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Average distance, minimum degree, and irregularity index

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A B S T R A C T

Let $G = (V, E)$ be a connected graph of order n. The distance, $d_G(x, y)$, between vertices x and y in G is defined as the length of a shortest x-y path in G. The average distance, $\mu(G)$, of G is defined as $\mu(G) = \binom{n}{2}^{-1} \sum_{\{x,y\} \subseteq V} d_G(x, y)$. We give an upper bound on the average distance of a connected graph of given order and minimum degree where irregularity index is prescribed. Our results are a strengthening of the classical theorems by Kouider and Winkler (1997) [9] and by Dankelmann and Entringer (2000) [5] on average distance and minimum degree if the number of distinct terms in the degree sequence of the graph is prescribed.

1. Introduction

Let $G = (V, E)$ be a connected graph of order n. The *average distance* $\mu(G)$ of G is defined as $\mu(G) = \binom{n}{2}^{-1} \sum_{\{x,y\} \subseteq V} d_G(x, y)$, where $d_G(x, y)$ denotes the distance between vertices x and y in G. Introduced in graph theory by Doyle and Graver [6] as a measure of the compactness of a graph, the average distance, apart from being an interesting graph theoretical parameter in its own right, plays an important role in analysing communication networks (see for example, [3]). In such networks the time delay or signal disgradation for sending a message from one point to another is often proportional to the number of edges a message must travel. The average distance can be used to indicate the average performance of the network.

A lot of research effort, for instance [2–9], [12], [14–17], has been invested in gaining an understanding of the mathematical properties of average distance. Perhaps the first intriguing speculation to appear was that on the relationship between average distance and another graph parameter, the independence number, put forward by Fajtlowicz and Waller's computer programme, GRAFFITI [10], which sorts through various graphs and looks for simple relations among parameters, which, for a human mathematician are otherwise difficult to see. GRAFFITI conjectured again in 1987 [11] that for every δ-regular connected graph G of order n,

$$\mu(G) \le \frac{n}{\delta}.$$

This conjecture was answered only ten years later by Kouider and Winkler [9] who proved that if G is a connected graph of order n and minimum degree δ, then

$$\mu(G) \le \frac{n}{\delta + 1} + 2. \tag{1}$$

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Almost during the same time, Beezer, Riegsecker and Smith [2], in 2000, provided a similar upper bound on average distance in terms of order, size and minimum degree. In the same year Dankelmann and Entringer [5], using different methods and as an improvement to the work of Entringer, Kleitman and Szekely [8], considered the problem of finding an upper bound on the average distance for graphs of given minimum degree in connection with the problem of finding a spanning tree of the graph with small average distance. Their results implied the bound

$$\mu(G) \le \frac{n}{\delta+1} + 5, \tag{2}$$

for a connected graph G of order n and minimum degree δ.

Interestingly, the extremal graphs illustrating the tightness of bounds (1) and (2) are close to being regular, that is, with all vertices having the same degrees. A graph parameter which gives an indication of how far or near a graph is from being regular was introduced in [14] and called there the irregularity index. Precisely, the *irregularity index* $t(G)$ of graph G is the number of distinct terms in the degree sequence of G. The extremal graphs for the above bounds, (1) and (2), have irregularity index at most 2. It is therefore natural to ask if the bounds (1) and (2) can be improved when the irregularity index is prescribed. In this instalment to literature, we demonstrate that if G is a connected graph of order n, minimum degree δ and irregularity index t, then

$$\mu(G) \le \frac{n-t}{\delta+1} + O(1). \tag{3}$$

Further, we illustrate that (3) is best possible. Our results are an improvement of bounds (1) and (2) if irregularity index is prescribed. We note here that research on improving (2), if irregularity index is prescribed, has been independently reported on recently (see, for instance [1], where graphs of diameter not equal to three or four are dealt with).

We will use the following notation. Let S be a subset of $V(G)$. We denote the subgraph of G *induced* by S by $G[S]$. If u is a vertex in G, then the distance between u and S, $d_G(u,S)$, is defined as $\min_{v\in S} d_G(u,v)$. The *closed neighbourhood* $N_G[S]$ of S is the set $\cup_{u\in S} N_G[u]$, where $N_G[u]$ is the closed neighbourhood of the vertex u in G, that is, $\{x \in V(G) | d_G(x,u) \le 1\}$. The *open neighbourhood* of a vertex u in G is $N_G(u) = \{x \in V(G) | d_G(x,u) = 1\}$. A *2-packing* of G is a subset $A \subseteq V(G)$ with $d_G(u,v) > 2$ for all $u, v \in A$. The k^{th}*-power* of G, G^k, is the graph with vertex set $V(G)$, in which two distinct vertices u and v are adjacent if $d_G(u,v) \le k$. The *degree* of a vertex v in G is denoted by $\deg_G(v)$. We denote the number of end vertices in a tree T by $s(T)$.

Closely related to average distance is the Wiener index, $W(G) = \sum_{\{x,y\}\subseteq V(G)} d_G(x,y)$, of graph G. We will sometimes find it convenient to work with the Wiener index of the graph in stead of its average distance. The following result is folklore and will be handy.

Fact 1.1. ([6,7,13]) Let G be a connected graph of order n. Then $W(G) \le \frac{n(n-1)(n+1)}{6}$ with equality iff G is a path.

2. Results

Theorem 2.1. *Let G be a connected graph of order n, minimum degree δ and irregularity index t. Then*

$$W(G) \le \frac{(n-t)^3}{2(\delta+1)} + O(n^2).$$

Moreover, the bound is best possible.

Proof. Since the degree sequence of G has t distinct terms, let $\{v_1, v_2, \dots, v_t\}$ be a set of t vertices such that $\delta \le \deg(v_1) < \deg(v_2) < \cdots < \deg(v_t)$. Then

$$|N[v_t]| \ge \delta + t. \tag{4}$$

Obtain a maximal 2-packing A of G as follows: Let $A = \{v_t\}$. If $d_G(u, A) = 3$ for some u in G, add u to A until each vertex not in A is within distance 2 of A. Denote the cardinality $|A|$ of A by a.

Let H' be the forest with vertex set $N[A]$ and edge set all edges incident with a vertex in A. By our construction of H', there exist $|A| - 1$ edges in G, each of them joining two components of H', and whose addition to H' yields a tree $T_1 \le G$.

Claim 2.1.

$$\sum_{\{x,y\}\subseteq A} d_{T_1}(x,y) \le \frac{a^3}{2} + O(n^2).$$

Proof of Claim 2.1. Note that $T_1^3[A]$ is connected. An application of Fact 1.1 yields, $W(T_1^3[A]) \le \frac{a(a-1)(a+1)}{6}$. By our construction of A, if $x, y \in A$, then

$$d_{T_1}(x,y) \le 3d_{T_1^3[A]}(x,y). \tag{5}$$

Hence

$$\sum_{\{x,y\}\subseteq A} d_{T_1}(x,y) \le 3 \sum_{\{x,y\}\subseteq A} d_{T_1^3[A]}(x,y) = 3W(T_1^3[A]) \le \frac{a^3}{2} + O(n^2),$$

and the claim is proven.

Note that since G has minimum degree δ, in conjunction with (4), we get

$$\begin{aligned} n &\ge |\cup_{x\in A} N[x]| \\ &\ge |N[v_t]| + |\cup_{x\in A-\{v_t\}} N[x]| \\ &\ge \delta + t + (a-1)(\delta+1) \\ &= a(\delta+1) + t - 1. \end{aligned}$$

Therefore,

$$a \le \frac{n-t+1}{\delta+1}. \tag{6}$$

We construct a subgraph H of T_1 as follows: From (4) denote the set of any arbitrarily chosen $\delta+t-1$ neighbours of v_t by $M(v_t)$ and set $M[v_t] = M(v_t) \cup \{v_t\}$. Further, for any $x \in A - \{v_t\}$, denote the set of any arbitrarily chosen δ neighbours of x by $M(x)$ and set $M[x] = M(x) \cup \{x\}$. Let H be the subgraph of T_1 with vertex set $V(H) = \cup_{x\in A} M[x]$ and edge set all those edges of T_1 incident with vertices in $\cup_{x\in A} M[x]$, that is, $H = T_1[\cup_{x\in A} M[x]]$. Then

$$|V(H)| = |\cup_{x\in A} M[x]| = a(\delta+1) + t - 1. \tag{7}$$

Claim 2.2.

$$\sum_{\{x,y\}\subseteq V(H)} d_{T_1}(x,y) \le \frac{a^3}{2}(\delta+1)^2 + O(n^2).$$

Proof of Claim 2.2. Define a function $f: V(H) \longrightarrow A$ by $f(u) = u'$, where u' is the unique vertex in A that is within distance 1 of u. Hence

$$d_{T_1}(x,y) \le d_{T_1}(f(x), f(y)) + 2$$

for any $x, y \in V(H)$. It follows that

$$\begin{aligned} \sum_{\{x,y\}\subseteq V(H)} d_{T_1}(x,y) &\le \sum_{\{x,y\}\subseteq V(H)} [d_{T_1}(f(x), f(y)) + 2] \\ &= \sum_{\{x,y\}\subseteq V(H)} d_{T_1}(f(x), f(y)) + |V(H)|(|V(H)|-1). \end{aligned} \tag{8}$$

We now consider $\sum_{\{x,y\}\subseteq V(H)} d_{T_1}(f(x), f(y))$. Partition $V(H)$ as $V(H) = M[v_t] \cup \left(\cup_{x\in A-\{v_t\}} M[x]\right) = M[v_t] \cup Q$, say. Since $d_{T_1}(f(x), f(y)) \le 4$ for all $x, y \in M[v_t]$, we get

$$\sum_{\{x,y\}\subseteq M[v_t]} d_{T_1}(f(x), f(y)) \le 4\binom{\delta+t}{2} = O(n^2).$$

It follows that

$$\begin{aligned} \sum_{\{x,y\}\subseteq V(H)} d_{T_1}(f(x), f(y)) = &\sum_{\{x,y\}\subseteq M[v_t]} d_{T_1}(f(x), f(y)) + \sum_{\{x,y\}\subseteq Q} d_{T_1}(f(x), f(y)) \\ &+ \sum_{(x,y)\in M[v_t]\times Q} d_{T_1}(f(x), f(y)) \end{aligned}$$

$$= \sum_{\{x,y\}\subseteq Q} d_{T_1}(f(x), f(y)) + \sum_{(x,y)\in M[v_t]\times Q} d_{T_1}(f(x), f(y)) + O(n^2). \tag{9}$$

We look at each term in (9) separately. First, consider $\sum_{\{x,y\}\subseteq Q} d_{T_1}(f(x), f(y))$. Note that for each pair $\{u, v\} \subseteq A - \{v_t\}$, there are $(\delta+1)\times(\delta+1)$ pairs $\{x, y\} \subseteq Q$ such that $f(x) = u$ and $f(y) = v$. Therefore,

$$\sum_{\{x,y\}\subseteq Q} d_{T_1}(f(x), f(y)) = (\delta+1)^2 \sum_{\{u,v\}\subseteq A-\{v_t\}} d_{T_1}(u, v). \tag{10}$$

We now turn to $\sum_{(x,y)\in M[v_t]\times Q} d_{T_1}(f(x), f(y))$. Note that each $v \in A - \{v_t\}$, there are $(\delta+t)\times(\delta+1)$ pairs $(x, y) \in M[v_t]\times Q$ such that $f(x) = v_t$ and $f(y) = v$. Therefore,

$$\sum_{(x,y)\in M[v_t]\times Q} d_{T_1}(f(x), f(y)) = (\delta+t)(\delta+1) \sum_{v\in A-\{v_t\}} d_{T_1}(v_t, v). \tag{11}$$

Combining (9), (10) and (11), we get

$$\begin{aligned}\sum_{\{x,y\}\subseteq V(H)} d_{T_1}(f(x), f(y)) &= (\delta+1)^2 \sum_{\{u,v\}\subseteq A-\{v_t\}} d_{T_1}(u, v) + (\delta+t)(\delta+1) \sum_{u\in A-\{v_t\}} d_{T_1}(u, v_t) \\ &= (\delta+1)^2 \sum_{\{u,v\}\subseteq A} d_{T_1}(u, v) \\ &\quad + (t-1)(\delta+1) \sum_{u\in A-\{v_t\}} d_{T_1}(u, v_t). \end{aligned} \tag{12}$$

Now consider $\sum_{u\in A-\{v_t\}} d_{T_1}(u, v_t)$. From (5), and since $T_1^3[A]$ is connected, we get

$$\begin{aligned}\sum_{u\in A-\{v_t\}} d_{T_1}(u, v_t) &\le 3 \sum_{u\in A-\{v_t\}} d_{T_1^3[A]}(v_t, u) \\ &\le 3(1+2+\cdots+[a-1]) \\ &= \frac{3a(a-1)}{2}.\end{aligned}$$

This, in conjunction with (12), yields

$$\sum_{\{x,y\}\subseteq V(H)} d_{T_1}(f(x), f(y)) \le (\delta+1)^2 \sum_{\{u,v\}\subseteq A} d_{T_1}(u, v) + (t-1)(\delta+1)\frac{3a(a-1)}{2}.$$

It follows from (8), an application of Claim 2.1, and the fact that t and δ are fixed, that

$$\begin{aligned}\sum_{\{x,y\}\subseteq V(H)} d_{T_1}(x, y) &\le \sum_{\{x,y\}\subseteq V(H)} d_{T_1}(f(x), f(y)) + |V(H)|(|V(H)|-1) \\ &\le (\delta+1)^2 \sum_{\{u,v\}\subseteq A} d_{T_1}(u, v) + (t-1)(\delta+1)\frac{3a(a-1)}{2} + |V(H)|(|V(H)|-1) \\ &\le (\delta+1)^2\left[\frac{a^3}{2} + O(n^2)\right] + (t-1)(\delta+1)\frac{3a(a-1)}{2} + |V(H)|(|V(H)|-1), \\ &= \frac{a^3}{2}(\delta+1)^2 + O(n^2),\end{aligned}$$

and the claim is proven.

Recall that $V(T_1) = N[A]$ and that $H \le T_1$ is a subgraph of T_1 with $V(H) = \cup_{x\in A} M[x]$. Let $B = V(T_1) - H$ and denote the cardinality of B by b. Every vertex u of G that is not in T_1 is adjacent to some end vertex u' of T_1. Let T be a spanning tree of G with edge set $E(T_1) \cup \{uu' : u \in V(G) - V(T_1)\}$. Denote the set of vertices in G not in T_1 by C, that is, $C = V(G) - V(T_1)$, and let $c = |C|$. Hence $n = |V(H)| + b + c$ and from (7), we get

$$n = a(\delta+1) + t - 1 + b + c. \tag{13}$$

Denote the set of all neighbours in T of vertices in C by $\{u_1, u_2, \ldots, u_k\}$, that is, $N_T(C) = \cup_{u\in C} N_T(u) = \{u_1, u_2, \ldots, u_k\}$, where $N_T(u) = \{u'\}$. We now compute the number of end vertices of T.

Claim 2.3.

$$s(T) = a(\delta - 2) + b + c - k + t + 1.$$

Proof of Claim 2.3. By our construction of T,

$$\begin{aligned} s(T) &= s(T_1) + c - |\{u_1, u_2, \dots, u_k\}| \\ &= s(T_1) + c - k \\ &= s(H) + b + c - k \\ &= a(\delta - 2) + t + 1 + b + c - k, \end{aligned}$$

and the claim is proven.

We are now ready to bound the total distance of T.

Claim 2.4.

$$W(T) \le \frac{a^3}{2}(\delta + 1)^2 + \frac{3}{4}a[n^2 - a^2(\delta + 1)^2] + O(n^2).$$

Proof of Claim 2.4. Note that $V(T) = V(H) \cup B \cup C$. Denote $B \cup C$ by D. Then

$$\begin{aligned} W(T) &= \sum_{\{x,y\} \subseteq V(T)} d_T(x, y) \\ &= \sum_{\{x,y\} \subseteq V(H)} d_T(x, y) + \sum_{(x,y) \in V(H) \times D} d_T(x, y) + \sum_{\{x,y\} \subseteq D} d_T(x, y) \\ &= \sum_{\{x,y\} \subseteq V(H)} d_{T_1}(x, y) + \sum_{(x,y) \in V(H) \times D} d_T(x, y) + \sum_{\{x,y\} \subseteq D} d_T(x, y). \end{aligned} \tag{14}$$

We now consider $\sum_{(x,y) \in V(H) \times D} d_T(x, y)$. Each $x \in D$ is within distance 2 of a vertex x_A in A. Since $T_1^3[A]$ is connected, and each vertex in A is adjacent to δ neighbours in H, we get

$$\begin{aligned} \sum_{y \in V(H)} d_{T_1}(x, y) &\le 3(\delta + 1) \sum_{y \in A - \{v_t\}} [3 + d_{T_1^3[A]}(x_A, y)] + 3(\delta + t)[3 + d_{T_1^3[A]}(x_A, v_t)] \\ &\le 3(\delta + 1)[3(a - 1) + 1 + 2 + 3 + \cdots + (a - 1)] + 3(\delta + t)(3 + a - 1) \\ &= \frac{3}{2}a^2(\delta + 1) + O(n). \end{aligned}$$

It follows that

$$\sum_{(x,y) \in V(H) \times D} d_T(x, y) \le |D| \left(\frac{3}{2}a^2(\delta + 1) + O(n) \right) = \frac{3}{2}a^2(\delta + 1)(b + c) + O(n^2). \tag{15}$$

We now consider $\sum_{\{x,y\} \subseteq D} d_T(x, y)$ by looking at the edges of T. Denote the two branches of T obtained by removing an edge e from T by T'_e and T''_e. Define an integer-valued function θ on the set of edges of T by $\theta(e) = n_1 n_2$, where $n_1 = |D \cap V(T'_e)|$ and $n_2 = |D \cap V(T''_e)|$. Then

$$\sum_{\{x,y\} \subseteq D} d_T(x, y) = \sum_{e \in E(T)} \theta(e).$$

For each edge e incident with an end vertex of T, $\theta(e) \le |D| - 1$. For each of the k edges $e = u_i x_A$ with one end in $\{u_1, u_2, \dots, u_k\}$ and another end in A, we have

$$\theta(e) \le [\deg_T(u_i) - 1][|D| - \deg_T(u_i) + 1].$$

The contribution of all these k edges to $\sum_{e \in E(T)} \theta(e)$ being

$$\sum_{i=1}^{k}[\deg_T(u_i)-1][|D|-\deg_T(u_i)+1]=|D|\sum_{i=1}^{k}[\deg_T(u_i)-1]-\sum_{i}^{k}[\deg_T(u_i)-1]^2$$
$$<|D|c-\sum_{i}^{k}[\deg_T(u_i)-1]^2$$
$$\le O(n^2).$$

Since $n_1+n_2=|D|$, we have $\theta(e)\le\frac{|D|^2}{4}$ for each of the remaining edges e of T. It follows that

$$\sum_{\{x,y\}\subseteq D} d_T(x,y)=\sum_{e\in E(T)}\theta(e)\le s(T)[|D|-1]+O(n^2)+[|E(T)|-k-s(T)]\frac{|D|^2}{4},$$

and so from Claim 2.3,

$$\sum_{\{x,y\}\subseteq D} d_T(x,y)\le[|E(T)|-k-s(T)]\frac{|D|^2}{4}+O(n^2)$$
$$=[n-1-k-(a(\delta-2)+b+c-k+t+1)]\frac{|D|^2}{4}+O(n^2)$$
$$=[n-a(\delta+1)-b-c-t+1+3(a-1)]\frac{|D|^2}{4}+O(n^2).$$

From (13) this gives

$$\sum_{\{x,y\}\subseteq D} d_T(x,y)\le 3(a-1)\frac{|D|^2}{4}+O(n^2)=\frac{3}{4}a(b+c)^2+O(n^2).$$

Combining this with (14) and (15), we get

$$W(T)\le\sum_{\{x,y\}\subseteq V(H)} d_{T_1}(x,y)+\frac{3}{2}a^2(\delta+1)(b+c)+\frac{3}{4}a(b+c)^2+O(n^2).$$

An application of Claim 2.2, and using (13) to eliminate $b+c$, yields the claim.

Now define a single variable function g of a by

$$g(a):=\frac{a^3}{2}(\delta+1)^2+\frac{3}{4}a[n^2-a^2(\delta+1)^2].$$

We maximise g subject to (6), that is, subject to

$$a\le\frac{n-t+1}{\delta+1}.$$

A simple differentiation shows that g is maximised for $a=\frac{n-t+1}{\delta+1}$ to give

$$g(a)\le\frac{(n-t)^3}{2(\delta+1)}+O(n^2).$$

Together with Claim 2.4, we deduce that

$$W(T)\le\frac{(n-t)^3}{2(\delta+1)}+O(n^2),$$

from where the bound of the theorem is proven by applying the inequality $W(G)\le W(T)$.

To see that the bound is best possible, consider integers n,t,δ and k with $n=k(\delta+1)+(t-1)$. Let H_t be the graph with vertex set $\{v_1,v_2,\dots,v_t\}$ and edge set $\{v_iv_j|i+j>t\}$. Let $G_1=K_\delta$ and $G_2,G_3,\dots,G_k$ be disjoint copies of the complete graph $K_{\delta+1}$ and for each G_i, let a_ib_i be an edge in $E(G_i)$. Let $G_{n,\delta,t}$ be the graph obtained from the union of graphs

$$H_t,G_1-a_1b_1,G_2-a_2b_2,G_3-a_3b_3,\dots,G_{k-1}-a_{k-1}b_{k-1},G_k$$

by adding edges

$$\{uv | u \in V(H_t), v \in V(G_1)\} \cup \{b_i a_{i+1} | 1, 2, \ldots, k-1\}.$$

Then G has order n, minimum degree δ and irregularity index t. A simple calculation shows that

$$W(G_{n,\delta,t}) > \frac{(n-t)^2}{2(\delta+1)}. \quad \square$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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