neq \{e\}$. By Theorem 2.5.2, H has nilpotency class that is strictly less than c. $\square$

2.6 Rings and Subrings

In this section, we recall the definitions of ring and a subring. We prove some results that follow immediately from them. Lastly, we will recall the definition of integral domains and some results relating to them. Some of the basic proofs can be found in [3] and [4].

We start by recalling the definition of a ring.

Definition 2.6.1 (Ring)
A ring $\langle R, +, \cdot \rangle$ is a set R together with two binary operations $+ : R \to R$ (called addition) and $\cdot : R \to R$ (called multiplication) such that the following hold for any $a, b, c \in R$:

(i) *$\langle R, + \rangle$ is an abelian group,*

(ii) *multiplication is closed and associative, that is $ab \in R$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and*

(iii) *the distribution law holds:*

- *left distribution law: $a \cdot (b + c) = a \cdot b + a \cdot c$ and*
- *right distribution law: $(b + c) \cdot a = b \cdot a + c \cdot a$.*

Notation 2.6.2
For a ring R we write the multiplication product $a \cdot b$ as ab and we will indicate addition as $a + b$, for any $a, b \in R$.

The proof of the following characterizations of rings can be found in [3, Theorem 18.8].

Proposition 2.6.3
Let R be a ring with the additive identity 0. Then for any $v, w \in R$ we have the following hold:

(a) *$v0 = 0v = 0$,*

(b) *$(-v)(w) = (v)(-w) = -(vw)$ and*

(c) *$(-v)(-w) = vw$.*

Now, we recall the concept of a subring and state a theorem which serves as a test for a subring. The proof is basic and can be found in [4, Theorem 12.3].

Definition 2.6.4 (Subring)
Let R be a ring with binary operations $+$ and $\cdot$. A non-empty subset S of R is called subring if S is a ring under the operators $+$ and $\cdot$.

Theorem 2.6.5 (Test For Subring)
Let R be a ring with binary operations $+$ and $\cdot$. A non-empty subset S of R is a subring of R if and only if the following holds for all $v, w \in S$:

(a) *$v + (-w) \in R$ and*

(b) *$vw \in R$.*

Next, we recall integral domains. We will state the cancellation theorem in an integral domain. Lastly, we will conclude this section by recalling the concept of a field and state that every field is an integral domain.

Definition 2.6.6 (Commutative Ring)
A ring R is called a commutative ring if multiplication is commutative.

Definition 2.6.7 (Ring with Unity)
A ring R is called a ring with unity if it contains a multiplicative identity.

Notation 2.6.8
We take note of the following:

(i) *We call the multiplicative identity of a ring the unity and we will denote it by* 1.

(ii) *For the sake of notation, additive inverse will be represented as* $-a$ *and multiplicative inverse will be represented as* a^{-1} *for any* $a \in R$ *where* R *is a ring with unity.*

Definition 2.6.9 (Unit)
Let R be a ring with unity $1 \neq 0$. An element $u \in R$ is called a unit if it has a multiplicative inverse in R.

Proposition 2.6.10
Let R be a ring with unity $1 \neq 0$. If $u, v \in R$ are units, then uv is also a unit.

Proof. Since $u, v \in R$ are units, then $u^{-1} \in R$ and $v^{-1} \in R$. Firstly, $uv(v^{-1}u^{-1}) = u(vv^{-1})u^{-1} = u(1)u^{-1} = uu^{-1} = 1$. In addition, we have $(v^{-1}u^{-1})uv = v^{-1}(u^{-1}u)v = v^{-1}(1)v = v^{-1}v = 1$. Hence, $uv(v^{-1}u^{-1}) = 1 = (v^{-1}u^{-1})uv$ and $(uv)^{-1} = v^{-1}u^{-1}$. Therefore, uv is a unit in R. □

Definition 2.6.11 (Divisors of 0)
Let R be a ring and let $a, b \in R$ be non-zero elements. The elements a and b are called divisors of 0 *if $ab = 0$.*

Definition 2.6.12 (Integral Domain)
Let R be a commutative ring with unity $1 \neq 0$. Then R is called an integral domain if R contains no divisors of 0.

Proposition 2.6.13 (Cancellation in Integral Domain)
[3, Theorem 19.5] Let R be an integral domain. The cancellation laws hold in an integral domain, that is:

(a) *if $ab = ac$ for some $a, b, c \in R$ such that $a \neq 0$, then we have $b = c$ and*

(b) *if $ba = ca$ for some $a, b, c \in R$ such that $a \neq 0$, then we have $b = c$.*

Proposition 2.6.14
Let R be an integral domain and let $a, b \in R$. If ab is a unit, then we have a and b are units.

Proof. Let R be an integral domain with unity and additive identity 0 and let $a, b \in R$, such that ab is a unit. Since ab is a unit, we have that ab has a multiplicative inverse. Let u be the multiplicative inverse of ab. Then, $u(ab) = (ab)u = 1$.
We have $1 = u(ab) = (ua)b$. Consequently, $ua = 1(ua) = ((ua)b)(ua) = (ua)(b(ua))$, since multiplica-

tion is associative in R. Therefore, $ua = (ua)(b(ua))$ and

$$\begin{aligned} ua - (ua)(b(ua)) = 0 &\Rightarrow ua + (ua)(-b(ua)) = 0 \text{ by Proposition 2.6.3} \\ &\Rightarrow ua(1) + (ua)(-b(ua)) = 0 \\ &\Rightarrow ua(1 - b(ua)) = 0 \text{ by left distribution in } R. \end{aligned}$$

Thus $ua(1 - b(ua)) = 0$. Since R is an integral domain we have R contains no divisors of 0. We must have that either $ua = 0$ or $1 - b(ua) = 0$.
If $ua = 0$, then $1 = (ua)b = 0b = 0$, a contradiction. Hence, $1 - b(ua) = 0$ and $b(ua) = 1$.

Consequently, $(ua)b = b(ua) = 1$, $b^{-1} = ua$ and b is a unit. In a similar manner, one can show a is a unit. □

Definition 2.6.15 (Division Ring)
Let R be a ring with unity $1 \neq 0$. Then R is called a division ring if every non-zero element is a unit in R.

Definition 2.6.16 (Field)
Let R be a division ring. Then R is called a field if R is commutative.

Proposition 2.6.17
[3, Theorem 19.9] Every field F is an integral domain.

2.7 Principal Ideal Domain (PID) and Unique Factorization Domain (UFD)

In this section, we will recall the concepts of principal ideal domains (PID) and unique factorization domains (UFD). We begin with principal ideal domains, by firstly recalling the concept of an ideal for a ring, which is analogous to a normal subgroup of a group. Then, we recall the quotient ring and state the definition of a principal ideal and a principal ideal domain. Lastly, we recall a unique factorization domain and how it relates to a principal ideal domain. The concepts recalled in this section can be found in [3] and [4].

Definition 2.7.1 (Ideal)
Let R be a ring and let N be an additive subgroup of R. Then, N is called an ideal of R if the following hold for all $a, b \in R$:

(*i*) *$aN \subseteq N$ and*

(*ii*) *$Nb \subseteq N$.*

Similarly to quotient groups, we can create quotient rings using ideals of a ring. The proof of the following can be found in [3, Corollary 26.14].

Proposition 2.7.2 (Quotient Ring)
Let R be a ring and let N be an ideal of R. Then, the additive cosets of N form a ring $R/N = \{a+N : a \in R\}$ with binary operation defined as follows: For $(a+N), (b+N) \in R/N$

(*i*) *addition: $(a+N)+(b+N) = (a+b)+N$ and*

(*ii*) *multiplication: $(a+N)(b+N) = (ab)+N$.*

We call the ring R/N the quotient ring (or factor ring) of R by N.

The next proposition tells us that the set of multiples of an element a in a ring R is an ideal of R.

Proposition 2.7.3 (Principal Ideal Generated by a)
Let R be a commutative ring with unity 1 and let $a \in R$. The set $\{ra : r \in R\}$ of all multiples of a is an ideal of R. We call this the principal ideal generated by a and we denote it by $\langle a \rangle$.

Proof. Let R be a commutative ring with unity 1 and let $a \in R$.
Let $N = \{ra : r \in R\}$. We will show that N is an additive subgroup of R.
Since, $1 \in R$, we have $a = 1a \in N$, thus, N is non-empty.
Let $x, y \in N$ such that $x = r_1 a$ and $y = r_2 a$ for some $r_1, r_2 \in R$. We have:

$$\begin{aligned} x - y &= r_1 a - (r_2 a) \\ &= r_1 a + (-r_2)(a) \text{ by Proposition 2.6.3 } (b) \\ &= (r_1 + (-r_2))(a) \text{ by right distribution in } R. \end{aligned}$$

Then, $x - y = (r_1 + (-r_2))(a) \in N$ since $r_1 - r_2 \in R$.
Therefore, $x - y \in N$ for any $x, y \in N$. By Theorem 2.1.6, we have, N is an additive subgroup of R.

Now, we show that the two conditions of Definition 2.7.1 hold to conclude that N is an ideal.
Firstly, let $q \in R$ and let $x' \in qN$. Then $x' = qn$ for some $n \in N$. Since, $n \in N$, we have $n = ra$ for

some $r \in R$. Thus, $x' = q(ra) = (qr)a \in N$ and $qN \subseteq N$.

If $y' \in Nq$, then $y' = nq$ for some $n \in N$. Now, $n \in N$, we have $n = ra$ for some $r \in R$. Thus,

$$\begin{aligned} y' &= (ra)q \\ &= r(aq) \text{ since multiplication is associative} \\ &= r(qa) \text{ since } R \text{ is a commutative ring} \\ &= (rq)a \text{ since multiplication is associative.} \end{aligned}$$

Hence, $y' = (rq)a \in N$ since $q, r \in R$ and $rq \in R$. Therefore, $Nq \subseteq N$.

Consequently, $N = \{ra : r \in R\}$ is an additive subgroup of R, $qN \subseteq N$ and $Nq \subseteq N$ for any $q \in R$. By Definition 2.7.1, we have $N = \langle a \rangle = \{ra : r \in R\}$ is an ideal of R. □

We recall the definitions of a principal ideal and a principal ideal domain.

Definition 2.7.4 (Principal Ideal)
Let R be a commutative ring with unity 1. An ideal N of R is called a principal ideal if $N = \langle a \rangle$ for some $a \in R$.

Definition 2.7.5 (Principal Ideal Domain)
An integral domain R is called a principal ideal domain (PID) if every ideal of R is a principal ideal.

Now, we will recall the definition of a unique factorization domain. We start by recalling the definition of an irreducible element in R and the definition of associative element.

Definition 2.7.6 (Irreducible Element)
Let R be an integral domain. A non-zero non-unit element $a \in R$ is called an irreducible element if whenever $a = bc$ we have that either b or c is a unit.

Definition 2.7.7 (Associative Element)
Let R be an integral domain and let $a, b \in R$. The elements a and b are associative if $a = bu$ for some unit $u \in R$. We denote two associative elements a, b by $a \sim b$.

Proposition 2.7.8
Let R be an integral domain. For any $a, b \in R$, the relation:

$$a \sim b \text{ if and only if } a = bu \text{ for some unit } u \in R$$

is an equivalence relation.

Proof. Let R be an integral domain and let a, b and $c \in R$. We will show the conditions of an equivalence relation (Definition 2.3.8) hold.

(i) Since 1 is a unit in R, then $a = a1$ and $a \sim a$ for all $a \in R$.

(ii) Assume $a \sim b$. By Definition 2.7.7, $a = bu$ for some unit $u \in R$. Since u is a unit, then $u^{-1} \in R$ and $b = au^{-1}$. Since u^{-1} is a unit in R, we have $b \sim a$.
Assume $b \sim a$. By Definition 2.7.7, $b = av$ for some unit $v \in R$. Since v is a unit, then $v^{-1} \in R$ and $a = bv^{-1}$. Since v^{-1} is a unit in R, we have $a \sim b$. Thus, $a \sim b$ if and only if $b \sim a$.

(*iii*) Assume $a \sim b$ and $b \sim c$. By Definition 2.7.7, $a = bu$ and $b = cv$ for some units $u, v \in R$. Then, $a = (cv)u = c(vu)$ and by Proposition 2.6.10, we have vu is a unit and $a \sim c$.

We conclude that the associative relation in Definition 2.7.7 is an equivalence relation. □

We state the definition of a unique factorization domain.

Definition 2.7.9 (Unique Factorization Domain)
Let R be an integral domain. Then R is called a unique factorization domain (UFD) if the following conditions hold:

(*i*) *for every non-zero element $a \in R$ which is not a unit, we have $a = b_1 b_2 \dots b_k$ for some irreducible elements $b_1, b_2, \dots, b_k \in R$ and*

(*ii*) *if $b_1, b_2, \dots, b_k \in R$ and $c_1, c_2, \dots, c_l \in R$ are irreducible elements such that $b_1 b_2 \dots b_k = c_1 c_2 \dots c_l$, then we have $k = l$, and for all $b_i \in \{b_1, b_2, \dots, b_k\}$, there exists some $c_j \in \{c_1, c_2, \dots, c_l\}$ such that $b_i \sim c_j$.*

We state the following result that every principal ideal domain is also a unique factorization domain and the proof can be found at [4, Theorem 18.3].

Proposition 2.7.10
Every principal ideal domain (PID) is a unique factorization domain (UFD).

2.8 Primes, Greatest Common Divisor and Lowest Common Multiple

In this section, we recall prime elements of a ring. In particular, we show that an element of unique factorization domain (UFD) is prime if and only if it is irreducible. Furthermore, we define the set $\mathbf{P}_{members}$, for a set of primes P in R. We show that $\mathbf{P}_{members}$ is closed under multiplication. Lastly, we show that the greatest common divisor and lowest common multiple can be defined in a principal ideal domain.

The definitions and results on primes can be found in [2], [4] and [12].

Firstly, we recall the definition on division of elements in an integral domain and show how this related to the associativity property.

Definition 2.8.1 (Divides)
Let R be an integral domain and let $a, b \in R$. We say that a divides b if $b = ac$ for some $c \in R$. If a divides b, we write $a \mid b$.

Proposition 2.8.2
Let R be an integral domain and let $a, b \in R$. We have that $a \sim b$ if and only if a divides b and b divides a.

Proof. Let R be an integral domain and let $a, b \in R$.
Assume that $a \sim b$. By Proposition 2.7.8, we have that $\sim$ is an equivalence relation, so we have $b \sim a$. Since $a \sim b$, we have, by Definition 2.7.7, that $a = bu$ for some unit $u \in R$. Since $a = bu$, then b divides a.
In addition, $b \sim a$ and by Definition 2.7.7, $b = av$ for some unit $v \in R$. Since $b = av$, then a divides b. Hence, a divides b and b divides a.

For the converse, assume a divides b and b divides a. Since a divides b, we have $b = ac$ for some $c \in R$. Since b divides a, we have $a = bd$ for some $d \in R$. Then,

$$a = (ac)d \Rightarrow a = a(cd) \Rightarrow a - a(cd) = a(cd) - a(cd) \Rightarrow a(1) - a(cd) = 0 \Rightarrow a(1 - cd) = 0.$$

Since R is an integral domain (thus, R does not have zero divisors) we must have either $a = 0$ or $1 - cd = 0$.

- If $a = 0$ then $b = ac = 0c = 0$. Therefore, $a = 0 = b1$ and we have $a \sim b$.
- If $1 - cd = 0$ we have $cd = 1$. Hence c and d are units. Thus, d is a unit since $a = bd$. We have $a \sim b$.

□

Now, we define a prime element of an integral domain and we show that every prime element is irreducible in any integral domain.

Definition 2.8.3 (Prime Element)
Let R be an integral domain and let $p \in R$ be a non-zero element and p is not a unit. The element p is called a prime element of R if whenever p divides bc then either p divides b or p divides c.

Proposition 2.8.4
Let R be an integral domain. If $p \in R$ is a prime element in R, then p is an irreducible element in R.

Proof. Let R be an integral domain and let $p \in R$ be a prime element in R.
Let $p = bc$ for some $b, c \in R$. Since, $(bc) = p1$ we have $p \mid bc$ and $p \mid b$ or $p \mid c$ since p is prime.

In addition, $b \mid p$ since $p = bc$. We will consider the following cases:

Case I: If $p \mid b$ then, $p \mid b$ and $b \mid p$. Thus, by Proposition 2.8.2, $p \sim b$ and $p = bu$ for some unit $u \in R$.
Consequently, $bc = p = bu$ and $c = u$ since primes are non-zero. Therefore, c is a unit and p is irreducible.

Case II: If $p \mid c$, then $p = bc$ and $p = cb$ since R is commutative. Therefore, $c \mid p$. Now, $p \mid c$ and $c \mid p$. By Proposition 2.8.2 we have $p \sim c$ and $p = cu$ for some unit $u \in R$. Then $cb = p = cu$ and $b = u$. Thus, b is a unit and p is irreducible.

We conclude that prime elements in an integral domain are irreducible elements. □

The converse of Proposition 2.8.4 is not true in general. However, in a unique factorization domain, it is true. The proof of the following proposition can be found at [4, Theorem 18.2].

Proposition 2.8.5
Let R be a unique factorization domain (UFD). Then every irreducible element is a prime element.

Thus, we can think of decompositions in a UFD as prime decompositions. Now, we consider a set of primes and elements created by products of primes.

Definition 2.8.6 (P-members)
[12, Definition 3.1] Let R be a unique factorization domain (UFD) and let P be a set of primes in R. Let $a \in R$ be a non-zero and non-unit element of R. Then a is called a member of P, or a P-member, if all prime divisors of a are in P up to associativity.
We will denote the set of P-members in R by $\boldsymbol{P}_{members}$.

Lemma 2.8.7
Let R be a unique factorization domain (UFD) and let P be a set of primes in R. If $a, b \in \boldsymbol{P}_{members}$, then $ab \in \boldsymbol{P}_{members}$.

Proof. Let R be a unique factorization domain (UFD) and let P be a set of primes in R.
Let $a, b \in \mathbf{P}_{members}$, then $a = p_1 p_2 \dots p_k$, $b = q_1 q_2 \dots q_r$ with each p_i, q_s primes from $\mathbf{P}_{members}$ for all $i = 1, 2, \dots k$ and $s = 1, 2, \dots r$.
Since $ab \in R$ and R is a unique factorization domain, we have that the decomposition $ab = d_1 d_2 \dots d_t$ with each d_j being irreducible for all $j = 1, 2, \dots, t$. By Proposition 2.8.5, d_j is prime for all $j = 1, 2, \dots, t$, since in a unique factorization domain, irreducible elements are prime elements.

Then, $d_1 d_2 \dots d_t = ab = (p_1 p_2 \dots p_k)(q_1 q_2 \dots q_r)$. By the uniqueness of decompositions in a unique factorization domain (Property (ii) of Definition 2.7.9), we have that for every $j = 1, 2, \dots, t$ either $d_j \sim p_i$ or $d_j \sim q_s$ for some $i = 1, 2, \dots, k$, $j = 1, 2, \dots, r$. Therefore, each prime divisor of ab is in $\mathbf{P}_{members}$ up to associative elements. Hence, $ab \in \mathbf{P}_{members}$. □

Now, we define the greatest common divisor and lowest common multiple in a principal ideal domain (PID). Furthermore, we show that for any two non-zero elements a and b of a principal ideal domain (PID), the greatest common divisor of a and b exists. We will also show that, the lowest common multiple of a and b exists in a PID. The results on greatest common divisor can be found at [8, Chapter 4 Section 2].

Definition 2.8.8 (Greatest Common Divisor)
Let R be a principal ideal domain (PID) and let $a, b \in R$ be non-zero elements. The greatest common divisor of a and b, denoted by $\gcd(a, b)$, is an element $gcd(a, b) \in R$ such that:

(*i*) $\gcd(a, b)$ *divides a,*

(*ii*) $\gcd(a, b)$ *divides b and*

(*iii*) *if $d \in R$ divides a and b then d also divides* $\gcd(a, b)$.

Theorem 2.8.9
Let R be a principal ideal domain (PID) and let $a, b \in R$ be non-zero elements. Then, there exists a greatest common divisor of a and b which is unique up to associativity. Furthermore, there exist $s, t \in R$ such that $\gcd(a, b) = as + bt$.

Proof. Let R be a principal ideal domain (PID) and let $a, b \in R$ be non-zero elements. Let $I = \{ap+bq : p, q \in R\}$. We show I is an ideal of R. Firstly, $0 = a0 + b0 \in I$ and I is non-empty. For any $ap_1 + bq_1$, $ap_2 + bq_2 \in I$, we have:

$$\begin{aligned}(ap_1 + bq_1) - (ap_2 + bq_2) &= ap_1 + bq_1 - bq_2 - ap_2 \\ &= ap_1 - ap_2 + bq_1 - bq_2 \text{ since } R \text{ is abelian under addition} \\ &= a(p_1 - p_2) + b(q_1 - q_2) \in I \text{ by left distribution law.}\end{aligned}$$

Thus, by Theorem 2.1.6, we have I is an additive subgroup of R.

Let $x \in R$. For any $x(ap_3 + bq_3) \in xI$, we have $x(ap_3 + bq_3) = xap_3 + xbq_3 = axp_3 + bxq_3 = a(xp_3) + b(xq_3) \in I$.

Hence, $xI \subseteq I$. Furthermore, for any $(ap_4 + bq_4)x \in Ix$, we have $(ap_4 + bq_4)x = ap_4x + bq_4x = a(p_4x) + b(q_4x) \in I$.

Hence, $Ix \subseteq I$. Thus, by Definition 2.7.1, we have I is an ideal of R.

Since R is a principal ideal domain (PID), then there exists $d \in R$ such that $\langle d\rangle = I$, where $\langle d\rangle = \{rd : r \in R\} = \{dr : r \in R\}$. We show that d is the greatest common divisor of a and b.
Firstly, $d \in \langle d\rangle = I$ so there exist $s, t \in R$ such that $d = as + bt$.
In addition, $a = a1 + b0 \in I = \langle d\rangle$. Thus, $a = dr_1$ for some $r_1 \in R$ and d divides a.
Similarly, $b = a0 + b1 \in I = \langle d\rangle$. Thus, $a = dr_2$ for some $r_2 \in R$ and d divides b.

Let $h \in R$ such that h divides a and b. Then, there exist $c_1, c_2 \in R$ such that $a = hc_1$ and $b = hc_2$. Then, we have that $d = as + bt = (hc_1)s + (hc_2)t = h(c_1s) + h(c_2t) = h(c_1s + c_2t)$.

Then, h divides d, d divides a and d divides b. If h divides a and b then h divides d. By Definition 2.8.8, we have d is the greatest common divisor of a and b.

Now, assume there are two greatest common divisors of a and b, say d_1 and d_2. By Definition 2.8.8, we have d_1 divides a and b. Thus, d_1 divides d_2. Similarly, by Definition 2.8.8, we have d_2 divides a and b. Therefore, d_2 divides d_1. By Proposition 2.8.2, $d_1 \sim d_2$ and the greatest common divisor is unique to up associativity. □

Now, we recall the lowest common multiple of two elements of principal ideal domain (PID). Definition 2.8.10 and Theorem 2.8.11 is taken from [2] and [18].

Definition 2.8.10 (Lowest Common Multiple)
Let R be a principal ideal domain (PID) and let $a, b \in R$ be non-zero elements. The lowest common multiple of a and b, denoted by $lcm(a, b)$, is an element $lcm(a, b) \in R$ such that:

(i) a divides $lcm(a, b)$,

(ii) b divides $lcm(a, b)$ and

(iii) if there exists $l \in R$ such that a divides l and b divides l, then $lcm(a, b)$ also divides l.

Theorem 2.8.11
Let R be a principal ideal domain (PID) and let $a, b \in R$ be non-zero elements. Then there exists a lowest common multiple of a and b which is unique up to associativity.

Proof. Let R be a principal ideal domain (PID) and let $a, b \in R$ be non-zero elements. Let $I = \langle a \rangle \cap \langle b \rangle$ where, $\langle a \rangle = \{ra : r \in R\} = \{ar : r \in R\}$ and $\langle b \rangle = \{rb : r \in R\} = \{br : r \in R\}$. We show that I is an ideal of R. By Proposition 2.7.3, we have $\langle a \rangle$ and $\langle b \rangle$ are ideals. Thus, $\langle a \rangle$ and $\langle b \rangle$ are additive subgroups of R and by Proposition 2.1.7, $\langle a \rangle \cap \langle b \rangle$ is an additive subgroup of R.

Let $x \in R$. Since, $\langle a \rangle$ is a ideal of R, we have $x \langle a \rangle \subseteq \langle a \rangle$ and $\langle a \rangle x \subseteq \langle a \rangle$. Similarly, $x \langle b \rangle \subseteq \langle b \rangle$ and $\langle b \rangle x \subseteq \langle b \rangle$. Therefore, $x \langle a \rangle \cap x \langle b \rangle \subseteq \langle a \rangle \cap \langle b \rangle$ and $\langle a \rangle x \cap \langle b \rangle x \subseteq \langle a \rangle \cap \langle b \rangle$. By Definition 2.7.1, we have $\langle a \rangle \cap \langle b \rangle$ is an ideal of R.

Since, R is a principal ideal domain (PID), then there exists $c \in R$ such that $\langle c \rangle = I = \langle a \rangle \cap \langle b \rangle$. Since $c \in \langle c \rangle = \langle a \rangle \cap \langle b \rangle$, we have that $c \in \langle a \rangle$ and $c \in \langle b \rangle$. Thus, $c = ar_1$ and $c = ar_2$ for some $r_1, r_2 \in R$ and, by Definition 2.8.1, a divides c and b divides c.

Now, assume that there exists $l \in R$ such that $a \mid l$ and $b \mid l$. Then, $l = as_1$ and $l = bs_2$ for some $s_1, s_2 \in R$. Then, $l \in \langle a \rangle$ and $l \in \langle b \rangle$ and $l \in \langle a \rangle \cap \langle b \rangle = \langle c \rangle$. Consequently, $l = ct$ for some $t \in R$ and c divides d.

Therefore, c divides a, c divides b and if there exists $l \in R$ such that a divides l and b divides l, then c also divides l. Hence, by Definition 2.8.10, we have c is the lowest common divisor of a and b.

Assume there are two lowest common multiples of a and b, say c_1 and c_2. By Definition 2.8.10, a and b divides c_1, thus c_2 divides c_1. Similarly, by Definition 2.8.10, a and b divides c_2, thus c_1 divides c_2. By Proposition 2.8.2, we have $c_1 \sim c_2$ and the lowest common multiple is unique to up associativity. □

2.9 Ring Isomorphisms

In this section, we define homomorphisms for rings. There are equivalent Isomorphism Theorems for ring homomorphism. However, we will only need the First Isomorphism Theorem, to prove the main result of this section, which states that a ring with characteristic 0 has a subring which is isomorphic to the set of integers $\mathbb{Z}$. The information in this section can be found in [3] and [4].

We begin by stating the definitions of a ring homomorphism and ring isomorphism.

Definition 2.9.1 (Ring Homomorphism)
Let R and S be rings. A mapping $\varphi : R \to S$ is called a ring homomorphism if for all $a, b \in R$ we have:

(*i*) $\varphi(a + b) = \varphi(a) + \varphi(b)$ *and*

(*ii*) $\varphi(ab) = \varphi(a)\varphi(b)$.

Definition 2.9.2 (Ring Isomorphism)
Let R and S be rings. A mapping $\varphi : R \to S$ is called a ring isomorphism, denoted by $R \cong S$, if φ is a bijection and a ring homomorphism.

We also note without proof that the image of a ring homomorphism $\varphi : R \to S$ is a subring of S.

Proposition 2.9.3
Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism. Then $Im\varphi = \{\varphi(a) \,|\, a \in R\}$ is a subring of S.

We state without proof the 1^{st} Isomorphism Theorem For Rings. We refer the reader to [3, Theorem 26.17] for the proof.

Theorem 2.9.4 (1^{st} Isomorphism Theorem For Rings)
Let R and S be rings. If $\varphi : R \to S$ is a ring homomorphism with $Ker\varphi = \{a \in R : \varphi(a) = 0_S\}$, then we have:

(*a*) $Ker\varphi$ *is an ideal of* R,

(*b*) $G/Ker\varphi \cong Im\varphi$ *and*

(*c*) *if φ is a ring epimorphism, then $R/Ker\varphi \cong S$.*

Now, we recall the characteristic of a ring and we prove that a ring of characteristic 0 has a subring isomorphism to the set of integers $\mathbb{Z}$

Definition 2.9.5 (characteristic of a Ring)
Let R be a ring. A positive integer n is called the characteristic of R if n is the smallest positive integer such that $n \cdot a = a + a + \ldots (n\text{-times}) \cdots + a = 0$ for all $a \in R$.

If there is no positive integer n such that $n \cdot a = 0$ for all $a \in R$, then R is said to be of characteristic 0.

Note 2.9.6
In a ring R with unity 1, if $n \cdot 1 \neq 0$ for any $n \in \mathbb{Z}^+$, then R has characteristic 0.

Proposition 2.9.7
Let R be a ring with unity 1_R and has characteristic 0. Then R has a subring Z which is isomorphic

to the set of integers $\mathbb{Z}$.

Proof. Let $\phi : \mathbb{Z} \to R$ be defined as: $\phi(n) = n \cdot 1_R = \begin{cases} 1_R + 1_R + \overset{n-times}{\cdots} + 1_R, & \text{if } n > 0. \\ (-1_R) + (-1_R) + \overset{|n|-times}{\cdots} + (-1_R), & \text{if } n < 0. \\ 0_R & \text{if } n = 0. \end{cases}$

It can be shown that ϕ is a ring homomorphism. By the 1^{st} Isomorphism Theorem for rings (Theorem 2.9.4), we have $\mathbb{Z}/Ker\phi \cong Im\phi$.

However, $Ker\phi = \{m \in \mathbb{Z} : \phi(m) = 0_R\} = \{0_R\}$ since R is of characteristic 0.

Hence, $\mathbb{Z}/Ker\phi = \mathbb{Z}/\{0_R\} = \{z + 0_R : z \in \mathbb{Z}\} \cong \mathbb{Z}$ and $\mathbb{Z} \cong \mathbb{Z}/Ker\phi \cong Im\phi$. By Proposition 2.9.3, $Im\phi$ is a subring of R.

Thus, by letting $Z = Im\phi$ we have shown R has subring Z which is isomorphic to the integers $\mathbb{Z}$. □

2.10 R-Module

In this section, we give a quick overview of module theory and state a few propositions needed for this study. For a more in-depth study of R-modules, interested readers can consult [18].

We start by recalling the definition of a left R-module.

Definition 2.10.1 (Left R-Module)
Let R be a ring. A left R-module is an additive abelian group M with a binary operation $\cdot : R \times M \to M$, called scalar multiplication, such that the following hold for all $r, s \in R$ and $a, b \in M$:

(i) $r \cdot (a + b) = r \cdot a + r \cdot b$,

(ii) $(r + s) \cdot a = r \cdot a + s \cdot a$,

(iii) $r \cdot (s \cdot a) = (rs) \cdot a$ *and*

(iv) *if R has unity* 1, *then* $1 \cdot a = a$.

If property (iv) holds, then M is called a unitary left R-module.

Note 2.10.2
Right R-module can be defined in a similar manner, but for sake of simplification we will refer to a left R-module as a R-module.

Similar to groups and rings, one could define submodules and quotient modules.

Definition 2.10.3 (Submodule)
Let R be a ring and let M be a R-module. A subset $N \subseteq M$ is called a R-submodule of M, if N is an additive subgroup of M.

Proposition 2.10.4 (Quotient Module)
Let R be a ring, M be a R-module and let N be a R-submodule of M. The quotient group of cosets of N in M, defined as $M/N = \{m + N : m \in M\}$ is an R-module with scalar multiplication $\cdot : R \times M/N \to M/N$ defined as $r \cdot (m + N) = (r \cdot m) + N$.

The following is taken from [18]. We recall when a R-module is called a torsion module.

Definition 2.10.5 (Torsion Module)
Let R be a ring and let M be a R-module.

(i) An element $m \in M$ is called a torsion element if there exists $a \in R$ with $a \neq 0$, such that $a \cdot m = 0$.

(ii) If all elements of M are torsion elements, then M is called a torsion module.

(iii) If the only torsion element of M is 0, then M is called a torsion-free module.

Below, we give some results that will be noted later. We refer the readers to [18] for the proofs.

Proposition 2.10.6
[18, Proposition 9.1] Let R be an integral domain and let M be a R-module. Then the set of all torsion elements in M, denoted by $\tau(M)$, is a submodule of M.

Proposition 2.10.7
[18, Proposition 9.2] Let R be an integral domain and let M be a R-module. Then the quotient module $M/\tau(M)$ is torsion-free.

Definition 2.10.8 (Free Module)
[3, Definition 38.2] Let R be a ring and M be a R-module. Then M is called a free module, if there exists a subset $X \subseteq M$ such that the following conditions hold:

(*i*) *For any non-zero m in M, we can express it as a finite sum of distinct $x_i \in X$ as $m = n_1x_1 + n_2x_2 + \cdots + n_rx_r$ for non-zero $n_1, n_2, \ldots n_r \in R$. The expression is unique up to ordering.*

(*ii*) *$n_1x_1 + n_2x_2 + \ldots n_rx_r = 0$ for $x_1, x_2, \ldots x_r \in X$ if and only if $n_1 = n_2 = \cdots = n_r = 0$.*

The set X is called the basis of M and the number of elements in X is called the rank of M.

Theorem 2.10.9
[18, Theorem 9.3] Let R be an principal ideal domain and let M be a R-module. If M is a finitely generated torsion-free R-module, then M is a free module.

2.11 Binomial Ring

Lastly, we explore the binomial ring, we state the definition of a binomial ring. We also take note that every binomial ring has a subring isomorphic to the set of integers. Finally, we prove some identities of the binomial coefficients. The information for this section was taken from [21].

Definition 2.11.1 (Binomial Ring)
Let R be an integral domain of characteristic zero. We call R a binomial ring, if R has unity 1 such that for any $r \in R$ and $k \in \mathbb{Z}^+$, we have that:

$$\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!} \in R.$$

Note 2.11.2
We note that every binomial ring contains a subring which is isomorphic to the set of integers $\mathbb{Z}$, since a binomial ring has characteristic 0 (by Proposition 2.9.7). We will denote this subring by Z and we will denote Z^+ the subring isomorphic to the positive integers.

Proposition 2.11.3
[21, Theorem 2] Let R be a binomial ring and let Z be the subring of R isomorphic to $\mathbb{Z}$. Then we have the following identities:

(a) $\binom{a}{1} = a$ *for any* $a \in R$.

(b) $\binom{1}{n} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$

(c) $\binom{z}{z} = 1$ *for all* $z \in Z^+$.

(d) $\binom{z}{k} = 0$ *for all* $z \in Z^+$ *and* $k \in \mathbb{Z}^+$ *such that* $k > z$.

Proof. Let R be a binomial ring and let Z be the subring of R isomorphic to $\mathbb{Z}$.

(a) Let $a \in R$. Then $\binom{a}{1} = \frac{(a-1+1)}{1!} = \frac{(a)}{1} = a$. Hence, $\binom{a}{1} = a$ for any $a \in R$.

(b) For $n = 1$, we have by (a) that $\binom{1}{1} = 1$. Now, for $n \geq 2$, we have $\binom{1}{n} = \frac{(1)(1-1)\dots(1-n+1)}{n!} = \frac{(1)(0)\dots(1-n+1)}{n!} = \frac{0}{n!} = 0(\frac{1}{n!}) = 0$. Therefore, we have $\binom{1}{n} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$

(c) Let $z \in Z^+$ and let $q = \binom{z}{z}$. We will show $q = 1$. We have $q = \binom{z}{z} \Rightarrow q = \frac{(z)(z-1)\dots(z-z+1)}{z!} \Rightarrow q = \frac{(z)(z-1)\dots(1)}{z!} \Rightarrow q(z!) = (z)(z-1)\dots(1) \Rightarrow q(z!) = z \Rightarrow 0 = z! - q(z!) \Rightarrow 0 = (1-q)(z!)$. Thus, $0 = (1-q)(z!)$. Since R is a binomial ring and thus, is a integral domain, we have that either $(1-q) = 0$ or $z! = 0$. We have $z! > 0$ and we must have $(1-q) = 0$. Hence, $\binom{z}{z} = q = 1$.

(d) Let $z \in Z^+$ and let $k \in \mathbb{Z}^+$ such that $k > z$. We have $z - k < 0$ and $z - k + 1 < 1$, hence we have $\binom{z}{k} = \frac{z(z-1)\dots(z-k+1)}{k!} = \frac{z(z-1)\dots(z-k)(z-k-1)\dots0\dots(z-k+1)}{k!} = \frac{0}{k!} = 0$. Hence, $\binom{z}{k} = 0$ for all $z, k \in Z^+$ such that $k > z$.

□

This concludes our collection of preliminaries on groups and rings. We are now ready to begin our investigation on the nilpotent R-powered group.

Chapter 3

Nilpotent R-Powered Group

In this chapter, we explore the nilpotent R-powered group that was first introduced by P. Hall in [5]. The nilpotent R-powered group extends a nilpotent group G by a binomial ring R, to define a unique R-exponentiation g^α for each $g \in G$ and $\alpha \in R$. We will define the nilpotent R-powered group in section 3.1.

In section 3.2, we will be following [1] and [9] to introduce notions like R-subgroup, R-generated and some basic properties related to these concepts. We will also show that the quotient of a nilpotent R-powered group is also a nilpotent R-powered group.

In section 3.3, we will prove that the Isomorphisms Theorems also hold in the class of nilpotent R-powered groups.

In Section 3.4, we will show that abelian R-powered groups are just R-modules and restate some classic results relating to R-modules. Lastly, in Section 3.5, we will explore exact R-sequences.

3.1 Nilpotent R-Powered Group

We give the definition of a nilpotent R-powered group, which was first given by P. Hall in [5]. We will prove some basic properties that can be derived from the definition of a nilpotent R-powered group. We conclude with two lemmas that use the Hall-Petresco axiom and showing the direct product of nilpotent R-powered groups is again a nilpotent R-powered group.

The definition of a nilpotent R-powered group was first given by P. Hall in [5]. We will however, adopt ours from the definition given in [1].

Definition 3.1.1 (Nilpotent R-Powered Group)
[1, Definition 4.11] Let R be a binomial ring with unity 1. A nilpotent group G is called a nilpotent R-powered group or a $\mathcal{N}_R$-group if the R-exponentiation x^α is uniquely defined for all $x \in G$ and $\alpha \in R$, such that the following properties hold for any $x, y \in G$ and $\alpha, \beta \in R$:

(i) $x^1 = x$,

(ii) $x^\alpha x^\beta = x^{\alpha+\beta}$,

(iii) $(x^\alpha)^\beta = x^{\alpha\beta}$,

(iv) $y^{-1}x^\alpha y = (y^{-1}xy)^\alpha$ *and*

(v) *each product of the form* $x_1^\alpha x_2^\alpha \ldots x_n^\alpha$, *for some* $x_1, x_2, \ldots, x_n \in G$, *can be expressed as the following product of Hall-Petresco words* $\tau_i(x_1, x_2, \ldots, x_n)$:
$x_1^\alpha x_2^\alpha \ldots x_n^\alpha = \tau_1(x_1, x_2, \ldots, x_n)^\alpha \tau_2(x_1, x_2, \ldots, x_n)^{\binom{\alpha}{2}} \ldots \tau_k(x_1, x_2, \ldots, x_n)^{\binom{\alpha}{k}}$, *where* k *is the nilpotent class of the group* $\langle x_1, x_2, \ldots, x_n \rangle$.

Note 3.1.2

- *Traditionally,* R*-exponentiation is defined as an* R*-action (or ring action) from* R *onto* G*. However, we note that Properties* $(i), (ii)$ *and* (iii) *are the three properties of a ring action.*

- *Property* (v) *of Definition 3.1.1 is called the Hall-Petresco axiom. This property tells us that all products of the form* $x_1^\alpha x_2^\alpha \ldots x_n^\alpha$ *can be rewritten in a particular ordering scheme. Interested reader may consult Chapter 4.1 in [1].*

The next proposition shows that some results that can be deduced immediately from the definition of $\mathcal{N}_R$-group.

Proposition 3.1.3
[1, Lemma 4.6] Let G *be a* $\mathcal{N}_R$*-group. Then the following hold for any* $x \in G$ *and* $\alpha \in R$*:*

(a) $x^0 = 1$ *and*

(b) $(x^\alpha)^{-1} = x^{-\alpha}$.

Proof. Let G be a $\mathcal{N}_R$-group, $x \in G$ and let $\alpha \in R$.

(a) Since, $0 = 0+0$, we have $x^0 = x^{0+0} = x^0 x^0$ by property (ii) in Definition 3.1.1. Therefore, $1 = x^0$.

(b) Using property (ii) in Definition 3.1.1, we get $x^\alpha x^{-\alpha} = x^{\alpha+(-\alpha)} = x^0 = 1$. In addition, using property (ii) in Definition 3.1.1, we also get $x^{-\alpha} x^\alpha = x^{-\alpha+\alpha} = x^0 = 1$.

Therefore, $x^\alpha x^{-\alpha} = x^{-\alpha} x^\alpha = 1$ and $(x^\alpha)^{-1} = x^{-\alpha}$.

□

The next lemmas, Lemma 3.1.4 and Lemma 3.1.5 use the Hall-Petresco axiom. The first one shows that the Hall-Petresco words $\tau_1(x_1, x_2, \ldots, x_n)$ can be calculated by substituting $\alpha = 1$ in the Hall-Petresco axiom. Lemma 3.1.5, shows that if two elements x and y commutes then we can distribute the R-exponentiation in the product $(xy)^\alpha$.

Lemma 3.1.4 is given as note without proof in [1]. Here, we give a formal proof.

Lemma 3.1.4
Let G *be a* $\mathcal{N}_R$*-group and let* $x_1, x_2, \ldots, x_n \in G$*. Then* $\tau_1(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n$.

Proof. Let G be a $\mathcal{N}_R$-group and let $x_1, x_2, \ldots, x_n \in G$. By the Hall-Petresco Axiom we have that, for any $\alpha \in R$, $x_1^\alpha x_2^\alpha \ldots x_n^\alpha = \tau_1(x_1, x_2, \ldots, x_n)^\alpha \tau_2(x_1, x_2, \ldots, x_n)^{\binom{\alpha}{2}} \ldots \tau_k(x_1, x_2, \ldots, x_n)^{\binom{\alpha}{k}}$, where k is the nilpotency class of the group $\langle x_1, x_2, \ldots, x_n \rangle$.
Setting $\alpha = 1$, we get $x_1^1 x_2^1 \ldots x_n^1 = \tau_1(x_1, x_2, \ldots, x_n)^1 \tau_2(x_1, x_2, \ldots, x_n)^{\binom{1}{2}} \ldots \tau_k(x_1, x_2, \ldots, x_n)^{\binom{1}{k}}$. By Proposition 2.11.3 (d), we have $\binom{1}{r} = 0$ for all $r > 1$.
Thus, $x_1 x_2 \ldots x_n = \tau_1(x_1, x_2, \ldots, x_n)^1 \tau_2(x_1, x_2, \ldots, x_n)^0 \ldots \tau_k(x_1, x_2, \ldots, x_n)^0 = \tau_1(x_1, x_2, \ldots, x_n)$. □

Lemma 3.1.5

[11, Lemma 2.1] Let G be a $\mathcal{N}_R$-group. If $x, y \in G$ such that $xy = yx$, then we have by the Hall-Petresco axiom that $(xy)^\alpha = x^\alpha y^\alpha$ for any $\alpha \in R$.

Proof. Let G be a $\mathcal{N}_R$-group and let $x, y \in G$ such that $xy = yx$. Let $H = \langle x, y \rangle$. Since, $xy = yx$ we have H is abelian and hence has nilpotency class 1.
By the Hall-Petresco axiom we have that $x^\alpha y^\alpha = \tau_1(x,y)^\alpha \tau_2(x,y)^{\binom{\alpha}{2}} \ldots \tau_k(x,y)^{\binom{\alpha}{k}}$ for any $\alpha \in R$.
Since H has nilpotency class 1, we have $k = 1$ and thus, $x^\alpha y^\alpha = \tau_1(x,y)^\alpha$. By Lemma 3.1.4 we have that $\tau_1(x,y) = xy$. Therefore, $x^\alpha y^\alpha = (xy)^\alpha$. □

Corollary 3.1.6 is deduced from Lemma 3.1.5 and gives an alternative to the direct proof given in [1, Lemma 4.6].

Corollary 3.1.6

Let G be a $\mathcal{N}_R$-group. Then for any $\alpha \in R$, we have $1^\alpha = 1$.

Proof. For any $x \in G$ we have, $x1 = 1x$. Thus, by Lemma 3.1.5, $(x)^\alpha = (x1)^\alpha = x^\alpha 1^\alpha$ for any $\alpha \in R$ and $1 = 1^\alpha$ for any $\alpha \in R$. □

In [1, Definition 4.16] and [9, Section 2.4], the authors give the definition of a direct product of $\mathcal{N}_R$-group. We will expand on this and prove in Proposition 3.1.7 and Corollary 3.1.8 that the direct product of $\mathcal{N}_R$-groups is again a $\mathcal{N}_R$-group.

Proposition 3.1.7

Let G and H be $\mathcal{N}_R$-groups. Then the direct product $G \times H$ is a $\mathcal{N}_R$-group, with R-exponentiation defined as $(g,h)^\alpha = (g^\alpha, h^\alpha)$.

Proof. Let G and H be $\mathcal{N}_R$-groups.
Firstly, since G and H are nilpotent, then by Proposition 2.5.7, that the direct product $G \times H$ is also nilpotent.
Now, we show that the R-exponentiation defined as $(g,h)^\alpha = (g^\alpha, h^\alpha)$ is uniquely defined for each $(g,h) \in G \times H$ and $\alpha \in R$.
Let $(g_1,h_1),(g_2,h_2) \in G \times H$ such that $(g_1,h_1) = (g_2,h_2)$ and let $\alpha \in R$. Then, $(g_1,h_1) = (g_2,h_2) \Rightarrow g_1 = g_2$ and $h_1 = h_2 \Rightarrow g_1^\alpha = g_2^\alpha$ and $h_1^\alpha = h_2^\alpha \Rightarrow (g_1^\alpha, h_1^\alpha) = (g_2^\alpha, h_2^\alpha) \Rightarrow (g_1,h_1)^\alpha = (g_2,h_2)^\alpha$.

Therefore, $(g_1,h_1) = (g_2,h_2)$ implies $(g_1,h_1)^\alpha = (g_2,h_2)^\alpha$. Thus, each $(g,h)^\alpha$ is uniquely determined.

Now, we show that the axioms of Definition 3.1.1 hold for any $(g_1,h_1),(g_2,h_2) \in G \times H$ and $\alpha, \beta \in R$.

(i) We observe that, $(g_1,h_1)^1 = (g_1^1, h_1^1) = (g_1,h_1)$ since G and H are $\mathcal{N}_R$-groups. Hence property (i) in Definition 3.1.1 holds.

(ii) We have $(g_1,h_1)^\alpha(g_1,h_1)^\beta = (g_1^\alpha, h_1^\alpha)(g_1^\beta, h_1^\beta) = (g_1^\alpha g_1^\beta, h_1^\alpha h_1^\beta) = (g_1^{\alpha+\beta}, h_1^{\alpha+\beta}) = (g_1,h_1)^{\alpha+\beta}$, since G and H are $\mathcal{N}_R$-groups.

We conclude that property (ii) in Definition 3.1.1 holds.

(iii) Now, $[(g_1,h_1)^\alpha]^\beta = (g_1^\alpha, h_1^\alpha)^\beta = ([g_1^\alpha]^\beta, [h_1^\alpha]^\beta) = (g_1^{\alpha\beta}, h_1^{\alpha\beta}) = (g_1,h_1)^{\alpha\beta}$, since G and H are $\mathcal{N}_R$-groups. We conclude that property (iii) in Definition 3.1.1 holds.

(*iv*) Now,

$$\begin{aligned}[(g_2,h_2)^{-1}(g_1,h_1)(g_2,h_2)]^\alpha &= [(g_2^{-1},h_2^{-1})(g_1g_2,h_1h_2)]^\alpha \\ &= (g_2^{-1}g_1g_2,h_2^{-1}h_1h_2)^\alpha \\ &= ((g_2^{-1}g_1g_2)^\alpha,(h_2^{-1}h_1h_2)^\alpha) \\ &= (g_2^{-1}g_1^\alpha g_2,h_2^{-1}h_1^\alpha h_2) \text{ since } G \text{ and } H \text{ are } \mathcal{N}_R\text{-groups} \\ &= (g_2^{-1},h_2^{-1})(g_1^\alpha g_2,h_1^\alpha h_2) \\ &= (g_2,h_2)^{-1}(g_1^\alpha,h_1^\alpha)(g_2,h_2) \\ &= (g_2,h_2)^{-1}(g_1,h_1)^\alpha(g_2,h_2).\end{aligned}$$

We conclude that property (*iv*) in Definition 3.1.1 holds.

(*v*) For any $(g_1,h_1),(g_2,h_2),\dots(g_n,h_n) \in G\times H$ and $\alpha \in R$, we can express the product $(g_1,h_1)^\alpha(g_2,h_2)^\alpha\cdots(g_n,h_n)^\alpha$ as follows:

$$\begin{aligned}(g_1,h_1)^\alpha(g_2,h_2)^\alpha\cdots(g_n,h_n)^\alpha &= (g_1^\alpha,h_1^\alpha)(g_2^\alpha,h_2^\alpha)\cdots(g_n^\alpha,h_n^\alpha) \\ &= (g_1^\alpha g_2^\alpha\dots g_n^\alpha,h_1^\alpha h_2^\alpha\dots h_n^\alpha) \\ &= \left(\tau_1(\overline{g})^\alpha\tau_2(\overline{g})^{\binom{\alpha}{2}}\dots\tau_k(\overline{g})^{\binom{\alpha}{k}},\tau_1(\overline{h})^\alpha\tau_2(\overline{h})^{\binom{\alpha}{2}}\dots\tau_k(\overline{h})^{\binom{\alpha}{k}}\right) \\ &= \left(\tau_1(\overline{g})^\alpha,\tau_1(\overline{h})^\alpha\right)\left(\tau_2(\overline{g})^{\binom{\alpha}{2}},\tau_2(\overline{h})^{\binom{\alpha}{2}}\right)\dots\left(\tau_k(\overline{g})^{\binom{\alpha}{k}},\tau_k(\overline{h})^{\binom{\alpha}{k}}\right) \\ &= \left(\tau_1(\overline{g}),\tau_1(\overline{h})\right)^\alpha\left(\tau_2(\overline{g}),\tau_2(\overline{h})\right)^{\binom{\alpha}{2}}\dots\left(\tau_k(\overline{g}),\tau_k(\overline{h})\right)^{\binom{\alpha}{k}}\end{aligned}$$

where $\overline{g}=\{g_1,g_2,\dots,g_n\}$ and $\overline{h}=\{h_1,h_2,\dots,h_n\}$.

Therefore, each product of the form $(g_1,h_1)(g_2,h_2)\dots(g_n,h_n)$, can be expressed a product of Hall-Petresco words $\left(\tau_i(\overline{g}),\tau_i(\overline{h})\right)$. Hence, property (*v*) in Definition 3.1.1 holds.

Thus, $G\times H$ is a $\mathcal{N}_R$-group. □

Corollary 3.1.8
Let $\{G_i : i \in I\}$ be a family of $\mathcal{N}_R$-groups for some non-empty index set I. Then the direct product $G=\prod_{i\in I} G_i$ is also a $\mathcal{N}_R$-group.

Proof. Let $\{G_i : i\in I\}$ be a family of $\mathcal{N}_R$-groups for some non-empty index set I. We will proceed by induction on the size of the index set $|I|$.

If $|I|=2$, then, by Proposition 3.1.7, $G_1\times G_2$ is a $\mathcal{N}_R$-group. Assume the statement is true for $|I|=k$, that is the direct product $\prod_{i\in I}^{k} G_i$ is a $\mathcal{N}_R$-group. We show that the statement is true for $|I|=k+1$. We have, $\prod_{i\in I}^{k+1} G_i=\prod_{i\in I}^{k} G_i\times G_{k+1}$. By, induction hypothesis, $\prod_{i\in I}^{k} G_i$ is a $\mathcal{N}_R$-group and by assumption, G_{k+1} is a $\mathcal{N}_R$-group. Thus, by Proposition 3.1.7, $\prod_{i\in I}^{k+1} G_i=\prod_{i\in I}^{k} G_i\times G_{k+1}$ is a $\mathcal{N}_R$-group and $G=\prod_{i\in I} G_i$ is a $\mathcal{N}_R$-group. □

3.2 R-Subgroups and Normal R-Subgroups

In this section, we will explore R-subgroups and normal R-subgroups of $\mathcal{N}_R$-groups, which are analogous to the usual subgroups and normal subgroups of a regular nilpotent group.

Firstly, we state the definition of a R-subgroup taken from [9].

Definition 3.2.1 (R-Subgroups)
[9, Definition 2.1.2] Let G be a $\mathcal{N}_R$-group and let H be a subgroup of G. Then, H is called a R-subgroup of G, denoted by $H \leq_R G$, if $g^\alpha \in H$ for all $g \in H$ and $\alpha \in R$.

Now, we prove, via a lemma, that the intersection of R-subgroups is again a R-subgroup. This will be needed to verify that the R-generated group $\langle S \rangle_R$ is the smallest R-subgroup that contains S, for any $S \subseteq G$.

Lemma 3.2.2
[9, Lemma 2.1.1] Let G be a $\mathcal{N}_R$-group and let $\{H_i : i \in I\}$ be a family of R-subgroups of G for some index set I. Then, the intersection $H = \bigcap_{i \in I} H_i$ is also a R-subgroup of G.

Proof. Let G be a $\mathcal{N}_R$-group and let $\{H_i : i \in I\}$ be a family of R-subgroups of G for some index set I. Let $H = \bigcap_{i \in I} H_i$. Since, each H_i is a R-subgroup of G, we have, by Definition 3.2.1, that each H_i is a subgroup of G for all $i \in I$. Thus, by Proposition 2.1.7, $H = \bigcap_{i \in I} H_i$ is a subgroup of G.

Furthermore, since each H_i is a R-subgroup of G, then $x^\alpha \in H_i$ for each $x \in H_i$, $\alpha \in R$ and $i \in I$. Therefore, $x^\alpha \in H = \bigcap_{i \in I} H_i$ for each $x \in H$ and $\alpha \in R$. By Definition 3.2.1, $H = \bigcap_{i \in I} H_i$ is a R-subgroup of G. □

Proposition 3.2.3 is mentioned and explained as a note in [1]. Here, we give a formal statement and its proof.

Proposition 3.2.3 (R-Subgroup R-Generated by S)
Let G be a $\mathcal{N}_R$-group and let $S \subseteq G$. Then, the intersection of all R-subgroups containing S, denoted by $\langle S \rangle_R$, is a R-subgroup of G. Moreover, $\langle S \rangle_R$ is the smallest R-subgroup of G containing S. We call S the set of R-generators of $\langle S \rangle_R$.

Proof. Let G be a $\mathcal{N}_R$-group and let $S \subseteq G$.
Let $\mathbb{H} = \{H \mid H \leq_R G \text{ such that } S \subseteq H\}$ be the set of all R-subgroups of G containing S. By Lemma 3.2.2, we have $\langle S \rangle_R = \bigcap_{H \in \mathbb{H}} H$ is a R-subgroup of G.
Assume $\langle S \rangle_R$ is not the smallest R-subgroup of G containing S.
Let Q be the smallest R-subgroup of G containing S. We have that $Q \in \mathbb{H}$ and thus, $\langle S \rangle_R = \left(\bigcap_{H \in \mathbb{H}} H \right) \bigcap Q$. Consequently, $\langle S \rangle_R \subseteq Q$, which contradicts the assumption that Q is the smallest R-subgroup of G containing S.

Therefore, $\langle S \rangle_R$ is the smallest R-subgroup of G containing S. □

Definition 3.2.4 (R-generated)
[9, Definition 2.1.5] Let G be a $\mathcal{N}_R$-group. A group G is said to be R-generated if there exists $S \subseteq G$ such that $G = \langle S \rangle_R$.
We say that G is finitely R-generated if $|S|$ is finite.

The next proposition is from a note (without a proof) given in [1], relating R-generation by S, to the usual group generated by the subset S. Here, we give a formal proof.

Proposition 3.2.5
Let G be a $\mathcal{N}_R$-group and let $S \subseteq G$.
Then, $\langle S \rangle_R = \bigcup_{i=0}^{\infty} S_i$ where:

(a) $S_0 = \langle S \rangle$ *and*

(b) $S_{i+1} = \langle g^\alpha : g \in S_i,\ \alpha \in R \rangle$.

Proof. Let G be a $\mathcal{N}_R$-group, $S \subseteq G$, $I = \{0, 1, 2, \dots\}$ and let $a \in S_i$.
By Proposition 3.1.3, $a = a^1 \in S_{i+1}$, therefore, $S_i \subseteq S_{i+1}$. We have an increasing sequence $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_i \subseteq \cdots$.
We will show that each S_i is a subgroup of G for all $i \in I$, using induction on i.

For i = 0, we have, by Proposition 2.1.10, that $S_0 = \langle S \rangle$ is a subgroup of G.
Assume that the statement is true for $i = k$, that is, $S_k \leq G$.
We need to show that the statement holds for $i = k+1$. We have that each S_{k+1} is generated by a set $S'_{k+1} = \{g^\alpha : g \in S_k,\ \alpha \in R\}$. By the definition of S_k, we have $g \in S_k \leq G$ for each element g^α in S'_{k+1}.
By the inductive hypothesis, $S_k \leq G$. Thus, $g \in G$. However, $g^\alpha \in G$ since G is a $\mathcal{N}_R$-group.
Therefore, $S'_{k+1} = \{g^\alpha : g \in S_k,\ \alpha \in R\}$ is a subset of G. By Proposition 2.1.10, $S_{k+1} = \langle S'_{k+1} \rangle$ is a subgroup of G.

Therefore, by principle of mathematical induction we have $S_i \leq G$ for all $i \in I$.

Since $S_i \leq G$ for all $i \in I$ and $S_0 \subseteq S_1 \subseteq S_2 \subseteq S_3 \cdots$ we have, by Proposition 2.1.8, that $\bigcup_{i=0}^{\infty} S_i \leq G$.

Now, we show that $\bigcup_{i=0}^{\infty} S_i \leq G$ is closed under R-exponentiation. Let $x \in \bigcup_{i=0}^{\infty} S_i$ and let $\alpha \in R$.

Since $x \in \bigcup_{i=0}^{\infty} S_i$, then $x \in S_k$ for some $k \in I$. By the definition of S_{k+1}, $x^\alpha \in S_{k+1}$.

Therefore, $x^\alpha \in \bigcup_{i=0}^{\infty} S_i$ for any $x \in \bigcup_{i=0}^{\infty} S_i$ and $\alpha \in R$. By Definition 3.2.1, $\bigcup_{i=0}^{\infty} S_i$ is a R-subgroup of G.

Hence, $\bigcup_{i=0}^{\infty} S_i$ is a R-subgroup of G which contain $S \subseteq \langle S \rangle = S_0 \subseteq \bigcup_{i=0}^{\infty} S_i$ and $\langle S \rangle_R \subseteq \bigcup_{i=0}^{\infty} S_i$, since $\langle S \rangle_R$ is the intersection of R-subgroups of G containing S.

For the reverse containment, let $x \in \bigcup_{i=0}^{\infty} S_i$.

Since $x \in \bigcup_{i=0}^{\infty} S_i$, we have that $x \in S_k$ for some $k \in I$. We will prove that if $x \in S_k$ then $x \in \langle S \rangle_R$ by using induction on k.

If $k = 0$, then $x \in S_0 = \langle S \rangle$. Since $\langle S \rangle_R$ is the smallest R-subgroup of G containing all elements of S, then $x \in \langle S \rangle_R$.
Assume the statement is true for $k = t$, that is, if $x \in S_t$, then $x \in \langle S \rangle_R$.
Consider the case for $k = t + 1$. If $x \in S_{t+1}$, then $x = g_1^{\alpha_1} \dots g_r^{\alpha_r}$ for some $g_1, \dots, g_r \in S_t$ and $\alpha_1, \dots, \alpha_r \in R$, where $r \geq 1$.
By the inductive hypothesis, $g_1, \dots, g_r \in S_t$ implies that $g_1, \dots, g_r \in \langle S \rangle_R$.
However, $\langle S \rangle_R$ is a R-subgroup of G, so $g_1^{\alpha_1}, \dots, g_r^{\alpha_r} \in \langle S \rangle_R$.
Therefore, $x = g_1^{\alpha_1} \dots g_r^{\alpha_r} \in \langle S \rangle_R$, since $\langle S \rangle_R$ is closed as $\langle S \rangle_R$ is a R-subgroup of G.

Therefore, by the principle of mathematical induction, we conclude that if $x \in S_i$ for some $i \in I$ then $x \in \langle S \rangle_R$.
Thus, $\bigcup_{i=0}^{\infty} S_i \subseteq \langle S \rangle_R$ and $\langle S \rangle_R = \bigcup_{i=0}^{\infty} S_i$. □

We can deduce Corollary 3.2.6 from Proposition 3.2.5 and Proposition 2.1.11. Firstly, using Proposition 3.2.5, we represent a R-subgroup R-generated by a subset S in terms of the regular generated subgroups. Next, we use Proposition 2.1.11 to evaluate an explicit formulation of the R-Subgroup R-generated by S.

Corollary 3.2.6
Let G be a $\mathcal{N}_R$-group and let $S \subseteq G$.
Then, $\langle S \rangle_R = \{s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \mid s_i \in S,\ \alpha_i \in R,\ i = 1, 2, \dots, k\}$.

Proof. Let G be a $\mathcal{N}_R$-group and let $S \subseteq G$.
By Proposition 3.2.5, we have that $\langle S \rangle_R = \bigcup_{i=0}^{\infty} S_i$ where:

(*a*) $S_0 = \langle S \rangle$ and

(*b*) $S_{i+1} = \langle g^\alpha : g \in S_i,\ \alpha \in R \rangle$.

By Proposition 2.1.11, we have:

- $S_0 = \langle S \rangle = \{s_1^{r_1} s_2^{r_2} \dots s_k^{r_k} \mid s_i \in S \text{ and } r_i \in \{1, -1\},\ i = 1, 2, \dots, k\}$,
- $S_{i+1} = \langle g^\alpha : g \in S_i,\ \alpha \in R \rangle = \{(g_1^{\alpha_1})^{r_1} (g_2^{\alpha_2})^{r_2} \dots (g_k^{\alpha_k})^{r_k} \mid g_i \in S_i, \alpha_i \in R \text{ and } r_i \in \{1, -1\},\ i = 1, 2, \dots, k\}$.

However, G is a $\mathcal{N}_R$-group, so, $(g_i^{\alpha_i})^{r_i} = g_i^{\alpha_i r_i}$. Since R is a binomial ring and $r_i \in \{1, -1\}$, so $\alpha_i r_i \in R$ and we have the following:

- $S_0 = \langle S \rangle = \{s_1^{r_1} s_2^{r_2} \dots s_k^{r_k} \mid s_i \in S \text{ and } r_i \in \{1, -1\},\ i = 1, 2, \dots, k\}$,
- $S_{i+1} = \langle g^\alpha : g \in S_i,\ \alpha \in R \rangle = \{g_1^{\zeta_1} g_2^{\zeta_2} \dots g_k^{\zeta_k} \mid g_i \in S_i \text{ and } \zeta_i \in R,\ i = 1, 2, \dots, k\}$.

Let $x \in \langle S \rangle_R$, then, by Proposition 3.2.5, $x \in S_m$ for some $m = 0, 1, 2, \dots$ and by above, $x = g_1^{\zeta_1} g_2^{\zeta_2} \dots g_k^{\zeta_k}$ for some $g_i \in S_i$ and $\zeta_i \in R$. Then, $x \in \{s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \mid s_i \in S \text{ and } \alpha_i \in R,\ i = 1, 2, \dots, k\}$.
Thus, $\langle S \rangle_R \subseteq \{s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \mid s_i \in S \text{ and } \alpha_i \in R,\ i = 1, 2, \dots, k\}$.

Now, let $y \in \{s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k} \mid s_i \in S \text{ and } \alpha_i \in R,\ i = 1, 2, \dots, k\}$ such that $y = g_1^{\zeta_1} g_2^{\zeta_2} \dots g_k^{\zeta_k}$ for some $g_i \in S_i$ and $\zeta_i \in R$ for $i = 1, 2, \dots k$. We have $y = g_1^{\zeta_1} g_2^{\zeta_2} \dots g_k^{\zeta_k} = (g_1^{\zeta_1 1})(g_2^{\zeta_2 1}) \dots (g_k^{\zeta_k 1}) = (g_1^{\zeta_1})^1 (g_2^{\zeta_2})^1 \dots (g_k^{\zeta_k})^1$.

Consequently, $y \in S_m$ for some $m = 0, 1, \ldots,$ and $y \in \bigcup_{i=0}^{\infty} S_i = \langle S \rangle_R$.

Therefore, $\{s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_k^{\alpha_k} \mid s_i \in S$ and $\alpha_i \in R,\ i = 1, 2, \ldots, k\} \subseteq \langle S \rangle_R$ and $\langle S \rangle_R = \{s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_k^{\alpha_k} \mid s_i \in S$ and $\alpha_i \in R,\ i = 1, 2, \ldots, k\}$. □

We define the commutator R-subgroup of a $\mathcal{N}_R$-group and show that it is a normal R-subgroup.

Definition 3.2.7 (Commutator R-Subgroup)
[9, Definition 2.1.6] Let G be a $\mathcal{N}_R$-group and let M and N be R-subgroups of G. The commutator R-subgroup generated by M and N, denoted by $[M, N]_R$, is defined as $[M, N]_R = \langle [m, n] \mid m \in M, n \in N \rangle_R$. If $G = M = N$, then $[G, G]_R$ is called the commutator R-subgroup of G.

Note 3.2.8
By Corollary 3.2.6, $[G, G]_R = \langle [x, y] \mid x, y \in G \rangle_R = \{[x_1, y_1]^{\alpha_1} [x_2, y_2]^{\alpha_2} \ldots [x_k, y_k]^{\alpha_k} : x_i, y_i \in G,\ \alpha_i \in R,\ i = 1, 2, \ldots, k\}$.

Notation 3.2.9
For sake of simplicity, we will denote $[G, G]_R$ as just $[G, G]$ for the remainder of our study.

Lemma 3.2.10
Let G be a $\mathcal{N}_R$-group. The commutator R-subgroup $[G, G]$ is a normal R-subgroup of G.

Proof. Let G be a $\mathcal{N}_R$-group and let $[G, G]$ be the commutator R-subgroup of G. By definition 3.2.7, $[G, G]$ is a R-subgroup of G.
Let $g \in G$ and let $gug^{-1} \in g[G, G]g^{-1}$. Then, $gug^{-1} = uu^{-1}(gug^{-1}) = u(u^{-1}gug^{-1}) = u[u^{-1}, g] \in [G, G]$.

Therefore, $[G, G]$ is a normal R-subgroup of G. □

In a similar manner to ordinary groups, we define when a $\mathcal{N}_R$-group is said to be cyclic.

Definition 3.2.11 (Cyclic R-powered group)
Let G be a $\mathcal{N}_R$-group. Then, G is called a cyclic $\mathcal{N}_R$-group, if there exists $g \in G$ such that $G = \langle g \rangle_R$.

Next, we now introduce the concept of normal R-subgroup as it is presented in [9].

Definition 3.2.12 (Normal R-Subgroups)
[9, Definition 2.1.4] Let G be a $\mathcal{N}_R$-group and let H be a R-subgroup of G. Then H is called a normal R-subgroup of G, denoted by $H \trianglelefteq_R G$, if H is a normal subgroup of G.

The following lemma is extracted from the proof of Theorem 4.16 in [1] where the authors showed that the quotient of a nilpotent of R-powered group is again a $\mathcal{N}_R$-group. We extracted this part of the proof, as we will need this lemma again in the proof of the Second Isomorphism's Theorem for R-homomorphisms.

Lemma 3.2.13
Let G be a $\mathcal{N}_R$-group and let N be a normal R-subgroup of G. Then, for any $h \in G$, $n \in N$ and $\alpha \in R$, $h^\alpha n^\alpha = (hn)^\alpha n'$ for some $n' \in N$.

Proof. Let G be a $\mathcal{N}_R$-group, N a normal R-subgroup of G and let $h \in G$, $n \in N$, $\alpha \in R$.

From the Hall-Petresco Axiom, $h^\alpha n^\alpha = \tau_1(h,n)^\alpha \tau_2(h,n)^{\binom{\alpha}{2}} \ldots \tau_{k-1}(h,n)^{\binom{\alpha}{k-1}} \tau_k(h,n)^{\binom{\alpha}{k}}$ where k is the nilpotency class of the group $\langle h, n\rangle$. From Lemma 3.1.4, $\tau_1(h,n) = hn$. Therefore, $h^\alpha n^\alpha = (hn)^\alpha \tau_2(h,n)^{\binom{\alpha}{2}} \ldots \tau_{k-1}(h,n)^{\binom{\alpha}{k-1}} \tau_k(h,n)^{\binom{\alpha}{k}}$.
We obtain the equation:

$$(hn)^{-\alpha} h^\alpha n^\alpha = \tau_2(h,n)^{\binom{\alpha}{2}} \ldots \tau_{k-1}(h,n)^{\binom{\alpha}{k-1}} \tau_k(h,n)^{\binom{\alpha}{k}}. \tag{3.1}$$

We will show that each $\tau_i(h,n) \in N$ for all $i = 2, \ldots k$, by using induction on i and showing that $(hn)^i h^i n^i \in N$ for all $i = 2, \ldots k$.

For $i = 2$, equation (3.1) becomes $(hn)^{-2}h^2n^2 = \tau_2(h,n)$ since $\binom{2}{2} = 1$ and $\binom{2}{s} = 0$ for all $s > 2$. Also, $(hn)^{-2}h^2n^2 = [(hn)^2]^{-1}h^2n^2 = [hnhn]^{-1}h^2n^2 = n^{-1}h^{-1}n^{-1}h^{-1}h^2n^2 = n^{-1}h^{-1}n^{-1}hn^2 = n^{-1}(h^{-1}n^{-1}h)n^2$.

However, $h^{-1}n^{-1}h \in N$ since N is normal in G. Therefore, $n^{-1}(h^{-1}n^{-1}h)n^2 \in N$ and $\tau_2(h,n) = (hn)^{-2}h^2n^2 = n^{-1}(h^{-1}n^{-1}h)n^2 \in N$.

Assume the following is true for $j \leq i < k$:

(*a*) $\tau_j \in N$ and

(*b*) $(hn)^{-j}h^j n^j \in N$.

We will show that $\tau_i(h,n) \in N$ holds for $i = j+1$. Firstly, we note that:

$$\begin{aligned}
(hn)^{-(j+1)}h^{j+1}n^{j+1} &= (hn)^{-j-1}h^{j+1}n^{j+1} \\
&= (hn)^{-j}(hn)^{-1}h^{1+j}n^{j+1} \\
&= (hn)^{-j}n^{-1}h^{-1}h^1h^jn^{j+1} \\
&= (hn)^{-j}(1)n^{-1}h^jn^{j+1} \\
&= (hn)^{-j}(h^0)n^{-1}h^jn^{j+1} \\
&= (hn)^{-j}(h^{j-j})n^{-1}h^jn^{j+1} \\
&= (hn)^{-j}(h^j 1 h^{-j})n^{-1}h^jn^{j+1} \\
&= (hn)^{-j}(h^jn^0h^{-j})n^{-1}h^jn^{j+1} \\
&= (hn)^{-j}(h^jn^{j-j}h^{-j})n^{-1}h^jn^{j+1} \\
&= (hn)^{-j}(h^jn^jn^{-j}h^{-j})n^{-1}h^jn^{j+1} \\
&= ((hn)^{-j}h^jn^j)n^{-j}(h^{-j}n^{-1}h^j)n^{j+1}.
\end{aligned}$$

By the inductive hypothesis, we have $(hn)^{-j}h^jn^j \in N$. Furthermore, $h^{-j}n^{-1}h^j \in N$ since N is normal in G. Therefore, $(hn)^{-(j+1)}h^{j+1}n^{j+1} = ((hn)^{-j}h^jn^j)n^{-j}(h^{-j}n^{-1}h^j)n^{j+1} \in N$.

Applying equation (3.1), we obtain the following:
$(hn)^{-(j+1)}h^{j+1}n^{j+1} = \tau_2(h,n)^{\binom{\alpha}{2}} \ldots \tau_j(h,n)^{\binom{j+1}{j}} \tau_{j+1}(h,n)$. This can be rewritten as:
$\tau_{j+1}(h,n) = \tau_j(h,n)^{-\binom{j+1}{j}} \ldots \tau_2(h,n)^{-\binom{\alpha}{2}}(hn)^{-(j+1)}h^{j+1}n^{j+1}$.

By the inductive hypothesis, we have $\tau_j(h,n) \in N$ for all $j \leq i$.
Then, $\tau_{j+1}(h,n) = \tau_j(h,n)^{-\binom{j+1}{j}} \ldots \tau_2(h,n)^{-\binom{\alpha}{2}}(hn)^{-(j+1)}h^{j+1}n^{j+1} \in N$ and $\tau_i(h,n) \in N$ for all $i = 1, 2, \ldots, k$.

We recall equation (3.1):

$$(hn)^{-\alpha}h^{\alpha}n^{\alpha} = \tau_2(h,n)^{\binom{\alpha}{2}} \dots \tau_{k-1}(h,n)^{\binom{\alpha}{k-1}}\tau_k(h,n)^{\binom{\alpha}{k}}.$$

Each $\tau_i(h,n) \in N$ for all $i = 1,2,\dots,k$ and we have $\tau_2(h,n)^{\binom{\alpha}{2}} \dots \tau_{k-1}(h,n)^{\binom{\alpha}{k-1}}\tau_k(h,n)^{\binom{\alpha}{k}} = n'$ for some $n' \in N$.

Therefore, $(hn)^{-\alpha}h^{\alpha}n^{\alpha} = n'$ and $h^{\alpha}n^{\alpha} = (hn)^{\alpha}n'$. □

Now, we are ready to prove that the quotient of a $\mathcal{N}_R$-group is again a $\mathcal{N}_R$-group.

Proposition 3.2.14
[1, Theorem 4.16] Let G be a $\mathcal{N}_R$-group and let N be a normal R-subgroup of G. The quotient group G/N is a $\mathcal{N}_R$-group with R-exponentiation defined as $(gN)^{\alpha} = g^{\alpha}N$.

Proof. Let G be a $\mathcal{N}_R$-group and let N be a normal R-subgroup of G. By Corollary 2.5.6, G/N is a nilpotent group.
Firstly, we show that the R-exponentiation defined as $(gN)^{\alpha} = g^{\alpha}N$ is uniquely defined for each $gN \in G/N$ and $\alpha \in R$.
Let $gN, hN \in G/N$ such that $gN = hN$ and let $\alpha \in R$. By Proposition 2.2.6, $g = hn$ for some $n \in N$, so, $g^{\alpha} = (hn)^{\alpha}$.
From Lemma 3.2.13, we have $h^{\alpha}n^{\alpha} = (hn)^{\alpha}n'$ for some $n' \in N$, so, we can rearrange the equation as follows:

$$\begin{aligned} h^{\alpha}n^{\alpha} = (hn)^{\alpha}n' &\Rightarrow h^{\alpha} = [(hn)^{\alpha}n']n^{-\alpha} \text{ right multiplication by } n^{-\alpha} \\ &\Rightarrow h^{\alpha} = (hn)^{\alpha}(n'n^{-\alpha}) \Rightarrow h^{\alpha}N = (hn)^{\alpha}N \text{ by Proposition 2.2.6} \\ &\Rightarrow h^{\alpha}N = g^{\alpha}N \text{ since } g^{\alpha} = (hn)^{\alpha}. \end{aligned}$$

Therefore, $gN = hN$ implies $(gN)^{\alpha} = g^{\alpha}N = h^{\alpha}N = (hN)^{\alpha}$ and each $(gN)^{\alpha}$ is uniquely determined.

We need to show the axioms of Definition 3.1.1 hold for any $xN, yN \in G/N$ and $\alpha, \beta \in R$, by making use of the fact that G is a $\mathcal{N}_R$-group.

- (i) We observe that, $(xN)^1 = x^1N = xN$.
- (ii) $(xN)^{\alpha}(xN)^{\beta} = (x^{\alpha}N)(x^{\beta}N) = (x^{\alpha}x^{\beta})N = (x^{\alpha+\beta})N = (xN)^{\alpha+\beta}$.
- (iii) $((xN)^{\alpha})^{\beta} = (x^{\alpha}N)^{\beta} = (x^{\alpha})^{\beta}N = x^{\alpha\beta}N = (xN)^{\alpha\beta}$.
- (iv) $[(yN)^{-1}(xN)(yN)]^{\alpha} = [(y^{-1}N)(xyN)]^{\alpha} = [(y^{-1}xy)N]^{\alpha} = (y^{-1}xy)^{\alpha}N = (y^{-1}x^{\alpha}x)N$
$= (y^{-1}N)(x^{\alpha}N)(yN) = (yN)^{-1}(xN)^{\alpha}(yN)$.
- (v) For any $x_1N, x_2N, \dots x_nN \in G/N$ and $\alpha \in R$, we can express the product $(x_1N)^{\alpha}(x_2N)^{\alpha}\cdots(x_nN)^{\alpha}$

as follows:

$$\begin{aligned}(x_1N)^{\alpha}\cdots(x_nN)^{\alpha} = (x_1^{\alpha}N)\cdots(x_n^{\alpha}N) &= (x_1^{\alpha}\cdots x_n^{\alpha})N \\ &= \left(\tau_1(x_1,\dots,x_n)^{\alpha}\dots\tau_k(x_1,\dots,x_n)^{\binom{\alpha}{k}}\right)N \\ &= (\tau_1(x_1,\dots,x_n)^{\alpha}N)\dots\left(\tau_k(x_1,\dots,x_n)^{\binom{\alpha}{k}}N\right) \\ &= (\tau_1(x_1,\dots,x_n)N)^{\alpha}\dots(\tau_k(x_1,\dots,x_n)N)^{\binom{\alpha}{k}}.\end{aligned}$$

Therefore, each product of the form $(x_1N)^{\alpha}\cdots(x_nN)^{\alpha}$, can be expressed a product of Hall-Petresco words $\tau_i(x_1,x_2,\dots,x_n)N$.

Thus, G/N is a $\mathcal{N}_R$-group. □

Lastly, we show that if a $\mathcal{N}_R$-group is finitely R-generated, then the quotient is also finitely R-generated. This has been done for ordinary nilpotent groups and their quotients, which can be found at [16, Result 368 (i)].

Lemma 3.2.15
Let G be a $\mathcal{N}_R$-group and let N be a normal R-subgroup of G. If G is finitely R-generated then, G/N is also finitely R-generated.

Proof. Let G be a $\mathcal{N}_R$-group and let N be a normal R-subgroup of G.
Assume G is finitely R-generated. Then, there exists a finite subset $X=\{x_1,x_2,\dots,x_r\}\subseteq G$ such that $G=\langle X\rangle_R$.
Let $X'=\{x_1N,x_2N,\dots,x_rN\}$, we will show $G/N=\langle X'\rangle_R$.
From Corollary 3.2.6, we have:

- $\langle X\rangle_R=\{x_1^{\alpha_1}x_2^{\alpha_2}\dots x_r^{\alpha_r}\mid x_i\in X,\ \alpha_i\in R,\ i=1,2,\dots,r\}$ and
- $\langle X'\rangle_R=\{(x_1N)^{\alpha_1}(x_2N)^{\alpha_2}\dots(x_rN)^{\alpha_r}\mid x_iN\in X',\ \alpha_i\in R,\ i=1,2,\dots,r\}$.

Let $yN\in G/N$. We have $y\in G$. Since $G=\langle X\rangle_R$, then $y=x_1^{\beta_1}x_2^{\beta_2}\dots x_k^{\beta_k}$ for some $\beta_j\in R$, $j=1,2,\dots,k$. Then, $yN=(x_1^{\beta_1}x_2^{\beta_2}\dots x_k^{\beta_k})N=(x_1^{\beta_1}N)(x_2^{\beta_2}N)\dots(x_k^{\beta_k}N)=(x_1N)^{\beta_1}(x_2N)^{\beta_2}\dots(x_kN)^{\beta_k}$.

Therefore, $yN=(x_1N)^{\beta_1}(x_2N)^{\beta_2}\dots(x_kN)^{\beta_k}\in\langle X'\rangle_R$ and $G/N\subseteq\langle X'\rangle_R$.

For any $g\in\langle X'\rangle_R$, we have $g=(x_1N)^{\alpha_1}(x_2N)^{\alpha_2}\dots(x_rN)^{\alpha_r}$, for some $\alpha_i\in R$ for $i=1,2,\dots,r$.
Since, G/N is a $\mathcal{N}_R$-group, by closure we have $g=(x_1N)^{\alpha_1}(x_2N)^{\alpha_2}\dots(x_rN)^{\alpha_r}\in G/N$.
Hence, $\langle X'\rangle_R\subseteq G/N$ and $G/N=\langle X'\rangle_R$. By construction, X' is a finite subset of G/N, so, G/N is finitely R-generated. □

3.3 Isomorphism Theorems for R-Homomorphisms

In this section, we introduce the concept of R-homomorphisms, R-injections, R-surjections and R-isomorphisms. We prove that the three Isomorphism Theorems still hold in the context of $\mathcal{N}_R$-groups and R-homomorphisms. Furthermore, we state and prove the universal property of quotient groups in the context of $\mathcal{N}_R$-groups. We explore the concepts of R-characteristic subgroups and when a R-subgroup is R-complement of another R-subgroup. Lastly, we give some lemmas which will be needed for Section 3.5 on exact R-sequences.

First, we define a R-homomorphism.

Definition 3.3.1 (R-Homomorphism)
[1, Definition 4.15] Let G and H be $\mathcal{N}_R$-groups. A homomorphism $\varphi : G \to H$ is called a R-homomorphism if $\varphi(g^\alpha) = (\varphi(g))^\alpha$ for all $g \in G$ and $\alpha \in R$. Further, we have:

(i) If φ is injective then, φ is called a R-monomorphism or R-injective.

(ii) If φ is surjective then, φ is called a R-epimorphism or R-surjective.

(iii) If φ is a bijection then, φ is called a R-isomorphism and we denoted $G \cong_R H$.

The author in [9] makes note that the kernel and image can be defined in the usual way. We thus introduce the kernel of a R-homomorphism and show that a R-homomorphism is R-injective if and only if its kernel is trivial.

Definition 3.3.2 (Kernel of a R-Homomorphism)
Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. The kernel of φ, denoted by $Ker\varphi$, is defined as $Ker\varphi = \{x \in G \,|\, \varphi(x) = 1_H\}$.

Lemma 3.3.3 follows similarly to [3, Corollary 13.18], but re-contextualized it in terms of R-homomorphisms.

Lemma 3.3.3
Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. Then $\varphi : G \to H$ is R-injective if and only if $Ker\varphi = \{1\}$.

Proof. Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism.
If $\varphi : G \to H$ is R-injective, then by Definition 3.3.1, we have $\varphi : G \to H$ is injective.
Thus, for any $x \in Ker\varphi$, we have $\varphi(x) = 1 = \varphi(1)$ and $x = 1$ since φ is injective.
Thus, $Ker\varphi = \{1\}$.

For the converse, assume $Ker\varphi = \{1\}$.
Let $x, y \in G$ such that $\varphi(x) = \varphi(y)$. Then we have that $\varphi(x) = \varphi(y) \Rightarrow \varphi(x)\varphi(y)^{-1} = 1 \Rightarrow \varphi(x)\varphi(y^{-1}) = 1 \Rightarrow \varphi(xy^{-1}) = 1 \Rightarrow xy^{-1} \in Ker\varphi = \{1\} \Rightarrow xy^{-1} = 1 \Rightarrow x = y$.

Therefore, φ is injective and a R-homomorphism. Thus, φ is R-injective. □

Next, we introduce the image of a R-homomorphism $\varphi : G \to H$ and show that it is a R-subgroup of H. Furthermore, we prove if G is finitely R-generated then, the image of φ is also finitely R-generated.

Definition 3.3.4 (Image of a R-homomorphism)
Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. The image of φ, denoted by $Im(\varphi)$, is defined as $Im(\varphi) = \{\varphi(x) \,|\, x \in G\}$.

Lemma 3.3.5 is also a re-contextualization of Theorem 13.12 in [9].

Lemma 3.3.5
Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. Then $Im(\varphi)$ is a R-subgroup of H.

Proof. Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. Since, $1_H = \varphi(1_G) \in Im(\varphi)$, we have $Im(\varphi)$ is non-empty.

Let $x, y \in Im(\varphi)$ such that $x = \varphi(g)$ and $y = \varphi(h)$ for some $g, h \in G$. We have that $xy^{-1} = \varphi(g)\varphi(h)^{-1} = \varphi(g)\varphi(h^{-1}) = \varphi(gh^{-1})$ since φ is a R-homomorphism.

Thus, $xy^{-1} = \varphi(gh^{-1}) \in Im(\varphi)$. By Theorem 2.1.6, we have $Im(\varphi)$ is a subgroup of G.

Furthermore, for any $\alpha \in R$, we have $x^\alpha = \varphi(g)^\alpha = \varphi(g^\alpha)$.

Since $g^\alpha \in G$, we have $x^\alpha = \varphi(g^\alpha) \in Im(\varphi)$. Thus, by Definition 3.2.1, we have $Im(\varphi)$ is a R-subgroup of H. □

Now, we prove that the image of a finitely R-generated $\mathcal{N}_R$-group is again finitely R-generated.

Lemma 3.3.6
Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. If G is finitely R-generated then $Im(\varphi)$ is also finitely R-generated.

Proof. Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. In addition, assume that G is finitely R-generated, that is, there exists a finite subset $S \subseteq G$ such that $G = \langle S \rangle_R$.
We will show that $Im(\varphi) = \langle \varphi(S) \rangle_R$, where $\varphi(S) = \{\varphi(s) : s \in S\}$.
If $x \in Im(\varphi)$ then, $x = \varphi(g)$ for some $g \in G$. By Corollary 3.2.6, we have $g = s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k}$ for some $s_i \in S$, $\alpha_i \in R$ and $i = 1, 2, \dots, k$.
Then, $x = \varphi(g) = \varphi(s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k}) = \varphi(s_1^{\alpha_1})\varphi(s_2^{\alpha_2}) \dots \varphi(s_k^{\alpha_k}) = \varphi(s_1)^{\alpha_1}\varphi(s_2)^{\alpha_2} \dots \varphi(s_k)^{\alpha_k}$ since φ is a R-homomorphism. Thus, by Corollary 3.2.6, we have $x = \varphi(s_1)^{\alpha_1}\varphi(s_2)^{\alpha_2} \dots \varphi(s_k)^{\alpha_k} \in \langle \varphi(S) \rangle_R$. Hence, $Im(\varphi) \subseteq \langle \varphi(S) \rangle_R$.

Now, let $y \in \langle \varphi(S) \rangle_R$. By Corollary 3.2.6, we have $y = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_k^{\alpha_k}$ for some $z_i \in \varphi(S)$, $\alpha_i \in R$ and $i = 1, 2, \dots, k$.
Since, each $z_i \in \varphi(S)$, we have that each $z_i = \varphi(s_i)$ for some $s_i \in S$. Therefore, $y = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_k^{\alpha_k} = \varphi(s_1)^{\alpha_1}\varphi(s_2)^{\alpha_2} \dots \varphi(s_k)^{\alpha_k} = \varphi(s_1^{\alpha_1})\varphi(s_2^{\alpha_2}) \dots \varphi(s_k^{\alpha_k}) = \varphi(s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k})$ since φ is a R-homomorphism.

Thus, $y = \varphi(s_1^{\alpha_1} s_2^{\alpha_2} \dots s_k^{\alpha_k}) \in Im(\varphi)$ and $\langle \varphi(S) \rangle_R \subseteq Im(\varphi)$. We conclude that $Im(\varphi) = \langle \varphi(S) \rangle_R$.

In addition, S is finite, so $\varphi(S)$ is also finite.

Therefore, we have a finite subset $\varphi(S)$ of $Im(\varphi)$ such that $Im(\varphi) = \langle \varphi(S) \rangle_R$ and, by Definition 3.2.4, we have $Im(\varphi)$ is finitely R-generated. □

The authors in [1] and [9] state without proof the three Isomorphisms Theorem for R-homomorphisms. Here, we will give their proofs.

Theorem 3.3.7 (1^{st} Isomorphism Theorem for R-homomorphisms)
[9, Theorem 2.2.1] Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. Then,

(a) *$Ker\varphi \trianglelefteq_R G$,*

(b) *$G/Ker\varphi \cong_R Im\varphi$ and*

(c) *if φ is a R-surjective, then $G/Ker\varphi \cong_R H$.*

Proof. Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. Denote $Ker\varphi$ as K.

(a) By the First Isomorphism Theorem for groups (Theorem 2.3.5), we have $K \trianglelefteq G$. For any $x \in K$ and $\alpha \in R$, we have:

$$\begin{aligned} \varphi(x^\alpha) &= (\varphi(x))^\alpha \text{ since } \varphi \text{ is a } R\text{-homomorphism} \\ &= (1_H)^\alpha \text{ since } x \in K = Ker\varphi \\ &= 1_H \text{ by Corollary 3.1.6.} \end{aligned}$$

Therefore, $x^\alpha \in K$ and $K \leq_R G$.

Since $K \leq_R G$ and $K \trianglelefteq G$, then, by Definition 3.2.12, $K \trianglelefteq_R G$.

(b) Let $\theta : G/K \to Im(\varphi)$ be defined as $\theta(gK) = \varphi(g)$. By the First Isomorphism Theorem (Theorem 2.3.5), we know that θ is a well defined group homomorphism that is a bijection. We only need to show that θ is a R-homomorphism.

For any $gK \in G/K$ and $\alpha \in R$, we have $\theta((gK)^\alpha) = \theta(g^\alpha K) = \varphi(g^\alpha) = (\varphi(g))^\alpha = (\theta(gK))^\alpha$ since φ is a R-homomorphism.

Thus, since θ is a bijection and R-homomorphism, we have that θ is a R-isomorphism. We can write $G/Ker\varphi \cong_R Im(\varphi)$.

(c) Now, φ is a R-surjective and $Im(\varphi) = H$. Using (b), we get $G/Ker(\varphi) \cong_R H$.

□

In the following, we prove the 2^{nd} and 3^{rd} Isomorphism Theorems for R-homomorphisms with both proofs relying on the 1^{st} Isomorphism Theorem for R-homomorphisms.

Theorem 3.3.8 (2^{nd} Isomorphism Theorem for R-Homomorphisms)
[9, Theorem 2.2.2] Let G be a $\mathcal{N}_R$-group. Let H be a R-subgroup of G and let N be a normal R-subgroup of G. Then we have the following:

(a) *$HN = \langle H, N\rangle_R$ is a R-subgroup of G,*

(b) *N is a normal R-subgroup of HN,*

(c) *$H \cap N$ is a normal R-subgroup of H and*

(d) *$HN/N \cong_R H/(H \cap N)$.*

Proof. Let G be a $\mathcal{N}_R$-group. Let H be a R-subgroup of G and let N be a normal R-subgroup of G.

(a) We show $HN = \{hn : h \in H \text{ and } n \in N\}$ is a R-subgroup of G.
Firstly, by the Second Isomorphism Theorem for groups (Theorem 2.3.6), HN is a subgroup of G. Now, for any $\alpha \in R$, we have, by Lemma 3.2.13, that $h^\alpha n^\alpha = (hn)^\alpha q$ for some $q \in N$.
Therefore, $(x)^\alpha = (hn)^\alpha = (h^\alpha n^\alpha)q^{-1} = h^\alpha(n^\alpha q^{-1}) \in HN$.
Thus, by Definition 3.2.1, HN is a R-subgroup of G.
In addition, for any $h \in H$ and $n \in N$, we have $h = h(1_N) \in HN$ and $n = (1_H)n \in HN$. So, $H \subseteq HN$ and $N \subseteq HN$. Therefore, HN is a R-subgroup of G containing H and N. Thus, $\langle H, N\rangle_R \subseteq HN$, since $\langle H, N\rangle_R$ is the intersection of all R-subgroup of G containing H and N.

In addition, by Proposition 3.2.5, we have:
$\langle H, N\rangle_R = \bigcup_{i=0}^{\infty} S_i$ with $S_0 = \langle H, N\rangle$ and $S_i = \langle g^\alpha : g \in S_{i-1},\, \alpha \in R\rangle$ for $i \geq 1$.

Let $x \in HN$ such that $x = hn$ for some $h \in H$ and $n \in N$. Then, $hn \in \langle H, N\rangle = S_0$ and $x = hn \in \bigcup_{i=0}^{\infty} S_i = \langle H, N\rangle_R$.
Therefore, $HN \subseteq \langle H, N\rangle_R$ and $HN = \langle H, N\rangle_R$ is a R-subgroup of G.

(b) For any $n \in N$, $n = (e_H)n \in HN$, so, $N \subseteq HN$.
However, N is a R-subgroup of G and we have that N is a group under the same operators as HN since HN is also a R-subgroup of G. Hence N is a R-subgroup of HN.
By the Second Isomorphism Theorem for groups (Theorem 2.3.6), N is a normal subgroup of HN and by Definition 3.2.12, N is a normal R-subgroup of HN.

(c) By the Second Isomorphism Theorem for groups (Theorem 2.3.6), $H \cap N$ is a normal subgroup of H. In addition, for any $\alpha \in R$ and $x_1 \in H \cap N$, we have:

$$\begin{aligned} x_1 \in H \cap N &\Rightarrow x_1 \in H \text{ and } x_1 \in N \\ &\Rightarrow x_1^\alpha \in H \text{ and } x_1^\alpha \in N \text{ since } H \text{ and } N \text{ are } R\text{-subgroups of } G \\ &\Rightarrow x_1^\alpha \in H \cap N. \end{aligned}$$

Therefore, by Definition 3.2.12, $H \cap N$ is a normal R-subgroup of G.

(d) By (a) and (b), we have N is a normal R-subgroup of HN and by Proposition 3.2.14, we have HN/N is a $\mathcal{N}_R$-group.

Let $\varphi : H \to HN/N$ be defined as $\varphi(h) = hN$. We will show that φ is a R-surjective R-homomorphism.
For any $h_1, h_2 \in H$, we have that $\varphi(h_1h_2) = h_1h_2N = (h_1N)(h_2N) = \varphi(h_1)\varphi(h_2)$. Hence, φ is a group homomorphism.
Also, for any $h \in H$ and $\alpha \in R$ we have $\varphi(h^\alpha) = h^\alpha N = (hN)^\alpha = \varphi(h)^\alpha$. Therefore, φ is a R-homomorphism.
Furthermore, for any $(hn)N \in HN/N$, there exists $h \in N$ such that $(hn)N = h(nN) = h(N) = \varphi(h)$.

Hence, φ is surjective and $\varphi : H \to HN/N$ is a R-surjective R-homomorphism. By the 1^{st} Isomorphism Theorem for R-homomorphisms (Theorem 3.3.7), we have that $H/Ker\varphi \cong_R HN/N$.

We can calculate $Ker\varphi$ as follows:

$Ker\varphi = \{h \in H : \varphi(h) = 1_{HN/N}\} = \{h \in H : \varphi(h) = N\} = \{h \in H : hN = N\} = \{h \in H : h \in N\} = H \cap N$.

Therefore, $H/H \cap N \cong_R HN/N$.

□

Theorem 3.3.9 (3^{rd} Isomorphism Theorem for R-Homomorphism)
[9, Theorem 2.2.3] Let G be a $\mathcal{N}_R$-group and let H and N be normal R-subgroups of G such that $H \leq_R N$. Then, $(G/H)/(N/H) \cong_R G/N$.

Proof. Let G be a $\mathcal{N}_R$-group and let H and N be normal R-subgroups of G such that $H \leq_R N$.

Since H and N are normal R-subgroups of G, then, by Proposition 3.2.14, we have that G/H and G/N are $\mathcal{N}_R$-groups.

Let $\phi : G/H \to G/N$ be defined by $\phi(gH) = gN$. We will show that ϕ is a R-surjective R-homomorphism.

For any $aH, bH \in G/H$, $\phi((aH)(bH)) = \phi(abH) = abN = (aN)(bN) = \phi(aH)\phi(bH)$.

Hence, ϕ is a group homomorphism.

In addition, for any $\alpha \in R$ and $aH \in G/H$, $\phi((aH)^\alpha) = \phi((a^\alpha H)) = a^\alpha H = (aH)^\alpha = \phi(aH)^\alpha$ and ϕ is a R-homomorphism.

Furthermore, for all $gN \in G/N$ there exists gH such that $\phi(gH) = gN$. Therefore, ϕ is surjective and $\phi : G/H \to G/N$ is a R-surjective R-homomorphism. By the 1^{st} isomorphism theorem (Theorem 3.3.7), $(G/H)/Ker\phi \cong_R G/N$.

We have $Ker\phi = \{xH \in G/H : \phi(xH) = 1_{G/N}\} = \{xH \in G/H : xN = N\} = \{xH \in G/H : x \in N\} = \{xH : x \in N\} = N/H$.

Therefore, $(G/H)/(N/H) \cong_R G/N$. □

The next theorem (Theorem 3.3.10) is adapted from [16, Result 3.24] where we will give a proof in the context of $\mathcal{N}_R$-groups. Theorem 3.3.10 has many names in the literature such as the Fundamental Theorem on Homomorphisms or the Universal Property of Quotient Groups, we have picked the latter as it is more adapted to our context.

Theorem 3.3.10 (Universal Property of Quotient Groups)
Let G and H be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ be a R-homomorphism. Let K be a normal R-subgroup of G such that $K \subseteq Ker(\varphi)$. Then, there exists a unique R-homomorphism $\overline{\varphi} : G/K \to H$ defined as $\overline{\varphi}(gK) = \varphi(g)$ such that $\varphi = \overline{\varphi} \circ \pi$, where $\pi : G \to G/K$ is the canonical map $\pi(g) = gK$.

The map $\overline{\varphi}$ is called the map induced by φ.

Proof. Let G and H be $\mathcal{N}_R$-groups, $\varphi : G \to H$ a R-homomorphism and let K be a normal R-subgroup of G such that $K \subseteq Ker(\varphi)$.

Let $\overline{\varphi} : G/K \to H$ be defined as $\overline{\varphi}(gK) = \varphi(g)$ and let $\pi : G \to G/K$ be the canonical map $\pi(g) = gK$. The above information can be represented in the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/K \\ & \searrow^{\varphi} & \downarrow{\overline{\varphi}} \\ & & H \end{array}$$

Consider for $gK, hK \in G/K$ with $gK = hK$. By Proposition 2.2.6, $g = hk$ for some $k \in K$. Then

$$\begin{aligned} \overline{\varphi}(gK) = \varphi(g) &= \varphi(hk) \\ &= \varphi(h)\varphi(k) \text{ since } \varphi \text{ is } R\text{-homomorphism} \\ &= \varphi(h)1 \text{ since } k \in K \subseteq Ker(\varphi) \\ &= \varphi(h) = \overline{\varphi}(hK). \end{aligned}$$

Therefore, $gK = hK$ implies that $\overline{\varphi}(gK) = \overline{\varphi}(hK)$ and $\overline{\varphi}$ is well-defined.

For any $xK, yK \in G/K$, $\overline{\varphi}((xK)(yK)) = \overline{\varphi}((xy)K) = \varphi(xy) = \varphi(x)\varphi(y) = \overline{\varphi}(xK)\overline{\varphi}(yK)$. Furthermore, for any $\alpha \in R$ we have $\overline{\varphi}((xK)^\alpha) = \overline{\varphi}(x^\alpha K) = \varphi(x^\alpha) = \varphi(x)^\alpha = [\overline{\varphi}(x)]^\alpha$. Therefore, $\overline{\varphi}$ is a R-homomorphism.

In addition, for any $g \in G$, we have that $\overline{\varphi} \circ \pi(g) = \overline{\varphi}(\pi(g)) = \overline{\varphi}(gK) = \varphi(g)$. Thus, $\varphi = \overline{\varphi} \circ \pi$.

Lastly, assume there are two R-homomorphisms $\overline{\varphi_1}$ and $\overline{\varphi_2}$ such that $\varphi = \overline{\varphi_1} \circ \pi$ and $\varphi = \overline{\varphi_2} \circ \pi$. Then, $\overline{\varphi_1} \circ \pi = \overline{\varphi_2} \circ \pi$. Since $\pi : G \to G/K$ is R-surjective, then $\overline{\varphi_1} = \overline{\varphi_2}$.

Hence, there exists a unique R-homomorphism $\overline{\varphi}$ such that $\varphi = \overline{\varphi} \circ \pi$. □

We define R-characteristic R-subgroups and state when a R-subgroup is a R-complement of another R-subgroup. Definition 3.3.11 is an adapted version of Definition 3.1 in [16] and Lemma 3.3.12 is a test for R-characteristic.

Definition 3.3.11 (R-Characteristic R-Subgroup)
Let G be a $\mathcal{N}_R$-group. A R-subgroup H of G is called a R-characteristic R-subgroup of G, denoted by H R-char G, if $\phi(H) = H$ for all $\phi \in Aut_R(G) = \{\phi : G \to G \mid \phi$ are R-isomorphism$\}$.

Lemma 3.3.12
Let G be a $\mathcal{N}_R$-group and let H be a R-subgroup of G. Then H is a R-characteristic R-subgroup of G if and only if $\phi(H) \subseteq H$ for all $\phi \in Aut_R(G)$.

Proof. Let G be a $\mathcal{N}_R$-group and let H be a R-subgroup of G.
Assume H is a R-characteristic R-subgroup of G. By Definition 3.3.11, we have $\phi(H) = H$ for all $\phi \in Aut_R(G)$. Thus, $\phi(H) \subseteq H$ for all $\phi \in Aut_R(G)$.

Conversely, assume $\phi(H) \subseteq H$ for all $\phi \in Aut_R(G)$. For any $\phi \in Aut_R(G)$, we have $\phi^{-1} \in Aut_R(G)$. Thus, by assumption, we have that $\phi^{-1}(H) \subseteq H$ and $\phi(\phi^{-1}(H)) \subseteq \phi(H)$.
Then, $H = I_G(H) = (\phi \circ \phi^{-1})(H) = \phi(\phi^{-1}(H)) \subseteq \phi(H) \subseteq H$. Therefore, we have $H \subseteq \phi(H) \subseteq H$ and $\phi(H) = H$. Since, ϕ was arbitrarily chosen, then $\phi(H) = H$ for all $\phi \in Aut_R(G)$. Consequently, H is a R-characteristic R-subgroup of G. □

Definition 3.3.13 restated Definition 9.11 in [16], in the context of $\mathcal{N}_R$-groups and R-subgroups.

Definition 3.3.13 (R-Complement R-Subgroup)
Let G be a $\mathcal{N}_R$-group and let H and K be R-subgroups of G. The subgroup H is called the R-complement R-subgroup of K if the following hold:

(i) $H \cap K = \{1\}$ *and*

(ii) $G = HK = \{hk : h \in H \text{ and } k \in K\}$.

Corollary 3.3.14 is a result of the Second Isomorphism Theorem for R-homomorphisms (Theorem 3.3.8) and Definition 3.3.13.

Corollary 3.3.14
Let G be a $\mathcal{N}_R$-group, N a normal R-subgroup of G and let K be a R-subgroup of G. If K is the R-complement R-subgroup of N, then $G/N \cong_R K$.

Proof. Let G be a $\mathcal{N}_R$-group, N a normal R-subgroup of G and let K be a R-subgroup of G. Assume K to be the R-complement R-subgroup of N. By the Second Isomorphism Theorem for R-homomorphisms (Theorem 3.3.8), we have $KN/N \cong_R K/(K \cap N)$. By Definition 3.3.13, $KN = G$ and $K \cap N = \{1\}$. Thus, $G/N \cong_R K/\{1\} \cong_R K$.

□

Lastly, Lemma 3.3.15, Lemma 3.3.16 and Lemma 3.3.17 are needed for our exploration of exact R-sequences.

Lemma 3.3.15
Let G, H and K be $\mathcal{N}_R$-groups. If $\varphi : G \to H$ and $\theta : H \to K$ are R-homomorphisms, then $\theta \circ \varphi : G \to K$ is also a R-homomorphism.

Proof. Let G, H and K be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\theta : H \to K$ be R-homomorphisms.
Since the composition of two group homomorphisms is a homomorphism, we have that $\theta \circ \varphi : G \to K$ is a homomorphism.
In addition, for any $g \in G$ and $\alpha \in R$, we have the following:

$$\begin{aligned}\theta \circ \varphi(g^\alpha) &= \theta(\varphi(g^\alpha)) \\ &= \theta(\varphi(g)^\alpha) \text{ since } \varphi \text{ is a } R\text{-homomorphism} \\ &= \theta(\varphi(g))^\alpha \text{ since } \theta \text{ is a } R\text{-homomorphism} \\ &= [\theta \circ \varphi(g)]^\alpha.\end{aligned}$$

Therefore, $\theta \circ \varphi : G \to K$ is a R-homomorphism. □

Lemma 3.3.16
Let G, H and K be $\mathcal{N}_R$-groups. If $\varphi : G \to H$ and $\theta : H \to K$ are R-injective R-homomorphisms then, $\theta \circ \varphi : G \to K$ is also a R-injective R-homomorphism.

Proof. Let G, H and K be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\theta : H \to K$ be R-injective R-homomorphisms.

By Lemma 3.3.15, we have $\theta \circ \varphi$ is a R-homomorphism. In addition, for any $g, h \in G$ we have the following:

$$\begin{aligned} \theta \circ \varphi(g) = \theta \circ \varphi(h) &\Rightarrow \theta(\varphi(g)) = \theta(\varphi(h)) \\ &\Rightarrow \varphi(g) = \varphi(h) \text{ since } \theta \text{ is } R\text{-injective} \\ &\Rightarrow g = h \text{ since } \varphi \text{ is } R\text{-injective.} \end{aligned}$$

Thus, $\theta \circ \varphi : G \to K$ is R-injective. □

Lemma 3.3.17
Let G, H and K be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\theta : H \to K$ be R-homomorphisms. If $\theta \circ \varphi : G \to K$ is R-injective, then $\varphi : G \to H$ is also R-injective.

Proof. Let G, H and K be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\theta : H \to K$ be R-homomorphisms such that $\theta \circ \varphi : G \to K$ is R-injective. We have the following for any $x, y \in G$ that $\varphi(x) = \varphi(y) \Rightarrow \theta(\varphi(x)) = \theta(\varphi(y)) \Rightarrow \theta \circ \varphi(x) = \theta \circ \varphi(y) \Rightarrow x = y$ since $\theta \circ \varphi$ is R-injective. Thus, $\varphi : G \to H$ is R-injective. □

3.4 Abelian R-powered group

Motivated by the fact that every abelian group is nilpotent, in this section, we will explore abelian R-powered groups. We will show that abelian R-powered groups are just R-modules. Thus, all properties that hold for R-modules also hold for abelian R-powered groups. We will define a poly-R-cyclic series of an abelian R-powered group A and show that the rank of A is equal to the length of this series if A is finitely R-generated abelian group. This will be needed in our investigation of the free centre in Section 5.2.

The author in [1], mentioned how abelian R-powered group is related to be a R-module. We start by defining an abelian R-powered group and showing it is a R-module.

Definition 3.4.1 (Abelian R-powered group)
A $\mathcal{N}_R$-group A is called an abelian R-powered group if for any $g, h \in A$, we have $gh = hg$.

Proposition 3.4.2
Let A be an abelian R-powered group. Then A is a R-module with usual addition and multiplication $\cdot : R \times A \to A$ defined as $\alpha \cdot x = x^{\alpha}$ for $\alpha \in R$ and $x \in A$.

Proof. Let A be an abelian R-powered group. For any $x, y \in A$ and $\alpha, \beta \in R$, by Definition 3.1.1 (ii), (iii) and (i), we have:

(i) Since $xy = yx$, thus, by Lemma 3.1.5, $\alpha \cdot (x + y) = (x + y)^{\alpha} = x^{\alpha} + y^{\alpha} = \alpha \cdot x + \alpha \cdot y$.

(ii) $(\alpha + \beta) \cdot x = x^{\alpha+\beta} = x^{\alpha} + x^{\beta} = \alpha \cdot x + \beta \cdot y$,

(iii) $\alpha \cdot (\beta \cdot x) = (\beta \cdot x)^{\alpha} = (x^{\beta})^{\alpha} = x^{\beta\alpha} = x^{\alpha\beta} = (\alpha\beta) \cdot x$ and

(iv) $1 \cdot a = a^1 = a$.

Therefore, A is a R-module. □

Lemma 3.4.3
Let A be an abelian R-powered group and let N be a normal R-subgroup of A. We have the quotient A/N is an abelian R-powered group.

Proof. Let A be an abelian R-powered group and let N be a normal R-subgroup of A.
By Proposition 3.2.14, A/N is $\mathcal{N}_R$-group.
In addition, for any $aN, bN \in A/N$, $(aN)(bN) = (ab)N = (baN) = (bN)(aN)$.
Hence, A/N is an abelian R-powered group. □

Now, we give the definition of a free R-module, as taken from [18, Chapter 7.4]. We restate it in the context of an abelian R-powered group.

Definition 3.4.4 (Free Abelian R-Powered Group)
An abelian R-powered group A is called free if there exists a subset $X \subseteq A$ such that the following conditions hold:

(i) *For any non-zero $a \in A$, we can express it as a finite sum of distinct $x_i \in X$, $a = n_1 \cdot x_1 + n_2 \cdot x_2 + \cdots + n_r \cdot x_r = x_1^{n_1} + x_2^{n_2} + \cdots + x_r^{n_r}$ for non-zero $n_1, n_2, \ldots n_r \in R$. This expression is unique up to ordering.*

(ii) For any $x_1, x_2, \dots x_r \in X$, $n_1 \cdot x_1 + n_2 \cdot x_2 + \dots n_r \cdot x_r = 0$ if and only if $n_1 = n_2 = \cdots = n_r = 0$.

The set X is called the basis of A and the number of elements in X is called the rank of A and is denoted by $rank_R(A)$.

Lastly, we define a poly-R-cyclic series for an abelian R-powered group A and show the length of this series is equal to the rank of A if A is finitely R-generated.

Definition 3.4.5 (Poly-R-Cyclic)
[1, Definition 4.19] Let A be an abelian R-powered group. A subnormal R-series

$$0 = A_0 \trianglelefteq_R A_1 \trianglelefteq_R A_2 \trianglelefteq_R \cdots \trianglelefteq_R A_n = A$$

of A is called a poly-R-cyclic series, if each A_{i+1}/A_i is a cyclic R-powered group.

Theorem 3.4.6
[1, Theorem 4.22] Let A be an abelian R-powered group. If A is finitely R-generated then A has a poly-R-cyclic series.

Definition 3.4.7 (Hirsch-R-Length)
[1, Definition 4.20] Let A be a finitely R-generated abelian R-powered group. The minimal length of all poly-R-cyclic series of A is called the Hirsch-R-length and is denoted by $h_R(A)$.

The following result was used in [14] and [6] in their discussion of the free centre of finitely generated nilpotent groups. We prove it in the context of finitely R-generated free abelian R-powered groups.

Proposition 3.4.8
Let A be a finitely R-generated free abelian R-powered group. Then, the rank of A is equal to the Hirsch-R-length of A.

Proof. Let A be a finitely R-generated free abelian R-powered group and let $h = h_R(A)$.
Let $0 = A_0 \trianglelefteq_R A_1 \trianglelefteq_R A_2 \trianglelefteq_R \cdots \trianglelefteq_R A_h = A$ be the minimal poly-R-cyclic series of length $h = h_R(A)$.

By Definition 3.4.5, each A_i/A_{i-1} is cyclic. By Definition 3.2.11, $A_i/A_{i-1} = \langle a_i + A_{i-1}\rangle_R$ for some $a_i \in A_i$ for each $i = 1, 2, \dots, h$.

We will show, by induction, that $A_j = \langle a_1, a_2, \dots, a_j\rangle_R$ for $j = 1, 2, \dots, h$.
For $j = 1$, $\langle a_1\rangle_R = \langle a_1 + 0\rangle_R = \langle a_1 + A_0\rangle_R = A_1/A_0 = A_1/0 = A_1$ and $A_1 = \langle a_1\rangle_R$. Assume that the statement is true for $j = k \leq h$, that is, $A_k = \langle a_1, a_2, \dots, a_k\rangle_R$. Consider the case when $j = k+1 \leq h$. Let $x \in A_{k+1}$. We have $x + A_k = (a_{k+1} + A_k)^\alpha$ for some $\alpha \in R$ and $x + A_k = a_{k+1}^\alpha + A_k$. By Proposition 2.2.6, $x = a_{k+1} + q$ for some $q \in A_k$.
By inductive hypothesis, $A_k = \langle a_1, a_2, \dots, a_k\rangle_R$ and by Corollary 3.2.6, $q = (a_1)^{\alpha_1} + (a_2)^{\alpha_2} + \dots (a_k)^{\alpha_k}$, for some $\alpha_1, \alpha_2, \dots, \alpha_k \in R$. Therefore, $x = a_{k+1}^\alpha + [(a_1)^{\alpha_1} + (a_2)^{\alpha_2} + \dots (a_k)^{\alpha_k}] \in \langle a_1, a_2, \dots, a_k, a_{k+1}\rangle_R$ and $A_{k+1} \subseteq \langle a_1, a_2, \dots, a_k, a_{k+1}\rangle_R$.

Since $0 = A_0 \trianglelefteq_R A_1 \trianglelefteq_R A_2 \trianglelefteq_R \cdots \trianglelefteq_R A_k \trianglelefteq_R A_{k+1}$, then $a_i \in A_{k+1}$ for all $i = 1, 2, \dots, k+1$. Therefore, $\{a_1, a_2, \dots, a_k, a_{k+1}\}$ is a subset of A_{k+1} and by Proposition 3.2.3, $\langle a_1, a_2, \dots, a_k, a_{k+1}\rangle_R \subseteq A_{k+1}$.
Therefore, $A_{k+1} = \langle a_1, a_2, \dots, a_k, a_{k+1}\rangle_R$ and $A_j = \langle a_1, a_2, \dots, a_j\rangle_R$ for $j = 1, 2, \dots, h$.
Consequently, $A = A_h = \langle a_1, a_2, \dots, a_h\rangle_R$ and we conclude that A can be generated by h elements. Hence, $rank(A) = h = h_R(A)$. □

3.5 Exact R-sequences

In this section, we explore exact sequences of $\mathcal{N}_R$-groups. We start by giving the definitions of exact R-sequences and short R-sequences. Furthermore, we will show that there exist short R-sequences involving the direct product of any two $\mathcal{N}_R$-groups. Lastly, we will show in a specific commutative diagram that if we have two R-isomorphisms, then the third one should also be a R-isomorphism.

This section was inspired by previous studies on exact sequences of groups and R-modules in [15, Chapter 11], [2, Section 10.5] and [18, Section 7.1]. In particular, we have taken some results and prove that they also hold true in the context of $\mathcal{N}_R$-groups.

Firstly, we give the definition of a exact R-sequence and short R-sequences.

Definition 3.5.1 (Exact R-Sequence)
Let k be a positive integer greater than 3. The sequence:

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3 \xrightarrow{\phi_3} \dots \xrightarrow{\phi_{k-1}} G_{k-1} \xrightarrow{\phi_k} G_k$$

of $\mathcal{N}_R$-groups $G_1, G_2, \dots, G_k$, with R-homomorphisms $\phi_i : G_i \to G_{i+1}$ for all $i = 1, 2, \dots, k-1$, is called an exact R-sequence if $Im(\phi_i) = Ker(\phi_{i+1})$ for all $i = 1, 2, \dots, k-1$.

Definition 3.5.2 (Short Exact R-Sequence)
An exact R-sequence is called a short exact R-sequence if it is of the form:

$$1 \xrightarrow{\gamma} G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \xrightarrow{\phi} 1$$

where G_1, G_2, G_3 are $\mathcal{N}_R$-groups and φ and ψ are R-homomorphisms.

The following proposition gives a test when a R-sequence is exact and is an adpated version of [2, Corollary 23].

Proposition 3.5.3
Let G_1, G_2, G_3 be $\mathcal{N}_R$-groups. The R-sequence

$$1 \xrightarrow{\gamma} G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \xrightarrow{\phi} 1$$

is exact if and only if the following hold

(a) *φ is R-injective,*

(b) *$Im(\varphi) = Ker(\psi)$ and*

(c) *ψ is R-surjective.*

In this case we can omit the R-homomorphisms γ and ϕ and just write the sequence as:

$$1 \to G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \to 1.$$

Proof. Let G_1, G_2, G_3 be $\mathcal{N}_R$-groups.
If $1 \xrightarrow{\gamma} G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \xrightarrow{\phi} 1$ is an exact R-sequence, then, $Im(\gamma) = Ker(\varphi)$, $Im(\varphi) = Ker(\psi)$ and $Im(\psi) = Ker(\phi)$.

(a) Since, γ is a trivial inclusion, then, $Ker(\varphi) = Im(\gamma) = \{1\}$. By Lemma 3.3.3, we have φ is R-injective.

(b) By definition of exact R-sequences, we have $Im(\varphi) = Ker(\psi)$.

(c) Since all R-homomorphisms are well-defined, then $\phi(x) = 1$ for all $x \in G_3$ and $Ker(\phi) = G_3$. Consequently, $Im(\psi) = Ker(\phi) = G_3$ and ψ is R-surjective.

Hence, φ is R-injective, $Im(\varphi) = Ker(\psi)$ and ψ is R-surjective.

Conversely, assume φ is R-injective, $Im(\varphi) = Ker(\psi)$ and ψ is R-surjective.

(a) Since, φ is R-injective, we have by Lemma 3.3.3, that $Ker(\varphi) = \{1\}$. In addition, $Im(\gamma) = \{1\}$. Therefore, $Im(\gamma) = Ker(\varphi) = \{1\}$.

(b) By assumption, $Im(\varphi) = Ker(\psi)$.

(c) Since, ψ is R-surjective, we have that $Im(\psi) = G_3$. Now, since $\phi : G_3 \to \{1\}$ is a well defined R-homomorphism, we have that $\phi(x) = 1$ for all $x \in G_3$. Therefore, $Ker(\phi) = G_3$ and $Im(\psi) = Ker(\phi) = G_3$.

Thus, $Im(\gamma) = Ker(\varphi)$, $Im(\varphi) = Ker(\psi)$ and $Im(\psi) = Im(\phi)$ and the R-sequence $1 \xrightarrow{\gamma} G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\psi} G_3 \xrightarrow{\phi} 1$ is exact. □

In the following, we show that there exists an exact R-sequence involving the direct product of any two $\mathcal{N}_R$-groups. This was given as an example in [2, Example (1), page 379] for the case of R-modules.

Proposition 3.5.4
Let G and H be $\mathcal{N}_R$-groups. The following R-sequence

$$1 \to G \xrightarrow{i} G \times H \xrightarrow{\pi} H \to 1$$

is exact with

- *$i : G \to G \times H$ defined as $i(g) = (g, 1)$ and*
- *$\pi : G \times H \to H$ defined as $\pi(g, h) = h$.*

Proof. Let G and H be $\mathcal{N}_R$-groups and let $1 \to G \xrightarrow{i} G \times H \xrightarrow{\pi} H \to 1$ be a R-sequence with $i : G \to G \times H$ defined as $i(g) = (g, 1)$ and $\pi : G \times H \to H$ defined as $\pi(g, h) = h$.

By Proposition 3.5.3, we need to show that

(a) i is R-injective,

(b) π is R-surjective and

(c) $Im(i) = Ker(\pi)$.

Firstly, we show that i is R-injective.

(a) For any $g_1, g_2 \in G$ we have $i(g_1) = i(g_2) \Rightarrow (g_1, 1) = (g_2, 1) \Rightarrow g_1 = g_2$. Thus, i is R-injective.

(b) Secondly, for any $h \in H$, there exists $(1, h) \in G \times H$ such that $\pi(1, h) = h$. Thus, π is R-surjective.

(c) Thirdly, we show that $Im(i) = Ker(\pi)$. Let $x \in Im(i)$ such that $x = i(g)$ for some $g \in G$. Then, $\pi(x) = \pi(i(g)) = \pi((g, 1)) = 1$ and $x \in Ker(\pi)$. Hence, $Im(i) \subseteq Ker(\pi)$.

Let $y \in Ker(\pi)$ Then, $y = (g, 1) = \pi(g)$ for some $g \in G$. Thus, $y \in Im(i)$, $Ker(\pi) \subseteq Im(i)$ and $Im(i) = Ker(\pi)$.

Therefore, $1 \to G \xrightarrow{i} G \times H \xrightarrow{\pi} H \to 1$ is a short exact R-sequence, for any $\mathcal{N}_R$-groups G and H.

□

Finally, we show that in the following commutative diagram if two homomorphisms are R-isomorphisms then the third is also a R-isomorphism. This is an adaptation of the Short Five Lemma given in [2, Proposition 24].

Proposition 3.5.5
Let

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_1 & \xrightarrow{\varphi_1} & G_1 & \xrightarrow{\psi_1} & Q_1 & \longrightarrow & 1 \\ & & \downarrow{\theta_K} & & \downarrow{\theta_G} & & \downarrow{\theta_Q} & & \\ 1 & \longrightarrow & K_2 & \xrightarrow{\varphi_2} & G_2 & \xrightarrow{\psi_2} & Q_2 & \longrightarrow & 1 \end{array} \tag{3.2}$$

be a commutative diagram with each row being a short exact R-sequence.

If θ_G and θ_Q are R-isomorphisms, then θ_K is also a R-isomorphism.

Proof. We need to show that θ_K is R-injective and R-surjective.

Using Proposition 3.5.3 and Definition 3.3.1, we have by Diagram (3.2):

- φ_1, φ_2, θ_G and θ_Q are R-injective R-homomorphisms,
- ψ_1, ψ_2, θ_G and θ_Q are R-surjective R-homomorphisms and
- $Im(\varphi_1) = Ker(\psi_1)$ and $Im(\varphi_2) = Ker(\psi_2)$.

Let $k, k_* \in K_1$ such that $\theta_K(k) = \theta_K(k_*)$. Since θ_G and φ_1 are R-injective, then, by Lemma 3.3.16, $\theta_G \circ \varphi_1$ is R-injective and

$$\begin{aligned} \theta_K(k) = \theta_K(k_*) &\Rightarrow \varphi_2(\theta_K(k)) = \varphi_2(\theta_K(k_*)) \\ &\Rightarrow \varphi_2 \circ \theta_K(k) = \varphi_2 \circ \theta_K(k_*) \\ &\Rightarrow \theta_G \circ \varphi_1(k) = \theta_G \circ \varphi_1(k_*) \text{ by commutative diagram (3.2)} \\ &\Rightarrow k = k_* \text{ since } \theta_G \circ \varphi_1 \text{ is } R\text{-injective.} \end{aligned}$$

Thus, $\theta_K(k) = \theta_K(k_*) \Rightarrow k = k_*$ and θ_K is R-injective.

Now, we show that θ_K is R-surjective. Let $k_2 \in K_2$. We have that $\varphi_2(k_2) \in G_2$. Since θ_G is R-surjective, then there exists $g_1 \in G_1$ such that

$$\varphi_2(k_2) = \theta_G(g_1). \tag{3.3}$$

Further,

$$\begin{aligned}\theta_Q \circ \psi_1(g_1) &= \psi_2 \circ \theta_G(g_1) \text{ by the commutative diagram (3.2)}\\ &= \psi_2(\theta_G(g_1))\\ &= \psi_2(\varphi_2(k_2)) \text{ by Equation (3.3)}\\ &= 1 \text{ since } Im(\varphi_2) = Ker(\psi_2)\\ &= \theta_Q(1).\end{aligned}$$

Since, θ_Q is R-injective, then $\psi_1(g_1) = 1$ and $g_1 \in Ker(\psi_1) = Im(\varphi_1)$. Therefore, there exists $k_1 \in K$ such that:

$$g_1 = \varphi_1(k_1). \tag{3.4}$$

Then, we have the following

$$\begin{aligned}\varphi_2(k_2\theta_K(k_1)^{-1}) &= \varphi_2(k_2)\varphi(\theta_K(k_1)^{-1})\\ &= \varphi_2(k_2)\varphi_2(\theta_K(k_1))^{-1}\\ &= \theta_G(g_1)\varphi_2(\theta_K(k_1))^{-1} \text{ by Equation (3.3)}\\ &= \theta_G(g_1)[\varphi_2 \circ \theta_K(k_1)]^{-1}\\ &= \theta_G(g_1)[\theta_G \circ \varphi_1(k_1)]^{-1} \text{ by the commutative diagram (3.2)}\\ &= \theta_G(g_1)\theta_G(\varphi_1(k_1))^{-1}\\ &= \theta_G(g_1)\theta_G(g_1)^{-1} \text{ by Equation (3.4)}\\ &= 1.\end{aligned}$$

Therefore, $k_2\theta_K(k_1)^{-1} \in Ker(\varphi_2)$. By Lemma 3.3.3, $Ker(\varphi_2) = \{1\}$, hence, $k_2\theta_K(k_1)^{-1} = 1$ and $k_2 = \theta_K(k_1)$. Since, $k_2 \in K_2$ was arbitrarily chosen, θ_K is R-surjective and θ_K is a R-isomorphism. □

This concludes our introductory investigation on $\mathcal{N}_R$-groups. We began with an investigation on R-subgroups and normal R-subgroups and then observed that the Isomorphism Theorem still holds true in the context of $\mathcal{N}_R$-groups. Lastly, we explored abelian $\mathcal{N}_R$-groups as R-modules and concluded by investigating the exact sequence of $\mathcal{N}_R$-groups. Now, we move on to explore the P-localization of $\mathcal{N}_R$-groups.

Chapter 4

P-localization of a Nilpotent R-powered Group

To continue towards our study of the genus, we will need to take a tour of P-localization of $\mathcal{N}_R$-groups, as outlined in [13], for some set of primes P in R. The work done in [13] relies on root extraction of $\mathcal{N}_R$-groups done in [1], [11] and [12]. We will follow in a similar manner, by firstly investigating root extraction in Section 4.1, Section 4.2 and Section 4.3, before investigating P-localization of $\mathcal{N}_R$-groups in the later Sections. In particular, we will work towards proving the Fundamental Theorem of P-localization in Section 4.7 and in Section 4.8, we prove that, in a specific ring, a map P-localize if and only if it is a P-isomorphism. Lastly, in Section 4.9 we prove some results which will be needed in our study of the genus of a $\mathcal{N}_R$-group in Chapter 5.

4.1 R-Torsion, $\mathscr{U}_R$-Groups and R-Radicable

Firstly, we state the definition of a α^{th}-root of a $\mathcal{N}_R$-group and a $\mathscr{U}_R$-group G. We define when a $\mathcal{N}_R$-group is called R-torsion, R-torsion-free and said to be of finite type. We will give a proof when a $\mathcal{N}_R$-group is a $\mathscr{U}_R$-group. Lastly, we state some results relating to abelian R-powered groups and in conclusion we define a R-radicable group.

This section is based on [11] and [12] which explores root extraction for $\mathcal{N}_R$-groups and [1, Section 2.3.2] which is an exploration of root extract of ordinary nilpotent groups.

Definition 4.1.1 (α^{th} root)
[11] Let G be a $\mathcal{N}_R$-group and let $\alpha \in R$. An element $g \in G$ is said to have an α^{th} root in G, if there exists $h \in G$ such that $g = h^{\alpha}$.

Definition 4.1.2 ($\mathscr{U}_R$-group)
[11] Let G be a $\mathcal{N}_R$-group. The group G is called a $\mathscr{U}_R$-group if every element $g \in G$ has a unique α^{th} root (if it exists) for every $\alpha \in R$.

Proposition 4.1.3
[1, Theorem 4.26] Let G be a $\mathcal{N}_R$-group. Then G is a $\mathscr{U}_R$-group if and only if $g^{\alpha} = h^{\alpha}$, for some $\alpha \in R$, implies that $g = h$ for all $g, h \in G$.

Proof. Assume that G is a $\mathscr{U}_R$-group. Let $g, h \in G$ such that $g^{\alpha} = h^{\alpha}$ for some $\alpha \in R$. Since G is a

$\mathcal{N}_R$-group, then $g^\alpha \in G$ and there exists $x \in G$ such that $x = g^\alpha$. Then x has an α^{th}-root. However, $x = g^\alpha = h^\alpha$, so, $g = h$ since G is a $\mathscr{U}_R$-group and each element has a unique α^{th}-root, if it exists. Then, for every $g, h \in G$ such that $g^\alpha = h^\alpha$ for some $\alpha \in R$, we have $g = h$.
Conversely, assume that for every $g, h \in G$ and some $\alpha \in R$ such that $g^\alpha = h^\alpha$, we have that $g = h$. Let $x \in G$. Assume x has two α^{th}-roots $g_*, h_* \in G$. Then, $x = g_*^\alpha$, $x = h_*^\alpha$ and $g_*^\alpha = x = h_*^\alpha$. From the assumption, we have $g_* = h_*$ and x has a unique α^{th}-root if it exists. Therefore, G is a $\mathscr{U}_R$-group. □

Now, we state the definition of R-torsion and when a group is R-torsion-free. We also prove that a subgroup of a R-torsion-free group is also R-torsion-free.

Definition 4.1.4 (R-Torsion)
[11, Definition 2.3] Let G be a $\mathcal{N}_R$-group.

(*i*) *An element $g \in G$ is called a R-torsion if there exists $\alpha \in R$ such that $g^\alpha = 1$.*

(*ii*) *If all elements of G are R-torsion elements, then G is called a R-torsion group.*

(*iii*) *If the only R-torsion element of G is 1, then G is called a R-torsion-free group.*

Lemma 4.1.5
Let G be a R-torsion-free $\mathcal{N}_R$-group. Every R-subgroup of G is also R-torsion-free.

Proof. Let G be a R-torsion-free $\mathcal{N}_R$-group and let H be a R-subgroup of G.
Let $g \in H$ be a R-torsion element. Since H is a R-subgroup of G, then $g \in G$. Furthermore, G is R-torsion-free, so $g = 1$.
Therefore, the only R-torsion element of H is 1 and H is R-torsion-free. □

The next proposition tells us that the set of all R-torsion elements in G is a normal R-subgroup of G and the proof of this proposition can be found in [1, Theorem 4.24 (*ii*)]

Proposition 4.1.6 (R-Torsion Subgroup)
[1, Theorem 4.24 (ii)] Let G be a $\mathcal{N}_R$-group. The set of all R-torsion elements in G, denoted by $\tau(G)$, is a normal R-subgroup of G.

Now, we give the definition of a finite type $\mathcal{N}_R$-group.

Definition 4.1.7 (Group of Finite Type)
[12, Definition 2.6] A $\mathcal{N}_R$-group G is said to be of finite type if

(*i*) *G is finitely R-generated and*

(*ii*) *G is a R-torsion group.*

We are now ready to prove the main result of this section, that a $\mathcal{N}_R$-group is R-torsion-free if and only if it is a $\mathscr{U}_R$-group.

The proof of Proposition 4.1.8 is based on [1, Theorem 2.7], which is the case on ordinary nilpotent groups.

Proposition 4.1.8
[1, Theorem 4.26 (ii)] Let G be a $\mathcal{N}_R$-group. Then G is a R-torsion-free group if and only if G is a $\mathscr{U}_R$-group.

Proof. Let G be a $\mathcal{N}_R$-group of nilpotency class c.
Assume that G is a R-torsion-free group. We will show that G is a $\mathscr{U}_R$-group by using induction on the nilpotency class of G. We show that, for every $g, h \in G$ such that $g^\alpha = h^\alpha$ for some $\alpha \in R$ we have $g = h$.
If $c = 1$, we have, by Note 2.5.3 that G is an abelian group. Assume $g, h \in G$ are such that $g^\alpha = h^\alpha$ for some $\alpha \in R$. Then

$$\begin{aligned} g^\alpha = h^\alpha \Rightarrow g^\alpha h^{-\alpha} = 1 \Rightarrow g^\alpha (h^{-1})^\alpha = 1 \\ &\Rightarrow (gh^{-1})^\alpha = 1 \text{ since } G \text{ is abelian and using Corollary 3.1.5} \\ &\Rightarrow gh^{-1} = 1 \text{ since } G \text{ is } R\text{-torsion-free} \\ &\Rightarrow g = h. \end{aligned}$$

Hence, $g^\alpha = h^\alpha$ implies that $g = h$ and G is a $\mathscr{U}_R$-group.
Assume that the statement is true for $c = k > 1$, that is, if G is a R-torsion-free group with nilpotency class of at most k then, G is a $\mathscr{U}_R$-group.
Let G be a $\mathcal{N}_R$-group which is R-torsion-free and of nilpotency class of $k+1$. Let $g, h \in G$ such that $g^\alpha = h^\alpha$ for some $\alpha \in R$ and consider $H = \langle g, \gamma_1(G) \rangle_R$. By Proposition 3.2.5, H is a R-subgroup of G and by Lemma 2.5.9 H has nilpotency class that is less than or equal to k.
Also, by Lemma 4.1.5, H is a R-torsion-free, since H is a R-subgroup of G.
Thus, H meets the requirements of the inductive hypothesis. We note that the product $hg^{-1}h^{-1} = g^{-1}[g, h] \in H$, since $H = \langle g, \gamma_1(G) \rangle_R$. Then,

$$\begin{aligned} g^\alpha = h^\alpha \Rightarrow g^\alpha = (hhh^{-1})^\alpha \\ &\Rightarrow g^\alpha = hh^\alpha h^{-1} \text{ by Definition 3.1.1 } (iv) \\ &\Rightarrow g^\alpha = hg^\alpha h^{-1} \text{ since } g^\alpha = h^\alpha \\ &\Rightarrow g^\alpha = (hgh^{-1})^\alpha \text{ by Definition 3.1.1 } (iv) \\ &\Rightarrow g = hgh^{-1} \text{ by induction hypothesis since } g, hgh^{-1} \in H \\ &\Rightarrow gh = hg \Rightarrow ghg^{-1}h^{-1} = 1 \Rightarrow [g, h] = 1 \\ &\Rightarrow g^\alpha h^\alpha = (gh)^\alpha \text{ by Lemma 3.1.5.} \end{aligned}$$

We have that $[g, h] = 1$ and by Lemma 2.4.3, we have $[g, h]^{-1} = [g, h^{-1}] = 1$. Then, since $[g, h^{-1}] = 1$, by Lemma 3.1.5, we have $g^\alpha (h^{-1})^\alpha = (gh^{-1})^\alpha$ and

$$\begin{aligned} g^\alpha = h^\alpha \Rightarrow g^\alpha h^{-\alpha} = 1 \Rightarrow g^\alpha (h^{-1})^\alpha = 1 \\ &\Rightarrow (gh^{-1})^\alpha = 1 \Rightarrow gh^{-1} = 1 \text{ since } G \text{ is } R\text{-torsion-free} \\ &\Rightarrow g = h. \end{aligned}$$

Hence, $g^\alpha = h^\alpha$ implies that $g = h$. By Proposition 4.1.3, G is a $\mathscr{U}_R$-group.
Conversely, if G is a $\mathscr{U}_R$-group and g is an R-torsion-element, then by Definition 4.1.4, there exists $\alpha \in R$ such that $g^\alpha = 1$. By Corollary 3.1.6, $1^\alpha = 1$ and $g^\alpha = 1^\alpha$. Since G is U_R-group, then, by Proposition 4.1.3, $g = 1$.
Therefore, the only R-torsion element in G is 1 and G is a R-torsion-free group. □

An abelian R-powered group is just a R-module so, one can use results proven for R-module. We now recall a few results from [18] proven for R-modules. Here the results are adpated to abelian R-powered groups.

Proposition 4.1.9
[18, Proposition 9.2] Let R be a integral domain and let A be an abelian R-powered group. Then the quotient module $A/\tau(A)$ is R-torsion-free.

Theorem 4.1.10
[18, Theorem 9.3] Let R be a principal ideal domain and let A be an abelian R-powered group. If A is finitely R-generated and R-torsion-free, then A is a free abelian R-powered group.

Lastly, we state the definition of a R-radicable group.

Definition 4.1.11 (R-Radicable Group)
[11] A $\mathcal{N}_R$-group G is called a R-radicable group if each $g \in G$ has at least one α^{th}-root in G for every $\alpha \in R$.

Note 4.1.12
A group is R-radicable $\mathscr{U}_R$-group if and only if every element $g \in G$ has a unique α^{th}-root for each $\alpha \in R$.

4.2 P-Torsion, $\mathscr{U}_P$-Groups and P-Radicable

In this section, we recall the concept of P-members from Section 2.8 for a set of primes P in a binomial ring R. However, we need to put an extra constraint on the ring R. We will require that the binomial ring R to be a unique factorization domain (UFD). This is needed so that each non-zero and non-unit element of R has a decomposition of prime elements which is unique up to associativity.
We define a P-torsion element and when a $\mathcal{N}_R$-group is said to be of finite P-type. In addition, we define a $\mathscr{U}_P$-group and show that a $\mathcal{N}_R$-group is P-torsion-free if and only if G is a $\mathscr{U}_P$-group. Next, we will state the definition of a P-radicable group and explore the p-primary components of a $\mathcal{N}_R$-group, which are analogous to p-subgroups for regular nilpotent groups.

Definition 4.2.1 (P-members)
[11, Definition 2.4] Let R be a UFD binomial ring, P be a set of primes in R and let $\alpha \in R$ be a non-zero and non-unit element of R. The element α is called a member of P or a P-member if $\alpha = 1$ or all prime divisors of α are in P up to associativity.
We will denote the set of P-members in R by $\boldsymbol{P}_{members}$.

For the remainder of this section, R is a unique factorization domain (UFD) binomial ring, unless otherwise stated and P is a set of primes in R.

Definition 4.2.2 (P-Torsions)
[12, Definition 3.1] Let G be a $\mathcal{N}_R$-group.

(*i*) *An element $g \in G$ is called a P-torsion if there exists $\alpha \in \boldsymbol{P}_{members}$ such that $g^\alpha = 1$.*

(*ii*) *If all elements of G are P-torsion elements, then G is called a P-torsion group.*

(*iii*) *If the only P-torsion element of G is 1, then G is called a P-torsion-free group.*

The next proposition tells us that the set of all P-torsion elements in G is a normal R-subgroup of G. The proof of this proposition can be found in [1, Theorem 4.24 (*i*)].

Proposition 4.2.3 (P-Torsion Subgroup)
[1, Theorem 4.24 (i)] Let G be a $\mathcal{N}_R$-group. The set of all P-torsion elements in G, denoted by $\tau_P(G)$, is a normal R-subgroup of G.

We state the definition of a finite P-type $\mathcal{N}_R$-group.

Definition 4.2.4 (Group of Finite P-Type)
[12, Definition 3.2] A $\mathcal{N}_R$-group G is said to be of finite P-type if

(*i*) *G is finitely R-generated and*

(*ii*) *G is a P-torsion group.*

Definition 4.2.5 ($\mathscr{U}_P$-group)
[12] Let G be a $\mathcal{N}_R$-group. The group G is called a $\mathscr{U}_P$-group if every element $g \in G$ has a unique α^{th} root (if it exists) for every $\alpha \in \boldsymbol{P}_{members}$.

The following results are similar to the results in Section 4.1. Proofs to Proposition 4.2.6, Corollary 4.2.7 and Proposition 4.2.8 follow similarly to the ones done in Proposition 4.1.3, Lemma 4.1.5 and Proposition 4.1.8 respectively.

Proposition 4.2.6

[1, Theorem 4.26 (i)] Let G be a $\mathcal{N}_R$-group. Then, G is a $\mathscr{U}_P$-group if and only if for every $g, h \in G$ with $g^\alpha = h^\alpha$ for some $\alpha \in \boldsymbol{P}_{members}$, we have that $g = h$.

Corollary 4.2.7

[12, Lemma 4.1]. Let G be a $\mathcal{N}_R$-group. Every R-subgroup of $\mathscr{U}_P$-group is also a $\mathscr{U}_P$-group.

Proposition 4.2.8

[12, Theorem 4.5] Let G be a $\mathcal{N}_R$-group. Then, G is a P-torsion-free group if and only if G is a $\mathscr{U}_P$-group.

Now, we will state the definition of a P-radicable group.

Definition 4.2.9 (P-Radicable Group)

[12] A $\mathcal{N}_R$-group G is called a P-radicable group if each $g \in G$ has at least one α^{th}-root in G for every $\alpha \in \boldsymbol{P}_{members}$.

Note 4.2.10

We have that a $\mathcal{N}_R$-group is P-radicable $\mathscr{U}_P$-group if and only if every element $g \in G$ has a unique α^{th}-root for each $\alpha \in \boldsymbol{P}_{members}$.

Lastly, we define the p-primary component of G. In [12], the authors show that if a $\mathcal{N}_R$-group is of finite type, then it is R-isomorphic to a direct product of all its p-primary components. We will also prove a lemma relating the P'-torsion subgroup and the p-primary component of a $\mathcal{N}_R$-group, where P' is a set of all primes in R not contained in P.

Definition 4.2.11

[12, Definition 2.7] Let G be a $\mathcal{N}_R$-group and let p be a prime in R. The p-primary component of G, denoted by $G^{(p)}$, is the set $G^{(p)} = \{g \in G : g^\alpha = 1 \text{ for some } \alpha \in \{p\}_{members}\}$.

Proposition 4.2.12

[12, Theorem 2.13] Let G be a $\mathcal{N}_R$-group and let p be a prime in R. The p-primary component of G, $G^{(p)}$, is a normal R-subgroup of G.

Theorem 4.2.13

[12, Theorem 2.14] Let G be a $\mathcal{N}_R$-group and let $\mathbb{P}_R$ be the set of all prime elements of R. If G is of finite type, then $G \cong_R \prod_{p \in \mathbb{P}_R} G^{(p)}$.

We prove Lemma 4.2.14 as will use it to show that the p-localization of a $\mathcal{N}_R$-group of finite type is R-isomorphic to the p-component of G.

Lemma 4.2.14

Let R be a PID binomial ring and let G be a $\mathcal{N}_R$-group of finite type. Let p be a prime in R, $P = \{p\}$ and let P' be the set of primes in R not contained in P. Then, $G^{(p)} \cong_R G/\tau_{P'}(G)$.

Proof. Let R be a PID binomial ring and let G be a $\mathcal{N}_R$-group of finite type. Let p be a prime in R, $P = \{p\}$ and let P' be the set of primes in R not contained in P.

We will that show $G^{(p)}$ is a R-complement R-subgroup of $\tau_{P'}(G)$. Let $x \in G^{(p)} \cap \tau_{P'}(G)$. Then, there exist $\alpha \in \mathbf{P}_{members}$ and $\beta \in \mathbf{P}'_{members}$ such that $x^\alpha = 1$ and $x^\beta = 1$. Since $\gcd(\alpha, \beta) = 1$, so, there exist

$\mu, \omega \in R$ such that $1 = \alpha\mu + \beta\omega$.
Then, $x = x^1 = x^{\alpha\mu+\beta\omega} = x^{\alpha\mu}x^{\beta\omega} = (x^\alpha)^\mu(x^\beta)^\omega = (1)^\mu(1)^\omega = 1$ and $G^{(p)} \cap \tau_{P'}(G) = \{1\}$.

We need to show that $G = G^{(p)}\tau_{P'}(G)$. Now, $G^{(p)}$ and $\tau_{P'}(G)$ are R-subgroups of G, so $G^{(p)}\tau_{P'}(G) \subseteq G$.

Let $g \in G$. Since G is of finite type, there exists $\upsilon \in R$ such that $g^\upsilon = 1$. We have the following cases:

> Case I: If $\upsilon \in \mathbf{P}_{members}$ then $g = g1 \in G^{(p)}\tau_{P'}(G)$.
>
> Case II: If $\upsilon \notin \mathbf{P}_{members}$ then, $\upsilon \in \tau_{P'}(G)$ and $g = 1g \in G^{(p)}\tau_{P'}(G)$, since $1^\alpha = 1$ for any $\alpha \in \{p\}_{members}$ and $g^\upsilon = 1$.

Hence, $G \subseteq G^{(p)}\tau_{P'}(G)$ and $G = G^{(p)}\tau_{P'}(G)$.

By Definition 3.3.13, $G^{(p)}$ is a R-complement of $\tau_{P'}(G)$.

By Proposition 4.2.3, $\tau_{P'}(G)$ is a normal R-subgroup. Therefore, by Corollary 3.3.14, $G^{(p)} \cong_R G/\tau_{P'}(G)$.

□

4.3 Residually of Finite P-Type

This section was adapted from [1, Section 5.2.12], which is a study on the residual properties for regular nilpotent groups. We will expand on this and show that the residual properties hold for $\mathcal{N}_R$-groups. In particular, we will show that a $\mathcal{N}_R$-group is residually of finite P-type if it is a subdirect product of groups that are all of finite P-type. Further, we take note of a result from [13], that a $\mathcal{N}_R$-group is residually of finite P-type if and only if G is P'-torsion-free.

In this section, R is a unique factorization domain (UFD) binomial ring, unless otherwise stated and P is a set of primes in R.

Firstly, we define residually of finite P-type $\mathcal{N}_R$-group and prove an equivalent property.

Definition 4.3.1 (Residually of Finite P-Type)
[1, Definition 5.14] A $\mathcal{N}_R$-group G is said to be residually of finite P-type if for every $g \in G$, with $g \neq 1$, there exists a normal R-subgroup N_g of G such that

(i) *$g \notin N_g$ and*

(ii) *G/N_g is of finite P-type.*

Proposition 4.3.2
[1, Lemma 5.32] Let G be a $\mathcal{N}_R$-group. Let I be a non-empty index set. Then G is residually of finite P-type if and only if there exists a family $\{N_i : i \in I\}$ of normal R-subgroups of G such that:

(a) *G/N_i is of finite P-type, for each $i \in I$ and*

(b) $\displaystyle\bigcap_{i\in I} N_i = \{1\}$.

Proof. Let G be a $\mathcal{N}_R$-group.

Assume that G is of residually of finite P-type. By Definition 4.3.1, we have that, for each $1 \neq g \in G$, there exists a normal R-subgroup N_g of G such that:

(i) $g \notin N_g$ and

(ii) G/N_g is of finite P-type.

Consider the family of normal R-subgroups of G, $\{N_g : g \in G\}$. By (ii), G/N_g is of finite P-type for each $g \in G$. We need to show that $\displaystyle\bigcap_{i\in I} N_i = \{1\}$.

Assume $\displaystyle\bigcap_{g\in G} N_g \neq \{1\}$ and let $x \in \displaystyle\bigcap_{g\in G} N_g$ such that $x \neq 1$. Since each N_g is a normal R-subgroup of G, then $x \in G$ with $x \neq 1$. Since G is of residually of finite P-type, then there exists a normal R-subgroup N_x of G such that $x \notin N_x$ and $x \notin \displaystyle\bigcap_{g\in G} N_g$, which is a contradiction. Therefore, we must have that $\displaystyle\bigcap_{g\in G} N_g = \{1\}$.

Conversely, assume there exists a family $\{N_i : i \in I\}$ of normal R-subgroups of G such that:

(a) G/N_i is of finite P-type, for each $i \in I$ and

(b) $\displaystyle\bigcap_{i\in I} N_i = \{1\}$.

Let $g \in G$ such that $g \neq 1$ and let $N_k \in \{N_i : i \in I\}$ for some $k \in I$. We have the following cases:

Case I: If $g \notin N_k$, then there exists a normal R-subgroup of G such that $g \notin N_k$ and G/N_k is of finite P-type.

Case II: If $g \in N_k$, by assumption we have that $\bigcap_{i \in I} N_i = \{1\}$. Then, $g \notin N_j$ for all $j \in I$ with $j \neq k$. Let $l \in I$ such that $l \neq k$. Then, there is a normal R-subgroup N_l of G such that $g \notin N_l$ and G/N_l is of finite P-type.

Since $g \in G$ with $g \neq 1$ was arbitrarily chosen, then, for any $g \in G$ with $g \neq 1$, there exists a normal R-subgroup N_g of G such that $g \in N_g$ and G/N_g is of finite P-type. Therefore, by Definition 4.3.1, we have G is of residually of finite P-type.

□

Now, we define a subdirect product and show that a $\mathcal{N}_R$-group which is residually of finite P-type is just a subdirect product of $\mathcal{N}_R$-groups of finite P-type.

Definition 4.3.3 (Subdirect Product)
[1, Definition 5.15] Let $\{G_i : i \in I\}$ be a family of $\mathcal{N}_R$-groups for some non-empty index set I.

(*i*) *The map $\pi_i : \prod_{j \in I} G_j \to G_i$, defined as $\pi_i(g_1, g_2, \ldots, g_i, \ldots) = g_i$ is called a projective map (which is R-surjective).*

(*ii*) *Let H be a R-subgroup of $\prod_{i \in I} G_i$. Then the restriction map $\pi_i \,|_{\,H} : H \to G_i$ is called the projection of H to G_i.*

(*iii*) *Let H be a R-subgroup of $\prod_{i \in I} G_i$. Then H is called subdirect product, if $\pi_i(H) = G_i$ for all $i \in I$.*

Lemma 4.3.4
[1, Lemma 5.32] Let G be a $\mathcal{N}_R$-group. If G is residually of finite P-type, then G is a subdirect product of $\mathcal{N}_R$-groups of finite P-type.

Proof. Let G be a $\mathcal{N}_R$-group.

Since G is residually of finite P-type, by Proposition 4.3.2, there exists a family $\{N_i : i \in I\}$ of normal R-subgroups of G such that:

(*i*) G/N_i is of finite P-type for each $i \in I$ and

(*ii*) $\bigcap_{i \in I} N_i = \{1\}$.

Let $\varphi : G \to \prod_{i \in I}(G/N_i)$ be defined as $\varphi(g) = \prod_{i \in I}(gN_i) = (gN_1, gN_2, \ldots, gN_i, \ldots)$.

For any $g, h \in G$, we have:

$$\begin{aligned}\varphi(gh) = \prod_{i\in I}(gh)N_i &= ((gh)N_1, (gh)N_2, \ldots, (gh)N_i, \ldots) \\ &= ((gN_1)(hN_1), (gN_2)(hN_2), \ldots, (gN_i)(hN_i), \ldots) \\ &= (gN_1, gN_2, \ldots, gN_i, \ldots)(hN_1, hN_2, \ldots, hN_i, \ldots) \\ &= \left(\prod_{i\in I} gN_i\right)\left(\prod_{i\in I} hN_i\right) = \varphi(g)\varphi(h).\end{aligned}$$

In addition, for any $\alpha \in R$, we have:

$$\begin{aligned}\varphi(g^\alpha) = \prod_{i\in I}(g^\alpha)N_i &= (g^\alpha N_1, g^\alpha N_2, \ldots, g^\alpha N_i, \ldots) \\ &= ((gN_1)^\alpha, (gN_2)^\alpha, \ldots, (gN_i)^\alpha, \ldots) \\ &= (gN_1, N_2, \ldots, gN_i, \ldots)^\alpha \text{ by Corollary 3.1.8} \\ &= \left(\prod_{i\in I} gN_i\right)^\alpha = \varphi(g)^\alpha.\end{aligned}$$

Therefore, $\varphi : G \to \prod_{i\in I}(G/N_i)$ is a R-homomorphism. Furthermore

$$\begin{aligned}\varphi(g) = \varphi(h) \Rightarrow \prod_{i\in I} gN_i = \prod_{i\in I} hN_i &\Rightarrow gN_i = hN_i \text{ for all } i \in I \\ &\Rightarrow (gN_i)(h^{-1}N_i) = N_i \text{ for all } i \in I \\ &\Rightarrow gh^{-1}N_i = N_i \text{ for all } i \in I \\ &\Rightarrow gh^{-1} \in N_i \text{ for all } i \in I \\ &\Rightarrow gh^{-1} \in \bigcap_{i\in I} N_i = \{1\} \text{ by assumption} \\ &\Rightarrow gh^{-1} = 1 \Rightarrow g = h.\end{aligned}$$

Therefore, $\varphi : G \to \prod_{i\in I}(G/N_i)$ is R-injective.

By the First Isomorphism Theorem for R-homomorphisms, $G/Ker(\varphi) \cong_R Im(\varphi)$.

Since φ is R-injective, by Lemma 3.3.3, $Ker(\varphi) = \{1\}$ and $G \cong_R G/\{1\} \cong_R Im(\varphi)$. By Lemma 3.3.5, $Im(\varphi)$ is a R-subgroup of $\prod_{i\in I}(G/N_i)$ and G is a R-subgroup of $\prod_{i\in I}(G/N_i)$.

Lastly, $\pi_i(G) = \{\pi_i(g) : g \in G\} = \{gN_i : g \in G\} = G/N_i$. By Definition 4.3.3, G is a subdirect product of $\{G/N_i : i \in I\}$. By assumption, each G/N_i is of finite P-type and G is a subdirect product of $\mathcal{N}_R$-groups of finite P-type. □

The next theorem is taken from [13] and will play an important role in our proof of Theorem 4.8.2, one of the major result needed in our work. Also, we take note that the main constraint on the binomial ring R in Theorem 4.8.2 can be traced back to this theorem.

Theorem 4.3.5

[13, Theorem 5.9] Let R be a principal ideal domain (PID) which has a subring isomorphic to the set of rational numbers and let G be a finitely R-generated $\mathcal{N}_R$-group. Then, G is residually of finite P-type if and only if G is P'-torsion-free.

4.4 P-Local Groups

Let P be a set of primes in R. In this section, we define a P-local $\mathcal{N}_R$-group. We will show that a $\mathcal{N}_R$-group is P-local if and only if it is a P'-radicable $\mathscr{U}_{P'}$-group, where P' is the complementary set of P. We show that a direct product of P-local groups is again P-local. We give a proposition that gives conditions on when a R-subgroup of a P-local $\mathcal{N}_R$-group is P-local. Lastly, we show that a $\mathcal{N}_R$-group of finite P-type is P-local.

Firstly, we define the complement set of a set of primes and state the definition of a P-local group. The notation in this section is adapted from [12] and [13].

Definition 4.4.1 (Complementary Set of P)
Let R be a UFD binomial ring and let P be a set of primes in R. The complementary set of P, denoted by P', is the set of all primes in R that are not in P.

For the remainder of this section, R is a unique factorization domain (UFD) binomial ring and P is a set of primes in R.

Definition 4.4.2 (P-local Group)
[13, Definition 3.1] A $\mathcal{N}_R$-group G is called P-local if for every $\alpha \in \mathbf{P}'_{members}$, the mapping $\phi_\alpha : G \to G$ defined by $\phi_\alpha(g) = g^\alpha$ is a bijective mapping.

Next, we show that a group is P-local if and only if it is a P'-radicable $\mathscr{U}_{P'}$-group. The authors in [13, Remark 3.2] make note of this result as a remark. However, we state and prove the result.

Proposition 4.4.3
[13, Remark 3.2] Let G be a $\mathcal{N}_R$-group. Then, G is P-local if and only if G is a P'-radicable $\mathscr{U}_{P'}$-group.

Proof. Let G be a $\mathcal{N}_R$-group. Assume that G is a P-local group and let $\alpha \in \mathbf{P}'_{members}$. The group G is a P-local group so for $\alpha \in \mathbf{P}'_{members}$, there exists a bijection $\phi_\alpha : G \to G$ defined by $\phi_\alpha(x) = x^\alpha$.
The map $\phi_\alpha : G \to G$ is surjective (since it is a bijection), so for any $g \in G$, there exists $h \in G$ such that $h^\alpha = \phi_\alpha(h) = g$. Hence, g has a α^{th}-root.
Suppose that g has two α^{th}-roots s and t. Then, $g = s^\alpha = \phi_\alpha(s)$, $g = t^\alpha = \phi_\alpha(t)$ and $\phi_\alpha(s) = g = \phi_\alpha(t)$. Therefore, $t = s$ since ϕ_α is injective.
Thus, every element $g \in G$ has an unique α^{th}-root for any $\alpha \in \mathbf{P}'_{members}$ and G is a P'-radicable $\mathscr{U}_{P'}$-group.
Conversely, assume G is a P'-radicable $\mathscr{U}_{P'}$-group and let $\alpha \in \mathbf{P}'_{members}$.
We will show that the map $\phi_\alpha : G \to G$ defined by $\phi_\alpha(x) = x^\alpha$ is a bijection.
For any $g, h \in G$ such that $\phi_\alpha(g) = \phi_\alpha(h)$ we have that $\phi_\alpha(g) = \phi_\alpha(h) \Rightarrow g^\alpha = h^\alpha \Rightarrow g = h$ since G is $\mathscr{U}_{P'}$-group. Therefore, ϕ_α is injective.
In addition, G is a P'-radicable group and we have that, every $g \in G$ has at least one α^{th}-root.
Therefore, $g = k^\alpha = \phi_\alpha(k)$ for some $k \in G$ and ϕ_α is surjective. Consequently, ϕ_α is a bijective. Since $\alpha \in \mathbf{P}'_{members}$ was arbitrarily chosen, then $\phi_\alpha : G \to G$ defined by $\phi_\alpha(x) = x^\alpha$ is a bijection for any $\alpha \in \mathbf{P'}_{members}$.
Therefore, by Definition 4.4.2, G is a P-local group. □

The next corollary follows immediately from Proposition 4.4.3 and Proposition 4.2.8.

Corollary 4.4.4
Let G be a $\mathcal{N}_R$-group. If G is P-local then G is P'-torsion-free.

Proof. If G is P-local, then, by Proposition 4.4.3, G is a $\mathscr{U}_{P'}$-group. Using Proposition 4.2.8, we conclude that G is a P'-torsion-free group. □

Now, we show that a direct product of P-local groups is again P-local. The result and its proof can be found in [13, Lemma 5.2]. Here we present a more elaborate proof of the result.

Lemma 4.4.5
[13, Lemma 5.2] Let $\{G_i : i \in I\}$ be a family of P-local $\mathcal{N}_R$-groups for some index set I. Then the direct product $G = \prod_{i\in I} G_i$ is also a P-local $\mathcal{N}_R$-group.

Proof. Let $\{G_i : i \in I\}$ be a family of P-local $\mathcal{N}_R$-groups for some index set I and let $G = \prod_{i\in I} G_i$. By Corollary 3.1.8, G is a $\mathcal{N}_R$-group. We only need to show that G is P-local.

Firstly, each G_i is P-local. By Proposition 4.4.3, each G_i is a P'-radicable $\mathscr{U}_{P'}$-group, so, for any $g_i \in G_i$ and $\alpha \in \mathbf{P}'_{members}$, there exists a unique α^{th}-root h_i.

Therefore, for any $(g_1, g_2, \ldots, g_k, \ldots)$, there exists a unique $(h_1, h_2, \ldots, h_k, \ldots)$ such that $(h_1, h_2, \ldots, h_k, \ldots)^\alpha = (h_1^\alpha, h_2^\alpha, \ldots, h_k^\alpha, \ldots) = (g_1, g_2, \ldots, g_k, \ldots)$. Hence, G is a P'-radicable $\mathscr{U}_{P'}$-group and $G = \prod_{i\in I} G_i$ is P-local. □

We recall the following definition from [12, Definition 3.3]. We restate a remark from [13, Remark 3.3] as a proposition and give its proof.

Definition 4.4.6 (P-isolated)
[12, Definition 3.3] Let G be a $\mathcal{N}_R$-group. A R-subgroup H of G is said to P-isolated in G if for every $g \in G$ and $\alpha \in \boldsymbol{P}_{members}$ such that $g^\alpha \in H$ we have that $g \in H$.

Proposition 4.4.7
[13, Remark 3.3] Let G be a $\mathcal{N}_R$-group. A R-subgroup H of G is P-local if and only if H is P'-isolated in G.

Proof. Let G be a $\mathcal{N}_R$-group.
Assume that H is P-local. Then, by Proposition 4.4.3, H is a P'-radicable $\mathscr{U}_{P'}$-group.
Let $g \in G$ such that $g^\alpha \in H$ for some $\alpha \in \mathbf{P}'_{members}$. We will show that $g \in H$.
Since H is a P'-radicable, then each element in H has an α^{th}-root. Thus, $g^\alpha \in H$ has an α^{th}-root, say $x \in H$. Then, $g^\alpha = x^\alpha$ implies $g = x \in H$ since H is a $\mathscr{U}_{P'}$-group.
Therefore, for any $g \in G$ such that $g^\alpha \in H$ for some $\alpha \in \mathbf{P}'_{members}$, we have $g \in H$ and H is P'-isolated in G.

Conversely, assume H is P'-isolated in G. We will show that H is P'-radicable $\mathscr{U}_{P'}$-group.
Firstly, G is P-local, so G is a P'-radicable $\mathscr{U}_{P'}$-group. By Corollary 4.2.7, we have that every R-subgroup of a $\mathscr{U}_{P'}$-group is a $\mathscr{U}_{P'}$-group. Thus, H is $\mathscr{U}_{P'}$-group.
We show that H is P'-radicable. Let $g \in H$ and let $\alpha \in \mathbf{P}'_{members}$. Since G is P'-radicable and $g \in H \subseteq G$, then, there exists $x \in G$ such that $g = x^\alpha$ and $x^\alpha = g \in H$. Therefore, $x \in G$ and $x^\alpha \in H$. Using the assumption that H is P'-isolated in G, we have $x \in H$.
Thus, for any $g \in H$ and $\alpha \in \mathbf{P}'_{members}$, there exists $x \in H$ such that $g = x^\alpha$ and each $g \in H$ has at

least one α^{th}-root for any $\alpha \in \mathbf{P}'_{members}$ and H is a P'-radicable $\mathscr{U}_{P'}$-group. By Proposition 4.2.6, H is P-local. □

Lastly, we show that a $\mathcal{N}_R$-group of finite P-type is P-local. This result and its proof can be found in [13, Theorem 3.14]. Here we present a more elaborate proof of the result.

Lemma 4.4.8

[13, Theorem 3.14] Let G be a $\mathcal{N}_R$-group. If G is of finite P-type then, G is P-local.

Proof. Let G be a $\mathcal{N}_R$-group of finite P-type.

Firstly, we show that G is a $\mathscr{U}_{P'}$-group. Let $g, h \in G$ such that $g^\alpha = h^\alpha$ for some $\alpha \in \mathbf{P}'_{members}$. Since, G is of finite P-type, it is a P-torsion group. Therefore, there exist $\beta_1, \beta_2 \in \mathbf{P}_{members}$ such that $g^{\beta_1} = 1$ and $h^{\beta_2} = 1$. Set $\beta = \beta_1\beta_2$, then, $g^\beta = g^{\beta_1\beta_2} = (g^{\beta_1})^{\beta_2} = (1)^{\beta_2} = 1 = (1)^{\beta_1} = (h^{\beta_2})^{\beta_1} = h^{\beta_2\beta_1} = h^{\beta_1\beta_2} = h^\beta$.

However, α and β are relatively prime since $\alpha \in \mathbf{P}'_{members}$ and $\beta \in \mathbf{P}_{members}$, so, $\gcd(\alpha, \beta) = 1$. Then, there exists $\mu, \omega \in R$ such that $1 = \alpha\mu + \beta\omega$ and $g = g^1 = g^{\alpha\mu+\beta\omega} = g^{\alpha\mu}g^{\beta\omega} = (g^\alpha)^\mu(g^\beta)^\omega = (h^\alpha)^\mu(g^\beta)^\omega = (h^\alpha)^\mu(h^\beta)^\omega = h^{\alpha\mu}h^{\beta\omega} = h^{\alpha\mu+\beta\omega} = h^1$
$= h$ since $g^\alpha = h^\alpha$ and $g^\beta = h^\beta$.

Therefore, $g^\alpha = h^\alpha$ implies $g = h$ and G is a $\mathscr{U}_{P'}$-group.

Let $\alpha_* \in \mathbf{P}'_{members}$ and let $x \in G$. Since G is a P-torsion group, then there exists $\beta_* \in \mathbf{P}_{members}$ such that $x^{\beta_*} = 1$. Since $\gcd(\alpha_*, \beta_*) = 1$ then $1 = \alpha_*\mu_* + \beta_*\omega_*$ for some $\mu_*, \omega_* \in R$ and $x = x^1 = x^{\alpha_*\mu_*+\beta_*\omega_*} = x^{\alpha_*\mu_*}x^{\beta_*\omega_*} = x^{\alpha_*\mu_*}(x^{\beta_*})^{\omega_*} = x^{\alpha_*\mu_*}(1)^{\omega_*} = x^{\alpha_*\mu_*} = x^{\mu_*\alpha_*} = (x^{\mu_*})^{\alpha_*}$.

Therefore, $x = (x^{\mu_*})^{\alpha_*}$ and x has a α_*^{th}-root. Thus, every element in G has at least one α_*^{th}-root for any $\alpha_* \in \mathbf{P}'_{members}$ and G is a P'-radicable group.

We conclude that G is a P'-radicable $\mathscr{U}_{P'}$-group and G is P-local. □

4.5 R-Generated P-Local Group

In this section, we study R-subgroups generated as P-local groups. The material is needed for the proof of the Fundamental Theorem of P-localization of a $\mathcal{N}_R$-group. Furthermore, we will give some results, which can be found in [13], relating the R-subgroups generated as P-local groups to the P-isolator.

Firstly, we will prove that an intersection of all P-local R-subgroups containing a subset S is also a P-local subgroup. Following [13], we define what is meant for a group to be called R-generated as a P-local group.

Proposition 4.5.1 (R-Subgroup Generated by S as a P-Local Group)
Let G be a P-local $\mathcal{N}_R$-group and let $S \subseteq G$. Then, the intersection of all P-local R-subgroups containing S, denoted by $\langle S\rangle_R^{P\text{-local}}$, is a P-local R-subgroup of G. Moreover, $\langle S\rangle_R^{P\text{-local}}$ is the smallest P-local R-subgroup of G containing S.

Proof. Let G be a P-local $\mathcal{N}_R$-group and let $S \subseteq G$.
Let $\mathbb{H} = \{H \mid H \leq_R G,\ S \subseteq H,\ H\ P\text{-local}\}$ be the set of all P-local R-subgroups of G containing S. By Lemma 3.2.2, $\langle S\rangle_R^{P\text{-local}} = \bigcap_{H\in\mathbb{H}} H$ is a R-subgroup of G.

Since $\langle S\rangle_R^{P\text{-local}} = \bigcap_{H\in\mathbb{H}} H$ is a R-subgroup of G, we have, by Proposition 4.4.7, that $\langle S\rangle_R^{P\text{-local}}$ is P-local if and only if $\langle S\rangle_R^{P\text{-local}}$ is P'-isolated in G.

Let $x \in G$ such that $x^\alpha \in \langle S\rangle_R^{P\text{-local}} = \bigcap_{H\in\mathbb{H}} H$ for some $\alpha \in \mathbf{P}'_{members}$. We will show that $x \in \langle S\rangle_R^{P\text{-local}}$.
Now, $x^\alpha \in \langle S\rangle_R^{P\text{-local}} = \bigcap_{H\in\mathbb{H}} H$ implies that $x^\alpha \in H$ for all $H \in \mathbb{H}$ and this implies $x \in H$ for all $H \in \mathbb{H}$ since each $H \in \mathbb{H}$ is P-local (and in particular is P'-isolated in G). Therefore, $x \in \bigcap_{H\in\mathbb{H}} H = \langle S\rangle_R^{P\text{-local}}$.

We have $x \in G$ such that $x^\alpha \in \langle S\rangle_R^{P\text{-local}} = \bigcap_{H\in\mathbb{H}} H$ for some $\alpha \in \mathbf{P}'_{members}$ and $x \in \langle S\rangle_R^{P\text{-local}}$. Therefore, $\langle S\rangle_R^{P\text{-local}}$ is P'-isolated in G and by Proposition 4.4.7, $\langle S\rangle_R^{P\text{-local}}$ is P-local.

We conclude that $\langle S\rangle_R^{P\text{-local}}$ is a P-local R-subgroup of G.
Now, we show that $\langle S\rangle_R^{P\text{-local}}$ is the smallest such P-local R-subgroup of G. Assume that $\langle S\rangle_R^{P\text{-local}}$ is not the smallest P-local R-subgroup of G containing S.
Let Q be the smallest P-local R-subgroup of G containing S. Then, $Q \in \mathbb{H}$ and $\langle S\rangle_R^{P\text{-local}} = \bigcap_{H\in\mathbb{H}} H \bigcap Q$.
Consequently, $\langle S\rangle_R^{P\text{-local}} \subseteq Q$, which contradicts the assumption that Q is the smallest P-local R-subgroup of G containing S.

Therefore, $\langle S\rangle_R^{P\text{-local}}$ is the smallest P-local R-subgroup of G containing S. □

Definition 4.5.2 (R-Generated as a P-Local Group)
[13, Definition 3.7] A $\mathcal{N}_R$-group G is said to R-generated as a P-local group if there exists $S \subseteq G$ such that $G = \langle S\rangle_R^{P\text{-local}}$.
We say that G is finitely R-generated as a P-local group if $|S|$ is finite.

Lemma 4.5.3
If G is a P-local $\mathcal{N}_R$-group, then, $\left\langle \langle S\rangle_R^{P\text{-}local} \right\rangle_R^{P\text{-}local} = \langle S\rangle_R^{P\text{-}local}$, *where* $S \subseteq G$.

Proof. Let G be a P-local $\mathcal{N}_R$-group and let $S \subseteq G$.
By Proposition 4.5.1, we have that $\left\langle \langle S\rangle_R^{P\text{-local}} \right\rangle_R^{P\text{-local}}$ is the smallest P-local R-subgroup of G containing $\langle S\rangle_R^{P\text{-local}}$ and $\langle S\rangle_R^{P\text{-local}} \subseteq \left\langle \langle S\rangle_R^{P\text{-local}} \right\rangle_R^{P\text{-local}}$.

Let $x \in \left\langle \langle S\rangle_R^{P\text{-local}} \right\rangle_R^{P\text{-local}}$ and let $\mathbb{H} = \{H \mid H \leq_R G,\ \langle S\rangle_R^{P\text{-local}} \subseteq H,\ H\ P\text{-local}\}$ be the set of all P-local R-subgroups of G containing $\langle S\rangle_R^{P\text{-local}}$. Then, $x \in \langle\langle S\rangle_R\rangle_R^{P\text{-local}} = \bigcap_{H\in\mathbb{H}} H$ and $x \in H$ for each $H \in \mathbb{H}$. However, $S \subseteq \langle S\rangle_R^{P\text{-local}}$, so, $S \subseteq \langle S\rangle_R^{P\text{-local}} \subseteq H$ for any $H \in \mathbb{H}$.
Therefore, $\mathbb{H} \subseteq \mathbb{H}_* = \{H \mid H \leq_R G,\ S \subseteq H,\ H\ P\text{-local}\}$ and $x \in \langle S\rangle_R^{P\text{-local}}$.
Therefore, $\langle S\rangle_R^{P\text{-local}} = \left\langle \langle S\rangle_R^{P\text{-local}} \right\rangle_R^{P\text{-local}}$. □

Furthermore, we define the P-isolator of a subset of a $\mathcal{N}_R$-group. The next Theorem taken from [13] gives an explicit equality of a P-isolator of a R-subgroup H of G. We state a corollary that relates the R-subgroup generated as a P-local group by a R-subgroup H, with its P-isolator. We refer the reader to [12] and [13] for the proofs of the results that follow.

Definition 4.5.4 (P-isolator)
[12, Definition 3.4] Let R be a UFD binomial ring, G a $\mathcal{N}_R$-group and let P be a set of primes in R. Let S be a subset of G. The P-isolator of S in G, denoted by $I_P(S,G)$, is the intersection of all R-subgroup of G containing S which are P-isolated in G.

Theorem 4.5.5
[13, Theorem 3.9] Let R be a UFD binomial ring, G a $\mathcal{N}_R$-group and let P be a set of primes in R. If H is a R-subgroup of G, then $I_P(H,G) = \{g \in G : g^\alpha \in H$ for some $\alpha \in \boldsymbol{P}_{members}\}$.

Corollary 4.5.6
[13, Corollary 3.10] Let G be a P-local $\mathcal{N}_R$-group, let H a R-subgroup of G and let P be a set of primes in R. Then $\langle H\rangle_R^{P\text{-}local} = I_{P'}(H,G) = \{g \in G : g^\alpha \in H$ for some $\alpha \in \boldsymbol{P}'_{members}\}$.

Lemma 4.5.7
[13, Lemma 3.12] Let G be a P-local $\mathcal{N}_R$-group and let H be a R-subgroup of G. If $G = \langle H\rangle_R^{P\text{-}local}$, then G and H have the same nilpotency class.

Lemma 4.5.8
[13, Lemma 3.13] Let G and H be P-local $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\psi : G \to H$ be R-homomorphisms. If N is a R-subgroup of G such that $G = \langle N\rangle_R^{P\text{-}local}$ and $\varphi(x) = \psi(x)$ for all $x \in N$, then $\varphi = \psi$.

Proof. Let G and H be P-local $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\psi : G \to H$ be R-homomorphisms. Let N be a R-subgroup of G such that $G = \langle N\rangle_R^{P\text{-local}}$ and assume that $\varphi(x) = \psi(x)$ for all $x \in N$.
For any $g \in G = \langle N\rangle_R^{P\text{-local}}$, we have, by Corollary 4.5.6, that $\langle N\rangle_R^{P\text{-local}} = \{g \in G : g^\alpha \in N$ for some $\alpha \in \mathbf{P}'_{members}\}$. Then, there exists some $\alpha \in \mathbf{P}'_{members}$ such that $g^\alpha \in N$.

Then,

$$\begin{aligned}\varphi(g)^\alpha &= \varphi(g^\alpha) \text{ since } \varphi \text{ is a } R\text{-homomorphism} \\ &= \psi(g^\alpha) \text{ since } \varphi(x) = \psi(x) \text{ for all } x \in N \\ &= \psi(g)^\alpha \text{ since } \psi \text{ is a } R\text{-homomorphism.}\end{aligned}$$

Thus, $\varphi(g)^\alpha = \psi(g)^\alpha$. Since H is P-local, then, H is $\mathscr{U}_{P'}$-group (by Proposition 4.4.3). Therefore, $\varphi(g) = \psi(g)$ and $\varphi = \psi$. □

4.6 P-Injective, P-Surjective and P-Isomorphism

In this section, we introduce the concepts of P-injective, P-surjective and P-isomorphism for a set of primes P in R. Firstly, we define when a R-homomorphism is P-injective. We will show that for a R-homomorphism between two P-local $\mathcal{N}_R$-groups that, being P-injective is equivalent to being R-injective. We will follow the same procedure for P-surjection. This section is based on work done in [13].

In the following, R will be consider to be a UFD binomial ring and P to be a set of primes in R.

Definition 4.6.1 (P-injective)
[13, Definition 4.1 (i)] Let G and H be $\mathcal{N}_R$-groups and let $\phi : G \to H$ be a R-homomorphism. The R-homomorphism ϕ is called a P-injective, if $Ker\phi = \{g \in G : g \text{ is a } P'\text{-torsion element } \} = \{g \in G : g^{\alpha} = 1 \text{ for some } \alpha \in \boldsymbol{P'}_{members}\}$.

The following lemma was given in [13, Proposition 4.2] without proof. Here we give its proof.

Lemma 4.6.2
[13, Proposition 4.2] Let G, H and K be $\mathcal{N}_R$-groups. If $\varphi : G \to H$ and $\psi : H \to K$ are P-injective R-homomorphisms, then, $\psi \circ \varphi : G \to K$ is also P-injective.

Proof. Let G, H and K be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\psi : H \to K$ be P-injective R-homomorphisms.
Since $\varphi : G \to H$ and $\psi : H \to K$ are P-injective, we have:

($\star$) $Ker\varphi = \{g \in G \,|\, g$ is a P'-torsion element$\}$ and

($\star\star$) $Ker\psi = \{h \in H \,|\, h$ is a P'-torsion element$\}$.

We will show that $Ker(\psi \circ \varphi) = \{g \in G \,|\, g$ is a P'-torsion element$\}$.
If $x \in Ker(\psi \circ \varphi)$, then $\psi \circ \varphi(x) = 1$ and hence, $\varphi(x) \in Ker\psi$.
By ($\star\star$), $\varphi(x) \in Ker\psi$ and $\varphi(x)$ is a P'-torsion element.
Then, there exists $\alpha \in \mathbf{P'}_{members}$ such that $\varphi(x)^{\alpha} = 1$ and $\varphi(x^{\alpha}) = \varphi(x)^{\alpha} = 1$, since φ is a R-homomorphism. Therefore $x^{\alpha} \in Ker\varphi$.
By ($\star$), $x^{\alpha} \in Ker\varphi$ and x^{α} is a P'-torsion element.
Then there exists $\beta \in \mathbf{P'}_{members}$ such that $(x^{\alpha})^{\beta} = 1$ and $\alpha\beta \in \mathbf{P'}_{members}$ with $x^{\alpha\beta} = (x^{\alpha})^{\beta} = 1$. Therefore, x is P'-torsion element and $x \in \{g \in G \,|\, g$ is a P'-torsion element$\}$. Hence, $Ker(\psi \circ \varphi) \subseteq \{g \in G \,|\, g$ is a P'-torsion element$\}$.

If $y \in \{g \in G \,|\, g$ is a P'-torsion element$\}$, then $y \in G$ and is a P'-torsion element. By ($\star$), $y \in Ker\varphi = \{g \in G \,|\, g$ is a P'-torsion element$\}$.
In addition, $\varphi(y) \in H$. Since y is a P'-torsion element, then, there exists $\omega \in \mathbf{P'}_{members}$ such that $y^{\omega} = 1$. Since ψ is a R-homomorphism, we have $\psi(y)^{\omega} = \psi(y^{\omega}) = \psi(1) = 1$. Therefore, $\varphi(y) \in H$ and $\varphi(y)$ is a P'-torsion element. Consequently, we have, by ($\star\star$), that $\varphi(y) \in \{h \in H \,|\, h$ is a P'-torsion element$\} = Ker\psi$.
Since $\varphi(y) \in Ker\psi$, then $\psi \circ \varphi(y) = \psi[\varphi(y)] = 1$.
Therefore, $y \in Ker(\psi \circ \varphi)$ and $\{g \in G \,|\, g$ is a P'-torsion element$\} \subseteq Ker(\psi \circ \varphi)$.
Thus, $Ker(\psi \circ \varphi) = \{g \in G \,|\, g$ is a P'-torsion element$\}$ and $\psi \circ \varphi$ is P-injective. □

The following lemma was given in [13, Lemma 4.5 (i)] without proof. Here we give its proof.

Lemma 4.6.3

[13, Lemma 4.5 (i)] Let G and H be P-local $\mathcal{N}_R$-groups. A R-homomorphism $\phi : G \to H$ is P-injective if and only if $\phi : G \to H$ is R-injective.

Proof. Let G and H be P-local $\mathcal{N}_R$-groups and let $\phi : G \to H$ be R-homomorphism.
If $\phi : G \to H$ is P-injective, then $Ker\phi = \{g \in G \,|\, g \text{ is a } P'\text{-torsion element}\}$.
Since G is P-local, we have by Corollary 4.4.4, that, G is a P'-torsion-free group.
Therefore, the only P'-torsion element is 1, $Ker\phi = \{1\}$ and by Lemma 3.3.3, $\phi : G \to H$ is R-injective.

Conversely, assume that $\phi : G \to H$ is R-injective. We need to show that $Ker\phi = \{g \in G \,|\, g$ is a P'-torsion element$\}$.
Firstly, ϕ is R-injective and by Lemma 3.3.3, $Ker\phi = \{1\}$. However, 1 is a P'-torsion element. Hence, $Ker\phi \subseteq \{g \in G \,|\, g \text{ is a } P'\text{-torsion element}\}$.
If $x \in \{g \in G \,|\, g$ is a P'-torsion element$\}$, then, there exists $\alpha \in \mathbf{P'}_{members}$ such that $x^\alpha = 1$ and we have $\phi(x)^\alpha = \phi(x^\alpha) = \phi(1) = 1 = 1^\alpha = \phi(1^\alpha) = \phi(1)^\alpha$.
Since H is P-local (in particular a $\mathscr{U}_{P'}$-group), we have $\phi(x) = \phi(1)$ and ϕ is a R-injective. Therefore, $x = 1$ and $x \in Ker\phi = \{1\}$. Consequently, $\{g \in G \,|\, g$ is a P'-torsion element$\} \subseteq Ker\phi$, $Ker\phi = \{g \in G \,|\, g$ is a P'-torsion element$\}$ and ϕ is a P-injective. □

Similarly, we define a P-surjective R-homomorphism and show that the composition of two P-surjective R-homomorphisms is still P-surjective. We show that for a R-homomorphism between two P-local $\mathcal{N}_R$-groups that, being P-surjective is equivalent to being R-surjective.

Definition 4.6.4 (P-Surjective)
[13, Definition 4.1 (ii)] Let G and H be $\mathcal{N}_R$-groups. A R-homomorphism $\phi : G \to H$ is called P-surjective if, for every $h \in H$, there exists $\alpha \in \mathbf{P'}_{members}$ such that $h^\alpha \in Im(\phi)$.

The following lemma was given in [13, Proposition 4.2] without proof. Here we give its proof.

Lemma 4.6.5

[13, Proposition 4.2] Let G, H and K be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\psi : H \to K$ be R-homomorphisms. If $\varphi : G \to H$ and $\psi : H \to K$ are P-surjective, then $\psi \circ \varphi : G \to K$ is also P-surjective.

Proof. Let G, H and K be $\mathcal{N}_R$-groups and let $\varphi : G \to H$ and $\psi : H \to K$ be P-surjective R-homomorphisms.
For any $k \in K$, there exists $\alpha \in \mathbf{P'}_{members}$ such that $k^\alpha \in Im(H)$ since $\psi : H \to K$ is P-surjective.
Now, $k^\alpha \in Im(\psi)$ and $k^\alpha = \psi(h)$ for some $h \in H$.
Since $\varphi : G \to H$ is P-surjective, there exists $\beta \in \mathbf{P'}_{members}$ such that $h^\beta \in Im(\varphi)$ and $h^\beta = \varphi(g)$ for some $g \in G$.
Using the above and the fact that the product of two P-members is again a P-member (Lemma 2.8.7), we have $k^\alpha = \psi(h) \Rightarrow (k^\alpha)^\beta = \psi(h)^\beta \Rightarrow (k^\alpha)^\beta = \psi(h^\beta) \Rightarrow (k^\alpha)^\beta = \psi(\varphi(g)) \Rightarrow k^{\alpha\beta} = (k^\alpha)^\beta = \psi(\varphi(g)) = \psi \circ \varphi(g) \Rightarrow k^{\alpha\beta} \in Im(\psi \circ \varphi) \Rightarrow k^\omega \in Im(\psi \circ \varphi)$ for some $\omega \in \mathbf{P'}_{members}$. Hence, for any $k \in K$, there exists $\omega \in \mathbf{P'}_{members}$ such that $k^\omega \in Im(\psi \circ \varphi)$. Thus, $\psi \circ \varphi$ is P-surjective. □

The following lemma was given in [13, Lemma 4.5 (ii)] without proof. Here we give its proof.

Lemma 4.6.6

[13, Lemma 4.5 (ii)] Let G and H be P-local $\mathcal{N}_R$-groups and let $\phi : G \to H$ be a R-homomorphism. Then $\phi : G \to H$ is P-surjective if and only if $\phi : G \to H$ is R-surjective.

Proof. Let G and H be P-local $\mathcal{N}_R$-groups and let $\phi : G \to H$ be a R-homomorphism.
Assume that $\phi : G \to H$ is P-surjective.
Since G and H are P-local, we have by Proposition 4.4.3, that G and H are P'-radicable $\mathscr{U}_{P'}$-groups. For any $h \in H$, there exists $\alpha \in \mathbf{P}'_{members}$ such that $h^\alpha \in Im(\phi)$ and $h^\alpha = \phi(x)$ for some $x \in G$, since ϕ is P-surjective. Since G is a P'-radicable, then x has an α^{th}-root, say, $x = y^\alpha$ for some $y \in G$ and $h^\alpha = \phi(x) = \phi(y^\alpha) = \phi(y)^\alpha$. Now, H is a $\mathscr{U}_{P'}$-group and by Proposition 4.2.6, we have $h = \phi(y)$.
Therefore, there exists $y \in G$, such that $h = \phi(y)$ and $\phi : G \to H$ is R-surjective.
Conversely, assume that $\phi : G \to H$ is R-surjective and let $h \in H$. We need to show that $h^\alpha \in Im(\phi)$ for some $\alpha \in \mathbf{P}'_{members}$. Since, $h \in H$ and $\phi : G \to H$ is R-surjective then, there exists $g \in G$ such that $h = \phi(g)$. Then, $h^\alpha = \phi(g)^\alpha = \phi(g^\alpha) \in Im(\phi)$ for any $\alpha \in \mathbf{P}'_{members}$ since ϕ is a R-homomorphism. Therefore, $\phi : G \to H$ is P-surjective. □

We take note of the following lemma that, if a composition of two R-homomorphisms is P-surjective, then one of the R-homomorphisms is also P-surjective.

Lemma 4.6.7

[13, Proposition 4.4] Let G, H and K be $\mathcal{N}_R$-groups and let $\phi : G \to H$ and $\psi : H \to K$ be R-homomorphisms. If $\psi \circ \phi : G \to K$ is P-surjective, then $\psi : H \to K$ is also P-surjective.

Proof. Let G, H and K be $\mathcal{N}_R$-groups and let $\phi : G \to H$ and $\psi : H \to K$ be R-homomorphisms. Let $\psi \circ \phi : G \to K$ be P-surjective. Since, $\psi : G \to K$ is P-surjective, for all $k \in K$, there exists $\alpha \in \mathbf{P}'_{members}$ such that $k^\alpha \in Im(\psi \circ \phi)$.
Since, $k^\alpha \in Im(\psi \circ \phi)$, then, $k^\alpha = \psi \circ \phi(x)$ for some $x \in G$.
Therefore, $k^\alpha = \psi \circ \phi(x) = \psi(\phi(x)) \in Im(\psi) = \{\psi(h) : h \in H\}$.
Thus, for any $k \in K$, there exists $\alpha \in \mathbf{P}'_{members}$, such that $k^\alpha \in Im(\psi)$. Hence, ψ is a P-surjective. □

Lastly, we define P-isomorphisms and collect some results to conclude that for a R-homomorphism between two P-local $\mathcal{N}_R$-groups that, being P-isomorphism is equivalent to being R-isomorphism.

Definition 4.6.8 (P-Isomorphism)

[13, Definition 4.1 (iii)] Let G and H be $\mathcal{N}_R$-groups. A R-homomorphism $\phi : G \to H$ is called a P-isomorphism, if it is both P-injective and P-surjective.

Corollary 4.6.9

[13, Proposition 4.2] Let G, H and K be $\mathcal{N}_R$-groups. If $\varphi : G \to H$ and $\psi : H \to K$ are P-isomorphisms, then $\psi \circ \varphi : G \to K$ is also a P-isomorphism.

Proof. Follows from Lemma 4.6.2 and Lemma 4.6.5. □

Corollary 4.6.10

Let G and H be P-local $\mathcal{N}_R$-groups. A R-homomorphism $\phi : G \to H$ is a P-isomorphism if and only if $\phi : G \to H$ is a R-isomorphism.

Proof. Follows from Lemma 4.6.3 and Lemma 4.6.6. □

4.7 Fundamental Theorem of P-localization

In this section, we will show that every $\mathcal{N}_R$-group has a P-localization map which is unique up to R-isomorphisms. This section follows the work done in [13] on the P-localization of a $\mathcal{N}_R$-group and the inspiration is taken from [7] where P-localization of ordinary nilpotent groups was discussed.

Firstly, we start by defining the P-localization map of a $\mathcal{N}_R$-group.

Definition 4.7.1 (P-localization)
[13, Definition 5.1] Let R be a UFD binomial ring, G and G_P be $\mathcal{N}_R$-groups and let P be a set of primes in R. A R-homomorphism $e : G \to G_P$ is called a P-localization map of G, if

- *(i) G_P is a P-local $\mathcal{N}_R$-group and*
- *(ii) for any P-local $\mathcal{N}_R$-group K and a R-homomorphism $\psi : G \to K$, there exists a unique R-homomorphism $\alpha : G_P \to K$ such that $\psi = \alpha \circ e$.*

Note 4.7.2
Property (ii) is called the universal property of a P-localization map and is pictorially represented in the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{e} & G_P \\ & \searrow^{\psi} & \downarrow{\scriptstyle \alpha} \\ & & K \end{array}$$

Now, we prove the Fundamental Theorem of P-localization of a $\mathcal{N}_R$-group. The proof is an expanded version of the one given in [13, Theorem 5.3].

Theorem 4.7.3 (Fundamental Theorem of P-Localization)
[13, Theorem 5.3] Let R be a UFD binomial ring and let G be a $\mathcal{N}_R$-group. For any set of primes P in R there exists a P-localization map of G. In addition, we have the following:

- *(a) the P-localization map of G is P-surjective,*
- *(b) if G has nilpotency class c, then G_P has nilpotency class of at most c and*
- *(c) if G is finitely R-generated, then G_P is also finitely R-generated.*

Proof. Let G be a $\mathcal{N}_R$-group with R being a UFD binomial ring.
First, we define the pair $S_i = (H_i, \psi_i)$ as follows:

- (i) H_i is P-local and
- (ii) $\psi_i : G \to H_i$ is a R-homomorphism such that $H_i = \langle Im(\psi_i)\rangle_R^{P\text{-local}}$.

Let $\mathbb{S} = \{S_i : i \in I\} = \{(H_i, \psi_i) : i \in I\}$ be the set of pairs as defined above. We show that $\mathbb{S}$ is non-empty. We will do this by showing that $H_1 = \{1\}$ and $\psi_1 : G \to \{1\}$ defined as $\psi_1(g) = 1$ satisfies the condition above.

By Corollary 3.1.6, $1^\alpha = 1$ for any $\alpha \in \mathbf{P}'_{members}$ and $H_1 = \{1\}$ is a P'-radicable group. In addition, if 1 had two α^{th}-roots we would have $x^\alpha = 1 = y^\alpha$ for some $x, y \in H_1$. However, $H_1 = \{1\}$ thus, $x = y = 1$ and H_1 is a $\mathscr{U}_{P'}$-group. Since $H_1 = \{1\}$ is P'-radicable $\mathscr{U}_{P'}$-group, we have, by Proposition 4.4.3, that $H_1 = \{1\}$ is P-local.

Now, $\psi_1(gh) = 1 = 1 \cdot 1 = \psi_1(g)\psi_1(h)$ and $\psi(g^\omega) = 1 = 1^\omega = \psi(g)^\omega$, for any $g, h \in G$ and $\omega \in R$, so, ψ_1 is a R-homomorphism.

By Corollary 4.5.6, we have $\langle H_1 \rangle_R^{P\text{-local}} = \{g \in G : g^\alpha \in H_1 \text{ for some } \alpha \in \mathbf{P}'_{members}\}$. However, $g^\alpha \in H_1 = \{1\}$ implies that $g^\alpha = 1 = 1^\alpha$. Since H_1 is a $\mathscr{U}_{P'}$-group, we have $g = 1$.
Therefore, $\langle H_1 \rangle_R^{P\text{-local}} = \{g \in G : g^\alpha \in H_1 \text{ for some } \alpha \in \mathbf{P}'_{members}\} = \{1\} = H_1$.

Consequently, there exists at least one $S_1 = (H_1, \psi_1)$ for any $\mathcal{N}_R$-group and $\mathbb{S}$ is non-empty.

We define $H = \prod_{i \in I} H_i$ to be the direct product of each H_i and let $e : G \to H$ be given by $e(g) = (\psi_1(g), \psi_2(g), \ldots, \psi_k(g), \ldots)$.

We show that $e : G \to H$ is a R-homomorphism. For any $g, h \in G$

$$\begin{aligned} e(gh) &= (\psi_1(gh), \psi_2(gh), \ldots, \psi_k(gh), \ldots) \\ &= (\psi_1(g)\psi_1(h), \psi_2(g)\psi_2(h), \ldots, \psi_k(g)\psi_k(h), \ldots) \text{ since each } \psi_i \text{ are } R\text{-homomorphisms} \\ &= (\psi_1(g), \psi_2(g), \ldots, \psi_k(g), \ldots)(\psi_1(h), \psi_2(h), \ldots, \psi_k(h), \ldots) \\ &= e(g)e(h) = \phi(g)\phi(h). \end{aligned}$$

In addition, for any $g \in G$ and $\alpha \in R$, we also have:

$$\begin{aligned} e(g^\alpha) &= (\psi_1(g^\alpha), \psi_2(g^\alpha), \ldots, \psi_k(g^\alpha), \ldots) \\ &= (\psi_1(g)^\alpha, \psi_2(g)^\alpha, \ldots, \psi_k(g)^\alpha, \ldots) \text{ since each } \psi_i \text{ are } R\text{-homomorphisms} \\ &= (\psi_1(g), \psi_2(g), \ldots, \psi_k(g), \ldots)^\alpha \\ &= e(g)^\alpha. \end{aligned}$$

Hence, $e : G \to H$ is a R-homomorphism.

Let $G_P = \langle Im(e) \rangle_R^{P-Local}$. We will show that $e : G \to G_P$, defined as $e(g) = (\psi_1(g), \psi_2(g), \ldots, \psi_k(g), \ldots)$ is a P-localization map.

By Proposition 4.5.1, G_P is P-local, since G_P is R-generated as a P-local group. We only need to show that the universal property of a P-localization map hold for $e : G \to G_P$. We will show that, for any P-local $\mathcal{N}_R$-group K and R-homomorphism $\varphi : G \to K$, there exists a unique R-homomorphism $\alpha : G_P \to K$ such that $\varphi = \alpha \circ e$.

Let K be a P-local $\mathcal{N}_R$-group and let $\varphi : G \to K$ be a R-homomorphism.

Note that $(\langle Im(\varphi) \rangle_R^{P-Local}, \varphi) \in \mathbb{S}$, so, there exists $k \in I$ such that $(\langle Im(\varphi) \rangle_R^{P-Local}, \varphi) = (H_k, \psi_k)$. Define $\mu : G_P \to K$ as $\mu(g_1, g_2, \ldots g_k, \ldots,) = g_k$. Then, we have $(\mu \cdot e)(g) = \mu(e(g)) = \mu((\psi_1(g), \psi_2(g), \ldots, \psi_k(g), \ldots$ $\psi_k(g) = \varphi(g)$.

Therefore, for any P-local $\mathcal{N}_R$-group K and R-homomorphism $\varphi : G \to K$, there exists a R-homomorphism $\mu : G_P \to K$ such that $\mu \cdot e = \varphi$.

For the uniqueness, assume that there is another map $\zeta : G_P \to K$ such that $\zeta \cdot e = \varphi$. We show that $\mu = \zeta$.

For any $g \in G$, $\mu \cdot e(g) = \varphi(g) = \zeta \cdot e(g)$ implies that $\mu(e(g)) = \zeta(e(g))$. Therefore, $\mu(x) = \zeta(x)$ for all

$x \in Im(e)$. In addition, $G_P = \langle Im(e) \rangle_R^{P-Local}$ and we have $\mu = \zeta$, by Lemma 4.5.8.
Therefore, for any P-local $\mathcal{N}_R$-group K and R-homomorphism $\varphi : G \to K$, there exists a unique R-homomorphism $\mu : G_P \to K$ such that $\mu \cdot e = \varphi$.
Thus, $e : G \to G_P$ defined as $e(g) = (\psi_1(g), \psi_2(g), \ldots, \psi_k(g), \ldots)$ is a P-localization map of G.

Lastly,

(a) We will show that the P-localization map $e : G \to G_P$ is P-surjective. Since, $Im(e)$ is a R-subgroup of $G_P = \langle Im(e) \rangle_R^{P-Local}$, we have, by Corollary 4.5.6, that $\langle Im(e) \rangle_R^{P\text{-local}} = \{g \in G_P : g^\alpha \in Im(e) \text{ for some } \alpha \in \mathbf{P}'_{members}\}$. Then, for any $g \in G_P$, there exists $\alpha \in \mathbf{P}'_{members}$ such that $g^\alpha \in Im(e)$. Hence, $e : G \to G_P$ is P-surjective.

(b) Assume that G has nilpotency class c. Let $\phi : G \to Im(e)$ be defined as $\phi(g) = e(g)$. Now, ϕ is a surjective R-homomorphism and by Proposition 2.5.5, we have $Im(e)$ is a nilpotent group with nilpotency class of at most c. In addition, $G_P = \langle Im(e) \rangle_R^{P-Local}$, so, by Lemma 4.5.7, G_P and $Im(e)$ have the same nilpotency class.
Thus, G_P is of nilpotency class of at most c.

(c) Assume that G is finitely R-generated. Since $e : G \to G_P$ is a R-homomorphism, we have by Lemma 3.3.6, that $Im(e)$ is finitely R-generated as a P-local group. Therefore, there exists a finite subset $S \subseteq Im(e)$ such that $Im(e) = \langle S \rangle_R$. Then, $G_P = \langle Im(e) \rangle_R^{P\text{-local}} = \langle \langle S \rangle_R \rangle_R^{P\text{-local}}$. By Lemma 4.5.3, we have $\langle \langle S \rangle_R \rangle_R^{P\text{-local}} = \langle S \rangle_R^{P\text{-local}}$. Thus, $G_P = \langle S \rangle_R^{P\text{-local}}$ and G_P is finitely R-generated, since S is finite.

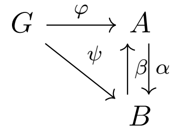
□

Now, we prove that the P-localization map is unique up to R-isomorphisms. We will do this by showing that if there exist two localization maps on the same $\mathcal{N}_R$-group, then the P-localizations must be R-isomorphic. Further, if there is a group which is R-isomorphic to the P-localization, then it also a P-localization.

For the remainder of this section, R is a unique factorization domain (UFD) binomial ring and P is a set of primes in R.

Proposition 4.7.4
Let G be a $\mathcal{N}_R$-group. If $\varphi : G \to A$ and $\psi : G \to B$ are P-localization maps of G then, $A \cong_R B$.

Proof. Consider the following diagram:



Let $\varphi : G \to A$ and $\psi : G \to B$ be P-localization maps of G. By Definition 4.7.1, we have that A and B are P-local. Since B is P-local and $\psi : G \to B$ is a R-homomorphism, then, by the universal property of φ, there exists $\alpha : A \to B$ such that $\psi = \alpha \circ \varphi$.
Similarly, since A is P-local and there is a R-homomorphism $\varphi : G \to A$, then, by the universal property of ψ, there exists $\beta : B \to A$ such that $\varphi = \beta \circ \psi$.

Thus, $\psi = \alpha \circ \varphi = \alpha \circ (\beta \circ \psi) = (\alpha \circ \beta) \circ \psi$. Since $\psi = (\alpha \circ \beta) \circ \psi$, we conclude that $\alpha \circ \beta = I_A$.
Similarly, $\varphi = \beta \circ \psi = \beta \circ \alpha \circ \varphi = (\beta \circ \alpha) \circ \varphi$. Since $\varphi = (\beta \circ \alpha) \circ \varphi$, then $\beta \circ \alpha = I_B$.

Therefore, $\alpha \circ \beta = I_A$ and $\beta \circ \alpha = I_B$ and α and β are invertible.
Consequently, α and β are bijections. Therefore, $\alpha : A \to B$ is a R-isomorphism and $A \cong_R B$. □

The, next proposition informs us that if any P-local $\mathcal{N}_R$-group is R-isomorphic to another $\mathcal{N}_R$-group's P-localization, then we can construct a new P-localization map. This, together with Proposition 4.7.4 allow us to conclude that a P-localization is unique up to R-isomorphisms.

Proposition 4.7.5
Let G be a $\mathcal{N}_R$-group. Let $e : G \to G_P$ be a P-localization map of G.
If there exists a P-local $\mathcal{N}_R$-group Q such that $Q \cong_R G_P$, then there exists a P-localization map $\phi : G \to Q$.

Proof. Let G be a $\mathcal{N}_R$-group and let $e : G \to G_P$ be a P-localization map of G.
Let $\sigma : G_P \to Q$ be a R-isomorphism. We will show that the map given by $\sigma \circ e : G \to Q$ is a P-localization map.
By assumption Q is P-local, so we only need to show the universal property of a P-localization map.
Let K be a P-local $\mathcal{N}_R$-group and let $\psi : G \to K$ be a R-homomorphism. Since $e : G \to G_P$ is a P-localization map of G, then, there exists a unique R-homomorphism $\theta : G_P \to K$ such that $\psi = \theta \circ e$.
We define $\theta_Q = \theta \circ \sigma^{-1}$ and we have $\theta_Q \circ (\sigma \circ e) = (\theta \circ \sigma^{-1}) \circ (\sigma \circ e) = \theta \circ (\sigma^{-1} \circ \sigma) \circ e = \theta \circ e = \psi$. Since θ and σ^{-1} is unique and the composite of two unique maps is unique, then we must that $\theta_Q = \theta \circ \sigma^{-1}$ is unique.

Therefore, for any P-local group K and $\psi : G \to K$ R-homomorphism, there exists a unique R-homomorphism θ_Q, such that $\theta_Q \circ (\sigma \circ e) = \psi$ and $\sigma \circ e : G \to Q$ is a P-localization map of G. □

We have shown that the P-localization map $e : G \to G_P$ of a $\mathcal{N}_R$-group is unique up to R-isomorphisms. Since G_P is unique up to R-isomorphisms, we call G_P the P-localization of G and if $P = \{p\}$, we write G_P as G_p.

4.8 P-Localization in a PID

In this section, we relate P-Localization and P-Isomorphism. We will show that, in a unique factorization domain, if a map is P-isomorphic, then it is a P-localization map. However, the converse is only true in a principal ideal domain which has a subring isomorphic to the set of rational numbers.

Proposition 4.8.1

Let R be a UFD binomial ring, G and Q be $\mathcal{N}_R$-groups and let P be a set of primes in R. If Q is P-local and $\phi : G \to Q$ is a P-isomorphism, then $\phi : G \to Q$ is a P-localization map of G.

Proof. Let G and Q be $\mathcal{N}_R$-groups, Q be P-local and let $\phi : G \to Q$ be a P-isomorphism.
By the Fundamental Theorem of P-localization of a $\mathcal{N}_R$-groups, there exists a P-localization map $e : G \to G_P$. We show that $Q \cong_R G_P$ and use Proposition 4.7.5 to conclude that ϕ is a P-localization map. Since Q and G_P are both P-local, by Corollary 4.6.10, we only need to show that Q and G_P are P-isomorphic in order to conclude that Q and G_P are R-isomorphic.

Now, Q is P-local and $\phi : G \to Q$ is a R-homomorphism, so, by the universal property of $e : G \to G_P$, there exists a unique R-homomorphism $\theta : G \to G_P$, such that $\phi = \theta \circ e$.
By the Fundamental Theorem of P-localization, we have e is P-surjective, since the P-localization map is always P-surjective.
Since, $\phi : G \to Q$ is a P-isomorphism, then ϕ is P-injective and P-surjective.
Now, $\phi = \theta \circ e$ is P-surjective, so, by Lemma 4.6.7, we must have that θ is P-surjective.

In order to show that $\theta : G_P \to Q$ is P-injective, we need to show that $Ker(\theta) = \{v \in G_P : v$ is a P'-torsion element$\}$. If $x \in Ker(\theta)$, then, $\theta(x) = 1$. Since $e : G \to G_P$ is P-surjective and $x \in G_P$, then there exists $\alpha \in \mathbf{P'}_{members}$ such that $x^\alpha \in Im(e)$. Therefore, $x^\alpha = e(g)$ for some $g \in G$ and we have the following:

$$\begin{aligned} \theta \circ e(g) = \theta(e(g)) = \theta(x^\alpha) &= (\theta(x))^\alpha \text{ since } \theta \text{ is a } R\text{-homomorphism} \\ &= 1^\alpha \text{ since } x \in Ker(\theta) \\ &= 1. \end{aligned}$$

Therefore, $g \in Ker(\theta \circ e)$. However, $\theta \circ e$ is P-injective, then, we must have $g \in Ker(\theta \circ e) = \{y \in G : y$ is a P'-torsion element$\}$.
Since g is a P'-torsion element, then, there exists $\beta \in \mathbf{P'}_{members}$ such that $g^\beta = 1$.
Therefore,

$$\begin{aligned} x^{\alpha\beta} = (x^\alpha)^\beta &= (e(g))^\beta \text{ since } x^\alpha = e(g) \\ &= e(g^\beta) \text{ since } e \text{ is } R\text{-homomorphism} \\ &= e(1) = 1. \end{aligned}$$

In addition, $\alpha, \beta \in \mathbf{P'}_{members}$, so $\alpha\beta \in \mathbf{P'}_{members}$ and x is a P'-torsion element and $Ker(\theta) \subseteq \{v \in G_P : v$ is a P'-torsion element$\}$.

Let $y \in \{v \in G_P : v$ is a P'-torsion element$\}$. Since y is a P'-torsion element, then there exists $\omega \in \mathbf{P'}_{members}$ such that $y^\omega = 1$ and we have $\theta(y)^\alpha = \theta(y^\alpha) = \theta(1) = 1 = 1^\alpha$.

Therefore, $\theta(y)^\alpha = 1^\alpha$. However, Q is P-local, so Q is a $\mathscr{U}_{P'}$-group and by Proposition 4.2.6, we have

$\theta(y)^\alpha = 1^\alpha$ implies that $\theta(y) = 1$.
Thus, $y \in Ker(\theta)$, $\{v \in G_P : v \text{ is a } P'\text{-torsion element}\} \subseteq Ker(\theta)$ and θ is P-injective.

Consequently, $\theta : G_P \to Q$ is P-injective and P-surjective and $\theta : G_P \to Q$ is a P-isomorphism.

However, G_P and Q are P-local groups, so, by Corollary 4.6.10, $G_P \cong_R Q$ and by Proposition 4.7.5, $\theta : G \to Q$ is a P-localization map. □

The converse of Proposition 4.8.1 is still an open problem for a general UFD ring R and a general $\mathcal{N}_R$-group and is given as a conjecture in [13, Conjecture 6.1]. For the case when the ring R is a principal ideal domain which has a subring isomorphic to the set of rational numbers and the $\mathcal{N}_R$-group G is finitely R-generated, we have that the converse is true. This extra constraint is needed as the authors in [13] needed this constraint to prove that if G is finitely R-generated P'-torsion-free then it is residually of finite P-type.

Theorem 4.8.2

[13, Theorem 5.10] Let R be a principal ideal domain (PID) which has a subring isomorphic to the set of rational numbers and let G be a finitely R-generated $\mathcal{N}_R$-group. Then the P-localization map $e : G \to G_P$ is a P-isomorphism.

Proof. Let G be a finitely R-generated $\mathcal{N}_R$-group.

Let $e : G \to G_P$ be the P-localization map. By Definition 4.7.1, G_P is P-local, thus, we only need to show that e is a P-isomorphism.

By the Fundamental Theorem of P-localization (Theorem 4.7.3), e is P-surjective.

We need to show $Ker(e) = \{g \in G : g^\alpha = 1 \text{ for some } \alpha \in \mathbf{P}'_{members}\}$.

Let $x \in \{g \in G : g^\alpha \text{ for some } \alpha \in \mathbf{P}'_{members}\}$. Then, there exists $\alpha \in \mathbf{P}'_{members}$ such that $x^\alpha = 1$. Then, $e(x)^\alpha = e(x^\alpha) = e(1) = 1 = 1^\alpha$ and $e(x) = 1$ since G_P is P-local. Therefore, $x \in Ker(e)$ and $\{g \in G : g^\alpha = 1 \text{ for some } \alpha \in \mathbf{P}'_{members}\} \subseteq Ker(e)$.

Let $T = \{g \in G : g^\alpha = 1 \text{ for some } \alpha \in \mathbf{P}'_{members}\}$. Since $T \subseteq Ker(e)$, we have, by the Universal Property of Quotient Groups (Theorem 3.3.10), that there exists a R-homomorphism $\overline{e} : G/T \to G_P$, defined as $\overline{e}(gT) = e(T)$ such that $e = \overline{e} \circ \pi$ where $\pi : G \to G/T$ is the canonical map.

Let $gT \in G/T$ be a P'-torsion element. Then there exists $\alpha \in \mathbf{P}'_{members}$ such that $(gT)^\alpha = T$ and we have $g^\alpha T = (gT)^\alpha = T$. Therefore, $g^\alpha \in T = \{g \in G : g^\alpha = 1 \text{ for some } \alpha \in \mathbf{P}'_{members}\}$ and there exists $\beta \in \mathbf{P}'_{members}$ such that $g^{\alpha\beta} = (g^\alpha)^\beta = 1$. By Lemma 2.8.7, $\alpha\beta \in \mathbf{P}'_{members}$, so, $g \in T$ and $gT = T$. Since gT was arbitrarily chosen, the only P'-torsion element in G/T is the identity T. Therefore, by Definition 4.2.2, G/T is P'-torsion-free.

Now, G/T is P'-torsion-free and, by Theorem 4.3.5 we have that G/T is residually of finite P-type. Then, by Lemma 4.3.4, G/T is a subdirect product of $\mathcal{N}_R$-groups of finite P-type, say $\{H_j : j \in I\}$. So, by Definition 4.3.3, G/T is a R-subgroup of $H_1 \times H_2 \times \cdots \times H_k \times \ldots$ such that for each $j \in I$:

(*i*) H_j is of finite P-type and

(*ii*) $\pi_j(G/T) = H_j$.

By Lemma 4.4.8, each H_j is P-local and by Lemma 4.4.5, $H_1 \times H_2 \times \cdots \times H_k \times \cdots$ is P-local.

Let $H = H_1 \times H_2 \times \cdots \times H_k \times \cdots$. Let $i : G/T \to H$ be the inclusion map (since G/T is a R-subgroup of H). Then, we have a R-homomorphism $i \circ \pi : G \to H$ from G to a P-local group H and by the universal property of the P-localization map e (Definition 4.7.1), there exists a R-homomorphism $\phi : G_P \to H$ such that $i \circ \pi = \phi \circ e$.

However, $e = \overline{e} \circ \pi$ so, $i \circ \pi = \phi \circ \overline{e} \circ \pi$ and $i = \phi \circ \overline{e}$.

Since, i is R-injective, we have $\phi \circ \overline{e}$ is R-injective. By Lemma 3.3.17, we have $\overline{e}$ is R-injective and by Lemma 3.3.3, we have $Ker(\overline{e}) = \{T\}$.

If $y \in Ker(e)$, then, $\overline{e}(yT) = e(y) = 1$ and $yT \in Ker(\overline{e}) = \{T\}$ so, $yT = T$ and $y \in T$. Therefore, $Ker(e) \subseteq T$, e is P-injective and e is a P-isomorphism.
The following diagram is a visual representation of the proof:

$$\begin{array}{ccc} G & \xrightarrow{e} & G_P \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \overline{e}} & \downarrow{\scriptstyle \phi} \\ G/T & \xrightarrow{i} & H \end{array}$$

□

The following theorem is a combination of the results from Proposition 4.8.1 and Theorem 4.8.2.

Theorem 4.8.3
Let R be a principal ideal domain (PID) which has a subring isomorphic to the set of rational numbers and let G be a finitely R-generated $\mathcal{N}_R$-group. Let $\theta : G \to Q$ be a R-homomorphism. Then, $\theta : G \to Q$ is a P-localization map if and only if Q is P-local and $\theta : G \to Q$ is a P-isomorphism.

Proof. The proof follows from Proposition 4.8.1 and Theorem 4.8.2. □

4.9 Applications of P-Localization

In this section, we give some applications of P-localizations that will be needed in Chapter 5, where we give our major results on the description of the genus of a $\mathcal{N}_R$-group. Firstly, we will look at how the property of finite type affect the P-localization of a $\mathcal{N}_R$-group. We investigate the P-localization of a direct product of two finitely R-generated $\mathcal{N}_R$-groups. Lastly, we evaluate the P-localization of the commutator R-subgroup of a finitely R-generated $\mathcal{N}_R$-group.

For this section, R will be considered as a principal ideal domain (PID) with a subring isomorphic to the set of rational numbers and P is a set of primes in R.

Firstly, we show that, if G is of finite type then, G_P is also of finite type and the P-localization map is R-surjective.

Lemma 4.9.1
Let G be a $\mathcal{N}_R$-group. If G is of finite type then, the P-localization G_P is also of finite type.

Proof. Let G be a $\mathcal{N}_R$-group of finite type. By Definition 4.1.7, G is finitely R-generated and is a R-torsion group.
Let $e : G \to G_P$ be the P-localization map. Since G is finitely R-generated, by Fundamental Theorem of P-Localization (Theorem 4.7.3 (c)), we have G_P is also finitely R-generated.

Now, $e : G \to G_P$ is a P-localization map, then, by Theorem 4.8.3, e is P-isomorphism and for any $x \in G_P$ there exists $\alpha \in \mathbf{P}'_{members}$ such that $x^\alpha \in Im(e)$, since e is P-surjective. Hence, $x^\alpha = e(g)$ for some $g \in G$.
Furthermore, G is a R-torsion group, so there exists $\beta \in R$ such that $g^\beta = 1$ and $x^{\alpha\beta} = (x^\alpha)^\beta = (e(g))^\beta = e(g^\beta) = e(1) = 1$.

Therefore, for any $x \in G_P$ there exists $\alpha\beta \in R$ such that $x^{\alpha\beta} = 1$ and G_P is a R-torsion group.

Since G_P is finitely R-generated and a R-torsion group, then, by Definition 4.1.7, we have that G_P is of finite type. □

We have shown in the Fundamental Theorem of P-localization that the P-localization map is P-surjective. In the next proposition, we prove that the P-localization map of a $\mathcal{N}_R$-group of finite type is actually a R-surjective map.

Proposition 4.9.2
Let G be a $\mathcal{N}_R$-group and let $e : G \to G_P$ be the P-localization map. If G is of finite type, then, $e : G \to G_P$ is R-surjective.

Proof. Let G be a $\mathcal{N}_R$-group of finite type and let $e : G \to G_P$ be the P-localization map.

By Lemma 4.9.1, we have that G_P is of finite type so, G_P is a R-torsion group. For any $x \in G_P$ there exists $\alpha \in R$ such that $x^\alpha = 1$ since G_P is a R-torsion group.
Now, $e : G \to G_P$ is P-surjective so, there exists $\beta \in \mathbf{P}'_{members}$ such that $x^\beta \in Im(e)$. Consequently, there exists $g \in G$ such that $x^\beta = e(g)$.

We have the following cases:

Case (I): If $\alpha \notin \mathbf{P}'_{members}$ then, $\gcd(\alpha,\beta)=1$ and by Theorem 2.8.9, there exists $\mu,\omega \in R$ such that $1=\alpha\mu+\beta\omega$. So, $x = x^1 = x^{\alpha\mu+\beta\omega} = x^{\alpha\mu}x^{\beta\omega} = (x^\alpha)^\mu (x^\beta)^\omega = (1)^\mu (e(g))^\omega = e(g)^\omega = e(g^\omega)$ and there exists $g^\omega \in G$ such that $x = e(g^\omega)$.

Case (II): If $\alpha \in \mathbf{P}'_{members}$ then, $\alpha\beta \in \mathbf{P}'_{members}$ and we have that $x^{\alpha\beta} = (x^\alpha)^\beta = (1)^\beta = 1 = 1^{\alpha\beta} = e(1)^{\alpha\beta}$. Since G_P is P-local then, $x^{\alpha\beta} = e(1)^{\alpha\beta}$ implies $x = e(1)$.

Therefore, for any $x \in G_P$ there exists $g \in G$ such that $x = e(g)$ and e is R-surjective. □

In the next corollary, we will show that for any $\mathcal{N}_R$-group of finite type G and prime p in R, the p-localization G_p is R-isomorphic to the p-primary component of G.

Corollary 4.9.3
Let G be a $\mathcal{N}_R$-group of finite type and let p be a prime in R. Then, $G^{(p)} \cong_R G_p$, where $G^{(p)}$ is the p-primary component of G.

Proof. Let G be a $\mathcal{N}_R$-group of finite type. Let $e : G \to G_p$ be the p-localization map. By Proposition 4.9.2, e is R-surjective and by the First Isomorphism Theorem for R-Homomorphism (Theorem 3.3.7), $G/Ker(e) \cong_R G_p$.
Let P' be set the primes in R which are not p. By Theorem 4.8.3, we have e is P-injective and $Ker(e) = \{g \in G : g^\alpha = 1 \text{ for some } \alpha \in \mathbf{P}'_{members}\} = \tau_{P'}(G)$.

Hence, $G/\tau_{P'}(G) \cong_R G_p$ and by Lemma 4.2.14, $G/\tau_{P'}(G) \cong_R G^{(p)}$ and $G^{(p)} \cong_R G_p$. □

Now, we investigate the P-localization of a direct product of two finitely R-generated $\mathcal{N}_R$-groups.

Proposition 4.9.4
Let G and H be finitely R-generated $\mathcal{N}_R$-groups. Then, $(G\times H)_P \cong_R G_P \times H_P$.

Proof. Let G and H be finitely R-generated $\mathcal{N}_R$-groups.

By the fundamental theorem of P-localization of $\mathcal{N}_R$-groups, there exists P-localization maps $e_G : G \to G_P$ and $e_H : H \to H_P$. By Theorem 4.8.3, e_G and e_H are P-isomorphisms.

We define $e : G \times H \to G_P \times H_P$ by $e(g,h) = (e_G(g), e_H(h))$. We will show that this a P-localization map.
Let $(w,z) \in G_P \times H_P$. Since, $e_G : G \to G_P$ and $e_H : H \to H_P$ are P-surjective, then, there exists $\alpha,\beta \in \mathbf{P}'_{members}$ such that $w^\alpha \in Im(e_G)$ and $z^\beta \in Im(e_H)$. Then, $w^\alpha = e_G(g)$ and $z^\beta = e_H(h)$ for some $g \in G$ and $h \in H$. Therefore, $(w,z)^{\alpha\beta} = (w^{\alpha\beta}, z^{\alpha\beta}) = (w^{\alpha\beta}, z^{\beta\alpha}) = ((w^\alpha)^\beta, (z^\beta)^\alpha) = ((e_G(g))^\beta, (e_H(h))^\alpha) = (e_G(g^\beta), e_H(h^\alpha)) = e(g^\beta, h^\alpha) \in Im(e)$. Hence, for any $(w,z) \in G_P \times H_P$, there exists $\alpha\beta \in \mathbf{P}'_{members}$ such that $(w,z)^{\alpha\beta} \in Im(e)$ and $e : G\times H \to G_P \times H_P$ is P-surjective.

Now, we will show that e is P-injective, by showing that
$Ker(e) = \{(x,y) \in G \times H : (x,y) \text{ is a } P'\text{-torsion element}\}$.
Let $(g,h) \in Ker(e)$. Since $e(g,h) = (1,1)$, then, $(e_G(g), e_H(h)) = (1,1)$ and $e_G(g) = 1$ and $e_H(h) = 1$. Since, e_G and e_H are P-injective, we have that there exists $\alpha,\beta \in \mathbf{P}'_{members}$ such that $g^\alpha = 1$ and $h^\beta = 1$.
Then, $(g,h)^{\alpha\beta} = (g^{\alpha\beta}, h^{\alpha\beta}) = ((g^\alpha)^\beta, (h^\alpha)^\beta) = ((1)^\beta, (1)^\beta) = (1,1)$ and (g,h) is a P'-torsion element.

Hence, $(g,h) \in \{(x,y) \in G \times H : (x,y) \text{ is a } P'\text{-torsion element}\}$ and $Ker(e) \subseteq \{(x,y) \in G \times H : (x,y) \text{ is a } P'\text{-torsion element}\}$.

Let $(g_1, h_1) \in \{(x, y) \in G \times H : (x, y)$ is a P'-torsion element$\}$. Then, (g_1, h_1) is a P'-torsion element and there exists $\alpha \in \mathbf{P}'_{members}$ such that $(g_1, h_1)^\alpha = (1, 1)$. Then,

$$\begin{aligned}(g_1, h_1)^\alpha = (1,1) &\Rightarrow (g_1^\alpha, h_1^\alpha) = (1,1)\\ &\Rightarrow g_1^\alpha = 1 \text{ and } h_1^\alpha = 1\\ &\Rightarrow g_1 \text{ is a } P'\text{-torsion element and } h_1 \text{ is a } P'\text{-torsion element}\\ &\Rightarrow g_1 \in \{g \in G : g \text{ is a } P'\text{-torsion element }\} = Ker(e_G) \text{ and}\\ &\qquad h \in \{h \in H : h \text{ is a } P'\text{-torsion element }\} = Ker(e_H)\\ &\Rightarrow e_G(g_1) = 1 \text{ and } e_H(h_1) = 1\\ &\Rightarrow e(g_1, h_1) = (e_G(g_1), e_H(h_1)) = (1,1)\\ &\Rightarrow (g_1, h_1) \in Ker(e).\end{aligned}$$

Therefore, $\{(x, y) \in G \times H : (x, y)$ is a P'-torsion element$\} \subseteq Ker(e)$ and $Ker(e) = \{(x, y) \in G \times H : (x, y)$ is a P'-torsion element$\}$. Thus, e is a P-injective. By Lemma 4.4.5, $G_P \times H_P$ is P-local.

Since $G_P \times H_P$ is P-local and $e : G \times H \to G_P \times H_P$ is a P-isomorphism, then, by Theorem 4.8.3, $e : G \times H \to G_P \times H_P$ is a P-localization map.

Now, $G \times H$ is a $\mathcal{N}_R$-group so, by the fundamental theorem of P-localization of $\mathcal{N}_R$-groups, there exists a P-localization map $e' : G \times H \to (G \times H)_P$.
Consequently, by Proposition 4.7.4, we have that $(G \times H)_P \cong_R G_P \times H_P$. □

Lastly, we investigate the P-localization of the commutator R-subgroup of finite type.

Proposition 4.9.5
Let G be a finitely R-generated $\mathcal{N}_R$-group with a commutator R-subgroup $[G, G]$. If $[G, G]$ is of finite type then, $[G, G]_P = [G_P, G_P]$.

Proof. Let G be a finitely R-generated $\mathcal{N}_R$-group such that the commutator R-subgroup $[G, G]$ is of finite type. Let $e' : G \to G_P$ be the P-localization map of G and let $e : [G, G] \to [G, G]_P$ be the P-localization map by restriction e'. By Lemma 4.9.1, $[G, G]_P$ is of finite type.
In addition, by The Fundamental Theorem of P-localization (Theorem 4.7.3), $[G, G]_P = \langle e([G, G])\rangle_R^{P\text{-local}} = \langle [e'(G), e'(G)]\rangle_R^{P\text{-local}} = \langle [G_P, G_P]\rangle_R^{P\text{-local}}$.
By Lemma 3.2.10, $[G_P, G_P]$ is normal R-subgroup of G_P. Therefore, by Corollary 4.5.6, we have $\langle [G_P, G_P]\rangle_R^{P\text{-local}} = I_{P'}([G_P, G_P], G_P) = \{g \in G_P : g^\alpha \in [G_P, G_P]$ for some $\alpha \in \mathbf{P}'_{members}\}$. Let $V = \{g \in G_P : g^\alpha \in [G_P, G_P]$ for some $\alpha \in \mathbf{P}'_{members}\}$.

Then, $[G, G]_P = V$. Since $[G, G]_P$ is of finite type, then V is also of finite type.

Now, we show that $[G_P, G_P] = V$.

Let $x \in V$. Then, $x \in G_P$ and $x^\alpha \in [G_P, G_P]$ for some $\alpha \in \mathbf{P}'_{members}$. Since V is of finite type, then, there exists $\beta \in R$ such that $x^\beta = 1$. We have the following cases:

Case (I): If $\beta \notin \mathbf{P}'_{members}$, then $\gcd(\alpha, \beta) = 1$ and by Theorem 2.8.9, there exists $\mu, \omega \in R$ such that $1 = \alpha\mu + \beta\omega$. Then, $x = x^1 = x^{\alpha\mu+\beta\omega} = x^{\alpha\mu}x^{\beta\omega} = (x^\alpha)^\mu(x^\beta)^\omega = (x^\alpha)^\mu(1)^\omega = (x^\alpha)^\mu = (h)^\mu$ for some $h \in [G_P, G_P]$. Since, $[G_P, G_P]$ is a R-subgroup of G, we have $h^\mu \in [G_P, G_P]$.

Hence, $x \in [G_P, G_P]$ and $V \subseteq [G_P, G_P]$.

Case (II): If $\beta \in \mathbf{P}'_{members}$, then $\alpha\beta \in \mathbf{P}'_{members}$.

We show that $[G_P, G_P]$ is P-local. Firstly, by Proposition 4.4.7, $[G_P, G_P]$ is P-local if and only if $[G_P, G_P]$ is P'-isolated.

Let $g \in G_P$ and $\alpha \in \mathbf{P}'_{members}$ such that $g^\alpha \in [G_P, G_P]$. Assume that $g \notin [G_P, G_P]$. Then, $g \notin [G_P, G_P] \Rightarrow g[G_P, G_P] \neq [G_P, G_P] \Rightarrow (g[G_P, G_P])^\alpha \neq ([G_P, G_P])^\alpha \Rightarrow g^\alpha[G_P, G_P] \neq [G_P, G_P] \Rightarrow g^\alpha \notin [G_P, G_P]$. This ia a contradiction, so, $g \in [G_P, G_P]$. Therefore, $[G_P, G_P]$ is P'-isolated in G_P and $[G_P, G_P]$ is P-local.

Then, $x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha = 1^\alpha = 1 = 1^{\beta\alpha} = 1^{\alpha\beta}$. Since $[G_P, G_P]$ is P-local (in particular a $\mathscr{U}_P$-group), we have $x^{\alpha\beta} = 1^{\alpha\beta}$ implies that $x = 1$. Therefore, $x = 1 \in [G_P, G_P]$ and $V \subseteq [G_P, G_P]$.

Let $y \in [G_P, G_P]$. We have, $y \in G_P$ and $y^1 \in [G_P, G_P]$ since $[G_P, G_P]$ is a R-subgroup of G_P. Therefore, $y \in V$ and $[G_P, G_P] \subseteq V$. Consequently, $[G, G]_P = V = [G_P, G_P]$. □

This concludes our chapter on P-localization of $\mathcal{N}_R$-groups. We gave a more detailed proof of the Fundamental Theorem of P-localization on $\mathcal{N}_R$-groups and we showed that in a specific ring a map P-localize if and only if it is a P-isomorphism. Lastly we proved some results which will be needed in our next chapter on the genus of a $\mathcal{N}_R$-group.

Chapter 5

Genus of a Nilpotent R-Powered Group

Finally, in this chapter, we study the genus of a finitely R-generated $\mathcal{N}_R$-group G. In Section 5.1, we define the genus of G and give some results that follows immediately from the definition of the genus $\mathcal{G}(G)$. Then, we show that, if $\mathcal{G}(G \times A) = \mathcal{G}(H \times A)$ for some finitely R-generated abelian R-powered group A, then $\mathcal{G}(M) = \mathcal{G}(N)$. Furthermore, we show that, if G and H are finitely R-generated $\mathcal{N}_R$-groups with commutator R-subgroups of finite type, then their commutator R-subgroups are R-isomorphic.

Lastly, in Section 5.2, we show that the free centre can be defined for a finitely R-generated $\mathcal{N}_R$-group. This definition will be crucial for further studies on the genus of a finitely R-generated $\mathcal{N}_R$-group.

The results we present here, are a generalization of results proven by Mislin in [14] for the case for finitely generated nilpotent groups.

In this chapter, R is a principal ideal domain (PID) binomial ring with a subring isomorphic to the set of rational numbers. The set of prime elements of R will be denoted by $\mathbb{P}_R$.

5.1 Genus of a Nilpotent R-Powered Group

Firstly, we define the genus of a finitely R-generated $\mathcal{N}_R$-group G.

Definition 5.1.1 (Genus)
Let G be a finitely R-generated $\mathcal{N}_R$-group. The genus of G, denoted by $\mathcal{G}(G)$, is the set of all R-isomorphism classes of finitely R-generated $\mathcal{N}_R$-group H such that G_p is R-isomorphic to H_p for all $p \in \mathbb{P}_R$.

Proposition 5.1.2
Let G and H be finitely R-generated $\mathcal{N}_R$-groups. Then, the following hold:

(*a*) $[G] \in \mathcal{G}(G)$ *and*

(*b*) *if* $[H] \in \mathcal{G}(G)$, *then* $H_p \cong_R G_p$ *for all* $p \in \mathbb{P}_R$.

Proof. Let G and H be finitely R-generated $\mathcal{N}_R$-groups.

(a) Since the P-localization of G is unique up to R-isomorphisms, then $G_p \cong_R G_p$ for any $p \in \mathbb{P}_R$ and $[G] \in \mathcal{G}(G)$.

(b) Follows immediately from Definition 5.1.1.

□

Next, we prove a result on the genus of two finitely R-generated $\mathcal{N}_R$-groups. In the subsequent, we will show that if two finitely R-generated $\mathcal{N}_R$-groups are R-isomorphic then they have equivalent genus.

Proposition 5.1.3
Let G and H be finitely R-generated $\mathcal{N}_R$-groups. Then, $\mathcal{G}(G) = \mathcal{G}(H)$ if and only if $G_p \cong H_p$ for all $p \in \mathbb{P}_R$.

Proof. Assume $\mathcal{G}(G) = \mathcal{G}(H)$. Then, $\mathcal{G}(G) \subseteq \mathcal{G}(H)$ and if $[M] \in \mathcal{G}(G)$, then, $[M] \in \mathcal{G}(H)$. Since $[M] \in \mathcal{G}(G)$, then $G_p \cong_R M_p$ for any $p \in \mathbb{P}_R$. Similarly, if $[M] \in \mathcal{G}(H)$, then, $M_p \cong_R H_p$ for any $p \in \mathbb{P}_R$.
Therefore, $G_p \cong_R M_p \cong_R H_p$ for any $p \in \mathbb{P}_R$.

Conversely, assume that $G_p \cong H_p$ for all $p \in \mathbb{P}_R$ and let $[M] \in \mathcal{G}(G)$. Then, $G_p \cong_R M_p$ for all $p \in \mathbb{P}_R$ and $M_p \cong_R G_p \cong_R H_p$ for all $p \in \mathbb{P}_R$. Therefore $[M] \in \mathcal{G}(H)$ and $\mathcal{G}(G) \subseteq \mathcal{G}(H)$.
Similarly, if $[N] \in \mathcal{G}(H)$, then $N_p \cong_R H_p \cong_R G_p$ and $[N] \in G(G)$ for all $p \in \mathbb{P}_R$.
Thus, $\mathcal{G}(H) \subseteq \mathcal{G}(G)$ and hence $\mathcal{G}(G) = \mathcal{G}(H)$. □

Corollary 5.1.4
Let G and H be finitely R-generated $\mathcal{N}_R$-groups. If $G \cong_R H$, then $\mathcal{G}(G) = \mathcal{G}(H)$.

Proof. Let G and H be finitely R-generated $\mathcal{N}_R$-groups such that $G \cong_R H$. By Proposition 5.1.2 (a), we have $[G] \in \mathcal{G}(G)$. Since $G \cong_R H$, then $[G] = [H]$. Therefore, $[H] = [G] \in \mathcal{G}(G)$. Consequently, by Definition 5.1.1, we have that $G_p \cong_R H_p$ for all $p \in \mathbb{P}_R$.
Therefore, $G_p \cong_R H_p$ for all $p \in \mathbb{P}_R$ and $\mathcal{G}(G) = \mathcal{G}(H)$ by Proposition 5.1.3. □

Now, we show that for any finitely R-generated abelian R-powered group A, if $\mathcal{G}(G \times A) = \mathcal{G}(H \times A)$, then $\mathcal{G}(G) = \mathcal{G}(H)$ for any finitely R-generated $\mathcal{N}_R$-groups G and H. This is a generalization of [14, Theorem 1] in the context of $\mathcal{N}_R$-groups.

Theorem 5.1.5
Let G and H be finitely R-generated $\mathcal{N}_R$-groups and let A be a finitely R-generated abelian R-powered group. If $\mathcal{G}(G \times A) = \mathcal{G}(H \times A)$ then $\mathcal{G}(G) = \mathcal{G}(H)$.

Proof. Let G and H be finitely R-generated $\mathcal{N}_R$-groups and let A be a finitely generated abelian group such that $\mathcal{G}(G \times A) = \mathcal{G}(H \times A)$. Since, $\mathcal{G}(G \times A) = \mathcal{G}(H \times A)$, then, we have by Proposition 5.1.3, that $(G \times A)_p \cong_R (H \times A)_p$ for all $p \in \mathbb{P}_R$.
Let $p \in \mathbb{P}_R$. By Proposition 4.9.4, $(G \times A)_p \cong_R G_p \times A_p$ and $(H \times A)_p \cong_R H_p \times A_p$. Consequently, $G_p \times A_p \cong_R H_p \times A_p$.

By Proposition 3.5.4, we have the following two exact sequences:

$$1 \to G_p \xrightarrow{i} G_p \times A_p \xrightarrow{\pi} A_p \to 1 \tag{5.1}$$

$$1 \to H_p \xrightarrow{i} H_p \times A_p \xrightarrow{\pi} A_p \to 1 \tag{5.2}$$

We can combine the exact sequences (5.1) and (5.2) to get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_p & \xrightarrow{i} & G_p \times A_p & \xrightarrow{\pi} & A_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong_R & & \downarrow \cong_R & & \\ 0 & \longrightarrow & H_p & \xrightarrow{i} & H_p \times A_p & \xrightarrow{\pi} & A_p & \longrightarrow & 0 \end{array} \tag{5.3}$$

By Proposition 3.5.5, we have $G_p \cong_R H_p$. Since, $p \in \mathbb{P}_R$ was arbitrarily chosen, then $G_p \cong_R H_p$ for all $p \in \mathbb{P}_R$ and by Proposition 5.1.3, we have $\mathcal{G}(G) = \mathcal{G}(H)$. □

Lastly, we show that if G and H are finitely R-generated $\mathcal{N}_R$-groups with commutator R-subgroups of finite type and have the same genus, then their commutator R-subgroups are R-isomorphic. This is a generalization of [14, Lemma 2] in the context of $\mathcal{N}_R$-groups.

Theorem 5.1.6
Let G and H be finitely R-generated $\mathcal{N}_R$-groups with commutator R-subgroups of finite type. If $\mathcal{G}(G) = \mathcal{G}(H)$, then $[G, G] \cong_R [H, H]$.

Proof. Let G and H be finitely R-generated $\mathcal{N}_R$-groups with commutator R-subgroups of finite type. Assume $\mathcal{G}(G) = \mathcal{G}(H)$. Then, $G_p \cong_R H_p$ for all $p \in \mathbb{P}_R$. By Proposition 4.9.5, we have $[G, G]_p \cong_R [G_p, G_p]$ and $[H, H]_p \cong_R [H_p, H_p]$. Therefore,

$$\begin{aligned} [G, G] &\cong_R \prod_{p \in \mathbb{P}_R} [G, G]^{(p)} \text{ by Theorem 4.2.13} \\ &\cong_R \prod_{p \in \mathbb{P}_R} [G, G]_p \text{ by Corollary 4.9.3} \\ &\cong_R \prod_{p \in \mathbb{P}_R} [G_p, G_p] \text{ by Proposition 4.9.5} \\ &\cong_R \prod_{p \in \mathbb{P}_R} [H_p, H_p] \text{ since } G_p \cong_R H_p \text{ for all } p \in \mathbb{P}_R \\ &\cong_R \prod_{p \in \mathbb{P}_R} [H, H]_p \text{ by Proposition 4.9.5} \\ &\cong_R \prod_{p \in \mathbb{P}_R} [H, H]^{(p)} \text{ by Theorem 4.2.13} \\ &\cong_R [H, H] \end{aligned}$$

Hence, $[G, G] \cong_R [H, H]$.

□

5.2 Free Centre

In this section, we show that the free centre, as defined in [14] for regular finitely generated nilpotent groups, can be defined for a finitely R-generated $\mathcal{N}_R$-group. Firstly we take note, from [1], that all R-subgroups of a finitely R-generated $\mathcal{N}_R$-groups (where R is a PID) are also finitely R-generated. This will allow us to conclude that the R-subgroup $\tau(Z(G))$ is of finite order if G is a finitely R-generated $\mathcal{N}_R$-group. Furthermore, following [9], we can define the exponent of a $\mathcal{N}_R$-group of finite type. We define the free centre using the exponent of the R-subgroup $\tau(Z(G))$. Lastly, we will prove some properties of the free centre.

Firstly, we define when a $\mathcal{N}_R$-group is said to be R-max and state a result, taken from [1], that G is Max-R if G is a finitely R-generated $\mathcal{N}_R$-group and R is a PID.

Definition 5.2.1 (Max-R)
[1, Definition 4.18] A $\mathcal{N}_R$-group G is said to be Max-R if every R-subgroup of G is finitely R-generated.

Theorem 5.2.2
[1, Theorem 4.21] Let R be principal ideal domain (PID) and let G be a $\mathcal{N}_R$-group. If G is finitely R-generated group, then G is a Max-R group.

In [10], the authors introduce the concept of the order of an element of a $\mathcal{N}_R$-group of finite type. Firstly, we consider the set $I_g = \{\alpha \in R : g^\alpha = 1\}$ and show that it is an ideal of R. This is needed, because the definition of the order of an element depends on the set I_g. After this, we will give the definition of the exponent of a $\mathcal{N}_R$-group of finite type.

Proposition 5.2.3
Let G be a $\mathcal{N}_R$-group of finite type and let $g \in G$. The set $I_g = \{\alpha \in R : g^\alpha = 1\}$ is an ideal of R.

Proof. Let G be a $\mathcal{N}_R$-group of finite type and let $g \in G$. Since, G is of finite type, then, there exists $\beta \in R$ such that $g^\beta = 1$. Hence, $\beta \in I_g$ and thus I_g is non-empty.
Further, for any $\alpha, \beta \in I_g$, we have $g^{\alpha-\beta} = g^\alpha g^{-\beta} = g^\alpha (g^\beta)^{-1} = (1)(1)^{-1} = 1$.

Hence, for any $\alpha, \beta \in I_g$, $\alpha - \beta \in I$ and by Theorem 2.1.6, I_g is an additive subgroup of R.

Let $\mu \in R$ and let $\mu\omega \in \mu I_g$. Then $g^{\mu\omega} = g^{\omega\mu} = (g^\omega)^\mu = 1^\mu = 1$. Therefore, $\mu I_g \subseteq I_g$.

Similarly, for $\zeta\mu \in I_g\mu$, we have $g^{\zeta\mu} = (g^\zeta)^\mu = 1^\mu = 1$ since $\zeta \in I_g$.

Therefore, $\zeta\mu \in I_g$ and $I_g\mu \subseteq I_g$.

Hence, $\mu I_g \subseteq I_g$ and $I_g\mu \subseteq I_g$ for any $\mu \in R$. By Definition 2.7.1, we have that I_g is an ideal of R. □

Note 5.2.4
If R is a principal ideal domain (PID), then $I_g = \{\alpha \in R : g^\alpha = 1\}$ is an ideal of R and we have that there exists $\sigma \in R$ such that $\langle \sigma \rangle = I_g$. Therefore, we may define the order of g as any elements associative with σ.

Definition 5.2.5 (Exponent)
[10, Definition 2.21] Let R be a binomial ring which is a principal ideal domain and let G be a $\mathcal{N}_R$-group of finite type. The exponent of G is the lowest common multiple of all the orders of elements in G.

Now, we take note that the R-subgroup $\tau(Z(G))$ is of finite order if G is a finitely R-generated $\mathcal{N}_R$-group. For the remainder of the section, we let R be a principal ideal domain (PID) binomial ring which has a subring isomorphic to the set of rational numbers and we let the set of prime elements of R be denoted by $\mathbb{P}_R$.

Proposition 5.2.6
Let G be a finitely R-generated $\mathcal{N}_R$-group. Then, the R-subgroup $\tau(Z(G))$ is of finite type.

Proof. Let G be a finitely R-generated $\mathcal{N}_R$-group. By Theorem 5.2.2, the R-subgroup $\tau(Z(G))$ is finitely R-generated. Further, by Proposition 4.1.6, $\tau(Z(G))$ is a R-torsion group. Thus, by Definition 4.1.7, $\tau(Z(G))$ is of finite type. □

Since $\tau(Z(G))$ is of finite type, we can get the exponent of $\tau(Z(G))$. The exponent of $\tau(Z(G))$ is needed for the definition of the free centre. The definition is adapted from [14], where the free centre of ordinary nilpotent group was given.

Definition 5.2.7 (Free Centre)
Let G be a finitely R-generated $\mathcal{N}_R$-group. Let η be the exponent of $\tau(Z(G))$. The free centre of G, denoted by $FZ(G)$, is defined as
$FZ(G) = \{x \in Z(G) : x \text{ has a } \eta^{th}\text{-root in } Z(G)\} = \{x \in Z(G) : x = y^\eta \text{ for some } y \in Z(G)\}$.

Lastly, we give some properties relating to the free centre. These properties are inspired by Mislin in [14].

Proposition 5.2.8
Let G be a finitely R-generated $\mathcal{N}_R$-group and let $FZ(G)$ be the free centre of G. Then, we have the following

(*a*) *$FZ(G)$ is a R-characteristic R-subgroup of G,*

(*b*) *$FZ(G)$ is a free abelian R-powered group and*

(*c*) *$FZ(G)$ has rank that is equal to the Hirsch-R-length of G.*

Proof. Let G be a finitely R-generated $\mathcal{N}_R$-group and let $FZ(G)$ be the free centre of G. Let η be the exponent of $\tau(Z(G))$. By Definition 5.2.7, $FZ(G) = \{x \in Z(G) : x \text{ has a } \eta^{th}\text{-root in } Z(G)\} = \{x \in Z(G) : x = y^\eta \text{ for some } y \in Z(G)\}$.

(*a*) We show that $FZ(G)$ is a R-characteristic R-subgroup of G.
Firstly, we show that $FZ(G)$ is a R-subgroup. By Corollary 3.1.6, we have $1 = 1^\eta$. Hence $1 \in FZ(G)$ and $FZ(G)$ is non-empty. Let $x, y \in FZ(G)$, then $x, y \in Z(G)$ with $x = g^\eta$ and $y = h^\eta$ for some $g, h \in Z(G)$. Then, $xy^{-1} = g^\eta(h^\eta)^{-1} = g^\eta(h^{-1})^\eta = (gh^{-1})^\eta$, by Lemma 3.1.5 and the fact that g and h commute.

Therefore, xy^{-1} has a η^{th}-root in $Z(G)$ and $xy^{-1} \in FZ(G)$.
By Theorem 2.1.6, $FZ(G)$ is a subgroup of G.

Now, for any $\alpha \in R$, $x^\alpha = (y^\eta)^\alpha = (y^\alpha)^\eta$.

Since $y \in Z(G)$ and $Z(G)$ being a R-subgroup of G, then $y^\alpha \in Z(G)$.
Therefore, x^α has a η^{th}-root in $Z(G)$ and $x^\alpha \in FZ(G)$.
Thus, $FZ(G)$ is a R-subgroup of G.

Now, we show that $FZ(G)$ is a R-characteristic subgroup of G.
Let $\phi \in Aut_R(G) = \{\psi : G \to G \,|\, \psi \text{ is a } R\text{-isomorphism}\}$.
If $z \in \phi(FZ(G))$, then $z = \phi(z_*)$ for some $z_* \in FZ(G)$. We need to show that $z \in Z(G)$ and z has a η^{th}-root in $Z(G)$.

Let $n \in G$. Since ϕ is a R-isomorphism (and thus is R-surjective), then, there exists $n_* \in G$ such that $n = \phi(n_*)$.
Therefore, we have $nz = \phi(n_*)\phi(z_*) = \phi(n_*z_*) = \phi(z_*n_*) = \phi(z_*)\phi(n_*) = zn$, since $z_* \in FZ(G) \subseteq Z(G)$. Thus, $z \in Z(G)$.

Since, $z_* \in FZ(G)$, then, there exists $t \in Z(G)$ such that $z_* = t^\eta$ and we have $z = \phi(z_*) = \phi(t^\eta) = \phi(t)^\eta$.

Similarly, as above, we have $n\phi(t) = \phi(n_*)\phi(t) = \phi(n_*t) = \phi(tn_*) = \phi(t)\phi(n_*) = \phi(t)n$, since $t \in Z(G)$. Then, $\phi(t) \in Z(G)$, $z \in Z(G)$ and $z = \phi(t)^\eta$ for some $\phi(t) \in Z(G)$. Consequently, $z \in FZ(G)$ and $\phi(FZ(G)) \subseteq FZ(G)$. Since, ϕ was arbitrarily chosen, then $\phi(FZ(G)) \subseteq FZ(G)$ for all $\phi \in Aut_R(G)$ and by Lemma 3.3.12, $FZ(G)$ is a R-characteristic R-subgroup of G.

(*b*) Now, we show that $FZ(G)$ is a free abelian R-powered group.

Let $\psi : Z(G) \to FZ(G)$ be defined by $\psi(x) = x^\eta$. For any $x, y \in Z(G)$, we have, by Lemma 3.1.5, that $\psi(xy) = (xy)^\eta = x^\eta y^\eta = \psi(x)\psi(y)$. Also, for any $\alpha \in R$, $\psi(x^\alpha) = (x^\alpha)^\eta = x^{\alpha\eta} = x^{\eta\alpha} = (x^\eta)^\alpha = \psi(x)^\alpha$. Therefore, ψ is a R-homomorphism.
In addition, for any $x \in FZ(G)$, we have, by Definition 5.2.7, that there exists $y \in Z(G)$ such that $x = y^\eta = \psi(y)$. Consequently, ψ is a R-surjective R-homomorphism and by the First Isomorphism Theorem for R-homomorphisms (Theorem 3.3.7), we have $Z(G)/Ker(\phi) \cong_R FZ(G)$.

However, $Ker(\phi) = \{x \in Z(G) : \psi(x) = 1\} = \{x \in Z(G) : x^\eta = 1\} = \tau(Z(G))$, since η is the exponent of $\tau(Z(G))$. Therefore, $FZ(G) \cong_R Z(G)/\tau(Z(G))$.

Now, we show that $Z(G)/\tau(Z(G))$ is free. Since G is finitely R-generated $\mathcal{N}_R$-group and R is a principal ideal domain we have, by Theorem 5.2.2, that G is max-R. Hence, all R-subgroups of G are finitely R-generated. Consequently, $Z(G)$ is finitely R-generated.

By Lemma 3.2.15, we have $Z(G)/\tau(Z(G))$ is finitely R-generated.

Now, $Z(G)$ is an abelian R-powered group, so, by Lemma 3.4.3, that $Z(G)/\tau(Z(G))$ is an abelian R-powered group. Further, by Proposition 4.1.9, $Z(G)/\tau(Z(G))$ is R-torsion-free and we have $Z(G)/\tau(Z(G))$ is a finitely R-generated R-torsion-free abelian R-powered group. Using Theorem 4.1.10, we conclude that $Z(G)/\tau(Z(G))$ is a free and $FZ(G) \cong_R Z(G)/\tau(Z(G))$ is free.

(*c*) From above, $FZ(G)$ is a finitely R-generated free abelian R-powered group. Thus, by Proposition 3.4.8, $rank(FZ(G)) = h_R(FZ(G))$.

□

Notation 5.2.9
We will denote the quotient $G/FZ(G)$ as $Q(G)$.

Thus, we have shown the genus can be defined for a $\mathcal{N}_R$-group. We have also shown that some classical results on the genus can be carried over to context of $\mathcal{N}_R$-groups. Moreover, we showed that the free centre $FZ(G)$ and the quotient $Q(G)$ can be defined for $\mathcal{N}_R$-groups, with both of these playing a critical role in further studies of the genus of ordinary nilpotent groups. This indicates that further studies on the genus of nilpotent R-powered groups are possible.

Chapter 6

Conclusion and Further Work

In [1], [9], [11] and [13], many results of ordinary nilpotent groups are shown to carry over to nilpotent R-powered groups. This leads us pose a question, "Do the results proven on the genus for ordinary nilpotent groups carry over to nilpotent R-powered groups?".

Mislin's exploration of the genus of a finitely generated nilpotent group, in [14], relies heavily on the P-localization of a nilpotent group and later on a subgroup called the free centre.

Thus, we firstly studied and investigated P-localization of nilpotent R-powered groups G, guided by the work done by Majewicz and Zyman in [13]. We give more detailed proofs to many results given in [13] for understanding and completeness. This includes a proof to the Fundamental Theorem of P-Localization (Theorem 4.7.3). We prove, in Theorem 4.8.3, that in the case when R is a principal ideal domain with subring isomorphic to the set of rational numbers, a map P-localizes if and only if it is a P-isomorphism. This allowed us to prove some new results on P-localization of a nilpotent R-powered groups.

In Lemma 4.9.1, we have shown that for a nilpotent R-powered group of finite type G, the P-localization of G is also of finite type. Furthermore, we showed, in Proposition 4.9.4, that P-localization of a direct product $G \times H$ is R-isomorphic to the direct product of the P-localizations of G and H, $G_P \times H_P$. Lastly, in Proposition 4.9.5, we showed that if the commutator R-subgroup $[G, G]$ is of finite type, then we have $[G, G]_p \cong_R [G_p, G_p]$ for any prime p in R.

In Chapter 5, we define the genus of a finitely R-generated nilpotent R-powered group. We prove some results on the genus.

Our main results are Theorem 5.1.5 and Theorem 5.1.6. Theorem 5.1.5 states that for any two finitely R-generated nilpotent R-powered groups G and H, if $\mathcal{G}(G \times A) = \mathcal{G}(H \times A)$ for some finitely R-powered abelian R-powered group A, then we have $\mathcal{G}(G) = \mathcal{G}(H)$. Theorem 5.1.6 states that if G and H have commutator R-subgroup of finite type and have the same genus, then we have their commutator R-subgroups are R-isomorphic.

Lastly, in Section 5.2, we have shown that the free centre can be defined for nilpotent R-powered groups. This will assist in further studies on the genus of nilpotent R-powered groups, as the free centre played a critical role in further studies of the genus of ordinary nilpotent groups. In particular, we give a conjecture:

Conjecture 6.0.1
Let G and H be finitely R-generated nilpotent R-powered groups with commutator R-subgroups of finite type. If $G \in \mathcal{G}(H)$, then we have $FZ(H) \cong_R FZ(G)$ and $Q(H) \cong_R Q(G)$.

Since we have the free centre $FZ(G)$ and the quotient group $Q(G)$ can be defined in a nilpotent R-powered group, a similar approach could be taken to prove that there is an abelian group structure on the genus of a finitely R-generated nilpotent R-powered group with commutator R-subgroup of finite type, as undertaken in [6]. Thus, we give the following conjecture:

Conjecture 6.0.2
Let G be finitely R-generated nilpotent R-powered group with commutator R-subgroup of finite type. Let σ be the exponent of the quotient $Q(G)/[Q(G), Q(G)]$. Then, we have $\mathcal{G}(G) \cong_R (R/\langle\sigma\rangle)^*/\{1, -1\}$.

Bibliography

[1] Clement A., Majewicz S., Zyman, M., The theory of nilpotent groups, Birkhäuser, Cham, Switzerland, 2017, 1-217.

[2] Dummit S., Foote R.M., Abstract Algebra, John Wiley & Sons, Inc., Hoboken, New Jersey, United States of America, 2004, Third Edition.

[3] Fraleigh, J.B., *A first course in abstract algebra*, Pearson Education Limited, Harlow, 2014, 11-265.

[4] Gallian J.A., Contemporary Abstract Algebra, Brooks/Cole, Belmont, United States of America, 2010, Seventh Edition.

[5] Hall P., The Edmonton notes on nilpotent groups (Queen Mary College Mathematics Notes, 1969).

[6] Hilton P., Mislin G., On the genus of a nilpotent group with finite commutator subgroup, *Mathematische Zeitschrift*, 1967, 146(3), 201-211.

[7] Hilton P., Mislin G., Roitberg J., Localization of nilpotent groups in: L. Nachbin(ED), *Localization of nilpotent groups and spaces*, North-Holland, Amsterdam, 2016, 1-43.

[8] Lang S., Undergraduate Algebra, Springer Science+Business Media Inc., New York, United States of America, 2005, Third Edition.

[9] Majewicz S., Nilpotent $\mathbb{Q}[x]$-powered groups and $\mathbb{Z}[x]$-groups, 2004, City University of New York.

[10] Majewicz S., Zyman, M., Power-commutative nilpotent R-powered groups, *Groups Complexity Cryptology*, 2009, 1(2), 297-309.

[11] Majewicz S., Zyman, M., On the extraction of roots in exponential A-groups, *Groups, Geometry, and Dynamics*, 2010, 4(4), 835-846.

[12] Majewicz S., Zyman, M., On the extraction of roots in exponential A-groups II, *Communications in Algebra*, 2012, 40(1), 64-86.

[13] Majewicz S., Zyman, M., Localization of nilpotent R-powered groups, *Journal of Group Theory*, 2012, 15(1), 119-135.

[14] Mislin G., Nilpotent groups with finite commutator subgroups, in: P. HILTON (ED), *Localization in Group Theory and Homotopy Theory*, Lecture Notes in Mathematics, Springer-Verlag Berlin, 1974, 418, 103-120.

[15] Robinson D.J.S, A Course in the Theory of Groups, *Graduate Texts in Mathematics 148*, Springer-Verlag, New York, United States of America, 1996, Second Edition.

[16] Rose J.S., A course on group theory, Dover Publications Inc., Mineola, New York, 1994, 1-50.

[17] Rotman J.J., An Introduction to the Theory of Groups, *Graduate Texts in Mathematics 80*, Springer-Verlag, New York, United States of America, 1991, Fourth Edition.

[18] Rotman J.J., *Advanced Modern Algebra*, Prentice Hall, New Jersey, United States of America, 2003, 423-441; 646-663.

[19] Warfield R., Genus and cancellation for groups with finite commutator subgroup, *Journal of Pure and Applied Algebra*, 1975, 6(2), 125-132.

[20] Witbooi P., Generalizing the Hilton-Mislin genus group, *J. Algebra*, 2001, 239(1), 327-339.

[21] Xantcha Q.R., Binomial rings: axiomatisation, transfer and classification, *arXiv*, 2018, 1104 (1931v4), 1-18.