MATRIX POLAR COORDINATES

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Declaration

I, the undersigned, hereby declare that the work contained in this dissertation is my own original work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not previously in its entirety or in part been submitted for any degree or examination in any other University.

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Abstract

Matrix models feature prominently when studying string theory. In this project we extend well known single matrix model results to two matrix models. The two matrix model is represented using polar coordinates and then used to compute the kinetic piece of the quantum mechanical Hamiltonian operator of two, space indexed, hermitian matrices with a radially invariant potential. As an extension of these matrix polar coordinates, we determine the form(s) of the Laplacian(s) that act on invariant states. The radially dependent Hamiltonian operator is shown to be equivalent to a system of non interacting (2+1) dimensional fermions. Further on, we consider the integral of the two matrix model in polar coordinates to show the standard solution which emulates the Wigner distribution in the free case, when $g_{YM}^2 = 0$. Also in the large N limit we find that the polar coordinate matrix model, when solved using perturbation theory, agrees with the well known result of perturbation theory up to order λ, where λ is the 't Hooft coupling constant.

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Contents

Chapter 1

Introduction

The relationship between large N gauge theories and string theory has given birth to numerous ideas in Physics of which the AdS/CFT correspondence is the most well known. The breakthrough was the insight by 't Hooft [1] who realized that large N gauge theories are related to strings, and it was this insight that eventually contributed to the development of "Gauge-Gravity" dualities.

The gauge theory description is applicable in the investigation of the high energy behavior of strong interactions, but its application becomes limited when investigating low energy phenomena such as confinement and chiral symmetry. The study of strong interactions (Quantum Chromodynamics) contains, within its theory, an effective coupling constant that becomes large at large distances where perturbation theory is not applicable. To investigate the low energy dynamics where perturbation theory is not applicable, 't Hooft proposed the use of the $1/N$ expansion where N is the number of quark colours/parameter of $SU(N)$ for large N. The study of QCD in the $1/N$ expansion results in planar diagrams which can be topologically mapped to Feynman diagrams in perturbation theory. The $SU(N)$ gauge theory can be considered as a matrix model of $N \times N$ unitary matrices since the gauge fields appear in the adjoint representation of the group. The leading contribution to any correlator comes from planar diagrams.

Matrix models first appeared in the study of nuclear physics, when studying energy levels of atomic nuclei, and statistical physics. The single hermitian matrix model was the first and simplest model to be solved by [2] in the large N limit i.e. the planar limit. The authors of [2] used this single matrix model to obtain the combinatorics of planar diagrams and the ground state energy of a one dimensional oscillator with a ϕ^4 interaction. For the quantum mechanical Hamiltonian, the authors of [2] showed that a fermionic picture emerges.

The single matrix model is also used in the formulation of string theory in two dimensions [4], when considered in the double scaling limit. Generalizations can be made to map higher matrix models to string theory [4] but this includes the risk of a tachyon appearing in the theory.

The study of the large N limit of multi-matrix models is of great interest. For instance, multi-matrix models have been used in defining M theory [7] and in the context of the AdS/CFT correspondence, they are an ingredient in deriving correlators of supergravity and 1/2 BPS states incorporating supersymmetry and conformal invariance [8]. The plane-wave matrix theory derived from the $\mathcal{N} = 4$ Super Yang-Mills (SYM) is another well known matrix model related to the dilation operator [9]. Recently, matrix models have been studied in the context of a possible mechanism for the "emergence of gravity" [5] [6].

The main purpose of this project is to study matrix models in matrix valued polar coordinates, with further motivation provided at the end of the second chapter. The dissertation is structured as follows: Chapter two introduces the background and some of the key ideas in string theory. In the third Chapter, we review the single hermitian matrix model, and some of its properties. After the two matrix model has been introduced in terms of polar coordinates, Chapter four is dedicated to computing the Hamiltonian operator and Laplacian using two parameterizations that we define, referred to as parameterization I and II. In Chapter five we introduce invariant states and describe the form of the Laplacian when acting on invariant states. Further on the fermionisation of the two matrix model is shown in Chapter six. In Chapter seven we obtain a solution for the two matrix model when studied in the absence of interactions. We also show the results of the two matrix model that agree with perturbation theory in the large N limit. Lastly, Chapter eight concludes the work done in our project and questions that need to be investigated in future research.

Chapter 2

Gauge/Gravity duality

2.1 String Theory: An Overview

The objective of string theory is to unify all the known four fundamental forces of nature into a single consistent mathematical framework, hence being termed the theory of everything. This objective also means that string theory should be able to unify quantum mechanics with general relativity to obtain the theory of quantum gravity.

String theory, a theory whose elementary particles consists of "strings", was a theory that was originally proposed to capture and describe the physics of strongly interacting particles, namely mesons and hadrons. These particles, were to be interpreted as variations of the oscillation spectrum of a string, making string theory a probable description for the dual resonance model, which is a model that describes the strongly interacting hadrons and mesons.

Due to various limitations (attempts to use relativistic strings to describe hadrons), an alternative description of strong interactions was adopted, and that description was Quantum Chromodynamics (QCD). QCD is a renormalizable quantum field theory of the gauge group $SU(3)$ of the Yang-Mills and is also an asymptotically free theory, which means that the effective coupling constant of the theory decreases as the energy increases, but QCD becomes strongly coupled at low energies. It was later realized that these dual resonance models are more effective for a different purpose, that is, they were later realized to be quantum theories of relativistic vibrating fundamental strings.

The fundamental strings of string theory have dynamics described either by the Nambu-Goto or Polyakov action. These actions describe how relativistic strings propagate through spacetime and further describe the string's kinetic energy and tension in the absence of interactions.

String theorists postulate that the types of particles observed in nature, which are also observed from the mass spectrum of the respective string, should emerge from the string's oscillation modes. In string theory, there can be either open strings or closed strings. A closed string, which is topologically identical to a circle with no end points, can include in its spectrum of particles, a graviton (spin 2 messenger particle). The open string, with two end points, is topologically equivalent to a segment of a line, includes in its spectrum of particles a photon (spin 1 messenger particle). The appearance of closed strings is a natural consequence, since by joining two end points of an open string you can form a closed string, and these strings, open and closed, interact by joining and splitting. As candidates for the unification of all of the four fundamental forces of nature, these strings have typical length of order Planck length l_p approximated as

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \cong 1.61624 \times 10^{-35} m,$$

where the Planck's constant $\hbar = 6.58211899 \times 10^{-16}$eVs, Newtons gravitational constant $G \approx 6.6748 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ and the speed of light $c \approx 3.0 \times 10^8\text{m.s}^{-1}$.

Thus far, (from what has been written above), we have referred numerous times to the relativistic string. Before we introduce the idea of the relativistic string, it is best we first describe the non relativistic string, which will allow the subtle properties of the relativistic string to be more appreciable. This explanation is mostly deduced from the work of [11].

If we consider a stretched non relativistic string in the (x, y) plane, then this would be a string that can display both longitudinal and transverse modes of oscillation. In our case, we require our non relativistic string to adhere to the following conditions: the x-coordinate of the string does not change in time and its transverse oscillations are along the y-coordinate. These non relativistic strings can have the following adjustable parameters: the tension T_0 and the mass per unit length μ_0. When stretched, the tension of the non relativistic string experiences a change that is so small that it is assumed to be approximately constant. The mass of this non relativistic string does not change during stretching. If this string is stretched to a length ϵ then its total mass is $M_0 = \mu_0\epsilon$ and the change in energy is the work done to stretch the string, that is $E_0 = T_0\epsilon$. If it is assumed that at any point along the static non relativistic string experiencing small oscillations $\partial y/\partial x$, that

$$\frac{\partial y}{\partial x} \ll 1$$

when a small piece is stretched from x to $x + \mathrm{d}x$ and at a certain time t, its transverse oscillations are $y(t, x)$ at x and $y(t, x + \mathrm{d}x)$ at $x + \mathrm{d}x$, then this string will experience a net force, this means that the tension on the string also changes. Thus the net vertical force is approximately

$$\mathrm{d}F_{net} \cong T_0\frac{\partial^2 y}{\partial x^2}\mathrm{d}x.$$

The total mass of the stretched string is $\mathrm{d}m = \mu_0\mathrm{d}x$, so using Newtons second law, it follows that

$$T_0\frac{\partial^2 y}{\partial x^2}\mathrm{d}x = \mu_0\mathrm{d}x\frac{\partial^2 y}{\partial t^2},$$

and naturally, cancelling $\mathrm{d}x$ on both sides of the equation we find

$$T_0\frac{\partial^2 y}{\partial x^2} = \mu_0\frac{\partial^2 y}{\partial t^2}. \tag{2.1.1}$$

When equation (2.1.1) above is compared to the standard wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2}, \tag{2.1.2}$$

and where v_0 is the velocity of the transverse waves, the following equivalence, between equations (2.1.1) and (2.1.2), can be made

$$v_0 = \sqrt{T_0/\mu_0} \tag{2.1.3}$$

$$\Rightarrow T_0 = v_0^2 \mu_0. \tag{2.1.4}$$

Equation (2.1.4) above is true for the case of the static (no motion along the x-axis) non relativistic stretched string. The velocity of the transverse waves v_0 is related to the the tension T_0 and the mass per unit length μ_0. So in essence, the non relativistic string can be described as having a tension when stretched and its mass does **not** change during stretching.

The strings of string theory are more fundamental than the above mentioned classical non relativistic strings. In string theory, relativistic strings are what defines the theory. These relativistic strings are fundamental in themselves, that is, they are not made up of smaller constituent particles. When these relativistic strings vibrate, they only display transverse modes of oscillation and not longitudinal modes [11]. The action of our relativistic string is given by the previously mentioned Nambu-Goto string action, which is written below as

$$\begin{aligned} S_{N.G} &= -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} \\ &= -\frac{T_0}{c}\int d\tau d\sigma \sqrt{-\gamma}, \end{aligned} \tag{2.1.5}$$

where $\gamma = \det(\gamma_{\alpha\beta})$ such that $\gamma_{\alpha\beta}$ is the induced metric on the world-sheet swept out by the string.

The variables appearing in equation (2.1.5) represent particular dynamics of the string and we proceed to give their meaning. When a relativistic point particle moves in spacetime, it maps out a world line, but a string oscillating in spacetime traces out a two dimensional surface known as a "worldsheet". The Nambu-Goto string action (appearing above) computes the area of the string's worldsheet. The worldsheet swept out by the relativistic string is parameterized by the coordinates $(\sigma^1 \equiv \tau, \sigma^2 \equiv \sigma)$. The string's worldsheet is embedded in the target spacetime where X^μ for $\mu = 0, 1, 2, ..., D-1$, gives the string's path in spacetime i.e. it is the string's coordinate. The parameter τ is related to the time on the string and the parameter σ is related to the positions along the string. Also, the following identifications are made

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \qquad X^{\mu'} \equiv \frac{\partial X^\mu}{\partial \sigma}.$$

The second line in equation (2.1.5) is the Nambu-Goto string action expressed in a reparameterization invariant form, whose derivation is explained more formally in [11].

The potential for a static (no kinetic energy) open stretched relativistic string of length q is $V = T_0 q$ [11]. Thus as you do work to pull the string, you give the string energy,that is , pulling the string creates rest energy or rest mass. The rest mass per unit length of the relativistic string is μ_0, so it can be shown that

$$T_0 = \mu_0 c^2 = \frac{V}{q} \Rightarrow \mu_0 = \frac{T_0}{c^2}. \tag{2.1.6}$$

From the equation (2.1.6) above, the tension of the string remains constant but the mass of the string experiences a change either due to a change in the string's length or due to excitations of the vibrational modes of the string. We can compare equations (2.1.4) where the tension T_0 of the non relativistic string was proportional to the square of the velocity of the transverse waves i.e. v_0^2, and equation (2.1.6) which is the tension for the relativistic string, where the transverse velocity has been replaced by the speed of light c^2.

Another crucial parameter, related to the tension T_0, that is worth mentioning is the slope parameter α' or the Regge slope, which we will later see emerge in one of the most crucial mathematical constructs in string theory: the AdS/CFT correspondence.

To illustrate how α' comes about, an example from [11] is used. If a straight rigidly rotating open string on the (x, y) plane is considered, then the angular momentum J of the relativistic string is related to its energy E as

$$J = (\alpha' \hbar) E^2. \tag{2.1.7}$$

The dimensions of α' are the inverse of energy squared. This Regge slope can be related to the previously mentioned string tension T_0. For the open string considered above, the following can be deduced

$$J = \left(\frac{1}{2\pi T_0 c} \right) E^2. \tag{2.1.8}$$

From equations (2.1.7) and (2.1.8), it follows that

$$\alpha' \hbar = \frac{1}{2\pi T_0 c} \Rightarrow T_0 = \frac{1}{2\pi \alpha' \hbar c}. \tag{2.1.9}$$

When the tension of the string is known, its length can be obtained using the Regge slope with the following equation

$$l_s = \hbar c \sqrt{\alpha'}, \tag{2.1.10}$$

where l_s is known as the characteristic string length.

From all that has been explained so far, we can deduce that the non relativistic string is a classical string that has distinguishing properties from the relativistic string. The relativistic string features more prominently in string theory but unfortunately it is not the bread and butter of string theory. Bosonic String Theory (BST) is formulated using relativistic strings, but at best BST is a toy model because of its limited capabilities to describe nature as we observe it i.e. BST is only limited to the description of bosons. Below we explain some of the problems of BST and introduce an alternative theory that is considered as a possibility to be a candidate of the theory of everything.

The reparameterization invariant Nambu-Goto action is an action of the relativistic BST. Some of the limitations that render BST as an unsuitable candidate toward becoming the theory of everything are:

(i) BST only predicts the existence of bosons, messenger particles in nature, contrary to our physical observations which include fermions. As an example, some of the quantum states of BST that represent particle states have particles such as the photon and the graviton.

(ii) BST is a theory that contains/predicts a particle whose speed exceeds that of the photon and has an imaginary mass. This particle is known as the "tachyon", and renders the theory unstable.

(iii) This is a theory that is consistent in 26-spacetime dimensions and not in the $(3+1)$-dimensional universe that we observe.

(iv) Unphysical particle states known as "ghost's" are predicted in BST, but these are absent in 26-spacetime dimensions where the theory is consistent.

With all these unfortunate negatives for BST, an alternative direction was taken for the pursuit of the theory of everything that will unite general relativity with quantum mechanics. This led to the Superstring theory.

Super(symmetric)string theory is a relativistic string theory that incorporates supersymmertry and fermions in its formulation. Fermions are the particles that make up the matter that we observe in nature. Supersymmetry is a symmetry that relates bosons to fermions, and unites matter with the forces of nature. In a theory with supersymmetry, bosons and fermions are grouped into multiplets of equal mass.

There are five types of superstring theories: open type I and closed type I, closed type IIA, closed type IIB, and the heterotic string which is separated into two groups, the group with symmetry $SO(32)$ and the group with $E_8 \times E_8$ symmetry. At first sight, these superstring theories were considered to be distinct, but it was only realized later that they are an underlying elementary description of a more fundamental theory, known as M- theory, where the M is yet to be defined. Surprisingly M-theory is not a string theory, it is a theory whose mathematical composition is that of membranes i.e. 2-branes and 5-branes and not D-branes. These five superstring theories are related to each other through mathematical transformations known as dualities. Dualities are types of symmetries that bring about a relationship between theories that have different descriptions in separate systems but illustrate the same physics. For example, duality symmetries that relate theories in large and small distance scales are known as T-dualities and those that relate the theories in the strong and weak coupling are known as S-dualities. By combining the T-duality and the S-duality, this gives the U-duality, which is the symmetry of M theory.

D-Branes

Open strings have boundary conditions at their end points. When the end point of the open string is **fixed** , this boundary condition is referred to as a Dirichlet boundary condition. These Dirichlet boundary conditions are applicable to spatial extensions only. Alternatively another boundary condition is the one that does not require any constraints to be imposed on the string's end point. This means that the string's end point can move in any direction, in this case the boundary condition is referred to as Neuman boundary condition.

The objects that the end points of a string are attached to are determined by the motion of the string's end point in space and these objects are known as membranes or more commonly, branes. The formulation [12] [13] of mathematical objects known as "Dirichlet-branes" or D-branes, alternatively "Dirichlet p-branes" or Dp-branes, was a paramount conceptualization that permitted further developments of string theory. String theory describes Dp-branes as $(p+1)$-dimensional spacetime hypersurfaces where strings can attach [14]. The D in Dp-brane [12] represents boundary conditions that arise as a result of the string end-points that are attached to a physical object and the p represents the number of spatial dimensions of the brane. A closed string, which has no endpoints, moving in the bulk, can touch a Dp-brane and then open up to become an open string. Thus the Dp-brane [12] [15] can be viewed as a source for closed strings and this phenomenon is related to the "T-duality".

If a string is stretched between two points, then the objects that are attached to the string end points are considered to be D0-branes. When the string end points are allowed to be confined vertically along the "y- axis", not permitting horizontal motion, then the end points of the string are said to be attached to a D1-brane. Boundary conditions that are extended over a membrane give a D2-brane. The D3-brane, a membrane of higher dimensional spatial extension, can also be obtained by describing the motion of the string's end points.

Dp-branes have computable properties like energy density, volume [12] and other interesting quantities. At first, before the D-branes, p-branes [12] [13] [15] were classical solutions to supergravity, which is a low energy limit of (type IIB) string theory. A more complete description of these p-branes was obtained by defining Dp-branes and this was proposed by Polchinski [13] [15]. Below we give a more physical short description on how Dp-branes appear in string theory.

A conformal field theory [12] [14], is a theory that is invariant under conformal transformations on a $(1+1)$-dimensional string worldsheet and defines a particular superstring theory. This conformal field theory contains within it bosonic fields X^{μ} which define the coordinates of the string in 10-dimensional spacetime. Using supersymmetry, these bosonic fields can be related to their fermionic partners ψ^{μ} also in the 10-dimensional spacetime. The boundary [14] conditions on the bosonic fields X^{μ}, can either be Neuman boundary conditions or the Dirichlet boundary conditions. Each bosonic field has distinct boundary conditions. With the Neumann boundary conditions, the open string end points can move freely in $(9-p)$ dimensions. The Dp-brane arises when $(9-p)$ of the fields have Dirichlet boundary conditions, giving rise to string end points that are constrained to lie on a p-dimensional hypersurface and this hypersurface being the Dp-brane [13].

For the fermion fields [14] on an string, different boundary conditions can also be considered. In this case, the open string boundary conditions correspond to integer and half integer modes that are referred to as Ramond (denoted R) and Neveu-Schwarz (denoted NS) boundary conditions. For the closed string, there can be the periodic and anti-periodic boundary conditions corresponding to the left and right moving fermions. This gives rise to four distinct sectors for the closed string, which are: NS-NS, R-R, NS-R and R-NS [14].

All the five superstring theories that were previously mentioned can each be described by a different assembly of Dp-branes, each collection specified by different Dirichlet boundary conditions on the end points of a string. The [14] type IIA(IIB) superstring theories have Dp-branes with even(odd) p-dimensions. Also with each superstring theory there is a fundamental string and a Neveu-Schwarz(NS)-five brane. The Dp-branes of one type of superstring theory can be related to the Dp-brane of another type of superstring theory through duality transformations and with the sufficient dualities this further suggest's non-uniqueness of the branes, in that the branes of one (superstring) theory can be mapped to the branes of another (superstring) theory and further mapped to a string.

By stacking N parallel Dp-branes [12], N^2 different open strings emerge. These N^2 different open strings [12] arise because of the different branes the end points of the open strings can be attached too. The N^2 is the dimensions of the adjoint representation of the gauge group $U(N)$ that gives the maximally supersymmetric $U(N)$ gauge theory on the worldvolume of the Dp-brane. The expectation values of the scalar fields determine the relative separations of the parallel Dp-branes in $(9 - p)$ transverse directions. So Dp-branes can be described in terms of a $U(N)$ supersymmetric gauge theory on its world volume. When the expectation values of the scalar fields vanish and N is taken to be large ($N \to \infty$), the N-stacked parallel Dp-branes become a heavy object and it is this heavy object that is a source for closed strings, whose description contains gravity [12]. In simpler terms, the open strings living on the Dp-brane (gauge theory) become gravitational sources ($\Rightarrow$ closed strings living in the bulk), this phenomena couples a gauge theory to a description of gravity, and is considered as a type of duality. Hence the term "Gauge-gravity" duality.

The most renowned example [16] of a gauge-gravity duality is a stack of D3-branes which can be described using either the open string description or the closed string description. In the low energy limit of this stack of the D3-branes, one can consider the limit where $l_s \to 0$, where we had previously defined l_s as the string length. In this limit, the open string description of the D3-branes is reduced to an $\mathcal{N} = 4$ Super Yang-Mills theory (Superconformal field theory) and the closed string description of the D3-branes reduces to a (super) string theory on $AdS_5 \times S^5$ with a gravitational description.

Large N Gauge Theories

It is only natural that we proceed to link large N gauge theories to string theories, because to describe strong interactions, string theory was the original candidate but a gauge theory (QCD) proved to be more suitable, which left the question: Is there a relationship between string theory and gauge theory? Gauge theories proved to be sound when studying the high-energy behavior of strong interactions, but not so at low energies [15]. At these low energies, phenomena such as chiral symmetry breaking and confinement exist and gauge theory is not an appropriate tool to characterize the physics of such phenomena. Could there be a theory that has a duality description such that, at one limit the theory is strongly coupled and equivalently at the other limit the theory is weakly coupled and vice versa? Does QCD have a dual description, that is, can we find two related theories that describe QCD from different limits, the strong and the weak limit, with these two theories related by duality transformations? Physicist's have been hinting that string theory could be a dual description of QCD [15].

In a paper published by 't Hooft [1], the author proposed that a gauge theory that has gauge group $U(N)$ representing quarks that have a colour index which runs from one to N, should be considered in the large N limit with $g_{YM}^2 N = \lambda$ held fixed , where λ is known as the 't Hooft's coupling constant. The dimensionless coupling constant g_{YM} of the theory can be adjusted to be weakly coupled ($g_{YM} \ll 1$) or be strongly coupled ($g_{YM} \gg 1$) and N is an integer number which serves as a parameter of the theory from the gauge group $U(N)$. When the above large N limit gauge theory proposed by 't Hooft is considered in the strong coupling limit ($\lambda \gg 1$), it suggests a theory of strings. These large N gauge theories can be shown to be equivalent to string theory [15].

By attempting to part quark and anti-quark pairs [15] [1], which are objects of QCD, flux tubes or Wilson lines are formed between these quark anti-quark pairs and these flux tubes are considered to be strings. This means that the large N gauge theory could be described by strings. Below a very schematic overview is given, obtained from [15], to indicate a relationship between gauge theory and string theory.

A correspondence between a theory of strings and a $U(N)$ gauge theory can be manifested in the large N limit through a perturbative expansion of the $U(N)$ gauge theory. A schematic Lagrangian is first considered

$$\mathcal{L} \sim \mathrm{tr}\left(\mathrm{d}\Phi_i \mathrm{d}\Phi_j\right) + g_{YM} c^{ijk} \mathrm{tr}\left(\Phi_i \Phi_j \Phi_k\right) + g_{YM}^2 d^{ijkl} \mathrm{tr}\left(\Phi_i \Phi_j \Phi_k \Phi_l\right), \qquad (2.1.11)$$

whose interactions are $SU(N)$ invariant.

The above schematic Lagrangian is for some arbitrary field Φ_i^a, whose upper index a is in the adjoint representation of $SU(N)$ and i represents the type of field that is being considered. If the field is a quark, then i would be the flavor index. By assuming that the Lagrangian above has fields whose three point vertices are proportional to g_{YM} and four-point vertices proportional to g_{YM}^2, the fields can then be rescaled by the following [1]

$$\Phi_i \equiv \frac{1}{g_{YM}} \Psi_i. \qquad (2.1.12)$$

Thus the Lagrangian in equation (2.1.11) becomes (for $c^{ijk}, d^{ijkl} \equiv$ constants)

$$\mathcal{L} \sim \frac{1}{g_{YM}^2} \left[\mathrm{tr}\left(\mathrm{d}\Psi_i \mathrm{d}\Psi_i\right) + c^{ijk} \mathrm{tr}\left(\Psi_i \Psi_j \Psi_k\right) + d^{ijkl} \mathrm{tr}\left(\Psi_i \Psi_j \Psi_k \Psi_l\right)\right]. \qquad (2.1.13)$$

In the large N gauge theory, fields that carry double indices are known as the matrix valued adjoint fields Φ^a [15]. These adjoint fields can be represented as a product of fundamental and anti-fundamental fields Φ_j^i. The double index notation on Φ_j^i reproduce the double lines of the ribbon graphs of large N gauge theories, whilst the Feynman diagrams which are obtained from the theory of equation (2.1.11) can also be represented using double line notation from these adjoint fields. Correlators of SU(N) will take the following form

$$\left\langle \Phi^i_j \Phi^k_l \right\rangle \propto \left(\delta^i_l \delta^j_k - \frac{1}{N} \delta^i_j \delta^k_l \right). \tag{2.1.14}$$

The Kronecker delta terms are the ones that contribute to the planar diagrams whilst the terms that are sub-leading in N i.e. $1/N$ are the terms that are neglected because they are non-planar. From the Lagrangian of equation (2.1.11), the respective Feynman diagrams may be viewed as a particular simplicial decomposition of a surface with V vertices, E propagators and F loops. A planar diagram, a diagram with V vertices, E propagators and F loops, is associated with a coefficient proportional to the following

$$N^{V-E+F} \lambda^{E-V} = N^{\chi} \lambda^{E-V} \qquad \chi = V - E + F, \tag{2.1.15}$$

where χ is the Euler characteristic of the surface that is associated with the respective diagram. The Euler characteristic describes the topology of the surface, thus for closed oriented surfaces we get

$$\chi = 2 - 2g, \tag{2.1.16}$$

where g is the genus (the number of handles) on the topological surface. As an example, a sphere would have genus zero, a doughnut would have genus one etc. In the case of planar diagrams, their genus is zero.

In the large N limit, the topological expansion of the planar diagrams may be written as a double expansion of the following form

$$Z = \sum_{g=0}^{\infty} N^{2-2g} f_g (\lambda), \tag{2.1.17}$$

where f_g is some polynomial. The Feynman loop diagrams of string theory are given by the following loop expansion

$$Z = \sum_{g=0} g_s^{2g-2} Z_g. \tag{2.1.18}$$

Equations (2.1.17) and (2.1.18) are comparable since the string coupling constant, g_s, is related to the rank N of the gauge group by

$$g_s \equiv \frac{1}{N}. \tag{2.1.19}$$

The above schematic illustration from [15] does indicate that the Feynman diagrams can be topologically mapped to planar diagrams of the large N gauge theory, and also by reconciling the parameters of string theory and those of gauge theory, a strong-weak duality correspondence in the expansion formulas of both the string theory, equation (2.1.18), and the gauge theory, equation (2.1.17), is manifest.

2.2 The AdS/CFT Correspondence

The gauge-gravity duality provides a concrete example of how a theory of strings can be related to gauge theories as proposed by ’t Hooft [1]. The AdS/CFT correspondence [16][17], sometimes referred to as the gauge/gravity duality, is also a duality theory that relates theories with gravity in d-dimensions and those without gravity in (d-1)-dimensions. The best known example of the AdS/CFT correspondence was conjectured by Juan Maldacena.

Maldacena [16] starts with type IIB string theory and considers a stack of D3 branes separated by a distance r in the decoupling limit

$$\acute{\alpha} \rightarrow 0 \qquad\qquad U \equiv \frac{r}{\acute{\alpha}},$$

where U is held fixed, r is defined as the separation distance of N parallel D3 branes and $\acute{\alpha}$ was first introduced in equation (2.1.7). The type IIB string theory becomes weakly coupled in this limit, leading to supergravity solutions when $\lambda = g_{YM}^2 N \gg 1$.

Thus, Maldacena proposes that in the near extremal black D3 brane solution in the decoupling limit of the large N ($N \rightarrow \infty$) limit, the $D = 4$, $\mathcal{N} = 4$ $U(N)$ SYM theory has in its Hilbert space the states of type IIB supergravity on $AdS_5 \times S^5$. This means that type IIB supergravity contains gravitons propagating on $AdS_5 \times S^5$. This led Maldacena to conjecture that “Type IIB string theory on $AdS_5 \times S^5$ plus some appropriate boundary conditions is dual to $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory".

The above conjecture is also a manifestation of the strong-weak coupling nature of the duality. The $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory is a conformally invariant gauge theory that lives in lower dimensions with respect to the type IIB string theory found on $AdS_5 \times S^5$. In the AdS/CFT correspondence the parameters of the strings in AdS can be matched to those of the $\mathcal{N} = 4$ SYM theory that lives on the boundary of the AdS. The string theory living on the $AdS_5 \times S^5$ spacetime background has the following parameters [15] [16] [17] : $g_s \equiv$ dimensionless string coupling constant, $l_s \equiv$ string length that determines the size of the fluctuations of the string world sheet and $R \equiv$ curvature radius of $AdS_5 \times S^5$. On the boundary of $AdS_5 \times S^5$ where the four dimensional $\mathcal{N} = 4$ SYM theory lives, we can identify the following parameters: $N \equiv$ rank of the gauge group $U(N)$, $g_{YM} \equiv$ dimensionless coupling constant. Thus the following identifications are made

$$g_s = g_{YM}^2, \; \left(\frac{R}{l_s}\right)^4 = 4\pi g_{YM}^2 N = 4\pi\lambda. \tag{2.2.1}$$

In equation (2.2.1) above we see that if we take $\lambda \to \infty$, which corresponds to the strong coupling limit of the $D = 4$, $\mathcal{N} = 4$ SYM theory, the radius of the $AdS_5 \times S^5$ correspondingly becomes large. In this limit, the weakly coupled type IIB string theory becomes supergravity states with the usual GR type interactions. When the effective coupling $g_{YM}^2 N = \lambda$ becomes large the perturbative calculations cannot be trusted in the Yang-Mills theory [16], but the computations on the supergravity side on $AdS_5 \times S^5$ can be trusted as $R \to \infty$. The possibility of extending this duality from supergravity states to the "stringy" states was discussed in [18] by considering type IIB string theory in a maximally supersymmetric plane-wave background which has been shown to be dual to $\mathcal{N} = 4$ large N $U(N)$ Super Yang-Mills gauge theory in $(3 + 1)$-dimensional Minkowski spacetime. This extension will be delved into at a later stage.

2.3 Holography

Holography [19] is an idea that is deeply linked with string theory and the AdS/CFT correspondence. It is known that for a black hole, the entropy is proportional to the area of its event horizon as a consequence of black hole thermodynamics. But this type of relation is not restricted to black holes, it is a property of any quantum theory of gravity, that is, holography asserts that any quantum theory of gravity is completely equivalent to a quantum field theory in fewer dimensions.

Black holes are thermodynamical systems with temperature T and entropy S where the entropy is also equivalent to the degrees of freedom of the black hole. The entropy and temperature of a black hole are defined by

$$S = \frac{A}{4G}, \qquad\qquad T = \frac{\kappa}{2\pi},$$

where A is the area of the event horizon, G is the Newtonian constant and κ is the surface gravity.

Principally, the holographic principle relates the information of the black hole and all the constituents that fall within it and requires that the information that fell within the black hole be described by the surface fluctuations of the event horizon. This principle was proposed to solve the information paradox which basically says that the description of the low-dimensional surface oscillations of the gravitational event horizon will describe the information of the higher dimensional thermodynamical body like a black hole.

In general, the concept of holography [17] [19] is applicable also when gravity in d-dimensions is related to a local field theory in $(d-1)$ dimensions. In particular this can be interpreted to mean that quantum gravity in five-dimensions can be rendered to be equivalent to the local field theory of the $(3+1)$-dimensional Minkowski space.

Holography is an idea [20] [21] that is not only limited to black holes. Further evidence for Maldacena's conjecture is when symmetry groups are considered in both strong and weak coupling limits of the correspondence.

The isometry groups of the anti de Sitter space in $(d+1)$-dimensions is the *same* as the conformal symmetry group in d-dimensions [12]. A more specific example is the isometry [12] group of AdS_5, $SO(2,4)$ which matches the conformal symmetry group of the gauge theory in $(3+1)$ dimensional Minkowski space. Further, [12] the isometries of S^5 which form SU(4) $\sim$ SO(6) are the R-symmetries of $\mathcal{N}=4$ SYM that rotate the six Higgs fields into each other [16]. In the $\mathcal{N}=4$ SYM theory, there appears an R-symmetry which is identical to the latter isometry of S^5. As previously mentioned, when including supersymmetry in a theory it implies the inclusion of a fermionic description of the theory. The appearance of the fermionic generators of the string theory means that the $AdS_5 \times S^5$ background has a complete isometry supergroup which is $SU(2,2|4)$, and this isometry supergroup is equivalent to the $\mathcal{N}=4$ superconformal symmetry [12]. This spells [12] out a duality, and such dualities are also referred to as holographic since a string theory in AdS_{d+1} defined in $(d+1)$ dimensions is equivalent to a quantum field theory that lives in lower d-dimensions, the CFT_d.

2.4 A Short Description Of The Anti de Sitter (AdS) Space

The de Sitter and anti de Sitter spaces represent solutions that are obtained from solving pure gravity equations that have a corresponding cosmological constant [22]. The de Sitter, Anti de Sitter and flat spaces are maximally symmetric, that is, they have the largest possible isometry group allowed. The authors of [22] consider the following action of pure gravity

$$S = -s\frac{1}{16\pi G_D}\int d^D\sqrt{|g|}\,(R+\Lambda). \qquad (2.4.1)$$

In the above action of pure gravity given by equation (2.4.1), the first term s can either be $s=1$ or $s=-1$. When $s=1$, it means that computations were performed using a Minkowski metric $g_{\mu\nu}$ whose first entry on the main diagonal is a positive and the rest of the entries are negative. Alternatively, $s = 1$ corresponds to a Euclidean metric. When we have $s = -1$, the Minkowski metric has a matrix whose first entry on the main diagonal is negative and the rest of the terms on the main diagonal are positive. Factors appearing in the above action for pure gravity are: $\Lambda \equiv$ cosmological constant, $R \equiv$ scalar curvature, $G_D \equiv$ Newtonian constant in D-dimensions and $g \equiv$ determinant of a spacetime metric. For the above action of pure gravity, the Einstein equations [22] of motions in General relativity are shown to be

$$R_{\mu\nu} = \frac{\Lambda}{(2-D)} g_{\mu\nu}. \tag{2.4.2}$$

Above in equation (2.4.2), the de Sitter space corresponds to a positive cosmological constant $\Lambda > 0$, and the anti de Sitter space corresponds to negative cosmological constant, $\Lambda < 0$. There are many ways to represent the AdS space [22] [23], using various parameterizations. This being said, the AdS space can be constructed using Poincarè coordinates. We are going to embed the AdS space in a higher dimensional flat space. Consider [24] [25] the five-dimensional anti de Sitter (AdS_5) manifold embedded on a six-dimensional Euclidean space, then by intersecting the (AdS_5) with hyperplanes defined by

$$X_4 + X_5 = e^n,$$

the Poincarè coordinates are introduced on AdS_5, where each slice of the intersecting hyperplanes is represented by Π_n which is a copy of Minkowski spacetime.

The coordinates of the AdS_5 can be considered to be Poincarè coordinates because the points that lie on each slice Π_n can be parameterized by the $(3+1)$ dimensional Minkowskian coordinates, by rescaling them using e^n on the hyperplane. The metric of the embedding space has the following form

$$\mathrm{d}s^2 = \left[(\mathrm{d}X_0)^2 - (\mathrm{d}X_1)^2 - (\mathrm{d}X_2)^2 - (\mathrm{d}X_3)^2 - (\mathrm{d}X_4)^2 + (\mathrm{d}X_5)^2\right].$$

The AdS metric is obtained by computing the induced metric on the surface

$$(X_0)^2 + (X_5)^2 - (X_1)^2 - (X_2)^2 - (X_3)^2 - (X_4)^2 = R^2 \qquad (2.4.3)$$

where R is the radius of curvature of the AdS space.

After rescaling it with the coordinates on the hyperplanes, using the (3+1) Minkowskian coordinates, the metric takes the following form:

$$\mathrm{d}s^2 = R^2 \left(e^{2n}\left((\mathrm{d}x_0)^2 - (\mathrm{d}x_1)^2 - (\mathrm{d}x_2)^2 - (\mathrm{d}x_3)^2\right) - \mathrm{d}n^2\right),$$

where R is a constant radius of the five-dimensional anti de Sitter space.

Since each copy of the hyperplane Π_n of AdS_5 is a copy of (3+1) Minkowski spacetime, rescaling the points of the hyperplane by e^{2n} introduces the (v, x_0, x_1, x_2, x_3) coordinates on AdS_5, which are Poincarè coordinates. When we consider the $n \to \infty$ limit, this takes us to the boundary of the anti de Sitter space, which is at space like infinity. The boundary of the anti de Sitter space can be thought of as a $(3+1)$-dimensional Minkowskian spacetime, where the conformal theory lives, making the metric

$$\mathrm{d}s^2 \propto \left((\mathrm{d}x_0)^2 - (\mathrm{d}x_1)^2 - (\mathrm{d}x_2)^2 - (\mathrm{d}x_3)^2\right).$$

This purely illustrates that maps can be defined between the string theory states that are defined on the bulk and the conformally invariant operators of the dual field theory that are defined on the Minkowski space, which is the boundary of the bulk.

2.5 The BMN Limit

We now return to Maldacena's conjecture and how the supergravity modes can be extended to the "stringy" states. The authors of [18] propose an AdS/CFT correspondence between the $D = 4$, $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory and type IIB string theory living in the plane-wave background which is a Penrose limit of the $AdS_5 \times S^5$. To start, the metric of the $AdS_5 \times S^5$ space, which is defined as

$$ds^2 = R^2 \left[-dt^2 \cosh^2 \rho + \sinh^2 \rho d\Omega_3^2 + d\rho^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3'^2 \right] \tag{2.5.1}$$

is considered.

The above metric in equation (2.5.1) [18][26][27] of $AdS_5 \times S^5$ is written in terms of the global coordinates of the AdS space, for a particle moving along the ψ direction and sitting at $\rho = 0$ and $\theta = 0$. By introducing light-cone coordinates $\tilde{x}^\pm = \frac{1}{2}(t \pm \psi)$ and performing a rescaling by introducing $x^\pm$, r, y in the scaling limit R, the following is defined

$$x^+ = \frac{\tilde{x}^+}{\mu} \quad x^- = \mu R^2 \tilde{x}^- \quad \rho = \frac{r}{R} \quad \theta = \frac{y}{R} \quad R \to \infty,$$

where R is the radius of the AdS space.

In the $R \to \infty$ limit [18], which is defined as the Penrose limit, the metric of the $AdS_5 \times S^5$, becomes the metric of the plane-wave background which is given by

$$ds^2 = -4dx^+dx^- - \mu^2 \vec{z}^2 dx^{+2} + d\vec{z}^2, \tag{2.5.2}$$

where μ is the mass parameter [26] that maintains the canonical length dimensions for the rescaled coordinates. By taking $\mu \to 0$ in the above metric, the $(3+1)$-dimensional Minkowski metric is recovered which corresponds to the strong coupling limit on the gauge theory side of the duality. Also in the Penrose limit, [18] obtains

$$2p^- = \Delta - J \qquad\qquad 2p^+ = \frac{\Delta + J}{R^2}, \tag{2.5.3}$$

where p^- and p^+ are the conjugate momenta of $x^\pm$. The parameter E, $(E = i\partial_t)$, in the global coordinates t of the AdS space is identified with the scaling dimension Δ of the Super Yang-Mills operator. The angular momentum J, $(J = -i\partial_\psi)$, also in global coordinates ψ of the AdS space corresponds to the charge of a $U(1)$ subgroup of the $SO(2)$ R-symmetry group of the $\mathcal{N} = 4$ Super Yang-Mills. In the AdS space, states that have non-zero momentum p^+

and $R \rightarrow \infty$ in the light cone coordinate transformation are states with large angular momentum in the global coordinates of the AdS space such that

$$J \sim R^2 \sim N^{1/2}. \tag{2.5.4}$$

In the Penrose limit, when $N \rightarrow \infty$ with g_{YM} held fixed, in addition to this, J^2/N and the conformal dimension $\Delta - J$ are also held fixed, this corresponds to the BMN limit. On the gauge theory side, only states projected from the BMN limit are considered to construct a duality with the closed strings of type IIB string theory living in the plane-wave geometry of $AdS_5 \times S^5$. The closed strings which have harmonic oscillator modes $n = 0$, when excited, give the spectrum of massless supergravity modes propagating about the plane-wave geometry, these are the same supergravity modes that were observed in the conjecture of [16]. The operator $\mathrm{tr}\left(Z^J\right)$ [18] [26] [27], defined as a chiral primary operator, is a single trace operator that is associated with $\Delta - J = 0$, J is taken to be an $SO(2)$ generator that rotates the plane defined by the two scalars ϕ_5 and ϕ_6 and Z is defined by

$$Z \equiv \phi^5 + i\phi^6,$$

with the trace taken over N colour indices [18]. The $\Delta - J = 0$ states correspond to the supergravity states obtained by [16] in the low energy limit of type IIB string theory. The normalized chiral operator, with dimension J at weak coupling, is dual to the vacuum state in the light cone gauge. Below in equation (2.5.5) is a correspondence for the ground state (non-excited) supergravity mode with the $n = 0$ closed string oscillator modes on the string theory side

$$*(\text{Gauge theory side}) * \frac{1}{\sqrt{J}N^{J/2}}\mathrm{tr}\left(Z^J\right) \longleftrightarrow |0, p_+\rangle_{lc} * (\text{String theory side}) * . \tag{2.5.5}$$

In equation (2.5.5) above, we see an AdS/CFT correspondence (at weak coupling) between the single trace states of SYM theory on $R \times S^3$, which is a gauge theory or a spectrum of dimensions of single trace operators of the theory on R^4, dual to a vacuum state in the light cone gauge which represents the pp-wave limit. The strings in the above plane-wave/gauge correspondence in equation (2.5.5) are a particular description of a mode of the ten-dimensional (ground state) supergravity in a particular wave function [18]. The operator $\mathrm{tr}\left(Z^J\right)$ is the only one that has conformal dimension $\Delta - J = 0$. Thus to generate the rest of the massless supergravity modes, the operator ϕ^r is inserted into $\mathrm{tr}\left(Z^J\right)$, for $r = 1, 2, 3, 4$

$$\frac{1}{N^{J/2+1/2}}\mathrm{tr}\left(\phi^r Z^J\right). \tag{2.5.6}$$

The above state in equation (2.5.6) is normalized in the planar limit ($N \to \infty$) and ϕ is a scalar that is neutral under the rotations of J. The zero momentum oscillators a_0^i with $i = 1, 2, 3, ..., 8$ and S_0^b with $b = 1, 2, 3, ..., 8$ act on the light cone vacuum state $|0, p_+\rangle_{lc}$. These two actions give the rest of the massless supergravity modes on the string theory side. The i and b superscripts count the fermionic operators which constitute the sixteen component gaugino χ. An example of a plane-wave/gauge duality with $\Delta - J = 1$, is defined below for a state with two excitations [18]

$$\frac{1}{N^{J/2+1}}\frac{1}{\sqrt{J}}\sum_{l=1}^{J}\mathrm{tr}\left(\phi^r Z^l \psi^b_{J=1/2} Z^{J-l}\right) \longleftrightarrow a_0^{\dagger k} S_0^{\dagger b}|0, p_+\rangle_{l.c}, \tag{2.5.7}$$

where $\psi^b_{J=1/2}$ is one of eight fermionic operators of the sixteen component gaugino χ which have $J = 1/2$, the other eight have $J = -1/2$.

Operators given by the trace of Z, with the scalar fields inserted in the trace can be associated with planar diagrams [18], in which the scalar fields are contracted and operators which resulted from the insertion of fields with $\Delta - J = 1$ and the planar diagrams that are associated to these fields give rise to $(1 + 1)$-dimensional fields for each oscillator mode of the closed string at weak 't Hooft coupling. When $\lambda \neq 0$, the fields acquire mass, making them heavier, thus the operators that are associated with these fields have large conformal dimensions. An example of a correspondence involving a "stringy" state, when $\Delta - J = 2$ for non-supergravity modes, which means we have non-zero oscillator modes ($n \neq 0$) acting on the light cone Hamiltonian is given as

$$\frac{1}{\sqrt{J}} \sum_{l=1}^{J} \frac{1}{N^{J/2+1}} \mathrm{tr}\left(\phi^3 Z^l \phi^4 Z^{J-l}\right) e^{\frac{2\pi i n l}{J}} \longleftrightarrow a_n^{\dagger 8} a_{-n}^{\dagger 7} |0, p_+\rangle_{l.c.}. \tag{2.5.8}$$

The stringy state appears on the left hand of the above equation. The above correspondence in equation (2.5.8) was shown by [18] for a certain limit of type IIB string states living on the plane-wave geometry to be dual to a particular sector of states of the $D = 4$, $\mathcal{N} = 4$ $U(N)$ SYM on the gauge side by defining the BMN limit. The [9] gauge theory that is dual to the type IIB plane-wave superstring theory is argued through the principle of holography [28] to be an effective quantum mechanical one-dimensional plane-wave matrix model that arises from the Kaluza-Klein reduction of the $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3$.

2.6 Plane Wave Matrix Theory

Reference [9] illustrates a self-contained and consistent framework to infer plain-wave matrix theory, showing that by performing a Kaluza-Klein truncation on the $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory on $R \times S^3$, a consistent definition of plane-wave matrix theory can be obtained. Also by the application of perturbation theory, it is further shown in [9], that the one-loop anomalous dimension of pure scalar operators (which is equivalent to the first order energy shift of states of the plane-wave matrix model) is reproduced in the perturbative extension plane-wave matrix theory.

To derive the plane-wave matrix model, the authors of [9] first perform a dimensional reduction by taking $\mathcal{N} = 1$ Supersymmetric Yang-Mills in ten-dimensions on a six torus and then dimensionally reducing it to $D = 4$, $\mathcal{N} = 4$ Supersymmetric Yang Mills. The fields of the $D = 4$, $\mathcal{N} = 4$ Superconformal Yang-Mills theory consists of a vector field A_μ, six real scalar fields ϕ_i as well as four Weyl spinors $\lambda_{\alpha A}$, in the adjoint representation of the gauge group. The fields (all four dimensional) are expressed in spherical harmonics on S^3 in the Coulomb gauge $\nabla_a A^a$. The spherical harmonics on S^3 come in irreducible representations (m_L, m_R) of the $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ isometry group.

The mode expansions of the fields are inserted into the action of $D = 4$, $\mathcal{N} = 4$ SYM. The action is integrated over the field on which the harmonic expansion is being performed to give a one-dimensional theory with an infinite number of fields, whose mass spectrum is investigated. After using special properties of the spherical harmonics, [9] obtains a mass spectrum which is given in figure 1 on page 7 of [9], that represents a particle spectrum of $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3$. This mass spectrum is the Kaluza-Klein mass tower. Various states in the Kaluza-Klein tower may be reached by acting on the states of the tower using two super-charges Q_L and Q_R. The supercharge Q_L takes you to the upper left and Q_R to the upper right of the Kaluza-Klein mass tower. The entire Kaluza-Klein mass tower is viewed as a single irreducible representation of the superconformal theory.

From the mass tower spectrum, [9] considers the lowest lying supermultiplet $(\mathbf{1}, \mathbf{1}, \mathbf{6}) + (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{3}, \mathbf{1}, \mathbf{1})$ which corresponds to $\frac{1}{2}BPS$ states, the second lowest lying supermultiplet corresponds to $\frac{1}{4}BPS$ states etc. The truncation performed on the tower of Kaluza-Klein modes is restricted to the lowest lying supermultiplet. States appearing in this particle spectrum are labeled by $SU(2)_L \otimes SU(2)_R \otimes SU(4)$ representations, which represent modes of the spectrum. From the particle spectrum, the plane wave-matrix theory one dimensional Lagrangian L is derived by truncating the infinitely large spectrum of the Kaluza-Klein state modes down to the lowest lying supermultiplet. The one dimensional Lagrangian is given by the following equation

$$
\begin{aligned}
L &= \operatorname{tr}\left(\frac{1}{2}\left(D_t X_I\right)^2 - \frac{1}{2}\left(\frac{m}{6}\right)^2 X_a^2 - \frac{1}{2}\left(\frac{m}{6}\right)^2 X_i^2\right) \\
&+ \operatorname{tr}\left(\frac{m}{3} i\varepsilon_{abc} X_a X_b X_c + \frac{1}{4}\left[X_I; X_J\right]^2\right) \\
&- \operatorname{tr}\left(2i\theta^\dagger D_t\theta + \frac{m}{2}\theta^\dagger\theta - 2\theta^\dagger\sigma^a\left[X_a;\theta\right] + \theta^\dagger i\sigma^2\rho_i\left[X_i;\theta^*\right]\right) \\
&- \operatorname{tr}\left(\theta^T i\sigma^2\rho_i^2\left[X_i;\theta\right]\right).
\end{aligned} \tag{2.6.1}
$$

For the one-dimensional Lagrangian in equation (2.6.1) above, the variables appear as follows: $X_I \equiv$ one-dimensional bosonic hermitian $N \times N$ matrix, $\theta \equiv$ one-dimensional fermionic hermitian $N \times N$ matrix, $D_t \equiv$ covariant derivative with respect to time t, $\sigma^a \equiv$ usual Pauli matrices, $\rho_i \equiv$ Clebsch-Gordon coefficients, $m \equiv$ mass parameter of the plane-wave matrix model, and the spatial indices $a, b, c \equiv 1, 2, 3$. The indices I and J count the elements of a particular representation, taking the values from one to the dimension of the representation. In the Lagrangian above, $J, I = 1, ...9$ (transverse $SO(9)$ index).

The mass parameter m of the plane-wave matrix model theory is defined by $m = \frac{6}{R}$ where R is the radius of the three sphere. When deriving the one dimensional Lagrangian, the relation between m and g_{YM} is shown to be

$$
\left(\frac{m}{3}\right)^3 = \frac{32\pi^2}{g_{YM}^2}. \tag{2.6.2}
$$

The $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3$ has been thus linked to the plane-wave matrix model theory using supersymmetry and by truncating Kaluza-Klein modes on the mass tower spectrum. The authors of [9] further show that through perturbation theory, the one-loop effective vertex, that determines the first order energy shift of states in the plane-wave matrix model is equivalent to the one-loop dilatation operator of $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills [29]. The equivalence [9] of the one-loop effective vertex and one-loop dilatation operator relates a field theory of $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3$ to a matrix quantum mechanics (the plane-wave matrix model theory). This relation between the scaling dimensions of Super Yang Mills operators on R^4 and the corresponding states in the plane-wave matrix theory is a consequence of the state operator map of conformal field theory. The equivalence of the scaling dimensions shown by [9] was proved for protected multiplets which exist in both the gauge theory and the plane-wave matrix theory in the SO(6) sector. The protected multiplets of the gauge theory are chiral primary operators and those of the plane-wave matrix theory are energy states, the former being represented by pure multi-scalar operators with vanishing anomalous dimensions and the latter is represented by symmetric traceless excitations.

The result [9] [30] that equates the one-loop effective vertex of the plane wave matrix theory to the one-loop contribution of the dilatation operator implies that the $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory can be integrable. Thus, the one-loop dilatation operator of $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills in the large N limit can be considered as the Hamiltonian (the one used to obtain the one-loop effective vertex of the plane-wave matrix theory) of an integrable $SO(6)$ spin chain model, and this view suggests the integrability of the $D = 4$, $\mathcal{N} = 4$ Super Yang-Mills theory. The integrability of large N plane wave matrix theory is shown in [30] where the Hamiltonian of the plane wave matrix theory is considered with all fields being $N \times N$ traceless hermitian matrices from the gauge group $SU(N)$. Integrability [30] means that there exists a conserved charge $U(\Lambda)$ in the theory such that

$$U(\Lambda) = \sum_{k=1}^{\infty} \Lambda^k U_{2k}, \tag{2.6.3}$$

where

$$\Lambda := G^2 N \qquad\qquad G := \frac{g_{YM}}{4\pi}, \tag{2.6.4}$$

and this charge commutes with the dilatation operator of $D = 4$, $\mathcal{N} = 4$ SYM. Integrability implies an infinite number of conserved charges, one for each degree of freedom in the theory.

By performing perturbation theory on the plane-wave matrix theory Hamiltonian, the energy operator is obtained up to three loop order. In [9], the two-loop dilatation operator on $\mathcal{N} = 4$ Super Yang-Mills could not be shown to agree with the two-loop effective vertex of the plane-wave matrix theory, but this is corrected in [30] by renormalizing the plane-wave matrix theory mass parameter m for $M = \frac{m}{3}$ such that

$$\frac{1}{M^3} = \frac{g_{YM}^2}{32\pi^2}\left(1 + \frac{7}{32\pi^2} g_{YM}^2 N - \frac{11}{252\pi^4} g_{YM}^4 N^2 + \mathcal{O}(g_{YM}^6)\right), \tag{2.6.5}$$

obtained from the path integral of higher Kaluza-Klein modes of the gauge theory on S^3.

As mentioned in [9], and as it is evidenced in equation (2.6.1), the six scalar fields that appear in the derivation of the plane-wave matrix theory generally couple through a Yang-Mills coupling constant as

$$-g_{YM}^2 \sum_{i<j} [X_i, X_j]^2 , \tag{2.6.6}$$

where X_i and X_j are two of the six scalar fields of [9]. These scalar fields can be considered as the Higgs fields or matrix valued coordinates observed in [31], who shows matrix models maps in the context of the AdS/CFT correspondence.

In general, the Higgs [1] [7] [9] of these matrix models always couple through a Yang-Mills interaction. Clearly the study of matrix models with this type of interaction, equation (2.6.6), is important in the understanding of the AdS/CFT correspondence. Of particular relevance is the emergence of geometric degrees of freedom from Matrix valued theories. In [6], it is proposed that this could be understood in terms of matrices which become mutually commuting in the strong coupling limit.

The work contained in this project is mainly based on the two matrix model. In the work of [31], the two matrix model is treated asymmetrically in following sense: one of the matrices, which generates the large N planar background, is treated in the coordinate basis and the other matrix, which is decomposed into "impurity operators", is treated in a creation/annihilation basis. The asymmetric treatment of these two matrices within the frame of reference of 1/2 BPS states, and the dual free harmonic Hamiltonian with the former matrix being the holomorphic component of a complex matrix, results in a mapping between a collective density description of the dynamics of this matrix and the droplet description of the LLM droplet metric [33] obtained in [31]. Further examples of the asymmetrical treatment of the two matrix model are shown in [34] [36] [40].

In our project, a different approach is taken. The quantum mechanics of two hermitian matrices X_1 and X_2, with spatial indices $(1, 2)$ are treated symmetrically by rewriting them as matrix valued polar coordinates which will hopefully make the transition from matrices to geometry easier to trace.

Chapter 3

Matrix Valued Polar Coordinates and a Review of the Single Matrix Model

3.1 Background and Motivation

In this dissertation, I will concentrate on the matrix model of two hermitian matrices. There are many reasons why systems of two hermitian matrices are of interest. For instance, the zero dimensional integral of two matrices coupled by Yang-Mills interaction was used in [6] to study the distribution of the density of eigenvalues, and to discuss a possible emergent geometry in this context.

In [33] and [35], it was also shown that a BMN type spectrum can be obtained by considering the Hamiltonian of two hermitian matrices in a supersymmetric background. In [38], a study of a non-symmetric Hamiltonian model of two matrices was undertaken.

In [30], [33], [35] and [38], the treatment of the two matrices is antisymmetric in the sense that one generating a large N background, is treated in position space "exactly", and the other is treated in a creation/annihilation basis, with the idea of adding impurities to the large N background of the other. The motivation for this project is to introduce a more symmetric description of two matrix models which reflect the expected global space time symmetries and to observe what type of dynamics will evolve by treating the two matrix model symmetrically.

3.2 Introducing Matrix Polar Coordinates

In this project we will study the kinetic piece of the Hamiltonian and the path integral of the two matrix valued (hermitian) coordinates.

We define the non-commuting matrix coordinates

$$X_1 \equiv (X_1)_{ij} \qquad\qquad X_2 \equiv (X_2)_{ij}. \tag{3.2.1}$$

X_1 and X_2 are two of the six $N \times N$ hermitian bosonic matrices/Higgs scalars which are obtained in the dimensional reduction of the $\mathcal{N} = 1$ SYM, from $D = 10$ to $\mathcal{N} = 4$, $D = 4$ dimensions [9]. X_1 and X_2 can also define a plane on the bosonic sector of $\mathcal{N} = 4$ SYM on $S^3 \times R$ [34], and these matrices can be grouped to define a complex scalar field

$$Z = X_1 + iX_2. \tag{3.2.2}$$

We introduce the indices in equation (3.2.1) to indicate the matrix nature of X_1 and X_2, where i and j represent entries in the matrix such that $i, j = 1, 2, ..., N$.

For ordinary (commuting) cartesian coordinates x_1 and x_2, it is well known that one can introduce complex coordinates

$$z = x_1 + ix_2 = re^{i\theta} \tag{3.2.3}$$

$$z^* = x_1 - ix_2 = re^{-i\theta}, \tag{3.2.4}$$

where r and θ are the standard real polar coordinates such that

$$x_1 = r\cos\theta \qquad\qquad x_2 = r\sin\theta. \tag{3.2.5}$$

How far can one take this analogy in the case of the non-commuting matrix coordinates X_1 and X_2? We define a complex matrix as

$$Z = X_1 + iX_2 = RU, \tag{3.2.6}$$

where R is a hermitian $N \times N$ matrix and U is a $N \times N$ unitary matrix. We write the matrix Z as shown in equation (3.2.6) because the right hand side takes a form that embodies the matrix polar coordinate nature of Z which is analogous to the real polar coordinates of equation (2.2.3) and (2.2.4). This is not to say that X_1 and X_2 can be written explicitly in terms of $\sin\theta$ and $\cos\theta$ and we cannot map $U \to e^{i\theta}$ and $R \to r$, this would be mapping $N \times N$ matrices to real numbers, which is incorrect. Of course, to be explicit $U \neq e^{i\theta}$ because our work assumes that the unitary matrix U is some arbitrary "angular" piece of Z and R is the radial part of Z. The first observation is that this parameterization correctly preserves the number of degrees of freedom as both R and U have N^2 independent real degrees of freedom, and therefore Z has $2N^2$ independent degrees of freedom.

Since

$$Z = RU \qquad\qquad Z^\dagger = U^\dagger R \tag{3.2.7}$$

then

$$2X_1 = (Z + Z^\dagger) = RU + U^\dagger R \tag{3.2.8}$$

$$2iX_2 = Z - Z^\dagger = RU - U^\dagger R, \tag{3.2.9}$$

and one needs to consistently make sure not to commute R and U. In this dissertation, we will want to obtain the Hamiltonian for the two matrices by diagonalizing the hermitian " radial matrix " R.

We will ask ourselves if the properties/dynamics computed for the single matrix can be generalized to a larger number of matrices. In this dissertation we answer this by calculating similar features as the single matrix model for the parameterized two matrix structure, $Z = RU$. It may be nice to do larger number of matrices, this is a goal we will pursue in future research. Before we proceed, we will revise the single hermitian matrix.

3.3 Reviewing The Single Hermitian Matrix

In this section we compute some properties of the single matrix model of an $N \times N$ hermitian matrix M. These properties will be a crucial guideline in sections that follow because from them, we will be able to deduce certain general properties for models with a higher number of matrices i.e. the two matrix model in polar coordinates. Objectives are as follows:

(1) Calculate the infinitesimal line element $\mathrm{d}M$ for the single matrix picture M.

(2) From (1), determine the Laplacian $\nabla^2_{S.M}$ for the single matrix picture.

(3) Introduce the fermionisation picture of the single matrix.

(4) Compute the conjugate momentum $P_{ij} = \partial/\partial M_{ji}$.

To accomplish our goals, we first start by considering a Hamiltonian that represents the dynamics of a single hermitian matrix. This Hamiltonian takes the following form

$$\begin{aligned}
\hat{H} &= -\frac{1}{2}\mathrm{tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + \mathrm{tr}\left(K(M)\right) \\
&= -\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial M_{ij}}\frac{\partial}{\partial M_{ji}} + \mathrm{tr}\left(K(M)\right) \\
&= -\frac{1}{2}\nabla^2_{S.M} + \mathrm{tr}\left(K(M)\right).
\end{aligned}$$

where the indices i and j are the matrix indices that run from 1 to the size of the matrix, in this case N. Later on we will construct an eigenvalue/Schödinger equation using the Hamiltonian above. The $N \times N$ unitary matrices V and $V^\dagger$ are introduced such that $V, V^\dagger \in U(N)$ [43]. These unitary matrices are helpful because they are the angular variables that diagonalize M into a matrix with N eigenvalues and $N^2 - N$ angular degrees of freedom, allowing it to be shown in a form proportional to the eigenvalue matrix of M, such that

$$M = V^{\dagger} r V.$$

The term r, in the equation above, is an $N \times N$ diagonal matrix whose entries are the eigenvalues of M, $r = diag(r_1, r_2, ..., r_N)$. If the potential $\mathrm{tr}\,(K(M))$ is invariant under the similarity transformation ($SU(N)$ rotations),

$$M \rightarrow U^{\dagger} M U \tag{3.3.1}$$

then the potential depends only on the eigenvalues of M. This similarity transformation, a global $SU(N)$ gauge symmetry, is important because when it is identified with a local symmetry within a matrix model, then this identification ensures us that, had we computed physical observables in our theory, for instance correlators, then these would be invariant when treated under this similarity transformation.

Since the matrix M is hermitian i.e. $\Rightarrow M = M^{\dagger}$, it follows that $\Rightarrow \mathrm{d}M = \mathrm{d}M^{\dagger}$. We also have that r is hermitian i.e. $r = r^*$ (matrix with real entries).

We now proceed to compute the variation $\mathrm{d}M$ from which the Laplacian can be obtained. For the analogue of the line element $\mathrm{d}M$ in terms of matrices we find

$$\begin{aligned} \mathrm{d}M &= \mathrm{d}V^{\dagger} r V + V^{\dagger} \mathrm{d}r V + V^{\dagger} r \mathrm{d}V \\ &= V^{\dagger} \left(V \mathrm{d}V^{\dagger} r + \mathrm{d}r + r \mathrm{d}V V^{\dagger} \right) V. \end{aligned} \tag{3.3.2}$$

Above, in equation (3.3.2), we introduce the definition: $\mathrm{d}S = \mathrm{d}V V^{\dagger}$. The differential $\mathrm{d}S$ is a $N \times N$ anti-hermitian traceless matrix that is identified with the angular degrees of freedom of the variation, with the following condition: $\mathrm{d}S = -\mathrm{d}S^{\dagger}$. To see this, we take note of the following property:

$$\mathrm{d}S = \mathrm{d}V V^{\dagger} = -V \mathrm{d}V^{\dagger} = -\mathrm{d}S^{\dagger}, \tag{3.3.3}$$

where $\mathrm{d}S^{\dagger} = V \mathrm{d}V^{\dagger}$ and we note the property $V \mathrm{d}V^{\dagger} = -\mathrm{d}V V^{\dagger}$.

When we use the definition of the anti-hermitian differential $\mathrm{d}S$ in $\mathrm{d}M$, it follows that

$$
\begin{aligned}
\mathrm{d}M &= V^\dagger\left(V\mathrm{d}V^\dagger r + \mathrm{d}r + r\mathrm{d}VV^\dagger\right)V \\
&= V^\dagger\left(-\mathrm{d}VV^\dagger r + \mathrm{d}r + r\mathrm{d}VV^\dagger\right)V \\
&= V^\dagger\left(\mathrm{d}r + r\mathrm{d}VV^\dagger - \mathrm{d}VV^\dagger r\right)V \\
&= V^\dagger\left(\mathrm{d}r + [r, \mathrm{d}S]\right)V.
\end{aligned}
\tag{3.3.4}
$$

So in summary, for the variation $\mathrm{d}M$, we find that

$$
\mathrm{d}M = V^\dagger\left(\mathrm{d}r + [r, \mathrm{d}S]\right)V. \tag{3.3.5}
$$

Now that the first objective has been accomplished, we now need to put the structures that are going to be needed to compute the Laplacian of the single matrix model, $\nabla^2_{S.M}$, in place. To start off, we need to determine the square of the line element $\mathrm{d}s^2$ in the matrix language. The square of the infinitesimal line element $\mathrm{d}s^2$ can be similarly represented in the matrix language to be

$$
\mathrm{tr}\left(\mathrm{d}M^2\right) \equiv \mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{3.3.6}
$$

The metric tensor $g_{\mu\nu}$, which features prominently in the definition of the Laplacian, will enable us to compute the Jacobian under change of variables from matrices to eigenvalues and unitary matrices.

To calculate the the square of the line element in terms of matrices, we need $\mathrm{d}M$ and we also need

$$
\begin{aligned}
\mathrm{d}M^\dagger &= V^\dagger\left(\mathrm{d}r + \left[\mathrm{d}S^\dagger, r\right]\right)V \\
&= V^\dagger\left(\mathrm{d}r + [r, \mathrm{d}S]\right)V \\
&= \mathrm{d}M,
\end{aligned}
\tag{3.3.7}
$$

then $\mathrm{tr}\left(\mathrm{d}M^2\right)$ can be expressed in terms of commutators as

$$\begin{aligned} \operatorname{tr}\left(\mathrm{d}M^2\right) &= \operatorname{tr}\left(\mathrm{d}M\mathrm{d}M^\dagger\right) \\ &= \operatorname{tr}\left(V^\dagger\left(\mathrm{d}r + [r, \mathrm{d}S]\right) V V^\dagger \left(\mathrm{d}r + [r, \mathrm{d}S]\right) V\right) \\ &= \operatorname{tr}\left((\mathrm{d}r)^2 + 2\mathrm{d}r[r, \mathrm{d}S] + ([r, \mathrm{d}S])^2\right) \\ &= \operatorname{tr}\left((\mathrm{d}r)^2 + [r, \mathrm{d}S]^2\right). \end{aligned} \tag{3.3.8}$$

Below, in equation (3.3.9) is an identity that uses the cyclicity of the trace with the matrices A, B, C,

$$\operatorname{tr}\left([A, B]C\right) = \operatorname{tr}\left([B, C]A\right) = \operatorname{tr}\left([C, A]B\right). \tag{3.3.9}$$

We used the above identity in equation (3.3.8) as follows

$$\operatorname{tr}\left(\mathrm{d}r\,[r, \mathrm{d}S]\right) = \operatorname{tr}\left(\mathrm{d}S\,[\mathrm{d}r, r]\right) = 0, \tag{3.3.10}$$

since

$$[\mathrm{d}r, r] = 0, \tag{3.3.11}$$

to get rid of the middle term in the second line of equation (3.3.8).

Thus in component form, equation (3.3.8) becomes

$$\begin{aligned} \operatorname{tr}\left(\mathrm{d}M^2\right) &= \sum_i \left(\mathrm{d}M^2\right)_{ii} \\ &= \operatorname{tr}\left((\mathrm{d}r)^2\right) + \operatorname{tr}\left([r, \mathrm{d}S]^2\right) \\ &= \sum_i \left(d(r)_i\right)^2 - \sum_{i,j} (\mathrm{d}S)_{ij} (\mathrm{d}S)_{ji} \left((r)_i - (r)_j\right)^2. \end{aligned} \tag{3.3.12}$$

To take into account the entire degrees of freedom of the trace of the square of the variation, we need to include the complex conjugates of the differentials, it follows that

$$
\begin{aligned}
\operatorname{tr}\left(\mathrm{d}M^2\right) &= \sum_i (\mathrm{d}(r)_i)^2 - \sum_{i,j} (\mathrm{d}S)_{ij} (\mathrm{d}S)_{ji} ((r)_i - (r)_j)^2 \\
&= \sum_i (\mathrm{d}(r)_i)^2 + \sum_{i\neq j} (\mathrm{d}S)_{ij} \left(\mathrm{d}S^*_{ij}\right) ((r)_i - (r)_j)^2 \\
&= \sum_i (\mathrm{d}(r)_i)^2 + \frac{1}{2}\sum_{i\neq j} ((r)_i - (r)_j)^2 \left\{(\mathrm{d}S^*)_{ij} (\mathrm{d}S)_{ij}\right\} \\
&+ \frac{1}{2}\sum_{i\neq j} ((r)_i - (r)_j)^2 \left\{(\mathrm{d}S)_{ij} (\mathrm{d}S^*)_{ij}\right\} \\
&= \sum_i (\mathrm{d}(r)_i)^2 + \sum_{i>j} ((r)_i - (r)_j)^2 \left\{(\mathrm{d}S^*)_{ij} (\mathrm{d}S)_{ij}\right\} \\
&+ \sum_{i>j} ((r)_i - (r)_j)^2 \left\{(\mathrm{d}S)_{ij} (\mathrm{d}S^*)_{ij}\right\} \\
&= g_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu.
\end{aligned} \tag{3.3.13}
$$

The complex conjugates of dS changes the positions of the indices, this means that

$$
\mathrm{d}S_{ij} = -\mathrm{d}S^*_{ji}. \tag{3.3.14}
$$

Taking the complex conjugate of dS means taking the complex conjugate of each element inside the matrix. From equation (3.3.14) we could equivalently have

$$
\frac{\partial}{\partial S_{ij}} = -\frac{\partial}{\partial S^*_{ji}}. \tag{3.3.15}
$$

The angular degrees of freedom appearing in the fourth line of equation (3.3.13) can be rewritten in terms of the generators of $SU(N)$ which involve the diagonal generators of the Cartan subalgebra [4].

From the last line of equation (3.3.13) the metric tensor $g_{\mu\nu}$ is identified as

$$
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (r_i - r_j)^2 & 0 \\ 0 & 0 & (r_i - r_j)^2 \end{pmatrix}, \tag{3.3.16}
$$

for $\mu, \nu = 0, 1, 2$ which specify $\left(\mathrm{d}r, \mathrm{d}S_{ij}, \mathrm{d}S^*_{ij}\right)$.

So naturally from this we can compute the determinant of the metric tensor in order for us to calculate the Laplacian for the single matrix model, and we find

$$\det g_{\mu\nu} = \prod_{i<j} (r_i - r_j)^4, \tag{3.3.17}$$

and the inverse of $g_{\mu\nu}$, which we call $g^{\nu\mu}$ is

$$g^{\nu\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(r_i - r_j)^2} & 0 \\ 0 & 0 & \frac{1}{(r_i - r_j)^2} \end{pmatrix}. \tag{3.3.18}$$

Also what is required to compute $\nabla^2_{S.M}$ is the factor G, which is given by

$$G = \sqrt{\det g_{\mu\nu}} = \prod_{i<j} (r_i - r_j)^2 = \Delta^2, \tag{3.3.19}$$

where

$$\Delta = \prod_{i<j} (r_i - r_j), \tag{3.3.20}$$

is the usual Vandermonde determinant and G is the Jacobian under change of variables $M \to (V, r)$.

Now using the definition of the Laplacian and the terms computed in equations (3.3.18) and (3.3.19) in equation (3.3.21)

$$\nabla^2 = \frac{1}{\sqrt{\det g_{\mu\nu}}} \frac{\partial}{\partial X^\nu} g^{\nu\mu} \sqrt{\det g_{\mu\nu}} \frac{\partial}{\partial X^\mu}, \tag{3.3.21}$$

the Laplacian for the single matrix model denoted by $\nabla^2_{S.M}$, is given by

$$
\begin{aligned}
\nabla^2_{S.M} &= \frac{1}{\sqrt{\det g_{\mu\nu}}}\frac{\partial}{\partial X^\mu}\sqrt{\det g_{\mu\nu}}g^{\mu\nu}\frac{\partial}{\partial X^\nu} \qquad (3.3.22)\\
&= \frac{1}{\prod_{i<j}(r_i-r_j)^2}\frac{\partial}{\partial r_i}\left[\prod_{i<j}(r_i-r_j)^2\right]\frac{\partial}{\partial r_i}\\
&+ \frac{1}{\prod_{i<j}(r_i-r_j)^2}\frac{\partial}{\partial S_{ij}}\left[\prod_{i<j}(r_i-r_j)^2\sum_{i>j}\frac{1}{(r_i-r_j)^2}\right]\frac{\partial}{\partial S^*_{ij}}\\
&+ \frac{1}{\prod_{i<j}(r_i-r_j)^2}\frac{\partial}{\partial S^*_{ij}}\left[\prod_{i<j}(r_i-r_j)^2\sum_{i>j}\frac{1}{(r_i-r_j)^2}\right]\frac{\partial}{\partial S_{ij}}\\
&= \frac{1}{\prod_{i<j}(r_i-r_j)^2}\frac{\partial}{\partial r_i}\left[\prod_{i<j}(r_i-r_j)^2\right]\frac{\partial}{\partial r_i}\\
&- \frac{2}{\prod_{i<j}(r_i-r_j)^2}\frac{\partial}{\partial S_{ij}}\left[\prod_{i<j}(r_i-r_j)^2\sum_{i>j}\frac{1}{(r_i-r_j)^2}\right]\frac{\partial}{\partial S_{ji}}\\
&= \frac{1}{\Delta^2}\frac{\partial}{\partial r_i}\left[\Delta^2\right]\frac{\partial}{\partial r_i} - \sum_{i\neq j}\frac{1}{(r_i-r_j)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S_{ji}}.
\end{aligned}
$$

Using the Laplacian in equation (3.3.22) our Hamiltonian operator now takes the following form

$$
\begin{aligned}
\hat{H} &= -\frac{1}{2}\nabla^2_{S.M} + \mathrm{tr}\,(K(M)) \qquad (3.3.23)\\
&= -\frac{1}{2}\frac{1}{\Delta^2}\frac{\partial}{\partial r_i}\left[\Delta^2\right]\frac{\partial}{\partial r_i} - \sum_{i\neq j}\frac{1}{(r_i-r_j)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S_{ji}} + \mathrm{tr}\,(K(M))\,.
\end{aligned}
$$

The first term in last line of equation (3.3.23) is the kinetic term that depends on the eigenvalues r_i of the hermitian matrix M. The second term in equation (3.3.23) is the "angular" component of the kinetic term that preserves the angular degrees of freedom and represents the nonsinglet $SU(N)$ angular momentum degrees of freedom [42] of the Hamiltonian operator. The "angular" component of the kinetic term are the generators of left rotations, these will be seen later in equations (4.1.31) and (4.1.32) for the two matrix model. The constraints are such that, when this angular component of the kinetic term acts on ground state wavefunctions in the singlet sector of the $SU(N)$ representation, we must get zero. These singlet wavefunctions Φ will be independent of the angular variables V and $V^\dagger$ and should be symmetric wavefunctions of the eigenvalues r_i of M. So this means that we are reduced to solving the ground state energy of the Hamiltonian operator for a symmetric singlet wavefunction

$$\hat{H}\Phi = E\Phi. \tag{3.3.24}$$

Now that we have shown the variation dM and derived $\nabla^2_{S.M}$, we can proceed to introduce the fermion picture in the section that follows.

3.4 Reviewing The Fermionic Framework Of The Single Matrix

Our third objective is to introduce the free fermion picture for the single matrix model, which is what we proceed to do in this section. To start off, we use the analogy to the Schrödinger equation for some spectrum of eigenvalues E [1]

$$\left(\frac{1}{\Delta^2}\sum_i \frac{\partial}{\partial r_i}\Delta^2\frac{\partial}{\partial r_i}\right)\Phi = 2E\Phi. \tag{3.4.1}$$

Above in equation (3.4.1), Φ is a symmetric wavefunction that depends on the eigenvalues r_i of M and is invariant under the transformation $\Phi \to \Phi\left(UMU^\dagger\right)$.

We can define the following wave function that also depends on the eigenvalues r_i of M,

$$\Psi(r_i) = \Delta(r_i)\Phi(r_i). \tag{3.4.2}$$

Equation (3.4.2) is an anti-symmetric function, because the right hand side is a product of a symmetric function and the Vandermonde determinant, which is antisymmetric under exchange of any two eigenvalues.

[1] The potential term is trivial for this discussion, provided it only depends on the eigenvalues.

The function Φ is now rewritten in terms of Ψ and Δ in equation (3.4.1) to get the equation (3.4.3) below

$$-\frac{1}{\Delta^2}\sum_i \frac{\partial}{\partial r_i}\Delta^2\frac{\partial}{\partial r_i}\frac{\Psi}{\Delta} = -\frac{1}{\Delta^2}\left(\sum_i \frac{\partial}{\partial r_i}\Delta^2\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right)\Psi = 2E\frac{\Psi}{\Delta},$$

$$\Rightarrow -\left(\frac{1}{\Delta}\sum_i \frac{\partial}{\partial r_i}\Delta\right)\left(\Delta\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right)\Psi = 2E\Psi. \qquad (3.4.3)$$

To simplify equation (3.4.3), we apply the following identity,

$$\begin{aligned}\frac{1}{\Delta}\frac{\partial}{\partial r_i}\Delta &= \frac{\partial}{\partial r_i}\ln\Delta = \frac{\partial}{\partial r_i}\sum_{i<k}\ln(r_j - r_k) \qquad (3.4.4)\\ &= \sum_{k>i}\frac{1}{r_i - r_k} - \sum_{j<i}\frac{1}{r_j - r_i} = \sum_{k\neq i}\frac{1}{r_i - r_k}\\ \Rightarrow \frac{1}{\Delta}\frac{\partial}{\partial r_i}\Delta &= \sum_{k\neq i}\frac{1}{r_i - r_k}\end{aligned}$$

So in total we have the identity

$$\frac{1}{\Delta}\left(\frac{\partial}{\partial r_i}\Delta\right) = \frac{\partial \ln\Delta}{\partial r_i} = \sum_{k\neq i}\frac{1}{(r_i - r_k)} \qquad (3.4.5)$$

and

$$\begin{aligned}\Delta\left(\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right) &= -\frac{\Delta}{\Delta^2}\frac{\partial\Delta}{\partial r_i} = -\frac{1}{\Delta}\frac{\partial\Delta}{\partial r_i} = -\frac{\partial}{\partial r_i}\ln\Delta \qquad (3.4.6)\\ &= -\sum_{j\neq i}\frac{1}{r_i - r_j}.\end{aligned}$$

Then using equation (3.4.5) and (3.4.6) we can derive the following result

$$
\begin{aligned}
& \sum_i \left(\frac{1}{\Delta}\frac{\partial}{\partial r_i}\Delta\right)\left(\Delta\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right) \\
= & \sum_i \left(\frac{\partial}{\partial r_i} + \sum_{k\neq i}\frac{1}{r_i - r_k}\right)\left(\frac{\partial}{\partial r_i} - \sum_{j\neq i}\frac{1}{r_i - r_j}\right) \\
= & \sum_i \frac{\partial^2}{\partial r_i^2} + \sum_{i\neq j}\frac{1}{(r_i - r_j)^2} - \sum_{i\neq j}\frac{1}{(r_i - r_j)}\frac{\partial}{\partial r_i} + \sum_k \frac{1}{r_i - r_k}\frac{\partial}{\partial r_i} \\
- & \sum_i \sum_{j\neq i, k\neq i}\frac{1}{(r_i - r_k)}\frac{1}{(r_i - r_j)} \\
= & \sum_i \frac{\partial^2}{\partial r_i^2} - \sum_{i\neq k\neq j}\frac{1}{(r_i - r_k)}\frac{1}{(r_i - r_j)}.
\end{aligned}
$$

So far we can conclude that

$$
\left(\frac{1}{\Delta}\sum_i \frac{\partial}{\partial r_i}\Delta\right)\left(\Delta\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right) = \sum_i \frac{\partial^2}{\partial r_i^2} - \sum_{i\neq k\neq j}\frac{1}{(r_i - r_k)}\frac{1}{(r_i - r_j)}. \tag{3.4.7}
$$

On the right hand side of equation (3.4.7) above, the second term has indices that satisfy the condition $i \neq j \neq k$. As a result it can be shown to vanish choosing any three distinct eigenvalues, say r_1, r_2, r_3.

$$
\begin{aligned}
& \sum_{i\neq k, i\neq j}\frac{1}{(r_i - r_k)}\frac{1}{(r_i - r_j)} = \frac{1}{(r_1 - r_2)(r_1 - r_3)} \\
+ & \frac{1}{(r_2 - r_1)(r_2 - r_3)} + \frac{1}{(r_3 - r_1)(r_3 - r_2)} \\
= & \frac{1}{(r_1 - r_2)}\left(\frac{1}{(r_1 - r_3)} - \frac{1}{(r_2 - r_3)}\right) + \frac{1}{(r_1 - r_2)(r_1 - r_3)} \\
= & \frac{1}{(r_1 - r_2)}\left(\frac{r_2 - r_3 - r_1 + r_3}{(r_1 - r_3)(r_3 - r_3)}\right) + \frac{1}{(r_3 - r_1)(r_3 - r_2)} \\
= & \frac{1}{(r_1 - r_2)}\frac{(r_1 - r_2)}{(r_1 - r_3)(r_2 - r_3)} + \frac{1}{(r_3 - r_1)(r_3 - r_2)} \\
= & -\frac{1}{(r_1 - r_2)}\frac{(r_1 - r_2)}{(r_1 - r_3)(r_2 - r_3)} + \frac{1}{(r_3 - r_1)(r_3 - r_2)} \\
= & -\frac{1}{(r_1 - r_3)(r_2 - r_3)} + \frac{1}{(r_3 - r_1)(r_3 - r_2)} = 0.
\end{aligned}
$$

Then this means that

$$-\left(\frac{1}{\Delta}\sum_i \frac{\partial}{\partial r_i}\Delta\right)\left(\Delta\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right)\Psi = \sum_i \frac{\partial^2}{\partial r_i^2}\Psi = 2E\Psi. \tag{3.4.8}$$

Now we have reduced the eigenvalue problem to solving

$$\left(\sum_i^N \chi_i\right)\Psi(r_i) = E\Psi(r_i) \tag{3.4.9}$$

where

$$\chi_i = -\frac{1}{2}\frac{\partial^2}{\partial r_i^2} + \mathrm{tr}\left(K(r_i)\right). \tag{3.4.10}$$

What does the result in equation (3.4.9) mean? On the left hand side of equation (3.4.9), the operator χ_i acting on the wavefunction $\Psi(r_i)$ is a sum of single-particle non-relativistic Hamiltonians. Our eigenvalue problem has now been reduced to solving a fermion problem with N degrees of freedom. This ground state of free non-relativistic fermions move in the potential $\mathrm{tr}\left(K(r_i)\right)$. The result derived in equation (3.4.9) is a well known result also obtained by the authors of [2] [4].

3.5 The Conjugate Momentum For The Single Matrix

The objective for this section of the work is to obtain the conjugate momentum $P_{ij} = \partial/\partial M_{ji}$ found in the Hamiltonian operator $\hat{H}$. We first define the following commutator

$$[P_{ji}, M_{ab}] = \delta_{jb}\delta_{ai} \tag{3.5.1}$$

where P_{ji}, the conjugate momentum of M_{ab}, is given by

$$P_{ji} = \frac{\partial}{\partial M_{ij}}.$$

We write the matrix M_{ab} in component form as follows:

$$M_{ab} = \sum_{\alpha} V^{\dagger}_{a\alpha} r_{\alpha} V_{\alpha b}.$$

The expression of the conjugate momentum P_{ji} is defined using partial differential equations as

$$P_{ji} = \frac{\partial}{\partial M_{ij}} = \sum_{k\gamma} \frac{\partial V_{k\gamma}}{\partial M_{ij}} \frac{\partial}{\partial V_{k\gamma}} + \sum_{k} \frac{\partial r_k}{\partial M_{ij}} \frac{\partial}{\partial r_k} \tag{3.5.2}$$

for $i, j, k, \gamma = 1, 2, \ldots, N$.

The terms

$$\frac{\partial V_{k\gamma}}{\partial M_{ij}} \qquad \frac{\partial r_k}{\partial M_{ij}}$$

are to be considered as coefficients in the partial differential equation of the conjugate momentum. To solve for these coefficients, we consider dM in equation (3.3.5) written in terms of indices

$$\begin{aligned} \mathrm{d}M_{ij} &= \sum_{kq} V^{\dagger}_{ik} \left(dr + [r, \mathrm{d}S]\right)_{kq} V_{qj} \qquad (3.5.3) \\ &= \sum_{kq} V^{\dagger}_{ik} V_{qj} \delta_{qk} \mathrm{d}r_k + \sum_{kq} V^{\dagger}_{ik} V_{qj} \mathrm{d}S_{kq} (r_k - r_q), \end{aligned}$$

this is rewritten as

$$\begin{aligned} V_{ai} \mathrm{d}M_{ij} V^{\dagger}_{jb} &= \sum_{kq} V_{ai} V^{\dagger}_{ik} \delta_{qk} \mathrm{d}r_k V_{qj} V^{\dagger}_{jb} + \sum_{kq} V_{ai} V^{\dagger}_{ik} \mathrm{d}S_{kq} \left(r_k - r_q\right) V_{qj} V^{\dagger}_{jb} \\ &= \delta_{ak} \delta_{qb} \delta_{qk} \mathrm{d}r_k + \sum_{kq} \delta_{ak} \delta_{qb} \mathrm{d}S_{kq} \left(r_k - r_q\right) \qquad (3.5.4) \\ &= \delta_{ba} \mathrm{d}r_a + \mathrm{d}S_{ab} \left(r_a - r_b\right). \end{aligned}$$

We now consider two different cases for the equation dM_{ij} in equation (3.5.4): case(I) is when $a = b$ and case(II) is when $a \neq b$. For case I, we find that the equation for dM_{ij} allows us to obtain the coefficient

$$\frac{\partial r_k}{\partial M_{ij}} = \sum_{k} V^{\dagger}_{ik} V_{kj}, \tag{3.5.5}$$

and for case (II), the equation for $\mathrm{d}M_{ij}$ allows us to obtain the second coefficient

$$\frac{\partial V_{k\gamma}}{\partial M_{ij}} = \sum_{k\neq q} \frac{V^\dagger_{jq} V_{q\gamma}}{r_k - r_q} V_{ki}. \tag{3.5.6}$$

Using equations (3.5.5) and (3.5.6) in equation (3.5.2) the complete expression of the conjugate momentum becomes

$$P_{ji} = \frac{\partial}{\partial M_{ij}} = \sum_{\gamma} \sum_{k\neq q} \frac{V^\dagger_{jq} V_{q\gamma}}{r_k - r_q} V_{ki} \frac{\partial}{\partial V_{k\gamma}} + \sum_k V^\dagger_{ik} V_{kj} \frac{\partial}{\partial r_k}. \tag{3.5.7}$$

By defining the conjugate momentum as we did above, we can verify the commutator in equation (3.5.1).

Now that the framework is in place, the question follows: how far can we extend the analogy to a matrix model with more than a single matrix? This question we answer in the chapter that follows using a model with two matrices.

Chapter 4

The Laplacians For The Two Matrix Model

4.1 The Hamiltonian And The Laplacian: Parameterization I

Because of the double index structure of the matrix degrees of freedom that we consider, we try to identify suitable parameterizations of the complex matrix coordinate Z in equation (3.2.6) which result in a diagonal metric in the matrix indices $A \equiv (ij)$. This will enable us to obtain the corresponding Laplacian operator, whose definition is given by equation (3.3.21).

The objectives for this part of the project are as follows:

(1) Compute the infinitesimal line element $\mathrm{d}Z$ for the complex matrix coordinate.

(2) Obtain the two matrix model Laplacian ∇_I^2 in polar coordinates using parameterization I.

In all of these parameterizations, R will be diagonalized. This is because ultimately it is hoped that an effective theory in terms of the density of eigenvalues of the radial coordinate is obtained. We will describe two such parameterizations, which we will denote by parameterization I and II.

We begin by performing an angular parameterization of R, such that:

$$R = V^{\dagger} r V, \tag{4.1.1}$$

where r is the $N \times N$ diagonal matrix of the eigenvalues of R and $V, V^{\dagger}$ are unitary $N \times N$ non-hermitian matrices which are the angular variables of R. From the above notation, we can re-write the matrix Z as follows:

$$Z = RU = \left(V^{\dagger} r V\right) U = V^{\dagger} r V U. \tag{4.1.2}$$

Parameterization I, considered in this section allows us to retain the variables r, V and U. The matrix coordinates are defined up to $V \rightarrow DV$, $V^{\dagger} \rightarrow V^{\dagger} D^{\dagger}$ for the diagonal matrix D, thus preserving the number of degrees of freedom of the system.

The first step is to calculate the analogue to square of the line element

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}, \tag{4.1.3}$$

using our newly defined matrix spherical coordinate Z. The analogy of the square of the line element in our matrix coordinate is represented by

$$\begin{aligned} \mathrm{tr}\left(\mathrm{d}Z^{\dagger} \mathrm{d}Z\right) &= \eta_{\mu\nu} \mathrm{d}X_I^{\mu} \mathrm{d}X_I^{\nu} \\ &\equiv \mathrm{d}s^2, \end{aligned} \tag{4.1.4}$$

where $\mathrm{d}X_I$ are the matrix differential variables that are obtained using parameterization I and $\eta_{\mu\nu}$ is the metric tensor in our two matrix model for parameterization I.

To start of, we compute $\mathrm{d}Z$, the distance along Z by taking the infinitesimal differential of Z

$$\begin{aligned} \mathrm{d}Z &= \mathrm{d}\left(V^{\dagger} r \left(VU\right)\right) \\ &= \left(\mathrm{d}V^{\dagger} r V U + V^{\dagger} \mathrm{d}r V U + V^{\dagger} r \mathrm{d}V U + V^{\dagger} r V \mathrm{d}U\right) \\ &= V^{\dagger}\left(V \mathrm{d}V^{\dagger} r + \mathrm{d}r + r \mathrm{d}V V^{\dagger} + r V \mathrm{d}U \left(U^{\dagger} V^{\dagger}\right)\right) VU. \end{aligned} \tag{4.1.5}$$

From the equation above we used the unitarity property of the matrices U, $U_{ab}U^\dagger_{bc} = \delta_{ac}$, and the property

$$V\mathrm{d}V^\dagger = -\mathrm{d}VV^\dagger. \tag{4.1.6}$$

Thus, using the two properties above in the expression for $\mathrm{d}Z$ and regrouping terms, we obtain:

$$\begin{aligned}\mathrm{d}Z &= V^\dagger\left(\mathrm{d}r + r\mathrm{d}VV^\dagger - \mathrm{d}VV^\dagger r + rV\mathrm{d}UU^\dagger V^\dagger\right)VU \\ &= V^\dagger\left(\mathrm{d}r + \left[r, \mathrm{d}VV^\dagger\right] + rV\mathrm{d}UU^\dagger V^\dagger\right)VU.\end{aligned} \tag{4.1.7}$$

Using a similar approach to obtain $\mathrm{d}Z$, we compute $\mathrm{d}Z^\dagger$, and this is shown to be

$$\mathrm{d}Z^\dagger = U^\dagger V^\dagger\left(\mathrm{d}r + \left[r, \mathrm{d}VV^\dagger\right] - VU\mathrm{d}U^\dagger V^\dagger r\right)V. \tag{4.1.8}$$

The expressions $\mathrm{d}Z$ and $\mathrm{d}Z^\dagger$ suggest that we introduce the anti-hermitian, Lie-algebra differential matrices:

$$\mathrm{d}S = \mathrm{d}VV^\dagger = -\mathrm{d}S^\dagger \qquad\qquad \mathrm{d}X = V\mathrm{d}UU^\dagger V^\dagger = -\mathrm{d}X^\dagger. \tag{4.1.9}$$

Thus, the expressions for $\mathrm{d}Z$ and $\mathrm{d}Z^\dagger$ can now be written as

$$\mathrm{d}Z = V^\dagger\left(\mathrm{d}r + [r, \mathrm{d}S] + r\mathrm{d}X\right)VU \tag{4.1.10}$$

and

$$\mathrm{d}Z^\dagger = U^\dagger V^\dagger\left(\mathrm{d}r + [r, \mathrm{d}S] - \mathrm{d}Xr\right)V. \tag{4.1.11}$$

Equations (4.1.10) and (4.1.11) fulfill the first objective for this section. We now proceed to our second objective.

With the expressions for $\mathrm{d}Z$ and $\mathrm{d}Z^\dagger$, we obtain the analogue of the square of the line element in the language of our spherical coordinates

$$\mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) = \mathrm{tr}\left(\left(\mathrm{d}r + [r, \mathrm{d}S] - \mathrm{d}Xr\right)\left(\mathrm{d}r + [r, \mathrm{d}S] + r\mathrm{d}X\right)\right) \quad (4.1.12)$$

$$\begin{aligned} \mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) &= \mathrm{tr}\left(\mathrm{d}r^2 + \mathrm{d}r[r, \mathrm{d}S] + \mathrm{d}rr\mathrm{d}X + [r, \mathrm{d}S]\mathrm{d}r\right) \quad (4.1.13) \\ &+ \mathrm{tr}\left([r, \mathrm{d}S]^2 + [r, \mathrm{d}S]r\mathrm{d}X - \mathrm{d}Xr\mathrm{d}r - \mathrm{d}Xr[r, \mathrm{d}S] - \mathrm{d}Xr^2\mathrm{d}X\right). \end{aligned}$$

As before, since $[\mathrm{d}r, r] = 0$, and using the property of equation (3.3.9) it follows that

$$\mathrm{tr}\left(\mathrm{d}r\,[r, \mathrm{d}S]\right) = \mathrm{tr}\left(\mathrm{d}S\,[\mathrm{d}r, r]\right) = 0. \quad (4.1.14)$$

Therefore

$$\mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) = \mathrm{tr}\left(\mathrm{d}r^2 + [r, \mathrm{d}S]\,[r, \mathrm{d}S] + [r, \mathrm{d}S]\,[r, \mathrm{d}X] - r^2\left(\mathrm{d}X\right)^2\right).$$

In component form

$$\begin{aligned} \mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) &= \sum_i \left(\mathrm{d}r_i\right)^2 + \sum_{ij} [r, \mathrm{d}S]_{ij}\,[r, \mathrm{d}S]_{ji} + \sum_{ij} [r, \mathrm{d}S]_{ij}\,[r, \mathrm{d}X]_{ji} \\ &- \sum_{ij} r_i^2 \mathrm{d}X_{ij}\mathrm{d}X_{ji}. \quad (4.1.15) \end{aligned}$$

In equation (4.1.15) above, we see commutators that mix radial and angular degrees of freedom as observed for the single matrix model. The last term is also a product of anti-hermitian traceless matrices. The terms with angular degrees of freedom, just like the single matrix model, can also be rewritten in terms of $SU(N)$ generators.

It is important that the equation (4.1.15) above be written in component form because this will allow us to obtain the entries in the matrix of the metric tensor $\eta_{\mu\nu}$ such that the coefficients of the product of the differentials, $\mathrm{d}X_I\mathrm{d}X_I$, will be entries to the matrix of the metric tensor $\eta_{\mu\nu}$. Thus it follows that

$$\begin{aligned}\operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) &= \sum_{i}\left(\mathrm{d}r_i\right)^2-\sum_{ij}\left(r_i-r_j\right)^2\mathrm{d}S_{ij}\mathrm{d}S_{ji} \\ &- \frac{1}{2}\sum_{ij}\left(r_i-r_j\right)^2\left\{\mathrm{d}S_{ij}\mathrm{d}X_{ji}+\mathrm{d}X_{ij}\mathrm{d}S_{ji}\right\} \\ &- \frac{1}{2}\sum_{ij}\left(r_i^2+r_j^2\right)\mathrm{d}X_{ij}\mathrm{d}X_{ji}.\end{aligned} \tag{4.1.16}$$

In equation (4.1.16) above, commutators are written in terms of the eigenvalues as follows:

$$[r,\mathrm{d}S]_{ij}=(r\mathrm{d}S)_{ij}-(\mathrm{d}Sr)_{ij}=r_i\mathrm{d}S_{ij}-\mathrm{d}S_{ij}r_j=(r_i-r_j)\mathrm{d}S_{ij}. \tag{4.1.17}$$

In a similar fashion as above

$$[r,\mathrm{d}S]_{ji}=(r_j-r_i)\mathrm{d}S_{ji}=-(r_i-r_j)\mathrm{d}S_{ji} \tag{4.1.18}$$

and

$$[r,\mathrm{d}X]_{ji}=(r_j-r_i)\mathrm{d}X_{ji}. \tag{4.1.19}$$

We separate equation (4.1.16) into $i=j$ and $i\neq j$ terms, and using the anti-hermiticity of $\mathrm{d}X$ and $\mathrm{d}S$, $\operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right)$ becomes

$$\begin{aligned}\operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) &= \sum_{i}\left\{\left(\mathrm{d}r_i\right)^2+\left(r_i\right)^2\mathrm{d}X_{ii}\mathrm{d}X_{ii}^*\right\} \\ + &\sum_{i\neq j}\left\{\left(r_i-r_j\right)^2\mathrm{d}S_{ij}\mathrm{d}S_{ij}^*+\frac{1}{2}\left(r_i-r_j\right)^2\left[\mathrm{d}S_{ij}dX_{ij}^*+\mathrm{d}X_{ij}\mathrm{d}S_{ij}^*\right]\right\} \\ + &\sum_{i\neq j}\left\{\frac{1}{2}\left(r_i^2+r_j^2\right)\mathrm{d}X_{ij}\mathrm{d}X_{ij}^*\right\}\end{aligned} \tag{4.1.20}$$

or equivalently

$$\begin{aligned}
\mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) &= \sum_{i}\left\{(\mathrm{d}r_i)^2 + (r_i)^2\,\mathrm{d}X_{ii}\mathrm{d}X_{ii}^{*}\right\} \\
&+ \sum_{i<j}\left\{2\,(r_i - r_j)^2\,\mathrm{d}S_{ij}\mathrm{d}S_{ij}^{*} + (r_i - r_j)^2\left[\mathrm{d}S_{ij}dX_{ij}^{*} + \mathrm{d}X_{ij}\mathrm{d}S_{ij}^{*}\right]\right\} \\
&+ \sum_{i<j}\left\{\left(r_i^2 + r_j^2\right)\mathrm{d}X_{ij}\mathrm{d}X_{ij}^{*}\right\}.
\end{aligned} \tag{4.1.21}$$

We remember that we drew the analogy: $\mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) \equiv \mathrm{d}s^2$. Equation (4.1.21) can now be rewritten as in equation (4.1.4) with the following differential variables: $\left(\mathrm{d}r_i, \mathrm{d}X_{ii}, \mathrm{d}S_{ij(i<j)}, \mathrm{d}X_{ij(i<j)}, \mathrm{d}S^{*}_{ij(i<j)}, \mathrm{d}X^{*}_{ij(i<j)}\right)$.

Due to the nature of $\mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right)$, it has terms that have complex conjugates and we need to consider the entire degrees of freedom of the system. From equation (4.1.21), the matrix of metric tensor $\eta_{\mu\nu}$ of $\mathrm{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right)$ in equation (4.1.4) is thus:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & r_i^2 & 0 & 0 & 0 & 0 \\
0 & 0 & (r_i - r_j)^2 & \frac{1}{2}(r_i - r_j)^2 & 0 & 0 \\
0 & 0 & \frac{1}{2}(r_i - r_j)^2 & \frac{1}{2}(r_i^2 + r_j^2) & 0 & 0 \\
0 & 0 & 0 & 0 & (r_i - r_j)^2 & \frac{1}{2}(r_i - r_j)^2 \\
0 & 0 & 0 & 0 & \frac{1}{2}(r_i - r_j)^2 & \frac{1}{2}(r_i^2 + r_j^2)
\end{pmatrix}. \tag{4.1.22}$$

Using the above matrix in equation (4.1.22), we compute its determinant in order for us to obtain the expression for the Laplacian ∇_I^2 using parameterization I. The determinant of the matrix $\eta_{\mu\nu}$ is then:

$$\det \eta_{\mu\nu} = \prod_i (r_i^2) \prod_{i<j} \left\{\frac{1}{2}(r_i^2 + r_j^2)(r_i - r_j)^2 - \frac{1}{4}(r_i - r_j)^2 (r_i - r_j)^2\right\}^2. \tag{4.1.23}$$

After some algebra, the expression above takes the form

$$\det \eta_{\mu\nu} = \prod_i r_i^2 \left(\prod_{i<j} \frac{1}{4}(r_i^2 - r_j^2)^2\right)^2. \tag{4.1.24}$$

We now introduce new notation Δ^2_{MR} such that

$$\Delta^2_{MR} = \frac{1}{4}\prod_{i<j}(r_i^2 - r_j^2)^2. \tag{4.1.25}$$

Equation (4.1.24) is rewritten using our new notation, equation (4.1.25), to become

$$\det \eta_{\mu\nu} = \prod_i r_i^2 (\Delta^2_{MR})^2. \tag{4.1.26}$$

From equation (4.1.26), the following can be noted: Δ^2_{MR} takes a form that resembles the square of the Vandermonde determinant that we saw previously in equation (3.3.20) for the single matrix model. The eigenvalues, when compared to those of equation (3.3.20), are now replaced by r_i^2 instead of being r_i. This is interesting as it points to similarities with the single matrix model. When working with matrices, the Vandermonde determinant appears naturally.

From the definition of the Laplacian, we compute the variable G, which represents a Jacobian obtained under change of variables, from $J\mathrm{d}Z\mathrm{d}Z^\dagger \rightarrow \mathrm{d}r\mathrm{d}S\mathrm{d}X$:

$$G = \sqrt{\det \eta_{\mu\nu}} = \prod_k r_k \Delta^2_{MR}, \tag{4.1.27}$$

noticeably G also depends on the the eigenvalues of R only.

The inverse of the matrix $\eta_{\mu\nu}$ i.e. $\eta^{\nu\mu}$ takes the following form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r_i^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4(r_i^2+r_j^2)}{(r_i-r_j)^2(r_i+r_j)^2} & \frac{-2}{(r_i+r_j)^2} & 0 & 0 \\ 0 & 0 & \frac{-2}{(r_i+r_j)^2} & \frac{4}{(r_i+r_j)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4(r_i^2+r_j^2)}{(r_i-r_j)^2(r_i+r_j)^2} & \frac{-2}{(r_i+r_j)^2} \\ 0 & 0 & 0 & 0 & \frac{-2}{(r_i+r_j)^2} & \frac{4}{(r_i+r_j)^2} \end{pmatrix}. \tag{4.1.28}$$

Thus using the definition of the Laplacian in equation (3.3.21), the definition in equation (4.1.27) and the inverse matrix $\eta^{\nu\mu}$ the Laplacian in the polar coordinates of the two matrix model is

$$\begin{aligned}
\nabla_I^2 &= \frac{1}{\prod_k r_k}\frac{1}{\Delta_{MR}^2}\frac{\partial}{\partial r_i}\left[\prod_k r_k\Delta_{MR}^2\right]\frac{\partial}{\partial r_i}+\frac{\partial}{\partial X^A}g^{AB}\frac{\partial}{\partial X^B}(A\neq B)\\
&= \frac{1}{\prod_k r_i}\frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\prod_k r_k\right\}\Delta_{MR}^2\frac{\partial}{\partial r_i}+\frac{1}{\prod_k r_k}\frac{1}{\Delta_{MR}^2}\prod_k r_k\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i}\\
&+ \frac{\prod_k r_k\Delta_{MR}^2}{\prod_k r_k\Delta_{MR}^2}\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}+\frac{\partial}{\partial X^A}g^{AB}\frac{\partial}{\partial X^B}(A\neq B)\\
&= \frac{1}{\prod_k r_k}\left\{\frac{\partial}{\partial r_i}\prod_k r_k\right\}\frac{\partial}{\partial r_i}+\frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i}+\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}\\
&+ \frac{\partial}{\partial X^A}g^{AB}\frac{\partial}{\partial X^B}(A\neq B). \qquad (4.1.29)
\end{aligned}$$

With more detail, below we include all the terms that have $A \neq B$ to get the following

$$\begin{aligned}
\nabla_I^2 &= \frac{1}{\prod_k r_k}\left\{\frac{\partial}{\partial r_i}\prod_k r_k\right\}\frac{\partial}{\partial r_i}+\frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i}+\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}\\
&+ \left\{\frac{1}{r_i^2}\frac{\partial}{\partial X_{ii}}\frac{\partial}{\partial X_{ii}^*}\right\}+\sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S_{ij}^*} \qquad (4.1.30)\\
&- \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial X_{ij}^*}+\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial S_{ij}^*}\right\}+\sum_{i\neq j}\frac{4}{(r_i+r_j)^2}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X_{ij}^*}.
\end{aligned}$$

In principle, what can be observed is that the Laplacian in equation (4.1.30) has, in the first line, radial operators and in the second and third lines are the operators with angular degrees of freedom. As in the case for the single matrix Laplacian, they do not mix. The two matrix model has more degrees of freedom than the single matrix model, this would explain why we see much more terms in the Laplacian of equation (4.1.30). Matrix models constructed using more than two matrices would result in more complicated Laplacians, and this could be explored in future research.

In equation (4.1.30), the partial differentials

$$E^{(S)}_{ji} = \frac{\partial}{\partial S_{ij}} = \sum_b V_{jb} \frac{\partial}{\partial V_{ib}} \tag{4.1.31}$$

and

$$E^{(X)}_{ji} = \frac{\partial}{\partial X_{ij}} = \sum_\gamma \sum_\alpha \sum_\theta V^\dagger_{\gamma i} V_{j\theta} U_{\theta\alpha} \frac{\partial}{\partial U_{\gamma\alpha}} \tag{4.1.32}$$

are identified with the $SU(N)$ generators of left rotations.

In total, the Laplacian is thus

$$\begin{aligned}
\nabla^2_I &= \sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i} + 16\sum_{i<j}\left(\frac{r_i - r_j}{r_i^2 - r_j^2}\right)\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \\
&+ \left\{\frac{1}{r_i^2}\frac{\partial}{\partial X_{ii}}\frac{\partial}{\partial X^*_{ii}}\right\} + \sum_{i\neq j}\frac{2(r_i^2 + r_j^2)}{(r_i^2 - r_j^2)}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S^*_{ij}} \\
&- \sum_{i\neq j}\frac{2}{(r_i + r_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial X^*_{ij}} + \frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial S^*_{ij}}\right\} + \sum_{i\neq j}\frac{4}{(r_i + r_j)^2}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X^*_{ij}}.
\end{aligned} \tag{4.1.33}$$

Now we can define the Hamiltonian operator using our new parameterization as follows

$$\begin{aligned}
\hat{H}_I &= -\frac{1}{2}\frac{\partial}{\partial Z_{ij}}\frac{\partial}{\partial Z_{ji}} + \mathrm{tr}\,(K(Z)) \\
&= -\frac{1}{2}\nabla^2_I + \mathrm{tr}\,(K(Z))\,.
\end{aligned} \tag{4.1.34}$$

We observe a consistent uniformity that is from the Laplacian of the single matrix model in the last line of equation (3.3.22) to the Laplacian in equation (4.1.33) which is: the Hamiltonian operator in equation (4.1.34) will have an isolated kinetic piece whose operators are in terms of the eigenvalues r_i and it will also have an angular dependent kinetic piece. There are no terms that appear to mix radial and angular dependent operators. Due to the form that the Hamiltonian operator takes in equation (4.1.34) as a result equation (4.1.33), it means that if we decide to construct the Schrödinger equation

$$\hat{H}_I \zeta_{symm} = E\zeta_{symm}, \tag{4.1.35}$$

the problem in equation (4.1.35) will be reduced to solving an eigenvalue problem for some symmetric function ζ_{symm}. We will now proceed to introduce the second parameterization in the section that follows.

4.2 The Hamiltonian And The Laplacian: Parameterization II

Now that we have seen the form of the Laplacian in equation (4.1.34) that was derived using parameterization I, where the variables V and U were preserved, we now compute the Laplacian using parameterization II, which is our objective in this section. For parameterization II, we start as before with

$$Z = RU. \tag{4.2.1}$$

After the angular re-parameterization of R, $R = V^\dagger r V$
we obtain

$$Z = RU = \left(V^\dagger r V\right) U = V^\dagger r \left(VU\right) \equiv V^\dagger r W. \tag{4.2.2}$$

Noticeably, we have introduced a new variable $W = VU$ into the matrix valued polar coordinate Z. In terms of this second parameterization, we again compute $\mathrm{d}Z$ and $\mathrm{d}Z^\dagger$. We start with $\mathrm{d}Z$, which is

$$\begin{aligned}
\mathrm{d}Z &= \mathrm{d}V^\dagger r W + V^\dagger \mathrm{d}r W + V^\dagger r \mathrm{d}W \\
&= V^\dagger \left(\mathrm{d}r + V\mathrm{d}V^\dagger r + r\mathrm{d}WW^\dagger\right) W \\
\Rightarrow \mathrm{d}Z &= V^\dagger \left(\mathrm{d}r - \mathrm{d}VV^\dagger r + r\mathrm{d}WW^\dagger\right) W.
\end{aligned} \tag{4.2.3}$$

Again, by introducing $\mathrm{d}S$, and defining the new quantity $\mathrm{d}T = \mathrm{d}WW^\dagger$, which is also an anti-hermitian differential, equation (4.2.3) becomes

$$\mathrm{d}Z = V^\dagger \left(\mathrm{d}r + r\mathrm{d}T - \mathrm{d}Sr\right) W. \tag{4.2.4}$$

Using a similar approach to obtain equation (4.2.4), we calculate $\mathrm{d}Z^\dagger$ for $Z^\dagger = W^\dagger r V$ to get

$$\begin{aligned} \mathrm{d}Z^\dagger &= \mathrm{d}W^\dagger r V + W^\dagger \mathrm{d}r V + W^\dagger r \mathrm{d}V \\ &= W^\dagger \left(\mathrm{d}r + W\mathrm{d}W^\dagger r + r\mathrm{d}V V^\dagger\right) V. \end{aligned} \tag{4.2.5}$$

The definitions $\mathrm{d}S$ and $\mathrm{d}T$ are introduced into the last line of equation (4.2.5) to obtain

$$\mathrm{d}Z^\dagger = W^\dagger \left(\mathrm{d}r + r\mathrm{d}S - \mathrm{d}T r\right) V. \tag{4.2.6}$$

In equation (4.2.6) above, we have again used the properties of the anti-hermitian differentials

$$\mathrm{d}T = \mathrm{d}W W^\dagger = -W\mathrm{d}W^\dagger = -\mathrm{d}T^\dagger \tag{4.2.7}$$

$$\mathrm{d}S = \mathrm{d}V V^\dagger = -V\mathrm{d}V^\dagger = -\mathrm{d}S^\dagger. \tag{4.2.8}$$

We now consider the following definitions :

$$\mathrm{d}Y^+ = \frac{1}{\sqrt{2}}\left(\mathrm{d}T + \mathrm{d}S\right) \qquad \mathrm{d}Y^- = \frac{1}{\sqrt{2}}\left(\mathrm{d}T - \mathrm{d}S\right), \tag{4.2.9}$$

where $\mathrm{d}Y^+$ and $\mathrm{d}Y^-$ are introduced to diagonalize the Laplacian calculated previously in equation (4.1.34), using parameterization I.

The above expressions in equation (4.2.9) are inverted to give

$$\mathrm{d}T = \frac{1}{\sqrt{2}}\left(\mathrm{d}Y^+ + \mathrm{d}Y^-\right) \qquad \mathrm{d}S = \frac{1}{\sqrt{2}}\left(\mathrm{d}Y^+ - \mathrm{d}Y^-\right). \tag{4.2.10}$$

The new variables introduced in equation (4.2.10) are used to rewrite $\mathrm{d}Z$ and $\mathrm{d}Z^\dagger$, thus for our newly defined expressions we get

$$\mathrm{d}Z = V^\dagger \left(\mathrm{d}r + \frac{1}{\sqrt{2}}\left[r, \mathrm{d}Y^+\right] + \frac{1}{\sqrt{2}}\left\{r, \mathrm{d}Y^-\right\}\right) W \tag{4.2.11}$$

and of course

$$\mathrm{d}Z^{\dagger} = W^{\dagger}\left(\mathrm{d}r + \frac{1}{\sqrt{2}}\left[r, \mathrm{d}Y^{+}\right] - \frac{1}{\sqrt{2}}\left\{r, \mathrm{d}Y^{-}\right\}\right)V^{\dagger}. \tag{4.2.12}$$

Proceeding as previously shown for the parameterization I, we would like to compute the analogy of the square of the line element

$$\begin{aligned}\operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) &\equiv \mathrm{d}s^2 = g_{\gamma\alpha}\mathrm{d}x^{\gamma}\mathrm{d}x^{\alpha} \\ &\equiv G_{(AB)}\mathrm{d}X^{A}_{II}\mathrm{d}X^{B}_{II}.\end{aligned}$$

using parameterization II. The terms $\mathrm{d}X_{II}$ are the matrix variables for this parameterization. It then follows that $\operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right)$ is the product

$$\operatorname{tr}\left(V^{\dagger}\left(\mathrm{d}r + \frac{1}{\sqrt{2}}\left[r, \mathrm{d}Y^{+}\right] - \frac{1}{\sqrt{2}}\left\{r, \mathrm{d}Y^{-}\right\}\right)WW^{\dagger}\left(\mathrm{d}r + \frac{1}{\sqrt{2}}\left[r; \mathrm{d}Y^{+}\right] + \frac{1}{\sqrt{2}}\left\{r, \mathrm{d}Y^{-}\right\}\right)V\right). \tag{4.2.13}$$

We multiply out the brackets,

$$\Rightarrow \operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) = \operatorname{tr}\left(\mathrm{d}r^2\right) + \frac{1}{2}\operatorname{tr}\left(\left[r, \mathrm{d}Y^{+}\right]\left[r, \mathrm{d}Y^{+}\right]\right) - \frac{1}{2}\operatorname{tr}\left(\left\{r, \mathrm{d}Y^{-}\right\}\left\{r, \mathrm{d}Y^{-}\right\}\right) \tag{4.2.14}$$

The above form of $\operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right)$ in equation (4.2.14) can be obtained in component form for terms that have $i = j$ and those that have $i \neq j$ such that

$$\begin{aligned}\operatorname{tr}\left(\mathrm{d}Z^{\dagger}\mathrm{d}Z\right) &= \sum_i \left(\mathrm{d}r_i\right)^2 + 2\sum_i \left(r_i\right)^2 \mathrm{d}Y^{-}_{ii}\left(\mathrm{d}Y^{-}\right)^{*}_{ii} \qquad (4.2.15)\\ &+ \frac{1}{2}\sum_{i\neq j}\left(r_i - r_j\right)^2 \mathrm{d}Y^{+}_{ij}\left(\mathrm{d}Y^{+}\right)^{*}_{ij} + \frac{1}{2}\sum_{i\neq j}\left(r_i + r_j\right)^2 \mathrm{d}Y^{-}_{ij}\left(\mathrm{d}Y^{-}\right)^{*}_{ij}.\end{aligned}$$

We consider a similar approach used for parameterization I, when computing the Laplacian for parameterization II. Using equation (4.2.15), the matrix of the metric tensor $G_{(AB)}$ is thus given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2r_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(r_i - r_j\right)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(r_i + r_j\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(r_i - r_j)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(r_i + r_j)^2 \end{pmatrix}. \tag{4.2.16}$$

Simply enough, the determinant of the matrix in equation (4.2.16) above is given by the product of the elements on the main diagonal and it is

$$\begin{aligned} \det G_{(AB)} &= 2\prod_i r_i^2 \frac{1}{16} \prod_{i<j} (r_i - r_j)^2 (r_i + r_j)^2 (r_i - r_j)^2 (r_i + r_j)^2 \\ &= 2\prod_k r_k^2 \left(\Delta_{MR}^2\right)^2 . \end{aligned} \tag{4.2.17}$$

To use the definition of the Laplacian, we further define

$$G = \sqrt{\det G_{(AB)}} = \sqrt{2\prod_k r_k^2 \left(\Delta_{MR}^2\right)^2} = \sqrt{2}\prod_k r_k \Delta_{MR}^2. \tag{4.2.18}$$

Equation (4.2.19) is exactly the same as equation (4.1.28) up to a factor of $\sqrt{2}$.

Again using the elementary definition of the inverse of the matrix, the matrix of the inverse metric tensor $G^{(BA)}$ is thus

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2r_i^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{(r_i - r_j)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{(r_i + r_j)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{(r_i - r_j)^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{(r_i + r_j)^2} \end{pmatrix}. \tag{4.2.19}$$

We again compute the Laplacian using information from our metric G^{BA} that takes all the degrees of freedom into account, we then get

$$\nabla^2_{II} = \frac{1}{\sqrt{\det G_{(AB)}}} \frac{\partial}{\partial X^A} \sqrt{\det G_{(AB)}} G^{AB} \frac{\partial}{\partial X^B} \tag{4.2.20}$$

$$\begin{aligned}
\nabla^2_{II} &= \frac{1}{\sqrt{2}\prod_k r_k \Delta^2_{MR}} \frac{\partial}{\partial r_i} \left[\sqrt{2}\prod_k r_k \Delta^2_{MR}\right] \frac{\partial}{\partial r_i} \\
&+ \frac{1}{\sqrt{2}\prod_k r_k \Delta^2_{MR}} \frac{\partial}{\partial Y^-_{ii}} \sqrt{2}\prod_k r_k \Delta^2_{MR} \frac{1}{2r_i^2} \frac{\partial}{\partial (Y^-)^*_{ii}} \\
&+ \frac{1}{\sqrt{2}\prod_k r_k \Delta^2_{MR}} \frac{\partial}{\partial Y^+_{ij}} \sqrt{2}\prod_k r_k \Delta^2_{MR} \frac{2}{(r_i - r_j)^2} \frac{\partial}{\partial (Y^+)^*_{ij}} \\
&+ \frac{1}{\sqrt{2}\prod_k r_k \Delta^2_{MR}} \frac{\partial}{\partial Y^-_{ij}} \sqrt{2}\prod_k r_k \Delta^2_{MR} \frac{2}{(r_i + r_j)^2} \frac{\partial}{\partial (Y^-)^*_{ij}}.
\end{aligned} \tag{4.2.21}$$

After differentiating and canceling terms the Laplacian above in equation (4.2.22) takes the final form

$$\begin{aligned}
\nabla^2_{II} &= \frac{1}{\prod_k r_k} \left\{\frac{\partial}{\partial r_i}\prod_k r_k\right\} \frac{\partial}{\partial r_i} + \frac{1}{\Delta^2_{MR}} \left\{\frac{\partial}{\partial r_i}\Delta^2_{MR}\right\} \frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \\
&+ \sum_i \frac{1}{2r_i} \frac{\partial}{\partial Y^-_{ii}} \frac{\partial}{\partial (Y^-)^*_{ii}} + \sum_{i\neq j} \frac{2}{(r_i - r_j)^2} \frac{\partial}{\partial Y^+_{ij}} \frac{\partial}{\partial (Y^+)^*_{ij}} \\
&+ \sum_{i\neq j} \frac{2}{(r_i + r_j)^2} \frac{\partial}{\partial Y^-_{ij}} \frac{\partial}{\partial (Y^-)^*_{ij}}.
\end{aligned} \tag{4.2.22}$$

If we observe and compare equations (4.2.22), (4.1.33) and the last line of equation (3.3.22), the operators that depend on the eigenvalues r_i appear consistently in all three frameworks. In addition to this, the operator terms that show angular variable dependence are isolated and do not mix with those operators that have radial dependence. Due to the form taken by the Laplacian in equation (4.2.22), we can also use the same argument as that of the Laplacian derived in the previous section using parameterization I to motivate how the fermionic picture can be introduced if the Laplacian in equation (4.2.22) becomes the kinetic piece of some Hamiltonian operator.

4.3 The Generator Of $O(2)$ Rotations

We now proceed to tackle our next objective, which is to calculate the angular momentum i.e. the generator of $O(2)$ rotations using the already defined matrix valued polar coordinates Z and $Z^\dagger$.

The definition of the angular momentum operator $\hat{L}$ in terms of the hermitian matrices X_1, X_2 and their momentum conjugates, $\partial/\partial X_1$ and $\partial/\partial X_2$, is given by

$$\begin{aligned}\hat{L} &= \frac{1}{i}\mathrm{tr}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_1}\right) \\ &= \frac{1}{i}\left\{\sum_{ab}(X_1)_{ab}\left(\frac{\partial}{\partial X_2}\right)_{ab} - \sum_{ab}(X_2)_{ab}\left(\frac{\partial}{\partial X_1}\right)_{ab}\right\}.\end{aligned} \tag{4.3.1}$$

The partial equations in equation (4.3.1) are re-written as follows:

$$\left(\frac{\partial}{\partial X_1}\right)_{ab} = \sum_{ij}\frac{\partial Z_{ij}}{(\partial X_1)_{ab}}\frac{\partial}{\partial Z_{ij}} + \sum_{ij}\frac{\partial Z^\dagger_{ij}}{(\partial X_1)_{ab}}\frac{\partial}{\partial Z^\dagger_{ij}} \tag{4.3.2}$$

and

$$\left(\frac{\partial}{\partial X_2}\right)_{ab} = \sum_{ij}\frac{\partial Z_{ij}}{(\partial X_2)_{ab}}\frac{\partial}{\partial Z_{ij}} + \sum_{ij}\frac{\partial Z^\dagger_{ij}}{(\partial X_2)_{ab}}\frac{\partial}{\partial Z^\dagger_{ij}}. \tag{4.3.3}$$

The next natural step is to determine the coefficients

$$\frac{\partial Z_{ij}}{(\partial X_1)_{ab}} \qquad \frac{\partial Z^\dagger_{ij}}{(\partial X_1)_{ab}} \qquad \frac{\partial Z_{ij}}{(\partial X_2)_{ab}} \qquad \frac{\partial Z^\dagger_{ij}}{(\partial X_2)_{ab}} \tag{4.3.4}$$

in the partial differential equations of equations (4.3.2) and (4.3.3), but these are solved in the appendix B. To solve equation (4.3.4), the hermitian matrices X_1 and X_2 are written in terms of the polar coordinates Z and $Z^\dagger$ as

$$Z = X_1 + iX_2 \qquad\qquad Z^\dagger = X_1 - iX_2. \tag{4.3.5}$$

Reorganizing the terms above in equation (4.3.5), we write the equations in terms of X_1 and X_2

$$X_1 = \frac{1}{2}\left(Z + Z^\dagger\right) \qquad\qquad X_2 = \frac{1}{2i}\left(Z - Z^\dagger\right). \tag{4.3.6}$$

Equation (4.3.6) is used to solve for the coefficients in equation (4.3.4), this would then permit us to rewrite the definition of the angular momentum in terms of the polar coordinates as follows

$$\frac{1}{i}\mathrm{tr}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_2}\right) = \sum_{ab}(Z)_{ab}\frac{\partial}{\partial (Z)_{ab}} - \sum_{ab}(Z^\dagger_{ab})\frac{\partial}{\partial (Z^\dagger)_{ab}}. \tag{4.3.7}$$

Using partial differential equations, in equation (4.3.7), we define the following:

$$\frac{\partial}{\partial Z_{ab}} = \sum_i \frac{\partial r_i}{\partial Z_{ab}}\frac{\partial}{\partial r_i} + \sum_{i\neq j}\frac{\partial Y^+_{ij}}{\partial Z_{ab}}\frac{\partial}{\partial Y^+_{ij}} + \sum_{i\neq j}\frac{\partial Y^-_{ij}}{\partial Z_{ab}}\frac{\partial}{\partial Y^-_{ij}} \tag{4.3.8}$$

and

$$\frac{\partial}{\partial Z^\dagger_{ab}} = \sum_i \frac{\partial r_i}{\partial Z^\dagger_{ab}}\frac{\partial}{\partial r_i} + \sum_{i\neq j}\frac{\partial Y^+_{ij}}{\partial Z^\dagger_{ab}}\frac{\partial}{\partial Y^+_{ij}} + \sum_{i\neq j}\frac{\partial Y^-_{ij}}{\partial Z^\dagger_{ab}}\frac{\partial}{\partial Y^-_{ij}}. \tag{4.3.9}$$

In equation (4.3.8), we would now want to determine the following coefficients

$$\frac{\partial r_i}{\partial Z_{ab}} \qquad\qquad \frac{\partial Y^+_{ij}}{\partial Z_{ab}} \qquad\qquad \frac{\partial Y^-_{ij}}{\partial Z_{ab}}. \tag{4.3.10}$$

Looking at equation (4.3.9), we also determine its coefficients.

To calculate the coefficients of equations (4.3.8) and (4.3.9), we use the expressions of $\mathrm{d}Z$ and $\mathrm{d}Z^\dagger$ obtained in the previous section for parameterization II, and these were shown to be

$$\mathrm{d}Z = V^\dagger\left(\mathrm{d}r + \frac{1}{\sqrt{2}}\left[r, \mathrm{d}Y^+\right] + \frac{1}{\sqrt{2}}\left\{r, \mathrm{d}Y^-\right\}\right)W, \tag{4.3.11}$$

$$\mathrm{d}Z^\dagger = W^\dagger\left(\mathrm{d}r + \frac{1}{\sqrt{2}}\left[r, \mathrm{d}Y^+\right] - \frac{1}{\sqrt{2}}\left\{r, \mathrm{d}Y^-\right\}\right)V. \tag{4.3.12}$$

After some algebra (shown in appendix B) and using equations (4.3.11) and (4.3.12), the coefficients in equations (4.3.8) and (4.3.9) are shown to be

$$\frac{\partial r_i}{\partial Z_{ab}} = \sum_i V_{ia} W^\dagger_{bi} \qquad \frac{\partial r_i}{\partial Z^\dagger_{ab}} = \sum_i W_{ia} V^\dagger_{bi}, \tag{4.3.13}$$

$$\frac{\partial Y^+_{ij}}{\partial Z_{ab}} = \sqrt{2}\sum_{i\neq j} \frac{V_{ia} W^\dagger_{bj}}{(r_i - r_j)} \qquad \frac{\partial Y^+_{ij}}{\partial Z^\dagger_{ab}} = \sqrt{2}\sum_{i\neq j} \frac{W_{ia} V^\dagger_{bj}}{(r_i - r_j)}, \tag{4.3.14}$$

$$\frac{\partial Y^-_{ij}}{\partial Z_{ab}} = \sqrt{2}\sum_{i\neq j} \frac{V_{ia} W^\dagger_{bj}}{(r_i + r_j)} \qquad \frac{\partial Y^-_{ij}}{\partial Z^\dagger_{ab}} = -\sqrt{2}\sum_{i\neq j} \frac{W_{ia} V^\dagger_{bj}}{(r_i + r_j)}, \tag{4.3.15}$$

$$\frac{\partial Y^-_{ii}}{\partial Z_{ab}} = \frac{1}{\sqrt{2}}\sum_{i} \frac{V_{ia} W^\dagger_{bj}}{(r_i)} \qquad \frac{\partial Y^-_{ii}}{\partial Z^\dagger_{ab}} = \frac{-1}{\sqrt{2}}\sum_{i} \frac{W_{ia} V^\dagger_{bj}}{(r_i)}. \tag{4.3.16}$$

Equations (4.3.13), (4.3.14), (4.3.15) and (4.3.16) are substituted back into equations (4.3.8) and (4.3.9) to obtain the complete expressions for $\partial/\partial Z_{ab}$ and $\partial/\partial Z^\dagger_{ab}$. The following equations are obtained

$$\frac{\partial}{\partial Z_{ab}} = \sum_i V_{ia} W^\dagger_{bi} \frac{\partial}{\partial r_i} + \sqrt{2}\sum_{i\neq j} \frac{V_{ia} W^\dagger_{bj}}{(r_i - r_j)} \frac{\partial}{\partial Y^+_{ij}} + \sqrt{2}\sum_{ij} \frac{V_{ia} W^\dagger_{bj}}{(r_i + r_j)} \frac{\partial}{\partial Y^-_{ij}} \tag{4.3.17}$$

and

$$\frac{\partial}{\partial Z^\dagger_{ab}} = \sum_i W_{ia} V^\dagger_{bi} \frac{\partial}{\partial r_i} + \sqrt{2}\sum_{i\neq j} \frac{W_{ia} V^\dagger_{bj}}{(r_i - r_j)} \frac{\partial}{\partial \left(Y^+_{ij}\right)} - \sqrt{2}\sum_{ij} \frac{W_{ia} V^\dagger_{bj}}{(r_i + r_j)} \frac{\partial}{\partial \left(Y^-_{ij}\right)}. \tag{4.3.18}$$

The expressions $\partial/\partial Z_{ab}$ and $\partial/\partial Z^\dagger_{ab}$, in equation (4.3.17) and (4.3.18), are used in equation (4.3.7) to obtain the full expression of the angular momentum, which is given by

$$\hat{L} = \frac{1}{i}\mathrm{tr}\left(X_1 \frac{\partial}{\partial X_2} - X_2 \frac{\partial}{\partial X_2}\right) = \sqrt{2}\sum_i \frac{\partial}{\partial Y^-_{ii}}, \tag{4.3.19}$$

where $\hat{L}$ is the angular momentum or equivalently the generator of $U(1) \sim SO(2)$ rotations.

$\hat{L}$ is an $N \times N$ matrix operator with entries on the main diagonal that depend on the variables that we saw in the Laplacian of the second parameterization in equation (4.2.22). We would have hoped to see terms that are proportional to the square of the angular momentum, with both diagonal and off-diagonal entries in the matrix, appearing in equation (4.2.22). When one considers the case of the Laplacian operator in ordinary spherical coordinates, for $\hbar = 1$, one can see the square of the angular momentum operator inside the equation of the Laplacian operator. This would mean that the Laplacian in ordinary spherical coordinates would be partitioned into a radially dependent piece and the part with the square of the angular momentum operator multiplied by some radial factor. We would have hoped to see a similar construct in our Laplacian from parameterization II after computing the angular momentum operator using our matrix polar coordinates. This is not the case, as the angular part of the Laplacian is associated with $U(N)$ "rotations".

Chapter 5

Gauge Invariant States

5.1 Form of the Laplacians acting on Invariant States

We now wish to consider the action of the Laplacians, equations (4.1.30) and (4.2.22), arrived at in the previous chapters, when acting on the states that depend on the matrix variables

$$Q = VUV^{\dagger}, \tag{5.1.1}$$

$$Q^{\dagger} = VU^{\dagger}V^{\dagger} \tag{5.1.2}$$

and the eigenvalue r.

First it would be best to explain where this variable comes from.

We remind ourselves that for a diagonalized hermitian matrix $R = V^{\dagger}rV$ and the unitary matrix U, the matrix $Z = RU$ with its conjugate $Z^{\dagger}$ were defined as

$$Z = V^{\dagger}rVU \qquad\qquad Z^{\dagger} = U^{\dagger}V^{\dagger}rV. \tag{5.1.3}$$

Consider states constructed from the trace of the matrices Z and $Z^{\dagger}$, in equation (5.1.3) above

$$\mathrm{tr}\,(Z) = \mathrm{tr}\left(V^\dagger r V U\right) = \mathrm{tr}\left(U V^\dagger r V\right) = \mathrm{tr}\left(V U V^\dagger r\right) = \mathrm{tr}\,(Qr) \qquad (5.1.4)$$

and similarly for the conjugate $Z^\dagger$

$$\mathrm{tr}\left(Z^\dagger\right) = \mathrm{tr}\left(U^\dagger V^\dagger r V\right) = \mathrm{tr}\left(V U^\dagger V^\dagger r\right) = \mathrm{tr}\left(Q^\dagger r\right). \qquad (5.1.5)$$

Equations (5.1.4) and (5.1.5) signal the first appearance of Q and $Q^\dagger$ through the use of the cyclic property of the trace and these states will be referred to as "invariant states".

In this project, the Laplacians are going to act on functions constructed as a product of Z and $Z^\dagger$ terms under the trace, and these functions will consequently be proportional to either Q or $Q^\dagger$. That is the reason why we would prefer changing the current representation of variables that appear in the Laplacians in parameterization I and II to variables in equations (5.1.4) and (5.1.5). Functions on which our Laplacians will act on could take any of the following probable forms

$$\mathrm{tr}\,(ZZ) = \mathrm{tr}\left(V^\dagger r V U V^\dagger r V U\right) = \mathrm{tr}\left(V U V^\dagger r V U V^\dagger r\right) = \mathrm{tr}\,(QrQr)$$

and similarly for conjugate $Z^\dagger$,

$$\mathrm{tr}\left(Z^\dagger Z^\dagger\right) = \mathrm{tr}\left(U^\dagger V^\dagger r V U^\dagger V^\dagger r V\right) = \mathrm{tr}\left(V U^\dagger V^\dagger r V U^\dagger V^\dagger r\right) = \mathrm{tr}\left(Q^\dagger r Q^\dagger r\right).$$

We could further have

$$\mathrm{tr}\left(Z^4\right) = \mathrm{tr}\,(QrQrQrQr)$$

and

$$\mathrm{tr}\left((Z^\dagger)^4\right) = \mathrm{tr}\left(Q^\dagger r Q^\dagger r Q^\dagger r Q^\dagger r\right).$$

The above two expressions can be generalized to order n of product of Z or $Z^\dagger$ terms as follows,

$$\mathrm{tr}\left(Z^n\right) = \mathrm{tr}\left(QrQr.....Qr\right) \tag{5.1.6}$$

and

$$\mathrm{tr}\left((Z^\dagger)^n\right) = \mathrm{tr}\left(Q^\dagger rQ^\dagger r......Q^\dagger r\right). \tag{5.1.7}$$

In equation (5.1.6), the product Qr appears n-times, corresponding to the order of Z in the trace, the same rule is true for the conjugate $Z^\dagger$.

We now would naturally ask ourselves if there are products in the trace that mix the Q and $Q^\dagger$ terms. An example of such a state is the invariant "quartic" state. Some examples of general invariant states are given below

$$\mathrm{tr}\left(ZZ^\dagger\right) = \mathrm{tr}\left(r^2\right) \qquad \mathrm{tr}\left(ZZ^\dagger Z\right) = \mathrm{tr}\left(Qr^3\right) \qquad \mathrm{tr}\left(Z^\dagger ZZ^\dagger\right) = \mathrm{tr}\left(Q^\dagger r^3\right),$$

$$\mathrm{tr}\left(ZZZ^\dagger ZZ^\dagger ZZZ\right) = \mathrm{tr}\left(QrQr^5QrQr\right) \qquad \mathrm{tr}\left(Z^\dagger Z^\dagger ZZ^\dagger Z^\dagger Z\right) = \mathrm{tr}\left(r^3Q^\dagger r^3Q^\dagger\right),$$

$$\mathrm{tr}\left(ZZ^\dagger ZZ^\dagger ZZ^\dagger\right) = \mathrm{tr}\left(r^6\right) \qquad \mathrm{tr}\left(ZZZZ^\dagger ZZ\right) = \mathrm{tr}\left(QrQrQr^3Qr\right).$$

We can also construct "quartic" invariant state as follows:

$$Z^2 = V^\dagger rQrQV \qquad \text{and} \qquad (Z^\dagger)^2 = V^\dagger Q^\dagger rQ^\dagger rV, \tag{5.1.8}$$

thus using the trace, the quartic state is

$$\begin{aligned} \mathrm{tr}\left(Z^2(Z^\dagger)^2\right) &= \mathrm{tr}\left(V^\dagger rQrQVV^\dagger Q^\dagger rQ^\dagger rV\right) \\ &= \mathrm{tr}\left(rQrQQ^\dagger rQ^\dagger r\right) \\ &= \mathrm{tr}\left(r^2Qr^2Q^\dagger\right). \end{aligned} \tag{5.1.9}$$

The above invariant states represent the type of objects that can be constructed for our Laplacian to act on using both parameterizations I and II. So from the brief discussion above, the following deductions can be made:
(i) Q is a variable inherent inside the "wavefunctions", this forces us to rewrite the Laplacians so as to operate on these newly formulated wavefunctions
(ii) for these "wavefunctions" or invariant sates, we can get invariant states that mix Q and $Q^\dagger$ terms and
(iii) depending on the order of Z or $Z^\dagger$ appearing in the trace, we get different orders of r appearing at different positions inside the trace.

Now that we have introduced and established where the variable Q comes from, why the variable Q is important and why were these the objects which motivated the change in representation of the Laplacians, we proceed to rewrite the Laplacian from parameterization I in terms of the new variables. It is best to pick up where we left off, the Laplacian from parameterization I was shown to be

$$\begin{aligned}\nabla_I^2 &= \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \\ &+ \sum_i\left\{\frac{1}{r_i^2}\frac{\partial}{\partial X_{ii}}\frac{\partial}{\partial X_{ii}^*}\right\} + 2\sum_{i\neq j}\frac{(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S_{ij}^*} \\ &- \frac{2}{(r_i+r_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial X_{ij}^*} + \frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial S_{ij}^*}\right\} + \frac{4}{(r_i+r_j)^2}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X_{ij}^*}.\end{aligned}$$

Using the anti-hermitian property of the differentials, it follows that

$$\mathrm{d}X_{ji}^* = -\mathrm{d}X_{ij} \Rightarrow \frac{\partial}{\partial X_{ji}^*} = -\frac{\partial}{\partial X_{ij}}, \tag{5.1.10}$$

so our Laplacian changes signs as above to become

$$\begin{aligned}\nabla_I^2 &= \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \\ &- \left\{\frac{1}{r_i^2}\frac{\partial}{\partial X_{ii}}\frac{\partial}{\partial X_{ii}}\right\} - \frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S_{ji}} \\ &+ \frac{2}{(r_i+r_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial X_{ji}} + \frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial S_{ji}}\right\} - \frac{4}{(r_i+r_j)^2}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X_{ji}}.\end{aligned}$$

We first define $\mathrm{d}P = \mathrm{d}QQ^\dagger$ and also define

$$\frac{\partial}{\partial S_{ij}} = \sum_{\alpha,\beta} \frac{\partial Q_{\alpha\beta}}{\partial S_{ij}} \frac{\partial}{\partial Q_{\alpha\beta}} \qquad \frac{\partial}{\partial X_{ij}} = \sum_{\alpha,\beta} \frac{\partial Q_{\alpha\beta}}{\partial X_{ij}} \frac{\partial}{\partial Q_{\alpha\beta}}. \tag{5.1.11}$$

From equation (5.1.11) above, we need to calculate the coefficients $\partial Q_{\alpha\beta}/\partial S_{ij}$ and $\partial Q_{\alpha\beta}/\partial X_{ij}$. We thus proceed as follows

$$\begin{aligned} \mathrm{d}Q_{\alpha\beta} &= \mathrm{d}(VUV^\dagger)_{\alpha\beta} \\ &= (\mathrm{d}VUV^\dagger + V\mathrm{d}UV^\dagger + VU\mathrm{d}V^\dagger)_{\alpha\beta} \\ &= \sum_{k,q} \mathrm{d}V_{\alpha\beta} U_{kq} V^\dagger_{q\beta} + \sum_{k',q'} V_{\alpha k'} \mathrm{d}U_{k'q'} V^\dagger_{q'\beta} + \sum_{a,b} U_{ab} \mathrm{d}V^\dagger_{b\beta}. \end{aligned} \tag{5.1.12}$$

In the line that follows we use the property of unitary matrices such that $\sum_b V_{ab} V^\dagger_{bc} = \delta_{ac}$, we then get

$$\begin{aligned} \mathrm{d}Q_{\alpha\beta} &= \sum_{kq} (\mathrm{d}VV^\dagger V)_{\alpha k} U_{kq} V^\dagger_{q\beta} + \sum_{k',q'} V_{\alpha k'} \mathrm{d}U_{k'q'} V^\dagger_{q'\beta} \\ &+ \sum_{a,b} V_{\alpha a} U_{ab} (V^\dagger V \mathrm{d}V^\dagger)_{b\beta} \\ &= \sum_{a',k,q} (\mathrm{d}VV^\dagger)_{\alpha a'} V_{a'k} U_{kq} V^\dagger_{q\beta} + \sum_{k',q'} V_{\alpha k'} \mathrm{d}U_{k'q'} V^\dagger_{q'\beta} \\ &- \sum_{a,b} V_{\alpha a} U_{ab} (V^\dagger \mathrm{d}VV^\dagger)_{b\beta} \\ &= \sum_{a',k,q} (\mathrm{d}S)_{\alpha a'} V_{a'k} U_{kq} V^\dagger_{q\beta} + \sum_{k'q'} V_{\alpha k'} \mathrm{d}U_{k'q'} V^\dagger_{q'\beta} \\ &- \sum_{e,a,b} V_{\alpha a} U_{ab} V^\dagger_{be} \mathrm{d}S_{e\beta}. \end{aligned} \tag{5.1.13}$$

In the first line of equation (5.1.13) we use the anti-hermiticity of the differentials such that $\mathrm{d}VV^\dagger = -V\mathrm{d}V^\dagger$. Using known definitions, the above equation can be compactly expressed by utilizing some of the summations as

$$\begin{aligned} \mathrm{d}Q_{\alpha\beta} &= \sum_{a'} (\mathrm{d}S)_{\alpha a'} (VUV^\dagger)_{a'\beta} + \sum_{k',q'} V_{\alpha k'} \mathrm{d}U_{k'q'} V^\dagger_{q'\beta} - \sum_e (VUV^\dagger)_{\alpha e} (\mathrm{d}S)_{e\beta} \\ &= \sum_{a'} (\mathrm{d}S)_{\alpha a'} (Q)_{a'\beta} + \sum_{k',q'} V_{\alpha k'} \mathrm{d}U_{k'q'} V^\dagger_{q'\beta} - \sum_e (Q)_{\alpha e} (\mathrm{d}S)_{e\beta}. \end{aligned} \tag{5.1.14}$$

Equation (5.1.14) looks promising because we see the differential dS appearing which we consider to be an old variable, multiplied by the new variable Q, this clearly signals that we are heading in the right direction. In equation (5.1.14), we introduce the inverse of dS_{ij} to obtain the following property

$$\frac{\partial S_{a'b'}}{\partial S_{ab}} = \delta_{a'a}\delta_{bb'}. \tag{5.1.15}$$

It then follows that from equation (5.1.14) we can deduce

$$\begin{aligned}
\frac{\partial Q_{\alpha\beta}}{\partial S_{ij}} &= \sum_{a'} \frac{\partial S_{\alpha a'}}{\partial S_{ij}} Q_{a'\beta} + \sum_{k',q'} V_{ak'} \frac{\partial U_{k'q'}}{\partial S_{ij}} V^{\dagger}_{q'\beta} - \sum_{e} Q_{\alpha e} \frac{\partial S_{e\beta}}{\partial S_{ij}} \\
&= \sum_{a'} \delta_{\alpha i}\delta_{ja'} Q_{a'\beta} - \sum_{e} \delta_{ei}\delta_{j\beta} Q_{\alpha e} \\
\Rightarrow \frac{\partial Q_{\alpha\beta}}{\partial S_{ij}} &= \delta_{\alpha i} Q_{j\beta} - \delta_{j\beta} Q_{\alpha i}. \qquad (5.1.16)
\end{aligned}$$

Using equation (5.1.16), we can now solve for $\partial/\partial S_{ij}$, and this gives

$$\begin{aligned}
\frac{\partial}{\partial S_{ij}} &= \sum_{\alpha\beta} \frac{\partial Q_{\alpha\beta}}{\partial S_{ij}} \frac{\partial}{\partial Q_{\alpha\beta}} = \sum_{\alpha,\beta} (\delta_{\alpha i} Q_{j\beta} - \delta_{j\beta} Q_{\alpha i}) \frac{\partial}{\partial Q_{\alpha\beta}} \qquad (5.1.17) \\
&= \sum_{\alpha\beta} \delta_{\alpha i} Q_{j\beta} \frac{\partial}{\partial Q_{\alpha\beta}} \frac{\partial}{\partial Q_{\alpha\beta}} - \sum_{\alpha\beta} \delta_{j\beta} Q_{\alpha i} \frac{\partial}{\partial Q_{\alpha\beta}} \\
&= \sum_{\beta} Q_{j\beta} \frac{\partial}{\partial Q_{i\beta}} - \sum_{\alpha} Q_{\alpha i} \frac{\partial}{\partial Q_{\alpha j}}.
\end{aligned}$$

So, in total we have

$$\frac{\partial}{\partial S_{ij}} = \sum_{\beta} Q_{j\beta} \frac{\partial}{\partial Q_{i\beta}} - \sum_{\alpha} Q_{\alpha i} \frac{\partial}{\partial Q_{\alpha j}}. \tag{5.1.18}$$

We now continue to find the coefficient for $\partial/\partial X_{ij}$, so continuing from equation (5.1.14), we have

$$\begin{aligned}
\mathrm{d}Q_{\alpha\beta} &= \sum_{a'}(\mathrm{d}S)_{\alpha a'}(Q)_{a'\beta} + \sum_{k',q'} V_{\alpha k'}\mathrm{d}U_{k'q'}V^{\dagger}_{q'\beta} - \sum_{e}(Q)_{\alpha e}(\mathrm{d}S)_{e\beta} \\
&= \sum_{a'}(\mathrm{d}S)_{\alpha a'}(Q)_{a'\beta} + \sum_{k',q'} V_{\alpha k'}(\mathrm{d}UU^{\dagger}V^{\dagger}VU)_{k'q'}V^{\dagger}_{q'\beta} \\
&- \sum_{e}(Q)_{\alpha e}(\mathrm{d}S)_{e\beta} \qquad (5.1.19)\\
&= \sum_{a'}(\mathrm{d}S)_{\alpha a'}(Q)_{a'\beta} + \sum_{\eta}(V\mathrm{d}UU^{\dagger}V^{\dagger})_{\alpha\eta}(VUV^{\dagger})_{\eta\beta} \\
&- \sum_{e}(Q)_{\alpha e}(\mathrm{d}S)_{e\beta} \\
\Rightarrow \mathrm{d}Q_{\alpha\beta} &= \sum_{a'}(\mathrm{d}S)_{\alpha a'}(Q)_{a'\beta} + \sum_{\eta}\mathrm{d}X_{\alpha\eta}Q_{\eta\beta} - \sum_{e}(Q)_{\alpha e}(\mathrm{d}S)_{e\beta}.
\end{aligned}$$

For the middle term in the second line of equation (5.1.19), we again used the unitarity identity of matrices such that $U^{\dagger}V^{\dagger}VU = 1$ and the usual definition of $\mathrm{d}X$. Similarly, we divide by the inverse of the differential $\mathrm{d}X$, from the last line of equation (5.1.19) we obtain the following

$$\begin{aligned}
\frac{\partial Q_{\alpha\beta}}{\partial X_{ij}} &= \sum_{\alpha}\frac{\partial S_{\alpha a'}}{\partial X_{ij}}Q_{a'\beta} + \sum_{\eta}\frac{\partial X_{\alpha\eta}}{\partial X_{ij}}Q_{\eta\beta} - \sum_{e}\frac{\partial S_{e\beta}}{\partial X_{ij}} \qquad (5.1.20)\\
&= \sum_{\eta}\delta_{\alpha i}\delta_{j\eta}Q_{\eta\beta}, \\
\Rightarrow \frac{\partial Q_{\alpha\beta}}{\partial X_{ij}} &= \delta_{\alpha i}Q_{j\beta}.
\end{aligned}$$

The first line of equation (5.1.20) uses the fact that

$$\frac{\partial S_{\alpha a'}}{\partial X_{ij}} = \frac{\partial S_{e\beta}}{\partial X_{ij}} = 0 \qquad \text{and} \qquad \frac{\partial X_{\alpha\eta}}{\partial X_{ij}} = \delta_{\alpha i}\delta_{j\eta}. \qquad (5.1.21)$$

It then follows that from the definition in equation (5.1.11) we have that

$$\frac{\partial}{\partial X_{ij}} = \sum_{\alpha,\beta}\frac{\partial Q_{\alpha\beta}}{\partial X_{ij}}\frac{\partial}{\partial Q_{\alpha\beta}} = \sum_{\alpha,\beta}\delta_{\alpha i}Q_{j\beta}\frac{\partial}{\partial Q_{\alpha\beta}} = \sum_{\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}. \qquad (5.1.22)$$

When the indices i and j are swapped around, the following equations can also be derived

$$\frac{\partial}{\partial S_{ji}} = \sum_{\beta} Q_{i\beta}\frac{\partial}{\partial Q_{j\beta}} - \sum_{\alpha} Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}} \quad \text{and} \quad \frac{\partial}{\partial X_{ji}} = \sum_{\beta} Q_{i\beta}\frac{\partial}{\partial Q_{j\beta}}. \tag{5.1.23}$$

We can now introduce the new notation into our Laplacian from parameterization I using equations (5.1.18), (5.1.22) and (5.1.23) to get the following new expression

$$\begin{aligned}
\nabla_I^2 = {} & \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \\
- \; & \frac{1}{r_i^2}\sum_{\beta} Q_{i\beta}\frac{\partial}{\partial Q_{i\beta}}\sum_{\alpha} Q_{i\alpha}\frac{\partial}{\partial Q_{i\alpha}} \\
- \; & \frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}\left(\sum_{\beta} Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}} - \sum_{\beta} Q_{\beta i}\frac{\partial}{\partial Q_{\beta j}}\right)\times \\
& \left(\sum_{\alpha} Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} - \sum_{\alpha} Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}}\right) \\
+ \; & \frac{2}{(r_i+r_j)^2}\left(\sum_{\beta} Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}} - \sum_{\alpha} Q_{\beta i}\frac{\partial}{\partial Q_{\beta j}}\right)\sum_{\alpha} Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} \\
+ \; & \frac{2}{(r_i+r_j)^2}\sum_{\beta} Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}\left(\sum_{\alpha} Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} - \sum_{\alpha} Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}}\right) \\
- \; & \frac{4}{(r_i+r_j)^2}\sum_{\beta} Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}\sum_{\alpha} Q_{i\alpha}\frac{\partial}{\partial Q_{i\alpha}}.
\end{aligned} \tag{5.1.24}$$

In equation (5.1.24), we multiply out all the brackets and regroup the terms to get

$$
\begin{aligned}
\nabla_I^2 \;=\; & \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \qquad (5.1.25)\\
- & \frac{1}{r_i^2}\sum_\beta Q_{i\beta}\frac{\partial}{\partial Q_{i\beta}}\sum_\alpha Q_{i\alpha}\frac{\partial}{\partial Q_{i\alpha}}\\
- & \sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2+r_j^2)}\left(\sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} - \sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}}\right)\\
+ & \sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2+r_j^2)}\left(\sum_{\alpha,\beta}Q_{\beta i}\frac{\partial}{\partial Q_{\beta j}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} - \sum_{\alpha,\beta}Q_{\beta i}\frac{\partial}{\partial Q_{\beta j}}Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}}\right)\\
+ & \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\left(\sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} - \sum_{\alpha,\beta}Q_{\beta i}\frac{\partial}{\partial Q_{\beta j}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}}\right)\\
+ & \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\left(\sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} - \sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}}\right)\\
- & \sum_{i\neq j}\frac{4}{(r_i+r_j)^2}\sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}}.
\end{aligned}
$$

The terms appearing in the Laplacian given in equation (5.1.25) are regrouped according to their coefficients, then equation (5.1.25) becomes

$$
\begin{aligned}
& \nabla_I^2 = \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta_{MR}^2}\left\{\frac{\partial}{\partial r_i}\Delta_{MR}^2\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}\\
- & \frac{1}{r_i^2}\sum_{\beta,i} Q_{i\beta}\frac{\partial}{\partial Q_{i\beta}}\sum_\alpha Q_{i\alpha}\frac{\partial}{\partial Q_{i\alpha}}\\
+ & \sum_{i\neq j}\left(\frac{-2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2} + \frac{2}{(r_i+r_j)^2} + \frac{2}{(r_i+r_j)^2} - \frac{4}{(r_i+r_j)^2}\right)\times\\
& \sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}} \qquad (5.1.26)\\
+ & \sum_{i\neq j}\left(\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2} - \frac{2}{(r_i+r_j)^2}\right)\sum_{\alpha,\beta}Q_{j\beta}\frac{\partial}{\partial Q_{i\beta}}Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}}\\
+ & \sum_{i\neq j}\left(\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2} - \frac{2}{(r_i+r_j)^2}\right)\sum_{\alpha,\beta}Q_{\beta i}\frac{\partial}{\partial Q_{\beta j}}Q_{i\alpha}\frac{\partial}{\partial Q_{j\alpha}}\\
- & \sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}\sum_{\alpha,\beta}Q_{\beta i}\frac{\partial}{\partial Q_{\beta j}}Q_{\alpha j}\frac{\partial}{\partial Q_{\alpha i}}.
\end{aligned}
$$

We know introduce the following new notation [2] to make the representation

[2] We could have introduced these generators at an earlier stage. This would have resulted

easier

$$E^L_{ji} \equiv \sum_b Q_{jb} \frac{\partial}{\partial Q_{ib}} \qquad\qquad E^R_{ji} \equiv Q_{ai} \frac{\partial}{\partial Q_{aj}}. \tag{5.1.27}$$

Equation (5.1.27) is the generator of left and right $SU(N)$ rotations. So now our Laplacian from parameterization I in the new variables takes its final form to become

$$\begin{aligned} \nabla^2_I &= \frac{1}{\prod_i r_i} \left\{ \frac{\partial}{\partial r_i} \prod_i r_i \right\} \frac{\partial}{\partial r_i} + \frac{1}{\Delta^2_{MR}} \left\{ \frac{\partial}{\partial r_i} \Delta^2_{MR} \right\} \frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} - \frac{1}{r_i^2} E^L_{ii} E^L_{ii} \\ &- \sum_{i \neq j} \frac{2(r_i^2 + r_j^2)}{(r_i^2 - r_j^2)^2} E^L_{ji} E^L_{ij} + \sum_{i \neq j} \frac{4 r_i r_j}{(r_i^2 - r_j^2)} \left(E^L_{ji} E^R_{ji} + E^R_{ij} E^L_{ij} \right) \\ &- \sum_{i \neq j} \frac{2(r_i^2 + r_j^2)}{(r_i^2 - r_j^2)} \left(E^R_{ij} E^R_{ji} - E^L_{ji} E_{ij} \right). \end{aligned} \tag{5.1.28}$$

This procedure is also performed for the parameterization II in which the Laplacian was shown to be

$$\begin{aligned} \nabla^2_{II} &= \frac{1}{\prod_i r_i} \left\{ \frac{\partial}{\partial r_i} \prod_i r_i \right\} \frac{\partial}{\partial r_i} + \frac{1}{\Delta^2_{MR}} \left\{ \frac{\partial}{\partial r_i} \Delta^2_{MR} \right\} \frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \\ &+ \frac{1}{2r_i} \frac{\partial}{\partial Y^-_{ii}} \frac{\partial}{\partial (Y^-)^*_{ii}} + \frac{2}{(r_i - r_j)^2} \frac{\partial}{\partial Y^+_{ij}} \frac{\partial}{\partial (Y^+)^*_{ij}} \\ &+ \frac{2}{(r_i + r_j)^2} \frac{\partial}{\partial Y^-_{ij}} \frac{\partial}{\partial (Y^-)^*_{ij}}. \end{aligned} \tag{5.1.29}$$

We need to start again from the definition of Q. In parameterization II we had earlier introduced the variable $W = VU$, this means that we can now rewrite Q as

$$Q = VUV^\dagger = WV^\dagger, \tag{5.1.30}$$

and then we can write

in a simple form of the derivation with the new variables from a notational point of view.

$$\begin{aligned}
\mathrm{d}Q &= \mathrm{d}WV^\dagger + W\mathrm{d}V^\dagger = \mathrm{d}W(W^\dagger W)V^\dagger + W(V^\dagger V)\mathrm{d}V^\dagger \\
&= (\mathrm{d}WW^\dagger)(WV^\dagger) + (WV^\dagger)(V\mathrm{d}V^\dagger) \\
&= (\mathrm{d}WW^\dagger)(WV^\dagger) - (WV^\dagger)(\mathrm{d}VV^\dagger) \\
\Rightarrow \mathrm{d}Q &= \mathrm{d}TQ - Q\mathrm{d}S. \qquad (5.1.31)
\end{aligned}$$

Again in equation (5.1.31), we used the fact that $\mathrm{d}T = \mathrm{d}WW^\dagger$, which was only defined for parameterization II earlier on. We use the definitions from equation (4.2.10) to rewrite the last line of equation (5.1.31) to get

$$\begin{aligned}
\mathrm{d}Q &= \frac{1}{\sqrt{2}}(\mathrm{d}Y^+ + \mathrm{d}Y^-)Q - \frac{1}{\sqrt{2}}Q(\mathrm{d}Y^+ - \mathrm{d}Y^-) \qquad (5.1.32) \\
&= \frac{1}{\sqrt{2}}(\mathrm{d}Y^+Q - Q\mathrm{d}Y^+) + \frac{1}{\sqrt{2}}(\mathrm{d}Y^-Q + Q\mathrm{d}Y^-).
\end{aligned}$$

Inserting the indices in equation (5.1.32) we get

$$\begin{aligned}
\Rightarrow \mathrm{d}Q_{ab} &= \frac{1}{\sqrt{2}}\left(\sum_c \mathrm{d}Y^+_{ac}Q_{cb} - \sum_c Q_{ac}\mathrm{d}Y^+_{cb}\right) \\
&+ \frac{1}{\sqrt{2}}\left(\sum_c \mathrm{d}Y^-_{ac}Q_{cb} + \sum_c Q_{ac}\mathrm{d}Y^-_{cb}\right).
\end{aligned}$$

We again use the method that was applied in the previous section, that is, we introduce the inverse of $\mathrm{d}Y^+$ and $\mathrm{d}Y^-$ to get the following

$$\frac{\partial Q_{ab}}{\partial Y^+_{ij}} = \frac{1}{\sqrt{2}}\left(\sum_c \left(\frac{\partial Y^+_{ac}}{\partial Y^+_{ij}}\right) Q_{cb} - \sum_c Q_{ac}\left(\frac{\partial Y^+_{cb}}{\partial Y^+_{ij}}\right)\right) \qquad (5.1.33)$$

and

$$\frac{\partial Q_{ab}}{\partial Y^-_{ij}} = \frac{1}{\sqrt{2}}\left(\sum_c \left(\frac{\partial Y^-_{ac}}{\partial Y^-_{ij}}\right) Q_{cb} + \sum_c Q_{ac}\left(\frac{\partial Y^-_{cb}}{\partial Y^-_{ij}}\right)\right). \qquad (5.1.34)$$

Equations (5.1.33) and (5.1.34) use the same property, which is

$$\frac{\partial Y^{+/-}_{\gamma\theta}}{\partial Y^{+/-}_{\gamma'\theta'}} = \delta_{\gamma\gamma'}\delta_{\theta'\theta} \qquad \frac{\partial Y^{+/-}_{\gamma\theta}}{\partial Y^{-/+}_{\gamma'\theta'}} = 0. \qquad (5.1.35)$$

If we use the definitions of equation (5.1.33) in equations (5.1.34) and (5.1.35) it then follows that

$$\frac{\partial Q_{ab}}{\partial Y^{+}_{ij}} = \frac{1}{\sqrt{2}} \left(\sum_c \delta_{ai}\delta_{jc}Q_{cb} - \sum_c Q_{ac}\delta_{ci}\delta_{jb} \right) = \frac{1}{\sqrt{2}} \left(\delta_{ai}Q_{jb} - Q_{ai}\delta_{jb} \right) \quad (5.1.36)$$

and

$$\frac{\partial Q_{ab}}{\partial Y^{+}_{ij}} = \frac{1}{\sqrt{2}} \left(\sum_c \delta_{ai}\delta_{jc}Q_{cb} - \sum_c Q_{ac}\delta_{ci}\delta_{jb} \right) = \frac{1}{\sqrt{2}} \left(\delta_{ai}Q_{jb} - Q_{ai}\delta_{jb} \right). \quad (5.1.37)$$

With this information, equations (5.1.36) and (5.1.37), we can rewrite the Laplacian from the second parameterization using our new variable Q. From this Laplacian we only consider terms that have $i < j$

$$
\begin{aligned}
\nabla^2_{II:i<j} &= \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\frac{\partial}{\partial Y^-_{ij}}\frac{\partial}{(\partial Y^-_{ij})^*}+\sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\frac{\partial}{\partial Y^+_{ij}}\frac{\partial}{(\partial Y^+_{ij})^*}\\
&= -\sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\frac{\partial}{\partial Y^-_{ij}}\frac{\partial}{\partial Y^-_{ji}}-\sum_{i\neq j}\frac{2}{(r_i-r_j)^2}\frac{\partial}{\partial Y^+_{ij}}\frac{\partial}{\partial Y^+_{ji}}\\
&= -\sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\sum_{a,b}\frac{\partial Q_{ab}}{\partial Y^-_{ij}}\frac{\partial}{\partial Q_{ab}}\sum_{c,d}\frac{\partial Q_{cd}}{\partial Y^-_{ji}}\frac{\partial}{\partial Q_{cd}}\\
&- \sum_{i\neq j}\frac{2}{(r_i-r_j)^2}\sum_{a,b}\frac{\partial Q_{ab}}{\partial Y^+_{ij}}\frac{\partial}{\partial Q_{ab}}\sum_{c,d}\frac{\partial Q_{cd}}{\partial Y^+_{ji}}\frac{\partial}{\partial Q_{cd}}\\
&= -\sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\sum_{a,b}\frac{1}{\sqrt{2}}\left(\delta_{ai}Q_{jb}+Q_{ai}\delta_{jb}\right)\times\\
&\quad \frac{\partial}{\partial Q_{ab}}\sum_{c,d}\frac{1}{\sqrt{2}}\left(\delta_{cj}Q_{id}+Q_{cj}\delta_{di}\right)\frac{\partial}{\partial Q_{cd}}\\
&- \sum_{i\neq j}\frac{2}{(r_i-r_j)^2}\sum_{a,b}\frac{1}{\sqrt{2}}\left(\delta_{ai}Q_{jb}-Q_{ai}\delta_{jb}\right)\times\\
&\quad \frac{\partial}{\partial Q_{ab}}\sum_{c,d}\frac{1}{\sqrt{2}}\left(\delta_{cj}Q_{id}-Q_{cj}\delta_{id}\right)\frac{\partial}{\partial Q_{cd}}\\
&= -\sum_{i\neq j}\frac{1}{(r_i+r_2)^2}\left(\sum_b Q_{jb}\frac{\partial}{\partial Q_{ib}}+\sum_a Q_{ai}\frac{\partial}{\partial Q_{aj}}\right)\times\\
&\quad \left(\sum_d Q_{id}\frac{\partial}{\partial Q_{jd}}+\sum_c Q_{cj}\frac{\partial}{\partial Q_{ci}}\right)\\
&- \sum_{i\neq j}\frac{1}{(r_i-r_2)^2}\left(\sum_b Q_{jb}\frac{\partial}{\partial Q_{ib}}-\sum_a Q_{ai}\frac{\partial}{\partial Q_{aj}}\right)\times\\
&\quad \left(\sum_d Q_{id}\frac{\partial}{\partial Q_{jd}}-\sum_c Q_{cj}\frac{\partial}{\partial Q_{ci}}\right)\\
&= -\sum_{i\neq j}\frac{1}{(r_i+r_2)^2}\left(\sum_a Q_{ja}\frac{\partial}{\partial Q_{ia}}+\sum_a Q_{ai}\frac{\partial}{\partial Q_{aj}}\right)\times\\
&\quad \left(\sum_b Q_{ib}\frac{\partial}{\partial Q_{jb}}+\sum_b Q_{bj}\frac{\partial}{\partial Q_{bi}}\right)\\
&- \sum_{i\neq j}\frac{1}{(r_i-r_2)^2}\left(\sum_a Q_{ja}\frac{\partial}{\partial Q_{ia}}-\sum_a Q_{ai}\frac{\partial}{\partial Q_{aj}}\right)\times\\
&\quad \left(\sum_b Q_{ib}\frac{\partial}{\partial Q_{jb}}-\sum_b Q_{bj}\frac{\partial}{\partial Q_{bi}}\right). \qquad (5.1.38)
\end{aligned}
$$

In equation (5.1.38), after multiplying out terms and regrouping we get the following

$$\begin{aligned}
\nabla^2_{II:i<j} &= -\sum_{i\neq j}\left(\frac{1}{(r_i+r_j)^2}+\frac{1}{(r_i-r_j)^2}\right)\sum_a\sum_b Q_{ja}\frac{\partial}{\partial Q_{ia}}Q_{ib}\frac{\partial}{\partial Q_{jb}} \\
&- \sum_{i\neq j}\left(\frac{1}{(r_i+r_j)^2}+\frac{1}{(r_i-r_j)^2}\right)\sum_a\sum_b Q_{ai}\frac{\partial}{\partial Q_{aj}}Q_{bj}\frac{\partial}{\partial Q_{bi}} \\
&- \sum_{i\neq j}\left(\frac{1}{(r_i+r_j)^2}-\frac{1}{(r_i-r_j)^2}\right)\sum_a\sum_b Q_{ja}\frac{\partial}{\partial Q_{ia}}Q_{bj}\frac{\partial}{\partial Q_{bi}} \\
&- \sum_{i\neq j}\left(\frac{1}{(r_i+r_j)^2}-\frac{1}{(r_i-r_j)^2}\right)\sum_a\sum_b Q_{ai}\frac{\partial}{\partial Q_{aj}}Q_{ib}\frac{\partial}{\partial Q_{jb}}.
\end{aligned} \tag{5.1.39}$$

In equation (5.1.39) we can include all the other terms i.e. terms that have $i = j$, but first there is the pure imaginary term that we would prefer to specify in a similar fashion as the other terms that involve the variable Q, that is

$$\begin{aligned}
\frac{1}{2r^2}\frac{\partial}{\partial Y^-_{ii}}\frac{\partial}{\partial Y^-_{ii}} &= \frac{1}{4}\frac{1}{r^2}\left(\sum_a Q_{ai}\frac{\partial}{\partial Q_{aj}}+\sum_b Q_{jb}\frac{\partial}{\partial Q_{ib}}\right)\times \\
&\left(\sum_b Q_{jb}\frac{\partial}{\partial Q_{ib}}+\sum_a Q_{ai}\frac{\partial}{\partial Q_{aj}}\right).
\end{aligned} \tag{5.1.40}$$

Again at this point we prefer to introduce a more compact notation to represent the left, E^L, and right, E^R multiplication of the variables given in equation (5.1.27). From all the above information and using all our calculations for the terms that have $i = j$ and those that have $i \neq j$ and making the required substitution, the complete expression for our Laplacian in the second parameterization is thus

$$\begin{aligned}
\nabla^2_{II} &= \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta^2_{MR}}\left\{\frac{\partial}{\partial r_i}\Delta^2_{MR}\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \qquad (5.1.41)\\
&- \frac{1}{2r_i^2}\frac{\partial}{\partial Y^-_{ii}}\frac{\partial}{\partial Y^-_{ii}} - \frac{2}{(r_i-r_j)^2}\frac{\partial}{\partial Y^+_{ij}}\frac{\partial}{\partial Y^+_{ji}} - \frac{2}{(r_i+r_j)^2}\frac{\partial}{\partial Y^-_{ij}}\frac{\partial}{\partial Y^-_{ji}}\\
&= \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta^2_{MR}}\left\{\frac{\partial}{\partial r_i}\Delta^2_{MR}\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}\\
&- \frac{1}{4r_i^2}(E^R_{ii}+E^L_{ii})(E^R_{ii}+E^L_{ii})\\
&- \sum_{i\neq j}\left(\frac{1}{(r_i+r_j)^2}+\frac{1}{(r_i-r_j)^2}\right)\{E^L_{ji}E^L_{ij}+E^R_{ij}E^R_{ij}\}\\
&- \sum_{i\neq j}\left(\frac{1}{(r_i+r_j)^2}-\frac{1}{(r_i-r_j)^2}\right)\{E^L_{ji}E^R_{ji}+E^R_{ij}E^R_{ij}\}\\
\nabla^2_{II} &= \frac{1}{\prod_i r_i}\left\{\frac{\partial}{\partial r_i}\prod_i r_i\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta^2_{MR}}\left\{\frac{\partial}{\partial r_i}\Delta^2_{MR}\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i}\\
&- \frac{1}{4r_i^2}(E^R_{ii}+E^L_{ii})(E^R_{ii}+E^L_{ii})\\
&- \sum_{i\neq j}\left\{\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}(E^L_{ij}E^L_{ji}+E^R_{ij}E^R_{ji}) - \frac{4r_ir_j}{(r_i^2-r_j^2)^2}\left(E^L_{ij}E^R_{ji}+E^R_{ij}E^L_{ji}\right)\right\}.
\end{aligned}$$

From the Laplacian of equation (4.1.30) obtained using parameterization I we see that $\partial/\partial S_{ii}$ is absent, so in terms of our new invariant states we would require

$$0 = \frac{\partial}{\partial S_{ii}} = E^L_{ii} - E^R_{ii} \Rightarrow E^L_{ii} = E^R_{ii}. \qquad (5.1.42)$$

Similarly in equation (5.1.29) we see that $\partial/\partial Y^+_{ii}$ is absent, this enforces the condition

$$0 = \frac{\partial}{\partial Y^+_{ii}} = \frac{1}{\sqrt{2}}(E^L_{ii} - E^R_{ii}) \Rightarrow E^L_{ii} = E^R_{ii}. \qquad (5.1.43)$$

When the condition in equation (5.1.42) is used in equation (5.1.38) and the condition in equation (5.1.43) is used in equation (5.1.41), we find that

$$\nabla^2_I = \nabla^2_{II}. \qquad (5.1.44)$$

Section 5.1 served to illustrate the importance of introducing the new variables and how these arise naturally. In both our Laplacians represented in the new variables, Q and $Q^\dagger$, we saw their natural forms being preserved: the Laplacians were separated into their radial and angular parts and no mixing of operators occurred. We could have extended the new variables to rewriting the angular momentum $\hat{L}$, but this would not have contributed crucially to our project. Questions of what type of physical objects would emerge had our Laplacians from equation (5.1.44) acted on the newly defined invariant states would be saved for future research.

Chapter 6

Polar Matrix Coordinates: Fermionisation

6.1 Fermionic Picture

We notice that the Laplacians from equations (5.1.28) and (5.1.41) both have an identical radial piece, that only depends on the eigenvalues of R. This means that changing parameterizations only affects the angular piece of the Laplacians and the radial piece is independent of this change. The objective of this section is to observe how fermionisation manifests itself for our chosen two matrix model by considering the radial piece of both Laplacians. The radial part of the Laplacians is

$$\begin{aligned}\nabla^2_{radial} &= \frac{1}{\Delta^2_{MR}(r_i^2)}\frac{1}{\prod_k r_k}\sum_i \frac{\partial}{\partial r_i}(\prod_k r_k)\Delta^2_{MR}(r_i^2)\frac{\partial}{\partial r_i} \\ &= \frac{1}{\Delta^2_{MR}}\sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i} r_i \Delta^2_{MR}(r_i^2)\frac{\partial}{\partial r_i}.\end{aligned} \tag{6.1.1}$$

In the second line of equation (6.1.1), we introduce the variable $\rho_i = r_i^2$, then we find

$$\nabla^2_{radial} = \frac{4}{\Delta^2(\rho_i)}\sum_i \frac{\partial}{\partial \rho_i}\rho_i \Delta^2(\rho_i)\frac{\partial}{\partial \rho_i}, \tag{6.1.2}$$

since

$$\Delta^2(\rho_i) = \Delta^2_{MR}(r_i^2). \tag{6.1.3}$$

Equation (6.1.3) was previously observed to be the modified Vandermonde determinant given by equation (4.1.25).

To start off, we consider the following equation

$$\begin{aligned}
&\frac{4}{\Delta(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\Delta(\rho_i)\rho_i\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)} = \\
&4\sum_i\left(\frac{1}{\Delta(\rho_i)}\frac{\partial}{\partial\rho_i}\Delta(\rho_i)\right)\rho_i\left(\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)}\right) \\
= \; & 4\sum_i\left(\frac{\partial}{\partial\rho_i}+\sum_{k\neq i}\frac{1}{\rho_i-\rho_k}\right)\rho_i\left(\frac{\partial}{\partial\rho_i}-\sum_{j\neq i}\frac{1}{\rho_i-\rho_j}\right) \\
= \; & 4\left\{\sum_i\left(\frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i}-\sum_{j\neq i}\frac{1}{\rho_i-\rho_j}+\sum_{j\neq i}\frac{\rho_i}{(\rho_i-\rho_j)^2}\right)\right\} \\
+ \; & 4\sum_i\left(\sum_{k\neq i}\frac{1}{\rho_i-\rho_k}\rho_i\frac{\partial}{\partial\rho_i}-\rho_i\sum_{j\neq i}\frac{1}{\rho_i-\rho_j}\rho_i\frac{\partial}{\partial\rho_i}\right) \\
- \; & 4\sum_i\left(\sum_{j\neq i,k\neq i}\frac{\rho_i}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right).
\end{aligned} \tag{6.1.4}$$

To simplify equation (6.1.4) we note that

$$\sum_{i\neq j}\frac{1}{\rho_i-\rho_j} = 0, \tag{6.1.5}$$

this then leaves us with

$$\begin{aligned}
&\frac{4}{\Delta(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\Delta(\rho_i)\rho_i\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)} = \\
&4\left\{\sum_i\frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i}+\sum_{i\neq j}\frac{\rho_i}{(\rho_i-\rho_j)^2}-\sum_{j\neq i\neq k}\frac{\rho_i}{(\rho_i-\rho_j)(\rho_i-\rho_k)}\right\}.
\end{aligned} \tag{6.1.6}$$

We now want to show in equation (6.1.6) that the difference of the second and third term is effectively zero. We note that

$$\sum_{i\neq j}\frac{\rho_i}{(\rho_i-\rho_j)^2}-\sum_{i\neq j\neq k}\frac{\rho_i}{(\rho_i-\rho_k)(\rho_i-\rho_j)}=\sum_{i\neq j\neq k}\frac{\rho_i}{(\rho_i-\rho_k)(\rho_i-\rho_j)}. \tag{6.1.7}$$

Equation (6.1.7) can be shown to vanish identically by considering any three eigenvalues ρ_1, ρ_2, ρ_3. We have

$$\begin{aligned}
\frac{\rho_1}{(\rho_1-\rho_2)(\rho_1-\rho_3)} + \frac{\rho_2}{(\rho_2-\rho_1)(\rho_2-\rho_3)} + \frac{\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)} &= \\
\frac{1}{(\rho_1-\rho_2)}\left(\frac{\rho_1}{(\rho_1-\rho_3)} - \frac{\rho_2}{(\rho_2-\rho_3)}\right) + \frac{\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)} &= \\
\frac{1}{(\rho_1-\rho_2)}\left(\frac{\rho_1\rho_2-\rho_1\rho_3-\rho_2\rho_1+\rho_2\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)}\right) + \frac{\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)} &= \\
\frac{1}{(\rho_1-\rho_2)}\frac{\rho_3(\rho_2-\rho_1)}{(\rho_1-\rho_3)(\rho_2-\rho_3)} + \frac{\rho_3}{(\rho_1-\rho_3)(\rho_2-\rho_3)} &= \\
-\frac{\rho_3}{(\rho_1-\rho_3)(\rho_2-\rho_3)} + \frac{\rho_3}{(\rho_1-\rho_3)(\rho_2-\rho_3)} &= 0
\end{aligned} \tag{6.1.8}$$

For the symmetric wavefunction ϕ the following can be deduced

$$\begin{aligned}
\frac{4}{\Delta^2(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\rho_i\Delta^2(\rho_i)\frac{\partial}{\partial\rho_i}\phi &= E\phi \\
\Rightarrow -4\left(\sum_i \xi_i\right)\Psi &= E\Psi,
\end{aligned} \tag{6.1.9}$$

where

$$\xi_i = \frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i} + \mathrm{tr}\left(K(\rho_i)\right). \tag{6.1.10}$$

The second line of equation (6.1.9) is true if

$$\Psi \equiv \Delta(\rho_i)\phi, \tag{6.1.11}$$

where $\Delta(\rho_i)$ is the Vandermonde determinant and ϕ is a one dimensional symmetric wavefunction that depends on the eigenvalues of R.

The framework used in the fermionisation picture for the polar coordinate Z is a framework that is analogous to the single matrix model fermion picture. Using equation (6.1.11) to rewrite equation (6.1.8) gives us equation (6.1.4). The wavefunction Ψ, which was obtained by construct, is antisymmetric and only depends on the eigenvalues r_i. For the eigenvalue problem in equation (6.1.10), we see ξ_i which represents N Hamiltonians for free non-relativistic fermions moving in the potential $K(r_i)$. Thus equation (6.1.9) is the Schödinger equation for N non-interacting $2+1$ dimensional non-relativistic fermions.

This means that for our project we were also able to construct a fermion picture using the polar coordinate Z. The next natural question would be: Can the fermionisation picture be extended to matrix models with more than two matrices and how would the ground state energies differ?

Chapter 7

The Integral In Polar Coordinates

7.1 Computing The Two Matrix Model Integral

In this chapter we discuss the path integral of the two hermitian matrices in terms of our polar parameterization.

The case of the two matrix model has been considered by the authors of [3], who study correlators and the free energy at both the strong and weak coupling limits. This work uses an auxiliary field in a supersymmetric setting to obtain a determinant expression in terms of

$$\phi = \frac{1}{\sqrt{2}}\left(X_3 + iX_4\right), \tag{7.1.1}$$

where X_3 and X_4 are two of the four bosonic matrices.

The result of [3] is used by [6] to study local emergent geometry. The authors of [6] consider a two hermitian matrix model where one of the matrices is integrated out exactly. The local emergent geometry in [6] is due to the eigenvalue distribution density of one of the matrices. The eigenvalue density function, determined by the saddle point equations in the large N limit (strong coupling limit), is considered in the continuum limit. The authors of [6] establish that in the strong coupling limit, the two-matrix model can be interpreted as emergent two dimensional geometry and further show the commuting and non-commuting properties of the hermitian matrices at strong coupling. At strong coupling, this eigenvalue density function maps out a hemisphere distribution which is interpreted as an emergent local geometry of the two-matrix model. The authors of [6] have argued that matrix models with more than two matrices i.e. bosonic and fermionic matrix models, show no emergent local geometry. When the authors of [6] include supersymmetry in the matrix models (matrix models with more than two matrices), it is argued that the emergence of geometry does not appear convincingly and that this geometry is not clearly defined.

Using our definition of the radial matrix coordinates, our objective for this section would be to compute the saddle point equation, and make comments on the two matrix model at the weak coupling limit as this would allow us to make comments about our method used to represent the matrix model.

We start by defining the integral of the two matrix model, using our newly defined matrix polar coordinates. The integral is

$$B = \int \mathrm{d}Z\mathrm{d}Z^{\dagger} e^{-\frac{\omega^2}{2}\mathrm{tr}\left(ZZ^{\dagger}\right)-g_{YM}^2\mathrm{tr}\left(\left[Z,Z^{\dagger}\right]^2\right)} \tag{7.1.2}$$

where $Z = RU$ and naturally $Z^{\dagger} = U^{\dagger}R$, as previously defined, U is the unitary matrix and R is an hermitian matrix.

Substituting our definitions into the integral, we get

$$B = \int \mathrm{d}Z\mathrm{d}Z^{\dagger} e^{-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)-2g_{YM}^2\mathrm{tr}\left(R^4\right)+2g_{YM}^2\mathrm{tr}\left(R^2UR^2U^{\dagger}\right)} \equiv \int \mathrm{d}Z\mathrm{d}Z^{\dagger} e^{-V(Z,Z^{\dagger})}. \tag{7.1.3}$$

The potential in the integral above is

$$V(Z,Z^{\dagger}) = \frac{\omega^2}{2}\mathrm{tr}\left(R^2\right) + 2g_{YM}^2\mathrm{tr}\left(R^4\right) - 2g_{YM}^2\mathrm{tr}\left(R^2UR^2U^{\dagger}\right).$$

The expression $\mathrm{tr}\left(\mathrm{d}Z\mathrm{d}Z^{\dagger}\right)$, was previously shown to be proportional with the variables $\mathrm{d}S, \mathrm{d}r$ and $\mathrm{d}X$ also $\mathrm{tr}\left(\mathrm{d}Z\mathrm{d}Z^{\dagger}\right)$ was defined as the square of the line element. It then naturally follows that

$$\begin{aligned}
B &= \int \mathrm{d}r\mathrm{d}S\mathrm{d}X \prod_i r_i \Delta_{MR}^2 e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)-2g_{YM}^2\mathrm{tr}\left(R^4\right)+2g_{YM}^2\mathrm{tr}\left(R^2UR^2U^\dagger\right)\right)} \\
&= \frac{1}{4}\int \mathrm{d}r\mathrm{d}S\mathrm{d}X \prod_i r_i \prod_{i<j}(r_i^2-r_j^2)^2 \times \\
& \quad e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)-2g_{YM}^2\mathrm{tr}\left(R^4\right)+2g_{YM}^2\mathrm{tr}\left(R^2UR^2U^\dagger\right)\right)}, \qquad (7.1.4)
\end{aligned}$$

where $\mathrm{d}S = \mathrm{d}VV^\dagger$ and $\mathrm{d}X = V\mathrm{d}UU^\dagger V^\dagger$ and we can see the Jacobian J which is the Vandermonde determinant multiplied by a product of eigenvalues r_i

$$J = \prod_i r_i \Delta_{MR}^2.$$

We proceed further to substitute the definitions of $\mathrm{d}r$, $\mathrm{d}S$ and $\mathrm{d}X$ into equation (7.1.4), we then get

$$\begin{aligned}
B &= \frac{1}{4}\int \mathrm{d}r(\mathrm{d}VV^\dagger)(V\mathrm{d}UU^\dagger V^\dagger) \prod_i r_i \prod_{i<j}(r_i^2-r_j^2)^2 \\
&\times \quad e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)-2g_{YM}^2\mathrm{tr}\left(R^4\right)+2g_{YM}^2\mathrm{tr}\left(R^2UR^2U^\dagger\right)\right)}. \qquad (7.1.5)
\end{aligned}$$

In equation (7.1.5), the measure is invariant under the action of matrices in the adjoint representation, such that

$$V(\mathrm{d}UU^\dagger)V^\dagger \longrightarrow \mathrm{d}UU^\dagger, \qquad (7.1.6)$$

which then gives

$$B = \frac{1}{4}\int \mathrm{d}r(\mathrm{d}VV^\dagger)(\mathrm{d}UU^\dagger) \prod_i r_i \prod_{i<j}(r_i^2-r_j^2)^2 e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)-2g_{YM}^2\mathrm{tr}\left(R^4\right)+2g_{YM}^2\mathrm{tr}\left(R^2UR^2U^\dagger\right)\right)}.$$

Again in equation (7.1.5), the integral is invariant in the measure such that

$$\mathrm{d}UU^\dagger \longrightarrow \mathrm{d}U. \qquad (7.1.7)$$

The invariance of the measure above in equation (7.1.7) is explained below:

$$\mathrm{d}F_{ab} = \mathrm{d}U_{ad}U^{\dagger}_{db} = U^{\dagger}_{db}\mathrm{d}U_{ad} = \left(U^{\dagger}_{db}\delta_{ac}\right)\mathrm{d}U_{cd} = M_{ab,cd}\mathrm{d}U_{cd}$$

where

$$M_{ab,cd} \equiv \delta_{ac}U^{\dagger}_{db}. \tag{7.1.8}$$

$M_{ab,cd}$ is an $N^2 \times N^2$ matrix whose entries are the $U^{\dagger}$ matrices on the main diagonal and zero everywhere else. Under the change of variables in equation (7.1.7), the Jacobian $\det(\partial F/\partial U^{\dagger})$ is a diagonal matrix whose entries are a product of $U^{\dagger}$ matrices. Since $U^{\dagger}$ is a unitary matrix, it means that the Jacobian is one.

The integral B now takes the following form

$$\begin{aligned} B &= \frac{1}{4}\int \mathrm{d}r(\mathrm{d}VV^{\dagger})\prod_i r_i \prod_{i<j}(r_i^2 - r_j^2)^2 e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)\right)} \times \\ &\quad \int \mathrm{d}UU^{\dagger}e^{\left(g_{YM}^2\mathrm{tr}\left((R^2-UR^2U^{\dagger})^2\right)\right)} \\ &= \frac{1}{4}\int \mathrm{d}r(\mathrm{d}VV^{\dagger})\prod_i r_i \prod_{i<j}(r_i^2 - r_j^2)^2 e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)\right)} \times \\ &\quad \int \mathrm{d}Ue^{\left(g_{YM}^2\mathrm{tr}\left((R^2-UR^2U^{\dagger})^2\right)\right)} \\ &= \frac{const}{4g_{YM}^2}\int \mathrm{d}r(\mathrm{d}V)\prod_i r_i \prod_{i<j}(r_i^2 - r_j^2)^2 e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(r^2\right)\right)} \times \\ &\quad \det\left(e^{(g_{YM}^2(r_i^2-r_j^2)^2)}\right)\left[\prod_{i<j}(r_i^2 - r_j^2)\right]^{(-2)}. \end{aligned} \tag{7.1.9}$$

In the second line equation (7.1.9), up to a constant factor, we see the Harish Chandra-Itzykson Zuber integral [41] where $\mathrm{d}U$ is the normalized Haar measure on the unitary group $U(N)$,

$$\int \mathrm{d}Ue^{\left(g_{YM}^2 tr(R^2-UR^2U^{\dagger})^2\right)} = \det\left(e^{(g_{YM}^2(r_i^2-r_j^2)^2)}\right)\left[\prod_{i<j}(r_i^2 - r_j^2)\right]^{(-2)}. \tag{7.1.10}$$

The last line of equation (7.1.9) shows the differential dV which does not depend on any of the other variables, thus it can be integrated out, to give unity i.e.

$$\int \mathrm{d}V = 1,$$

and for the *const* factor we have

$$const = (g_{YM}^2)^{-N(N-1)/2} \prod_{1}^{N-1} p!. \tag{7.1.11}$$

The final form of the two matrix integral in polar coordinates thus becomes

$$B = \frac{const}{16 g_{YM}^2} \int \mathrm{d}r \prod_i r_i e^{-\frac{\omega^2}{2}\mathrm{tr}\left(r^2\right) + \ln\det\left(e^{g_{YM}^2 (r_i^2 - r_j^2)^2}\right)}. \tag{7.1.12}$$

7.2 The Free Case

For the free case, we set the coupling constant $g_{YM}^2 = 0$, thus equation (7.1.2) becomes

$$B = \int \mathrm{d}Z \mathrm{d}Z^\dagger e^{-\mathrm{tr}\left(ZZ^\dagger\right)}. \tag{7.2.1}$$

When solved, the integral B, in equation (7.2.1), in this free limit becomes

$$B = \frac{1}{4} \int \mathrm{d}r \prod_i r_i \prod_{i<j} (r_i^2 - r_j^2)^2 e^{-\frac{\omega^2}{2}\mathrm{tr}\left(r^2\right)}. \tag{7.2.2}$$

We set $\rho_i = r_i^2$, this implies that

$$B = \frac{1}{8} \int \prod_i d\rho_i \prod_{i<j} (\rho_i - \rho_j)^2 e^{-\frac{\omega^2}{2}\sum_i \rho_i}. \tag{7.2.3}$$

Equation (7.2.3) is rewritten in a form that will allow us to compute the saddle point equation such that

$$B = \frac{1}{8} \int \prod_i \mathrm{d}\rho_i e^{\sum_{i\neq j} \ln(\rho_i - \rho_j) - \frac{1}{2}\omega^2 \sum_i \rho_i}. \tag{7.2.4}$$

The above result is used to compute the saddle point equation, which is given by [3]

$$\frac{\partial}{\partial \rho_k}\left(\sum_{i\neq j}\ln(\rho_i-\rho_j)^2-\frac{1}{2}\omega^2\sum_i \rho_i\right)=0. \tag{7.2.5}$$

Proceeding from equation (7.2.5) above we get

$$\begin{aligned}
\sum_{i\neq i}\frac{\partial}{\partial\rho_k}\ln(\rho_i-\rho_j)-\frac{1}{2}\omega^2\sum_i\frac{\partial}{\partial\rho_k}\rho_i &= 0\\
\sum_{i\neq j}\left(\frac{1}{\rho_i-\rho_j}\left(\frac{\partial\rho_i}{\partial\rho_k}-\frac{\partial\rho_j}{\partial\rho_k}\right)\right)-\frac{1}{2}\omega^2\sum_i\frac{\partial\rho_i}{\partial\rho_k} &= 0\\
\sum_{i\neq j}\left(\frac{1}{\rho_i-\rho_j}(\delta_{ik}-\delta_{jk})\right)-\frac{1}{2}\omega^2\sum_i\delta_{ik} &= 0\\
\sum_{k\neq j}\frac{1}{\rho_k-\rho_j}+\sum_{k\neq j}\frac{1}{\rho_k-\rho_j}-\frac{1}{2}\omega^2 &= 0.
\end{aligned}$$

Therefore

$$\omega^2=4\sum_k\frac{1}{\rho_k-\rho_j}, \tag{7.2.6}$$

summing over all k except $k=j$ only. For now we will not comment on the result of equation (7.2.6), we will return to it later. Instead, we will first discuss a particular solution of the single matrix model briefly in order for us to understand the significance of equation (7.2.6).

The standard free single matrix model is a model that gives the Wigner distribution, a solution that we are also interested in. For the single matrix we have the following

$$A=\int \mathrm{d}X e^{-\frac{1}{2}\omega^2\mathrm{tr}\left(X^2\right)}=\int \mathrm{d}\lambda_i\prod_{i<j}(\lambda_i-\lambda_j)^2 e^{-\frac{1}{2}\omega^2\sum_i\lambda_i^2}. \tag{7.2.7}$$

Equation (7.2.7) is rewritten as

$$A=\int \mathrm{d}\lambda_i e^{\sum_{i\neq j}\ln(\lambda_i-\lambda_j)-\frac{1}{2}\omega^2\sum_i\lambda_i^2}. \tag{7.2.8}$$

[3]The large N scaling can be explicitly included by rescaling $\rho_i\to N\rho_i$ and $\sum_i\to N\sum_i$

The first logarithm appearing in exponential term of equation (7.2.8) is the Vandermonde determinant that arises when digonalising X. The saddle point equation for the single matrix model in equation (7.2.8) is

$$\frac{\partial}{\partial\lambda_k}\left(\sum_{i<j}\ln(\lambda_i-\lambda_j)-\frac{1}{2}\omega^2\sum_i\lambda_i^2\right)=0 \tag{7.2.9}$$

$$\sum_{i\neq j}\frac{\partial}{\partial\lambda_k}\ln(\lambda_i-\lambda_j)-\frac{1}{2}\omega^2\sum_i\frac{\partial\lambda_i^2}{\partial\lambda_k}=0. \tag{7.2.10}$$

Equation (7.2.10) becomes

$$2\sum_{j\neq k}\frac{1}{\lambda_k-\lambda_j}=\omega^2\lambda_k. \tag{7.2.11}$$

In equation (7.2.11) the standard solution, taken in the large N limit gives the Wigner distribution [34] [40], and this solution is given by

$$\int \mathrm{d}y\frac{\phi(y)}{x-y}=\frac{\omega^2}{2}x. \tag{7.2.12}$$

We seem to see the emergence of a similar type of solution for equation (7.2.6) in our work. Previously we obtained the following equation for the saddle point equation (7.2.6)

$$2\sum_{j\neq k}\frac{1}{\rho_k-\rho_j}=\frac{1}{2}\omega^2. \tag{7.2.13}$$

If we go back to our old notation, to rewrite the above expression, we get

$$2\sum_{j\neq k}\frac{1}{r_k^2-r_j^2}=\frac{1}{2}\omega^2. \tag{7.2.14}$$

We then multiply r_k on either side of the equation to obtain

$$2\sum_{j\neq k}\frac{r_k}{r_k^2-r_j^2}=\frac{1}{2}\omega^2 r_k. \tag{7.2.15}$$

Previously we defined $\rho_i = r_i^2$, this then enforces the requirement that $\rho_i > 0$ for $\pm r_i$. From equation (7.2.15), we can rewrite the left hand side as follows

$$\frac{2r_k}{r_k^2 - r_j^2} = \frac{(r_k + r_j) + (r_k - r_j)}{(r_k - r_j)(r_k + r_j)} = \frac{1}{r_k - r_j} + \frac{1}{r_k + r_j}. \tag{7.2.16}$$

So from equation (7.2.16) above we have

$$2\sum_{j\neq k} \frac{r_k}{r_k^2 - r_j^2} = \sum_{j\neq k} \left(\frac{1}{r_k - r_j} + \frac{1}{r_k + r_j} \right) = \frac{1}{2}\omega^2 r_k, \tag{7.2.17}$$

which is true for $r_j > 0$ and $r_j < 0$, we can define

$$\sum_{r_j} \frac{1}{r_k - r_j} = \frac{1}{2}\omega^2 r_k, \tag{7.2.18}$$

for $r_k \neq r_j$.

The above form of equation (7.2.18) resembles the solution for the single matrix model in equation(7.2.11), which gave the Wigner distribution. Equation (7.2.18) is true for $r_j > 0$ and $r_j < 0$ and is thus seen to be an extension of the solution seen for the single matrix model. Due the condition $\rho_i > 0$, this requires that our solution in equation (7.2.18) include both positive and negative eigenvalues and this inclusion of entire spectrum of eigenvalues is what makes our solution unique from that of the single matrix model.

7.3 Two Matrices: Perturbation Theory

We now continue to explore perturbation theory in our (polar coordinate) matrix model. The objective of this section is to show that the matrix model that we have defined agrees in the large N limit with standard perturbation theory.

To start off, we consider equation (7.1.9) in our work, where in the second line we saw the integral over unitary matrices which we denote by ζ:

$$\zeta = \int \mathrm{d}U e^{-2g_{YM}^2 \mathrm{tr}\left(R^2 U R^2 U^\dagger\right)}. \tag{7.3.1}$$

Equation (7.3.1) is identical to the expression of equation (3.2) of [41], which is

$$I(M_1, M_2) = \int dU \exp[\beta \mathrm{tr}\left(M_1 U M_2 U^\dagger\right)]. \tag{7.3.2}$$

If Λ_1 is the diagonal matrix of eigenvalues obtained when diagonalizing the hermitian matrix M_1 and Λ_2 is the diagonal matrix of eigenvalues obtained when diagonalizing the hermitian matrix M_2 then [41] shows the following equivalence:

$$\begin{aligned} \int dU \exp\left[-\frac{1}{2t}\mathrm{tr}\left(\Lambda_1 - U\Lambda_2 U^\dagger\right)^2\right] &= t^{N(N-1)/2}\prod_N^1 p! \times \\ &\quad \frac{\det[\exp - (1/2t)(\lambda_{1,i} - \lambda_{2,j})^2]}{\Delta(\Lambda_1)\Delta(\Lambda_2)} \\ &= \beta^{-N(N-1)/2}\prod_1^{N-1} p! \frac{\det(e^{\beta\lambda_{1,i}\lambda_{2,j}})}{\Delta(\Lambda_1)\Delta(\Lambda_2)} \\ &= I(M_1, M_2). \end{aligned} \tag{7.3.3}$$

On the right hand side of equation (7.3.3), above $\lambda_{1,i}$ and $\lambda_{2,j}$ are elements of the matrices Λ_1 and Λ_2. In the second line of equation (7.3.3), β is a coupling constant and so is t.

In [41], the expansion of $\ln[I(M_1, M_2)]$ is considered in the large N limit, and the eigenvalues of each of the matrices M_1 and M_2 are to be rescaled by $\sqrt{N}$. Let $\Lambda_1 = \sqrt{N}a$ and $\Lambda_2 = \sqrt{N}b$ with Λ_1 and Λ_2 being diagonal matrices of order unity, the authors of [41] consider the following expansion

$$\begin{aligned} X(\Lambda_1, \Lambda_2, \beta) &= \lim_{N\to\infty} \frac{1}{N^2} \ln\left[I(\sqrt{N}a, \sqrt{N}b)\right] \\ &= \lim_{N\to\infty} \frac{1}{N^2} \ln\left[\int dU e^{N\beta \mathrm{tr}\left(\Lambda_1 U \Lambda_2 U^\dagger\right)}\right]. \end{aligned} \tag{7.3.4}$$

The quantity $X(\Lambda_1, \Lambda_2, \beta)$ is expressed as a power series expansion in β as

$$X(\Lambda_1, \Lambda_2, \beta) = \sum_1^\infty \frac{\beta^k}{k} X_k(\Lambda_1, \Lambda_2), \tag{7.3.5}$$

where $X_k(\Lambda_1, \Lambda_2) = X_k(\Lambda_2, \Lambda_1)$ is a symmetric function of a_i and b_i homogeneous of degree k. For our work we need to consider the first order expansion

($k = 1$) such that[4]

$$X(\Lambda_1, \Lambda_2, g_{YM}) = \beta X_{k=1} = \beta\langle\Lambda_1\rangle\langle\Lambda_2\rangle \tag{7.3.6}$$

where

$$\langle\Lambda_1^p\rangle \equiv \frac{1}{N}\mathrm{tr}\,(\Lambda_2^p) \tag{7.3.7}$$

and similarly

$$\langle\Lambda_2^p\rangle \equiv \frac{1}{N}\mathrm{tr}\,(\Lambda_2^p)\,. \tag{7.3.8}$$

Below we are going to make clear why and how the work/calculation of [41] is required and utilized in our project.

We start off from the second line of equation (7.1.9)

$$\begin{aligned}
B &= \frac{1}{16}\int \mathrm{d}r(\mathrm{d}VV^\dagger)\prod_i r_i \prod_{i<j}(r_i^2 - r_j^2)^2 e^{\left(-\frac{\omega^2}{2}\mathrm{tr}\left(R^2\right)\right)} \times \\
&\quad \int \mathrm{d}U e^{\left(g_{YM}^2 \mathrm{tr}\left((R^2 - UR^2U^\dagger)^2\right)\right)} \\
&= \frac{1}{16}\int \mathrm{d}r \prod_i r_i \prod_{i<j}(r_i^2 - r_j^2)^2 e^{-\frac{\omega^2}{2}\mathrm{tr}\left(r^2\right) + 2g_{YM}^2 \mathrm{tr}\left(r^4\right)} \times \\
&\quad \int \mathrm{d}U e^{-2g_{YM}^2 \mathrm{tr}\left(Ur^2U^\dagger r^2\right)} \\
&= \frac{1}{16}\int \mathrm{d}r \prod_i r_i \prod_{i<j}(r_i^2 - r_j^2)^2 e^{-\frac{\omega^2}{2}\mathrm{tr}\left(r^2\right) + 2g_{YM}^2 \mathrm{tr}\left(r^4\right)} \times \zeta
\end{aligned} \tag{7.3.9}$$

In the last line of equation (7.3.9), the integral over unitary matrices ζ is identical to equation (3.2) of [41], for $\beta = -2g_{YM}$. We rescale the eigenvalues r such that $r_i \to \sqrt{N} r_i$ and introduce a new variable $r_i^2 = \rho_i$. Equation (7.3.9) is then taken in the large N limit, with ζ being identical to equation (7.3.4) when taken in the large N limit. So it follows that equation (7.3.9) now becomes

[4] The expansion for higher orders for k can be seen in TABLE II. The coefficients $X_k(A, B)$ in [41] on page 417.

$$
\begin{aligned}
B &= \frac{1}{32}\int \mathrm{d}\rho_i \prod_{i<j}(\rho_i^2-\rho_j^2)^2 e^{-\frac{\omega^2}{2}\mathrm{tr}(\rho)+2g_{YM}^2\mathrm{tr}\left(\rho^2\right)} e^{\ln[\zeta]} \qquad (7.3.10)\\
&= \frac{1}{32}\int \mathrm{d}\rho_i e^{\sum_{i\neq j}\ln|\rho_i-\rho_j|-\frac{\omega^2}{2}\sum_i\rho_i+2g_{YM}^2\sum_i\rho_i^2} e^{\ln\zeta}\\
&= \frac{1}{32}\int \mathrm{d}\rho_i e^{\sum_{i\neq j}\ln|\rho_i-\rho_j|-\frac{\omega^2}{2}\sum_i\rho_i+2g_{YM}^2\sum_i\rho_i^2+X(r^2,r^2,\beta)}
\end{aligned}
$$

where

$$
\begin{aligned}
X(r^2,r^2,\beta) &= \lim_{N\to\infty}\frac{1}{N^2}\ln[\zeta] \qquad (7.3.11)\\
&= \lim_{N\to\infty}\frac{1}{N^2}\ln\left[\int \mathrm{d}U e^{-2g_{YM}^2N^2\mathrm{tr}\left(Ur^2U^\dagger r^2\right)}\right].
\end{aligned}
$$

We can see that equation (7.3.11) in our work is identical to equation (3.23) from [41]. We now want to consider the first order expansion of equation (7.3.11) which is the same equation (3.24) of [41], and this gives

$$
\begin{aligned}
X(r^2,r^2,\beta) &= \beta X_{k=1}(r^2,r^2) \qquad (7.3.12)\\
&= \frac{\beta}{N^2}\mathrm{tr}\left(r^2\right)\mathrm{tr}\left(r^2\right)\\
&= \frac{\beta}{N^2}\left(\sum_i\rho_i\right)^2.
\end{aligned}
$$

From equation (7.3.10), we now have

$$
B = \frac{1}{32}\int \mathrm{d}\rho_i e^{-S_{eff}}, \qquad (7.3.13)
$$

where the effective action S_{eff} given by

$$
S_{eff} = -\sum_{i\neq j}\ln|\rho_i-\rho_j| + \frac{\omega^2}{2}\sum_i\rho_i - 2g_{YM}^2N\sum_i\rho_i^2 - 2g_{YM}^2\left(\sum_i\rho_i\right)^2. \qquad (7.3.14)
$$

From the equation (7.3.14) above, the saddle point equation for the effective action is

$$\begin{aligned}\frac{\partial S_{eff}}{\partial \rho_i} &= \sum_{i\neq j}\frac{2}{\rho_i-\rho_j}-\frac{1}{2}\omega^2-4\lambda\rho_i+4\lambda\sum_k\rho_k=0 \\ \Rightarrow \sum_{j\neq i}\frac{2}{\rho_i-\rho_j} &= \frac{\omega^2}{2}+\lambda\rho_i-4\lambda\left(\sum_k\rho_k\right) \\ \Rightarrow \sum_{j\neq i}\frac{2}{r_i^2-r_j^2} &= \frac{\omega^2}{2}+\lambda r_i^2-4\lambda\left(\omega_2\right),\end{aligned} \tag{7.3.15}$$

where $\omega_2 = \sum_k \rho_k = \sum_k r_k^2$ and we have also introduced the t'Hooft coupling constant $\lambda = g_{YM}^2 N$. The last line of equation (7.3.15) is an equation of eigenvalues that gives the stationary condition.

Equation (7.3.15) may be solved in the large N limit but more importantly we can extend the definition of the density of eigenvalue $\eta(x)$ to include negative eigenvalues. In the free case given by equation (7.2.17), we see the extension of the eigenvalues which can also be written for equation (7.3.15). This allows us to introduce the density eigenvalues $\eta(x) = \sum_i \delta(r-r_i)$ with the condition $\eta(-x) = \eta(x)$. As it is standard we introduce an analytic function $F(z)$ for the last line of equation (7.3.15)

$$F(z) = \int \mathrm{d}y \frac{\eta(y)}{z-y}. \tag{7.3.16}$$

In the Free case of the previous section, two important results [5] emerge: the normalization of the eigenvalue density function

$$\int_{-\infty}^{\infty} \mathrm{d}y\eta(y) = 2 \tag{7.3.17}$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}y y^2\eta(y) = 2\omega_2. \tag{7.3.18}$$

For a more general formula for equations (7.3.17) and (7.3.18) [6], the following equation is true

[5] Clarity on the derivation of the proceeding results will be given in the appendix C.

[6] See appendix C for the derivation.

$$2\sum_i r_i^{2p} = \int_{-\infty}^{\infty} \mathrm{d}y y^{2p}\eta(y). \tag{7.3.19}$$

In equation (7.3.19), we see that we recover equation (7.3.17) for $p = 0$ and we recover equation (7.3.18) for $p = 1$. The generalization to higher order factors can be obtained and these become the coefficients in the expansion of equation (7.3.16). The analytic function $F(z)$ can be written as an expansion whose coefficient is $1/z^n$ for $n = 1, 2, 3, ...$ outside some finite support $(-a, a)$, such that

$$\begin{aligned} F(z) &= \int_{-a}^{a} \mathrm{d}y \frac{\eta(y)}{z-y} = \frac{1}{z}\int_{-a}^{a} \mathrm{d}y \frac{\eta(y)}{1-y/z} \\ &= \frac{1}{z}\int_{-a}^{a} \mathrm{d}y\eta(y) + \frac{1}{z^2}\int_{-a}^{a} \mathrm{d}yy\eta(y) + \frac{1}{z^3}\int_{-a}^{a} \mathrm{d}y^2 y\eta(y) + ... \\ &= \frac{2}{z} + \frac{2\omega_2}{z^3} + ... \end{aligned} \tag{7.3.20}$$

The analytic function $F(z)$ can also be written as a function that satisfies certain properties [7], allowing it to be constructed uniquely as

$$F(z) = \left(\frac{1}{2}\omega^2 - 4\lambda\omega_2\right) z + 4\lambda z^3 - \left(4\lambda z^2 + d\right)\sqrt{z^2 - a^2}. \tag{7.3.21}$$

By defining $\bar{\omega} = \omega^2 - 8\lambda\omega_2$, an expansion is also performed on equation (7.3.21), and the coefficients of $1/z$, $1/z^3$ etc are equated with those of equation (7.3.20), in turn, this allows us to solve for the unknown constants a and d in equation (7.3.21) and the following equations are obtained [8]

$$d = \frac{1}{2}\bar{\omega}^2 + 2a^2\lambda, \tag{7.3.22}$$

$$2 = \frac{1}{2}a^4\lambda + \frac{1}{2}a^2 d \tag{7.3.23}$$

and

[7] These properties are given for the single matrix model in [2]. We have generalized these properties to the two matrix model.

[8] See appendix C for derivation.

$$2\omega_2 = \frac{1}{4}a^6\lambda + \frac{1}{8}a^4 d. \tag{7.3.24}$$

Using algebra, and computing up to order λ, the following result can be obtained

$$\omega_2 = \frac{2}{\omega^2} - \frac{32}{\omega^6}\lambda. \tag{7.3.25}$$

The result in equation (7.3.25) is the perturbative weak coupling expansion for the two point correlator $\langle \mathrm{tr}\left(ZZ^\dagger\right)\rangle$ up to order λ which can be derived from the integral B in equation (7.1.2), using standard Feynman diagram perturbation expansion for the two point function $\langle \mathrm{tr}\,(X_1^2)\rangle$. We can deduce that our matrix valued spherical coordinates agrees (as illustrated above) with the the correct results of perturbation theory. It would be interesting to explore higher order correction of λ using perturbation theory on our matrix model with spherical polar coordinates and the strong coupling regime.

Chapter 8

Conclusion

In this dissertation we introduced matrix valued polar coordinates in the description of matrix models of two hermitian matrices and described properties of this parameterization in the context of quantum mechanical and path integral systems.

After the introductory chapter and a review of some of the properties of the single matrix model, Chapter four introduces the matrix valued polar coordinate. The Laplacian operator appearing in the kinetic energy term of the Hamiltonian has been obtained explicitly for two parameterizations which are described in this chapter. For both parameterizations, the Laplacians consists of a piece involving the eigenvalues of the "radial" matrix plus an angular part.

The form of these Laplacians on gauge invariant states was described in Chapter five, and were unique. For potentials that depend only on the radial eigenvalues, it was established in Chapter six that the corresponding Hamiltonian has a description in terms of higher dimensional $(2+1)$ fermions. This generalizes the well known fermionic picture of the single matrix model.

In chapter seven we studied the integral of the two matrix model in polar coordinates. When the free case ($g_{YM}^2 = 0$) of the integral was considered, its solution in the large N limit gave the standard Wigner distribution. We showed that the integral of the two matrix model gives the correct result of perturbation theory in the weak coupling expansion, equation (7.3.25), up to order λ. When looking at the result of equation (7.3.25), in the future we would like to do computations to higher orders of λ and see how consistent the result is with perturbation theory and to also investigate the strong coupling limit using our two matrix model in polar coordinates.

We found that the questions tackled in this project compelled us to ask further questions that might be of interest for future research. Some of the questions are as follows:

(1) Can we generalize the two matrix model, without any coupling between the angular and radial terms, to a more interesting three matrix model, which is QCD, to compute rich dynamics of the model.

(2) With the three and higher dimensional matrix models, can we still speak of "fermionisation", if this is so, this might hint in the direction of quantum gravity.

(3) With higher matrix models, is perturbation theory still a viable tool to obtain higher orders of λ, or would we need a new mathematical model to answer more complex questions? The higher the number matrices in a model, the complexity of computations also increases.

Numerous questions, as a result of the work done in this project, still need to be investigated. This project opens up a possible new approach to the study of matrix models and the rich discoveries that have yet to be made in string theory.

Appendix A

The Single Hermitian Matrix

A.1 Laplacian In Real Coordinates

The Laplacian for 2-dimensional real cartesian coordinates (x, y) is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{A.1.1}$$

If we define polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, then the Laplacian in polar coordinates becomes

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{A.1.2}$$

Even in real coordinates the Laplacian is separated into radial and angular parts.

A.2 Single Matrix Model: Conjugate Momentum

In this appendix we explain, in detail, how the equations (3.5.5) and (3.5.6) were derived. Below we show how to compute the co-efficients

$$\frac{\partial (r)_k}{\partial M_{ij}} = \sum_k V^{\dagger}_{ik} V_{kj} \tag{A.2.1}$$

for case I whose matrix elements lie on the main diagonal and for case II whose matrix elements are off-diagonal

$$\frac{\partial V_{k\gamma}}{\partial M_{ij}} = \sum_{k\neq q} \frac{V^{\dagger}_{jq} V_{q\gamma} V_{ki}}{(r)_k - (r)_q}. \tag{A.2.2}$$

The conjugate momentum in partial equations was previously defined as

$$P_{ji} = \frac{\partial}{\partial M_{ij}} = \sum_{k\gamma} \frac{\partial V_{k\gamma}}{\partial M_{ij}} \frac{\partial}{\partial V_{k\gamma}} + \sum_{k} \frac{\partial (r)_k}{\partial M_{ij}} \frac{\partial}{\partial (r)_k}. \tag{A.2.3}$$

To compute the co-efficients in equation (A.2.3) we start with the definition of with dM

$$\begin{aligned} \mathrm{d}M_{ij} &= \left\{ V^{\dagger} \left(\mathrm{d}r + [r, \mathrm{d}S] \right) V \right\}_{ij} \\ &= \sum_{k,q} V^{\dagger}_{ik} \left(\mathrm{d}r + [r, \mathrm{d}S] \right)_{kq} V_{qj} \\ &= \sum_{kq} V^{\dagger}_{ik} V_{qj} \left(\delta_{qk} (\mathrm{d}r)_k + [r, \mathrm{d}S]_{kq} \right). \end{aligned} \tag{A.2.4}$$

Previously we had shown how to compute commutators with index variables, thus $[r, \mathrm{d}S]_{kq} = \left((r)_k - (r)_q \right) \mathrm{d}S_{kq}$, with this, equation (A.2.4) becomes

$$\mathrm{d}M_{ij} = \sum_{kq} V^{\dagger}_{ik} V_{qj} \delta_{qk} (r)_k + \sum_{kq} V^{\dagger}_{ik} V_{qj} \mathrm{d}S_{kq} \left((r)_k - (r)_q \right). \tag{A.2.5}$$

To solve for the co-efficients in equation (A.2.3), we consider equation (A.2.5) for the first case: $q = k$

$$\Rightarrow \mathrm{d}M_{ij} = \sum_{kq} V^{\dagger}_{ik} V_{qj} (\mathrm{d}r)_k = \sum_{k} V^{\dagger}_{ik} (\mathrm{d}r)_k V_{kj}, \tag{A.2.6}$$

it then follows that the first co-efficient is given by

$$\frac{\partial (r)_k}{\partial M_{ij}} = \sum_{k} V^{\dagger}_{ik} V_{kj}. \tag{A.2.7}$$

We know proceed to case II in equation (A.2.5): $q \neq k$

$$\Rightarrow \mathrm{d}M_{ij} = \sum_{kq} V^{\dagger}_{ik} V_{qj} \mathrm{d}S_{kq} \left((r)_k - (r)_q\right). \tag{A.2.8}$$

In equation (A.2.8) we use $\mathrm{d}S = \mathrm{d}VV^{\dagger}$, which was defined earlier on, we then get

$$\begin{aligned} \mathrm{d}M_{ij} &= \sum_{kq} V^{\dagger}_{ik} V_{qj} (\mathrm{d}VV^{\dagger})_{kq} \left((r)_k - (r)_q\right) \\ &= \sum_{\gamma kq} V^{\dagger}_{ik} V_{qj} \mathrm{d}V_{k\gamma} V^{\dagger}_{\gamma q} \left((r)_k - (r)_{q}.\right). \end{aligned} \tag{A.2.9}$$

Thus, it naturally follows that

$$\frac{\partial V_{k\gamma}}{\partial M_{ij}} = \sum_{k \neq q} \frac{V_{ki} V^{\dagger}_{jq} V_{q\gamma}}{\left((r)_k - (r)_q\right)}. \tag{A.2.10}$$

This completes our expression for the conjugate momentum, with the coefficients calculated in equations (A.2.7) and (A.2.10), the conjugate momentum is given by

$$P_{ji} = \frac{\partial}{\partial M_{ij}} = \sum_{\gamma} \sum_{k \neq q} \left\{ \frac{V^{\dagger}_{jq} V_{q\gamma} V_{ki}}{(r)_k - (r)_q} \right\} \frac{\partial}{\partial V_{k\gamma}} + \sum_{k} \left\{ V^{\dagger}_{ik} V_{kj} \right\} \frac{\partial}{\partial (r)_k}, \tag{A.2.11}$$

which satisfies equation equation (3.5.7).

In the calculation that follows we prove the statement

$$\frac{\partial M_{ab}}{\partial M_{ij}} = \delta_{ai} \delta_{jb}. \tag{A.2.12}$$

First we define

$$M_{ab} = \left(V^{\dagger} r V\right)_{ab} = \sum_{\alpha} V^{\dagger}_{a\alpha} r_{\alpha} V_{\alpha b}. \tag{A.2.13}$$

Using the definition of the conjugate momentum it follows that

$$\frac{\partial M_{ab}}{\partial M_{ij}} = \left(\sum_{\gamma} \sum_{k \neq q} \left\{ \frac{V^{\dagger}_{jq} V_{q\gamma} V_{ki}}{(r)_k - (r)_q} \right\} \frac{\partial}{\partial V_{k\gamma}} + \sum_{k} \left\{ V^{\dagger}_{ik} V_{kj} \right\} \frac{\partial}{\partial (r)_k} \right) \sum_{\alpha} V^{\dagger}_{a\alpha} r_{\alpha} V_{\alpha b}. \tag{A.2.14}$$

$$\begin{aligned}\frac{\partial M_{ab}}{\partial M_{ij}} &= \left(\sum_{\alpha}\sum_{\gamma}\sum_{k\neq q}\left\{\frac{V^{\dagger}_{jq}V_{q\gamma}V_{ki}}{(r)_k-(r)_q}\right\}\frac{\partial V^{\dagger}_{a\alpha}}{\partial V_{k\gamma}}\lambda_{\alpha}V_{\alpha b}\right) \\ &+ \left(\sum_{\alpha}\sum_{\gamma}\sum_{k\neq q}\left\{\frac{V^{\dagger}_{jq}V_{q\gamma}V_{ki}}{(r)_k-(r)_q}\right\}V^{\dagger}_{a\alpha}r_{\alpha}\frac{\partial V_{\alpha b}}{\partial V_{k\gamma}}\right) \\ &+ \sum_{\alpha}\sum_{k}\left\{V^{\dagger}_{ik}V_{kj}\right\}V^{\dagger}_{a\alpha}\frac{\partial r_{\alpha}}{\partial r_k}V_{\alpha b}.\end{aligned} \tag{A.2.15}$$

We make use of the following definitions:

$$\frac{\partial V^{\dagger}_{a\alpha}}{\partial V_{k\gamma}} = -V^{\dagger}_{ak}V^{\dagger}_{\gamma\alpha} \qquad \frac{\partial V_{\alpha b}}{\partial V_{k\gamma}} = \delta_{\alpha k}\delta_{\gamma b} \qquad \frac{\partial r_{\alpha}}{\partial r_k} = \delta_{\alpha k}, \tag{A.2.16}$$

we then obtain

$$\begin{aligned}\frac{\partial M_{ab}}{\partial M_{ij}} &= \sum_{\alpha}\sum_{\gamma k}\sum_{k\neq q}\frac{V^{\dagger}_{jq}V_{q\gamma}V_{ki}}{(r_k-r_q)}V^{\dagger}_{a\alpha}r_{\alpha}\delta_{\alpha k}\delta_{\gamma b} \\ &- \sum_{\alpha}\sum_{\gamma k}\sum_{k\neq q}\frac{V^{\dagger}_{jq}V_{q\gamma}V_{ki}}{(r_k-r_q)}V^{\dagger}_{ak}V^{\dagger}_{\gamma\alpha}r_{\alpha}V_{\alpha b} \\ &+ \sum_{\alpha}\sum_{k}V^{\dagger}_{ik}V_{kj}V^{\dagger}_{a\alpha}\delta_{k\alpha}V_{\alpha b}.\end{aligned} \tag{A.2.17}$$

We sum over the Kronecker delta functions to obtain a change of variables in the expression above, this then gives

$$\frac{\partial M_{ab}}{\partial M_{ij}} = \sum_{k\neq q}\frac{V^{\dagger}_{jq}V_{qb}V_{ki}V^{\dagger}_{ak}r_k}{(r_k-r_q)} - \sum_{k\neq q}\frac{V^{\dagger}_{jq}V_{qb}V_{ki}V^{\dagger}_{ak}r_q}{(r_k-r_q)} + \sum_{k}V^{\dagger}_{ik}V_{kj}V^{\dagger}_{ak}V_{kb} \tag{A.2.18}$$

$$\frac{\partial M_{ab}}{\partial M_{ij}} = \sum_{k\neq q}V^{\dagger}_{jq}V_{qb}V^{\dagger}_{ak}V_{ki} + \sum_{k}V^{\dagger}_{ik}V_{kj}V^{\dagger}_{ak}V_{kb}. \tag{A.2.19}$$

The first term above in equation $(A.2.19)$ is separated into a summation that has $k=q$ and $k\neq q$. Due to the restriction when calculating the coefficients, we are only interested in terms that have $k\neq q$ and we subtract those that have $k=q$. This then gives the equation

$$\frac{\partial M_{ab}}{\partial M_{ij}} = \underbrace{(\sum_k V^\dagger_{ak} V_{ki})(\sum_q V^\dagger_{jq} V_{qb})}_{k \neq q} - \underbrace{\sum_k V^\dagger_{jk} V_{kb} V^\dagger_{ak} V_{ki}}_{\sum_{k=q}} + \sum_k V_{ki} V^\dagger_{jk} V^\dagger_{ak} V_{kb}, \tag{A.2.20}$$

$$\Rightarrow \frac{\partial M_{ab}}{\partial M_{ij}} = \delta_{ai}\delta_{jb}. \tag{A.2.21}$$

Appendix B

The Algebra of Parameterization II

B.1 The Conjugate Momentum in Polar Coordinates

Below we show how to construct the conjugate momentum for the two matrix model using our polar coordinates. We first start of with the partial differential equations to $\partial/\partial Z_{ab}$ given by

$$\frac{\partial}{\partial Z_{ab}} = \sum_i \frac{\partial r_i}{\partial Z_{ab}} \frac{\partial}{\partial r_i} + \sum_{i \neq j} \frac{\partial Y_{ij}^+}{\partial Z_{ab}} \frac{\partial}{\partial Y_{ij}^+} + \sum_{i \neq j} \frac{\partial Y_{ij}^-}{\partial Z_{ab}} \frac{\partial}{\partial Y_{ij}^-}, \tag{B.1.1}$$

and also define $\partial/\partial Z_{ab}^\dagger$ as

$$\frac{\partial}{\partial Z_{ab}^\dagger} = \sum_i \frac{\partial r_i}{\partial Z_{ab}^\dagger} \frac{\partial}{\partial r_i} + \sum_{i \neq j} \frac{\partial Y_{ij}^+}{\partial Z_{ab}^\dagger} \frac{\partial}{\partial Y_{ij}^+} + \sum_{i \neq j} \frac{\partial Y_{ij}^-}{\partial Z_{ab}^\dagger} \frac{\partial}{\partial Y_{ij}^-}. \tag{B.1.2}$$

A starting point for us is to use the previously defined equations for dZ and $dZ^\dagger$, which are

$$dZ = V^\dagger \left(dr + \frac{1}{\sqrt{2}} \left[r, dY^+ \right] + \frac{1}{\sqrt{2}} \left\{ r, dY^- \right\} \right) W \tag{B.1.3}$$

$$dZ^{\dagger} = W^{\dagger}\left(dr + \frac{1}{\sqrt{2}}\left[r, dY^{+}\right] - \frac{1}{\sqrt{2}}\left\{r, dY^{-}\right\}\right)V. \tag{B.1.4}$$

We rewrite equations (B.1.3) and (B.1.4) as

$$VdZY^{\dagger} = \left(dr + \frac{1}{\sqrt{2}}\left[r, dY^{+}\right] + \frac{1}{\sqrt{2}}\left\{r, dY^{-}\right\}\right) \tag{B.1.5}$$

$$WdZ^{\dagger}V^{\dagger} = \left(dr + \frac{1}{\sqrt{2}}\left[r, dY^{+}\right] - \frac{1}{\sqrt{2}}\left\{r, dY^{-}\right\}\right). \tag{B.1.6}$$

Equations (B.1.5) and (B.1.6) are summed to give

$$\sum_{ij}(VdZW^{\dagger})_{ij} + (YdZ^{\dagger}V^{\dagger})_{ij} = 2\sum_{ij}\delta_{ji}dr_i + \sqrt{2}\sum_{i<j}[r, dY^{+}]_{ij} \tag{B.1.7}$$

$$\sum_{ab}\sum_{ij}V_{ia}dZ_{ab}W^{\dagger}_{bj} + \sum_{ab}\sum_{ij}W_{ia}dZ^{\dagger}_{ab}V^{\dagger}_{bj} = 2\sum_{ij}\delta_{ji}dr_i + \sqrt{2}(r_i - r_j)dY^{+}_{ij}.$$

Just like in the first parameterization, we consider the case where $i = j$ in the second line of equation (B.1.7) above, and when $i = j$, it follows that

$$\sum_{ab}\sum_{i}V_{ia}dZ_{ab}W^{\dagger}_{bi} + \sum_{ab}\sum_{i}W_{ia}dZ^{\dagger}_{ab}V^{\dagger}_{bi} = 2\sum_{ij}\delta_{ii}dr_i. \tag{B.1.8}$$

Equation (B.1.8) then gives us our first set of coefficients, which are

$$\frac{\partial r_i}{\partial Z_{ab}} = \sum_{i}V_{ia}W^{\dagger}_{bi} \qquad \frac{\partial r_i}{\partial Z^{\dagger}_{ab}} = \sum_{i}W_{ia}V^{\dagger}_{bi}. \tag{B.1.9}$$

Again using the second line of equation (B.1.7), when we consider the case when $i \neq j$, the following equation comes about

$$\sum_{ij}\sum_{ab}V_{ia}dZ_{ab}W^{\dagger}_{bj} + \sum_{ij}\sum_{ab}W_{ia}dZ^{\dagger}_{ab}V^{\dagger}_{bj} = \sqrt{2}\sum_{i<j}(r_i - r_j)dY^{+}_{ij}. \tag{B.1.10}$$

From equation (B.1.10) above, the following coefficient are deduced

$$\frac{\partial Y^{+}_{ij}}{\partial Z_{ab}} = \sqrt{2}\sum_{i<j}\frac{V_{ia}W^{\dagger}_{bj}}{(r_i - r_j)} \qquad \frac{\partial Y^{+}_{ij}}{\partial Z^{\dagger}_{ab}} = \sqrt{2}\sum_{i<j}\frac{W_{ia}V^{\dagger}_{bj}}{(r_i - r_j)}. \tag{B.1.11}$$

To obtain the rest of the coefficients, we now subtract our two previous equations, that is we subtract (B.1.6) from (B.1.5) to obtain the new equation expressed with indices

$$\sum_{ij}(VdZW^\dagger)_{ij} - \sum_{ij}(WdZV^\dagger)_{ij} = \sqrt{2}\sum_{ij}\{dr, dY^-\}_{ij} \tag{B.1.12}$$

$$\sum_{ab}\sum_{ij} V_{ia}dZ_{ab}W^\dagger_{bj} - \sum_{ij}\sum_{ab} W_{ia}dZ^\dagger_{ab}V_{bj} = \sqrt{2}\sum_{ij}(r_i + r_j)dY^-_{ij}.$$

From the second line of equation (B.1.12) above, we again consider the case where $i = j$, and in this instance we get the following equation

$$\sum_{i}\sum_{ab} V_{ia}dZ_{ab}W^\dagger_{bi} - \sum_{i}\sum_{ab} W_{ia}dZ^\dagger_{ab}V_{bj} = 2\sqrt{2}r_i dY^-_{ij}, \tag{B.1.13}$$

and the coefficients are defined as

$$\frac{\partial Y^-_{ii}}{\partial Z_{ab}} = \frac{1}{\sqrt{2}}\sum_i \frac{V_{ia}W^\dagger_{bi}}{r_i} \qquad \frac{\partial Y^-_{ii}}{\partial Z^\dagger_{ab}} = -\frac{1}{\sqrt{2}}\sum_i \frac{Y_{ia}V^\dagger_{bi}}{r_i}. \tag{B.1.14}$$

Now, when we consider the instance when $i \neq j$ in equation (B.1.12), this gives the equation

$$\sum_{ab}\sum_{ij} V_{ia}dZ_{ab}W^\dagger_{bj} - \sum_{ij}\sum_{ab} W_{ia}dZ^\dagger_{ab}V^{bj} = \sqrt{2}\sum_{ij}(r_i + r_j)dY^-_{ij}. \tag{B.1.15}$$

The coefficients obtained from equation (B.1.15) above are

$$\frac{\partial Y^-_{ij}}{\partial Z_{ab}} = \sqrt{2}\sum_{ij} \frac{V_{ia}W^\dagger_{bj}}{(r_i + r_j)} \qquad \frac{\partial Y^-_{ij}}{\partial Z^\dagger_{ab}} = -\sqrt{2}\sum_{ij} \frac{W_{ia}V^\dagger_{bj}}{(r_i + r_j)}. \tag{B.1.16}$$

With these coefficients being defined, a more complete form of the conjugate momentum $\partial/\partial Z_{ab}$ in equation (B.1.1) and $\partial/\partial Z^\dagger_{ab}$ in equation (B.1.2) is

$$\frac{\partial}{\partial Z_{ab}} = \sum_i V_{ia}W^\dagger_{bi}\frac{\partial}{\partial r_i} + \sqrt{2}\sum_{i\neq j}\frac{V_{ia}W^\dagger_{bj}}{(r_i - r_j)}\frac{\partial}{\partial Y^+_{ij}} + \sqrt{2}\sum_{ij}\frac{V_{ia}W^\dagger_{bj}}{(r_i + r_j)}\frac{\partial}{\partial W^-_{ij}} \tag{B.1.17}$$

and

$$\frac{\partial}{\partial Z^\dagger_{ab}} = \sum_i W_{ia} V^\dagger_{bi} \frac{\partial}{\partial r_i} + \sqrt{2} \sum_{i \neq j} \frac{W_{ia} V^\dagger_{bj}}{(r_i - r_j)} \frac{\partial}{\partial \left(Y^+_{ij}\right)} - \sqrt{2} \sum_{ij} \frac{W_{ia} V^\dagger_{bj}}{(r_i + r_j)} \frac{\partial}{\partial \left(Y^-_{ij}\right)}. \tag{B.1.18}$$

B.2 Deriving the angular momentum

In this section of the appendix the objective is to show how the angular momentum i.e. the generator of $U(1) \sim SO(2)$ rotations in equation (4.3.19) was calculated. The angular momentum had already been defined, and is given by the equation

$$\hat{L} \equiv \frac{1}{i} \mathrm{tr} \left(X_1 \frac{\partial}{\partial X_2} - X_2 \frac{\partial}{\partial X_1} \right), \tag{B.2.1}$$

such that

$$\frac{1}{i} \mathrm{tr} \left(X_1 \frac{\partial}{\partial X_2} - X_2 \frac{\partial}{\partial X_1} \right) = \sum_{ab} (X_1)_{ab} \left(\frac{\partial}{\partial X_2} \right)_{ab} - \sum_{ab} (X_2)_{ab} \left(\frac{\partial}{\partial X_1} \right)_{ab}. \tag{B.2.2}$$

Above, in equation (B.2.2), we solve the following

$$\left(\frac{\partial}{\partial X_1} \right)_{ab} = \sum_{ij} \frac{\partial Z_{ij}}{(\partial X_1)_{ab}} \frac{\partial}{\partial Z_{ij}} + \sum_{ij} \frac{\partial Z^\dagger_{ij}}{(\partial X_1)_{ab}} \frac{\partial}{\partial Z^\dagger_{ij}} \tag{B.2.3}$$

and

$$\left(\frac{\partial}{\partial X_2} \right)_{ab} = \sum_{ij} \frac{\partial Z_{ij}}{(\partial X_2)_{ab}} \frac{\partial}{\partial Z_{ij}} + \sum_{ij} \frac{\partial Z^\dagger_{ij}}{(\partial X_2)_{ab}} \frac{\partial}{\partial Z^\dagger_{ij}}, \tag{B.2.4}$$

using

$$(Z)_{ij} = (X_1)_{ij} + i(X)_{ij} \qquad (Z^\dagger)_{ij} = (X_1)_{ij} - i(X_2)_{ij}. \tag{B.2.5}$$

Equation (B.2.5) can be re-written in terms of the matrices X_1 and X_2 as

$$(X_1)_{ij} = \frac{1}{2}\left((Z)_{ij} + (Z^\dagger)_{ij}\right) \qquad (X_2)_{ij} = \frac{1}{2i}\left((Z)_{ij} - (Z^\dagger)_{ij}\right). \quad \text{(B.2.6)}$$

Through elementary differentiation with respect to $(X_1)_{ij}$ and $(X_2)_{ij}$, in the equation (B.2.6), the following coefficients are obtained

$$\frac{\partial(Z)_{ij}}{\partial(X_1)_{ab}} = \delta_{ai}\delta_{bj} \qquad \frac{\partial(Z^\dagger)_{ij}}{\partial(X_1)_{ab}} = \delta_{ai}\delta_{bj} \quad \text{(B.2.7)}$$

and

$$\frac{\partial(Z)_{ij}}{\partial(X_2)_{ab}} = i\delta_{ia}\delta_{bj} \qquad \frac{\partial(Z^\dagger)_{ij}}{\partial(X_2)_{ab}} = -i\delta_{ai}\delta_{bj}. \quad \text{(B.2.8)}$$

These coefficients computed in equation (B.2.7) and (B.2.8) are then re-substitute back into equations (B.2.3) and (B.2.4), and this gives

$$\begin{aligned}
\left(\frac{\partial}{\partial X_1}\right)_{ab} &= \sum_{ij}\frac{\partial Z_{ij}}{(\partial X_1)_{ab}}\frac{\partial}{\partial Z_{ij}} + \sum_{ij}\frac{\partial Z^\dagger_{ij}}{(\partial X_1)_{ab}}\frac{\partial}{\partial Z^\dagger_{ij}} \\
&= \sum_{ij}\delta_{ai}\delta_{bj}\frac{\partial}{\partial(Z)_{ij}} + \sum_{ij}\delta_{ai}\delta_{bj}\frac{\partial}{\partial(Z^\dagger)_{ij}} \\
&\Rightarrow \left(\frac{\partial}{\partial X_1}\right)_{ab} = \frac{\partial}{\partial(Z)_{ab}} + \frac{\partial}{\partial(Z^\dagger)_{ab}} \qquad \text{(B.2.9)}
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial}{\partial X_2}\right)_{ab} &= \sum_{ij}\frac{\partial Z_{ij}}{(\partial X_2)_{ab}}\frac{\partial}{\partial Z_{ij}} + \sum_{ij}\frac{\partial Z^\dagger_{ij}}{(\partial X_2)_{ab}}\frac{\partial}{\partial Z^\dagger_{ij}} \\
&= i\sum_{ij}\delta_{ia}\delta_{bj}\frac{\partial}{\partial(Z)_{ij}} - i\sum_{ij}\delta_{ai}\delta_{bj}\frac{\partial}{\partial(Z^\dagger)_{ij}} \\
&\Rightarrow \frac{\partial}{\partial(X_2)_{ab}} = i\left(\frac{\partial}{\partial(Z)_{ab}} - \frac{\partial}{\partial(Z^\dagger)_{ab}}\right). \qquad \text{(B.2.10)}
\end{aligned}$$

We can know proceed to represent the angular momentum in terms of the non-hermitian matrix Z as follows

$$\hat{L} = \frac{1}{i}\mathrm{tr}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_1}\right) = \frac{1}{i}\sum_{ab}\left\{(X_1)_{ab}\frac{\partial}{\partial(X_2)_{ab}} - (X_2)_{ab}\frac{\partial}{\partial(X_1)_{ab}}\right\} \quad \text{(B.2.11)}$$

$$
\begin{aligned}
\hat{L} &= \frac{1}{i}\mathrm{tr}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_1}\right) \\
&= \frac{1}{i}\sum_{ab}\frac{1}{2}(Z+Z^\dagger)_{ab}\; i\left(\frac{\partial}{\partial (Z)_{ab}} - \frac{\partial}{\partial (Z^\dagger)_{ab}}\right) \\
&- \frac{1}{i}\sum_{ab}\frac{1}{2i}(Z-Z^\dagger)_{ab}\left(\frac{\partial}{\partial (Z)_{ab}} + \frac{\partial}{\partial (Z^\dagger)_{ab}}\right)
\end{aligned}
\tag{B.2.12}
$$

After some algebra and reorganizing some terms we get

$$
\hat{L} = \frac{1}{i}\mathrm{tr}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_1}\right) = \sum_{ab}(Z)_{ab}\frac{\partial}{\partial (Z)_{ab}} - \sum_{ab}(Z^\dagger)_{ab}\frac{\partial}{\partial (Z^\dagger)_{ab}}.
\tag{B.2.13}
$$

Using all the information that has been calculated thus far, we can obtain the expression of the angular momentum in equation (B.2.1) explicitly in terms of the variables that were defined when calculating the Laplacian. We first define the non-hermitian $N \times N$ matrix Z using indices as

$$
(Z)_{ab} = \left(V^\dagger r W\right)_{ab} = \sum_p W^\dagger_{ap} r_p V_{pb} (Z^\dagger)_{ab} = (W^\dagger r V)_{ab} = \sum_p W^\dagger_{ap} r_p V_{pb}
\tag{B.2.14}
$$

$\partial/\partial(Z)_{ab}$ and $\partial/\partial(Z^\dagger)_{ab}$ have already been calculated previously. The angular momentum is computed to be

$$
\begin{aligned}
\hat{L} &= \frac{1}{i}\mathrm{tr}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_1}\right) \\
&= \sum_{ab}(Z)_{ab}\frac{\partial}{\partial (Z)_{ab}} - \sum_{ab}(Z^\dagger)_{ab}\frac{\partial}{\partial (Z^\dagger)_{ab}} \\
&= \sum_{ab}\sum_p V^\dagger_{ap} r_p W_{pb}\left(\sum_i V_{ia}W^\dagger_{bi}\frac{\partial}{\partial r_i} + \sqrt{2}\sum_{i<j}\frac{V_{ia}W^\dagger_{bj}}{(r_i - r_j)}\frac{\partial}{\partial Y^+_{ij}}\right) \\
&+ \sqrt{2}\sum_{ij}\sum_{ab}\sum_p V^\dagger_{ap} r_p W_{pb}\left(\frac{V_{ia}W^\dagger_{bj}}{(r_i + r_j)}\frac{\partial}{\partial Y^-_{ij}}\right) \\
&- \sum_{ab}\sum_p W^\dagger_{ap} r_p V_{pb}\left(\sum_i W_{ia}V^\dagger_{bi}\frac{\partial}{\partial r_i} + \sqrt{2}\sum_{i<j}\frac{W_{ia}V^\dagger_{bj}}{(r_i - r_j)}\frac{\partial}{\partial Y^+_{ij}}\right) \\
&- \sqrt{2}\sum_{ij}\sum_{ab}\sum_p W^\dagger_{ap} r_p V_{pb}\left(\frac{W_{ia}V^\dagger_{bj}}{(r_i + r_j)}\frac{\partial}{\partial Y^-_{ij}}\right)
\end{aligned}
\tag{B.2.15}
$$

$$\begin{aligned}
\Rightarrow \hat{L} &= \frac{1}{i}\mathrm{tr}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_1}\right) \\
&= \sum_p\sum_i(\sum_a V_{ia}V^\dagger_{ap})r_p(\sum_b W_{pb}W^{bi})\frac{\partial}{\partial r_i} \\
&+ \sqrt{2}\sum_{i<j}\sum_p(\sum_a V_{ia}V^\dagger_{ap})(\sum_b W_{pb}W^\dagger_{bj})\frac{r_p}{(r_i-r_j)}\frac{\partial}{\partial Y^+_{ij}} \\
&+ \sqrt{2}\sum_{ij}\sum_p(\sum_a V_{ia}V^\dagger_{ap})(\sum_b W_{pb}W^\dagger_{bj})\frac{r_p}{(r_i+r_j)}\frac{\partial}{\partial Y^-_{ij}} \\
&- \sum_p\sum_i(\sum_a W_{ia}W^\dagger_{ap})(\sum_b V_{pb}V^\dagger_{bi})r_p\frac{\partial}{\partial r_i} \\
&- \sqrt{2}\sum_{i<j}\sum_p(\sum_a W_{ia}W^\dagger_{ap})(\sum_b V_{pb}V^\dagger_{bj})\frac{r_p}{(r_i-r_j)}\frac{\partial}{\partial Y^+_{ij}} \\
&+ \sqrt{2}\sum_{ij}\sum_p(\sum_a W_{ia}W^\dagger_{ap})(\sum_b V_{pb}V^\dagger_{bj})\frac{r_p}{(r_i+r_j)}\frac{\partial}{\partial Y^-_{ij}}
\end{aligned} \tag{B.2.16}$$

In equation (B.2.16), angular terms that appear in the brackets sum out to become Kronecker delta functions. An example of the summation would be

$$(\sum_a V_{ia}V^\dagger_{ap})(\sum_b W_{pb}W^\dagger_{bi}) = (\delta_{ip})(\delta_{pi}). \tag{B.2.17}$$

So, to move forward a step further, the expression of the angular momentum, thus becomes

$$\begin{aligned}
\hat{L} &= \sum_i\sum_p \delta_{ip}\delta_{pi}r_p\frac{\partial}{\partial r_i} + \sqrt{2}\sum_{i<j}\sum_p \delta_{ip}\delta_{pj}\frac{r_p}{(r_i-r_j)}\frac{\partial}{\partial Y^+_{ij}} \\
&+ \sqrt{2}\sum_{ij}\sum_p \delta_{ip}\delta_{pj}\frac{r_p}{(r_i+r_j)}\frac{\partial}{\partial Y^-_{ij}} - \sum_i\sum_p \delta_{ip}\delta_{pi}r_p\frac{\partial}{\partial r_i} \\
&- \sqrt{2}\sum_{i<j}\sum_p \delta_{ip}\delta_{pj}\frac{r_p}{(r_i-r_j)}\frac{\partial}{\partial Y^+_{ij}} + \sqrt{2}\sum_{ij}\sum_p \delta_{ip}\delta_{pj}\frac{r_p}{(r_i+r_j)}\frac{\partial}{\partial Y^-_{ij}}.
\end{aligned} \tag{B.2.18}$$

Terms naturally cancel out above in equation (B.2.18), and what remains is

$$\hat{L} = 2\sqrt{2}\sum_{ij}\sum_p \delta_{ip}\delta_{pj}\frac{r_p}{(r_i+r_j)}\frac{\partial}{\partial Y^-_{ij}}. \tag{B.2.19}$$

In equation (B.2.19) above we sum out the p index, by doing this all p indices become i i.e. $p \to i$,

$$\hat{L} = 2\sqrt{2}\sum_{ij}\delta_{ij}\frac{r_i}{(r_i + r_j)}\frac{\partial}{\partial Y^-_{ij}}. \tag{B.2.20}$$

We perform a summation in equation (B.2.20) for all i and j indices including $i = j$, this in turn gives the final form of the angular momentum

$$\hat{L} = \sqrt{2}\sum_{i}\frac{\partial}{\partial Y^-_{ii}}. \tag{B.2.21}$$

Equation (B.2.21) is the generator of $U(1) \sim SO(2)$ rotations.

B.3 The Commutator In Polar Coordinates

In this section we would like to show how the commutator

$$[P_{ba}, Z_{\gamma\beta}] = \delta_{\gamma a}\delta_{b\beta} \tag{B.3.1}$$

is computed, because if the above commutator is true then the expression for the conjugate momentum P_{ba} is also true. We define the conjugate momentum in terms of our polar coordinate Z as

$$P_{ba} = \frac{\partial}{\partial Z_{ab}}. \tag{B.3.2}$$

We start of with the definition of the conjugate momentum in terms of partial equations

$$\frac{\partial}{\partial Z_{ab}} = \sum_i V_{ia}W^\dagger_{bi}\frac{\partial}{\partial r_i} + \sqrt{2}\sum_{i\neq j}\frac{V_{ia}W^\dagger_{bj}}{(r_i - r_j)}\frac{\partial}{\partial Y^+_{ij}} + \sqrt{2}\sum_{ij}\frac{V_{ia}W^\dagger_{bj}}{(r_i + r_j)}\frac{\partial}{\partial W^-_{ij}}. \tag{B.3.3}$$

We introduce indices on Z as

$$Z_{\gamma\beta} = \sum_m V^\dagger_{\gamma m} r_m W_{m\beta}, \tag{B.3.4}$$

and we also define the following partials

$$\frac{\partial}{\partial Y_{ij}^{+}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial T_{ij}} + \frac{\partial}{\partial S_{ij}}\right) \qquad \frac{\partial}{\partial Y_{ij}^{-}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial T_{ij}} - \frac{\partial}{\partial S_{ij}}\right). \quad \text{(B.3.5)}$$

Using the definition $\mathrm{d}T = \mathrm{d}WW^{\dagger}$ we have

$$\frac{\partial}{\partial T_{ij}} = \sum_{k'} W_{jk'} \frac{\partial}{\partial W_{ik'}}. \quad \text{(B.3.6)}$$

Similarly, using $\mathrm{d}S = \mathrm{d}VV^{\dagger}$, we find

$$\frac{\partial}{\partial S_{ij}} = -\sum_{k} V_{ki}^{\dagger} \frac{\partial}{\partial V_{kj}^{\dagger}}. \quad \text{(B.3.7)}$$

Equations (B.3.5), (B.3.6) and (B.3.7) are substituted into equation (B.3.3) to give

$$\begin{aligned}
\frac{\partial}{\partial Z_{ab}} &= \sum_{i} V_{ia} W_{bi}^{\dagger} \frac{\partial}{\partial r_i} + \sum_{i<j} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i - r_j)} \sum_{k'} W_{jk'} \frac{\partial}{\partial W_{ik'}} \\
&+ \sum_{ij} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i + r_j)} \sum_{k'} W_{jk'} \frac{\partial}{\partial W_{ik'}} - \sum_{i<j} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i - r_j)} \sum_{k} V_{ki}^{\dagger} \frac{\partial}{\partial V_{kj}^{\dagger}} \\
&+ \sum_{ij} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i + r_j)} \sum_{k} V_{ki}^{\dagger} \frac{\partial}{\partial V_{kj}^{\dagger}}.
\end{aligned} \quad \text{(B.3.8)}$$

We allow this operator in equation (B.3.8) to act on the the previously defined matrix $Z_{\gamma\beta}$ from equation (B.3.4), it then follows that

$$\begin{aligned}
\frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} &= \sum_{i} \sum_{m} V_{ia} W_{bi}^{\dagger} V_{\gamma m}^{\dagger} \frac{\partial r_m}{\partial r_i} W_{m\beta} \\
&+ \sum_{i<j} \sum_{m} \sum_{k'} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i - r_j)} W_{jk'} V_{\gamma m}^{\dagger} r_m \frac{\partial W_{m\beta}}{\partial W_{ik'}} \\
&+ \sum_{ij} \sum_{k'} \sum_{m} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i + r_j)} W_{jk'} V_{\gamma m}^{\dagger} r_m \frac{\partial W_{m\beta}}{\partial W_{ik'}} \\
&- \sum_{i<j} \sum_{k} \sum_{m} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i - r_j)} V_{ki}^{\dagger} \frac{\partial V_{\gamma m}^{\dagger}}{\partial V_{kj}^{\dagger}} r_m W_{m\beta} \\
&+ \sum_{ij} \sum_{k} \sum_{m} \frac{V_{ia} W_{bj}^{\dagger}}{(r_i + r_j)} V_{ki}^{\dagger} \frac{\partial V_{\gamma m}^{\dagger}}{\partial V_{kj}^{\dagger}} r_m W_{m\beta}.
\end{aligned} \quad \text{(B.3.9)}$$

In equation (B.3.9) we use the following identities

$$\frac{\partial r_m}{\partial r_i} = \delta_{mi} \frac{\partial W_{m\beta}}{\partial W_{ik'}} = \delta_{mi}\delta_{k'\beta} \qquad \frac{\partial V^\dagger_{\gamma m}}{\partial V^\dagger_{kj}} = \delta_{\gamma k}\delta_{jm}. \tag{B.3.10}$$

Equation (B.3.9) now becomes

$$\begin{aligned} \frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} &= \sum_i \sum_m V_{ia} W^\dagger_{bi} V^\dagger_{\gamma m} \delta_{mi} W_{m\beta} \qquad \text{(B.3.11)} \\ &+ \sum_{i<j} \sum_m \sum_{k'} \frac{V_{ia} W^\dagger_{bj}}{(r_i - r_j)} W_{jk'} V^\dagger_{\gamma m} r_m \delta_{mi} \delta_{k'\beta} \\ &+ \sum_{ij} \sum_m \sum_{k'} \frac{V_{ia} W^\dagger_{bj}}{(r_i + r_j)} W_{jk'} V^\dagger_{\gamma m} r_m \delta_{mi} \delta_{k'\beta} \\ &- \sum_{i<j} \sum_m \sum_k \frac{V_{ia} W^\dagger_{bj}}{(r_i - r_j)} V^\dagger_{ki} \delta_{\gamma k} \delta_{jm} r_m W_{m\beta} \\ &+ \sum_{ij} \sum_m \sum_k \frac{V_{ia} W^\dagger_{bj}}{(r_i + r_j)} V^\dagger_{ki} \delta_{\gamma k} \delta_{jm} r_m W_{m\beta}. \end{aligned}$$

In equation (B.3.11) above we sum out the Kronecker delta functions, this results in some of the terms changing their indices, so we get

$$\begin{aligned} \frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} &= \sum_i \sum_m V_{ia} W^\dagger_{bi} V^\dagger_{\gamma m} \delta_{mi} W_{m\beta} + \sum_{i<j} \sum_m \sum_{k'} \frac{V_{ia} W^\dagger_{bj}}{(r_i - r_j)} W_{jk'} V^\dagger_{\gamma m} r_m \delta_{mi} \delta_{k'\beta} \\ &+ \sum_{ij} \sum_{k'} \sum_m \frac{V_{ia} W^\dagger_{bj}}{(r_i + r_j)} W_{jk'} V^\dagger_{\gamma m} r_m \delta_{mi} \delta_{k'\beta} \\ &- \sum_{i<j} \sum_k \sum_m \frac{V_{ia} W^\dagger_{bj}}{(r_i - r_j)} V^\dagger_{ki} \delta_{\gamma k} \delta_{jm} r_m W_{m\beta} \\ &+ \sum_{ij} \sum_k \sum_m \frac{V_{ia} W^\dagger_{bj}}{(r_i + r_j)} V^\dagger_{ki} \delta_{\gamma k} \delta_{jm} r_m W_{m\beta}. \end{aligned} \tag{B.3.12}$$

Equation (B.3.12) implies the following

$$\begin{aligned} \frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} &= \sum_i V_{ia} W^\dagger_{bi} V^\dagger_{\gamma i} W_{i\beta} + \sum_{i<j} \frac{V_{ia} W^\dagger_{bj}}{(r_i - r_j)} W_{j\beta} V^\dagger_{\gamma i} r_i + \sum_{i<j} \frac{V_{ia} W^\dagger_{bj}}{r_i + r_j} W_{j\beta} V^\dagger_{\gamma i} r_i \\ &- \sum_{i<j} \frac{V_{ia} W^\dagger_{bj}}{r_i - r_j} V^\dagger_{\gamma i} r_j W_{j\beta} + \sum_{ij} \frac{V_{ia} W^\dagger_{bj}}{r_i + r_j} W_{j\beta} V^{\gamma i} r_i. \end{aligned} \tag{B.3.13}$$

In equation (B.3.13), after canceling the common factors that appear in the denominator and the numerator for all terms except for the first term we then get

$$\frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} = \sum_i V_{ia}W^\dagger_{bi}V^\dagger_{\gamma i}W_{i\beta} + \sum_{i<j} V_{ia}W^\dagger_{bj}W_{j\beta}V^\dagger_{\gamma} + \sum_i\sum_j V_{ia}W^\dagger_{bj}W_{j\beta}V^\dagger_{\gamma i}. \tag{B.3.14}$$

It is only the middle term that is of major importance as it comes with a restriction. For the middle term, we sum over all i and j indices except terms that have $i = j$, as this will allow an infinity factor appearing within our calculation, so then we get

$$\begin{aligned} \frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} &= \sum_i V_{ia}W^\dagger_{bi}V^\dagger_{\gamma i}W_{i\beta} + \left(\sum_i V^\dagger_{\gamma i}V_{ia}\right)\left(\sum_j W^\dagger_{bj}W_{j\beta}\right) \\ &- \sum_i V_{ia}W^\dagger_{bi}W_{i\beta}V^\dagger_{\gamma i} + \left(\sum_i V^\dagger_{\gamma i}V_{ia}\right)\left(\sum_j W^\dagger_{bj}W_{j\beta}\right) \end{aligned} \tag{B.3.15}$$

$$\Longrightarrow \frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} = 2\left(\sum_i V^\dagger_{\gamma i}V_{ia}\right)\left(\sum_j W^\dagger_{bj}W_{j\beta}\right). \tag{B.3.16}$$

Contracting out the indices, the final form of equation (B.3.16) becomes

$$\frac{\partial Z_{\gamma\beta}}{\partial Z_{ab}} = 2\delta_{\gamma a}\delta_{b\beta}, \tag{B.3.17}$$

satisfying our previous definition of the commutator of the conjugate momentum (except for the factor 2 appearing in the front of the right hand side).

Appendix C

Perturbation for two matrix model

In this appendix we will explain how to derive the general formula of equation (7.3.19) which is a generalization of equation (7.3.17) and equation (7.3.18).

In the free case (section 7.2) we showed an extension of the of eigenvalues such that $\phi(r) = \phi(-r)$. For some finite support we have

$$\int_{-a}^{a} \frac{\phi(y)}{r^2 - y^2} = \int_{0}^{a} \frac{2\phi(y)}{r^2 - y^2} = \frac{1}{2}\omega^2, \tag{C.0.1}$$

which is the solution to equation (7.2.6). Let $\grave{\rho} = y^2$ and $\mathrm{d}\grave{\rho} = 2y\mathrm{d}y$, so this implies that

$$\begin{aligned} \int_0^{a^2} \mathrm{d}\grave{\rho} \frac{\Phi(\grave{\rho})}{\rho - \grave{\rho}} &= \frac{1}{2}\omega^2 \\ &= \int_0^{a^2} 2y\mathrm{d}y \frac{\Phi(y^2)}{\rho - y^2}. \end{aligned} \tag{C.0.2}$$

When we refer back to equation (C.0.1), we find that $\phi(y) \equiv 2y\Phi(y^2)$ and $\phi(\sqrt{\rho}) \equiv 2\sqrt{\rho}\Phi(\rho)$. We can then define the following function

$$\Phi(\rho) = \sum_i \delta(\rho - \rho_i), \tag{C.0.3}$$

then for $\rho = y^2$ and $\mathrm{d}\rho = 2y\mathrm{d}y$

$$\begin{aligned}
\sum_i r_i^{2p} &= \int_0^\infty \Phi(\rho)\rho^p \mathrm{d}\rho \\
&= \int_0^\infty 2y\Phi(y^2)y^{2p}\mathrm{d}y \\
&= \int_0^\infty \left(2y\Phi(y^2)\right) y^{2p}\mathrm{d}y \\
&= \int_0^\infty \phi(y)y^{2p}\mathrm{d}y \\
&= \frac{1}{2}\int_{-\infty}^\infty \phi(y)y^{2p}\mathrm{d}y.
\end{aligned} \tag{C.0.4}$$

So in total this gives

$$2\sum_i r_i^{2p} = \int_{-\infty}^\infty \phi(y)y^{2p}\mathrm{d}y. \tag{C.0.5}$$

Equation (C.0.5) is the generalized derivation of equation (7.3.19).

We now wish to explain how the the equations (7.3.22), (7.3.23) and (7.3.24) were obtained. We start off with the analytic function of equation (7.3.21)

$$\begin{aligned}
F(z) &= \left(\frac{1}{2}\omega^2 - 4\lambda\omega_2\right) z + 4\lambda z^3 - \left(4\lambda z^2 + d\right)\sqrt{z^2 - a^2} \\
&= \left(\frac{1}{2}\omega^2 - 4\lambda\omega_2\right) z + 4\lambda z^3 - \left(4\lambda z^3 + dz\right)\sqrt{1 - \frac{a^2}{z^2}}.
\end{aligned} \tag{C.0.6}$$

A binomial expansion is performed on the the square root term appearing in equation (C.0.6) above such that

$$\sqrt{1 - \frac{a^2}{z^2}} = 1 - \frac{1}{2}\frac{a^2}{z^2} + \frac{1}{2}\frac{1}{2}\frac{-1}{2}\frac{a^4}{z^4} + \frac{1}{3!}\frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{-a^6}{z^6} + ... \tag{C.0.7}$$

Equation (C.0.7) is substituted back into equation (C.0.6), and in total the following is obtained

$$
\begin{aligned}
F(z) &= \frac{1}{2}\bar{\omega}^2 z + 2a^2\lambda z - dz + \frac{1}{2}a^4\lambda\frac{1}{z} + \frac{1}{2}a^2 d\frac{1}{z} \\
&+ \frac{1}{4}a^6\lambda\frac{1}{z^3} + \frac{1}{8}a^4 d\frac{1}{z^3} \\
&= \left(\frac{1}{2}\bar{\omega}^2 + 2a^2\lambda - d\right) z + \left(\frac{1}{2}a^4\lambda + \frac{1}{2}a^2 d\right)\frac{1}{z} \\
&+ \left(\frac{1}{4}a^6\lambda + \frac{1}{8}a^4 d\right)\frac{1}{z^3}.
\end{aligned} \tag{C.0.8}
$$

We will know compare and equate the coefficients of z, $1/z$ and $1/z^3$ from the last line of equation (7.3.20) and those from the last line equation (C.0.8) and what we find is the following

$$\left(\frac{1}{2}\bar{\omega}^2 + 2a^2\lambda - d\right) = 0, \tag{C.0.9}$$

$$\left(\frac{1}{2}a^4\lambda + \frac{1}{2}a^2 d\right) = 2, \tag{C.0.10}$$

and

$$\left(\frac{1}{4}a^6\lambda + \frac{1}{8}a^4 d\right) = 2\omega_2 \tag{C.0.11}$$

where $\omega_2 = \sum_i r_i^2$.

The constant d is made the subject of equation (C.0.9) and is substituted into equations (C.0.10) and (C.0.11), and the following equations are obtained

$$6a^4\lambda + a^2\bar{\omega}^2 - 8 = 0 \tag{C.0.12}$$

or

$$6a^4\lambda + a\omega^2 - 8a^2\lambda\omega_2 - 8 = 0 \tag{C.0.13}$$

and

$$8a^6\lambda + a^2\left(a^2\bar{\omega}^2\right) = 32\omega_2 \tag{C.0.14}$$

or

$$8a^6\lambda + a^4\omega^2 - 8\lambda\omega_2 a^4 = 32\omega_2. \tag{C.0.15}$$

for $\bar{\omega} = \omega^2 - 8\lambda\omega_2$. From equation (C.0.12), we see that

$$a^2\bar{\omega}^2 = 8 - 6\lambda a^4, \tag{C.0.16}$$

when equation (C.0.16) above is substituted in equation (C.0.14), the following is derived

$$16\omega_2 = a^6\lambda + 4a^2. \tag{C.0.17}$$

With the equations (C.0.13) and (C.0.14), the following quartic equation is also derived

$$8a^4\lambda + 2a^2\omega^2 - a^8\lambda^2 - 16 = 0. \tag{C.0.18}$$

The roots of the above quartic equation are approximated to the first order in λ such that

$$a^2 = \frac{8}{\omega^2} + A\lambda + O(\lambda^2), \tag{C.0.19}$$

and terms that have order of λ greater than 1 are neglected. To solve for the constant A we use equation (C.0.18) up to order λ, which is our quartic equation and with this we get

$$\begin{aligned} 0 &= 8\lambda\frac{8^2}{\omega^4} + 2\omega^2\left(\frac{8}{\omega^2} + A\lambda\right) - 16 \\ 0 &= 8^3\frac{\lambda}{\omega^4} + 16 + 2\omega^2 A\lambda - 16 \\ \Longrightarrow A &= -\frac{8^2 4}{\omega^6}. \end{aligned} \tag{C.0.20}$$

Now that we know the constant A, we continue to compute the result of perturbation theory using equation (C.0.17), to get the following

$$\begin{aligned}
\omega_2 &= \frac{1}{4}a^2 + \frac{1}{16}\lambda a^6 \qquad \text{(C.0.21)}\\
&= \frac{1}{4}\left(\frac{8}{\omega^2} - \frac{8^2 4}{\omega^6}\lambda\right) + \frac{1}{16}\lambda\frac{8^3}{\omega^6}\\
\Longrightarrow \omega_2 &= \frac{2}{\omega^2} - \frac{8^2}{2}\frac{\lambda}{\omega^6}\\
&= \frac{2}{\omega^2} - \frac{32}{\omega^6}\lambda.
\end{aligned}$$

In the last line of equation (C.0.21) we see the results of perturbation theory up to order λ.

Below we will answer why the result (C.0.21) is important and what does this result mean. We first start with a perturbative calculation for the correlator of X_1^2, that is

$$\langle X_1 X_1 \rangle = \int \mathrm{d}X_1 \int \mathrm{d}X_1 \mathrm{tr}\left(X_1^2\right) e^{-V(X_1,X_2)} \qquad \text{(C.0.22)}$$

where

$$\begin{aligned}
V(X_1,X_2) &= \frac{\omega^2}{2}\mathrm{tr}\left(X_1^2\right) + \frac{\omega^2}{2}\mathrm{tr}\left(X_2^2\right) - 4\mathrm{tr}\left([X_1,X_2]^2\right) \qquad \text{(C.0.23)}\\
&= \frac{\omega^2}{2}\mathrm{tr}\left(X_1^2\right) + \frac{\omega^2}{2}\mathrm{tr}\left(X_2^2\right) - 8g_{YM}^2\mathrm{tr}\left(X_1X_2X_1X_2\right)\\
&- \mathrm{tr}\left(X_1^2\right)\mathrm{tr}\left(X_2^2\right).
\end{aligned}$$

It can be shown that for the two point correlator in equation (C.0.22) it is true that

$$\begin{aligned}
\langle \mathrm{tr}\left(X_1^2\right)\rangle &= \left(-8g_{YM}^2\right) 2\left(\frac{1}{\omega^2}\right)^3 N^3 \qquad \text{(C.0.24)}\\
&= -\frac{16}{\omega^6}\lambda.
\end{aligned}$$

Using perturbation theory, the planar diagrams, leading in N, with a single vertex arises from the term

$$e^{-8g_{YM}^2\mathrm{tr}\left(X_1^2X_2^2\right)}, \qquad \text{(C.0.25)}$$

and the non-planar diagrams that are sub leading in N would emerge from the expression

$$e^{-8g_{YM}^2 \mathrm{tr}(X_1 X_2 X_1 X_2)}. \tag{C.0.26}$$

In the first line equation (C.0.24) the first term, $(-8g_{YM}^2)$, is due to the single vertex of the planar diagram, the second factor is the symmetry factor. The factor $(1/\omega^2)^3$ is due to the three propagators that can be constructed from a Feynman diagram with a single vertex. The last factor N^3 comes from the three closed loops that are formed from a planar diagram with a single vertex. To see the result that agrees with the our calculation in equation (C.0.21), we consider the sum of the correlators of X_1^2 and X_2^2 such that

$$\langle \mathrm{tr}\left(ZZ^\dagger\right)\rangle \equiv \mathrm{tr}\left(X_1^2\right) + \mathrm{tr}\left(X_2^2\right) = -\frac{32}{\omega^6}\lambda. \tag{C.0.27}$$

The above result in equation (C.0.27) is true for $Z = X_1 + iX_2 \equiv RU$. Equation (C.0.27), which appears in the standard perturbation theory for the weak coupling expansion is the same as equation (7.3.25) whose results was obtained using the two matrix model in spherical coordinates.

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