On identities of Euler type and partitions with initial repetitions

by

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

In this research, we explore partition functions related to identities of Euler type and partitions with initial repetitions.

On identities of Euler type, we mainly focus on identities due to P. A. MacMahon, G. E. Andrews and M. V. Subbarao. We give new bijections for partition theorems due to MacMahon and Andrews and provide a generalization of Subbarao's finitization theorem. For the original Euler's identity which gave rise to identities of Euler type, Andrews and M. Merca found a new combinatorial interpretation for the total number of even parts in all partitions of n into distinct parts. We genaralize their result and establish more variations with connections to some of the work of S. Fu and D. Tang.

By conjugating partitions into distinct parts, we obtain partitions without gaps. Andrews extended the notion of partitions without gaps to partitions with initial repetitions. We study partitions with initial repetitions and give several Legendre theorems. These theorems provide partition-theoretic interpretations of some well known q-series identities of Rogers-Ramanujan type. We further deduce parity formulas for partition functions associated with this class of partitions.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Expressing a positive integer as a sum of positive integers is the basis of the theory of integer partitions. Partitions were first considered by G.W. Leibniz [11] but significant discoveries were made in the eighteenth century when L. Euler proved numerous results. Many more mathematicians have since been fascinated by integer partitions, namely; S. Ramanujan, L. J. Rogers, P. A. MacMahon and G. E. Andrews, to name a few. Their contributions have propelled the advancement of integer partitions research.

A partition of a non-negative integer n is a sequence $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, $\lambda_i \in \mathbb{Z}_{\geq 1}$ $\forall i$ and $\sum_{i=1}^{r} \lambda_i = n$. The number n is called the weight of λ which is denoted by $|\lambda|$ and the summands λ_i are called parts. An alternative representation of partitions is $(\mu_1^{m_1}, \mu_2^{m_2}, \cdots, \mu_s^{m_s})$ with $\mu_1 > \mu_2 > \ldots > \mu_s$, m_i is the multiplicity of μ_i and $|\mu| = \sum_{i=1}^{s} \mu_i m_i$. For example, the partitions of 5 are $(5), (4, 1), (3, 2), (3, 1^2), (2^2, 1), (2, 1^3)$ and (1^5) which are 7 in total. To count all possible partitions we use generating functions where the generating function for the number of unrestricted partitions of n, denoted $p(n)$, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \text{ where } |q| < 1 \ . \tag{1.1}$$

Note that

$$\sum_{n=0}^{\infty} p(n)q^n = \left(1 + q^{1.1} + q^{2.1} + \cdots\right)\left(1 + q^{1.2} + q^{2.2} + \cdots\right)$$

$$\times \left(1+q^{1.3}+q^{2.3}+\cdots\right)\left(1+q^{1.4}+q^{2.4}+\cdots\right)\cdots.$$

The expansion $(1+q^j+q^{2j}+q^{3j}+\cdots)$ represents the appearance of a part j in a partition or its non-appearance (represented by 1). Now

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{1-q^1}\frac{1}{1-q^2}\frac{1}{1-q^3}\frac{1}{1-q^4}\cdots = \prod_{n=1}^{\infty}\frac{1}{1-q^n}.$$

When we place restrictions or conditions on parts or the occurrence of the parts, we call these restricted partitions. One example of such is the number of partitions on n into distinct parts. This number is denoted by $d(n)$. For instance, $d(5)=3$ and the $d(5)$-partitions are: $(5), (4,1)$ and $(3,2)$.

A partition can be graphically represented using a plot called the Ferrers graph. Each part in the partition $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ is represented by a string of dots, thus the Ferrers graph is a left-justified array of dots with λ_i dots in the i^{th} row. For example, the Ferrers graph of $(16, 12, 9, 8, 5, 3)$ a partition of 53 is given as follows:

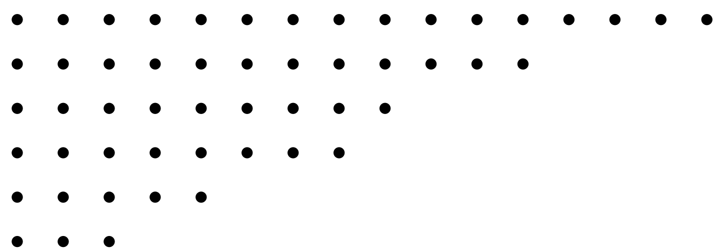

Figure 1.1: The Ferrers graph of $(16, 12, 9, 8, 5, 3)$.

Reading the parts as the number of dots in the columns of a Ferrers graph gives a new partition called the *conjugate* denoted by λ'. An alternative plot is a k-modular diagram. Instead of representing each part λ_i as a string of dots, λ_i is represented by its residues modulo k as a column. Each λ_i is expressed as $\lambda_i = |\lambda_i|_k.k + r$ where r is the least positive residue of λ_i modulo k and $|\lambda_i|_k$ is the largest multiple of k. For instance the 3-modular diagram of $(16, 12, 9, 8, 5, 3)$ is given in Figure 1.2.

We also consider a few algebraic operations on partitions. The union of two partitions λ and μ is simply the multiset union $\lambda \cup \mu$ where λ and μ are treated as multisets. The sum of two partitions $\lambda + \mu$ is the partition whose i^{th} part is equal to the sum of the i^{th} part of λ and the i^{th} part of μ. If the lengths are not equal we

$$
\begin{array}{cccccc}
1 & 3 & 3 & 2 & 2 & 3 \\
3 & 3 & 3 & 3 & 3 & \\
3 & 3 & 3 & 3 & & \\
3 & 3 & & & & \\
3 & & & & & \\
3 & & & & &
\end{array}
$$

Figure 1.2: The 3-modular diagram of $(16, 12, 9, 8, 5, 3)$

append zeros to the partition with smaller length. For example, let $\lambda = (10, 4^3, 2, 1^4)$ and $\mu = (13^2, 9^3, 6, 2^2, 1)$. Then

$$\lambda \cup \mu = (13^2, 10, 9^3, 6, 4^3, 2^3, 1^5)$$

and

$$\lambda + \mu = (23, 17, 13^2, 11, 7, 3^2, 2).$$

The conjugate of a partition λ' can also be computed algebraically as follows. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \cdots, \lambda_{r-1}^{m_{r-1}}, \lambda_r^{m_r})$. Then λ' is given by

$$\lambda' = \left(\left(\sum_{i=1}^{r} m_i \right)^{\lambda_r}, \left(\sum_{i=1}^{r-1} m_i \right)^{\lambda_{r-1} - \lambda_r}, \cdots, (m_1)^{\lambda_1 - \lambda_2} \right).$$

If a partition is equal to its conjugate, i.e. $\lambda' = \lambda$, the partition is called *self-conjugate* and the Ferrers graph is symmetric about the main diagonal. We further find that conjugation is an involution since it is bijective and $(\lambda')' = \lambda$.

1.1 Preliminaries

1.1.1 q-Series

We use the following notation. For $a, q \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$,

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) = \prod_{i=0}^{n-1}(1-aq^i)$$

where $(a;q)_0 = 1$. Also, for $|q| < 1$, we have

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.2}$$

We define a ${}_2\phi_1$ q-hypergeometric series by

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} z^n. \tag{1.3}$$

Using the q-series notation, the generating function for $p(n)$ is given as

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1.4}$$

The following result called the q-binomial theorem was obtained by A. Cauchy.

Theorem 1 (q-Binomial Theorem, [5]). *If $|q|, |t| < 1$, then*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n t^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n}. \tag{1.5}$$

It is a q-analog of the binomial theorem owing to the fact that if $a := q^\beta$ for $\beta \in \mathbb{Z}_{>0}$, then

$$1 + \sum_{k=1}^{\infty} \binom{\beta+k-1}{k} t^k = (1-t)^{-\beta}, \text{ as } q \to 1^-.$$

The proof of Theorem 1 mirrors that of the next corollary due to Euler.

Corollary 1 (Euler). *For $|t|, |q| < 1$,*

$$\sum_{n=0}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} = \prod_{n=0}^{\infty}(1+tq^n). \tag{1.6}$$

Proof. Let

$$(1+t)(1+tq)(1+tq^2)\cdots = \sum_{n=0}^{\infty} c_n t^n \tag{1.7}$$

where $c_n = c_n(q)$. Replacing t by tq yields

$$(1+tq)(1+tq^2)(1+tq^3)\cdots = \sum_{n=0}^{\infty} q^n c_n t^n. \tag{1.8}$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty} t^n c_n &= (1+t)\sum_{n=0}^{\infty} t^n q^n c_n \quad \text{(by (1.7) and (1.8))}\\ &= \sum_{n=0}^{\infty} t^n q^n c_n + \sum_{n=0}^{\infty} t^{n+1} q^n c_n\\ &= c_0 + \sum_{n=1}^{\infty} t^n q^n c_n + \sum_{n=1}^{\infty} t^n q^{n-1} c_{n-1}\\ &= c_0 + \sum_{n=1}^{\infty} t^n (q^n c_n + q^{n-1} c_{n-1}).\end{aligned} \tag{1.9}$$

Since c_0 is the constant term of the Maclaurin series (1.7), we have that $c_0 = 1$. Comparing coefficients of t^n for $n \geq 1$ in (1.9) yields

$$c_n = q^n c_n + q^{n-1} c_{n-1}$$

i.e.

$$c_n - q^n c_n = q^{n-1} c_{n-1}$$

so that

$$c_n = \frac{q^{n-1}}{(1-q^n)} c_{n-1}. \tag{1.10}$$

If we continue through with the iteration, c_n can be expressed in terms of q as follows

$$c_n = \frac{q^{\frac{(n^2-n)}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{1.11}$$

Hence

$$(1+t)(1+tq)(1+tq^2)\cdots = \sum_{n=0}^{\infty} \frac{t^n q^{\frac{(n^2-n)}{2}}}{(1-q)\cdots(1-q^n)}. \tag{1.12}$$

□

Cauchy also proved the following identity, see [5].

Proposition 1. *For* $|z|, |q| < 1$,

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n} z^n}{(q;q)_n (z;q)_n} = \prod_{n=0}^{\infty} (1 - zq^n)^{-1}. \tag{1.13}$$

J. Jacobi proved the following result which shall be useful in our work.

Theorem 2 (Jacobi's triple product identity, [5]). *For $t \neq 0$ and $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} t^n q^{n^2} = \prod_{n=0}^{\infty} \left(1 - q^{2n+2}\right)\left(1 + tq^{2n+1}\right)\left(1 + t^{-1}q^{2n+1}\right). \tag{1.14}$$

Proof. Note that Corollary 1 can be written as

$$(-t;q)_\infty = \sum_{n=0}^{\infty} \frac{t^n q^{\frac{(n^2-n)}{2}}}{(q;q)_n}. \tag{1.15}$$

Replacing q by q^2 and t by tq in (1.15) yields

$$(-tq;q^2)_\infty = \sum_{n=0}^{\infty} \frac{t^n q^{n^2}}{(q^2;q^2)_n},$$

i.e.

$$(1+tq)(1+tq^3)(1+tq^5)\cdots = \sum_{n=0}^{\infty} \frac{t^n q^{n^2}}{(q^2;q^2)_n}. \tag{1.16}$$

Suppose that $q \neq 0$ replacing t by tq^{-2k} in (1.16), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n q^{n^2-2kn}}{(q^2;q^2)_n} &= (1+tq^{1-2k})(1+tq^{3-2k})(1+tq^{5-2k})\cdots \\ &= (1+tq^{1-2k})(1+tq^{3-2k})\cdots(1+tq^{-1})(1+tq)\cdots.\end{aligned} \tag{1.17}$$

Multiplying (1.17) by $q^{k^2} = q^{(2k-1)+(2k-3)+\cdots+3+1}$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n q^{n^2-2kn+k^2}}{(q^2;q^2)_n} &= (t+q^{2k-1})(t+q^{2k-3})\cdots(t+q)(1+tq)(1+tq^3)\cdots \\ &= (t+q)(t+q^3)\cdots(t+q^{2k-1})(1+tq)(1+tq^3)\cdots.\end{aligned} \tag{1.18}$$

We now divide (1.18) by t^k to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^{n-k} q^{(n-k)^2}}{(q^2;q^2)_n} &= (1+t^{-1}q)(1+t^{-1}q^3)(1+t^{-1}q^5)\cdots \\ &\quad \cdots(1+t^{-1}q^{2k-1})(1+tq)(1+tq^3)(1+tq^5)\cdots\end{aligned} \tag{1.19}$$

Note that (1.19) can be written as

$$(-t^{-1}q;q^2)_n(-tq;q^2)_\infty = \sum_{n=-k}^{\infty} \frac{t^n q^{n^2}}{(q^2;q^2)_{n+k}} \tag{1.20}$$

and letting $k \to \infty$ for each fixed n,

$$(-t^{-1}q;q^2)_\infty(-tq;q^2)_\infty = \sum_{n=-\infty}^{\infty} \frac{t^n q^{n^2}}{(q^2;q^2)_\infty}$$

i.e.

$$(-t^{-1}q;q^2)_\infty(-tq;q^2)_\infty(q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} t^n q^{n^2}$$

which is Jacobi's triple product identity.

□

The following theorem is a special case of Theorem 2 and it is due to Gauss.

Corollary 2.

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} \frac{1-q^n}{1+q^n}. \tag{1.21}$$

Proof. Setting $t = -1$ in Theorem 2, we obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} &= (q;q^2)_\infty(q;q^2)_\infty(q^2;q^2)_\infty \\ &= \frac{(q;q^2)_\infty(q;q^2)_\infty(q;q)_\infty}{(q;q^2)_\infty} \\ &= (q;q^2)_\infty(q;q)_\infty \\ &= \frac{(q;q)_\infty}{(-q;q)_\infty}.\end{aligned}$$

□

Replacing q by $q^{\frac{3}{2}}$ and t by $-q^{-\frac{1}{2}}$ in Theorem 2 gives rise to the following theorem by Euler.

Theorem 3 (Euler's Pentagonal Number Theorem, [5])**.** *For* $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = \prod_{n=1}^{\infty} (1-q^n). \tag{1.22}$$

Proof. Recall Theorem 2:

$$(-t^{-1}q;q^2)_\infty(-tq;q^2)_\infty(q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} t^n q^{n^2}.$$

Replacing q by $q^{\frac{3}{2}}$ and then t by $-q^{\frac{1}{2}}$, we find

$$(q;q^2)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}},$$

i.e.

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}}.$$

□

Corollary 3. *For* $|q| < 1$,

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty}(1-q^n)^3. \tag{1.23}$$

Proof. Recall Theorem 2:

$$(-t^{-1}q;q^2)_\infty(-tq;q^2)_\infty(q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} t^n q^{n^2}.$$

Replacing q by $q^{\frac{1}{2}}$ and t by $-tq^{\frac{1}{2}}$, we obtain

$$(t^{-1};q)_\infty(tq;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n t^n q^{\frac{n^2+n}{2}}$$

so that

$$(1-t^{-1})(t^{-1}q;q)_\infty(tq;q)_\infty(q;q)_\infty = \sum_{n=0}^{\infty}(-1)^n\left(t^n - t^{-n-1}\right)q^{\frac{n^2+n}{2}}.$$

Multiplying by $t^{\frac{1}{2}}$ yields

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})(t^{-1}q;q)_\infty(tq;q)_\infty(q;q)_\infty = \sum_{n=0}^{\infty}(-1)^n\left(t^{n+\frac{1}{2}} - t^{-n-\frac{1}{2}}\right)q^{\frac{n^2+n}{2}}$$

Suppose $t \neq 1$ and divide by $t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ we obtain

$$\begin{aligned}(t^{-1}q;q)_\infty(tq;q)_\infty(q;q)_\infty &= 1 + \sum_{n=1}^{\infty}(-1)^n \left(\frac{t^{n+\frac{1}{2}} - t^{-n-\frac{1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right) q^{\frac{n^2+n}{2}} \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n \left(t^n + t^{n-1} + \cdots + t^{-n}\right) q^{\frac{n^2+n}{2}}\end{aligned}$$

if we let $t \to 1$, we obtain

$$(q;q)_\infty^3 = 1 + \sum_{n=1}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}.$$

□

1.1.2 Dissections of power series

Suppose we have a series,

$$F(q) = \sum_{n=0}^{\infty} f(n)q^n \tag{1.24}$$

and $m \in \mathbb{Z}_{>1}$. Then the m-dissection of $F(q)$ is an expression of the form of

$$F(q) = F_0(q) + F_1(q) + F_2(q) + \cdots + F_{m-1}(q), \tag{1.25}$$

where for each r with $0 \leq r \leq m-1$,

$$F_r(q) = \sum_{n=0}^{\infty} f(mn+r)q^{mn+r}. \tag{1.26}$$

Observe that

$$\begin{aligned}F(q) &= \sum_{r=0}^{m-1} F_r(q) \\ &= \sum_{r=0}^{m-1}\sum_{n=0}^{\infty} f(mn+r)q^{mn+r}.\end{aligned}$$

For instance, consider $m = 5$. The 5-dissection of $\sum_{n=0}^{\infty} a_n q^{\frac{n(n+1)}{2}}$ is given by

$$\sum_{n=0}^{\infty} a_n q^{\frac{n(n+1)}{2}} = \sum_{n=0}^{\infty} f(n)q^n = \sum_{r=0}^{4} f(5n+r)q^{5n+r}$$

where

$$f(n) = \begin{cases} a_j & n = \frac{j(j+1)}{2}; \\ o & otherwise. \end{cases}$$

So,

$$\sum_{n=0}^{\infty} a_n q^{\frac{n(n+1)}{2}} = K_0 + K_1 + K_3$$

where K_i consists of all terms in which the power of q is congruent to i modulo 5. Since $\frac{n(n+1)}{2} \equiv 0 \pmod 5 \iff n \equiv 0, 4 \pmod 5$, we have that

$$\begin{aligned} K_0 &= \sum_{\substack{n=0 \\ n\equiv 0,4 \pmod 5}}^{\infty} a_n q^{\frac{n(n+1)}{2}} \\ &= \sum_{n=0}^{\infty} a_{5n} q^{\frac{5n(5n+1)}{2}} + \sum_{n=0}^{\infty} a_{5n+4} q^{\frac{(5n+4)(5n+5)}{2}}. \end{aligned}$$

Similarly,

$$\begin{aligned} K_1 &= \sum_{\substack{n=0 \\ n\equiv 1,3 \pmod 5}}^{\infty} a_n q^{\frac{n(n+1)}{2}} \\ &= \sum_{n=0}^{\infty} a_{5n+1} q^{\frac{(5n+1)(5n+2)}{2}} + \sum_{n=0}^{\infty} a_{5n+3} q^{\frac{(5n+3)(5n+4)}{2}} \end{aligned}$$

and

$$\begin{aligned} K_3 &= \sum_{\substack{n=0 \\ n\equiv 2 \pmod 5}}^{\infty} a_n q^{\frac{n(n+1)}{2}} \\ &= \sum_{n=0}^{\infty} a_{5n+2} q^{\frac{(5n+2)(5n+3)}{2}}. \end{aligned}$$

1.1.3 Rogers-Ramanujan identities

We shall introduce the much celebrated Rogers-Ramanujan identities. Unaware of each other's work, L. J. Rogers in 1894 and S. Ramanujan in 1913 independently discovered the q-series identities. Ramanujan discovered Rogers' proof of the first identity and they jointly proved the second identity. These identities are called the first (rep. second) Rogers-Ramanujan identity. The identities are as follows:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.27}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.28}$$

We shall prove identities (1.27) and (1.28) using the concept of Bailey pairs due to W. N. Bailey.

Definition 1. *Suppose the sequences* $\{\alpha_k\}_{k=0}^{\infty}$ *and* $\{\beta_k\}_{k=0}^{\infty}$ *satisfy the condition*

$$\beta_k = \sum_{j=0}^{k} \frac{\alpha_j}{(q;q)_{k-j}(cq;q)_{k+j}}. \tag{1.29}$$

Then (α_k, β_k) *is called a Bailey pair.*

Theorem 4 (Bailey's lemma)**.** *Let* (α_k, β_k) *be a Bailey pair. Then (subject to convergence conditions)*

$$\sum_{k=0}^{\infty} q^{k^2} c^k \beta_k = \frac{1}{(cq;q)_\infty} \sum_{k=0}^{\infty} q^{k^2} c^k \alpha_k. \tag{1.30}$$

Proof. By Definition (1), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} q^{k^2} c^k \sum_{j=0}^{k} \frac{\alpha_j}{(q;q)_{k-j}(cq;q)_{k+j}} \\
&= \sum_{j=0}^{\infty} \alpha_j \sum_{k=j}^{\infty} \frac{q^{k^2} c^k}{(q;q)_{k-j}(cq;q)_{k+j}} \\
&= \sum_{j=0}^{\infty} q^{j^2} c^j \alpha_j \sum_{k=0}^{\infty} \frac{q^{k(k+2j)} c^k}{(q;q)_k (cq;q)_{k+2j}}.
\end{aligned}$$

For the inner sum, setting $z := cq^{2j}$ in the left-hand side of (1.13) yields

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{z^k q^{k^2}}{(q;q)_k (zq;q)_k} &= \sum_{k=0}^{\infty} \frac{q^{k^2+2kj} c^k (cq;q)_{2j}}{(q;q)_k (cq;q)_{k+2j}} \\
&= (cq;q)_{2j} \sum_{k=0}^{\infty} \frac{q^{k^2+2kj} c^k}{(q;q)_k (cq;q)_{k+2j}}.
\end{aligned}$$

And setting $z := cq^{2j}$ in the right-hand side of (1.13) yields

$$\frac{1}{(zq;q)_\infty} = \frac{1}{(cq^{2j+1};q)_\infty}.$$

By equating the left-hand side to the right-hand side, the result follows. Thus

$$\sum_{k=0}^{\infty} q^{k^2} c^k \sum_{j=0}^{k} \frac{\alpha_j}{(q;q)_{k-j}(cq;q)_{k+j}} = \sum_{j=0}^{\infty} q^{j^2} c^j \alpha_j \frac{1}{(cq;q)_\infty}.$$

□

Lemma 1. *If (α_k, β_k) is a Bailey pair, then*

$$\alpha_k = \frac{(1-cq^{2k})}{1-c} \sum_{j=0}^{k} \frac{(c;q)_{k+j}(-1)^{k-j} q^{\binom{k-j}{2}} \beta_j}{(q;q)_{k-j}}. \tag{1.31}$$

Proof. By (1.30), we have

$$\begin{aligned}
&\frac{(1-cq^{2k})}{1-c} \sum_{j=0}^{k} \frac{(c;q)_{k+j}(-1)^{k-j} q^{\binom{k-j}{2}} \beta_j}{(q;q)_{k-j}} \\
&= \frac{(1-cq^{2k})}{1-c} \sum_{j=0}^{k} \frac{(c;q)_{k+j}(-1)^{k-j} q^{\binom{k-j}{2}}}{(q;q)_{k-j}} \sum_{h=0}^{j} \frac{\alpha_h}{(q;q)_{j-h}(cq;q)_{j+h}} \\
&= \frac{(1-cq^{2k})}{1-c} \sum_{h=0}^{k} \alpha_h \sum_{j=h}^{k} \frac{(c;q)_{k+j}(-1)^{k-j} q^{\binom{k-j}{2}}}{(q;q)_{k-j}(q;q)_{j-h}(cq;q)_{j+h}} \\
&= \frac{(1-cq^{2k})}{1-c} \sum_{h=0}^{k} \alpha_h \sum_{j=0}^{k-h} \frac{(c;q)_{k+j+h}(-1)^{k-j-h} q^{\binom{k-j-h}{2}}}{(q;q)_{k-j-h}(q;q)_{j}(cq;q)_{j+2h}} \\
&= \frac{(1-cq^{2k})}{1-c} \sum_{h=0}^{k} \alpha_h \frac{(c;q)_{k+h}(-1)^{k-h} q^{\binom{k-h}{2}}}{(q;q)_{k-h}(cq;q)_{2h}} \\
&\qquad \times \sum_{j=0}^{k-h} \frac{(q^{k-h-j+1};q)_j (cq^{k+h};q)_j (-1)^j q^{-j(k-h)+\binom{j+1}{2}}}{(q;q)_j (cq^{2h+1};q)_j}.
\end{aligned}$$

Denoting the inner sum by $T(h,q)$, we have

$$\begin{aligned}
T(h,q) &= \sum_{j=0}^{k-h} \frac{(q^{k-h-j+1};q)_j (cq^{k+h};q)_j (-1)^j q^{-j(k-h)+\binom{j+1}{2}}}{(q;q)_j (cq^{2h+1};q)_j} \\
&= \sum_{j=0}^{k-h} \frac{(q^{-(k-h)};q)_j (cq^{k+h};q)_j q^j}{(q;q)_j (cq^{2h+1};q)_j} \\
&= {}_2\phi_1(q^{-(k-h)}, cq^{k+h}; cq^{2h+1}; q, q)
\end{aligned}$$

$$= \frac{(q^{-k+h+1};q)_{k-h}}{(q^{2h+1};q)_{k-h}}.$$

Note that

$$T(h,q) = \begin{cases} 1, & \text{if } h = k; \\ 0, & \text{if } h < k. \end{cases}$$

Thus,

$$\frac{(1-cq^{2k})}{1-c} \sum_{j=0}^{k} \frac{(c;q)_{k+j}(-1)^{k-j} q^{\binom{k-j}{2}} \beta_j}{(q;q)_{k-j}} = \frac{(1-cq^{2k})}{1-c} \alpha_k \frac{(c;q)_{2k}}{(cq;q)_{2k}} = \alpha_k,$$

which completes the proof. □

Consider $\beta_k = \frac{1}{(q;q)_k}$. To determine α_k, we have

$$\begin{aligned}
\alpha_k &= \frac{(1-cq^{2k})}{1-c} \sum_{j=0}^{k} \frac{(c;q)_{k+j}(-1)^{k-j} q^{\binom{k-j}{2}} \beta_j}{(q;q)_{k-j}} \\
&= \frac{(1-cq^{2k})}{1-c} \sum_{j=0}^{k} \frac{(c;q)_{k+j}(-1)^{k-j} q^{\binom{k-j}{2}} \beta_j}{(q;q)_{k-j}(q;q)_j} \\
&= \frac{(1-cq^{2k})}{1-c} \frac{(c;q)_k(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \sum_{j=0}^{k} \frac{(cq^k;q)_j(-1)^j q^{\left(\frac{-kj+j+1}{2}\right)} (q^{k-j+1};q)_j}{(q;q)_j} \\
&= \frac{(1-cq^{2k})}{1-c} \frac{(c;q)_k(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \sum_{j=0}^{k} \frac{(cq^k;q)_j(q^{-k};q)_j q^j}{(q;q)_j} \\
&= \frac{(1-cq^{2k})}{1-c} \frac{(c;q)_k(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \ {}_2\phi_1(q^{-k}, cq^k; 0; q, q) \\
&= \frac{(1-cq^{2k})}{1-c} \frac{(c;q)_k(-1)^k q^{\binom{k}{2}} c^k q^{k^2}}{(q;q)_k} \\
&= \frac{(1-cq^{2k})}{1-c} \frac{(c;q)_k(-1)^k q^{\frac{k(3k-1)}{2}} c^k}{(q;q)_k}.
\end{aligned}$$

We are now ready to prove the Rogers-Ramanujan identities.

For proof of the first identity (1.27), we proceed as follows:

By Theorem 4, with

$$\beta_n = \frac{1}{(q;q)_n} \quad \text{and} \quad \alpha_n = \frac{(1-cq^{2n})}{1-c}\frac{(c;q)_n(-1)^n q^{\frac{n(3n-1)}{2}} c^n}{(q;q)_n},$$

we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2}c^n}{(q;q)_n} = \frac{1}{(cq;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-cq^{2n})(c;q)_n(-1)^n c^{2n} q^{\frac{5n^2-n}{2}}}{(1-c)(q;q)_n}. \tag{1.32}$$

Setting $c=1$ gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n(1+q^n)q^{\frac{5n^2-n}{2}}\right) \\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}} \\
&= \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \\
&= \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}.
\end{aligned}$$

For proof of the second identity (1.28), we set $c=q$ in (1.32) so that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n(1-q^{2n+1})q^{\frac{5n^2+3n}{2}} \\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}} \\
&= \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \\
&= \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.
\end{aligned}$$

It is worth mentioning that several identities that are in conformity to the two Rogers-Ramanujan identities (in terms of appearance and structure) have been established. These identities are called identities of Rogers-Ramanujan type, or sometimes, simply Rogers-Ramanujan identities when there is no conflict of terminology. MacMahon gave a partition theoretic-interpretation of the two original Rogers-Ramanujan identities, and thus a discovery of more q-identities of this type necessitated the need to find equivalent combinatorial interpretations of partition-theoretic interest. One case in point is how Lucy J. Slater discovered more q-identities of Rogers-Ramanujan type and some of her identities were given interpretation by Andrews and Agarwal, just to mention a few. In our work, we shall also consider some of Slater's identities of Rogers-Ramanujan type.

1.1.4 Congruences

P. A MacMahon calculated $p(n)$ for $1 \leq n \leq 200$ which he displayed in a table with columns of five numbers.

1	7	42	176	$\cdots$
1	11	56	231	$\cdots$
2	15	77	297	$\cdots$
3	22	101	385	$\cdots$
5	30	135	490	$\cdots$

S. Ramanujan observed that each number in the bottom row is a multiple of 5, i.e. for all $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod{5}. \tag{1.33}$$

This subsequently led to further congruences:
For all $n \geq 0$,

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.34}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.35}$$

Before proving (1.33), (1.34) and (1.35), we recall the following:

Theorem 5. *Let $r \in \mathbb{N}$ and $k = p^r$ a prime power. Then*

$$(q;q)_n^k \equiv \left(q^k;q^k\right)_n \pmod{p}.$$

Proof. Observe that

$$(q;q)_n^k = \left(\prod_{i=1}^{n} \left(1-q^i\right)\right)^k = \prod_{i=1}^{n} \left(1-q^i\right)^k.$$

By the binomial theorem,

$$(q;q)_n^k = \prod_{i=1}^{n} \left(1-q^i\right)^k$$

$$
\begin{aligned}
&= \prod_{i=1}^{n}\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}q^{ij}\right)\\
&= \prod_{i=1}^{n}\left(1+\sum_{j=1}^{k-1}\binom{k}{j}(-1)^{k-j}q^{ij}+(-1)^{k}q^{ki}\right)\\
&\equiv \prod_{i=1}^{n}\left(1+(-1)^{k}q^{ki}\right) \pmod p
\end{aligned}
$$

since $\binom{k}{j} \equiv 0 \pmod p$ $\forall\, 1 \leq j \leq k-1$. We will now consider two cases.
Case I: $p = 2$

$$(-1)^k = 1 \equiv \pm 1 \pmod 2.$$

Case II: p is odd

$$(-1)^k = -1.$$

Thus $(-1)^k \equiv -1 \pmod p$ and

$$
\begin{aligned}
\prod_{i=1}^{n}\left(1+(-1)^{k}q^{ki}\right) &\equiv \prod_{i=1}^{n}\left(1-q^{ki}\right) \pmod p\\
&= \left(q^k;q^k\right)_n.
\end{aligned}
$$

Therefore, $(q;q)_n^{p^r} \equiv \left(q^{p^r};q^{p^r}\right)_n \pmod p$. □

Proof of (1.33). In Jacobi's result, Theorem 2, note that $\frac{n(n+1)}{2} \equiv 0, 1, 3 \pmod 5$, $\frac{n(n+1)}{2} \equiv 3 \pmod 5$ if and only if $n \equiv 2 \pmod 5$ and the coefficient $2n+1 \equiv 0 \pmod 5$. Thus

$$
\begin{aligned}
&(q;q)_\infty^3\\
&= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}\\
&= 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} - 15q^{28}\\
&\quad + 17q^{36} - 19q^{45} + 21q^{55} - \cdots\\
&\equiv 1 - 3q + 3q^6 - q^{10} - q^{15} + 3q^{21} - 3q^{36} + q^{45} + q^{55} - \cdots \pmod 5\\
&= (1 - q^{10} - q^{15} + q^{45} + \cdots) - 3q(1 - q^5 - q^{20} + q^{35} + q^{65} - \cdots)\\
&= J_0 + J_1
\end{aligned}
$$

where J_i contains terms wherein the power of q is congruent to $i \pmod 5$. Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q;q)_\infty} \\
&= \frac{(q;q)_\infty^9}{(q;q)_\infty^{10}} \\
&= \frac{((q;q)_\infty^3)^3}{((q;q)_\infty^5)^2} \\
&\equiv \frac{(J_0+J_1)^3}{(q^5;q^5)_\infty^2} \pmod 5 \\
&= \frac{J_0^3+3J_0^2J_1+3J_0J_1^2+J_1^3}{(q^5;q^5)_\infty^2}.
\end{aligned}$$

Exponents of q on the right hand side are not congruent to $4 \pmod 5$ and so (1.33) follows. □

The proof of (1.34) follows similarly to that of (1.33).

Proof of (1.34). In Theorem 2, note that $\frac{n(n+1)}{2} \equiv 0,1,3,6 \pmod 7$, $\frac{n(n+1)}{2} \equiv 6 \pmod 7$ if and only if $n \equiv 3 \pmod 7$ and the coefficient $2n+1 \equiv 0 \pmod 7$. We find that

$$\begin{aligned}
&(q;q)_\infty^3 \\
&= \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}} \\
&= 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} - 15q^{28} \\
&\quad + 17q^{36} - 19q^{45} + 21q^{55} - \cdots \\
&\equiv 1 - 3q + 5q^3 - 5q^{10} - 3q^{15} - q^{21} - q^{28} + 3q^{36} - 5q^{45} + \cdots \pmod 7 \\
&= (1 - q^{21} - q^{28} + \cdots) - 3q(1 - q^{14} - q^{35} + \cdots) \\
&\quad + 5q^3(1 - q^7 - q^{42} + \cdots) \\
&= J_0 + J_1 + J_3
\end{aligned}$$

where J_i contains terms wherein the power of q is congruent to $i \pmod 7$. Thus

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

$$
\begin{aligned}
&= \frac{(q;q)_\infty^6}{(q;q)_\infty^7} \\
&= \frac{\left((q;q)_\infty^3\right)^2}{(q;q)_\infty^7} \\
&\equiv \frac{(J_0+J_1+J_3)^2}{(q^7;q^7)_\infty} \pmod 7 \\
&= \frac{J_0^1+2J_0J_1+J_1^2+2J_0J_3+2J_1J_3+J_3^2}{(q^7;q^7)_\infty}.
\end{aligned}
$$

Exponents of q on the right hand side are not congruent to $5 \pmod 7$ and so (1.34) follows. □

Proof of (1.35). In Theorem 2, note that $\frac{n(n+1)}{2} \equiv 0,1,3,4,6,10 \pmod{11}$, $\frac{n(n+1)}{2} \equiv 4 \pmod{11}$ if and only if $n \equiv 5 \pmod{11}$ and the coefficient $2n+1 \equiv 0 \pmod{11}$. Expanding Corollary 3 and grouping coefficients modulo 11 as done in previous proofs, we obtain

$$(q;q)_\infty^3 \equiv J_0 + J_1 + J_3 + J_6 + J_{10} \pmod{11}$$

where J_i contains terms wherein the exponent of q is $i \pmod{11}$. Thus we have,

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{(q;q)_\infty} \\
&= \frac{(q;q)_\infty^{21}}{(q;q)_\infty^{22}} \\
&= \frac{\left((q;q)_\infty^3\right)^7}{\left((q;q)_\infty^{11}\right)^2} \\
&\equiv \frac{(J_0+J_1+J_3+J_6+J_{10})^7}{(q^{11};q^{11})_\infty^2} \pmod{11}.
\end{aligned}
$$

If we extract those terms in which the exponent on q is $6 \pmod{11}$, we find that

$$\sum_{n=0}^{\infty} p(11n+6)q^{11n+6} \equiv \frac{P}{(q^{11};q^{11})_\infty^2} \pmod{11} \tag{1.36}$$

where

$$
\begin{aligned}
P \equiv\ & 7J_0^6J_6 + 10J_0^5J_3^2 + J_0^4J_1J_6J_{10} + 8J_0^3J_1^3J_3 + 2J_0^3J_1J_3^2J_{10} \\
& + 8J_0^3J_6^3J_{10} + 3J_0^2J_1^2J_3J_6^2 + 3J_0^2J_1^2J_6J_{10}^2 + 3J_0^2J_3^2J_6^2J10 \\
& + 2J_0^2J_3J_6J_{10}^3 + 10J_0^2J_{10}^5 + 7J_0J_1^6 + J_0J_1^4J_3J_{10} + 2J_0J_1^2J_3^3J_6 \\
& + 3J_0J_1^2J_3^2J_{10}^2 + J_0J_1J_3J_6^4 + 2J_0J_1J_6^3J_{10}^2 + J_0J_3^4J_6J_{10} \\
& + 8J_0J_3^3J_{10}^3 + 10J_1^5J_6^2 + 2J_1^3J_3J_6^2J_{10} + 8J_1^3J_6J_{10}^3 + 10J_1^2J_3^5 \\
& + 8J_1J_3^3J_6^3 + 3J_1J_3^2J_6^2J_{10}^2 + J_1J_3J_6J_{10}^4 + 7J_1J_{10}^6 + 7J_3^6J_{10} \\
& + 7J_3J_6^6 + 10J_6^5J_{10}^2.
\end{aligned}
$$

It turns out that $P \equiv 0 \pmod{11}$ (see [17]). $\square$

1.1.5 Multivariable generating functions

Sometimes we not only want to study the total number of partitions of a given weight but also track the number of parts. In this case, we introduce a second variable z to track the number of parts. For instance, the generating function for the number of partitions into distinct parts of a given weight and number of parts is

$$F(z,q) = \prod_{n=1}^{\infty}(1 + zq^n).$$

Setting $z = 1$ yields the generating function for $d(n)$. Upon expanding $F(z,q)$, we can arrange the terms as

$$F(z,q) = \left(za_1 + z^3a_3 + z^5a_5 + \cdots\right) + \left(a_0 + z^2a_2 + z^4a_4 + \cdots\right)$$

where we have separated the terms according to parity of the exponents of z. Further, a_i's are functions of q. Let

$$F_1(z) = za_1 + z^3a_3 + z^5a_5 + \cdots$$

and

$$F_2(z) = a_0 + z^2a_2 + z^4a_4 + \cdots$$

so that

$$F(z,q) = F_1(z) + F_2(z). \tag{1.37}$$

We interpret $F_1(1)$ and $F_2(1)$ to be the generating functions for the number of $d(n)$-partitions in which the number of parts is odd and even, respectively. In order to find $F_1(1)$ and $F_2(1)$, we take advantage of the behaviour of $F(z,q)$ at $z=-1$. Now

$$F_1(z)_{|z=-1} = F_1(-1) = -F_1(1)$$

since the exponents of z are odd. Similarly,

$$F_2(z)_{|z=-1} = F_2(-1) = F_2(1).$$

Thus

$$F_1(-1) + F_1(1) = 0, \tag{1.38}$$

$$F_2(-1) - F_2(1) = 0. \tag{1.39}$$

In (1.37), we have

$$F_1(-1) + F_2(-1) = F(-1,q), \tag{1.40}$$

$$F_1(1) + F_2(1) = F(1,q). \tag{1.41}$$

The idea is to express $F_1(1)$ and $F_2(1)$ in terms of $F(1,q)$ or $F(-1,q)$ by using (1.38)-(1.41). We have

$$F_1(1) = \frac{1}{2}\left[F(1,q) - F(-1,q)\right]$$

and

$$F_2(1) = \frac{1}{2}\left[F(1,q) + F(-1,q)\right].$$

Let $d^e(n)$ (resp. $d^o(n)$) be the number of $d(n)$-partitions of n into an even (resp. odd) number of parts. From $F_1(1)$ and $F_2(n)$, we can construct the generating function for $d^o(n) - d^e(n)$. Clearly,

$$\begin{aligned}\sum_{n=0}^{\infty}\left(d^o(n) - d^e(n)\right)q^n &= F_1(1) - F_2(1)\\ &= \frac{1}{2}\left[F(1,q) - F(-1,q)\right] - \frac{1}{2}\left[F(1,q) + F(-1,q)\right]\\ &= -F(-1,q)\\ &= -\prod_{n=1}^{\infty}(1-q^n)\end{aligned}$$

so that

$$\sum_{n=1}^{\infty}(d^e(n) - d^o(n))q^n = \prod_{n=1}^{\infty}(1 - q^n).$$

In fact this partition-theoretic interpretation of $\prod_{n=1}^{\infty}(1 - q^n)$ was given by A. M. Legendre [5] and we record his theorem below.

Theorem 6 (Legendre,[5])**.** *For all $n \geq 0$,*

$$d^e(n) - d^o(n) = \begin{cases} (-1)^j, & \text{if } n = \frac{j(3j\pm 1)}{2}\ j \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.42}$$

Any partition theorem which conforms to the above theorem is referred to as a Legendre theorem or an identity of Euler-pentagonal type.

We can further place a restriction on the parts we would like to track. For example, let $a(n)$ denote the total number of ones in all partitions of n and denote $a_{n,m}$ be the number of partitions of n having m ones. First, observe that

$$\begin{aligned} a(n) &= \sum_{m=0}^{\infty} m a_{n,m} \\ &= \sum_{m=1}^{\infty} m a_{n,m}. \end{aligned}$$

The generating function for $a_{n,m}$ is

$$F(z, q) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m} z^m q^n = \frac{1}{1 - zq}\prod_{n=2}^{\infty}\frac{1}{1 - q^n}$$

which implies

$$\frac{\partial}{\partial z}F(z, q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} m a_{n,m} z^{m-1} q^n$$

so that

$$\begin{aligned} \frac{\partial}{\partial z}F(z, q)_{|_{z=1}} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} m a_{n,m} q^n \\ &= \sum_{n=0}^{\infty} a(n) q^n. \end{aligned}$$

But

$$\frac{\partial}{\partial z}F(z,q) = \frac{\partial}{\partial z}\left(\frac{1}{1-zq}\prod_{n=2}^{\infty}\frac{1}{1-q^n}\right)$$
$$= \frac{q}{(1-zq)^2}\prod_{n=2}^{\infty}\frac{1}{1-q^n}$$

which implies

$$\frac{\partial}{\partial z}F(z,q)_{|z=1} = \frac{q}{(1-q)^2}\prod_{n=2}^{\infty}\frac{1}{1-q^n}.$$

Hence,

$$\sum_{n=0}^{\infty}a(n)q^n = \frac{q}{(1-q)^2}\prod_{n=2}^{\infty}\frac{1}{1-q^n}.$$

In chapter 2, we will look at the work done that has motivated our research: starting with what G. Andrews, P. A. MacMahon and M. V. Subbarao have done on identities of Euler type and then discussing partitions with initial repetitions, originally introduced by G. Andrews. More specifically, in Chapter 3, we present our results on identities of Euler type. These results range from new bijections to new generalizations of well known partition theorems. Work in this chapter has been published and appears as two journal articles. In Chapter 4, we generalize some work of Andrews and Merca on partitions that focus on the total number of certain parts in partitions related to identities of Euler type. Chapter 5 presents new results on partitions with initial repetitions; ranging from Legendre theorems to parity formulas. Among the results in this chapter, of particular interest is a Legendre theorem on Andrews' partitions with initial 2-repetitions. For some additional studies in partition theory, see [1], [2], [23], [22], [24], [25].

Chapter 2

Background

In this chapter, we present relevant literature on identities of Euler type and partitions with initial repetitions.

2.1 Partition identities of Euler type

In 1748 L. Euler proved the following famous result - a result that is important in this thesis.

Theorem 7 (Euler, [5]). *For all n, the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

Notice that partitions into distinct parts have a restriction on the number of times a part can occur and partitions into odd parts have a restriction on the set the part belongs to. Partition identities of this form are referred to as identities of Euler type. A beautiful bijective proof and its generalization were given by J. W. L. Glaisher.

Theorem 8 (Glaisher, [14]). *Let k be a positive integer. For all n, the set of partitions of n in wherein parts appear at most $k-1$ times is equinumerous with the set of partitions of n in which parts are not multiples of k.*

Proof. The generating function for the number of partitions of n where parts occur at most $k-1$ times is

$$\prod_{n=1}^{\infty}\left(1+q^{n}+q^{2n}+q^{3n}+\cdots+q^{(k-1)n}\right).$$

We note that

$$\prod_{n=1}^{\infty}\left(1+q^n+q^{2n}+q^{3n}+\cdots+q^{(k-1)n}\right)=\prod_{n=1}^{\infty}\frac{1-q^{kn}}{1-q^n}$$
$$=\prod_{n=1}^{\infty}\prod_{i=0}^{k-1}\frac{1-q^{kn}}{1-q^{kn-i}}$$
$$=\prod_{n=1}^{\infty}\prod_{i=1}^{k-1}\frac{1}{1-q^{kn-i}}$$

which is the generating function for the cardinality of the set of partitions of n in which parts are not multiples of k.. $\square$

We now describe a bijective proof for Theorem 8. This is a proof given by Glaisher and we shall call it Glaisher's bijection.

Let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_r^{m_r})$ be a partition of n in which no part is divisible by k. The notation for μ implies that $\mu_1 > \mu_2 > \ldots$ are parts having multiplicities $m_1, m_2, \ldots$, respectively. For each m_i, we find its k-ary expansion, i.e.

$$m_i = b_{i0} + b_{i1}k + b_{i2}k^2 + b_{i3}k^3 + \cdots + b_{il_i}k^{l_i}$$

where $0 \leq b_{ij} \leq k-1$ and $1 \leq j \leq l_i$.

We then map $\mu_i^{m_i}$ to $\bigcup_{j=0}^{l_i}(k^j\mu_i)^{b_{ij}}$, where now $k^j\mu_i$ is a part with multiplicity b_{ij}. The image of μ, which we shall denote by $\phi_k(\mu)$, is given by

$$\bigcup_{i=1}^{r}\bigcup_{j=0}^{l_i}(k^j\mu_i)^{b_{ij}}.$$

It is clear that this is a partition of n in which each part appears at most $k-1$ times.

Conversely, suppose that $\lambda = (\lambda_1^{c_1}, \lambda_2^{c_2}, \ldots)$ is a partition of n in which each part appears at most $k-1$ times. Write $\lambda_i = k^{r_i} \cdot b_i$ where $k \nmid b_i$ and then map $\lambda_i^{c_i}$ to $(b_i)^{k^{r_i}c_i}$ for each i, where now b_i is a part with multiplicity $k^{r_i}c_i$. The inverse of ϕ is defined as:

$$\phi_k^{-1}(\lambda) = \bigcup_{i\geq 1}(b_i)^{k^{r_i}c_i}.$$

Note that the parts of the resulting partition are not multiples of k.

Euler's theorem (Theorem 7) is a special case of Glaisher's theorem with $k = 2$. The following theorems, which are due to MacMahon and Subbarao, are examples of identities of Euler type.

Theorem 9 (MacMahon, [20]). *The set of partitions of n in which odd multiplicities are greater than 1 is equinumerous with the set of partitions of n in which odd parts are congruent to* $3 \pmod 6$.

Andrews, Eriksson, Petrov, and Romik [6] gave the first bijective proof of Theorem 9 which we describe in the sequel.

Let $A(n,1)$ denote the set of partitions of n in which odd multiplicities are greater than 1 and let $C(n,1)$ denote the set of partitions of n in which odd parts are congruent to $3 \pmod 6$.

Suppose $\lambda = (l^{m_l}, (l-1)^{m_{l-1}}, \ldots, 3^{m_3}, 2^{m_2}, 1^{m_1}) \in A(n,1)$ in which m_i is the multiplicity of i. Clearly, $m_i \in \{0, 2, 3, 4, \ldots\}$. Split m_i as

$$m_i = h_i + p_i, \text{ where } h_i \in \{0, 3\}, \; p_i \in \{0, 2, 4, 6, 8, \ldots\}.$$

For $j \geq 1$, define d_j in the following way:

$$\begin{aligned}
d_{6t+1} &= d_{6t+5} = 0, \\
d_{6t+2} &= \frac{1}{2} p_{3t+1}, \\
d_{6t+4} &= \frac{1}{2} p_{3t+2}, \\
d_{6t+3} &= \frac{1}{3} h_{2t+1} + p_{6t+3}, \\
d_{6t+6} &= \frac{1}{3} h_{2t+2} + p_{6t+6},
\end{aligned}$$

where $t = 0, 1, 2, \ldots$

The partition $(s^{d_s}, (s-1)^{d_{s-1}}, \ldots, 2^{d_2}, 1^{d_1})$ is in $C(n,1)$ and the transformation is invertible. We shall call the above mapping the *Andrews-Eriksson-Petrov-Romik bijection.*

Andrews gave the following generalization of Theorem 9.

Theorem 10 (Andrews, [4]). *Let* $r \in \mathbb{Z}_{\geq 0}$. *The set of partitions of n in which parts with odd multiplicity appear at least* $2r + 1$ *times is equinumerous with the set of partitions of n in which parts are even or congruent to* $2r + 1 \pmod{4r + 2}$.

Theorem 10 is a generalization of MacMahon's theorem since the case where $r = 1$ gives Theorem 9. Subbarao finitized Theorem 10 by placing further restriction on the odd multiplicities and introducing new restrictions on the even multiplicities (see Theorem 11).

Theorem 11 (Subbarao, [28])**.** *Let $m > 1, r \geq 0$ be integers, and let $A_{m,r}(n)$ be the set of partitions of n in which parts with even multiplicities appear less than $2m$ times, and parts with odd multiplicities appear at least $2r+1$ and at most $2(m+r)-1$ times. Let $B_{m,r}(n)$ be the set of partitions of n in which parts are either even and not congruent to $0 \pmod{2m}$, or odd and congruent to $2r+1 \pmod{4r+2}$. Then $|A_{m,r}(n)| = |B_{m,r}(n)|$.*

S. Fu and J. A. Sellers in [12] gave the following generalization.

Theorem 12 (Fu-Sellers, [12])**.** *Let $k, m, l \in \mathbb{Z}$ such that $k \geq 2, m \geq 2, l \geq 1$ and l is not a multiple of k. Denote by $E_{m,l,k}(n)$ be the set of partitions of n in which the multiplicity of each part is either congruent to 0 modulo k and at most $km - 1$ or else congruent to l modulo k and at least l but at most $l + k(m - 1)$. Denote by $F_{m,l,k}(n)$ the set of partitions of n into parts with two colors, say blue and red, where all parts in blue must be congruent to $0 \pmod{k}$ but not congruent to $0 \pmod{km}$ and all parts in red must be congruent to $l \pmod{l}$. Then the two sets $E_{m,l,k}(n)$ and $F_{m,l,k}(n)$ are equinumerous.*

Stemming from Theorem 7, Andrews on a joint work with M. Merca in [10] investigated the number of even parts in all partitions of n into distinct parts, i.e. the number of even parts in $d(n)$-partitions. This investigation resulted in the following theorem.

Theorem 13 (Andrews-Merca, [10])**.** *Let $a(n)$ denote the number of partitions of n in which the set of even parts is a singleton and let $b(n)$ be the difference between the number of parts in partitions of n into odd parts and the number of parts in partitions of n into distinct parts. Further, let $c(n)$ be the number of partitions of n where exactly one part is repeated. Then $a(n) = b(n) = c(n)$ for all $n \geq 1$.*

One of the generalizations of Theorem 13 due to S. Fu and D. Tang (see [13]) is stated below.

Theorem 14 (Fu and Tang, [13]). *Let $O_{j,k}(n)$ denote the number of partitions of n in which there are exactly j different parts congruent to 0 modulo k and $D_{j,k}(n)$ denote the number of partitions of n in which exactly j different parts are repeated at least k times. Then for all $j, n \geq 0$ and $k \geq 1$, $O_{j,k}(n) = D_{j,k}(n)$.*

Note that $j = 1, k = 2$ gives $a(n) = c(n)$ in Theorem 13.

Proof. Let w track the number of parts congruent to 0 (mod k) or parts repeated at least k times. Consider the following generating functions.

$$O_k(w,q) := \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} O_{j,k}(n) w^j q^n$$

and

$$D_k(w,q) := \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} D_{j,k}(n) w^j q^n.$$

Then

$$\begin{aligned} O_k(w,q) &= \prod_{m=1}^{\infty} \left(1 + wq^{km} + wq^{2km} + wq^{3km} + \cdots\right) \prod_{\substack{n=1 \\ n \not\equiv 0 \pmod{k}}}^{\infty} \frac{1}{1-q^n} \\ &= \prod_{m=1}^{\infty} \left(1 + \frac{wq^{km}}{1-q^{km}}\right) \prod_{n=1}^{\infty} \frac{1-q^{kn}}{1-q^n} \end{aligned}$$

and

$$\begin{aligned} D_k(w,q) &= \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \cdots + q^{(k-1)n} + wq^{kn} + wq^{(k+1)n} + \cdots) \\ &= \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \cdots + q^{(k-1)n})(1 + wq^{kn} + wq^{(2k)n} + \cdots) \\ &= \prod_{n=1}^{\infty} \frac{1-q^{kn}}{1-q^n} \prod_{m=1}^{\infty} \left(1 + \frac{wq^{km}}{1-q^{km}}\right). \end{aligned}$$

The coefficients of $w^j q^n$ for $O_k(w,q)$ and $D_k(w,q)$ are clearly the same. □

Fu and Tang described a bijection for Theorem 14 which can be stated as follows:

Proof. For $k \geq 1$, define a map $\varphi_k : O_{j,k}(n) \to D_{j,k}(n)$ as follows. Let $\pi = (\pi_1^{t_1}, \pi_2^{t_2}, \cdots, \pi_r^{t_r}) \in O_{j,k}(n)$.

$$\pi_i^{t_i} \longmapsto \begin{cases} \left(\frac{\pi_i}{k}\right)^{kt_i} & \text{if } \pi_i \equiv 0 \pmod{k}, \\ \phi_k(\pi_i^{t_i}) & \text{if } \pi_i \not\equiv 0 \pmod{k}, \end{cases}$$

where ϕ_k is the Glaisher map. The image of π under the map φ_k is given by

$$\bigcup_{i \geq 1} \varphi_k(\pi_i^{t_i}).$$

Note that for the case when $\pi_i \equiv 0 \pmod{k}$, the image parts are repeated at least k times and for the case $\pi_i \not\equiv 0 \pmod{k}$, the image parts repeat not more than $k-1$ times. □

2.2 Partitions with initial repetitions

It is believed that N. J. Fine was the first to consider partitions without gaps [7]. A classic example are the conjugates of partitions into distinct parts. Andrews used the construction of partitions without gaps and extended the concept to what he called partitions with initial k-repetitions [7].

Definition 2. *A partition with initial k-repetitions is a partition in which if any j appears at least k times as a part, then each positive integer less than j appears at least k times as a part.*

Note that when $k = 1$ we obtain partitions without gaps. The following theorem relates partitions with initial repetitions to partitions enumerated by Glaisher's generalization of Theorem 7.

Theorem 15 (Andrews,[7])**.** *The set of partitions of n with initial k-repetitions is in one-to-one correspondence with the set of partitions of n in which parts are not multiples of $2k$, which in turn, is equinumerous with the set of partitions of n in which parts appear at most $2k-1$ times.*

Proof. The generating function for the number of partitions of n with initial k-repetitions is

$$\sum_{n=0}^{\infty} \frac{q^{k.1+k.2+\cdots+kn}}{(q;q)_n} \times \prod_{j=n+1}^{\infty} (1 + q^j + q^{2j} + \cdots + q^{(k-1)j})$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{q^{\frac{kn(n+1)}{2}}}{(q;q)_n} \times \prod_{j=n+1}^{\infty} \frac{(1-q^{kj})}{(1-q^j)} \\
&= \sum_{n=0}^{\infty} \frac{q^{\frac{kn(n+1)}{2}}}{(q;q)_n} \times \frac{(q^k;q^k)_\infty (q;q)_n}{(q^k;q^k)_n (q;q)_\infty} \\
&= \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\frac{kn(n+1)}{2}}}{(q^k;q^k)_n} \\
&= \frac{(q^k;q^k)_\infty (-q^k;q^k)_\infty}{(q;q)_\infty} \\
&= \frac{(q^{2k};q^{2k})_\infty}{(q;q)_\infty} \\
&= \prod_{j=1}^{\infty} (1+q^j+q^{2j}+\cdots+q^{(2k-1)j}).
\end{aligned}
$$

The last two equalities give the number of partitions of n into parts not divisible by $2k$ and the number of partitions of n in which no part is repeated more than $2k-1$ times. □

Relating to partitions with initial 2-repetitions, Andrews discovered the following Legendre theorem which we state without proof.

Theorem 16. *Let $\mathcal{D}_e(m,n)$ be the set of partitions of n with initial 2-repetitions, with m different parts and an even number of distinct parts. Similarly, let $\mathcal{D}_o(m,n)$ be the set of partitions of n with initial 2-repetitions, with m different parts and an odd number of distinct parts. Then*

$$
|\mathcal{D}_e(m,n)| - |\mathcal{D}_o(m,n)| = \begin{cases} (-1)^j, & \text{if } m = j, n = \frac{j(j+1)}{2}; \\ 0, & \text{otherwise.} \end{cases}
$$

In [19], W. Keith provided a bijective proof of Theorem 15 in which he used the k-modular diagrams. We discuss his map in the sequel.

Definition 3. *Given a partition $\mu = (\mu_1, \mu_2, \ldots)$, a k-strip of length i is a removable row of k-units which is obtained by subtracting k from all parts μ_1 to μ_i if $\mu_i - \mu_{i+1} \geq k$ and $|\mu_i|_k - |\mu_{i+1}|_k \geq 1$.*

Definition 4. *A partition $\mu = (\mu_1, \mu_2, \cdots, \mu_l)$ is called k-flat if $\mu_l < k$ and $\mu_i - \mu_{i+1} < k$ for all $1 \leq i \leq l$.*

Keith's bijection

Let μ be a partition in which parts appear at most $2k-1$ times. The bijection is as follows:

- Compute μ'.
- Remove all k-strips from μ', obtaining (π, δ) where the partition π is k-flat and δ has distinct parts of k multiples.
- Deduce $\alpha = \pi + \delta$ by vector addition. Assume the convention that the shorter of π or δ is filled out with parts of size 0.
- Conjugate α.

Note that α' is a unique partition with initial k-repetitions.

Example 1. *Consider $n = 83$ and $k = 4$.*

The partition $(7^3, 6^2, 5, 4^5, 3^3, 2^7, 1^2)$ has no part appearing more than 7 times and its conjugate is $(23, 21, 14, 11, 6, 5, 3)$. Thus 4-modular diagram with S_i k-strips, $1 \leq i \leq 2$ is given below.

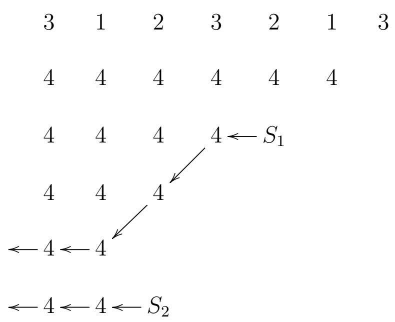

Figure 2.1: 4-modular diagram of $(23, 21, 14, 11, 6, 5, 3)$

From the 4-modular diagram we obtain that

$$\pi = (15, 13, 10, 7, 6, 5, 3) \quad \text{and} \quad \delta = (16, 8)$$

which gives that $\alpha = (31, 21, 10, 7, 6, 5, 3)$. Taking the conjugate we obtain that

$$\alpha' = (7^3, 6^2, 5, 4, 3^3, 2^{11}, 1^{10})$$

which is a partition with initial 4-repetitions.

A.O. Munagi and D. Nyirenda in [23] gave a simplified version of Keith's bijection above. It bypasses the use of k-modular diagrams and uses simple algebraic operations to determine π and δ. Their bijection requires the following definition and proposition.

Definition 5. *Let L_k be an operator from the set of partitions of n wherein every part appears with multiplicity k to the set of partitions of n into distinct parts that are multiples of k:*

$$L_k : (\mu_1^k, \mu_2^k, \cdots) \mapsto (k\mu_1, k\mu_2, \cdots).$$

Then L_k is an injective map.

Proposition 2. *Let μ be a partition in which parts appear at most $2k-1$ times and $\mu = \upsilon_0 \cup \upsilon$ such that $\upsilon_0 := (\mu_j^k : m_{\mu_j}(\mu) \geq k)$ and υ is the sub-partition which results from removing the parts of υ_0 from μ. Then*

$$\pi = \upsilon' \quad \text{and} \quad \delta = L_k(\upsilon_0),$$

where $m_{\mu_j}(\mu)$ is the multiplicity of μ_j in μ.

Munagi-Nyirenda bijection

Let μ be a partition in which no part appears more than $2k-1$ times. Then execute the following steps:

- Decompose $\mu = \upsilon_0 \cup \upsilon$ where $\upsilon_0 := (\mu_j^k : m_{\lambda_j}(\mu) \geq k)$.
- Set $\delta := L_k(\upsilon_0)$ and $\pi := \upsilon'$.
- Work out $\alpha = \pi + \delta$.
- Find α. Note that α' is a partition with initial k-repetitions.

Andrews in [9] introduced partitions with early conditions thereby extending partitions with initial repetitions to a broader class. In this thesis, we find new identities on partitions with initial repetitions and provide partition-theoretic interpretations of some identities of Rogers-Ramanujan type.

Chapter 3

Identities of Euler type

In this chapter, we study Theorems 10 and 11 due to Andrews and Subbarao. New bijections and generalizations of their bijections are given. Work in this chapter has been published in the articles:

- B. Mugwangwavari, D. Nyirenda, A note on the Andrews-Eriksson-Petrov-Romik bijection for MacMahon's partition theorem, *Journal of Integer Sequences*, **24** (2021), Article 21.5.6.
- D. Nyirenda, B. Mugwangwavari, On Generalizations of Theorems of MacMahon and Subbarao. *Ann. Comb.* (2022). https://doi.org/10.1007/s00026-022-00592-5

which are [21] and [26], respectively.

3.1 Andrews-Eriksson-Petrov-Romik bijection for MacMahon's partition theorem

Until the publication of the article [6], there appeared to be no bijection that naturally extends Andrews-Eriksson-Petrov-Romik bijection to prove Andrews' generalization of MacMahon's partition theorem (Theorem 10). For this, you may consult [12] and the references therein. In this section, we supply a bijection for Theorem 10, which naturally generalizes Andrews-Eriksson-Petrov-Romik bijection and is different from the ones given in the literature.

3.1.1 The bijection

Let $A_r(n)$ denote the set of partitions of n wherein parts appearing an odd number of times actually appear at least $2r+1$ times and let $B_r(n)$ denote the set of partitions of n wherein odd parts are congruent to $2r+1 \pmod{4r+2}$. We need to establish the one-to-one correspondence between the sets $A_r(n)$ and $B_r(n)$.
Let $(l^{h_l}, (l-1)^{h_{l-1}}, \ldots, 3^{h_3}, 2^{h_2}, 1^{h_1}) \in A_r(n)$. In this notation, h_i is the multiplicity of i. Note that h_i can be zero for some i. Clearly, $h_i \in \{0, 2, 4, \ldots, 2r, 2r+1, 2r+2, \ldots\}$. Write h_i as

$$h_i = k_i + g_i \text{ where } k_i \in \{0, 2r+1\} \text{ and } g_i \in \{0, 2, 4, 6, 8, 10, \ldots\}$$

This decomposition is unique since k_i and g_i can be made explicit, i.e.

$$k_i = (2r+1)\left(\frac{1-(-1)^{h_i}}{2}\right) \text{ and } g_i = h_i - (2r+1)\left(\frac{1-(-1)^{h_i}}{2}\right).$$

For $j \geq 1$, define d_j as follows:

$$\begin{aligned} d_{(4r+2)t+2j-1} &= 0 \text{ for } j \in \{1, 2, \ldots, 2r+1\} \setminus \{r+1\}, \\ d_{(4r+2)t+2j} &= \frac{1}{2} g_{(2r+1)t+j} \text{ for } j \in \{1, 2, \ldots, 2r\}, \\ d_{(4r+2)t+(2r+1)j} &= \frac{1}{2r+1} k_{2t+j} + g_{(4r+2)t+(2r+1)j} \text{ for } j \in \{1, 2\}, \end{aligned}$$

where $t = 0, 1, 2, \ldots$.
The image is thus given by

$$(\ldots, f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1}).$$

We claim that $(\ldots, f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1}) \in B_r(n)$. In order to show that

$$(l^{h_l}, (l-1)^{h_{l-1}}, \ldots, 3^{h_3}, 2^{h_2}, 1^{h_1}) \mapsto (\ldots, f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1})$$

defines a bijection from $A_r(n)$ onto $B_r(n)$, it suffices to show that $\sum_{i\geq 1} id_i = n$.
Thus

$$\begin{aligned} \sum_{i\geq 1} id_i = &\sum_{j=1}^{r}\sum_{t=0}^{\infty}((4r+2)t+2j-1)d_{(4r+2)t+2j-1} + \sum_{j=r+2}^{2r+1}\sum_{t=0}^{\infty}((4r+2)t+2j-1)d_{(4r+2)t+2j-1} \\ &+ \sum_{j=1}^{2r}\sum_{t=0}^{\infty}((4r+2)t+2j)d_{(4r+2)t+2j} + \sum_{j=1}^{2}\sum_{t=0}^{\infty}((4r+2)t+(2r+1)j)d_{(4r+2)t+(2r+1)j} \end{aligned}$$

$$= \sum_{j=1}^{2r} \sum_{t=0}^{\infty} ((2r+1)t + j) g_{(2r+1)t+j} + \sum_{j=1}^{2} \sum_{t=0}^{\infty} ((4r+2)t + (2r+1)j) g_{(4r+2)t+(2r+1)j}$$
$$+ \sum_{j=1}^{2} \sum_{t=0}^{\infty} (2t+j) k_{2t+j}.$$

Using the notation

$$S(a,b) = \sum_{t=0}^{\infty} (at+b) g_{at+b} \quad \text{and} \quad T(a,b) = \sum_{t=0}^{\infty} (at+b) k_{at+b},$$

observe that

$$\sum_{t=0}^{\infty} ((2r+1)t+j) g_{(2r+1)t+j} = \sum_{\ell=0,2r+1} \sum_{t=0}^{\infty} ((4r+2)t + j + \ell) g_{(4r+2)t+j+\ell}$$
$$= S(4r+2, j) + S(4r+2, 2r+1)$$

and

$$\sum_{t=0}^{\infty} (2t+j) k_{2t+j} = \sum_{t=0}^{\infty} \sum_{\ell=0}^{2r} ((4r+2)t + j + 2\ell) k_{(4r+2)t+j+2\ell}$$
$$= \sum_{\ell=0}^{2r} T(4r+2, j+2\ell).$$

Hence

$$\sum_{i \geq 1} i d_i = \sum_{j=1}^{2r} S(4r+2, j) + \sum_{j=1}^{2r} S(4r+2, j+2r+1) + \sum_{j=1}^{2} S(4r+2, (2r+1)j)$$
$$+ \sum_{\ell=0}^{2r} T(4r+2, 1+2\ell) + \sum_{\ell=0}^{2r} T(4r+2, 2+2\ell)$$
$$= \sum_{j=1}^{2r} (S(4r+2, j) + S(4r+2, j+2r+1) + T(4r+2, 1+2j) + T(4r+2, 2+2j))$$
$$+ \sum_{j=1}^{2} S(4r+2, (2r+1)j) + T(4r+2, 1) + T(4r+2, 2)$$
$$= \sum_{j=1}^{2r+1} (S(4r+2, j) + S(4r+2, j+2r+1)) + \sum_{j=0}^{2r} (T(4r+2, 1+2j) + T(4r+2, 2+2j))$$

$$
\begin{aligned}
&= \sum_{j=1}^{2r+1} (S(4r+2,j) + S(4r+2,j+2r+1)) + \sum_{j=1}^{2r+1} (T(4r+2,2j-1) + T(4r+2,2j)) \\
&= \sum_{j=1}^{4r+2} S(4r+2,j) + \sum_{j=1}^{4r+2} T(4r+2,j) \\
&= \sum_{j=1}^{4r+2} (S(4r+2,j) + T(4r+2,j)) \\
&= \sum_{j=1}^{4r+2} \sum_{t=0}^{\infty} ((4r+2)t+j)(g_{(4r+2)t+j} + k_{(4r+2)t+j}) \\
&= \sum_{t=0}^{\infty} \sum_{j=1}^{4r+2} ((4r+2)t+j)h_{(4r+2)t+j} \\
&= \sum_{i=1}^{\infty} ih_i \\
&= n.
\end{aligned}
$$

Remark 1. *Setting* $r = 1$ *in the bijection yields the Andrews-Eriksson-Petrov-Romik bijction [6].*

For example, consider $n = 15$ and $r = 2$. Table 3.1 demonstrates the correspondence. The inverse of our map is described in the following section.

$A_2(15)$	$\longrightarrow$	$B_2(15)$
(3,3,3,3,3)	$\mapsto$	(15)
(2,2,2,2,2,1,1,1,1,1)	$\mapsto$	(10,5)
(4,4,1,1,1,1,1,1,1)	$\mapsto$	(8,5,2)
(3,3,2,2,1,1,1,1,1)	$\mapsto$	(6,5,4)
(3,3,1,1,1,1,1,1,1,1,1)	$\mapsto$	(6,5,2,2)
(5,5,1,1,1,1,1)	$\mapsto$	(5,5,5)
(2,2,2,2,1,1,1,1,1,1,1)	$\mapsto$	(5,4,4,2)
(2,2,1,1,1,1,1,1,1,1,1,1,1)	$\mapsto$	(5,4,2,2,2)
(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	$\mapsto$	(5,2,2,2,2,2)

Table 3.1: The map $A_r(n) \to B_r(n)$ for $r = 2, n = 15$.

3.1.2 The inverse mapping

We now describe the inverse to our bijection.
Given that $(\ldots, f^{d_f}, (f-1)^{d_{f-1}}, \ldots, 2^{d_2}, 1^{d_1}) \in B_r(n)$. Define g_i and k_i as

$$g_{(2r+1)t+j} = 2d_{(4r+2)t+2j}, j = 1, 2, \ldots 2r, t = 0, 1, 2, \ldots$$

For the remaining cases, we have, for $t = 0, 1, 2, \ldots$,

$$g_{(4r+2)t+(2r+1)j} = \begin{cases} d_{(4r+2)t+(2r+1)j}, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 0 \pmod 2,\ j = 1, 2; \\ d_{(4r+2)t+(2r+1)j} - 1, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 1 \pmod 2,\ j = 1, 2, \end{cases}$$

and

$$k_{2t+j} = \begin{cases} 0, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 0 \pmod 2,\ j = 1, 2; \\ 2r+1, & \text{if } d_{(4r+2)t+(2r+1)j} \equiv 1 \pmod 2,\ j = 1, 2. \end{cases}$$

Then the partition $(\ldots, 3^{g_3+k_3}, 2^{g_2+k_2}, 1^{g_1+k_1})$ is in $A_r(n)$.
For example, Table 3.3 demonstrates the inverse mapping when $n = 15$.

$B_2(15)$	$\longrightarrow$	$A_2(15)$
(10,5)	$\mapsto$	(2,2,2,2,2,1,1,1,1,1)
(8,5,2)	$\mapsto$	(4,4,1,1,1,1,1,1,1)
(6,5,4)	$\mapsto$	(3,3,2,2,1,1,1,1,1)
(6,5,2,2)	$\mapsto$	(3,3,1,1,1,1,1,1,1,1,1)
(5,4,4,2)	$\mapsto$	(2,2,2,2,1,1,1,1,1,1,1)
(5,4,2,2,2)	$\mapsto$	(2,2,1,1,1,1,1,1,1,1,1,1,1)
(5,2,2,2,2,2)	$\mapsto$	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)

Table 3.2: The inverse map $B_r(n) \to A_r(n)$ for $n = 15$, $r = 2$.

3.2 On generalizations of theorems of MacMahon and Subbarao

Bijective proofs have been given for Theorems 10 and 11 (see [12, 15, 18]).
Our goal in this section is threefold: to provide a new explicit bijection for Theorem

10, to generalize Theorem 11 and consequently extend the bijective maps of Fu and Sellers [12], Kanna, Dharmendra, Sridhara, and Kumar [18] and to derive some congruence and recurrence relations for related partition functions.

3.2.1 A new bijection for Theorem 10

Let $A(n,r)$ denote the set of partitions of n in which odd multiplicities are greater than or equal to $2r+1$. Further, denote by $C(n,r)$ the set of partitions of n in which odd parts are congruent to $2r+1 \pmod{4r+2}$.

For $r \geq 1$, define the map $\beta_r : A(n,r) \to C(n,r)$ as follows. Let $(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots) \in A(n,r)$.

If $\lambda_i \equiv 0 \pmod 2$, then

$$\lambda_i^{m_i} \longmapsto \begin{cases} \lambda_i^{m_i-(2r+2v+2)}, (2\lambda_i)^{r+v+1}, & \text{if } m_i \equiv 2v+1 \pmod{2r+1}, \\ & 0 \leq v \leq r-1, \\ \\ \lambda_i^{m_i-2v}, (2\lambda_i)^{v}, & \text{if } m_i \equiv 2v \pmod{2r+1}, \\ & 0 \leq v \leq r. \end{cases}$$

If $\lambda_i \equiv 1 \pmod 2$, then

$$\lambda_i^{m_i} \longmapsto \begin{cases} ((2r+1)\lambda_i)^{\frac{m_i-(2r+2v+2)}{2r+1}}, (2\lambda_i)^{r+v+1}, & \text{if } m_i \equiv 2v+1 \pmod{2r+1}, \\ & 0 \leq v \leq r-1 \\ \\ ((2r+1)\lambda_i)^{\frac{m_i-2v}{2r+1}}, (2\lambda_i)^{v}, & \text{if } m_i \equiv 2v \pmod{2r+1}, \\ & 0 \leq v \leq r. \end{cases}$$

The image of $(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots)$ under the map β_r is then given by

$$\bigcup_{i \geq 1} \beta_r(\lambda_i^{m_i}).$$

It is not difficult to see that β_r defines a bijection for Theorem 10 and that setting $r = 1$ in β_r gives rise to the mapping β_1.

$A(n,r)$	$\longrightarrow$	$C(n,r)$
$(5^2,1^7)$	$\mapsto$	$(10,7)$
$(4^2,1^9)$	$\mapsto$	$(8,7,2)$
$(3^2,2^2,1^7)$	$\mapsto$	$(7,6,4)$
$(3^2,1^{11})$	$\mapsto$	$(7,6,2^2)$
$(2^4,1^9)$	$\mapsto$	$(7,4^2,2)$
$(2^2,1^{13})$	$\mapsto$	$(7,4,2^3)$
(1^{17})	$\mapsto$	$(7,2^5)$

Table 3.3: The map $A(n,r) \to C(n,r)$ for $r=3, n=17$.

Table 3.3 shows an example for $r=3$ and $n=17$.
The inverse of β_r is described as follows:

Let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \cdots) \in C(n,r)$. Then

$$\mu_i^{m_i} \mapsto \begin{cases} \left(\frac{\mu_i}{2r+1}\right)^{(2r+1)m_i}, & \mu_i \equiv 2r+1 \pmod{4r+2}; \\ \\ \mu_i^{(2r+1)\lfloor\frac{m_i}{2r+1}\rfloor}, \left(\frac{\mu_i}{2}\right)^{2(m_i-(2r+1)\lfloor\frac{m_i}{2r+1}\rfloor)}, & \mu_i \equiv 0 \pmod{2}. \end{cases}$$

Then map μ to $\bigcup_{i\geq 1} \beta_r^{-1}(\mu_i^{m_i})$.

Note that when $n=30$ and $r=2$ the partition (3^{10}) maps to (6^5) by β_2 and (15^2) by the bijection in Section 3.1 showing that the two maps are different.

3.2.2 Generalization of Theorem 11

Unless otherwise specified, in this this section and the one afterwards, we assume that a and p are positive integers such that $\gcd(a,p)=1$.
For integers $m, r \geq 1$, let $B_{p,r,a,m}(n)$ denote the set of partitions of n in which multiplicities which are congruent to $ja \mod p$ are greater than or equal to $j(pr+a)$ and less than or equal to $j(pr+a)+p(m-1)$ where $j=0,1,2,\ldots,p-1$.
Furthermore, let $E_{p,r,a,m}(n)$ denote the set of partitions of n wherein parts divisible by p are not divisible by pm and those not divisible by p are congruent to $-s(pr+a)$ $\pmod{p^2r+pa}$ where $s=1,2,\ldots,p-1$. Then we have the following theorem.

Theorem 17. *For all integers $n \geq 0$,*

$$|B_{p,r,a,m}(n)| = |E_{p,r,a,m}(n)|.$$

Proof. Observe that, if a part of a partition in $E_{p,r,a,m}(n)$ is divisible by p, then it is not divisible by pm. To account for such, we have the generating function:

$$\prod_{j\geq 1, j\equiv 0 \pmod p, j\not\equiv 0 \pmod{pm}} \frac{1}{1-q^j}.$$

To account for other parts which are not divisible by p, we use

$$\prod_{s=1}^{p-1}\prod_{n=1}^{\infty} \frac{1}{1-q^{(p^2r+ap)n-s(pr+a)}}.$$

Hence, we have

$$\sum_{n=0}^{\infty} |E_{p,r,a,m}(n)|q^n = \prod_{s=1}^{p-1}\prod_{n=1}^{\infty} \frac{1}{1-q^{(p^2r+ap)n-s(pr+a)}} \prod_{j\geq 1, j\equiv 0 \pmod p, j\not\equiv 0 \pmod{pm}} \frac{1}{1-q^j}$$

The multiplicity of a part of a partition in $B_{p,r,a,m}(n)$ is of the form

$$\alpha(pr+a)+\beta p \text{ where } \alpha = 0,1,2,\dots p-1 \text{ and } \beta = 0,1,2,\dots,m-1.$$

So we have the generating function:

$$\begin{aligned}
&\sum_{n=0}^{\infty} |B_{p,r,a,m}(n)|q^n \\
&= \prod_{n=1}^{\infty} \big(1+q^{pn}+q^{2pn}+q^{3pn}+\cdots+q^{(m-1)pn} \\
&\quad + q^{(pr+a)n}+q^{(pr+a+p)n}+q^{(pr+a+2p)n}+q^{(pr+a+3p)n}+\cdots+q^{(pr+a+(m-1)p)n}\big) \\
&\quad + q^{2(pr+a)n}+q^{(2(pr+a)+p)n}+q^{(2(pr+a)+2p)n}+q^{(2(pr+a)+3p)n}+\cdots+q^{(2(pr+a)+(m-1)p)n} \\
&\quad + q^{3(pr+a)n}+q^{(3(pr+a)+p)n}+q^{(3(pr+a)+2p)n}+q^{(3(pr+a)+3p)n}+\cdots+q^{(3(pr+a)+(m-1)p)n} \\
&\qquad \vdots \\
&\qquad \vdots \\
&\quad + q^{(p-1)(pr+a)n}+q^{((p-1)(pr+a)+p)n}+q^{((p-1)(pr+a)+2p)n}+\cdots+q^{((p-1)(pr+a)+(m-1)p)n}\big) \\
&= \prod_{n=1}^{\infty}\left(\sum_{j=0}^{m-1} q^{jpn} + q^{(pr+a)n}\sum_{j=0}^{m-1} q^{jpn} + q^{2(pr+a)n}\sum_{j=0}^{m-1} q^{jpn} + \cdots + q^{(p-1)(pr+a)n}\sum_{j=0}^{m-1} q^{jpn}\right)
\end{aligned}$$

$$= \prod_{n=1}^{\infty} \sum_{j=0}^{p-1} q^{(j(pr+a)n)} \sum_{i=0}^{m-1} q^{ipn}$$
$$= \prod_{n=1}^{\infty} \frac{(1-q^{p(pr+a)n})(1-q^{pmn})}{(1-q^{(pr+a)n})(1-q^{pn})}$$
$$= \prod_{s=1}^{p-1} \prod_{n=1}^{\infty} \frac{1}{1-q^{(pr+a)(pn-s)}} \prod_{j\geq 1, j\equiv 0 \pmod{p}, j\not\equiv 0 \pmod{pm}} \frac{1}{1-q^j}$$
$$= \prod_{s=1}^{p-1} \prod_{n=1}^{\infty} \frac{1}{1-q^{(p^2r+ap)n-s(pr+a)}} \prod_{j\geq 1, j\equiv 0 \pmod{p}, j\not\equiv 0 \pmod{pm}} \frac{1}{1-q^j}$$
$$= \sum_{n=0}^{\infty} |E_{p,r,a,m}(n)| q^n.$$

□

Remark 2. *Note that with $p = 2$, $a = 1$, Theorem 17 reduces to Subbarao's finitization, Theorem 11.*

Introducing the following notation for a positive integer v,

$$\mathrm{ord}_v(j) := \max\{i \in \mathbb{Z}_{\geq 0} : v^i \mid j\},$$

we proceed to describe a bijection for Theorem 17.
Let $\mu = (\mu_1^{\omega_1}, \mu_2^{\omega_2}, \cdots, \mu_t^{\omega_t}) \in E_{p,r,a,m}(n)$. Define a map $\gamma : E_{p,r,a,m}(n) \to B_{p,r,a,m}(n)$ as follows.

Case 1: $\mu_i \not\equiv 0 \pmod{pr+a}$.
We write ω_i as

$$\omega_i = m^{\rho_1} + m^{\rho_2} + m^{\rho_3} + \cdots + m^{\rho_l} + \eta \tag{3.1}$$

where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_l > 0$ and $0 \leq \eta < m$. The representation of ω_i in (3.1) is unique and arises from the m-ary expansion of ω_i. For instance, if $\omega_i = 1 + 2 \cdot 4 + 2 \cdot 4^2 + 3 \cdot 4^4$, we rewrite ω_i as $1 + (4 + 4) + (4^2 + 4^2) + (4^4 + 4^4 + 4^4)$ so that $\eta = 1, \rho_1 = \rho_2 = \rho_3 = 4, \rho_4 = \rho_5 = 2, \rho_6 = \rho_7 = 1$.
Then construct a partition

$$x_i = \left(m^{\rho_1} \times \frac{\mu_i}{p}\right)^p \cup \left(m^{\rho_2} \times \frac{\mu_i}{p}\right)^p \cup \cdots \cup \left(m^{\rho_l} \times \frac{\mu_i}{p}\right)^p.$$

Thus

$$\mu_i^{\omega_i} \mapsto \begin{cases} x_i, & \text{if } \eta = 0; \\ x_i \cup \left(\frac{\mu_i}{p}\right)^{p\eta}, & \text{if } 0 < \eta < m. \end{cases}$$

Case 2: $\mu_i \equiv 0 \pmod{pr+a}$. We write ω_i as

$$\omega_i = p^{\rho_1} + p^{\rho_2} + p^{\rho_3} + \cdots + p^{\rho_l} + \zeta \tag{3.2}$$

where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_l > 0$ and $0 \leq \zeta < p$. Construct a partition

$$y_i = \left(p^{\rho_1} \times \frac{\mu_i}{pr+a}\right)^{pr+a} \cup \left(p^{\rho_2} \times \frac{\mu_i}{pr+a}\right)^{pr+a}$$
$$\cup \left(p^{\rho_3} \times \frac{\mu_i}{pr+a}\right)^{pr+a} \cup \cdots \cup \left(p^{\rho_l} \times \frac{\mu_i}{pr+a}\right)^{pr+a}.$$

Thus

$$\mu_i^{\omega_i} \mapsto \begin{cases} y_i, & \text{if } \zeta = 0; \\ y_i \cup \left(\frac{\mu_i}{pr+a}\right)^{(pr+a)\zeta}, & \text{if } 0 < \zeta < p. \end{cases}$$

The image is then defined as

$$\gamma(\mu) = \bigcup_{i \geq 1} \gamma(\mu_i^{\omega_i}).$$

To prove that $\gamma(\mu) \in B_{p,r,a,m}(n)$, we consider a couple of things. First, note that μ and $\gamma(\mu)$ have the same size. To see why this is the case, assume that

$$\mu_i \not\equiv 0 \pmod{pr+a} \quad \text{for} \quad i = 1, 2, 3, \ldots, h$$

and

$$\mu_i \equiv 0 \pmod{pr+a} \quad \text{for} \quad i = h+1, h+2, \ldots, t.$$

Then

$$\begin{aligned} |\gamma(\mu)| &= \sum_{i=1}^{t} |\gamma(\mu_i^{\omega_i})| \\ &= \sum_{i=1}^{h} |\gamma(\mu_i^{\omega_i})| + \sum_{i=h+1}^{t} |\gamma(\mu_i^{\omega_i})|. \end{aligned}$$

Considering the two sums on the preceding right-hand side separately, we have

$$\begin{aligned}
\sum_{i=1}^{h} |\gamma(\mu_i^{\omega_i})| &= \sum_{i=1}^{h} \left| x_i \cup \left(\frac{\mu_i}{p}\right)^{p\eta} \right| \\
&= \sum_{i=1}^{h} |x_i| + p\eta\frac{\mu_i}{p} \\
&= \sum_{i=1}^{h} |x_i| + \eta\mu_i \\
&= \sum_{i=1}^{h} \left[p\left(\sum_{j=1}^{\ell} \frac{\mu_i}{p} m^{\rho_j}\right) + \eta\mu_i \right] \\
&= \sum_{i=1}^{h} \mu_i \left(\sum_{j=1}^{\ell} m^{\rho_j} + \eta\right) \\
&= \sum_{i=1}^{h} \mu_i\omega_i \quad \text{(by (3.1))}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=h+1}^{t} |\gamma(\mu_i^{\omega_i})| &= \sum_{i=h}^{t} \left| y_i \cup \left(\frac{\mu_i}{pr+a}\right)^{(pr+a)\zeta} \right| \\
&= \sum_{i=h}^{t} |y_i| + (pr+a)\zeta\frac{\mu_i}{pr+a} \\
&= \sum_{i=h}^{t} |y_i| + \zeta\mu_i \\
&= \sum_{i=h}^{t} \left[(pr+a)\left(\sum_{j=1}^{\ell} \frac{\mu_i}{pr+a} p^{\rho_j}\right) + \zeta\mu_i \right] \\
&= \sum_{i=h}^{t} \mu_i \left(\sum_{j=1}^{\ell} p^{\rho_j} + \zeta\right) \\
&= \sum_{i=h}^{t} \mu_i\omega_i \quad \text{(by (3.2))}.
\end{aligned}$$

Thus

$$|\gamma(\mu)| = \sum_{i=1}^{h} \mu_i\omega_i + \sum_{i=h}^{t} \mu_i\omega_i$$

$$= \sum_{i=1}^{t} \mu_i \omega_i$$
$$= |\mu|.$$

Second, we need to show that the multiplicities of parts in $\gamma(\mu)$ indeed satisfy the description of multiplicities for partitions in $B_{p,r,a,m}(n)$. Using (3.1) and (3.2), it is not difficult to see that the multiplicity of $m^{\rho_j} \times \frac{\mu_i}{p}$ is cp where c is the coefficient of m^{ρ_j} in the base m expansion of ω_i. Thus $c = 0, 1, 2, \ldots, m-1$. Similarly, the multiplicity of $p^{\rho_j} \times \frac{\mu_i}{pr+a}$ is $d(pr+a)$ where d is the coefficient of p^{ρ_j} in the base p expansion of ω_i. Hence, $d = 0, 1, 2, \ldots, p-1$. Clearly, the multiplicity of parts are thus $\equiv ja \mod p$, at least $j(pr+a)$ and at most $j(pr+a) + p(m-1)$ for some $0 \leq j \leq p-1$.
Finally, the uniqueness of the representation of ω_i in (3.1) and (3.2) implies that γ is injective. In fact γ is surjective and we construct its inverse in the following section.

The Inverse of γ

We now give the inverse of γ, i.e. γ^{-1}. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_1}, \ldots) \in B_{p,r,a,m}(n)$. Define a map $\gamma^{-1} : B_{p,r,a,m}(n) \to E_{p,r,a,m}(n)$ as follows.

Case I: $m_i \equiv 0 \pmod p$

$$\lambda_i^{m_i} \longmapsto \left(\frac{p\lambda_i}{m^r}\right)^{\frac{m_i}{p} m^r}, \text{ where } r = \operatorname{ord}_m(\lambda_i).$$

Case II: $m_i \not\equiv 0 \pmod p$

Thus $m_i \equiv ja \pmod p$ for some $j \in \{1, 2, \ldots, p-1\}$. We have

$$\lambda_i^{m_i} \longmapsto \begin{cases} \left(((pr+a)\lambda_i)^j, \lambda_i^{m_i - j(pr+a)}\right), & \text{if } \lambda_i \not\equiv 0 \pmod p \\ \\ \left((\frac{(pr+a)\lambda_i}{p^t})^{jp^t}, \lambda_i^{m_i - j(pr+a)}\right), & \text{if } \lambda_i \equiv 0 \pmod p \end{cases}$$

where $t = \operatorname{ord}_p(\lambda_i)$.
In Case II, if $m_i - j(pr+a) > 0$, you apply Case I to the subpartition $\lambda_i^{m_i - j(pr+a)}$.

The image is then defined as

$$\gamma^{-1}(\lambda) = \bigcup_{i\geq 1} \gamma^{-1}(\lambda_i^{m_i}).$$

Note that $\gamma^{-1}(\lambda) \in E_{p,r,a,m}(n)$ because of the following:
In Case I, since $\frac{\lambda_i}{m^r}$ is not divisible by m, it follows that the image parts $p\frac{\lambda_i}{m^r}$ are divisible by p, but not divisible by pm. Such parts define subpartitions in $E_{p,r,a,m}(n)$.
In Case II, if $\lambda_i \not\equiv 0 \pmod p$, then $(pr+a)\lambda_i = (pr+a)(pq-s)$ for some $s = 1, 2, \ldots, p-1$ and $q \geq 1$. Thus

$$\begin{aligned}(pr+a)\lambda_i &= (pr+a)pq - s(pr+a)\\ &= (p^2r+ap)q - s(pr+a)\\ &\equiv -s(pr+a) \pmod{p^2r+ap}.\end{aligned}$$

On the other hand, if $\lambda_i \equiv 0 \pmod p$, then $\frac{\lambda_i}{p^t} \not\equiv 0 \pmod p$, and by the previous arguments, we must have $(pr+a)\frac{\lambda_i}{p^t} \equiv -s(pr+a) \pmod{p^2r+ap}$.
Clearly, in any case the image parts are subpartitions in $E_{p,r,a,m}(n)$. This in turn means $\gamma^{-1}(\lambda) \in E_{p,r,a,m}(n)$.

Example 2. *Consider* $n = 42, p = 5, m = 4, a = 2$ *and* $r = 4$.

We have $B_{5,4,2,4}(42) = \{(4^5, 1^{22}), (3^5, 1^{27}), (2^{10}, 1^{22}), (2^5, 1^{32})\}$.
The sub-partitions 2^5 and 1^{32} of $(2^5, 1^{32})$ have multiplicities congruent to $0 \pmod 5$ and $2 \pmod 5$, respectively. Applying the map γ gives

$$2^5 \mapsto 10$$

$$1^{32} \mapsto (22, 5^2).$$

Hence, taking the union of the image parts we obtain that $(2^5, 1^{32}) \mapsto (22, 10, 5^2)$.
Similarly, applying τ to the remaining partitions gives

$$(4^5, 1^{22}) \mapsto (22, 5^4)$$

$$(3^5, 1^{27}) \mapsto (22, 15, 5)$$

$$(2^{10}, 1^{22}) \mapsto (22, 10^2)$$

which are partitions in $E_{5,4,2,4}(42)$.

Conversely, note that $E_{5,4,2,4}(42) = \{(22, 5^4), (22, 15, 5), (22, 10^2), (22, 10, 5^2)\}$. The sub-partitions 22 and 5^4 of $(22, 5^4)$ have parts congruent to 0 (mod 22) and 5 (mod 22), respectively. Applying the map τ^{-1} gives

$$22 \mapsto 1^{22}$$

$$5^4 \mapsto 4^5.$$

Taking the union of the image parts yields $(22, 5^4) \mapsto (4^5, 1^{22})$. Similarly, applying τ^{-1} to the remaining partitions gives

$$(22, 15, 5) \mapsto (3^5, 1^{27})$$

$$(22, 10^2) \mapsto (2^{10}, 1^{22})$$

$$(22, 10, 5^2) \mapsto (2^5, 1^{32})$$

which are partitions in $B_{5,4,2,4}(42)$.

Henceforth, we shall denote the cardinality $|B_{p,r,a,m}(n)|$ by $b_{p,r,a,m}(n)$ and $|E_{p,r,a,m}(n)|$ by $e_{p,r,a,m}(n)$. Furthermore, let

$$b_{p,r,a,\infty}(n) = \lim_{m\to\infty} b_{p,r,a,m}(n),$$

$$e_{p,r,a,\infty}(n) = \lim_{m\to\infty} e_{p,r,a,m}(n).$$

Observe that:
$b_{p,r,a,\infty}(n)$ is the number of partitions of n in which multiplicities that are congruent to $ja \mod p$ are actually greater than or equal to $j(pr+a)$ where $j = 0, 1, 2, \ldots, p-1$. Let the set of such partitions be denoted by $B_{p,r,a,\infty}(n)$. Also, $e_{p,r,a,\infty}(n)$ is the number of partitions of n wherein parts not divisible by p are congruent to $-s(pr+a) \pmod{p^2r + pa}$ where $s = 1, 2, \ldots, p-1$. Let the set of such partitions be denoted by $E_{p,r,a,\infty}(n)$. Consequently, we have the following result.

Corollary 4. *For all $n \geq 0$, we have*

$$b_{p,r,a,\infty}(n) = e_{p,r,a,\infty}(n).$$

It is immediately noticeable that this corollary is a new extension of Andrews' theorem, Theorem 10 (set $p = 2, a = 1$). We give a bijective proof which extends Fu and Sellers' bijection in [12].

Proof. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots) \in B_{p,r,a,\infty}(n)$. The bijection is given as follows.

- If m_i is congruent to $0 \pmod p$, then
$$\lambda_i^{m_i} \longmapsto (p\lambda_i)^{\frac{m_i}{p}}.$$
Each of these new parts is congruent to $0 \pmod p$.

- If m_i is congruent to $ja \pmod p$ for some $1 \le j \le p-1$, then do the following: We know that $m_i \ge j(pr + a)$. Thus, we split off $j(pr + a)$ copies of the part λ_i and combine any of the remaining as was done in the previous step of the algorithm. This now leaves us with $j(pr + a)$ copies of each of the parts λ_i which had multiplicity $ja \pmod p$ in the original partition. We now take j copies of each such part and realize that these define a subpartition wherein parts appear at most $p - 1$ times. We now apply Glaisher's map to obtain a subpartition wherein parts are not divisible by p. Finally, to get back the size of n, we multiply each of the parts in this subpartition wherein parts are not divisible by p by $pr + a$.

To reverse the transformation, let $\mu = (\mu_1^{\omega_1}, \mu_2^{\omega_2}, \cdots, \mu_t^{\omega_t}) \in E_{p,r,a,\infty}(n)$. Then

- If μ_i is congruent to $0 \pmod p$, then
$$\mu_i^{\omega_i} \longmapsto \left(\frac{\mu_i}{p}\right)^{p\omega_i}.$$
Each new part will have multiplicity congruent to $0 \pmod p$.

- If μ_i congruent to $-s(pr + a) \pmod{p^2 r + pa}$, then
we divide each part by $pr + a$ and realize that these define a subpartition wherein parts are not divisible by p. We now apply Glaisher's map to obtain a subpartition wherein parts appear at most $p - 1$ times. Finally, to get back the size of n, we repeat each part $pr + a$ times in the subpartition wherein parts appear at most $p - 1$ times.

□

In the next section, we consider a more general set $B_{v,p,r,a,m}(n)$, where $m, v, p, a, r \geq 1$ are integers, $\gcd(a,p) = 1$ and $v \leq p$. We define $B_{v,p,r,a,m}(n)$ to be the set of partitions of n in which multiplicities which are congruent to $ja \mod p$ are greater than or equal to $j(pr+a)$ and less than or equal to $j(pr+a) + p(m-1)$ where $j = 0, 1, 2, \ldots, v-1$.
We also let $b_{v,p,r,a,m}(n) = |B_{v,p,r,a,m}(n)|$. Note that $B_{p,r,a,m}(n)$ in Theorem 17 is actually $B_{p,p,r,a,m}(n)$.

3.2.3 Arithmetic properties

We recall (1.22), (1.23) and (1.4).

Theorem 18. *For an integer n, If $\gcd(v,p) \nmid n$, then*

$$b_{v,p,r,a,m}(n) = \sum_{j=1}^{n} (-1)^{j+1} b_{v,p,r,a,m}(n - w(j))$$

where $w(j) = \frac{(pr+a)j(3j\pm1)}{2}$ for $1 \leq j \leq n$ and $b_{v,p,r,a,m}(0) := 1$ and $b_{v,p,r,a,m}(n) = 0$ for all $n < 0$.

Proof. By a similar manipulation as in the proof of Theorem 17, one can show that

$$\sum_{n=0}^{\infty} b_{v,p,r,a,m}(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{v(pr+a)n})(1-q^{pmn})}{(1-q^{(pr+a)n})(1-q^{pn})}$$

so that

$$\prod_{n=1}^{\infty}(1-q^{(pr+a)n}) \sum_{n=0}^{\infty} b_{v,p,r,a,m}(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{v(pr+a)n})(1-q^{pmn})}{1-q^{pn}}.$$

By invoking (1.22), we have

$$(1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{n(pr+a)(3n\pm1)}{2}}) \sum_{n=0}^{\infty} b_{v,p,r,a,m}(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{v(pr+a)n})(1-q^{pmn})}{1-q^{pn}}. \quad (3.3)$$

The exponents of q in the power series representation of the right-hand side of (3.3) are all divisible by $\gcd(v,p)$. Using the notation $[q^n](f(q))$ for the coefficient of q^n in the power series representation of $f(q)$, observe that

$$[q^n]\left((1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{n(pr+a)(3n\pm1)}{2}}) \sum_{n=0}^{\infty} b_{v,p,r,a,m}(n) q^n\right) = 0$$

for all n not divisible by $\gcd(v, p)$. □

Theorem 19. *Let $p > 3$ be prime. Then*

$$b_{2,p,r,a,m}(pn + t) \equiv 0 \pmod 2,\ n \geq 0$$

where $24ta^{-1} + 1$ is a quadratic nonresidue modulo p. Here, a^{-1} is the inverse of a modulo p.

Proof. From the proof of Theorem 18, the generating function of the sequence $b_{2,p,r,a,m}(0), b_{2,p,r,a,m}(1), \ldots$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{2,p,r,a,m}(n)q^n &= \prod_{n=1}^{\infty} \frac{(1 - q^{2(pr+a)n})(1 - q^{pmn})}{(1 - q^{(pr+a)n})(1 - q^{pn})} \\
&= \prod_{n=1}^{\infty} \frac{(1 + q^{(pr+a)n})(1 - q^{pmn})}{1 - q^{pn}} \\
&\equiv \prod_{n=1}^{\infty} \frac{(1 - q^{(pr+a)n})(1 - q^{pmn})}{1 - q^{pn}} \pmod 2 \\
&= \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{s(pr+a)(3s+1)}{2}} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{jpm(3j+1)}{2}} \sum_{k=0}^{\infty} p(k)q^{pk} \quad \text{(by (1.4) and (1.22))} \\
&\equiv \sum_{s=-\infty}^{\infty} q^{\frac{s(pr+a)(3s+1)}{2}} \sum_{j=-\infty}^{\infty} q^{\frac{jpm(3j+1)}{2}} \sum_{k=0}^{\infty} p(k)q^{pk} \pmod 2. \qquad (3.4)
\end{aligned}$$

Comparing the exponents, we have

$$pn + t = \frac{s(pr + a)(3s + 1)}{2} + \frac{pmj(3j + 1)}{2} + pk$$

where $s, j \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. Reducing the equation modulo p, we get

$$t \equiv \frac{sa(3s + 1)}{2} \pmod p.$$

We have

$$\frac{2t}{3a} \equiv s^2 + \frac{s}{3} \equiv \left(s + \frac{1}{6}\right)^2 - \frac{1}{36} \pmod p,$$

i.e.

$$\frac{24t}{a} \equiv (6s + 1)^2 - 1 \pmod p,$$

i.e.

$$24ta^{-1}+1 \equiv (6s+1)^2 \pmod{p}.$$

So if $24ta^{-1}+1$ is a quadratic nonresidue modulo p, then the coefficient of q^n in the right-hand side of (3.4) must be 0. This completes the proof. □

Theorem 20. *Let $p \geq 5$ be prime. Then*

$$b_{4,p,r,a,m}(pn+t) \equiv 0 \pmod{2}, \; n \geq 0$$

where $8ta^{-1}+1$ is a quadratic nonresidue modulo p.

Proof. Observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_{4,p,r,a,m}(n)q^n \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{4(pr+a)n})(1-q^{pmn})}{(1-q^{(pr+a)n})(1-q^{pn})} \\
&\equiv \prod_{n=1}^{\infty} \frac{(1-q^{(pr+a)n})^4(1-q^{pmn})}{(1-q^{(pr+a)n})(1-q^{pn})} \pmod{2} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{(pr+a)n})^3(1-q^{pmn})}{1-q^{pn}} \\
&= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{(pr+a)n(n+1)}{2}} \sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{jpm(3j+1)}{2}} \sum_{k=0}^{\infty} p(k)q^{pk} \;\text{(by (1.4), (1.22) and (1.23))} \\
&\equiv \sum_{n=0}^{\infty} q^{\frac{(pr+a)n(n+1)}{2}} \sum_{j=-\infty}^{\infty} q^{\frac{jpm(3j+1)}{2}} \sum_{k=0}^{\infty} p(k)q^{pk} \pmod{2},
\end{aligned}$$

from which the result follows by a similar reasoning as in the proof of Theorem 19. □

Chapter 4

Extensions and variations of Andrew-Merca identities

In this chapter, we use the following notation for our partition functions:
$d_e(n)$: the number of partitions of n in which exactly one even part is repeated and odd parts are unrestricted;
$d_o(n)$: the number of partitions of n in which exactly one odd part is repeated and even parts are unrestricted;
$a(n)$: the total number of even parts in all partitions of n into distinct parts;
$c_e(n)$: the number of partitions in which exactly one part is repeated and this part is even;
$c_o(n)$: the number of partitions in which exactly one part is repeated and this part is odd;
$b_o(n)$: the number of partitions of n into an odd number of parts in which the set of even parts has only one element;
$b_e(n)$: the number of partitions of n into an even number of parts in which the set of even parts has only one element;
$b(n) = b_o(n) - b_e(n)$ and $c(n) = c_o(n) - c_e(n)$.
We also recall, for the sake of emphasis, the function $d(n)$, which denotes the number of partitions of n into distinct parts.

Partition identities involving various classes of partitions into distinct parts have been studied. Recently, Andrews and Merca [10] obtained the following result.

Theorem 21. *For all $n \geq 1$,*

$$a(n) = c(n) = (-1)^n b(n).$$

Note that, if $b'(n)$ denotes the number of partitions of n in which the set of even parts is singleton, then $b'(n) = b_o(n) + b_e(n)$. Hence, one of the implications of the theorem is that

$$2 \mid a(n) - b'(n). \tag{4.1}$$

In this chapter, we generalize the identitity $a(n) = c(n)$ of Theorem 21 and also generalize (4.1). We then consider some variations. As a consequence, some connections with the work of Fu and Tang in [13] are highlighted and bijective proofs are provided in such cases. This work has been submitted for publication in an ISI-accredited journal.

4.1 Generalizing the identity $a(n) = c(n)$ of Theorem 21

We first observe the following.

Theorem 22. *Suppose that p and r are non-negative integers such that $p \geq r + 2$. Denote by $a_r(n,p)$ the total number of parts congruent to $-r \pmod{p}$ in partitions of n into distinct parts. Let $g_r(n,p)$ denote the number of partitions in which exactly one part is repeated and the multiplicity of this repeated part is at least $p - r$ and congruent to $-r, -r+1 \pmod{p}$. Let $g_{r,o}(n,p)$ (resp. $g_{r,e}(n,p)$) denote the number of $g_r(n,p)$-partitions in which the repeated part is odd (resp. even) and let $c_r(n,p) = g_{r,o}(n,p) - g_{r,e}(n,p)$. Then*

$$a_r(n,p) = c_r(n,p).$$

It is important to realize that setting $p = 2$ and $r = 0$ yields the first equality in Andrews and Merca's theorem, Theorem 21.

Proof. Let $F(z,q)$ be the multivariable generating function for $c_r(n,p)$ in which z tracks the repeated part. Then

$$F(z,q) = \sum_{n=1}^{\infty} z^n (q^{(p-r)n} + q^{(p-r+p)n} + q^{(p-r+2p)n} + \cdots + q^{(p-r+1)n} + q^{(p-r+1+p)n}$$

$$+ q^{(p-r+1+2p)n} + \cdots) \prod_{j\neq n, j=1}^{\infty} (1+q^j)$$

$$= \sum_{n=1}^{\infty} z^n (q^{(p-r)n}(1+q^{pn}+q^{2pn}+q^{3pn}+\cdots) + q^{(p-r+1)}(1+q^{pn}$$

$$+ q^{2pn} + q^{3pn} + \cdots)) \prod_{j\neq n, j=1}^{\infty} (1+q^j)$$

$$= \sum_{n=1}^{\infty} z^n q^{(p-r)n}(1+q^n)(1+q^{pn}+q^{2pn}+q^{3pn}+\cdots) \prod_{j\neq n, j=1}^{\infty} (1+q^j)$$

$$= \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}(1+q^n)}{1-q^{pn}} \prod_{j\neq n, j=1}^{\infty} (1+q^j)$$

$$= \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}}{1-q^{pn}} \prod_{j=1}^{\infty}(1+q^j)$$

$$= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}}{1-q^{pn}}.$$

Using the fact that $\sum_{n=0}^{\infty}(g_{r,o}(n,p) - g_{r,e}(n,p))q^n = -F(-1,q)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} c_r(n,p)q^n &= -F(-1,q) \\
&= -(-q;q)_\infty \sum_{n=1}^{\infty} \frac{(-1)^n q^{(p-r)n}}{1-q^{pn}} \\
&= -(-q;q)_\infty \sum_{n=1}^{\infty} (-1)^n q^{-rn} \frac{q^{pn}}{1-q^{pn}} \\
&= -(-q;q)_\infty \sum_{n=1}^{\infty} (-1)^n q^{-rn} \sum_{m=1}^{\infty} q^{pnm}
\end{aligned}$$

$$
\begin{aligned}
&= -(-q;q)_\infty \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n q^{-rn}q^{pnm} \\
&= -(-q;q)_\infty \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-q^{-r+pm})^n \\
&= -(-q;q)_\infty \sum_{m=1}^{\infty}\frac{-q^{-r+pm}}{1+q^{pm-r}} \\
&= (-q;q)_\infty \sum_{m=1}^{\infty}\frac{q^{-r+pm}}{1+q^{pm-r}}.
\end{aligned}
$$

On the other hand,

$$
\sum_{n=0}^{\infty} a_r(n,p)q^n = \frac{\partial}{\partial z}\Big|_{z=1}\prod_{n=1}^{\infty}(1+zq^{pn-r})\prod_{i=1,i\neq r}^{p}\prod_{n=1}^{\infty}(1+q^{pn-i})
$$

$$
= \prod_{i=1,i\neq r}^{p}\prod_{n=1}^{\infty}(1+q^{pn-i})\prod_{n=1}^{\infty}(1+zq^{pn-r})\Big|_{z=1}\frac{\partial}{\partial z}\Big|_{z=1}\sum_{n=1}^{\infty}\log(1+zq^{pn-r})
$$

$$
\begin{aligned}
&= \prod_{n=1}^{\infty}(1+q^n)\sum_{n=1}^{\infty}\frac{q^{pn-r}}{1+q^{pn-r}} \\
&= (-q;q)_\infty \sum_{n=1}^{\infty}\frac{q^{-r+pn}}{1+q^{pn-r}}.
\end{aligned}
$$

Thus

$$
\sum_{n=0}^{\infty} a_r(n,p)q^n = \sum_{n=0}^{\infty} c_r(n,p)q^n
$$

and the theorem follows. □

The above theorem provides a new combinatorial interpretation of the total number of parts congruent to $-r$ modulo p in partitions of n into distinct parts.

Recall the function $O_{p,j}(n)$ in [13]. For $j = 1$, we shall denote this function by $o_p(n)$ which counts partitions of n in which the set of parts congruent to 0 modulo p is singleton. Define $a(n,p)$ to be the total number of parts congruent to 0 modulo p in all partitions of n into parts appearing at most $p-1$ times. Clearly

$a(n,2) = a(n) = a_0(n,2)$ and $o_2(n) = b'(n)$. We have the following property which extends (4.1).

Theorem 23. *For all $n \geq 0$,*

$$p \mid a(n,p) - o_p(n).$$

Proof. The generating function for $o_p(n)$ is given by

$$\sum_{n=0}^{\infty} o_p(n) q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}}$$

and the generating function for $a(n,p)$ is given by Herden, Sepanski, et al. [16]

$$\frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn} + 2q^{2pn} + 3q^{3pn} + \ldots (p-1)q^{(p-1)pn}}{1 + q^{pn} + q^{2pn} + \ldots + q^{(p-1)pn}}.$$

Recall the identity:

$$\sum_{k=1}^{n} kx^k = \frac{x(1-x^{n+1}) - (n+1)(1-x)x^{n+1}}{(1-x)^2}. \tag{4.2}$$

Manipulating the generating function, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a(n,p) q^n &= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{\sum_{j=1}^{p-1} j q^{pnj}}{\frac{1-q^{p^2 n}}{1-q^{pn}}} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{\frac{q^{pn}(1-q^{p^2 n}) - p(1-q^{pn})q^{p^2 n}}{(1-q^{pn})^2}}{\frac{1-q^{p^2 n}}{1-q^{pn}}} \quad \text{(by (4.2) with } n = p-1, x = q^{pn}) \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{1-q^{pn}}{1-q^{p^2 n}} \frac{q^{pn}(1-q^{p^2 n}) - p(1-q^{pn})q^{p^2 n}}{(1-q^{pn})^2} \\
&\equiv \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{1-q^{pn}}{1-q^{p^2 n}} \frac{q^{pn}(1-q^{p^2 n})}{(1-q^{pn})^2} \pmod{p} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}} \\
&= \sum_{n=0}^{\infty} o_p(n) q^n.
\end{aligned}$$

□

Let $f(q) = \sum\limits_{n=1}^{\infty} \frac{q^n}{1-q^n} - p \sum\limits_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}}$ and $s(n) = p(n)$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} a(n,p)q^n &= (q^p;q^p)_\infty \left(\sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}} - p \sum_{n=1}^{\infty} \frac{q^{p^2 n}}{1-q^{p^2 n}} \right) \frac{1}{(q;q)_\infty} \\ &= (q^p;q^p)_\infty f(q^p) \sum_{n=0}^{\infty} s(n)q^n \\ &= \sum_{j=0}^{p-1} q^j (q^p;q^p)_\infty f(q^p) \sum_{n=0}^{\infty} s(pn+j)q^{pn}\end{aligned}$$

so that

$$\sum_{j=0}^{p-1} q^j \sum_{n=0}^{\infty} a(pn+j,p)q^{pn} = \sum_{j=0}^{p-1} q^j (q^p;q^p)_\infty f(q^p) \sum_{n=0}^{\infty} s(pn+j)q^{pn}$$

which implies that, for a fixed $0 \le j \le p-1$,

$$\sum_{n=0}^{\infty} a(pn+j,p)q^{pn+j} = (q^p;q^p)_\infty f(q^p) \sum_{n=0}^{\infty} s(pn+j)q^{pn+j}$$

so that

$$\sum_{n=0}^{\infty} a(pn+j,p)q^{pn} = (q^p;q^p)_\infty f(q^p) \sum_{n=0}^{\infty} s(pn+j)q^{pn}.$$

Replacing q with $q^{1/p}$ yields

$$\sum_{n=0}^{\infty} a(pn+j,p)q^{n} = (q;q)_\infty f(q) \sum_{n=0}^{\infty} s(pn+j)q^{n}.$$

Thus, for $n \ge 0$,

$$s(pn+j) \equiv 0 \pmod{p} \;\Rightarrow\; a(pn+j,p) \equiv 0 \pmod{p}.$$

In particular, applying the Ramanujan's partition congruences: $s(5n+4) \equiv 0 \pmod{5}$, $s(7n+5) \equiv 0 \pmod{7}$ and $s(11n+6) \equiv 0 \pmod{11}$, we have the following result.

Theorem 24. *For all $n \ge 0$,*

$$a(5n+4,5) \equiv 0 \pmod{5},$$

$$a(7n+5,7) \equiv 0 \pmod{7}$$

and

$$a(11n+6,11) \equiv 0 \pmod{11}.$$

Let $o_{p,e}(n)$ (resp. $o_{p,o}(n)$) be the number of $o_p(n)$-partitions in which the number of parts congruent to 0 modulo p is even (resp. odd) and

$$H(z,q) = \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{zq^{pn}}{1-zq^{pn}}.$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty} o_{p,o}(n)q^n &= \frac{1}{2}\left(H(1,q) - H(-1,q)\right) \\
&= \frac{(q^p;q^p)_\infty}{2(q;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^{pn}}{1-q^{pn}} + \sum_{n=1}^{\infty}\frac{q^{pn}}{1+q^{pn}}\right) \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{pn}}{1-q^{2pn}} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty} q^{pn}\sum_{m=0}^{\infty} q^{2pnm} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} q^{(p+2pm)n} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\frac{q^{p+2pm}}{1-q^{2pm+p}} \qquad (4.3)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\sum_{n=0}^{\infty} o_{p,e}(n)q^n &= \frac{1}{2}\left(H(1,q) + H(-1,q)\right) \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2pn}}{1-q^{2pn}}. \qquad (4.4)
\end{aligned}$$

Thus, if for $i = 0, p$, $h_i(n,p)$ is the number of partitions in which parts are congruent to $1, 2, 3, \ldots, p-1$ modulo p or congruent to i modulo $2p$ and the set of parts $\equiv i \pmod{2p}$ is singleton, then

$$o_{p,o}(n) = h_p(n,p) \quad \text{and} \quad o_{p,e}(n) = h_0(n,p).$$

4.2 Further variations

For $i = 0, 2$, let $f_i(n)$: the number of partitions of n in which the set of parts congruent to $i \pmod 4$ is singleton. Then we have the following theorem:

Theorem 25. *For $n \geq 1$,*

$$d_e(n) = f_0(n) \quad \textit{and} \quad d_o(n) = f_2(n).$$

Proof. Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} d_e(n)q^n &= \frac{1}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n+2n}}{1-q^{2n}} \prod_{j\neq n, j=1}^{\infty} (1+q^{2j}) \\
&= \frac{1}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}(-q^2;q^2)_\infty}{(1-q^{2n})(1+q^{2n})} \\
&= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \\
&= \frac{(-q^2;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \\
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \\
&= \frac{1}{(q;q^4)_\infty (q^2;q^4)_\infty (q^3;q^4)_\infty} \sum_{n=1}^{\infty} (q^{4n} + q^{4n+4n} + q^{4n+4n+4n} + \ldots) \\
&= \sum_{n=0}^{\infty} f_0(n)q^n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{n=0}^{\infty} d_o(n)q^n &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n-1+2n-1}}{1-q^{2n-1}} \prod_{j\neq n, j=1}^{\infty} (1+q^{2j-1}) \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}(-q;q^2)_\infty}{(1-q^{2n-1})(1+q^{2n-1})} \\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{1-q^{4n-2}}
\end{aligned}$$

$$= \frac{(-q;q^2)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty (q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{1-q^{4n-2}}$$

$$= \frac{(q^2;q^4)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{1-q^{4n-2}}$$

$$= \frac{1}{(q;q^4)_\infty (q^3;q^4)_\infty (q^4;q^4)_\infty} \sum_{n=1}^{\infty} (q^{4n-2} + q^{2(4n-2)} + q^{3(4n-2)} + \dots)$$

$$= \sum_{n=0}^{\infty} f_2(n) q^n.$$

□

We also note the following bijective proof which is uniform for both identities in the theorem.

The bijection

Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_l^{m_l})$ be a partition enumerated by $f_2(n)$ or $f_0(n)$. Then

$$\lambda_i^{m_i} \mapsto \begin{cases} \left(\dfrac{\lambda_i}{2}\right)^{2m_i}, & \lambda_i \equiv 0, 2 \pmod 4; \\ \\ \phi_2(\lambda_i^{m_i}), & \lambda_i \equiv 1, 3 \pmod 4. \end{cases}$$

To invert the mapping, let $\mu = (\mu_1^{s_1}, \mu_2^{s_2}, \dots, \mu_l^{s_l})$ be a partition enumerated by $d_o(n)$ or $d_e(n)$. Then

$$\mu_i^{s_i} \mapsto (2\mu_i)^{\lfloor \frac{s_i}{2} \rfloor} \cup \phi_2^{-1}(\mu_i^{s_i - 2\lfloor \frac{s_i}{2} \rfloor}).$$

Recall that

$$\sum_{n=0}^{\infty} d_e(n) q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}}$$

so that

$$(q;q)_\infty \sum_{n=0}^{\infty} d_e(n) q^n = (q^4;q^4)_\infty \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}}. \tag{4.5}$$

Using (1.22), we have

$$(q;q)_\infty \sum_{n=0}^{\infty} d_e(n)q^n = \left(1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{n(3n\pm1)}{2}}\right)\sum_{n=0}^{\infty} d_e(n)q^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_k d_e(n-k)q^n \tag{4.6}$$

where

$$c_k = \begin{cases} (-1)^j, & \text{if } k = \frac{j(3j\pm1)}{2}, j \in \mathbb{Z}_{\geq 0};\\ 0, & \text{otherwise.}\end{cases}$$

Note that the right-hand side of (4.6) is equal to

$$\sum_{n=0}^{\infty}\left[d_e(n) + \sum_{j=1}^{\lfloor\frac{1+\sqrt{24n+1}}{6}\rfloor}(-1)^j\left(d_e\left(n-\frac{j(3j+1)}{2}\right)+d_e\left(n-\frac{j(3j-1)}{2}\right)\right)\right]q^n.$$

Invoking (1.22) and the identity

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} = \sum_{n=1}^{\infty}\sigma_0(n)q^n$$

where $\sigma_0(n)$ is the number of positive divisors of n, (4.5) becomes

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}\cdot\sum_{n=0}^{\infty} d_e(n)q^n$$
$$= \sum_{n=-\infty}^{\infty}(-1)^n q^{2n(3n+1)}\sum_{n=1}^{\infty}\sigma_0(n)q^{4n}.$$

Hence, we have the following.

Corollary 5. *For all integers $r \geq 0$, let*

$$A(r) = \{j \in \mathbb{Z}_{>0} : 2j(3j+1) \equiv r \pmod 4 \text{ and } j \leq \left\lfloor\frac{\sqrt{6r+1}-1}{6}\right\rfloor\}$$

and

$$B(r) = \{j \in \mathbb{Z}_{>0} : 2j(3j-1) \equiv r \pmod 4 \text{ and } j \leq \left\lfloor\frac{1+\sqrt{6r+1}}{6}\right\rfloor\}.$$

Then for a positive integer n,

$$d_e(n) = \begin{cases} \sum_{j=1}^{\lfloor \frac{1+\sqrt{24n+1}}{6} \rfloor} (-1)^{j+1} \left(d_e\left(n - \frac{j(3j+1)}{2}\right) + d_e\left(n - \frac{j(3j-1)}{2}\right)\right), & \text{if } n \not\equiv 0 \pmod 4 \\ \sum_{j=1}^{\lfloor \frac{1+\sqrt{24n+1}}{6} \rfloor} (-1)^{j+1} \left(d_e\left(n - \frac{j(3j+1)}{2}\right) + d_e\left(n - \frac{j(3j-1)}{2}\right)\right) + \gamma(n), & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

where

$$\gamma(n) = \sigma_0(n/4) + \sum_{j \in A(n)} (-1)^j \sigma_0\left(\frac{n - 2j(3j+1)}{4}\right) + \sum_{j \in B(n)} (-1)^j \sigma_0\left(\frac{n - 2j(3j-1)}{4}\right).$$

In this case, we have a relationship between $d_e(n)$ and $\sigma_0(n)$.

Example 3. *Consider $n = 8$.*

The $d_e(8)$-partitions are:

$$4^2, (4, 2^2), (3, 2^2, 1), 2^4, (2^3, 1^2), (2^2, 1^4)$$

We now apply the recurrence. Since $8 \equiv 0 \pmod 4$, we have

$$\begin{aligned} d_e(8) &= d_e(8-2) + d_e(8-1) - d_e(8-7) - d_e(8-5) + \gamma(8) \\ &= d_e(6) + d_e(7) - d_e(1) - d_e(3) + \gamma(8) \\ &= d_e(4) + 2d_e(5) - d_e(1) + d_e(6) - 2d_e(2) - d_e(1) + \gamma(8) \\ &= d_e(2) + 3d_e(3) + 3d_e(4) + d_e(5) - 3d_e(1) + \gamma(4) + \gamma(8) \\ &= 4d_e(1) + 6d_e(2) + 4d_e(3) + d_e(4) + 4\gamma(4) + \gamma(8) \\ &= 10d_e(1) + 5d_e(2) + d_e(3) + 5\gamma(4) + \gamma(8) \\ &= 6d_e(1) + d_e(2) + 5\gamma(4) + \gamma(8) \\ &= d_e(1) + 5\gamma(4) + \gamma(8) \\ &= 5\gamma(4) + \gamma(8). \end{aligned}$$

Now, $A(8) = \{1\}, B(8) = \{1\}, A(4) = \{\}$ and $B(4) = \{1\}$ and so

$$\begin{aligned} d_e(8) &= 5\gamma(4) + \gamma(8) \\ &= 5[\sigma_0(1) - \sigma_0(0)] + \sigma_0(2) + \sigma_0(0) - \sigma_0(1) \end{aligned}$$

$$
\begin{aligned}
&= \sigma_0(2) + 4\sigma_0(1) - 5\sigma_0(0) \\
&= 2 + 4(1) - 5(0) \\
&= 6.
\end{aligned}
$$

We also note that

$$
\begin{aligned}
\sum_{n=0}^{\infty} d_o(n)q^n &= \frac{(q^2;q^4)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{1-q^{4n-2}} \\
&= \frac{1}{(-q^2;q^2)_\infty (q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{1-q^{4n-2}} \\
&\equiv \frac{1}{(q^2;q^2)_\infty (q;q)_\infty} \left(\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \right) \pmod 2 \\
&\equiv \frac{1}{(q;q)_\infty^3} \left(\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \right) \pmod 2
\end{aligned}
$$

so that

$$
\sum_{n=0}^{\infty} d_o(n)q^n \sum_{n=0}^{\infty} q^{n(n+1)/2} \equiv \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \pmod 2 \tag{4.7}
$$

where we have used (3) for $(q;q)_\infty^3$.
But the right-hand side of (4.7) is

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \sigma_0(n)q^{2n} - \sum_{n=1}^{\infty} \sigma_0(n)q^{4n} \\
&= \sum_{n>1, n\equiv 0 \pmod 4} \sigma_0(n/2)q^n + \sum_{n>1, n\equiv 2 \pmod 4}^{\infty} \sigma_0(n/2)q^n - \sum_{n>1, n\equiv 0 \pmod 4}^{\infty} \sigma_0(n/4)q^n \\
&= \sum_{n>1, n\equiv 0 \pmod 4} (\sigma_0(n/2) - \sigma_0(n/4))q^n + \sum_{n>1, n\equiv 2 \pmod 4}^{\infty} \sigma_0(n/2)q^n.
\end{aligned}
$$

If $n \equiv 0 \pmod 4$, we can write $n = 2^m b$ for some positive integers $m \geq 2$ and b odd. Then

$$
\begin{aligned}
\sigma_0(n/2) - \sigma_0(n/4) &= \sigma_0(2^m b/2) - \sigma_0(2^m b/4) \\
&= \sigma_0(2^{m-1}b) - \sigma_0(2^{m-2}b) \\
&= \sigma_0(2^{m-1})\sigma_0(b) - \sigma_0(2^{m-2})\sigma_0(b) \quad (\sigma_0 \text{ is multiplicative})
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma_0(b)(\sigma_0(2^{m-1}) - \sigma_0(2^{m-2})) \\
&= \sigma_0(b)(m - (m-1)) \\
&= \sigma_0(b).
\end{aligned}
$$

On the other hand, if $n \equiv 2 \pmod 4$, we have $n = 2b$ for some odd positive integer b. Thus,

$$\sigma_0(n/2) = \sigma_0(b).$$

Denoting by $v_2(n)$, the 2-adic valuation of n, we obtain the following result.

Corollary 6. *For $n > 0$, we have*

$$\sum_{j=0}^{\lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor} d_o(n - j(j+1)/2) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \equiv 1 \pmod 2; \\ \sigma_0\left(n/2^{v_2(n)}\right) \pmod 2, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

4.3 Generalizations

The partition function $d_o(n)$ can be generalized as follows: Let $k, p \geq 2$ and $0 \leq r < p$ be integers and define $d_p(n, k, r)$ as the number of partitions of n in which exactly one part $\equiv r \pmod p$ appears at least k times. In this definition, parts not congruent to $r \pmod p$ appear with unrestricted multiplicity. Similarly, let $f_p(n, k, r)$ denote the number of partitions of n in which parts $\equiv kr \pmod{pk}$ form a singleton set.

If $r \neq 0$, we have

$$\sum_{n=0}^{\infty} d_p(n,k,r)q^n = \frac{(q^r;q^p)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{k(pn+r)}}{1-q^{pn+r}} \prod_{j\neq n, j=0}^{\infty} (1 + q^{pj+r} + q^{2(pj+r)} + \cdots + q^{(k-1)(pj+r)})$$

$$
\begin{aligned}
&= \frac{(q^r;q^p)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{k(pn+r)}}{(1-q^{pn+r}) \sum_{i=0}^{k-1} q^{(pn+r)i}} \prod_{j=0}^{\infty} \frac{1-q^{k(pj+r)}}{1-q^{pj+r}} \\
&= \frac{(q^r;q^p)_\infty (q^{kr};q^{kp})_\infty}{(q^r;q^p)_\infty (q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{pkn+kr}}{1-q^{pkn+kr}}
\end{aligned}
$$

$$= \frac{(q^{kr};q^{kp})_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{pkn+kr}}{1-q^{pkn+kr}}$$
$$= \sum_{n=0}^{\infty} f_p(n,k,r)q^n.$$

If $r=0$, then

$$\sum_{n=0}^{\infty} d_p(n,k,0)q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{kpn}}{1-q^{pn}}\prod_{j\neq n, j=1}^{\infty}(1+q^{pj}+q^{2pj}+\cdots+q^{(k-1)pj})$$

which simplifies to

$$\frac{(q^{kp};q^{kp})_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{pkn}}{1-q^{pkn}} = \sum_{n=0}^{\infty} f_p(n,k,0)q^n.$$

Thus, we have:

Theorem 26. *Let $p \geq 2$. For an integer $n \geq 1$,*

$$f_p(n,k,r) = d_p(n,k,r).$$

A bijection (which extends the one for $d_e(n)/f_0(n)$ and $d_o(n)/f_2(n)$), is given as follows:

Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_l^{m_l})$ be a partition enumerated by $f_p(n,k,r)$. Then

$$\lambda_i^{m_i} \mapsto \begin{cases} \left(\dfrac{\lambda_i}{k}\right)^{km_i}, & \text{if } \lambda_i \equiv 0, k, 2k, \ldots, (p-1)k \pmod{pk}; \\ \\ \phi_k(\lambda_i^{m_i}), & \text{otherwise.} \end{cases}$$

From λ, note that there is only one part congruent to $kr \pmod{pk}$ (with unrestricted multiplicity). Under the bijection, this part is mapped to the exactly one part which is congruent to $r \pmod p$ repeated at least k times. For other parts of λ that are not congruent to $kr \pmod{kp}$, there are two cases.
Case 1: If the part is divisible by k, it can be shown that such a part must be congruent to $jk \pmod{pk}$ where $j \in \{0,1,2,\ldots,p-1\} \setminus \{r\}$. This part is mapped to a part not congruent to $r \pmod p$ with multiplicity at least k.

Case 2: If that part of λ is not divisible by k, we apply ϕ_k.

Conversely, let $\mu = (\mu_1^{s_1}, \mu_2^{s_2}, \ldots, \mu_l^{s_l})$ be a partition enumerated by $d_p(n,k,r)$. Then

$$\mu_i^{s_i} \mapsto (k\mu_i)^{\lfloor \frac{s_i}{k} \rfloor} \cup \phi_k^{-1}(\mu_i^{s_i - k\lfloor \frac{s_i}{k} \rfloor}) \tag{4.8}$$

represents the inverse mapping.

Example 4. *Consider $n = 9$, $p = 3$, $k = 4$ and $r = 1$.*

The $d_3(9,4,1)$-partitions are:

$$(5,1^4), (4,1^5), (3,2,1^4), (3,1^6), (2^2,1^5), (2,1^7), (1^9).$$

To find the image of (1^9), we perform the transformation:

$$\left\lfloor \frac{9}{4} \right\rfloor = 2 \text{ and } 9 - 4\left\lfloor \frac{9}{4} \right\rfloor = 1.$$

Applying the inverse map in (4.8) yields

$$1^9 \mapsto (4^2, 1).$$

Similarly, we have

$$(5,1^4) \mapsto (5,4)$$
$$(4,1^5) \mapsto (4,1^5)$$
$$(3,2,1^4) \mapsto (4,3,2)$$
$$(3,1^6) \mapsto (4,3,1^2)$$
$$(2^2,1^5) \mapsto (4,2^2,1)$$
$$(2,1^7) \mapsto (4,2,1^3).$$

4.3.1 Connection with the work of Fu and Tang

Recall that $o_k(n)$ denotes the number of partitions of n in which the set of parts congruent to 0 modulo k is singleton. Fu and Tang [13] showed that if $d_k(n)$ is the number of partitions of n where exactly one part is repeated at least k times, then

$$o_k(n) = d_k(n). \tag{4.9}$$

Setting $k = 4$ in (4.9), $o_4(n) = d_4(n)$. On the other hand, we have shown that $f_0(n) = d_e(n)$ in Theorem 25. Since $f_0(n) = o_4(n)$, we have the identity

$$d_4(n) = d_e(n). \tag{4.10}$$

Note that (4.10) is not 'trivial'. As a preparation for a more generalized result, we describe its bijective proof.
Let λ be enumerated by $d_4(n)$ and j be the part that is repeated at least 4 times. Then $\lambda = \bar{\lambda} \cup j^m$ where m is the multiplicity of j in λ and $\bar{\lambda}$ is the subpartition of λ consisting of parts that are repeated at most 3 times. Write m as

$$m = 4q + i \quad \text{where} \ \ 0 \leq i \leq 3 \ \text{ and } \ q \geq 1.$$

One writes $j^m = j^{4q} \cup j^i$ and converts j^{4q} into $(2j)^{2q}$.
Apply Glaisher map ϕ_4^{-1} on $\bar{\lambda} \cup j^i$ so that the image $\phi_4^{-1}(\bar{\lambda} \cup j^i)$ is a partition into parts not divisible by 4.
Decompose the partition as

$$\phi_4^{-1}(\bar{\lambda} \cup j^i) = \lambda' \cup \lambda''$$

where λ' is the subpartition consisting of odd parts and λ'' is the subpartition consisting of parts $\equiv 2 \pmod 4$. Divide each part of λ'' by 2 and apply ϕ_2 on the resulting partition. What follows is a partition into distinct parts. Then multiply every part of this partition by 2 and call the resulting partition β.
Note that

$$(2j)^{2q} \cup \beta \cup \lambda'$$

is a partitition enumerated by $d_e(n)$.

The inverse will be demonstrated in the general map. We require the following lemma in the subsequent work.

Lemma 2. *Let p, k and x be positive integers such that $pk \geq 2$. Suppose that $x \not\equiv 0 \pmod{pk}$ and $x \equiv 0 \pmod p$. Then $\frac{x}{p} \not\equiv 0 \pmod k$.*

Now, setting $r = 0$ in Theorem 26, observe that $d_p(n, k, 0)$ is the number of partitions of n in which exactly one part $\equiv 0 \pmod p$ appears at least k times. By the above result of Fu and Tang, the following generalization of (4.10) follows:

Theorem 27. *For $n \geq 0$,*

$$d_{pk}(n) = d_p(n, k, 0).$$

Our interest in this theorem is in its bijective construction, which extends the given mapping for the special case $k = 2$ and $p = 2$.

Let λ be enumerated by $d_{pk}(n)$ and j be the part that is repeated at least pk times. Then $\lambda = \bar{\lambda} \cup j^m$ where is the multiplicity of j in λ and $\bar{\lambda}$ is the subpartition of λ consisting of parts that are repeated at most $pk - 1$ times. Write m as

$$m = pkq + i \quad \text{where} \quad 0 \leq i \leq pk - 1 \quad \text{and} \quad q \geq 1.$$

One rewrites j^m as $j^m = j^{pkq} \cup j^i$ and converts j^{pkq} into $(pj)^{kq}$.

Apply Glaisher map ϕ_{pk}^{-1} on $\bar{\lambda} \cup j^i$ so that the image $\phi_{pk}^{-1}(\bar{\lambda} \cup j^i)$ is a partition into parts not divisible by pk.

Decompose this partition as

$$\phi_{pk}^{-1}(\bar{\lambda} \cup j^i) = \lambda' \cup \lambda''$$

where λ' is the subpartition consisting of parts $\not\equiv 0 \pmod{p}$ and λ'' is the subpartition consisting of parts $\equiv 0 \pmod{p}$. Divide each part of λ'' by p and note that, by Lemma 2, the resulting parts are not divisible by k. Apply ϕ_k on the resulting partition. What follows is a partition into parts that appear at most $k - 1$ times . Then multiply every part of this partition by p and call the resulting partition β. Note that

$$(pj)^{kq} \cup \beta \cup \lambda'$$

is a partitition enumerated by $d_p(n, k, 0)$.

Example 5. *Consider $n = 433$, $p = 3$ and $k = 4$.*

Consider the $d_{12}(433)$-partition:

$$(13^{10}, 10^5, 7^{30}, 6^2, 4^5, 1^{11})$$

which is decomposed as:

$$(13^{10}, 10^5, 6^2, 4^5, 1^{11}) \cup (7^{30})$$

where $j = 7$ and $m = 30 = 12 \cdot 2 + 6$. Thus $7^{30} = (7^{24}) \cup (7^6)$ and so

$$(7^{24}) \mapsto (21^8).$$

Applying the Glaisher's map to the remaining parts yields

$$\phi_{12}^{-1}(13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{11}) = (13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{11}).$$

Note that $6 \equiv 0 \pmod 3$ and thus

$$\beta = 3 \times \phi_4\left(\frac{6}{3}, \frac{6}{3}\right) = (6^2).$$

Taking the union of all the image parts gives

$$(21^8, 13^{10}, 10^5, 6^2, 4^5, 1^{11})$$

which is a $d_3(433, 4, 0)$-partition.

The inverse is described as follows:
Let μ be enumerated by $d_p(n, k, 0)$. Decompose the partition as

$$\mu = s^m \cup \beta \cup \lambda'$$

where s is that one part $\equiv 0 \pmod p$ whose multiplicity m is at least k times, β is the subpartition consisting of parts $\equiv 0 \pmod p$ whose multiplicities are $\leq k-1$ and λ' is the subpartition consisting of parts $\not\equiv 0 \pmod p$. Write m as

$$m = kt + f \quad \text{where} \quad 0 \leq f \leq k-1 \quad \text{and} \quad t \geq 1.$$

One rewrites s^m as $s^m = s^{kt} \cup s^f$ and converts s^{kt} into $\left(\frac{s}{p}\right)^{pkt}$.
Divide each part of $\beta \cup s^f$ by p and then apply ϕ_k^{-1} on the resulting partition. Then multiply every part of the obtained partition by p and denote the resulting partition by μ'. Note that μ' is a partition whose parts are not divisible by pk but divisible by p. Thus the parts of $\lambda' \cup \mu'$ are not divisible by pk. Now compute

$$\mu'' = \phi_{pk}(\lambda' \cup \mu')$$

so that μ'' is a partition with parts appearing at most $pk-1$ times. Clearly,

$$\left(\frac{s}{p}\right)^{pkt} \cup \mu''$$

is a partition enumerated by $d_{pk}(n)$.

Reversing the previous example, recall the $d_3(433,4,0)$-partition:

$$(21^8, 13^{10}, 10^5, 6^2, 4^5, 1^{11}).$$

We decompose it follows:

$$(21^8) \cup (6^2) \cup (13^{10}, 10^5, 4^5, 1^{11}).$$

Thus,

$$(21^8) \mapsto (7^{24}),$$

$$3 \times \phi_4^{-1}\left(\frac{6}{3}, \frac{6}{3}\right) = (6^2).$$

Applying the Glaisher's map to $(13^{10}, 10^5, 4^5, 1^{11}) \cup (6^2)$ gives

$$\phi_{12}(13^{10}, 10^5, 6^2, 4^5, 1^{11}) = (13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{11}).$$

Now, taking the union of all the image parts yields

$$(13^{10}, 10^5, 7^{30}, 6^2, 4^5, 1^{11})$$

which is a $d_{12}(433)$-partition.

4.3.2 A slightly different partition function

For $\alpha \geq k$, let $g(n, \alpha, k, p)$ denote the number of partitions of n in which only one part appears at least α times and its multiplicity is congruent to $\alpha + j$ modulo p where $0 \leq j \leq k-1$ and all other parts appear at most $k-1$ times. Denote by $g_o(n, \alpha, k, p)$ (resp. $g_e(n, \alpha, k, p)$ the number of $g(n, \alpha, k, p)$-partitions in which the part repeated at least α times is odd (resp. even). Then we have:

Theorem 28.

$$\sum_{n=0}^{\infty} g_o(n, \alpha, k, p) - g_e(n, \alpha, k, p)q^n = \frac{(q^k; q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{pn+\alpha}}{1+q^{pn+\alpha}}. \tag{4.11}$$

Proof. If z tracks the part repeated at least α times, observe that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g(n, \alpha, k, p)q^n z^m$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \big(z^n q^{\alpha n} + z^n q^{(\alpha+p)n} + z^n q^{(\alpha+2p)n} + z^n q^{(\alpha+3p)n} + z^n q^{(\alpha+4p)n} + \\
&\qquad \cdots + z^n q^{(\alpha+1+p)n} + z^n q^{(\alpha+1+2p)n} + z^n q^{(\alpha+1+3p)n} + z^n q^{(\alpha+1+4p)n} + \\
&\qquad \cdots + z^n q^{(\alpha+2+p)n} + z^n q^{(\alpha+2+2p)n} + z^n q^{(\alpha+2+3p)n} + z^n q^{(\alpha+2+4p)n} + \\
&\qquad \cdots + z^n q^{(\alpha+3+p)n} + z^n q^{(\alpha+3+2p)n} + z^n q^{(\alpha+3+3p)n} + z^n q^{(\alpha+3+4p)n} + \\
&\qquad\qquad \vdots \\
&\qquad \cdots + z^n q^{(\alpha+k-2+p)n} + z^n q^{(\alpha+k-2+2p)n} + z^n q^{(\alpha+k-2+3p)n} + z^n q^{(\alpha+k-2+4p)n} + \\
&\qquad \cdots + z^n q^{(\alpha+k-1+p)n} + z^n q^{(\alpha+k-1+2p)n} + z^n q^{(\alpha+k-1+3p)n} + z^n q^{(\alpha+k-1+4p)n} + \\
&\qquad \cdots \big) \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}) \\
&= \sum_{n=1}^{\infty} z^n \sum_{i=0}^{k-1} \sum_{m=0}^{\infty} q^{(\alpha+i+mp)n} \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}) \\
&= \sum_{n=1}^{\infty} z^n q^{\alpha n} \sum_{i=0}^{k-1} q^{in} \sum_{m=0}^{\infty} q^{mpn} \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}) \\
&= \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1 - q^{pn}} \sum_{i=0}^{k-1} q^{in} \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}). \\
&= \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1 - q^{pn}} \prod_{j=1}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}). \\
&= \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1 - q^{pn}} \prod_{j=1}^{\infty} \frac{1 - q^{kj}}{1 - q^j} \\
&= \frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1 - q^{pn}}
\end{aligned}
$$

so that

$$
\begin{aligned}
\sum_{n=0}^{\infty} (g_o(n, \alpha, k, p) - g_e(n, \alpha, k, p)) q^n &= -\frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\alpha n}}{1 - q^{pn}} \\
&= -\frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{\alpha n} \sum_{m=0}^{\infty} q^{pnm} \\
&= -\frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-q^{\alpha+pm})^n
\end{aligned}
$$

$$= -\frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\frac{-q^{\alpha+pm}}{1+q^{\alpha+pm}}$$
$$= \frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\frac{q^{\alpha+pm}}{1+q^{\alpha+pm}}.$$

□

We have the following relationship between $h(n,p)$ and $g(n,p,p,p)$.

Proposition 3. *For all $n \geq 0$,*

$$g_o(n,p,p,p) = h_p(n,p) \ \ \textit{and}\ \ g_e(n,p,p,p) = h_0(n,p).$$

Proof. From the fact that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g(n,p,p,p)q^n z^m = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{z^n q^{pn}}{1-q^{pn}},$$

recall that z tracks the part which is repeated at least p times. Then

$$\sum_{n=0}^{\infty} g_o(n,p,p,p)q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1, n\equiv 1 \pmod 2}^{\infty}\frac{q^{pn}}{1-q^{pn}}$$
$$= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2pn-p}}{1-q^{2pn-p}}$$
$$= \sum_{n=0}^{\infty} h_p(n,p)q^n$$

and on the same note,

$$\sum_{n=0}^{\infty} g_e(n,p,p,p)q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1, n\equiv 0 \pmod 2}^{\infty}\frac{q^{pn}}{1-q^{pn}}$$
$$= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2pn}}{1-q^{2pn}}$$
$$= \sum_{n=0}^{\infty} h_0(n,p)q^n.$$

□

Chapter 5

Legendre theorems and parity for partitions with initial repetitions

In this chapter, we study partitions with initial repetitions. This class of partitions was introduced by George Andrews. We find several Legendre theorems, including a new one for Andrews' partitions with 2-initial repetitions. Our identities provide a partition-theoretic interpretation of the following identities of Rogers-Ramanujan type due to L. J. Slater [27]:

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n}=\prod_{n=1}^{\infty}(1-q^{4n})(1-q^{4n-1})(1-q^{4n-3}), \tag{5.1}$$

$$\prod_{n=1}^{\infty}(1-q^{2n})\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(q;q)_{2n+1}}=\prod_{n=1}^{\infty}(1-q^{4n})(1+q^{4n-1})(1+q^{4n-3}), \tag{5.2}$$

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\prod_{n=1}^{\infty}(1-q^{5n})(1-q^{5n-1})(1-q^{5n-4}), \tag{5.3}$$

$$\prod_{n=1}^{\infty}(1-q^{2n})\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}}=\prod_{n=1}^{\infty}(1-q^{8n})(1+q^{8n-1})(1+q^{8n-7}), \tag{5.4}$$

$$\prod_{n=1}^{\infty}(1-q^{4n})\sum_{n=0}^{\infty}\frac{q^{4n^2}(q;q^2)_{2n}}{(q^4;q^4)_{2n}}=\prod_{n=1}^{\infty}(1-q^{12n})(1-q^{12n-5})(1-q^{12n-7}), \tag{5.5}$$

$$\prod_{n=1}^{\infty}(1-q^{4n})\sum_{n=0}^{\infty}\frac{q^{4n(n+1)}(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}}=\prod_{n=1}^{\infty}(1-q^{12n})(1-q^{12n-1})(1-q^{12n-11}), \tag{5.6}$$

$$\prod_{n=1}^{\infty}(1-q^{4n})\sum_{n=0}^{\infty}\frac{q^{4n(n+1)}(-q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}} = \prod_{n=1}^{\infty}(1-q^{12n})(1+q^{12n-1})(1+q^{12n-11}), \quad (5.7)$$

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1+q^{2n-1})}\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_{n+1}(-q^2;q^4)_n}{(q^2;q^2)_{2n+1}} = \prod_{n=1}^{\infty}(1-q^{16n})(1-q^{16n-4})(1+q^{16n-12}). \quad (5.8)$$

Furthermore, we derive parity for countless partition functions in this category and illustrate that our approach is fruitful for many of the listed Slater identities of Rogers-Ramanujan type. Work in this chapter has been submitted for publication to *Ramanujan Journal*.

Some of the useful tools we use include the following identities (see [27]):

$$\prod_{n=1}^{\infty}(1-q^{n})\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}(1-q^{5n})(1-q^{5n-2})(1-q^{5n-3}), \quad (5.9)$$

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n-1})}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty}(1-q^{6n})(1-q^{6n-2})(1-q^{6n-4}), \quad (5.10)$$

$$\prod_{n=1}^{\infty}(1-q^{2n})\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q;q^2)_n}{(q;q^2)_{n+1}(q^4;q^4)_n} = \prod_{n=1}^{\infty}(1-q^{6n})(1+q^{6n-1})(1+q^{6n-5}), \quad (5.11)$$

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1+q^{2n-1})}\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}} = \prod_{n=1}^{\infty}(1-q^{6n})(1+q^{6n-2})(1+q^{6n-4}), \quad (5.12)$$

$$\prod_{n=1}^{\infty}(1-q^{2n})\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \prod_{n=1}^{\infty}(1-q^{7n})(1-q^{7n-1})(1-q^{7n-6}), \quad (5.13)$$

$$\prod_{n=1}^{\infty}(1-q^{2n})\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \prod_{n=1}^{\infty}(1-q^{7n})(1-q^{7n-2})(1-q^{7n-5}), \quad (5.14)$$

$$\prod_{n=1}^{\infty}(1-q^{2n})\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \prod_{n=1}^{\infty}(1-q^{7n})(1-q^{7n-3})(1-q^{7n-4}), \quad (5.15)$$

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1+q^{2n+1})}\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \prod_{n=1}^{\infty}(1-q^{8n})(1-q^{8n-3})(1-q^{8n-5}), \quad (5.16)$$

$$\prod_{n=1}^{\infty}(1-q^{n})\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty}(1-q^{12n})(1-q^{12n-2})(1-q^{12n-10}). \quad (5.17)$$

5.1 On Andrews' partitions with 2-initial repetitions

Our first result is as follows:

Lemma 3. *For* $|q| < 1$,

$$\sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q;q)_{2n-1}} = (-q;q)_\infty \sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2}, \tag{5.18}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q;q)_{2n}} = (-q;q)_\infty \sum_{n=0}^{\infty} (-1)^{n} q^{n^2}. \tag{5.19}$$

Proof. $d(n) = d^e(n) - d^o(n) + 2d^o(n)$ so that

$$\sum_{n=0}^{\infty} d(n)q^n = \sum_{n=0}^{\infty} (d^e(n) - d^o(n))q^n + 2\sum_{n=0}^{\infty} d^o(n)q^n.$$

$$\begin{aligned}
2\sum_{n=0}^{\infty} d^o(n)q^{\infty} &= (-q;q)_\infty - \sum_{n=0}^{\infty} (d^e(n) - d^o(n))q^n \\
&= (-q;q)_\infty - (q;q)_\infty \\
&= (-q;q)_\infty \left(1 - \frac{(q;q)_\infty}{(-q;q)_\infty}\right) \\
&= (-q;q)_\infty \left(1 - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right) \quad \text{(by (1.21))} \\
&= 2(-q;q)_\infty \sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2}.
\end{aligned}$$

It can also be shown that

$$\sum_{n=0}^{\infty} d^o(n)q^n = \sum_{n=1}^{\infty} \frac{q^{n(2n-1)}}{(q;q)_{2n-1}}$$

from which (5.18) follows. For (5.19), we have

$$\sum_{n=0}^{\infty} d^e(n)q^n = \sum_{n=0}^{\infty} d(n)q^n - \sum_{n=0}^{\infty} d^o(n)q^n$$

$$= (-q;q)_\infty + (-q;q)_\infty \sum_{n=1}^{\infty} (-1)^n q^{n^2} \text{ by (5.18)}$$
$$= (-q;q)_\infty \sum_{n=0}^{\infty} (-1)^n q^{n^2}.$$

It can be shown that
$$\sum_{n=0}^{\infty} d^e(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n}}$$
and so (5.19) follows. □

Theorem 29. *Let $b^e(n)$ be the number of partitions of n with initial 2-repetitions in which either all parts are distinct or the largest repeated part is even. Similarly, let $b^o(n)$ denote the number of partitions of n with initial 2-repetitions in which at least one part is repeated and the largest repeated part is odd. Then*
$$b^e(n) - b^o(n) = \begin{cases} 1, & \text{if } n = \frac{j(j+1)}{2}, j \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Note that
$$\sum_{n=0}^{\infty} b^e(n) q^n = \prod_{j=1}^{\infty} (1+q^j) + \sum_{m=1}^{\infty} \frac{q^{2(1+2+3+\ldots+2m)}}{(q;q)_{2m}} \prod_{j=2m+1}^{\infty} (1+q^j)$$
$$= \sum_{m=0}^{\infty} \frac{q^{2(1+2+3+\ldots+2m)}}{(q;q)_{2m}} \prod_{j=2m+1}^{\infty} (1+q^j)$$
$$= \sum_{m=0}^{\infty} \frac{q^{2m(2m+1)}}{(q;q)_{2m}} \prod_{j=2m+1}^{\infty} \frac{(1-q^{2j})}{(1-q^j)}$$
$$= \sum_{m=0}^{\infty} \frac{q^{2m(2m+1)}}{(q;q)_{2m}(q^{2m+1};q)_\infty} \prod_{j=2m+1}^{\infty} (1-q^{2j})$$
$$= \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{2m(2m+1)} \frac{\prod_{j=1}^{\infty}(1-q^{2j})}{\prod_{j=1}^{2m}(1-q^{2j})}$$
$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{2m(2m+1)}}{(q^2;q^2)_{2m}}$$

and
$$\sum_{n=0}^{\infty} b^o(n) q^n = \sum_{m=1}^{\infty} \frac{q^{2(1+2+3+\ldots+2m-1)}}{(q;q)_{2m-1}} \prod_{j=2m}^{\infty} (1+q^j)$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{q^{2(1+2+3+\ldots+2m-1)}}{(q;q)_{2m-1}} \prod_{j=2m}^{\infty} \frac{1-q^{2j}}{1-q^j} \\
&= \sum_{m=1}^{\infty} \frac{q^{2m(2m-1)}}{(q;q)_{2m-1}(q^{2m};q)_\infty} \prod_{j=2m}^{\infty} (1-q^{2j}) \\
&= \frac{1}{(q;q)_\infty} \sum_{m=1}^{\infty} q^{2m(2m-1)} \frac{\prod_{j=1}^{\infty}(1-q^{2j})}{\prod_{j=1}^{2m-1}(1-q^{2j})} \\
&= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{q^{2m(2m-1)}}{(q^2;q^2)_{2m-1}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{n=0}^{\infty}(b^e(n)-b^o(n))q^n &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{2n(2n+1)}}{(q^2;q^2)_{2n}} - \sum_{n=1}^{\infty}\frac{q^{2n(2n-1)}}{(q^2;q^2)_{2n-1}}\right) \\
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{2n^2} - \sum_{n=1}^{\infty}(-1)^{n+1}q^{2n^2}\right) \\
&\quad \text{(By (5.18) and (5.19))} \\
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right) \\
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty}\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \text{ (by (1.21))} \\
&= \frac{(q^4;q^4)_\infty(-q;q)_\infty}{(-q^2;q^2)_\infty} \\
&= (q^4;q^4)_\infty(-q;q^4)_\infty(-q^3;q^4)_\infty \\
&= \sum_{n=0}^{\infty} q^{2n^2+n} \text{ (by (1.14))} \\
&= \sum_{n=0}^{\infty} q^{n(n+1)/2}.
\end{aligned}
$$

□

5.2 Related partition functions

In this section, we give several Legendre theorems for partitions with initial repetitions. In turn, these provide partition-theoretic interpretation of equations (5.1), (5.2), (5.3), (5.4), (5.5), (5.6), (5.7) and (5.8).

Let $c_1(n)$ denote the number of partitions of n in which either

(a) all parts are distinct, the only odd part, if any, is 1 and even parts are at least 8 and divisible by 4
or

(b) the largest repeated even part $2j$ appears exactly 4 times, all positive even integers $< 2j$ appear exactly 4 times, even parts $> 2j$ are at least $8j + 8$, distinct and divisible by 4, odd parts are distinct and at most $4j + 1$.

Let $c_{1,e}(n)$ (resp. $c_{1,o}(n)$) denote the number of $c_1(n)$-partitions in which the number of distinct even parts is even (resp. odd). Then we have

Theorem 30. *For all $n \geq 0$,*

$$c_{1,e}(n) - c_{1,o}(n) = \begin{cases} 1, & \text{if } n = j(6j+5), j \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since

$$\sum_{n=0}^{\infty} c_1(n) q^n = \sum_{n=0}^{\infty} q^{4(2+4+6+\ldots+2n)} (-q;q^2)_{2n+1} (-q^{8n+8};q^4)_\infty,$$

we must have

$$\begin{aligned} \sum_{n=0}^{\infty} (c_{1,e}(n) - c_{1,o}(n)) q^n &= \sum_{n=0}^{\infty} q^{4(2+4+6+\ldots+2n)} (-q;q^2)_{2n+1} (q^{8n+8};q^4)_\infty \\ &= \sum_{n=0}^{\infty} q^{4n(n+1)} (-q;q^2)_{2n+1} (q^{4(2n+2)};q^4)_\infty \\ &= \sum_{n=0}^{\infty} q^{4n(n+1)} (-q;q^2)_{2n+1} \frac{(q^4;q^4)_\infty}{(q^4;q^4)_{2n+1}} \\ &= (q^4;q^4)_\infty \sum_{n=0}^{\infty} (-q;q^2)_{2n+1} \frac{q^{4n(n+1)}}{(q^4;q^4)_{2n+1}} \\ &= \prod_{n=1}^{\infty} (1+q^{12n-11})(1+q^{12n-1})(1-q^{12n}) \quad \text{(by (5.7))} \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} q^{n(6n+5)}. \quad \text{(by (1.14))}$$

□

Let $c_2(n)$ denote the number of partitions of n in which either

(a) even parts are distinct and the only odd part, if any, is 1
or

(b) there exists $j \geq 1$ such that an even part $2j$ appears twice, all positive even integers $< 2j$ appear twice, any even part $> 2j$ is distinct and the largest odd part is at most $2j + 1$.

Furthermore, let $c_{2,e}(n)$ (resp. $c_{2,o}(n)$) be $c_2(n)$-partitions with an even (resp. odd) number of distinct even parts. Then we have the following.

Theorem 31. *For all $n \geq 0$,*

$$c_{2,e}(n) - c_{2,o}(n) = \begin{cases} 1 & n = 4j^2 + 3j, j \in \mathbb{Z}; \\ 0, & \textit{otherwise.} \end{cases}$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty} c_2(n)q^n &= \frac{(-q^2;q^2)_\infty}{1-q} + \sum_{n=1}^{\infty} \frac{q^{2+2+4+4+6+6+\cdots+2n+2n}}{(1-q)(1-q^3)\ldots(1-q^{2n+1})}(-q^{2n+2};q^2)_\infty \\ &= \sum_{n=0}^{\infty} \frac{q^{2+2+4+4+6+6+\cdots+2n+2n}}{(1-q)(1-q^3)\ldots(1-q^{2n+1})}(-q^{2n+2};q^2)_\infty,\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}(c_{2,e}(n) - c_{2,o}(n))q^n &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q^{2n+2};q^2)_\infty}{(q;q^2)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q^2;q^2)_\infty}{(q;q^2)_{n+1}(q^2;q^2)_n} \\ &= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}(q^2;q^2)_n}\end{aligned}$$

$$= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} \quad \text{(by (5.4))}$$
$$= \prod_{n=1}^{\infty} \left(1+q^{8n-1}\right)\left(1+q^{8n-7}\right)\left(1-q^{8n}\right)$$
$$= \sum_{n=-\infty}^{\infty} q^{4n^2+3n}.$$

Example 6. *Consider* $n = 10$.

The $c_2(10)$-partitions are:

$$(10), (8,2), (8,1^2), (6,4), (6,2^2), (6,2,1^2), (6,1^4), (4,2^2,1^2), (4,2,1^4), (4,1^6), (3^2,2^2), (3,2^2,1^3),$$
$$(2^2,1^6), (2,1^8), (1^{10})$$

From the above, note that $c_{2,e}(10)$-partitions are: $(8,2), (6,4), (6,2,1^2), (4,2,1^4), (3^2,2^2), (3,2^2,1^3)$, $(2^2,1^6), (1^{10})$ and $c_{2,o}(10)$-partitions are: $(10), (8,1^2), (6,2^2), (6,1^4), (4,2^2,1^2), (4,1^6), (2,1^8)$. Thus,

$$c_{2,e}(10) - c_{2,o}(10) = 1.$$

This agrees with the theorem because $10 = 4(-2)^2 + 3(-2)$.

Let $c_3(n)$ denote the number of partitions of n in which the largest odd part $2j+1 (j \geq 0)$ occurs at least j times, even parts are distinct and greater than $2j+1$. let $c_{3,e}(n)$ (resp. $c_{3,o}(n)$) be $c_3(n)$-partitions with an even (resp. odd) number of distinct even parts. Then:

Theorem 32. *For all* $n \geq 0$,

$$c_{3,e}(n) - c_{3,o}(n) = \begin{cases} 1 & n = 2j^2 + j, j \in \mathbb{Z}; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We have

$$\sum_{n=0}^{\infty} c_3(n) q^n = \sum_{n=0}^{\infty} \frac{q^{\overbrace{(2n+1)+(2n+1)+(2n+1)+\ldots+(2n+1)}^{n\text{ times}}}}{(1-q)(1-q^3)\ldots(1-q^{2n+1})} (-q^{2n+2};q^2)_\infty$$

and thus,

$$\sum_{n=0}^{\infty}(c_{3,e}(n)-c_{3,o}(n))q^n = \sum_{n=0}^{\infty} \frac{q^{\overbrace{(2n+1)+(2n+1)+(2n+1)+\ldots+(2n+1)}^{n \text{ times}}}}{(1-q)(1-q^3)\ldots(1-q^{2n+1})}(q^{2n+2};q^2)_\infty$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}(q^{2n+2};q^2)_\infty}{(q;q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}(q^2;q^2)_\infty}{(q;q^2)_{n+1}(q^2;q^2)_n}$$

$$= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n+1}}$$

$$= \prod_{n=1}^{\infty}(1+q^{4n-1})(1+q^{4n-3})(1-q^{4n}) \text{ (by (5.2))}$$

$$= \sum_{n=-\infty}^{\infty} q^{2n^2+n}.$$

□

Let $c_4(n)$ is the number of partitions of n in which either

(a) all parts are distinct
or

(b) the largest repeated part j appears twice, all positive integers less than j appear twice and any part greater than j is distinct.

Similar to the previous formulations, let $c_{4,e}(n)$ (resp.$c_{4,o}(n)$) be $c_4(n)$-partitions with an even (resp. odd) number of distinct parts.

Theorem 33. *For all $n \geq 0$,*

$$c_{4,e}(n) - c_{4,o}(n) = \begin{cases} (-1)^j & n = (5j^2+3j)/2, j \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Clearly,

$$\sum_{n=0}^{\infty} c_4(n)q^n = \sum_{n=0}^{\infty} q^{1+1+2+2+3+3+\cdots+n+n}(-q^{n+1};q)_\infty = \sum_{n=0}^{\infty} q^{n(n+1)}(-q^{n+1};q)_\infty$$

so that

$$\begin{aligned}
\sum_{n=0}^{\infty}(c_{4,e}(n)-c_{4,o}(n))q^n &= \sum_{n=0}^{\infty} q^{n(n+1)}(q^{n+1};q)_\infty \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q;q)_\infty}{(q;q)_n} \\
&= \prod_{n=1}^{\infty}\left(1-q^{5n-1}\right)\left(1-q^{5n-4}\right)\left(1-q^{5n}\right) \quad (\text{by} \quad (5.3)) \\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}}.
\end{aligned}$$

□

Example 7. *Consider* $n = 7$.

The $c_4(7)$-partitions are:

$$(7), (6,1), (5,2), (5,1^2), (4,3), (4,2,1), (3,2,1^2).$$

$c_{4,e}(7)$-partitions are:

$$(6,1), (5,2), (4,3), (3,2,1^2)$$

and the $c_{4,o}(7)$-partitions are:

$$(7), (5,1^2), (4,2,1).$$

Thus, $c_{4,e}(7) - c_{4,o}(7) = 1$. Indeed, this verifies the theorem as $7 = 5(-2)^2 + 3(-2)$.

Let $c_5(n)$ be the number of partitions of n in which either

(a) all parts are distinct
or

(b) there exists $j \geq 1$ such that all odd positive integers $\leq j$ appear twice or thrice and other odd parts are distinct, all even positive integers $\leq j$ appear twice, even parts $> 2j$ are distinct and no even integer in the interval $(j, 2j]$ appears.

Let $c_{5,e}(n)$ (resp. $c_{5,o}(n)$) denote the number of $c_5(n)$-partitions with an even (resp. odd) number of even distinct parts. Then

$$\sum_{n=0}^{\infty} c_5(n)q^n = \sum_{n=0}^{\infty} q^{1+1+2+2+\cdots n+n}(-q^{2n+2};q^2)_\infty(-q;q^2)_\infty$$
$$= \sum_{n=0}^{\infty} q^{n(n+1)}(-q^{2n+2};q^2)_\infty(-q;q^2)_\infty$$

and

$$\sum_{n=0}^{\infty}(c_{5,e}(n) - c_{5,o}(n))q^n = \sum_{n=0}^{\infty} q^{n(n+1)}(q^{2n+2};q^2)_\infty(-q;q^2)_\infty.$$

Observe that

$$\sum_{n=0}^{\infty}(c_{5,e}(n) - c_{5,o}(n))(-q)^n = \sum_{n=0}^{\infty}(-q)^{n(n+1)}((-q)^{2n+2};q^2)_\infty(-(-q);(-q)^2)_\infty$$
$$= (q;q^2)_\infty \sum_{n=0}^{\infty} q^{n(n+1)}(q^{2n+2};q^2)_\infty$$
$$= (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^2;q^2)_\infty}{(q^2;q^2)_n}$$
$$= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n}$$
$$= \prod_{n=1}^{\infty}\left(1-q^{4n-1}\right)\left(1-q^{4n-3}\right)\left(1-q^{4n}\right) \quad \text{(by (5.1))}$$
$$= \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}.$$

We have the following result:

Theorem 34. *For all $n \geq 0$,*

$$c_{5,e}(n) - c_{5,o}(n) = \begin{cases} (-1)^{n+j}, & n = 2j^2+j, j \in \mathbb{Z}; \\ 0 & \textit{otherwise.} \end{cases}$$

Let $c_6(n)$ be the number of partitions of n in which either

(a) all parts are even, distinct and divisible by 4
    or

(b) the largest repeated part is $2j-1$ (for some $j \geq 1$) and appears exactly 4 times or 5 times, all positive odd integers $< 2j-1$ appear four times or 5 times, any other odd part is distinct and is at most $4j-1$ in part size, even parts are $\geq 8j+4$, distinct and divisible by 4.

Let $c_{6,e}(n)$ (resp. $c_{6,o}(n)$) denote the number of $c_6(n)$-partitions with an even (resp. odd) number of even distinct parts. Then

$$\sum_{n=0}^{\infty} c_6(n)q^n = \sum_{n=0}^{\infty} q^{4(1+3+5+\cdots+2n-1)}(-q^{8n+4};q^4)_\infty(-q;q^2)_{2n}$$

so that

$$\sum_{n=0}^{\infty}(c_{6,e}(n) - c_{6,o}(n))q^n = \sum_{n=0}^{\infty} q^{4n^2}(q^{8n+4};q^4)_\infty(-q;q^2)_{2n}$$

which implies

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n(c_{6,e}(n) - c_{6,o}(n))q^n &= \sum_{n=0}^{\infty} q^{4n^2}(q^{8n+4};q^4)_\infty(q;q^2)_{2n} \\
&= \sum_{n=0}^{\infty} \frac{q^{4n^2}(q^4;q^4)_\infty(q;q^2)_{2n}}{(q^4;q^4)_{2n}} \\
&= (q^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{q^{4n^2}(q;q^2)_{2n}}{(q^4;q^4)_{2n}} \quad \text{(by (5.5))} \\
&= \prod_{n=1}^{\infty}\left(1-q^{12n-5}\right)\left(1-q^{12n-7}\right)\left(1-q^{12n}\right) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{6n^2+n}.
\end{aligned}$$

Hence, we have:

Theorem 35. *For all $n \geq 0$,*

$$c_{6,e}(n) - c_{6,o}(n) = \begin{cases} (-1)^{n+j}, & n = 6j^2+j, j \in \mathbb{Z}; \\ \\ 0 & \textit{otherwise.} \end{cases}$$

Let $c_7(n)$ is the number of partitions of n in which either

(a) the smallest odd part, if it appears, is at least 3 and even parts are distinct and at least 4

or

(b) $\exists j \geq 1$ such that j appears three or four times if $j \equiv 2 \pmod 4$ and appears exactly three times if $j \not\equiv 2 \pmod 4$, all positive integers $i < j$ appear exactly twice or thrice if $i \equiv 2 \pmod 4$ and appear exactly twice if $i \not\equiv 2 \pmod 4$, an even part greater than j but less than $4j$ is distinct and congruent to 2 (mod 4), any other even part is $\geq 4j + 4$ and distinct, odd parts $> j$ are actually at least $2j + 3$.

Let $c_{7,e}(n)$ (resp. $c_{7,o}(n)$) be the number of $c_7(n)$-partitions in which the number of distinct even parts if (a) holds is even (resp. odd) or the number of distinct even parts that are $\geq 4j + 4$ is even (resp. odd) if (b) holds. Clearly,

$$\begin{aligned}
\sum_{n=0}^{\infty} c_7(n)q^n &= \frac{(-q^4;q^2)_\infty}{(q^3;q^2)_\infty} + \sum_{n=1}^{\infty} \frac{q^{1+1+2+2+\cdots+n-1+n-1+n+n+n}(-q^2;q^4)_n(-q^{4n+4};q^2)_\infty}{(q^{2n+3};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^2;q^4)_n(-q^{4n+4};q^2)_\infty}{(q^{2n+3};q^2)_\infty}
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{n=0}^{\infty}(c_{7,e}(n) - c_{7,o}(n))q^n &= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^2;q^4)_n(q^{4n+4};q^2)_\infty}{(q^{2n+3};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^2;q^4)_n(q;q^2)_{n+1}(q^2;q^2)_\infty}{(q;q^2)_\infty(q^2;q^2)_{2n+1}} \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^2;q^4)_n(q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}} \quad \text{(by (5.8))} \\
&= \prod_{n=1}^{\infty} \left(1 - q^{16n-4}\right)\left(1 - q^{16n-12}\right)\left(1 - q^{16n}\right) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{8n^2+4n}.
\end{aligned}$$

This leads to the theorem below.

Theorem 36. *For all $n \geq 0$,*

$$c_{7,e}(n) - c_{7,o}(n) = \begin{cases} (-1)^j & n = 8j^2 + 4j, j \in \mathbb{Z}; \\ \\ 0, & \textit{otherwise.} \end{cases}$$

From the theorem, it can be observed that $c_{7,e}(2n+1) - c_{7,o}(2n+1) = 0$ for all $n \geq 0$ so that

$$c_7(2n+1) \equiv c_{7,e}(2n+1) - c_{7,o}(2n+1) \equiv 0 \pmod{2}.$$

We record this result below.

Corollary 7. *For all $n \geq 0$, we have*

$$c_7(2n+1) \equiv 0 \pmod{2}.$$

Let $c_8(n)$ be the number of partitions of n in which either

(a) even parts are distinct, greater than 6 and divisible by 4 and the only odd part that may appear is 1 and is distinct.
or

(b) the largest repeated part is $2j$ (for some $j \geq 1$) and appears exactly 4 times, all positive even integers $< 2j$ appear four times, any even part $> 2j$ is actually at least $8j+8$, distinct and divisible by 4, odd parts are distinct and are at most $4j+1$ in part size.

Let $c_{8,e}(n)$ (resp. $c_{8,o}(n)$) denote the number of $c_8(n)$-partitions with an even (resp. odd) number of even distinct parts. Then

$$\sum_{n=0}^{\infty} c_8(n)q^n = \sum_{n=0}^{\infty} q^{4(2+4+6+\cdots+2n)}(-q^{8n+8};q^4)_\infty(-q;q^2)_{2n+1}$$

so that

$$\sum_{n=0}^{\infty}(c_{8,e}(n) - c_{8,o}(n))q^n = \sum_{n=0}^{\infty} q^{4n(n+1)}(q^{8n+8};q^4)_\infty(-q;q^2)_{2n+1}$$

which implies

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n(c_{8,e}(n)-c_{8,o}(n))q^n &= \sum_{n=0}^{\infty} q^{4n(n+1)}(q^{8n+8};q^4)_\infty(q;q^2)_{2n+1}\\ &= \sum_{n=0}^{\infty} \frac{q^{4n(n+1)}(q^4;q^4)_\infty(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}}\\ &= (q^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{q^{4n(n+1)}(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}} \quad (\text{by} \quad (5.6))\\ &= \prod_{n=1}^{\infty}\left(1-q^{12n-1}\right)\left(1-q^{12n-11}\right)\left(1-q^{12n}\right)\\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+5n}.\end{aligned}$$

Hence, we have:

Theorem 37. *For all* $n \geq 0$,

$$c_{8,e}(n)-c_{8,o}(n) = \begin{cases}(-1)^{n+j}, & n = 6j^2+5j, j \in \mathbb{Z};\\ \\ 0 & \text{otherwise.}\end{cases}$$

Example 8. *Consider* $n = 39$.

The $c_8(39)$-partitions are:

$$(28,3,2^4),(9,5,4^4,2^4,1),(7,5,4^4,3,2^4).$$

$c_{8,e}(39)$-partitions are:

$$(9,5,4^4,2^4,1),(7,5,4^4,3,2^4)$$

and the $c_{8,o}(39)$-partitions are:

$$(28,3,2^4).$$

Thus, $c_{8,e}(39)-c_{8,o}(39) = 1$. Indeed, this verifies the theorem as $39 = 6(-3)^2+5(-3)$ and $39-3 = 36$.

5.3 Parity

In this section, we derive parity formulas for partition functions that enumerate certain partitions with initial repetitions.

Theorem 38. *Let $c_9(n)$ be the number of partitions of n in which either*

(a) all parts are distinct
or

(b) there is an odd repeated part $2j-1$ and all positive integers less than $2j-1$ appear as repeated parts, any part greater than $2j$ is distinct.

Then

$$c_9(n) \equiv 1 \iff n = j(j+1)/2, j \geq 0.$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} c_9(n)q^n &= \prod_{j=1}^{\infty}(1+q^j) + \sum_{n=1}^{\infty} \frac{q^{1+1+2+2+\ldots+(2n-1)+(2n-1)}}{(q;q)_{2n}} \prod_{j=2n+1}^{\infty}(1+q^j) \\
&= \sum_{n=0}^{\infty} \frac{q^{1+1+2+2+\ldots+(2n-1)+(2n-1)}}{(q;q)_{2n}} \prod_{j=2n+1}^{\infty}(1+q^j) \\
&= \sum_{n=0}^{\infty} \frac{q^{4n^2-2n}}{(q;q)_{2n}} \frac{\prod_{j=1}^{\infty}(1+q^j)}{\prod_{j=1}^{2n}(1+q^j)} \\
&= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{4n^2-2n}}{(q;q)_{2n}(-q;q)_{2n}} \\
&= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{4n^2-2n}}{(q^2;q^2)_{2n}} \\
&= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{4n^2-2n}}{(q^4;q^4)_n(q^2;q^4)_n} \\
&= (-q;q)_\infty \prod_{n=0}^{\infty} \frac{1}{1-q^{2+4n}} \quad (z=q^2, q:=q^4 \text{ in } (1.13)) \\
&= \frac{(-q;q^2)_\infty(-q^2;q^2)_\infty}{(-q;q^2)_\infty(q;q^2)_\infty}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \\
&\equiv (q;q)^3 \pmod 2 \\
&\equiv \sum_{n=0}^{\infty} q^{n(n+1)/2}.
\end{aligned}
$$

□

Note that $\sum_{n=0}^{\infty} c_9(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}$, we conclude that $c_9(n)$ satisfies further congruences. For more of such, see [8].

Let $c_{10}(n)$ be the number of partitions of n in which there exists $j \geq 1$ such that even parts are distinct and $> 2j$ and all odd parts $< 2j$ appear exactly once, odd parts $\geq 2j+1$ appear unrestricted. Then we have the following theorem.

Theorem 39. *For all $n \geq 0$,*

$$c_{10}(5n+2) \equiv 0 \pmod 2.$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_{10}(n)q^n &= \sum_{n=1}^{\infty} \frac{q^{1+3+5+\cdots+(2n-1)}(-q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} \\
&= \sum_{n=1}^{\infty} \frac{q^{n^2}(-q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} - \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \\
&\equiv \sum_{n=0}^{\infty} \frac{q^{n^2}(q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty} + (q^2;q^2)_\infty(-q;q)_\infty \pmod 2 \\
&\equiv \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n(q^2;q^2)_\infty}{(-q;q^2)_\infty(q^2;q^2)_n} + (q;q)_\infty^3 \pmod 2 \\
&= \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} + \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \\
&\equiv \prod_{n=1}^{\infty}\left(1-q^{8n-3}\right)\left(1-q^{8n-5}\right)\left(1-q^{8n}\right) + \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 2 \text{ (by (5.16))}
\end{aligned}
$$

$$= \sum_{n=-\infty}^{\infty} q^{4n^2+n} + \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 2 \quad (\text{by } (1.14)).$$

Since there is no integer n such that $4n^2+n \equiv 2 \pmod 5$ or $n(n+1)/2 \equiv 2 \pmod 5$, it must follow that $c_{10}(5n+2) \equiv 0 \pmod 2$.
For instance, there are two partitions of 7 enumerated by $c_{10}(7)$. These are: $(6,1)$ and $(3^2,1)$. Thus $c_{10}(7) \equiv 0 \pmod 2$.

Let $c_{11}(n)$ be the number of partitions of n in which, there is a positive integer j such that 1 appears with multiplicities j^2 or j^2+1, odd parts > 1 are distinct, all even parts are distinct and those $> 2j$ are at least $4j+4$ in size and divisible by 4, no even integer in the interval $(j, 2j]$ appears as a part. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{11}(n)q^n &= \sum_{n=1}^{\infty} q^{1+1+\cdots+1(n^2\text{ times})}(1+q)(-q^3;q^2)_\infty(-q^2;q^2)_n(-q^{4n+4};q^4)_\infty \\
&= \sum_{n=1}^{\infty} q^{n^2}(1+q)(-q^3;q^2)_\infty(-q^2;q^2)_n(-q^{2n+2};q^2)_\infty(-q^{2n+2};q^2)_\infty \pmod 2 \\
&= \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_\infty(-q^2;q^2)_\infty(-q^{2n+2};q^2)_\infty \pmod 2 \\
&= \sum_{n=1}^{\infty} q^{n^2}(-q;q)_\infty(-q^{2n+2};q^2)_\infty \pmod 2 \\
&= (-q;q)_\infty \sum_{n=1}^{\infty} q^{n^2} \frac{(-q^2;q^2)_\infty}{q^2;q^2)_n} \pmod 2 \\
&\equiv \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} \pmod 2 \\
&\equiv \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} \pmod 2 \\
&\equiv \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} - 1 \right) \pmod 2 \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} - \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\
&\equiv (q^2;q^6)_\infty(q^4;q^6)_\infty(q^6;q^6)_\infty + \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \pmod 2 \quad (\text{by } (5.10))
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} q^{n(3n+1)} + \sum_{n=0}^{\infty} q^{n(n+1)/2} \quad (\text{by} \quad (1.14)).$$

Note that, for all $n \in \mathbb{Z}$, we have $n(3n+1) \equiv 0, 2, 4 \pmod 5$ and for all $n \in \mathbb{Z}_{\geq 0}$, $n(n+1)/2 \equiv 0, 1, 3 \pmod 5$. More specifically, we have $n(3n+1) \equiv 0 \pmod 5 \iff n \equiv 0, 3 \pmod 5$, $n(3n+1) \equiv 2 \pmod 5 \iff n \equiv 4 \pmod 5$, $n(3n+1) \equiv 4 \pmod 5 \iff n \equiv 1, 2 \pmod 5$, $n(n+1)/2 \equiv 0 \pmod 5 \iff n \equiv 0, 4 \pmod 5$, $n(n+1)/2 \equiv 1 \pmod 5 \iff n \equiv 1, 3 \pmod 5$ and $n(n+1)/2 \equiv 3 \pmod 5 \iff n \equiv 2 \pmod 5$.

Hence

$$\sum_{n=0}^{\infty} c_{11}(5n+1) q^{5n+1} \equiv \sum_{n \geq 1, n \equiv 1,3 \pmod 5} q^{n(n+1)/2} \pmod 2,$$

$$\sum_{n=0}^{\infty} c_{11}(5n+2) q^{5n+2} \equiv \sum_{n \equiv 4 \pmod 5} q^{n(3n+1)} = \sum_{n=-\infty}^{\infty} q^{(5n+4)(15n+13)} \pmod 2,$$

$$\sum_{n=0}^{\infty} c_{11}(5n+3) q^{5n+3} \equiv \sum_{n > 1, n \equiv 2 \pmod 5} q^{n(n+1)/2} = \sum_{n=0}^{\infty} q^{(5n+2)(5n+3)/2} \pmod 2$$

and

$$\sum_{n=0}^{\infty} c_{11}(5n+4) q^{5n+4} \equiv \sum_{n \equiv 1,2 \pmod 5} q^{n(3n+1)} \pmod 2$$

so that we have the following.

Theorem 40. *For all $n \geq 0$,*

$$c_{11}(5n+1) \equiv 1 \pmod 2 \ \text{ iff } \ n = \frac{j(j+1)-2}{10}, \ j \geq 1 \ \text{ and } \ j \equiv 1, 3 \pmod 5,$$

$$c_{11}(5n+2) \equiv 1 \pmod 2 \ \text{ iff } \ n = \frac{(5j+4)(15j+13)-2}{5}, \ j \in \mathbb{Z}.$$

$$c_{11}(5n+3) \equiv 1 \pmod 2 \ \text{ iff } \ n = \frac{(5j+2)(5j+3)-6}{10}, \ j \geq 0,$$

$$c_{11}(5n+4) \equiv 1 \pmod 2 \ \text{ iff } \ n = \frac{j(3j+1)-4}{5}, \ j \in \mathbb{Z} \ \text{ and } \ j \equiv 1, 2 \pmod 5.$$

Recall that

$$\sum_{n=0}^{\infty} c_{11}(n) q^n \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)} + \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Observe that none of the exponents $n(3n+1)$ or $n(n+1)/2$ is congruent to 5, 7, 9 modulo 11. Thus, we have:

Theorem 41. *For all $n \geq 0$,*

$$c_{11}(11n+5) \equiv 0 \pmod 2,$$

$$c_{11}(11n+7) \equiv 0 \pmod 2,$$

$$c_{11}(11n+9) \equiv 0 \pmod 2.$$

More generally, it can be shown by completing the square in Theorems 39 and 41 that the following statements hold.

Proposition 4. *If N is such that $8N+1$ is not a square and $16N+1$ is not a square, then*

$$c_{10}(N) \equiv 0 \pmod 2.$$

Proof. Suppose $N = 4m^2 + m$, then

$$\begin{aligned} N &= 4(m+\frac{1}{8})^2 - \frac{1}{16} \\ &= \frac{(8m+1)^2}{16} - \frac{1}{16} \end{aligned}$$

i.e.

$$16N + 1 = (8m+1)^2.$$

Similarly, suppose $N = \frac{v(v+1)}{2}$ then

$$\begin{aligned} N &= \frac{(v+\frac{1}{2})^2}{2} - \frac{1}{8} \\ &= \frac{(2v+1)^2}{8} - \frac{1}{8} \end{aligned}$$

i.e.

$$8N + 1 = (2v+1)^2.$$

Hence if $8N+1$ and $16N+1$ are both not squares, then the coefficient of q^N must be 0 on the right-hand side. Thus $c_{10}(N) \equiv 0 \pmod 2$.

□

The proof of the following result mirrors that of Proposition 4.

Proposition 5. *If N is such that $8N+1$ is not a square and $12N+1$ is not a square, then*

$$c_{11}(N) \equiv 0 \pmod 2.$$

Let $c_{12}(n)$ denote the number of partitions of n in which either

(a) all parts are even and distinct or

(b) there is an even part $2j$ which appears twice, all positive even integers $< 2j$ appear twice, any even part larger than $2j$ is actually $\geq 4j+2$ and distinct, odd parts are distinct and at most $2j-1$ in part size.

Then we have the following theorem.

Theorem 42. *For all $n \geq 0$,*

$$c_{12}(n) \equiv \begin{cases} 1 \pmod 2, & n = (7j^2+3j)/2, j \in \mathbb{Z}; \\ \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof. Note that

$$\begin{aligned}
\sum_{n\geq 0} c_{12}(n)q^n &= (-q^2;q^2)_\infty + \sum_{n\geq 1} q^{2+2+4+4+\cdots+2n+2n} \prod_{i=1}^{n}(1+q^{2i-1}) \prod_{j=1}^{\infty}(1+q^{4n+2j}) \\
&= \sum_{n\geq 0} q^{2n(n+1)}(-q;q^2)_n(-q^{4n+2};q^2)_\infty \\
&\equiv \sum_{n\geq 0} q^{2n(n+1)}(q;q^2)_n(q^{4n+2};q^2)_\infty \pmod 2 \\
&= \sum_{n\geq 0} q^{2n(n+1)}(q;q^2)_n \frac{(q^2;q^2)_{2n}}{(q^2;q^2)_{2n}}(q^{4n+2};q^2)_\infty \\
&= \sum_{n\geq 0} q^{2n(n+1)}(q;q^2)_n \frac{(q^2;q^2)_\infty}{(q^2;q^2)_{2n}} \\
&= (q^2;q^2)_\infty \sum_{n\geq 0} q^{2n(n+1)} \frac{(q;q^2)_n}{(q;q)_{2n}(-q;q)_{2n}} \\
&= (q^2;q^2)_\infty \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} \\
&\equiv \prod_{n\geq 1}(1-q^{7n})(1+q^{7n-2})(1+q^{7n-5}) \pmod 2 \quad (\text{by } (5.14)) \\
&\equiv \sum_{n=-\infty}^{\infty} q^{(7n^2+3n)/2}.
\end{aligned}$$

□

Let $c_{13}(n)$ denote the number of partitions of n in which either

(a) all parts are even and distinct
or

(b) odd parts appear twice or thrice and are gap-free, even parts are distinct and the smallest even part is ≥ 2(the largest odd part) + 4.

Then parity of $c_{13}(n)$ can be deduced from the following.

Theorem 43. *For all $n \geq 0$,*

$$c_{13}(n) \equiv \begin{cases} 1 \pmod 2, & n = (7j^2+j)/2, j \in \mathbb{Z}; \\ \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof. It is clear that $\sum_{n\geq 0} c_{13}(n)q^n = (-q^2;q^2)_\infty + \sum_{n\geq 1} q^{1+1+3+3+\ldots+2n-1}(-q;q^2)_n(-q^{4n+2};q^2)_\infty$, and by a similar manipulation as in Theorem 42, we have

$$\begin{aligned}\sum_{n\geq 0} c_{13}(n)q^n &\equiv (q^2;q^2)_\infty \sum_{n\geq 0} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} \pmod 2 \\ &\equiv \prod_{n\geq 1}(1-q^{7n})(1+q^{7n-3})(1+q^{7n-4}) \pmod 2 \quad (\text{by } (5.15)) \\ &\equiv \sum_{n=-\infty}^{\infty} q^{(7n^2+n)/2} \pmod 2.\end{aligned}$$

□

Let $c_{14}(n)$ denote the number of partitions of n in which either

(a) 1 is the only odd part that may appear and even parts are distinct
or

(b) there is an even part $2j$ that appears twice, all positive even integers $< 2j$ appear twice, any even part larger than $2j$ is actually $\geq 4j+2$ and distinct, odd parts are $\leq 2j+1$ and those $\leq 2j-1$ are distinct.

We have:

Theorem 44. *For all $n \geq 0$,*

$$c_{14}(n) \equiv \begin{cases} 1 \pmod 2, & n = (7j^2+5j)/2, j \in \mathbb{Z}; \\ \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof. The generating function for the partition function in question is

$$\sum_{n\geq 0} c_{14}(n)q^n = \frac{(-q^2;q^2)_\infty}{1-q} + \sum_{n\geq 1} q^{2+2+4+4+\ldots 2n+2n}\frac{(-q;q^2)_n}{1-q^{2n+1}}(-q^{4n+2};q^2)_\infty.$$

However, by a similar manipulation as in the proof of Theorem 42, we find that

$$\begin{aligned}\sum_{n\geq 0} c_{14}(n)q^n &\equiv (q^2;q^2)_\infty \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} \pmod 2 \\ &\equiv \prod_{n\geq 1}(1-q^{7n})(1+q^{7n-1})(1+q^{7n-6}) \pmod 2 \quad (\text{by } (5.13)) \\ &\equiv \sum_{n=-\infty}^{\infty} q^{(7n^2+5n)/2}.\end{aligned}$$

□

Let $c_{15}(n)$ denote the number of partitions of n in which either

(a) even parts are distinct and congruent 2 (mod 4) and 1 is the only odd part that may appear
or

(b) the largest even part $2j$ appears exactly twice if $2j \equiv 0 \pmod 4$, and appears twice or thrice if $2j \equiv 2 \pmod 4$, all positive even integers $< 2j$ and $\equiv 0 \pmod 4$ are repeated exactly twice, and those $< 2j$ and $\equiv 2 \pmod 4$ appear twice or thrice, any even part larger than $2j$ that is divisible by 4 is actually atleast $4j+4$ in part size and distinct, even parts that are $> 2j$ and $\equiv 2 \pmod 4$ are distinct and $2j+1$ is the only odd integer that may appear.

The following result follows:

Theorem 45. *For all $n \geq 0$,*

$$c_{15}(n) \equiv \begin{cases} 1 \pmod 2, & n = 3j^2+2j, j \in \mathbb{Z}; \\ \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof. Note that

$$\begin{aligned}
\sum_{n\geq 0} c_{15}(n)q^n &= \frac{(-q^2;q^4)_\infty}{1-q} + (-q^2;q^4)_\infty \sum_{n\geq 1} \frac{q^{2(2+4+6+\cdots+2n)}}{1-q^{2n+1}}(-q^{4n+4};q^4)_\infty \\
&\equiv (q^2;q^4)_\infty \sum_{n\geq 0} \frac{q^{2n(n+1)}}{1-q^{2n+1}}(q^{4n+4};q^4)_\infty \pmod 2 \\
&= \sum_{n\geq 0} \frac{q^{2n(n+1)}}{1-q^{2n+1}}(q^2;q^4)_\infty \frac{(q^4;q^4)_\infty}{(q^4;q^4)_n} \pmod 2 \\
&= \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(1-q^{2n+1})(q^4;q^4)_n}(q^2;q^2)_\infty \\
&= \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(1-q^{2n+1})} \frac{(q;q^2)_n}{(q;q^2)_n(q^4;q^4)_n}(q^2;q^2)_\infty \\
&\equiv (q^2;q^2)_\infty \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} \frac{(-q;q^2)_n}{(q^4;q^4)_n} \pmod 2 \\
&= \prod_{n\geq 1}(1-q^{6n})(1+q^{6n-5})(1+q^{6n-1}) \ \ (\text{by } \ (5.11)) \\
&\equiv \sum_{n=-\infty}^{\infty} q^{3n^2+2n}.
\end{aligned}$$

□

Let $c_{16}(n)$ denote the number of partitions of n in which either

(a) all parts are even and distinct
or

(b) there is the largest odd part $2j-1$ which appears once, all positive odd integers $\leq j$ appear once or twice , all positive odd integers $> j$ appear once, even parts $\leq j$ are distinct, and those $> j$ are distinct and actually $\geq 2j+2$ in size.

We obtain the following theorem.

Theorem 46. *For all $n \geq 0$,*

$$c_{16}(49n+r) \equiv 0 \pmod 2,$$

where $r = 6, 20, 27, 34, 41, 48$.

Proof. It is not difficult to see that

$$\begin{aligned}
\sum_{n\geq 0} c_{16}(n)q^n &= (-q^2;q^2)_\infty + \sum_{n\geq 1} q^{1+3+5+\ldots+2n-1}(1+q)(1+q^2)\ldots(1+q^n)(-q^{2n+2};q^2)_\infty \\
&= \sum_{n\geq 0} q^{n^2}(-q;q)_n(-q^{2n+2};q^2)_\infty \\
&= \sum_{n\geq 0} q^{n^2}\frac{(q;q)_n}{(q;q)_n}(-q;q)_n(-q^{2n+2};q^2)_\infty \\
&\equiv \sum_{n\geq 0} q^{n^2}\frac{(q^2;q^2)_n}{(q;q)_n}(q^{2n+2};q^2)_\infty \pmod 2 \\
&= (q^2;q^2)_\infty \sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n} \\
&\equiv (q;q)_\infty \prod_{n\geq 1}(1-q^{5n})(1+q^{5n-2})(1+q^{5n-3}) \pmod 2 \;\; (\text{by } (5.9)) \\
&= \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \sum_{n=-\infty}^{\infty} q^{n(5n+1)/2} \\
&= \sum_{n=-\infty}^{\infty}\left(q^{n(6n+1)} + q^{(2n+1)(3n+2)}\right)\sum_{n=-\infty}^{\infty}\left(q^{n(10n+1)} + q^{(2n+1)(5n+3)}\right).
\end{aligned}$$

Since the exponents in $\sum_{n=-\infty}^{\infty}\left(q^{n(6n+1)} + q^{(2n+1)(3n+2)}\right)$ are congruent to

$$0, 1, 2, 5, 7, 8, 12, 14, 15, 19, 21, 22, 26, 28, 29, 33, 35, 36, 40, 42, 43, 47$$

modulo 49 and the exponents in $\sum_{n=-\infty}^{\infty}\left(q^{n(10n+1)} + q^{(2n+1)(5n+3)}\right)$ are congruent to

$$0, 2, 3, 7, 9, 10, 11, 14, 16, 17, 21, 23, 24, 28, 30, 31, 35, 37, 38, 42, 44, 45$$

modulo 49, it follows that the product
$\sum_{n=-\infty}^{\infty}\left(q^{n(6n+1)} + q^{(2n+1)(3n+2)}\right)\sum_{n=-\infty}^{\infty}\left(q^{n(10n+1)} + q^{(2n+1)(5n+3)}\right)$ has no exponent congruent to 6, 20, 27, 34, 41, 48 modulo 49. Thus

$$\sum_{n\geq 0} c_{16}(49n+r)q^{49n+r} \equiv 0 \pmod 2$$

where $r = 6, 20, 27, 34, 41, 48$. □

Let $c_{17}(n)$ be the number of partitions of n in which either

(a) all parts are distinct and greater than 1
or

(b) there exists $j \geq 2$ such that 1 appears exactly j^2 times and parts > 1 are at least $j+1$ in size and distinct.

Then, we have:

Theorem 47. *For all $n \geq 0$,*

$$c_{17}(n) \equiv \begin{cases} 1 \pmod 2, & n = (5j^2+j)/2, j \in \mathbb{Z}; \\ \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof. We have:

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{17}(n) q^n &= (-q^2;q)_\infty + \sum_{n=2}^{\infty} q^{n^2}(-q^{n+1};q)_\infty \\
&\equiv (1+2q)(-q^2;q)_\infty + \sum_{n=2}^{\infty} q^{n^2}(-q^{n+1};q)_\infty \pmod 2 \\
&= (1+q)(-q^2;q)_\infty + q(-q^2;q)_\infty + \sum_{n=2}^{\infty} q^{n^2}(-q^{n+1};q)_\infty \\
&= (-q;q)_\infty + q(-q^2;q)_\infty + \sum_{n=2}^{\infty} q^{n^2}(-q^{n+1};q)_\infty \\
&= \sum_{n=0}^{\infty} q^{n^2}(-q^{n+1};q)_\infty \\
&\equiv \sum_{n=0}^{\infty} q^{n^2}(q^{n+1};q)_\infty \pmod 2 \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_\infty}{(q;q)_n} \\
&= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \\
&\equiv \prod_{n=1}^{\infty} \left(1+q^{5n-2}\right)\left(1+q^{5n-3}\right)\left(1-q^{5n}\right) \pmod 2 \ \text{(by (5.9))}
\end{aligned}$$

$$\equiv \sum_{n=-\infty}^{\infty} q^{\frac{5n^2+n}{2}} \pmod 2.$$

□

Example 9. *Consider* $n = 11$.

The $c_{17}(11)$-paritItions are:

$$11, (9,2), (8,3), (7,4), (7,1^4), (6,5), (6,3,2), (5,4,2), (4,3,1^4)$$

and so $c_{17}(11) = 9 \equiv 1 \pmod 2$. Indeed this is true since $11 = \frac{5(2)^2+2}{2} (j = 2)$ in the theorem.

Let $c_{18}(n)$ be the number of partitions of n in which either

(a) all parts are distinct
 or

(b) there exists $j \geq 1$ such that all positive odd integers $\leq j$ appear twice or thrice and other odd parts are distinct, all positive even integers $\leq j$ appear twice, even parts $> 2j$ are distinct and no even integer in the interval $(j, 2j]$ appears.

Then, we have:

Theorem 48. *For all* $n \geq 0$,

$$c_{18}(n) \equiv \begin{cases} 1 \pmod 2, & n = 2j^2 + j, j \in \mathbb{Z}; \\ \\ 0 \pmod 2 & \textit{otherwise}. \end{cases}$$

Proof. The generating function for $c_{18}(n)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} c_{18}(n) q^n &= \sum_{n=0}^{\infty} q^{1+1+2+2+\cdots+n+n} (-q^{2n+2}; q^2)_\infty (-q; q^2)_\infty \\ &\equiv (q; q^2)_\infty \sum_{n=0}^{\infty} q^{n(n+1)} (q^{2n+2}; q^2)_\infty \pmod 2 \\ &= (q; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)} (q^2; q^2)_\infty}{(q^2; q^2)_n} \end{aligned}$$

$$= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n}$$
$$= \prod_{n=1}^{\infty} \left(1-q^{4n-1}\right)\left(1-q^{4n-3}\right)\left(1-q^{4n}\right) \quad \text{(by (5.1))}$$
$$\equiv \sum_{n=-\infty}^{\infty} q^{2n^2+n} \pmod 2.$$

□

Let $c_{19}(n)$ be the number of partitions of n in which there is $j \geq 1$ such that $2j-1$ appears and all positive odd integers $< 2j-1$ appear, even parts are $> 2j$ and distinct, odd parts $> 2j$ may appear. Then

$$\sum_{n=0}^{\infty} c_{19}(n)q^n = \sum_{n=1}^{\infty} \frac{q^{1+3+5+\cdots+(2n-1)}(-q^{2n+2};q^2)_\infty}{(q;q^2)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{q^{n^2}(-q^{2n+2};q^2)_\infty}{(q;q^2)_\infty}$$
$$\equiv \sum_{n=1}^{\infty} \frac{q^{n^2}(q^2;q^2)_\infty}{(q^2;q^2)_n(q;q^2)_\infty} \pmod 2$$
$$= \sum_{n=1}^{\infty} \frac{q^{n^2}(q^2;q^2)_\infty(q;q^2)_n}{(q^2;q^2)_n(q;q^2)_\infty(q;q^2)_n}$$
$$\equiv \sum_{n=1}^{\infty} \frac{q^{n^2}(q^2;q^2)_\infty(-q;q^2)_n}{(q^2;q^2)_n(-q;q^2)_\infty(q;q^2)_n} \pmod 2$$
$$\equiv \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n(q;q^2)_n} - \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \pmod 2$$
$$\equiv \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}} + (q;q)_\infty^3 \pmod 2$$
$$\equiv \prod_{n=1}^{\infty} \left(1+q^{6n-2}\right)\left(1+q^{6n-4}\right)\left(1-q^{6n}\right) + \sum_{n=0}^{\infty} q^{n(n+1)/2} \quad \text{(by (5.12))}$$
$$\equiv \sum_{n=-\infty}^{\infty} q^{3n^2+n} + \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 2.$$

The series on the right-hand side is the same as the series expansion for $\sum_{n=0}^{\infty} c_{11}(n)q^n$

and so Theorem 41 is valid for $c_{19}(n)$, i.e.

$$c_{19}(11n+5) \equiv 0 \pmod 2,$$

$$c_{19}(11n+7) \equiv 0 \pmod 2$$

and

$$c_{19}(11n+9) \equiv 0 \pmod 2.$$

Let $c_{20}(n)$ denote the number of partitions of n in which either

(a) all parts are distinct and greater than or equal to 2
or

(b) the largest part j appears exactly three times if $j \equiv 0 \pmod 2$, and appears three or four times if $j \equiv 1 \pmod 2$, all positive even integers $< j$ appear exactly twice, all positive odd integers $< j$ appear two or three times, all even parts $> j$ are actually at least $2j+2$ in part size and distinct, odd parts $> j$ are distinct and no odd part is equal to $2j+1$.

We have:

Theorem 49. *For all $n \geq 0$,*

$$c_{20}(n) \equiv \begin{cases} 1 \pmod 2, & n = 6j^2+4j, j \in \mathbb{Z}; \\ \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \sum_{n=0}^{\infty} c_{20}(n)q^n &= (-q^2;q)_\infty + \sum_{n=1}^{\infty} q^{1+1+2+2+\cdots+(n-1)+(n-1)+n+n+n}(-q;q^2)_n(-q^{2n+2};q)_\infty \\ &\equiv \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^{2n+2};q^2)_\infty (q;q^2)_\infty}{(1-q^{2n+1})} \pmod 2 \\ &= (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^2;q^2)_\infty}{(q^2;q^2)_n(1-q^{2n+1})} \\ &= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^2;q^2)_n(1-q^{2n+1})} \end{aligned}$$

$$
\begin{aligned}
&= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q;q^2)_n}{(q;q^2)_n(q^2;q^2)_n(1-q^{2n+1})} \\
&\equiv (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q;q^2)_{n+1}(q^2;q^2)_n} \pmod 2 \\
&= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q;q)_{2n+1}} \\
&= \prod_{n=1}^{\infty} \left(1-q^{12n-2}\right)\left(1-q^{12n-10}\right)\left(1-q^{12n}\right) \quad (\text{by} \quad (5.17)) \\
&\equiv \sum_{n=-\infty}^{\infty} q^{6n^2+4n} \pmod 2.
\end{aligned}
$$

The following corollary is immediately disposable.

Corollary 8. *For all $n \geq 0$, we have,*

$$c_{20}(2n+1) \equiv 0 \pmod 2.$$

Chapter 6

Conclusion

In this dissertation, we reviewed relevant theorems in the literature that introduce several concepts, namely, identites of Euler type (Theorems 9, 10 and 11), partitions with initial repetitions and identities of Rogers-Ramanujan type.

In Chapter 2, we studied several bijective proofs one of which is due to Andrews, Eriksson, Petrov and Romik for Theorem 9 and a new class of partitions introduced by Andrews called partitions with initial repetitions.

In Chapter 3, we gave a new generalized bijection of the Andrews-Eriksson-Petrov-Romik bijection for Andrews' partition theorem (Theorem 10), generalized Subbarao's partition theorem (Theorem 11) and subsequently generalized Theorem 10.

In Chapter 4, we extended Andrews and Merca's theorem (Theorem 13). We derived recurrence formulas for related functions. New partition identities were obtained and we also studied their connection to the work of Fu and Tang.

Lastly, in Chapter 5, we discovered identities of Euler-pentagonal type for some partitions with initial repetitions as well as obtained infinite families of modulo 2 Ramanujan congruences for countless related partition functions.

The work in Chapter 3 has been published in journals: *Integer Sequences*, *Annals of Combinatorics* and the work in Chapters 4 and 5 has been submitted to two ISI-accredited journals (*Revista de la Real Academia de Ciencias Exactas, Fsicas y*

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Appendix

SAGE codes:

```
# p=5 r=4 a=2 m=4
def bprma(n):# outputs B_{5,4,4,2}(n)
    b = []
    for j in Partitions(n).list():
        b1 = []
        b2 = []
        b3 = []
        b4 = []
        b5 = []
        for t in j:
            if list(j).count(t)%5 == 0 and 0 <= list(j).count(t) <= 15:
                b1.append(t)
        for s in j:
            if list(j).count(s)%5 == 2 and 22 <= list(j).count(s) <= 37:
                b2.append(s)
        for k in j:
            if list(j).count(k)%5 == 4 and 44 <= list(j).count(k) <= 59:
                b3.append(k)
        for l in j:
            if list(j).count(l)%5 == 1 and 66 <= list(j).count(l) <= 81:
                b4.append(l)
        for h in j:
            if list(j).count(h)%5 == 3 and 88 <= list(j).count(h) <= 103:
                b5.append(h)
```

```
        if len(b1) + len(b2) + len(b3) + len(b4) + len(b5) == len(j):
            b.append(j)
    return len(b), b

def eprma(n):# outputs E_{5,4,4,2}(n)
    e0 = []
    for j in Partitions(n).list():
        e1 = []
        e2 = []
        for t in j:
            if t%20 in [5,10,15]:
                e1.append(t)
        for s in j:
            if s%110 in [22,44,66,88]:
                e2.append(t)
        if len(e1) + len(e2) == len(j):
            e0.append(j)
    return len(e0), e0

def arminuscr(n):# outputs a_{1}(n,3), c_{1}(n,3) and a_{1}(n,3) - c_{1}(n,3)
    ar  = []
    ar1 = []
    cr1 = []
    cr2 = []
    cr3 = []
    for j in Partitions(n).list():
        D = [p for p in j if list(j).count(p) == 1]
        R = [q for q in j if (list(j).count(q))%3 in [0,2]]
        if len(j) == len(D):
            ar.append(j)
        if len(R) > 0 and len(D) + len(R) == len(j) and
                                Set(R).cardinality() == 1:
            cr1.append(j)
    for t in ar:
        for s in t:
```

```
            if s%3 == 2:
                ar1.append(s)
    for s in cr1:
        for t in s:
            if (list(s).count(t))%3 in [0,2] and t%2 == 1:
                cr2.append(s)
            if (list(s).count(t))%3 in [0,2] and t%2 == 0:
                cr3.append(s)
    return len(ar1), (len(Set(cr2)) - len(Set(cr3))),
           len(ar1) - (len(Set(cr2)) - len(Set(cr3)))

def de(n):# outputs the number of partitions of n in which exactly
          # one even part is repeated and odd parts are unrestricted
    d = []
    for j in Partitions(n).list():
        E = [p for p in j if p%2 == 0]
        O = [p for p in j if p%2 == 1]
        d1 = []
        d2 = []
        d3 = []
        d4 = []
        for s in E:
            if list(j).count(s) > 1:
                d1.append(s)
                if Set(d1).cardinality() == 1:
                    d2.append(d1)
        for s in E:
            if list(j).count(s) == 1:
                d3.append(s)
        for k in O:
            if list(j).count(k) >= 0:
                d4.append(k)
        if len(d2) != 0 and len(d2) + len(d3) + len(d4) == len(j):
                d.append(j)
    return len(d), d
```

```
def fo(n):# outputs the number of partitions of n in which
             # the set of parts congruentto 0 mod 4 is singleton
    f = []
    for j in Partitions(n).list():
        E = [p for p in j if p%4 == 0]
        F = [p for p in j if p%4 in [1,2,3]]
        f1 = []
        f2 = []
        f3 = []
        for t in E:
            if list(j).count(t) > 0:
                f2.append(t)
                if Set(f2).cardinality() == 1:
                    f3.append(f2)
        for k in F:
            if list(j).count(k) >= 0:
                f1.append(k)
        if len(f3) != 0 and len(f3) + len(f1) == len(j):
            f.append(j)
    return len(f), f

def do(n):# outputs the number of partitions of n in which exactly
              # one odd part is repeated and even parts are unrestricted
    d0 = []
    for j in Partitions(n).list():
        E = [p for p in j if p%2 == 0]
        O = [p for p in j if p%2 == 1]
        d1 = []
        d2 = []
        d3 = []
        d4 = []
        for s in O:
            if list(j).count(s) > 1:
                d1.append(s)
```

```
            if Set(d1).cardinality() == 1:
                d2.append(d1)
    for s in O:
        if list(j).count(s) == 1:
            d3.append(s)
    for s in E:
        if list(j).count(s) >= 0:
            d4.append(s)
    if len(d2) != 0 and len(d2) + len(d3) + len(d4) == len(j):
        d0.append(j)
return len(d0), d0


def f2(n):# outputs the number of partitions of n in which
          # the set of parts congruent to 2 mod 4 is singleton
    f0 = []
    for j in Partitions(n).list():
        E = [p for p in j if p%4 == 2]
        F = [p for p in j if p%4 in [0, 1, 3]]
        f1 = []
        f3 = []
        f4 = []
        for t in E:
            if list(j).count(t) > 0:
                f1.append(t)
                if Set(f1).cardinality() == 1:
                    f4.append(f1)
        for t in F:
            if list(j).count(t) >= 0:
                f3.append(t)
        if len(f4) != 0 and len(f4) + len(f3) == len(j):
            f0.append(j)
    return len(f0), f0

def o4(n):# outputs the number of partitions of n in which the set
```

```
        # of parts congruent to 0 mod 4
    o0 = []
    for j in Partitions(n).list():
        E = [p for p in j if p%4 == 0]
        O = [p for p in j if p%4 in [1,2,3]]
        o1 = []
        o2 = []
        o3 = []
        for t in E:
            if list(j).count(t) >= 0:
                o1.append(t)
                if Set(o1).cardinality() == 1:
                    o2.append(o1)
        for s in O:
            if list(j).count(s) >= 0:
                o3.append(s)
        if len(o2) != 0 and len(o2) + len(o3) == len(j):
            o0.append(j)
    return len(o0), o0

def d4(n):# outputs the number of partitions of n where exactly one part
        # is repeated at least 4 times
    d0 = []
    for j in Partitions(n).list():
        d = [p for p in j if p >0]
        d1 = []
        d2 = []
        d3 = []
        for t in d:
            if list(j).count(t) < 4:
                    d1.append(t)
        for s in d:
            if list(j).count(s) > 3:
                    d2.append(s)
                    if Set(d2).cardinality() == 1:
```

```
                    d3.append(d2)
    if len(d3) != 0 and len(d1) + len(d3) == len(j):
        d0.append(j)
return len(d0), d0
```