Double reduction of partial differential equations with applications to laminar jets and wakes

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A Dissertation submitted to the Faculty of Science,
University of the Witwatersrand, Johannesburg,
in fulfilment of the requirements for the degree of Master of Science.

Declaration

I declare that this project is my own, unaided work. The content of the project is original except where due references have been made. It is being submitted as partial fulfillment of the requirements for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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July 13, 2015

Abstract

Invariant solutions for two-dimensional free and wall jets are derived by considering the Lie point symmetry associated with the appropriate conserved vectors of Prandtl's boundary layer equations for the jets. For the two-dimensional jets we also consider the comparison, advantages and disadvantages between the standard method that uses a linear combination of all the Lie point symmetries of Prandtl's boundary layer equations to generate the invariant solution with the new method explored in this paper which uses the Lie point symmetry associated with a conserved vector to generate the invariant solution. Invariant solutions for two-dimensional classical and self-propelled wakes are also derived by considering the Lie point symmetry associated with the appropriate conserved vectors of Prandtl's boundary layer equations for the wakes. We also consider and discuss the standard method that uses a linear combination of all the Lie point symmetry of Prandtl's boundary layer equations to generate the invariant solutions for the classical and self-propelled wakes.

Acknowledgments

First I would like to thank God for his blessings, guidance in my life and his grace in enabling me to complete my dissertation.

I would like to express my sincere appreciation to my supervisor Professor D.P. Mason for his tireless guidance, assistance and encouragement. The time he took to share his advice and for his support and kindness are greatly appreciated. Without his supervision, this work would not have been possible.

I must acknowledge the CAM's staff members and higher degree students for their outstanding academic and social contribution which made the CAM School an excellent place. Very special thanks to Batho, Unathi and Nkululeko for their willingness to help and their selflessness. To my friends around Gauteng and those globally, thank you for all the cheerful and happy times that kept me going.

I highly recognize the contribution given by my supportive family who have always been there for me. My dear mom Matshepo Kokela who has tried everything in her power and encouraged me to work hard is my pillar of strength and everything I have accomplished is indebted to her. To my dad Lucas Kokela and brother Zweli Kokela, thanks for their unconditional encouragement and love. To the Kekana and Kokela family at large, thank you for always understanding and for all the words of wisdom. I appreciate everything you have done for me.

Finally, I thank the School of Computational and Applied Mathematics and the University of the Witwatersrand for their financial support with the Postgraduate Merit Award and the NRF contributions to making it possible.

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Chapter 1

Introduction

The concept of point symmetries introduced by Lie and conservation laws play a vital role in the study of partial differential equations.

Recently Sjoberg [1] established a double reduction theorem for partial differential equations with two independent variables. In the first reduction the partial differential equation is reduced to an ordinary differential equation of the same order. In the second reduction the ordinary differential equation is integrated once to give an ordinary differential equation of order one less. In the double reduction theorem the partial differential equation has a conservation law and a Lie point symmetry of this partial differential equation associated with the conservation law is used to reduce it to an ordinary differential equation.

Sjoberg applied the theorem to the linear heat equation, the Benjamin-Bona-Mahony (BBM) equation for modelling small amplitude waves with long wavelength, the sine-Gordon equation and lastly to a system of three equations from one-dimensional gas dynamics.

Some problems in fluid mechanics require a conserved quantity to complete their solutions. These problem generally have homogeneous boundary conditions in which the right hand side of the boundary condition is zero. Examples are jet flow problems and problems concerning wakes. Recently it was shown how the conserved quantity for jet flows can be derived systematically by considering the conservation laws for the partial differential equation [4].

We will state the double reduction theorem of Sjoberg in two independent variables. We will then consider new applications of the theorem to jet flow problems and wakes. By associating a Lie point symmetry with the conserved vector we will investigate how double reduction can be used to solve these problems.

An outline of the dissertation is as follows. An overview is given in Chapter 2 of the methods used in the research. The chapter includes the statement of the double reduction theorem and there will be a brief explanation of conservation laws and associated Lie point symmetries. In Chapter 3, we introduce the two-dimensional one-fluid free and wall jets. We consider the solution of the two-dimensional two-fluid free jet in Chapter 4. In Chapter 5, we investigate how the double reduction theorem applies to the wake. We consider the one-fluid wake behind a fixed and self-propelled body. In the final Chapter, the conclusions drawn from the analysis done in this research are summarized.

Chapter 2

Research methodology

2.1 Lie point symmetry and group invariant solution

Consider the m^{th} order partial differential equation (PDE) in n independent variables,

$$F(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(m)}) = 0, \tag{2.1.1}$$

where $x = (x^1, x^2, \cdots, x^n)$ and $u_{(i)}$ denotes the collection of i^{th} order partial derivatives. Then

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}, \tag{2.1.2}$$

with summation over repeated indices, where

$$\xi^i = \xi^i(x^1, \cdots, x^n), \quad \eta = \eta(x^1, \cdots, x^n), \tag{2.1.3}$$

is a **Lie point symmetry** the of PDE (2.1.1) provided

$$X^{[m]} F|_{F=0} = 0, \tag{2.1.4}$$

where $X^{[m]}$ is the m^{th} order prolongation of X.

We will require the prolongation of X only up to order three and for two independent variables:

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial u_1} + +\zeta_2 \frac{\partial}{\partial u_2} + +\zeta_{11} \frac{\partial}{\partial u_{11}} + +\zeta_{12} \frac{\partial}{\partial u_{12}}$$
$$+ \zeta_{22} \frac{\partial}{\partial u_{22}} + \zeta_{111} \frac{\partial}{\partial u_{111}} + \cdots + +\zeta_{222} \frac{\partial}{\partial u_{222}},$$

where

$$\zeta_i = D_i(\eta) + \psi_s D_i(\xi^s), \tag{2.1.5}$$

$$\zeta_{ij} = D_j(\eta_i) + \psi_{is} D_j(\xi^s), \tag{2.1.6}$$

$$\zeta_{ijk} = D_k(\eta_{ij}) + \psi_{ijs} D_k(\xi^s), \tag{2.1.7}$$

and D_i is the operator of total differentiation with respect to x^i,

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ji} \frac{\partial}{\partial u_j} + \cdots . \tag{2.1.8}$$

with the summation over repeated indices.

If the partial differential equation (2.1.1) admits a Lie point symmetry X the **group invariant solution** $u = \Phi(x^1, \cdots, x^n)$ of (2.1.1) generated by X is obtained by solving the first order partial differential equation

$$X(u - \Phi(x^1, \cdots, x^n)) \mid_{u=\Phi} = 0. \tag{2.1.9}$$

The partial differential equation (2.1.1) of order m is reduced to an ordinary equation, also of order m.

2.2 Conservation laws and conserved vectors

A **conservation law** for the partial differential equation (2.1.1) is defined by

$$D_i(T^i) \mid_{PDE} = 0, \tag{2.2.1}$$

where there is summation over the repeated index and D_i denotes the i^{th} total derivative defined by (2.1.8). The vector $\mathbf{T} = (T^1, T^2, \cdots, T^n)$ is referred to as a **conserved vector**.

Conservation laws for a partial differential equation can be derived by several methods. We will derive the conservation laws using the direct method and the multiplier method.

In the direct method equation (2.2.1) is used as the determining equation for the components of the conserved vector $\mathbf{T} = (T^1, T^2, \cdots, T^n)$. Equation (2.2.1) is separated by equating the coefficients of like powers and products of the partial derivatives of u.

A multiplier Λ for the partial differential equation(2.1.1) has the property

$$\Lambda F = D_i(T^i) \tag{2.2.2}$$

for all functions u and not only for solutions of the partial differential equation (2.1.1). The right hand side of (2.2.2) is a divergence expression. The Euler operator E_u annihilates divergence expressions. The determining equation for the multiplier Λ is

$$E_u[\Lambda F] = 0. \tag{2.2.3}$$

In two independent variables (x, y) the Euler operator is defined as

$$E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - \cdots . \tag{2.2.4}$$

Equation (2.2.3) is solved for Λ by equating like powers and products of the partial derivatives of u. We then suppose that u is a solution of the partial differential (2.1.1). Equation (2.2.2) becomes

$$D_i T^i|_{PDE} = \Lambda F|_{PDE} = 0. \tag{2.2.5}$$

The conserved vectors are found by performing elementary manipulations on the equation

$$\Lambda F|_{PDE} = 0. \tag{2.2.6}$$

We know that for the multiplies Λ which have been found, (2.2.6) can be expressed in the form of a conservation law (2.2.1). The multiplier Λ can depend on $x^1, x^2, \cdots, x^n$, on u and on the partial derivatives of u. The more variables included in Λ the greater the chance of finding a new conserved vector but the calculations become longer as more variables are considered.

New conserved vectors may be generated from known conserved vectors and Lie point symmetries of the PDE [5,6].

If T^i are the components of a conserved vector of the partial differential equation (2.1.1) and X is a Lie point symmetry of (2.1.1) then

$$T^i_* = X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i), \qquad i = 1, 2, \cdots, n \tag{2.2.7}$$

are the components of a conserved vector of (2.1.1).

In (2.2.7), X is prolongated to as many partial derivatives as is required. The conserved vector (2.2.7) may be a new conserved vector but it may also be a linear combination of known conserved vectors or a trivial conserved vector for which equation (2.2.1) is identically satisfied without imposing the PDE. It may also be zero.

A Lie point symmetry of the partial differential equation (2.1.1) is said to be ***associated with the conservation law*** *for (2.1.1) with components T^i if [5,6]*

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1, 2, \cdots, n. \tag{2.2.8}$$

2.3 Double reduction theorem

Sjoberg [1] developed a new theory that explains how a Lie point symmetry which is associated with a conserved vector for a partial differential equation can be used to perform a double reduction of the partial differential equation. Her study shows how the second reduction, due to the association of the Lie point symmetry with a conserved vector for the partial differential equation, can be performed systematically.

We will outline the results of Sjoberg's paper [1] and state the double reduction theorem. Sjoberg used independent variables t and x and dependent variable u. We will express the results obtained by Sjoberg [1] and her double reduction theorem in terms of the independent variables x, y and dependent variable ψ used in boundary layer theory. We will first summarise the results that we need before stating the theorem.

Consider the PDE

$$F(x, y, \psi, \psi_{(1)}, \cdots, \psi_{(q)}) = 0, \tag{2.3.1}$$

where $\psi_{(k)}$ denotes the collection of k^{th} order partial derivatives. Suppose that the partial differential equation (2.3.1) admits a Lie point symmetry, X, that is associated with the conserved vector $T = (T^1, T^2)$, where T satisfies the conservation law

$$D_x T^x + D_y T^y|_{PDE} = 0. \tag{2.3.2}$$

Let r, s, w be canonical variables such that

$$X = \frac{\partial}{\partial s}. \tag{2.3.3}$$

A solution which is invariant under the Lie point symmetry X is of the form $w = w(r)$. It satisfies an ODE of the same order ,q, as of the PDE. We define a non-local variable v by

$$\frac{\partial v}{\partial y} = T^x \quad , \quad \frac{\partial v}{\partial x} = -T^y. \tag{2.3.4}$$

In terms of the canonical variables

$$T^r = \frac{\partial v}{\partial s}, \qquad T^s = -\frac{\partial v}{\partial r} \tag{2.3.5}$$

and the conservation law becomes

$$D_r T^r + D_s T^s = 0. \tag{2.3.6}$$

Sjoberg showed that the components T^r and T^s can be expressed in terms of the original boundary layer coordinates (x, y) as

$$T^s = \frac{T^x D_x(s) + T^y D_y(s)}{D_x(r)D_y(s) - D_y(r)D_x(s)}, \tag{2.3.7}$$

$$T^r = \frac{T^x D_x(r) + T^y D_y(r)}{D_x(r)D_y(s) - D_y(r)D_x(s)}. \tag{2.3.8}$$

The components T^x, T^y depend on $(x, y, \psi, \psi_{(1)}, \cdots, \psi_{(q)})$. Since $w = w(r)$ it follows that T^s, T^r depend on $(s, r, w, w_r, \cdots, w_{r^{q-1}})$ where the solution $w(r)$ is invariant under the Lie point symmetry X. Thus equation (2.3.6) becomes

$$D_r T^r + \frac{\partial T^s}{\partial s} = 0. \tag{2.3.9}$$

Now the Lie point symmetry X is associated with the conserved vector T. Sjoberg showed that this implies that

$$XT^r = 0, \qquad XT^s = 0 \tag{2.3.10}$$

and therefore since X is given in canonical form by (2.3.3),

$$\frac{\partial T^r}{\partial s} = 0, \quad \frac{\partial T^s}{\partial s} = 0. \tag{2.3.11}$$

Thus

$$T = T^r(r, w, w_r, w_{rr}, \cdots, w_{r^{q-1}}) \tag{2.3.12}$$

and the conservation law (2.3.9) reduces to

$$D_r T^r = 0. \tag{2.3.13}$$

Hence

$$T^r(r, w, w_r, w_{rr}, \cdots, w_{r^{q-1}}) = k, \tag{2.3.14}$$

where k is a constant and T^r is given by (2.3.8). Equation (2.3.14) is an ODE of order $q-1$.

We have the following theorem due to Sjoberg [1].

Double reduction theorem

A PDE of order q,

$$F(x, y, \psi, \psi_{(1)}, \cdots, \psi_{(q)}) = 0, \tag{2.3.15}$$

in two independent variables (x, y) which admits a Lie point symmetry X that is associated with a conserved vector T of the PDE, is reduced to an ODE of order $q-1$ which is

$$T^r = k, \tag{2.3.16}$$

where $T^r(x, y, w_r, w_{rr}, \cdots, w_{r^{q-1}})$ is given by equation (2.3.8).

Hence a partial differential equation of order q in two independent variables which admits a Lie point symmetry that is associated with a conserved vector T for the partial differential equation is reduced to an ordinary differential equation of order $q-1$.

The double reduction theorem provides a unifying thread that runs through the dissertation.

Chapter 3

Two-dimensional one-fluid free and wall jets

3.1 Introduction

The theory of laminar jets has many applications in science and engineering. There are many different kinds of jets which includes free jets, wall jets and liquid jets. Jet flows may take several forms such as two-dimensional, axisymmetric and radial. There are jets which consist of one fluid and also two-fluid jets. In this chapter we will consider two-dimensional free and wall jets consisting of one fluid. In Chapter 4 a two-fluid free jet will be considered. In jet flow problems the conserved quantity plays a central role. This is because in jet flow problems the boundary conditions are homogeneous and as we know from the derivation of a similarity solution we will require a conserved quantity to determine the remaining unknown exponent and therefore solve the problem. The physical significance of the conserved quantity is that it is a measure the strength of the jet. The conserved quantity for the free jet can be derived by integrating Prandtl's momentum boundary layer equation across the jet and by applying also the continuity equation [3]. The conserved quantity for the wall jet can be derived in a similar way although the derivation is much more difficult. It requires a considerable amount of physical insight and can be described as challenging. The conserved quantity for the wall jet was first derived by Glauert [4]. A systematic approach to finding the conserved quantities for jet flows was introduced by Naz, Mason and Mohamed [2]. It consists of first funding the conservation laws the partial differential equation. The conserved quantities are than obtained by integrating the conservation laws across the jet and by applying the boundary conditions. We will apply this method in Chapter 5 to give an alternative derivation of the conserved quantity for a two-dimensional classical wake and a wake behind a self-propelled body.

The laminar free and wall jets are described by Prandtl's momentum boundary layer equation because there is a region of rapid change perpendicular to the axes of the jets. The continuity equation is also used and the jets is subject to homogeneous boundary conditions.

The new approach that this dissertation uses was first suggested by Kara and Mohamed [5,6], where they found the relationship between conservation laws and Lie point symmetries of a partial differential equation. They introduced the concept of the association of a Lie point symmetry with a conserved vector for a partial differential equation. The method consists in first deriving the conservation laws and therefore the conserved vectors for the partial differential equation. Instead of using a linear combination of all the Lie point symmetries of the partial differential equation to reduce the partial differential equation to an ordinary differential equation they used the Lie point symmetry associated with one of the conserved vectors to perform the reduction.

By the double reduction theorem of Sjoberg [1] it is certain that the ordinary partial differential equation can be integrated at least once. The condition for a Lie point symmetry to be associated with a conserved vector is the invariance condition that is used to derive the Lie point symmetry. The conserved vector that is chosen is determined by the boundary conditions for the problem. For a jet flow problem the corresponding conservation law gives the conserved quantity for the jet. It takes less work to derive the Lie point symmetry associated with the partial differential equation. The conserved quantity is satisfied without further conditions. In this chapter we will apply this method to solve the problem of the two-dimensional free jet and wall jet. This approach was used by Anthonyrajah and Mason [7] to solve the problem of turbulent compressible flow in a channel. Mason and Hill [8] also used the method to solve the problem of an axisymmetric turbulent free jet.

A large amount of research has been done on the two-dimensional free jet and wall jet. The two-dimensional free jet was first solved by Schlichting [3]. The free jet was solved by using a linear combination of Lie point symmetries of the partial differential equation for the stream function to reduce it to an ordinary differential equation by Mason [9]. Glauert [4] formulated and solved the laminar and turbulent wall jet by deriving a similarity solution. Ruscic [10] solved the wall jet by using a linear combination of the Lie point symmetries of Prandtl's boundary layer equation for the stream function.

An outline of this chapter is as follows. The mathematical model of the free jet and wall jet in terms of the velocity components and stream function using Prandtl's boundary layer partial differential equation is formulated in Sections 3.2 and 3.3. In Section 3.4 the conserved quantities for the free jet and wall jet derived from the conservation laws for Prandtl's boundary layer equation for the stream function. The free jet is considered in Sections 3.5 and 3.6. The associated Lie point symmetry is calculated and the invariant solution for the free jet is derived. In Sections 3.7 and 3.8 the wall jet is considered. The associated Lie point symmetry is derived and the invariant solution is obtained. In Section 3.9 a comparison of the methods of solution for jet flow problems is made and conclusions are drawn in Section 3.10.

3.2 Mathematical model

Consider the two-dimensional free and wall jets. The fluid in the jets is viscous and incompressible and is the same as the surrounding fluid which is at rest far from the jets. A free jet is formed when fluid emerges from a long narrow orifice in a wall into the same viscous, incompressible fluid at rest [3]. The axis of symmetry of the jet is the x-axis. The y-axis is perpendicular to the jet in the plane of the jet. The free jet is illustrated in Figure 3.1

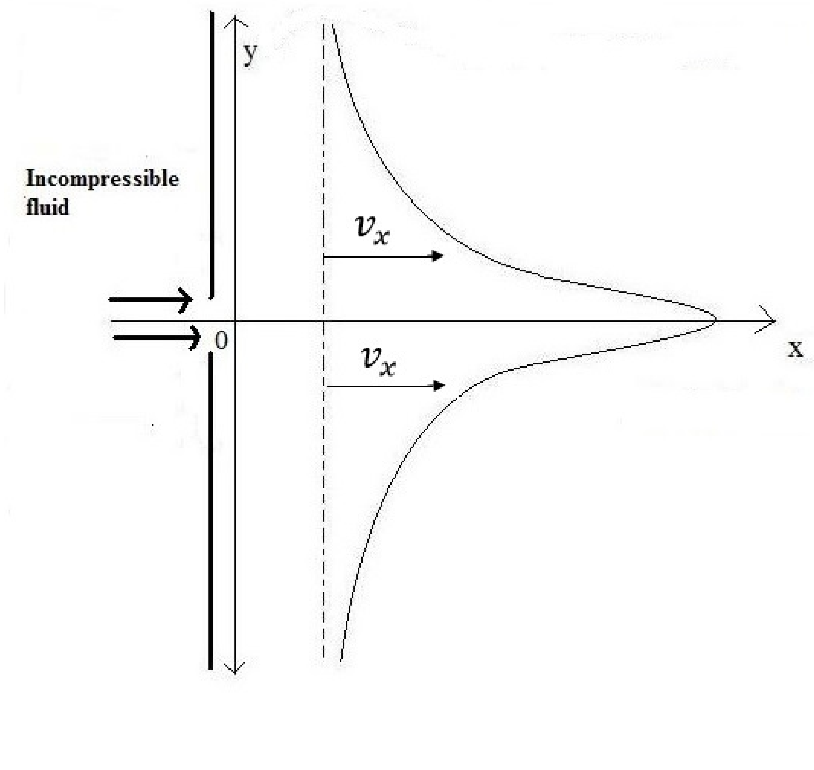

Figure 3.1: Velocity profile of a two-dimensional free jet.

A two-dimensional wall jet is formed when a two-dimensional jet strikes a fixed surface at right angles and spreads out over it [3]. A two-dimensional jet of water falling into partly full sink and spreading out over the base is a wall jet. The fluid velocity tends to zero at the outer edge of the jet. The flow of fluid due to a jet falling on an empty sink is not a wall jet because there is then a free surface with a constant pressure boundary condition.

A wall jet is also formed from a flow of fluid along a boundary. This can occur when the level of fluid is different in two parts of a canal. When a sluice gate separating two sections of a canal is slightly raised the fluid will flow from the part with higher water level to the part with lower water level. This flow along the base forms as wall jet [4]. The wall jet is illustrated in Figure 3.2

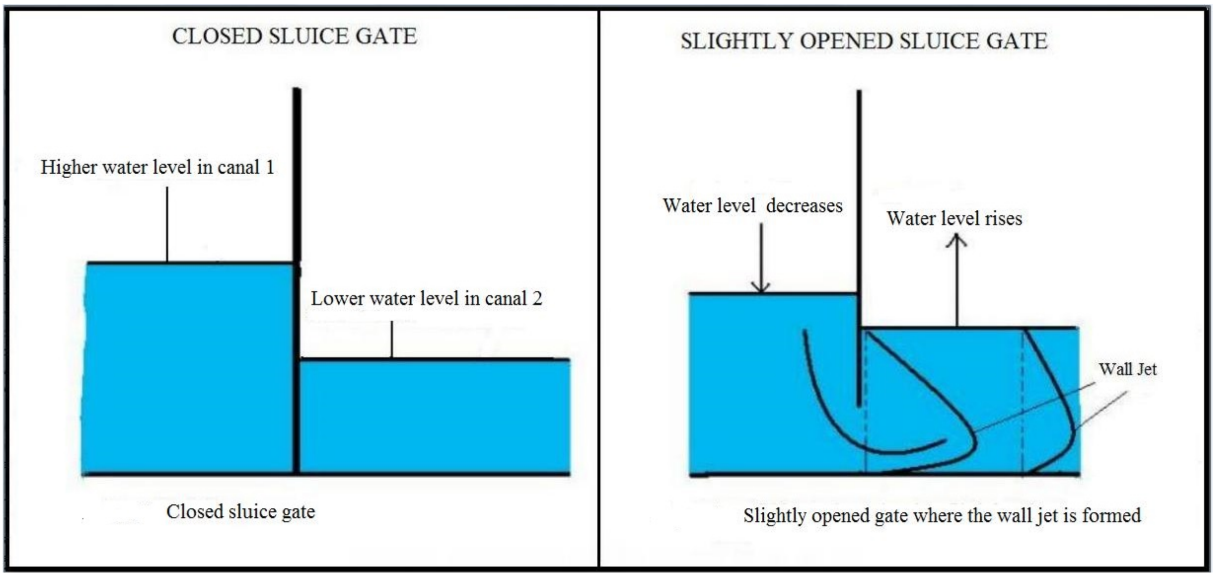

Figure 3.2: Wall jet formed by opening a sluice gate.

3.3 Velocity components

The flow in the two-dimensional free jet and wall jet is described by

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}, \tag{3.3.1}$$

which is known as the Prandtl's boundary layer equation [3]. The continuity equation is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \tag{3.3.2}$$

where v_x and v_y are the velocity components in the x- and y-directions, respectively and ν is the kinematic viscosity of the fluid.

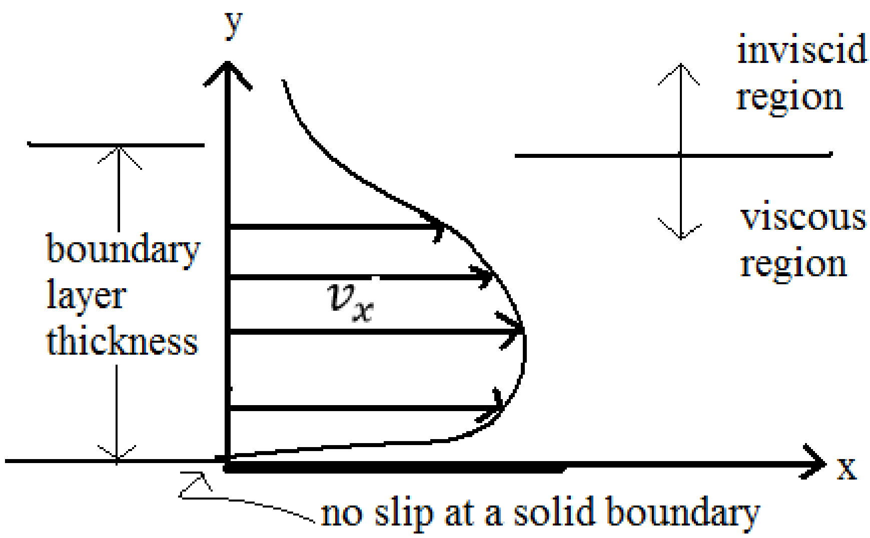

Figure 3.3: Velocity profile of a two-dimensional wall jet.

Stream function

Since we are working with two-dimensional incompressible fluid jets we can introduce a stream function. The stream function $\psi(x, y)$ be defined by

$$v_x = \frac{\partial \psi}{\partial y} \quad , \quad v_y = -\frac{\partial \psi}{\partial x}. \tag{3.3.3}$$

Equation (3.3.1) becomes the third order partial differential equation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}. \tag{3.3.4}$$

The continuity equation (3.3.2) is identically satisfied.

Stream function of a free jet

The partial differential equation (3.3.4) is solved subject to the following boundary conditions:

$$v_y = -\frac{\partial \psi}{\partial x} = 0 \qquad at \quad y = 0, \tag{3.3.5}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad at \quad y = 0, \tag{3.3.6}$$

$$v_x = \frac{\partial \psi}{\partial y} = 0 \qquad at \quad y = \pm\infty, \tag{3.3.7}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad at \quad y = \pm\infty. \tag{3.3.8}$$

The boundary conditions are illustrated in Figure 3.4.

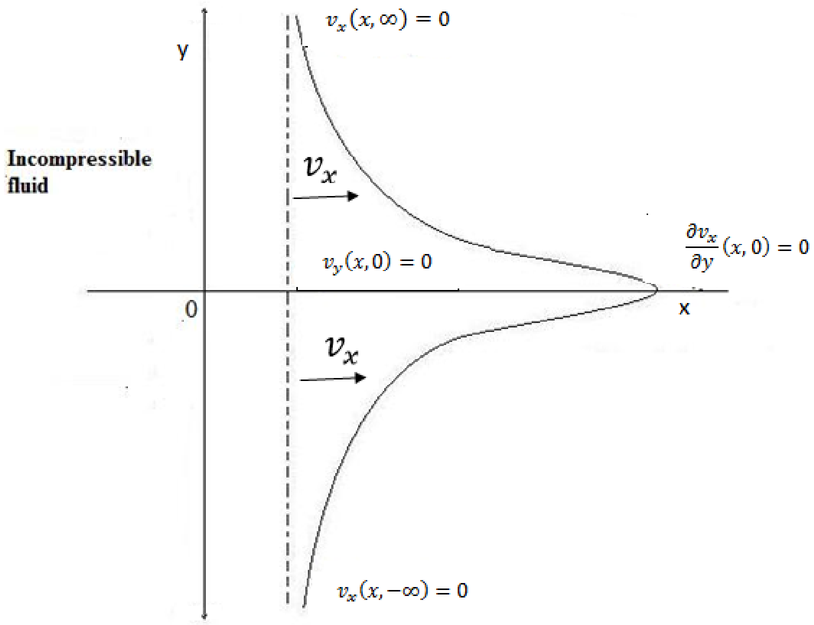

Figure 3.4: Two-dimensional free jet with boundary conditions.

The conserved quantity is [9,4]

$$E = \int_{-\infty}^{\infty} \left(\frac{\partial \psi(x,y)}{\partial y} \right)^2 \mathrm{d}y. \tag{3.3.9}$$

E is a constant independent of x. We will outline the derivation of E when conservation laws are considering in Section 3.4

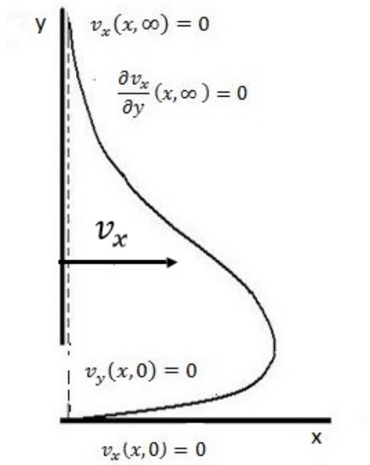

Figure 3.5: Two-dimensional wall jet with boundary conditions.

Stream function in a wall jet

The wall jet is determined by solving equation (3.3.4) subject to the following boundary conditions:

$$v_x = -\frac{\partial \psi}{\partial y} = 0 \qquad at \qquad y = 0, \tag{3.3.10}$$

$$v_y = -\frac{\partial \psi}{\partial x} = 0 \qquad at \qquad y = 0, \tag{3.3.11}$$

$$v_x = \frac{\partial \psi}{\partial y} = 0 \qquad at \qquad y = \infty, \tag{3.3.12}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad at \qquad y = \infty. \tag{3.3.13}$$

The boundary conditions are illustrated in Figure 3.5. Equation (3.3.10) is the no slip condition for a viscous fluid at a stationary boundary. Equation (3.3.11) states that the normal component of fluid velocity vanishes at the boundary because the boundary is not porous.

The stream function $\psi(x, y)$ for the free jet and wall jet satisfy one further condition. From the boundary condition (3.3.5) for the free jet and (3.3.11) for the wall jet.

$$\frac{\partial \psi}{\partial x}(x, 0) = 0 \tag{3.3.14a}$$

and therefore

$$\psi(x,0) = \psi_0, \qquad 0 \leq x \leq \infty \tag{3.3.14b}$$

where ψ_0 is a constant. Careful definition of the stream function shows that ψ_0 is an additive constant [2]. Since an additive constant in the stream function does not contribute to the velocity components defined by (3.3.3) we choose $\psi_0 = 0$ and therefore

$$\psi(x,0) = 0, \qquad 0 \leq x \leq \infty \tag{3.3.14c}$$

This result is required in the derivation of the conserved quantity for the wall jet. The conserved vector in terms of the stream function can be rewritten as [2]:

$$C = \int_0^\infty \psi(x,y) \left(\frac{\partial \psi(x,y)}{\partial y} \right)^2 dy, \tag{3.3.15}$$

where C is a constant independent of x. The derivation of C will be outlined when conversation laws are considered in the next section.

3.4 Conservation law

Kara and Mohamed [5,6] derived the relationship between conservation laws and the Lie point symmetries of a partial differential equation. Conservation laws for the partial differential equation (3.3.4) were found by Naz et al.[2]. The conservation laws for the partial differential equation (3.3.4) take the form

$$D_1 T^1 + D_2 T^2|_{PDE} = 0, \tag{3.4.1a}$$

where

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + \cdots, \tag{3.4.1b}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \cdots \tag{3.4.1c}$$

where the suffices x and y denote partial differentiation with respect to x and y, When x,y,ψ and the partial derivatives of ψ are regarded as independent variables, then partial differentiation will be denoted by a suffix, for example, by ψ_x,ψ_{yx}. When x and y are regarded as the only independent variables then partial differentiation will be denoted by, for example, $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y\partial x}$.

Naz et al [2] used the multiplier method to show that any conserved vector of the third order partial differential equation (3.3.4) with multiplier $\Lambda(x, y, \psi, \psi_x, \psi_y)$ is a linear combination of the two conserved vectors.

$$T^1 = \psi_y^2, \qquad T^2 = -\psi_x\psi_y - \nu\psi_{yy}, \tag{3.4.2a}$$

and

$$T^1 = \psi\psi_y^2, \qquad T^2 = -\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}. \tag{3.4.2b}$$

The Lie point symmetry

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi))\frac{\partial}{\partial\psi} \tag{3.4.3}$$

of the partial differential equation (3.3.4) is said to be associated with the conserved vector $T = (T^1, T^2)$ for (3.3.4) if

$$X(T^i) + T^i(D_k\xi^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2. \tag{3.4.4}$$

where k is summed from 1 to 2 and the total derivatives D_1 and D_2 are defined by equations (3.4.1b) and (3.4.1c). The Lie point symmetry X in (3.4.4) is prolonged as required if T^i depends on the derivatives of ψ. The determining equation for the Lie point symmetry X associated with the conserved vector $T = (T^1, T^2)$ is equation (3.4.4). It has two components

$$i = 1: \qquad X(T^1) + T^1D_2\xi^2 - T^2D_2\xi^1 = 0, \tag{3.4.5a}$$

$$i = 2: \qquad X(T^2) + T^2D_1\xi^1 - T^1D_1\xi^2 = 0. \tag{3.4.5b}$$

The conserved quantities E and C, for the free jet and wall jet can be calculated from the conserved vectors. The conserved vector chosen depends on the boundary conditions. When $x, y, \psi, \psi_x, \psi_y$ and higher derivatives are regarded as independent variables the conservation law is written in terms of the total derivatives as equation (3.4.1a). When x and y are regarded as the independent variables then

$$\frac{\partial T^1}{\partial x}(x, y, \psi(x,y), \psi_x(x,y), \psi_y(x,y), \ldots) = D_x T^1, \tag{3.4.6}$$

$$\frac{\partial T^2}{\partial y}(x, y, \psi(x,y), \psi_x(x,y), \psi_y(x,y), \ldots) = D_y T^2, \tag{3.4.7}$$

and the conservation law can be written in terms of partial derivatives as

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{3.4.8}$$

Consider first the free jet. Choose the first conserved vector (3.4.2a) and regard x and y as independent variables. Equation (3.4.8) becomes

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial \psi}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left[-\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \nu\frac{\partial^2 \psi}{\partial y^2}\right] = 0 \tag{3.4.9}$$

Integrate (3.4.9) with respect to y from $y = -\infty$ to $y = +\infty$ keeping x constant. Then

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[\left(\frac{\partial \psi}{\partial y}\right)^2\right] dy + \int_{-\infty}^{\infty}\frac{\partial}{\partial y}\left[-\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \nu\frac{\partial^2 \psi}{\partial y^2}\right] dy = 0 \tag{3.4.10}$$

and therefore

$$\frac{d}{dx}\int_{-\infty}^{\infty}\left[\left(\frac{\partial \psi}{\partial y}\right)^2\right] dy + \left[-\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \nu\frac{\partial^2 \psi}{\partial y^2}\right]_{y=-\infty}^{y=\infty} dy = 0. \tag{3.4.11}$$

Using the boundary conditions (3.3.7) and (3.3.8) for a free jet it follows that

$$\frac{d}{dx}\int_{-\infty}^{\infty}\left[\left(\frac{\partial \psi}{\partial y}\right)^2\right] dy = 0 \tag{3.4.12}$$

and therefore

$$E = \frac{d}{dx}\int_{-\infty}^{\infty}\left(\frac{\partial\psi}{\partial y}\right)^2 dy = \textit{constant independent of } x. \tag{3.4.13}$$

Consider next the wall jet. Choose the second conserved vector (3.4.2b) with x and y as independent variables. Then (3.4.8) becomes

$$\frac{\partial}{\partial x}\left[\psi\left(\frac{\partial\psi}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left[-\psi\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\nu}{2}\left(\frac{\partial\psi}{\partial y}\right)^2 - \nu\psi\frac{\partial^2\psi}{\partial y^2}\right] = 0 \tag{3.4.14}$$

Integrate (3.4.14) with respect to y from $y = 0$ to $y = +\infty$ keeping x constant. This gives

$$\int_0^{\infty}\frac{\partial}{\partial x}\left[\psi\left(\frac{\partial\psi}{\partial y}\right)^2\right]dy + \int_0^{\infty}\frac{\partial}{\partial y}\left[-\psi\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\nu}{2}\left(\frac{\partial\psi}{\partial y}\right)^2 - \nu\psi\frac{\partial^2\psi}{\partial y^2}\right] = 0 \tag{3.4.15}$$

and hence

$$\frac{d}{dx}\int_0^{\infty}\left[\psi(x,y)\left(\frac{\partial\psi}{\partial y}\right)^2\right]dy + \left[-\psi\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\nu}{2}\left(\frac{\partial\psi}{\partial y}\right)^2 - \nu\psi\frac{\partial^2\psi}{\partial y^2}\right]_{y=0}^{y=\infty} = 0. \tag{3.4.16}$$

Using the boundary conditions (3.3.10), (3.3.12), (3.3.14a) and condition (3.3.14c) we obtain

$$\frac{d}{dx}\int_0^{\infty}\left[\psi(x,y)\left(\frac{\partial\psi}{\partial y}\right)^2\right]dy = 0 \tag{3.4.17}$$

and therefore

$$C = \int_0^{\infty}\left[\psi(x,y)\left(\frac{\partial\psi}{\partial y}\right)^2\right]dy = \textit{constant independent of } x. \tag{3.4.18}$$

We now outline the steps in the established method deriving on invariant solution for a partial differential equation using Lie point symmetries and conservation laws and then outline a new approach introduced recently [5]. These method apply to problems with homogeneous boundary conditions, such as jet flow and wake problem, for which a conserved quantity occurs in their formulation and is required to complete the similarity solution.

The established method consist of the following steps:

- All the Lie point symmetries of the partial differential equation are derived.
- A linear combinations of all the Lie point symmetries is used to reduce the partial differential equation to an ordinary differential equation. The boundary conditions and conserved quantity are written in terms of the transformed variables. The transformation contains an undetermined parameter α.
- The parameter α is obtained from the conserved quantity.
- The ordinary differential equation is solved subjected to the boundary conditions and the conserved quantity.

The modified approach using the associated Lie point symmetry consist of the following steps:

- The conservation laws and corresponding conserved vectors for the partial differential equation are derived.
- The conserved quantity is derived from one of the conservation laws and the boundary conditions. The boundary conditions determine which conservation law to use.
- The Lie point symmetry associated with the conserved vector used to obtain the conserved quantity is derived from the determining equation (3.4.4).
- The associated Lie point symmetry is used to reduce the partial differential equation. The boundary conditions and conserved quantity are explained in terms of the transformed variables. There is not an undetermined parameter in the transformation.

- The ordinary differential equation is solved subjected to the boundary condition and conserved quantity. By the Double Reduction Theorem of Sjoberg [1] the ordinary differential equation can be integrated at least one (reduced in order to one order less) because the transformation was generated by the associated Lie point symmetry.

3.5 Associated Lie point symmetry for the free jet

The conserved vector (3.4.2a) was used to derive the conserved quantity for the free jet. We therefore use the Lie point symmetry associated with this conserved vector to reduce the partial differential equation(3.3.4) to an ordinary differential equation. We now derive the Lie point symmetry associated with the conserved vector (3.4.2a).

The two components of the determining equation for the associated Lie point symmetry:

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial \psi} \tag{3.5.1}$$

are given by (3.4.5a) and (3.4.5b). Now using the prolongation of X we have

$$X^{[1]} = \left[X + \zeta_2\frac{\partial}{\partial \psi_y}\right]\left(\psi_y^2\right) = 2\zeta_2\psi_y \tag{3.5.2}$$

where

$$\zeta_2 = \frac{\partial \eta}{\partial y} + \psi_y\frac{\partial \eta}{\partial \psi} - \psi_x\frac{\partial \xi^1}{\partial y} - \psi_x\psi_y\frac{\partial \xi^1}{\partial \psi} - \psi_y\frac{\partial \xi^2}{\partial y} - \psi_y^2\frac{\partial \xi^2}{\partial \psi}. \tag{3.5.3}$$

Also

$$T^1D_2\xi^2 = \psi_y^2\frac{\partial \xi^2}{\partial y} + \psi_y^3\frac{\partial \xi^2}{\partial \psi}. \tag{3.5.4}$$

$$T^2D_2\xi^1 = -\psi_x\psi_y\frac{\partial \xi^1}{\partial y} - \psi_x\psi_y^2\frac{\partial \xi^1}{\partial \psi} - \nu\left(\psi_{yy}\frac{\partial \xi^1}{\partial y} + \psi_y\psi_{yy}\frac{\partial \xi^1}{\partial \psi}\right). \tag{3.5.5}$$

The first component, (3.4.5a), of the determining equation when expanded in full is

$$\begin{aligned}
&2\psi_y \frac{\partial \eta}{\partial y} + 2\psi_y^2 \frac{\partial \eta}{\partial \psi} - 2\psi_x \psi_y \frac{\partial \xi}{\partial y} - 2\psi_x \psi_y^2 \frac{\partial \xi}{\partial \psi} \\
&- 2\psi_y^2 \frac{\partial \xi^2}{\partial y} - 2\psi_y^3 \frac{\partial \xi^2}{\partial y} + \psi_y^2 \frac{\partial \xi^2}{\partial \psi} + \psi_y^3 \frac{\partial \xi^2}{\partial \psi} \\
&+ \psi_x \psi_y \frac{\partial \xi^1}{\partial y} + \psi_x \psi_y^2 \frac{\partial \xi^1}{\partial \psi} + \nu \psi_{yy} \frac{\partial \xi^1}{\partial y} + \nu \psi_y \psi_{yy} \frac{\partial \xi^2}{\partial \psi} = 0.
\end{aligned} \tag{3.5.6}$$

Separating (3.5.6) by the powers and products of the partial derivatives of ψ gives:

$$\psi_y \psi_{yy} : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.5.7}$$

$$\psi_{yy} : \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{3.5.8}$$

$$\psi_x \psi_y^2 : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.5.9}$$

$$\psi_x \psi_y : \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{3.5.10}$$

$$\psi_y^3 : \qquad \frac{\partial \xi^2}{\partial \psi} = 0, \tag{3.5.11}$$

$$\psi_y^2 : \qquad 2\frac{\partial \eta}{\partial \psi} - \frac{\partial \xi^2}{\partial y} = 0, \tag{3.5.12}$$

$$\psi_y : \qquad \frac{\partial \eta}{\partial y} = 0, \tag{3.5.13}$$

From equations (3.5.7) and (3.5.8)

$$\xi^1 = \xi^1(x). \tag{3.5.14}$$

from (3.5.11)

$$\xi^2 = \xi^2(x, y). \tag{3.5.15}$$

from (3.5.13)

$$\eta = \eta(x, \psi). \tag{3.5.16}$$

One equation remains ,(3.5.12), which becomes

$$2\frac{\partial \eta}{\partial \psi}(x,\psi) = \frac{\partial \xi^2}{\partial y}(x,y) \tag{3.5.17}$$

Differentiate (3.5.17) with respect to ψ. This gives

$$\frac{\partial^2 \eta}{\partial \psi^2}(x,\psi) = 0 \tag{3.5.18}$$

and therefore

$$\eta(x,\psi) = \psi A(x) + B(x). \tag{3.5.19}$$

Differentiate (3.5.17) with respect to y. This gives

$$\frac{\partial^2 \xi^2}{\partial y^2}(x,y) = 0 \tag{3.5.20}$$

and hence

$$\xi^2(x,y) = yP(x) + Q(x). \tag{3.5.21}$$

Substitute (3.5.19) and (3.5.21) back into (3.5.17) which gives

$$P(x) = 2A(x). \tag{3.5.22}$$

Hence from the first determining equation we obtain

$$\xi^1 = \xi^1(x), \quad \xi^2(x,y) = 2yA(x) + Q(x)., \quad \eta(x,\psi) = \psi A(x) + B(x). \tag{3.5.23}$$

The second determining equation is (3.4.5b). Now using the second prolongation of X and (3.4.2a) for T^2

$$\begin{aligned} X^{[2]}T^2 &= \left[X + \zeta_1 \frac{\partial}{\partial \psi_x} + \zeta_2 \frac{\partial}{\partial \psi_y} + \zeta_{22}\frac{\partial}{\partial \psi_{yy}}\right](-\psi_x\psi_y - \nu\psi_{yy}) \\ &= -\psi_y\zeta_1 - \psi\zeta_2 - \nu\zeta_{22} \end{aligned} \tag{3.5.24}$$

But expanding (2.1.5)

$$\zeta_1 = \frac{\partial \eta}{\partial x} + \psi_x \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial x} - \psi_x^2 \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial x} - \psi_x \psi_y \frac{\partial \xi^2}{\partial \psi}. \tag{3.5.25}$$

which using (3.5.23) reduces to

$$\zeta_1 = \psi \frac{dA}{dx} + \frac{dB}{dx} - \psi_x A(x) - \psi_x \frac{d\xi^1}{dx} - 2y \frac{A}{dx} \psi_y^2 - \frac{dQ}{dx} \psi_y. \tag{3.5.26}$$

Also

$$\zeta_2 = \frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_x \psi_y \frac{\partial \xi^1}{\partial \psi} - \psi_y \frac{\partial \xi^2}{\partial y} - \psi_y^2 \frac{\partial \xi^2}{\partial \psi} \tag{3.5.27}$$

which using (3.5.23) becomes

$$\zeta_2 = -\psi_y A(x), \tag{3.5.28}$$

expanding (2.1.6) gives

$$\begin{aligned}
\zeta_{22} = {} & \frac{\partial^2 \eta}{\partial y^2} + 2\psi_y \frac{\partial^2 \eta}{\partial y \partial \psi} - \psi_x \frac{\partial^2 \xi^1}{\partial y^2} - 2\psi_x \psi_y \frac{\partial^2 \xi^1}{\partial y \partial \psi} \\
& - \psi_y \frac{\partial^2 \xi^2}{\partial y^2} - 2(\psi_y)^2 \frac{\partial^2 \xi^2}{\partial y \partial \psi} + (\psi_y)^2 \frac{\partial^2 \eta}{\partial \psi^2} - \psi_x (\psi_y)^2 \frac{\partial^2 \xi^1}{\partial \psi^2} \\
& - (\psi_y)^3 \frac{\partial^2 \xi^2}{\partial \psi^2} - 2\psi_{xy} \frac{\partial \xi^1}{\partial y} - 2\psi_y \psi_{xy} \frac{\partial \xi^1}{\partial \psi} + \psi_{yy} \frac{\partial \eta}{\partial \psi} \\
& - \psi_x \psi_{yy} \frac{\partial \xi^1}{\partial \psi} - 2\psi_{yy} \frac{\partial \xi^2}{\partial y} - 3\psi_y \psi_{yy} \frac{\partial \xi^2}{\partial \psi}
\end{aligned} \tag{3.5.29}$$

which using (3.5.23) reduces to

$$\zeta_{22} = -3\psi_{yy} A(x), \tag{3.5.30}$$

Thus from (3.5.24)

$$\begin{aligned}
X^{[2]}(T^2) = {} & -\psi \frac{dA}{dx} \psi_y - \frac{dB}{dx} \psi_y + \frac{d\xi^1}{dx} \psi_x \psi_y + 2y \frac{dA}{dx} \psi_y^2 \\
& + \frac{dQ}{dx} \psi_y^2 + 3\nu A(x) \psi_{yy}.
\end{aligned} \tag{3.5.31}$$

Also

$$T^2 D_1 \xi^1 = -\psi_x \psi_y \frac{d\xi^1}{dx} - \nu \psi_{yy} \frac{d\xi^1}{dx}, \tag{3.5.32}$$

$$T^1 D_1 \xi^2 = \psi_y^2 \left(2y \frac{dA}{dx} + \frac{dQ}{dx} \right). \tag{3.5.33}$$

The second determining equation becomes

$$\psi \frac{dA}{dx}\psi_y + \frac{dB}{dx}\psi_y - 3\nu A(x)\psi_{yy} + \nu \frac{d\xi^1}{dx}\psi_{yy} = 0. \tag{3.5.34}$$

Separating (3.5.34) by the powers and products of the derivatives of ψ gives

$$\psi_{yy}: \qquad \frac{d\xi^1}{dx} - 3A(x) = 0, \tag{3.5.35}$$

$$\psi_y: \qquad \psi \frac{dA}{dx} + \frac{dB}{dx} = 0. \tag{3.5.36}$$

Separating (3.5.36) in powers of ψ gives

$$\psi: \qquad \frac{dA}{dx} = 0, \tag{3.5.37}$$

$$\psi^0: \qquad \frac{dB}{dx} = 0, \tag{3.5.38}$$

and therefore

$$A(x) = A_0, \qquad B(x) = B_0 \tag{3.5.39}$$

where A_0 and B_0 are constants. Thus from (3.5.35),

$$\xi^1(x) = 3A_0 x + D_0, \tag{3.5.40}$$

where D_0 is a constant. From (3.5.23)

$$\xi^2(x,y) = 2A_0 y + Q(x), \quad \eta(x,\psi) = A_0\psi + B_0. \tag{3.5.41}$$

Renaming the constants and the arbitrary function by $D_0 = c_1$,$A_0 = c_2$,$B_0 = c_3$ and $Q(x) = g(x)$ we have

$$\xi^1(x) = c_1 + 3c_2 x, \tag{3.5.42}$$

$$\xi^2(x,y) = 2c_2 y + g(x), \tag{3.5.43}$$

$$\eta(\psi) = c_3 + c_2\psi. \tag{3.5.44}$$

The Lie point symmetry associated with elementary conserved vector (3.4.2a) is

$$X = (c_1 + 3c_2x)\frac{\partial}{\partial x} + (2c_2y + g(x))\frac{\partial}{\partial y} + (c_3 + c_2\psi)\frac{\partial}{\partial \psi}. \tag{3.5.45}$$

The derivation of the associated Lie point symmetry is easier than the derivation of the Lie point symmetries of the partial differential equation. To calculate the associated Lie point symmetry ,(3.5.45), the prolongation of X only to second order is required and only ζ_1,ζ_2 and ζ_{22} were needed. To calculate the Lie point symmetries of partial differential equation ,(3.3.4), the prolongation of X to third order is required and ζ_{222} is needed which contains many terms when expanded. There are two determining equations for the associated Lie point symmetry which contain fewer terms than the one large determining equation for the Lie point symmetries of the partial differential equation. The results from the first determining equation can be used to greatly simplify the second determining equation as we saw in (3.5.34). The associated Lie point symmetry can readily be calculated by hand without the aid of computer algebra.

3.6 Invariant solution for the free jet

The stream function $\psi = \Phi(x, y)$ is an invariant solution generated by the Lie point symmetry (3.5.45) associated with the conserved vector(3.4.2a) provided

$$X(\psi - \Phi(x, y)) \mid_{\psi=\Phi(x,y)}= 0, \tag{3.6.1}$$

Consider solutions with $c_2 \neq 0$, Then (3.6.1) is satisfied provided

$$(c_1^* + 3x)\frac{\partial \Phi}{\partial x} + (2y + g^*(x))\frac{\partial \Phi}{\partial y} = c_{3^*} + \Phi. \tag{3.6.2}$$

where

$$c_1^* = \frac{c_1}{c_2} \qquad ,c_3^* = \frac{c_3}{c_2}, \qquad g^*(x) = \frac{g(x)}{c_2}. \tag{3.6.3}$$

The differential equations of the characteristic curves of the first order linear differential equation (3.6.2) are

$$\frac{dx}{c_1^* + 3x} = \frac{dy}{2y + g^*(x)} = \frac{d\Phi}{c_3^* + \Phi}. \tag{3.6.4}$$

The first pair of terms give

$$\frac{dy}{dx} - \frac{2}{3(a_1 + x)} y = \frac{g^*(x)}{3(a_1 + x)} \tag{3.6.5}$$

where

$$a_1 = \frac{c_1^*}{3} \tag{3.6.6}$$

Hence

$$\frac{y}{(a_1 + x)^{\frac{2}{3}}} - G(x) = b_1 \tag{3.6.7}$$

where b_1 is a constant and

$$G(x) = \frac{1}{3} \int^x \frac{g^*(x) dx}{(a_1 + x)^{\frac{5}{3}}} \tag{3.6.8}$$

The first and last pair in (3.6.4)

$$\frac{c_3^* + \Phi}{(a_1 + x)^{\frac{1}{3}}} = b_2 \tag{3.6.9}$$

where b_2 is a constant. The general solution of the PDE (3.6.2) is

$$b_2 = F(b_1) \tag{3.6.10}$$

where F is an arbitary function. Since $\psi = \Phi$ we obtain

$$\psi(x, y) = (a_1 + x)^{\frac{1}{3}} F(\xi) - c_3^* \tag{3.6.11}$$

where

$$\xi = \frac{y}{(a_1 + x)^{\frac{2}{3}}} - G(x). \tag{3.6.12}$$

Since $g^*(x)$ is arbitrary we choose $g^*(x) = 0$ to give $\xi = 0$ when $y = 0$. Thus

$$\xi = \frac{y}{(a_1 + x)^{\frac{2}{3}}}. \tag{3.6.13}$$

Substituting (3.6.11) and (3.6.13) into the PDE (3.3.4) gives the ODE

$$3\nu\frac{d^3F}{d\xi^3} + \frac{d}{d\xi}\left[F\frac{dF}{d\xi}\right] = 0. \tag{3.6.14}$$

The boundary conditions (3.3.5) to (3.3.7) become

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\pm\infty) = 0. \tag{3.6.15}$$

The boundary condition (3.3.9) is not required to solve the ODE (3.6.14). The conserved quantity(3.3.9) becomes

$$E = 2\int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi. \tag{3.6.16}$$

The conserved quantity (3.6.16) is independent of x without further conditions because it is generated by the associated Lie point symmetry. Once the ODE (3.6.14) has been solved for $F(\xi)$ the velocity components are given by

$$v_x(x,y) = \frac{1}{(x+a_1)^{\frac{1}{3}}}\frac{dF}{d\xi}, \tag{3.6.17}$$

$$v_y(x,y) = \frac{1}{3(x+a_1)^{\frac{2}{3}}}\left[2\xi\frac{dF}{d\xi} - F(\xi)\right]. \tag{3.6.18}$$

Consider the constant a_1. The long narrow orfice in the wall is assumed to be infinitely thin. In order for the volume of flow to be finite and the momentum to be finite it is necessary to assume an infinite fluid velocity at the orifice. Now from (3.6.17) , $\frac{dF}{d\xi}(0)$ is finite because $v_x(x,0)$ is finite for $x > 0$. Thus we take $a_1 = 0$ to make $v_x(x,0) = \infty$ at $x = 0$.

Consider next c_3^*. From (3.3.13) the stream function satisfies the condition

$$\psi(x,0) = 0 \tag{3.6.19}$$

Since $F(0) = 0$ it follows from (3.6.11) that $c_3^* = 0$.

Consider now the solution of the ODE (3.6.14). Integrating (3.6.14) once with respect to ξ gives

$$3\nu\frac{d^2F}{d\xi^2} + F\frac{dF}{d\xi} = A \tag{3.6.20}$$

where A is a constant Since $\frac{df}{d\xi}(0)$ is finite the boundary condition (3.6.15) at $\xi = 0$ gives $A = 0$. Equation (3.6.20) becomes

$$3\nu\frac{d^2F}{d\xi^2} + \frac{1}{2}\frac{d}{d\xi}\left(F^2\right) = 0 \tag{3.6.21}$$

and integrating again we obtain

$$3\nu\frac{dF}{d\xi} + \frac{1}{2}F^2 = \frac{B^2}{2} \tag{3.6.22}$$

where B^2 is a positive constant. The constant is positive because $\frac{dF}{d\xi} > 0$ since $v_x(x, y) > 0$. Equation (3.6.22) is a variable separable ODE. Its general solution is

$$F(\xi) = B\ \tanh\left(\frac{B}{6\nu}\xi + \alpha\right), \tag{3.6.23}$$

where α is a constant. Since $F(0) = 0$ it follows that $\alpha = 0$. The remaining boundary condition in (3.6.20) at $\xi = \pm\infty$ is identically satisfied. The constant B is obtained from the conserved quantity (3.6.16) which gives

$$E = \frac{B^3}{6\nu}\int_0^\infty \mathsf{sech}^4\eta\ d\eta. \tag{3.6.24}$$

Since

$$\int \mathsf{sech}^4\eta\ d\eta = \tanh\ \eta - \frac{1}{3}\tanh^2\ \eta \tag{3.6.25}$$

we find that

$$B = (9\nu E)^{\frac{1}{3}}\,. \tag{3.6.26}$$

The stream function (3.6.11) therefore is

$$\psi(x, y) = (9\nu Ex)^{\frac{1}{3}}\tanh\left[\left(\frac{E}{24\nu^2}\right)^{\frac{1}{3}}\xi\right] \tag{3.6.27}$$

where

$$\xi = \frac{y}{x^{\frac{2}{3}}} \tag{3.6.28}$$

and the x and y components of the velocity, (3.6.17) and (3.6.18) ,are

$$v_x(x,y) = \left[\frac{3E^2}{8\nu x}\right]^{\frac{1}{3}} \mathsf{sech}^2\left[\left(\frac{E}{24\nu^2}\right)^{\frac{1}{3}}\xi\right], \tag{3.6.29}$$

$$v_y(x,y) = (9\nu E)^{\frac{1}{3}}\left[2\left(\frac{E}{24\nu^2}\right)^{\frac{1}{3}}\xi\ \mathsf{sech}^2\left[\left(\frac{E}{24\nu^2}\right)^{\frac{1}{3}}\xi\right] + \tanh\ \left[\left(\frac{E}{24\nu^2}\right)^{\frac{1}{3}}\xi\right]\right]. \tag{3.6.30}$$

The results of this section agree with the results derived by Schlichting [3] using a similarity solution and by Mason [9] using a linear combination of all the Lie point symmetries of the PDE (3.3.4).

3.7 Associated Lie point symmetry for the wall jet

In a similar way we solve for the wall jet using the conserved vector (3.4.2b). We first calculate the Lie point symmetry

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi)\frac{\partial}{\partial \psi} \tag{3.7.1}$$

associated with the conserved vector (3.4.2b) using the two components, (3.4.5a) and (3.4.5b), of the determining equation. Consider first (3.4.5a). Using (3.5.3) for ζ_2 in the prolongation of X we have

$$\begin{aligned} X^{[1]}(T^1) &= \left(X + \zeta_2\frac{\partial}{\partial \psi_y}\right)(\psi\psi_y^2) \\ &= \eta\psi_y^2 + 2\psi\psi_y\zeta_y \end{aligned} \tag{3.7.2}$$

Also

$$T^1 D_2(\xi^2) = \psi\frac{\partial \xi^2}{\partial y}\psi_y^2 + \psi\frac{\partial \xi^2}{\partial \psi}\psi_y^3, \tag{3.7.3}$$

$$\begin{aligned} T^2 D_2(\xi^1) = &- \psi \frac{\partial \xi^1}{\partial y} \psi_x \psi_y + \frac{1}{2} \nu \frac{\partial \xi^1}{\partial y} \psi_y^2 - \nu \psi \frac{\partial \xi^1}{\partial y} \psi_{yy} \\ &- \psi \frac{\partial \xi^1}{\partial \psi} \psi_x \psi_y^2 + \frac{1}{2} \nu \frac{\partial \xi^1}{\partial \psi} \psi_y^3 - \nu \psi \frac{\partial \xi^1}{\partial \psi} \psi_y \psi_{yy}. \end{aligned} \tag{3.7.4}$$

The first determining equation (3.4.5a) yields

$$\begin{aligned} &\eta \psi_y^2 + 2\psi \frac{\partial \eta}{\partial y} \psi_y + 2\psi \frac{\partial \eta}{\partial \psi} \psi_y^2 - 2\psi \frac{\partial \xi^1}{\partial y} \psi_x \psi_y - 2\psi \frac{\partial \xi^1}{\partial \psi} \psi_x \psi_y^2 - 2\psi \frac{\partial \xi^2}{\partial y} \psi_y^2 - 2\psi \frac{\partial \xi^2}{\partial \psi} \psi_y^3 \\ &+ \psi \frac{\partial \xi^2}{\partial y} \psi_y^2 + \psi \frac{\partial \xi^2}{\partial \psi} \psi_y^3 + \psi \frac{\partial \xi^1}{\partial y} \psi_x \psi_y - \frac{1}{2} \nu \frac{\partial \xi^1}{\partial y} \psi_y^2 + \nu \psi \frac{\partial \xi^1}{\partial y} \psi_{yy} \\ &+ \psi \frac{\partial \xi^1}{\partial \psi} \psi_x \psi_y^2 - \frac{1}{2} \nu \frac{\partial \xi^1}{\partial \psi} \psi_y^3 + \nu \psi \frac{\partial \xi^1}{\partial \psi} \psi_y \psi_{yy} = 0. \end{aligned} \tag{3.7.5}$$

We assume that $\nu \neq 0$. Separating (3.7.5) by the powers and products of the partial derivatives of ψ gives:

$$\psi_y \psi_{yy} : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.7.6}$$

$$\psi_{yy} : \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{3.7.7}$$

$$\psi_x \psi_y^2 : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.7.8}$$

$$\psi_x \psi_y : \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{3.7.9}$$

$$\psi_y^3 : \qquad \psi \frac{\partial \xi^2}{\partial \psi} + \frac{1}{2} \nu \frac{\partial \xi^1}{\partial \psi} = 0, \tag{3.7.10}$$

$$\psi_y^2 : \qquad 2\psi \frac{\partial \eta}{\partial \psi} + \eta - \psi \frac{\partial \xi^2}{\partial y} - \frac{1}{2} \nu \frac{\partial \xi^1}{\partial y} = 0. \tag{3.7.11}$$

$$\psi_y : \qquad \frac{\partial \eta}{\partial y} = 0. \tag{3.7.12}$$

From (3.7.6) and (3.7.7),

$$\xi^1 = \xi^1(x). \tag{3.7.13}$$

Equation (3.7.10) reduces to

$$\frac{\partial \xi^2}{\partial \psi} = 0, \tag{3.7.14}$$

and therefore

$$\xi^2 = \xi^2(x, y). \tag{3.7.15}$$

From (3.7.12)

$$\eta = \eta(x, \psi). \tag{3.7.16}$$

The remaining equation, (3.7.11), becomes

$$2\psi \frac{\partial \eta}{\partial \psi}(x, \psi) + \eta(x, \psi) - \psi \frac{\partial \xi^2}{\partial y}(x, y) = 0. \tag{3.7.17}$$

Differentiate (3.7.17) by y. This gives

$$\frac{\partial^2 \xi^2}{\partial y^2}(x, y) = 0 \tag{3.7.18}$$

and therefore

$$\xi^2(x, y) = yA(x) + B(x). \tag{3.7.19}$$

Equation (3.7.17) becomes

$$\frac{\partial \eta}{\partial \psi}(x, \psi) + \frac{1}{2\psi}\eta(x, \psi) = \frac{1}{2}A(x). \tag{3.7.20}$$

The solution of the first order differential equation (3.7.20) is

$$\eta(x, \psi) = \frac{1}{3}\psi A(x) + \frac{D(x)}{\psi^{\frac{1}{2}}} \tag{3.7.21}$$

where $D(x)$ is an arbitrary function. In summary the first component of the determining equation yields

$$\xi^1 = \xi^1(x), \quad \xi^2(x, y) = yA(x) + B(x), \quad \eta(x, \psi) = \frac{1}{3}\psi A(x) + \frac{D(x)}{\psi^{\frac{1}{2}}}. \tag{3.7.22}$$

Consider next the second component of the determining equation (3.4.5b). We will derive it for

$$\xi^1 = \xi^1(x), \qquad \xi^2 = \xi^2(x, y), \qquad \eta(x, \psi) = \eta(x, \psi) \tag{3.7.23}$$

and impose the results in (3.7.22) after it has been separated by powers and products of the partial derivatives of ψ. Using the second prolongation of X and (3.4.2b) for T^2 we obtain

$$\begin{aligned} X^{[2]}T^2 &= \left[X + \zeta_1 \frac{\partial}{\partial \psi_x} + \zeta_2 \frac{\partial}{\partial \psi_y} + \zeta_{22} \frac{\partial}{\partial \psi_{yy}}\right] \left(-\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy}\right) \\ &= -\eta\psi_x\psi_y - \nu\eta\psi_{yy} - \psi\psi_y\zeta_1 - \psi\psi_x\zeta_2 + \nu\psi_y\zeta_2 - \nu\psi\zeta_{22}. \end{aligned} \tag{3.7.24}$$

But ζ_1,ζ_2 and ζ_{22} are given by (3.5.25), (3.5.27) and (3.5.29). Using (3.7.23) we obtain

$$\zeta_1 = \frac{\partial \eta}{\partial x} + \psi_x \left(\frac{\partial \eta}{\partial \psi} - \frac{d\xi^1}{dx} \right) - \psi_y \frac{\partial \xi^2}{\partial x}, \tag{3.7.25}$$

$$\zeta_2 = \psi_y \left(\frac{\partial \eta}{\partial \psi} - \frac{\partial \xi^2}{\partial y} \right), \tag{3.7.26}$$

$$\zeta_{22} = -\psi \frac{\partial^2 \xi^2}{\partial y^2} + \psi_y^2 \frac{\partial^2 \eta}{\partial \psi^2} + \psi_{yy} \left(\frac{\partial \eta}{\partial \psi} - 2\frac{\partial \xi^2}{\partial y} \right). \tag{3.7.27}$$

Equation(3.7.24) becomes

$$\begin{aligned} X^{[2]}(T^2) =& - \eta\psi_x\psi_y - \nu\eta\psi_{yy} - \psi\frac{\partial \eta}{\partial x}\psi_y - \psi\frac{\partial \eta}{\partial \psi}\psi_x\psi_y + \psi\frac{d\xi^1}{dx}\psi_x\psi_y + \psi\frac{\partial \xi^2}{\partial x}\psi_y^2 \\ & - \psi\frac{\partial \eta}{\partial \psi}\psi_x\psi_y + \psi\frac{\partial \xi^2}{\partial y}\psi_x\psi_y + \nu\frac{\partial \eta}{\partial \psi}\psi_y^2 - \nu\frac{\partial \xi^2}{\partial y}\psi_y^2 + \nu\psi\frac{\partial^2 \xi^2}{\partial y^2}\psi_y \\ & - \nu\psi\frac{\partial^2 \eta}{\partial \psi^2}\psi_y^2 - \nu\psi\frac{\partial \eta}{\partial \psi}\psi_{yy} + 2\nu\psi\frac{\partial \xi^2}{\partial y}\psi_{yy}. \end{aligned} \tag{3.7.28}$$

We also have

$$T^2 D_1(\xi^1) = -\psi\frac{d\xi^1}{dx}\psi_x\psi_y + \frac{\nu}{2}\frac{d\xi^1}{dx}\psi_y^2 - \nu\psi\frac{d\xi^1}{dx}\psi_{yy} \tag{3.7.29}$$

and

$$T^1 D_1(\xi^2) = \psi\frac{\partial \xi^1}{\partial x}\psi_y^2. \tag{3.7.30}$$

The second component of the determining equation, (3.4.5b) becomes

$$\begin{aligned} & - \eta\psi_x\psi_y - \nu\eta\psi_{yy} - \psi\frac{\partial \eta}{\partial x}\psi_y - \psi\frac{\partial \eta}{\partial \psi}\psi_x\psi_y + \psi\frac{d\xi^1}{dx}\psi_x\psi_y + \psi\frac{\partial \xi^2}{\partial x}\psi_y^2 \\ & - \psi\frac{\partial \eta}{\partial \psi}\psi_x\psi_y + \psi\frac{\partial \xi^2}{\partial y}\psi_x\psi_y + \nu\frac{\partial \eta}{\partial \psi}\psi_y^2 - \nu\frac{\partial \xi^2}{\partial y}\psi_y^2 + \nu\psi\frac{\partial^2 \xi^2}{\partial y^2}\psi_y - \nu\psi\frac{\partial^2 \eta}{\partial \psi^2}\psi_y^2 \\ & - \nu\psi\frac{\partial \eta}{\partial \psi}\psi_{yy} + 2\nu\psi\frac{\partial \xi^2}{\partial y}\psi_{yy} - \psi\frac{d\xi^1}{dx}\psi_x\psi_y + \frac{\nu}{2}\frac{d\xi^1}{dx}\psi_y^2 - \nu\psi\frac{d\xi^1}{dx}\psi_{yy} \\ & - \psi\frac{\partial \xi^1}{\partial x}\psi_y^2 = 0 \end{aligned} \tag{3.7.31}$$

Equation (3.7.31) was derived assuming only (3.7.23).

We separate (3.7.31) by powers and products of the partial derivatives of ψ.

$$\psi_{yy}: \quad \eta + \psi\frac{\partial \eta}{\partial \psi} - 2\psi\frac{\partial \xi^2}{\partial y} + \psi\frac{d\xi^1}{dx} = 0, \tag{3.7.32}$$

$$\psi_x\psi_y: \quad \eta + 2\psi\frac{\partial\eta}{\partial\psi} - \psi\frac{\partial\xi^2}{\partial y} = 0, \tag{3.7.33}$$

$$\psi_y^2: \quad \frac{\partial\eta}{\partial\psi} - \psi\frac{\partial^2\eta}{\partial\psi^2} - \frac{\partial\xi^2}{\partial y} + \frac{1}{2}\frac{d\xi^1}{dx} = 0, \tag{3.7.34}$$

$$\psi_y: \quad \frac{\partial\eta}{\partial x} - \nu\frac{\partial^2\xi^2}{\partial y^2} = 0. \tag{3.7.35}$$

We now use the results in (3.7.22). Equation (3.7.33) is identically satisfied. Equations (3.7.32), (3.7.34) and (3.7.35) become

$$-\frac{4}{3}\psi A(x) + \frac{D(x)}{2\psi^{\frac{1}{2}}} + \psi\frac{d\xi^1}{dx} = 0, \tag{3.7.36}$$

$$-\frac{4}{3}\psi A(x) + \frac{5}{2}\frac{D(x)}{\psi^{\frac{1}{2}}} + \psi\frac{d\xi^1}{dx} = 0, \tag{3.7.37}$$

$$-\frac{1}{3}\psi^{\frac{3}{2}}\frac{dA}{dx} + \frac{dD(x)}{dx} = 0. \tag{3.7.38}$$

Separate (3.7.38) in powers of ψ

$$\psi^{\frac{3}{2}}: \quad \frac{dA}{dx} = 0, \tag{3.7.39}$$

$$\psi^0: \quad \frac{dD}{dx} = 0. \tag{3.7.40}$$

Hence $A(x) = A_0$ and $D_x = D_0$ where A_0 and D_0 are constants. Separating (3.7.36) in powers of ψ gives

$$\psi: \quad \frac{d\xi^1}{dx} - \frac{4}{3}A_0 = 0, \tag{3.7.41}$$

$$\psi^{-\frac{1}{2}}: \quad D_0 = 0. \tag{3.7.42}$$

Thus

$$\xi^1(x) = \frac{4}{3}A_0 x + E_0, \tag{3.7.43}$$

where E_0 is a constant. Equation (3.7.37) is identically satisfied. From (3.7.22),

$$\xi^2(x,y) = A_0 y + B(x), \quad \eta(\psi) = \frac{1}{3}A_0\psi. \tag{3.7.44}$$

Renaming the constants and the arbitrary function by $E_0 = c_1$, $\frac{1}{3}A = c_2$, $B(x) = g(x)$ we obtain

$$\xi^1(x) = c_1 + 4c_2 x, \tag{3.7.45}$$

$$\xi^2(x,y) = 3c_2y + g(x), \tag{3.7.46}$$

$$\eta(\psi) = c_2\psi. \tag{3.7.47}$$

The Lie point symmetry associated with the second conserved vector(3.4.2b) is

$$X = (c_1 + 4c_2x)\frac{\partial}{\partial x} + (3c_2y + g(x))\frac{\partial}{\partial y} + c_2\psi\frac{\partial}{\partial \psi}. \tag{3.7.48}$$

3.8 Invariant solution for the wall jet

An invariant solution $\psi = \Phi(x,y)$ of the partial differential equation(3.3.4) and generated by the Lie point symmetry (3.7.48) associated with the conserved vector (3.4.2b) satisfies

$$X(\psi - \Phi(x,y)) \mid_{\psi=\Phi(x,y)} = 0. \tag{3.8.1}$$

We will consider solutions with $c_2 \neq 0$. Then (3.8.1) is satisfied provided

$$(a_1 + x)\frac{\partial \Phi}{\partial x} + \left(\frac{3}{4}y + g^*(x)\right)\frac{\partial \Phi}{\partial y} = \frac{1}{4}\Phi \tag{3.8.2}$$

where

$$a_1 = \frac{c_1}{4c_4}, \qquad , g^*(x) = \frac{g(x)}{4c_2} \tag{3.8.3}$$

The differential equation of the characteristic curves of (3.8.2) are

$$\frac{dx}{a_1 + x} = \frac{dy}{\frac{3}{4}y + g^*(x)} = 4\frac{d\Phi}{\Phi}. \tag{3.8.4}$$

The first pair of terms in (3.8.4) lead to the ordinary differential equation

$$\frac{dy}{dx} - \frac{3}{4(a_1 + x)}y = \frac{g^*(x)}{(a_1 + x)}. \tag{3.8.5}$$

Thus

$$\frac{y}{(a_1 + x)^{\frac{3}{4}}} - G(x) = b_1 \tag{3.8.6}$$

where b_1 is a constant and

$$G(x) = \int^x \frac{g^*(x)dx}{(a_1 + x)^{\frac{7}{4}}}. \tag{3.8.7}$$

The first and last terms in (3.8.4) give

$$\frac{\Phi}{(a_1+x)^{\frac{1}{4}}} = b_2 \tag{3.8.8}$$

where b_2 is a constant. The general solution of (3.8.2) is

$$b_2 = F(b_1) \tag{3.8.9}$$

where F is an arbitrary function. But $\psi = \Phi$ and therefore

$$\psi(x,y) = (a_1+x)^{\frac{1}{4}} F(\xi) \tag{3.8.10}$$

where

$$\xi = \frac{y}{(a_1+x)^{\frac{3}{4}}} - G(x). \tag{3.8.11}$$

Since $g^*(x)$ is arbitrary we choose $g^*(x) = 0$. Then $G(x) = 0$ and $\xi = 0$ when $y = 0$. Thus

$$\xi = \frac{y}{(a_1+x)^{\frac{3}{4}}}. \tag{3.8.12}$$

We now substitute (3.8.10) and (3.8.12) into the PDE (3.3.4). The PDE reduces to the ODE

$$4\nu\frac{d^3F}{d\xi^3} + F\frac{d^2F}{d\xi^2} + 2\left(\frac{dF}{d\xi}\right)^2 = 0. \tag{3.8.13}$$

The boundary conditions (3.3.10) to (3.3.13) for the wall jet become

$$\frac{dF}{d\xi} = 0 \quad , F(0) = 0, \quad \frac{dF}{d\xi}(\infty) = 0. \tag{3.8.14}$$

As was the case for the free jet the boundary conditions are all homogeneous conditions with zero right hand side. The conserved quantity (3.3.15) becomes

$$C = \int_0^\infty F(\xi)\left(\frac{dF}{d\xi}\right)^2 d\xi. \tag{3.8.15}$$

No further condition has to be imposed to make C independent of x because it was generated by a Lie point symmetry associated with a conserved vector for the PDE (3.3.4) , namely (3.4.2b). After $F(\xi)$ has been obtained the fluid velocity component are given by

$$v_x(x,y) = \frac{\partial \psi}{\partial y} = \frac{1}{(a_1+x)^{\frac{1}{2}}}\frac{dF}{d\xi}, \tag{3.8.16}$$

$$v_y(x,y) = -\frac{\partial \psi}{\partial x} = \frac{1}{4(a_1+x)^{\frac{3}{4}}}\left[3\xi\frac{dF}{d\xi} - F(\xi)\right]. \tag{3.8.17}$$

Consider first the constant a_1 which occurs in ξ, v_x and v_y. The long narrow orifice in the wall at $x=0$, $y=0$ is assumed to be infinitely thin. In order for the volume flux to be finite and the momentum to be finite, the fluid velocity $v_x(x,y)$ must be infinite at the orifice. This only occurs if $a_1 = 0$.

We now outline the solution by Glauert [4] of the ODE (3.8.13). We first multiply (3.8.13) by F and combine the second and third terms. This gives

$$4\nu F\frac{d^3F}{d\xi^3} + \frac{d}{d\xi}\left(F^2\frac{dF}{d\xi}\right) = 0, \tag{3.8.18}$$

which can be rewritten as

$$4\nu\left[\frac{d}{d\xi}\left(F\frac{d^2F}{d\xi^2}\right) - \frac{1}{2}\frac{d}{d\xi}\left(\frac{dF}{d\xi}\right)^2\right] + \frac{d}{d\xi}\left(F^2\frac{dF}{d\xi}\right) = 0. \tag{3.8.19}$$

Integrating with respect to ξ gives

$$4\nu\left[F\frac{d^2F}{d\xi^2} - \frac{1}{2}\left(\frac{dF}{d\xi}\right)^2\right] + F^2\frac{dF}{d\xi} = B, \tag{3.8.20}$$

where B is a constant. To obtain B we impose the boundary conditions in (3.8.14) at $\xi = 0$. We observe that $F''(0)$ is finite because the shear stress at the boundary $\tau_{yx}(x,0)$ is finite and

$$\tau_{yx}(x,0) = \mu\left(\frac{\partial v_x}{\partial y}(x,0) + \frac{\partial v_y}{\partial x}(x,0)\right) = \frac{\mu}{x^{\frac{5}{4}}}F''(0), \tag{3.8.21}$$

Hence $B = 0$. In order to integrate (3.8.20) we multiply it by $F^\alpha(\xi)$ where α is still to be chosen and write it in the form [10]

$$4\nu\left[\frac{d}{d\xi}\left(F^{1+\alpha}\frac{dF}{d\xi}\right) - \left(\frac{3}{2}+\alpha\right)F^\alpha\left(\frac{dF}{d\xi}\right)^2\right] + \frac{1}{(3+\alpha)}\frac{d}{d\xi}\left(F^{3+\alpha}\right) = 0. \tag{3.8.22}$$

Choosing $\alpha = -\frac{3}{2}$, equation (3.8.22) reduces to

$$4\nu \frac{d}{d\xi}\left(F^{-\frac{1}{2}}\frac{dF}{d\xi}\right) + \frac{2}{3}\frac{d}{d\xi}\left(F^{\frac{3}{2}}\right) = 0 \tag{3.8.23}$$

and hence

$$4\nu F^{-\frac{1}{2}}\frac{dF}{d\xi} + \frac{2}{3}F^{\frac{3}{2}} = D, \tag{3.8.24}$$

where D is a constant. To obtain D we cannot impose the boundary conditions at $\xi = 0$ because the ratio $F'(0)/F^{\frac{1}{2}}(0)$ is undetermined. We therfore impose the boundary condition at $\xi = \infty$. Since $F(\infty)$ is not known we let $F(\infty) = A$. We will see that A is determined from the conserved quantity C. Thus

$$D = \frac{2}{3}A^{\frac{3}{2}} \tag{3.8.25}$$

and (3.8.24) becomes

$$\frac{1}{F^{\frac{1}{2}}}\frac{dF}{d\xi} = \frac{A^{\frac{3}{2}}}{6\nu}\left[1 - \left(\frac{F}{A}\right)^{\frac{3}{2}}\right]. \tag{3.8.26}$$

Let

$$\frac{F}{A} = g^2. \tag{3.8.27}$$

Then (3.8.26) becomes

$$\frac{dg}{d\xi} = \frac{A}{12\nu}\left(1 - g^3\right) \tag{3.8.28}$$

which is a variable separable ODE. It can be written as

$$\left[\frac{1}{1-g} + \frac{g+2}{1+g+g^2}\right]dg = \frac{A}{4\nu}d\xi. \tag{3.8.29}$$

Now

$$\int \frac{g+2}{1+g+g^2}dg = \frac{1}{2}\ln\left(1+g+g^2\right) + \sqrt{3}\tan^{-1}\left(\frac{1+2g}{\sqrt{3}}\right). \tag{3.8.30}$$

Integrating (3.8.29), imposing the boundary condition $g(0) = 0$ and using the identity [11]

$$\tan^{-1}\theta - \tan^{-1}\phi = \tan^{-1}\left(\frac{\theta - \phi}{1 + \theta\phi}\right) \tag{3.8.31}$$

gives

$$\xi = \frac{4\nu}{A}\left[\ln\left(\frac{(1+g+g^2)^{\frac{1}{2}}}{1-g}\right) + \sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}g}{2+g}\right)\right]. \tag{3.8.32}$$

The constant A has still to be determined. The conserved quantity C ,given by (3.8.15) can expressed in terms of g as

$$C = 4A^3 \int_0^1 g^4 \frac{dg}{d\xi} dg \tag{3.8.33}$$

since $g(\infty) = 1$. Using (3.8.28), equation (3.8.33) becomes

$$C = \frac{A^4}{3\nu} \int_0^1 g^4(1-g^3)dg \tag{3.8.34}$$

and therefore

$$A = [40\nu C]^{\frac{1}{4}}. \tag{3.8.35}$$

The solution for the stream function and the velocity components can be written in parametric form with g as parameter, where $0 \leq g \leq 1$. From (3.8.32) and (3.8.35) the similarity variable is given by

$$\xi = \frac{y}{x^{\frac{3}{4}}} = \left(\frac{32\nu^3}{5c}\right)^{\frac{1}{4}}\left[\ln\left(\frac{(1+g+g^2)^{\frac{1}{2}}}{1-g}\right) + \sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}g}{2+g}\right)\right]. \tag{3.8.36}$$

From (3.8.27) and (3.8.35),

$$F(\xi) = [40\nu c]^{\frac{1}{4}} g^2 \tag{3.8.37}$$

and therefore

$$\psi(x,y) = [40\nu Cx]^{\frac{1}{4}} g^2. \tag{3.8.38}$$

From (3.8.37) and (3.8.28),

$$\frac{dF}{d\xi} = \left[\frac{10\ C}{9\nu}\right]^{\frac{1}{2}} g(1-g^3) \tag{3.8.39}$$

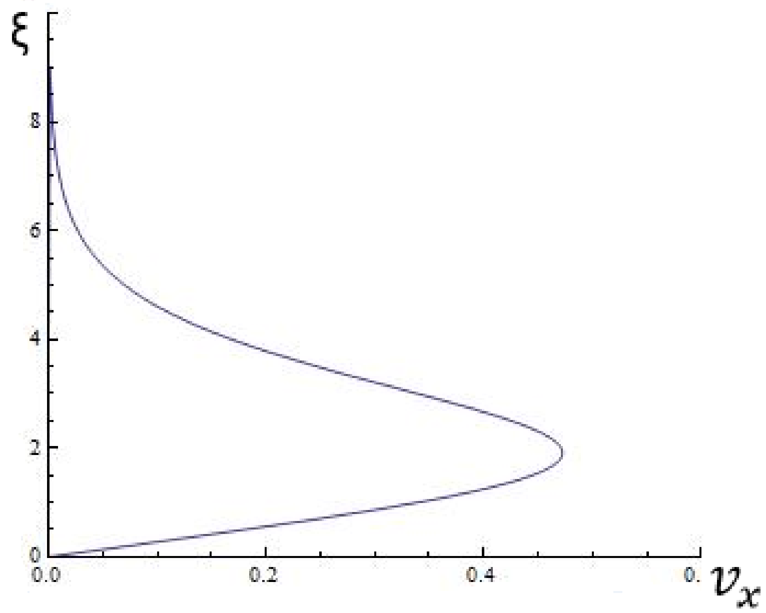

Figure 3.6: Parametric Solution for v_x for the wall jet plotted against x for $C = 1$ and $\nu = 1$ at $x = 1$. At $x = 1$ and $\xi = y$.

and therefore

$$v_x(x, y) = \left[\frac{10\ C}{9\nu x}\right]^{\frac{1}{2}} g(1 - g^3). \tag{3.8.40}$$

The solution (3.8.17) for $v_y(x, y)$ can be expressed in terms of g from the results for ξ, $F(\xi)$ and $F'(\xi)$.

The parametric solution for $v_x(x, y)$, for example, is obtained from (3.8.36) and (3.8.40) by giving the parameter g the values $0 \leq g \leq 1$. It is illustrated in Figure 3.6.

The solution for the wall jet again illustrates the double reduction theorem of Sjoberg [1]. The third order ODE obtained from the reduction of the PDE to an ODE (first reduction) could be integrated once to give a second order ODE (second reduction). It could be integrated further to give a parametric solution. We also saw again that not all of the constants of integration could be obtained from the boundary conditions. The conserved quantity was required to complete the solution and determine $F(\infty)$. An indication that a further condition such as a conserved quantity would be required was that all of the boundary conditions were homogeneous boundary conditions.

3.9 Comparison of results and of methods of solution

The results that we obtained for the free jet and the wall jet using an associated Lie point symmetry will be compared with the results obtained using a linear combination of all the Lie point symmetries of the PDE. The way the conserved quantity was used in the two methods will be compared. The Lie point symmetries of the PDE (3.3.4) are [8].

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad , \quad X_2 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi},$$
$$X_3 = \frac{\partial}{\partial x} \quad , \quad X_4 = \frac{\partial}{\partial \psi} \quad , \quad X_h = h(x)\frac{\partial}{\partial x}. \tag{3.9.1}$$

Then the linear combination of the Lie point symmetries is

$$\begin{aligned} X_L &= c_1X_1 + c_2X_3 + c_3X_3 + c_4X_4 + c_hX_h \\ &= [(c_1+c_2)x + c_3]\frac{\partial}{\partial x} + (c_1y + D(x))\frac{\partial}{\partial y} + (c_2\psi + c_4)\frac{\partial}{\partial \psi}. \end{aligned} \tag{3.9.2}$$

We will require the prolongation coefficients

$$\zeta_1 = -c_1\psi_x - \frac{dh}{dx}\psi_y, \tag{3.9.3}$$

$$\zeta_2 = (c_2 - c_1)\psi_y, \tag{3.9.4}$$

$$\zeta_{22} = (c_2 - 2c_1)\psi_{yy}, \tag{3.9.5}$$

which can be derived using the formulae (3.5.25), (3.5.27) and (3.5.29). We will show that the Lie point symmetry (3.5.45) for the free jet,

$$X_F = (c_1 + 3c_2)x\frac{\partial}{\partial x} + (2c_1y + h(x))\frac{\partial}{\partial y} + (c_3 + c_2\psi)\frac{\partial}{\partial \psi} \tag{3.9.6}$$

and (3.7.48) for the wall jet,

$$X_W = (c_1 + 4c_2 x)\frac{\partial}{\partial x} + (3c_1 y + h(x))\frac{\partial}{\partial y} + c_2\psi\frac{\partial}{\partial \psi} \tag{3.9.7}$$

are special cases of (3.9.2). We will do that two ways. Firstly , by associating (3.9.2) with the conserved vectors (3.4.2a) and (3.4.2b). Secondly , by finding the conditions on the constants c_1, c_2, c_3 and c_4 for the conserved quantities (3.4.13) and (3.4.18) to be independent of x.

3.9.1 The free jet

The conditions for X_L to be associated with a conserved vector are (3.4.5a) and (3.4.5b). We find that X_L is associated with the elementary conserved vector (3.4.2a) provided

$$(2c_2 - c_1)T^1 = 0, \qquad (2c_2 - c_1)T^2 = 0, \tag{3.9.8}$$

that is provided

$$c_1 = 2c_2. \tag{3.9.9}$$

The Lie point symmetry (3.9.2) becomes

$$X = (c_3 + 3c_2 x)\frac{\partial}{\partial x} + (2c_2 y + h(x))\frac{\partial}{\partial y} + (c_4 + c_2\psi)\frac{\partial}{\partial \psi}. \tag{3.9.10}$$

By setting $c_3 = c_1$ and $c_4 = c_3$, (3.9.10) agrees with (3.9.6) for the free jet.

Consider now the conserved quantity for the free jet (3.4.13). We saw in Section 3.6 that when the associated Lie point symmetry is used to reduce the PDE (3.3.4) to an ODE, the conserved quantity (3.6.16) is independent of x identically. When X_L is used to reduce the PDE it is found that [9],

$$\psi(x,y) = \left[x + \frac{c_3}{c_1 + c_2}\right]^{\frac{c_2}{c_1+c_2}} F(\xi) - \frac{c_4}{c_2}, \tag{3.9.11}$$

where

$$\xi = \frac{y}{\left(x + \frac{c_3}{c_1+c_2}\right)^{\frac{c_1}{c_1+c_2}}} - G(x) \tag{3.9.12}$$

and therefore

$$E = \left[x + \frac{c_3}{c_1 + c_2}\right]^{\frac{2c_2 - c_1}{c_1 + c_2}} \int_{-\infty}^{\infty} \left(\frac{dF}{d\xi}\right)^2 d\xi. \qquad (3.9.13)$$

Unlike the conserved quantity (3.6.16) obtained using the associated Lie point symmetry, (3.9.13)depends on x. It is independent of x provided $c_1 = 2c_2$. This condition is the same as the condition(3.9.9) for X_L to be associated with the elementary conserved vector.

3.9.2 The wall jet

Using conditions (3.4.5a) and (3.4.5b) it can be shown that X_L is associated with the second conserved vector (3.4.2b) provided

$$c_4 T^1_{(1)} + (3c_2 - c_1)T^1_{(2)} = 0 \ , \qquad c_4 T^2_{(1)} + (3c_2 - c_1)T^2_{(2)} = 0, \qquad (3.9.14)$$

where $(T^1_{(1)}, T^2_{(1)})$ are the components of the elementary conserved vector (3.4.2a) and $(T^1_{(2)}, T^2_{(2)})$ are the components of the second conserved vector (3.4.2b). Thus X_L is associated with the second conserved vector (3.4.2b) provided

$$c_4 = 0, \quad and \qquad c_1 = 3c_2 \qquad (3.9.15)$$

The Lie point symmetry X_L becomes:

$$X = (c_3 + 4c_2 x)\frac{\partial}{\partial x} + (3c_2 y + h(x))\frac{\partial}{\partial y} + c_2\psi\frac{\partial}{\partial \psi}. \qquad (3.9.16)$$

If we set $c_3 = c_1$ then (3.9.16) agrees with the Lie point symmetry (3.9.7) derived directly from the determining equations (3.4.5a) and (3.4.5b) and which generated the invariant solution for the wall jet.

Consider now the conserved quantity, for the wall jet, (3.4.18). The condition (3.3.14c), namely

$$\psi(x, 0) = 0, \qquad (3.9.17)$$

was used in its derivation. From the boundary conditions (3.8.14) for the wall jet,

$$F(0) = 0 \tag{3.9.18}$$

Imposing conditions (3.9.17) and (3.9.18) on the stream function (3.9.11) gives

$$c_4 = 0. \tag{3.9.19}$$

By using (3.9.11)and (3.9.12) the conserved quantity (3.4.18) becomes [10]

$$C = \left[x + \frac{c_3}{c_1 + c_2}\right]^{\frac{3c_3 - c_1}{c_1 + c_2}} \int_0^\infty F(\xi) \left(\frac{dF}{d\xi}\right)^2 d\xi. \tag{3.9.20}$$

Equation (3.9.20) compares with (3.9.20) which is independent of x and was derived using the associated Lie point symmetry (3.9.7). Equation (3.9.20) is independent of x provided

$$c_1 = 3c_2. \tag{3.9.21}$$

Conditions (3.9.19) and (4.2.1) agree with condition (3.9.15) for X_L to be associated with the second conserved vector.

3.10 Conclusions

The method which uses a linear combination of all the Lie point symmetries of the PDE is preferred if several problems have to be solved described by the PDE. The same reduction of the PDE to an ODE can be used for several problems because it contains undetermined constants. This method also applies to boundary layer problems for which there is no conserved quantity or conserved vector. If only one or two problems have to be solved which have conserved vectors such as jet flow problems then the derivation of the associated Lie point symmetry and its application to reduce the PDE to an ODE is a more direct and less laborious way to derive the invariant solution.The conserved vector depends on partial derivatives which are at least one order less than the order of the PDE and therefore simpler prolongation formulae of at least one order less are required. Also, the two smaller determining equations for a Lie point symmetry of the PDE to be associated with a conserved vector are easier to analyse than the one large determining equation for the Lie point symmetries of the PDE.

The double reduction theorem of Sjoberg ensures that the ODE can be integrated at least once. We have seen that the condition for the conserved quantity to be independent of x is equivalent to using an associated Lie point symmetry to derive the invariant solution. The double reduction theorem of Sjoberg will therefore apply to both methods of solution.

Chapter 4

Two-dimensional two-fluid free jet

4.1 Introduction

In this chapter we will investigate the two-dimensional two-fluid free jet. The Lie point symmetry associated with the elementary conservation law for each jet will be used to derive the invariant solution.

Two-fluid two-dimensional and axisymmetric jets were studied recently by Herczynsk, Weidman and Burde [12] who derived a conserved quantity for the two-fluid jets and similarity solutions. The two jets are not independent but are related by boundary conditions at the interface. We propose to derive the conserved quantity for the two-fluid jet from conservation laws for the boundary layer equations for each jet. We will obtain the interface $y = \phi(x)$ between the two fluids in the process of deriving the group invariant solutions for the stream functions and velocity components. We saw for the one-fluid free jet in Chapter 3 that the associated Lie point symmetry contained an arbitrary function $g(x)$. We will see that the arbitrary functions $g_1(x)$ and

$g_2(x)$ which occur in the Lie point symmetries for the two-fluid jet are closely related to the interface $\phi(x)$ between the two fluids.

An aim of this chapter is to investigate if the systematic method introduced by Naz et al. [2] to derive a conserved quantity for a jet can be extended to the two-fluid (and therefore multi-fluid) jet. There are significant differences between the one-fluid and two-fluid jets. For the one fluid jet the conserved quantity can be obtained by integrating the conservation law for the PDE across the jet and imposing boundary conditions. For the two-fluid jet there is no conservation law for the whole jet because there is not a PDE that describes the whole jet. There are two PDE's, one for the upper fluid jet and one for the lower fluid jet. For the two-fluid jet the conserved quantity applied for the combined jet consisting of the two fluids. There is a conservation law for the PDE of each fluid and instead of boundary conditions at the interface between the fluids there are matching conditions.

4.2 Mathematical model

Upper and lower layer fluids are simultaneously discharged into the surrounding medium consisting of the two fluid layers at rest as shown in Figure 4.1. The fluids are incompressible. To keep the notation simple we let $v_x = u$ and $v_y = v$. The fluid variables in the upper layer are denoted by a suffix 1 and are

$$u_1 = u_1(x, y), \quad v_1 = v_1(x, y), \quad p_1 = p_1(x, y), \quad \rho_1 \tag{4.2.1}$$

while the fluid variables in the lower layer are denoted by a suffix 2 and are

$$u_2 = u_2(x, y), \quad v_2 = v_2(x, y), \quad p_2 = p_2(x, y), \quad \rho_2 \tag{4.2.2}$$

where the fluid densities, ρ_1 and ρ_2, are constants. For hydrodynamic stability the heavier fluid must lie below the lighter fluid and therefore

$$\rho_2 > \rho_1. \tag{4.2.3}$$

It is assumed that the Reynolds number is sufficiently large that Prandtl's boundary layer equations apply to the jet flow in each layer. The flow is steady. Expressed in terms of the velocity components the boundary layer equations for the upper and lower fluids are as follows:

Upper fluid :

$$u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = \nu_1 \frac{\partial^2 u_1}{\partial y^2}, \tag{4.2.4}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{4.2.5}$$

$$\nu_1 = \frac{\mu_1}{\rho_1}. \tag{4.2.6}$$

Lower fluid :

$$u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} = \nu_2 \frac{\partial^2 u_2}{\partial y^2}, \tag{4.2.7}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0. \tag{4.2.8}$$

$$\nu_2 = \frac{\mu_2}{\rho_2}. \tag{4.2.9}$$

The interface between the two fluids is

$$y = \phi(x). \tag{4.2.10}$$

We will first state the boundary conditions and then give a brief explanation of each boundary condition.

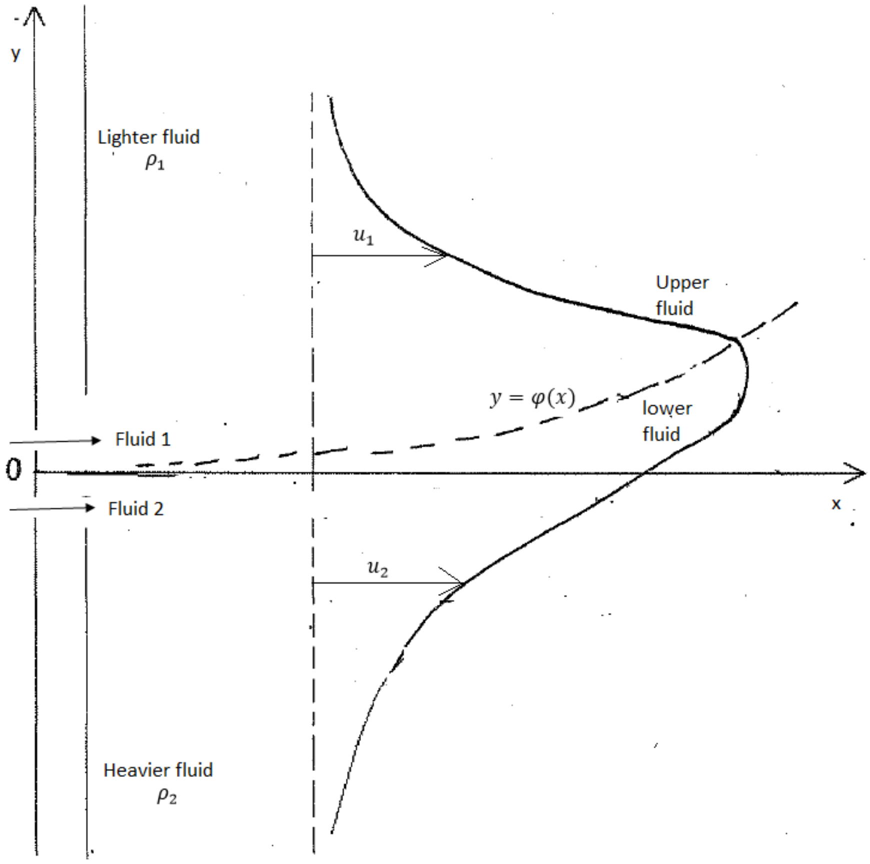

Figure 4.1: Velocity profile for a two-fluid free jet.

Boundary conditions

$$y = +\infty : \quad u_1 = 0, \tag{4.2.11}$$

$$y = \phi(x) : \quad u_1 = u_2, \tag{4.2.12}$$

$$y = \phi(x) : \quad v_1 = v_2, \tag{4.2.13}$$

$$y = \phi(x) : \quad \mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y}, \tag{4.2.14}$$

$$y = \phi(x) : \quad -p_1 + 2\mu_1 \frac{\partial v_1}{\partial y} = -p_2 + 2\mu_2 \frac{\partial v_2}{\partial y}, \tag{4.2.15}$$

$$y = -\infty : \quad u_2 = 0. \tag{4.2.16}$$

Conditions (4.2.11) and (4.2.16) state that the x-component of the fluid velocity vanishes far from the interface. The y-component of velocity does not necessarily vanish as $y \to \pm\infty$ but is determined as part of the solution.

Equations (4.2.12) and (4.2.13) state that the velocity is continuous at the interface. Equation (4.2.14) states that the tangential stress is continuous at the interface. The tangential stress is approximately in the x-direction and is

$$\tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \simeq \mu \frac{\partial u}{\partial y} \tag{4.2.17}$$

in the boundary layer approximation. Continuity of the tangential stress

$$\tau_{yx}^{(1)}(x, \phi(x)) = \tau_{yx}^{(2)}(x, \phi(x)) \tag{4.2.18}$$

therefore gives (4.2.14), Equation (4.2.15) states that the normal stress is continuous at the interface. The normal stress is approximately in the y-direction and is

$$\tau_{yy} = -p + 2\mu \frac{\partial v}{\partial y}. \tag{4.2.19}$$

Continuity of the normal stress

$$\tau_{yy}^{(1)}(x, \phi(x)) = \tau_{yy}^{(2)}(x, \phi(x)) \tag{4.2.20}$$

therefore gives (4.2.15).

A property of the interface is that a fluids particle on the interface remains on the interface. Thus

$$\frac{D}{Dt} \left(y - \phi(x) \right) |_{y=\phi(x)} = 0 \tag{4.2.21}$$

and therefore

$$\frac{Dy}{Dt}|_{y=\phi(x)} - u(x, \phi(x)) \frac{d\phi}{dx} = 0 \tag{4.2.22}$$

which is

$$v(x, \phi(x)) - u(x, \phi(x)) \frac{d\phi}{dx} = 0. \tag{4.2.23}$$

Equation (4.2.23) applies to both fluid and therefore

$$v_1(x, \phi(x)) - u_1(x, \phi(x)) \frac{d\phi}{dx} = 0, \tag{4.2.24}$$

$$v_2(x, \phi(x)) - u_2(x, \phi(x)) \frac{d\phi}{dx} = 0. \tag{4.2.25}$$

These two results will be used when deriving the conserved quantity for the two-fluid jet. They can also be interpreted as the condition that the normal velocity component at the interface vanishes.

Since the jets are two-dimensional and the fluids are incompressible we can introduce a stream function, $\psi_1(x,y)$ and $\psi_2(x,y)$, for each fluid defined by:

$$u_i = \frac{\partial \psi}{\partial y}\ , \quad v_i = -\frac{\partial \psi}{\partial x}, \qquad i = 1, 2. \tag{4.2.26}$$

The continuity equations (4.2.5) and (4.2.8) are identically satisfied. Expressed in terms of the stream functions the two-fluid jet problem can be stated as follows:

$$\text{upper fluid :} \quad \frac{\partial \psi_1}{\partial y}\frac{\partial^2 \psi_1}{\partial x \partial y} - \frac{\partial \psi_1}{\partial x}\frac{\partial^2 \psi_1}{\partial y^2} = \nu_1 \frac{\partial^3 \psi_1}{\partial y^3}. \tag{4.2.27}$$

$$\text{lower fluid :} \quad \frac{\partial \psi_2}{\partial y}\frac{\partial^2 \psi_2}{\partial x \partial y} - \frac{\partial \psi_2}{\partial x}\frac{\partial^2 \psi_2}{\partial y^2} = \nu_2 \frac{\partial^3 \psi_2}{\partial y^3}. \tag{4.2.28}$$

boundary conditions:

$$y = +\infty : \quad \frac{\partial \psi_1}{\partial y} = 0, \tag{4.2.29}$$

$$y = \phi(x) : \quad \frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial y}, \tag{4.2.30}$$

$$y = \phi(x) : \quad \frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x}, \tag{4.2.31}$$

$$y = \phi(x) : \quad \mu_1 \frac{\partial^2 \psi_1}{\partial y^2} = \mu_2 \frac{\partial^2 \psi_2}{\partial y^2}, \tag{4.2.32}$$

$$y = \phi(x) : \quad -p_1 + 2\mu_1 \frac{\partial^2 \psi_1}{\partial y \partial x} = -p_2 + 2\mu_2 \frac{\partial^2 \psi_2}{\partial y \partial x}, \tag{4.2.33}$$

$$y = -\infty : \quad \frac{\partial \psi_2}{\partial y} = 0. \tag{4.2.34}$$

interface conditions:

$$\frac{\partial \psi_1}{\partial x}(x, \phi(x)) + \frac{\partial \psi_1}{\partial y}(x, \phi(x)) \frac{d\phi}{dx} = 0, \tag{4.2.35}$$

$$\frac{\partial \psi_2}{\partial x}(x, \phi(x)) + \frac{\partial \psi_2}{\partial y}(x, \phi(x)) \frac{d\phi}{dx} = 0. \tag{4.2.36}$$

To complete the mathematical model the conserved quantity for the two-fluid jet is required. The conserved quantity will be derived in the next section.

4.3 Conserved quantity for the two-fluid jet

For the free jet described by Prandtl's boundary layer equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3} \tag{4.3.1}$$

the conserved vector is from (3.4.2a),

$$T^1 = \psi_y^2 \ , \qquad T^2 = -\psi_x\psi_y - \frac{\mu}{\rho}\psi_{yy} \tag{4.3.2}$$

and the associated Lie point symmetry is, from (3.5.45)

$$X = (c_1 + 3c_2x)\frac{\partial}{\partial x} + (2c_2y + g(x))\frac{\partial}{\partial y} + (c_3 + c_2\psi)\frac{\partial}{\partial\psi}. \tag{4.3.3}$$

Because the boundary condition (4.2.31) is expressed in terms of the dynamic viscosity μ and not in terms of the kinematic velocity ν, the conserved vector (3.4.2a) is multiplied by the constant ρ. Thus

$$T^1 = \rho\psi_y^2, \qquad T^2 = -\rho\psi_x\psi_y - \mu\psi_{yy}, \tag{4.3.4}$$

We consider the general case in which $c_2 \neq 0$. In order to simplify the subsequent calculations we divide the associated Lie point symmetry (4.3.3) by c_2. This gives

$$X^* = 3(c_1^* + x)\frac{\partial}{\partial x} + 2(y + g^*(x))\frac{\partial}{\partial y} + (c_3^* + \psi)\frac{\partial}{\partial\psi}, \tag{4.3.5}$$

where

$$X^* = \frac{1}{c_2}X, \qquad c_1^* = \frac{c_1}{3c_2}, \quad c_3^* = \frac{c_3}{c_2}. \tag{4.3.6}$$

We will denote the constants in the associated Lie point symmetry of the upper fluid by a_1 and a_3 and in the lower fluid by b_1 and b_3. We therefore have for each fluid:

upper fluid

$$T_1^1 = \rho_1\psi_{1y}^2, \qquad T_1^2 = -\rho_1\psi_{1x}\psi_{1y} - \mu_1\psi_{1yy}, \tag{4.3.7}$$

$$X_1 = 3(a_1 + x)\frac{\partial}{\partial x} + 2\left(y + g_1(x)\right)\frac{\partial}{\partial y} + (a_3 + \psi_1)\frac{\partial}{\partial\psi_1}. \tag{4.3.8}$$

lower fluid

$$T_2^1 = \rho_2 \psi_{2y}^2, \qquad T_2^2 = -\rho_2 \psi_{2x}\psi_{2y} - \mu_2 \psi_{2yy}, \tag{4.3.9}$$

$$X_2 = 3(b_1 + x)\frac{\partial}{\partial x} + 2\left(y + g_2(x)\right)\frac{\partial}{\partial y} + (b_3 + \psi_2)\frac{\partial}{\partial \psi_2}. \tag{4.3.10}$$

When $x, y, \psi, \psi_x, \psi_y, \ldots$ are regarded as independent variables the conservation laws are written in terms of the total derivatives as

$$D_x T^1 + D_y T^2|_{\mathsf{PDE}} = 0. \tag{4.3.11}$$

When x and y are regarded as the independent variables then

$$\frac{\partial T^1}{\partial x}\left(x, y, \psi(x,y), \psi_x(x,y), \psi_y(x,y), \ldots\right) = D_x T^1, \tag{4.3.12}$$

$$\frac{\partial T^2}{\partial y}\left(x, y, \psi(x,y), \psi_x(x,y), \psi_y(x,y), \ldots\right) = D_y T^2, \tag{4.3.13}$$

and the conservation law can be written in terms of partial derivatives as

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y}|_{\mathsf{PDE}} = 0. \tag{4.3.14}$$

We integrate the conservation laws across both jets from $y = -\infty$ to $y = +\infty$ at constant x. Now

$$\text{upper fluid :} \qquad \frac{\partial T_1^1}{\partial x} + \frac{\partial T_1^2}{\partial y} = 0, \qquad \phi(x) \le y \le \infty, \tag{4.3.15}$$

$$\text{lower fluid :} \qquad \frac{\partial T_2^1}{\partial x} + \frac{\partial T_2^2}{\partial y} = 0, \qquad -\infty \le y \le \phi(x). \tag{4.3.16}$$

Thus

$$\int_{-\infty}^{\phi(x)} \frac{\partial T_2^1}{\partial x}(x,y)dy + \int_{-\infty}^{\phi(x)} \frac{\partial T_2^2}{\partial y}(x,y)dy$$
$$+ \int_{\phi(x)}^{\infty} \frac{\partial T_1^1}{\partial x}(x,y)dy + \int_{\phi(x)}^{\infty} \frac{\partial T_1^2}{\partial y}(x,y)dy = 0. \tag{4.3.17}$$

Now

$$\int_{-\infty}^{\phi(x)} \frac{\partial T_2^1}{\partial x}(x,y)dy = \int_{-\infty}^{\phi(x)} \frac{\partial}{\partial x}\left[\rho_2 \left(\frac{\partial \psi_2}{\partial y}(x,y)\right)^2\right] dy. \tag{4.3.18}$$

But the general result for differentiation under the integral sign is [13]

$$\frac{d}{dx}\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial x}(x,y)dy + f(x,\phi_2(x))\frac{d\phi_2}{dx} - f(x,\phi_1(x))\frac{d\phi_1}{dx}. \tag{4.3.19}$$

Thus if β is a large positive constant

$$\frac{d}{dx}\int_{-\beta}^{\phi(x)} \rho_2\left(\frac{\partial\psi_2}{\partial y}(x,y)\right)^2 dy = \int_{-\beta}^{\phi(x)} \frac{\partial}{\partial x}\left(\rho_2\left(\frac{\partial\psi_2}{\partial y}(x,y)\right)^2\right) dy + \rho_2\left(\frac{\partial\psi_2}{\partial y}(x,\phi(x))\right)^2\frac{d\phi}{dx} \tag{4.3.20}$$

and letting $\beta \to \infty$ we have

$$\int_{-\infty}^{\phi(x)} \frac{\partial}{\partial x}\left[\rho_2\left(\frac{\partial\psi_2}{\partial y}(x,y)\right)^2\right] dy = \frac{d}{dx}\int_{-\infty}^{\phi(x)} \rho_2\left(\frac{\partial\psi_2}{\partial y}(x,y)\right)^2 dy - \rho_2\left(\frac{\partial\psi_2}{\partial y}(x,\phi(x))\right)^2\frac{d\phi}{dx} \tag{4.3.21}$$

and therefore

$$\int_{-\infty}^{\phi(x)} \frac{\partial}{\partial x}T_2^1(x,y)dy = \frac{d}{dx}\int_{-\infty}^{\phi(x)} \rho_2\left(\frac{\partial\psi_2}{\partial y}(x,y)\right)^2 dy - \rho_2\left(\frac{\partial\psi_2}{\partial y}(x,\phi(x))\right)^2\frac{d\phi}{dx}. \tag{4.3.22}$$

Similarly,

$$\int_{\phi(x)}^{\infty} \frac{\partial T_1^1}{\partial x}(x,y)dy = \frac{d}{dx}\int_{\phi(x)}^{\infty} \rho_1\left(\frac{\partial\psi_1}{\partial y}(x,y)\right)^2 dy + \rho_1\left(\frac{\partial\psi_1}{\partial y}(x,\phi(x))\right)^2\frac{d\phi}{dx}. \tag{4.3.23}$$

Also

$$\begin{aligned}\int_{-\infty}^{\phi(x)} \frac{\partial T_2^2}{\partial x}(x,y)dy &= \left[T_2^2(x,y)\right]_{-\infty}^{\phi(x)} \\ &= \left[-\rho_2\frac{\partial\psi_2}{\partial x}(x,y)\frac{\partial\psi_2}{\partial y}(x,y) - \mu_2\frac{\partial^2\psi_2}{\partial y^2}(x,y)\right]_{-\infty}^{\phi(x)} \\ &= -\rho_2\frac{\partial\psi_2}{\partial x}(x,\phi(x))\frac{\partial\psi_2}{\partial y}(x,\phi(x)) - \mu_2\frac{\partial^2\psi_2}{\partial y^2}(x,\phi(x)) \\ &\quad + \rho_2\frac{\partial\psi_2}{\partial x}(x,-\infty)\frac{\partial\psi_2}{\partial y}(x,-\infty) + \mu_2\frac{\partial^2\psi_2}{\partial y^2}(x,-\infty).\end{aligned} \tag{4.3.24}$$

Now from the boundary condition (4.2.34)

$$\frac{\partial \psi_2}{\partial y}(x,-\infty) = 0 \tag{4.3.25}$$

and we assume further that

$$\frac{\partial u}{\partial y}(x,-\infty) = \frac{\partial^2 \psi_2}{\partial y^2}(x,-\infty) = 0 \tag{4.3.26}$$

Thus (4.3.24) reduces to

$$\int_{-\infty}^{\phi(x)} \frac{\partial T_2^2}{\partial y}(x,y)dy = -\rho_2 \frac{\partial \psi_2}{\partial x}(x,\phi(x))\frac{\partial \psi_2}{\partial y}(x,\phi(x)) - \mu_2 \frac{\partial^2 \psi_2}{\partial y^2}(x,\phi(x)). \tag{4.3.27}$$

Similarly

$$\int_{\phi(x)}^{\infty} \frac{\partial T_1^2}{\partial y}(x,y)dy = \rho_1 \frac{\partial \psi_1}{\partial x}(x,\phi(x))\frac{\partial \psi_1}{\partial x}(x,\phi(x)) + \mu_1 \frac{\partial^2 \psi_1}{\partial y^2}(x,\phi(x)). \tag{4.3.28}$$

Hence substituting (4.3.22), (4.3.23), (4.3.27) and (4.3.28) into (4.3.17) we obtain

$$\begin{aligned}
&\frac{d}{dx}\int_{-\infty}^{\phi(x)} \rho_2 \left(\frac{\partial \psi_2}{\partial y}\right)^2 dy + \frac{d}{dx}\int_{\phi(x)}^{\infty} \rho_1 \left(\frac{\partial \psi_1}{\partial y}\right)^2 dy \\
&+ \rho_1 \frac{\partial \psi_1}{\partial y}(x,\phi(x)) \left[\frac{\partial \psi_1}{\partial x}(x,\phi(x)) + \frac{\partial \psi_1}{\partial y}(x,\phi(x))\frac{d\phi}{dx}\right] \\
&- \rho_2 \frac{\partial \psi_2}{\partial y}(x,\phi(x)) \left[\frac{\partial \psi_2}{\partial x}(x,\phi(x)) + \frac{\partial \psi_2}{\partial y}(x,\phi(x))\frac{d\phi}{dx}\right] \\
&+ \mu_1 \frac{\partial^2 \psi_1}{\partial y^2}(x,\phi(x)) - \mu_2 \frac{\partial^2 \psi_2}{\partial y^2}(x,\phi(x)) = 0.
\end{aligned} \tag{4.3.29}$$

By using the interfacial conditions (4.2.35) and (4.2.36) and the tangential stress boundary condition (4.2.32) at the interface, (4.3.29) reduces to

$$\frac{d}{dx}\left[\int_{-\infty}^{\phi(x)} \rho_2 \left(\frac{\partial \psi_2}{\partial y}(x,y)\right)^2 dy + \int_{\phi(x)}^{\infty} \rho_1 \left(\frac{\partial \psi_1}{\partial y}(x,y)\right)^2 dy\right] = 0. \tag{4.3.30}$$

Hence

$$\int_{-\infty}^{\phi(x)} \rho_2 \left(\frac{\partial \psi_2}{\partial y}(x,y)\right)^2 dy + \int_{\phi(x)}^{\infty} \rho_1 \left(\frac{\partial \psi_1}{\partial y}(x,y)\right)^2 dy = J \tag{4.3.31}$$

where J is independent of x and therefore constant.

The conserved quantity for the two-fluid jet is J. Its physical significance is that it is the total momentum flux in the streamwise x-direction.

4.4 Invariant solution for two-fluid free jet

Consider first the upper fluid. The associated Lie point symmetry, X_1, is given by (4.3.8). The function $\psi_1 = \Phi_1(x, y)$ is an invariant solution generated by (4.3.8) provided

$$X_1 \left(\psi_1 - \Phi_1(x, y)\right)|_{\psi_1 = \Phi_1(x,y)} = 0, \tag{4.4.1}$$

which is satisfied provided

$$3(a_1 + x)\frac{\partial \Phi_1}{\partial x} + 2(y + g_1(x))\frac{\partial \Phi_1}{\partial y} = a_3 + \psi_1. \tag{4.4.2}$$

The differential equations of the characteristic curves of the first order PDE (4.4.2) are

$$\frac{dx}{3(a_1 + x)} = \frac{dy}{2(y + g_1(x))} = \frac{d\psi_1}{a_3 + \psi_1}. \tag{4.4.3}$$

The first pair of terms gives

$$\frac{dy}{dx} - \frac{2}{3(a_1 + x)}y = \frac{2g_1(x)}{3(a_1 + x)}. \tag{4.4.4}$$

The integrating factor is $(a_1 + x)^{-\frac{2}{3}}$. The solution of (4.4.4) is

$$\frac{y - G_1(x)}{(a_1 + x)^{\frac{2}{3}}} = k_1 \tag{4.4.5}$$

where k_1 is a constant and

$$G_1(x) = \frac{2}{3}(a_1 + x)^{\frac{2}{3}} \int^x \frac{g_1(x)dx}{(a_1 + x)^{\frac{5}{3}}}. \tag{4.4.6}$$

The first and last terms in (4.4.3) give

$$\frac{\psi_1 + a_3}{(a_1 + x)^{\frac{1}{3}}} = k_2, \tag{4.4.7}$$

where k_2 is a constant. The general solution of the PDE (4.4.2) is

$$k_2 = F_1(k_1) \tag{4.4.8}$$

where F_1 is an arbitrary function. Since $\psi_1 = \Phi_1(x, y)$ it follows that

$$\psi_1(x, y) = (a_1 + x)^{\frac{1}{3}} F_1(\xi_1) - a_3 \tag{4.4.9}$$

where

$$\xi_1 = \frac{y - G_1(x)}{(a_1 + x)^{\frac{2}{3}}} \tag{4.4.10}$$

and $G_1(x)$ is given by (4.4.6).

Similarly it can be shown that for the lower fluid,

$$\psi_2(x, y) = (b_1 + x)^{\frac{1}{3}} F_1(\xi_2) - b_3 \tag{4.4.11}$$

where

$$\xi_2 = \frac{y - G_2(x)}{(b_1 + x)^{\frac{2}{3}}} \tag{4.4.12}$$

and

$$G_2(x) = \frac{2}{3}(b_1 + x)^{\frac{2}{3}} \int^x \frac{g_2(x)dx}{(b_1 + x)^{\frac{5}{3}}}. \tag{4.4.13}$$

The PDEs (4.2.27) and (4.2.28) for $\psi_1(x, y)$ and $\psi_2(x, y)$ are transformed to ODEs for $F_1(\xi_1)$ and $F_2(\xi_2)$. The velocity components (4.2.26) are also expressed in terms of $F_1(\xi_1)$ and $F_2(\xi_2)$. We obtain

upper fluid

$$3\frac{\mu_1}{\rho_1}\frac{d^3F_1}{d\xi_1^3} + \frac{d}{d\xi_1}\left(F_1\frac{dF_1}{d\xi_1}\right) = 0, \tag{4.4.14}$$

$$u_1(x, y) = (a_1 + x)^{-\frac{1}{3}}\frac{dF_1}{d\xi_1}, \tag{4.4.15}$$

$$v_1(x, y) = \frac{1}{3}(a_1 + x)^{-\frac{2}{3}}\left[2\xi_1\frac{dF_1}{d\xi_1} - F_1(\xi_1)\right]. \tag{4.4.16}$$

lower fluid

$$3\frac{\mu_2}{\rho_2}\frac{d^3F_2}{d\xi_2^3} + \frac{d}{d\xi_2}\left(F_2\frac{dF_2}{d\xi_2}\right) = 0, \tag{4.4.17}$$

$$u_2(x, y) = (b_1 + x)^{-\frac{1}{3}}\frac{dF_2}{d\xi_2}, \tag{4.4.18}$$

$$v_2(x, y) = \frac{1}{3}(b_1 + x)^{-\frac{2}{3}}\left[2\xi_2\frac{dF_2}{d\xi_2} - F_2(\xi_2)\right]. \tag{4.4.19}$$

Consider now the coordinates ξ_1 and ξ_2 defined by (4.4.10) and (4.4.12). We choose the origin of the coordinates to be at the interface $y = \phi(x)$. Thus

$$y = \phi(x): \quad \xi_1 = \frac{\phi(x) - G_1(x)}{(a_1 + x)^{\frac{2}{3}}} = 0, \tag{4.4.20}$$

$$\xi_2 = \frac{\phi(x) - G_2(x)}{(b_1 + x)^{\frac{2}{3}}} = 0. \tag{4.4.21}$$

and therefore

$$G_1(x) = G_2(x) = \phi(x), \tag{4.4.22}$$

where $G_1(x)$ and $G_2(x)$ are defined by (4.4.6) and (4.4.13). This relates the arbitrary function $g_1(x)$ and $g_2(x)$ in the associated Lie point symmetries to the interface $\phi(x)$. For the one-fluid jet we chose $g(x) = 0$. Consider next the constants a_1 and b_1. Now

$$\delta_1(x) = (a_1 + x)^{\frac{2}{3}}, \qquad \delta_2(x) = (b_1 + x)^{\frac{2}{3}}, \tag{4.4.23}$$

are a measure of the thickness of the jet in each fluid layer. We expect each jet thickness to tend to zero as $x \to 0$:

$$\lim_{x \to 0} \delta_1(x) = a_1^{\frac{2}{3}} = 0, \qquad \lim_{x \to 0} \delta_2(x) = b_1^{\frac{2}{3}} = 0. \tag{4.4.24}$$

We therefore take $a_1 = 0$ and $b_1 = 0$. This will also ensure that the velocities $u_1(x, \phi(x))$ and $u_2(x, \phi(x))$ are infinite at $x = 0$ to give a finite volume flux at the infinitely thin orifice in the wall. Thus

$$\xi_1 = \xi_2 = \xi = \frac{y - \phi(x)}{x^{\frac{2}{3}}} \tag{4.4.25}$$

and $g_1(x) = g_2(x) = g(x)$ where

$$\frac{2}{3} x^{\frac{2}{3}} \int^x \frac{g(x^*) dx^*}{x^{*\frac{2}{3}}} = \phi(x). \tag{4.4.26}$$

The boundary conditions (4.2.29) to (4.2.34), the interfacial conditions (4.2.35) and (4.2.36) and the conserved quantity (4.3.31) can now be written in terms of the similarity variables. The problem can be summarised as follows.

Similarity variables

$$\psi_1(x, y) = x^{\frac{1}{3}} F_1(\xi) - a_3, \tag{4.4.27}$$

$$\psi_2(x, y) = x^{\frac{1}{3}} F_2(\xi) - b_3, \tag{4.4.28}$$

$$\xi = \frac{y - \phi(x)}{x^{\frac{2}{3}}}. \tag{4.4.29}$$

Ordinary differential equations

$$\frac{3\mu_1}{\rho_1} \frac{d^3 F_1}{d\xi^3} + \frac{d}{d\xi}\left(F_1 \frac{dF_1}{d\xi}\right) = 0, \tag{4.4.30}$$

$$\frac{3\mu_2}{\rho_2} \frac{d^3 F_2}{d\xi^3} + \frac{d}{d\xi}\left(F_2 \frac{dF_2}{d\xi}\right) = 0. \tag{4.4.31}$$

Boundary conditions

$$y = +\infty : \qquad \frac{dF_1}{d\xi}(\infty) = 0, \tag{4.4.32}$$

$$y = \phi(x) : \qquad \frac{dF_1}{d\xi}(0) = \frac{dF_2}{d\xi}(0), \tag{4.4.33}$$

$$y = \phi(x) : \qquad F_1(0) = F_2(0), \tag{4.4.34}$$

$$y = \phi(x) : \qquad \mu_1 \frac{d^2 F_1}{d\xi^2}(0) = \mu_2 \frac{d^2 F_2}{d\xi^2}(0), \tag{4.4.35}$$

$$y = \phi(x) : \qquad p_1 - p_2 = \frac{2}{3}\left[\mu_1 \frac{dF_1}{d\xi}(0) - \mu_2 \frac{dF_2}{d\xi}(0)\right] x^{-\frac{4}{3}}, \tag{4.4.36}$$

$$y = -\infty : \qquad \frac{dF_2}{d\xi}(-\infty) = 0. \tag{4.4.37}$$

Interfacial conditions

$$F_1(0) + 3x^{\frac{1}{3}} \frac{dF_1}{d\xi}(0) \frac{d\phi}{dx} = 0, \tag{4.4.38}$$

$$F_2(0) + 3x^{\frac{1}{3}} \frac{dF_2}{d\xi}(0) \frac{d\phi}{dx} = 0. \tag{4.4.39}$$

Conserved quantity

$$J = \int_{-\infty}^{0} \rho_2 \left(\frac{dF_2}{d\xi}\right)^2 d\xi + \int_{0}^{\infty} \rho_1 \left(\frac{dF_1}{d\xi}\right)^2 d\xi, \tag{4.4.40}$$

Fluid velocity components

$$u_1(x,y) = x^{-\frac{1}{3}}\frac{dF_1}{d\xi}, \tag{4.4.41}$$

$$v_1(x,y) = \frac{1}{3}x^{-\frac{2}{3}}\left[2\xi\frac{dF_1}{d\xi} - F_1(\xi)\right], \tag{4.4.42}$$

$$u_2(x,y) = x^{-\frac{1}{3}}\frac{dF_2}{d\xi}, \tag{4.4.43}$$

$$v_2(x,y) = \frac{1}{3}x^{-\frac{2}{3}}\left[2\xi\frac{dF_2}{d\xi} - F_2(\xi)\right]. \tag{4.4.44}$$

Consider now the solution of the problem. Consider first the upper fluid layer. Integrating (4.4.30) once with respect to ξ gives

$$3\frac{\mu_1}{\rho_1}\frac{d^2F_1}{d\xi^2} + F_1\frac{dF_1}{d\xi} = A_1 \tag{4.4.45}$$

where A_1 is a constant. For the one-fluid jet the boundary conditions at $\xi = 0$ gave $A_1 = 0$. For the two-fluid jet A_1 cannot be obtained from the conditions at $\xi = 0$ because they are matching conditions. Equation (4.4.44) can be written as

$$\frac{3\mu_1}{\rho_1}\frac{d^2F_1}{d\xi^2} + \frac{1}{2}\frac{d}{d\xi}(F_1^2) = A_1 \tag{4.4.46}$$

and integrating again gives

$$3\frac{\mu_1}{\rho_1}\frac{dF_1}{d\xi} + \frac{1}{2}F_1^2 = A_1\xi + c_1. \tag{4.4.47}$$

Now the left hand side of (4.4.47) is finite at $\xi = \infty$ from (4.4.32) because $v_1(x,\infty)$ is finite. Thus $A_1 = 0$. The constant $c_1 > 0$ because $\frac{dF_1}{d\xi} > 0$ since $u_1(x,y) > 0$. We write $c_1 = B^2/2$. Equation (4.4.47) becomes

$$\frac{6\mu_1}{\rho_1}\frac{dF_1}{d\xi} = B_1^2 - F_1^2 \tag{4.4.48}$$

which is variables separable. The general solution is

$$F_1(\xi) = B_1\tanh\left(\frac{B_1\rho_1}{6\mu_1}\xi + \alpha_1\right), \tag{4.4.49}$$

where α_1 is a constant.

Similarly, for the lower fluid layer

$$F_2(\xi) = B_2 \tanh\left(\frac{B_2 \rho_2}{6\mu_2}\xi + \alpha_2\right), \tag{4.4.50}$$

where B_2 and α_2 are constants. The general solutions (4.4.49) and (4.4.50) for the two-fluid jet are the same as (3.6.18) for the one-fluid.

Consider now the boundary conditions (4.4.32) to (4.4.37). The boundary condition (4.4.32) at $\xi = -\infty$ and (4.4.37) at $\xi = +\infty$ are identically satisfied. The boundary conditions (4.4.33) to (4.4.36) at $y = \phi(x)$ become

$$\frac{B_1^2 \rho_1}{\mu_1 \cosh^2 \alpha_1} = \frac{B_2^2 \rho_2}{\mu_2 \cosh^2 \alpha_2}, \tag{4.4.51}$$

$$B_1 \tanh \alpha_1 = B_2 \tanh \alpha_2, \tag{4.4.52}$$

$$\frac{B_1^3 \rho_1^2}{\mu_1} \frac{\sinh \alpha_1}{\cosh^3 \alpha_1} = \frac{B_2^3 \rho_2^2}{\mu_2} \frac{\sinh \alpha_2}{\cosh^3 \alpha_2}, \tag{4.4.53}$$

$$p_1 - p_2 = \frac{1}{9}\left[\frac{B_1^2 \rho_1}{\cosh^2 \alpha_1} - \frac{B_2^2 \rho_2}{\cosh^2 \alpha_2}\right] x^{-\frac{4}{3}}. \tag{4.4.54}$$

The interfacial conditions (4.4.38) and (4.4.39) become

$$\frac{d\phi}{dx} = -2\frac{\mu_1}{B_1 \rho_1} \sinh \alpha_1 \cosh \alpha_1 \; x^{-\frac{1}{3}}, \tag{4.4.55}$$

$$\frac{d\phi}{dx} = -2\frac{\mu_2}{B_2 \rho_2} \sinh \alpha_2 \cosh \alpha_2 \; x^{-\frac{1}{3}}. \tag{4.4.56}$$

For a non-trivial solution for $F_1(\xi)$ and $F_2(x)$ we require $B_1 \neq 0$ and $B_2 \neq 0$. If $\alpha_1 = 0$ but $\alpha_2 \neq 0$ then from (4.4.52)

$$B_2 = B_1 \frac{\tanh \alpha_1}{\tanh \alpha_2} = 0, \tag{4.4.57}$$

while if $\alpha_1 \neq 0$ but $\alpha_2 = 0$ then

$$B_1 = B_2 \frac{\tanh \alpha_2}{\tanh \alpha_1} = 0. \tag{4.4.58}$$

Thus $\alpha_1 = 0$, $\alpha_2 \neq 0$ and $\alpha_1 \neq 0$, $\alpha_2 = 0$ lead to trivial solutions. For non-trivial solutions with $B_1 \neq 0$ and $B_2 \neq 0$ we therefore need to consider $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_1 = 0$, $\alpha_2 = 0$.

Consider first $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Then from (4.4.52)

$$\frac{B_1}{B_2} = \frac{\tanh \alpha_2}{\tanh \alpha_1} = \frac{\sinh \alpha_2 \cosh \alpha_1}{\cosh \alpha_2 \sinh \alpha_2} \tag{4.4.59}$$

and from (4.4.51) and using (4.4.59),

$$\frac{\mu_1}{\mu_2} = \frac{\rho_1}{\rho_2} \frac{\sinh^2 \alpha_2}{\sinh^2 \alpha_1}. \tag{4.4.60}$$

Substituting (4.4.59) and (4.4.60) into (4.4.53) gives

$$\rho_1 = \rho_2. \tag{4.4.61}$$

But we are considering a heavier fluid with density ρ_2 below a lighter fluid with density ρ_1 and therefore $\rho_2 > \rho_1$. We will not consider the case $\rho_1 = \rho_2$.

Consider now $\alpha_1 = 0$ and $\alpha_2 = 0$. Then the boundary conditions (4.4.52) and (4.4.53) are identically satisfied. The boundary condition (4.4.51) reduces to

$$\frac{B_1}{B_2} = \left(\frac{\rho_2}{\rho_1} \frac{\mu_1}{\mu_2} \right)^{\frac{1}{2}}. \tag{4.4.62}$$

The remaining condition relating B_1 and B_2 is the conserved quantity J given by (4.4.40). Substituting (4.4.49) and (4.4.50) with $\alpha_1 = \alpha_2 = 0$ into (4.4.40) gives

$$J = \frac{1}{6} \left[\frac{B_2^3 \rho_2^2}{\mu_2} + \frac{B_1^3 \rho_1^2}{\mu_1} \right] \int_0^\infty \mathsf{sech}^4\, w \; dw. \tag{4.4.63}$$

But

$$\int_0^\infty \mathsf{sech}^4\, w \; dw = \left[\tanh w - \frac{1}{3} \tanh^2 w \right]_0^\infty = \frac{2}{3} \tag{4.4.64}$$

and therefore

$$J = \frac{1}{9} \left[\frac{B_2^3 \rho_2^2}{\mu_2} + \frac{B_1^3 \rho_1^2}{\mu_1} \right]. \tag{4.4.65}$$

Eliminating B_2 using (4.4.62) gives

$$J = \frac{B_1^3 \rho_1^2}{9\mu_1} \left[1 + \left(\frac{\rho_2 \mu_2}{\rho_1 \mu_1} \right)^{\frac{1}{2}} \right] \tag{4.4.66}$$

and hence

$$B_1 = \left(\frac{9\mu_1 J}{\rho_1^2} \right)^{\frac{1}{3}} \left[1 + \left(\frac{\rho_2 \mu_2}{\rho_1 \mu_1} \right)^{\frac{1}{2}} \right]^{-\frac{1}{3}}. \tag{4.4.67}$$

In summary we have found that

$$F_1(\xi) = B_1 \tanh \left[\frac{B_1 \rho_1}{6\mu_1} \xi \right], \tag{4.4.68}$$

$$F_2^(\xi) = B_2 \tanh \left[\frac{B_2 \rho_2}{6\mu_2} \xi \right], \tag{4.4.69}$$

$$\xi = \frac{y - \phi(x)}{x^{\frac{2}{3}}}, \tag{4.4.70}$$

where B_1 and B_2 are given in terms of J by (4.4.67) and (4.4.62). The velocity components (4.4.41) to (4.4.44) become

$$u_1(x,y) = \frac{B_1^2 \rho_1}{6\mu_1} x^{-\frac{1}{3}} \mathsf{sech}^2 \left[\frac{B_1 \rho_1}{6\mu_1} \xi \right], \tag{4.4.71}$$

$$u_2(x,y) = \frac{B_1^2 \rho_1}{6\mu_1} x^{-\frac{1}{3}} \mathsf{sech}^2 \left[\frac{B_1 \rho_1}{6\mu_1} \left(\frac{\mu_1}{\mu_2} \frac{\rho_2}{\rho_1} \right)^{\frac{1}{2}} \xi \right], \tag{4.4.72}$$

$$v_1(x,y) = \frac{1}{3} x^{-\frac{2}{3}} \left[\frac{B_1^2 \rho_1}{3\mu_1} \xi \mathsf{sech}^2 \left(\frac{B_1 \rho_1}{6\mu_1} \xi \right) - B_1 \tanh \left(\frac{B_1 \rho_1}{6\mu_1} \xi \right) \right], \tag{4.4.73}$$

$$v_2(x,y) = \frac{1}{3} x^{-\frac{2}{3}} \left[\frac{B_1^2 \rho_1}{3\mu_1} \xi \ \mathsf{sech}^2 \left(\frac{B_1 \rho_1}{6\mu_1} \left(\frac{\mu_1}{\mu_2} \frac{\rho_2}{\rho_1} \right)^{\frac{1}{2}} \xi \right) - B_1 \left(\frac{\rho_1 \mu_2}{\rho_2 \mu_1} \right)^{\frac{1}{2}} \tanh \left(\frac{B_1 \rho_1}{6\mu_1} \left(\frac{\mu_1}{\mu_2} \frac{\rho_2}{\rho_1} \right)^{\frac{1}{2}} \xi \right) \right] \tag{4.4.74}$$

The transverse velocity v_y does not varnish as $y \to \pm\infty$ for,

$$v_1(x, +\infty) = -\frac{B_1}{3} x^{-\frac{2}{3}}, \tag{4.4.75}$$

$$v_2(x, -\infty) = \frac{B_1}{3} \left(\frac{\rho_1 \mu_2}{\rho_2 \mu_1} \right)^{\frac{1}{2}} x^{-\frac{2}{3}}. \tag{4.4.76}$$

In the boundary layer approximation there is inflow in the transverse direction at $y = \pm\infty$ to maintain conservation of mass in the two-fluid jet.

The continuity of normal stress at the interface (4.4.54) and the interfacial conditions (4.4.55) and (4.4.56) have not yet been used. In the next section they will be used to determine the equation of the interface $y = \phi(x)$.

4.5 Interface

We first describe how Herczynski et al. [12] obtained the interface $y = \phi(x)$. We will then suggest an alternative derivation

4.5.1 Pressure difference equation

Herczynski et al. [12] first derived an expression for the pressure difference ,$p_1 - p_2$, at the interface. The interface is illustrated in Figure 4.2.

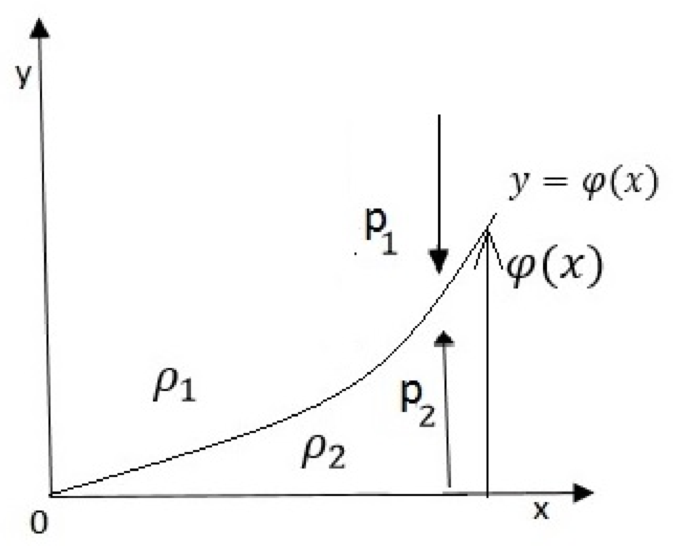

Figure 4.2: The interface $y = \phi(x)$ between two fluids of a two-fluid plane jet

Now

Upthrust on surface element dA of interface = weight of fluid displaced below dA

$$= (\rho_2 - \rho_1) g \phi(x) dA \tag{4.5.1}$$

Thus the hydrostatic pressures at the interface satisfy

$$(p_1 - p_2) dA = (\rho_2 - \rho_1) g \phi(x) dA \tag{4.5.2}$$

and therefore

$$p_1 - p_2 = (\rho_2 - \rho_1) g \phi(x). \tag{4.5.3}$$

A similar equation for $p_1 - p_2$ was derived by Lock [14] for the interface between parallel stream of fluid. Equation (4.4.54) becomes

$$(\rho_2 - \rho_1) g \phi(x) = \frac{1}{9} \left[\frac{B_1 \rho_1}{\cosh^2 \alpha_1} - \frac{B_2 \rho_2}{\cosh^2 \alpha_2} \right] x^{-\frac{4}{3}} \tag{4.5.4}$$

But $\alpha_1 = \alpha_2 = 0$ and using (4.4.62) we obtain

$$\phi(x) = \frac{B_1^2 \frac{\rho_1}{\rho_2} \frac{\mu_2}{\mu_1} \left[\frac{\mu_1}{\mu_2} - 1 \right]}{9 \left(1 - \frac{\rho_1}{\rho_2} \right) g} x^{-\frac{4}{3}}. \tag{4.5.5}$$

Finally using (4.4.67) for B_1, (4.5.5) can be written as

$$\phi(x) = \frac{\left(\frac{\mu_1 J}{3 \rho_1^2} \right)^{\frac{2}{3}} \left(\frac{\rho_1}{\rho_2} \right)^2 \left(\frac{\mu_1}{\mu_2} - 1 \right)}{g \left(1 - \frac{\rho_1}{\rho_2} \right) \left[1 + \left(\frac{\rho_1 \mu_1}{\rho_2 \mu_2} \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \left(\frac{\rho_1 \mu_1}{\rho_2 \mu_2} \right)^{\frac{2}{3}} x^{\frac{4}{3}}} \tag{4.5.6}$$

which agrees with $\phi(x)$ given by Herczynski et al. [12] for the two-dimensional two-fluid jet.

The boundary approximate solution for the two-fluid jet applies far from the origin. The solution behaves like

$$\phi(x) \sim x^{-\frac{4}{3}} \tag{4.5.7}$$

which is not satisfied for a small x. Since $\alpha_1 = \alpha_2 = 0$ the interfacial conditions (4.4.55) and (4.4.56) are not satisfied by (4.5.6).

4.5.2 Interfacial conditions

We now present an alternative derivation of the interface $\phi(x)$. We use the interfacial conditions (4.4.55) and (4.4.56). Since $\alpha_1 = \alpha_2 = 0$ they reduce to

$$\frac{d\phi}{dx} = 0 \tag{4.5.8}$$

and therefore

$$\phi(x) = c \tag{4.5.9}$$

where c is a constant. The interface will pass through the orifice in the wall provided $\phi(0) = 0$. Thus $c = 0$ and

$$\phi(x) = 0. \tag{4.5.10}$$

With this derivation the interface in the two-fluid plane jet is flat. Equation (4.4.54) gives the pressure difference at the interface. Since $\alpha_1 = \alpha_2 = 0$ and using (4.4.62) we obtain

$$p_1 - p_2 = \frac{\rho_1 B_1^2}{9}\left(1 - \frac{\mu_2}{\mu_1}\right) x^{-\frac{4}{3}} \tag{4.5.11}$$

Using (4.4.67) for B_1^2, (4.5.11) becomes

$$p_1 - p_2 = \rho_1 \left(\frac{\mu_1 J}{3\rho_1^2}\right)^{\frac{2}{3}} \frac{\left(\frac{\rho_1}{\rho_2}\right)\left(\frac{\mu_1}{\mu_2} - 1\right)}{\left(\frac{\rho_1\mu_1}{\rho_2\mu_2}\right)^{\frac{2}{3}}\left[1 + \left(\frac{\rho_1\mu_1}{\rho_2\mu_2}\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} x^{\frac{4}{3}}}. \tag{4.5.12}$$

Since $\phi(x) = 0$, we have from (4.4.29),

$$\xi = \frac{y}{x^{\frac{2}{3}}}. \tag{4.5.13}$$

4.6 Results

The velocity profiles in each layer are obtained from (4.4.71) and (4.4.72). Equation (4.4.71) can be written as

$$u_1(x, y) = \frac{B_1^2 \rho_1}{6\mu_1} x^{-\frac{1}{3}} \operatorname{sech}^2\left[\frac{y - \phi(x)}{\frac{6\mu_1}{B_1\rho_1} x^{\frac{2}{3}}}\right] \tag{4.6.1}$$

on the interface $y = \phi(x)$,

$$u_1(x, \phi(x)) = \frac{B_1^2 \rho_1}{6\mu_1} x^{-\frac{1}{3}}. \tag{4.6.2}$$

We also define

$$\delta_1(x) = \frac{6\mu_1}{B_1 \rho_1} x^{\frac{2}{3}} \tag{4.6.3}$$

Equation (4.6.1) becomes

$$u_1(x, y) = u_1(x, \phi(x)) \mathsf{sech}^2 \left[\frac{y - \phi(x)}{\delta_1(x)} \right] \tag{4.6.4}$$

Similarly , (4.4.72) can be expressed as

$$u_2(x, y) = u_2(x, \phi(x)) \mathsf{sech}^2 \left[\frac{y - \phi(x)}{\delta_2(x)} \right] \tag{4.6.5}$$

where

$$\delta_2(x) = \frac{6\mu_2}{B_2 \rho_2} x^{\frac{2}{3}} \tag{4.6.6}$$

But from the matching conditions at the interface

$$u_1(x, \phi(x)) = u_2(x, \phi(x)) = u(x, \phi(x)). \tag{4.6.7}$$

Equations (4.6.4) and (4.6.5) can be written in normalised form as

$$\frac{u_1(x, y)}{u(x, \phi(x))} = \mathsf{sech}^2 \left(\frac{y - \phi(x)}{\delta_1(x)} \right), \tag{4.6.8}$$

$$\frac{u_2(x, y)}{u(x, \phi(x))} = \mathsf{sech}^2 \left(\frac{y - \phi(x)}{\delta_2(x)} \right). \tag{4.6.9}$$

We see from (4.6.8) and (4.6.9) that $\delta_1(x)$ and $\delta_2(x)$ are measures of the width of the jet in each layer of fluid. Since from (4.4.62)

$$B_2 = \left(\frac{\rho_1 \mu_2}{\rho_2 \mu_1} \right)^{\frac{1}{2}} B_1 \tag{4.6.10}$$

it follows that [12]

$$\frac{\delta_1(x)}{\delta_2(x)} = \left(\frac{\nu_1}{\nu_2} \right)^{\frac{1}{2}}, \tag{4.6.11}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. Thus the width of the jet in the upper and lower fluid layers is the same if the kinematic viscosity of the fluids is the same. When $\nu_1 = \nu_2$ the velocity profiles (4.6.8) and (4.6.9) are symmetric about the interface $y = \phi(x)$. This result also follows from (4.4.71) and (4.4.72) and we see further that $\nu_1 = \nu_2$ then $v_1(x, y) = v_2(x, y)$.

Using (4.4.67) for B_1 and (4.6.10) for B_2 the width of the jet in each layer can be written as

$$\delta_1(x) = \frac{6\mu_1}{\rho_1}\left(\frac{\rho_1^2}{9\mu_1 J}\right)^{\frac{1}{3}}\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} x^{\frac{2}{3}}, \tag{4.6.12}$$

$$\delta_2(x) = \frac{6\mu_2}{\rho_2}\left(\frac{\rho_2^2}{9\mu_2 J}\right)^{\frac{1}{3}}\left[1 + \left(\frac{\rho_1\mu_1}{\rho_2\mu_2}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} x^{\frac{2}{3}}. \tag{4.6.13}$$

The interfacial velocity (4.6.2) becomes

$$u(x, \phi(x)) = \frac{\rho_1}{6\mu_1}\left(\frac{9\mu_1 \jmath}{\rho_1^2}\right)^{\frac{2}{3}} \frac{x^{-\frac{1}{3}}}{\left[1 + \left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{\frac{1}{2}}\right]^{\frac{2}{3}}}. \tag{4.6.14}$$

Although the dependence of $\delta(x)$ and $u(x, \phi(x))$ on ρ and μ is complicated the dependence on $\jmath$ and x is clear. In each layer , as J increases the jet becomes narrower while as x increases it becomes broader. As J increases the interfacial velocity increases while as x moves downstream the interfacial velocity decreases.

Consider now the momentum flux in each layer. In the upper layer the momentum flux is

$$J_1 = \int_{\phi(x)}^{\infty} \rho_1 u_1^2(x, y)dy \tag{4.6.15}$$

Using (4.4.71) for $u_1(x, y)$ and transforming from y to ξ we obtain

$$J_1 = \frac{B_1^4\rho_1^3}{36\mu_1^2}\int_0^{\infty} \mathsf{sech}^4\left(\frac{B_1\rho_1}{6\mu_1}\xi\right) d\xi \tag{4.6.16}$$

which can be expressed as

$$J_1 = \frac{B_1^3 \rho_1^2}{6\mu_1} \int_0^{\infty} \sinh^4 \ w dw \tag{4.6.17}$$

The integral is evaluated in (4.4.64). Hence

$$J_1 = \frac{B_1^3 \rho_1^2}{9\mu_1} \tag{4.6.18}$$

and using (4.4.67) for B_1 we obtain

$$J_1 = \frac{J}{\left[1 + \left(\frac{\rho_2 \mu_2}{\rho_1 \mu_1}\right)^{\frac{1}{2}}\right]}. \tag{4.6.19}$$

The momentum flux in the lower fluid layer is

$$J_2 = \int_{-\infty}^{\phi(x)} \rho_2 u_2(x, y) dy \tag{4.6.20}$$

and it can be shown similarly that

$$J_2 = \frac{J}{\left[1 + \left(\frac{\rho_1 \mu_1}{\rho_2 \mu_2}\right)^{\frac{1}{2}}\right]}. \tag{4.6.21}$$

Since J is constant we see that the momentum flux of each fluid layer, J_1 and J_2, are separately constant [12]. This is a consequence of the solution and cannot be imposed at the start of the problem. Also, $J_1 = J_2$ provided

$$\frac{\rho_1 \mu_1}{\rho_2 \mu_2} = 1 \tag{4.6.22}$$

Herczynski et al. [12] plotted the normalised velocity profile of a two-fluid jet for air over water, silicone oil over water, air over silicone oil and , for comparison, water over water. We will investigate graphically the effects of varying the conserved momentum flux J on the normalised velocity profiles. We will consider air over water and silicone oil over water. The values of the physical quantities are listed in Table 4.1 and are the same as used by Herczynski et

Fluid	ρ g/cm^3	μ g/cms	ν cm^2/s
Air	1.205×10^{-3}	1.8075×10^{-4}	1.5×10^{-1}
Oil	9.762×10^{-1}	10.931	11.197
Water	9.982×10^{-1}	1.0022×10^{-2}	1.004×10^{-2}

Table 4.1: Density, shear viscosity and kinematic viscosity of air, silicone oil and water at 20°C.

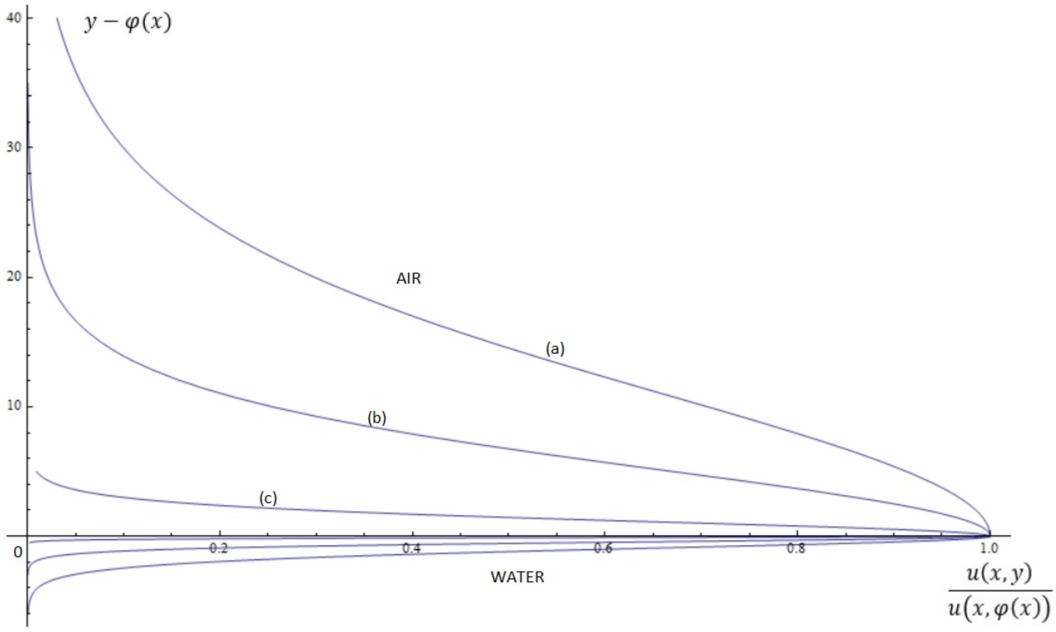

Figure 4.3: Velocity profile of two-dimensional two-fluid jet, upper fluid layer air, lower fluid layer water , at $x = 100\ cm$ downstream and (a)$J = 10g/s$, (b) $J = 10^2 g/s^2$, (c) $J = 10^3 g/s$.

al. [12]. The velocity profile are graphed at $x = 100\ cm$ downstream and J takes the values 10, 10^2 and $10^3 gs^{-2}$. In Figure 4.3 the upper fluid is air and the lower fluid is water while in Figure 4.4 the upper fluid is silicone oil and the lower fluid is water. In both Figures we see that increasing J decreases the width of the jet in each layer consistent with the width estimates (4.6.12)

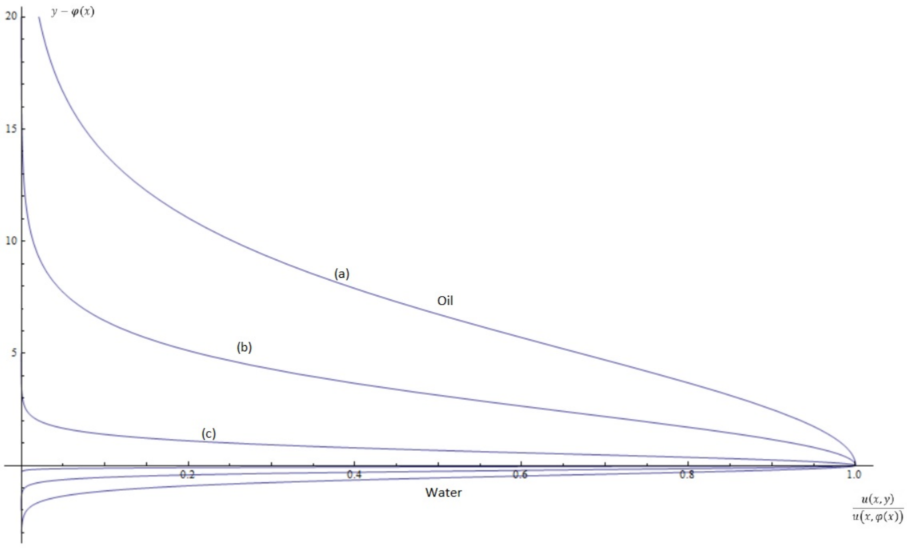

Figure 4.4: Velocity profile of two-dimensional two-fluid jet ,upper fluid layer silicone oil, lower fluid layer water , at $x = 100\ cm$ downstream and (a) $J = 10g/s^2$, (b) $J = 10^2 g/s^{-2}$, (c) $J = 10^3 g/s^{-2}$..

and (4.6.13). From Table 4.1 and (4.6.11) for air over water

$$\frac{\delta_1(x)}{s_2(x)} = 3.87 \tag{4.6.23}$$

while for oil over water

$$\frac{\delta_1(x)}{s_2(x)} = 33.4. \tag{4.6.24}$$

These results on the ratio of the width in each layer are consistent with Figures 4.3 and 4.4.

4.7 Conclusion

The Lie point symmetry associated with the conserved vector for each fluid layer was used to reduce the PDE to an ODE in each layer. The derivation of a conserved vector for a PDE does not depend on the boundary conditions. The conserved quantity does depend on the boundary conditions. For the two-fluid jet on the conserved quantity was the combined momentum flux of each fluid layer. It was found that after the problem had been solved that the momentum flux of each fluid is a conserved quantity. This result could not be derived at the start of the problem because at the interface only matching conditions are given.

The double reduction theorem of Sjoberg [1] was again satisfied. The PDE for each fluid layer was reduces to an ODE which could be integrated once. The ODE could be integrated further and an analytical solution was derived. The solution in each fluid layer is the same as for a one-fluid free jet because a constant of integration which is shown to be zero from the boundary condition on the axis of the one-fluid jet can be shown to be zero from the boundary condition at $\pm\infty$ for the two-fluid jet.

An alternative way to calculate the equation of the interface,$y = \phi(x)$, was proposed using the interfacial condition for each layer. The physical significance of these two conditions is that the fluid velocity normal to the interface in each layer vanishes. These two conditions were also used in the derivation of the conserved quantity J for the two-fluid jet. The matching condition for continuity of the normal stress at the interface gave the pressure difference across the interface. The boundary conditions at $y = \phi(x)$ and the matching conditions at $y = \phi(x)$ were all satisfied.

Chapter 5

Two-dimensional one-fluid wakes behind fixed and self-propelled bodies

5.1 Introduction

The motion of a body through a fluid creates a wake downstream of the body. In this Chapter we investigate the wake behind a slender symmetric two-dimensional body. The first wake, referred to the classical wake, is the wake downstream of a fixed body. It was first investigated by Goldstein [15]. The second kind of wake is that behind a self-propelled body. It has been investigated by Birkhoff and Zarantonello [16]. We will investigate these two laminar wakes using conservation laws and associated Lie point symmetries. In wake problems the conserved quantity plays a central role. This is because the boundary conditions are homogeneous and as we know from the derivation of a similarity solution we will require a conserved quantity to complete the solution of the problem.

Kara and Mahomed [5,6] derived a relation between the conservation laws an the Lie point symmetries of a partial differential equation. We will use their relation to obtain the associated Lie point symmetry which is used to derive the invariant solution. Naz, Mason and Mahomed [9] showed how a conservation law for a partial differential equation could be used to derive the conserved quantity for jet flow problems. They also showed how the mulitiplier method could be used to derive the conservations laws for Prandtl's boundary layer equations.

We will apply two methods, the direct method and the multiplier method, to derive the conservation laws for the partial differential equation describing two-dimensional wakes. The methods will be briefly compared. We will use the conservation laws to derive the conserved quantities for the classical wake and the wake behind a self-propelled body. We will then use the Lie point symmetries associated with the corresponding conserved vectors to derive the invariant solutions.

An outline of this chapter is as follows. The mathematical models of the classical and self-propelled wakes are presented in Section 5.2. The derivation of the conservation laws for the partial differential equation describing a two-dimensional wake is given in Section 5.3. In Section 5.4 the Lie point symmetry associated with the conserved vector for the classical wake is derived, and the invariant solution for the classical wake is obtained. In Section 5.5 the Lie point symmetry associated with the conserved vector for the self-propelled wake is obtained and the invariant solution for the self-propelled wake is derived. In Section 5.6 a comparison is made between the derivation of the invariant solution for the two wakes using the associated Lie point symmetry and using a linear combination of all the Lie point symmetries of the partial differential equation for the two-dimensional wake. Finally, conclusions are presented in

Section 5.7. .

5.2 Mathematical Model

Consider the two-dimensional wake downstream of a slender symmetric plane body aligned with a uniform flow.The two-dimensional wake is illustrated in Figure 5.1.

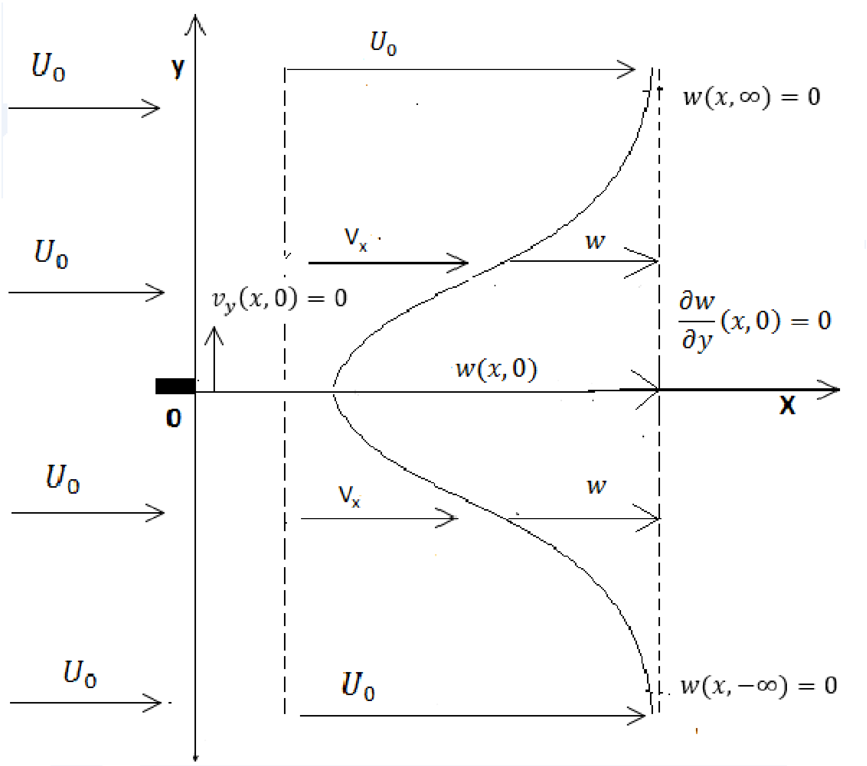

Figure 5.1: Two-dimensional wake.

The speed of the uniform flow is U_0. The velocity components are

$$v_x(x, y) = U_0 - w(x, y), \tag{5.2.1}$$

$$v_y(x, y) = 0 + v(x, y), \tag{5.2.2}$$

where $w(x, y)$ is referred to as the velocity deficit in the x-direction.We assume that

$$w(x, y) \ll U_0, \qquad |v(x, y)| \ll U_0, \tag{5.2.3}$$

The quantities $w(x, y)$ and $v_y(x, y)$ are first order in smallness. Prandtl's boundary layer equations for constant mainstream velocity U_0 are

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}, \tag{5.2.4}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{5.2.5}$$

Using (5.2.1) and (5.2.2), equation (5.2.4) becomes

$$-U_0 \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial x} - v_y \frac{\partial w}{\partial y} = -\nu \frac{\partial^2 w}{\partial y^2}. \tag{5.2.6}$$

The terms of second order in smallness are neglected. Equation (5.2.6) reduces to

$$\frac{\partial w}{\partial x} = \frac{\nu}{U_0} \frac{\partial^2 w}{\partial y^2}. \tag{5.2.7}$$

Substitute (5.2.1) and (5.2.2) into (5.2.5). The continuity equation reduces to

$$-\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.2.8}$$

The velocity deficit therefore satisfies the linear diffusion equation, From the continuity equation (5.2.8) a stream function $\psi(x, y)$ can be introduced defined by

$$w = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \tag{5.2.9}$$

Equation (5.2.7) becomes

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\nu}{U_0} \frac{\partial^3 \psi}{\partial y^3}. \tag{5.2.10}$$

The boundary conditions for a two-dimensional classical wake and a wake behind a self propelled body are the same. The boundary condition for a two dimensional wake are:

$$y = \infty: \quad w = 0, \qquad \frac{\partial \psi}{\partial y} = 0, \tag{5.2.11}$$

$$y = \infty: \quad \frac{\partial w}{\partial y} = 0, \qquad \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{5.2.12}$$

$$y = 0: \quad \frac{\partial w}{\partial y} = 0, \qquad \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{5.2.13}$$

$$y = 0: \quad v = 0, \qquad \frac{\partial \psi}{\partial x} = 0, \tag{5.2.14}$$

$$y = 0: \quad \frac{\partial w}{\partial y} = 0, \qquad \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{5.2.15}$$

$$y = -\infty: \quad w = 0, \qquad \frac{\partial \psi}{\partial y} = 0. \tag{5.2.16}$$

The boundary conditions (5.2.12) and (5.2.15) at $y - \pm\infty$ are needed to derive the conserved quantities. Equation (5.2.13) is the condition for $w(x, y)$ to have an extremum at $y = 0$. All four boundary conditions are homogeneous since the right hand side vanishes. A conserved quantity is therefore required to complete the solution of the problem.

For the two-dimensional classical wake behind a fixed body the conserved quantity may be derived by integrating the linear diffusion equation (5.2.7) across the wake from $y = -\infty$ to $y = \infty$ at any position x. This gives

$$\int_{-\infty}^{\infty} w(x, y) dy = J, \tag{5.2.17}$$

where J is a constant independent of x. Now the drag on the body is the total momentum deficit per unit span [12]

$$D = U_0 \rho \int_{-\infty}^{\infty} w(x, y) dx, \tag{5.2.18}$$

which from (5.2.17) is independent of x. The drag D is usually used as the conserved quantity for the classical wake.

The conserved quantity for the wake behind a self-propelled body is obtained by multiplying equation (5.2.7) by y^2 and then integrating across the wake [16]. This gives

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2 w(x,y)dy = \frac{\nu}{U_0}\int_{-\infty}^{\infty} y^2\frac{\partial^2 w}{\partial y^2}dy \tag{5.2.19}$$

and integrating the right hand side twice by parts and using the boundary conditions for a wake gives

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2 w(x,y)dy = \frac{2\nu}{U_0}\int_{-\infty}^{\infty} w(x,y)dy = \frac{2\nu D}{\rho U_0^2} \tag{5.2.20}$$

But for a self-propelled body the drag D on the body is zero. Thus from (5.2.20) the conserved quantity for the wake behind a self-propelled body is

$$K = U_0\rho\int_{-\infty}^{\infty} y^2 w(x,y)dy \tag{5.2.21}$$

where K is a constant independent of x. The K represents the second moment of the momentum deficit of the flow [12]. Since the drag D is the total momentum deficit of the flow per unit span the wake behind a self propelled body is sometimes referred to as a momentumless wake [12].

The reason why the PDE (5.2.7) was first multiplied by y^3 was because the integral on the right hand side of (5.2.19) had to be integrated twice to obtain the drag D which is zero for the momentumless wake. This required considerable physical insight by Birkhoff and Zarantonello [16]. We now show that this can be deduced systematically by first deriving the conservation laws for the PDE (5.2.10). This approach also unifies the derivation of the conserved quantities for the classical and momentumless wakes.

5.3 Conservation laws

The conservation laws for PDE

$$\frac{\partial^2\psi}{\partial x\partial y} = K\frac{\partial^3\psi}{\partial y^3}, \qquad K = \frac{\nu}{U_0}, \tag{5.3.1}$$

for the stream function $\psi(x, y)$ are of the form

$$D_1 T^1 + D_2 T^2|_{PDE} = 0, \tag{5.3.2}$$

where

$$D_1 = D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + \psi_{xx} \frac{\partial}{\partial \psi_x} + \psi_{yx} \frac{\partial}{\partial \psi_y} + \cdots, \tag{5.3.3}$$

$$D_2 = D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + \psi_{yy} \frac{\partial}{\partial \psi_y} + \psi_{xy} \frac{\partial}{\partial \psi_x} + \cdots \tag{5.3.4}$$

The notation introduced in Section 3.5 will be used in which suffices will be used to denote partial differentiation when x,y, ψ and the partial derivatives of ψ are regarded as independent variables.

Two systematic methods will be used to derive the conservation laws for the PDE (5.3.1), the direct method and the multiplier method.

5.3.1 Direct method

In the direct method equation (5.3.2) is used as the determining equation for the components of the conserved vector $T = (T^1, T^2)$. Equation (5.3.2) is separated by equating the coefficients of like powers and products of the partial derivatives of ψ. Consider a conserved vector of the form

$$T^1 = T^1(y, \psi, \psi_y) \quad T^2 = T^2(y, \psi, \psi_y, \psi_{yy}). \tag{5.3.5}$$

Equation(5.3.1) becomes

$$\left(\psi_x \frac{\partial T^1}{\partial \psi} + \psi_{xy} \frac{\partial T^1}{\partial \psi_y} + \frac{\partial T^2}{\partial y} + \psi_y \frac{\partial T^2}{\partial \psi} + \psi_{yy} \frac{\partial T^2}{\partial \psi_y} + \psi_{yyy} \frac{\partial T^2}{\partial \psi_{yy}}\right)|_{PDE} = 0. \tag{5.3.6}$$

But from the PDE (5.3.1)

$$\psi_{xy} = K \psi_{yyy} \tag{5.3.7}$$

and replacing ψ_{xy} in (5.3.6) gives

$$\psi_x \frac{\partial T^1}{\partial \psi} + \kappa \psi_{yyy} \frac{\partial T^1}{\partial \psi_y} + \frac{\partial T^2}{\partial y} + \psi_y \frac{\partial T^2}{\partial \psi} + \psi_{yy} \frac{\partial T^2}{\partial \psi_y} + \psi_{yyy} \frac{\partial T^2}{\partial \psi_{yy}} = 0. \tag{5.3.8}$$

Separating (5.3.8) by ψ_{yyy} we obtain:

$$\psi_{yyy}: \qquad K\frac{\partial T^1}{\partial \psi_y} + \frac{\partial T^2}{\partial \psi_{yy}} = 0, \tag{5.3.9}$$

$$R: \qquad \psi_x\frac{\partial T^1}{\partial \psi} + \frac{\partial T^2}{\partial y} + \psi_y\frac{\partial T^2}{\partial \psi} + \psi_{yy}\frac{\partial T^2}{\partial \psi_y} = 0, \tag{5.3.10}$$

where R denotes the remainder. We integrate (5.3.9) with respect to ψ_{yy} to obtain T^2. The conserved vector now is

$$T^1 = T^1(y, \psi, \psi_y), \quad T^2 = -\kappa\psi_{yy}\frac{\partial T^1}{\partial \psi_y}(y, \psi, \psi_y) + A(y, \psi, \psi_y), \tag{5.3.11}$$

where A depends only on y, ψ and ψ_y. Substituting the conserved vector into the reminder (5.3.10) gives

$$\begin{aligned} &\psi_x\frac{\partial T^1}{\partial \psi} - K\psi_{yy}\frac{\partial^2 T^1}{\partial y \partial \psi_y} + \frac{\partial A}{\partial y} - K\psi_y\psi_{yy}\frac{\partial^2 T^1}{\partial \psi \partial \psi_y} \\ &\psi_y\frac{\partial A}{\partial \psi} - K\psi_{yy}^2\frac{\partial^2 T^1}{\partial \psi_y^2} + \psi_{yy}\frac{\partial A}{\partial \psi_y} = 0 \end{aligned} \tag{5.3.12}$$

Since T^1 and A depend on ψ_y we cannot separate (5.3.12) by ψ_y. Separating (5.3.12) by the other derivatives we obtain

$$\psi_{yy}^2: \qquad \frac{\partial^2 T^1}{\partial \psi_y^2}(y, \psi, \psi_y) = 0, \tag{5.3.13}$$

$$\psi_{yy}: \qquad -K\frac{\partial^2 T^1}{\partial y \partial \psi_y}(y, \psi, \psi_y) - K\psi_y\frac{\partial^2 T^1}{\partial \psi \partial \psi_y}(y, \psi, \psi_y) + \frac{\partial A}{\partial \psi_y}(y, \psi, \psi_y) = 0, \tag{5.3.14}$$

$$\psi_x: \qquad \frac{\partial T^1}{\partial \psi}(y, \psi, \psi_y) = 0, \tag{5.3.15}$$

$$R: \qquad \frac{\partial A}{\partial y}(y, \psi, \psi_y) + \psi_y\frac{\partial A}{\partial \psi}(y, \psi, \psi_y) = 0. \tag{5.3.16}$$

From (5.3.13) and (5.3.15) it follows that

$$T^1 = B(y)\psi_y + C(y). \tag{5.3.17}$$

From(5.3.14) we find that

$$\frac{\partial A}{\partial \psi_y}(y, \psi, \psi_y) = K\frac{dB}{dy}(y) \tag{5.3.18}$$

and therefore

$$A(y,\psi,\psi_y) = K\frac{dB}{dy}\psi_y + D(y,\psi). \tag{5.3.19}$$

Finally, (5.3.16) gives

$$D = D(x\psi) \tag{5.3.20}$$

and

$$K\frac{d^2B}{dy^2}(y) = -\frac{dD}{d\psi}(\psi). \tag{5.3.21}$$

Using separation of variables in (5.3.21) we obtain

$$B(y) = \frac{1}{K}\left(\frac{\alpha_1}{2}y^2 + \alpha_2 y + \alpha_3\right), \tag{5.3.22}$$

$$D(\psi) = -\alpha_1\psi + \alpha_4, \tag{5.3.23}$$

where α_1 to α_4 are constants.

The conserved vector therefore is

$$T^1 = \frac{1}{K}\left(\frac{\alpha_1}{2}y^2 + \alpha_2 y + \alpha_3\right)\psi_y + C(y), \tag{5.3.24}$$

$$T^2 = -\left(\frac{\alpha_1}{2}y^2 + \alpha_2 y + \alpha_3\right)\psi_{yy} + (\alpha_1 y + \alpha_2)\psi_y - \alpha_1\psi + \alpha_4. \tag{5.3.25}$$

The conserved vector

$$T^1 = C(y), \quad T^2 = \alpha_4,$$

is a trivial conserved vector in which (5.3.2) is satisfied independently of the PDE. It is put equal to zero. The conserved vector , (5.3.24) and (5.3.25), is multiplied by K to put it in a more convenient form. Equations (5.3.24) and (5.3.25) is a linear combination of three conserved vectors.

$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 \neq 0$:

$$T^1 = \psi_y, \qquad T^2 = -K\psi_{yy}. \tag{5.3.26}$$

$\alpha_1 = 0, \alpha_2 \neq 0, \alpha_3 = 0$:

$$T^1 = y\psi_y, \qquad T^2 = -K(y\psi_{yy} - \psi_y). \tag{5.3.27}$$

$\alpha_1 \neq 0, \alpha_2 = 0, \alpha_3 = 0$:

$$T^1 = y^2\psi_y, \qquad T^2 = -K\left(y^2\psi_{yy} + 2y\psi_y - 2\psi\right). \tag{5.3.28}$$

The boundary conditions of a problem determine which conserved vector is used to calculate the conserved quantity. Equation (5.3.26) is the conserved vector for the classical wake. It is called the elementary conserved vector. Equation (5.3.28) is the conserved vector for the wake behind a self-propelled body.

5.3.2 Multiplier method

If Λ is a multiplier for the PDE:

$$\psi_{xy} - K\psi_{yyy} = 0 \tag{5.3.29}$$

then

$$\Lambda\left(\psi_{xy} - K\psi_{yyy}\right) = D_1T^1 + D_2T^2 \tag{5.3.30}$$

for all functions $\psi(x, y)$ where $D_1 = D_x$, $D_2 = D_y$ are the total derivatives defined by (5.3.3) and (5.3.4). The right hand side of (5.3.30) is a divergence expression and $T = (T^1, T^2)$ is a conserved vector. When (5.3.29) is satisfied. Consider a multiplier of form

$$\Lambda = \Lambda(y, \psi, \psi_x, \psi_y). \tag{5.3.31}$$

We do not include x in Λ because the PDE depends mainly on the derivatives with respect to y. The determining equation for the multiplier Λ is

$$E_\psi\left[\Lambda(y, \psi, \psi_x, \psi_y)\left(\psi_{xy} - K\psi_{yyy}\right)\right] = 0, \tag{5.3.32}$$

where E_ψ the Euler operator defined by

$$E_\psi = \frac{\partial}{\partial\psi} - D_x\frac{\partial}{\partial\psi_x} - D_y\frac{\partial}{\partial\psi_y} + D_x^2\frac{\partial}{\partial\psi_{xx}} + D_xD_y\frac{\partial}{\partial\psi_{xy}} + D_y^2\frac{\partial}{\partial\psi_{yy}} - . \tag{5.3.33}$$

The Euler operator annihilated the divergence on the right hand side of (5.3.30). When expanded , equation (5.3.32) becomes

$$\frac{\partial \Lambda}{\partial \psi}\left(\psi_{xy}-K\psi_{yyy}\right)-D_x\left[\frac{\partial \Lambda}{\partial \psi_x}(\psi_{xy}-K\psi_{yyy})\right]$$
$$-D_y\left[\frac{\partial \Lambda}{\partial \psi_y}(\psi_{xy}-K\psi_{yyy})\right]+D_xD_y[\Lambda]+KD_y^3[\Lambda]=0. \quad (5.3.34)$$

Consider first the fourth order partial derivatives terms in (5.3.34). Now

$$\frac{\partial \Lambda}{\partial \psi}\left(\psi_{xy}-K\psi_{yyy}\right)=\text{third and lower order partial derivative terms}, \quad (5.3.35)$$

$$-D_x\left[\frac{\partial \Lambda}{\partial \psi_x}(\psi_{xy}-K\psi_{yyy})\right]=K\frac{\partial \Lambda}{\partial \psi_x}\psi_{yyyx} \quad (5.3.36)$$
$$+\text{third and lower order partial derivative terms},$$

$$-D_y\left[\frac{\partial \Lambda}{\partial \psi_y}\left(\psi_{xy}-K\psi_{yyy}\right)\right]=K\frac{\partial \Lambda}{\partial \psi_y}\psi_{yyyy}, \quad (5.3.37)$$
$$+\text{third and lower order partial derivative terms},$$

$$D_xD_y[\Lambda]=\text{third and lower order partial derivative terms}, \quad (5.3.38)$$

$$KD_y^3[\Lambda]=K\left[\frac{\partial \Lambda}{\partial \psi_x}\psi_{xyyy}+\frac{\partial \Lambda}{\partial \psi_y}\psi_{yyyy}\right] \quad (5.3.39)$$
$$+\text{third and lower order partial derivative terms}.$$

The determining equation (5.3.34) becomes

$$2K\frac{\partial \Lambda}{\partial \psi_x}\psi_{xyyy}+2K\frac{\partial \Lambda}{\partial \psi_y}\psi_{yyyy} \quad (5.3.40)$$
$$+\text{third and lower order partial derivative terms}$$

Hence equating to zero the coefficients of the independent derivatives ψ_{xyyy} and ψ_{yyyy} we obtain

$$\psi_{xyyy}: \quad \frac{\partial \Lambda}{\partial \psi_x}=0, \quad (5.3.41)$$

$$\psi_{yyyy}: \quad \frac{\partial \Lambda}{\partial \psi_y}=0 \quad (5.3.42)$$

and therefore

$$\Lambda = \Lambda(y, \psi). \qquad (5.3.43)$$

The determining equation (5.3.34) reduces to

$$\frac{\partial \Lambda}{\partial \psi}(\psi_{xy} + K\psi_{yyy}) + D_x D_y[\Lambda] + K D_y^3[\Lambda] = 0. \qquad (5.3.44)$$

consider next the coefficient of ψ_{xy}. Now

$$D_x D_y[\Lambda(y,\psi)] = \psi_x \frac{\partial^2 \Lambda}{\partial y \partial \psi} + \psi_x \psi_y \frac{\partial^2 \Lambda}{\partial \psi^2} + \psi_{xy} \frac{\partial \Lambda}{\partial \psi} \qquad (5.3.45)$$

and

$$D_y^3[\Lambda(y,\psi)] = \text{terms independent of } \psi_{xy}. \qquad (5.3.46)$$

Equating to zero the coefficient of ψ_{xy} in (5.3.44) therefore gives

$$\frac{\partial \Lambda}{\partial \psi}(y, \psi) = 0 \qquad (5.3.47)$$

and hence

$$\Lambda = \Lambda(y) \qquad (5.3.48)$$

The determining equations (5.3.44) becomes

$$\frac{d^3 \Lambda}{dy^3} = 0 \qquad (5.3.49)$$

and therefore the multiplier is of the form

$$\Lambda(y) = \alpha_1 y^2 + \alpha_2 y + \alpha_3, \qquad (5.3.50)$$

where α_2, α_2 and α_3 are constants.

Equation (5.3.30) can now be written as follows , after making elementary manipulations:

$$\begin{aligned}(\alpha_1 y^2 + \alpha_2 y + \alpha_3)(\psi_{xy} - K\psi_{yyy}) &= D_x[\alpha_1 y^2 \psi_y + \alpha_2 y \psi_y + \alpha_3 \psi_y] \\ &+ D_y[\alpha_1 K(-y^2 \psi_{yy} + 2y\psi_y + 2\psi) \\ &+ \alpha_2 K(-y\psi_{yy} + \psi_y) + \alpha_3 K(-\psi_{yy})],\end{aligned} \qquad (5.3.51)$$

which holds for arbitrary functions $\psi(x,y)$. When $\psi(x,y)$ is a solution of the PDE (5.3.29), equation (5.3.51) reduces to

$$D_x[\alpha_1 y^2\psi_y + \alpha_2 y\psi_y + \alpha_3\psi_y]$$
$$+ D_y[\alpha_1 K(-y^2\psi_{yy} + 2y\psi_y - 2\psi) + \alpha_2 K(-y\psi_{yy} + \psi_y) + \alpha_3 K(-\psi_{yy})] = 0 \tag{5.3.52}$$

Thus any conserved vector of the PDE (5.3.29) with multiplier $\Lambda(y,\psi,\psi x,\psi_y)$ is a linear combination of the three conserved vectors

$$T^1 = \psi_y, \quad T^2 = -K\psi_{yy}, \tag{5.3.53}$$

$$T^1 = y\psi_y, \quad T^2 = -K(y\psi_{yy} - \psi_y), \tag{5.3.54}$$

$$T^1 = y^2\psi_y, \quad T^2 = -K(y^2\psi_{yy} - 2y\psi_y + 2\psi), \tag{5.3.55}$$

which agrees with the conserved vectors (5.3.26) to (5.3.28) derived by the direct method.

The application of conservation laws to derive the conserved quantities unifies the classical and self-propelled wake problems because the conservation laws are obtained in the same calculation. The appearance of y^2 in the multiplier and conserved vector occurred in a natural way and did not require deep physical insight.

In the direct method the variables on which the components of the conserved vector depend have to be chosen while in the multiplier method the variables on which the multiplier depends have to be chosen, both at the start of the calculation. The larger the class of variables considered the larger the class of conserved vectors that may be found. For the variables chosen the direct method and the multiplier method gave the same conserved vectors. The direct method also gave trivial conserved vectors while they were not generated by the multiplier method. The derivation of the conserved vectors using the multiplier method can be greatly simplified by first picking out the coefficients of the highest derivatives. The multiplier obtained was quite simple. The direct method gave the conserved vectors in the final step while in the multiplier method a further step is required to obtain the conserved vectors. This final step can be done by elementary manipulations but requires experience.

5.4 Invariant solution for the classical wake

In this section we will derive the invariant solution for the two-dimensional classical wake. We first derive the Lie point symmetry associated with the elementary conserved vector and then use this Lie point symmetry to derive the invariant solution.

We first explain why we choose the Lie point symmetry associated with the elementary conserved vector. Consider the conserved quantity for the classical wake. It was shown in Section 4.3 that when x and y are regarded as independent variables, a conservation law can be written in terms of partial derivatives as

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y}|_{PDE} = 0. \tag{5.4.1}$$

Integrate (5.4.1) across the wake from $y = -\infty$ and $y = +\infty$. Then

$$\int_{-\infty}^{\infty} \frac{\partial T^1}{\partial x}(x,y)dy + \int_{-\infty}^{\infty} \frac{\partial T^2}{\partial y}(x,y)dy = 0 \tag{5.4.2}$$

and therefore

$$\frac{d}{dx}\int_{-\infty}^{\infty} T^1(x,y)dy + \left[T^2(x,y)\right]_{y=-\infty}^{y=\infty} = 0. \tag{5.4.3}$$

Consider the elementary conserved vector

$$T^1(x,y) = \frac{\partial \psi}{\partial y} \quad , \quad T^2(x,y) = -K\frac{\partial^2 \psi}{\partial y^2}. \tag{5.4.4}$$

Equation (5.4.3) becomes

$$\frac{d}{dx}\int_{-\infty}^{\infty} \frac{\partial \psi}{\partial y}(x,y)dy - K\left[\frac{\partial^2 \psi}{\partial y^2}(x,y)\right]_{y=-\infty}^{y=\infty} = 0. \tag{5.4.5}$$

Using the boundary conditions (5.2.12) and (5.2.15) we obtain

$$\frac{d}{dx}\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial y}(x,y)dy = 0 \tag{5.4.6}$$

and therefore

$$\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial y}(x,y)dy = J, \tag{5.4.7}$$

where J is a constant. As explained in Section 5.2, we let

$$\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial y}(x,y)dy = \frac{D}{\rho U_0}, \tag{5.4.8}$$

where D is the Drag on the body. The conserved quantity must be non-zero in order to obtain the remaining unknown quantities in the solution. For flow past a fixed body the drag is non-zero. (The drag is zero for a self-propelled body.) The conserved vector which is used to obtain the conserved quantity for the classical wake is therefore the elementary conserved vector (5.4.4). In order to derive an invariant solution for the classical wake, the Lie point symmetry associated with the elementary conserved vector (5.4.4) is therefore used.

The conditions for a Lie point symmetry,

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi))\frac{\partial}{\partial\psi} \tag{5.4.9}$$

to be associated with a conserved vector $T = (T^1, T^2)$ is (3.4.4). It has two components (3.4.5a) and (3.4.5b):

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{5.4.10}$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0, \tag{5.4.11}$$

where X is prolonged as required if T^i depends on the derivatives of ψ.

We derive the Lie point symmetry associated with the elementary conserved vector (5.4.4). Consider first the condition (5.4.10). Then

$$X(T^1) = X^{[1]}(\psi_y) = \zeta_2 \tag{5.4.12}$$

where

$$\zeta_2 = \frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_x \psi_y \frac{\partial \xi^2}{\partial y} - \psi_y \frac{\partial \xi^2}{\partial y} - \psi_y^2 \frac{\partial \xi^1}{\partial \psi}. \tag{5.4.13}$$

Also

$$T^1 D_2(\xi^2) = \psi_y \frac{\partial \xi^2}{\partial y} + \psi_y^2 \frac{\partial \xi^2}{\partial \psi}, \tag{5.4.14}$$

and

$$T^2 D_2(\xi^1) = -K\psi_{yy} \frac{\partial \xi^1}{\partial y} - K\psi_y \psi_{yy} \frac{\partial \xi^1}{\partial \psi}. \tag{5.4.15}$$

The condition (5.4.10) becomes

$$\frac{\partial \eta}{\partial y} + \psi_y \frac{\partial \eta}{\partial \psi} - \psi_x \frac{\partial \xi^1}{\partial y} - \psi_x \psi_y \frac{\partial \xi^1}{\partial \psi} + K\psi_{yy} \frac{\partial \xi^1}{\partial y} + K\psi_y \psi_{yy} \frac{\partial \xi^1}{\partial \psi} = 0. \tag{5.4.16}$$

Separating (5.4.16) by the derivatives of ψ and their products gives

$$\psi_y \psi_{yy} : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{5.4.17}$$

$$\psi_{yy} : \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.4.18}$$

$$\psi_x \psi_y : \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{5.4.19}$$

$$\psi_y : \qquad \frac{\partial \eta}{\partial \psi} = 0, \tag{5.4.20}$$

$$\psi_x : \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.4.21}$$

$$R : \qquad \frac{\partial \eta}{\partial y} = 0, \tag{5.4.22}$$

It follows directly from (5.4.17) to (5.4.22) that

$$\xi^1 = \xi^1(x), \qquad \xi^2 = \xi^2(x, y, \psi), \qquad \eta = \eta(x). \tag{5.4.23}$$

Consider next the second determining equation (5.4.11). Now

$$X(T^2) = X^{[2]}(-K\psi_{yy}) = -K\zeta_{22}, \tag{5.4.24}$$

where ζ_{22} is defined by (3.5.29). Using (5.4.23), ζ_{22} reduces to

$$\zeta_{22} = \psi_y \frac{\partial^2 \xi^2}{\partial y^2} - 2\psi_y^2 \frac{\partial^2 \xi^2}{\partial y \partial \psi} - \psi_y^3 \frac{\partial^2 \xi^2}{\partial \psi^2} - 2\psi_{yy} \frac{\partial \xi^2}{\partial y} - 3\psi_y \psi_{yy} \frac{\partial \xi^2}{\partial \psi}. \tag{5.4.25}$$

Also

$$T^2 D_1(\xi^1) = -K\psi_{yy} \frac{d\xi^1}{dx}, \tag{5.4.26}$$

and

$$T^1 D_1(\xi^2) = \psi_y \frac{\partial \xi^2}{\partial x} + \psi_x \psi_y \frac{\partial \xi^2}{\partial \psi}. \tag{5.4.27}$$

Equation (5.4.11) becomes

$$K \left[\psi_y \frac{\partial^2 \xi^2}{\partial y^2} + 2\psi_y^2 \frac{\partial^2 \xi^2}{\partial y \partial \psi} + \psi_y^3 \frac{\partial^2 \xi^2}{\partial \psi^2} + 2\psi_{yy} \frac{\partial \xi^2}{\partial y} + 3\psi_y \psi_{yy} \frac{\partial \xi^2}{\partial \psi} \right]$$
$$- K\psi_{yy} \frac{d\xi^1}{dx} - \psi_y \frac{\partial \xi^2}{\partial x} - \psi_x \psi_y \frac{\partial \xi^2}{\partial \psi} = 0. \tag{5.4.28}$$

Separating the determining equation (5.4.28) by the derivatives of ψ and the product and powers of the derivatives gives

$$\psi_y \psi_{yy} : \qquad \frac{\partial \xi^2}{\partial \psi} = 0, \tag{5.4.29}$$

$$\psi_{yy} : \qquad 2\frac{\partial \xi^2}{\partial y} - \frac{d\xi^1}{dx} = 0, \tag{5.4.30}$$

$$\psi_x \psi_y : \qquad \frac{\partial \xi^2}{\partial \psi} = 0, \tag{5.4.31}$$

$$\psi_y^3 : \qquad \frac{\partial^2 \xi^2}{\partial \psi^2} = 0, \tag{5.4.32}$$

$$\psi_y^2 : \qquad \frac{\partial^2 \xi^2}{\partial y \partial \psi} = 0, \tag{5.4.33}$$

$$\psi_y : \qquad \kappa \frac{\partial^2 \xi^2}{\partial y^2} - \frac{\partial \xi^2}{\partial x} = 0. \tag{5.4.34}$$

Equation (5.4.29) and (5.4.31) gives that

$$\xi^2 = \xi^2(x, y). \tag{5.4.35}$$

The system (5.4.29) to (5.4.34) reduces to

$$2\frac{\partial \xi^2}{\partial y}(x, y) - \frac{d\xi^1}{dx}(x) = 0, \tag{5.4.36}$$

$$K\frac{\partial^2\xi^2}{\partial y^2}(x,y) - \frac{\partial \xi^2}{\partial x}(x,y) = 0. \tag{5.4.37}$$

Integrating equation (5.4.36) with respect to y gives

$$\xi^2(x,y) = \frac{1}{2}\frac{d\xi^1}{dx}y + A(x) \tag{5.4.38}$$

where A is a function that depends only on x. We substitute (5.4.38) into (5.4.37) which gives

$$\frac{1}{2}\frac{d^2\xi^2}{dx^2}y + \frac{dA(x)}{dx} = 0. \tag{5.4.39}$$

Separating (5.4.39) with respect to y we obtain

$$y: \qquad \frac{d^2\xi^2}{dx^2} = 0, \tag{5.4.40}$$

$$R: \qquad \frac{dA(x)}{dx} = 0, \tag{5.4.41}$$

From equation (5.4.40)

$$.\xi^1 = c_1 + c_2 x, \tag{5.4.42}$$

where c_1 and c_2 are constants while from the Remainder (5.4.41)

$$A(x) = c_3, \tag{5.4.43}$$

where c_3 is a constant. Equation (5.4.38) becomes

$$\xi^2(y) = \frac{1}{2}c_2 y + c_3. \tag{5.4.44}$$

The infinitesimals therefore are

$$\xi^1(x) = c_1 + c_2 x, \quad \xi^2(x,y) = \frac{1}{2}c_1 y + c_3, \quad \eta(x) = h(x), \tag{5.4.45}$$

where h is an arbitrary function that depends only on x. The Lie point symmetry generator associated with the elementary conserved vector (5.4.4) is therefore

$$X = (c_1 + c_2 x)\frac{\partial}{\partial x} + (c_3 + \frac{1}{2}c_2 y)\frac{\partial}{\partial y} + h(x)\frac{\partial}{\partial \psi}, \tag{5.4.46}$$

where c_1,c_2 and c_3 are constants. It is a linear combination of three Lie point symmetries each associated with the elementary conserved vector.

Consider now the invariant solution generated by the associated Lie point symmetry (5.4.46). We consider the general case in which $c_2 \neq 0$. In order to simply the calculations we multiply (5.4.46) by $\frac{2}{c_2}$ to obtain

$$X^* = 2(c_1^* + x)\frac{\partial}{\partial x} + (c_3^* + y)\frac{\partial}{\partial y} + h^*\frac{\partial}{\partial \psi} \tag{5.4.47}$$

where

$$X^* = \frac{2}{c_2}X, \quad c_1^* = \frac{c_1}{c_2}, \quad c_3^* = \frac{2c_3}{c_2}, \quad h^*(x) = \frac{2}{c_2}h(x). \tag{5.4.48}$$

We suppress the stars to keep the notation simple. The function $\psi = \Psi(x, y)$ is an invariant solution of the PDE (5.3.1) generated by the Lie point symmetry associated with the elementary conserved vector (5.4.4) provided

$$X(\psi - \Psi(x, y))|_{\psi=\Psi(x,y)} = 0, \tag{5.4.49}$$

that is, provided

$$2(c_1 + x)\frac{\partial \Psi}{\partial x} + (c_3 + y)\frac{\partial \Psi}{\partial y} = h(x). \tag{5.4.50}$$

The differential equations of the characteristic curves of the PDE (5.4.50) are

$$\frac{dx}{2(c_1 + x)} = \frac{dy}{c_3 + y} = \frac{d\Psi}{h(x)}. \tag{5.4.51}$$

The first pair of terms give

$$\frac{y + c_3}{(c_1 + x)^{\frac{1}{2}}} = a_1 \tag{5.4.52}$$

where a_1 is a constant. The first and last terms in (5.4.51) give

$$\Psi - H(x) = a_2, \tag{5.4.53}$$

where a_2 is a constant and

$$H(x) = \frac{1}{2}\int^x \frac{h(x)dx}{(c_1 + x)}. \tag{5.4.54}$$

The general solution of the PDE (5.4.50) is

$$a_2 = F(a_1), \tag{5.4.55}$$

where F is an arbitrary function. Since $\psi = \Psi(x, y)$ we have

$$\psi(x, y) = F(\xi) + H(x), \tag{5.4.56}$$

where

$$\xi = \frac{y + c_3}{(c_1 + x)^{\frac{1}{2}}}. \tag{5.4.57}$$

Substitution of (5.4.56) and (5.4.57) into the PDE (5.4.1) yields a third order ordinary differential equation

$$2K\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} = 0. \tag{5.4.58}$$

Consider now the boundary conditions. We choose $c_3 = 0$ to make $\xi = 0$ when $y = 0$. The velocity deficit and the velocity component in the y-direction are

$$w(x, y) = \frac{\partial\psi}{\partial y} = \frac{1}{(c_1 + x)^{\frac{1}{2}}}\frac{dF}{d\xi}, \tag{5.4.59}$$

$$v(x, y) = \frac{\partial\psi}{\partial x} = -\frac{1}{2(c_1 + x)}\xi\frac{dF}{d\xi} + \frac{dH}{d\xi}. \tag{5.4.60}$$

Expand in terms of $F(\xi)$ the boundary conditions (5.2.11) and (5.2.16) become

$$y = +\infty: \quad w(x, \infty = 0), \quad \frac{dF}{d\xi}(\infty) = 0, \tag{5.4.61}$$

$$y = +\infty: \quad \frac{\partial w}{\partial y}(x, \infty) = 0, \quad \frac{d^2F}{d\xi^2}(\infty) = 0, \tag{5.4.62}$$

$$y = 0: \quad \frac{\partial w}{\partial y}(x, 0) = 0, \quad \frac{d^2F}{d\xi^2}(0) = 0, \tag{5.4.63}$$

$$y = 0: \quad v(x, 0) = 0, \quad \frac{dH}{dx} = 0, \tag{5.4.64}$$

$$y = -\infty: \quad \frac{\partial w}{\partial y}(x, -\infty) = 0, \quad \frac{d^2F}{d\xi^2}(-\infty) = 0, \tag{5.4.65}$$

$$y = -\infty: \quad \frac{\partial w}{\partial y}(x, -\infty) = 0, \quad \frac{dF}{d\xi}(-\infty) = 0. \tag{5.4.66}$$

From (5.4.64)

$$H(x) = H_0, \tag{5.4.67}$$

where H_0 is a constant. Thus

$$\psi(x,y) = F(\xi) + H_0, \qquad \xi = \frac{y}{(c_1+x)^{\frac{1}{2}}} \tag{5.4.68}$$

and

$$v(x,y) = -\frac{1}{2(c_1+x)}\xi\frac{dF}{d\xi}. \tag{5.4.69}$$

Expressed in terms of $F(\xi)$ the conserved quantity,

$$D = \rho U_0 \int_{-\infty}^{\infty} w(x,y)dy, \tag{5.4.70}$$

becomes

$$\int_0^{\infty} \frac{dF}{d\xi}d\xi = \frac{D}{2\rho U_0}. \tag{5.4.71}$$

We see that the problem can be expressed in terms of $\frac{dF}{d\xi}$. The function $F(\xi)$ does not need to be obtained.

Consider now the solution of the ODE (5.4.58). It can be written as

$$2K\frac{d^3F}{d\xi^3} + \frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0 \tag{5.4.72}$$

and integrating once we obtain

$$2K\frac{d^2F}{d\xi^2} + \xi\frac{dF}{d\xi} = A, \tag{5.4.73}$$

where A is a constant. Now $\frac{dF}{d\xi}(0)$ is finite because $w(x,0)$ is finite. Imposing the boundary condition (5.4.63) at $\xi = 0$ therefore gives $A = 0$. Equation (5.4.73) is a first order ODE in $\frac{dF}{d\xi}$. The solution is

$$\frac{dF}{d\xi} = C\exp\left[-\frac{\xi^2}{4K}\right] \tag{5.4.74}$$

where C is a constant. All the boundary conditions (5.4.61) to (5.4.66) are identically satisfied by (5.4.74). The constant C is obtained from the conserved quantity (5.4.71). Substituting (5.4.74) into (5.4.71) and using properties of the Gamma function [13] we have

$$\int_0^\infty \exp\left(-\frac{\xi^2}{4K}\right) = \sqrt{K}\int_0^\infty s^{-\frac{1}{2}}\exp(-s)ds = \sqrt{K}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi K} \quad (5.4.75)$$

and therefore

$$C = \frac{D}{2\sqrt{\pi K}\rho U_0}. \quad (5.4.76)$$

Hence

$$\frac{dF}{d\xi} = \frac{D}{2\sqrt{\pi K}\rho U_0}\exp\left[-\frac{\xi^2}{4K}\right]. \quad (5.4.77)$$

It remains to obtain the constant c_1. Now $w(x, y)$ is given by (5.4.59) and using (5.4.68)for ξ and (5.4.77) we obtain

$$w(x, y) = \frac{D}{2\sqrt{\pi K}\rho U_0}\frac{1}{(c_1 + x)^{\frac{1}{2}}}\exp\left[-\frac{y^2}{4K(c_1 + x)}\right]. \quad (5.4.78)$$

The wake extends to infinity for both $y > 0$ and $y < 0$ but for large values of $|y|$ the velocity deficit is small. We define the effective width of the wake, $W(x)$, to be twice the value of y for which the argument in the exponential is -1. Thus

$$W(x) = 4\sqrt{K}(c_1 + x)^{\frac{1}{2}}. \quad (5.4.79)$$

We can expect that the effective width vanishes as $x \to 0$. Thus

$$\lim_{x\to 0} W(x) = 4\sqrt{kc_1} = 0. \quad (5.4.80)$$

We therefore take $c_1 = 0$.

We obtain the following results:

$$\xi = \frac{y}{x^{\frac{1}{2}}}, \quad (5.4.81)$$

$$\frac{dF}{d\xi} = \frac{D}{2\sqrt{\pi K}\rho U_0}\exp\left[-\frac{y^2}{4Kx}\right] \quad (5.4.82)$$

and therefore

$$w(x,y) = \frac{D}{2\sqrt{\pi K}\rho U_0}\frac{1}{x^{\frac{1}{2}}}\exp\left[-\frac{y^2}{4Kx}\right], \tag{5.4.83}$$

$$v(x,y) = -\frac{D}{4\sqrt{\pi K}\rho U_0}\frac{y}{x^{\frac{3}{2}}}\exp\left[-\frac{y^2}{4Kx}\right]. \tag{5.4.84}$$

We do not require $F(\xi)$ but it can be obtained by integrating (5.4.77). It is found that

$$F(\xi) = \frac{D}{2\rho U_0}\mathsf{erf}\left[\frac{\xi}{2\sqrt{K}}\right] \tag{5.4.85}$$

where we took $F(0) = 0$ and the error function is defined by

$$\mathsf{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)du. \tag{5.4.86}$$

Hence

$$\psi(x,y) = \frac{D}{2\rho U_0}\mathsf{erf}\left[\frac{y}{2\sqrt{Kx}}\right] + H_0. \tag{5.4.87}$$

The results agree with the results obtained by Goldstein [15] and Herczynski et al [12]. Since

$$K = \frac{\nu}{U_0} \tag{5.4.88}$$

the effective width of the wake is

$$W(x) = 4\left(\frac{\nu x}{U_0}\right)^{\frac{1}{2}}. \tag{5.4.89}$$

The width increases downstream as $x^{\frac{1}{2}}$. It also increases with kinematic viscosity as $\nu^{\frac{1}{2}}$ because of the increase in diffusion of vorticity by viscosity. The maximum value of the velocity deficit is on the x-axis and is

$$w(x,0) = \frac{D}{2\sqrt{\pi\mu}\rho U_0}\frac{1}{x^{\frac{1}{2}}}. \tag{5.4.90}$$

The maximum velocity deficit decreases downstream as $x^{-\frac{1}{2}}$ and the wake becomes weaker the further downstream from the body.

The x-component of the fluid velocity in the wake can be written in dimensionless variable as

$$v_x^*(x^*, y^*) = 1 - \frac{D^*}{2\sqrt{\pi}} \frac{1}{x^{*\frac{1}{2}}} \exp\left[-\frac{y^{*2}}{4x^*}\right], \tag{5.4.91}$$

where

$$v_x^* = \frac{v_x}{U_0}, \quad D^* = \frac{D}{\rho U_0 \nu}, \quad x^* = \frac{x}{\nu} U_0, \quad y^* = \frac{y}{\nu} U_0.$$

The ratio $\frac{\nu}{U_0}$ is the characteristic length and D^* is the dimensionless drag per unit span. In Figure 5.2, v_x^* is plotted against y^* at $x^* = 1$ for $D^* = 2$ and $D^* = 4$. We see that as D^* increases the maximum value of the velocity deficit increases. We also see that the width of the wake does not depend on the drag D. The width depends on ν/U_0.

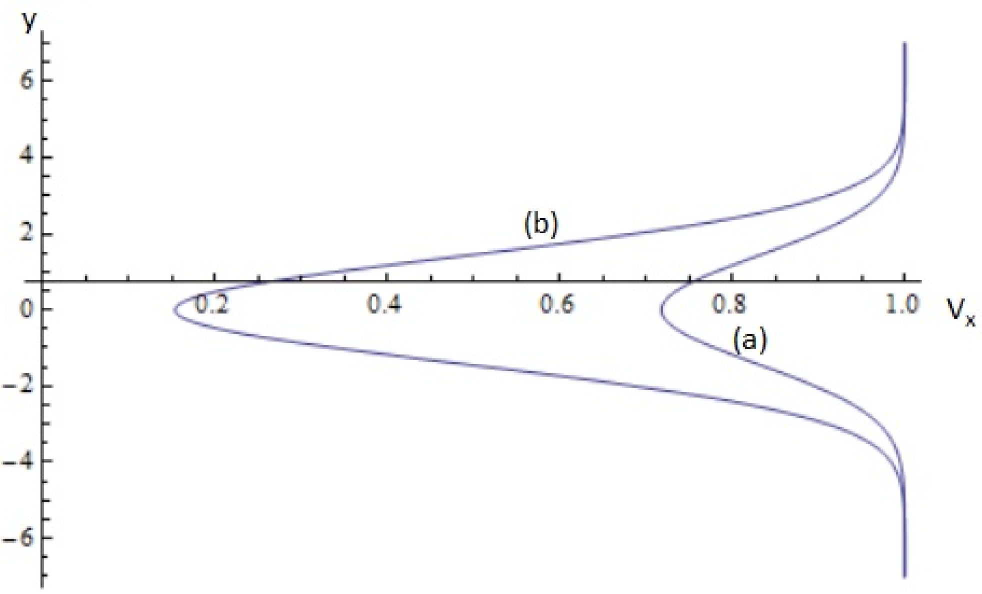

Figure 5.2: Velocity profile of the classical wake behind a fixed body at $x^* = 1$ and (a) $D^* = 1$, (b) $D^* = 3$

5.5 Invariant solution for the wake behind a self-propelled body

In this section we will derive the invariant solution for the wake behind a self-propelled body using an associated Lie point symmetry. We first motivate why we choose the Lie point symmetry associated with the conserved vector (5.3.28).

When x and y are regarded as the independent variables a conservation law can be written in terms of partial derivatives as (5.4.1). Integrating (5.4.1) across the wake gives

$$\frac{d}{dx}\int_{-\infty}^{\infty} T^1(x,y)dy + \int_{-\infty}^{\infty} \frac{\partial T^2}{\partial y}(x,y)dy = 0. \tag{5.5.1}$$

But when x and y are regarded as the independent variables the conserved vector (5.3.28) can be written as

$$T^1 = y^2\frac{\partial\psi}{\partial y}, \qquad T^2 = -K\left(y^2\frac{\partial^2\psi}{\partial y^2} - 2y\frac{\partial\psi}{\partial y} + 2\psi\right). \tag{5.5.2}$$

By substituting (5.5.2) into (5.5.1) we obtain

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^2\frac{\partial\psi}{\partial x}(x,y)dy = K\int_{-\infty}^{\infty} y^2\frac{\partial^3\psi}{\partial y^3}(x,y)dy. \tag{5.5.3}$$

This equation is the same as (5.2.19) obtained by first multiplying the PDE (5.2.7) by y^2 and then integrating across the wake. By using the conserved vector (5.5.2) the y^2 occurred naturally in the integrand and did not need to be introduced. For a self-propelled body the drag per unit span D is zero. It is shown in Section 5.2 that by integrating (5.5.3) twice by parts the conserved quantity , S, for the wake behind a self-propelled body is obtained:

$$S = U_0\rho\int_{-\infty}^{\infty} y^2\frac{\partial\psi}{\partial y}(x,y)dy. \tag{5.5.4}$$

We therefore use the Lie point symmetry associated with the conserved vector (5.5.2) to derive the invariant solution for the wake behind a self-propelled body.

The Lie point symmetry

$$X = \xi^1(x,y,\psi)\frac{\partial}{\partial x} + \xi^2(x,y,\psi)\frac{\partial}{\partial y} + \eta(x,y,\psi))\frac{\partial}{\partial \psi} \tag{5.5.5}$$

is associated with the conserved vector $T = (T^1, T^2)$ provided (3.4.5a) and (3.4.5b) are satisfied, that is provided

$$X(T^1) + T^1 D_2\xi^2 - T^2 D_2\xi^1 = 0, \tag{5.5.6}$$

$$X(T^2) + T^2 D_1\xi^1 - T^1 D_1\xi^2 = 0. \tag{5.5.7}$$

The variables x, y, ψ and the partial derivatives of ψ are independent, The conserved vector is given by (5.3.28).

Consider first (5.5.6). Now

$$X(T^1) = X^{[1]}(y^2\psi_y) = 2y\xi^2\psi_y + y^2\zeta_2 \tag{5.5.8}$$

where ζ_2 is given by (5.4.13). Also

$$T^1 D_2(\xi^2) = y^2\psi_y\frac{\partial \xi^2}{\partial y} + y^2\psi_y^2\frac{\partial \xi^2}{\partial \psi} \tag{5.5.9}$$

and

$$T^2 D_2(\xi^1) = -K\left(y^2\psi_{yy} - 2y\psi_y + 2\psi\right)\left[\frac{\partial \xi^1}{\partial y} + \psi_y\frac{\partial \xi^1}{\partial \psi}\right]. \tag{5.5.10}$$

When expanded, condition (5.5.6) is

$$\begin{aligned}
&2y\xi^2\psi_y + y^2\frac{\partial \eta}{\partial y} + y^2\frac{\partial \eta}{\partial \psi}\psi_y - y^2\frac{\partial \xi^1}{\partial y}\psi_x - y^2\frac{\partial \xi^1}{\partial \psi}\psi_x\psi_y \\
&- y^2\frac{\partial \xi^2}{\partial y}\psi_y - y^2\frac{\partial \xi^2}{\partial \psi}\psi_y^2 + y^2\frac{\partial \xi^2}{\partial y}\psi_y - y^2\frac{\partial \xi^2}{\partial \psi}\psi_y^2 \\
&+ K\left[y^2\frac{\partial \xi^1}{\partial y}\psi_{yy} + y^2\frac{\partial \xi^1}{\partial \psi}\psi_y\psi_{yy} - 2y\frac{\partial \xi^1}{\partial y}\psi_y - 2y\frac{\partial \xi^2}{\partial \psi}\psi_y^2 + 2\psi\frac{\partial \xi^1}{\partial y} + 2\psi\frac{\partial \xi^1}{\partial \psi}\psi_y\right] = 0.
\end{aligned} \tag{5.5.11}$$

Separating the determining equation (5.5.11) by the powers and products of the derivatives of ψ gives

$$\psi_y\psi_{yy}: \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{5.5.12}$$

$$\psi_{yy}: \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.5.13}$$

$$\psi_y^2: \qquad \frac{\partial \xi^1}{\partial \psi} = 0, \tag{5.5.14}$$

$$\psi_y: \qquad 2y\xi^2 + y^2\frac{\partial \eta}{\partial \psi} + 2\kappa\left(-y\frac{\partial \xi^1}{\partial y} + \psi\frac{\partial \xi^1}{\partial \psi}\right) = 0, \tag{5.5.15}$$

$$\psi_x: \qquad \frac{\partial \xi^1}{\partial y} = 0, \tag{5.5.16}$$

$$R: \qquad y^2\frac{\partial \eta}{\partial y} + 2K\psi\frac{\partial \xi^1}{\partial y} = 0. \tag{5.5.17}$$

From (5.5.12) and (5.5.13)

$$\xi^1 = \xi^1(x). \tag{5.5.18}$$

Equation (5.5.17) now gives

$$\eta = \eta(x, \psi). \tag{5.5.19}$$

The system (5.5.12) to (5.5.17) reduces to

$$\xi^2(x, y, \psi) = -\frac{y}{2}\frac{\partial \eta}{\partial \psi}(x, \psi). \tag{5.5.20}$$

From the first determining equation , (5.5.6), we therefore obtain

$$\xi^1 = \xi^1(x) \qquad \xi^2(x, y, \psi) = -\frac{y}{2}\frac{\partial \eta}{\partial \psi}(x, \psi), \qquad \eta = \eta(x, \psi). \tag{5.5.21}$$

Consider next the second determining equation (5.5.7) with ξ^1, ξ^2 and η given by (5.5.21). Now

$$X^{[2]}(T^2) = -K\left[-y^2\frac{\partial \eta}{\partial \psi}\psi_{yy} + y\frac{\partial \eta}{\partial \psi}\psi_y + 2\eta - 2y\zeta_2 + y^2\zeta_{22}\right]. \tag{5.5.22}$$

But from (5.4.13) for ζ_2 we obtain

$$\zeta_2 = \frac{3}{2}\frac{\partial\eta}{\partial\psi}\psi_y + \frac{y}{2}\frac{\partial^2\eta}{\partial\psi^2}\psi_y^2 \tag{5.5.23}$$

and using (3.5.29) for ζ_{22} we have

$$\zeta_{22} = 2\frac{\partial\eta}{\partial\psi}\psi_{yy} + 2\frac{\partial^2\eta}{\partial\psi^2}\psi_y^2 + \frac{3}{2}y\frac{\partial^2\eta}{\partial\psi^2}\psi_y\psi_{yy} + \frac{y}{2}\frac{\partial^3\eta}{\partial\psi^3}\psi_y^3. \tag{5.5.24}$$

Also

$$T^2 D_1(\xi^1) = -K\left[y^2\frac{d\xi^1}{dx}\psi_{yy} - 2y\frac{d\xi^1}{dx}\psi_y + 2\psi\frac{d\xi^1}{dx}\right] \tag{5.5.25}$$

and

$$T^1 D_1(\xi^2) = -\frac{y^3}{2}\frac{\partial^2\eta}{\partial x\partial\psi}\psi_y - \frac{y^3}{2}\frac{\partial^2\eta}{\partial\psi^2}\psi_x\psi_y. \tag{5.5.26}$$

Equation (5.5.7) becomes

$$\begin{aligned}
&-K\left[y^2\frac{\partial\eta}{\partial\psi}\psi_{yy} - 2y\frac{\partial\eta}{\partial\psi}\psi_y + y^2\frac{\partial^2\eta}{\partial\psi^2}\psi_y^2 + \frac{y^3}{2}\frac{\partial^3\eta}{\partial\psi^3}\psi_y^3\right]\\
&-K\left[\frac{3}{2}y^3\frac{\partial^2\eta}{\partial\psi^2}\psi_y\psi_{yy} + 2\eta + y^2\frac{d\xi^1}{dx}\psi_{yy} - 2y\frac{d\xi^1}{dx}\psi_y + 2\psi\frac{d\xi^1}{dx}\right]\\
&+\frac{y^3}{2}\frac{\partial^2\eta}{\partial x\partial\psi}\psi_y + \frac{y^3}{2}\frac{\partial^2\eta}{\partial\psi^2}\psi_x\psi_y = 0.
\end{aligned} \tag{5.5.27}$$

Separating the determining equation (5.5.27) by the powers and products of the partial derivatives of ψ we obtain

$$\psi_y\psi_{yy}: \quad \frac{\partial^2\eta}{\partial\psi^2} = 0, \tag{5.5.28}$$

$$\psi_{yy}: \quad \frac{\partial\eta}{\partial\psi} + \frac{d\xi^1}{dx} = 0, \tag{5.5.29}$$

$$\psi_y^3: \quad \frac{\partial^3\eta}{\partial\psi^3} = 0, \tag{5.5.30}$$

$$\psi_y^2: \quad \frac{\partial^2\eta}{\partial\psi^2} = 0, \tag{5.5.31}$$

$$\psi_y: \quad -2K\left(\frac{\partial\eta}{\partial\psi} + \frac{d\xi^1}{dx}\right) + \frac{y^2}{2}\frac{\partial^2\eta}{\partial x\partial\psi} = 0, \tag{5.5.32}$$

$$\psi_x\psi_y: \quad \frac{\partial^2\eta}{\partial\psi^2} = 0, \tag{5.5.33}$$

$$R: \qquad \eta + \psi \frac{d\xi^1}{dx} = 0. \tag{5.5.34}$$

From (5.5.28)

$$\eta(x, \psi) = B(x)\psi + C(x) \tag{5.5.35}$$

where B and C depend only on x. The system (5.5.28) to (5.5.34) becomes

$$B(x) + \frac{d\xi^1}{dx} = 0, \tag{5.5.36}$$

$$-2K\left(B(x) + \frac{d\xi^1}{dx}\right) + \frac{y^2}{2}\frac{dB}{dx} = 0, \tag{5.5.37}$$

$$\left(B(x) + \frac{d\xi^1}{dx}\right)\psi + C(x) = 0. \tag{5.5.38}$$

Substituting (5.5.36) into (5.5.37) gives

$$\frac{dB}{dx} = 0 \tag{5.5.39}$$

and therefore

$$B(x) = B_0 \tag{5.5.40}$$

where B_0 is a constant. Substituting (5.5.36) into (5.5.38) gives

$$C(x) = 0 \tag{5.5.41}$$

and therefore

$$\eta(x) = B_0\psi. \tag{5.5.42}$$

From (5.5.21),

$$\xi^2(y) = -\frac{B_0}{2}y. \tag{5.5.43}$$

The system (5.5.36) to (5.5.38) reduces to

$$\frac{d\xi^1}{dx} = -B_0 \tag{5.5.44}$$

and therefore

$$\xi^1(x) = -B_0 x + B_1 \tag{5.5.45}$$

where B_1 is a constant.

Thus from the first and second determining equations, (5.5.6) and (5.5.7), we obtain

$$\xi^1(x) = -B_0 x + B_1, \quad \xi^2(y) = -\frac{B_0}{2}y, \quad \eta(\psi) = B_0\psi. \tag{5.5.46}$$

We let $B_1 = c_1$ and $B_0 = 2c_2$. The Lie point symmetry associated with the conserved vector (5.3.28) is

$$X = (c_1 + 2c_2 x)\frac{\partial}{\partial x} + c_2 y\frac{\partial}{\partial y} - 2c_2\psi\frac{\partial}{\partial \psi}. \tag{5.5.47}$$

The Lie point symmetry generator (5.5.47) is a linear combination of two Lie point symmetries associated with the conserved vector (5.3.28). Unlike the Lie point symmetry (5.4.46) associated with the elementary conserved vector it does not contain an arbitrary function.

Consider now the invariant solution generated by the Lie point symmetry (5.5.47). Consider the general case in which $c_2 \neq 0$ and divide (5.5.47) by c_2. Then

$$X^* = 2(c_1^* + x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi} \tag{5.5.48}$$

where

$$X^* = \frac{1}{c_2}X, \quad c_1^* = \frac{c_1}{2c_2}. \tag{5.5.49}$$

The stars are suppressed to keep the notation simple. Now $\psi = \Psi(x, y)$ is an invariant solution of the PDE (5.3.1) generated by the Lie point symmetry associated with the conserved vector (5.3.28) provided

$$X(\psi - \Psi(x, y))|_{\psi=\Psi} = 0, \tag{5.5.50}$$

that is, provided

$$2(c_1 + x)\frac{\partial \Psi}{\partial x} + y\frac{\partial \Psi}{\partial y} = -2\Psi. \tag{5.5.51}$$

The differential equations of the characteristic curves of (5.5.54) are

$$\frac{dx}{2(c_1 + x)} = \frac{dy}{y} = -\frac{d\Psi}{2\Psi}. \tag{5.5.52}$$

The first pair of terms give

$$\frac{y}{(c_1 + x)^{\frac{1}{2}}} = b_1, \tag{5.5.53}$$

where b_1 is a constant. The first and last terms give

$$(c_1 + x)\Psi = b_2, \tag{5.5.54}$$

where b_2 is a constant. The general solution of the PDE (5.5.53) is

$$b_2 = F(b_1) \tag{5.5.55}$$

where F is an arbitrary function . Since $\psi = \Psi(x, y)$ the general solution for $\psi(x, y)$ is

$$\psi(x, y) = \frac{1}{(c_1 + x)} F(\xi) \tag{5.5.56}$$

where

$$\xi = \frac{y}{(c_1 + x)^{\frac{1}{2}}}. \tag{5.5.57}$$

Substitution of (5.5.56) into the PDE (5.4.1)yields the third order ordinary differential equation

$$2K\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + 3\frac{dF}{d\xi} = 0. \tag{5.5.58}$$

The velocity deficit and the velocity component in the y-direction are

$$w(x, y) = \frac{\partial\psi}{\partial y} = \frac{1}{(c_1 + x)^{\frac{3}{2}}}\frac{dF}{d\xi}, \tag{5.5.59}$$

$$v(x, y) = \frac{\partial\psi}{\partial x} = -\frac{1}{2(c_1 + x)^2}\left[2F(\xi) + \xi\frac{dF}{d\xi}\right]. \tag{5.5.60}$$

The boundary conditions are again (5.2.11) to (5.2.16). Expressed in terms of $F(\xi)$ they are:

$$y = +\infty : \quad w(x, \infty) = 0, \quad \frac{dF}{d\xi}(\infty) = 0, \tag{5.5.61}$$

$$y = +\infty: \quad \frac{\partial w}{\partial y}(x, \infty) = 0, \quad \frac{d^2 F}{d\xi^2}(\infty) = 0, \tag{5.5.62}$$

$$y = 0: \quad \frac{\partial w}{\partial y}(x, 0) = 0, \quad \frac{d^2 F}{d\xi^2}(0) = 0, \tag{5.5.63}$$

$$y = 0: \quad v(x, 0) = 0, \quad F(0) = 0, \tag{5.5.64}$$

$$y = -\infty: \quad \frac{\partial w}{\partial y}(x, -\infty) = 0, \quad \frac{d^2 F}{d\xi^2}(-\infty) = 0, \tag{5.5.65}$$

$$y = -\infty: \quad w(x, -\infty) = 0, \quad \frac{dF}{d\xi}(-\infty) = 0. \tag{5.5.66}$$

The only difference between the boundary conditions for a classical wake and the wake behind a self-propelled body is (5.5.64), The condition $F(0) = 0$ is not satisfied by a classical wake. The conserved quantity , (5.5.4), expressed in terms of $F(\xi)$ is

$$S = 2\rho U_0 \int_0^\infty \xi^2 \frac{dF}{d\xi} d\xi. \tag{5.5.67}$$

Unlike the classical wake, the wake behind a self-propelled body cannot be expressed in terms of $\frac{dF}{d\xi}$ because of (5.5.60) and (5.5.64). The function $F(\xi)$ needs to be determined.

In order to solve the ODE (5.5.58) it is first rewritten as

$$2K \frac{d^3 F}{dF^3} + \frac{d}{d\xi}\left(\xi \frac{dF}{d\xi}\right) + 2\frac{dF}{d\xi} = 0. \tag{5.5.68}$$

Integrating once we obtain

$$2K \frac{d^2 F}{d\xi^2} + \xi \frac{dF}{d\xi} + 2F = A, \tag{5.5.69}$$

where A is a constant. Now $\frac{dF}{d\xi}(0)$ is finite since $w(x, 0)$ is finite. The boundary conditions (5.5.63) and (5.5.64) at $\xi = 0$ give $A = 0$. By multiplying (5.5.69) by ξ it can be written as

$$2K\xi \frac{d^2 F}{d\xi^2} + \frac{d}{d\xi}\left(\xi^2 \frac{dF}{d\xi}\right) = 0. \tag{5.5.70}$$

. But

$$\xi \frac{d^2F}{d\xi^2} = \frac{d}{d\xi}\left(\xi \frac{dF}{d\xi}\right) - \frac{dF}{d\xi} \tag{5.5.71}$$

and therefore (5.5.70) becomes

$$2K\left[\frac{d}{d\xi}\left(\xi \frac{dF}{d\xi}\right) - \frac{dF}{d\xi}\right] + \frac{d}{d\xi}\left(\xi^2 \frac{dF}{d\xi}\right) = 0. \tag{5.5.72}$$

Integrating once we obtain

$$2K\xi \frac{dF}{d\xi} + (\xi^2 - 2K)F = B, \tag{5.5.73}$$

where B is a constant. Imposing the boundary condition (5.5.64) at $\xi = 0$ gives $B = 0$. Equation (5.5.73) is the first order ODE

$$\frac{dF}{d\xi} + \left(\frac{\xi}{2K} - \frac{1}{\xi}\right) F = 0 \tag{5.5.74}$$

The general solution is

$$F(\xi) = C\xi \exp\left[-\frac{\xi^2}{4K}\right], \tag{5.5.75}$$

where C is a constant.

The boundary conditions (5.5.61) to (5.5.66) are all identically satisfied by (5.5.75). The constant C is obtained from the conserved quantity (5.5.67). Substituting (5.5.75) into (5.5.67) gives

$$S = 8\rho U_0 K^{\frac{3}{2}} C \int_0^\infty \left(u^{\frac{1}{2}} - 2u^{\frac{3}{2}}\right) \exp(-u)du. \tag{5.5.76}$$

But the Gamma function is [13]

$$\Gamma(n) = \int_0^\infty u^{n-1} \exp(-u)du \tag{5.5.77}$$

and

$$\Gamma(n) = (n-1)\Gamma(n-1) \quad if \quad n - 1 > 0, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{5.5.78}$$

Thus,

$$\int_0^\infty u^{\frac{1}{2}} \exp(-u)du = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \tag{5.5.79}$$

$$\int_0^\infty u^{\frac{3}{2}} \exp(-u)du = \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \tag{5.5.80}$$

and therefore

$$C = -\frac{S}{8\sqrt{\pi}\rho U_0 K^{\frac{3}{2}}}. \tag{5.5.81}$$

Thus

$$F(\xi) = \frac{S}{8\sqrt{\pi}\rho U_0 K^{\frac{3}{4}}}\xi \exp\left[-\frac{\xi^2}{4K}\right]. \tag{5.5.82}$$

The remaining constant c_1 is obtained in the same way as for a classical wake. The effective width of the wake behind a self-propelled body and the classical wake are the same because the similarity variable ξ and the argument in the exponential in the solution are the same. The effective width is given by (5.4.79) and since we expect it to vanish at $x = 0$ we take $c_1 = 0$.

We obtain the following results:

$$\xi = \frac{y}{\sqrt{x}}, \tag{5.5.83}$$

$$F(\xi) = -\frac{S}{8\sqrt{\pi}\rho U_0 K^{\frac{3}{2}}} \frac{y}{x^{\frac{1}{2}}} \exp\left[-\frac{y^2}{4Kx}\right], \tag{5.5.84}$$

$$\psi(x, y) = -\frac{S}{8\sqrt{\pi}\rho U_0 K^{\frac{3}{2}}} \frac{y}{x^{\frac{3}{2}}} \exp\left[-\frac{y^2}{4Kx}\right] \tag{5.5.85}$$

and therefore

$$w(x, y) = -\frac{S}{8\sqrt{\pi}\rho U_0 K^{\frac{3}{2}}} \frac{1}{x^{\frac{3}{2}}} \left(1 - \frac{y^2}{2Kx}\right) \exp\left[-\frac{y^2}{4Kx}\right] \tag{5.5.86}$$

$$v_x(x, y) = U_0\left[1 + \frac{S}{8\sqrt{\pi}\rho U_0^2 K^{\frac{3}{2}}} \frac{1}{x^{\frac{3}{2}}} \left(1 - \frac{y^2}{2Kx}\right) \exp\left[-\frac{y^2}{4Kx}\right]\right], \tag{5.5.87}$$

$$v_y(x, y) = \frac{S}{16\sqrt{\pi}\rho U_0 K^{\frac{3}{2}}} \frac{y}{x^{\frac{5}{2}}} \left(3 - \frac{y^2}{2Kx}\right) \exp\left[-\frac{y^2}{4K}\right]. \tag{5.5.88}$$

The results agree with those of Herczynski et al [12]. The following points on the y- axis of the wake are of interest:

$$\begin{aligned} \textit{velocity deficit vanishes at} \quad & y = \pm\sqrt{2}(Kx)^{\frac{1}{2}}, \\ \textit{end points of the effective width are} \quad & y = \pm 2(Kx)^{\frac{1}{2}}, \\ \textit{y components of the velocity vanishes at} \quad & y = \pm\sqrt{6}(Kx)^{\frac{1}{2}}. \end{aligned}$$

Since $K = \frac{\nu}{U_0}$ these points move further from the x-axis as the viscosity increases due to an increase in diffusion. The velocity deficit is negative for

$$-\sqrt{2}(Kx)^{\frac{1}{2}} < y < \sqrt{2}(Kx)^{\frac{1}{2}}. \tag{5.5.89}$$

On the x-axis

$$w(x,0) = -\frac{SU_0^{\frac{1}{2}}}{8\sqrt{\pi}\rho\nu^{\frac{3}{2}}}\frac{1}{x^{\frac{3}{2}}}, \tag{5.5.90}$$

$$v_y(x,0) = U_0\left[1 + \frac{S}{8\sqrt{\pi}\rho U_0^{\frac{1}{2}}\nu^{\frac{3}{2}}}\frac{1}{x^{\frac{3}{2}}}\right]. \tag{5.5.91}$$

The magnitude of $w(x,0)$ decreases with viscosity and distance as $\nu^{-\frac{3}{2}}$ and $x^{-\frac{3}{2}}$ which is faster than a classical wake which decreases as $\nu^{-\frac{1}{2}}$ and $x^{-\frac{1}{2}}$.

Expressed in dimensionless variables the x-component of the velocity is

$$v_x^*(x^*, y^*) = 1 + \frac{S^*}{8\sqrt{\pi}}\frac{1}{x^{*\frac{3}{2}}}\left[1 - \frac{y^{*2}}{2x^*}\right]\exp\left(-\frac{y^{*2}}{4x^*}\right) \tag{5.5.92}$$

where

$$v_x^* = \frac{v_x}{U_0}, \quad S^* = \frac{S}{\rho U_0^2 K^3}, \quad x^* = \frac{x}{K}, \quad y^* = \frac{y}{K}, \tag{5.5.93}$$

where $K = \frac{\nu}{U_0}$ and S^* is the dimensionless second moment of the momentum deficit of the flow per unit span. In Figure 5.3, $v_x^*(x,y)$ is plotted against y^* at $x^* = 1$ for $S^* = 1$ and $S^* = 10$. The graphs show that as S^* increases the magnitude of the velocity deficit increases. We also see that the width of the wake does not depend on S. The width is determined by the diffusion constant K.

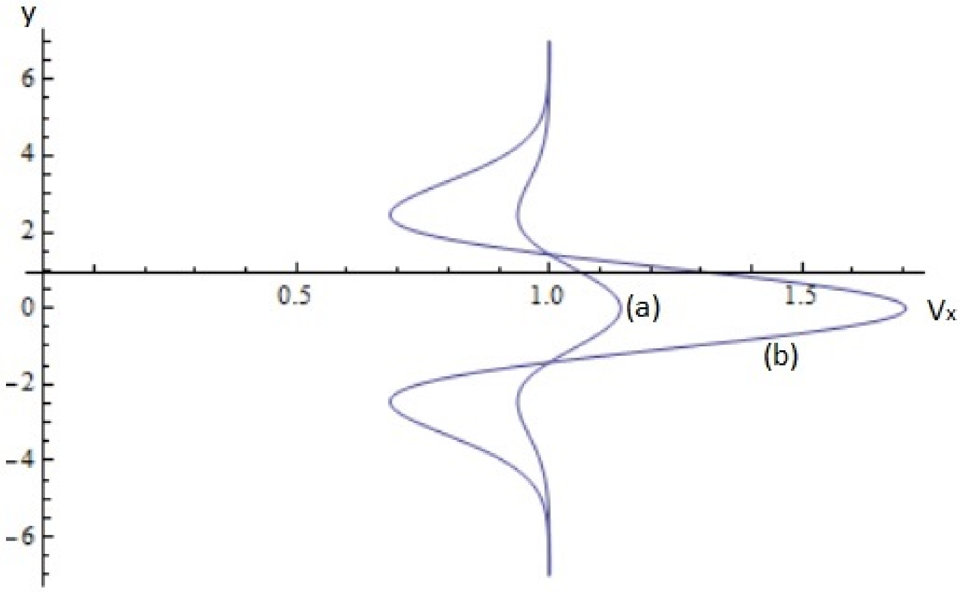

Figure 5.3: Velocity profile of the wake behind a self-propelled body at $x^* = 1$ and (a) $S^* = 2$, (b) $S^* = 10$

.

5.6 Comparison of the results and methods of solution for the two-dimensional wake

The results obtained for the classical wake and for the wake behind a self propelled body using associated Lie point symmetries will be compared with the results obtained using a linear superposition of all the Lie point symmetries of PDE (5.3.1) of the wake. Since the second method of solution does not appear in the literature the derivation of Lie point symmetries of (5.3.1) and the derivation of the group invariant solution is given in Appendix A.

It is shown in Appendix A that a linear superposition of all the Lie point symmetries of the PDE (5.3.1) can be expressed in the form

$$X_L = 2(c_1 + x)\frac{\partial}{\partial x} + (c_3 + y)\frac{\partial}{\partial y} + (c_4\psi + B(x,y))\frac{\partial}{\partial \psi}, \tag{5.6.1}$$

where $B(x,y)$ is any solution of (5.3.1):

$$\frac{\partial^2 B}{\partial x \partial y} = K\frac{\partial^3 B}{\partial y^3}. \tag{5.6.2}$$

This form of X_L is the most convenient to use to compare with the Lie point symmetry for the classical wake (5.4.47),

$$X_C = 2(c_1 + x)\frac{\partial}{\partial x} + (c_3 + y)\frac{\partial}{\partial y} + h(x)\frac{\partial}{\partial \psi}, \tag{5.6.3}$$

where $h(x)$ is an arbitrary function and for the wake behind a self-propelled body (5.5.48)

$$X_{SP} = 2(c_1 + x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\psi\frac{\partial}{\partial \psi}. \tag{5.6.4}$$

The following prolongation coefficients for will be required which can be calculated from the formulae (3.5.27) and (3.5.29):

$$\zeta_2 = (c_4 - 1)\psi_y + \frac{\partial B}{\partial y}(x,y), \tag{5.6.5}$$

$$\zeta_{22} = (c_4 - 2)\psi_{yy} + \frac{\partial^2 B}{\partial y^2}. \tag{5.6.6}$$

Now $B = B(x)$ is solution of (5.6.2) for any function $B(x)$. It is shown in Appendix A that when $B = B(x)$ the group invariant solution of the PDE (5.3.1) generated by X_L is

$$\psi(x,y) = (c_1 + x)^{\frac{1}{2}c_4}F(\xi) + H(x), \tag{5.6.7}$$

where

$$\xi = \frac{c_3 + y}{(c_1 + x)^{\frac{1}{2}}}, \tag{5.6.8}$$

$$H(x) = \frac{1}{2}(c_1 + x)^{\frac{1}{2}c_4}\int^x \frac{B(x)dx}{(c_1 + x)^{1+\frac{c_2}{2}}}. \tag{5.6.9}$$

The ODE and boundary conditions satisfied by $F(\xi)$ are also given in Appendix A.

From (5.3.53) to (5.3.55) the conserved vectors for the PDE (5.3.1) are

$$T^1_{(1)} = \psi_y, \quad T^2_{(1)} = -K\psi_{yy}, \tag{5.6.10}$$

$$T^1_{(2)} = y\psi_y, \quad T^2_{(2)} = -K(y\psi_{yy} - \psi_y), \tag{5.6.11}$$

$$T^1_{(3)} = y^2\psi_y, \quad T^2_{(3)} = -K(y^2\psi_{yy} - 2y\psi_y + 2\psi). \tag{5.6.12}$$

The condition for a Lie point symmetry

$$X = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial\psi} \tag{5.6.13}$$

to be associated with a conserved vector $T = (T^1, T^2)$ are (3.4.5a) and (3.4.5b):

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{5.6.14}$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{5.6.15}$$

We will show that X_C and X_{SP} are obtained from X_L in two ways , firstly by associating X_L with the conserved vectors T_1 and T_2 and secondly by determining the conditions on the constants c_1,c_3,c_4 and $B(x)$ for the conserved quantities (5.4.8) and (5.5.4) to be independent of x.

5.6.1 The classical wake

With the Lie point symmetry X_L and the conserved vector $T_{(1)}$ given by (5.6.10), the conditions (5.6.14) and (5.6.15) become

$$c_4T^1 + \frac{\partial B}{\partial y}(x, y) = 0, \tag{5.6.16}$$

$$c_4T^2 - K\frac{\partial^2 B}{\partial y^2}(x, y) = 0. \tag{5.6.17}$$

Thus X_L is associated with $T_{(1)}$ if

$$c_4 = 0 \quad and \quad B = B(x) \tag{5.6.18}$$

and X_L reduces to X_c.

Consider next the conserved quantity for the classical wake, (5.4.8):

$$D = \rho U_0 \int_{-\infty}^{\infty} \frac{\partial \psi}{\partial y}(x, y) dy. \tag{5.6.19}$$

Consider $B = B(x)$ so that (5.6.2) is satisfied. Expressed in terms of the invariant solution, (5.6.7) and (5.6.8), which is valid for $B = B(x)$, D becomes

$$D = \rho U_0 (c_1 + x)^{\frac{1}{2}c_4} \int_{-\infty}^{\infty} \frac{dF}{d\xi} d\xi, \tag{5.6.20}$$

which is independent of x provided $c_4 = 0$. Again X_L reduces to X_C. In both cases the boundary value problem for $F(\xi)$ in Appendix A reduces to the boundary value problem for the classical wake.

5.6.2 The wake behind a self-propelled body

Using the Lie point symmetry X_L and the conserved vector $T_{(3)}$ given by (5.6.12) the conditions (5.6.14) and (5.6.15) become

$$(c_4 + 2)T^1_{(3)} + 2c_3 T^1_{(2)} + \frac{\partial B}{\partial y} = 0, \tag{5.6.21}$$

$$(c_4 + 2)T^2_{(3)} + 2c_3 T^2_2 + y^2 \frac{\partial^2 B}{\partial y^2} - 2y \frac{\partial B}{\partial y} + 2B = 0. \tag{5.6.22}$$

Condition (5.6.21) is satisfied if

$$c_4 = -2, \quad c_3 = 0, \qquad B = B(x) \tag{5.6.23}$$

and (5.6.22) is satisfied provided the further condition

$$B(x) = 0 \tag{5.6.24}$$

holds. Thus X_L reduces to X_{SP}.

Consider now the conserved quantity for the wake behind a self-propelled body, (5.5.4) :

$$S = \rho U_0 \int_{-\infty}^{\infty} y^2 \frac{\partial \psi}{\partial y} dy. \tag{5.6.25}$$

Again consider $B = B(x)$ so that condition (5.6.2) is satisfied. Expressed in terms of the invariant solution, (5.6.7) and (5.6.8), S becomes

$$S = \rho U_0 \left[(c_1 + x)^{\frac{c_4}{2}+1} \int_{-\infty}^{\infty} \xi^2 \frac{dF}{d\xi} d\xi - 2c_3 (c_1 + x)^{\frac{1}{2}(c_4+1)} \int_{-\infty}^{\infty} \xi \frac{dF}{d\xi} d\xi + c_3^2 (c_1 + x)^{\frac{c_4}{2}} \int_{-\infty}^{\infty} \frac{dF}{d\xi} d\xi \right] \tag{5.6.26}$$

The second and third integrals in (5.6.25) cannot both be zero. Thus S is independent of x provided

$$c_3 = 0, \qquad c_4 = -2. \tag{5.6.27}$$

Unlike the condition for X_L to be associated with $T_{(3)}$ a condition on $B(x)$ is not obtained. However, since $B(x) = 0$ satisfies (5.6.2) we can choose $B(x) = 0$ to keep the problem simple. For Example the boundary condition (A.2.17) in Appendix A reduces to $F(0) = 0$. Thus X_L again reduces to X_{SP}.

5.7 Conclusions

The solution of the classical wake and the wake behind a self-propelled body both illustrate the double reduction theorem of Sjoberg [1]. For the classical wake, the PDE for the wake, (5.3.1), was reduced to the ODE (5.4.58). This ODE could be rewritten in form (5.4.72) which could be immediately integrated once thus making the double reduction. It could be integrated further and by imposing the boundary conditions an analytical solution was obtained. For the wake behind a self-propelled body the PDE for the wake was first reduced to the ODE (5.5.58) which could be rewritten in the form (5.5.68) and immediately integrated once. The double reduction was therefore again performed. The differential equation could again be integrated further and by using the boundary conditions an analytical solution to the problem was derived.

Like the jet problems the wake problems have a conserved quantity which is required to complete the solution. This is a special feature of problems with homogeneous boundary conditions. The conserved quantities can be obtained with the aid of the conservation laws for the PDE. It was easier to derive the Lie point symmetry associated with a conserved vector than all the Lie point symmetries of the PDE. If only one or two wake problems have to be solved then it is easier to use an associated Lie point symmetry. If all the Lie point symmetries of the PDE for the wake have been derived then the invariant solution for a range of wake problems can be considered

For the classical wake and the wake behind a self-propelled body the condition that the conserved quantity is independent of x was essentially equivalent to using a Lie point symmetry associated with the corresponding conserved vector to derive the invariant solution.

Chapter 6

Conclusions

When a problem requires a conserved quantity to complete the mathematical formulation there are two general methods to derive the invariant solution. The first method, which is well established, is to derive all the Lie point symmetries of the partial differential equation and to use a linear combination of all the Lie point symmetries to reduce the partial differential equation to an ordinary differential equation. The ordinary differential equation will contain undetermined parameters. One of these parameters is obtained by insisting that the conserved quantity is independent of all variables. The conserved quantity can be obtained systematically from one of the conservation laws for the partial differential equation and the boundary conditions. The second method consists in deriving the Lie point symmetry associated with the corresponding conserved vector and using ,this Lie point symmetry to reduce the partial differential equation to an ordinary differential equation. The conserved quantity , when expressed in terms of the similarity variable is found to be a constant

If only one jet flow or wake problem is considered the second method is more direct and less laborious. Since the order of the derivatives in the conserved vector is less than the order of the partial differential equation the prolongation formulae are less complicated. Instead of one large determining there are two smaller determining equations which together are easier to solve. For a partial differential equation with n variables there will be n determining equations one for each component of the association condition. If more than one or two problems have to be solved it is generally easier to use the first method and calculate all the Lie point symmetries of the partial differential equation. This approach can also be used to derive the invariant solution for boundary layers for which a conserved quantity does not exist.

The double reduction theorem of Sjoberg was very well illustrated by the jet flow and wake problems considered. All of the ordinary differential equations obtained by reducing the partial differential equation using an associated Lie point symmetry could be integrated once. In all cases they could be integrated further and an exact solution obtained. The condition for the conserved quantity to be independent of the variables is equivalent to using the associated Lie point symmetry to derive the invariant solution. The double reduction theorem will therefore apply to both methods. A conserved quantity does not exist in boundary layer problems and the double reduction theorem therefore does not apply which may explain why analytical solutions cannot generally be derived.

The two-fluid jet illustrated the double reduction theorem for two-fluid problems. The conserved quantity was the combined momentum flux of each layer because of matching conditions at the interface. The double reduction theorem was satisfied for each fluid layer because the partial differential equation for each layer was reduced by an associated Lie point symmetry. The solution in each fluid layer turned out to be the same as for a one fluid free jet because a constant of integration is zero in both solutions although as a result of imposing different boundary conditions.

We proposed an alternative way to calculate the equation of the interface between the two fluids of the two-fluid jet by using the condition, which is well established in the literature [17], that a fluid particle on the interface must remain on the interface as the fluid evolves. This condition is equivalent to the requirement that the component of the fluid velocity normal to the interface vanishes in each layer.

Two-fluid classical and self-propelled wakes have been considered by Herczynski et al. [12]. Unlike jet flows, the wake is an approximate solution to the boundary layer equations. The two-fluid wakes can also be solved using Lie group analysis and conservation laws as described in this dissertation.

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Appendix A

Lie point symmetries of the PDE for the wake and invariant solutions

A.1 Lie point symmetries of the PDE for wake

In this section we will outline the derivation of the Lie point symmetries of the PDE for the wake,

$$\frac{\partial^2 \psi}{\partial x \partial y} = K \frac{\partial^3 \psi}{\partial y^3}. \tag{A.1.1}$$

Now

$$X = \xi^1(x, y, \psi) \frac{\partial}{\partial x} + \xi^2(x, y, \psi) \frac{\partial}{\partial y} + \eta(x, y, \psi) \frac{\partial}{\partial \psi} \tag{A.1.2}$$

is a Lie point symmetry of the PDE (A.1.1) provided

$$X^{[3]}(\psi_{xy} - K\psi_{yyy})|_{PDE} = 0 \tag{A.1.3}$$

where only the following terms in the prolongation $X^{[3]}$ are required,

$$X^{[3]} = X + \zeta_{12} \frac{\partial}{\partial \psi_{xy}} + \zeta_{222} \frac{\partial}{\partial \psi_{yyy}}. \tag{A.1.4}$$

Equation (A.1.3) reduces to

$$\zeta_{12} - \zeta_{222}|_{PDE} = 0. \tag{A.1.5}$$

Expressions for ζ_{12} and ζ_{22} can be obtained by expanding (2.1.6) and (2.1.7.) The PDE is imposed on (A.1.5) by replacing ψ_{xy} in ζ_{12} and ζ_{222} by $K\psi_{yyy}$. In expanded form the determining equation (A.1.5) is

$$\begin{aligned}
&\frac{\partial^2\eta}{\partial x\partial y} + \psi_x\frac{\partial^2\eta}{\partial y\partial\psi} - \psi_x\frac{\partial^2\xi^1}{\partial y\partial x} - \psi_x^2\frac{\partial^2\xi^1}{\partial y\partial\psi} - \psi_y\frac{\partial^2\xi^2}{\partial x\partial y} - \psi_x\psi_y\frac{\partial^2\xi^2}{\partial y\partial\psi}\\
&+ \psi_y\frac{\partial^2\eta}{\partial\psi\partial x} + \psi_x\psi_y\frac{\partial^2\eta}{\partial\psi^2} - \psi_x\psi_y\frac{\partial^2\xi^1}{\partial x\partial\psi} - \psi_x^2\psi_y\frac{\partial^2\xi^1}{\partial\psi^2} - \psi_y^2\frac{\partial^2\xi^2}{\partial\psi\partial x} - \psi_x\psi_y^2\frac{\partial^2\xi^2}{\partial\psi^2}\\
&+ K\frac{\partial\eta}{\partial\psi}\psi_{yyy} - K\frac{\partial\xi^1}{\partial x}\psi_{yyy} - 2K\frac{\partial\xi^1}{\partial\psi}\psi_x\psi_{yyy} - K\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yyy}\\
&- \psi_{yy}\frac{\partial\xi^2}{\partial x} - \psi_x\psi_{yy}\frac{\partial\xi^2}{\partial\psi} - \psi_{xx}\frac{\partial\xi^1}{\partial y} - \psi_{xx}\psi_y\frac{\partial\xi^1}{\partial\psi} - K\frac{\partial\xi^2}{\partial y}\psi_{yyy} - K\frac{\partial\xi^2}{\partial\psi}\psi_y\psi_{yyy}\\
&+ K[-\frac{\partial^3\eta}{\partial y^3} - 3\psi_y\frac{\partial^3\eta}{\partial\psi\partial y^2} + \psi_x\frac{\partial^3\xi^1}{\partial y^3} + 3\psi_x\psi_y\frac{\partial^3\xi^1}{\partial\psi\partial y^2} + \psi_y\frac{\partial^3\xi^2}{\partial y^3} + 3\psi_y^2\frac{\partial^3\xi^2}{\partial\psi\partial y^2}\\
&- 3\psi_y^2\frac{\partial^3\eta}{\partial y\partial\psi^2} + 3\psi_x\psi_y^2\frac{\partial^3\xi^1}{\partial y\partial\psi^2} + 3\psi_y^3\frac{\partial^3\xi^2}{\partial y\partial\psi^2} + 3K\frac{\partial^2\xi^1}{\partial y^2}\psi_{yyy} + 6K\frac{\partial^2\xi^1}{\partial y\partial\psi}\psi_y\psi_{yyy}\\
&- 3\psi_{yy}\frac{\partial^2\eta}{\partial y\partial\psi} + 3\psi_x\psi_{yy}\frac{\partial^2\xi^1}{\partial y\partial\psi} + 3\psi_{yy}\frac{\partial^2\xi^2}{\partial y^2} + 9\psi_y\psi_{yy}\frac{\partial^2\xi^2}{\partial y\partial\psi} - \psi_y^3\frac{\partial^3\eta}{\partial\psi^3}\\
&+ \psi_x\psi_y^3\frac{\partial^3\xi^1}{\partial\psi^3} + \psi_y^4\frac{\partial^3\xi^2}{\partial\psi^3} + 3K\frac{\partial^2\xi^1}{\partial\psi^2}\psi_y^2\psi_{yyy} - 3\psi_y\psi_{yy}\frac{\partial^2\eta}{\partial\psi^2} + 3\psi_x\psi_y\psi_{yy}\frac{\partial^2\xi^1}{\partial\psi^2}\\
&+ 6\psi_y^2\psi_{yy}\frac{\partial^2\xi^2}{\partial\psi^2} + 3K\frac{\partial\xi^1}{\partial\psi}\psi_{yy}\psi_{yyy} + 3\psi_{yy}^2\frac{\partial\xi^1}{\partial\psi} + 3\psi_{xyy}\frac{\partial\xi^1}{\partial y} + 3\psi_{xyy}\frac{\partial\xi^2}{\partial\psi}\\
&- \psi_{yyy}\frac{\partial\eta}{\partial\psi} + 3\psi_{yyy}\frac{\partial\xi^2}{\partial y} + \psi_x\psi_{yyy}\frac{\partial\xi^1}{\partial\psi} + 4\psi_y\psi_{yyy}\frac{\partial\xi^2}{\partial\psi}]. = 0
\end{aligned} \tag{A.1.6}$$

We first equate to zero the coefficients of the following three partial derivatives of ψ.

$$\psi_{yy}\psi_{yyy}: \qquad \frac{\partial\xi^1}{\partial\psi} = 0. \tag{A.1.7}$$

$$\psi_{xyy}: \qquad \frac{\partial\xi^1}{\partial y} = 0, \tag{A.1.8}$$

$$\psi_{yy}^2: \qquad \frac{\partial\xi^2}{\partial\psi} = 0, \tag{A.1.9}$$

Thus

$$\xi^1 = \xi^1(x), \quad \xi^2 = \xi^2(x, y), \quad \eta = \eta(x, y, \psi) \tag{A.1.10}$$

The determining equation (A.1.6) is greatly simplified. It is separated by the remaining powers and products of the partial derivatives of ψ:

$$\psi_{yyy}: \qquad 2\frac{\partial \xi^2}{\partial y} - \frac{d\xi^1}{dx} = 0, \tag{A.1.11}$$

$$\psi_y\psi_{yy} \ and \ \psi_x\psi_y: \qquad \frac{\partial^2 \eta}{\partial \psi^2} = 0, \tag{A.1.12}$$

$$\psi_{yy}: \qquad 3K\frac{\partial^2 \xi^2}{\partial y^2} - 3K\frac{\partial^2 \eta}{\partial y \partial \psi} - \frac{\partial \xi^2}{\partial x} = 0, \tag{A.1.13}$$

$$\psi_y: \qquad K\frac{\partial^3 \xi^2}{\partial y^3} - \frac{\partial^2 \xi^2}{\partial x \partial y} - 3K\frac{\partial^3 \eta}{\partial \psi \partial y^2} + \frac{\partial^2 \eta}{\partial \psi \partial x} = 0, \tag{A.1.14}$$

$$\psi_x: \qquad \frac{\partial^2 \eta}{\partial y \partial \psi} = 0, \tag{A.1.15}$$

$$R: \qquad \frac{\partial^2 \eta}{\partial x \partial y} - K\frac{\partial^3 \eta}{\partial y^3} = 0. \tag{A.1.16}$$

From (A.1.11),

$$\xi^2(x, y) = \frac{y}{2}\frac{d\xi^1}{dx} + C(x), \tag{A.1.17}$$

where $C(x)$ depends only on x. From (A.1.12) and (A.1.15) we obtain

$$\eta(x, y, \psi) = A(x)\psi + B(x, y). \tag{A.1.18}$$

Next substitute (A.1.17) and (A.1.18) into (A.1.13). This gives

$$\frac{1}{2}\frac{d^2\xi^1}{dx^2}y + \frac{dC}{dx} = 0 \tag{A.1.19}$$

and separating by y we obtain

$$y: \qquad \frac{d^2\xi^1}{dx^2} = 0, \tag{A.1.20}$$

$$y^0: \qquad \frac{dC}{dx} = 0. \tag{A.1.21}$$

Thus

$$\xi^1(x) = c_1 + c_2 x \tag{A.1.22}$$

and

$$C(x) = c_3, \tag{A.1.23}$$

where c_1, c_2 and c_3 are constants. Thus (A.1.17) becomes

$$\xi^2(y) = \frac{1}{2}c_2 y + c_3. \tag{A.1.24}$$

Next substitute (A.1.18) and (A.1.24) for η and $\xi^2(y)$ into (A.1.14) which reduces to

$$\frac{dA}{dx} = 0. \tag{A.1.25}$$

Thus $A(x) = c_4$ where c_4 is a constant and (A.1.18) becomes

$$\eta(x, y, \psi) = c_4\psi + B(x, y). \tag{A.1.26}$$

Finally, substituting (A.1.26) into (A.1.16)gives

$$\frac{\partial^2 B(x,y)}{\partial x \partial y} - K\frac{\partial^3 B(x,y)}{\partial y^3} = 0 \tag{A.1.27}$$

which is the original PDE (A.1.1). Thus

$$\xi^1(x) = c_1 + c_2 x, \quad \xi^2(y) = c_3 + \frac{1}{2}c_2 y, \quad \eta(x, y, \psi) = c_4\psi + B(x, y), \tag{A.1.28}$$

where $B(x, y)$ is any solution of (A.1.27) and therefore

$$X = (c_1 + c_2 x)\frac{\partial}{\partial x} + (c_3 + \frac{1}{2}c_2 y)\frac{\partial}{\partial y} + (c_4\psi + B(x, y))\frac{\partial}{\partial \psi}. \tag{A.1.29}$$

The Lie point symmetries of the PDE (A.1.1) and therefore

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial y},$$
$$X_4 = \psi\frac{\partial}{\partial \psi}, \quad X_B = B(x, y)\frac{\partial}{\partial}. \tag{A.1.30}$$

Thus X consist of the sum of X_B and a linear combination of X_1,X_2,X_3 and X_4.

In Chapter 4 we considered the general case in which $c_2 \neq 0$ and multiplied the Lie point symmetries associated with the conserved vectors by $2/c_2$. In order to compare with these results we multiply (A.1.29) by $2/c_2$. This gives

$$X = 2(c_1^* + x)\frac{\partial}{\partial x} + (c_3^* + y)\frac{\partial}{\partial y} + (c_4^*\psi + B^*(x, y))\frac{\partial}{\partial \psi} \tag{A.1.31}$$

where

$$X^* = \frac{2}{c_2}X, \qquad c_1^* = \frac{c_1}{c_2}, \quad c_3^* = \frac{2c_3}{c_2} \qquad c_4^* = \frac{2c_4}{c_2}, \qquad B^*(x,y) = \frac{2}{c_2}B(x,y). \tag{A.1.32}$$

The star will be suppressed to keep the notation simple.

A.2 Invariant solution for wake

Now $\psi = \Psi(x,y)$ is an invariant solution of the PDE (A.1.1) generated by the Lie point symmetry (A.1.31) provided

$$X(\psi - \Psi(x,y))|_{\psi=\Psi} = 0, \tag{A.2.1}$$

that is , provided

$$2(c_1 + x)\frac{\partial \Psi}{\partial x} + (c_3 + y)\frac{\partial \Psi}{\partial y} = c_4\Psi + B(x,y). \tag{A.2.2}$$

The differential equations of the characteristic curves of the PDE (A.2.2) are

$$\frac{dx}{2(c_1+x)} = \frac{dy}{c_3+y} = \frac{d\Psi}{c_4\Psi + B(x,y)}. \tag{A.2.3}$$

The first pair of terms in (A.2.3) give

$$\frac{c_3+y}{(c_1+x)^{\frac{1}{2}}} = a_1, \tag{A.2.4}$$

where a_1 is a constant. Consider next the first and third terms in (A.2.3). The function $B(x,y)$ must satisfy the PDE (A.1.27) for any function $B(x)$. Now in the associated Lie point symmetry for the classical wake ,(5.4.47), $B = B(x)$ while in the associated Lie point symmetry for the wake behind a self-propelled body, (5.5.48), $B = 0$. In order to compare (A.1.31) with these two solutions we therefore consider $B = B(x)$. The first and last terms in (A.2.3) are

$$\frac{dx}{2(c_1+x)} = \frac{d\Psi}{c_4\Psi + B(x)}. \tag{A.2.5}$$

The general solution of (A.2.5) is

$$\frac{\Psi}{(c_1+x)^{\frac{c_4}{2}}} - \frac{1}{2}\int^x \frac{B(x)dx}{(c_1+x)^{1+\frac{c_4}{2}}} = a_2, \tag{A.2.6}$$

where F is an arbitrary function and since $\psi = \Psi$ we obtain the group invariant solution

$$\psi(x,y) = (c_1+x)^{\frac{c_4}{2}} F(\xi) + H(x) \tag{A.2.7}$$

where a_2 is constant.

The general solution of the PDE (A.2.2) with $B = B(x)$ is

$$a_2 = F(a_1), \tag{A.2.8}$$

where

$$\xi = \frac{c_3+y}{(c_1+x)^{\frac{1}{2}}} \tag{A.2.9}$$

and

$$H(x) = \frac{1}{2}(c_1+x)^{\frac{c_4}{2}} \int^x \frac{B(x)}{(c_1+x)^{1+\frac{c_4}{2}}}. \tag{A.2.10}$$

Substituting (A.2.7) into the PDE (A.1.1) gives the ODE

$$2K\frac{d^3F}{d\xi^3} + \xi\frac{d^2F}{d\xi^2} + (1-c_4)\frac{dF}{d\xi} = 0. \tag{A.2.11}$$

The velocity deficit and the y-component of the fluid velocity in the wake are

$$w(x,y) = \frac{\partial\psi}{\partial y} = (c_1+x)^{\frac{1}{2}(c_4-1)}\frac{dF}{d\xi}, \tag{A.2.12}$$

$$v_y = \frac{\partial\psi}{\partial x} = \frac{1}{2}(c_1+x)^{\frac{c_4}{2}-1}\left[c_4F(\xi) - \xi\frac{dF}{d\xi}\right] + \frac{dH}{dx}. \tag{A.2.13}$$

The boundary conditions for the classical wake and for the wake behind a self-propelled body are the same. They are

$$y = \infty : \quad w(x, \infty) = 0, \quad \frac{dF}{d\xi}(\infty) = 0, \tag{A.2.14}$$

$$y = 0 : \quad \frac{\partial w}{\partial y}(x, 0) = 0, \quad \frac{d^2 F}{d\xi^2}(0) = 0, \tag{A.2.15}$$

$$y = 0 : \quad v_y(x, 0) = 0, \quad \frac{c_4}{2}(c_1 + x)^{\frac{c_4}{2} - 1} F(0) + \frac{dH}{dx} = 0, \tag{A.2.16}$$

$$y = -\infty : \quad w(x, -\infty) = 0, \quad \frac{dF}{d\xi}(-\infty) = 0. \tag{A.2.17}$$

The conserved quantities for the classical wake and for the wake behind a self-propelled body are different. This separates the two problems. The conserved quantities are considered in Section 5.2.