The symmetry structures of curved manifolds and wave equations

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Declaration

I, the undersigned, declare that the contents of this thesis is my original work, except where due references have been made.

Jean Juste harrisson Bashingwa, February 2017

Dedication

To my son McJan Jisar

Acknowledgements

First of all I would like to thank God the creator of universes, for many blessings on me.

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Abstract

Killing vectors are widely used to study conservation laws admitted by spacetime metrics or to determine exact solutions of Einstein field equations (EFE) via Killing's equation. Its solutions on a manifold are in one-to-one correspondence with continuous symmetries of the metric on that manifold. Two well known spherically symmetric static spacetime metrics in Relativity that admit maximal symmetry are given by Minkowski and de-Sitter metrics. Some other spherically symmetric metrics forming interesting solutions of the EFE are known as Schwarzschild, Kerr, Bertotti-Robinson and Einstein metrics. We study the symmetry properties and conservation laws of the geodesic equations following these metrics as well as the wave and Klein-Gordon (KG) type equations constructed using the covariant d'Alembertian operator on these manifolds. As expected, properties of reduction procedures using symmetries are more involved than on the well known flat (Minkowski) manifold.

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Chapter 1

Introduction

Symmetry method is one of the cornestones of the geometric study of differential equations (DEs) [1, 2, 3]. It has been developed in the nineteenth century by the prominent Norwegian mathematician Sophus Lie (1842-1899). His works on continuous groups of transformations leaving invariants DEs introduced what is known as symmetry analysis of DEs.

A symmetry of a given system of DEs is a transformation that maps every solution of the system of DEs into another solution of the same DEs. Lie's approach remained unexploited for half a century. After his death, G. Birkhoff [4], I. Sedov [5] L.V. Ovsiannikov [3], H Weyl [6] , etc worked on Lie groups and contributed significantly in the application of the symmetry analysis of DEs in mathematical physics.

Another important implication of symmetry in physics is the link between symmetries and conservation laws. In 1918, a German mathematician Emmy Noether (1882-1935), with encouragement from Felix Klein, noticed a connection between continuous symmetries and conservation laws [7]. Noether symmetries, which are variational symmetries are associated with DEs which possess a Lagrangian. There are also methods which provide conservation laws independent of a Lagrangian (eg. direct construction method) [1]. However such methods may be cumbersome and require the usage of computer software.

In this thesis, we perform the symmetry analysis for ordinary differential equations (ODEs) and partial differential equations (PDEs) arising from some manifolds of interest. We study the association of symmetries, conservation laws, reduction and integrability by usage of invariants. This thesis contains compilation of several published articles.

Outline of the thesis

In chapter 2, we provide necessary tools for our investigation.

In chapter 3, we perform a symmetry analysis for the Euler-Lagrange (EL) equations arising from the Anti-self-duality (ASD) Einstein metrics. This metric is associated to the second heavenly equation (HE). Using Lie point symmetries, we reduce and find the exact solution for the heavenly equation. We also find the conservation laws using Noether theorem or direct construction.

The results presented in this chapter have been published [42]

In chapter 4, we study the invariance properties generated by some well-known metrics of neutral signatures. As the metrics depend on solutions of PDEs, we construct exact solutions of the PDEs using Lie group methods. We determine the isometries and the variational symmetries of the underlying metrics and corresponding Euler–Lagrange equations for both (Einstein Weyl structures and the corresponding four dimension metric obtained using the Jones-Tod construction) and establish relationships between these and conservation laws.

The results presented in this chapter have been published [64]

In chapter 5, we perform a symmetry analysis for a class of KG and wave equations that arise in Einstein and Kerr spacetimes. Using the underlying point symmetry, we reduce the wave equations by the invariant method, and obtain some exact solutions. Noether symmetries and conservation laws are obtained in each case, for the wave equations. Finally, we construct the high order symmetries and determine the corresponding conserved quantities.

The results presented in this chapter have been accepted for publication [66]

In chapter 6, we study diffusion equations in curved manifolds. These equations are constructed using the Laplace-Beltrami operators. In the first part, the Lie point symmetry for each equation is determined. In the second part, we compute the generalized symmetries with focus on evolutionary vector fields. We show that we can recover some of the geometric symmetries through these generalized symmetries.

The results presented in this chapter have been submitted for publication [67]

Chapter 2

Groundwork

2.1 Introduction

We will introduce some basic tools for the symmetry analysis of DEs.

2.2 Analysis of DEs via Symmetry

Basic Definitions

2.2.1 Definition. An m-dimensional manifold M is a topological space covered by countable collection of of subsets $W_\alpha \subset M$ called coordinate charts, and one-to-one maps

$\chi_\alpha : W_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$ called local coordinates on M

2.2.2 Definition. A Lie group is a smooth manifold which is also a group , such that the group multiplication $(g, h) \to g.h$ and inversion $g \to g^{-1}$ define smooth maps.

2.2.3 Definition. Let M be a smooth manifold. A local group of transformations acting on M is given by a (local) Lie group G, an open subset $\mathcal{U}$, with

$$\{e\} \times M \subset \mathcal{U} \subset G \times M,$$

which is the domain of definition of the group action, and a smooth map $\psi : \mathcal{U} \mapsto M$ with the following properties:

- If $(h, x) \in \mathcal{U}, (g, \psi(h, x)) \in \mathcal{U}$, and also $(g \cdot h, x) \in \mathcal{U}$, then
$$\psi(g, \psi(h, x)) = \psi(g \cdot h, x).$$
- For all $x \in M$,
$$\psi(e, x) = x.$$
- If $(g, x) \in \mathcal{U}$, then $(g^{-1}, \psi(g, x)) \in \mathcal{U}$ and
$$\psi(g^{-1}, \psi(g, x)) = x.$$

In this thesis, we will consider both point transformations and generalized symmetries in which infinitesimals involve derivatives of the dependent variables.

2.3 Symmetries of DEs

The symmetry group of a system of DEs is the largest local group of transformations that map solutions onto other solutions.

Consider an rth-order system of DEs

$$\Delta^{\nu}(x, u_{(r)}) = 0, \quad \nu = 1, ..., m \tag{2.3.1}$$

where

$x = (x^1, x^2, ..., x^p)$, $u = (u^1, u^2, ..., u^q)$ are respectively independent, dependent variables and $u_{(r)}$ denoting the derivatives of the u's with respect to the x's up to order r.

$$u_i^{\alpha} = D_i(u^{\alpha}), \qquad u_{ij}^{\alpha} = D_i D_j(u^{\alpha})$$

with

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha}\frac{\partial}{u_j^{\alpha}} + ... \quad i = 1, ..., p. \tag{2.3.2}$$

Let us consider a one-parameter (ϵ) Lie group

$$\psi(x, u) = (\psi_1(x, u), \psi_2(x, u)) = (\bar{x}, \bar{u}). \tag{2.3.3}$$

By expanding (2.3.3) to first-order in ϵ and by letting

$$\xi^i(x, u) = \tfrac{d}{d\epsilon}\psi_1^i(x, u)|_{\epsilon=0}, \qquad i = 1, \ldots, p,$$

$$\varphi^j(x, u) = \tfrac{d}{d\epsilon}\psi_2^j(x, u))|_{\epsilon=0}, \qquad i = 1, \ldots, q,$$

where $\psi_1 = (\psi_1^1, \psi_1^2, \ldots, \psi_1^p)$ and $\psi_2 = (\psi_2^1, \psi_2^2, \ldots, \psi_2^p)$, the infinitesimal transformations take the form

$$\bar{x} = x + \epsilon\xi(x, u)$$

$$\bar{u} = u + \epsilon\phi(x, u).$$

$\xi = (\xi^1, \ldots, \xi^p)$ and $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^q)$ are called the infinitesimals of (2.3.3) and the corresponding vector field is

$$X = \xi^i \frac{\partial}{\partial x^i} + \varphi^j \frac{\partial}{\partial u^j}$$

This operator is an abbreviated form of the following infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \varphi^j \frac{\partial}{\partial u^j} + \sum_{s \geq 1} \zeta^\alpha_{i1 \ldots is} \frac{\partial}{\partial u^\alpha_{i1 \ldots is}} \tag{2.3.4}$$

where

$$\zeta_i^\alpha = D_i(\mathcal{Q}^\alpha) + \xi^j u^\alpha_{ij}$$
$$\zeta^\alpha_{i1 \ldots is} = D_{i1 \ldots is}(\mathcal{Q}^\alpha) + \xi^j u^\alpha_{i1 \ldots is}, \quad s > 1$$

where $\mathcal{Q}^\alpha$ are Lie characteristic functions given by

$$\mathcal{Q}^\alpha = \varphi^\alpha - \sum_i \xi^i u_i^\alpha$$

2.3.1 Theorem (Infinitesimal criterion)**.** [2] A connected group of transformation G is a symmetry group of a system of DEs of maximal rank (2.3.1) if, and only if, the classical infinitesimal symmetry conditions

$$X[\Delta^\nu(x, u_{(r)})] = 0, \quad \text{whenever} \quad \Delta^\nu(x, u_{(r)}) = 0 \tag{2.3.5}$$

holds for every infinitesimal generator X of G.

2.4 Other operators

2.4.1 Definition. The Euler operator, is defined by

$$\frac{\delta}{\delta u^j} = \frac{\partial}{\partial u^j} + \sum_{s \geq 1} (-1)^s D_{i1 \ldots is} \frac{\partial}{\partial u^j_{i1 \ldots is}}, \quad j = 1, \ldots, q \tag{2.4.1}$$

The terms Euler operator and variational derivative are equivalents. The goal is to find the extrema of a *functional*

$$\mathcal{L}[u] = \int_\Omega L(x, u_{(n)}) dx$$

where $\Omega \subset X$ is an open, connected subset on $\mathbb{R}^n$, $L(x, u_(n))$ is called the Lagrangian.

2.4.2 Definition. An operator X of the form (2.3.4) is called a Noether symmetry corresponding to a Lagrangian L, up to a gauge function $B^i = (B^1, ..., B^p)$, if it satisfies the Killing-type equation

$$X(L) + LD_i(\xi^i) = D_i(B^i) \tag{2.4.2}$$

For any Noether symmetry X there exists a flux $\Phi^i = (\Phi^1, ..., \Phi^p)$, defined by

$$\Phi^i = B^i - N^i(L) \tag{2.4.3}$$

which is a conserved current of the Euler-Lagrange equations, N^i is given by

$$N^i = \xi^i + \mathcal{Q}^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i1...is}(\mathcal{Q}^\alpha) \frac{\delta}{\delta u_{i1...is}^\alpha} \tag{2.4.4}$$

There are many different ways to construct conservation laws, these include the partial Noether theorem [8] and the *multiplier method* for non-variational problems

Consider the system (2.3.1) . A flux $\Phi^i = (\Phi^1, ..., \Phi^p)$, is conserved if

$$D_i \Phi^i = 0$$

along solution of (2.3.1). It can be shown [9] that every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, ...)$ such that

$$Q_\mu \Delta^\mu = D_i \Phi^i \tag{2.4.5}$$

Then we have

$$\frac{\delta}{\delta u}(Q_\mu \Delta^\mu) = \frac{\delta}{\delta u}(D_i \Phi^i) = 0. \tag{2.4.6}$$

By the homotopy operator we can invert the divergence operator D_i and obtain the conserved current. [10, 1].

Chapter 3

Einstein manifolds

3.1 Introduction

The dispersionless integrable systems in 3+1 dimensions do not admit soliton solutions. However, such systems may be described in terms of ASD Quaternion-Kahler four-manifolds. (see [11] and [12] and references therein). If the Ricci-flat condition is imposed on top of ASD, the work of Plebanski [13] implies the existence of a local coordinate system (x, y, z, w) and a function u such that any ASD flat metric, g is given by

$$\mathrm{d}s^2 = 2(\mathrm{d}z\mathrm{d}y + \mathrm{d}w\mathrm{d}x - u_{xx}\mathrm{d}z^2 - u_{yy}\mathrm{d}w^2 + 2u_{xy}\mathrm{d}w\mathrm{d}z) \tag{3.1.1}$$

where $u(x, y, z, w)$ satisfies the second heavenly equation (HE)

$$u_{wx} - u_{zy} + u_{xx}u_{yy} - u_{xy}^2 = 0. \tag{3.1.2}$$

In this chapter we mainly refer to the results of the phenomenal work by Plebanski who, in a number of papers, e.g., [13], showed that the EFE that lead to ASD metrics are reducible to the elliptic complex Monge-Amperé equation (CMA). For example, the complex manifold with a Kähler metric

$$\mathrm{d}s^2 = u_{i\bar{k}}\mathrm{d}z^i\mathrm{d}\bar{z}^k. \tag{3.1.3}$$

with a metric potential $u(z^1, z^2, \bar{z}^1, \bar{z}^2)$ that satisfies the CMA equation

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = 0. \tag{3.1.4}$$

Versions of (3.1.4) are sometimes referred to as the 'first heavenly equation and, via a number of transformations, can be transformed to the Boyer-Finley (BF) equation which has been studied by a number of people, inter alia, Calderbank and Todd [14] and Martina, Sheftel and Winternitz [15].

Recently, Nutku and Sheftel [16] have, in detail, considered transformations of (3.1.4) and the corresponding metric (3.1.3) based on some solutions of the HE.

- A solution is given by

$$w = \ln\left[\frac{(p+a(z))b'(z)}{1+|b(z)|^2}\right]^2, \tag{3.1.5}$$

  the metric (3.1.1) becomes

$$\mathrm{d}s^2 = \frac{4r^4}{r^4+\alpha^2}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{r^4+\alpha^2}{r^2}(\mathrm{d}t + (1+\cos\theta)\mathrm{d}\phi)^2 \tag{3.1.6}$$

  which looks like an Eguchi-Hanson [17] metric

$$\mathrm{d}s^2 = \frac{r^4}{r^4-\alpha^2}\mathrm{d}r^2 + \frac{1}{4}r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{r^4-\alpha^2}{4r^2}(\mathrm{d}t + \cos\theta\mathrm{d}\phi)^2 \tag{3.1.7}$$

- A non invariant solution

$$w = \ln(p^2 + p(z+\bar{z}) + |z|^2 - 2\ln(1+|z|^2), \tag{3.1.8}$$

  of the BF equation leads to the metric

$$\begin{aligned}\mathrm{d}s^2 \;&= \frac{r^2}{r^4+\cot^2(\frac{1}{2}\theta)\sin^2\phi}(2r\mathrm{d}r + \cot(\tfrac{1}{2}\theta)\sin\phi\mathrm{d}\phi + \frac{\cos\phi}{2\sin^2(\frac{1}{2}\theta)}\mathrm{d}\theta)^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\ &+ \frac{r^4+\cot^2(\frac{1}{2}\theta)\sin^2\phi}{r^2}[dt + (1+\cos\theta)\mathrm{d}\phi \\ &+ \frac{(r^2-\cot(\frac{1}{2}\theta)\sin\phi)\sin\phi\mathrm{d}\theta - (r^2\cos\phi + \cot^2(\frac{1}{2}\theta)\sin^2\phi)\sin\theta\mathrm{d}\phi}{2\sin^2(\frac{1}{2}\theta)(r^4+\cot^2(\frac{1}{2}\theta)\sin^2\phi)}]^2.\end{aligned} \tag{3.1.9}$$

3.2 Ricci-flat metric

Reduction of the heavenly equation

We, firstly, determine exact solutions for the HE.

The generators of the Lie point symmetry of (3.1.2) are

$$\begin{array}{l}\mathcal{X}_1 = x\partial_u, \quad \mathcal{X}_2 = f^3(w,z)\partial_y, \quad \mathcal{X}_3 = yf^4(w,z) + x\int f^4 \mathrm{d}w)\partial_u,\\ \mathcal{X}_4 = xf^5(z)\partial_u, \quad \mathcal{X}_5 = 2w\partial_y - xy\partial_u, \quad \mathcal{X}_6 = w\partial_w + y\partial_y + u\partial_u,\\ \mathcal{X}_7 = 2w\partial_w - x\partial_x + y\partial_y - u\partial_u, \quad \mathcal{X}_8 = -2f^6(z)\partial_y + x^2 f^6_z\partial_u,\\ \mathcal{X}_9 = 2f^1(z,w)\partial_x - 2\int f^1_z \mathrm{d}w + (x^2\int f^1_{zz}\mathrm{d}w + y(f^1_w + 2xf^1_z))\partial_u,\\ \mathcal{X}_{10} = 6f^7(z)\partial_w + 6f^7_z\partial_y - x^2 f^7_{zz}\partial_u,\\ \mathcal{X}_{11} = 6f^2(z,w)\partial_z - 6(x\int f^2_{zz}\mathrm{d}w + yf^2_z)\partial_y + 6(yf^2_w + xf^2_z)\partial_x - 6\int f^2_z \mathrm{d}w\partial_w\\ +(x^3\int f^2_{zzz} + y(y^2 f^2_{ww} + 3x(yf^2_{zw} + xf^2_{zz})))\partial_u\end{array}$$

- In $\mathcal{X}_{11}$, if $f^2 = -\frac{1}{6}z$, we get the symmetry generator $\mathcal{X} = w\partial_w - x\partial_x + y\partial_y - z\partial_z$, we have the following transformed variables

  $$X = xw, \quad Y = y/w, \quad Z = zw, \quad U = u, \quad U = U(X,Y,Z)$$

  Relation (3.1.2) becomes

  $$U_X + XU_{XX} - YU_{XY} + ZU_{XZ} - U_{YZ} + U_{XX}U_{YY} - U^2_{XY} = 0, \tag{3.2.1}$$

  which admits, inter alia, the symmetry generator

  $\mathcal{X}^1_{11} = 2\partial_X + Y^2\partial_U$ and the scaling generator $\mathcal{X}^2_{11} = X\partial_X + Y\partial_Y + 3U\partial_U$.

  Via $\mathcal{X}^1_{11}$, we see that (3.2.1) reduces to the following equation

  $$\bar{U}_{YZ} + \frac{3}{2}Y^2 = 0,$$

  where $\bar{U} = U - \frac{1}{2}XY^2$ $(\bar{U} = \bar{U}(Y,Z))$. Hence, a solution of the HE is

  $$u = -\frac{1}{2}\frac{zy^3}{w^2} + \frac{1}{2}\frac{xy^2}{w}. \tag{3.2.2}$$

  Similarly, (3.2.1) may be transformed using $\mathcal{X}^2_{11}$, i.e., $\alpha = Y/X$, $Z = Z$ and $\bar{U} = U/X^3$ with $\bar{U} = \bar{U}(\alpha, Z)$. That is,

  $$2\alpha^2\bar{U}_{\alpha\alpha} - (z\alpha + 1)\bar{U}_{\alpha Z} - 7\alpha\bar{U}_\alpha + 3z\bar{U}_Z + 15\bar{U} + 4\bar{U}^2_\alpha = 0.$$

- For an alternative reduction of (3.2.1), one may take a linear combination of $\mathcal{X}_{10}$ and $\mathcal{X}_{11}$ with $F^7 = \frac{1}{6}z$ and $-\frac{1}{6}t$, respectively.

- $\mathcal{X}_6 + \mathcal{X}_7$ leads to similarity variables $\alpha = tx^3$, $\beta = yx^2$, $z = z$ and $U = u$ with $U = U(\alpha, \beta, z)$.

  The reduced PDE may be analysed further using a Lie symmetry reduction and a large class of invariant and exact solution are obtainable.

Lie & Noether symmetries and Killing vectors

The Lagrangian is given by

$$L = z'y' + w'x' - u_{xx}z'^2 - u_{yy}w'^2 + 2u_{xy}w'z', \tag{3.2.3}$$

where $'$ is the derivative with respect to the arclength variable s. The Euler-Lagrange (EL) equations are:

$$\begin{aligned}
&-w'' - w'^2u_{xyy} + 2w'z'u_{xxy} - z'^2u_{xxx} = 0,\\
&-z'' - w'^2u_{yyy} + 2w'z'u_{xyy} - z'^2u_{xxy} = 0,\\
&-y'' - 2w''u_{xy} - w'^2\left(u_{yyz} + 2u_{xyw}\right) + 2z''u_{xx} + z'^2u_{xxz} + 2y'z'u_{xxy}\\
&-2w'\left(y'u_{xyy} - z'u_{xxw} + x'u_{xxy}\right) + 2x'z'u_{xxx} = 0,\\
&-x'' + 2w''u_{yy} + w'^2u_{yyw} + 2w'z'u_{yyz} + 2w'y'u_{yyy} - 2z''u_{xy} - 2z'^2u_{xyz}\\
&+2w'x'u_{xyy} - 2y'z'u_{xyy} - z'^2u_{xxw} - 2x'z'u_{xxy} = 0.
\end{aligned} \tag{3.2.4}$$

For $u = -\frac{1}{2}\frac{zy^3}{w^2} + \frac{1}{2}\frac{xy^2}{w}$ given in (3.2.2), the EL (geodesic) equations are given by

$$\begin{aligned}
0 &= \frac{w'^2}{w} + w'',\\
0 &= -z'' + \frac{2w'z'}{w} + \frac{3zw'^2}{w^2}\\
0 &= -y'' - \frac{2w''y}{w} + \frac{5w'^2y}{w^2} - \frac{2w'y'}{w}\\
0 &= -x'' + 2w''\left(\frac{x}{w} - \frac{3yz}{w^2}\right) + w'^2\left(-\frac{x}{w^2} + \frac{6yz}{w^3}\right) - \frac{6w'z'y}{w^2} - \frac{6w'y'z}{w^2} - \frac{2}{w}\left(z''y - w'x' + y'z'\right)
\end{aligned} \tag{3.2.5}$$

The generators of Lie group that leave invariant the system (3.2.5) are:

$$\begin{aligned}
&X_1 = -(\sqrt{2}-1)yw^{-2+\sqrt{2}}\partial_x + w^{-1+\sqrt{2}}\partial_z, \qquad X_2 = yw^{-22-\sqrt{2}}(1+\sqrt{2})\partial_x + w^{-1-\sqrt{2}}\partial_z,\\
&X_3 = \partial_s, \qquad X_4 = s\partial_s, \qquad X_5 = -(-1+\sqrt{2})zw^{\sqrt{2}}\partial_x + w^{1+\sqrt{2}}\partial_y,\\
&X_6 = zw^{-\sqrt{2}}(1+\sqrt{2})\partial_x + w^{1-\sqrt{2}}\partial_y, \qquad X_7 = \frac{s}{w}\partial_x,\\
&X_8 = \frac{\ln w}{w}\partial_x, \qquad X_9 = \frac{1}{w}\partial_x, \qquad X_{10} = \ln w\partial_s,\\
&X_{11} = w\partial_w - x\partial_x, \qquad X_{12} = x\partial_x + y\partial_y, \qquad X_{13} = x\partial_x + z\partial_z.
\end{aligned} \tag{3.2.6}$$

The Noether symmetry generators X wich satisfies the equation (2.4.2) is given by:

$$\begin{array}{ll} X_1 = 2s\partial_s + w\partial_w + x\partial_x + y\partial_y + z\partial_z, & X_2 = \left(-\frac{\sqrt{7}}{3} + \frac{2}{3}\right) z^{2+\sqrt{7}}\partial_w + yz^{1+\sqrt{7}}\partial_x, \\ X_6 = z\partial_x, & X_4 = -(2+\sqrt{7})wz^{1+\sqrt{7}}\partial_x + z^{\sqrt{7}}\partial_y, \\ X_5 = wz^{-1-\sqrt{7}}(\sqrt{7}-2)\partial_x + z^{-\sqrt{7}}\partial_y, & X_3 = \left(\frac{2}{3} + \frac{\sqrt{7}}{3}\right) z^{2-\sqrt{7}}\partial_w + yz^{1-\sqrt{7}}\partial_x, \\ X_7 = sz\partial_x, & X_8 = \partial_s. \end{array} \tag{3.2.7}$$

Each one lead to a conservation law for the geodesic equations by Noether theorem. We list some for illustrative purposes.

i. X_8: $T_8 = \frac{w'^2(xw-3yz)-2yww'z'-w^2(z'y'+x'w')}{w^2}$

ii. X_5: $T_5 = (\sqrt{7}-2)wz^{-1-\sqrt{7}}w' + z^{-\sqrt{7}}z'$

iii. X_1: $T_1 = wx' - w'x + \frac{8w'yz}{w} + 3yz' + zy' - 2s\left[w'x' - w'^2\left(\frac{x}{w} - \frac{3yz}{w^2}\right) + \frac{2yz'}{w} + y'z'\right]$

The algebra of Killing vectors is generated by generated by

$$X_2, \quad X_3, \quad X_4, \quad X_5, \quad X_6.$$

3.3 ASD metrics solutions of the EFE

The Minkowski spacetime , is governed in polar coordinates, by the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \tag{3.3.1}$$

it admits a 10-dimensional algebra of Killing vectors, viz., $SO(3)\bigotimes\mathbb{R}^4$.

In [20], authors showed that the algebra of symmetries for the ELE via the Lagrangian

$$L = t'^2 - r'^2 - r^2\theta'^2 - r^2\sin^2\theta\phi'^2 \tag{3.3.2}$$

admits a 17-dimensional algebra including the Killing vectors.

In fact, using the multiplier approach, we got 17 multipliers (first derivative dependent), with $Q = (Q^r, Q^\theta, Q^\phi, Q^t)$, viz.,

$$
\begin{array}{ll}
(-t\sin\theta\sin\phi, -t\sin\phi\cos\theta, \frac{-t\cos\phi}{r\sin\theta}, r\sin\theta\sin\phi), & \\
(-t\sin\theta\cos\phi, -t\cos\phi\cos\theta, \frac{t\sin\phi}{r\sin\theta}, r\sin\theta\sin\phi), & \\
(-t\cos\phi, \frac{t\sin\theta}{r}, 0, r\cos\theta), & (\cos\theta, -\sin\theta/r, 0, 0), \\
(\sin\theta\sin\phi, \sin\phi\cos\theta/r, \frac{\cos\phi}{r\sin\theta}, 0), & (\sin\theta\cos\phi, \cos\phi\cos\theta/r, -\frac{\sin\phi}{r\sin\theta}, 0), \\
(s\sin\theta\sin\phi, s\sin\phi\cos\theta/r, \frac{s\cos\phi}{r\sin\theta}, 0), & (s\sin\theta\cos\phi, s\cos\phi\cos\theta/r, -\frac{s\sin\phi}{r\sin\theta}, 0), \\
(s\cos\theta, -s\sin\theta/r, 0, 0), & \\
(0, -\cos\phi, \cot\theta\sin\phi, 0), & (0, \sin\phi, \cot\theta\cos\phi, 0), \\
(0, 0, 1, 0), & (0, 0, 0, s), \\
(0, 0, 0, 1), & (-r', -\theta', -\phi', t'), \\
(sr - r's^2, -\theta's^2, -\phi's^2, st - t's^2), & (r - 2sr', -2s\theta', -2s\phi', t - 2st')
\end{array}
\tag{3.3.3}
$$

More details about the Lie point symmetry for this particular spacetime can be found in [21].

We note that all of the variational symmetries which are not Killing vectors include the s, the arclength variable. This, it will be emphasized in this paper, is not always the case as was previously thought.

Modified Eguchi-Hanson metric

In this subsection section, we are interested on metrics that arise from new ASD solutions of the EFE.

In (3.1.6), the Lagrangian is

$$
L = \frac{4r^4}{r^4+\alpha^2}r'^2 + r^2\theta'^2 + r^2\sin^2\theta\phi'^2 + \frac{r^4+\alpha^2}{r^2}(t'^2 + 2(1+\cos\theta))t'\phi' + (1+\cos\theta))^2\phi'^2).
\tag{3.3.4}
$$

The EL equations are

$$-\frac{1}{r^3}[2(-2(\alpha^2-r^4)r'(t'+(1+\cos(\theta))\phi')-r(\alpha^2+r^4)(\sin(\theta)\theta'\phi'-t''-(1+\cos(\theta))\phi''))]=0,$$

$$\frac{16r^7r'^2}{(\alpha^2+r^4)^2}-\frac{16r^3r'^2}{\alpha^2+r^4}+2r\theta'^2+2r\sin(\theta)^2\phi'^2+4r(t'+(1+\cos(\theta))\phi')^2$$
$$-\frac{2(\alpha^2+r^4)(t'+(1+\cos(\theta))\phi')^2}{r^3}-\frac{8r^4r''}{\alpha^2+r^4}=0,$$

$$-4rr'\theta'+r^2\sin(2\theta)\phi'^2-\frac{2(\alpha^2+r^4)\sin(\theta)\phi'(t'+(1+\cos(\theta))\phi')}{r^2}-2r^2\theta''=0,$$

$$\frac{1}{r^3}2\cos(\tfrac{\theta}{2})[4\cos(\tfrac{\theta}{2})r'((\alpha^2-r^4)t'+(\alpha^2(1+\cos(\theta))-2r^4)\phi')$$
$$+r(2(\alpha^2+r^4)\sin(\tfrac{\theta}{2})t'\theta'+4(\alpha^2(1+\cos(\theta))+r^4)\sin(\tfrac{\theta}{2})\theta'\phi'-2\cos(\tfrac{\theta}{2})((\alpha^2+r^4)t''$$
$$+(\alpha^2(1+\cos(\theta))+2r^4)\phi''))]=0. \tag{3.3.5}$$

The algebra of point variational symmetries, $X=\xi\partial_s+\tau\partial_t+\rho\partial_r+A\partial_\theta+B\partial_\phi$, are obtainable from (2.4.2) which expands, if each is set to zero, into the overdetermined system

$$\begin{aligned}
t_s^3 \quad &: -r^2\xi_t-\frac{\alpha^2\xi_t}{r^2}\\
r_s^3 \quad &: -\frac{4r^4\xi_r}{r^4+\alpha^2}\\
\theta_s^3 \quad &: -r^2\xi_\theta\\
\phi_s^3 \quad &: -r^2\xi_\phi-\frac{\alpha^2\xi_\phi}{r^2}-4r^2\cos(\theta)\xi_\phi-\frac{4\alpha^2\cos(\theta)\xi_\phi}{r^2}-2r^2\cos^2(\theta)\xi_\phi-\frac{2\alpha^2\cos^2(\theta)\xi_\phi}{r^2}\\
&\quad +2r^2\cos(\theta)\xi_\phi+\frac{2\alpha^2\cos(\theta)\xi_\phi}{r^2}+r^2\cos^2(\theta)\xi_\phi+\frac{\alpha^2\cos^2(\theta)\xi_\phi}{r^2}-r^2\sin^2(\theta)\xi_\phi\\
t_s^2 \quad &: 2r\rho-\frac{2\alpha^2\rho}{r^3}+2r^2B_t+\frac{2\alpha^2B_t}{r^2}+2r^2\cos(\theta)B_t+\frac{2\alpha^2\cos(\theta)B_t}{r^2}-r^2\xi_s-\frac{\alpha^2\xi_s}{r^2}+2r^2\tau_t+\frac{2\alpha^2\tau_t}{r^2}\\
r_s^2 \quad &: -\frac{16r^7\rho}{(r^4+\alpha^2)^2}+\frac{16r^3\rho}{r^4+\alpha^2}-\frac{4r^4\xi_s}{r^4+\alpha^2}+\frac{8r^4\rho_r}{r^4+\alpha^2}\\
\theta_s^2 \quad &: 2r\rho+2r^2A_\theta-r^2\xi_s
\end{aligned}$$

$$
\begin{aligned}
\phi_s^2 \quad &: 2r\rho - \tfrac{2\alpha^2\rho}{r^3} + 4r\rho\cos(\theta) - \tfrac{4\alpha^2\rho\cos(\theta)}{r^3} + 2r\rho\cos^2(\theta) - \tfrac{2\alpha^2\rho\cos^2(\theta)}{r^3} - 2Ar^2\sin(\theta) \\
&-\tfrac{2A\alpha^2\sin(\theta)}{r^2} - \tfrac{2A\alpha^2\cos(\theta)\sin(\theta)}{r^2} + 2r\rho\sin^2(\theta) + 2r^2B_\phi + \tfrac{2\alpha^2B_\phi}{r^2} + 4r^2\cos(\theta)B_\phi \\
&+\tfrac{4\alpha^2\cos(\theta)B_\phi}{r^2} + 2r^2\cos^2(\theta)B_\phi + \tfrac{2\alpha^2\cos^2(\theta)B_\phi}{r^2} + 2r^2\sin^2(\theta)B_\phi - r^2\xi_s - \tfrac{\alpha^2\xi_s}{r^2} - 4r^2\cos(\theta)\xi_s \\
&-\tfrac{4\alpha^2\cos(\theta)\xi_s}{r^2} - 2r^2\cos^2(\theta)\xi_s - \tfrac{2\alpha^2\cos^2(\theta)\xi_s}{r^2} \\
&+2r^2\cos(\theta)\xi_s + \tfrac{2\alpha^2\cos(\theta)\xi_s}{r^2} + r^2\cos^2(\theta)\xi_s + \tfrac{\alpha^2\cos^2(\theta)\xi_s}{r^2} - r^2\sin^2(\theta)\xi_s + 2r^2\tau_\phi + \tfrac{2\alpha^2\tau_\phi}{r^2} \\
&+2r^2\cos(\theta)\tau_\phi + \tfrac{2\alpha^2\cos(\theta)\tau_\phi}{r^2} \\
r_st_s \quad &: 2r^2B_r + \tfrac{2\alpha^2B_r}{r^2} + 2r^2\cos(\theta)B_r + \tfrac{2\alpha^2\cos(\theta)B_r}{r^2} + \tfrac{8r^4\rho_t}{r^4+\alpha^2} + 2r^2\tau_r + \tfrac{2\alpha^2\tau_r}{r^2} \\
\theta_st_s \quad &: 2r^2A_t + 2r^2B_\theta + \tfrac{2\alpha^2B_\theta}{r^2} + 2r^2\cos(\theta)B_\theta + \tfrac{2\alpha^2\cos(\theta)B_\theta}{r^2} + 2r^2\tau_\theta + \tfrac{2\alpha^2\tau_\theta}{r^2} \\
\phi_st_s \quad &: 4r\rho - \tfrac{4\alpha^2\rho}{r^3} + 4r\rho\cos(\theta) - \tfrac{4\alpha^2\rho\cos(\theta)}{r^3} - 2Ar^2\sin(\theta) - \tfrac{2A\alpha^2\sin(\theta)}{r^2} + 2r^2B_t \\
&+\tfrac{2\alpha^2B_t}{r^2} + 4r^2\cos(\theta)B_t + \tfrac{4\alpha^2\cos(\theta)B_t}{r^2} + 2r^2\cos^2(\theta)B_t + \tfrac{2\alpha^2\cos^2(\theta)B_t}{r^2} + 2r^2\sin^2(\theta)B_t + 2r^2B_\phi \\
&+\tfrac{2\alpha^2B_\phi}{r^2} + 2r^2\cos(\theta)B_\phi + \tfrac{2\alpha^2\cos(\theta)B_\phi}{r^2} - 2r^2\xi_s - \tfrac{2\alpha^2\xi_s}{r^2} - 4r^2\cos(\theta)\xi_s - \tfrac{4\alpha^2\cos(\theta)\xi_s}{r^2} \\
&+2r^2\cos(\theta)\xi_s + \tfrac{2\alpha^2\cos(\theta)\xi_s}{r^2} + 2r^2\tau_t + \tfrac{2\alpha^2\tau_t}{r^2} + 2r^2\cos(\theta)\tau_t + \tfrac{2\alpha^2\cos(\theta)\tau_t}{r^2} + 2r^2\tau_\phi + \tfrac{2\alpha^2\tau_\phi}{r^2} \\
r_s\theta_s \quad &: 2r^2A_r + \tfrac{8r^4\rho_\theta}{r^4+\alpha^2} \\
r_s\phi_s \quad &: 2r^2B_r + \tfrac{2\alpha^2B_r}{r^2} + 4r^2\cos(\theta)B_r + \tfrac{4\alpha^2\cos(\theta)B_r}{r^2} + 2r^2\cos^2(\theta)B_r + \tfrac{2\alpha^2\cos^2(\theta)B_r}{r^2} \\
&+2r^2\sin^2(\theta)B_r + \tfrac{8r^4\rho_\phi}{r^4+\alpha^2} + 2r^2\tau_r + \tfrac{2\alpha^2\tau_r}{r^2} + 2r^2\cos(\theta)\tau_r + \tfrac{2\alpha^2\cos(\theta)\tau_r}{r^2} \\
\theta_s\phi_s \quad &: 2r^2A_\phi + 2r^2B_\theta + \tfrac{2\alpha^2B_\theta}{r^2} + 4r^2\cos(\theta)B_\theta + \tfrac{4\alpha^2\cos(\theta)B_\theta}{r^2} + 2r^2\cos^2(\theta)B_\theta + \tfrac{2\alpha^2\cos^2(\theta)B_\theta}{r^2} \\
&+2r^2\sin^2(\theta)B_\theta + 2r^2\tau_\theta + \tfrac{2\alpha^2\tau_\theta}{r^2} + 2r^2\cos(\theta)\tau_\theta + \tfrac{2\alpha^2\cos(\theta)\tau_\theta}{r^2} \\
t_s \quad &: 2r^2B_s + \tfrac{2\alpha^2B_s}{r^2} + 2r^2\cos(\theta)B_s + \tfrac{2\alpha^2\cos(\theta)B_s}{r^2} - f_t + 2r^2\tau_s + \tfrac{2\alpha^2\tau_s}{r^2} \\
r_s \quad &: -f_r + \tfrac{8r^4\rho_s}{r^4+\alpha^2} \\
\theta_s \quad &: 2r^2A_s - f_\theta \\
\phi_s \quad &: 2r^2B_s + \tfrac{2\alpha^2B_s}{r^2} + 4r^2\cos(\theta)B_s + \tfrac{4\alpha^2\cos(\theta)B_s}{r^2} + 2r^2\cos^2(\theta)B_s + \tfrac{2\alpha^2\cos^2(\theta)B_s}{r^2} + 2r^2\sin^2(\theta)B_s \\
&-f_\phi + 2r^2\tau_s + \tfrac{2\alpha^2\tau_s}{r^2} + 2r^2\cos(\theta)\tau_s + \tfrac{2\alpha^2\cos(\theta)\tau_s}{r^2} \\
&: -f_s
\end{aligned}
\tag{3.3.6}
$$

Whilst this is a cumbersome approach, the multiplier method provides all the multipliers for (3.3.5), viz.,

$$
\begin{array}{lll}
(1,0,0,0), & (0,0,0,1), & (-t',-r',-\theta',-\phi'), \\
(-\frac{\cos\phi}{\tan(\frac{1}{2}\theta)},0,\sin\phi,\frac{\cos\phi}{\tan\theta}), & & (\frac{\sin\phi}{\tan(\frac{1}{2}\theta)},0,\cos\phi\,-\frac{\sin\phi}{\tan\theta}).
\end{array}
\tag{3.3.7}
$$

It turns out that each leads to a variational symmetry and the algebra is generated by

$$
\partial_t, \qquad \partial_\phi, \qquad \partial_s, \qquad -\frac{\cos\phi}{\tan(\frac{1}{2}\theta)}\partial_t+\sin\phi\partial_\theta+\frac{\cos\phi}{\tan\theta}\partial_\phi, \qquad \frac{\sin\phi}{\tan(\frac{1}{2}\theta)}\partial_t+\cos\phi\partial_\theta-\frac{\sin\phi}{\tan\theta}\partial_\phi.
\tag{3.3.8}
$$

It is easy to see that all of the above excluding the translation in s are isometries of the manifold.

Metrics with ultra hyperbolic signature

One choice of the metric here [16] is

$$\mathrm{d}s^2 = w_p(4e^w \mathrm{d}z\mathrm{d}\bar{z} - \mathrm{d}p^2) - \frac{1}{w_p}\left[\mathrm{d}t + i(w_z\mathrm{d}z - w_{\bar{z}}\mathrm{d}\bar{z})\right]^2 \tag{3.3.9}$$

in which case the EFE reduce to the hyperbolic BF equation

$$w_{z\bar{z}} - (e^w)_{pp} = 0 \tag{3.3.10}$$

for which a solution is

$$w = \ln\left[\frac{(p + a(z))b'(z)}{b(z) + \bar{b}(\bar{z})}\right]^2. \tag{3.3.11}$$

Then there are two metrics of which one is

- $$\begin{aligned} \mathrm{d}s_1^2 &= (2p + a + b)\left[\tfrac{4f'g'}{(f+g)^2}\mathrm{d}u\mathrm{d}v - \tfrac{1}{(p+a)(p+b)}\mathrm{d}p^2\right] \\ &+ \tfrac{(p+a)(p+b)}{(2p+a+b)}\left[\mathrm{d}t + (\tfrac{2g'}{f+g} - \tfrac{b'}{p+b} - \tfrac{g''}{g'})\mathrm{d}v - (\tfrac{2f'}{f+g} - \tfrac{a'}{p+a} - \tfrac{f''}{f'})\mathrm{d}u\right]^2, \end{aligned} \tag{3.3.12}$$

  where $a = a(u)$, $b = b(v)$, $f = f(u)$ and $g = g(v)$.

  In the case, a and b equal to constants with $f = u$, $g = v$, we get a large number of vector fields. We choose $a = b = 1$; the multipliers $Q = (Q^u, Q^v, Q^p, Q^t)$, are

$$\begin{aligned}
&Q^1 = (-\tfrac{su\sqrt{u+v}e^{\frac{1}{4}t}}{\sqrt{p}}, 0, \tfrac{\sqrt{p}sve^{\frac{1}{4}t}}{\sqrt{u+v}}, -2\tfrac{\sqrt{u+v}se^{\frac{1}{4}t}}{\sqrt{p}}), && Q^2 = (\tfrac{s\sqrt{u+v}e^{\frac{1}{4}t}}{\sqrt{p}}, 0, \tfrac{\sqrt{p}se^{\frac{1}{4}t}}{\sqrt{u+v}}, 0),\\
&Q^3 = (0, -\tfrac{sv\sqrt{u+v}e^{-\frac{1}{4}t}}{\sqrt{p}}, \tfrac{\sqrt{p}sue^{-\frac{1}{4}t}}{\sqrt{u+v}}, 2\tfrac{\sqrt{u+v}se^{-\frac{1}{4}t}}{\sqrt{p}}), && Q^4 = (0, \tfrac{s\sqrt{u+v}e^{-\frac{1}{4}t}}{\sqrt{p}}, \tfrac{\sqrt{p}se^{-\frac{1}{4}t}}{\sqrt{u+v}}, 0),\\
&Q^5 = (-\tfrac{u\sqrt{u+v}e^{\frac{1}{4}t}}{\sqrt{p}}, 0, \tfrac{\sqrt{p}ve^{\frac{1}{4}t}}{\sqrt{u+v}}, -2\tfrac{\sqrt{u+v}e^{\frac{1}{4}t}}{\sqrt{p}}), && Q^6 = (\tfrac{\sqrt{u+v}e^{\frac{1}{4}t}}{\sqrt{p}}, 0, \tfrac{\sqrt{p}e^{\frac{1}{4}t}}{\sqrt{u+v}}, 0),\\
&Q^7 = (0, -\tfrac{v\sqrt{u+v}e^{-\frac{1}{4}t}}{\sqrt{p}}, \tfrac{\sqrt{p}ue^{-\frac{1}{4}t}}{\sqrt{u+v}}, 2\tfrac{\sqrt{u+v}e^{-\frac{1}{4}t}}{\sqrt{p}}), && Q^8 = (0, \tfrac{\sqrt{u+v}e^{-\frac{1}{4}t}}{\sqrt{p}}, \tfrac{\sqrt{p}e^{-\frac{1}{4}t}}{\sqrt{u+v}}, 0),\\
&Q^9 = (0, (u+v)e^{-\frac{1}{2}t}, 0, -2e^{-\frac{1}{2}t}), && Q^{10} = (-\tfrac{1}{2}u^2, \tfrac{1}{2}v^2, 0, -(u+v)),\\
&Q^{11} = (u, v, 0, 0), && Q^{12} = (-1, 1, 0, 0),\\
&Q^{13} = ((u+v)e^{\frac{1}{2}t}, 0, 0, 2e^{\frac{1}{2}t}), && Q^{14} = (0, 0, 0, 1),\\
&Q^{15} = (-u', -v', -p', -t'), && Q^{16} = (-su', -sv', (p+1) - sp', -st').
\end{aligned} \tag{3.3.13}$$

In general, if a or b are not constant, only two symmetries (multipliers) are obtained, viz., ∂_s and ∂_t.

- Another related metric from (3.3.11) is given by

$$\begin{aligned} \mathrm{d}s_2^2 &= (2p+a+b)\left[\tfrac{4f'g'}{(1+fg)^2}\mathrm{d}u\mathrm{d}v + \tfrac{1}{(p+a)(p+b)}\mathrm{d}p^2\right] \\ &\quad - \tfrac{(p+a)(p+b)}{(2p+a+b)}\left[\mathrm{d}t + (\tfrac{2fg'}{1+fg} - \tfrac{b'}{p+b} - \tfrac{g''}{g'})\mathrm{d}v - (\tfrac{2gf'}{1+fg} - \tfrac{a'}{p+a} - \tfrac{f''}{f'})\mathrm{d}u\right]^2. \end{aligned} \tag{3.3.14}$$

For a and b constants we obtain 16 multipliers corresponding to 16 variational symmetries.

3.4 Conclusion

We obtain 10-dimensional killing vectors which suggest that the manifold is flat, for the two particular cases discussed in Section 3.3.

However, we cannot conclude whether they are equivalent to the Minkowksi or de Sitter metrics; the algebra of Noether symmetries in the latter two cases are 17- and 12- dimensional, respectively, whilst the two classes in Section 3.3 are 16-dimensional. Lastly, we see that the multiplier approach is more efficient and inclusive than the other approaches.

Chapter 4

Anti-self-duality (ASD) structures in neutral signature

4.1 Introduction

The concept of anti-self-duality in four dimension is very close to the integrability of specific field equations and manifolds. From a physicists point of view the differential equations leading to anti-self-dual hyper-Kähler or quaternion Kähler are similar in kind to those of Einstein's general relativity as they impose restrictions on the Riemann curvature of four manifolds [22]. The Riemann curvature tensor of a pseudo-Riemannian manifold can be decomposed into three part

$$R_{abcd} = S_{abcd} + E_{abcd} + W_{abcd}$$

where S_{abcd} is the scalar curvature, E_{abcd} is Ricci curvature and W_{abcd} is the Weyl curvature. In lower dimension $(n = 1, 2, 3)$ the Weyl tensor always vanishes. For $n = 4$ the Weyl curvature W decomposes further under $SO(4)$ into its self-dual and anti-self-dual part W^+, W^- . A metric is ASD if and only if $W^+ = 0$. [23]

The Jones-Tod construction

The construction relates ASD conformal structures in four dimension to Einstein-Weyl (EW) structures in three dimensions. In neutral signature it can be formulated as follows:

4.1.1 Theorem. Let $(M, [g])$ be a neutral ASD four manifold with a null conformal Killing vector K. An Einstein-Weyl structure on the space $\mathcal{W}$ of trajectories of K is defined by

$$h = |K|^{-2} g - |K|^{-4} \mathbb{K} \odot \mathbb{K}, \qquad \omega = 2|K|^{-2} *_g (\mathbb{K} \wedge \mathbb{K})$$

where $*_g$ is the Hogde-$*$ of g,$\odot$ is a symmetric tensor product, $|K|^2 = g(K, K)$, and $\mathbb{K} = g(K, .)$. All EW structures arise in this way. Conversely, let (h, ω) be a three dimensional Lorentzian EW structure on $\mathcal{W}$, and let (V, η) be a function and a 1-form on W satisfying the generalised monopole equation

$$*_h (\mathrm{d}V + \frac{1}{2}\omega V) = \mathrm{d}\eta \tag{4.1.1}$$

where $*_h$ is the Hodge-$*$ of h. Then

$$g = Vh - \frac{1}{V}(\mathrm{d}\phi + \eta)^2 \tag{4.1.2}$$

is an ASD metric with isometry ∂_ϕ. The details can be found in [29]

4.2 Scalar-flat Kähler

Let $(\mathcal{M}, g)$ be a scalar-flat Kähler metric in neutral signature with symmetry. LeBrun [26] had shown that the problem can be reduced to a pair of coupled PDEs: the $SU(\infty)$-Toda equation and its linearisation [28]. The metric takes the form

$$g = V(e^u(\mathrm{d}x^2 + \mathrm{d}y^2) - \mathrm{d}t^2) - \frac{1}{V}(\mathrm{d}\phi + \eta)^2 \tag{4.2.1}$$

where the function u satisfies the $SU(\infty)$-Toda equation

$$(e^u)_{tt} - u_{xx} - u_{yy} = 0 \tag{4.2.2}$$

and V is a solution to its linearisation. The corresponding EW space from the Jones-Tod construction is

$$h = e^u(\mathrm{d}x^2 + \mathrm{d}y^2) - \mathrm{d}t^2, \quad \omega = 2u_t \mathrm{d}t. \tag{4.2.3}$$

Reduction and exact solutions of $SU(\infty)$-Toda equation

The Lie point symmetry algebra of (4.2.2) is spanned by

$$\begin{aligned}
&\mathcal{X}_1 = \partial_t \\
&\mathcal{X}_2 = 2\partial_u + t\partial_t \\
&\mathcal{X}_3 = \partial_y \\
&\mathcal{X}_4 = -2f_{1,x}\partial_u + f_1(x,y)\partial_x + \int f_{1,x}\mathrm{d}y\partial_y
\end{aligned}$$

where the function f_1 satisfies the constraint $\int f_{1,xx}\mathrm{d}y + f_{1,y} = 0$

In $\mathcal{X}_4$ if $f_1 = x$ we have the following invariants and transformed variables

$$u = \ln y^{-2} + U, \quad X = \frac{x}{y}, \quad T = t, \quad U = U(X,T).$$

The equation (4.2.2) becomes

$$(U_T^2 + U_{TT})e^U - U_{XX} - 2XU_X - X^2U_{XX} - 2 = 0, \tag{4.2.4}$$

which admits the symmetries

$$\begin{aligned}
&\mathcal{X}_4^1 = \partial_T, \\
&\mathcal{X}_4^2 = 2\partial_U + T\partial_T, \\
&\mathcal{X}_4^3 = (1+X^2)\partial_X - 2X\partial_U, \\
&\mathcal{X}_4^4 = (1+X^2)\arctan(X)\partial_X - 2(1+X\arctan(X))\partial_U.
\end{aligned}$$

- Via $\mathcal{X}_4^3$, (4.2.4) is reduced to the ODE

  $$(\bar{U}_T^2 + \bar{U}_{TT})e^{\bar{U}} = 0$$

  where

  $$\ln(1+X^2) = -U + \bar{U}, \qquad \bar{U} = \bar{U}(T)$$

  Solving and replacing back we get a solution to (4.2.2)

  $$u = \ln\frac{-C_1t - C_2C_1}{x^2+y^2} \tag{4.2.5}$$

- Using $\mathcal{X}_4^2$, one can show that the solution is

$$u = \ln\left[\frac{\frac{C_1^2 t^2}{4}\left[\tan(\frac{C_1}{2}\arctan(\frac{x}{y}) + C_1 C_2)^2 + 1\right]}{x^2+y^2}\right]. \tag{4.2.6}$$

An alternative reduction can be done using $\mathcal{X}_2$ in which case

the equation (4.2.2) becomes

$$2e^U - U_{XX} - U_{YY} = 0$$

where $U = u - \ln t^2, \quad X = x, \quad Y = y, \quad U = U(X,Y)$.

Reducing again using its symmetries , solving and replacing back we have the solution

$$u = \ln\left[t^2\left(\frac{\tan(\frac{C_2+y-x}{2C_1^2})^2}{C_1^2}\right)\right]. \tag{4.2.7}$$

4.2.1 Remark. The metric (4.2.1) can be found explicitly from the monopole equation (4.1.1) for some solutions as follows [24]:

Rewrite the metric (4.2.3) in orthonormal triad $h = e_1^2 + e_2^2 - e_3^2$, where

$$e_1 = (e^u)^{1/2}\mathrm{d}x, \qquad e_2 = (e^u)^{1/2}\mathrm{d}y, \qquad e_3 = \mathrm{d}t$$

The duality relations

$$*_h e_1 = e_2 \wedge e_3, \qquad *_h e_2 = e_3 \wedge e_1, \qquad *_h e_3 = e_2 \wedge e_1$$

yield

$$*_h \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}t, \qquad *_h \mathrm{d}y = \mathrm{d}t \wedge \mathrm{d}x, \qquad *_h \mathrm{d}t = e^u \mathrm{d}y \wedge \mathrm{d}x$$

Take the special case $V = u_t$ (which leads to a pseudo hyperKähler metric), and use the above relations to write the monoplole equation (4.1.1) as

$$u_{tx}\mathrm{d}y \wedge \mathrm{d}t + u_{ty}\mathrm{d}t \wedge \mathrm{d}x + (u_{tt} + u_t^2)e^u \mathrm{d}y \wedge \mathrm{d}x = \mathrm{d}\eta$$

Using the solution (4.2.6) with a good choice of constant in the above relation gives $\eta = -\frac{2}{y}\mathrm{d}x$ or more generally $\eta = -\frac{2}{y}\mathrm{d}x + \mathrm{d}y + \mathrm{d}t$

and the metric (4.2.1) is

$$g = \frac{2t(\mathrm{d}x^2 + \mathrm{d}y^2)}{y^2} - \frac{2\mathrm{d}t^2}{t} - \frac{t}{2}\left[\mathrm{d}\phi - \frac{2}{y}\mathrm{d}x\right]^2. \tag{4.2.8}$$

Variational symmetries and conservation laws

The EW space leads to the Lagrangian

$$L = e^u(\dot{x}^2 + \dot{y}^2) - \dot{t}^2$$

where the dot refers to the derivative with respect to the arclength s.

- In solution (4.2.5) taking $C_1 = -1, C_2 = 0$ the Lagrangian becomes

$$L = \frac{t}{x^2+y^2}(\dot{x}^2 + \dot{y}^2) - \dot{t}^2.$$

The EL equations are

$$\begin{aligned} &(-2x^2\ddot{x} + (2\dot{x}^2 - 2\dot{y}^2)x - 2y^2\ddot{x} + 4y\dot{y}\dot{x})t - 2\dot{x}\dot{t}(x^2+y^2) = 0, \\ &(-2y^2\ddot{y} + (-2\dot{x}^2 + 2\dot{y}^2)y - 2x^2\ddot{y} + 4x\dot{x}\dot{y})t - 2\dot{y}\dot{t}(x^2+y^2) = 0, \\ &\tfrac{\dot{x}^2+\dot{y}^2}{x^2+y^2} + 2\ddot{t} = 0. \end{aligned} \tag{4.2.9}$$

We get the following generators of Noether symmetry,

$$\begin{aligned} X_1 &= \partial_s - \tfrac{4yF(x,y) - 4F_y(x^2+y^2)}{4x}\partial_x + F(x,y)\partial_y, \\ X_2 &= s\partial_s - \tfrac{4yF(x,y) + (x^2+y^2)(1-4F_y)}{4x}\partial_x + F(x,y)\partial_y + \tfrac{t}{2}\partial_t. \end{aligned} \tag{4.2.10}$$

Alternatively, the "multiplier" method provides all the multipliers for (4.2.9) , viz.,

$$\left(F(x,y), \frac{(x^2+y^2)F_x - xF(x,y)}{y}, 0\right)$$

where $F(x,y)$ satisfies the constraints

$$\begin{aligned} F_{xy} &= \tfrac{x(xF_x + yF_y - F(x,y))}{(x^2+y^2)y}, \\ F_{yy} &= \tfrac{xF_x + yF_y - F(x,y)}{x^2+y^2}, \\ F_{xx} &= \tfrac{-xF_x - yF_y + F(x,y)}{x^2+y^2}. \end{aligned} \tag{4.2.11}$$

- In solution (4.2.6) taking $C_1 = 2, C_2 = 0$ the Lagrangian becomes

$$L = \frac{t^2}{y^2}(\dot{x}^2 + \dot{y^2}) - \dot{t^2}.$$

The EL (geodesic) equations are given by

$$\begin{array}{l} -4t\dot{t}\dot{x} - 2t^2\ddot{x} + \frac{4t^2\dot{x}\dot{y}}{y} = 0, \\ (\dot{x^2} + y\ddot{y} - \dot{y})t^2 + 2y\dot{t}\dot{y}t = 0, \\ \frac{2t(\dot{x^2}+\dot{y^2})}{\dot{y^2}} + 2\ddot{t} = 0. \end{array} \qquad (4.2.12)$$

We consider the multiplier $M = (M^x, M^y, M^t)$ of 0^{th} order in derivatives. The equation (2.4.6) gives, after separating by powers of $\dot{x}, \dot{y}$ and $\dot{t}$ a system of PDEs

$$
\begin{aligned}
\dot{x}\dot{t} \quad & : -4t^2y^2g_{x,y} - 4ty^2g_x - 4t^2yf_t - f_yty^2 \\
& : -4t^2y^2fx,y + 4t^2yg_t - 4ty^2h_t - 8ty^2f_x - 4y^2h + 8tyg \\
& : h_{x,t} \\
\dot{y}\dot{t} \quad & : -4t^2y^2g_{t,y} + 4t^2yg_t - 8ty^2g_y - 4ty^2h_t - 4y^2h + 8ytg \\
& : -4t^2y^2f_{t,y} - 4ty^2g_x + 4t^2yf_t - 4ty^2f_y \\
& : h_{t,y} \\
\dot{x}\dot{y} \quad & : -4t^2y^2g_{x,y} + t^2yg_x - 4ty^2h_x \\
& : 4y^4h_{x,y} + ty^2g_x + 4tt^2yf_t + 4ty^2f_y \\
& : -4t^2y^2f_{x,y} + 4yt^2g_y - 4ty^2h_y + 8yt^2f_x + 8yth - 12t^2g \\
\dot{x^2} \quad & : -2t^2y^2f_{x,x} + 2t^2yg_x - 2ty^2h_x \\
& : 2y^4h_{x,x} - 2t^2yg_t + 2ty^2h_t + 4y^2tf_x + 2y^2h - 4tyg \\
& : -2t^2y^2g_{x,x} + 2ty^2h_y - 2yt^2g_y - 4yt^2f_x - 4yth + 6t^2g \\
\dot{y^2} \quad & : 2y^4h_{y,y} + 2t^2yg_t + 4ty^2g_y + 2ty^2h_t + 2y^2h - 4ytg \\
& : -2t^2y^2g_{y,y} + 6yt^2g_y - 2ty^2h_y + 4yth - 6t^2g \\
& : -2t^2y^2f_{y,y} + 2t^2yg_x + 4yt^2f_y + 2ty^2h_x \\
\dot{t^2} \quad & : -2t^2y^2f_{t,t} - 4ty^2f_t \\
& : -2t^2y^2g_{t,t} - 4ty^2g_t \\
& : h_{t,t} \\
\dot{x} \quad & : -4t^2y^2g_{s,x} - 4yt^2f_s \\
& : -4t^2y^2t_{s,x} + 4t^2yg_s - 4ty^2h_s \\
& : 4y^4h_{s,x} + ty^2f^s \\
\dot{y} \quad & : -4t^2y^2g_{s,y} + 4t^2yg_s - 4ty^2h_s \\
& : -4t^2y^2f_{s,y} + 4yt^2f_s \\
& : 4y^4h_{s,y} + ty^2g_s \\
\dot{t} \quad & : -4t^2y^2g_{s,t} - 4ty^2g_s \\
& : -4t^2y^2f_{s,t} - 4ty^2f_s \\
& : h_{s,t} \\
\ddot{x} \quad & : -2t^2y^2g_x - 2t^2y^2f_y \\
& : -4t^2y^2f_x - 4ty^2h + 4t^2yg \\
& : -2t^2y^2f_t + 2y^4h_x \\
\ddot{y} \quad & : -4t^2y^2g_y - 4ty^2h + 4t^2yg
\end{aligned}
$$

$$
\begin{aligned}
& : -2t^2y^2g_x - 2t^2y^2f_y \\
& : -2t^2y^2g_t + 2y^4h_y \\
\ddot{t} : \quad & -2t^2y^2f_t + 2y^4h_x \\
& : h_t \\
\dot{x^0} \quad & : f_{s,s} \\
& : g_{s,s} \\
& : h_{s,s}
\end{aligned}
$$

After some tedious calculations we get

$$
\begin{array}{ll}
M^1 = \left(-\frac{2sxy}{t}, \frac{s(x^2-y^2)}{t}, \frac{s(x^2+y^2)}{y}\right) & M^6 = \left(0, \frac{1}{t}, \frac{1}{y}\right) \\
M^2 = \left(\frac{-ys}{t}, \frac{sx}{t}, \frac{sx}{y}\right) & M^7 = (y^2 - x^2, -2xy, 0) \\
M^3 = \left(0, \frac{s}{t}, \frac{s}{y}\right) & M^8 = (x, y, 0) \\
M^4 = \left(\frac{-2xy}{t}, \frac{x^2-y^2}{t}, \frac{x^2+y^2}{y}\right) & M^9 = (1, 0, 0) \\
M^5 = \left(\frac{-y}{t}, \frac{x}{t}, \frac{x}{y}\right) &
\end{array}
$$

Each of M^i s leads to a variational symmetry and the corresponding algebra of symmetries is generated by

$$
\begin{array}{lll}
\frac{2sxy}{t}\partial_x + \frac{s(x^2-y^2)}{t}\partial_y + \frac{s(x^2+y^2)}{y}\partial_t, & \frac{-ys}{t}\partial_x + \frac{sx}{t}\partial_y + \frac{sx}{y}\partial_t, & \frac{s}{t}\partial_y + \frac{s}{y}\partial_t \\
\frac{-2xy}{t}\partial_x + \frac{x^2-y^2}{t}\partial_y + \frac{x^2+y^2}{y}\partial_t, & \frac{-y}{t}\partial_x + \frac{x}{t}\partial_y + \frac{x}{y}\partial_t, & \frac{1}{t}\partial_y + \frac{1}{y}\partial_t \\
(y^2 - x^2)\partial_x - 2xy\partial_y, & x\partial_x + y\partial_y, & \partial_x
\end{array} \tag{4.2.13}
$$

The complete set of variational symmetry is spanned by

$$
\begin{array}{lll}
\mathcal{X}_1 = \frac{2sxy}{t}\partial_x + \frac{s(x^2-y^2)}{t}\partial_y + \frac{s(x^2+y^2)}{y}\partial_t & , & \mathcal{X}_9 = \partial_x \\
\mathcal{X}_2 = \frac{-ys}{t}\partial_x + \frac{sx}{t}\partial_y + \frac{sx}{y}\partial_t & , & \mathcal{X}_{10} = s^2\partial_s + st\partial_t \\
\mathcal{X}_3 = \frac{s}{t}\partial_y + \frac{s}{y}\partial_t & , & \mathcal{X}_{11} = s\partial_s + \frac{t}{2}\partial_t \\
\mathcal{X}_4 = \frac{-2xy}{t}\partial_x + \frac{x^2-y^2}{t}\partial_y + \frac{x^2+y^2}{y}\partial_t & , & \mathcal{X}_{12} = \partial_s \\
\mathcal{X}_5 = \frac{-y}{t}\partial_x + \frac{x}{t}\partial_y + \frac{x}{y}\partial_t & & \\
\mathcal{X}_6 = \frac{1}{t}\partial_y + \frac{1}{y}\partial_t & & \\
\mathcal{X}_7 = (y^2 - x^2)\partial_x - 2xy\partial_y & & \\
\mathcal{X}_8 = \quad x\partial_x + y\partial_y & &
\end{array} \tag{4.2.14}
$$

Each of these or linear combination lead to a conservation law for the geodesic equations . We list some for illustrative purposes,

$$
\begin{array}{ll}
\mathcal{X}_4 : & T_4 = -\frac{2}{y^2}(tx^2\dot{y} - 2txy\dot{x} - ty^2\dot{y} - x^2y\dot{t} - y^3\dot{t}) \\
\mathcal{X}_5 : & T_5 = \frac{-2}{y^2}(tx\dot{y} - ty\dot{x} - xy\dot{t}) \\
\mathcal{X}_7 : & T_7 = \frac{t^2}{y^2}(x^2\dot{x} + 2xy\dot{y} - y^2\dot{x})
\end{array} \tag{4.2.15}
$$

- The associated four dimensional ASD manifold is given by the metric (4.2.8). The Lagrangian is

$$
L = \frac{2t(\dot{x}^2 + \dot{y}^2)}{y^2} - \frac{2\dot{t}^2}{t} - \frac{t}{2}\left(\dot{\phi} - \frac{2\dot{x}}{y}\right)^2.
$$

For which the Euler-Lagrange equations are

$$\begin{array}{l} -2ty\ddot{\phi} + 2t\dot{y}\dot{\phi} - 2y\dot{t}\dot{\phi} = 0, \\ ((-2\dot{x}\dot{\phi} - 4\ddot{y})t - 4\dot{y}\dot{t})t + 4t\dot{y}^2 = 0, \\ \frac{2\dot{x}^2+2\dot{y}^2}{y^2} - \frac{2\dot{t}^2}{t^2} - \frac{1}{2}\left(\dot{\phi} - \frac{2\dot{x}}{y}\right)^2 + \frac{4\ddot{t}}{t} = 0, \\ y(y\dot{t}\dot{\phi} - 2\dot{t}\dot{x}) - 2ty\ddot{x} + 2t\dot{x}\dot{y} + ty^2\ddot{\phi} = 0. \end{array} \tag{4.2.16}$$

The seventeen dimensional Lie algebra of Noether symmetries are

$$\begin{array}{lll} \mathcal{X}_1 = & \frac{s^2}{2}\partial_s + st\partial_t, & f = -4t \\ \mathcal{X}_2 = & s\partial_s + t\partial_t, & f = 0 \\ \mathcal{X}_3 = & \partial_s, & f = 0 \\ \mathcal{X}_4 = & \frac{y^2-x^2}{4}\partial_x - \frac{xy}{2}\partial_y + y\partial_\phi, & f = 0 \\ \mathcal{X}_5 = & x\partial_x + y\partial_y, & f = 0 \\ \mathcal{X}_6 = & \partial_x, & f = 0 \\ \mathcal{X}_7 = & \partial_\phi, & f = 0 \\ \mathcal{X}_8 = & \frac{y}{2}\sin\frac{\phi}{2}\partial_x + \frac{y}{2}\cos\frac{\phi}{2}\partial_y + \sin\frac{\phi}{2}\partial_\phi, & f = 0 \\ \mathcal{X}_9 = & \frac{y}{2}\cos\frac{\phi}{2}\partial_x - \frac{y}{2}\cos\frac{\phi}{2}\partial_y + \cos\frac{\phi}{2}\partial_\phi, & f = 0 \\ \mathcal{X}_{10} = & -\frac{\sqrt{y}}{\sqrt{t}}\cos\frac{\phi}{4}\partial_x + \frac{\sqrt{y}}{\sqrt{t}}\sin\frac{\phi}{4}\partial_y + \frac{\sqrt{t}}{\sqrt{y}}\sin\frac{\phi}{4}\partial_t, & f = 0 \\ \mathcal{X}_{11} = & \frac{\sqrt{y}}{\sqrt{t}}\sin\frac{\phi}{4}\partial_x + \frac{\sqrt{y}}{\sqrt{t}}\cos\frac{\phi}{4}\partial_y + \frac{\sqrt{t}}{\sqrt{y}}\cos\frac{\phi}{4}\partial_t, & f = 0 \\ \mathcal{X}_{12} = & -\frac{s\sqrt{y}}{\sqrt{t}}\cos\frac{\phi}{4}\partial_x + \frac{s\sqrt{y}}{\sqrt{t}}\sin\frac{\phi}{4}\partial_y + \frac{s\sqrt{t}}{\sqrt{y}}\sin\frac{\phi}{4}\partial_t, & f = -\frac{8\sqrt{t}}{\sqrt{y}}\sin\frac{\phi}{4} \\ \mathcal{X}_{13} = & \frac{s\sqrt{y}}{\sqrt{t}}\sin\frac{\phi}{4}\partial_x + \frac{s\sqrt{y}}{\sqrt{t}}\cos\frac{\phi}{4}\partial_y + \frac{s\sqrt{t}}{\sqrt{y}}\cos\frac{\phi}{4}\partial_t, & f = -\frac{8\sqrt{t}}{\sqrt{y}}\sin\frac{\phi}{4} \end{array}$$

$$\begin{array}{lll} \mathcal{X}_{14} = & \frac{y^{\frac{3}{2}}\sin\frac{\phi}{4}+x\sqrt{y}\cos\frac{\phi}{4}}{4\sqrt{t}}\partial_x + \frac{y^{\frac{3}{2}}\cos\frac{\phi}{4}-x\sqrt{y}\sin\frac{\phi}{4}}{4\sqrt{t}}\partial_y - & \\ & \frac{x\sqrt{t}\sin\frac{\phi}{4}+y\sqrt{t}\cos\frac{\phi}{4}}{4\sqrt{y}}\partial_t + \frac{\sqrt{y}}{\sqrt{t}}\sin\frac{\phi}{4}\partial_\phi, & f = 0 \\ \mathcal{X}_{15} = & \frac{y^{\frac{3}{2}}\cos\frac{\phi}{4}-x\sqrt{y}\sin\frac{\phi}{4}}{4\sqrt{t}}\partial_x - \frac{y^{\frac{3}{2}}\sin\frac{\phi}{4}+x\sqrt{y}\cos\frac{\phi}{4}}{4\sqrt{t}}\partial_y & \\ & -\frac{x\sqrt{t}\cos\frac{\phi}{4}-y\sqrt{t}\sin\frac{\phi}{4}}{4\sqrt{y}}\partial_t + \frac{\sqrt{y}}{\sqrt{t}}\cos\frac{\phi}{4}\partial_\phi, & f = 0 \\ \mathcal{X}_{16} = & \frac{sy^{\frac{3}{2}}\cos\frac{\phi}{4}-sx\sqrt{y}\sin\frac{\phi}{4}}{8\sqrt{t}}\partial_x - \frac{sy^{\frac{3}{2}}\sin\frac{\phi}{4}+sx\sqrt{y}\cos\frac{\phi}{4}}{8\sqrt{t}}\partial_y & \\ & -\frac{sx\sqrt{t}\cos\frac{\phi}{4}-sy\sqrt{t}\sin\frac{\phi}{4}}{8\sqrt{y}}\partial_t + \frac{s\sqrt{y}}{2\sqrt{t}}\cos\frac{\phi}{4}\partial_\phi, & f = \frac{-y\sqrt{t}\sin\frac{\phi}{4}+x\sqrt{t}\cos\frac{\phi}{4}}{\sqrt{y}} \\ \mathcal{X}_{17} = & \frac{sy^{\frac{3}{2}}\sin\frac{\phi}{4}+sx\sqrt{y}\cos\frac{\phi}{4}}{8\sqrt{t}}\partial_x + \frac{sy^{\frac{3}{2}}\cos\frac{\phi}{4}-sx\sqrt{y}\sin\frac{\phi}{4}}{8\sqrt{t}}\partial_y & \\ & -\frac{sx\sqrt{t}\sin\frac{\phi}{4}+sy\sqrt{t}\cos\frac{\phi}{4}}{8\sqrt{y}}\partial_t + \frac{\sqrt{y}}{2\sqrt{t}}\sin\frac{\phi}{4}\partial_\phi, & f = \frac{x\sqrt{t}\sin\frac{\phi}{4}+y\sqrt{t}\cos\frac{\phi}{4}}{\sqrt{y}} \end{array} \tag{4.2.17}$$

The ten dimensional algebra of Killing vectors is spanned by

$\mathcal{X}_4, \mathcal{X}_5, \mathcal{X}_6, \mathcal{X}_7, \mathcal{X}_8, \mathcal{X}_9, \mathcal{X}_{10}, \mathcal{X}_{11}, \mathcal{X}_{14}, \mathcal{X}_{15}$

4.3 ASD Null Kähler with symmetry. The dKP equation

In [40] it is demonstrated that an ASD null Kähler metric is given by

$$g = w_x(\mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4h_x\mathrm{d}t^2) - w_x^{-1}(\mathrm{d}\phi - w_x\mathrm{d}y - 2w_y\mathrm{d}t)^2 \tag{4.3.1}$$

where $h = h(x, y, t)$ and $w = w(x, y, t)$ are real valued function satisfying the following relations

$$h_{yy} - h_{xt} + h_x h_{xx} = 0, \tag{4.3.2}$$

and

$$w_{yy} - w_{xt} + (h_x w_x)_x = 0. \tag{4.3.3}$$

Redefining $u = h_x$, (4.3.2) becomes

$$(u_t - uu_x)_x = u_{yy} \tag{4.3.4}$$

which is the dispersionless Kadomtsev-Petviashvili (dKP) equation.

The corresponding EW Structure is

$$h = \mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4u\mathrm{d}t^2, \quad \omega = -4u_x\mathrm{d}t \tag{4.3.5}$$

Symmetry reductions and exact solutions of dKP equation

We, firstly, present a procedure to determine exact solutions of the equation (4.3.4).

A basis of the Lie point symmetry algebra of (4.3.4) is

$$\begin{aligned}
\mathcal{X}_1 &= -f_2'\partial_u + f_2(t)\partial_x \\
\mathcal{X}_2 &= 2x\partial_x + 4u\partial_u + y\partial_y \\
\mathcal{X}_3 &= 2f_3(t)\partial_y + yf_3'\partial_x - yf_3''\partial_u \\
\mathcal{X}_4 &= (2xf_1' + y^2f_1'')\partial_x + (-4uf_1' - 2xf_1'' - y^2f_1''')\partial_u + 4yf_1'\partial_y + 6f_1(t)\partial_t
\end{aligned}$$

Thus, a large class of reductions of (4.3.4) is obtainable.

Taking a linear combination $2\mathcal{X}_2 + \mathcal{X}_4$ where in $\mathcal{X}_4$, $f_1 = t$, we get the scaling symmetry generator $\mathcal{X}_c = 6x\partial_x + 6t\partial_t + 6y\partial_y$ leading to the invariants and transformed variables

$$X = \frac{x}{t}, \qquad Y = \frac{y}{t}, \qquad U = u, \qquad U = U(X, Y).$$

so that (4.3.4) becomes

$$U_X + X_{XX} + YU_{XY} + U_X^2 + UU_{XX} + U_{YY} = 0 \tag{4.3.6}$$

which admits, the symmetry generators

$$\begin{aligned}\mathcal{X}_c^1 &= -\partial_U + \partial_X \\ \mathcal{X}_c^2 &= 2\partial_Y + Y\partial_X \\ \mathcal{X}_c^3 &= 2U\partial_U + 2X\partial_X + Y\partial_Y\end{aligned}$$

Each of those symmetries leads to a second order ODE,

Via $\mathcal{X}_c^1 = -\partial_U + \partial_X$, it can be shown that (4.3.6) reduces to the canonical equation

$$\bar{U}_{YY} = 0,$$

where $\bar{U} = U + X, \quad \bar{U} = \bar{U}(Y)$. Hence, a solution of the equation (4.3.4) is

$$u = \frac{C_1 y - x + C_2 t}{t} \tag{4.3.7}$$

Similarly, (4.3.6) may be transformed using $\mathcal{X}_c^2 = 2\partial_Y + Y\partial_X$, we get

$$u = \frac{-2C_1C_2 + \sqrt{2C_1C_2 + C_2(\frac{y^2}{4t^2} - \frac{x}{t}) + 1} - 1}{C_2} \tag{4.3.8}$$

- For a alternative reduction one can take $\mathcal{X}_4$ with $f_1 = t$. The invariants and transformed variables are

$$X = \frac{x}{t^{1/3}}, \quad Y = \frac{y}{t^{2/3}}, \quad U = \frac{u}{t^{-1/3}}, \quad U = U(X, Y).$$

the equation (4.3.4) becomes

$$U_X + \frac{X}{3}U_{XX} + \frac{2Y}{3}U_{XY} + U_X^2 + UU_{XX} + U_Y Y = 0, \tag{4.3.9}$$

which admits, inter alia, the symmetry generator $\mathcal{X}_4^1 = -\partial_U + 3\partial_x$
and $\mathcal{X}_4^2 = 3Y\partial_X + 9\partial_Y + Y\partial_U$.
Via $\mathcal{X}_4^1$, (4.3.9) is reduced to the following second ODE

$$\bar{U}_{YY} = \frac{2}{9}$$

where $\bar{U} = U + \frac{X}{3}, \quad \bar{U} = \bar{U}(Y)$,
and the solution to dKP equation (4.3.4) is

$$u = \frac{y^2}{9t^2} - \frac{x}{3t} + \frac{C_1 y}{t^{4/3}} + \frac{C_2}{t^{2/3}}. \tag{4.3.10}$$

Via $\mathcal{X}_4^2$, we get the solution

$$u = \frac{y^2}{9t^2} - \frac{x}{3t} + \frac{1}{t^{4/3}}\sqrt{C_1(\frac{y^2}{6} - xt) + 6t^{4/3}C_2}. \tag{4.3.11}$$

Replacing back the solution (4.3.7) with $(C_1 = C_2 = 0)$ in equation (4.3.3) gives the PDE

$$w_{yy} - w_x t - \frac{1}{t}w_x - \frac{x}{t}w_{xx} = 0 \tag{4.3.12}$$

which is reducible using symmetry reduction.
A basis of the Lie point algebra of (4.3.12) is

$$\begin{aligned}
\mathcal{X}_1 &= f_1(x,y,t)\partial_w \\
\mathcal{X}_2 &= w\partial_w \\
\mathcal{X}_3 &= t\partial_x \\
\mathcal{X}_4 &= (-1+\tfrac{1}{t^2})x\partial_x + (\tfrac{1}{t}+t)\partial_t \\
\mathcal{X}_5 &= (1+\tfrac{1}{t^2})x\partial_x + (\tfrac{1}{t}-t)\partial_t \\
\mathcal{X}_6 &= \partial_y \\
\mathcal{X}_7 &= t^2\partial_y + ty\partial_x \\
\mathcal{X}_8 &= \tfrac{2x}{t}\partial_y + \tfrac{xy}{t^2}\partial_x + \tfrac{y}{t}\partial_t, \qquad t \neq 0 \\
\mathcal{X}_9 &= 2x\partial_x + y\partial_y \\
\mathcal{X}_{10} &= 2t^2y\partial_y - t^2w\partial_w + t^3\partial_t + t(tx+y^2)\partial_x \\
\mathcal{X}_{11} &= \tfrac{-2xw}{t}\partial_w + \tfrac{4xy}{t}\partial_y + \tfrac{x(4tx+y^2)}{t^2}\partial_x + \tfrac{y^2}{t}\partial_t \\
\mathcal{X}_{12} &= (2tx+y^2)\partial_y = 3xy\partial_x + ty\partial_t - wy\partial_w
\end{aligned}$$

With the constraint:

$$tf_{1,yy} - f_{1,x} - tf_{1,xt} - xf_{1,xx} = 0,$$

where $f_{1,yy} = \partial_{y,y}f_1$.

The generator $\mathcal{X}_5 = (1+\frac{1}{t^2})x\partial_x + (\frac{1}{t}-t)\partial_t$ leads to invariants and transformed variables

$$X = \frac{x(1-t^2)}{t}, \quad Y = y, \quad W = w, \quad W = W(X,Y).$$

The equation (4.3.12) becomes:

$$W_{YY} + 2XW_{XX} + 2W_X = 0 \tag{4.3.13}$$

which admits inter alia the generators $\mathcal{X}_5^1 = \partial_Y, \quad \mathcal{X}_5^2 = f_1(X,Y)\partial_W$ and the scaling symmetry $\mathcal{X}_5^3 = -2Y\partial_Y - 4X\partial_X + W\partial_W$ where f_1 satisfies the constraint

$$f_{1,YY} + 2t_1, X + 2Xf_{1,XX} = 0.$$

- Using the linear combination $\mathcal{X}_5^c = \mathcal{X}_5^1 + \mathcal{X}_5^2$ with $f_1 = y$ it can be shown that (4.3.13) becomes the second order ODE

$$1 + 2X\bar{W}_{XX} + 2\bar{W}_X = 0$$

  where $\bar{W} = W - \frac{Y^2}{2}, \quad \bar{W} = \bar{W}(X)$, its solution is

$$\bar{W} = C_1 \ln X - \frac{x}{2} + C_2$$

Substituting back the solution of (4.3.12) is

$$w = \frac{y^2}{2} + C_1 \ln(\frac{x}{t}(1-t^2)) - \frac{x(1-t^2)}{t} + C_2 \tag{4.3.14}$$

- Via the scaling symmetry one can show that the solution of (4.3.12) is

$$w = \frac{1}{y^{1/2}} \left[\frac{C_1 hypergeometric\left(\left[\frac{1}{4}, \frac{1}{4}\right], \left[\frac{1}{2}\right], \frac{-y^2 t}{x(1-t^2)}\right)}{\left[\frac{x(1-t^2)}{ty^2}\right]^{1/4}} + \frac{C_1 hypergeometric\left(\left[\frac{3}{4}, \frac{3}{4}\right], \left[\frac{3}{2}\right], \frac{-y^2 t}{x(1-t^2)}\right)}{\left[\frac{x(1-t^2)}{ty^2}\right]^{3/4}} \right] \tag{4.3.15}$$

For an alternative reduction of (4.3.12) on can use $\mathcal{X}_8 = \frac{2x}{t}\partial_y + \frac{xy}{t^2}\partial_x + \frac{y}{t}\partial_t$ we get

$$2YW_{YY} + 3W_Y + 2XW_{XY} \tag{4.3.16}$$

where

$$X = \frac{x}{t}, \quad Y = \frac{y^2}{2} - tx, \quad w = W, \quad W = W(T, X)$$

which admits, inter alia, the symmetry generator

$$\mathcal{X}_8^1 = 2f_2(X)\partial_X + \frac{2Y f_2(x)}{X}\partial_Y - \frac{W f_2(x)}{X}\partial_W.$$

In $\mathcal{X}_8^1$, if $f_2 = x$ we have a scaling symmetry, the equation (4.3.16) becomes

$$\bar{W}_{\bar{X}} = 0 \tag{4.3.17}$$

Where $\bar{X} = \frac{Y}{X}, \quad \bar{W} = \frac{W}{X^{-1/2}}, \quad \bar{X} = \bar{W}(\bar{X})$

Solving and replace back we get the solution of (4.3.12)

$$w = \left(\frac{x}{t}\right)^{-1/2}. \tag{4.3.18}$$

Noether symmetries and conservation laws

Consider the Einstein structure (4.3.5).

- The corresponding Lagrangian is after replacing the solution (4.3.7) $(C_1 = C_2 = 0)$

$$L = \dot{y}^2 - 4\dot{x}\dot{t} + \frac{4x\dot{t}^2}{t}.$$

The Euler-Lagrange equations are

$$\begin{array}{l} \frac{4\dot{t}^2}{t} + 4\ddot{t} = 0, \\ \ddot{y} = 0, \\ x\dot{t}^2 - 2xt\ddot{t} - 2t\dot{x}\dot{t} + 4t^2\ddot{x} = 0. \end{array} \tag{4.3.19}$$

The multiplier approach gives us all zero order multipliers, viz

$$\begin{array}{ll} M^1 = \left(\frac{sx}{t^2}, 0, \frac{s}{t}\right), & M^5 = (0, s, 0) \\ M^2 = \left(\frac{xy}{t^2}, \frac{2x}{t}, \frac{y}{t}\right), & M^6 = (0, 1, 0) \\ M^3 = \left(\frac{x}{t^2, 0, \frac{1}{t}}\right), & M^7 = \left(ty, t^2, 0\right) \\ M^4 = (-x, 0, t), & M^8 = (st, 0, 0) \end{array} \tag{4.3.20}$$

These, as symmetries are include in the complete set of Noether symmetries viz

$$\begin{array}{ll} \mathcal{X}_1 = \frac{s^2}{2}\partial_s + \frac{3sx}{4}\partial_x + \frac{sy}{2}\partial_y + \frac{st}{4}\partial_t, & f = -tx + \frac{y^2}{2} \\ \mathcal{X}_2 = s\partial_s + x\partial_x + \frac{y}{2}\partial_y, & f = 0 \\ \mathcal{X}_3 = \partial_s, & f = 0 \\ \mathcal{X}_4 = -\frac{sx}{4t^2}\partial_x - \frac{s}{4t}\partial_t, & f = \frac{x}{t} \\ \mathcal{X}_5 = \frac{s}{2}\partial_y, & f = y \\ \mathcal{X}_6 = -\frac{st}{2}\partial_x, & f = t^2 \\ \mathcal{X}_7 = \frac{xy}{t^2}\partial_x + \frac{2x}{t}\partial_y + \frac{y}{t}\partial_t, & f = 0 \\ \mathcal{X}_8 = \frac{x}{t^2}\partial_x + \frac{1}{t}\partial_t, & f = 0 \\ \mathcal{X}_9 = -x\partial_x + t\partial_t, & f = 0 \\ \mathcal{X}_{10} = \partial_y, & f = 0 \\ \mathcal{X}_{11} = ty\partial_x + t^2\partial_y, & f = 0 \\ \mathcal{X}_{12} = t\partial_x, & f = 0 \end{array} \tag{4.3.21}$$

where $\mathcal{X}_7, \mathcal{X}_8, \mathcal{X}_9, \mathcal{X}_{10}, \mathcal{X}_{10}, \mathcal{X}_{12}$ are Killing vectors and f the corresponding gauge. Each one or linear combination lead to a conservation law for the Euler-lagrange equations. Let give some illustrative examples

$$\begin{array}{ll} \mathcal{X}_1: & T_1 = -\frac{s}{t}(4st\dot{t}\dot{x} - st\dot{y}^2 - 4sx\dot{t}^2 - t^2\dot{x} - tx\dot{t} + ty\dot{y}) - \frac{s^2}{t}(-2t\dot{x}\dot{t} + \frac{t\dot{y}^2}{2} + 2x\dot{t}^2) - tx + \frac{y^2}{2} \\ \mathcal{X}_7: & T_7 = -\frac{4}{t^2}(tx\dot{y} - ty\dot{x} + xy\dot{t}) \\ \mathcal{X}_9: & T_9 = 4t\dot{x} - 12x\dot{t} \end{array}$$

- The solution (4.3.10) in (4.3.5) give the Lagrangian

$$L = \dot{y}^2 - 4\dot{x}\dot{t} - 4\left(\frac{y^2}{9t^2} - \frac{x}{3t}\right)\dot{t}^2$$

. The corresponding Euler-Lagrange equations are

$$\begin{aligned}
&\tfrac{4\dot{t}^2}{3t} + 4\ddot{t} = 0,\\
&-\tfrac{8y\dot{t}^2}{9t^2} - 2\ddot{y} = 0,\\
&36t^3\ddot{x} + (-24\dot{x}\dot{t} - 24x\ddot{t})t^2 + (12x\dot{t}^2 + 16y\dot{t}\dot{y} + 8y^2\ddot{t})t - 8y^2\dot{t}^2 = 0.
\end{aligned}$$

The zero gauge Noether symmetries are

$$\begin{aligned}
&2s\partial_s + 2x\partial_x + y\partial_y, \quad -x\partial_x + t\partial_t, \quad \tfrac{y}{3t^{1/3}}\partial_x + t^{2/3}\partial_y, \quad \tfrac{y(2\ln t+3)}{6t^{1/3}}\partial_x + t^{2/3}\ln t\partial_y\\
&t^{1/3}\partial_x, \quad \partial_s
\end{aligned} \tag{4.3.22}$$

while the multiplier approach gives us the following multipliers

$$(-x, 0, t); \quad \left(\tfrac{y}{3t^{1/3}}, t^{2/3}, 0\right); \quad \left(\tfrac{2y\ln t+3y}{6t^{1/3}}, t^{2/3}\ln t, 0\right); \quad \left(st^{1/3}, 0, 0\right); \quad (t^{1/3}, 0, 0). \tag{4.3.23}$$

The complete set of variational symmetries is given by (4.3.22) and (4.3.23) viz.

$$\begin{aligned}
&\mathcal{X}_1 = 2s\partial_s + 2x\partial_x + y\partial_y && \mathcal{X}_6 = \tfrac{2y\ln t+3y}{6t^{1/3}}\partial_x + t^{2/3}\ln t\partial_y\\
&\mathcal{X}_2 = -x\partial_x + t\partial_t && \mathcal{X}_7 = st^{1/3}\partial_x\\
&\mathcal{X}_3 = \tfrac{y}{3t^{1/3}}\partial_x + t^{2/3}\partial_y\\
&\mathcal{X}_4 = t^{1/3}\partial_x\\
&\mathcal{X}_5 = \partial_s.
\end{aligned} \tag{4.3.24}$$

The four dimension ASD null Kähler metric is given by (4.3.1). After replacing the solution (4.3.7) and (4.3.14) $(C_1 = C_2 = 0)$ the lagrangian is

$$L = \frac{1-t^2}{t}\left(\dot{y}^2 - 4\dot{x}\dot{t} - \frac{4x\dot{t}^2}{t}\right) - \frac{t}{1-t^2}\left(\dot{\phi} - \frac{1-t^2}{t}\dot{y} - 2y\dot{t}\right)^2.$$

The Euler-Lagrange equations are

$$\begin{aligned}
&-t^2\ddot{t} - 2t\dot{t}^2 + \ddot{t} = 0\\
&(4\ddot{t}y - 2\ddot{\phi})t^2 + (8y\dot{t}^2 - 4\dot{t}\dot{\phi})t - 4y\ddot{t} + 2\ddot{\phi} = 0\\
&-4t^8\ddot{x} + (8\dot{x}\dot{t} + 8x\ddot{t} + 4y\ddot{y} + 4\dot{y}^2)t^7 + (-16y\dot{t}\dot{y} - 8y^2\ddot{t} + 4y\ddot{\phi} + 4\dot{y}\dot{\phi} + 12\ddot{\phi})t^6 + (2y^2\dot{t}^2 - 24\dot{x}\dot{t}\\
&-24x\ddot{t} - 8y\ddot{y} - 8y^2 - \dot{\phi}^2)t^5 + (8x\dot{t}^2 + 16y\dot{t}\dot{y} + 8y^2\ddot{t} - 4y\ddot{\phi} - 4\dot{y}\dot{\phi} - 12\ddot{x})t^4 + (4y^2\dot{t}^2 + 24\dot{x}\dot{t} + 24x\ddot{t}\\
&+4y\ddot{y} + 4\dot{y}^2 - \dot{\phi}^2)t^3 + (-16x\dot{t}^2 + 4\ddot{x})t^2 + (-8\dot{t}\dot{x} - 8x\ddot{t})t + 8x\dot{t}^2 = 0\\
&-t^4\ddot{y} + (4\dot{t}\dot{y} + 4y\ddot{t} - 2\ddot{\phi})t^3 + (-4y\dot{t}^2 + 2\dot{t}\dot{\phi} + 4\ddot{y})t^2 + (-4\dot{t}\dot{y} - 4y\ddot{t} + 2\ddot{\phi})t - 4y\dot{t}^2 + 2\dot{t}\dot{\phi} - 2\ddot{y} = 0
\end{aligned}$$

The multipliers are

$$(0, \frac{1}{2}, 0, t); \quad (0, 0, 0, 1); \quad (yt^2 - \frac{t\phi}{2}, t^3 - t, 0, 0); \quad (yt, t^2 - 1, 0, 0); \quad (t, 0, 0, 0)$$

while the complete set of variational symmetries is

$$2s\partial_s+2x\partial_x+y\partial_y+\phi\partial_\phi; \quad \partial_s; \quad \frac{1}{2}\partial_y+t\partial_\phi; \quad \partial_\phi; \quad (t^y-\frac{t\phi}{2})\partial_x+(t^3-t)\partial_y; \quad ty\partial_x+(t^2-1)\partial_y; \quad t\partial_x; \quad \frac{st}{4}\partial_x$$

where the last generator is a no zero gauge symmetry.

4.4 Pseudo-hypercomplex with symmetry. The hyper-CR equation.

The EW structure is [27]

$$h = (\mathrm{d}y + u\mathrm{d}t)^2 - 4(\mathrm{d}x + w\mathrm{d}t)\mathrm{d}t, \qquad \omega = u_x\mathrm{d}y + (uu_x + 2u_y)\mathrm{d}t \tag{4.4.1}$$

where $u(x, y, t)$ and $w(x, y, t)$ satisfy

$$u_t + w_y + uw_x - wu_x = 0, \qquad u_y + w_x = 0 \tag{4.4.2}$$

Different classes of solutions to (4.4.2) which yield non-trivial EW structures are found.

- If we assume that u and w do not contain y as dependent variable, we can integrate easily the corresponding equations in in (4.4.2)

$$h = (\mathrm{d}y + f\mathrm{d}t)^2 - 4\mathrm{d}x\mathrm{d}t, \qquad \omega = f'\mathrm{d}y + ff'\mathrm{d}t \tag{4.4.3}$$

  where f=f(x) is an arbitrary function. A large class of complete solutions belong to this class. For example $f = a^2x$, where a is a non-zero constant leads to the Einstein-Weyl structure on Thurston's Nil manifold $S^1 \times \mathbb{R}^2$ [25]

- Looking for t-independent solutions (4.4.2) is reduced to a linear equation. To integrate this system we take advantage of the classical *hodograph transform* (see e.g [30] ,[31]) which achieved by interchanging the roles of the dependent and independent variables. To this end we consider $x = x(u, w), y = y(u, w)$ and compute the derivatives

$$\partial_w x = \frac{1}{J}u_y; \quad \partial_u x = -\frac{1}{J}w_y; \quad \partial_w y = -\frac{1}{J}u_x; \quad \partial_u y = \frac{1}{J}w_x$$

  where $J = \left|\frac{\partial(u,w)}{\partial(x,y)}\right|$ is the Jacobian. We assume $J \neq 0$ (important!)

Replacing back the second equation in (4.4.2) implies the existence of a potential $F = F(u,w)$ such that

$$x = F_u, \qquad y = -F_w$$

while the first leads to

$$F_{uu} + uF_{uw} + wF_{ww} = 0$$

By reduction a large class of solutions can be determined.

- Looking for the integrability condition of the system (4.4.2) one may assume the existence of a potential $v = v(x,y,t)$ such that the second equation yields

$$u = v_x, \qquad w = -v_y \tag{4.4.4}$$

and the first equation becomes

$$v_{xt} - v_{yy} - v_x v_{xy} + v_y v_{xx} = 0 \tag{4.4.5}$$

A basis of the Lie point symmetry algebra of (4.4.5) is given by the generators

$$\begin{aligned}
&\mathcal{X}_1 = f_4(t)\partial_v\\
&\mathcal{X}_2 = 2x\partial_v + y\partial_x\\
&\mathcal{X}_3 = f_3(t)\partial_x + yf_3'(t)\partial_v\\
&\mathcal{X}_4 = 2x\partial_x + 3v\partial_v + y\partial_y\\
&\mathcal{X}_5 = 2f_2(t)\partial_y + (2xf_2' + y^2 f_2'')\partial_v + 2yf_2'\partial_x\\
&\mathcal{X}_6 = 6f_1(t)\partial_t + (6vf_1' + 6xyf_1'' + y^3 f_1''')\partial_v + (6xf_1' + 3y^2 f_1'')\partial_x + 6yf_1'\partial_y
\end{aligned}$$

In $\mathcal{X}_6$, taking $f_1 = \frac{t}{6}$ we get a scaling symmetry $\mathcal{X}_6 = x\partial_x + y\partial_y + t\partial_t + v\partial_v$. The invariants and transformed variables are

$$X = \frac{x}{t}, \quad Y = \frac{y}{t}, \quad V = \frac{v}{t}, \quad V = V(X,Y).$$

In terms of the news variables the equation (4.4.5) becomes

$$-XV_{XX} - YV_{XY} - V_{YY} - V_X V_{XY} + V_{XX} V_Y = 0$$

which admit inter alia the symmetry generators $\mathcal{X}_6^1 = -\partial_Y + X\partial_V$ and

$\mathcal{X}_6^2 = \partial_X + Y\partial_V$

Through $\mathcal{X}_6^1$ the previous equation is reduced to

$$-2X\bar{V}_{XX} + \bar{V}_X = 0$$

where $\bar{V} = V + XY, \qquad \bar{V} = \bar{V}(X)$. The solution is $\bar{V} = C_1 + C_2 X^{3/2}$.

Replacing back we get a solution to (4.4.5)

$$v = C_2 t \left(\frac{x}{t}\right)^{1/3} - \frac{xy}{t} + C_1 t \tag{4.4.6}$$

(4.4.6) and (4.4.4) give solution to (4.4.2)

$$u = \frac{-y}{t} + \frac{3C_2}{2}\sqrt{\frac{x}{t}}, \qquad w = \frac{x}{t} \tag{4.4.7}$$

Using $\mathcal{X}_6^2$ we have the solution

$$u = \frac{y}{t}, \qquad w = -C_1 - \frac{x}{t} + \frac{y^2}{t^2} \tag{4.4.8}$$

- **hydrodynamic reduction**

The system (4.4.2) can be cast in general quasi-linear vector form

$$\mathbf{u_y} + \mathbf{A}(\mathbf{u})\mathbf{u_x} + \mathbf{B}(\mathbf{u})\mathbf{u_t} = \mathbf{0}, \tag{4.4.9}$$

where $\mathbf{u} = (u, w)^T$ is a vector whose componements depend on (x, y, t), and

$$A(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ -w & u \end{pmatrix}, \qquad B(\mathbf{u}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The method of hydrodynamic reductions consists of seeking multiphase solutions in the form

$$\mathbf{u}(x, y, t) = \mathbf{u}(R^1(x, y, t), \cdots, R^n(x, y, t))$$

where $R^i = R^i(x, y, t)$ (the so called Riemaniann invariants) satisfy a pair of commuting system of hydrodynamic type

$$R^i_y = \gamma^i(R)R^i_x, \qquad R^i_t = \mu^i(R)R^i_x, \qquad i = 1, 2, \cdots, N \tag{4.4.10}$$

The number of Riemann invariants is arbitrary! Solutions of this type were extensively investigated [33, 34, 35]. Later, they appeared in the context of the dispersionless KP hierarchy [36, 37, 38, 39]. We will call a multidimensional system *integrable* if it possesses sufficiently many $n-$ component reductions of the form (4.4.10)

The requirement of the commutativity of the flows is equivalent to the following restrictions on their characteristic speeds [32]

$$\frac{\partial_j \gamma^i}{\gamma^j - \gamma^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j, \quad \partial_j = \partial / \partial_{R^j} \tag{4.4.11}$$

(no summation!). Once these conditions are met, the general solution of (4.4.10) is given by the implicit 'generalized hodograph' formula [32]

$$v^i(R) = x + \gamma^i(R) y + \mu^i(R) t$$

where $v^i(R)$ are characteristic speeds of the general flow commuting with (4.4.10), that is the general solution of the linear system

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \gamma^i}{\gamma^j - \gamma^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j$$

Substituting $\mathbf{u}(R^1, ..., R^n)$ in (4.4.2) and using (4.4.10) one readily arrives

$$\partial_i w = -\gamma^i \partial_i u, \qquad \mu^i = \gamma^{i^2} + u\gamma + w \tag{4.4.12}$$

The compatibility condition $\partial_i \partial_j w = \partial_j \partial_i w$ implies

$$\partial_i \partial_j u = \frac{\partial_j \gamma^i}{\gamma^j - \gamma^i} \partial_i u + \frac{\partial_i \gamma^j}{\gamma^j - \gamma^i} \partial_j u \tag{4.4.13}$$

while (4.4.12) in the commutativity condition (4.4.11) results in

$$\partial_j \gamma^i = -\partial_j u \tag{4.4.14}$$

The substitution of (4.4.14) into (4.4.13) implies the Gibbons-Tsarev system for $u(R)$ and $\gamma(R)$

$$\partial_j \gamma^i = -\partial_j u, \qquad \partial_i \partial_j u = \frac{-2\partial_j u \partial_i u}{\gamma^j - \gamma^i} \tag{4.4.15}$$

$i \neq j$, which was first derived in [38, 39]. For any solution u, γ^i of the system (4.4.15), one can reconstruct μ^i and w by virtue of (4.4.12). In case of two components this system takes the form

$$\partial_1 \gamma^2 = -\partial_1 u, \quad \partial_2 \gamma^1 = -\partial_2 u, \quad \partial_1 \partial_2 u = \frac{-2\partial_1 u \partial_2 u}{\gamma^2 - \gamma^1}$$

The general solution of this system is parametrized by four arbitrary functions of a single argument. Moreover, the system (4.4.15) is invariant under the reparametrization

$R^1 \to f^1(R^1), R^2 \to f^2(R^2)$ where f^1, f^2 are arbitrary functions of their arguments.

4.4.1 Remark. A four dimension hypercomplex manifold can be constructed from (4.1.1), (4.1.2) and (4.4.1) as follows

rewrite the metric (4.4.1) in orthonormal triad $h = e_1^2 + e_2^2 - e_3^2$, where

$$e_1 = \mathrm{d}y + u\mathrm{d}t, \quad e_2 = \mathrm{d}x + (w-1)\mathrm{d}t, \quad e_3 = \mathrm{d}x + (w+1)\mathrm{d}t$$

The duality relations

$$*_h e_1 = e_2 \wedge e_3, \qquad *_h e_2 = e_3 \wedge e_1, \qquad *_h e_3 = e_2 \wedge e_1$$

yield

$$*_h \mathrm{d}t = \mathrm{d}y \wedge \mathrm{d}t, \quad *_h \mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}t + u\mathrm{d}x \wedge \mathrm{d}t + 2w\mathrm{d}t \wedge \mathrm{d}, \quad *_h \mathrm{d}y = 2\mathrm{d}x \wedge \mathrm{d}t - u\mathrm{d}y \wedge \mathrm{d}ty$$

Take the special case $V = u_x/2$ (which leads to pseudo-hyper-kähler with triholomorphic homothety), and use the above relation to write the monopole equation (4.1.1) as

$$\mathrm{d}y \wedge \mathrm{d}t \left(\frac{u_x u_y}{2} + \frac{u_{xt}}{2} - \frac{u u_{xy}}{2} - w u_{xx} \right) + \mathrm{d}x \wedge \mathrm{d}t \left(u_{xy} + \frac{u_x^2}{2} + \frac{u u_{xx}}{2} \right) + \frac{u_{xx}}{2} \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\eta$$

The simple solution $u = x$ (4.4.3) in the above relation results to $\eta = \mathrm{d}x + \mathrm{d}y + \frac{x}{2}\mathrm{d}t$ and the four dimension manifold is given by

$$g = \frac{1}{4} \left[(\mathrm{d}y + x\mathrm{d}t)^2 - 4\mathrm{d}x\mathrm{d}t \right] - \left(\mathrm{d}\phi + \mathrm{d}x + \mathrm{d}y + \frac{x}{2}\mathrm{d}t \right)^2. \qquad (4.4.16)$$

Variational symmetries and conservation laws

- The Lagrangian corresponding to the EW Structure (4.4.1) is after substitution of solution (4.4.7)

$$L = \left(\dot{y} - \frac{y\dot{t}^2}{t}\right)^2 - 4\left(\dot{x} + \frac{x\dot{t}}{t}\right)\dot{t}.$$

  The complete set of variational symmetries is obtained ,

$$-x\partial_x + t\partial_t; \quad 2s\partial_s + 2x\partial_x + y\partial_y; \quad t\partial_y; \quad -\frac{y}{t^2}\partial_x + \frac{1}{t}\partial_x; \quad \partial_s; \quad \frac{1}{t}\partial_x; \quad -\frac{s}{4t}\partial_x$$

  where the last symmetry is non zero gauge and all symmetries which are not involved the arc length s are Killing vectors.

- By the solution (4.4.8), the Lagrangian from the EW structure (4.4.1) is

$$L = \left(\dot{y} + \frac{y\dot{t}}{t}\right)^2 - 4\left(\dot{x} + \left(-\frac{x}{t} + \frac{y^2}{t^2}\right)\dot{t}\right)\dot{t}.$$

variational symmetries are

$$-x\partial_x + t\partial_t; \quad 2s\partial_s + 2x\partial_x + y\partial_y; \quad y\partial_x + t\partial_y; \quad y(\ln t + \tfrac{1}{2})\partial_x + t\ln t\partial_y; \quad t\partial_x$$
$$\partial_s; \quad -\tfrac{st}{2}.$$

where the last symmetry is non zero gauge symmetry. Note that if we use solution refereed to (4.4.3) we got five variational symmetries.

- The four dimensional pseudo-hyper-Kähler is given by (4.4.16). The corresponding Lagrangian is

$$L = \frac{1}{4}(\dot{y} + x\dot{t})^2 - \dot{t}\dot{x} - \left(\dot{\phi} + \dot{x} + \dot{y} + \frac{x\dot{t}}{2}\right)^2.$$

The Euler-Lagrange equations are

$$\begin{aligned}
&(2x+2)\ddot{t} + (-\dot{y} - 2\dot{\phi})\dot{t} + 4\ddot{x} + 4\ddot{y} + 4\ddot{\phi} = 0,\\
&x\ddot{t} + \dot{t}\dot{x} + 4\ddot{x} + 3\ddot{y} + 4\ddot{\phi} = 0,\\
&\dot{x}^2 + \tfrac{1}{2}(\dot{y} + 2\dot{\phi})\dot{x} + \tfrac{1}{2}(2\ddot{x} + \ddot{y} + 2\ddot{\phi})x + \ddot{x} = 0,\\
&\dot{x}\dot{t} + x\ddot{t} + 2\ddot{x} + 2\ddot{y} + 2\ddot{\phi} = 0.
\end{aligned}$$

Variational symmetries are

$$\begin{aligned}
&\mathcal{X}_1 = 2\partial_x - 2t\partial_y + t\partial_\phi; && \mathcal{X}_6 = \partial_\phi\\
&\mathcal{X}_2 = -2s\partial_s + x\partial_x - (2x+3y+2\phi)\partial_y - 3t\partial_t + \left(\tfrac{3y}{2} + \phi\right)\partial_\phi; && \mathcal{X}_7 = \partial_t\\
&\mathcal{X}_3 = -2xe^{(x+\frac{y}{2}+\phi)}\partial_y + 2e^{(x+\frac{y}{2}+\phi)}\partial_t + xe^{(x+\frac{y}{2}+\phi)}\partial_\phi; && \mathcal{X}_8 = \partial_y\\
&\mathcal{X}_4 = -e^{-(x+\frac{y}{2}+\phi)}\partial_x + e^{-(x+\frac{y}{2}+\phi)}\partial_\phi; && \mathcal{X}_9 = \partial_s\\
&\mathcal{X}_5 = -x\partial_x + t\partial_t + x\partial_\phi; && \mathcal{X}_{10} = -s\partial_y + \tfrac{s}{2}\partial_\phi\\
&\mathcal{X}_{11} = s\partial_s + (\phi + x + \tfrac{3y}{2})\partial_y + t\partial_t - \tfrac{1}{4}(2x+3y+2\phi)\partial_\phi.
\end{aligned}$$

where $\mathcal{X}_{10}$ is no zero gauge and $\mathcal{X}_1, \mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_5, \mathcal{X}_6, \mathcal{X}_7, \mathcal{X}_8$ are Killing vectors.

Some corresponding conserved quantities are

$$\begin{aligned}
\mathcal{X}_2: \quad T_2 = {}& x\dot{t} + 4s\dot{x}\dot{\phi} + 4s\dot{y}\dot{\phi} - \tfrac{3tx\dot{y}}{2} + 2s\dot{x}\dot{t} - 3tx(\dot{\phi} + \dot{x}) + 4s\dot{x}\dot{y} - x\dot{y} - 2x(\dot{\phi} + \dot{x}) + 2sx\dot{t}\dot{\phi}\\
& sx\dot{t}\dot{y} + 2sx\dot{t}\dot{x} - 3y\dot{x} - \tfrac{3y\dot{y}}{2} - 3y\dot{\phi} - 2\phi\dot{x} - \phi\dot{y} - 2\phi\dot{\phi} + 2s\dot{\phi}^2 - 3t\dot{x} + \tfrac{3s\dot{y}^2}{2} + 2s\dot{x}^2\\
\mathcal{X}_3: \quad T_3 = {}& 2\dot{x}e^{(x+\phi+\frac{y}{2})}\\
\mathcal{X}_4: \quad T_4 = {}& -\dot{t}e^{-(x+\phi+\frac{y}{2})}
\end{aligned}$$

4.5 Conclusion

We studied the invariance properties generated by some well-known metrics of neutral signatures. As the metrics depended on solutions of PDEs, we constructed exact solutions of the PDEs using Lie group methods. From the specific forms of the metrics, we determined the isometries and the variational symmetries of the underlying metrics and corresponding Euler–Lagrange equations for both (Einstein Weyl structures and the corresponding four dimension metric constructed using the Jones-Tod construction). We established relationships between the resultant Lie algebras, viz., the algebra of isometries are subalgebras of the algebras of variational symmetries. For illustration, we chose some cases for which we constructed some conservation laws via these symmetries or the "multiplier approach". The interesting result occurs in Section 4.3 where the Lagrangian obtained from the three dimension EW structure has more variational symmetries than its corresponding four dimension Lagrangian obtained by Jones-Toda construction.

Chapter 5

Wave equations on curved manifolds

5.1 Introduction

The standard wave equation in $(3+1)$-dimensions has been extensively studied in the literature. A detailed symmetry analysis of this equation is discussed in [41]. In particulary, the symmetry classification problem for a number of wave equations has been studied in flat space [49, 50, 51, 52, 53] and non flat space (non-zero constant curvature) [45, 46]. In this work we pursue an investigation of symmetries of the wave equation on some spacetimes with non diagonal metric g_{ij} and neutral signatures.

5.2 Wave equations on ASD-Einstein manifold

A detailed symmetry analysis of ASD-Einstein manifolds has been done in [42] .The metric on the ASD Ricci-flat is locally given by

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}y + \mathrm{d}t\mathrm{d}x - \left(-\frac{3z}{t^2}y + \frac{x}{t}\right)\mathrm{d}t^2 + \frac{2y}{t}\mathrm{d}t\mathrm{d}z$$

A Gordon type equation in a curved space or in curvilinear coordinates is given by the expression

$$\Box u \equiv \frac{1}{\sqrt{|g|}}\left(g^{ab}u_{,b}\sqrt{|g|}\right)_a = k(u) \tag{5.2.1}$$

g_{ab} being the metric of this space, $g = \parallel g_{ab} \parallel$ its determinant, g^{ab} the inverse of g_{ab} , $_{,b} = \partial_b$ and $\Box$ is the D Alembertian, sometimes called the "box" operator.

Consequently, the Gordon type equation on the ASD manifofd takes the form

$$\frac{xt-3yz}{t^2}u_{xx}-\frac{2y}{t}u_{xy}+u_{x,t}+u_{yz}-k(u)=0. \tag{5.2.2}$$

Lie symmetries

We consider the following cases for $k(u)$ in (5.2.2) given by

(i) $k(u)=0$ (wave equation)

(ii) $k(u)=u$

(iii) $k(u)=u^3$

(iv) $k(u)=u^n, \quad n\neq 0,1,3$

Case (i) $k(u)=0.$

In this case, (5.2.2) admits the following 15-dimensional Lie algebra,

$$\begin{aligned}
X_1 =&\quad u\partial_u\\
X_2 =&\quad f_1(x,y,z,t)\partial_u\\
X_3 =&\quad t\partial_x\\
X_4 =&\quad 3t^{-1-\sqrt{7}}z\partial_x+(2+\sqrt{7})t^{-\sqrt{7}}\partial_y\\
X_5 =&\quad 3t^{-1+\sqrt{7}}z\partial_x-(-2+\sqrt{7})t^{\sqrt{7}}\partial_y\\
X_6 =&\quad t\partial_t+y\partial_y\\
X_7 =&\quad 2y\partial_y+3t\partial_t-x\partial_x\\
X_8 =&\quad (-2+\sqrt{7})t^{1-\sqrt{7}}y\partial_x+t^{2-\sqrt{7}}\partial_z\\
X_9 =&\quad (2+\sqrt{7})t^{1+\sqrt{7}}y\partial_x-t^{2+\sqrt{7}}\partial_z\\
X_{10} =&\quad t^{-\sqrt{7}}\left[(1-\sqrt{7})uy\partial_u+\sqrt{7}y^2\partial_y+tx\partial_z-ty\partial_t+\tfrac{(-3+\sqrt{7})txy+3y^2z}{t}\partial_x\right]\\
X_{11} =&\quad t^{\sqrt{7}}\left[(-1-\sqrt{7})uy\partial_u+\sqrt{7}y^2\partial_y-tx\partial_z+ty\partial_t+\tfrac{(3+\sqrt{7})txy-3y^2z}{t}\partial_x\right]\\
X_{12} =&\quad 2t\partial_t+y\partial_y+z\partial_z\\
X_{13} =&\quad \tfrac{t^2x^2-txyz+3y^2z^2}{t^3}\partial_x-\tfrac{u(tx+yz)}{t^2}\partial_u+\tfrac{xz}{t}\partial_z+\tfrac{y(tx+2yz)}{t^2}\partial_y-\tfrac{yz}{t}\partial_t\\
X_{14} =&\quad t^{-2-\sqrt{7}}\left[(1-\sqrt{7})uz\partial_u+(\sqrt{7}-2)z^2\partial_z-tz\partial_t+(tx+2yz)\partial_y+\tfrac{(\sqrt{7}-3)txz+3yz^2}{t}\partial_x\right]\\
X_{15} =&\quad t^{-2+\sqrt{7}}\left[(-1-\sqrt{7})uz\partial_u+(2+\sqrt{7})z^2\partial_z+tz\partial_t-(tx+2yz)\partial_y+\tfrac{(3+\sqrt{7})txz-3yz^2}{t}\partial_x\right]
\end{aligned} \tag{5.2.3}$$

where

$$t^2f_{1,yz}+t^2f_{1,xt}-2tyf_{1,xy}+txf_{1,xx}-3yzf_{1,xx}=0$$

Case (ii) $k(u)=u.$

Equation (5.2.2) admits 8-dimensional Lie algebra :

$$\begin{aligned}
X_1 =&\ u\partial_u \\
X_2 =&\ f_1(x,y,z,t)\partial_u \\
X_3 =&\ t\partial_x \\
X_4 =&\ -\frac{z(2f_2[t]+tf_2')}{t} + f_2[t]\partial_y \\
X_5 =&\ (-2+\sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_6 =&\ -(2+\sqrt{7})t^{1+\sqrt{7}}y\partial_x + t^{2+\sqrt{7}}\partial_z \\
X_7 =&\ y\partial_y - z\partial_z \\
X_8 =&\ t\partial_t - x\partial_x
\end{aligned} \tag{5.2.4}$$

where

$$-t^2 f_1 + 2t^2 f_{1,yz} + 2t^2 f_{1,xt} - 4ty f_{1,xy} + 2tx f_{1,xx} - 6yz f_{1,xx} = 0$$

and

$$7f_2 - tf_2' - t^2 f_2'' = 0$$

Case (iii) $k(u) = u^3$.

Equation (5.2.2) admits 14-dimensional Lie algebra :

$$\begin{aligned}
X_1 =&\ t\partial_x \\
X_2 =&\ -f_1(t)\partial_y + z\left(\frac{2f_1(t)}{t} + f_1'\right)\partial_x \\
X_3 =&\ (-2+\sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_4 =&\ (2+\sqrt{7})t^{1+\sqrt{7}}y\partial_x - t^{2+\sqrt{7}}\partial_z \\
X_5 =&\ (1-\sqrt{7})t^{-\sqrt{7}}uy\partial_u + \sqrt{7}t^{-\sqrt{7}}y^2\partial_y + t^{1-\sqrt{7}}x\partial_z - t^{1-\sqrt{7}}y\partial_t + t^{-1-\sqrt{7}}y((\sqrt{7}-3)xt + 3yz)\partial_x \\
X_6 =&\ -(1+\sqrt{7})t^{\sqrt{7}}uy\partial_u + \sqrt{7}t^{\sqrt{7}}y^2\partial_y - t^{1+\sqrt{7}}x\partial_z + t^{1+\sqrt{7}}y\partial_t + t^{-1+\sqrt{7}}y((3+\sqrt{7})xt - 3yz)\partial_x \\
X_7 =&\ y\partial_y - z\partial_z \\
X_8 =&\ 2z\partial_z + t\partial_t - u\partial_u + x\partial_x \\
X_9 =&\ 2x\partial_x + 2z\partial_z - u\partial_u \\
X_{10} =&\ t^{-2\sqrt{7}}\left[(-1+\sqrt{7})u\partial_u - (-2+\sqrt{7})z\partial_z - \sqrt{7}y\partial_y + \frac{tx+2(-7+2\sqrt{7})yz}{t}\partial_x + t\partial_t\right] \\
X_{11} =&\ t^{2\sqrt{7}}\left[(1+\sqrt{7})u\partial_u - (2+\sqrt{7})z\partial_z - \sqrt{7}y\partial_y + \frac{-tx+2(7+2\sqrt{7})yz}{t}\partial_x - t\partial_t\right] \\
X_{12} =&\ t^{-2-\sqrt{7}}\left[-(-1+\sqrt{7})uz\partial_u + (-2+\sqrt{7})z^2\partial_z - tz\partial_t + (tx+2yz)\partial_y + \frac{z((-3+\sqrt{7})tx+3yz)}{t}\partial_x\right] \\
X_{13} =&\ t^{-2+\sqrt{7}}\left[-(1+\sqrt{7})uz\partial_u + (2+\sqrt{7})z^2\partial_z + tz\partial_t - (tx+2yz)\partial_y + t^{-1}z((3+\sqrt{7})tx - 3yz)\partial_x\right] \\
X_{14} =&\ \frac{t^2x^2 - txyz + 3y^2z^2}{t^3}\partial_x + \frac{u(tx+yz)}{t^2}\partial_u + \frac{xz}{t}\partial_z + \frac{y(tx+2yz)}{t^2}\partial_y - \frac{yz}{t}\partial_t
\end{aligned} \tag{5.2.5}$$

Case (iv) $k(u) = u^n, \quad n \neq 0, 1, 3.$

Equation (5.2.2) admits 8-dimensional Lie algebra :

$$\begin{aligned}
X_1 &= t\partial_x \\
X_2 &= 3t^{-1-\sqrt{7}}z\partial_x + (2+\sqrt{7})t^{-\sqrt{7}}\partial_y \\
X_3 &= 3t^{-1+\sqrt{7}}z\partial_x - (-2+\sqrt{7})t^{\sqrt{7}}\partial_y \\
X_4 &= -\tfrac{u}{-1+n}\partial_u + t\partial_t + y\partial_y \\
X_5 &= -\tfrac{2u}{-1+n}\partial_u + 3t\partial_t + 2y\partial_y - x\partial_x \\
X_6 &= -\tfrac{2u}{-1+n}\partial_u + 2t\partial_t + y\partial_y + z\partial_z \\
X_7 &= (-2+\sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_8 &= (2+\sqrt{7})t^{1+\sqrt{7}}y\partial_x - t^{2+\sqrt{7}}\partial_z
\end{aligned} \tag{5.2.6}$$

Symmetry reduction

Since $[X,Y] = 0$ with $X = t\partial_t + y\partial_y \quad Y = t\partial_t - x\partial_x$, where X and Y appear as Lie symmetries (also Noether symmetries as will be seen later) in all the above cases (even not explicitly but as linear combinations), we may start reducing with X.

The characteristic equations are

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u}{0}$$

Integrating we get $\alpha = \frac{y}{t}$ and (5.2.2) becomes

$$(2x - 6\alpha z)u_{xx} - 6\alpha u_{x\alpha} + 2u_{z\alpha} = 0 \tag{5.2.7}$$

with $u = u(x, \alpha, z)$.

If we further reduce (5.2.7) by Y, we obtain $\bar{Y} = -\alpha\partial_\alpha - x\partial_x$. Consequently, the characteristic equations,

$$\frac{\mathrm{d}x}{-x} = \frac{\mathrm{d}\alpha}{-\alpha} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0}$$

By integration, we obtain $\beta = \frac{x}{\alpha}$ and (5.2.7) reduces to

$$(4\beta - 3z)u_{\beta\beta} + 3u_\beta - \beta u_{\beta z} = 0 \tag{5.2.8}$$

with $u = u(\beta, z)$

Equation (5.2.8) may be reduced using the underlying symmetries. The Lie point symmetries are given by

$$(f_1(\beta, z) + uf_2(z))\partial_u + f_4(z)\partial_z + f_3(\beta, z)\partial_\beta$$

where

$$\begin{aligned}
&3zf_3 - 3\beta f_4 - 3z\beta f_4' + 4\beta^2 f_4' + \beta^2 f_{3,z} + 3z\beta f_{3,\beta} - 4\beta^2 f_{3,\beta} = 0,\\
&-3f_{1,\beta} + \beta f_{1,\beta z} + 3zf_{1,\beta\beta} - 4\beta f_{1,\beta\beta} = 0,\\
&12f_3 - 9f_4 - 3z\beta f_2' + 4\beta^2 f_2' + 3\beta f_{3,z} + 9zf_{3,\beta} - 12\beta f_{3,\beta} + (3z\beta - 4\beta^2)f_{3,\beta z}+\\
&(9z^2 - 24z\beta + 16\beta^2)f_{3,\beta\beta} = 0
\end{aligned}$$

Take $f_1 = 0 \quad f_2 = 1, \quad f_3 = \beta \quad f_4 = z$, we have the characteristic equations

$$\frac{\mathrm{d}z}{z} = \frac{\mathrm{d}\beta}{\beta} = \frac{\mathrm{d}u}{u}$$

By integrating we obtain $\gamma = \frac{\beta}{z}$, $u = \frac{U}{z}, \quad U = U(\gamma)$ and (5.2.8) reduces to the second order ODE

$$(\gamma^2 + 4\gamma - 3)U_{\gamma\gamma} + (2\gamma + 3)U_\gamma = 0.$$

The solution is

$$U(\gamma) = C_1 + \exp\left[-\frac{1}{7}\operatorname{arctan h}\left(\frac{1}{7}(x+2)\sqrt{7}\right)\right] C_2$$

where C_1, C_2 are constant

Noether symmetries

Consider the wave equation (5.2.2). Since it is variational, the corresponding Lagrangian is given by

$$L = \frac{tx - 3yz}{t^2}u_x^2 - \frac{2y}{t}u_y u_x + u_t u_x + u_z u_y - 2h(u) \tag{5.2.9}$$

where $h(u) = \int k(u)\mathrm{d}u$

Case (i) $h(u) = 0$

For the Lagrangian (5.2.9), we obtain after separation by derivatives of u in the relation (2.4.2), the following overdetermined system

$$
\begin{array}{ll}
u_x^3 & : \xi_u \\
u_x^2 u_y & : \eta_u \\
u_x^2 u_t & : \tau_u \\
u_x u_t u_z & : \gamma_u \\
u_x u_y & : -2yt^2\tau_t - 4t^2 y\phi_u - t\left(2xt - 6yz\right)\eta_x - \eta_t t^3 - 2t^2 y\gamma_z - t^3\xi_z + 2yt\tau - 2t^2\eta \\
u_x u_z & : -t\left(2xt - 6yz\right)\gamma_x - \gamma_t t^3 + 2t^2 y\gamma_y - t^3\xi_y \\
u_x u_t & : 2t^3\phi_u - t\left(2xt - 6yz\right)\tau_x + 2t^2 y\tau_y + t^3\gamma_z + t^3\eta_y \\
u_y u_z & : 2\gamma_x t^2 y + 2t^3\phi_u + \tau_t t^3 + \xi_x t^3 \\
u_y u_t & : -\eta_x t^3 + 2\tau_x t^2 y - \tau_z t^3 \\
u_z u_t & : -\gamma_x t^3 - \tau_y t^3 \\
u_x^2 & : \left(xt - 3yz\right)\left(2t\phi_u + t\tau_t + t\gamma_z + t\eta_y - t\xi_x\right) + 2t^2 y\xi_y - t^3\xi_t - \left(xt - 6yz\right)\tau \\
 & + \left(-3z\eta + t\xi - 3y\gamma\right)t \\
u_y^2 & : 2y\eta_x - \eta_z t \\
u_z^2 & -\gamma_y \\
u_t^2 & : \tau_x \\
u_z & \phi_y - f_{3,u} \\
u_x & : t\left(2xt - 6yz\right)\phi_x + \phi_t t^3 - f_{1,u}t^3 - 2t^2 y\phi_y \\
u_t & : \phi_x - f_{4,u} \\
u_y & : -2y\phi_x + \phi_z t - f_{2,u}t \\
1 & : f_{1,x} + f_{2,y} + f_{3,z} + f_{4,z}
\end{array}
$$

Therefore, the coefficients of infinitesimal generator are

$$
\begin{aligned}
\eta(x,y,z,t,u) = \quad & \tfrac{1}{2t^2}\Big[2\,t^{-2-\sqrt{7}}C_3\,t^2yz + \left(\left(C_5\,y^2 + 2\,C_7\right)t^2 + C_3\,xt\right)t^{-\sqrt{7}} + 2\,t^{-2+\sqrt{7}}C_2\,t^2yz + \\
& 2\,t^{-2\sqrt{7}}C_{10}\,t^2y + 2\,t^{2\sqrt{7}}C_9\,t^2y + \left(\left(C_4\,y^2 + 2\,C_6\right)t^2 + C_2\,tx\right)t^{\sqrt{7}} \\
& +2\,y\left(t^2C_8 + 1/2\,C_1\,tx + C_1\,yz\right)\Big] \\
\xi\,(x,y,z,t,u) = \quad & \Big[\left(\left(14\,C_{13}\,t^3y + 7\,C_3\,xz\right)\sqrt{7} - 28\,C_{13}\,t^3y - 21\,C_3\,xz\right)t^{1-\sqrt{7}} + 28yC_9\,z\left(2+\sqrt{7}\right)t^{2\sqrt{7}+2} \\
& +14\,z\left(\left(3/14\,C_5\,y^2 + C_7\right)\sqrt{7} - 2\,C_7\right)t^{-\sqrt{7}+2} + 28\,yC_{10}\,z\left(-2+\sqrt{7}\right)t^{-2\sqrt{7}+2} - \\
& + \left(\left(-14\,C_{12}t^3y - 7\,xz\_C2\right)\sqrt{7} - 28\,C_{12}\,t^3y - 21\,xzC_2\right)t^{1+\sqrt{7}} - \\
& 14\,z\left(\left(3/14\,C_4\,y^2 + C_6\right)\sqrt{7} + 2\,C_6\right)t^{2+\sqrt{7}} - 3\,\left(t^3x\sqrt{7}C_5 - \tfrac{7}{3}\,t^3xC_5 - 7\,z^2C_3\right)yt^{-\sqrt{7}} \\
& -2\,t^{-2\sqrt{7}}\sqrt{7}C_{10}\,xt^3 + 2t^{2\sqrt{7}}\sqrt{7}C_9\,xt^3 + 3\left(t^3x\sqrt{7}\_C4 + 7/3\,t^3xC_4 + 7\,z^2C_2\right)yt^{\sqrt{7}} \\
& +21\,y^2z^2C_1 - 7\,txyzC_1 + 14\,t^2\left(t^2C_{15} + x\left(C_8 + C_{11} - C_{14}\right)t + 1/2\,x^2C_1\right)\Big]\,\tfrac{1}{14t^3} \\
\tau\,(x,y,z,t,u) = \quad & \frac{\left(C_2\,t^{\sqrt{7}} + C_3\,t^{-\sqrt{7}} + C_1\,y\right)z}{-2t} + \tfrac{1}{\sqrt{7}}\left(t^{1+\sqrt{7}}C_4 - t^{1-\sqrt{7}}C_5\right)y - 1/7\,t^{-2\sqrt{7}+1}\sqrt{7}C_{10} \\
& +1/7\,t^{2\sqrt{7}+1}\sqrt{7}C_9 + tC_{14} \\
\gamma\,(x,y,z,t,u) = \quad & \tfrac{1}{14t^{1+2\sqrt{7}}}\Big[7\,t\left(t^{\sqrt{7}}\right)^2z^2C_3\,\left(-2+\sqrt{7}\right)t^{-2-\sqrt{7}} + \left(t^{\sqrt{7}}\right)^2\left(\sqrt{7}xC_5 + 14\,tC_{13}\right)t^{-\sqrt{7}+2} - \\
& 7\,t\left(t^{\sqrt{7}}\right)^2z^2C_2\,\left(2+\sqrt{7}\right)t^{-2+\sqrt{7}} + \left(t^{\sqrt{7}}\right)^2\left(-\sqrt{7}xC_4 + 14\,C_{12}\,t\right)t^{2+\sqrt{7}} - \\
& 4\,\left(-t\left(\sqrt{7}+\tfrac{7}{2}\right)C_9\,\left(t^{\sqrt{7}}\right)^4 + \left(\tfrac{-7}{2}\,C_{11}\,t - \tfrac{7}{4}\,xC_1\right)\left(t^{\sqrt{7}}\right)^2 + tC_{10}\,\left(\sqrt{7}-\tfrac{7}{2}\right)\right)z\Big] \\
f1\,(x,y,z,t,u) = \quad & \int\tfrac{\phi_tt^2 - 2ty\phi_y + 2\phi_xxt - 6\phi_xyz}{t^2}\,\mathrm{d}u + \\
& \int\tfrac{-\left(\int\phi_{x,t}\,\mathrm{d}u\right)t^2 - t^2\left(F_{2,t} + F_{1,z} + F_{3,y}\right) - \left(\int\phi_{y,z}\,\mathrm{d}u\right)t^2 + \left(\int 2y\phi_{y,x} + 2\phi_x - t\phi_{y,z}\,\mathrm{d}u\right)t + \left(\int(2xt - 6yz)\phi_{x,x}\,\mathrm{d}u\right)}{t^2}\mathrm{d}x \\
& + \int\tfrac{\int t\left(t\phi_{x,t} - 2y\phi_{y,x} + 2\phi_x\right)}{t^2}\mathrm{d}x + F_4 \\
f2\,(x,y,z,t,u) = \quad & \int -\tfrac{2\phi_xy - \phi_zt}{t}\,\mathrm{d}u + F_3 \\
f3\,(x,y,z,t,u) = \quad & \int\phi_y\,\mathrm{d}u + F_1 \\
& f4\,(x,y,z,t,u) = \int\phi_x\,\mathrm{d}u + F_2 \\
\phi(x,y,z,t,u)_{x,x} = \quad & -\tfrac{t\left(-2y\phi_{x,y} + t\left(\phi_{t,x} + \phi_{y,z}\right)\right)}{xt - 3\,yz} \\
\phi\,(x,y,z,t,u)_u = \quad & \tfrac{1}{196yt^{2+2\sqrt{7}}}\Big[\left((xt - 98\,yz)\sqrt{7} + 5\,xt + 98\,yz\right)\left(t^{\sqrt{7}}\right)^2t^2C_3\,t^{-2-\sqrt{7}} - \\
& C_2\left(t^{\sqrt{7}}\right)^2\left((xt - 98\,yz)\sqrt{7} - 5\,xt - 98\,yz\right)t^2t^{-2+\sqrt{7}} + \\
& 14\,\left(t^{\sqrt{7}}\right)^2\left(\left(ty^2C_5 - 1/14\,C_3\,x\right)\sqrt{7} - 7\,ty^2C_5 - \tfrac{5\,C_3\,x}{14}\right)tt^{-\sqrt{7}} - \\
& 98\,\left(t^{\sqrt{7}}\right)^2t^{-2\sqrt{7}}C_{10}\,t^2y - 98\,\left(t^{\sqrt{7}}\right)^2t^{2\sqrt{7}}C_9\,t^2y - 28\,\left(\sqrt{7} + 7/2\right)t^2C_9\,y\left(t^{\sqrt{7}}\right)^4 + \\
& \left(\left(-14\,C_4\,t^2y^2 + C_2\,tx\right)\sqrt{7} - 98\,C_4\,t^2y^2 - 5\,C_2\,tx\right)\left(t^{\sqrt{7}}\right)^3 - \\
& 98\,\left(\left(C_8 + C_{11}\right)t^2 + C_1\,tx + C_1\,yz\right)y\left(t^{\sqrt{7}}\right)^2 + 28\,\left(\sqrt{7} - 7/2\right)t^2C_{10}\,y\Big]
\end{aligned}
\tag{5.2.10}
$$

where C_i are constants and $F_i = F_i(x,y,z,t)$

And we set $F_1 = F =_2 = F_3 = F_4 = 0$

When we separate (5.2.10), we find that the Noether point symmetries are

$$
\begin{aligned}
X_1 = &\quad x\partial_x + y\partial_y - \tfrac{u}{2}\partial_u \\
X_2 = &\quad -x\partial_x + t\partial_t \\
X_3 = &\quad x\partial_x + z\partial_z - \tfrac{u}{2}\partial_u \\
X_4 = &\quad t^{\sqrt{7}}\left[-\tfrac{z(2+\sqrt{7})}{t}\partial_x + \partial_y\right] \\
X_5 = &\quad \tfrac{1}{t^{\sqrt{7}}}\left[\tfrac{z(-2+\sqrt{7})}{t}\partial_x + \partial_y\right] \\
X_6 = &\quad -t^{1+\sqrt{7}}y(2+\sqrt{7})\partial_x + t^{2+\sqrt{7}}\partial_z \\
X_7 = &\quad yt^{1-\sqrt{7}}(-2+\sqrt{7})\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_8 = &\quad t\partial_x \\
X_9 = &\quad \tfrac{t^2x^2-txyz+3y^2z^2}{2t^3}\partial_x + \tfrac{y(tx+2yz)}{2t^2}\partial_y + \tfrac{xz}{2t}\partial_z - \tfrac{yz}{2t}\partial_t - \tfrac{u(tx+yz)}{2t^2}\partial_u \\
X_{10} = &\quad t^{\sqrt{7}}\left[-\tfrac{z(\sqrt{7}tx+3xt-3yz)}{2t^3}\partial_x + \tfrac{tx+2yz}{2t^2}\partial_y - \tfrac{z^2(2+\sqrt{7})}{2t^2}\partial_z - \tfrac{z}{2t}\partial_t + \tfrac{zu(1+\sqrt{7})}{2t^2}\partial_u\right] \\
X_{11} = &\quad \tfrac{1}{t^{\sqrt{7}}}\left[\tfrac{z(\sqrt{7}tx-3xt+3yz)}{2t^3}\partial_x + \tfrac{tx+2yz}{2t^2}\partial_y + \tfrac{z^2(-2+\sqrt{7})}{2t^2}\partial_z - \tfrac{z}{2t}\partial_t - \tfrac{zu(-1+\sqrt{7})}{2t^2}\partial_u\right] \\
X_{12} = &\quad t^{\sqrt{7}}\left[\tfrac{y(3\sqrt{7}tx-3\sqrt{7}yz+7xt)}{14t}\partial_x + \tfrac{y^2}{2}\partial_y - \tfrac{\sqrt{7}tx}{14}\partial_z + \tfrac{\sqrt{7}ty}{14}\partial_t - \tfrac{yu(\sqrt{7}+7)}{14}\partial_u\right] \\
X_{13} = &\quad \tfrac{1}{t^{\sqrt{7}}}\left[\tfrac{-y(3\sqrt{7}tx-3\sqrt{7}yz-7xt)}{14t}\partial_x + \tfrac{y^2}{2}\partial_y + \tfrac{\sqrt{7}tx}{14}\partial_z - \tfrac{\sqrt{7}ty}{14}\partial_t + \tfrac{yu(\sqrt{7}-7)}{14}\partial_u\right] \\
X_{14} = &\quad t^{2\sqrt{7}}\left[\tfrac{\sqrt{7}tx-14\sqrt{7}yz-28yz}{7t}\partial_x + y\partial_y + \tfrac{z(7+2\sqrt{7})}{7}\partial_z + \tfrac{\sqrt{7}t}{7}\partial_t - \tfrac{7+\sqrt{7}}{7}u\partial_u\right] \\
X_{15} = &\quad \tfrac{1}{t^{2\sqrt{7}}}\left[-\tfrac{\sqrt{7}tx-14\sqrt{7}yz+28yz}{7t}\partial_x + y\partial_y - \tfrac{z(-7+2\sqrt{7})}{7}\partial_z - \tfrac{\sqrt{7}t}{7}\partial_t + \tfrac{-7+\sqrt{7}}{7}u\partial_u\right]
\end{aligned}
\tag{5.2.11}
$$

The conserved quantities are three form ω such that the four form $D\omega$ vanishes. Thus

$$\omega = \Phi^x \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t - \Phi^y \mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}t + \Phi^z \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t - \Phi^t \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

so that

$$D_x\Phi^x + D_y\Phi^y + D_z\Phi^z + D_t\Phi^t = 0$$

$$
\begin{aligned}
\Phi^t_1 =\ & u_x(2xu_x + 2yu_y + u)\\
\Phi^t_2 =\ & \tfrac{1}{t}[t^2u_yu_z + 2txu_x^2 - 2tyu_xu_y - 3yzu_x^2]\\
\Phi^t_3 =\ & -\tfrac{u_x}{2}(2xu_x + 2zu_z + u)\\
\Phi^t_4 =\ & u_x(u_xt^{-1+\sqrt{7}}z\sqrt{7} + 2u_xt^{-1+\sqrt{7}} - u_yt^{\sqrt{7}})\\
\Phi^t_5 =\ & -u_x(u_xt^{-1-\sqrt{7}}z\sqrt{7} - 2u_xt^{-1-\sqrt{7}} + u_yt^{-\sqrt{7}})\\
\Phi^t_6 =\ & u_x(u_xt^{1+\sqrt{7}}y\sqrt{7} + 2u_xt^{1+\sqrt{7}}y - u_zt^{2+\sqrt{7}})\\
\Phi^t_7 =\ & -u_x(u_xt^{1-\sqrt{7}}y\sqrt{7} - 2u_xt^{1-\sqrt{7}}y + u_zt^{2-\sqrt{7}})\\
\Phi^t_8 =\ & -{u_x}^2t\\
\Phi^t_9 =\ & -\tfrac{1}{2t^2}[tx^2{u_x}^2 + txyu_xu_y + txzu_xu_z + tyzu_yu_z + tuxu_x + uyzu_x] + \tfrac{u}{2t}\\
\Phi^t_{10} =\ & \tfrac{1}{-2}t^{-1+\sqrt{7}}zu_yu_z + t^{-2+\sqrt{7}}xz{u_x}^2 + \tfrac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}xz{u_x}^2 + \tfrac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}z^2u_xu_z +\\
& \tfrac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}zuu_x + t^{-2+\sqrt{7}}z^2u_xu_z - \tfrac{1}{2}t^{-1+\sqrt{7}}xu_xu_y + \tfrac{1}{2}t^{-2+\sqrt{7}}zuu_x\\
\Phi^t_{11} =\ & \tfrac{1}{2}\Big(-t^{-1-\sqrt{7}}zu_yu_z + 2t^{-2-\sqrt{7}}xz{u_x}^2 - t^{-2-\sqrt{7}}\sqrt{7}xz{u_x}^2 - t^{-2-\sqrt{7}}\sqrt{7}z^2u_xu_z -\\
& t^{-2-\sqrt{7}}\sqrt{7}zuu_x + 2t^{-2-\sqrt{7}}z^2u_xu_z + t^{-2-\sqrt{7}}zuu_x - t^{-1-\sqrt{7}}xu_xu_y\Big)\\
\Phi^t_{12} =\ & \tfrac{2}{7}\Big(\sqrt{7}t^{1+\sqrt{7}}yu_yu_z - 2t^{\sqrt{7}}\sqrt{7}xy{u_x}^2 - 2\sqrt{7}t^{\sqrt{7}}y^2u_xu_y + t^{1+\sqrt{7}}\sqrt{7}xu_xu_z -\\
& t^{\sqrt{7}}\sqrt{7}yuu_x - 14t^{\sqrt{7}}xy{u_x}^2 - 14t^{\sqrt{7}}y^2u_xu_y - 14t^{\sqrt{7}}yuu_x\Big)\\
\Phi^t_{13} =\ & \tfrac{1}{14}\Big(-t^{1-\sqrt{7}}\sqrt{7}yu_yu_z + 2t^{-\sqrt{7}}\sqrt{7}xy{u_x}^2 + 2\sqrt{7}t^{-\sqrt{7}}y^2u_xu_y + t^{-\sqrt{7}}\sqrt{7}yuu_x -\\
& 7t^{-\sqrt{7}}xy{u_x}^2 - 7t^{-\sqrt{7}}y^2u_xu_y - t^{1-\sqrt{7}}\sqrt{7}xu_xu_z - 7t^{-\sqrt{7}}yuu_x\Big)\\
\Phi^t_{14} =\ & \tfrac{1}{7}\Big(\sqrt{7}t^{2\sqrt{7}+1}u_yu_z - 2\sqrt{7}t^{2\sqrt{7}}yu_xu_y + 11t^{2\sqrt{7}-1}\sqrt{7}yz{u_x}^2 - 2t^{2\sqrt{7}}\sqrt{7}zu_xu_z +\\
& 28t^{2\sqrt{7}-1}yz{u_x}^2 - 7t^{2\sqrt{7}}\sqrt{7}uu_x - 7t^{2\sqrt{7}}yu_xu_y - 7t^{2\sqrt{7}}zu_xu_z - 7t^{2\sqrt{7}}uu_x\Big)\\
\Phi^t_{15} =\ & -\tfrac{1}{7}\Big(\sqrt{7}t^{1-2\sqrt{7}}u_yu_z - 2\sqrt{7}t^{-2\sqrt{7}}yu_xu_y + 11t^{-2\sqrt{7}-1}\sqrt{7}yz{u_x}^2 - 2t^{-2\sqrt{7}}\sqrt{7}zu_xu_z\\
& -28t^{-2\sqrt{7}-1}yz{u_x}^2 - t^{-2\sqrt{7}}\sqrt{7}u_x + 7t^{-2\sqrt{7}}yu_xu_y + 7t^{2\sqrt{7}}zu_xu_z - 7t^{-2\sqrt{7}}u_x\Big)\\
\Phi^z_1 =\ & u_y(2xu_x + 2yu_y + u)\\
\Phi^z_2 =\ & u_y(tu_t - xu_x)\\
\Phi^z_3 =\ & \tfrac{1}{2t^2}(2t^2xu_xu_y - 2t^2zu_xu_t - 2txzu_x^2 + 4tyzu_xu_y + 6yz^2u_x^2 + t^2uu_y)\\
\Phi^z_4 =\ & -u_y(u_xt^{-1+\sqrt{7}}z\sqrt{7} + 2u_xt^{-1+\sqrt{7}}z - u_yt^{\sqrt{7}})\\
\Phi^z_5 =\ & u_y(u_xt^{-1-\sqrt{7}}z\sqrt{7} - 2u_xt^{-1-\sqrt{7}}z + u_yt^{-\sqrt{7}})\\
\Phi^z_6 =\ & (-t^{1+\sqrt{7}}\sqrt{7}yu_xu_y - t^{2+\sqrt{7}}u_xu_t - t^{1+\sqrt{7}}x{u_x}^2 + 3t^{\sqrt{7}}yz{u_x}^2)\\
\Phi^z_7 =\ & (t^{1-\sqrt{7}}\sqrt{7}yu_xu_y - t^{2-\sqrt{7}}u_xu_t - t^{1-\sqrt{7}}x{u_x}^2 + 3t^{-\sqrt{7}}yz{u_x}^2)\\
\Phi^z_8 =\ & u_ytu_x\\
\Phi^z_9 =\ & \tfrac{1}{2t^3}[t^2x^2u_xu_y + t^2xy{u_y}^2 - t^2xzu_tu_x - t^2yzu_tu_y - tx^2z{u_x}^2 + txyzu_xu_y + 2\,ty^2z{u_y}^2\\
& +3xyz^2{u_x}^2 + 3y^2z^2u_xu_y + t^2uxu_y + tuyzu_y] + \tfrac{zu}{2t^2}
\end{aligned}
$$

$$\Phi^z_{10} = \tfrac{1}{2}t^{-2+\sqrt{7}}\sqrt{7}z^2u_xu_t + \tfrac{1}{2}t^{\sqrt{7}-3}\sqrt{7}xz^2{u_x}^2 - t^{\sqrt{7}-3}\sqrt{7}yz^2u_xu_y - \tfrac{3}{2}t^{-4+\sqrt{7}}\sqrt{7}yz^3{u_x}^2 + t^{-2+\sqrt{7}}z^2u_xu_t + t^{\sqrt{7}-3}xz^2{u_x}^2 - \tfrac{1}{2}t^{\sqrt{7}-3}yz^2u_xu_y - 3t^{-4+\sqrt{7}}yz^3{u_x}^2 - \tfrac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}xzu_xu_y - \tfrac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}zuu_y - \tfrac{3}{2}t^{-2+\sqrt{7}}xzu_xu_y + t^{-2+\sqrt{7}}yz{u_y}^2 + \tfrac{1}{2}t^{-1+\sqrt{7}}x{u_y}^2 - \tfrac{1}{2}t^{-1+\sqrt{7}}zu_yu_t - \tfrac{1}{2}t^{-2+\sqrt{7}}zuu_y$$

$$\Phi^z_{11} = \tfrac{1}{2}\Big(-t^{-2-\sqrt{7}}\sqrt{7}z^2u_xu_t - t^{-\sqrt{7}-3}\sqrt{7}xz^2{u_x}^2 + 2t^{-\sqrt{7}-3}\sqrt{7}yz^2u_xu_y + 3t^{-4-\sqrt{7}}\sqrt{7}yz^3{u_x}^2 + 2t^{-2-\sqrt{7}}z^2u_xu_t + 2t^{-\sqrt{7}-3}xz^2{u_x}^2 - t^{-\sqrt{7}-3}yz^2u_xu_y - 6t^{-4-\sqrt{7}}yz^3{u_x}^2 + t^{-2-\sqrt{7}}\sqrt{7}xzu_xu_y + t^{-2-\sqrt{7}}\sqrt{7}zuu_y - 3t^{-2-\sqrt{7}}xzu_xu_y + 2t^{-2-\sqrt{7}}yz{u_y}^2 - t^{-2-\sqrt{7}}zuu_y + t^{-1-\sqrt{7}}x{u_y}^2 - t^{-1-\sqrt{7}}zu_yu_t\Big)$$

$$\Phi^z_{12} = \tfrac{2}{7}\Big(\sqrt{7}t^{1+\sqrt{7}}xu_xu_t + \sqrt{7}t^{\sqrt{7}}x^2{u_x}^2 + t^{\sqrt{7}}\sqrt{7}xyu_xu_y - 3t^{-1+\sqrt{7}}\sqrt{7}xyz{u_x}^2 - 3t^{-1+\sqrt{7}}\sqrt{7}y^2zu_xu_y + t^{1+\sqrt{7}}\sqrt{7}yu_yu_t + t^{\sqrt{7}}\sqrt{7}yuu_y + 7t^{\sqrt{7}}xyu_xu_y + 7t^{\sqrt{7}}y^2{u_y}^2 + 7t^{\sqrt{7}}yuu_y\Big)$$

$$\Phi^z_{13} = \Big(3t^{-1-\sqrt{7}}\sqrt{7}xyz{u_x}^2 - t^{1-\sqrt{7}}\sqrt{7}xu_xu_t - t^{-\sqrt{7}}\sqrt{7}x^2{u_x}^2 - t^{-\sqrt{7}}\sqrt{7}xyu_xu_y + 3t^{-1-\sqrt{7}}\sqrt{7}y^2zu_xu_y - t^{-\sqrt{7}}\sqrt{7}yuu_y + 7t^{-\sqrt{7}}xyu_xu_y + 7t^{-\sqrt{7}}y^2{u_y}^2 - t^{1-\sqrt{7}}\sqrt{7}yu_yu_t + 7t^{-\sqrt{7}}yuu_y\Big)\tfrac{1}{14}$$

$$\Phi^z_{14} = \tfrac{1}{7}\Big(-2\sqrt{7}t^{2\sqrt{7}}zu_xu_t - 2\sqrt{7}t^{2\sqrt{7}-1}xz{u_x}^2 - 10t^{2\sqrt{7}-1}\sqrt{7}yzu_xu_y + 6t^{-2+2\sqrt{7}}\sqrt{7}yz^2{u_x}^2 - 7t^{2\sqrt{7}}zu_xu_t - t^{2\sqrt{7}-1}xz{u_x}^2 - 14t^{2\sqrt{7}-1}yzu_xu_y + 21t^{-2+2\sqrt{7}}yz^2{u_x}^2 + t^{2\sqrt{7}}\sqrt{7}xu_xu_y + t^{2\sqrt{7}}\sqrt{7}uu_y + 7t^{2\sqrt{7}}y{u_y}^2 + t^{2\sqrt{7}+1}\sqrt{7}u_yu_t + 7t^{2\sqrt{7}}uu_y\Big)$$

$$\Phi^z_{15} = \Big(2\sqrt{7}t^{-2\sqrt{7}}zu_xu_t + 2\sqrt{7}t^{-1-2\sqrt{7}}xz{u_x}^2 + 10t^{-1-2\sqrt{7}}\sqrt{7}yzu_xu_y - 6\sqrt{7}t^{-2-2\sqrt{7}}yz^2{u_x}^2 - 7t^{-2\sqrt{7}}zu_xu_t - 7t^{-1-2\sqrt{7}}xz{u_x}^2 - 14t^{-1-2\sqrt{7}}yzu_xu_y + 21t^{-2-2\sqrt{7}}yz^2{u_x}^2 - t^{-2\sqrt{7}}\sqrt{7}xu_xu_y + t^{-2\sqrt{7}}y{u_y}^2 - 1/7t^{1-2\sqrt{7}}\sqrt{7}u_yu_t - 1/7t^{-2\sqrt{7}}\sqrt{7}u_y + u_yt^{-2\sqrt{7}}\Big)\tfrac{1}{7}$$

$$\Phi^y_1 = \tfrac{1}{t^2}\left(2t^2xu_xu_z - 2t^2yu_tu_x - 6txyu_x^2 + 6y^2zu_x^2 + t^2uu_z - 2tuyu_x\right)$$

$$\Phi^y_2 = -\tfrac{1}{t}[(tu_z - 2yu_x)(tu_t - xu_x)]$$

$$\Phi^y_3 = -\tfrac{1}{2t}[(tu_z - 2yu_x)(2xu_x + 2zu_z + u)]$$

$$\Phi^y_4 = -\tfrac{1}{t}u_x(2t^{-1+\sqrt{7}}\sqrt{7}yzu_x + 7t^{-1+\sqrt{7}}yzu_x - t^{\sqrt{7}}\sqrt{7}zu_z - t^{\sqrt{7}}xu_x - 2t^{\sqrt{7}}zu_z - t^{1+\sqrt{7}}u_t)$$

$$\Phi^y_5 = -\tfrac{1}{t}u_x(-2t^{-1-\sqrt{7}}\sqrt{7}yzu_x + 7t^{-1-\sqrt{7}}yzu_x + t^{-\sqrt{7}}\sqrt{7}zu_z - t^{-\sqrt{7}}xu_x - 2t^{-\sqrt{7}}zu_z - t^{1-\sqrt{7}}u_t)$$

$$\Phi^y_6 = \tfrac{1}{t}\left(tu_z - 2yu_x\right)\left(u_xt^{1+\sqrt{7}}y\sqrt{7} + 2u_xt^{1+\sqrt{7}}y - u_zt^{2+\sqrt{7}}\right)$$

$$\Phi^y_7 = -\tfrac{1}{t}\left(tu_z - 2yu_x\right)\left(u_xt^{1-\sqrt{7}}y\sqrt{7} - 2u_xt^{1-\sqrt{7}}y + u_zt^{2-\sqrt{7}}\right)$$

$$\Phi^y_8 = -\left(tu_z - 2yu_x\right)u_x$$

$$\Phi^y_9 = \tfrac{1}{-2t^3}\left[t^2x^2u_xu_z - t^2xyu_tu_x + t^2xz{u_z}^2 - t^2yzu_tu_z - 3\,tx^2y{u_x}^2 - 3\,txyzu_xu_z + 3\,xy^2z{u_x}^2 + 3\,y^2z^2u_xu_z + t^2uxu_z - 2\,tuxyu_x + tuyzu_z - 2\,uy^2zu_x\right] - \tfrac{uy}{2t^2}$$

$$\Phi^y_{10} = \tfrac{1}{2t}\Big[\sqrt{7}t^{-1+\sqrt{7}}xzu_xu_z + 2t^{-1+\sqrt{7}}z^2{u_z}^2 + \sqrt{7}t^{-1+\sqrt{7}}zuu_z + t^{\sqrt{7}}zu_zu_t + t^{-1+\sqrt{7}}zuu_z + t^{\sqrt{7}}xu_xu_t + \sqrt{7}t^{-1+\sqrt{7}}z^2{u_z}^2 + t^{-1+\sqrt{7}}x^2{u_x}^2 - 7t^{-2+\sqrt{7}}xyz{u_x}^2 - 7t^{-2+\sqrt{7}}yz^2u_xu_z - 2t^{-2+\sqrt{7}}yzuu_x - 2\sqrt{7}t^{-2+\sqrt{7}}xyz{u_x}^2 - 2\sqrt{7}t^{-2+\sqrt{7}}yz^2u_xu_z - 2\sqrt{7}t^{-2+\sqrt{7}}yzuu_x + 3t^{-1+\sqrt{7}}xzu_xu_z\Big] - \tfrac{t^{\sqrt{7}}}{2t^2}u(1+\sqrt{7})$$

$$\begin{aligned}\Phi^y_{11} =\ & \tfrac{1}{2t}\Big(-7t^{-2-\sqrt{7}}xyzu_x{}^2 - 7t^{-2-\sqrt{7}}yz^2u_xu_z - 2t^{-2-\sqrt{7}}yzuu_x + 2t^{-1-\sqrt{7}}z^2u_z{}^2 \\ & +t^{-\sqrt{7}}zu_zu_t - t^{-1-\sqrt{7}}\sqrt{7}z^2u_z{}^2 + t^{-\sqrt{7}}xu_xu_t + t^{-1-\sqrt{7}}x^2u_x{}^2 + 3t^{-1-\sqrt{7}}xzu_xu_z + \\ & 2t^{-2-\sqrt{7}}\sqrt{7}xyzu_x{}^2 + 2t^{-2-\sqrt{7}}\sqrt{7}yz^2u_xu_z + 2t^{-2-\sqrt{7}}\sqrt{7}yzuu_x - t^{-1-\sqrt{7}}\sqrt{7}zuu_z - \\ & t^{-1-\sqrt{7}}\sqrt{7}xzu_xu_z + t^{-1-\sqrt{7}}zuu_z\Big) + \tfrac{uz(5+\sqrt{7})}{t^{3+\sqrt{7}}}\end{aligned}$$

$$\begin{aligned}\Phi^y_{12} =\ & \tfrac{-1}{7t}\Big[-7t^{1+\sqrt{7}}y^2u_xu_t - 21t^{\sqrt{7}}xy^2u_x{}^2 + 21t^{-1+\sqrt{7}}y^3zu_x{}^2 - 3t^{\sqrt{7}}\sqrt{7}y^2zu_xu_z + \\ & 6t^{-1+\sqrt{7}}\sqrt{7}y^3zu_x{}^2 + 5t^{1+\sqrt{7}}\sqrt{7}xyu_xu_z - 6t^{\sqrt{7}}\sqrt{7}xy^2u_x{}^2 - t^{2+\sqrt{7}}\sqrt{7}xu_z{}^2 + t^{2+\sqrt{7}}\sqrt{7}yu_zu_t \\ & -2t^{1+\sqrt{7}}\sqrt{7}y^2u_xu_t + t^{1+\sqrt{7}}\sqrt{7}yuu_z - 2t^{\sqrt{7}}\sqrt{7}y^2uu_x + 7t^{1+\sqrt{7}}xyu_xu_z + 7t^{1+\sqrt{7}}yuu_z \\ & -14t^{\sqrt{7}}y^2uu_x\Big] + \tfrac{ut^{\sqrt{7}}(7+\sqrt{7})}{14}\end{aligned}$$

$$\begin{aligned}\Phi^y_{13} =\ & \tfrac{-1}{14t}\Big[3t^{-\sqrt{7}}\sqrt{7}y^2zu_xu_z - 6t^{-1-\sqrt{7}}\sqrt{7}y^3zu_x{}^2 - 7t^{1-\sqrt{7}}y^2u_xu_t - 21t^{-\sqrt{7}}xy^2u_x{}^2 + 21t^{-1-\sqrt{7}}y^3zu_x{}^2 \\ & -5t^{1-\sqrt{7}}\sqrt{7}xyu_xu_z + 6t^{-\sqrt{7}}\sqrt{7}xy^2u_x{}^2 - t^{1-\sqrt{7}}\sqrt{7}yuu_z + 2t^{-\sqrt{7}}\sqrt{7}y^2uu_x + 7t^{1-\sqrt{7}}xyu_xu_z + \\ & t^{2-\sqrt{7}}\sqrt{7}xu_z{}^2 - t^{2-\sqrt{7}}\sqrt{7}yu_zu_t + 2t^{1-\sqrt{7}}\sqrt{7}y^2u_xu_t + 7t^{1-\sqrt{7}}yuu_z - 14t^{-\sqrt{7}}y^2uu_x\Big] - \tfrac{u(\sqrt{7}-7)}{14t^{\sqrt{7}}}\end{aligned}$$

$$\begin{aligned}\Phi^y_{14} =\ & \tfrac{1}{7t}\Big[18t^{2\sqrt{7}}\sqrt{7}yzu_xu_z - 28t^{2\sqrt{7}-1}\sqrt{7}y^2zu_x{}^2 + 7t^{2\sqrt{7}+1}yu_xu_t + 7t^{2\sqrt{7}}xyu_x{}^2 - 77t^{2\sqrt{7}-1}y^2zu_x{}^2 - \\ & t^{2\sqrt{7}+1}\sqrt{7}xu_xu_z - 2t^{2\sqrt{7}+1}\sqrt{7}zu_z{}^2 + 2t^{2\sqrt{7}}\sqrt{7}xyu_x{}^2 + 42t^{2\sqrt{7}}yzu_xu_z - t^{2\sqrt{7}+1}\sqrt{7}uu_z + \\ & 2t^{2\sqrt{7}}\sqrt{7}yuu_x - 7t^{2\sqrt{7}+1}zu_z{}^2 - t^{2\sqrt{7}+2}\sqrt{7}u_zu_t + 2t^{2\sqrt{7}+1}\sqrt{7}yu_xu_t - 7t^{2\sqrt{7}+1}uu_z + 14t^{2\sqrt{7}}yuu_x\Big]\end{aligned}$$

$$\begin{aligned}\Phi^y_{15} =\ & \tfrac{-1}{7t}\Big[18t^{-2\sqrt{7}}\sqrt{7}yzu_xu_z - 28t^{-1-2\sqrt{7}}\sqrt{7}y^2zu_x{}^2 - 7t^{1-2\sqrt{7}}yu_xu_t - 7t^{-2\sqrt{7}}xyu_x{}^2 + 77t^{-1-2\sqrt{7}}y^2zu_x{}^2 \\ & -t^{1-2\sqrt{7}}\sqrt{7}xu_xu_z - 2t^{1-2\sqrt{7}}\sqrt{7}zu_z{}^2 + 2t^{-2\sqrt{7}}\sqrt{7}xyu_x{}^2 - 42t^{-2\sqrt{7}}yzu_xu_z + 7t^{1-2\sqrt{7}}zu_z{}^2 - \\ & t^{2-2\sqrt{7}}\sqrt{7}u_zu_t + 2t^{1-2\sqrt{7}}\sqrt{7}yu_xu_t - t^{1-2\sqrt{7}}\sqrt{7}u_z + 2\sqrt{7}t^{-2\sqrt{7}}yu_x + 7t^{1-2\sqrt{7}}u_z - 14t^{-2\sqrt{7}}yu_x\Big]\end{aligned}$$

$$\begin{aligned}\Phi^x_1 =\ & \tfrac{1}{t^2}[2t^2xu_yu_z - 2t^2yu_tu_y - 2tx^2u_x^2 - 4txyu_xu_y + 4ty^2u_y^2 + 6xyzu_x^2 + 12y^2zu_xu_y - t^2uu_t \\ & -2tuxu_x + 2tyuu_y + 6uyzu_x]\end{aligned}$$

$$\Phi^x_2 = \tfrac{1}{t^2}[t^3u_t^2 + 2t^2xu_tu_x + t^2xu_yu_z - 2t^2yu_tu_y - tx^2u_x^2 - 6tyzu_tu_x + 3xyzu_x^2]$$

$$\begin{aligned}\Phi^x_3 =\ & \tfrac{1}{2t^2}[2t^2xu_yu_z - 2t^2zu_tu_z - 2tx^2u_x^2 - 4txzu_xu_z + 4tyzu_yu_z + 6xyzu_x^2 + 12yz^2u_xu_z - t^2uu_t \\ & -2tuxu_x + 2tuyu_y + 6uyzu_x]\end{aligned}$$

$$\begin{aligned}\Phi^x_4 =\ & \tfrac{t^{\sqrt{7}}}{t^2}\left[tzu_yu_z - \sqrt{7}xzu_x^2 + 3t^{-1}\sqrt{7}yz^2u_x^2 + 2tzu_yu_z - 2xzu_x^2 - 6yzu_xu_y + 6t^{-1}yz^2u_x^2\right. \\ & \left.+t^2u_yu_t + 2txu_xu_y - 2tyu_y^2\right]\end{aligned}$$

$$\begin{aligned}\Phi^x_5 =\ & \tfrac{t^{-\sqrt{7}}}{t^2}\left[-tzu_yu_z + \sqrt{7}xzu_x^2 - 3t^{-1}\sqrt{7}yz^2u_x^2 + 2tzu_yu_z - 2xzu_x^2 - 6yzu_xu_y + 6t^{-1}yz^2u_x^2\right. \\ & \left.+t^2u_yu_t + 2txu_xu_y - 2tyu_y^2\right]\end{aligned}$$

$$\begin{aligned}\Phi^x_6 =\ & \tfrac{1}{t^2}[t^{3+\sqrt{7}}\sqrt{7}yu_yu_z - t^{2+\sqrt{7}}\sqrt{7}xyu_x{}^2 + 3t^{1+\sqrt{7}}\sqrt{7}y^2zu_x{}^2 - 2t^{2+\sqrt{7}}xyu_x{}^2 + 6t^{1+\sqrt{7}}y^2zu_x{}^2 \\ & +t^{4+\sqrt{7}}u_zu_t + 2t^{3+\sqrt{7}}xu_xu_z - 6t^{2+\sqrt{7}}yzu_xu_z]\end{aligned}$$

$$\begin{aligned}\Phi^x_7 =\ & \tfrac{1}{-t^2}[\sqrt{7}t^{3-\sqrt{7}}yu_yu_z - \sqrt{7}t^{2-\sqrt{7}}xyu_x{}^2 + 3t^{1-\sqrt{7}}\sqrt{7}y^2zu_x2 + 2t^{2-\sqrt{7}}xyu_x{}^2 - 6t^{1-\sqrt{7}}y^2zu_x{}^2 \\ & -t^{4-\sqrt{7}}u_zu_t - 2t^{3-\sqrt{7}}xu_xu_z + 6t^{2-\sqrt{7}}yzu_xu_z]\end{aligned}$$

$$\Phi^x_8 = -\tfrac{1}{t}[u_yu_zt^2 - txu_x{}^2 + 3yzu_x{}^2]$$

$$\Phi^x_9 = \frac{-1}{2t^5}\big[t^4x^2u_yu_z - t^4xyu_tu_y - t^4xzu_tu_z + t^4yzu_t{}^2 - t^3x^3u_x{}^2 - 2\,t^3x^2yu_xu_y - 2\,t^3x^2zu_xu_z$$
$$+2t^3xy^2u_y{}^2 + 2\,t^3xyzu_tu_x + t^3xyzu_yu_z - 4\,t^3y^2zu_tu_y + 4\,t^2x^2yzu_x{}^2 + 2t^2xy^2zu_xu_y +$$
$$6t^2xyz^2u_xu_z + 4\,t^2y^3zu_y{}^2 - 6\,t^2y^2z^2u_tu_x + 3\,t^2y^2z^2u_yu_z - 6txy^2z^2u_x{}^2 + 12\,ty^3z^2u_xu_y +$$
$$9y^3z^3u_x{}^2 - t^4uxu_t - 2t^3ux^2u_x + 2\,t^3uxyu_y - t^3uyzu_t + 4\,t^2uxyzu_x + 2\,t^2uy^2zu_y +$$
$$6\,tuy^2z^2u_x\big] + \frac{u(tx-10yz)}{2t^3}$$

$$\Phi^x_{10} = -\frac{1}{t^2}\Big[12t^{-2+\sqrt{7}}y^2z^2u_xu_y - 6t^{-1+\sqrt{7}}yz^2u_xu_t - t^{\sqrt{7}}\sqrt{7}xzu_yu_z + 2t^{-1+\sqrt{7}}xzuu_x - 2t^{-1+\sqrt{7}}yzuu_y$$
$$-12t^{-2+\sqrt{7}}xyz^2u_x{}^2 - 4t^{\sqrt{7}}yzu_yu_t + 4t^{-1+\sqrt{7}}xz^2u_xu_z + t^{-1+\sqrt{7}}\sqrt{7}x^2zu_x{}^2 + \sqrt{7}t^{\sqrt{7}}z^2u_zu_t +$$
$$t^{1+\sqrt{7}}zu_t{}^2 + 2t^{\sqrt{7}}xzu_xu_t - 3t^{\sqrt{7}}xzu_yu_z - 2\sqrt{7}t^{-1+\sqrt{7}}yzuu_y + 2t^{-1+\sqrt{7}}xyzu_xu_y -$$
$$2\sqrt{7}t^{-1+\sqrt{7}}yz^2u_yu_z - 3\sqrt{7}t^{-2+\sqrt{7}}xyz^2u_x{}^2 + 2\sqrt{7}t^{-1+\sqrt{7}}xz^2u_xu_z - t^{-1+\sqrt{7}}yz^2u_yu_z -$$
$$6\sqrt{7}t^{-2+\sqrt{7}}yz^2uu_x + 2\sqrt{7}t^{-1+\sqrt{7}}xzuu_x - 6t^{-2+\sqrt{7}}yz^2uu_x + \sqrt{7}t^{\sqrt{7}}zuu_t - 6\sqrt{7}t^{-2+\sqrt{7}}yz^3u_xu_z -$$
$$t^{1+\sqrt{7}}xu_yu_t + t^{\sqrt{7}}zuu_t + 2t^{\sqrt{7}}z^2u_zu_t + 4t^{-1+\sqrt{7}}y^2zu_y{}^2 + 3t^{-1+\sqrt{7}}x^2zu_x{}^2 + 9t^{\sqrt{7}-3}y^2z^3u_x{}^2 -$$
$$2t^{\sqrt{7}}x^2u_xu_y + 2t^{\sqrt{7}}xyu_y{}^2 - 12t^{-2+\sqrt{7}}yz^3u_xu_z\Big] + \frac{t^{\sqrt{7}}}{2t^3}uz(-5+\sqrt{7})$$

$$\Phi^x_{11} = -\frac{1}{2t^2}\Big[3t^{-1-\sqrt{7}}x^2zu_x{}^2 + 9t^{-\sqrt{7}-3}y^2z^3u_x{}^2 + t^{-\sqrt{7}}zuu_t + 4t^{-1-\sqrt{7}}xz^2u_xu_z - t^{-1-\sqrt{7}}\sqrt{7}x^2zu_x{}^2$$
$$-t^{-\sqrt{7}}\sqrt{7}z^2u_zu_t - 12t^{-2-\sqrt{7}}yz^3u_xu_z - t^{-\sqrt{7}}\sqrt{7}zuu_t - 6t^{-2-\sqrt{7}}yz^2uu_x - 6t^{-1-\sqrt{7}}yz^2u_xu_t -$$
$$4t^{-\sqrt{7}}yzu_yu_t + 12t^{-2-\sqrt{7}}y^2z^2u_xu_y - t^{1-\sqrt{7}}xu_yu_t + 2t^{-\sqrt{7}}z^2u_zu_t + 4t^{-1-\sqrt{7}}y^2zu_y{}^2 + t^{1-\sqrt{7}}zu_t{}^2 +$$
$$2t^{-1-\sqrt{7}}xzuu_x - 2t^{-1-\sqrt{7}}yzuu_y - 12t^{-2-\sqrt{7}}xyz^2u_x{}^2 - t^{-1-\sqrt{7}}yz^2u_yu_z - 3t^{-\sqrt{7}}xzu_yu_z +$$
$$2t^{-\sqrt{7}}xzu_xu_t + 2t^{-\sqrt{7}}xyu_y{}^2 - 2t^{-\sqrt{7}}x^2u_xu_y + 6t^{-2-\sqrt{7}}\sqrt{7}yz^2uu_x - 2t^{-1-\sqrt{7}}\sqrt{7}xzuu_x +$$
$$2t^{-1-\sqrt{7}}\sqrt{7}yzuu_y + 3t^{-2-\sqrt{7}}\sqrt{7}xyz^2u_x{}^2 + 6t^{-2-\sqrt{7}}\sqrt{7}yz^3u_xu_z + t^{-\sqrt{7}}\sqrt{7}xzu_yu_z +$$
$$2t^{-1-\sqrt{7}}\sqrt{7}yz^2u_yu_z - 2t^{-1-\sqrt{7}}\sqrt{7}xz^2u_xu_z + 2t^{-1-\sqrt{7}}xyzu_xu_y\Big] - \frac{t^{-\sqrt{7}}}{2t^3}uz(5+\sqrt{7})$$

$$\Phi^x_{12} = \frac{2}{7t^2}\Big[9t^{-1+\sqrt{7}}\sqrt{7}y^3z^2u_x{}^2 - 21t^{\sqrt{7}}xy^2zu_x{}^2 - 42t^{\sqrt{7}}y^3zu_xu_y + 3t^{1+\sqrt{7}}\sqrt{7}x^2yu_x{}^2 - 7t^{2+\sqrt{7}}xyu_yu_z +$$
$$14t^{1+\sqrt{7}}xy^2u_xu_y - 2t^{2+\sqrt{7}}\sqrt{7}x^2u_xu_z - 2t^{2+\sqrt{7}}\sqrt{7}y^2u_yu_t - t^{3+\sqrt{7}}\sqrt{7}xu_zu_t - 42t^{\sqrt{7}}y^2zuu_x -$$
$$2t^{1+\sqrt{7}}\sqrt{7}y^2uu_y + 14t^{1+\sqrt{7}}xyuu_x + t^{2+\sqrt{7}}\sqrt{7}yuu_t + 3t^{1+\sqrt{7}}\sqrt{7}y^2zu_yu_z + 2t^{2+\sqrt{7}}\sqrt{7}xyu_xu_t +$$
$$t^{3+\sqrt{7}}\sqrt{7}yu_t{}^2 + 7t^{1+\sqrt{7}}x^2yu_x{}^2 + 7t^{2+\sqrt{7}}y^2u_yu_t + 7t^{2+\sqrt{7}}yuu_t - 14t^{1+\sqrt{7}}y^2uu_y + 6t^{1+\sqrt{7}}\sqrt{7}xyzu_xu_z$$
$$-14t^{1+\sqrt{7}}y^3u_y{}^2 - t^{2+\sqrt{7}}\sqrt{7}xyu_yu_z - 12t^{\sqrt{7}}\sqrt{7}xy^2zu_x{}^2 - 6t^{1+\sqrt{7}}\sqrt{7}y^2zu_xu_t + 2t^{1+\sqrt{7}}\sqrt{7}xyuu_x$$
$$-6t^{\sqrt{7}}\sqrt{7}y^2zuu_x\Big] + \frac{yut^{\sqrt{7}}(5\sqrt{7}-7)}{14t}$$

$$\Phi^x_{13} = \frac{1}{14t^2}\Big[14t^{1-\sqrt{7}}xyuu_x - 42t^{-\sqrt{7}}y^2zuu_x - 3t^{1-\sqrt{7}}\sqrt{7}y^2zu_yu_z + t^{2-\sqrt{7}}\sqrt{7}xyu_yu_z - 3t^{1-\sqrt{7}}\sqrt{7}x^2yu_x{}^2 -$$
$$7t^{2-\sqrt{7}}xyu_yu_z + 14t^{1-\sqrt{7}}xy^2u_xu_y + 2t^{2-\sqrt{7}}\sqrt{7}y^2u_yu_t - 6t^{1-\sqrt{7}}\sqrt{7}xyzu_xu_z + 6t^{-\sqrt{7}}\sqrt{7}y^2zuu_x -$$
$$2t^{1-\sqrt{7}}\sqrt{7}xyuu_x - 21t^{-\sqrt{7}}xy^2zu_x{}^2 - 42t^{-\sqrt{7}}y^3zu_xu_y + t^{3-\sqrt{7}}\sqrt{7}xu_zu_t + 2t^{1-\sqrt{7}}\sqrt{7}y^2uu_y -$$
$$14t^{1-\sqrt{7}}y^3u_y{}^2 + 2t^{2-\sqrt{7}}\sqrt{7}x^2u_xu_z - 9t^{-1-\sqrt{7}}\sqrt{7}y^3z^2u_x{}^2 - t^{2-\sqrt{7}}\sqrt{7}yuu_t + 7t^{1-\sqrt{7}}x^2yu_x{}^2 +$$
$$7t^{2-\sqrt{7}}y^2u_yu_t - t^{3-\sqrt{7}}\sqrt{7}yu_t{}^2 + 7t^{2-\sqrt{7}}yuu_t - 14t^{1-\sqrt{7}}y^2uu_y + 6t^{1-\sqrt{7}}\sqrt{7}y^2zu_xu_t -$$
$$2t^{2-\sqrt{7}}\sqrt{7}xyu_xu_t + 12t^{-\sqrt{7}}\sqrt{7}xy^2zu_x{}^2\Big] - \frac{yu(5\sqrt{7}+7)}{14t^{1+\sqrt{7}}}$$

$$
\begin{aligned}
\Phi^x_{14} = \; & \tfrac{-1}{7t^2}\Big[-4t^{2\sqrt{7}+1}\sqrt{7}xzu_xu_z + 6t^{2\sqrt{7}}\sqrt{7}yzuu_x - 14t^{2\sqrt{7}+1}xyu_xu_y - 14t^{2\sqrt{7}+1}xzu_xu_z + \\
& 28t^{2\sqrt{7}}xyzu_x{}^2 - 14t^{2\sqrt{7}+1}xuu_x - 2t^{2\sqrt{7}+1}\sqrt{7}xuu_x - 84t^{2\sqrt{7}-1}y^2z^2u_x{}^2 \\
& -2t^{2\sqrt{7}+2}\sqrt{7}zu_zu_t + 2t^{2\sqrt{7}+2}\sqrt{7}yu_yu_t - 10t^{2\sqrt{7}+1}\sqrt{7}yzu_yu_z - t^{2\sqrt{7}+2}\sqrt{7}uu_t + \\
& 14t^{2\sqrt{7}+1}yuu_y + 17t^{2\sqrt{7}}\sqrt{7}xyzu_x{}^2 + 12t^{2\sqrt{7}}\sqrt{7}yz^2u_xu_z + 6t^{2\sqrt{7}+1}\sqrt{7}yzu_xu_t - \\
& 7t^{2\sqrt{7}+2}uu_t - 14t^{2\sqrt{7}+1}yzu_yu_z - 42t^{2\sqrt{7}-1}\sqrt{7}y^2z^2u_x{}^2 + 42t^{2\sqrt{7}}y^2zu_xu_y - 2t^{2\sqrt{7}+2}\sqrt{7}xu_xu_t \\
& 7t^{2\sqrt{7}+2}yu_yu_t - 7t^{2\sqrt{7}+2}zu_zu_t - t^{2\sqrt{7}+1}\sqrt{7}x^2u_x{}^2 - t^{2\sqrt{7}+3}\sqrt{7}u_t{}^2 + t^{2\sqrt{7}+2}\sqrt{7}xu_yu_z \\
& +42t^{2\sqrt{7}}yz^2u_xu_z + 2t^{2\sqrt{7}+1}\sqrt{7}yuu_y + 14t^{2\sqrt{7}+1}y^2u_y{}^2 + 42t^{2\sqrt{7}}yzuu_x\Big] + \tfrac{2u(1+\sqrt{7})}{tt^{-2\sqrt{7}}} \\
\Phi^x_{15} = \; & \tfrac{1}{7t^2}\Big[6t^{-2\sqrt{7}}\sqrt{7}yzu_x - 14t^{1-2\sqrt{7}}yu_y + 14t^{1-2\sqrt{7}}xu_x - 42t^{-2\sqrt{7}}yzu_x - 2\sqrt{7}t^{2-2\sqrt{7}}xu_xu_t + \\
& \sqrt{7}t^{2-2\sqrt{7}}xu_yu_z - 10\sqrt{7}t^{1-2\sqrt{7}}yzu_yu_z + 17t^{-2\sqrt{7}}\sqrt{7}xyzu_x{}^2 + 12t^{-2\sqrt{7}}\sqrt{7}yz^2u_xu_z + \\
& 6t^{1-2\sqrt{7}}\sqrt{7}yzu_xu_t - 2t^{2-2\sqrt{7}}\sqrt{7}zu_zu_t + 7t^{2-2\sqrt{7}}u_t + 2t^{1-2\sqrt{7}}\sqrt{7}yu_y + 7t^{2-2\sqrt{7}}yu_yu_t + \\
& 7t^{2-2\sqrt{7}}zu_zu_t + 84t^{-1-2\sqrt{7}}y^2z^2u_x{}^2 - \sqrt{7}t^{1-2\sqrt{7}}x^2u_x{}^2 - t^{3-2\sqrt{7}}\sqrt{7}u_t{}^2 + 14t^{1-2\sqrt{7}}xzu_xu_z + \\
& 14t^{1-2\sqrt{7}}xyu_xu_y - 4t^{1-2\sqrt{7}}\sqrt{7}xzu_xu_z - 28t^{-2\sqrt{7}}xyzu_x{}^2 + 2t^{2-2\sqrt{7}}\sqrt{7}yu_yu_t - \\
& 42t^{-1-2\sqrt{7}}\sqrt{7}y^2z^2u_x{}^2 - 2t^{1-2\sqrt{7}}\sqrt{7}xu_x - 42t^{-2\sqrt{7}}yz^2u_xu_z + 14t^{1-2\sqrt{7}}yzu_yu_z - \\
& 42t^{-2\sqrt{7}}y^2zu_xu_y - t^{2-2\sqrt{7}}\sqrt{7}u_t - 14t^{1-2\sqrt{7}}y^2u_y{}^2\Big] - \tfrac{2u(\sqrt{7}-1)}{tt^{2\sqrt{7}}}
\end{aligned}
\tag{5.2.12}
$$

Case (ii) $h(u) = u^n, \quad n \neq 0, 1$

We obtain a 7-dimensional algebra of point symmetry generators namely

$$
\begin{aligned}
& X_1 = -x\partial_x + t\partial_t, \qquad X_2 = -y\partial_y + z\partial_z \qquad X_3 = -(2+\sqrt{7})t^{1+\sqrt{7}}y\partial_x + t^{2+\sqrt{7}}\partial_z \\
& X_4 = (-2+\sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z \qquad X_5 = -(2+\sqrt{7})t^{-1+\sqrt{7}}z\partial_x + t^{\sqrt{7}}\partial_y \\
& X_6 = (-2+\sqrt{7})t^{-1-\sqrt{7}}z\partial_x + t^{-\sqrt{7}}\partial_y \qquad X_7 = t\partial_x
\end{aligned}
\tag{5.2.13}
$$

The corresponding conserved quantities are

$$
\begin{aligned}
\Phi^t_1 = \; & \tfrac{1}{t}[-u_yu_zt^2 - 2\,txu_x{}^2 + 2\,yu_xu_yt + 3\,yzu_x{}^2 + u^nt^2] \\
\Phi^z_1 = \; & u_y\,(tu_t - xu_x) \\
\Phi^y_1 = \; & -\tfrac{1}{t}\,(tu_z - 2\,yu_x)\,(tu_t - xu_x) \\
\Phi^x_1 = \; & -\tfrac{1}{t^2}[-t^3u_t{}^2 - 2\,t^2xu_xu_t - t^2xu_yu_z + 2\,t^2yu_yu_t + tx^2u_x{}^2 + 6\,tyzu_xu_t - 3\,xyzu_x{}^2 + u^nt^2x] \\
\Phi^t_2 = \; & u_x\,(yu_y - zu_z)
\end{aligned}
$$

$$
\begin{aligned}
\Phi_2^z =\ & \tfrac{1}{t^2}[-t^2yu_y{}^2 - t^2zu_xu_t - txzu_x{}^2 + 2tyzu_xu_y + 3yz^2u_x{}^2 + u^nt^2z] \\
\Phi_2^y =\ & \tfrac{1}{t^2}[-t^2yu_xu_t - t^2zu_z{}^2 - txyu_x{}^2 + 2tyzu_xu_z + 3y^2zu_x{}^2 + u^nt^2y] \\
\Phi_2^x =\ & -\tfrac{1}{t^2}[\left(t^2u_t + 2txu_x - 2tyu_y - 6yzu_x\right)(yu_y - zu_z)] \\
\Phi_3^t =\ & -u_x\left(-u_xt^{1+\sqrt{7}}y\sqrt{7} - 2u_xt^{1+\sqrt{7}}y + u_zt^{2+\sqrt{7}}\right) \\
\Phi_3^z =\ & (3t^{\sqrt{7}}yzu_x{}^2 - \sqrt{7}t^{1+\sqrt{7}}yu_xu_y - t^{1+\sqrt{7}}xu_x{}^2 - t^{2+\sqrt{7}}u_xu_t + t^{2+\sqrt{7}}u^n) \\
\Phi_3^y =\ & -\tfrac{1}{t}(tu_z - 2yu_x)\left(-u_xt^{1+\sqrt{7}}y\sqrt{7} - 2u_xt^{1+\sqrt{7}}y + u_zt^{2+\sqrt{7}}\right) \\
\Phi_3^x =\ & \tfrac{1}{t^2}\Big(3\sqrt{7}t^{1+\sqrt{7}}y^2zu_x{}^2 + 6t^{1+\sqrt{7}}y^2zu_x{}^2 - t^{2+\sqrt{7}}\sqrt{7}xyu_x{}^2 - 2t^{2+\sqrt{7}}xyu_x{}^2 + \sqrt{7}t^{3+\sqrt{7}}yu_yu_z - \\
& \sqrt{7}u^nt^{3+\sqrt{7}}y + t^{4+\sqrt{7}}u_zu_t + 2t^{3+\sqrt{7}}xu_xu_z - 6t^{2+\sqrt{7}}yzu_xu_z - 2u^nt^{3+\sqrt{7}}y\Big) \\
\Phi_4^t =\ & -u_x\left(u_xt^{1-\sqrt{7}}y\sqrt{7} - 2u_xt^{1-\sqrt{7}}y + u_zt^{2-\sqrt{7}}\right) \\
\Phi_4^z =\ & (t^{1-\sqrt{7}}\sqrt{7}yu_xu_y + 3t^{-\sqrt{7}}yzu_x{}^2 - t^{1-\sqrt{7}}xu_x{}^2 - t^{2-\sqrt{7}}u_xu_t + t^{2-\sqrt{7}}u^n) \\
\Phi_4^y =\ & -\tfrac{1}{t}(tu_z - 2yu_x)\left(u_xt^{1-\sqrt{7}}y\sqrt{7} - 2u_xt^{1-\sqrt{7}}y + u_zt^{2-\sqrt{7}}\right) \\
\Phi_4^x =\ & -\tfrac{1}{t^2}\Big(3t^{1-\sqrt{7}}\sqrt{7}y^2zu_x{}^2 - 6t^{1-\sqrt{7}}y^2zu_x{}^2 - t^{2-\sqrt{7}}\sqrt{7}xyu_x{}^2 + t^{3-\sqrt{7}}\sqrt{7}yu_yu_z + \\
& 2t^{2-\sqrt{7}}xyu_x{}^2 - t^{3-\sqrt{7}}\sqrt{7}u^ny + 2t^{3-\sqrt{7}}u^ny - t^{4-\sqrt{7}}u_zu_t - 2t^{3-\sqrt{7}}xu_xu_z + 6t^{2-\sqrt{7}}yzu_xu_z\Big) \\
\Phi_5^t =\ & u_x\left(u_xt^{-1+\sqrt{7}}z\sqrt{7} + 2u_xt^{-1+\sqrt{7}}z - u_yt^{\sqrt{7}}\right) \\
\Phi_5^z =\ & -u_y\left(u_xt^{-1+\sqrt{7}}z\sqrt{7} + 2u_xt^{-1+\sqrt{7}}z - u_yt^{\sqrt{7}}\right) \\
\Phi_5^y =\ & -\tfrac{1}{t}(-t^{\sqrt{7}}\sqrt{7}zu_xu_z + 2t^{-1+\sqrt{7}}\sqrt{7}yzu_x{}^2 + 7t^{-1+\sqrt{7}}yzu_x{}^2 - t^{\sqrt{7}}xu_x{}^2 - 2t^{\sqrt{7}}zu_xu_z - \\
& t^{1+\sqrt{7}}u_xu_t + t^{1+\sqrt{7}}u^n) \\
\Phi_5^x =\ & \tfrac{1}{t^2}\Big[3t^{-1+\sqrt{7}}\sqrt{7}yz^2u_x{}^2 - \sqrt{7}t^{\sqrt{7}}xzu_x{}^2 + 6t^{-1+\sqrt{7}}yz^2u_x{}^2 + t^{1+\sqrt{7}}\sqrt{7}zu_yu_z - 2t^{\sqrt{7}}xzu_x{}^2 - \\
& 6t^{\sqrt{7}}yzu_xu_y - t^{1+\sqrt{7}}\sqrt{7}u^nz + 2t^{1+\sqrt{7}}zu_yu_z - 2t^{1+\sqrt{7}}u^nz + t^{2+\sqrt{7}}u_yu_t + 2t^{1+\sqrt{7}}xu_xu_y \\
& -2t^{1+\sqrt{7}}yu_y{}^2\Big] \\
\Phi_6^t =\ & -u_x\left(u_xt^{-1-\sqrt{7}}z\sqrt{7} - 2u_xt^{-1-\sqrt{7}}z + u_yt^{-\sqrt{7}}\right) \\
\Phi_6^z =\ & u_y\left(u_xt^{-1-\sqrt{7}}z\sqrt{7} - 2u_xt^{-1-\sqrt{7}}z + u_yt^{-\sqrt{7}}\right) \\
\Phi_6^y =\ & \tfrac{1}{t}(-7t^{-1-\sqrt{7}}yzu_x{}^2 - t^{-\sqrt{7}}\sqrt{7}zu_xu_z + 2t^{-1-\sqrt{7}}\sqrt{7}yzu_x{}^2 + t^{-\sqrt{7}}xu_x{}^2 + 2t^{-\sqrt{7}}zu_xu_z + \\
& t^{1-\sqrt{7}}u_xu_t - t^{1-\sqrt{7}}u^n) \\
\Phi_6^x =\ & -\tfrac{1}{t^2}\Big(3t^{-1-\sqrt{7}}\sqrt{7}yz^2u_x{}^2 - t^{-\sqrt{7}}\sqrt{7}xzu_x{}^2 - 6t^{-1-\sqrt{7}}yz^2u_x{}^2 + 2t^{-\sqrt{7}}xzu_x{}^2 + 6t^{-\sqrt{7}}yzu_xu_y + \\
& t^{1-\sqrt{7}}\sqrt{7}zu_yu_z - t^{1-\sqrt{7}}\sqrt{7}u^nz - 2t^{1-\sqrt{7}}zu_yu_z + 2t^{1-\sqrt{7}}u^nz - t^{2-\sqrt{7}}u_yu_t - 2t^{1-\sqrt{7}}xu_xu_y + \\
& 2t^{1-\sqrt{7}}yu_y{}^2\Big) \\
\Phi_7^t =\ & -u_x{}^2t \\
\Phi_7^z =\ & +u_ytu_x \\
\Phi_7^y =\ & -(tu_z - 2yu_x)u_x \\
\Phi_7^x =\ & \tfrac{1}{t}(-u_yu_zt^2 + txu_x{}^2 - 3yzu_x{}^2 + u^nt^2)
\end{aligned}
\tag{5.2.14}
$$

Variational symmetries, multipliers approach

Consider the wave equation (5.2.2) in ASD-Einstein spacetime with dependent variable $u = u(x, y, t)$, i.e., we project the system on z-axis

Let us choose $k(u) = -u$, we have

$$\frac{\delta}{\delta u}\left[\mathcal{Q}\left(\frac{x}{t}u_{xx} - \frac{2y}{t}u_{xy} + u_{x,t} + u\right)\right] = 0$$

where $\mathcal{Q} = \mathcal{Q}(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{xxx}, u_{xxy})$. Then

$$\left[\mathcal{Q}\left(\frac{x}{t}u_{xx} - \frac{2y}{t}u_{xy} + u_{x,t} + u\right)\right] = D_t\Phi^t + D_y\Phi^y + D_x\Phi^x$$

where (Φ^x, Φ^y, Φ^t) is the conserved flux. We obtain the set of multiplier $\mathcal{Q}_i$ together with the conserved densities, namely,

$$\begin{aligned}
\mathcal{Q}_1 &= \tfrac{u_x}{\sqrt{y}}\\
\Phi_1^x &= \tfrac{1}{8ty^{3/2}}[2yu_x\left(tu_t + (t-2y)u_y + 2xu_x\right) + u\left((t-2y)u_x + 2y\left(tu_{xt} + (t-2y)u_{xy}\right)\right)]\\
\Phi_1^y &= \tfrac{1}{4t\sqrt{y}}[(t-2y)\left({u_x}^2 - uu_{xtx}\right)]\\
\Phi_1^t &= \tfrac{{u_x}^2 - uu_{txtx}}{4\sqrt{y}}\\
\mathcal{Q}_2 &= \tfrac{1}{ty}(tu_x + xu_{xxx} + u_{xx})\\
\Phi_2^x &= \tfrac{1}{4t^2y^2}\left[y(2x\left(t{u_x}^2 - (tu_{xxt} + (t-2y)u_{xxy}\right)u_x + u_{xx}\left(tu_{xt} + (t-2y)u_{xy} + xu_{xx}\right)\right)\\
&\quad + tu_t\left(tu_x + u_{xx} + xu_{xxx}\right) + (t-2y)u_{\ y}\left(tu_x + u_{xx} + xu_{xxx}\right))\\
&\quad + u(yu_{xt}t^2 + yu_{xy}t^2 + (t-2y)u_{\ x}t - 2y^2u_{xy}t + u_{xx}t + yu_{xxt}t + yu_{xxy}t + xu_{xxx}t + xyu_{xxxt}t +\\
&\quad xyu_{xxxy}t + yu_{xx} - 2y^2u_{xxy} + xyu_{xxx} - 2xy^2u_{xxxy}\left.\right)]\\
\Phi_2^y &= \tfrac{1}{4t^2y}[(t-2y)\left(t{u_x}^2 + \left(u_{xx} + xu_{xxx}\right)u_x - u\left(tu_{xx} + 2u_{xxx} + xu_{xxx}\right)\right)]\\
\Phi_2^t &= \tfrac{1}{4ty}[t{u_x}^2 + \left(u_{xx} + xu_{xxx}\right)u_x - u\left(tu_{xx} + 2u_{xxx} + xu_{xxxx}\right)]\\
\mathcal{Q}_3 &= yu_y + \tfrac{1}{2}u\\
\Phi_3^x &= \tfrac{1}{4t}[yu_y\left(tu_t + (t-2y)u_y + 2xu_x\right) - u\left((t-2y)u_y + y\left(tu_{yt} + (t-2y)u_{yt} + 2xu_{xy}\right)\right)]\\
\Phi_3^y &= \tfrac{1}{4t}[y\left((t-2y)u_yu_x + u\left(2tu_{xt} + (t-2y)u_{xy} + 2xu_{xx}\right)\right)\\
\Phi_t^3 &= \tfrac{1}{4}y(u_yu_x - uu_{xy})]\\
\mathcal{Q}_4 &= \tfrac{1}{4t\sqrt{y}}(4tyu_y + 4xyu_{xxy} + tu + xu_{xx} + 2yu_{xy})
\end{aligned}$$

$$
\begin{aligned}
\Phi_4^x = & \frac{1}{96t^2y^{3/2}}[3t(t-2y)u^2 - \left(-48tu_{yy}y^3 + 16u_{xyy}y^3 + 16xu_{xxyy}y^3 + 24t^2u_{yt}y^2 + \right.\\
& 24t^2u_{yy}y^2 + 48txu_{xy}y^2 + 12u_{xy}y^2 - 8tu_{xyt}y^2 - 8tu_{xyy}y^2 + 32xu_{xxy}y^2 - 8txu_{xxyt}y^2 - \\
& 8txu_{xxyy}y^2 + 16x^2u_{xxxy}y^2 + 12t(t-2y)u_yy + 2tu_{xt}y + 8tu_{xy}y + 10xu_{xx}y + 2txu_{xxt}y + \\
& \left.14txu_{xxy}y + 8x^2u_{xxx}y - 3txu_{xx}\right)u + 2y\left(12ty(t-2y){u_y}^2 + \left(-8yu_{xx}x^2 + 24tyu_xx + \right.\right.\\
& \left.3tu_{xx}x - 16yu_{xx}x - 8tyu_{xxt}x - 8y^2u_{xxy}x + 4tyu_{xxy}x - 2tyu_{xt} - 8y^2u_{xy} + 4tyu_{xy}\right)u_y + \\
& 3tu_t\left(4tyu_y + 2yu_{xy} + x\left(u_{xx} + 4yu_{xxy}\right)\right) + 2x\left(8yu_{xy}\left(tu_{xt} + (t-2y)u_{xy} + xu_{xx}\right) + \right.\\
& \left.\left.u_x\left(8u_{xyy}y^2 - 4tu_{xyt}y - 4tu_{xyy}y + 8xu_{xxy}y + tu_{xt} + (t+12y)u_{xy} + 4xu_{xx}\right)\right)\right)]\\
\Phi_y^4 = & \frac{1}{48t^2\sqrt{y}}[8y{u_{xx}}^2x^2 - 8yu_xu_{xxx}x^2 + 3tu_xu_{xx}x - 16yu_xu_{xx}x + 8tyu_{xt}u_{xx}x - 16y^2u_{xy}u_{xx}x + \\
& 8tyu_{xy}u_{xx}x - 8tyu_xu_{xxt}x - 8y^2u_xu_{xxy}x + 4tyu_xu_{xxy}x + 12ty(t-2y)u_yu_x - 2tyu_xu_{xt} - \\
& 8y^2u_xu_{xy} + 4tyu_xu_{xy} + u\left(24yu_{xt}t^2 + 12y(t-2y)u_{xy}t + 24xyu_{xx}t - 3u_{xx}t + 10yu_{xxt}t - \right.\\
& 8yu_{xxy}t - 3xu_{xxx}t + 8xyu_{xxxt}t - 4xyu_{xxxy}t + 16yu_{xx} + 16y^2u_{xxy} + 32xyu_{xxx} + \\
& \left.8xy^2u_{xxxy} + 8x^2yu_{xxxx}\right)]\\
\Phi_t^4 = & \frac{1}{16t\sqrt{y}}[4tyu_x + \left(2y\mathsf{u}_{xy} + xu_{xx} + 4xyu_{xxy}\right)\mathsf{u}_x - u\left(4tyu_{xy} + u_{xx} + 6yu_{xxy} + xu_{xxx} + 4xyu_{xxxy}\right)]\\
\mathcal{Q}_5 = & \frac{u_{xxx}}{y^{2/3}}\\
\Phi_x^5 = & \frac{1}{12ty\sqrt[3]{y^2}}[-12u_{\mathsf{xy}}u_{xx}y^2 - 6u_yu_{xxx}y^2 - 6uu_{xxxy}y^2 + 6x{u_{xx}}^2y + 6tu_{xt}u_{xx}y + 6tu_{xy}u_{xx}y + \\
& 6u_x\left(-u_{xx} - tu_{xxt} - (t-2y)u_{xxy}\right)y + 8uu_{xxx}y + 3tu_tu_{xxx}y + 3tu_yu_{xxx}y + 3tuu_{xxxt}y + \\
& 3tuu_{xxxy}y + 2tuu_{xxx}]\\
\Phi_y^5 = & \frac{1}{4t\sqrt[3]{y^2}}[(t-2y)\left(u_xu_{xxx} - uu_{xxxx}\right)]\\
\Phi_t^5 = & \frac{1}{4\sqrt[3]{y^2}}[\mathsf{u}_xu_{xxx} - uu_{xxxx}]\\
\mathcal{Q}_6 = & u_{xxy}\\
\Phi_x^6 = & \frac{1}{12t}[4t{u_{xy}}^2 - 8y{u_{xy}}^2 + 4tu_{xt}u_{xy} + 4xu_{xx}u_{xy} - 2u_yu_{xx} - 2tu_yu_{xxt} - 4uu_{xxy} + 3tu_tu_{xxy} + \\
& tu_yu_{xxy} - 2yu_yu_{xxy} + u_x\left(4u_{xy} - 2t\mathsf{u}_{xyt} - 2tu_{xyy} + 4yu_{xyy} + 4xu_{xxy}\right) + tuu_{xxyt} + tuu_{xxyy} - \\
& 2yuu_{xxyy} - 2xu_yu_{xxx} - 2xuu_{xxxy}]\\
\Phi_y^6 = & \frac{1}{12t}[2x{u_{xx}}^2 + 2tu_{xt}u_{xx} + 2tu_{xy}u_{xx} - 4yu_{xy}u_{xx} + 4uu_{xxx} - u_x\left(2u_{xx} + 2tu_{xxt} - tu_{xxy} + \right.\\
& \left.2yu_{xxy} + 2xu_{xxx}\right) + 2tuu_{xxxt} - tuu_{xxxy} + 2yuu_{xxxy} + 2xuu_{xxxx}]\\
\Phi_t^6 = & \frac{1}{4}\left(u_xu_{xxy} - uu_{xxxy}\right)
\end{aligned}
\tag{5.2.15}
$$

5.3 Wave equations on Kerr spacetime

Introduction

A metric that describes a massive rotating object has been proposed by RP Kerr in 1963. Since then, many works have been done to investigate the structure and astrophysical applications of this spacetime . Exacts symmetries of this spacetime are investigated in [47]. In [48], the authors investigated the approximate symmetries on Kerr spacetime and found the rescaling factor for the energy. In this section, we analyse the symmetry structure of wave equation on Kerr spacetime. Noether approach and direct construction are used to find conserved densities on this spacetime.

The line element in this spacetime is given [43]

$$\mathrm{d}s^2 = \frac{\Delta}{\rho^2}\left[\mathrm{d}t - k\sin^2(\theta)\mathrm{d}\phi\right]^2 - \frac{\sin^2(\theta)}{\rho^2}[(r^2+k^2)\mathrm{d}\phi - k\mathrm{d}t]^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 \qquad (5.3.1)$$

where $\Delta = r^2 - 2Mr + k^2$ and $\rho^2 = r^2 + a^2\cos^2(\theta)$. M and k represent the mass and the rotation parameter, respectively. The angular momentum of the object is $J = Mk$

The Gordon type equation is given by (5.2.1)

$$\begin{array}{l}\frac{1}{(2Mr-k^2-r^2)}\left[4\left(Mr - \frac{k^2}{2} - \frac{r^2}{2}\right)^2 \sin^2 u_{r,r} - 2\left(Mr - \frac{k^2}{2} - \frac{r^2}{2}\right)\sin^2 u_{\theta,\theta}\right. \\ +2\sin^2\theta\left(k^2\left(Mr - \frac{k^2}{2} - \frac{r^2}{2}\right)\cos^2\theta - Mk^2r - \frac{k^2r^2}{2} - \frac{r^4}{2}\right)u_{t,t} + (k^2\cos^2\theta - 2Mr + r^2)u_{\phi,\phi} - \\ \left. 4\sin\theta(Mu_{t\phi}kr\sin\theta - (\sin\theta(M-r)u_r - \frac{1}{2}u_\theta\cos\theta)\left(Mr - \frac{k^2}{2} + \frac{r^2}{2}\right)\right] - \\ k(u)(r^2 + k^2\cos^2\theta)\sin^2\theta = 0\end{array} \qquad (5.3.2)$$

The corresponding Lagrangian is

$$\begin{array}{rl} L = & \frac{1}{2Mr-k^2-r^2}[\left(k^2\sin(\theta)(\cos(\theta))^2\left(Mr - \frac{1}{2}k^2 - \frac{1}{2}r^2\right) - Mk^2r\sin(\theta) - \frac{1}{2}k^2r^2\sin(\theta) - \right. \\ & \left.\frac{1}{2}\sin(\theta)r^4\right)u_t{}^2] - 2\frac{kMr\sin(\theta)u_tu_\phi}{2Mr-k^2-r^2} + \frac{1}{2}\sin(\theta)\left(2Mr - k^2 - r^2\right)u_r{}^2 - \frac{1}{2}\sin(\theta)u_\theta{}^2 + \\ & \frac{1}{2}\frac{\left(k^2(\cos(\theta))^2 - 2Mr + r^2\right)u_\phi{}^2}{\sin(\theta)(2Mr-k^2-r^2)} - h(u)\sin\theta(r^2 + k^2\cos\theta)\end{array} \qquad (5.3.3)$$

where $h(u) = \int k(u)\mathrm{d}u$

Symmetries of the waves equation-the Noether approach

We investigate the cases that yield zero gauge of (5.3.2)

The principal Noether algebra is for the case $h(u) = 0$ in (5.3.3)

$$\begin{array}{rl} X_1 = & \partial_u \\ X_3 = & \partial_t \\ X_2 = & \partial_\phi \end{array} \qquad (5.3.4)$$

The associated conserved vectors are

$$
\begin{aligned}
\Phi_1^\phi &= \frac{-2kMru_t+2M(\cos\theta)^2kru_t-2Mru_\phi+(\cos\theta)^2k^2u_\phi+r^2u_\phi}{\sin\theta(2Mr-k^2-r^2)}\\
\Phi_1^\theta &= \sin\theta u_\theta\\
\Phi_1^r &= \left(2Mr-k^2-r^2\right)u_r\sin\theta\\
\Phi_1^t &= -\frac{\sin\theta\left(2M(\cos\theta)^2k^2ru_t-(\cos\theta)^2k^4u_t-(\cos\theta)^2k^2r^2u_t-2Mk^2ru_t-k^2r^2u_t-u_tr^4-2kMru_\phi\right)}{2Mr-k^2-r^2}\\
\Phi_2^\phi &= -\frac{1}{2\sin\theta(2\,Mr-k^2-r^2)}\Big[-4\,M^2r^2{u_r}^2-4\,M\left(\cos\theta\right)^2k^2r{u_r}^2+2\,M\left(\cos\theta\right)^4k^2r{u_t}^2-\\
&4\,M\left(\cos\theta\right)^2k^2r{u_t}^2-k^2{u_\theta}^2-r^2{u_\theta}^2+r^4{u_t}^2-r^4{u_r}^2-k^4{u_r}^2+r^2{u_\phi}^2+2\,Mk^2r{u_t}^2-\\
&\left(\cos\theta\right)^4k^2r^2{u_t}^2+4\,M^2\left(\cos\theta\right)^2r^2{u_r}^2-4\,M\left(\cos\left(\theta\right)\right)^2r^3{u_r}^2+2\left(\cos\left(\theta\right)\right)^2k^2r^2{u_r}^2-\\
&2\,M\left(\cos\left(\theta\right)\right)^2r{u_\theta}^2+4\,Mk^2r{u_r}^2+2\,Mr{u_\theta}^2-2\,k^2r^2{u_r}^2+4\,Mr^3{u_r}^2-\left(\cos\left(\theta\right)\right)^4k^4{u_t}^2+\\
&\left(\cos\left(\theta\right)\right)^2k^4{u_r}^2-\left(\cos\left(\theta\right)\right)^2r^4{u_t}^2+\left(\cos\left(\theta\right)\right)^2r^4{u_r}^2+\left(\cos\left(\theta\right)\right)^2k^2{u_\theta}^2+\left(\cos\left(\theta\right)\right)^2r^2{u_\theta}^2+\\
&\left(\cos\left(\theta\right)\right)^2k^4{u_t}^2-2\,Mr{u_\phi}^2+\left(\cos\left(\theta\right)\right)^2k^2{u_\phi}^2+k^2r^2{u_t}^2\Big]\\
\Phi_2^\theta &= -\sin\left(\theta\right)u_\theta u_\phi\\
\Phi_2^r &= -\left(2\,Mr-k^2-r^2\right)u_r\sin\left(\theta\right)u_\phi\\
\Phi_2^t &= \frac{\sin(\theta)\left(2\,M(\cos(\theta))^2k^2ru_t-(\cos(\theta))^2k^4u_t-(\cos(\theta))^2k^2r^2u_t-2\,Mk^2ru_t-k^2r^2u_t-u_tr^4-2\,kMru_\phi\right)u_\phi}{2\,Mr-k^2-r^2}\\
\Phi_3^\phi &= -\frac{\left(-2\,kMru_t+2\,M(\cos(\theta))^2kru_t-2\,Mru_\phi+(\cos(\theta))^2k^2u_\phi+r^2u_\phi\right)u_t}{\sin(\theta)(2\,Mr-k^2-r^2)}\\
\Phi_3^\theta &= -\sin\left(\theta\right)u_\theta u_t\\
\Phi_3^r &= -\left(2\,Mr-k^2-r^2\right)u_r\sin\left(\theta\right)u_t\\
\Phi_3^t &= \frac{1}{2\sin(\theta)(2\,Mr-k^2-r^2)}\Big[-4\,M^2r^2{u_r}^2-4\,M\left(\cos\left(\theta\right)\right)^2k^2r{u_r}^2-2\,M\left(\cos\left(\theta\right)\right)^4k^2r{u_t}^2+\\
&4\,M\left(\cos\left(\theta\right)\right)^2k^2r{u_t}^2-k^2{u_\theta}^2-r^2{u_\theta}^2-r^4{u_t}^2-r^4{u_r}^2-k^4{u_r}^2-r^2{u_\phi}^2-2\,Mk^2r{u_t}^2+\\
&\left(\cos\left(\theta\right)\right)^4k^2r^2{u_t}^2+4\,M^2\left(\cos\left(\theta\right)\right)^2r^2{u_r}^2-4\,M\left(\cos\left(\theta\right)\right)^2r^3{u_r}^2+2\left(\cos\left(\theta\right)\right)^2k^2r^2{u_r}^2-\\
&2\,M\left(\cos\left(\theta\right)\right)^2r{u_\theta}^2+4\,Mk^2r{u_r}^2+2\,Mr{u_\theta}^2-2\,k^2r^2{u_r}^2+4\,Mr^3{u_r}^2+\left(\cos\left(\theta\right)\right)^4k^4{u_t}^2+\\
&\left(\cos\left(\theta\right)\right)^2k^4{u_r}^2+\left(\cos\left(\theta\right)\right)^2r^4{u_t}^2+\left(\cos\left(\theta\right)\right)^2r^4{u_r}^2+\left(\cos\left(\theta\right)\right)^2k^2{u_\theta}^2+\left(\cos\left(\theta\right)\right)^2r^2{u_\theta}^2-\\
&\left(\cos\left(\theta\right)\right)^2k^4{u_t}^2+2\,Mr{u_\phi}^2-\left(\cos\left(\theta\right)\right)^2k^2{u_\phi}^2-k^2r^2{u_t}^2\Big]
\end{aligned}
\tag{5.3.5}
$$

Symmetries of the wave equations-the multipliers approach

Consider the wave equation (5.3.2) with $k(u)=0$. We have

$$
\begin{aligned}
\frac{\delta}{\delta u}[\mathcal{Q} \quad & \frac{1}{(2Mr-k^2-r^2)}\Bigg[4\left(Mr-\frac{k^2}{2}-\frac{r^2}{2}\right)^2\sin^2 u_{r,r}-2\left(Mr-\frac{k^2}{2}-\frac{r^2}{2}\right)\sin^2 u_{\theta,\theta}\\
&+2\sin^2\theta\left(k^2\left(Mr-\frac{k^2}{2}-\frac{r^2}{2}\right)\cos^2\theta-Mk^2r-\frac{k^2r^2}{2}-\frac{r^4}{2}\right)u_{t,t}+(k^2\cos^2\theta-\\
&2Mr+r^2)u_{\phi,\phi}-4\sin\theta(Mu_{t\phi}kr\sin\theta-(\sin\theta(M-r)u_r-\tfrac{1}{2}u_\theta\cos\theta)\left(Mr-\frac{k^2}{2}+\frac{r^2}{2}\right)\Bigg]\\
&=D_t\Phi^t+D_r\Phi^r+D_\theta\Phi^\theta+D_\phi\Phi^\phi
\end{aligned}
$$

where $\mathcal{Q}=\mathcal{Q}(u_\phi,u_t,u_{\phi\phi},u_{tt},u_{t\phi},u_{\phi\phi t},u_{t\phi t},u_{\phi\phi\phi})$

After tedious calculations, we obtain a set of multipliers $\mathcal{Q}_i$ together with the conserved densities.

$$
\begin{aligned}
\mathcal{Q}_1 =\; & u_t \\
\Phi_1^t =\; & \tfrac{1}{2(k^2+r(r-2M))\sin(\theta)}[\sin(\theta)^2 u_{\,t}(2kMru_{\,\phi} + ((k^2+r(r-2M))\cos(\theta)k^2 + r(r^3+k^2(2M+ \\
& r)))u_{\,t}) - u(2k^2\cos(\theta)^2 + (k^2+r(r-2M))\sin(\theta)u_{\,\theta}\cos(\theta) + 2r(r-2M) + \sin(\theta)^2(u_{\,rr}k^4 \\
& +2r^2u_{\,rr}k^2 - 4Mru_{\,rr}k^2 - 2Mru_{\,t\phi}k + (k^2+r(r-2M))u_{\,\theta\theta} - 2(M-r)(k^2+r^2- \\
& 2Mr)u_{\,r} + r^4u_{\,rr} - 4Mr^3u_{\,rr} + 4M^2r^2u_{\,rr}))] \\
\Phi_1^r =\; & -\tfrac{1}{2}(k^2+r(r-2M))\sin(\theta)(u_{\,r}u_{\,t} - uu_{\,tr}) \\
\Phi_1^\theta =\; & \tfrac{1}{2}\left(-\cos(\theta)uu_{\,t} - \sin(\theta)u_{\,\theta}u_{\,t} + u\left(\sin u_{\,t} + \sin(\theta)u_{\,t\theta}\right)\right) \\
\Phi_1^\phi =\; & \tfrac{1}{k^2+r(r-2M)}[kMr\sin(\theta)\left(u_{\,t}{}^2 - uu_{\,tt}\right)] \\
\mathcal{Q}_2 =\; & 1 \\
\Phi_2^t =\; & \tfrac{1}{k^2+r(r-2M)}[\sin(\theta)\left(2kMru_{\,\phi} + \left(\left(k^2+r(r-2M)\right)\cos(\theta)k^2 + r\left(r^3+k^2(2M+r)\right)\right)u_{\,t}\right)] \\
\Phi_2^r =\; & -\left(k^2+r(r-2M)\right)\sin(\theta)u_{\,r} \\
\Phi_2^\theta =\; & -\cos(\theta)u(t,r,\theta,\phi) + \sin u - \sin(\theta)u_{\,\theta} \\
\Phi_2^\phi =\; & \tfrac{2kMr\sin(\theta)u_{\,t}}{k^2+r(r-2M)} \\
\mathcal{Q}_3 =\; & u_\phi \\
\Phi_3^t =\; & \tfrac{1}{2(k^2+r(r-2M))}[\sin(\theta)(2kMru_{\,\phi}{}^2 + ((k^2+r(r-2M))\cos(\theta)k^2 + r(r^3+k^2(2M+r)))u_{\,t}u_{\,\phi}- \\
& u(2kMru_{\phi\phi} + ((k^2+r(r-2M))\cos(\theta)k^2 + r(r^3+k^2(2M+r)))u_{\,t\phi}))] \\
\Phi_3^r =\; & -\tfrac{1}{2}\left(k^2+r(r-2M)\right)\sin(\theta)\left(u_{\,\phi}u_{\,r} - uu_{\,r\phi}\right) \\
\Phi_3^\theta =\; & \tfrac{1}{2}\left(-\cos(\theta)uu_{\,\phi} - \sin(\theta)u_{\,\theta}u_{\,\phi} + u\left(\sin u_\phi + \sin(\theta)u_{\,\theta\phi}\right)\right) \\
\Phi_3^\phi =\; & \tfrac{1}{2(k^2+r(r-2M))\sin(\theta)}[2kMr\sin(\theta)^2u_{\,\phi}u_{\,t} - u(2k^2\cos(\theta)^2 - (k^2+r(r- \\
& 2M))\sin(\theta)(k^2\sin(\theta)u_{\,tt} - u_{\,\theta})\cos(\theta) + 2r(r-2M) + \sin(\theta)^2(u_{\,rr}k^4 + 2r^2u_{\,rr}k^2
\end{aligned}
$$

$$-4Mru_{\,rr}k^2 - 2Mru_{\,t\phi}k + (k^2 + r(r-2M))u_{\,\theta\theta} - 2(M-r)(k^2 + r(r-2M))u_{\,r}$$
$$+r^4u_{\,rr} - 4Mr^3u_{\,rr} + 4M^2r^2u_{\,rr} - r(r^3 + k^2(2M+r))u_{tt}))]$$

$$\mathcal{Q}_4 = u_{tt\phi}$$

$$\Phi_4^t = \frac{1}{6(k^2+r(r-2M))\sin(\theta)}[-4k^2u_{\,t\phi}\cos(\theta)^2 + (k^2 + r(r-2M))\sin(\theta)(2\sin(\theta)u_{\,t\phi}u_{\,tt}k^2 +$$
$$2\sin(\theta)u_{\,t}u_{\,tt\phi}k^2 - \sin(\theta)u_{\,\phi}u_{\,ttt}k^2 - \sin(\theta)uu_{ttt\phi}k^2 + u_{\,\theta\phi}u_{\,t} - 2u_{\,\theta}u_{\,t\phi} + u_{\,\phi}u_{\,t\theta} -$$
$$2uu_{\,t\theta\phi})\cos(\theta) + 4(2M-r)ru_{\,t\phi} + \sin(\theta)^2(u_{\,rr\phi}u_{\,t}k^4 - 2u_{\,rr}u_{\,t\phi}k^4 + u_{\phi}u_{\,trr}k^4 -$$
$$2uu_{\,trr\phi}k^4 + 2r^2u_{\,rr\phi}u_{\,t}k^2 - 4Mru_{\,rr\phi}u_{\,t}k^2 - 2u_{\,\theta\theta}u_{\,t\phi}k^2 + 4Mu_{\,r}u_{\,t\phi}k^2 - 4ru_{\,r}u_{\,t\phi}k^2$$
$$-4r^2u_{\,rr}u_{\,t\phi}k^2 + 8Mru_{\,rr}u_{\,t\phi}k^2 + u_{\,\phi}u_{\,t\theta\theta}k^2 - 2uu_{\,t\theta\theta\phi}k^2 - 2Mu_{\,\phi}u_{\,tr}k^2 + 2ru_{\,\phi}u_{\,tr}k^2$$
$$+2r^2u_{\phi}u_{\,trr}k^2 - 4Mru_{\,\phi}u_{\,trr}k^2 - 4r^2uu_{\,trr\phi}k^2 + 8Mruu_{\,trr\phi}k^2 + 2r^2u_{\,t\phi}u_{\,tt}k^2$$
$$+2r^2u_{\,t}u_{\,tt\phi}k^2 + 4Mru_{\,t}u_{\,tt\phi}k^2 - r^2u_{\,\phi}u_{\,ttt}k^2 - 2Mru_{\,\phi}u_{\,ttt}k^2 - r^2uu_{\,ttt\phi}k^2$$
$$+8Mru_{\,t\phi}{}^2k - 4Mru_{\,t}u_{\,t\phi\phi}k + 2Mru_{\,\phi}u_{\,tt\phi}k + 2Mruu_{\,tt\phi\phi}k + (k^2 + r(r-$$
$$2M))u_{\,\theta\theta\phi}u_{\,t} - 2Mruu_{\,ttt\phi}k^2 - 2r^2u_{\,\theta\theta}u_{\,t\phi} - 6Mr^2u_{\,\phi}u_{\,tr} - r^4u_{\phi}u_{\,ttt}$$
$$-2(M-r)\left(k^2 + r^2 - 2Mr\right)u_{\,r\phi}u_{\,t} + r^4u_{\,rr\phi}u_{\,t} - 4Mr^3u_{\,rr\phi}u_{\,t} + 4M^2r^2u_{\,rr\phi}u_{\,t}$$
$$+4Mru_{\,\theta\theta}u_{t\phi} - 4r^3u_{\,r}u_{,\,t\phi} + 12Mr^2u_{\,r}u_{\,t\phi} - 8M^2ru_{\,r}u_{\,t\phi} - 2r^4u_{\,rr}u_{\,t\phi} + 8Mr^3u_{\,rr}u_{\,t\phi} -$$
$$8M^2r^2u_{\,rr}u_{\,t\phi} + r^2u_{\,\phi}u_{\,t\theta\theta} - 2Mru_{\,\phi}u_{\,t\theta\theta} - 2r^2uu_{\,t\theta\theta\phi} + 4Mruu_{t\theta\theta\phi} + 2r^3u_{\,\phi}u_{tr}$$
$$+4M^2ru_{\,\phi}u_{\,tr} - 4r^3uu_{\,tr\phi} + 12Mr^2uu_{\,tr\phi} - 8M^2ruu_{\,tr\phi} + r^4u_{\,\phi}u_{\,trr} - 4Mr^3u_{\,\phi}u_{\,trr} +$$
$$4M^2r^2u_{\,\phi}u_{\,trr} - 2r^4uu_{\,trr\phi} + 8Mr^3uu_{\,trr\phi} - 8M^2r^2uu_{\,trr\phi} + 2r^4u_{\,t\phi}u_{\,tt} + 2r^4u_{\,t}u_{\,tt\phi}$$
$$-r^4uu_{\,ttt\phi} + 4Muu_{\,tr\phi}k^2 - 4ruu_{\,tr\phi}k^2 + 4Mru_{\,t\phi}u_{\,tt}k^2)]$$

$$\Phi_4^r = -\tfrac{1}{2}\left(k^2 + r(r-2M)\right)\sin(\theta)\left(u_{\,r}u_{\,tt\phi} - uu_{\,ttr\phi}\right)$$

$$\Phi_4^\theta = \tfrac{1}{2}\left(-\cos(\theta)uu_{\,tt\phi} - \sin(\theta)u_{\,\theta}u_{\,tt\phi} + u\left(\sin u_{\,tt\phi} + \sin(\theta)u_{\,tt\theta\phi}\right)\right)$$

$$\Phi_4^\phi = \frac{1}{6(k^2+r(r-2M))\sin(\theta)}[-2k^2u_{,\,tt}\cos(\theta)^2 - \left(k^2 + r(r-2M)\right)\sin(\theta)(-\sin(\theta)u_{\,tt}{}^2k^2 -$$
$$\sin(\theta)uu_{\,tttt}k^2 + u_{\theta}u_{\,tt} + uu_{\,tt\theta} + u_{\,t}(k^2\sin(\theta)u_{\,ttt} - u_{\,t\theta}))\cos(\theta) + 2(2M-r)ru_{\,tt} +$$
$$\sin(\theta)^2(-u_{\,rr}u_{\,tt}k^4 - uu_{\,ttrr}k^4 + r^2u_{\,tt}{}^2k^2 + 2Mru_{\,tt}{}^2k^2 - u_{\,\theta\theta}u_{\,tt}k^2 + 2Mu_{\,r}u_{\,tt}k^2$$
$$-2ru_{\,r}u_{\,tt}k^2 - 2r^2u_{rr}u_{\,tt}k^2 + r^2uu_{\,tttt}k^2 - 2r^3u_{\,r}u_{\,tt} + 2Mruu_{tt\theta\theta}$$
$$+4Mru_{\,rr}u_{tt}k^2 - uu_{tt\theta\theta}k^2 + 2Muu_{\,ttr}k^2 - 2ruu_{,\,ttr}k^2 - 2r^2uu_{\,ttrr}k^2 + 4Mruu_{\,ttrr}k^2$$
$$+2Mruu_{\,tttt}k^2 + 4Mru_{\,t\phi}u_{tt}k - 2Mruu_{\,ttt\phi}k + r^4u_{\,tt}{}^2 - r^2u_{\,\theta\theta}u_{tt} + 2Mru_{\,\theta\theta}u_{\,tt}$$
$$+6Mr^2u_{\,r}u_{\,tt} - 4M^2ru_{\,r}u_{\,tt} - r^4u_{rr}u_{\,tt} + 4Mr^3u_{\,rr}u_{\,tt} - 4M^2r^2u_{\,rr}u_{\,tt} - r^2uu_{\,tt\theta\theta}$$
$$-2r^3uu_{\,ttr} + 6Mr^2uu_{\,ttr} - 4M^2ruu_{\,ttr} - r^4uu_{ttrr} + 4Mr^3uu_{,\,ttrr} - 4M^2r^2uu_{\,ttrr} + u_{\,t}(u_{\,trr}k^4$$
$$+2r^2u_{\,trr}k^2 - 4Mru_{\,trr}k^2 - r^2u_{\,ttt}k^2 - 2Mru_{\,ttt}k^2 + 2Mru_{\,tt\phi}k + \left(k^2 + r(r-2M)\right)u_{\,t\theta\theta} -$$
$$2(M-r)\left(k^2 + r^2 - 2Mr\right)u_{\,tr} + r^4u_{\,trr} - 4Mr^3u_{\,trr} + 4M^2r^2u_{\,trr} - r^4u_{\,ttt}) + r^4uu_{\,tttt})]$$

$$\mathcal{Q}_5 = u_{t\phi\phi}$$

$$\Phi_5^t = -\frac{1}{6(k^2+r(r-2M))\sin(\theta)}[u_{\,\phi\phi}(2k^2\cos(\theta)^2 - (k^2 + r(r-2M))\sin(\theta)(k^2\sin(\theta)u_{\,tt} - u_{\,\theta})\cos(\theta) +$$
$$2r(r-2M) + \sin(\theta)^2(u_{\,rr}k^4 + 2r^2u_{\,rr}k^2 - 4Mru_{\,rr}k^2 - r^2u_{\,tt}k^2 - 2Mru_{\,tt}k^2 - 4Mru_{\,t\phi}k +$$

$$
\begin{aligned}
& (k^2+r(r-2M))u_{\theta\theta} - 2(M-r)(k^2+r^2-2Mr)u_{r} + r^4 u_{rr} - 4Mr^3 u_{rr} + 4M^2r^2 u_{rr} \\
& -r^4 u_{tt})) + \sin(\theta)(\sin(\theta)(-3r(r^3+k^2(2M+r))u_t u_{t\phi\phi} - u_\phi(u_{rr\phi}k^4 + 2r^2 u_{rr\phi}k^2 - \\
& 4Mru_{rr\phi}k^2 - r^2u_{tt\phi}k^2 - 2Mru_{tt\phi}k^2 + 2Mru_{t\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi} - 2(M-r)(k^2 \\
& +r^2-2Mr)u_{r\phi} + r^4u_{rr\phi} - 4Mr^3u_{rr\phi} + 4M^2r^2u_{rr\phi} - r^4u_{tt\phi}) + u(u_{rr\phi\phi}k^4 + 2r^2u_{rr\phi\phi}k^2 \\
& -4Mru_{rr\phi\phi}k^2 + 2r^2u_{tt\phi\phi}k^2 + 4Mru_{tt\phi\phi}k^2 + 2Mru_{t\phi\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi\phi} - 2(M \\
& -r)(k^2+r^2-2Mr)u_{r\phi\phi} + r^4u_{rr\phi\phi} - 4Mr^3u_{rr\phi\phi} + 4M^2r^2u_{rr\phi\phi} + 2r^4u_{tt\phi\phi})) + (k^2+r(r \\
& -2M))\cos(\theta)(-3\sin(\theta)u_tu_{t\phi\phi}k^2 + u_\phi(k^2\sin(\theta)u_{tt\phi} - u_{\theta\phi}) + u(2\sin(\theta)u_{tt\phi\phi}k^2 + u_{\theta\phi\phi})))]
\end{aligned}
$$

$$
\begin{aligned}
\Phi_5^\phi = \; & \frac{1}{6(k^2+r(r-2M))\sin(\theta)}[-4k^2u_{t\phi}\cos(\theta)^2 - (k^2+r(r-2M))\sin\theta(-2\sin(\theta)u_{t\phi}u_{tt}k^2 + \sin(\theta)u_tu_{tt\phi}k^2 \\
& +\sin(\theta)u_\phi u_{ttt}k^2 - 2\sin(\theta)uu_{ttt\phi}k^2 - u_{\theta\phi}u_t + 2u_\theta u_{t\phi} - u_\phi u_{t\theta} + 2uu_{t\theta\phi})\cos(\theta)+ \\
& 4(2M-r)ru_{t\phi} + \sin(\theta)^2(u_{rr\phi}u_tk^4 - 2u_{rr}u_{t\phi}k^4 + u_\phi u_{trr}k^4 - 2uu_{trr\phi}k^4 + 2r^2u_{rr\phi}u_tk^2 - \\
& 4Mru_{rr\phi}u_tk^2 - 2u_{\theta\theta}u_{t\phi}k^2 + 4Mu_ru_{t\phi}k^2 - 4ru_ru_{t\phi}k^2 - 4r^2u_{rr}u_{t\phi}k^2 + 8Mru_{rr}u_{t\phi}k^2 + \\
& u_\phi u_{t\theta\theta}k^2 - 2uu_{t\theta\theta\phi}k^2 - 2Mu_\phi u_{tr}k^2 + 2ru_\phi u_{tr}k^2 + 4Muu_{tr\phi}k^2 - 4ruu_{tr\phi}k^2 + 2r^2u_\phi u_{trr}k^2 - \\
& 4Mru_\phi u_{trr}k^2 - 4r^2uu_{trr\phi}k^2 + 8Mruu_{trr\phi}k^2 + 2r^2u_{t\phi}u_{tt}k^2 + 4Mru_{t\phi}u_{tt}k^2 - r^2u_tu_{tt\phi}k^2 - \\
& 2Mru_tu_{tt\phi}k^2 - r^2u_\phi u_{ttt}k^2 - 2Mru_\phi u_{ttt}k^2 + 2r^2uu_{ttt\phi}k^2 + 4Mruu_{ttt\phi}k^2 + 8Mru_{t\phi}{}^2k + \\
& 2Mru_tu_{t\phi\phi}k - 4Mru_\phi u_{tt\phi}k + 2Mruu_{tt\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi}u_t - 2(M-r)(k^2+r^2- \\
& 2Mr)u_{r\phi}u_t + r^4u_{rr\phi}u_t - 4Mr^3u_{rr\phi}u_t + 4M^2r^2u_{rr\phi}u_t - 2r^2u_{\theta\theta}u_{t\phi} + 4Mru_{\theta\theta}u_{t\phi} \\
& +12Mr^2u_ru_{t\phi} - 8M^2ru_ru_{t\phi} - 2r^4u_{rr}u_{t\phi} + 8Mr^3u_{rr}u_{t\phi} - 8M^2r^2u_{rr}u_{t\phi} + r^2u_\phi u_{t\theta\theta} - \\
& 2Mru_\phi u_{t\theta\theta} - 2r^2uu_{t\theta\theta\phi} + 4Mruu_{t\theta\theta\phi} + 2r^3u_\phi u_{tr} - 6Mr^2u_\phi u_{tr} + 4M^2ru_\phi u_{tr} - 4r^3uu_{tr\phi} + \\
& 12Mr^2uu_{tr\phi} - 8M^2ruu_{tr\phi} + r^4u_\phi u_{trr} - 4Mr^3u_\phi u_{trr} + 4M^2r^2u_\phi u_{trr} - 2r^4uu_{trr\phi} + \\
& 8Mr^3uu_{trr\phi} - 8M^2r^2uu_{trr\phi} + 2r^4u_{t\phi}u_{tt} - r^4u_tu_{tt\phi} - r^4u_\phi u_{ttt} + 2r^4uu_{ttt\phi} - 4r^3u_ru_{t\phi})]
\end{aligned}
$$

$$\Phi_5^r = -\frac{1}{2}\left(k^2+r(r-2M)\right)\sin(\theta)\left(u_ru_{t\phi\phi} - uu_{tr\phi\phi}\right)$$

$$\Phi_5^\theta = \frac{1}{2}\left(-\cos(\theta)uu_{t\phi\phi} - \sin(\theta)u_\theta u_{t\phi\phi} + u\left(\sin u_{t\phi\phi} + \sin(\theta)u_{t\theta\phi\phi}\right)\right)$$

$$\mathcal{Q}_6 = u_{\phi\phi\phi}$$

$$
\begin{aligned}
\Phi_6^t = \; & \frac{1}{2(k^2+r(r-2M))}[\sin(\theta)(2kMru_\phi u_{\phi\phi\phi} + ((k^2+r(r-2M))\cos(\theta)k^2 + r(r^3+k^2(2M+r)))u_tu_{\phi\phi\phi} - \\
& u(2kMru_{\phi\phi\phi\phi} + ((k^2+r(r-2M))\cos(\theta)k^2 + r(r^3+k^2(2M+r)))u_{t\phi\phi\phi}))]
\end{aligned}
$$

$$\Phi_6^r = -\frac{1}{2}\left(k^2+r(r-2M)\right)\sin(\theta)\left(u_{\phi\phi\phi}u_r - uu_{r\phi\phi\phi}\right)$$

$$\Phi_6^\theta = \frac{1}{2}\left(-\cos(\theta)uu_{\phi\phi\phi} - \sin(\theta)u_\theta u_{\phi\phi\phi} + u\left(\sin u_{\phi\phi\phi} + \sin(\theta)u_{\theta\phi\phi\phi}\right)\right)$$

$$
\begin{aligned}
\Phi_6^\phi = \; & \frac{1}{2(k^2+r(r-2M))\sin(\theta)}[\sin\theta(\sin\theta(2kMru_{\phi\phi\phi}u_t + u_\phi(u_{rr\phi}k^4 + 2r^2u_{rr\phi}k^2 - 4Mru_{rr\phi}k^2 - r^2u_{tt\phi}k^2 - \\
& 2Mru_{tt\phi}k^2 - 4Mru_{t\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi} - 2(M-r)(k^2+r^2-2Mr)u_{r\phi} + r^4u_{rr\phi} - \\
& 4Mr^3u_{rr\phi} + 4M^2r^2u_{rr\phi} - r^4u_{tt\phi}) - u(u_{rr\phi\phi}k^4 + 2r^2u_{rr\phi\phi}k^2 - 4Mru_{rr\phi\phi}k^2 - r^2u_{tt\phi\phi}k^2 - \\
& 2Mru_{tt\phi\phi}k^2 - 2Mru_{t\phi\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi\phi} - 2(M-r)(k^2+r^2-2Mr)u_{r\phi\phi} + \\
& r^4u_{rr\phi\phi} - 4Mr^3u_{rr\phi\phi} + 4M^2r^2u_{rr\phi\phi} - r^4u_{tt\phi\phi})) - (k^2+r(r-2M))\cos(\theta)(u_\phi(k^2\sin(\theta)u_{tt\phi} - \\
& u_{\theta\phi}) + u(u_{\theta\phi\phi} - k^2\sin(\theta)u_{tt\phi\phi}))) - u_{\phi\phi}(2k^2\cos(\theta)^2 - (k^2+r(r-2M))\sin(\theta)(k^2\sin(\theta)u_{tt} - \\
& u_\theta)\cos(\theta) + 2r(r-2M) + \sin(\theta)^2(u_{rr}k^4 + 2r^2u_{rr}k^2 - 4Mru_{rr}k^2 - r^2u_{tt}k^2 - 2Mru_{tt}k^2 - \\
& 4Mru_{t\phi}k + (k^2+r(r-2M))u_{\theta\theta} - 2(M-r)(k^2+r^2-2Mr)u_r + r^4u_{rr} - 4Mr^3u_{rr} + \\
& 4M^2r^2u_{rr} - r^4u_{tt}))]
\end{aligned}
$$

5.4 conclusion

In section 5.2, we classified and and reduced the underlying equations (wave equations and Klein-Goldon equations) on ASD-Einstein metrics, we showed how the process leads to exact solutions by quadratures. A variational technique has been applied, and we found Noether's symmetries for those equations. In the cases discused, we found "fifteen" or "seven" dimensional Noether symmetries. Interestingly, all of them contain a scaling symmetry. Not all of those symmetries lead to physical conservation laws, but they all lead to mathematical conservation laws by Noether's theorem, and are useful in application, for example, reducing the underlying equations. Conserved quantities of the Klein-Gordon equations and wave equations are constructed in Section 5.2 & 5.3. Finally, some higher order symmetries are presented with their associated conservation laws.

Chapter 6

Generalized symmetries for the heat equations in (3+1)-dimension

6.1 Introduction

Generally speaking, a geometric quantity or a structure on manifold is evolved in a canonical way towards an optimal one. Furthermore, nonlinear equations have played an important role in differential geometry and topology over the last decades, in particular, the heat heat equation. It is now well known that the evolution equation defining the Ricci flow is a nonlinear diffusion equation [55, 56, 57]. For example, in [58], Hamilton introduced the Ricci-flow which deforms an initial metric in the direction of its Ricci tensor and in [54], the authors used the heat kernel equation to show that entropy is non-decreasing and moreover convex if the metric evolves under a super Ricci flow. The objective in the present Chapter is to perform symmetry analysis and obtain the higher order generalized symmetries of some linear diffusion equations constructed on curved manifolds.

Generalized symmetries have proved to be of importance in mathematical physics. In fact the existence of infinite number of such symmetries is a characterizing property of integrability. A vast amount of work has been done in the literature on the subject. In [59] the author classified symmetries arising from a two-dimensional linear diffusion equation with a nonlinear source term and determine those source term that admit nontivial symmetries. In [60, 61], the authors perform a symmetry analysis and find solution to a semilinear reaction-diffusion equation in multi-dimensions. We pursue an investigation of the generalized symmetries and focus on evolutionary vector field (see [2] for more details) of the heat equations on (3+1)-dimensional Anti-Self-Duality (ASD) manifolds in neutral signature. The study of ASD manifolds can be found in [62, 63].

The work is organised as follows:

In section 2, we construct the heat equations using the Laplace-Beltrami operator [65] on some manifolds (scalar-flat Kähler , null Kähler , pseudo-hypercomplex) and find the underlying Lie point symmetry for each case. The complete Lie analysis on those manifold have been done in [64].

In section 3, we compute the generalized symmetries with focus on *evolutionary* vector field.We find basis which generates all second order generalized symmetries and show that we recover some generators of the corresponding Lie point symmetry.

The Laplace-Beltrami operator is defined by

$$\triangle = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j).$$

6.2 Lie point symmetry

The heat equation in curvilinear coordinates is given by

$$U_t = \frac{1}{2}\triangle U.$$

Scalar-flat structures

The metric is [64]

$$h = e^u(\mathrm{d}x^2 + \mathrm{d}y^2) - \mathrm{d}z^2$$

where u satisfies

$$(e^u)_{zz} - u_{xx} - u_{yy} = 0$$

the heat equation is given by

$$U_t = \frac{y^2}{2z^2}U_{xx} - \frac{y}{z^2}U_y + \frac{y^2}{2z^2}U_{yy} - \frac{2}{z}U_z - \frac{1}{2}U_{zz}$$

for the solution $u = \ln\frac{z^2}{y^2}$ and the Lie algebra is spanned by

$$
\begin{aligned}
\mathcal{X}_1 = &\quad \partial_t \\
\mathcal{X}_2 = &\quad U\partial_U \\
\mathcal{X}_3 = &\quad F_5(x,y,z,t)\partial_U \\
\mathcal{X}_4 = &\quad \partial_x \\
\mathcal{X}_5 = &\quad x\partial_x + y\partial_y \\
\mathcal{X}_6 = &\quad 2Ux\partial_U + 2xy\partial_y + (x^2-y^2)\partial_x \\
X_7 = &\quad \tfrac{2Uy}{z}\partial_U + \tfrac{2xy}{z}\partial_x + \tfrac{-x^2+y^2}{z}\partial_y - \tfrac{x^2+y^2}{y}\partial_z \\
\mathcal{X}_8 = &\quad \tfrac{1}{y}\partial_z + \tfrac{1}{z}\partial_y \\
\mathcal{X}_9 = &\quad \tfrac{Uz}{y}\partial_U + \tfrac{t}{y}\partial_z + \tfrac{t}{z}\partial_y \\
\mathcal{X}_{10} = &\quad \tfrac{x}{y}\partial_z + \tfrac{x}{z}\partial_y - \tfrac{y}{z}\partial_x \\
\mathcal{X}_{11} = &\quad \tfrac{tx}{y}\partial_z + \tfrac{tx}{z}\partial_y - \tfrac{ty}{z}\partial_x + \tfrac{Uxz}{y}\partial_U \\
\mathcal{X}_{12} = &\quad \tfrac{2txy}{z}\partial_x + \tfrac{t(y^2-x^2)}{z}\partial_y - \tfrac{t(x^2+y^2)}{y}\partial_z + \left(\tfrac{2tUy}{z} - \tfrac{uz(x^2+y^2)}{y}\right)\partial_U \\
\mathcal{X}_{13} = &\quad 2t\partial_t + z\partial_z \\
\mathcal{X}_{14} = &\quad 2t^2\partial_t + 2tz\partial_z + u(-5t+z^2)\partial_U
\end{aligned}
\tag{6.2.1}
$$

where F_5 satisfies

$$2z^2F_{5,t} + 4zF_{5,z} + z^2F_{5,zz} + 2yF_{5,y} - y^2F_5 = 0$$

Null Kähler structures

The metric is given by

$$g = \mathrm{d}y^2 - \mathrm{d}x\mathrm{d}z - 4u\mathrm{d}z^2$$

where u satisfies the dKP equation (see [64])

$$(u_z - uu_x)_x = u_{yy}$$

- From the solution $u = -\frac{x}{z}$ the heat equation is

$$U_t = -\frac{U_x}{2z} - \frac{xU_{xx}}{2z} - \frac{U_{xz}}{2} + \frac{U_{yy}}{2}$$

The symmetry generators are

$$
\begin{aligned}
\mathcal{X}_1 &= \partial_t \\
\mathcal{X}_2 &= U\partial_U \\
\mathcal{X}_3 &= F_3(x,y,z,t)\partial_U \\
\mathcal{X}_4 &= z\partial_x \\
\mathcal{X}_5 &= tz\partial_x + Uz^2\partial_U \\
\mathcal{X}_6 &= \partial_y \\
\mathcal{X}_7 &= t\partial_y - Uy\partial_U \\
\mathcal{X}_8 &= 2t\partial_t + 2x\partial_x + y\partial_y \\
\mathcal{X}_9 &= yz\partial_x + z^2\partial_y \\
\mathcal{X}_{10} &= \left(\tfrac{1}{z} - z\right)\partial_z + x\left(1 + \tfrac{1}{z^2}\right)\partial_x \\
\mathcal{X}_{11} &= \tfrac{2Ux}{z}\partial_U - \tfrac{tx}{z^2}\partial_x + \tfrac{t}{z}\partial_z \\
\mathcal{X}_{12} &= 2t^2\partial_t + 2ty\partial_y + 3tx\partial_x + tz\partial_z - U(3t + y^2 - 2xz)\partial_U \\
\mathcal{X}_{13} &= \tfrac{2x}{z}\partial_y + \tfrac{xy}{z^2}\partial_x + \tfrac{y}{z}\partial_z \\
\mathcal{X}_{14} &= \left(z + \tfrac{1}{z}\right)\partial_z + x\left(-1 + \tfrac{1}{z^2}\right)\partial_x
\end{aligned}
$$

where F_3 satisfies

$$-2zF_{3,t} + zF_{3,yy} - F_{3,x} - zF_{3,xz} - xF_{3,xx} = 0$$

- From the second solution $u = \frac{y^2}{9z^2} - \frac{x}{3z}$ we got the heat equation

$$U_t = \frac{1}{18z^2}\left[(-3xz + y^2)U_{xx} - 9z^2U_{xz} + 9z^2U_{yy} - 3zU_x\right]$$

the Lie point symmetry generators are

$$
\begin{aligned}
\mathcal{X}_1 &= \partial_t \\
\mathcal{X}_2 &= U\partial_U \\
\mathcal{X}_3 &= F_4(x,y,z,t)\partial_U \\
\mathcal{X}_4 &= z^{1/3}\partial_x \\
\mathcal{X}_5 &= 2tz^{1/3}\partial_x + 3Uz^{4/3}\partial_U \\
\mathcal{X}_6 &= 2t\partial_t + 2x\partial_x + y\partial_y \\
\mathcal{X}_7 &= \tfrac{y}{z^{1/3}}\partial_x + 3z^{2/3}\partial_y \\
\mathcal{X}_8 &= \tfrac{y(3+2\ln(z))}{z^{1/3}}\partial_x + 6z^{2/3}\ln(z)\partial_y \\
\mathcal{X}_9 &= x\partial_x - z\partial_z
\end{aligned}
$$

Pseudo-hypercomplex structures

The metric is given [64]

$$g = (\mathrm{d}y + u\mathrm{d}z)^2 - 4(\mathrm{d}x + w\mathrm{d}z)\mathrm{d}z$$

where the functions u, w satisfy

$$u_z + w_y + uw_x - wu_x = 0, \quad u_y + w_x = 0$$

- The heat equation is given by

$$U_t = \frac{1}{2}\left[\frac{U_x}{2z} + \frac{xU_{xx}}{z} - \frac{yU_{xy}}{z} - U_{xz} + U_{yy}\right]$$

  for $u = \frac{-y}{z}, \quad w = \frac{x}{z}$

  The generators of the Lie point symmetry are

$$\begin{aligned}
\mathcal{X}_1 &= \partial_t \\
\mathcal{X}_2 &= u\partial_U \\
\mathcal{X}_3 &= F_1(x,y,z,t)\partial_U \\
\mathcal{X}_4 &= \tfrac{1}{z}\partial_x \\
\mathcal{X}_5 &= \tfrac{t}{z}\partial_x + 2u\ln(z)\partial_U \\
\mathcal{X}_6 &= 2t\partial_t + 2x\partial_x + y\partial_y \\
\mathcal{X}_7 &= \tfrac{y}{z^2}\partial_x + \left(\tfrac{-1}{z} + z\right)\partial_y \\
\mathcal{X}_8 &= -\tfrac{y}{z^2}\partial_x + \left(\tfrac{1}{z} + z\right)\partial_y \\
\mathcal{X}_9 &= x\partial_x - z\partial_z
\end{aligned}$$

  where $F_1(x,y,z,t)$ satisfies

$$-4zF_{1,t} + 2zF_{1,yy} + F_{1,x} - 2zF_{1,xz} - 2yF_{1,xy} + 2xF_{1,xx} = 0$$

- For the second solution $u = \frac{y}{z}, \quad w = -\frac{x}{z}$ the heat equation is

$$U_t = \frac{1}{2}\left[\frac{U_x}{2z} + \left(-\frac{x}{z} + \frac{y^2}{z^2}\right)U_{xx} + \frac{yU_{xy}}{z} - U_{xz} + U_{yy}\right]$$

  the basis of the Lie point symmetry is given by

$$\begin{aligned}
\mathcal{X}_1 &= \partial_t \\
\mathcal{X}_2 &= U\partial_U \\
\mathcal{X}_3 &= F_2(x,y,z,t)\partial_U \\
\mathcal{X}_4 &= z\partial_x \\
\mathcal{X}_5 &= tz\partial_x + uz^2\partial_U \\
\mathcal{X}_6 &= 2t\partial_t + 2x\partial_x + y\partial_y \\
\mathcal{X}_7 &= y\partial_x + z\partial_y \\
\mathcal{X}_8 &= 2z\ln(z)\partial_y + (y + 2y\ln(z))\partial_x \\
\mathcal{X}_9 &= x\partial_x - z\partial_z
\end{aligned}$$

where F_2 satisfies

$$-4z^2F_{2,t} + 2z^2F_{2,yy} + zF_{2,x} - 2z^2F_{2,xz} + 2yzF_{2,xy} + 2y^2F_{2,xx} - 2xzF_{2,xx} = 0$$

6.3 High-order symmetries scalar-flat structure

Among all generalized vector fields, we are interested in evolutionary vector field which are given by

$$V_{\mathcal{Q}} = \mathcal{Q}_\alpha[U]\partial_U \tag{6.3.1}$$

where [U] is includes the basis, fibre and jet variables that the characteristic $\mathcal{Q}$ depends on.

6.3.1 Definition. A *recursion* operator for a system of DEs Δ is a linear operator $\mathcal{R}$ such that whenever V_Q is an evolutionary symmetry of Δ, so is $V_{\tilde{Q}}$ with $\tilde{Q} = \mathcal{R}Q$

Let first compute the basis of all second order generalized symmetry of the diffusion equation in 1-dimension Euclidean space

$$U_t = U_{xx}$$

with characteristic $Q = Q(x, t, u, u_x, u_{xx})$

The invariance condition

$$D_tQ = D_x^2Q,$$

gives us after easy calculations

$$\begin{aligned} Q = \;& U_{xx}(UC_1 - xC_1 + C_2 + tC_3 + t^2C_4) + \tfrac{1}{4}(2U_x(xC_3 + 2txC_4 + 2C_5 + 2tC_6) \\ & +U(2tC_4 + x^2C_4 + 2xC_6 + 4C_7)) + f(x,t) \end{aligned}$$

where f satisfies $f_t = f_{xx}$

We have a seven dimension basis of characteristics

$$\begin{aligned}
\mathcal{Q}_1 =& \quad U_{xx}(U-x)\\
\mathcal{Q}_2 =& \quad U_{xx}\\
\mathcal{Q}_3 =& \quad tU_{xx}+\tfrac{1}{2}xU_x\\
\mathcal{Q}_4 =& \quad t^2U_{xx}+txU_x+\tfrac{1}{4}(2t+x^2)U\\
\mathcal{Q}_5 =& \quad U_x\\
\mathcal{Q}_6 =& \quad tU_x+\tfrac{1}{2}xU\\
\mathcal{Q}_7 =& \quad U
\end{aligned}$$

plus the infinite number of characteristic $Q_f = f(x,t)$. We are able to recover all second order generalised symmetries computed in [2] using recursion operators.

Secondly, we need to compute the basis of all the second order generalized symmetry of the heat equation on scalar-flat manifold

$$U_t = \frac{y^2}{2z^2}U_{xx} - \frac{y}{2z^2}U_y + \frac{y^2}{2z^2}U_{yy} - \frac{2}{z}U_z - \frac{1}{2}U_{zz}$$

The characteristic is given by

$$\mathcal{Q} = \mathcal{Q}(x,y,t,z,U,U_x,U_y,U_z,U_{xx},U_{xy},U_{xz},U_{yz},U_{yy},U_{zz})$$

we don't include U_t or second order derivatives which involve t to avoid dependencies, wherever it appears it is replaced by $\frac{y^2}{2z^2}U_{xx} - \frac{y}{2z^2}U_y + \frac{y^2}{2z^2}U_{yy} - \frac{2}{z}U_z - \frac{1}{2}U_{zz}$

$$\begin{aligned}
U_t &= \tfrac{y^2}{2z^2}U_{xx} - \tfrac{y}{2z^2}U_y + \tfrac{y^2}{2z^2}U_{yy} - \tfrac{2}{z}U_z - \tfrac{1}{2}U_{zz}\\
U_{xt} &= \tfrac{y^2}{2z^2}U_{xxx} - \tfrac{y}{2z^2}U_{xy} + \tfrac{y^2}{2z^2}U_{xyy} - \tfrac{2}{z}U_z - \tfrac{1}{2}U_{xzz}\\
U_{yt} &= \tfrac{y}{z^2}U_{xx} - \tfrac{1}{2z^2}U_y + \tfrac{y}{z^2}U_{yy} + \tfrac{y^2}{2z^2}U_{yxx} - \tfrac{y}{2z^2}U_{yy} + \tfrac{y^2}{2z^2}U_{yyy} - \tfrac{2}{z}U_z - \tfrac{1}{2}U_{yzz}\\
&\ \vdots
\end{aligned} \tag{6.3.2}$$

Applying the prolongation formula on the vector field (6.3.1) we got the infinitesimal criterion

$$D_t\mathcal{Q} = \frac{y^2}{2z^2}D_x^2\mathcal{Q} - \frac{y}{2z^2}D_y\mathcal{Q} + \frac{y}{2z^2}D_y^2\mathcal{Q} - \frac{2}{z}D_z\mathcal{Q} - \frac{1}{2}D_z^2\mathcal{Q} \tag{6.3.3}$$

where

$$
\begin{aligned}
D_t = \;& \partial_t + U_t\partial_U + U_{xt}\partial_{U_x} + U_{yt}\partial_{U_y} + U_{zt}\partial_{U_z} + Uxxt\partial_{U_{xx}} + Uxyt\partial_{U_{xy}} + Uxzt\partial_{U_{xz}} + Uyzt\partial_{U_{yz}} \\
& + Uyyt\partial_{U_{yy}} + Uzzt\partial_{U_{zz}} \\
D_x = \;& \partial_t + U_x\partial_U + U_{xx}\partial_{U_x} + U_{yx}\partial_{U_y} + U_{zx}\partial_{U_z} + Uxxx\partial_{U_{xx}} + Uxyx\partial_{U_{xy}} + Uxzx\partial_{U_{xz}} + Uyzx\partial_{U_{yz}} \\
& + Uyyx\partial_{U_{yy}} + Uzzx\partial_{U_{zz}} \\
D_y = \;& \partial_t + U_y\partial_U + U_{xy}\partial_{U_x} + U_{yy}\partial_{U_y} + U_{zy}\partial_{U_z} + Uxxy\partial_{U_{xx}} + Uxyy\partial_{U_{xy}} + Uxzy\partial_{U_{xz}} + Uyzy\partial_{U_{yz}} \\
& + Uyyy\partial_{U_{yy}} + Uzzy\partial_{U_{zz}}
\end{aligned}
\tag{6.3.4}
$$

Replacing back (6.3.4) in (6.3.3) and after tedious calculations, we obtain the following fifty "basic" characteristic

$$
\begin{aligned}
\mathcal{Q}_1 =& \ \tfrac{1}{6z}[(U_{xx}+U_{yy})\,y^3+\left((-U_{yz}z-5U_y)\,t+z^2U_y\right)y^2+\left(\left((-2\,xU_{x,z}+2\,U_z)\,z+(U_{xx}+U_{yy})\,x^2\right.\right.\\
&\left.\left.-6\,xU_x+6\,U\right)t-2\,z^2\,(-xU_x+U)\right)y+\left((U_{yz}z+U_y)\,t-z^2U_y\right)x^2]\\
\mathcal{Q}_2 =& \ \tfrac{1}{4z^2y^2}[\left(-U_{yy}y^6+(-4\,U_{xy}x+2\,U_{y,z}z+U_y)\,y^5+\left(-U_{zz}z^2+(4\,xU_{xz}-3\,U_z)\,z+2\,x^2\,(U_{yy}-\right.\right.\\
&\left.2\,U_{xx})\right)y^4+\left(4\,U_{xy}x^3+2\,x^2U_y\right)y^3+\left(-2\,U_{zz}x^2z^2+\left(4\,x^3U_{xz}-2\,x^2U_z\right)z-U_{yy}x^4\right)y^2-\\
&2\,\left(U_{yz}z-\tfrac{1}{2}U_y\right)x^4y-x^4z\,(zU_{zz}-U_z)\Big)\,t^2+4\,z^2\left(x^2+y^2\right)\left(-\tfrac{1}{2}\,U_yy^3+\left(-xU_x+\tfrac{1}{2}\,U_zz+U\right)y^2\right.\\
&\left.+\tfrac{1}{2}\,x^2yU_y+\tfrac{1}{2}x^2zU_z\right)t-U\,z^4\left(x^2+y^2\right)^2]\\
\mathcal{Q}_3 =& \ \tfrac{1}{2z^2y^2}[\left(\left(-U_{yy}y^2+(-2\,U_{y,z}z+U_y)\,y-U_{zz}z^2+U_zz\right)x^3+\left(3\,y^3U_{xy}+3\,y^2zU_{xz}\right)x^2+((U_{yy}-\right.\\
&\left.2\,U_{xx})\,y^2+U_yy-U_{zz}z^2-U_zz\big)\,y^2x-U_{xy}y^5+U_{xz}y^4z\right)t^2+2\,z^2\left((U_yy+U_zz)\,x^3-\tfrac{3}{2}x^2y^2U_x\right.\\
&\left.+y^2\,(U_zz+U)\,x-\tfrac{1}{2}U_xy^4\right)t-U\,xz^4\left(x^2+y^2\right)]\\
\mathcal{Q}_4 =& \ \tfrac{1}{z^2y^2}[\left(U_{xx}y^4+(-2\,U_{xy}x-U_y)\,y^3+\left(U_{yy}x^2-2\,xzU_{xz}-U_zz\right)y^2+2\,x^2\left(U_{yz}z-\tfrac{1}{2}U_y\right)y+\right.\\
&\left.x^2z\,(zU_{zz}-U_z)\right)t^2+z^2\left((2\,xU_x+U)\,y^2-2\,x^2yU_y-2\,x^2zU_z\right)t+U\,z^4x^2\\
\mathcal{Q}_5 =& \ \tfrac{1}{2zy}[-y^4U_{yy}+(U_{yz}z-3U_{xy}x)\,y^3+\left(x^2\,(U_{yy}-2\,U_{xx})+(zU_{x,z}-2\,U_x)\,x+U_zz+2\,U\right)y^2+\\
&x^2\,(U_{xy}x+U_{yz}z+2\,U_y)\,y+x^2z\,(xU_{xz}+U_z)]\\
\mathcal{Q}_6 =& \ \tfrac{1}{2}\,x^4U_{xx}+\tfrac{1}{2}\,(4\,yU_{xy}-2\,U_x)\,x^3+\tfrac{1}{2}\left((4\,U_{yy}-2\,U_{xx})\,y^2-2\,U_yy+2\,U\right)x^2-\tfrac{1}{2}\left(4\,y^3U_{xy}+2\,U_xy^2\right)x\\
&+\tfrac{1}{2}U_{xx}y^4-U_yy^3+U\,y^2\\
\mathcal{Q}_7 =& \ \tfrac{1}{2}x^3U_{x,x}+\tfrac{1}{2}\,(3\,yU_{xy}-2\,U_x)\,x^2+\tfrac{1}{2}\left((2\,U_{yy}-U_{xx})\,y^2-2\,U_yy+2\,U\right)x-\tfrac{1}{2}y^3U_{xy}\\
\mathcal{Q}_8 =& \ U\\
\mathcal{Q}_9 =& \ \tfrac{1}{z^2y^2}[\left((4\,U_{xx}-U_{yy})\,y^4+(-10\,U_{xy}x-3\,U_y)\,y^3+\left(U_{zz}z^2+(-10\,xU_{xz}-U_z)\,z+5\,U_{yy}x^2\right)y^2+\right.\\
&\left.10\,x^2\,(U_{y,z}z-1/2\,U_y)\,y+5\,x^2z\,(zU_{zz}-U_z)\right)t^2-10\,\left((-xU_x+1/5\,U_zz)\,y^2+x^2yU_y+x^2zU_z\right)z^2t\\
&+5\,z^4\left(x^2+1/5\,y^2\right)U]\\
\mathcal{Q}_{10} =& \ \tfrac{1}{2z^2y^2}[\left(-U_{zz}xz^2+\left((-2\,yU_{yz}+U_z)\,x+U_{xz}y^2\right)z+\left((-yU_{yy}+U_y)\,x+U_{xy}y^2\right)y\right)t^2+\left(2\,xU_zz^3+\right.\\
&\left.\left(2\,xyU_y-U_xy^2\right)z^2\right)t-U\,xz^4]\\
\mathcal{Q}_{11} =& \ \tfrac{1}{2z^2y^2}[\left(-U_{zz}z^2+(-2\,yU_{yz}+U_z)\,z-U_{y,y}y^2+U_yy\right)t^2+\left(2\,yz^2U_y+2\,U_zz^3\right)t-z^4U]\\
\mathcal{Q}_{12} =& \ \tfrac{1}{12yz}[\left(tyU_{xy}+tzU_{xz}-z^2U_x\right)x^4+\left(-2\,t\,(U_{xx}-U_{yy})\,y^2+\left(2\,tzU_{yz}-2\,z^2U_y\right)y+2\,U\,z^2-\right.\\
&\left.2\,zU_zt\right)x^3-6\,tx^2y^3U_{xy}+2\,y^2\left(t\,(U_{xx}-U_{yy})\,y^2+\left(tzU_{yz}-z^2U_y\right)y+U\,z^2-zU_zt\right)x+\\
&y^4\left(tyU_{xy}-tzU_{xz}+z^2U_x\right)]\\
\mathcal{Q}_{13} =& \ \tfrac{1}{6yz}[t\,(U_{xx}+U_{yy})\,y^4+\left((-U_{yz}z-2\,U_y)\,t+z^2U_y\right)y^3+\left(\left((-2\,xU_{xz}-U_z)\,z+(U_{xx}+U_{yy})\,x^2\right)t+\right.\\
&\left.z^2\,(2xU_x+U)\right)y^2+x^2\left((U_{yz}z-2\,U_y)\,t-z^2U_y\right)y+3\,x^2z\,(U\,z-tU_z)]\\
\mathcal{Q}_{14} =& \ \tfrac{1}{3yz}[\left(t\,(U_{xx}+U_{yy})\,x-tzU_{xz}+z^2U_x\right)y^2+x\left(tzU_{yz}-z^2U_y-2\,tU_y\right)y+3\,xz\,(U\,z-tU_z)]\\
\mathcal{Q}_{15} =& \ \tfrac{1}{zy}[t\,(U_{xx}+U_{yy})\,y^2+\left(tzU_{yz}-z^2U_y\right)y+U\,z^2-zU_zt]\\
\mathcal{Q}_{16} =& \ \tfrac{1}{3z}[(((-U_{yy}-U_{xx})\,y-U_{yz}z-U_y)\,x+y\,(zU_{xz}+3\,U_x))\,t+z^2\,(xU_y-yU_x)]\\
\mathcal{Q}_{17} =& \ \tfrac{1}{2zy}[y^3U_{xy}+(-2\,xU_{yy}+zU_{xz}+2\,U_x)\,y^2+\left(-x^2U_{xy}-2\,xzU_{yz}\right)y-zx\,(xU_{xz}-2\,U_z)]
\end{aligned}
$$

$$\mathcal{Q}_{18} = -x^3U_{xx} + (-3\,yU_{xy} + U_x)\,x^2 + y^2\,(U_{xx} - 2\,U_{yy})\,x + y^2\,(yU_{xy} + U_x)$$

$$\mathcal{Q}_{19} = \frac{1}{2}\,(U_{xx} + U_{yy})\,y^2 + xU_x$$

$$\mathcal{Q}_{20} = U_x$$

$$\mathcal{Q}_{21} = \frac{1}{6zy}[ty^4U_{yy} + ((3\,U_{xy}x - U_{yz}z - U_y)\,t + z^2U_y)\,y^3 - x\,(((U_{yy} - 2\,U_{xx})\,x + zU_{xz})\,t - z^2U_x)\,y^2 - x^2\,((U_{xy}x + U_{yz}z + U_y)\,t - z^2U_y)\,y - x^3z\,(tU_{xz} - zU_x)]$$

$$\mathcal{Q}_{22} = \frac{1}{6zy}[3\,ty^3U_{x,y} + (((-2\,U_{y,y} + 4\,U_{xx})\,x - zU_{xz})\,t + z^2U_x)\,y^2 - 2\,x\,((U_{yz}z + 3/2\,U_{xy}x + U_y)\,t - z^2U_y)\,y - 3\,x^2z\,(tU_{xz} - zU_x)]$$

$$\mathcal{Q}_{23} = \frac{1}{zy}[((-yU_{xy} - zU_{xz})\,x + U_{xx}y^2)\,t + z^2xU_x]$$

$$\mathcal{Q}_{24} = \frac{1}{zy}[-tyU_{x,y} - tzU_{xz} + z^2U_x]$$

$$\mathcal{Q}_{25} = \frac{1}{z}[(U_{yz}z + U_y + (U_{xx} + U_{yy})\,y)\,t - z^2U_y]$$

$$\mathcal{Q}_{26} = \frac{1}{z^2y^2}[(-z^2U_{xx} - tU_{yy})\,y^6 + (-4\,U_{xy}\,(-z^2 + t)\,x + 2\,tzU_{yz} + z^2U_y + tU_y)\,y^5 + (((-4\,U_{yy} + 2\,U_{xx})\,z^2 + 2\,t\,(U_{yy} - 2\,U_{xx}))\,x^2 + 4\,tU_{xz}xz - tU_{zz}z^2 + U_zz^3 - 3\,zU_zt)\,y^4 + 4\,x^2\,(U_{xy}\,(-z^2 + t)\,x + \frac{1}{2}\,U_y\,(z^2 + t))\,y^3 - x^2\,((z^2U_{xx} + tU_{yy})\,x^2 - 4\,tU_{xz}xz + 2\,tU_{zz}z^2 - 2\,U_zz^3 + 2\,zU_zt)\,y^2 - 2\,x^4\,(tzU_{yz} - 1/2\,z^2U_y - \frac{1}{2}tU_y)\,y - x^4z\,(tzU_{zz} - z^2U_z - tU_z)]$$

$$\mathcal{Q}_{27} = \frac{1}{z^2y^2}[-U_{xy}\,(-z^2 + t)\,y^5 + (((U_{xx} - 2\,U_{yy})\,z^2 + t\,(U_{yy} - 2\,U_{xx}))\,x + tzU_{xz})\,y^4 + 3\,x\,(U_{xy}\,(-z^2 + t)\,x + \frac{1}{3}U_y\,(z^2 + t))\,y^3 + ((-z^2U_{x,x} - tU_{y,y})\,x^3 + 3\,tU_{xz}zx^2 + (-tU_{zz}z^2 + U_zz^3 - zU_zt)\,x)\,y^2 - 2\,x^3\,(tzU_{yz} - \frac{1}{2}z^2U_y - \frac{1}{2}tU_y)\,y - x^3z\,(tzU_{zz} - z^2U_z - tU_z)]$$

$$\mathcal{Q}_{28} = \frac{1}{2z^2}[(U_{zz}z^2 + 4\,U_zz - ((U_{xx} + U_{yy})\,y - 2\,U_y)\,y)\,t - U_zz^3]$$

$$\mathcal{Q}_{29} = \frac{1}{4z^2y^2}[(((U_{xx} + U_{yy})\,z^2 + t\,(U_{yy} - U_{xx}))\,y^4 + 4txy^3U_{xy} + (U_zz^3 - tU_{zz}z^2 + 4\,(xU_{xz} - \frac{1}{2}U_z)\,tz - 2\,tU_{yy}x^2)\,y^2 - 4\,x^2\,(tzU_{yz} - \frac{1}{2}z^2U_y - \frac{1}{2}tU_y)\,y - 2\,x^2z\,(tzU_{zz} - z^2U_z - tU_z)]$$

$$\mathcal{Q}_{30} = \frac{1}{z^2y^2}[((-U_{yy}y^2 + (-2\,U_{yz}z + U_y)\,y - U_{zz}z^2 + U_zz)\,x + y^3U_{xy} + zU_{xz}y^2)\,t + xz^2\,(U_yy + U_zz)]$$

$$\mathcal{Q}_{31} = \frac{1}{z^2y^2}[U_zz^3 + (-tU_{zz} + U_yy)\,z^2 - 2\,(yU_{yz} - \frac{1}{2}U_z)\,tz - yt\,(yU_{yy} - U_y)]$$

$$\mathcal{Q}_{32} = -\frac{1}{2}\,((U_{xx} + U_{yy})\,y - 2\,U_y)\,y$$

$$\mathcal{Q}_{33} = \frac{1}{zy}[-y^4U_{yy} + (-3\,U_{xy}x + U_{yz}z + U_y)\,y^3 + (x^2\,(U_{yy} - 2\,U_{xx}) + zU_{xz}x)\,y^2 + x^2\,(U_{xy}x + U_{yz}z + U_y)\,y + zx^3U_{xz}]$$

$$\mathcal{Q}_{34} = \frac{1}{2zy}[y^3U_{xy} + (-2\,xU_{yy} + zU_{xz})\,y^2 - x\,(U_{xy}x + 2\,U_{yz}z - 2\,U_y)\,y - zx\,(xU_{xz} - 4\,U_z)]$$

$$\mathcal{Q}_{35} = \frac{1}{zy}[U_yy + U_zz]$$

$$\mathcal{Q}_{36} = \frac{1}{z^2y^2}[-y^6U_{yy} + (-4\,U_{xy}x + 2\,U_{yz}z + U_y)\,y^5 + ((2\,U_{yy} - 4\,U_{xx})\,x^2 + 4\,zU_{xz}x - U_{zz}z^2 - 3\,U_zz)\,y^4 + (4\,U_{xy}x^3 + 2\,x^2U_y)\,y^3 - x^2\,(U_{yy}x^2 - 4\,zU_{xz}x + 2\,U_{zz}z^2 + 2U_zz)\,y^2 - 2\,x^4\,(U_{yz}z - \frac{1}{2}U_y)\,y - x^4z\,(zU_{zz} - U_z)]$$

$$\mathcal{Q}_{37} = \frac{1}{z^2y^2}[-y^5U_{xy} + ((U_{yy} - 2\,U_{xx})\,x + zU_{xz})\,y^4 + (3x^2U_{xy} + xU_y)\,y^3 - (U_{yy}x^2 - 3\,zU_{xz}x + z\,(zU_{zz} + U_z))\,xy^2 - 2\,x^3\,(U_{yz}z - \frac{1}{2}U_y)\,y - x^3z\,(zU_{zz} - U_z)]$$

$$\mathcal{Q}_{38} = \frac{1}{2z^2}[(-U_{yy} - U_{xx})\,y^2 + 2\,U_yy + z\,(zU_{zz} + 4\,U_z)]$$

$$
\begin{aligned}
\mathcal{Q}_{39} = &\ \tfrac{1}{4z^2y^2}[(U_{yy}-U_{xx})\,y^4+4\,xy^3U_{xy}+\left(-2\,U_{yy}x^2+4\,zU_{xz}x-U_{zz}z^2-2\,U_zz\right)y^2-4\,x^2\,(U_{yz}z-\\
&\ \tfrac{1}{2}U_y)\,y-2\,x^2z\,(zU_{zz}-U_z)]\\
\mathcal{Q}_{40} = &\ \tfrac{1}{z^2y^2}[y^3U_{x,y}+(-xU_{yy}+zU_{xz})\,y^2-2\,(U_{yz}z-1/2\,U_y)\,xy-zx\,(zU_{zz}-U_z)]\\
\mathcal{Q}_{41} = &\ \tfrac{1}{z^2y^2}[-U_{yy}y^2+(-2\,U_{yz}z+U_y)\,y-U_{zz}z^2+U_zz]\\
\mathcal{Q}_{42} = &\ \tfrac{1}{2}x^2U_{xx}+xyU_{xy}+\tfrac{1}{2}U_{yy}y^2\\
\mathcal{Q}_{43} = &\ U_{xx}x+yU_{xy}\\
\mathcal{Q}_{44} = &\ U_{xx}\\
\mathcal{Q}_{45} = &\ \tfrac{1}{2zy}[-y^5U_{xy}+((-2\,U_{xx}+2U_{yy})\,x+zU_{xz})\,y^4+\left(6\,x^2U_{xy}-2\,xzU_{yz}\right)y^3+2\,\left((U_{xx}-U_{yy})\,x^2\right.\\
&\ \left.+U_zz\right)xy^2+\left(-U_{xy}x^4-2\,x^3zU_{y,z}\right)y-x^3z\,(xU_{xz}-2\,U_z)]\\
\mathcal{Q}_{46} = &\ \tfrac{1}{zy}[(U_{yy}-U_{xx})\,y^4+(6\,U_{xy}x-U_{yz}z)\,y^3+\left((3\,U_{xx}-3\,U_{yy})\,x^2+U_zz\right)y^2+\left(-2\,U_{xy}x^3-\right.\\
&\ \left.3\,x^2zU_{yz}\right)y-2\,\left(xU_{xz}-\tfrac{3}{2}U_z\right)x^2z]\\
\mathcal{Q}_{47} = &\ \tfrac{1}{zy}[(yU_{xy}+zU_{xz})\,x^2+\left((U_{yy}-U_{xx})\,y^2+U_{yz}yz-U_zz\right)x-y^3U_{xy}]\\
\mathcal{Q}_{48} = &\ \tfrac{1}{zy}[(-U_{yy}-U_{xx})\,y^2-U_{yz}yz+U_zz]\\
\mathcal{Q}_{49} = &\ \tfrac{1}{zy}[(xU_{xz}+yU_{yz}-U_z)\,z+xyU_{xy}+U_{y,y}y^2]\\
\mathcal{Q}_{50} = &\ \tfrac{1}{zy}[yU_{xy}+zU_{xz}]
\end{aligned}
$$

plus the infinite familly of characteristics

$$\mathcal{Q}_f = f(x,y,z,t)$$

.

Some of these characteristics, $\mathcal{Q}_8, \mathcal{Q}_{20}, \mathcal{Q}_{28}, \mathcal{Q}_{35}, \mathcal{Q}_{38}$ and the characteristic $\mathcal{Q}_f$ correspond to the geometric symmetries(generators of the Lie point symmetry) computed in (6.2.1). For example,

- $\mathcal{Q}_{35}$ is equivalent to

  $$\tilde{\mathcal{Q}_8} = \frac{1}{z}U_y + \frac{1}{y}U_z$$

  which corresponds to the vector field

  $$\mathcal{X}_8 = \frac{1}{y}\partial_z + \frac{1}{z}\partial_y$$

- $\mathcal{Q}_{38}$ is equivalent to

  $$\tilde{\mathcal{Q}_1} = U_t$$

  which corresponds to the vector field

  $$\mathcal{X}_1 = \partial_t$$

The first case on null Kähler manifold leads to a characteristic which is linear constant coefficient of fifty three "basic" characteristics while the pseudo-hypercomplex case, we got 25 "basic" characteristics.

6.4 Conclusion

In this Chapter, we performed the symmetry analysis of (3+1)-dimensional heat equations on ASD manifolds. In the first part, the Lie point symmetry for each equation was determined. Then, we computed the generalized symmetries with focus on evolutionary vector fields. We showed that we can recover some of the geometric symmetries through these generalized symmetries.

Chapter 7

Conclusion

Symmetry analysis is a powerfull tool to construct solutions of DEs. In this study, a special emphasis was placed on PDEs and ODEs arising from some ASD structures.

In chapter 3, we studied ODEs (Euler-Lagrange equations) obtained from various ASD Ricci-flat metrics. These metrics depend on a function which satisfies the second heavenly equation, a second order non linear PDE. We used symmetry reduction to construct exact solutions of the second HE. Variational symmetries were constructed using Noether theorem or multiplier approach. We also investigated some metrics with ultra hyperbolic signature. It turns out that even if some of these metrics have a 10-dimensional algebra of killing vectors like a flat manifold , they are not equivalent to Minkowksi or de Sitter metrics.

In chapter 4, we investigated the invariance properties generated by some well-known metrics of neutral signatures. As the metrics depended on solutions of PDEs, we constructed exact solutions of the PDEs using Lie group methods. From the specific forms of the metrics, we determined the isometries and the variational symmetries of the underlying metrics and corresponding Euler–Lagrange equations for both (Einstein Weyl structures and the corresponding four di mension metric constructed using the Jones-Tod construction). We established relationships between the resultant Lie algebras, viz., the algebra of isometries are subalgebras of the algebras of variational symmetries. For illustration, we chose some cases for which we constructed some conservation laws via these symmetries or the "multiplier approach". The interesting result occurs in Section 4.3 where the Lagrangian obtained from the three dimension EW structure had more variational symmetries than its corresponding four dimension Lagrangian obtained by Jones-Toda construction.

In chapter 5, we studied the Klein-Goldon and wave equations on Kerr and ASD-Einstein spacetimes. For the purpose, we have implemented the covariant d'Alembertian operator. Using the Lie symmetry generators in Section 5.2, we classified and and reduced the underlying equations , we showed how the process leads to exact solutions by quadratures. A variational technique has been applied, and we found Noether's symmetries for those equations. In the cases discussed, we found "fifteen" or "seven" dimensional Noether symmetries for ASD-Einsten spacetime and "three" for Kerr spacetime. Not all of those symmetries lead to physical conservation laws, but they all lead to mathematical conservation laws by Noether's theorem, and are useful in application, for example, reducing the underlying equations. Conserved quantities of the Klein-Gordon equations and wave equations are constructed in Section 5.2 & 5.3. Lastly, some higher order symmetries and their associated conservation laws are presented

In chapter 6, we studied the heat equations on manifolds with neutral signatures. For the purpose, we implemented the Laplace-Beltrami operator. We were interested in generalized symmetries i.e the infinitesimals may contain the derivatives of the dependent variables. In the first Section of this chapter, the Lie point symmetry for each equation was determined. In second section, we computed the generalized symmetries with focus on evolutionary vector fields. We showed that we can recover some of the geometric symmetries through these generalized symmetries.

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