J Opt (November 2024) 53(5):4113–4136
https://doi.org/10.1007/s12596-023-01392-7

RESEARCH ARTICLE

Optical solitons and conservation laws for the concatenation model in the absence of self-phase modulation

Ahmed H. Arnous[1] · **Anjan Biswas**[2,3,4,5] · **Abdul H. Kara**[6] · **Yakup Yıldırım**[7,8] · **Carmelia Mariana Balanica Dragomir**[4] · **Asim Asiri**[3]

Received: 1 August 2023 / Accepted: 26 August 2023 / Published online: 5 October 2023

Abstract This paper addresses the concatenation model with the absence of self-phase modulation. Two integration approaches, namely the generalized sine-Gordon equation method and the projective Riccati equation approach yielded a plethora of soliton solutions. The parameter constraints for the existence of such solitons are also presented. The conservation laws are also enumerated that are recovered with the multipliers approach.

Keywords Solitons · Sine-Gordon · Riccati · Concatenation · Self-phase modulation

✉ Anjan Biswas
biswas.anjan@gmail.com

1 Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El Shorouk Academy, Cairo, Egypt

2 Department of Mathematics and Physics, Grambling State University, Grambling, LA 71245-2715, USA

3 Mathematical Modeling and Applied Computation (MMAC) Research Group, Center of Modern Mathematical Sciences and their Applications (CMMSA), Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

4 Department of Applied Sciences, Cross-Border Faculty of Humanities, Economics and Engineering, Dunarea de Jos University of Galati, 111 Domneasca Street, Galati 800201, Romania

5 Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, South Africa

6 School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits, Johannesburg 2050, South Africa

7 Department of Computer Engineering, Biruni University, Istanbul 34010, Turkey

8 Department of Mathematics, Near East University, 99138 Nicosia, Cyprus

Introduction

The theory of optical solitons have made remarkable advances during the past half-a-century. The technology has advanced in unfathomable speed. There is still a lot to be achieved. Some of the factors that need improvements are the internet bottleneck effect, mitigating the noise effect and cross-talk during the soliton propagation, addressing the evolution of ghost pulses during soliton propagation and many others. There are a lot of means and measures that are continuously adopted to overcome these effects.

One of the new models that has been recently proposed is the conjunction of three well-known equation that dictate the propagation of solitons through optical fibers. These are the nonlinear Schrödinger's equation (NLSE), Lakshmanan–Porsezian–Daniel (LPD) equation and the Sasa–Satsuma equation (SSE). This is referred to as the concatenation model that was first conceived in 2014 [1, 2]. Later, a deluge of results have emerged and reported across the board. These range from the bifurcation analysis, numerical study of such solitons that emerged from the model, the Painleve analysis, conservation laws, quiescent solitons as well as studying the soliton dynamics with power-law of nonlinearity [3–14]. Subsequently, the model was extended to study it in birefringent fibers [6]. It is now time to move further on.

The current paper takes a look at the concatenation model that comes with the absence of self-phase modulation (SPM). Two integration approaches give way to a full spectrum of solitons that are enumerated in the paper. They are the generalized sine-Gordon equation approach and the projective Riccati equation method. The parameter constraints for the existence of such solitons also naturally fell out from the two schemes. Finally, the multipliers approach also led to the retrieval of the conservation laws, and the conserved quantities are computed with the usage of the bright solitons that emerged from the two integration schemes. The details are exhibited in the rest of the paper after a short ride through the introductory process.

Governing Model

The concatenation model in the absence of SPM is:

$$iq_t + aq_{xx} + c_1\left[\sigma_1 q_{xxxx} + \sigma_2 \left(q_x\right)^2 q^* + \sigma_3 \left|q_x\right|^2 q + \sigma_4 |q|^2 q_{xx} + \sigma_5 q^2 q^*_{xx} + \sigma_6 |q|^4 q\right] + ic_2\left[\sigma_7 q_{xxx} + \sigma_8 |q|^2 q_x + \sigma_9 q^2 q^*_x\right] = 0. \tag{1}$$

Equation (1) is the dimensionless form of the concatenation model that will be studied in this paper. Here in Eq. 4, $q(x, t)$ is the complex-valued function that represents the wave profile, where x and t are the two independent variables that represents the spatial and temporal coordinates, respectively. Also $i = \sqrt{-1}$. The first two terms in Eq. (1) is the NLSE with the missing SPM and the coefficient of a being the chromatic dispersion (CD). The first term is from the linear temporal evolution. The coefficient of c_1 comes from LPD equation, while the coefficient of c_2 comes from SSE. Thus, Eq. (1) is a concatenation of NLSE, LPD model and the SSE. This equation that will be studied in the rest of the paper to fetch its soliton solutions.

Assume the solution structure of Eq. (1) as follows:

$$q(x, t) = U(\xi)e^{i\phi(x,t)}, \tag{2}$$

where the wave variable ξ is given by

$$\xi = k(x - vt). \tag{3}$$

Here, $U(\xi)$ represents the amplitude component of the soliton solution and v is the speed of the soliton, while the phase component $\phi(x, t)$ is defined as

$$\phi(x, t) = -\kappa x + \omega t + \theta_0, \tag{4}$$

where ω is the frequency of the solitons, while κ represents the wave number, and θ_0 is the phase constant. Substituting (2) into (1) and then decomposing into real and imaginary parts

$$- k^2\left(a - 6c_1\kappa^2\sigma_1 + 3c_2\kappa\sigma_7\right)U'' + \left(a\kappa^2 - c_1\kappa^4\sigma_1 + c_2\kappa^3\sigma_7 + \omega\right)U - c_1k^4\sigma_1 U'''' - c_1k^2\left(\sigma_4 + \sigma_5\right)U^2U'' - c_1k^2\left(\sigma_2 + \sigma_3\right)UU'^2 + \kappa\left(c_1\kappa\left(\sigma_2 - \sigma_3 + \sigma_4 + \sigma_5\right) + c_2\left(\sigma_9 - \sigma_8\right)\right)U^3 - c_1\sigma_6 U^5 = 0, \tag{5}$$

and

$$k\left(2a\kappa - 4c_1\kappa^3\sigma_1 + 3c_2\kappa^2\sigma_7 + v\right)U' + k^3\left(4c_1\kappa\sigma_1 - c_2\sigma_7\right)U''' + k\left(2c_1\kappa\left(\sigma_2 + \sigma_4 - \sigma_5\right) - c_2\left(\sigma_8 + \sigma_9\right)\right)U^2U' = 0. \tag{6}$$

From the imaginary part, the soliton speed reaches

$$v = -2a\kappa + 4c_1\kappa^3\sigma_1 - 3c_2\kappa^2\sigma_7, \tag{7}$$

while the wave number reads

$$\kappa = \frac{c_2\sigma_7}{4c_1\sigma_1}, \tag{8}$$

with parametric restriction

$$2c_1\kappa\left(\sigma_2 + \sigma_4 - \sigma_5\right) - c_2\left(\sigma_8 + \sigma_9\right) = 0. \tag{9}$$

Equation (5) can be simplified as

$$k^2U'''' + s_6U^2U'' + s_5U'' + s_4UU'^2 + s_3U^5 + s_2U^3 + s_1U = 0. \tag{10}$$

where

$$\begin{cases} s_1 = -\dfrac{a\kappa^2 - c_1\kappa^4\sigma_1 + c_2\kappa^3\sigma_7 + \omega}{c_1k^2\sigma_1}, \\ s_2 = -\dfrac{\kappa\left(c_1\kappa\left(\sigma_2 - \sigma_3 + \sigma_4 + \sigma_5\right) + c_2\left(\sigma_9 - \sigma_8\right)\right)}{c_1k^2\sigma_1}, \\ s_3 = \dfrac{\sigma_6}{k^2\sigma_1}, \quad s_4 = \dfrac{\sigma_2 + \sigma_3}{\sigma_1}, \\ s_5 = \dfrac{a - 6c_1\kappa^2\sigma_1 + 3c_2\kappa\sigma_7}{c_1\sigma_1}, \quad s_6 = \dfrac{\sigma_4 + \sigma_5}{\sigma_1}. \end{cases} \tag{11}$$

An overview of the integration algorithms

Consider a governing model

$$F(u, u_x, u_t, u_{xt}, u_{xx}, \ldots) = 0, \tag{12}$$

where $u = u(x, t)$ denotes a wave profile, while t and x depict the time and space variables in sequence.

The relations

$$u(x, t) = U(\xi), \quad \xi = k(x - \upsilon t), \tag{13}$$

condense Eq. (12) to

$$P(U, -k\upsilon U', kU', k^2U'', \ldots) = 0, \tag{14}$$

where k is the wave width, ξ is the wave variable, and υ is the wave velocity.

Generalized sine-Gordon equation method

The performance steps for the generalized sine-Gordon equation method are:

Step 1 Assume Eq. (14) has the formal solution

$$U(\xi) = A_0 + \sum_{j=1}^{N} \cos[G(\xi)]^{j-1}\left(A_j \sin[G(\xi)] + B_j \cos[G(\xi)]\right), \tag{15}$$

along with the general sine-Gordon travelling wave reduction equation

$$G'(\xi) = \sqrt{\lambda + \mu \sin[G(\xi)]^2}. \tag{16}$$

Step 2 Equation (16) possesses the following cases:

Case-1: $\lambda = 0,\ \mu = 1$

$$G'(\xi) = \mp \sin[G(\xi)], \tag{17}$$

which gives the hyperbolic function solutions

$$\sin[G(\xi)] = \operatorname{sech}[\xi], \quad \text{and} \quad \cos[G(\xi)] = \pm \tanh[\xi], \tag{18}$$

or

$$\sin[G(\xi)] = \pm i \operatorname{csch}[\xi], \quad \text{and} \quad \cos[G(\xi)] = \pm \coth[\xi]. \tag{19}$$

Case-2: $\lambda = 1,\ \mu = -m^2$

$$G'(\xi) = \sqrt{1 - m^2 \sin[G(\xi)]^2}, \tag{20}$$

which gives the Jacobi's elliptic function solutions

$$\sin[G(\xi)] = \operatorname{sn}[\xi;m], \quad \text{and} \quad \cos[G(\xi)] = \operatorname{cn}[\xi;m], \tag{21}$$

or

$$\sin[G(\xi)] = \frac{1}{m}\operatorname{ns}[\xi;m], \quad \text{and} \quad \cos[G(\xi)] = -\frac{i}{m}\operatorname{ds}[\xi;m]. \tag{22}$$

Case-3: $\lambda = m^2,\ \mu = -1$

$$G'(\xi) = \sqrt{m^2 - \sin[G(\xi)]^2}, \tag{23}$$

which gives the Jacobi's elliptic function solutions

$$\sin[G(\xi)] = m \operatorname{sn}[\xi;m], \quad \text{and} \quad \cos[G(\xi)] = \operatorname{dn}[\xi;m], \tag{24}$$

or

$$\sin[G(\xi)] = \operatorname{ns}[\xi;m], \quad \text{and} \quad \cos[G(\xi)] = -i \operatorname{cs}[\xi;m], \tag{25}$$

where $i = \sqrt{-1}$.

Step 3 Substituting Eq. (15) along with Eq. (16) into Eq. (14), we get a polynomial in $\sin[G(\xi)]$ and $\cos[G(\xi)]$ which equal to zero. The obtained coefficients of this polynomial give the needed parameters in Eq. (13) and Eq. (15).

Remark 1 The convergence of Jacobi's elliptic functions toward their hyperbolic function limits as m tends to unity is listed as

$$\begin{aligned}
&\lim_{m\to1} \operatorname{cn}[\xi;m] = \lim_{m\to1} \operatorname{dn}[\xi;m] = \operatorname{sech}[\xi],\\
&\lim_{m\to1} \operatorname{sn}[\xi;m] = \tanh[\xi],\\
&\lim_{m\to1} \operatorname{cs}[\xi;m] = \lim_{m\to1} \operatorname{ds}[\xi;m] = \operatorname{csch}[\xi],\\
&\lim_{m\to1} \operatorname{ns}[\xi;m] = \coth[\xi].
\end{aligned} \tag{26}$$

Projective Riccati equation method

The algorithmic process of the projective Riccati equation's method is as follows:

Step 1 Assume Eq. (14) has the formal solution

$$U(\xi) = a_0 + \sum_{i=1}^{N} \psi^{i-1}(\xi)\Big(a_i\psi(\xi) + b_i\phi(\xi)\Big), \tag{27}$$

where $\psi(\xi)$ and $\phi(\xi)$ satisfy the following ODEs:

$$\begin{aligned}
&\psi'(\xi) = -\psi(\xi)\phi(\xi),\\
&\phi'(\xi) = 1 - \phi^2(\xi) - r\psi(\xi),
\end{aligned} \tag{28}$$

with

$$\phi(\xi)^2 = 1 - 2r\psi(\xi) + R(r)\psi(\xi)^2, \tag{29}$$

where r is constant and N a positive integer comes from the balancing principle in Eq. (14). Also, a_0, a_i and $b_i (i = 0, 1, ..., N)$ are constants.

Step 2 The solutions of Eq. (28) are listed as follows:

Case-1: $R(r) = 0$

$$\psi(\xi) = \frac{1}{2r}\operatorname{sech}^2\left[\frac{\xi}{2}\right], \quad \text{and} \quad \phi(\xi) = \tanh\left[\frac{\xi}{2}\right], \tag{30}$$

or

$$\psi(\xi) = -\frac{1}{2r}\operatorname{csch}^2\left[\frac{\xi}{2}\right], \quad \text{and} \quad \phi(\xi) = \coth\left[\frac{\xi}{2}\right]. \tag{31}$$

Case-2: $R(r) = \frac{24}{25}r^2$

$$\psi(\xi)=\frac{1}{r}\frac{5\ \mathrm{sech}[\xi]}{5\ \mathrm{sech}[\xi]\pm 1},\quad \text{and}\quad \phi(\xi)=\frac{\tanh[\xi]}{1\pm 5\ \mathrm{sech}[\xi]}. \tag{32}$$

Case-3: $R(r)=\frac{5}{9}r^2$

$$\psi(\xi)=\frac{1}{r}\frac{3\ \mathrm{sech}[\xi]}{3\ \mathrm{sech}[\xi]\pm 2},\quad \text{and}\quad \phi(\xi)=\frac{2}{2\ \coth[\xi]\pm 3\ \mathrm{csch}[\xi]}. \tag{33}$$

Case-4: $R(r)=r^2-1$

$$\psi(\xi)=\frac{4\ \mathrm{sech}[\xi]}{3\ \tanh[\xi]+4r\ \mathrm{sech}[\xi]+5},\quad \text{and}\quad \phi(\xi)=\frac{5\ \tanh[\xi]+3}{3\ \tanh[\xi]+4r\ \mathrm{sech}[\xi]+5}, \tag{34}$$

or

$$\psi(\xi)=\frac{\mathrm{sech}[\xi]}{r\ \mathrm{sech}[\xi]+1},\quad \text{and}\quad \phi(\xi)=\frac{\tanh[\xi]}{r\ \mathrm{sech}[\xi]+1}. \tag{35}$$

Case-5: $R(r)=r^2+1$

$$\psi(\xi)=\frac{\mathrm{csch}[\xi]}{r\ \mathrm{csch}[\xi]+1},\quad \text{and}\quad \phi(\xi)=\frac{\coth[\xi]}{r\ \mathrm{csch}[\xi]+1}. \tag{36}$$

Step 3: Inserting Eq. (27) along with Eq. (28) and Eq. (29) into Eq. (14), we get a polynomial of $\psi(\xi)$ and $\phi(\xi)$ which equal to zero. The obtained coefficients of this polynomial give the needed parameters in Eq. (13) and Eq. (27).

Optical solitons

The current section will apply the two integration algorithms to derive the soliton solutions to the concatenation model that is with no SPM. The subsequent two subsections detail the derivation.

The generalized sine-Gordon equation method

Balancing U'''' with U^5 in Eq. (10) gives $N=1$; accordingly, the solution takes the form

$$U(\xi)=A_0+A_1\sin[G(\xi)]+B_1\cos[G(\xi)]. \tag{37}$$

Inserting Eq. (37) together with Eq. (28) and Eq. (29) into Eq. (10), we get a system of algebraic equations

$$A_1\left(-A_1^2\left(10B_1^2s_3+\mu\left(s_4+2s_6\right)\right)+A_1^4s_3+3B_1^2\mu\left(s_4+2s_6\right)+5B_1^4s_3+24k^2\mu^2\right)=0, \tag{38}$$

$$2A_0A_1B_1\left(-10A_1^2s_3+10B_1^2s_3+\mu\left(s_4+4s_6\right)\right)=0, \tag{39}$$

$$A_1\left(A_1^2\left(-10A_0^2s_3+10B_1^2s_3+\lambda s_4+\lambda s_6+\mu s_4+3\mu s_6-s_2\right)+2\mu\left(A_0^2s_6-2k^2(5\lambda+7\mu)+s_5\right)\right.$$
$$\left.+B_1^2\left(30A_0^2s_3-3\lambda s_4-3\lambda s_6-4\mu s_4-5\mu s_6+3s_2\right)-2A_1^4s_3\right)=0, \tag{40}$$

$$2A_0A_1B_1\left(10A_0^2s_3+10A_1^2s_3-\lambda s_4-2\lambda s_6-\mu s_4-3\mu s_6+3s_2\right)=0, \tag{41}$$

$$A_1\left(A_0^2\left(10A_1^2s_3-s_6(\lambda+\mu)+3s_2\right)-A_1^2\lambda s_6-A_1^2\mu s_6+5A_0^4s_3+A_1^2s_2+A_1^4s_3\right.$$
$$\left.+B_1^2\lambda s_4+B_1^2\mu s_4+k^2\lambda^2+6k^2\lambda\mu+5k^2\mu^2-\lambda s_5-\mu s_5+s_1\right)=0, \tag{42}$$

$$B_1\left(-A_1^2\left(10B_1^2s_3+3\mu\left(s_4+2s_6\right)\right)+5A_1^4s_3+B_1^2\mu\left(s_4+2s_6\right)+B_1^4s_3+24k^2\mu^2\right)=0, \tag{43}$$

$$A_0\left(-A_1^2\left(30B_1^2s_3+\mu\left(s_4+4s_6\right)\right)+5A_1^4s_3+B_1^2\mu\left(s_4+4s_6\right)+5B_1^4s_3\right)=0, \tag{44}$$

$$B_1^2\left(10A_0^2s_3-\left(s_4+s_6\right)(\lambda+2\mu)+s_2\right)+2\mu\left(A_0^2s_6-10k^2(\lambda+2\mu)+s_5\right)-10A_1^4s_3$$
$$+B_1\left(A_1^2\left(-30A_0^2s_3+10B_1^2s_3+3\lambda s_4+3\lambda s_6+5\mu s_4+10\mu s_6-3s_2\right)\right)=0, \tag{45}$$

$$A_0\left(A_1^2\left(-10A_0^2s_3+30B_1^2s_3+\lambda s_4+2\lambda s_6+\mu s_4+6\mu s_6-3s_2\right)+B_1^2\left(10A_0^2s_3-\left(s_4+2s_6\right)(\lambda+2\mu)+3s_2\right)-10A_1^4s_3\right)=0, \tag{46}$$

$$B_1\left(A_0^2\left(30A_1^2s_3-s_6(\lambda+2\mu)+3s_2\right)-2A_1^2\lambda s_4-3A_1^2\lambda s_6-2A_1^2\mu s_4-4A_1^2\mu s_6+5A_0^4s_3+3A_1^2s_2+5A_1^4s_3+B_1^2\lambda s_4+B_1^2\mu s_4+k^2\lambda^2+16k^2\lambda\mu+16k^2\mu^2-\lambda s_5-2\mu s_5+s_1\right)=0, \tag{47}$$

$$A_0\left(-2A_1^2\lambda s_6-2A_1^2\mu s_6+A_0^4s_3+A_0^2\left(10A_1^2s_3+s_2\right)+3A_1^2s_2+5A_1^4s_3+B_1^2\lambda s_4+B_1^2\mu s_4+s_1\right)=0. \tag{48}$$

Solving these equations together yields the following results.

Case-1:

Result-1:

$$A_0=A_1=0,\ B_1=\pm\sqrt{\frac{6s_5-5s_1}{2s_2+s_4-4s_6}},\ k=\frac{1}{2}\sqrt{\frac{2s_5\left(s_2-s_4-2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)}{2\left(2s_2+s_4-4s_6\right)}}$$
$$s_3=\frac{\left(2s_2+s_4-4s_6\right)\left(s_1\left(3s_2-s_4+4s_6\right)-6s_2s_5\right)}{\left(5s_1-6s_5\right)^2}. \tag{49}$$

Plugging the obtained parameters in Eq. (49) with Eq. (18) or Eq. (19) into Eq. (37), as a consequence, we get dark and singular solitons with $6s_5-5s_1>0$, $2s_2+s_4--4s_6>0$, and $2s_5\left(s_2-s_4--2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)>0$, as shown below

$$q(x,t)=\pm\sqrt{\frac{6s_5-5s_1}{2s_2+s_4-4s_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{2s_5\left(s_2-s_4-2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)}{2\left(2s_2+s_4-4s_6\right)}}(x-vt)\right]\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}, \tag{50}$$

or

$$q(x,t)=\pm\sqrt{\frac{6s_5-5s_1}{2s_2+s_4-4s_6}}\coth\left[\frac{1}{2}\sqrt{\frac{2s_5\left(s_2-s_4-2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)}{2\left(2s_2+s_4-4s_6\right)}}(x-vt)\right]\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{51}$$

Result-2:

$$A_0=B_1=0,\ A_1=\pm\sqrt{-\frac{2\left(10s_1+9s_5\right)}{s_2+s_4+s_6}},\ k=\sqrt{-\left(s_1+s_5\right)},$$
$$s_3=\frac{\left(3s_5\left(4s_2+s_4-2s_6\right)+2s_1\left(6s_2+s_4-4s_6\right)\right)\left(s_2+s_4+s_6\right)}{2\left(10s_1+9s_5\right)^2}. \tag{52}$$

Substituting the parameters acquired in Eq. (52) with Eq. (18) or Eq. (19) into Eq. (37), as a consequence, we get bright soliton with $2\left(10s_1+9s_5\right)\left(s_2+s_4+s_6\right)<0$, $s_1+s_5<0$, and singular soliton with $2\left(10s_1+9s_5\right)\left(s_2+s_4+s_6\right)>0$, $s_1+s_5<0$, as presented below

$$q(x,t)=\pm\sqrt{-\frac{2\left(10s_1+9s_5\right)}{s_2+s_4+s_6}}\operatorname{sech}\left[\sqrt{-\left(s_1+s_5\right)}(x-vt)\right]e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}, \tag{53}$$

or

$$q(x,t)=\pm\sqrt{\frac{2\left(10s_1+9s_5\right)}{s_2+s_4+s_6}}\operatorname{csch}\left[\sqrt{-\left(s_1+s_5\right)}(x-vt)\right]e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{54}$$

Result-3:

$$A_0=0,\ A_1=\pm\sqrt{\frac{20s_1-6s_5}{8s_2+s_4-4s_6}},\ B_1=\pm\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}},$$
$$k=\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}},$$
$$s_3=\frac{\left(8s_2+s_4-4s_6\right)\left(s_1\left(12s_2-s_4+4s_6\right)-6s_2s_5\right)}{4\left(10s_1-3s_5\right)^2}. \tag{55}$$

Putting the derived parameters in Eq. (55) with Eq. (18) or Eq. (19) into Eq. (37), as a consequence, we get a complexiton, dark soliton and singular soliton solutions with $6s_5--20s_1>0$, $8s_2+s_4--4s_6>0$ and $4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4--2s_6\right)>0$, as given below

$$q(x,t)=\pm\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}}\left\{\tanh\left[\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}}(x-vt)\right]\right.$$
$$\left.\pm i\operatorname{sech}\left[\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}}(x-vt)\right]\right\}e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}, \tag{56}$$

or

$$q(x,t)=\pm\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}}(x-vt)\right]$$
$$\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}, \tag{57}$$

and

$$q(x,t)=\pm\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}}\coth\left[\frac{1}{2}\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}}(x-vt)\right]$$
$$\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{58}$$

Case-2:

Result-1:

$$A_0 = A_1 = 0, B_1 = \pm\sqrt{\frac{2m^2\left(10\left(2m^2-1\right)s_1+3\left(8m^4-8m^2+3\right)s_5\right)}{\left(-16m^4+16m^2-1\right)s_2+\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)s_4+\left(-16m^4+16m^2-1\right)s_6\right)}},$$
$$k = \sqrt{\frac{s_1\left(\left(2m^2-1\right)\left(s_4+s_6\right)+s_2\right)+s_5\left(\left(1-2m^2\right)^2 s_6+\left(2m^2-1\right)s_2+\left(2m^4-2m^2+1\right)s_4\right)}{\left(-16m^4+16m^2-1\right)s_2+\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)s_4+\left(-16m^4+16m^2-1\right)s_6\right)}},$$
$$s_3 = -\frac{1}{2\left(10\left(2m^2-1\right)s_1+3\left(8m^4-8m^2+3\right)s_5\right)^2}$$
$$\times\left(3s_5\left(\left(8m^2-4\right)s_2+s_4-2s_6\right)+2s_1\left(\left(2m^2-1\right)\left(s_4-4s_6\right)+6s_2\right)\right)$$
$$\times\left(\left(1-2m^2\left(3-4m^2\right)^2\right)s_6+\left(-16m^4+16m^2-1\right)s_2+\left(8m^6-12m^4+2m^2+1\right)s_4\right). \tag{59}$$

Inserting the parameters attained in Eq. (59) with Eq. (21) and Eq. (22) into Eq. (37), as a consequence, we get Jacobi's elliptic-type function solution

$$q(x,t) = \pm\sqrt{\frac{2m^2\left(10\left(2m^2-1\right)s_1+3\left(8m^4-8m^2+3\right)s_5\right)}{\left(-16m^4+16m^2-1\right)s_2+\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)s_4+\left(-16m^4+16m^2-1\right)s_6\right)}}$$
$$\text{cn}\left[\sqrt{\frac{s_1\left(\left(2m^2-1\right)\left(s_4+s_6\right)+s_2\right)+s_5\left(\left(1-2m^2\right)^2 s_6+\left(2m^2-1\right)s_2+\left(2m^4-2m^2+1\right)s_4\right)}{\left(-16m^4+16m^2-1\right)s_2+\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)s_4+\left(-16m^4+16m^2-1\right)s_6\right)}}(x-vt);m\right]$$
$$\times\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{60}$$

When the modulus of ellipticity approaches unity, we get a bright soliton solution with $2\left(10s_1+9s_5\right)\left(s_2+s_4+s_6\right)<0$, and $s_1+s_5<0$, as given below

$$q(x,t) = \pm\sqrt{-\frac{2\left(10s_1+9s_5\right)}{s_2+s_4+s_6}}\text{sech}\left[\sqrt{-\left(s_1+s_5\right)}(x-vt)\right]\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{61}$$

Also, we have the following Jacobi's elliptic-type function solution

$$q(x,t) = \mp\sqrt{-\frac{2\left(10\left(2m^2-1\right)s_1+3\left(8m^4-8m^2+3\right)s_5\right)}{\left(-16m^4+16m^2-1\right)s_2+\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)s_4+\left(-16m^4+16m^2-1\right)s_6\right)}}$$
$$\text{ds}\left[\sqrt{\frac{s_1\left(\left(2m^2-1\right)\left(s_4+s_6\right)+s_2\right)+s_5\left(\left(1-2m^2\right)^2 s_6+\left(2m^2-1\right)s_2+\left(2m^4-2m^2+1\right)s_4\right)}{\left(-16m^4+16m^2-1\right)s_2+\left(2m^2-1\right)\left(\left(4m^4-4m^2-1\right)s_4+\left(-16m^4+16m^2-1\right)s_6\right)}}(x-vt);m\right]$$
$$\times\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{62}$$

When the modulus of ellipticity approaches unity, we get a singular soliton solution with

$2\left(10s_1+9s_5\right)(s_2+s_4+s_6)>0$, and $s_1+s_5<0$, as indicated below

$$q(x,t)=\pm\sqrt{\frac{2\left(10s_1+9s_5\right)}{s_2+s_4+s_6}}\operatorname{csch}\left[\sqrt{-\left(s_1+s_5\right)}(x-vt)\right]e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{63}$$

Result-2:

$$\begin{aligned}
&A_0=B_1=0, A_1=\pm\sqrt{-\frac{2m^2\left(3\left(3m^4+2m^2+3\right)s_5-10\left(m^2+1\right)s_1\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}},\\
&k=\sqrt{\frac{s_1\left(s_2-\left(m^2+1\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^4+1\right)s_4+\left(m^2+1\right)^2s_6-\left(m^2+1\right)s_2\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}},\\
&s_3=-\frac{1}{2\left(10\left(m^2+1\right)s_1-3\left(3m^4+2m^2+3\right)s_5\right)^2}\\
&\quad\times 3s_5\left(\left(m^2-1\right)^2\left(s_4-2s_6\right)-4\left(m^2+1\right)s_2\right)+2s_1\left(6s_2-\left(m^2+1\right)\left(s_4-4s_6\right)\right)\\
&\quad\times\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2.
\end{aligned} \tag{64}$$

Plugging the parameters secured in Eq. (64) with Eq. (21) and Eq. (22) into Eq. (37), as a consequence, we get Jacobi's elliptic-type function solution

$$\begin{aligned}
q(x,t)=&\pm\sqrt{-\frac{2m^2\left(3\left(3m^4+2m^2+3\right)s_5-10\left(m^2+1\right)s_1\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}}\\
&\operatorname{sn}\left[\sqrt{\frac{s_1\left(s_2-\left(m^2+1\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^4+1\right)s_4+\left(m^2+1\right)^2s_6-\left(m^2+1\right)s_2\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}}(x-vt);m\right]\\
&\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}.
\end{aligned} \tag{65}$$

When the modulus of ellipticity approaches unity, we get a dark soliton solution with $6s_5-5s_1>0$, $2s_2+s_4--4s_6$, and $2s_5\left(s_2-s_4--2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)>0$, as described below

Also, we have the following Jacobi's elliptic-type function solution

$$\begin{aligned}
q(x,t)=&\pm\sqrt{\frac{6s_5-5s_1}{2s_2+s_4-4s_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{2s_5\left(s_2-s_4-2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)}{2\left(2s_2+s_4-4s_6\right)}}(x-vt)\right]\\
&\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}.
\end{aligned} \tag{66}$$

$$q(x,t)=\pm\sqrt{-\frac{2\left(3\left(3m^4+2m^2+3\right)s_5-10\left(m^2+1\right)s_1\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}}$$
$$\mathrm{ns}\left[\sqrt{\frac{s_1\left(s_2-\left(m^2+1\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^4+1\right)s_4+\left(m^2+1\right)^2 s_6-\left(m^2+1\right)s_2\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}}(x-vt);m\right]$$
$$\times\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{67}$$

When the modulus of ellipticity approaches unity, we get a singular soliton solution with $6s_5-5s_1>0, 2s_2+s_4--4s_6$, and $2s_5\left(s_2-s_4--2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)>0$, as shown below

$$q(x,t)=\pm\sqrt{\frac{6s_5-5s_1}{2s_2+s_4-4s_6}}\coth\left[\frac{1}{2}\sqrt{\frac{2s_5\left(s_2-s_4-2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)}{2\left(2s_2+s_4-4s_6\right)}}(x-vt)\right]$$
$$\times\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{68}$$

Result-3:

$$A_0=,\ A_1=\pm\sqrt{-\frac{2m^2\left(10\left(m^2-2\right)s_1+3\left(m^4-6m^2+6\right)s_5\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}},$$
$$B_1=\pm\sqrt{\frac{2m^2\left(10\left(m^2-2\right)s_1+3\left(m^4-6m^2+6\right)s_5\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}},$$
$$k=\sqrt{\frac{4s_1\left(\left(m^2-2\right)\left(s_4+s_6\right)+2s_2\right)+s_5\left(2\left(m^2-2\right)^2 s_6+4\left(m^2-2\right)s_2+\left(m^4-8m^2+8\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}},$$
$$s_3=-\frac{1}{4\left(10\left(m^2-2\right)s_1+3\left(m^4-6m^2+6\right)s_5\right)^2}$$
$$\times 3s_5\left(2\left(m^2-2\right)s_2-\left(m^2-1\right)\left(s_4-2s_6\right)\right)+s_1\left(\left(m^2-2\right)\left(s_4-4s_6\right)+12s_2\right)$$
$$\times 8\left(m^4-m^2+1\right)s_2-\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right). \tag{69}$$

Substituting the parameters achieved in Eq. (69) with Eq. (21) and Eq. (22) into Eq. (37), as a consequence, we get Jacobi's elliptic-type function solution

$$q(x,t) = \pm\sqrt{\frac{2m^2\left(10\left(m^2-2\right)s_1+3\left(m^4-6m^2+6\right)s_5\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}}$$
$$\Bigg(\mathrm{cn}\left[\sqrt{\frac{4s_1\left(\left(m^2-2\right)\left(s_4+s_6\right)+2s_2\right)+s_5\left(2\left(m^2-2\right)^2s_6+4\left(m^2-2\right)s_2+\left(m^4-8m^2+8\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}}(x-vt);m\right]$$
$$+\,i\,\mathrm{sn}\left[\sqrt{\frac{4s_1\left(\left(m^2-2\right)\left(s_4+s_6\right)+2s_2\right)+s_5\left(2\left(m^2-2\right)^2s_6+4\left(m^2-2\right)s_2+\left(m^4-8m^2+8\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}}(x-vt);m\right]\Bigg)$$
$$\times\, e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{70}$$

When the modulus of ellipticity approaches unity, we get a complexiton soliton solution with $6s_5-5s_1>0$, $2s_2+s_4--4s_6$, and $2s_5\left(s_2-s_4--2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)>0$, as presented below

When the modulus of ellipticity approaches unity, we get dark soliton solution with $6s_5-5s_1>0$, $2s_2+s_4--4s_6$, and $2s_5\left(s_2-s_4--2s_6\right)+s_1\left(2\left(s_4+s_6\right)-s_2\right)>0$, as given below

$$q(x,t) = \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}}\left\{-\tanh\left[\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}}(x-vt)\right]\right.$$
$$\left.+\,i\,\mathrm{sech}\left[\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}}(x-vt)\right]\right\} e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{71}$$

Also, we have the following Jacobi's elliptic-type function solution

$$q(x,t) = \pm\sqrt{-\frac{2\left(10\left(m^2-2\right)s_1+3\left(m^4-6m^2+6\right)s_5\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}}$$
$$\Bigg(\mathrm{ns}\left[\sqrt{\frac{4s_1\left(\left(m^2-2\right)\left(s_4+s_6\right)+2s_2\right)+s_5\left(2\left(m^2-2\right)^2s_6+4\left(m^2-2\right)s_2+\left(m^4-8m^2+8\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}}(x-vt);m\right]$$
$$-\,\mathrm{ds}\left[\sqrt{\frac{4s_1\left(\left(m^2-2\right)\left(s_4+s_6\right)+2s_2\right)+s_5\left(2\left(m^2-2\right)^2s_6+4\left(m^2-2\right)s_2+\left(m^4-8m^2+8\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4-4\left(m^4-m^2+1\right)s_6\right)-8\left(m^4-m^2+1\right)s_2}}(x-vt);m\right]\Bigg)$$
$$\times\, e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{72}$$

$$q(x,t)=\pm\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}}(x-vt)\right]\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{73}$$

Case-3:

Result-1:

$$A_0=A_1=0, B_1=\pm\sqrt{\frac{6\left(3m^4-8m^2+8\right)s_5-20\left(m^2-2\right)s_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4+\left(m^4-16m^2+16\right)s_6\right)-\left(m^4-16m^2+16\right)s_2}},$$
$$k=\sqrt{\frac{s_1\left(s_2-\left(m^2-2\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^2-2\right)^2s_6-\left(m^2-2\right)s_2+\left(m^4-2m^2+2\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4+\left(m^4-16m^2+16\right)s_6\right)-\left(m^4-16m^2+16\right)s_2}}, \tag{74}$$
$$s_3=-\frac{1}{2\left(10\left(m^2-2\right)s_1-3\left(3m^4-8m^2+8\right)s_5\right)^2}\times\left(2s_1\left(6s_2-\left(m^2-2\right)\left(s_4-4s_6\right)\right)+3s_5\left(m^4\left(s_4-2s_6\right)-4\left(m^2-2\right)s_2\right)\right)\times\left(\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4+\left(m^4-16m^2+16\right)s_6\right)-\left(m^4-16m^2+16\right)s_2\right).$$

Putting the obtained parameters in Eq. (74) with Eq. (24) and Eq. (25) into Eq. (37), as a consequence, we get Jacobi's elliptic-type function solution

$$q(x,t)=\pm\sqrt{\frac{6\left(3m^4-8m^2+8\right)s_5-20\left(m^2-2\right)s_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4+\left(m^4-16m^2+16\right)s_6\right)-\left(m^4-16m^2+16\right)s_2}}\left(\mathrm{dn}\left[\sqrt{\frac{s_1\left(s_2-\left(m^2-2\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^2-2\right)^2s_6-\left(m^2-2\right)s_2+\left(m^4-2m^2+2\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4+\left(m^4-16m^2+16\right)s_6\right)-\left(m^4-16m^2+16\right)s_2}}(x-vt);m\right]\right)\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{75}$$

When the modulus of ellipticity approaches unity, we get a bright soliton solution

$$q(x,t)=\pm\sqrt{-\frac{2\left(10s_1+9s_5\right)}{s_2+s_4+s_6}}\,\mathrm{sech}\left[\sqrt{-\left(s_1+s_5\right)}(x-vt)\right]e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{76}$$

Also, we have the following Jacobi's elliptic-type function solution

$$q(x,t)=\mp\sqrt{-\frac{6\left(3m^4-8m^2+8\right)s_5-20\left(m^2-2\right)s_1}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4+\left(m^4-16m^2+16\right)s_6\right)-\left(m^4-16m^2+16\right)s_2}}$$
$$\left(\operatorname{cs}\left[\sqrt{\frac{s_1\left(s_2-\left(m^2-2\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^2-2\right)^2s_6-\left(m^2-2\right)s_2+\left(m^4-2m^2+2\right)s_4\right)}{\left(m^2-2\right)\left(\left(m^4+4m^2-4\right)s_4+\left(m^4-16m^2+16\right)s_6\right)-\left(m^4-16m^2+16\right)s_2}}(x-vt);m\right]\right)$$
$$\times\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{77}$$

When the modulus of ellipticity approaches unity, we get a singular soliton solution

$$q(x,t)=\mp\sqrt{\frac{2\left(10s_1+9s_5\right)}{s_2+s_4+s_6}}\operatorname{csch}\left[\sqrt{-\left(s_1+s_5\right)}(x-vt)\right]\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{78}$$

Result-2:

$$A_0=B_1=0,A_1=\pm\sqrt{\frac{20\left(m^2+1\right)s_1-6\left(3m^4+2m^2+3\right)s_5}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}},$$
$$k=\sqrt{\frac{s_1\left(s_2-\left(m^2+1\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^4+1\right)s_4+\left(m^2+1\right)^2s_6-\left(m^2+1\right)s_2\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}},$$
$$s_3=-\frac{1}{2\left(10\left(m^2+1\right)s_1-3\left(3m^4+2m^2+3\right)s_5\right)^2}$$
$$\times\left(3s_5\left(\left(m^2-1\right)^2\left(s_4-2s_6\right)-4\left(m^2+1\right)s_2\right)+2s_1\left(6s_2-\left(m^2+1\right)\left(s_4-4s_6\right)\right)\right)$$
$$\times\left(\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2\right). \tag{79}$$

Inserting the parameters acquired in Eq. (79) with Eq. (24) and Eq. (25) into Eq. (37), as a consequence, we get Jacobi's elliptic-type function solution

$$q(x,t)=\pm\sqrt{\frac{m^2\left(20\left(m^2+1\right)s_1-6\left(3m^4+2m^2+3\right)s_5\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}}$$
$$\left(\operatorname{sn}\left[\sqrt{\frac{s_1\left(s_2-\left(m^2+1\right)\left(s_4+s_6\right)\right)+s_5\left(\left(m^4+1\right)s_4+\left(m^2+1\right)^2s_6-\left(m^2+1\right)s_2\right)}{\left(m^2+1\right)\left(\left(m^4-6m^2+1\right)s_4+\left(m^4+14m^2+1\right)s_6\right)-\left(m^4+14m^2+1\right)s_2}}(x-vt);m\right]\right)$$
$$\times\ e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{80}$$

When the modulus of ellipticity approaches unity, we get dark soliton solution

$$q(x,t) = \pm\sqrt{\frac{6s_5 - 5s_1}{2s_2 + s_4 - 4s_6}} \tanh\left[\frac{1}{2}\sqrt{\frac{s_5(-2s_2 + 2s_4 + 4s_6) + s_1(s_2 - 2(s_4 + s_6))}{2(4s_6 - s_4) - 4s_2}}(x - vt)\right] \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x + \omega t + \theta_0\right)}. \quad (81)$$

Also, we have the following Jacobi's elliptic-type function solution

$$q(x,t) = \pm\sqrt{\frac{20(m^2+1)s_1 - 6(3m^4+2m^2+3)s_5}{(m^2+1)\left((m^4-6m^2+1)s_4 + (m^4+14m^2+1)s_6\right) - (m^4+14m^2+1)s_2}} \left(\mathrm{ns}\left(\sqrt{\frac{s_1\left(s_2 - (m^2+1)(s_4+s_6)\right) + s_5\left((m^4+1)s_4 + (m^2+1)^2 s_6 - (m^2+1)s_2\right)}{(m^2+1)\left((m^4-6m^2+1)s_4 + (m^4+14m^2+1)s_6\right) - (m^4+14m^2+1)s_2}}(x - tv); m\right)\right) \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x + \omega t + \theta_0\right)}. \quad (82)$$

When the modulus of ellipticity approaches unity, we get singular soliton solution

$$q(x,t) = \pm\sqrt{\frac{6s_5 - 5s_1}{2s_2 + s_4 - 4s_6}} \coth\left[\frac{1}{2}\sqrt{\frac{s_5(-2s_2 + 2s_4 + 4s_6) + s_1(s_2 - 2(s_4 + s_6))}{2(4s_6 - s_4) - 4s_2}}(x - vt)\right] \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x + \omega t + \theta_0\right)}. \quad (83)$$

Result-3:

$$\begin{aligned}
A_0 &= 0, A_1 = \pm\sqrt{\frac{20(2m^2-1)s_1 - 6(6m^4-6m^2+1)s_5}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}},\\
B_1 &= \pm\sqrt{\frac{(20-40m^2)s_1 + 6(6m^4-6m^2+1)s_5}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}},\\
k &= \sqrt{\frac{4s_1\left(2s_2 - (2m^2-1)(s_4+s_6)\right) + s_5\left(2(1-2m^2)^2 s_6 + (4-8m^2)s_2 + (8m^4-8m^2+1)s_4\right)}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}},\\
s_3 &= \frac{1}{4\left(10(2m^2-1)s_1 - 3(6m^4-6m^2+1)s_5\right)^2}\\
&\quad \times \left(3s_5\left((4m^2-2)s_2 - m^2(m^2-1)(s_4-2s_6)\right) + s_1\left((2m^2-1)(s_4-4s_6) - 12s_2\right)\right)\\
&\quad \times \left((2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2\right).
\end{aligned} \quad (84)$$

Plugging the derived parameters in Eq. (84) with Eq. (24) and Eq. (25) into Eq. (37), as a consequence, we get Jacobi's elliptic-type function solution

$$
\begin{aligned}
q(x,t) = \pm & \sqrt{\frac{(20-40m^2)s_1 + 6(6m^4-6m^2+1)s_5}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}} \\
& \left(\mathrm{dn}\left[\sqrt{\frac{4s_1\left(2s_2 - (2m^2-1)(s_4+s_6)\right) + s_5\left(2(1-2m^2)^2 s_6 + (4-8m^2)s_2 + (8m^4-8m^2+1)s_4\right)}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}}(x-tv);m \right] \right. \\
& \left. + im\,\mathrm{sn}\left[\sqrt{\frac{4s_1\left(2s_2 - (2m^2-1)(s_4+s_6)\right) + s_5\left(2(1-2m^2)^2 s_6 + (4-8m^2)s_2 + (8m^4-8m^2+1)s_4\right)}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}}(x-vt);m \right] \right) \\
& \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}.
\end{aligned} \tag{85}
$$

When the modulus of ellipticity approaches unity, we get complexiton soliton solution

$$
\begin{aligned}
q(x,t) = \pm & \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}} \left\{ -\tanh\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt) \right] \right. \\
& \left. + i\,\mathrm{sech}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt) \right] \right\} e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}.
\end{aligned} \tag{86}
$$

Also, we have the following Jacobi's elliptic-type function solution

$$
\begin{aligned}
q(x,t) = \pm & \sqrt{\frac{(20-40m^2)s_1 + 6(6m^4-6m^2+1)s_5}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}} \\
& \left(i\mathrm{ns}\left[\sqrt{\frac{4s_1\left(2s_2 - (2m^2-1)(s_4+s_6)\right) + s_5\left(2(1-2m^2)^2 s_6 + (4-8m^2)s_2 + (8m^4-8m^2+1)s_4\right)}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}}(x-vt);m \right] \right. \\
& \left. - i\;\mathrm{cs}\left[\sqrt{\frac{4s_1\left(2s_2 - (2m^2-1)(s_4+s_6)\right) + s_5\left(2(1-2m^2)^2 s_6 + (4-8m^2)s_2 + (8m^4-8m^2+1)s_4\right)}{(2m^2-1)\left((4m^4-4m^2-1)s_4 + 4(m^4-m^2+1)s_6\right) - 8(m^4-m^2+1)s_2}}(x-vt);m \right] \right) \\
& \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}.
\end{aligned} \tag{87}
$$

When the modulus of ellipticity approaches unity, we get dark soliton solution

$$
\begin{aligned}
q(x,t) = \pm & \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}} \tanh\left[\frac{1}{2}\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt) \right] \\
& \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}.
\end{aligned} \tag{88}
$$

Projective Riccati equation method

Balancing U'''' with U^5 in Eq. (10) gives $N = 1$, accordingly the solution takes the form

$$
U(\xi) = a_0 + a_1\psi(\xi) + b_1\phi(\xi). \tag{89}
$$

Substituting Eq. (89) together with Eq. (28) and Eq. (29) into Eq. (10), we get a system of algebraic equations

$$a_1R(r)^2\left(5b_1^4s_3+3b_1^2\left(s_4+2s_6\right)+24k^2\right)+a_1^3\left(10b_1^2s_3+s_4+2s_6\right)R(r)+a_1^5s_3=0, \tag{90}$$

$$b_1\left(a_1^2\left(10b_1^2s_3+3s_4+6s_6\right)R(r)+5a_1^4s_3+R(r)^2\left(b_1^4s_3+b_1^2\left(s_4+2s_6\right)+24k^2\right)\right)=0, \tag{91}$$

$$\begin{aligned}&a_0b_1^2\left(5b_1^2s_3+s_4+4s_6\right)R(r)^2+a_1^3\left(5a_0a_1s_3-r\left(20b_1^2s_3+2s_4+3s_6\right)\right)\\&+a_1R(r)\left(a_0a_1\left(30b_1^2s_3+s_4+4s_6\right)-r\left(20b_1^4s_3+b_1^2\left(8s_4+17s_6\right)+60k^2\right)\right)=0,\end{aligned} \tag{92}$$

$$\begin{aligned}&-b_1R(r)\left(r\left(4b_1^4s_3+b_1^2\left(2s_4+5s_6\right)+36k^2\right)-2a_0a_1\left(10b_1^2s_3+s_4+4s_6\right)\right)\\&-b_1a_1^2\left(r\left(20b_1^2s_3+4s_4+7s_6\right)-20a_0a_1s_3\right)=0,\end{aligned} \tag{93}$$

$$\begin{aligned}&a_1^3\left(10a_0^2s_3+10b_1^2s_3+s_2+s_4+s_6\right)-2a_0a_1^2r\left(30b_1^2s_3+s_4+3s_6\right)\\&-2a_0b_1^2r\left(10b_1^2s_3+s_4+5s_6\right)R(r)+a_1\left(5r^2\left(4b_1^4s_3+b_1^2\left(s_4+2s_6\right)+6k^2\right)\right)\\&+R(r)\left(b_1^2\left(30a_0^2s_3+3s_2+2s_4+7s_6\right)+2\left(a_0^2s_6+10k^2+s_5\right)+10b_1^4s_3\right)=0,\end{aligned} \tag{94}$$

$$\begin{aligned}&b_1\left(R(r)\left(b_1^2\left(10a_0^2s_3+s_2+2s_6\right)+2\left(a_0^2s_6+4k^2+s_5\right)+2b_1^4s_3\right)-2a_0a_1r\left(20b_1^2s_3+s_4+4s_6\right)\right)\\&+b_1\left(a_1^2\left(30a_0^2s_3+10b_1^2s_3+3s_2+s_4+2s_6\right)+r^2\left(4b_1^4s_3+b_1^2\left(s_4+2s_6\right)+6k^2\right)\right)=0,\end{aligned} \tag{95}$$

$$\begin{aligned}&a_0b_1^2\left(R(r)\left(10a_0^2s_3+10b_1^2s_3+3s_2+4s_6\right)+r^2\left(20b_1^2s_3+s_4+4s_6\right)\right)+a_0a_1^2\left(10a_0^2s_3+30b_1^2s_3+3s_2+s_4+2s_6\right)\\&-a_1r\left(b_1^2\left(60a_0^2s_3+6s_2+2s_4+7s_6\right)+3\left(a_0^2s_6+5k^2+s_5\right)+20b_1^4s_3\right)=0,\end{aligned} \tag{96}$$

$$-b_1\left(a_0^2r\left(20b_1^2s_3+s_6\right)-2a_1a_0\left(10b_1^2s_3+3s_2+s_6\right)-20a_1a_0^3s_3+r\left(4b_1^4s_3+b_1^2\left(2s_2+s_6\right)+k^2+s_5\right)\right)=0, \tag{97}$$

$$\begin{aligned}&a_1\left(a_0^2\left(30b_1^2s_3+3s_2+s_6\right)+5a_0^4s_3+3b_1^2s_2+5b_1^4s_3+b_1^2s_6+k^2+s_1+s_5\right)\\&-2a_0b_1^2r\left(10a_0^2s_3+10b_1^2s_3+3s_2+s_6\right)=0,\end{aligned} \tag{98}$$

$$b_1\left(a_0^2\left(10b_1^2s_3+3s_2\right)+5a_0^4s_3+b_1^2\left(b_1^2s_3+s_2\right)+s_1\right)=0, \tag{99}$$

$$a_0\left(a_0^2\left(10b_1^2s_3+s_2\right)+a_0^4s_3+3b_1^2s_2+5b_1^4s_3+s_1\right)=0. \tag{100}$$

Solving these equations together yields the following results:

Case-1: $R(r)=0$

$$\begin{aligned}&a_0=a_1=0,\ b_1=\pm\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}},\ k=\sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}},\\&s_3=\frac{\left(8s_2+s_4-4s_6\right)\left(s_1\left(12s_2-s_4+4s_6\right)-6s_2s_5\right)}{4\left(10s_1-3s_5\right)^2}.\end{aligned} \tag{101}$$

Putting the parameters attained in Eq. (101) with Eq. (30) or Eq. (31) into Eq. (89), as a consequence, we get dark and singular solitons with $6s_5 - 20s_1 > 0, 8s_2 + s_4 - 4s_6 > 0$ and $4s_1(-2s_2 + s_4 + s_6) + s_5(4s_2 - s_4 - 2s_6) > 0$, as indicated below

$$q(x,t) = \pm\sqrt{\frac{6s_5 - 20s_1}{8s_2 + s_4 - 4s_6}}\tanh\left[\frac{1}{2}\sqrt{\frac{4s_1(-2s_2 + s_4 + s_6) + s_5(4s_2 - s_4 - 2s_6)}{8s_2 + s_4 - 4s_6}}(x - vt)\right] \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x + \omega t + \theta_0\right)}, \tag{102}$$

$$q(x,t) = \pm\sqrt{\frac{6s_5 - 20s_1}{8s_2 + s_4 - 4s_6}}\coth\left[\frac{1}{2}\sqrt{\frac{4s_1(-2s_2 + s_4 + s_6) + s_5(4s_2 - s_4 - 2s_6)}{8s_2 + s_4 - 4s_6}}(x - vt)\right] \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x + \omega t + \theta_0\right)}. \tag{103}$$

Case-2: $R(r) = \frac{24}{25}r^2$

Result-1:

$$a_0 = 0,\ a_1 = \frac{4}{5}\sqrt{\frac{3}{2}}b_1 r,\ b_1 = \sqrt{\frac{6s_5 - 20s_1}{8s_2 + s_4 - 4s_6}},\ k = \sqrt{\frac{4s_1(-2s_2 + s_4 + s_6) + s_5(4s_2 - s_4 - 2s_6)}{8s_2 + s_4 - 4s_6}},$$
$$s_3 = \frac{(8s_2 + s_4 - 4s_6)(s_1(12s_2 - s_4 + 4s_6) - 6s_2 s_5)}{4(10s_1 - 3s_5)^2}. \tag{104}$$

Inserting the parameters secured in Eq. (104) with Eq. (32) into Eq. (89), as a consequence, we get a straddled singular–singular soliton with $6s_5 - 20s_1 > 0, 8s_2 + s_4 - 4s_6 > 0$ and $4s_1(-2s_2 + s_4 + s_6) + s_5(4s_2 - s_4 - 2s_6) > 0$, as described below

$$q(x,t) = \sqrt{\frac{6s_5 - 20s_1}{8s_2 + s_4 - 4s_6}} \times \left\{\frac{\pm 2\sqrt{6}\operatorname{csch}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x - vt)\right] + 1}{\coth\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x - vt)\right] \pm 5\operatorname{csch}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x - vt)\right]}\right\} \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x + \omega t + \theta_0\right)}. \tag{105}$$

Result-2:

$$a_0 = b_1 = 0,\ a_1 = \frac{12\sqrt{2}}{5}r\sqrt{\frac{s_5}{2s_4 + 3s_6}},\ k = \sqrt{-\frac{s_5}{5}}, s_3 = -\frac{2s_4^2 + 11s_6 s_4 + 12s_6^2}{60s_5},$$
$$s_2 = \frac{1}{16}(6s_4 + 17s_6),\ s_1 = -\frac{4s_5}{5}. \tag{106}$$

Plugging the parameters achieved in Eq. (106) with Eq. (32) into Eq. (89), as a consequence, we get a straddled bright–bright soliton with $s_5 < 0$ and $2s_4 + 3s_6 < 0$, as shown below

$$q(x,t) = \left\{ \frac{12\sqrt{\frac{2s_5}{2s_4+3s_6}}\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]}{5\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right] \pm 1} \right\} e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{107}$$

Result-3:

$$\begin{aligned} &a_0 = a_1 = 0,\ b_1 = 2\sqrt{6}\sqrt{\frac{s_5}{4s_4+7s_6}}, \\ &k = \sqrt{\frac{2}{5}}\sqrt{-\frac{s_5(2s_4+3s_6)}{4s_4+7s_6}}, \\ &s_3 = -\frac{4s_4^2+23s_6s_4+28s_6^2}{120s_5}, \\ &s_2 = \frac{s_4}{3}+\frac{47s_6}{48},\ s_1 = -\frac{s_5(32s_4+43s_6)}{40s_4+70s_6}. \end{aligned} \tag{108}$$

Substituting the obtained parameters in Eq. (108) with Eq. (32) into Eq. (89), as a consequence, we get a straddled bright–dark soliton with $s_5 > 0$, $4s_4 + 7s_6 > 0$, and $2s_4 + 3s_6 < 0$, as presented below

$$q(x,t) = \left\{ \frac{2\sqrt{\frac{6s_5}{4s_4+7s_6}}\tanh\left[\sqrt{\frac{2}{5}}\sqrt{-\frac{s_5(2s_4+3s_6)}{4s_4+7s_6}}(x-vt)\right]}{1 \pm 5\operatorname{sech}\left[\sqrt{\frac{2}{5}}\sqrt{-\frac{s_5(2s_4+3s_6)}{4s_4+7s_6}}(x-vt)\right]} \right\} e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{109}$$

Case-3: $R(r) = \frac{5}{9}r^2$

Result-1:

$$\begin{aligned} &a_0 = 0,\ a_1 = \frac{\sqrt{5}}{3}b_1 r,\ b_1 = \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}},\ k = \sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}, \\ &s_3 = \frac{(8s_2+s_4-4s_6)(s_1(12s_2-s_4+4s_6)-6s_2s_5)}{4(10s_1-3s_5)^2}. \end{aligned} \tag{110}$$

Putting the parameters acquired in Eq. (110) with Eq. (33) into Eq. (89), as a consequence, we get a straddled singular–singular soliton with $6s_5 - -20s_1 > 0, 8s_2 + s_4 - -4s_6 > 0$ and $4s_1(-2s_2 + s_4 + s_6) + s_5(4s_2 - s_4 - -2s_6) > 0$, as given below

$$\begin{aligned} q(x,t) =& \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}} \\ &\times \left\{ \frac{\sqrt{5}\operatorname{csch}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right] \pm 2}{3\operatorname{csch}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right] \pm 2\coth\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]} \right\} \\ &\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \end{aligned} \tag{111}$$

Result-2:

$$a_0 = 0,\ a_1 = 2\sqrt{\frac{5}{3}}r\sqrt{\frac{s_5}{2s_4+3s_6}},\ b_1 = 0,\ k = \sqrt{-\frac{s_5}{5}},\ s_1 = -\frac{4s_5}{5},$$
$$s_2 = \frac{1}{15}\left(17s_4+33s_6\right),\ s_3 = -\frac{2s_4^2+11s_6s_4+12s_6^2}{60s_5}. \tag{112}$$

Inserting the derived parameters in Eq. (112) with Eq. (33) into Eq. (89), as a consequence, we get a straddled bright–bright soliton with $s_5 < 0$, and $2s_4 + 3s_6 < 0$, as indicated below

$$q(x,t) = \left\{\frac{2\sqrt{\frac{5}{3}}\sqrt{\frac{s_5}{2s_4+3s_6}}\left(3\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]\right)}{3\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]\pm 2}\right\}e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{113}$$

Result-3:

$$a_0 = a_1 = 0,\ b_1 = 2\sqrt{\frac{15s_5}{10s_4+63s_6}},\ k = \sqrt{-\frac{s_5\left(2s_4+3s_6\right)}{10s_4+63s_6}},$$
$$s_1 = \frac{4s_5\left(3s_6-2s_4\right)}{10s_4+63s_6},\ s_2 = \frac{s_4}{3}+\frac{3s_6}{5},\ s_3 = -\frac{10s_4^2+103s_6s_4+252s_6^2}{300s_5}. \tag{114}$$

Plugging the parameters attained in Eq. (114) with Eq. (33) into Eq. (89), as a consequence, we get a straddled singular–singular soliton with $s_5 > 0$, $4s_4 + 7s_6 > 0$, and $2s_4 + 3s_6 < 0$, as described below

$$q(x,t) = \left\{\frac{4\sqrt{\frac{15s_5}{10s_4+63s_6}}}{2\coth\left[\sqrt{-\frac{s_5(2s_4+3s_6)}{10s_4+63s_6}}(x-vt)\right]\pm 3\operatorname{csch}\left[\sqrt{-\frac{s_5(2s_4+3s_6)}{10s_4+63s_6}}(x-vt)\right]}\right\}$$
$$\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{115}$$

Case-4: $R(r) = r^2 - 1$

Result-1:

$$a_0 = 0,\ a_1 = b_1\sqrt{r^2-1},\ b_1 = \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}},\ k = \sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}},$$
$$s_3 = \frac{\left(8s_2+s_4-4s_6\right)\left(s_1\left(12s_2-s_4+4s_6\right)-6s_2s_5\right)}{4\left(10s_1-3s_5\right)^2}. \tag{116}$$

Substituting the parameters secured in Eq. (116) with Eq. (34) or Eq. (35) into Eq. (89), as a consequence, we get straddled bright–dark solitons and straddled singular–singular solitons with $6s_5 - -20s_1 > 0$, $8s_2 + s_4 - -4s_6 > 0$ and $4s_1\left(-2s_2 + s_4 + s_6\right) + s_5\left(4s_2 - s_4 - -2s_6\right) > 0$, as shown below

$$q(x,t)=\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}}\times\left\{\frac{4\sqrt{r^2-1}\operatorname{sech}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+5\tanh\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+3}{4r\operatorname{sech}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+3\tanh\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+5}\right\}\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}, \tag{117}$$

or

$$q(x,t)=\sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}}\times\left\{\frac{\sqrt{r^2-1}\operatorname{csch}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+1}{r\operatorname{csch}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+\coth\left(\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-tv)\right)}\right\}\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{118}$$

Result-2:

$$a_1=2\sqrt{\frac{3(r^2-1)s_5}{2s_4+3s_6}},\ a_0=b_1=0,\ k=\sqrt{-\frac{s_5}{5}},\ s_3=-\frac{2s_4^2+11s_6s_4+12s_6^2}{60s_5},\quad s_2=\frac{2(r^2+2)s_4+3(2r^2+1)s_6}{6(r^2-1)},\ s_1=-\frac{1}{5}(4s_5). \tag{119}$$

Putting the parameters achieved in Eq. (119) with Eq. (34) or Eq. (35) into Eq. (89), as a consequence, we get straddled bright–dark solitons and straddled bright–bright solitons with $s_5<0, r>1$, and $2s_4+3s_6<0$, as presented below

$$q(x,t)=\left\{\frac{8\sqrt{\frac{3(r^2-1)s_5}{2s_4+3s_6}}\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]}{4r\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]+3\tanh\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]+5}\right\}\times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}, \tag{120}$$

or

$$q(x,t) = \left\{ \frac{2\sqrt{\frac{3(r^2-1)s_5}{2s_4+3s_6}}\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]}{r\operatorname{sech}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]+1} \right\} \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{121}$$

Result-3:

$$a_0 = a_1 = 0, b_1 = 2\sqrt{\frac{3(r^2-1)s_5}{2(r^2-1)s_4+3(r^2+3)s_6}}, k = \sqrt{-\frac{(r^2-1)s_5(2s_4+3s_6)}{10(r^2-1)s_4+15(r^2+3)s_6}},$$
$$s_3 = -\frac{(s_4+4s_6)(2(r^2-1)s_4+3(r^2+3)s_6)}{60(r^2-1)s_5}, s_2 = \frac{(2r^2-3)s_6}{2(r^2-1)}+\frac{s_4}{3},$$
$$s_1 = -\frac{2s_5(4(r^2-1)s_4+3(2r^2-7)s_6)}{5(2(r^2-1)s_4+3(r^2+3)s_6)}. \tag{122}$$

Inserting the obtained parameters in Eq. (122) with Eq. (34) or Eq. (35) into Eq. (89), as a consequence, we get straddled bright–dark solitons with $(r^2-1)s_5 > 0$, $2(r^2--1)s_4+3(r^2+3)s_6 > 0$, and $2s_4+3s_6 < 0$, as given below

$$q(x,t) = 2\sqrt{\frac{3(r^2-1)s_5}{2(r^2-1)s_4+3(r^2+3)s_6}} \left\{ \frac{5\tanh\left[\sqrt{-\frac{(r^2-1)s_5(2s_4+3s_6)}{10(r^2-1)s_4+15(r^2+3)s_6}}(x-vt)\right]+3}{3\tanh\left[\sqrt{-\frac{(r^2-1)s_5(2s_4+3s_6)}{10(r^2-1)s_4+15(r^2+3)s_6}}(x-vt)\right]+4r\operatorname{sech}\left[\sqrt{-\frac{(r^2-1)s_5(2s_4+3s_6)}{10(r^2-1)s_4+15(r^2+3)s_6}}(x-vt)\right]+5} \right\} \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}, \tag{123}$$

or

$$q(x,t) = \left\{ \frac{2\sqrt{\frac{3(r^2-1)s_5}{2(r^2-1)s_4+3(r^2+3)s_6}}\tanh\left[\sqrt{-\frac{(r^2-1)s_5(2s_4+3s_6)}{10(r^2-1)s_4+15(r^2+3)s_6}}(x-vt)\right]}{r\operatorname{sech}\left[\sqrt{-\frac{(r^2-1)s_5(2s_4+3s_6)}{10(r^2-1)s_4+15(r^2+3)s_6}}(x-vt)\right]+1} \right\} \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{124}$$

Case-5: $R(r) = r^2+1$

Result-1:

$$a_0 = 0,\ a_1 = \sqrt{-\frac{2\left(r^2+1\right)\left(10s_1-3s_5\right)}{8s_2+s_4-4s_6}},\ b_1 = \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}},$$
$$k = \sqrt{\frac{4s_1\left(-2s_2+s_4+s_6\right)+s_5\left(4s_2-s_4-2s_6\right)}{8s_2+s_4-4s_6}},$$
$$s_3 = \frac{\left(8s_2+s_4-4s_6\right)\left(s_1\left(12s_2-s_4+4s_6\right)-6s_2s_5\right)}{4\left(10s_1-3s_5\right)^2}. \tag{125}$$

Plugging the parameters acquired in Eq. (125) with Eq. (36) into Eq. (89), as a consequence, we get a straddled bright–dark soliton with $6s_5 - 20s_1 > 0$, $8s_2 + s_4 - 4s_6 > 0$ and $4s_1\left(-2s_2+s_4+s_6\right) + v_5\left(4s_2-s_4-2s_6\right) > 0$, as indicated below

$$q(x,t) = \sqrt{\frac{6s_5-20s_1}{8s_2+s_4-4s_6}} \times \left\{ \frac{\sqrt{r^2+1}\,\mathrm{sech}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+1}{r\,\mathrm{sech}\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]+\tanh\left[\sqrt{\frac{4s_1(-2s_2+s_4+s_6)+s_5(4s_2-s_4-2s_6)}{8s_2+s_4-4s_6}}(x-vt)\right]} \right\} \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{126}$$

Result-2:

$$a_0 = a_1 = 0, b_1 = 2\sqrt{\frac{3\left(r^2+1\right)s_5}{2\left(r^2+1\right)s_4+3\left(r^2-3\right)s_6}}, k = \sqrt{-\frac{\left(r^2+1\right)s_5\left(2s_4+3s_6\right)}{10\left(r^2+1\right)s_4+15\left(r^2-3\right)s_6}},$$
$$s_3 = -\frac{\left(s_4+4s_6\right)\left(2\left(r^2+1\right)s_4+3\left(r^2-3\right)s_6\right)}{60\left(r^2+1\right)s_5}, s_2 = \frac{\left(2r^2+3\right)s_6}{2\left(r^2+1\right)}+\frac{s_4}{3},$$
$$s_1 = -\frac{2s_5\left(4\left(r^2+1\right)s_4+3\left(2r^2+7\right)s_6\right)}{5\left(2\left(r^2+1\right)s_4+3\left(r^2-3\right)s_6\right)}. \tag{127}$$

Substituting the derived parameters in Eq. (127) with Eq. (36) into Eq. (89), as a consequence, we get straddled a singular–singular soliton with $s_5 > 0$, $2(r^2+1)s_4 + 3(r^2 - -3)s_6 > 0$, and $2s_4 + 3s_6 < 0$, as described below

$$q(x,t) = \left\{ \frac{2\sqrt{\frac{3\left(r^2+1\right)s_5}{2\left(r^2+1\right)s_4+3\left(r^2-3\right)s_6}}\coth\left[\sqrt{-\frac{\left(r^2+1\right)s_5\left(2s_4+3s_6\right)}{10\left(r^2+1\right)s_4+15\left(r^2-3\right)s_6}}(x-vt)\right]}{r\,\mathrm{csch}\left[\sqrt{-\frac{\left(r^2+1\right)s_5\left(2s_4+3s_6\right)}{10\left(r^2+1\right)s_4+15\left(r^2-3\right)s_6}}(x-vt)\right]+1} \right\} \times e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{128}$$

Result-3:

$$a_1 = 2\sqrt{\frac{3\left(r^2+1\right)s_5}{2s_4+3s_6}},\ a_0 = b_1 = 0,\ k = \sqrt{-\frac{s_5}{5}},\ s_3 = -\frac{2s_4^2+11s_6s_4+12s_6^2}{60s_5},$$
$$s_2 = \frac{2\left(r^2-2\right)s_4+3\left(2r^2-1\right)s_6}{6\left(r^2+1\right)},\ s_1 = -\frac{4s_5}{5}. \tag{129}$$

Putting the parameters attained in Eq. (129) with Eq. (36) into Eq. (89), as a consequence, we get a straddled singular–singular soliton with $s_5 < 0$, and $2s_4 + 3s_6 < 0$, as shown below

we obtain

$$q(x,t) = \left\{ \frac{2\sqrt{\frac{3\left(r^2+1\right)s_5}{2s_4+3s_6}}\operatorname{csch}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]}{r\operatorname{csch}\left[\sqrt{-\frac{s_5}{5}}(x-vt)\right]+1} \right\} e^{i\left(-\left\{\frac{c_2\sigma_7}{4c_1\sigma_1}\right\}x+\omega t+\theta_0\right)}. \tag{130}$$

Conservation laws

For the conserved flow that renders a closed form of the respective PDE, we let $q = u + iv$ and split the PDE into a real system whose conserved vectors (T^t, T^x) satisfy the $D_tT^t + D_xT^x) = 0$ along the solutions of the PDEs. We present the final conserved densities for special cases of (1) as Φ^t. Subject to the condition

$$\sigma_4 = \sigma_2 + \sigma_5,$$

we have the following:

(a) Power (P) density:

$$\Phi_P^t = \frac{1}{2}|q|^2. \tag{131}$$

In addition to the above condition, if we have

$$\sigma_3 = 2\sigma_5, \quad \text{and} \quad \sigma_9 = 0, \tag{132}$$

(b) Linear momentum (M) density:

$$\Phi_M^t = \frac{1}{2}\mathfrak{I}\left(q^*q_x\right), \tag{133}$$

and

(c) Hamiltonian (H) density:

$$\begin{aligned}\Phi_H^t = {} & \frac{1}{2}a\mathfrak{R}\left(qq_{xx}^*\right) + c_1\Bigg[\frac{1}{2}\sigma_1\mathfrak{R}\left(qq_{xxxx}^*\right) \\ & + \frac{1}{4}\sigma_2\left\{\left(\mathfrak{R}\left(qq_x^*\right)\right)^2 + \left(\mathfrak{I}\left(q^*q_x\right)\right)^2 + |q|^2\mathfrak{R}\left(qq_{xx}^*\right) + |q|^2\mathfrak{R}\left(q_{xx}\right) + |q|^2\mathfrak{I}\left(q_{xx}\right) + 4\mathfrak{R}(q)\mathfrak{I}(q)\mathfrak{R}\left(q_x\right)\mathfrak{I}\left(q_x\right)\right\} \\ & + \frac{1}{2}\sigma_5\left\{|q|^2\mathfrak{R}\left(qq_{xx}^*\right) + |q|^2\left|q_x\right|^2\right\} + \frac{1}{6}|q|^6\Bigg] + c_2\left[-\frac{1}{2}\sigma_7\mathfrak{I}\left(q^*q_{xxx}\right) + \frac{1}{4}\sigma_8|q|^2\mathfrak{I}\left(q^*q_x\right)\right].\end{aligned} \tag{134}$$

Now, the bright 1-soliton solution is written as

$$q(x,t) = A \ \operatorname{sech}\left[B(x-vt)\right]e^{i\left(-\kappa x+\omega t+\sigma_0\right)}, \tag{135}$$

where A and B are the amplitude and inverse width of the soliton, respectively. From the phase component, the parameter ω is the frequency of the solitons, while κ is the wave number and σ_0 represents the phase constant. Therefore, the conserved quantities are:

$$P = \int_{-\infty}^{\infty} \Phi_P^t dx = \frac{A}{B}, \tag{136}$$

$$M = \int_{-\infty}^{\infty} \Phi_M^t dx = -\frac{\kappa A}{B}, \tag{137}$$

and

$$H=\int_{-\infty}^{\infty}\Phi_H^t dx=\frac{aA^2}{3B}\left(B^2+3\kappa^2\right)+c_1\left[\frac{\sigma_1 A^2}{15B}\left(7B^4+30\kappa^2B^2+15\kappa^4\right)\right.$$
$$+\sigma_2\left\{\frac{2A^4}{15B}\left(2B^2+5\kappa^2\right)+\frac{A^3}{4B}\int_{-\infty}^{\infty}\left\{\left(B^2-\kappa^2\right)\operatorname{sech}^3\tau-2B^2\operatorname{sech}^5\tau\right\}\cos\phi dx-\frac{\kappa A^3}{2}\int_{-\infty}^{\infty}\operatorname{sech}^3\tau\tanh\tau\sin\phi dx\right.$$
$$\left.+\frac{A^4}{4B}\int_{-\infty}^{\infty}\left\{\left(B^2-\kappa^2\right)\operatorname{sech}^4\tau-B^2\operatorname{sech}^6\tau\right\}\sin^2 2\phi dx+\frac{\kappa A^4}{4}\int_{-\infty}^{\infty}\operatorname{sech}\tau\tanh\tau\sin 4\phi dx\right\}$$
$$\left.+\sigma_5\left\{\frac{2A^4}{15B}\left(3B^2+5\kappa^2\right)+\frac{4A^6}{45B}\right\}\right]-c_2\left[\frac{\sigma_7\kappa A^2}{B}\left(B^2+\kappa^2\right)+\frac{\sigma_8\kappa A^4}{3B}\right], \tag{138}$$

where the notations

$$\tau=B(x-vt), \tag{139}$$

and

$$\phi=-\kappa x+\omega t+\sigma_0 \tag{140}$$

are implemented.

Conclusions

This paper studied the concatenation model that is conserved with the absence of SPM. Nevertheless, the model supported soliton solutions, and they are retrieved by the aid of two integration algorithms. A full spectrum of soliton solutions along with the parameter constraints are identified. These constraints guarantee the existence of such soliton solutions. Finally, the multiplier approach revealed the three conserved quantities for the model. The Hamiltonian came with quadratures. The results are nevertheless encouraging that would lead to several futures avenues to walk upon. The conservation laws would lead to the quasi-monochromatic dynamics of such solitons. The phenomenon of optical soliton cooling would be looked upon. The numerical simulations of such solitons by the Laplace–Adomian decomposition would also be considered. The quiescent solitons for the model are also another inquisitive issue. The results would sequentially emerge with time. These results would be recovered and aligned with the pre-existing works, with time [15–30].

References

1. A. Ankiewicz, N. Akhmediev, Higher-order integrable evolution equation and its soliton solutions. Phys Lett A **378**, 358–361 (2014)
2. A. Ankiewicz, Y. Wang, S. Wabnitz, N. Akhmediev, Extended nonlinear Schrödinger equation with higher-order odd and even terms and its rogue wave solutions. Phys Rev E **89**, 012907 (2014)
3. A.H. Arnous, A. Biswas, A.H. Kara, Y. Yildirim, L. Moraru, C. Iticescu, S. Moldovanu, A.A. Alghamdi, Optical solitons and conservation laws for the concatenation model with spatio-temporal dispersion (Internet traffic regulation). J Eur Opt Soc Rapid Publ **19**(2), 35 (2023)
4. A.H. Arnous, A. Biswas, A.H. Kara, Y. Yildirim, L. Moraru, C. Iticescu, S. Moldovanu, A.A. Alghamdi, Optical solitons and conservation laws for the concatenation model: Power-law nonlinearity. Appear Ain Shams Eng J (2023). https://doi.org/10.1016/j.asej.2023.102381
5. A. Biswas, J. Vega-Guzman, A.H. Kara, S. Khan, H. Triki, O. Gonzalez-Gaxiola, L. Moraru, P.L. Georgescu, Optical solitons and conservation laws for the concatenation model: undetermined coefficients and multipliers approach. Universe. **9**(1), Article 15 (2023)
6. A. Biswas, J. Vega-Guzman, Y. Yildirim, L. Moraru, C. Iticescu, A.A. Alghamdi, Optical solitons for the concatenation model with differential group delay: undetermined coefficients. Mathematics. **11**(9), 2012 (2023)
7. A. Biswas, J.M. Vega-Guzman, Y. Yildirim, S.P. Moshokoa, M. Aphane, A.A. Alghamdi, Optical solitons for the concatenation model with power-law nonlinearity: undetermined coefficients. Ukr J Phys Opt **24**(3), 185–192 (2023)
8. O. González-Gaxiola, A. Biswas, J.R.D. Chavez, A. Asiri, Bright and dark optical solitons for the concatenation model by the Laplace-Adomian decomposition scheme. Ukr J Phys Opt **24**(3), 222–234 (2023)
9. N.A. Kudryashov, A. Biswas, A.G. Borodina, Y. Yildirim, H.M. Alshehri, Painleve analysis and optical solitons for a concatenated model. Optik. **272**, 170255 (2023)
10. A. Kukkar, S. Kumar, S. Malik, A. Biswas, Y. Yildirim, S.P. Moshokoa, S. Khan, A.A. Alghamdi, Optical solitons for the concatenation model with Kudryashov's approaches. Ukr J Phys Opt **24**(2), 155–160 (2023)
11. L. Tang, A. Biswas, Y. Yildirim, A.A. Alghamdi, Bifurcation analysis and optical solitons for the concatenation model. Phys Lett A **480**, 128943 (2023)
12. H. Triki, Y. Sun, Q. Zhou, A. Biswas, Y. Yildirim, H.M. Alshehri, Dark solitary pulses and moving fronts in an optical medium with the higher-order dispersive and nonlinear effects. Chaos Solitons Fractals **164**, 112622 (2022)
13. M.-Y. Wang, A. Biswas, Y. Yıldırım, L. Moraru, S. Moldovanu, H.M. Alshehri, Optical solitons for a concatenation model by trial equation approach. Electronics **12**(1), Article 19 (2023)
14. Y. Yildirim, A. Biswas, L. Moraru, A.A. Alghamdi, Quiescent optical solitons for the concatenation model with nonlinear chromatic dispersion. Mathematics. **11**(7), Article 1709 (2023)
15. T. Han, Z. Li, C. Li, L. Zhao, Bifurcations, stationary optical solitons and exact solutions for complex Ginzburg-Landau equation with nonlinear chromatic dispersion in non-Kerr law media. J Opt (2023). https://doi.org/10.1007/s12596-022-01041-5
16. Z. Li, E. Zhu, Optical soliton solutions of stochastic Schrödinger-Hirota equation in birefringent fibers with spatiotemporal dispersion and parabolic law nonlinearity. J Opt (2023). https://doi.org/10.1007/s12596-023-01287-7

17. S. Nandy, V. Lakshminarayanan, Adomian decomposition of scalar and coupled nonlinear Schrödinger equations and dark and bright solitary wave solutions. J Opt **44**, 397–404 (2015)
18. L. Tang, Bifurcations and optical solitons for the coupled nonlinear Schrödinger equation in optical fiber Bragg gratings. J Opt (2023). https://doi.org/10.1007/s12596-022-00963-4
19. L. Tang, Phase portraits and multiple optical solitons perturbation in optical fibers with the nonlinear Fokas-Lenells equation. J Opt (2023). https://doi.org/10.1007/s12596-023-01097-x
20. S. Wang, Novel soliton solutions of CNLSEs with Hirota bilinear method. J Opt (2023). https://doi.org/10.1007/s12596-022-01065-x
21. S.A. AlQahtani, M.S. Al-Rakhami, M.A. Reham Shohib, M.E.M. Alngar, P. Pathak, Dispersive optical solitons with Schrödinger-Hirota equation using the Φ^6-model expansion approach. Opt Quantum Electron **55**(8), 701 (2023)
22. E.M. Zayed, R.M. Shohib, Solitons and other solutions to the improved perturbed nonlinear Schrodinger equation with the dual-power law nonlinearity using different techniques. Optik **171**, 27–43 (2018)
23. E.M. Zayed, R.M. Shohib, Optical solitons to the generalized nonlinear Schrödinger equations for pulse propagation using several different techniques. Optik **187**, 81–91 (2019)
24. E.M. Zayed, R.M. Shohib, A.G. Al-Nowehy, On solving the (3+ 1)-dimensional NLEQZK equation and the (3+ 1)-dimensional NLmZK equation using the extended simplest equation method. Comput Math Appl **78**(10), 3390–3407 (2019)
25. E.M. Zayed, R.M. Shohib, A.G. Al-Nowehy, Solitons and other solutions for higher-order NLS equation and quantum ZK equation using the extended simplest equation method. Comput Math Appl **76**(9), 2286–2303 (2018)
26. E.M. Zayed, R.M. Shohib, Optical solitons and other solutions to the dual-mode nonlinear Schrödinger equation with Kerr law and dual power law nonlinearities. Optik **208**, 163998 (2020)
27. E.M. Zayed, M.E. Alngar, R.M. Shohib, Cubic-quartic embedded solitons with χ (2) and χ (3) nonlinear susceptibilities having multiplicative white noise via Itô calculus. Chaos Solitons Fractals **168**, 113186 (2023)
28. E.M. Zayed, R.M. Shohib, M.E. Alngar, Dispersive optical solitons in birefringent fibers for stochastic Schrödinger-Hirota equation with parabolic law nonlinearity and spatiotemporal dispersion having multiplicative white noise. Optik **278**, 170736 (2023)
29. E.M. Zayed, M.E. Alngar, R.M. Shohib, Dispersive Optical Solitons to Stochastic Resonant NLSE with Both Spatio-Temporal and Inter-Modal Dispersions Having Multiplicative White Noise. Mathematics **10**(17), 3197 (2022)
30. E.M. Zayed, R.M. Shohib, M.E. Alngar, Cubic-quartic optical solitons in Bragg gratings fibers for NLSE having parabolic non-local law nonlinearity using two integration schemes. Opt Quantum Electron **53**(8), 452 (2021)