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A STRUCTURAL OBSERVATION ON PORT-HAMILTONIAN SYSTEMS*

RAINER H. PICARD†, SASCHA TROSTORFF‡, BRUCE WATSON§, AND MARCUS WAURICK¶

Abstract. We study port-Hamiltonian systems on a family of intervals and characterize all boundary conditions leading to m-accretive realizations of the port-Hamiltonian operator and thus to generators of contractive semigroups. The proofs are based on a structural observation that the port-Hamiltonian operator can be transformed to the derivative on a familiy of reference intervals by suitable congruence relations, allowing for studying the simpler case of a transport equation. Moreover, we provide well-posedness results for associated control problems without assuming any additional regularity of the operators involved.

Key words. port-Hamiltonian systems, m-accretive operators, congruences

MSC codes. 35F15, 46N20, 47B44

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1. Introduction. In this paper, we shall revisit port-Hamiltonian differential equations (going back to van der Schaft et al. [20, 21]), that is, a system of first order partial differential equations of the form

$$\begin{cases} \partial_t u + P_1 \partial_x \mathcal{H} u + P_0 \mathcal{H} u = 0 & \text{on }]0,\infty[\times I, \\ u(0,x) = u_0(x), & x \in I, \end{cases}$$

where I is a real interval, $u\colon]0,\infty[\times I \to \mathbb{R}^n$ is a vector field subject to suitable (linear) boundary conditions, $\mathcal{H}\colon I \to \mathbb{R}^{n\times n}$ is a matrix field attaining values in the symmetric positive definite matrices, $P_1 = P_1^* \in \mathbb{R}^{n\times n}$ is invertible, and $P_0 = -P_0^* \in \mathbb{R}^{n\times n}$.[1] There is a vast amount of literature addressing the well-posedness as well as other questions related to the equations at hand (see, e.g., the monograph [9], the survey [8], and the Ph.D. thesis [1] and the references therein). In particular, questions on the theory of boundary control and observation are treated within the framework of port-Hamiltonian systems. Also, higher-dimensional variants of port-Hamiltonian systems or port-Hamiltonian systems of higher order are discussed (see, e.g., [1, 5, 10]). As an intermediate step, several authors have dealt with port-Hamiltonian systems on networks (see [5, 6, 7], where networks are considered as examples, and [22] for a detailed study). More precisely, the interval I is replaced by a set of intervals. In this case,

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†Department of Mathematics, TU Dresden, Dresden, Saxony 01062, Germany (rainer.picard@tu-dresden.de).

‡Mathematisches Seminar, CAU Kiel, Kiel 24118, Germany (trostorff@math.uni-kiel.de).

§School of Mathematics, University of the Witwatersrand, Johannesburg 2000, South Africa (Bruce.Watson@wits.ac.za).

¶TU Bergakademie Freiberg, Freiberg 09599, Germany (Marcus.waurick@math.tu-freiberg.de).

[1]The theory developed in this article also works for complex matrices and complex-valued functions. However, since the complex case can always be reduced to the real case by considering copies of real spaces, we restrict ourselves to the real case.

the role of boundary conditions becomes more pronounced. In particular, as a result, there is an abundance of descriptions for boundary conditions leading to port-Hamiltonian operators, that is,

$$-P_1\partial_x\mathcal{H} - P_0\mathcal{H}$$

on a Hilbert space consisting of suitably many copies of L^2-type spaces that generate bounded or (quasi-) contractive semigroups (see [6] and [18] for nonlinear boundary conditions). If the operator $\mathcal{H}$ satisfies uniform boundedness conditions (from above and below), then $\mathcal{H} = 1$ can be assumed with no loss of generality; the desired boundary conditions result from a subtle interplay of P_1 and ∂_x. Note that P_0 is then dealt with by a standard perturbation argument. It appears to be commonly understood that reducing the port-Hamiltonian operator to

$$-P_1\partial_x$$

is the optimal way of treating port-Hamiltonian systems. The main tool provided in the paper at hand is the transformation of the latter operator (by suitable congruence transformations) to

$$\partial_x,$$

which is arguably the easiest case in which to discuss boundary conditions on networks. We do not rely on semi-group theory as our method to show existence, uniqueness, and continuous dependence on the data, and as such we can address well-posedness of equations of the form

$$(\partial_t M_0 + M_1 + P_1\partial_x + P_0)\, u = f,$$

on several copies of L^2-type spaces by explicit reduction to the case of

$$\big(\partial_t\widetilde{M}_0 + \widetilde{M}_1 + \partial_x\big)\widetilde{u} = \widetilde{f}.$$

Using the theory of evolutionary equations (see [11, 16] and Chapter 6 of [13]) to study the latter equation, we shall furthermore be able to treat partial differential algebraic equations; that is, we may allow $\widetilde{M}_0$ to have a proper nullspace, thus generalizing the class of port-Hamiltonian systems significantly.

The article is structured as follows. We consider the operator ∂_x on networks in section 2 and provide a characterization of all (linear) boundary conditions leading to m-accretive realizations of this operator on a suitable Hilbert space (and hence $-\partial_x$ would generate a contraction semigroup). After that, we show in section 3 how the abstract port-Hamiltonian operator $P_1\partial_x\mathcal{H}$ can be reduced to the case treated in section 2, which allows us to provide a new proof for the well-posedness of port-Hamiltonian systems in section 4. Moreover, using the framework of evolutionary equations instead of C_0-semigroups, we present a new approach to boundary control problems, which has the benefit that one does not need to assume smooth diagonalizability of $\mathcal{H}$, which is a standard assumption in the existing literature (see, e.g., [23]).

2. The operator ∂_x on networks. In this section, we briefly introduce the main operator of this paper. For this let $I_k \subseteq \mathbb{R}$ be a nonempty interval with nonempty complement (i.e., $I_k \neq]-\infty,\infty[$), where $k \in \{1,\dots,N\}$ for some $N \in \mathbb{N}$.

DEFINITION 2.1. *We define*

$$\partial_x \colon \bigoplus_{k=1}^{N} H^1(I_k) \subseteq \bigoplus_{k=1}^{N} L^2(I_k) \to \bigoplus_{k=1}^{N} L^2(I_k), \qquad (\phi_k)_k \mapsto ({\phi_k}')_k,$$

where $H^1(I_k)$ is the (standard) Sobolev space of $L^2(I_k)$-functions with weak derivative representable as an $L^2(I_k)$-function. Recall that $\bigoplus_{k=1}^{N} L^2(I_k)$ is the space $\bigtimes_{k=1}^{N} L^2(I_k)$ endowed with the inner product

$$\langle u, v\rangle_{\bigoplus_{k=1}^{N} L^2(I_k)} := \sum_{k=1}^{n} \langle u_k, v_k\rangle_{L^2(I_k)}.$$

With this operator at hand, we can consider the port-Hamiltonian operator $P_1\partial_x$ for a suitable matrix $P_1 \in \mathbb{R}^{N\times N}$ on $\bigoplus_{k=1}^{N} L^2(I_k)$. In order to get a well-defined object, we have to restrict the class of possible matrices P_1.

DEFINITION 2.2. *A matrix $P_1 \in \mathbb{R}^{N\times N}$ is called compatible if P_1 leaves the space $\bigoplus_{k=1}^{N} L^2(I_k)$ invariant.*

Remark 2.3. A typical example for a compatible matrix P_1 is a diagonal matrix. More generally, if we have several copies of one interval, say $I_1 = \cdots = I_j$ for some $j \in \{1, \ldots, N\}$, then P_1 could be block-diagonal. Indeed, this block-diagonal structure is equivalent to P_1 being compatible.

Before we come to a closer analysis of the port-Hamiltonian operator, we shall reduce the operator just introduced to a more managable reference case. For this, we put

$$\begin{aligned} N_{\mathrm{f}} &:= \{k \in \{1, \ldots, N\}\,;\, I_k \text{ bounded}\},\\ M_+ &:= \{k \in \{1, \ldots, N\}\,;\, \sup I_k = \infty\},\\ M_- &:= \{k \in \{1, \ldots, N\}\,;\, \inf I_k = -\infty\}. \end{aligned}$$

Moreover, for $n, m_+, m_- \in \mathbb{N}$ we define the space

$$L^2(n, m_+, m_-) := \left(L^2(]-1/2, 1/2[)\right)^n \oplus \left(L^2(]-1/2, \infty[)\right)^{m_+} \oplus \left(L^2(]-\infty, 1/2[)\right)^{m_-}.$$

Correspondingly, we introduce

$$\begin{aligned} &H^1(n, m_+, m_-)\\ &\quad := \left(H^1\left(]-1/2, 1/2[\right)\right)^n \oplus \left(H^1\left(]-1/2, \infty[\right)\right)^{m_+} \oplus \left(H^1\left(]-\infty, 1/2[\right)\right)^{m_-},\\ &H^1_0(n, m_+, m_-)\\ &\quad := \left(H^1_0\left(]-1/2, 1/2[\right)\right)^n \oplus \left(H^1_0\left(]-1/2, \infty[\right)\right)^{m_+} \oplus \left(H^1_0\left(]-\infty, 1/2[\right)\right)^{m_-}, \end{aligned}$$

where H^1_0 stands for the closure of smooth functions with compact support in the space H^1, that is, the space of Sobolev functions that vanish at the boundary. We now provide a congruence allowing us to transform the operator ∂_x on $\bigoplus_{k=1}^{N} L^2(I_k)$ to the standard space $L^2(\#N_{\mathrm{f}}, \#M_+, \#M_-)$.

PROPOSITION 2.4. *Let $a, b \in \mathbb{R}, a < b$.*

(a) *Consider*

$$\begin{aligned}\phi :\ ]-1/2,1/2[&\to\ ]a,b[\\ x &\mapsto -\left(x-\frac{1}{2}\right)a+\left(x+\frac{1}{2}\right)b.\end{aligned}$$

Then ϕ *is invertible*[2] *and*

$$\begin{aligned}\Phi : L^2\left(]a,b[\right) &\to L^2\left(]-1/2,1/2[\right),\\ u &\mapsto \sqrt{b-a}\,(u\circ\phi)\end{aligned}$$

is unitary.

(b) *Consider*

$$\begin{aligned}\phi^+ :\ ]-1/2,\infty[&\to\ ]a,\infty[,\\ x &\mapsto x+a+\frac{1}{2}.\end{aligned}$$

Then

$$\begin{aligned}\Phi^+ : L^2\left(]a,\infty[\right) &\to L^2\left(]-1/2,\infty[\right),\\ u &\mapsto u\circ\phi^+\end{aligned}$$

is unitary.

(c) *Let* $P_1 = P_1^* \in \mathbb{R}^{N\times N}$ *be a diagonal (hence compatible) matrix,* $n := \#N_{\mathrm{f}}, m_+ := \#M_+, m_- := \#M_-$, *and*

$$\begin{aligned}\partial_{x,\mathrm{n}} \colon H^1(n,m_+,m_-) \subseteq L^2(n,m_+,m_-) &\to L^2(n,m_+,m_-),\\ (\phi_k)_k &\mapsto (\phi'_k)_k,\end{aligned}$$

where the index n *serves as a reminder that we are working on the "normal" space* $L^2(n,m_+,m_-)$. *Then there exists a diagonal, real matrix* $\tilde{P}_1$ and a unitary operator $\Psi\colon \bigoplus_{k=1}^N L^2(I_k) \to L^2(n,m_+,m_-)$ *such that*

$$\Psi P_1 \partial_x \Psi^* = \widetilde{P}_1 \partial_{x,\mathrm{n}}.$$

Proof. (a) Let $u \in L^2(]a,b[)$. Then we compute

$$\int_{-1/2}^{1/2} \left|\sqrt{b-a}\,u\left(\phi\left(x\right)\right)\right|^2 \mathrm{d}x = \int_{-1/2}^{1/2} \left|u\left(\phi\left(x\right)\right)\right|^2 (b-a)\,\mathrm{d}x = \int_a^b \left|u\left(y\right)\right|^2 \mathrm{d}y.$$

Since Φ is also onto by the invertibility of ϕ, the assertion follows.

(b) The assertion follows with an elementary computation similar to (a).

(c) Without loss of generality, we assume that $\{1,\ldots,n\} = N_{\mathrm{f}}$ and that $\{n+1,\ldots,n+m_+\} = M_+$ as well as $\{n+m_++1,\ldots,N\} = M_-$. For $k\in\{1,\ldots,n\}$ we find $a_k, b_k \in \mathbb{R}$ such that $I_k =]a_k,b_k[$. Let ϕ_k be as in (a) with a_k, b_k replacing a, b. By the chain rule we have for all $k \in \{1,\ldots,n\}$

[2]It is easily verified that

$$\begin{aligned}\phi^{-1} :\ ]a,b[&\to\ ]-1/2,1/2[,\\ x &\mapsto \frac{1}{b-a}x - \frac{b+a}{2\,(b-a)}\end{aligned}$$

is the inverse of ϕ.

$$\partial_x (u \circ \phi_k) = (b_k - a_k)(\partial_x u) \circ \phi_k.$$

Hence,

$$\partial_{x,\mathrm{n}} \Phi_k = (b_k - a_k) \Phi_k \partial_x,$$

where we denoted Φ_k according to Φ as in (a) replacing ϕ by ϕ_k. Next, let $\Phi_{n+1}^+, \dots, \Phi_{m_+}^+$ be as in (b) with the lower bound a appropriately replaced in order that $\Phi_k^+ \colon L^2(I_k) \to L^2(]-1/2, \infty[)$ is unitary, $k \in \{n+1, \dots, n+m_+\}$. For $k \in \{n+m_++1, \dots, N\}$, we find, similar to (b), a unitary $\Phi_k^- \colon L^2(I_k) \to L^2(]-\infty, 1/2[)$. Thus, for all $k \in \{n+1, \dots, N\}$, we obtain

$$\partial_{x,\mathrm{n}} \Phi_k^{+/-} = \Phi_k^{+/-} \partial_x.$$

In consequence, denoting

$$\Psi := \begin{pmatrix} \Phi_1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & \Phi_n & \ddots & & \vdots \\ \vdots & & \ddots & \Phi_{n+1}^+ & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & \Phi_N^- \end{pmatrix} \text{ and}$$

$$\mathbf{b} - \mathbf{a} := \begin{pmatrix} b_1 - a_1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & b_n - a_n & \ddots & & \vdots \\ \vdots & & \ddots & 1 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix},$$

we deduce that Ψ is unitary, and since P_1 is diagonal and hence it commutes with Ψ, we compute

$$\begin{aligned} \Psi P_1 \partial_x \Psi^* &= \Psi P_1 (\mathbf{b} - \mathbf{a}) \Psi^* \partial_{x,\mathrm{n}} \\ &= P_1 (\mathbf{b} - \mathbf{a}) \partial_{x,\mathrm{n}} \\ &= \sqrt{(\mathbf{b} - \mathbf{a})} P_1 \sqrt{(\mathbf{b} - \mathbf{a})} \partial_{x,\mathrm{n}}. \end{aligned}$$

Thus, the assertion follows with

$$\widetilde{P}_1 = \sqrt{(\mathbf{b} - \mathbf{a})} P_1 \sqrt{(\mathbf{b} - \mathbf{a})}.$$

□

Remark 2.5. It can easily be seen from the proof that the statement in (c) remains true if $m_+ = m_- = 0$, $I_k =]a, b[$ for all $k \in \{1, \dots, N\}$, and $P_1 = P_1^*$ (so P_1 need not necessarily be diagonal). The matrix $\widetilde{P}_1$ claimed to exist then has the form

$$\widetilde{P}_1 = \sqrt{(b-a)} P_1 \sqrt{(b-a)}$$

and is not necessarily diagonal either.

If it is clear from the context, we will also write ∂_x instead of $\partial_{x,\mathrm{n}}$ (as it is defined in Proposition 2.4(c)).

We recall that by the Sobolev embedding theorem,

$$H^1(n, m_+, m_-) \subseteq (C([-1/2, 1/2]))^n \times (C_0([-1/2, \infty[))^{m_+} \times (C_0(]-\infty, 1/2]))^{m_-},$$

where we denote by $C_0(I)$ for an interval $I \subseteq \mathbb{R}$ the closure of $C_c(I)$ (continuous functions with compact support) with respect to the supremum-norm.

This fact will be used frequently throughout the paper.

A consequence of integration by parts and the Sobolev embedding theorem is the next proposition. Note that functions in $H^1(]a, b[)$ possess well-defined point evaluations at the boundary points a and b (if $a = -\infty$ or $b = \infty$, the respective evaluation is 0).

PROPOSITION 2.6. *Let* $u = \begin{pmatrix} u_{1/2} \\ u_\infty \\ u_{-\infty} \end{pmatrix}, v = \begin{pmatrix} v_{1/2} \\ v_\infty \\ v_{-\infty} \end{pmatrix} \in H^1(n, m_+, m_-)$. *Then we find*

$$\begin{aligned} &\langle \partial_x u, v\rangle + \langle u, \partial_x v\rangle \\ &= \left\langle u_{1/2}\left(\frac{1}{2}\right), v_{1/2}\left(\frac{1}{2}\right)\right\rangle - \left\langle u_{1/2}\left(-\frac{1}{2}\right), v_{1/2}\left(-\frac{1}{2}\right)\right\rangle \\ &\quad - \left\langle u_\infty\left(-\frac{1}{2}\right), v_\infty\left(-\frac{1}{2}\right)\right\rangle + \left\langle u_{-\infty}\left(\frac{1}{2}\right), v_{-\infty}\left(\frac{1}{2}\right)\right\rangle \\ &= \left\langle \begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix}, \begin{pmatrix} v_{1/2}(\frac{1}{2}) \\ v_{-\infty}(\frac{1}{2}) \end{pmatrix}\right\rangle - \left\langle \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix}, \begin{pmatrix} v_{1/2}(-\frac{1}{2}) \\ v_\infty(-\frac{1}{2}) \end{pmatrix}\right\rangle. \end{aligned}$$

Denote by $\mathring{\partial}_x$ the restriction of ∂_x to $H_0^1(n, m_+, m_-)$. A straightforward consequence of the previous observation is the following.

PROPOSITION 2.7. *The operators* ∂_x *and* $\mathring{\partial}_x$ *are densely defined and closed on* $L^2(n, m_+, m_-)$. *Moreover,*

$$\partial_x^* = -\mathring{\partial}_x \text{ and } -\mathring{\partial}_x^* = \partial_x,$$

where the adjoints are computed in $L^2(n, m_+, m_-)$.

The next statement summarizes the description of all linear, accretive operator extensions of the minimal operator $\mathring{\partial}_x$ (see also pages 18–22 of [17] for related material). We recall that for a symmetric matrix $K \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}$ the expression $K \leq c$ means that

$$\langle Kx, x\rangle \leq c\|x\|^2 \quad (x \in \mathbb{R}^n).$$

$K \geq c$ is defined analogously.

THEOREM 2.8. *Let* $\mathring{\partial}_x \subseteq D \subseteq \partial_x$ *be a linear operator on* $L^2(n, m_+, m_-)$.

(a) *Then the following statements are equivalent:*

(i) *D is accretive; that is,*

$$\langle Du, u\rangle \geq 0 \quad (u \in \mathrm{dom}(D)).$$

(ii) *There exists* $M \in \mathbb{R}^{(n+m_+)\times(n+m_-)}$ *with* $M^*M \leq 1$ *such that*

$$\begin{aligned} \mathrm{dom}(D) \subseteq \Big\{ &u = (u_{1/2}, u_\infty, u_{-\infty}) \in H^1(n, m_+, m_-)\,; \\ &M\begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix} + \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix} = 0 \Big\}. \end{aligned}$$

(b) *Let* $M \in \mathbb{R}^{(n+m_+)\times(n+m_-)}$ *be such that*

$$\operatorname{dom}(D) = \Big\{u = (u_{1/2}, u_\infty, u_{-\infty}) \in H^1(n, m_+, m_-)\,; \\ M\begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix} + \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix} = 0\Big\}.$$

Then[3] $\mathring{\partial}_x \subseteq -D^* \subseteq \partial_x$ *and*

$$\operatorname{dom}(D^*) = \Big\{v = (v_{1/2}, v_\infty, v_{-\infty}) \in H^1(n, m_+, m_-)\,; \\ \begin{pmatrix} v_{1/2}(\frac{1}{2}) \\ v_{-\infty}(\frac{1}{2}) \end{pmatrix} + M^*\begin{pmatrix} v_{1/2}(-\frac{1}{2}) \\ v_\infty(-\frac{1}{2}) \end{pmatrix} = 0\Big\}.$$

(c) D *is* maximal accretive*; that is,* D *is accretive and there exists no accretive relation extending* D *(or, equivalently,* D *and* D^* *are accretive) if and only if there exists* $M \in \mathbb{R}^{(n+m_+)\times(n+m_-)}$ *with* $M^*M \leq 1$ *such that*

$$\operatorname{dom}(D) = \Big\{u = (u_{1/2}, u_\infty, u_{-\infty}) \in H^1(n, m_+, m_-)\,; \\ M\begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix} + \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix} = 0\Big\}.$$

Proof. (a) Assume D is accretive. Then, by Proposition 2.6, we deduce that, for all $u \in \operatorname{dom}(D) \subseteq H^1(n, m_+, m_-)$,

$$0 \leq 2\langle Du, u\rangle \\ = \left\langle \begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix}, \begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix}, \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix} \right\rangle.$$

Hence,

$$\left\langle \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix}, \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix} \right\rangle \leq \left\langle \begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix}, \begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix} \right\rangle. \tag{2.1}$$

Therefore, for all $\begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix}$ with $u \in \operatorname{dom}(D)$, $M\begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix} := -\begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix}$ gives rise to a linear mapping, which is well defined by (2.1), defined on

$$R := \left\{ \begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix}; u \in \operatorname{dom}(D) \right\} \subseteq \mathbb{R}^{n+m_-}.$$

We put $M = 0$ on $R^{\perp_{\mathbb{R}^{n+m_-}}}$. Thus, $M \in L(\mathbb{R}^{n+m_-}, \mathbb{R}^{n+m_+})$ with $\|M\| \leq 1$ by (2.1). Hence, identifying M with its matrix representation $M \in \mathbb{R}^{(n+m_+)\times(n+m_-)}$ we obtain (ii). Next, assume (ii), and let M be as in (ii). Then we compute using Proposition 2.6 for all $u \in \operatorname{dom}(D)$

[3]Note that skew-self-adjointness of D requires M to be unitary, and so in particular $m_+ = m_-$.

$$\begin{aligned}
&2\langle Du,u\rangle\\
&\quad=\left\langle\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix},\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix}\right\rangle-\left\langle\begin{pmatrix}u_{1/2}(-\frac12)\\ u_{\infty}(-\frac12)\end{pmatrix},\begin{pmatrix}u_{1/2}(-\frac12)\\ u_{\infty}(-\frac12)\end{pmatrix}\right\rangle\\
&\quad=\left\langle\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix},\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix}\right\rangle-\left\langle M\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix},M\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix}\right\rangle\\
&\quad\geq 0
\end{aligned}$$

as $\|M\|\leq 1$.

(b) Since $\mathring{\partial}_x\subseteq D\subseteq\partial_x$ it follows that

$$-\mathring{\partial}_x=\partial_x^*\subseteq D^*\subseteq\mathring{\partial}_x^*=-\partial_x.$$

Hence, $\mathring{\partial}_x\subseteq -D^*\subseteq\partial_x$, and thus $H_0^1(n,m_+,m_-)\subseteq\operatorname{dom}(D^*)\subseteq H^1(n,m_+,m_-)$. Thus, for $u\in\operatorname{dom}(D)$ and $v\in H^1(n,m_+,m_-)$ we can use Proposition 2.6 and deduce

$$\begin{aligned}
&\langle Du,v\rangle+\langle u,\partial_x v\rangle\\
&\quad=\left\langle\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix},\begin{pmatrix}v_{1/2}(\frac12)\\ v_{-\infty}(\frac12)\end{pmatrix}\right\rangle-\left\langle\begin{pmatrix}u_{1/2}(-\frac12)\\ u_{\infty}(-\frac12)\end{pmatrix},\begin{pmatrix}v_{1/2}(-\frac12)\\ v_{\infty}(-\frac12)\end{pmatrix}\right\rangle\\
&\quad=\left\langle\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix},\begin{pmatrix}v_{1/2}(\frac12)\\ v_{-\infty}(\frac12)\end{pmatrix}\right\rangle+\left\langle M\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix},\begin{pmatrix}v_{1/2}(-\frac12)\\ v_{\infty}(-\frac12)\end{pmatrix}\right\rangle\\
&\quad=\left\langle\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix},\begin{pmatrix}v_{1/2}(\frac12)\\ v_{-\infty}(\frac12)\end{pmatrix}+M^*\begin{pmatrix}v_{1/2}(-\frac12)\\ v_{\infty}(-\frac12)\end{pmatrix}\right\rangle.
\end{aligned}$$

Next, let $(x,y)\in\mathbb{R}^n\times\mathbb{R}^{m_-}$. Then $M(x,y)\in\mathbb{R}^n\times\mathbb{R}^{m_+}$. Using suitable piecewise linear functions, it is not difficult to construct $u\in H^1(n,m_+,m_-)$ such that

$$\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}\ \text{ and }\ -M(x,y)=\begin{pmatrix}u_{1/2}(-\frac12)\\ u_{\infty}(-\frac12)\end{pmatrix}.$$

Hence,

$$\left\{\begin{pmatrix}u_{1/2}(\frac12)\\ u_{-\infty}(\frac12)\end{pmatrix}\,;\,u\in\operatorname{dom}(D)\right\}=\mathbb{R}^n\times\mathbb{R}^{m_-}.$$

As a consequence of this and the above computation, we deduce that $v\in\operatorname{dom}(D^*)$ if and only if $v\in H^1(n,m_+,m_-)$ and

$$\begin{pmatrix}v_{1/2}(\frac12)\\ v_{-\infty}(\frac12)\end{pmatrix}+M^*\begin{pmatrix}v_{1/2}(-\frac12)\\ v_{\infty}(-\frac12)\end{pmatrix}=0,$$

which establishes (b).

(c) At first we assume that $\operatorname{dom}(D)$ can be written as it is given in (c). Then, by (a), D is accretive. Moreover, by (b),

$$\operatorname{dom}(D^*) = \Big\{ v = (v_{1/2}, v_\infty, v_{-\infty}) \in H^1(n, m_+, m_-)\,;\\ \begin{pmatrix} v_{1/2}(\frac{1}{2}) \\ v_{-\infty}(\frac{1}{2}) \end{pmatrix} + M^* \begin{pmatrix} v_{1/2}(-\frac{1}{2}) \\ v_\infty(-\frac{1}{2}) \end{pmatrix} = 0 \Big\},$$

which means that D^* is accretive by arguing as in (a) (note that with $M^*M \leq 1$ we have $MM^* \leq 1$ and also that $-\partial_x \subseteq D^* \subseteq -\partial_x$). Since D is closed and densely defined, it follows that D is maximal accretive (see, e.g., [3, Corollary 3.17]). On the other hand, if D is maximal accretive, D is accretive, and therefore by (a), we find $M \in \mathbb{R}^{(n+m_+)\times(n+m_-)}$ with $M^*M \leq 1$ such that

$$\operatorname{dom}(D) \subseteq \Big\{ u = (u_{1/2}, u_\infty, u_{-\infty}) \in H^1(n, m_+, m_-)\,;\\ M \begin{pmatrix} u_{1/2}(\frac{1}{2}) \\ u_{-\infty}(\frac{1}{2}) \end{pmatrix} + \begin{pmatrix} u_{1/2}(-\frac{1}{2}) \\ u_\infty(-\frac{1}{2}) \end{pmatrix} = 0 \Big\}.$$

By the first part of the proof of (c), we have that ∂_x restricted to the right-hand side of this inclusion is maximal accretive. Hence, by the maximality of D, the inclusion is an equality, which establishes the assertion. ∎

Remark 2.9. The latter result is a special case of [22, Theorem 2.1], where a similar result is proved using the concept of boundary systems. However, we decided to provide this more direct and simpler proof above, in order to avoid this general theory and to keep the paper self-contained.

We remark here that the results in this section can be generalized also to infinite networks; that is, one considers operators on $\oplus_{j\in J} L^2(I_j)$ for an arbitrary index set J. Note, however, that in this case the mapping Ψ in Proposition 2.4(c) exists as a unitary mapping only if $\inf_{j\in J} |I_j| > 0$. For the general case we refer the reader to [22, section 7].

3. The congruence of ∂_x and $\mathbf{P_1}\partial_x\mathcal{H}$. Throughout, let $P_1 \in \mathbb{R}^{N\times N}$ be a compatible self-adjoint and invertible matrix, and let $\mathcal{H} : \mathbb{R} \to \mathbb{R}^{N\times N}$ be measurable and bounded such that $\mathcal{H}(x)$ is compatible for each $x \in \mathbb{R}$ and $\mathcal{H}$ attains values in the symmetric matrices and is uniformly positive definite; i.e., there exists $c > 0$ such that $\mathcal{H}(x) \geq c$ for all $x \in \mathbb{R}$. Moreover, we identify the function $\mathcal{H}$ with its induced multiplication operator on $\bigoplus_{k=1}^N L^2(I_k)$.[4] The aim of the present section is to identify the operator $P_1\partial_x\mathcal{H}$ on a suitable Hilbert space and the operator ∂_x on $L^2(n, m_+, m_-)$ as mutually congruent operators, if we choose $n, m_+, m_- \in \mathbb{N}$ suitably. Since maximal accretivity is preserved by conguences, Theorem 2.8 would then allow us to provide a characterization result for maximal accretivity of the operator $P_1\partial_x\mathcal{H}$. The reduction to the case $\mathcal{H} = 1$ is standard and well known in the theory of port-Hamiltonian systems (see, e.g., [9, Lemma 7.2.3]).

PROPOSITION 3.1. *Let $H := \bigoplus_{k=1}^N L^2(I_k)$ equipped with the inner product*

$$\langle u, v\rangle_H := \langle \mathcal{H}u, v\rangle \quad (u, v \in H).$$

[4]Note that it suffices to define $\mathcal{H}$ on $\bigcup_{k=1}^N I_k$. However, such a function can easily be extended to $\mathbb{R}$ by setting $\mathcal{H}(x) = E_N$ for $x \notin \bigcup_{k=1}^N I_k$.

Consider the mapping $B : H \to \bigoplus_{k=1}^{N} L^2(I_k)$ *given by* $Bu := \mathcal{H}u$. *Then*

$$B^* : \bigoplus_{k=1}^{N} L^2(I_k) \to H, \quad v \mapsto v,$$

where the adjoint is computed with respect to the inner products on H *and* $\bigoplus_{k=1}^{N} L^2(I_k)$.

Proof. First note that the inner product on H is well-defined due to the self-adjointness, boundedness, and positive definiteness of $\mathcal{H}$. Moreover, B is obviously linear and bounded, and for $u \in H, v \in \bigoplus_{k=1}^{N} L^2(I_k)$ we compute

$$\langle Bu, v\rangle = \langle \mathcal{H}u, v\rangle = \langle u, v\rangle_H,$$

which shows the asserted formula for B^*. ☐

The latter proposition shows that $P_1\partial_x\mathcal{H} = B^* P_1\partial_x B$ and hence, $P_1\partial_x\mathcal{H}$ and $P_1\partial_x$ are congruent as operators on H and $\bigoplus_{k=1}^{N} L^2(I_k)$, respectively. Next we show that $P_1\partial_x$ and ∂_x are congruent as well (a similar result was obtained in [2, Theorem 3.4] for one interval with finite length). The strategy is as follows. First, we diagonalize P_1 and then apply our congruence result Proposition 2.4(c). After that, we use the reflection operators σ_{-1} given by $(\sigma_{-1}u)(x) := u(-x)$ to obtain a congruent operator of the form $D\partial_x$ on $L^2(n, m_+, m_-)$, where D is a diagonal matrix with positive diagonal entries. Finally, using $\sqrt{D^{-1}}$ we obtain the asserted congruence to ∂_x. The precise statement is as follows.

THEOREM 3.2. *Let* $P_1 = P_1^* \in \mathbb{R}^{N\times N}$ *be invertible and compatible. Then there exist integers* $n, m_+, m_- \in \mathbb{N}_0$ *such that* $n+m_+ +m_- = N$ *and an invertible operator* $\mathcal{V}\colon \bigoplus_{k\in\{1,\ldots,N\}} L^2(I_k) \to L^2(n, m_+, m_-)$ *such that*

$$\mathcal{V}P_1\partial_x\mathcal{V}^* = \partial_x.$$

More precisely, $\mathcal{V}$ *is given as a product of constant matrices, the operator* Ψ *given in Proposition* 2.4(c), *and a diagonal operator consisting of identities and reflections on the diagonal.*

Proof. Since P_1 is self-adjoint, we find a unitary matrix $K \in \mathbb{R}^{N\times N}$ such that

$$KP_1K^* = D_1,$$

where $D_1 \in \mathbb{R}^{N\times N}$ is a diagonal matrix, whose diagonal entries are nonzero thanks to the invertibility of P_1. According to Proposition 2.4(c), we find a unitary mapping $\Psi\colon \bigoplus_{k\in\{1,\ldots,N\}} L^2(I_k) \to L^2(n, \widetilde{m}_+, \widetilde{m}_-)$ with suitable $n, \widetilde{m}_+, \widetilde{m}_- \in \mathbb{N}_0$ such that $n + \widetilde{m}_+ + \widetilde{m}_- = N$ and a diagonal matrix with real, nonzero entries $\widetilde{D}_1$ such that

$$\Psi D_1\partial_x\Psi^* = \widetilde{D}_1\partial_x.$$

Then, $\widetilde{D}_1$ can be written according to the latter block decomposition with diagonal matrices $\widetilde{D}_{1,\ell} \in \mathbb{R}^{\ell\times\ell}$ for $\ell \in \{n, \widetilde{m}_+, \widetilde{m}_-\}$:

$$\begin{pmatrix} \widetilde{D}_{1,n} & 0 & 0 \\ 0 & \widetilde{D}_{1,\widetilde{m}_+} & 0 \\ 0 & 0 & \widetilde{D}_{1,\widetilde{m}_-} \end{pmatrix}.$$

Let n^+ be the number of positive numbers in the diagonal of $\widetilde{D}_{1,n}$, $n^- := n - n^+$. Similarly we define $\widetilde{m}^{\pm}_{\pm}$. Next, we define an operator $W : \bigoplus_{k=1}^N L^2(\mathbb{R}) \to \bigoplus_{k=1}^N L^2(\mathbb{R})$ acting coordinatewise by

$$(W_k u)(x) := \begin{cases} u(x) & \text{if } \widetilde{D}_{1,kk} > 0, \\ u(-x) & \text{if } \widetilde{D}_{1,kk} < 0 \end{cases} \qquad (x \in \mathbb{R})$$

for each $u \in L^2(\mathbb{R})$ and $k \in \{1, \dots, N\}$. Moreover, it is clear that there exists a permutation matrix $Q \in \mathbb{R}^{N\times N}$ such that $QW[L^2(n, \widetilde{m}_+, \widetilde{m}_-)] \subseteq L^2(n, m_+, m_-)$, where $m_+ := \widetilde{m}^+_+ + \widetilde{m}^-_-$, $m_- := \widetilde{m}^-_+ + \widetilde{m}^+_-$. Setting $V := QW : L^2(n, \widetilde{m}_+, \widetilde{m}_-) \to L^2(n, m_+, m_-)$, we infer

$$V\left(\widetilde{D}_1\partial_x\right)V^* = D_2\partial_x$$

on the Hilbert space $L^2(n, m_+, m_-)$ where $D_2 \in \mathbb{R}^{N\times N}$ is a diagonal matrix with strictly positive diagonal entries. Indeed, if $\widetilde{D}_{1,kk}$ is positive, then W_k is just the identity, and if $\widetilde{D}_{1,kk}$ is negative, the reflection W_k yields a change of sign by the chain rule. With these transformations at hand, we obtain

$$\begin{aligned}
\partial_x &= \left(\sqrt{D_2}\right)^{-1} D_2\partial_x \left(\sqrt{D_2}\right)^{-1} \\
&= \left(\sqrt{D_2}\right)^{-1} V\left(\widetilde{D}_1\partial_x\right)V^*\left(\sqrt{D_2}\right)^{-1} \\
&= \left(\sqrt{D_2}\right)^{-1} V\Psi(D_1\partial_x)\Psi^*V^*\left(\sqrt{D_2}\right)^{-1} \\
&= \left(\sqrt{D_2}\right)^{-1} V\Psi K(P_1\partial_x)K^*\Psi^*V^*\left(\sqrt{D_2}\right)^{-1};
\end{aligned}$$

thus the assertion follows with $\mathcal{V} := (\sqrt{D_2})^{-1}V\Psi K$. □

Remark 3.3 (Boundary conditions for $P_1\partial_x$). Since $\mathcal{V}P_1\partial_x\mathcal{V}^* = \partial_x$ we infer that $\mathcal{V}^*$ is a bijection from $H^1(n, m_+, m_-)$ to $\oplus_{k=1}^N H^1(I_k)$. Note that a closer inspection of the proof of Theorem 3.2 reveals that also

$$\mathcal{V}P_1\mathring{\partial}_x\mathcal{V}^* = \mathring{\partial}_x,$$

where $\mathring{\partial}_x$ stands for the realization of ∂_x restricted to $\bigoplus_{k\in\{1,\dots,N\}} H^1_0(I_k)$ in the former case and to $H^1_0(n, m_+, m_-)$ in the latter case. The reason for this is that the transformation $\mathcal{V}$ maps smooth compactly supported functions into smooth compactly supported functions. In particular, this means that any choice of linear boundary conditions for $P_1\partial_x$ is in one-to-one correspondence to a boundary condition for ∂_x (see also Lemma 4.2 below). We have classified all boundary conditions for ∂_x leading to an m-accretive operator realization of ∂_x. Hence, we obtain a complete description of all boundary conditions for $P_1\partial_x$ leading to m-accretive operator realizations as a consequence of Theorem 3.2. We shall see in Lemma 4.2 below that the particular form of the congruence is not important. In fact, it turns out that boundary conditions for $P_1\partial_x$ can be rephrased into boundary conditions for ∂_x as long as $\mathcal{V}P_1\partial_x\mathcal{V}^* = \partial_x$ for *any* invertible operator $\mathcal{V}$.

Remark 3.4. Using the congruence in Theorem 3.2 and the fact that m-accretivity is preserved by such congruences, we can employ Theorem 2.8 to characterize all boundary conditions for the port-Hamiltonian operator $P_1\partial_x\mathcal{H}$ yielding an m-accretive operator. In this way, we can deal with rather general networks with finite and infinite edges in one unified framework. To the best of our knowledge, this has not been achieved in the literature before. In the next section, we will deal with the special case of finite edges with equal length and illustrate what this characterization may look like. In particular, we will recover known results from the literature in this case (see Theorem 4.1).

We conclude this section by looking at a variant of the above result under the assumption

$$-\infty < \inf_{k\in\{1,\ldots,N\}} I_k < \sup_{k\in\{1,\ldots,N\}} I_k < \infty \tag{3.1}$$

of finiteness of all intervals. Then $n = N, m_+ = m_- = 0$ in Theorem 3.2. We first provide another representation of ∂_x on $L^2(N, 0, 0) = L^2(]-1/2, 1/2[)^N$. It will be instrumental for the proof that the derivative of an odd function is even and that the derivative of an even function is odd.

THEOREM 3.5 ([12, pp. 61–62] and [18, pp. 2811–2812]). *Let* $\partial_x\colon H^1(]-1/2,1/2[)^N \subseteq L^2(]-1/2,1/2[)^N \to L^2(]-1/2,1/2[)^N$. *We define*

$$\iota_{\mathrm{e}}\colon L^2_{\mathrm{e}}(]-1/2,1/2[)^N \to L^2(]-1/2,1/2[)^N,$$
$$\iota_{\mathrm{o}}\colon L^2_{\mathrm{o}}(]-1/2,1/2[)^N \to L^2(]-1/2,1/2[)^N,$$

the canonical embeddings from the (componentwise) even and odd functions on $L^2(]-1/2,1/2[)$ *into* $L^2(]-1/2,1/2[)$. *Then*

$$\begin{pmatrix} \iota_{\mathrm{e}}^* \\ \iota_{\mathrm{o}}^* \end{pmatrix} \partial_x \begin{pmatrix} \iota_{\mathrm{e}} & \iota_{\mathrm{o}} \end{pmatrix} = \begin{pmatrix} 0 & \partial_{x,\mathrm{o}} \\ \partial_{x,\mathrm{e}} & 0 \end{pmatrix},$$

where

$$\partial_{x,\mathrm{e}}\colon H^1_{\mathrm{e}}(]-1/2,1/2[)^N \subseteq L^2_{\mathrm{e}}(]-1/2,1/2[)^N \to L^2_{\mathrm{o}}(]-1/2,1/2[)^N,$$
$$\phi \mapsto \phi',$$

and $H^1_{\mathrm{e}}(]-1/2,1/2[)^N := H^1(]-1/2,1/2[)^N \cap L^2_{\mathrm{e}}(]-1/2,1/2[)^N$*; similarly* $\partial_{x,\mathrm{o}}$ *and* $H^1_{\mathrm{o}}(]-1/2,1/2[)^N$.

Remark 3.6. Note that the previous theorem permits us to compute the form of the boundary conditions for $\mathring{\partial}_x \subseteq A \subseteq \partial_x$ m-accretive in terms of restrictions and extensions of $\left(\begin{smallmatrix} 0 & \partial_{x,\mathrm{e}} \\ \partial_{x,\mathrm{o}} & 0 \end{smallmatrix}\right)$. Indeed, let $\mathring{\partial}_x \subseteq A \subseteq \partial_x$ be maximal accretive, and let $M \in \mathbb{R}^{N\times N}$ be such that

$$\mathrm{dom}(A) = \left\{ u \in H^1(]-1/2,1/2[)^N\,;\, Mu\left(\frac{1}{2}\right) + u\left(-\frac{1}{2}\right) = 0 \right\}.$$

Note that a small computation (and invariance of smooth compactly supported functions) shows that

$$\begin{pmatrix} \iota_{\mathrm{e}}^* \\ \iota_{\mathrm{o}}^* \end{pmatrix} \mathring{\partial}_x \begin{pmatrix} \iota_{\mathrm{e}} & \iota_{\mathrm{o}} \end{pmatrix} = \begin{pmatrix} 0 & \mathring{\partial}_{x,\mathrm{o}} \\ \mathring{\partial}_{x,\mathrm{e}} & 0 \end{pmatrix}.$$

Also note that by $\iota_{\mathrm{e}}\iota_{\mathrm{e}}^* u(-1/2) = \iota_{\mathrm{e}}\iota_{\mathrm{e}}^* u(1/2)$ and $\iota_{\mathrm{o}}\iota_{\mathrm{o}}^* u(-1/2) = -\iota_{\mathrm{o}}\iota_{\mathrm{o}}^* u(1/2)$ we obtain

$$u\left(\frac{1}{2}\right) = (\iota_e\iota_e^* u + \iota_o\iota_o^* u)\left(\frac{1}{2}\right) = (\iota_e\iota_e^* u)\left(\frac{1}{2}\right) + (\iota_o\iota_o^* u)\left(\frac{1}{2}\right),$$
$$u\left(-\frac{1}{2}\right) = (\iota_e\iota_e^* u + \iota_o\iota_o^* u)\left(-\frac{1}{2}\right) = (\iota_e\iota_e^* u)\left(\frac{1}{2}\right) - (\iota_o\iota_o^* u)\left(\frac{1}{2}\right).$$

Hence,

$$\operatorname{dom}(A) = \Big\{u \in H^1(]-1/2,1/2[)^N; \\ M\left((\iota_e\iota_e^* u)\left(\frac{1}{2}\right) + (\iota_o\iota_o^* u)\left(\frac{1}{2}\right)\right) + (\iota_e\iota_e^* u)\left(\frac{1}{2}\right) - (\iota_o\iota_o^* u)\left(\frac{1}{2}\right) = 0\Big\}.$$

Thus, $\widetilde{A} := \begin{pmatrix} \iota_e^* \\ \iota_o^* \end{pmatrix} A \begin{pmatrix} \iota_e & \iota_o \end{pmatrix}$ is an m-accretive restriction of $\begin{pmatrix} 0 & \partial_{x,o} \\ \partial_{x,e} & 0 \end{pmatrix}$ with domain

$$\operatorname{dom}(\widetilde{A}) = \Big\{ \begin{pmatrix} u_e \\ u_o \end{pmatrix} \in H_e^1(]-1/2,1/2[)^N \oplus H_o^1(]-1/2,1/2[)^N; \\ \begin{pmatrix} M+I & M-I \end{pmatrix} \begin{pmatrix} u_e(1/2) \\ u_o(1/2) \end{pmatrix} = 0 \Big\}.$$

This leads to an alternative proof of one equivalence in [9, Theorem 7.2.4] (also pay attention to [9, Lemma 7.3.1]).

4. Well-posedness of port-Hamiltonian boundary control systems. In this section, we shall have a closer look at port-Hamiltonian systems in the simpler case $I_1 = \cdots = I_N =]a,b[$ for some $a,b \in \mathbb{R}$. Note that then $n = N, m_+ = m_- = 0$, and $L^2(N,0,0) = L^2(]-1/2,1/2[)^N = L^2(]-1/2,1/2[;\mathbb{R}^N)$. In fact, because of the structure theorem, which allows us to represent any port-Hamiltonian system as ∂_x, we are in a position to describe a rather general class of port-Hamiltonian systems and discuss related issues like well-posedness of associated boundary control systems. The foundation of this lies in the solution theory for evolutionary equations; see, e.g., [11, 13, 14, 16].

We briefly recall the setting of [8, section 5] (see also [9, section 7]). For this, let $a,b \in \mathbb{R}$, $a < b$, $P_1 = P_1^* \in \mathbb{R}^{N\times N}$ be invertible, $P_0 = -P_0^* \in \mathbb{R}^{N\times N}$, $\mathcal{H} \in L^\infty(]a,b[;\mathbb{R}^{N\times N})$. Assume there exists $m, M \in \mathbb{R}_{>0}$ such that

$$mI_{N\times N} \le \mathcal{H}(x) = \mathcal{H}(x)^* \le MI_{N\times N} \quad (\text{a.e. } x \in]a,b[).$$

Let

$$A\colon \operatorname{dom}(A) \subseteq L^2_{\mathcal{H}}(]a,b[)^N \to L^2_{\mathcal{H}}(]a,b[)^N, \\ u \mapsto P_1(\mathcal{H}u)' + P_0\mathcal{H}u,$$

where

$$\operatorname{dom}(\mathring{\partial}_x\mathcal{H}) \subseteq \operatorname{dom}(A) \subseteq \operatorname{dom}(\partial_x\mathcal{H}) \quad \text{and} \quad L^2_{\mathcal{H}}(]a,b[)^N := (L^2(]a,b[)^N; \langle\cdot,\mathcal{H}\cdot\rangle_{L^2(]a,b[)^N}).$$

In order to have a meaningful notion of well-posedness, classically, people focus on the generator properties of A. Generating a (C_0-)semigroup of contractions can be characterized as the closed densely defined operator being m-dissipative, by the Lumer–Philipps theorem. In the particular case of port-Hamiltonian systems this characterization can be reformulated in terms of the boundary conditions parametrized by some matrix W_B; see, e.g., [6, Theorem 1] or [8, Theorem 5.8].

THEOREM 4.1. *The following conditions are equivalent:*

(i) $-A$ *generates a semigroup of contractions on* $L^2_{\mathcal{H}}(]a,b[)^N$*; that is, A is maximal accretive;*

(ii) *A is accretive and there exists* $W_B \in \mathbb{R}^{N\times 2N}$ *such that*

$$\operatorname{dom}(A) = \left\{ u \in L^2(]a,b[)^N \,;\, \mathcal{H}u \in H^1(]a,b[)^N,\, W_B \begin{pmatrix} (\mathcal{H}u)(b) \\ (\mathcal{H}u)(a) \end{pmatrix} = 0 \right\};$$

(iii) *there exists* $W_B \in \mathbb{R}^{N\times 2N}$ *such that* $\operatorname{dom}(A)$ *is given as in* (ii) *and W_B has rank N and, in the sense of positive definiteness,*

$$W_B \begin{pmatrix} -P_1 & P_1 \\ I_{N\times N} & I_{N\times N} \end{pmatrix}^{-1} \begin{pmatrix} 0 & I_{N\times N} \\ I_{N\times N} & 0 \end{pmatrix} \left(W_B \begin{pmatrix} -P_1 & P_1 \\ I_{N\times N} & I_{N\times N} \end{pmatrix}^{-1} \right)^* \geq 0.$$

If any of the above holds, then there exist an invertible matrix $G \in \mathbb{R}^{2N\times 2N}$ *and matrices* $M, L \in \mathbb{R}^{N\times N}$ *with* $M^*M \leq 1$ *and L invertible such that*

$$W_B G = L \begin{pmatrix} M & 1 \end{pmatrix}.$$

We aim to prove this theorem with the help of our congruence result, Theorem 3.2, and the characterization result, Theorem 2.8. For this, we need to inspect how the transformation $\mathcal{V}$ of Theorem 3.2 acts on the boundary values of a function $u \in H^1(]a,b])^N$. It is remarkable that no knowledge of how $\mathcal{V}$ acts on functions with vanishing boundary values is needed in order to obtain that $H^1_0(]a,b])^N$-functions are mapped onto $H^1_0(]-1/2,1/2[)^N$-functions.

LEMMA 4.2. *Let* $P_1 = P_1^* \in \mathbb{R}^{N\times N}$ be *invertible. Moreover, let* $\mathcal{V} : L^2(]a,b[)^N \to L^2(]-1/2,1/2[)^N$ *be invertible such that*

$$\mathcal{V} P_1 \partial_x \mathcal{V}^* = \partial_x.$$

Then for $u \in H^1(]-1/2,1/2[)^N$ *we have that* $\mathcal{V}^*u \in H^1(]a,b[)^N$*, and there exists an invertible matrix* $G \in \mathbb{R}^{2N\times 2N}$ *such that*

$$\begin{pmatrix} (\mathcal{V}^*u)(b) \\ (\mathcal{V}^*u)(a) \end{pmatrix} = G \begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix} \quad (u \in H^1(]-1/2,1/2[)^N).$$

Moreover, G satisfies

$$G^* \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. From the congruence we see that $\mathcal{V}^*u \in H^1(]a,b[)^N$ for functions $u \in H^1(]-1/2,1/2[)^N$. Moreover, for $u, w \in H^1(]-1/2,1/2[)^N$ we compute by using the integration by parts formula twice

$$\left\langle \begin{pmatrix} (\mathcal{V}^*u)(b) \\ (\mathcal{V}^*u)(a) \end{pmatrix}, \begin{pmatrix} P_1(\mathcal{V}^*w)(b) \\ -P_1(\mathcal{V}^*w)(a) \end{pmatrix} \right\rangle_{\mathbb{R}^{2N}} \tag{4.1}$$

$$= \langle (\mathcal{V}^*u)(b), P_1(\mathcal{V}^*w)(b)\rangle_{\mathbb{R}^N} - \langle (\mathcal{V}^*u)(a), P_1(\mathcal{V}^*w)(a)\rangle_{\mathbb{R}^N}$$

$$= \langle \partial_x \mathcal{V}^* u, P_1 \mathcal{V}^* w\rangle_{L^2(]a,b[)^N} + \langle \mathcal{V}^* u, \partial_x P_1 \mathcal{V}^* w\rangle_{L^2(]a,b[)^N}$$

$$= \langle \mathcal{V} P_1 \partial_x \mathcal{V}^* u, w\rangle_{L^2(]-1/2,1/2[)^N} + \langle u, \mathcal{V} P_1 \partial_x \mathcal{V}^* w\rangle_{L^2(]-1/2,1/2[)^N}$$

$$= \langle \partial_x u, w\rangle_{L^2(]-1/2,1/2[)^N} + \langle u, \partial_x w\rangle_{L^2(]-1/2,1/2[)^N}$$

$$= \left\langle \begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix}, \begin{pmatrix} w(1/2) \\ -w(-1/2) \end{pmatrix} \right\rangle_{\mathbb{R}^{2N}}.$$

We consider now the binary relation

$$G := \left\{ \left(\begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix}, \begin{pmatrix} (\mathcal{V}^* u)(b) \\ (\mathcal{V}^* u)(a) \end{pmatrix} \right) ; u \in H^1(]-1/2,1/2[)^N \right\} \subseteq \mathbb{R}^{2N} \times \mathbb{R}^{2N}.$$

Obviously, G is linear. Moreover, G is a mapping: Indeed, due to the linearity, it suffices to prove that $\begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix} = 0$ implies $\begin{pmatrix} (\mathcal{V}^* u)(b) \\ (\mathcal{V}^* u)(a) \end{pmatrix} = 0$. For doing so we choose $u \in H^1(]-1/2,1/2[)^N$ with $\begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix} = 0$. By (4.1) it follows that

$$\left\langle \begin{pmatrix} (\mathcal{V}^* u)(b) \\ (\mathcal{V}^* u)(a) \end{pmatrix}, \begin{pmatrix} P_1(\mathcal{V}^* w)(b) \\ -P_1(\mathcal{V}^* w)(a) \end{pmatrix} \right\rangle = 0 \quad (w \in H^1(]-1/2,1/2[)^N).$$

Let now $x, y \in \mathbb{R}^N$ and define $f(t) := \frac{1}{b-a}\left((t-a)P_1^{-1}x + (t-b)P_1^{-1}y\right)$ for $t \in [a,b]$. Then $f \in H^1(]a,b[)^N$ and we set $w := (\mathcal{V}^*)^{-1} f \in H^1(]-1/2,1/2[)^N$ (this follows from $\mathcal{V} P_1 \partial_x \mathcal{V}^* = \partial_x$). Then, clearly, $P_1(\mathcal{V}^* w)(b) = x$ and $-P_1(\mathcal{V}^* w)(a) = y$, and so, we infer

$$\left\langle \begin{pmatrix} (\mathcal{V}^* u)(b) \\ (\mathcal{V}^* u)(a) \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = 0.$$

Since this holds for all $x, y \in \mathbb{R}^N$, we derive $\begin{pmatrix} (\mathcal{V}^* u)(b) \\ (\mathcal{V}^* u)(a) \end{pmatrix} = 0$. In the same way one shows that G is one-to-one. Moreover, it is obvious that the domain of G is $\mathbb{R}^{2N}$ (see also Lemma 4.10), and so G can be represented by an invertible matrix, which we again denote by $G \in \mathbb{R}^{2N \times 2N}$. Thus, we have

$$\begin{pmatrix} (\mathcal{V}^* u)(b) \\ (\mathcal{V}^* u)(a) \end{pmatrix} = G \begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix} \quad (u \in H^1(]-1/2,1/2[)^N).$$

Plugging this representation into (4.1), we obtain

$$\left\langle G \begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix}, \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} G \begin{pmatrix} w(1/2) \\ w(-1/2) \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} w(1/2) \\ w(-1/2) \end{pmatrix} \right\rangle$$

for all $u, w \in H^1(]-1/2,1/2[)^N$, from which we infer

$$G^* \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

□

Next, we present our alternative way of characterizing maximal accretivity of A in terms of the boundary conditions. Our approach makes use of the similarity result presented in the previous section.

Proof of Theorem 4.1. First, we note that we can assume without loss of generality that $P_0 = 0$ (note that $P_0\mathcal{H}$ is skew-self-adjoint on $L^2_{\mathcal{H}}(]a,b[)^N$ and thus, does not affect the maximal accretivity of A) and $\mathcal{H} = 1$, by Proposition 3.1.

(i) $\Rightarrow$ (ii): Assume that A is maximal accretive. We need to show that there exists $W_B \in \mathbb{R}^{N\times 2N}$ with

$$\operatorname{dom}(A) = \left\{u \in L^2(]a,b[)^N\,;\, \mathcal{H}u \in H^1(]a,b[)^N,\, W_B \begin{pmatrix} (\mathcal{H}u)(b) \\ (\mathcal{H}u)(a) \end{pmatrix} = 0\right\}.$$

By Theorem 3.2 we find $\mathcal{V} : L^2(]a,b[)^N \to L^2(]{-1/2},1/2[)^N$ invertible with $\mathcal{V}P_1\partial_x\mathcal{V}^* = \partial_x$. Hence, $D := \mathcal{V}A\mathcal{V}^*$ is maximal accretive and satisfies $\overset{\circ}{\partial}_x \subseteq D \subseteq \partial_x$ (note that $\mathcal{V}$ leaves $H^1_0(]a,b[)^N$ invariant by Lemma 4.2). By Theorem 2.8 we find a matrix $M \in \mathbb{R}^{N\times N}$ such that

$$\operatorname{dom}(D) = \{v \in H^1(]a,b[)^N\,;\, Mv(1/2) + v(-1/2) = 0\},$$

which in turn implies

$$\begin{aligned}
\operatorname{dom}(A) &= \{u \in L^2(]a,b[)^N\,;\, (\mathcal{V}^*)^{-1}u \in \operatorname{dom}(D)\}\\
&= \{u \in H^1(]a,b[)^N\,;\, M((\mathcal{V}^*)^{-1}u)(1/2) + ((\mathcal{V}^*)^{-1}u)(-1/2) = 0\}\\
&= \left\{u \in H^1(]a,b[)^N\,;\, \begin{pmatrix} M & 1\end{pmatrix} \begin{pmatrix} ((\mathcal{V}^*)^{-1})u(1/2) \\ ((\mathcal{V}^*)^{-1}u)(-1/2)\end{pmatrix} = 0\right\}\\
&= \left\{u \in H^1(]a,b[)^N\,;\, \begin{pmatrix} M & 1\end{pmatrix} G^{-1} \begin{pmatrix} u(b) \\ u(a)\end{pmatrix} = 0\right\},
\end{aligned}$$

where we have used Lemma 4.2 in the last equality. This establishes the implication (i) $\Rightarrow$ (ii) with $W_B := \begin{pmatrix} M & 1\end{pmatrix} G^{-1} \in \mathbb{R}^{N\times 2N}$.

(ii) $\Rightarrow$ (iii): Assume that A is accretive and that $\operatorname{dom}(A)$ is given as in (ii). Using Theorem 3.2 we find $\mathcal{V} : L^2(]a,b[)^N \to L^2(]-1/2,1/2[)^N$ invertible, such that

$$\mathcal{V}P_1\partial_x\mathcal{V}^* = \partial_x.$$

We set $D := \mathcal{V}A\mathcal{V}^*$ and obtain an accretive operator on $L^2(]-1/2,1/2[)^N$ satisfying $\overset{\circ}{\partial}_x \subseteq D \subseteq \partial_x$. Moreover, by Lemma 4.2 the domain of D is given by

$$\operatorname{dom}(D) = \left\{u \in H^1(]-1/2,1/2[)^N\,;\, W_B G \begin{pmatrix} u(1/2) \\ u(-1/2)\end{pmatrix} = 0\right\}$$

with $G \in \mathbb{R}^{2N\times 2N}$ as in Lemma 4.2. Moreover, by Theorem 2.8(a) there exists $M \in \mathbb{R}^N$ with $M^*M \leq 1$ such that

$$\operatorname{dom}(D) \subseteq \left\{u \in H^1(]-1/2,1/2[)^N\,;\, \begin{pmatrix} M & 1\end{pmatrix} \begin{pmatrix} u(1/2) \\ u(-1/2)\end{pmatrix} = 0\right\}.$$

Consider now the linear mappings

$$\begin{aligned}
F_1 : \mathbb{R}^N \times \mathbb{R}^N &\to H^1(]-1/2,1/2[)^N,\\
(x,y) &\mapsto (t \mapsto (1/2-t)y + (1/2+t)x)
\end{aligned}$$

and

$$\begin{aligned}
F_2 : H^1(]-1/2,1/2[)^N &\to \mathbb{R}^N \times \mathbb{R}^N,\\
u &\mapsto (u(1/2), u(-1/2)).
\end{aligned}$$

By definition $F_1[\ker W_B G] \subseteq \mathrm{dom}(D)$ and $F_2[\mathrm{dom}(D)] \subseteq \ker\begin{pmatrix} M & 1 \end{pmatrix}$, and thus,

$$\mathrm{id} = F_2 \circ F_1 : \ker W_B G \to \ker \begin{pmatrix} M & 1 \end{pmatrix}$$

is a well-defined linear injective mapping. Hence,

$$N \leq \dim\ker W_B G \leq \dim\ker \begin{pmatrix} M & 1 \end{pmatrix} = N,$$

and so, $\ker W_B G = \ker\begin{pmatrix} M & 1 \end{pmatrix}$. Thus, there is $L \in \mathbb{R}^{N\times N}$ invertible such that

$$W_B G = L \begin{pmatrix} M & 1 \end{pmatrix}. \tag{4.2}$$

In particular, W_B has rank N. It is left to show that

$$W_B \begin{pmatrix} -P_1 & P_1 \\ I_{N\times N} & I_{N\times N} \end{pmatrix}^{-1} \begin{pmatrix} 0 & I_{N\times N} \\ I_{N\times N} & 0 \end{pmatrix} \left(W_B \begin{pmatrix} -P_1 & P_1 \\ I_{N\times N} & I_{N\times N} \end{pmatrix}^{-1} \right)^* \geq 0 \tag{4.3}$$

and an easy computation shows that (4.3) is equivalent to

$$W_B \begin{pmatrix} -P_1^{-1} & 0 \\ 0 & P_1^{-1} \end{pmatrix} W_B^* \geq 0.$$

Using Lemma 4.2, we infer that

$$\begin{pmatrix} -P_1^{-1} & 0 \\ 0 & P_1^{-1} \end{pmatrix} = G \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G^*,$$

and so

$$\begin{aligned} W_B \begin{pmatrix} -P_1^{-1} & 0 \\ 0 & P_1^{-1} \end{pmatrix} W_B^* &= W_B G \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G^* W_B^* \\ &= L \begin{pmatrix} M & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M^* \\ 1 \end{pmatrix} L^* \\ &= L(-MM^* + 1)L^* \geq 0, \end{aligned}$$

which shows (ii) $\Rightarrow$(iii) as well as the formula $W_B G = L \begin{pmatrix} M & 1 \end{pmatrix}$.

(iii) $\Rightarrow$ (i): We again set $D := \mathcal{V}A\mathcal{V}^*$ and prove that D is maximal accretive. We recall that

$$\mathrm{dom}(D) = \left\{ u \in H^1(]-1/2, 1/2[)^N \,;\, W_B G \begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix} = 0 \right\}$$

with $G \in \mathbb{R}^{2N\times 2N}$ from Lemma 4.2, and from the computation above, we see that

$$W_B G \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G^* W_B^* \geq 0.$$

We set $K := W_B G = \begin{pmatrix} K_1 & K_2 \end{pmatrix} \in \mathbb{R}^{N\times 2N}$, which has rank N and satisfies $K_1 K_1^* \leq K_2 K_2^*$. Since K has rank N, the kernel of $\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ is trivial. Let now $x \in \ker K_2$. From $K_1 K_1^* \leq K_2 K_2^*$ it follows that $x \in \ker K_1$, and hence, $x \in \ker \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \{0\}$. Thus, K_2 is invertible, and we set $M := K_2^{-1} K_1$. Then $MM^* = K_2^{-1} K_1 K_1^* (K_2^{-1})^* \leq 1$, and hence $M^* M \leq 1$. Moreover, $\ker K = \ker\begin{pmatrix} M & 1 \end{pmatrix}$, and thus

$$\mathrm{dom}(D) = \left\{ u \in H^1(]-1/2, 1/2[)^N \,;\, \begin{pmatrix} M & 1 \end{pmatrix} \begin{pmatrix} u(1/2) \\ u(-1/2) \end{pmatrix} = 0 \right\}.$$

Thus, the maximal accretivity of D, and hence of A, follows from Theorem 2.8(c). □

The formulation of the generator property as in Theorem 4.1 is instrumental to understanding the well-posedness theorem for *boundary control systems* in connection with port-Hamiltonian systems, for which we will provide a different perspective below. We recall that in the context of evolutionary equations, the well-posedness of port-Hamiltonian boundary control systems has already been dealt with in [15, section 5.1]. In any case, we need the following notions. Let $W_{B,j} \in \mathbb{R}^{N_j \times 2N}$ and $N_j \in \mathbb{N}$, $j \in \{1,2\}$, with $N = N_1 + N_2, K \in \mathbb{N}$ and $W_C \in \mathbb{R}^{K\times 2N}$. We define

$$\begin{aligned}\mathfrak{A}\colon \operatorname{dom}(\mathfrak{A}) \subseteq L^2(]a,b[)^N &\to L^2(]a,b[)^N,\\ u &\mapsto P_1\left(\mathcal{H}u\right)' + P_0\mathcal{H}u,\\ \operatorname{dom}(\mathfrak{A}) &= \left\{u \in L^2(]a,b[)^N\,;\, \mathcal{H}u \in H^1(]a,b[)^N, W_{B,2}\begin{pmatrix}\mathcal{H}u(b)\\ \mathcal{H}u(a)\end{pmatrix} = 0\right\},\end{aligned}$$

as well as

$$\begin{aligned} B\colon \operatorname{dom}(\mathfrak{A}) &\to \mathbb{R}^{N_1},\\ u &\mapsto W_{B,1}\begin{pmatrix}\mathcal{H}u(b)\\ \mathcal{H}u(a)\end{pmatrix}, \text{ and}\\ C\colon \operatorname{dom}(\mathfrak{A}) &\to \mathbb{R}^{K},\\ u &\mapsto W_{C}\begin{pmatrix}\mathcal{H}u(b)\\ \mathcal{H}u(a)\end{pmatrix}.\end{aligned}$$

Theorem 4.3 ([23, Theorem 2.4] and [8, Theorem 7.7]). *Assume that for all $x \in]a,b[$ there exist an invertible matrix $S(x)$ and a diagonal matrix $\Delta(x)$ such that*

$$P_1\mathcal{H}(x) = S(x)^{-1}\Delta(x)S(x),$$

where S and Δ are continuously differentiable. Moreover, assume that $\operatorname{rank}\begin{pmatrix}W_{B,1}\\ W_{B,2}\end{pmatrix} = N$, $\operatorname{rank}\begin{pmatrix}W_{B,1}\\ W_{B,2}\\ W_C\end{pmatrix} = N + \operatorname{rank}(W_C)$ *and that $-A := -\mathfrak{A}|_{\ker(B)}$ is a generator of a C_0-semigroup on $L^2(]a,b[)^N$. Then for each $\tau > 0$ and all $v \in C^2([0,\tau])^{N_1}$, $u_0 \in \operatorname{dom}(\mathfrak{A})$ with $Bu_0 = v(0)$ there exists a unique classical solution $u \in C^1([0,\tau])^N$ of*

$$\begin{aligned}\dot u(t) &= -\mathfrak{A}u(t) = -P_1\left(\mathcal{H}u\right)' - P_0\mathcal{H}u, \quad u(0) = u_0,\\ v(t) &= Bu(t) = W_{B,1}\begin{pmatrix}\mathcal{H}u(t,b)\\ \mathcal{H}u(t,a)\end{pmatrix},\\ y(t) &= Cu(t) = W_{C}\begin{pmatrix}\mathcal{H}u(t,b)\\ \mathcal{H}u(t,a)\end{pmatrix}.\end{aligned}$$

Moreover, there exists a constant $m_\tau \geq 0$ (just depending on τ) such that

$$\|u(\tau)\|^2_{L^2(]a,b[)^N} + \int_0^\tau \|y(t)\|^2_{\mathbb{R}^K}\,\mathrm{d}t \leq m_\tau\left(\|u_0\|^2_{L^2(]a,b[)^N} + \int_0^\tau \|v(t)\|^2_{\mathbb{R}^{N_1}}\,\mathrm{d}t\right).$$

Owing to the flexibility of evolutionary equations, we are able to significantly improve the well-posedness result in as much as we do not need to impose any regularity conditions on $\mathcal{H}$. Further, we can address systems which are more general than the Cauchy problems of the previous theorem. In particular, we consider differential-algebraic equations. For this we consider equations of the following form.

Definition 4.4. *Let $P_1 = P_1^* \in \mathbb{R}^{N\times N}$ be invertible and $\mathcal{H} \in L^\infty(]a,b[;\mathbb{R}^{N\times N})$. Assume there exist $m, M \in \mathbb{R}_{>0}$ such that*

$$mI_{N\times N} \le \mathcal{H}(x) = \mathcal{H}(x)^* \le MI_{N\times N} \quad (a.e.\quad x \in]a,b[).$$

Let $M_0 = M_0^*, M_1 \in L(L^2(]a,b[)^N)$ such that $M_0 \ge 0$. An equation of the form

$$(\partial_t M_0 + M_1 + P_1\partial_x)\,\mathcal{H}U = F,$$

where $F\colon \mathbb{R}\times]a,b[\to \mathbb{R}^N$ is given and $U\colon \mathbb{R}\times]a,b[\to \mathbb{R}^N$ is the unknown, is called a differential-algebraic port-Hamiltonian equation; *here ∂_t is the derivative with respect to the $\mathbb{R}$-variable ("time"), and ∂_x is the coordinatewise derivative with respect to the spatial variable in $]a,b[$.*

Remark 4.5. The classical port-Hamiltonian operator is then covered by choosing $M_1 = P_0$ and $M_0 = \mathcal{H}^{-1}$.

We emphasize that in the literature different types of differential-algebraic port-Hamiltonian systems are treated which are not covered by the class above. Indeed, in [4] a port-Hamiltonian system modeling a viscoelastic nanorod is studied, and the resulting system is differential-algebraic in the sense that P_1 is nonregular. This is not covered by the class above. However, to deal with this example, the authors needed to restrict the state space to obtain a well-posed system, which is not needed for the class considered here.

We provide the counterpart of the generation property first. We will restrict ourselves to maximal accretive restrictions of $P_1\partial_x$ (corresponding to the case of a generator of a contraction semigroup) and note that the additional generality enters the problem via the operators M_0 and M_1. In order to formulate the well-posedness result, we need some notation from the theory of evolutionary equations.

Definition 4.6. *For a Hilbert space H and $\rho > 0$ we define*

$$L^2_\rho(\mathbb{R};H) := \left\{u\colon \mathbb{R}\to H\,;\, u \text{ Bochner-measurable}, \int_{\mathbb{R}} \|u(t)\|^2 \mathrm{e}^{-2\rho t}\,\mathrm{d}t < \infty\right\}$$

equipped with the natural inner product. Moreover, we define the operator ∂_t on $L^2_\rho(\mathbb{R};H)$ as the closure of

$$\begin{aligned} C_c^1(\mathbb{R};H) \subseteq L^2_\rho(\mathbb{R};H) &\to L^2_\rho(\mathbb{R};H),\\ \phi &\mapsto \phi', \end{aligned}$$

where $C_c^1(\mathbb{R};H)$ denotes the space of continuously differentiable functions with compact support on $\mathbb{R}$ taking values in H.

The domain of ∂_t is given by all functions $u \in L^2_\rho(\mathbb{R};H)$ such that there exists $v \in L^2_\rho(\mathbb{R};H)$ with

$$\int_{\mathbb{R}} u\phi' = -\int_{\mathbb{R}} v\phi \quad (\phi \in C_c^\infty(\mathbb{R}))$$

and in the latter case we have $v = \partial_t u$ (see [16, Proposition 4.1.1]). We remark that the so-defined operator ∂_t is continuously invertible on $L^2_\rho(\mathbb{R};H)$ (see, e.g., [13, 16]) and thus allows for the following definition.

Definition 4.7. *For $k \in \mathbb{N}$ and $\rho > 0$ we define the space $H_\rho^{-k}(\mathbb{R};H)$ as the completion of $L_\rho^2(\mathbb{R};H)$ with respect to the norm*

$$\|u\|_{\rho,-k} := \|\partial_t^{-k} u\|_{L_\rho^2}.$$

Remark 4.8. The space $H_\rho^{-k}(\mathbb{R};H)$ is a so-called extrapolation space for ∂_t. We refer the reader to[13, Chapter 2] for the general concept of Sobolev chains/lattices or to [3, Chapter II.5] for extrapolation spaces in the framework of C_0-semigroups. It is easy to see that ∂_t can be extended to a continuously invertible operator on $H_\rho^{-k}(\mathbb{R};H)$ and that each closed densely defined operator between two Hilbert spaces H_0, H_1 can be canonically extended to a closed and densely defined operator between $H_\rho^{-k}(\mathbb{R};H_0)$ and $H_\rho^{-k}(\mathbb{R};H_1)$.

We recall the well-posedness theorem for evolutionary equations in the form needed here; see also [16, Chapter 6].

Theorem 4.9 ([11, Solution Theory], [13, Theorem 6.2.5]). *Let H be a Hilbert space, $M_0, M_1 \in L(H)$ with $M_0 = M_0^* \geq 0$, and assume there exist $\rho_0 > 0$ and $c > 0$ such that*

$$\rho_0 \langle M_0 y, y\rangle + \langle M_1 y, y\rangle \geq c\|y\|^2$$

for all $y \in H$. Moreover, let $A\colon \operatorname{dom}(A) \subseteq H \to H$ be maximal accretive. Then for each $\rho \geq \rho_0$ the operator

$$\left(\overline{\partial_t M_0 + M_1 + A}\right)$$

is continuously invertible on $H_\rho^{-k}(\mathbb{R};H)$ for each $k \in \mathbb{N}_0$.

We give a short sketch of the proof of the above theorem. By the accretivity of A and the conditions on M_0, M_1 one confirms that $\partial_t M_0 + M_1 + A - c$ is accretive in $L_\rho^2(\mathbb{R};H)$ for ρ large enough. Using the spectral representation of ∂_t one can prove that the same holds for its adjoint. Thus, $\overline{\partial_t M_0 + M_1 + A}$ is boundedly invertible in $L_\rho^2(\mathbb{R};H)$, and invoking Remark 4.8, one obtains the assertion.

Before we can state the well-posedness result for boundary control problems for differential-algebraic port-Hamiltonian equations, we need the following prerequisite.

Lemma 4.10. *Let $N \in \mathbb{N}$. Then there exists $\eta\colon \mathbb{R}^{2N} \to H^1(]a,b[)^N$ continuous such that*

$$\gamma\left(\eta(v)\right) = v$$

for all $v \in \mathbb{R}^{2N}$, where

$$\gamma\colon H^1(]a,b[)^N \ni (u_k)_{k\in\{1,\ldots,N\}} \mapsto \begin{pmatrix} (u_k(b))_{k\in\{1,\ldots,N\}} \\ (u_k(a))_{k\in\{1,\ldots,N\}} \end{pmatrix}.$$

Proof. Let $v = (v_1, v_2) \in \mathbb{R}^N \times \mathbb{R}^N$. Then

$$\eta(v)(x) := \frac{1}{b-a}\left((x-a)v_1 + (b-x)v_2\right)$$

is a valid choice for η. The continuity properties are easily checked. ∎

The next result puts Theorem 4.3 into the perspective of evolutionary equations. Note that we do not assume any regularity condition on $\mathcal{H}$. As our main assumption, we shall assume the accretivity of the 'derivative part' of the port-Hamiltonian.

THEOREM 4.11. *Consider a differential-algebraic port-Hamiltonian equation as in Definition* 4.4. *Assume that there exists* $\rho_0 \geq 0$ *such that for all* $y \in L^2(]a,b[)^N$

$$\rho_0 \langle M_0 y, y\rangle + \langle M_1 y, y\rangle \geq c\langle y, y\rangle$$

for some $c > 0$. *Let* $\gamma\colon H^1(]a,b[)^N \to \mathbb{R}^{2N}$ *be given by*

$$\gamma(u_k)_{k\in\{1,\dots,N\}} = \begin{pmatrix} (u_k(b))_{k\in\{1,\dots,N\}} \\ (u_k(a))_{k\in\{1,\dots,N\}} \end{pmatrix}.$$

Let $W \in \mathbb{R}^{N\times 2N}$ *be such that*

$$\begin{aligned} A\colon \operatorname{dom}(A) \subseteq L^2(]a,b[)^N &\to L^2(]a,b[)^N, \\ u &\mapsto P_1 \partial_x u \end{aligned}$$

with

$$\operatorname{dom}(A) = \{u \in H^1(]a,b[)^N\,;\, W\gamma u = 0\}$$

being maximal accretive (cf. Theorem 4.1*). Furthermore, let* $v \in H_\rho^{-k}(\mathbb{R};\mathbb{R}^N)$ *for some* $k \in \mathbb{N}$ *and* $\rho \geq \rho_0$. *Then there exists a unique* $u \in H_\rho^{-k-1}(\mathbb{R}; L^2_{\mathcal{H}}(]a,b[)^N)$ *such that*

$$\begin{aligned} (\partial_t M_0 + M_1 + P_1 \partial_x)\, \mathcal{H}u &= 0, \\ W\gamma\mathcal{H}u &= v. \end{aligned}$$

Moreover, the mapping

$$H_\rho^{-k}(\mathbb{R};\mathbb{R}^N) \ni v \mapsto \mathcal{H}u \in H_\rho^{-k-1}(\mathbb{R}; L^2(]a,b[)^N) \cap H_\rho^{-k-2}(\mathbb{R}; H^1(]a,b[)^N)$$

is continuous. In particular,

$$v \mapsto C\gamma\mathcal{H}v \in H_\rho^{-k-2}(\mathbb{R};\mathbb{R}^K)$$

is continuous for each $C \in \mathbb{R}^{K\times 2N}$.

Remark 4.12. The continuity statements in the previous theorem are the proper replacements for the inequality asserted to hold in Theorem 4.3. However, Theorem 4.3 seems not to be a direct corollary of Theorem 4.11, since the continuous differentiability of u in Theorem 4.3 cannot be guaranteed in the general setting of differential-algebraic port-Hamiltonian equations and needs more assumptions on the operators involved (see, e.g., [19] for the interplay between evolutionary equations and C_0-semigroups). We emphasize that the previous theorem also deals with differential-algebraic equations as well as with rough $\mathcal{H}$. The price we have to pay is the regularity loss of the solution u and the observation $y = C\gamma\mathcal{H}u$. So far, we were not able to provide either an example or a theorem showing that the asserted regularity loss is really occurring. This is postponed to future research.

Proof of Theorem 4.11. We consider the equation

$$\begin{aligned} (\partial_t M_0 + M_1 + P_1 \partial_x)\, \mathcal{H}u &= 0, \\ W\gamma\mathcal{H}u &= v. \end{aligned}$$

Using that A is m-accretive, we apply Theorem 4.1 to find an invertible matrix $G \in \mathbb{R}^{2N\times 2N}$ and two matrices $L, M \in \mathbb{R}^{N\times N}$ such that L is invertible and $M^*M \leq 1$ with

$$WG = L\begin{pmatrix} M & 1 \end{pmatrix}.$$

Then

$$W\gamma\mathcal{H}u = v$$

is equivalent to

$$\begin{aligned} 0 &= L\begin{pmatrix} M & 1 \end{pmatrix} G^{-1}\gamma\mathcal{H}u - v \\ &= L\begin{pmatrix} M & 1 \end{pmatrix}\left(G^{-1}\gamma\mathcal{H}u - \begin{pmatrix} 0 \\ L^{-1}v \end{pmatrix}\right) \\ &= W\left(\gamma\mathcal{H}u - G\begin{pmatrix} 0 \\ L^{-1}v \end{pmatrix}\right). \end{aligned}$$

Let η be as in Lemma 4.10. Then the latter can equivalently be formulated by

$$\begin{aligned} 0 &= W\left(\gamma\mathcal{H}u - G\begin{pmatrix} 0 \\ L^{-1}v \end{pmatrix}\right) \\ &= W\left(\gamma\mathcal{H}u - \gamma\eta G\begin{pmatrix} 0 \\ L^{-1}v \end{pmatrix}\right) \\ &= W\gamma\left(\mathcal{H}u - \eta G\begin{pmatrix} 0 \\ L^{-1}v \end{pmatrix}\right), \end{aligned}$$

which in turn is equivalent to

$$\mathcal{H}u - \widetilde{v} \in \operatorname{dom}(A),$$

where $\widetilde{v} = \eta G\left(\begin{smallmatrix} 0 \\ L^{-1}v \end{smallmatrix}\right) \in H_\rho^{-k}(\mathbb{R}; H^1(]a,b[)^N)$. Thus, we obtain

$$\begin{aligned} (\partial_t M_0 + M_1 + P_1\partial_x)\,\mathcal{H}u &= 0, \\ W\gamma\mathcal{H}u &= v, \end{aligned}$$

which amounts to asking for

$$\overline{(\partial_t M_0 + M_1 + A)}(\mathcal{H}u - \widetilde{v}) = -\left(\partial_t M_0 + M_1 + P_1\partial_x\right)\widetilde{v} \in H_\rho^{-k-1}(\mathbb{R}; L^2(]a,b[)^N). \tag{4.4}$$

Next, Theorem 4.9 leads to unique existence of $\mathcal{H}u - \widetilde{v} \in H_\rho^{-k-1}(\mathbb{R}; L^2(]a,b[)^N)$, which shows the unique existence of $u \in H_\rho^{-k-1}(\mathbb{R}; L^2_{\mathcal{H}}(]a,b[)^N)$ (the computation above shows existence, and performing the steps backwards, we obtain uniqueness), solving the problem. Moreover, since the mapping

$$H_\rho^{-k}(\mathbb{R}; \mathbb{R}^N) \ni v \mapsto \widetilde{v} \in H_\rho^{-k}(\mathbb{R}; H^1(]a,b[)^N)$$

is continuous by Lemma 4.10 and

$$H_\rho^{-k}(\mathbb{R}; H^1(]a,b[)^N) \ni \widetilde{v} \mapsto (\partial_t M_0 + M_1 + P_1\partial_x)\,\widetilde{v} \in H_\rho^{-k-1}(\mathbb{R}; L^2(]a,b[)^N)$$

is easily seen to be continuous, Theorem 4.9 yields the continuity of

$$H_\rho^{-k}(\mathbb{R};\mathbb{R}^N) \ni v \mapsto \mathcal{H}u \in H_\rho^{-k-1}(\mathbb{R};L^2(]a,b[)^N).$$

Moreover, by (4.4) we have that

$$A(\mathcal{H}u-\widetilde{v}) = -\left(\partial_t M_0 + M_1 + P_1\partial_x\right)\widetilde{v} - (\partial_t M_0 + M_1)(\mathcal{H}u-\widetilde{v}) \in H_\rho^{-k-2}(\mathbb{R};L^2(]a,b[)^N),$$

which yields the continuity of the mapping

$$H_\rho^{-k}(\mathbb{R};\mathbb{R}^N) \ni v \mapsto \mathcal{H}u \in H_\rho^{-k-2}(\mathbb{R};H^1(]a,b[)^N)$$

since $\widetilde{v} \in H_\rho^{-k}(\mathbb{R};H^1(]a,b[)^N) \subseteq H_\rho^{-k-2}(\mathbb{R};H^1(]a,b[)^N)$. □

We illustrate the versatility of differential-algebraic port-Hamiltonian equations considered here in the following example.

Example 4.13. We consider a differential-algebraic port-Hamiltonian equation with $P_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and M_0 and M_1 given by

$$M_0 := \begin{pmatrix} \mathbb{1}_{I_h \cup I_p} & 0 \\ 0 & \mathbb{1}_{I_h} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \mathbb{1}_{I_e} & 0 \\ 0 & \mathbb{1}_{I_p \cup I_e} \end{pmatrix},$$

where $I_e, I_p, I_h \subseteq]a,b[$ are pairwise disjoint measurable subsets (not necessarily intervals) with $I_e \cup I_p \cup I_h =]a,b[$. The resulting equation then takes the form

$$\left(\partial_t \begin{pmatrix} \mathbb{1}_{I_h \cup I_p} & 0 \\ 0 & \mathbb{1}_{I_h} \end{pmatrix} + \begin{pmatrix} \mathbb{1}_{I_e} & 0 \\ 0 & \mathbb{1}_{I_p \cup I_e} \end{pmatrix} + \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}\right) \mathcal{H}U = F.$$

This problem satisfies the assumption of Theorem 4.11 for $\rho_0 = c = 1$. Hence, we can study boundary control problems like that in Theorem 4.11 with a boundary control of the form

$$W\gamma\mathcal{H}u = v,$$

where γ is the point evaluation at the boundaries b and a and $W \in \mathbb{R}^{4\times 4}$ satisfies $W^*W \leq 1$.

We note that the equation is of hyperbolic type on I_h, of parabolic type on I_p, and of elliptic type on I_e. Indeed, assuming for simplicity that $\mathcal{H}(x) = I_2$ and $F = \left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right)$ and decomposing U into $U = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$, we end up with the equation

$$\begin{aligned} u_1 + \partial_x u_2 &= f, \\ u_2 + \partial_x u_1 &= 0, \end{aligned}$$

on I_e, which can be written as the second order problem $u_1 - \partial_x^2 u_1 = f$, which is an equation of elliptic type. On I_p we obtain

$$\begin{aligned} \partial_t u_1 + \partial_x u_2 &= f, \\ u_2 + \partial_x u_1 &= 0, \end{aligned}$$

which results in the parabolic problem $\partial_t u_1 - \partial_x^2 u_1 = f$. Finally, on I_h, we have

$$\begin{aligned} \partial_t u_1 + \partial_x u_2 &= f, \\ \partial_t u_2 + \partial_x u_1 &= 0, \end{aligned}$$

which yields after applying the temporal derivative to the first equation and then substituting the second equation

$$\partial_t^2 u_1 - \partial_x^2 u_1 = \partial_t f,$$

which is a hyperbolic equation.

Remark 4.14. We emphasize that in the above equation no transmission conditions need to be imposed in order to apply Theorem 4.11. To the best of our knowledge, such boundary control systems have not been considered before in the literature.

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