Properties of Integer Partitions and Plane Partitions

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Professor Arnold Knopfmacher
and
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A thesis submitted in fulfillment of the requirements for the degree of
Doctor of Philosophy (by production of original research) in Mathematics

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Aubrey Blecher

Johannesburg, December 2012

Fixpoint

Dear reader:

This thesis has come towards the end of my university career, which according to a friend, is a fundamentally subversive act. Perhaps it is just part of an ongoing survival strategy.

At the outset, it seemed to me that the world of maths comprised individuals working on lonely problems.

However, my supervisors, Arnold Knopfmacher and Charlotte Brennan have inducted me into a vibrant community of ideas, intellectual challenge and brainstorming. They have created a fixed point in a world that is becoming increasingly tumultuous.

For this, I am grateful. And Arnold, I would like to thank you, also, for excusing my angst (or was it confusion?) of past years which led to many false starts before the focus of this thesis.

So, dear reader, if there is anything that the thesis has taught me (other than some mathematics), it is not to undervalue an intellectual haven in the midst of turmoil.

Thanks also to the School of Mathematics at the University of the Witwatersrand for granting me a reduced work load during this period of research.

Aubrey Blecher
Johannesburg
December, 2012.

Abstract

Generating functions and asymptotic analysis have been used in four different situations to establish new results for extremely well studied structures. Later in this thesis a more detailed individual abstract for each of these studies is provided. The four situations are:

A. Durfee square areas in integer partitions.

B. A study of the relationship between integer compositions and their constituent partitions by specifying the asymptotic expectation of the number of such partitions in arbitrary composition.

C. Similar to B above but focusing more on the generating functions rather than on the expectations derived therefrom.

D. In the area of plane partitions with additional structure imposed upon them.

Contents

1 Introduction to the thesis

This thesis is an account of the research done for the author's PhD. As such the greater part of this document is a record of peer reviewed research. Consequently the main content is contained in 4 sections, (6, 8, 10, 13), each of which is a reproduction of a different paper published in a journal of mathematics research. For those readers who are primarily concerned with the results and nature of the research, it is possible to take a shortcut by reading only those papers, as they are entirely self contained.

The other parts of these chapters will not however be redundant because their primary concern is to render the thesis understandable to an informed layperson (or mathematician from a different speciality) by explicating all the definitions, giving more extensive explanation than those carried in the papers themselves, providing some detail about the underlying ideas and mode of development of the concepts, an account of the developing train of thought of the author as well as suggesting open areas of research for the author or others to undertake.

The elements of this study are as follows:

1. Partitions of integers n, [1,3].
2. Compositions of integers n, [10].
3. Plane partitions of n, [1,2,3].
4. Durfee squares for a partition, [1,2].
5. Theory of generating functions, [24].
6. Theory of asymptotics, [8].

Elements 1 and 2 are *sine qua nons* for the whole of this study; element 3 is used exclusively for paper D; element 4 is used mainly for paper A; element 5 for papers B and C; and element 6 for papers A, B and C.

Because of the extensive use of generating functions, we may summarise this thesis as an application of the theory of generating functions to four specific situations detailed in the four papers A – D.

The reader should note that each paper that is presented has its own bibliography given immediately after that paper. All citations in that chapter refer to the list at the end of that paper. However, there is also an overall reference list at the end of the thesis. Any references that are not contained in the paper itself refer to the latter reference list.

2 Abstracts of Papers A – D

2.1 Paper A: Durfee square areas and associated partition identities

A partition of an integer n is a representation $n = a_1 + a_2 + \cdots + a_k$, with integer parts $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. The Durfee square is the largest square of points in the graphical representation of a partition. We consider generating functions for the sum of areas of the Durfee squares for various different classes of partitions of n. Throughout this paper the variable q is used as a marker for the size of the partition n. As a consequence, interesting partition identities are derived. The more general case of Durfee rectangles is also treated as well as the asymptotic growth of the mean area over all partitions of n.

2.2 Paper B: Compositions of positive integers n viewed as alternating sequences of increasing/decreasing partitions

Compositions and partitions of positive integers are often studied in separate frameworks where partitions are given by q-series and compositions exhibiting particular patterns are specified by generating functions for these patterns. Here, we view compositions as alternating sequences of partitions (i.e., alternating blocks) and obtain results for the asymptotic expectations of the number of such blocks (or parts per block) for different ways of defining the blocks.

2.3 Paper C: Compositions of n as alternating sequences of weakly increasing and strictly decreasing partitions

As in Paper B, we view compositions (more specifically here) as alternating sequences of weakly increasing and strictly decreasing partitions. But here, we obtain generating functions for the number of such partitions in terms of the size of the composition, the number of parts and the total number of "valleys" and "peaks". From this, we find the total number of "peaks" and "valleys" in the composition of n which have this pattern. We also obtain the generating function for compositions which split into just two partition blocks. Finally, we obtain the two generating functions for compositions of n that start either with a weakly increasing partition or a strictly decreasing partition.

2.4 Paper D: Geometry of totally symmetric plane partitions (TSPPs) with self-conjugate main diagonal

We show that a cyclically symmetric plane partition with self-conjugate main diagonal must be a TSPP and that any TSPP with self-conjugate main diagonal is uniquely determined by that diagonal. We also indicate by means of an example how this latter proof yields an algorithm for recovering the TSPP from its diagonal.

The following generating functions are developed. Namely:

1. A generating function for all TSPPs with self-conjugate main diagonal;
2. The generating function for 1-shell TSPPs (defined later).

3 Publication of the above papers

Three of the four papers in this thesis were published in 2012 in the journals specified. The remaining one (Paper A) has been submitted and is currently being reviewed.

1. **Paper A**
   A. Blecher, A. Knopfmacher, A. Munagi, Durfee Square Areas and Associated Partition Identities,
   Preprint. *Ars Combinatoria.* 50 percent contribution by A. Blecher.

2. **Paper B**
   A. Blecher, C. Brennan, T. Mansour, Compositions of positive integers n viewed as alternating sequences of increasing/decreasing partitions,
   Ars Combinatoria **106** (2012), no. 213–224. 33 percent contribution by A.Blecher.

3. **Paper C**
   A. Blecher, Compositions of n as alternating sequences of weakly increasing and strictly decreasing partitions,
   Cent. Eur. J. Math. **10**(2) (2012), 788–796.

4. **Paper D**
   A. Blecher, Geometry of totally symmetric plane partitions (TSPPs) with self-conjugate main diagonal,
   Utilitas Math. **88** (2012), 223–235 (July volume).

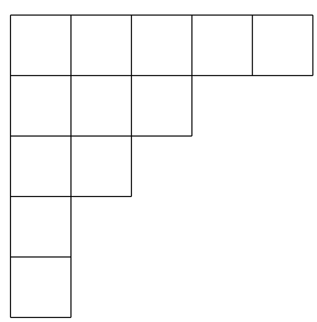
4 Basic concepts of the Papers A – D

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. If λ represents a partition of n, we will also write $\lambda \vdash n$ or $|\lambda| = n$. For example, $(20, 7, 5, 5, 4, 4, 1, 1, 1) \vdash 48$.

A partition $(\lambda_1, \ldots, \lambda_k)$ may be represented graphically by a *Ferrers diagram*, which is a left-justified array of dots, with λ_i dots in the i^{th} row. If the dots are replaced by squares, we obtain the *Young diagram* representation. These standard definitions as well as the one for a Durfee square below can be found, for example, in [1].

Example:

The Young diagram of the partition $(5, 3, 2, 1, 1)$ of 12 is



The largest square in the Ferrers diagram of a partition is called the Durfee square. If there are λ_i dots in the ith row then its size k can be described as the number of parts λ_i satisfying $\lambda_i \geq i$. In our example, the Durfee square has side of length 2 and area 4.

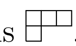
These papers require an understanding of the following concepts which are explicated here.

1. A *partition of a positive integer* n is a set of positive integers, $n_1 \geq n_2 \geq \cdots \geq n_k$ (each called a part of the partition), such that $n = n_1 + n_2 + \cdots + n_k$. The terminology is extended to allow us to say that $n_1 + n_2 + \cdots + n_k$ is a partition of n.

So for example the partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

There are precisely five such partitions. Now we may model a partition using a so-called Young's or Ferrers diagram. This is a left justified array of squares where row i has precisely n_i such squares.

For example the Young's diagram for $2+2$ is  and for $3+1$ is .

2. The largest square in the Young's diagram is called a Durfee square. So in the Young's diagram for $2+2$ above, the Durfee square is constituted by the whole diagram. Whereas as for $3+1$ above, the Durfee square is just a single square, $\square$.

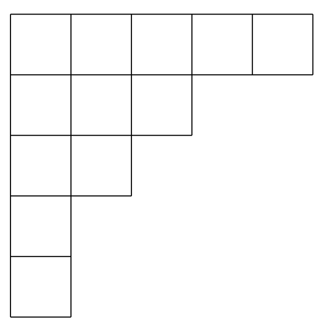
5 Paper A

5.1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. If λ represents a partition of n, we will also write $\lambda \vdash n$ or $|\lambda| = n$. For example, $(20, 7, 5, 5, 4, 4, 1, 1, 1) \vdash 48$.

A partition $(\lambda_1, \ldots, \lambda_k)$ may be represented graphically by a *Ferrers diagram*, which is a left-justified array of dots, with λ_i dots in the i^{th} row. If the dots are replaced by squares, we obtain the *Young diagram* representation. These standard definitions as well as the one for a Durfee square below are to be found, for example, in [1].

Example: The Young diagram of the partition $(5, 3, 2, 1, 1)$ of 12 is



The largest square in the Ferrers diagram of a partition is called the Durfee square. If there are λ_i dots in the ith row then its size k can be described as the number of parts λ_i satisfying $\lambda_i \geq i$. In our example, the Durfee square has size or length 2 and area 4.

Durfee squares in partitions have been studied previously with respect to the length of the side in [19, 20, 22] but the area does not seem to have been considered. In [20, 22] the emphasis was on asymptotic estimates for the length, whereas the main aim of the present paper is on deriving interesting partition identities.

In particular, by counting partitions in various classes according to the size of their Durfee squares we obtain the following partition identities.

Theorem 5.1 *For partitions into odd parts we have*

$$\sum_{k=0}^{\infty} q^{4k^2+2k}(1-q^{4k+2}+q^{6k+3}) \prod_{i=1}^{k+1} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty} \frac{1}{1-q^{2i-1}}. \quad (5.1)$$

For partitions into even parts we have

$$\sum_{k=0}^{\infty} q^{4k^2}(1-q^{4k+2}+q^{6k+2}) \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}} = \prod_{i=1}^{\infty} \frac{1}{1-q^{2i}}. \quad (5.2)$$

For partitions into distinct parts we have

$$\sum_{k=0}^{\infty} q^{\frac{3k^2-k}{2}} \frac{1+q^{2k}}{1+q^k} \prod_{i=1}^{k} \frac{1+q^i}{1-q^i} = \prod_{i=1}^{\infty}(1+q^i). \quad (5.3)$$

For partitions into distinct odd parts we have

$$\sum_{k=0}^{\infty} q^{8k^2} \frac{(1-q^{2k+1}+q^{6k+1}-q^{8k+2}+q^{10k+3})}{1-q^{2k+1}} \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}$$

$$= \prod_{i=1}^{\infty}(1+q^{2i-1}). \quad (5.4)$$

For partitions into distinct even parts we have

$$\sum_{k=0}^{\infty} q^{8k^2-2k}(1-q^{2k}+q^{4k}-q^{4k+2}+q^{6k+2}-q^{8k+2}+q^{10k+2}) \times$$

$$\times \prod_{i=1}^{k}(1+q^{2i}) \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}$$

$$= \prod_{i=1}^{\infty}(1+q^{2i}). \quad (5.5)$$

Next, let $a_P(n)$ denote the sum of the areas of the Durfee squares over all partitions of n belonging to a given class P of partitions, and let $A_P(x) = \sum_{n\geq 0} a_P(n)q^n$ be the associated generating function for the sum of the areas. For example, if P is the set of all partitions, then for $n = 4$ we find $a_P(n) = 1+1+4+1+1 = 8$ from the partitions (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$.

Theorem 5.2 *The generating functions $A_P(x)$ for partitions according to the sum of areas of their Durfee squares are as follows:*
For partitions into odd parts,

$$\sum_{k=0}^{\infty} q^{4k^2+2k} \prod_{i=1}^{k} \frac{1}{1-q^{2i-1}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} \left[4k^2 + \frac{(2k+1)^2 q^{2k+1}}{(1-q^{2k+1})(1-q^{4k+2})}\right]. \quad (5.6)$$

For partitions into even parts,

$$\sum_{k=0}^{\infty} q^{4k^2} \prod_{i=1}^{k} \frac{1}{1-q^{2i}} \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} \left[4k^2 + \frac{(2k+1)^2 q^{6k+2}}{1-q^{4k+2}}\right]. \quad (5.7)$$

For partitions into distinct parts,

$$\sum_{k=0}^{\infty} k^2 q^{\frac{3k^2-k}{2}} \frac{1+q^{2k}}{1+q^k} \prod_{i=1}^{k} \frac{1+q^i}{1-q^i}. \quad (5.8)$$

For partitions into distinct odd parts,

$$\sum_{k=0}^{\infty} q^{8k^2} \left(4k^2 + \frac{(2k+1)^2 q^{6k+1}\left(1-q^{2k+1}+q^{4k+2}\right)}{1-q^{2k+1}}\right) \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}. \quad (5.9)$$

For partitions into distinct even parts,

$$\sum_{k=0}^{\infty} q^{8k^2-2k} \left((2k+1)^2 q^{10k+2} + 4k^2(1-q^{2k}+q^{4k})(1-q^{4k+2})\right) \prod_{i=1}^{k}(1+q^{2i}) \times$$
$$\times \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}. \quad (5.10)$$

These results are established in Sections 2 and 3. In Section 4 we generalise the problem to the area of Durfee rectangles. Finally in Section 5 the asymptotic behaviour of the areas of Durfee squares is briefly discussed.

Some previous results on Durfee squares are to be found in [3],[4] and [6], however the emphasis in those papers was on asymptotics, not on partition identities.

5.2 The area of Durfee squares

5.2.1 Durfee square areas for unrestricted partitions

It is well known that the generating function for partitions with Durfee square of length k is given by
$$\frac{q^{k^2}}{\prod_{i=1}^{k}(1-q^i)^2}.$$
Thus the generating function for the sum of areas of Durfee squares for all partitions of n is given by

$$\sum_{k=1}^{\infty}\frac{k^2q^{k^2}}{\prod_{i=1}^{k}(1-q^i)^2}.$$

Similarly the generating function for the sum of areas of Durfee squares in self-conjugate partitions of n is given by

$$\sum_{k=1}^{\infty}\frac{k^2q^{k^2}}{\prod_{i=1}^{k}(1-q^{2i})}.$$

Durfee squares have also been studied with respect to basis partitions (see [20] p.784). If a partition λ has Durfee square of length k, then the rank vector of the partition is $(\lambda_1-\lambda'_1,\ldots,\lambda_k-\lambda'_k)$ where λ' is the conjugate partition. Although infinitely many partitions have a given rank vector, there is a unique such partition with a minimum sum of its parts. The generating function for basis partitions with Durfee square of size k is known to be

$$\frac{q^{k^2}\prod_{i=1}^{k}(1+q^i)}{\prod_{i=1}^{k}(1-q^i)}.$$

Hence the sum of areas of Durfee squares for basis partitions of n is

$$\sum_{k=1}^{\infty}\frac{k^2q^{k^2}\prod_{i=1}^{k}(1+q^i)}{\prod_{i=1}^{k}(1-q^i)}.$$

5.2.2 Durfee squares for partitions into odd parts

Consider partitions with odd parts with Durfee square of size $2k$. Then the partition π to the right of the Durfee square can be decomposed into a column of length $2k$ followed by a partition into at most $2k$ parts, all even numbers. Beneath the Durfee square we may place a partition into odd parts with largest part at most $2k-1$. The generating function for this case is then

$$q^{4k^2}q^{2k}\prod_{i=1}^{k}\frac{1}{1-q^{2i-1}}\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}.$$

Next consider partitions with odd parts with Durfee square of size $2k+1$. Here the partition π to the right of the Durfee square has at most $2k+1$ parts all even numbers. Beneath the Durfee square we may place a partition into odd parts with largest part at most $2k+1$. The generating function for this case is then

$$q^{(2k+1)^2}\prod_{i=1}^{k+1}\frac{1}{1-q^{2i-1}}\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}.$$

Combining the two disjoint cases and summing over k leads to the following identity for partitions into odd parts (equivalently, partitions into distinct parts)

$$\sum_{k=0}^{\infty}q^{4k^2+2k}(1-q^{4k+2}+q^{6k+3})\prod_{i=1}^{k+1}\frac{1}{1-q^{2i-1}}\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}=\prod_{i=1}^{\infty}\frac{1}{1-q^{2i-1}},$$

which establishes (5.1) above. Similarly the generating function for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty}q^{4k^2+2k}\prod_{i=1}^{k}\frac{1}{1-q^{2i-1}}\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}\left[4k^2+\frac{(2k+1)^2q^{2k+1}}{(1-q^{2k+1})(1-q^{4k+2})}\right]$$

as per (5.6) above.

5.2.3 Durfee squares for partitions into even parts

Consider partitions with even parts with Durfee square of size $2k$. Then the partition π to the right of the Durfee square is a partition into at most $2k$

parts, all even. Beneath the Durfee square we may place a partition into even parts with largest part at most $2k$. The generating function for this case is then

$$q^{4k^2}\prod_{i=1}^{k}\frac{1}{1-q^{2i}}\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}.$$

Next consider partitions with even parts with Durfee square of size $2k+1$. Here the partition π to the right of the Durfee square can be decomposed into a column or length $2k+1$ followed by a partition into at most $2k+1$ parts all even. Beneath the Durfee square we may place a partition into even parts with largest part at most $2k$. The generating function for this case is then

$$q^{(2k+1)^2}q^{(2k+1)}\prod_{i=1}^{k}\frac{1}{1-q^{2i}}\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}.$$

Combining the two disjoint cases and summing over k leads to the following identity for partitions into even parts

$$\sum_{k=0}^{\infty}q^{4k^2}(1-q^{4k+2}+q^{6k+2})\prod_{i=1}^{k}\frac{1}{1-q^{2i}}\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}=\prod_{i=1}^{\infty}\frac{1}{1-q^{2i}},$$

thus proving (5.2) above. Similarly the generating function for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty}q^{4k^2}\prod_{i=1}^{k}\frac{1}{1-q^{2i}}\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}\left[4k^2+\frac{(2k+1)^2q^{6k+2}}{1-q^{4k+2}}\right],$$

as per (5.7) above.

5.3 Partitions into distinct parts

We consider the same questions as above but now for partitions with distinct parts.

5.3.1 Durfee square areas

Consider partitions with distinct parts with Durfee square of size k. We need to consider two cases. In the first case, the partition π to the right of the Durfee square has exactly k parts. We decompose π into a triangular array of $1+2+\cdots+k$ squares from bottom to top followed by a partition into at most k parts. Beneath the Durfee square we may place a partion into distinct parts with largest part at most k. The generating function for this case is then

$$q^{k^2}\prod_{i=1}^{k}(1+q^i)q^{k(k+1)/2}\prod_{i=1}^{k}\frac{1}{1-q^i}.$$

In the second case the partition π to the right of the Durfee square has exactly $k-1$ parts. We decompose π into a triangular array of $1+2+\cdots+k-1$ squares from bottom to top followed by a partition into at most $k-1$ parts. Beneath the Durfee square we may place a partion into distinct parts with largest part at most $k-1$. The generating function for this case is then

$$q^{k^2}\prod_{i=1}^{k-1}(1+q^i)q^{k(k-1)/2}\prod_{i=1}^{k-1}\frac{1}{1-q^i}.$$

Combining the two disjoint cases and summing over k leads to the following identity for partitions into distinct parts

$$\sum_{k=0}^{\infty}q^{\frac{3k^2-k}{2}}\frac{1+q^{2k}}{1+q^k}\prod_{i=1}^{k}\frac{1+q^i}{1-q^i}=\prod_{i=1}^{\infty}(1+q^i)$$

This establishes (5.3) above. Consequently the generating function (5.8), for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty}k^2q^{\frac{3k^2-k}{2}}\frac{1+q^{2k}}{1+q^k}\prod_{i=1}^{k}\frac{1+q^i}{1-q^i}.$$

5.3.2 Durfee squares for partitions into distinct odd parts

Consider partitions with distinct odd parts with Durfee square of size $2k$. Then the partition π to the right of the Durfee square can be decomposed into a triangular array of $1+3+5+\cdots+(2(2k-1)+1)$ squares from bottom to top followed by a partition into at most $2k$ parts, all even numbers. Beneath the Durfee square we may place a partition into distinct odd parts with largest part at most $2k-1$. The generating function for this case is then

$$q^{4k^2}q^{4k^2}\prod_{i=1}^{k}(1+q^{2i-1})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}.$$

Next consider partitions with distinct odd parts with Durfee square of size $2k+1$. In the first case the partition π to the right of the Durfee square has exactly $2k$ parts. We decompose π into a triangular array of $0+2+4+\cdots+2(2k)$ squares from bottom to top followed by a partition into at most $2k$ parts all even numbers. Beneath the Durfee square we may place a partion into distinct odd parts with largest part at most $2k-1$. The generating function for this case is then

$$q^{(2k+1)^2}q^{2k(2k+1)}\prod_{i=1}^{k}(1+q^{2i-1})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}.$$

In the second case the partition π to the right of the Durfee square has exactly $2k+1$ parts. We decompose π into a triangular array of $2+4+\cdots+2(2k+1)$ squares from bottom to top followed by a partition into at most $2k+1$ parts all even numbers. Beneath the Durfee square we may place a partition into distinct odd parts with largest part at most $2k+1$. The generating function for this case is then

$$q^{(2k+1)^2}q^{(2k+1)(2k+2)}\prod_{i=1}^{k+1}(1+q^{2i-1})\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}.$$

Combining the three disjoint cases and summing over k gives

$$\sum_{k=0}^{\infty}q^{8k^2}\prod_{i=1}^{k}(1+q^{2i-1})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}\left(1+q^{6k+1}+\frac{q^{10k+3}(1+q^{2k+1})}{1-q^{4k+2}}\right).$$

After simplifying this leads to the following identity, (namely (5.4) above), for partitions into distinct odd parts

$$\sum_{k=0}^{\infty} q^{8k^2} \frac{(1-q^{2k+1}+q^{6k+1}-q^{8k+2}+q^{10k+3})}{1-q^{2k+1}} \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}$$
$$= \prod_{i=1}^{\infty}(1+q^{2i-1}).$$

Similarly the generating function, (5.9), for the total areas of the Durfee squares is

$$\sum_{k=0}^{\infty} q^{8k^2} \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}} \left(4k^2 + (2k+1)^2 \left[q^{6k+1} + \frac{q^{10k+3}(1+q^{2k+1})}{1-q^{4k+2}}\right]\right)$$

or equivalently

$$\sum_{k=0}^{\infty} q^{8k^2} \left(4k^2 + \frac{(2k+1)^2 q^{6k+1}\left(1-q^{2k+1}+q^{4k+2}\right)}{1-q^{2k+1}}\right) \prod_{i=1}^{k}(1+q^{2i-1}) \prod_{i=1}^{2k} \frac{1}{1-q^{2i}}.$$

Note that although the number of partitions with distinct odd parts equals the number of self-conjugate partitions of n, the sum of areas of the Durfee squares is different in each case.

5.3.3 Durfee squares for partitions into distinct even parts

Consider partitions with distinct even parts with Durfee square of size $2k+1$. Then the partition π to the right of the Durfee square can be decomposed into a triangular array of $1+3+5+\cdots+(2(2k)+1)$ squares from bottom to top followed by a partition into at most $2k+1$ parts, all even. Beneath the Durfee square we may place a partition into distinct even parts with largest part at most $2k$. The generating function for this case is then

$$q^{2(2k+1)^2} \prod_{i=1}^{k}(1+q^{2i}) \prod_{i=1}^{2k+1} \frac{1}{1-q^{2i}}.$$

Next consider partitions with distinct even parts with Durfee square of size $2k$. In the first case the partition π to the right of the Durfee square has exactly $2k-1$ parts. We decompose π into a triangular array of $0+2+4+\cdots+2(2k-1)$ squares from bottom to top followed by a partition into at most $2k-1$ parts all even. Beneath the Durfee square we may place a partition into distinct even parts with largest part at most $2k-2$. The generating function for this case is then

$$q^{8k^2-2k}\prod_{i=1}^{k-1}(1+q^{2i})\prod_{i=1}^{2k-1}\frac{1}{1-q^{2i}}.$$

In the second case the partition π to the right of the Durfee square has exactly $2k$ parts. We decompose π into a triangular array of $2+4+\cdots+2(2k)$ squares from bottom to top followed by a partition into at most $2k$ parts all even. Beneath the Durfee square we may place a partition into distinct even parts with largest part at most $2k$. The generating function for this case is then

$$q^{8k^2+2k}\prod_{i=1}^{k}(1+q^{2i})\prod_{i=1}^{2k}\frac{1}{1-q^{2i}}.$$

Combining the three disjoint cases and summing over k leads to the following identity (i.e. (5.5)), for partitions into distinct even parts

$$\sum_{k=0}^{\infty}q^{8k^2-2k}(1-q^{2k}+q^{4k}-q^{4k+2}+q^{6k+2}-q^{8k+2}+q^{10k+2})\times$$
$$\times\prod_{i=1}^{k}(1+q^{2i})\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}$$
$$=\prod_{i=1}^{\infty}(1+q^{2i}).$$

The generating function for the total areas of the Durfee squares is therefore given by (5.10) above, namely

$$\sum_{k=0}^{\infty}q^{8k^2-2k}\left((2k+1)^2q^{10k+2}+4k^2(1-q^{2k}+q^{4k})(1-q^{4k+2})\right)\prod_{i=1}^{k}(1+q^{2i})\times$$
$$\times\prod_{i=1}^{2k+1}\frac{1}{1-q^{2i}}.$$

5.4 The area of Durfee rectangles

We can extend the analysis in Section 2 to count areas of vertical and horizontal Durfee rectangles. A vertical (horizontal) Durfee rectangle is the largest filled rectangle in the Young diagram of a partition that contains the Durfee square where the width (height) is the same as that of the Durfee square. We use the decomposition of a partition as a Durfee square of sides k with firstly, a partition with at most k parts to the right of it. There is also a rectangle of $r \geq 0$ rows of width k below the Durfee square and beneath that a partition with parts of size at most $k-1$. The area of such a vertical Durfee rectangle is $k^2 + kr$. Then summing over k and r we find the generating function for the sum of vertical Durfee rectangle areas to be

$$\sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{(k^2 + kr) q^{k^2+kr} (1 - q^k)}{((q;q)_k)^2}.$$

By conjugation of partitions this is also the generating function for the sum of horizontal Durfee rectangle areas.

Now we consider the case of the general Durfee rectangle defined to be the maximum of the vertical and horizontal Durfee rectangles. We pair each partition with its conjugate and compute the rectangle area for that member of the pair for which the vertical rectangle is greater or equal to the horizontal rectangle. We refine the previous decomposition. To the right of the Durfee square we have a rectangle of height k and width l where $0 \leq l \leq r$ and then a partition with at most $k-1$ parts. First we consider the case where $l \neq r$. this gives

$$\sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \sum_{l=0}^{r-1} \frac{(k^2 + kr) q^{k^2+kr+kl}}{((q;q)_{k-1})^2}.$$

To count the area for both the partition and its conjugate we need to double the coefficients above. Next we consider the case where the horizontal and vertical rectangles are equal, that is, $l = r$. The area of a partition and its conjugate are both counted with this generating function:

$$\sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{(k^2 + kr) q^{k^2+2kr}}{((q;q)_{k-1})^2}.$$

Thus the sum of areas of the general Durfee rectangle over all partitions of

n has generating function

$$2\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{r-1}\frac{(k^2+kr)q^{k^2+kr+kl}}{((q;q)_{k-1})^{2}}+\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(k^2+kr)q^{k^2+2kr}}{((q;q)_{k-1})^{2}}.$$

5.5 Remarks on asymptotic estimates for the area of Durfee squares

In [19, 20, 22] a local limit theorem is established for the size of the Durfee square of a random partition, namely,

$$\mathbb{P}(length = c\sqrt{n}+tn^{1/4}) = \frac{n^{-1/4}}{\sqrt{2\pi\sigma^2}}e^{-t^2/(2\sigma^2)}(1+o(1)),$$

uniformly for $t = o(n^{1/12})$, with $c = \frac{\sqrt{6}\log 2}{\pi}$. More generally it is implicit in [19, 20] that a similar limit law holds for any of the special classes of partitions considered in Sections 2 and 3: Uniformly for $t = o(n^{1/12})$,

$$\mathbb{P}(length = c_F\sqrt{n}+tn^{1/4}) = \frac{n^{-1/4}}{\sqrt{2\pi\sigma_F^2}}e^{-t^2/(2\sigma_F^2)}(1+o(1)),$$

for certain constants c_F and σ_F^2 which depend on the particular case considered. In particular, this immediately implies a corresponding local limit law for the area in each of these cases. Uniformly for $t = o(n^{1/12})$,

$$\mathbb{P}(area = c_F^2 n+2c_F tn^{3/4}) = \frac{n^{-1/4}}{\sqrt{2\pi\sigma_F^2}}e^{-t^2/(2\sigma_F^2)}(1+o(1)),$$

with constants c_F and σ_F^2 as above. We remark that one has a parity restriction in the case of self-conjugate partitions: the area of the Durfee square must be congruent to $n \pmod 2$ (so that local limit theorem only holds for even/odd values, respectively).

A consequence of the local limit laws is that the mean area of the Durfee square tends to $c_F^2 n$ as $n \to \infty$. Or equivalently, the proportion of the area of the Young diagram that lies in the Durfee square tends to $c_F^2 n$ as $n \to \infty$.

Many of the constants c_F have been computed in [4]. Thus, for example we have, in the case of all partitions

$$c_F^2 = \frac{6\log^2 2}{\pi^2} \approx 0.29208$$

In the case of partitions into distinct parts

$$c_F^2 = \frac{12\log^2(1+\sqrt{5})/2}{\pi^2} \approx 0.28155.$$

In the case of self-conjugate partitions

$$c_F^2 = \frac{6\log^2 2}{\pi^2} \approx 0.29208.$$

In both the case of partitions into odd parts or partitions into even parts

$$c_F^2 = \frac{12\log^2(1+\sqrt{5})/2}{\pi^2} \approx 0.28155.$$

These constants give good agreement with numerical computations for $n = 1000$. For example, in the case of all partitions we find a mean area of $0.29764n$ when $n = 1000$.

Acknowledgement We thank Stephan Wagner for helpful advice concerning the application of the known asymptotic results on the size of the Durfee square to the case of the area considered above.

5.6 Bibliography for paper A

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6 Some background for Papers B and C

Historically, both compositions and partitions of integers have been thoroughly studied in separate contexts [1,3,8]. By this, I mean that the relationship between these studies has not received much attention.

The paper that follows was a first attempt (and Paper C after that, a second) to explore this relationship. More specifically these papers are an attempt to obtain the generating function for arbitrary compositions of n as concatenations of partitions (of various types) in terms of the number of such partitions. This allows us to obtain direct or asymptotic expressions (as n goes to infinity) for the average number of partitions contained in arbitrary compositions of n. The type of concatenations may force the parts to alternate between increasing and decreasing partitions or allow them not to alternate. Various types of partitions means separating partitions into those that are non decreasing (sometimes called weakly increasing), strictly increasing, non increasing (sometimes called weakly decreasing) and strictly decreasing. This is precisely defined in the papers that follow.

As preliminary examples, consider the composition $(1, 2, 3, 2, 2, 1)$ of 11. If the concatenation is **alternating** between **strictly increasing** and **strictly decreasing** partitions, it would be rendered as $(1, 2, 3)(2)(2)(1)$ where each of the partitions $(1, 2, 3)$,(2),(2),(1) is chosen in order from left to right to be of maximal size and **alternates** as

$(1, 2, 3)$ ...... maximal, strictly increasing,
(2) ...... maximal, strictly decreasing,
(2) ...... maximal, strictly increasing,
(1) ...... maximal, strictly decreasing.

If the same composition is to be split as **non-alternating** between **strictly increasing** and **strictly decreasing** partitions, it would be rendered as $(1, 2, 3)(2)(2, 1)$ where

$(1, 2, 3)$ ...... is maximal, strictly increasing,
(2) ...... is maximal, strictly increasing or decreasing (both are valid and it is in any case irrelevant as the non-alternating option was chosen),
$(2, 1)$ ...... is maximal, strictly decreasing,

which does **not** alternate between increasing and decreasing categories of partitions.

On the other hand, if the same composition is alternated between strictly increasing and non-increasing (weakly decreasing partitions), it would be rendered as $(1, 2, 3)(2, 2, 1)$ because

$(1, 2, 3)$ ...... is maximally, strictly increasing, and
$(2, 2, 1)$ ...... is maximally, non-increasing.

As a final example, we decompose the composition $(3, 2, 1, 2, 2, 1)$ of 11 as firstly **alternating** and then **non-alternating** concatenations of strictly increasing and non-increasing partitions. Firstly:
$(3, 2, 1)(2)(2, 1)$ alternates between non-increasing and strictly increasing, whereas $(3, 2, 1)(2, 2, 1)$ is non-alternating with two non-increasing partitions as components.

As the reader will no doubt have noticed from these examples, the first block from left to right may be any of the types of partitions allowed for the decomposition. In the alternating case, it has an effect on everything that follows because the next block must be maximal and alternate depending on the first block type.

7 Paper B

7.1 Introduction

Compositions and partitions of integers are well studied. For example, partition theory was extensively studied in [6] where the notion of compositions was also introduced. A simple survey of the history of partition theory is contained in [1] and a survey of the study of compositions is contained in [5].

Recently, some studies on compositions and partitions have focused on the relationship between them. In [2], Andrews develops q-series generating functions for certain specific types of compositions which consist of two adjoined partitions (concave compositions). In Section 2.5, Chapter 2 of his book [9], Stanley develops generating functions for another type of "unimodal" composition (consisting of two adjoined blocks of partitions).

This provides some of the motivation for the current study: i.e., presenting more general compositions as sequences of alternating blocks of partitions. However, we allow arbitrary compositions resulting in a decomposition into an arbitrary number of alternating blocks of partitions.

Roughly speaking, any composition of a fixed positive integer n may be viewed as being split up into blocks where each block is either increasing or decreasing. In other words, each block is a partition. (As a preliminary example the composition 1 4 3 2 1 2 3 of 16 may be split into partitions (14) (321) (23).) How compositions are split depends on precisely how the blocks are defined. In Sections 3 - 5 we make three (different) definitions for how to split compositions into alternating blocks and then make use of previous results on subword patterns in compositions (see [4],[5],[8]) to find generating functions for the number of such blocks contained within all the compositions of n.

In Theorem 1 (see Section 6), we find, using the aforementioned three definitions for the blocks, an asymptotic estimate as n tends to infinity for the expected number of partition blocks within an arbitrary composition of n as well as an estimate of the number of parts per block in each case.

7.2 Preliminaries

We need the following **definitions**.

A **composition** $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ of a positive integer n is an ordered collection of one or more positive integers whose sum is n. Each summand σ_i is called a part of the composition.

A **partition** of a positive integer n is either a non-increasing or non-decreasing sequence of positive integers whose sum is n. (So for example, for the partition $n = a_1+a_2+\cdots+a_k$ either $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ or $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$.)

Per the definition found in [7], let $[n] = \{1, 2, \ldots, n\}$ and let $[n]^\ell$ denote the set of words of length ℓ in the alphabet $[n]$. For any word σ in $[n]^m$, let $\mathrm{red}(\sigma)$ denote the member of $[n]^m$ obtained by replacing the smallest letter of σ by 1, replacing all letters corresponding to the second smallest element by 2 and so on. For example, if $\sigma = 42244 \in [4]^5$ then $\mathrm{red}(\sigma) = 21122$.

We call $\{\mathrm{red}(\sigma)\colon \sigma \in [n]^m,\ 1 \leq m \leq \ell\}$ the set of **subword patterns** in $[n]^\ell$. And we say that there is an occurrence of the pattern $\tau = \sigma_1, \sigma_2, \ldots, \sigma_m$ at index i in the composition (word) $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ if red $(\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+m-1}) = \tau$, $(i \leq s+1-m)$. **The number of occurrences of the pattern τ in** α is the number of different such indices i satisfying red $(\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+m-1}) = \tau$. For example, if $\alpha = 4243233$ then there are two occurrences of the pattern $\tau = 212$ (corresponding to 424 and 323) and one occurrence of the pattern $\tau = 11$ (corresponding to 33). Note that we require the letters within a given word corresponding to a pattern to be consecutive.

In accordance with the definitions in [5], an occurrence of any of the patterns

$$\left.\begin{array}{l} 12^{\ell-2}1 \\ 23^{\ell-2}1 \\ 13^{\ell-2}2 \end{array}\right\} \quad \text{integer } \ell \geq 3$$

in an arbitrary composition of n is known as a peak (strict if $\ell = 3$ and weak otherwise). Likewise, any occurrence of the patterns

$$\left.\begin{array}{l} 21^{\ell-2}2 \\ 21^{\ell-2}3 \\ 31^{\ell-2}2 \end{array}\right\} \quad \text{integer } \ell \geq 3$$

is known as a valley (strict if $\ell = 3$ and weak otherwise).

In each of the groups above, the generating function for the second pattern is the same as that for third (as can be seen from the reversal mapping applied to compositions).

We refer to [5], Theorem 4.39, p.120. In these formulae, we ignore the number of parts (i.e. set $y = 1$), and we choose our alphabet set as $A = \mathbb{N}$. We deal with occurrences of the patterns

$$\left.\begin{array}{rcl} \nu_1 & = & 12^{\ell-2}1 \\ \nu_2 & = & 23^{\ell-2}1 \\ \nu_2' & = & 13^{\ell-2}2 \\ \tau_1 & = & 21^{\ell-2}2 \\ \tau_2 & = & 21^{\ell-2}3 \\ \tau_2' & = & 31^{\ell-2}2 \end{array}\right\} \quad \ell \geq 3$$

$$\begin{array}{rcl} \nu_3 & = & 221 \\ \tau_3 & = & 211 \\ \nu_4 & = & 11 \end{array}$$

and obtain generating functions for the number of occurrences of the specified patterns:

$$C_{\mathbb{N}}^{\nu_2'} = C_{\mathbb{N}}^{\nu_2} = \frac{1}{1-\sum_{i\geq 1} x^i \prod_{j\geq i+1}\left(1+x^j(q-1)\sum_{\alpha\in\overline{\beta}_{j+1}} x^{\mathrm{ord}(\alpha)}\right)}$$

$$C_{\mathbb{N}}^{\tau_2'} = C_{\mathbb{N}}^{\tau_2} = \frac{1}{1-\sum_{i\geq 1} x^i \prod_{j=1}^{i-1}\left(1+x^j(q-1)\sum_{\alpha\in\beta_{j-1}} x^{\mathrm{ord}(\alpha)}\right)},$$

where $\overline{\beta}_{j+1}$ is the set of compositions α with parts in $\{j+1, j+2, \dots\}$ that are order isomorphic to the pattern $1^{\ell-2}$, and β_{j-1} is the set of compositions α with parts in $\{1, 2, \dots, j-1\}$ that are order isomorphic to the same pattern.

In the above formulae x marks the size n of the compositions and q marks the number of occurrences of the pattern under consideration.

So for example, we may write $C_{\mathbb{N}}^{\tau_2}$ as

$$C_{\mathbb{N}}^{\tau_2} = \sum_{n\geq 0}\sum_{b\geq 0} a(n,b)x^n q^b,$$

where $a(n,b)$ is the number of times that compositions of the integer n have exactly b occurrences of the pattern τ_2.

Simplifying the formulae above, we have:

$$\sum_{\alpha\in_{j+1}} x^{(\alpha)} = \sum_{r\geq j+1} (x^{\ell-2})^r = \frac{(x^{\ell-2})^{j+1}}{1-x^{\ell-2}}$$

and

$$\sum_{\alpha\in\beta_{j-1}} x^{(\alpha)} = \sum_{\alpha=1}^{j-1} (x^{\ell-2})^{\alpha} = \frac{x^{\ell-2}\left(1-(x^{\ell-2})^{j-1}\right)}{1-x^{\ell-2}}.$$

So

$$C_{\mathbb{N}}^{\nu_2} = \frac{1}{1-\sum\limits_{i\geq 1} x^i \prod\limits_{j\geq i+1}\left(1+x^j(q-1)\cdot\frac{(x^{\ell-2})^{j+1}}{1-x^{\ell-2}}\right)} \tag{7.1}$$

and

$$C_{\mathbb{N}}^{\tau_2} = \frac{1}{1-\sum\limits_{i\geq 1} x^i \prod\limits_{j=1}^{i-1}\left(1+x^j(q-1)\frac{x^{\ell-2}\left(1-x^{(\ell-2)(j-1)}\right)}{1-x^{\ell-2}}\right)}. \tag{7.2}$$

$$C_{\mathbb{N}}^{\nu_1} = \frac{1}{1-\sum_{j\geq 1}\frac{x^j}{1+x^j(1-q)\frac{(x^{\ell-2})^{j+1}}{1-x^{\ell-2}}}} \tag{7.3}$$

$$C_{\mathbb{N}}^{\tau_1} = \frac{1}{1-\sum_{j\geq 1}\frac{x^j}{1+x^j(1-q)\frac{x^{\ell-2}\left(1-(x^{\ell-2})^{j-1}\right)}{1-x^{\ell-2}}}}. \tag{7.4}$$

From [5] Theorem 4.3.5, p.115, again setting $y = 1$ and $A = \mathbb{N}$ and also $\ell = 3$, we obtain

$$C_{\mathbb{N}}^{\nu_3} = \frac{1}{1-\sum_{j\geq 1}x^j\prod_{i\geq j+1}\left(1-x^{2i}(1-q)\right)} \tag{7.5}$$

and

$$C_{\mathbb{N}}^{\tau_3} = \frac{1}{1-\sum_{j\geq 1}x^j\prod_{i=1}^{j-1}\left(1-x^{2i}(1-q)\right)}. \tag{7.6}$$

Finally from [5], Table 4.1, pg. 101, we obtain

$$C_{\mathbb{N}}^{\nu_4} = \frac{1}{1-\sum_{j=1}^{\infty}\frac{x^j}{1-x^j(q-1)}}. \tag{7.7}$$

The generating function for the total number of occurrences for the associated pattern in all compositions of n is found by differentiating the above formulae with respect to q and setting $q = 1$. We obtain

$$\begin{aligned}
\left.\frac{\partial C_{\mathbb{N}}^{\nu_1}}{\partial q}\right|_{q=1} &= \sum_{\ell\geq 3}\Big(1-\sum_{j\geq 1}x^j\Big)^{-2}\sum_{j\geq 1}x^j\Big[x^j\sum_{i\geq j+1}(x^{\ell-2})^i\Big]\\
&= \left(\frac{1-x}{1-2x}\right)^2\sum_{\ell\geq 3}\frac{x^{2\ell-2}}{(1-x^{\ell-2})(1-x^{\ell})}\\
\left.\left[\frac{\partial C_{\mathbb{N}}^{\nu_2'}}{\partial q}+\frac{\partial C_{\mathbb{N}}^{\nu_2}}{\partial q}\right]\right|_{q=1} &= 2\left.\frac{\partial C_{\mathbb{N}}^{\nu_2}}{\partial q}\right|_{q=1}\\
&= 2\sum_{\ell\geq 3}\Big(1-\sum_{i\geq 1}x^j\Big)\cdot\sum_{i\geq 1}x^i\sum_{j\geq i+1}\frac{x^j\cdot x^{(\ell-2)(j+1)}}{1-x^{\ell-2}}\\
&= 2\left(\frac{1-x}{1-2x}\right)^2\sum_{\ell\geq 3}\frac{x^{3(\ell-1)}}{(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})}\\
\left.\frac{\partial C_{\mathbb{N}}^{\tau_1}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\sum_{\ell\geq 3}\frac{x^{\ell+2}-x^{2\ell}}{(1-x^2)(1-x^{\ell-2})(1-x^{\ell})}\\
\left.\left[\frac{\partial C_{\mathbb{N}}^{\tau_2'}}{\partial q}+\frac{\partial C_{\mathbb{N}}^{\tau_2}}{\partial q}\right]\right|_{q=1} &= 2\left.\frac{\partial C_{\mathbb{N}}^{\tau_2}}{\partial q}\right|_{q=1}\\
&= 2\left(\frac{1-x}{1-2x}\right)^2\times\\
&\quad\times\sum_{\ell\geq 3}\frac{x^{\ell+3}-x^{2\ell+1}-x^{2\ell+2}+x^{3\ell}}{(1-x)(1-x^2)(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})}\\
\left.\frac{\partial C_{\mathbb{N}}^{\nu_3}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\cdot\frac{x^5}{(1-x^2)(1-x^3)}\\
\left.\frac{\partial C_{\mathbb{N}}^{\tau_3}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\cdot\frac{x^4}{(1-x)(1-x^3)}\\
\left.\frac{\partial C_{\mathbb{N}}^{\nu_4}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\frac{x^2}{1-x^2}.
\end{aligned}\tag{7.8}$$

7.3 Weakly increasing/weakly decreasing alternating blocks

We first split a composition of n into blocks on the basis that all of the parts in any particular block have the same relationship, namely, $\geq$ or $\leq$, to all the other parts of the same block. More precisely, for the arbitrary composition of n given by

$$n = x_1 + x_2 + \cdots + x_{q_s},$$

where either

$$x_1 \leq x_2 \leq \cdots \leq x_{q_1} > x_{q_1+1} \geq x_{q_1+2} \geq \cdots \geq x_{q_2} < x_{q_2+1} \leq \cdots x_{q_s} \quad (7.9)$$

or

$$x_1 \geq x_2 \geq \cdots \geq x_{q_1} < x_{q_1+1} \leq x_{q_1+2} \leq \cdots \leq x_{q_2} > x_{q_2+1} \geq \cdots x_{qs}, \quad (7.10)$$

the blocks are defined to be the s partitions

$$\begin{array}{l} x_1, x_2, \ldots x_{q_1} \\ x_{q_1+1}, x_{q_1+2} \ldots x_{q_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{q_{s-1}+1}, \ldots x_{q_s} \end{array}$$

derived from (9) or (10), where the value chosen for q_1 is maximal (and, subsequently, for the remaining q_i, $i > 1$). If all parts of the composition are equal, then there is only one block (which we may regard if we like as a weakly increasing partition).

The composition 133222 of 13 will be split into blocks (133) and (222) (where we might regard the first as being a weakly increasing partition and the second as weakly decreasing). (1), (33), (222) would not be correct because the first option above has the larger q_1 value.

With blocks as defined in this section, an occurrence of a peak or a valley, whether weak or strict in either case (i.e., any of the patterns ν_1, ν_2, ν_2', τ_1, τ_2, τ_2'), marks the beginning of a new block.

Hence for any particular composition the total number of blocks = total number of peaks + total number of valleys + 1.

Thus the generating function for the total number of alternating blocks is

$$\begin{aligned} G_1 &= \left[\frac{\partial C_{\mathbb{N}}^{\nu_1}}{\partial q} + \frac{2\partial C_{\mathbb{N}}^{\nu_2}}{\partial q} + \frac{\partial C_{\mathbb{N}}^{\tau_1}}{\partial q} + \frac{2\partial C_{\mathbb{N}}^{\tau_2}}{\partial q}\right]_{q=1} + \left(\frac{1-x}{1-2x}\right) \\ &= \left(\frac{1-x}{1-2x}\right)^2 \times \\ &\quad \times \sum_{\ell\geq 3} \frac{x^{\ell-3}\big(x^5 + x^{\ell+1} - 2x^{\ell+2} - x^{\ell+4} + x^{2\ell} - 2x^{2\ell+1} + 2x^{2\ell+2}\big)}{(1-x)^2(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})} \\ &\quad + \left(\frac{1-x}{1-2x}\right). \end{aligned}$$

The principal pole of the summand in

$$f(x) = \frac{1}{(1-2x)^2} \sum_{\ell\geq 3} \frac{x^{\ell-3}(x^5 + x^{\ell+1} - 2x^{\ell+2} - x^{\ell+4} + x^{2\ell} - 2x^{2\ell+1} + 2x^{2\ell+2}}{(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})}$$

occurs at $x = \frac{1}{2}$ (with multiplicity 2). Using the asymptotic analysis on page 257 of [3],

$$\begin{aligned} f(x) &\sim \left(\sum_{\ell\geq 3} \frac{\left(\frac{1}{2}\right)^{\ell-3}\left[\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^{\ell+1} - 2\left(\frac{1}{2}\right)^{\ell+2} - \left(\frac{1}{2}\right)^{\ell+4} + \left(\frac{1}{2}\right)^{2\ell} - 2\left(\frac{1}{2}\right)^{2\ell+1} + 2\left(\frac{1}{2}\right)^{2\ell+2}\right]}{\left(1-\left(\frac{1}{2}\right)^{\ell-2}\right)\left(1-\left(\frac{1}{2}\right)^{\ell-1}\right)\left(1-\left(\frac{1}{2}\right)^{\ell}\right)}\right) \\ &\quad \times \left(\frac{1}{1-2x}\right)^2. \end{aligned}$$

Hence

$$[x^n]f(x) \sim (0.136681\ldots)\,(n+1)2^n.$$

The total number of compositions of n is 2^{n-1}. Hence the expected number of blocks as $n \to \infty$ is $(0.273362\ldots)n + 0(1)$. Because the average number of parts of a composition of n is $\frac{n}{2}$, the expected number of parts in the blocks as $n \to \infty$ is $1.82908\ldots$.

Is the number $1.82908\ldots$ irrational?

7.4 Weakly increasing/strictly decreasing alternating blocks

Consider an arbitrary composition of $n = x_1 + x_2 + \cdots + x_{q_s}$. View the composition in one of the following manners

$$x_1 \le x_2 \le \cdots \le x_{q_1} > x_{q_1+1} > x_{q_1+2} > \cdots > x_{q_2} \le x_{q_2+1} \le \cdots \le x_{q_3} > \cdots x_{q_s}$$

or

$$x_1 > x_2 \cdots > x_{q_1} \le x_{q_1+1} \le x_{q_1+2} \le \cdots \le x_{q_2} > x_{q_2+1} \cdots > x_{q_3} \le \cdots x_{q_s}.$$

That is, all equalities are considered as part of a neighbouring (weakly) increasing block. So the compositions are split into partitions

$$\left.\begin{array}{l} x_1, x_2, \ldots, x_{q_1} \\ x_{q_1+1}, x_{q_1+2}, \ldots, x_{q_2} \\ \vdots \\ x_{q_{s-1}+1}, \ldots, x_{q_s} \end{array}\right\} \text{which alternately (weakly) increase or (strictly) decrease.}$$

As in Section 2, the sequence above which is used to determine the blocks is that one with the largest q_1 value.

Example 4.1. *The composition* 133222 *of 13 will be split into blocks (133), (2) and (22) (where we might regard the first and third as weakly increasing and the second as strictly decreasing).*

With blocks as defined in this section, an occurrence of any of the patterns 121, 231, 132, 221 (all peaks) or 212, 213, 312, 211 (all valleys) marks the beginning of a new block. These patterns are identical to those in Section 2 with the restriction that $\ell = 3$ only.

As before, for any particular composition, the total number of blocks = total number of peaks + total number of valleys + 1.

Thus the generating function for the total number of alternating blocks is

$$\begin{aligned} G_2 \;&=\; \left\{\left[\frac{\partial C_{\mathbb{N}}^{\nu_1}}{\partial q} + \frac{2\partial C_{\mathbb{N}}^{\nu_2}}{\partial q}\right] + \left[\frac{\partial C_{\mathbb{N}}^{\tau_1}}{\partial q} + \frac{2\partial C_{\mathbb{N}}^{\tau_2}}{\partial q}\right] + \left[\frac{\partial C_{\mathbb{N}}^{\nu_3}}{\partial q} + \frac{\partial C_{\mathbb{N}}^{\tau_3}}{\partial q}\right]\right\}\Bigg|_{\substack{\ell=3\\ q=1}} \\ &\quad + \left(\frac{1-x}{1-2x}\right). \end{aligned}$$

(First square bracket term represents strict peaks; second square bracket terms represent strict valleys and last term represents weak peaks or weak valleys)

$$\begin{aligned} G_2 &= \left(\frac{1-x}{1-2x}\right)^2 \frac{\left[x^4(1+x^2)\right]+[x^5+x^6]+[x^4+x^5-2x^6]}{(1-x)(1-x^2)(1-x^3)} + \left(\frac{1-x}{1-2x}\right) \\ &= \left(\frac{1-x}{1-2x}\right)^2 \frac{(1-2x-x^2+x^3+4x^4+3x^5-2x^6)}{(1-x)(1-x^2)(1-x^3)} \\ &= \frac{1}{2} + \frac{2}{3(1-x)} + \frac{1}{7(1-2x)^2} - \frac{13}{98(1-2x)} - \frac{2}{147}\frac{(13-2x-11x^2)}{(1-x^3)}. \end{aligned}$$

Extracting the coefficients of x^n from this equation, we obtain

$$[x^n] = \begin{cases} 1 & \text{if } n = 0 \\ \frac{48}{98} + \frac{2^n}{98} + \frac{7 \cdot 2^{n+1}}{98} n & \text{for } n \equiv 0 \mod 3 \text{ but } n \neq 0 \\ \frac{68}{98} + \frac{2^n}{98} + \frac{7 \cdot 2^{n+1}}{98} n & \text{for } n \equiv 1 \mod 3 \\ \frac{109}{147} + \frac{2^n}{98} + \frac{7 \cdot 2^{n+1}}{98} n & \text{for } n \equiv 2 \mod 3 \end{cases}.$$

Since there are 2^{n-1} compositions of integer n, the expected number of blocks is

$$E_1 = \begin{cases} 1 & \text{if } n = 0 \\ \frac{48}{98 \cdot 2^{n-1}} + \frac{1}{49} + \frac{2}{7} n & \text{if } n \equiv 0 \mod 3 \text{ but } n \neq 0 \\ \frac{68}{98 \cdot 2^{n-1}} + \frac{1}{49} + \frac{2}{7} n & \text{if } n \equiv 1 \mod 3 \\ \frac{109}{147 \cdot 2^{n-2}} + \frac{1}{49} + \frac{2}{7} n & \text{if } n \equiv 2 \mod 3 \end{cases}.$$

Asymptotically as $n \to \infty$, the expected number of blocks $= \frac{1}{49} + \frac{2}{7}n$.
It is well known that the average number of parts of each composition of n is $\frac{n}{2}$. Therefore the expected number of parts in the blocks forming the compositions of n is $1\frac{3}{4}$ as $n \to \infty$.

7.5 Strictly increasing/strictly decreasing alternating blocks

By analogy with Section 3, view an arbitrary composition $n = x_1 + x_2 + \cdots + x_{q_s}$ in one of the following manners

$$x_1 < x_2 < \cdots < x_{q_1} \geq x_{q_1+1} > x_{q_1+2} > \cdots > x_{q_2} \leq x_{q_2+1} < \cdots < x_{q_3} \geq \cdots x_{q_s}$$

or

$$x_1 > x_2 \cdots > x_{q_1} \leq x_{q_1+1} < x_{q_1+2} < \cdots < x_{q_2} \geq x_{q_2+1} > \cdots > x_{q_3} \leq \cdots x_{q_s}.$$

So the composition is split into partitions

$$\left.\begin{array}{l} x_1, x_2, \ldots, x_{q_1} \\ x_{q_1+1}, x_{q_1+2}, \ldots, x_{q_2} \\ \vdots \\ x_{q_{s-1}+1}, \ldots, x_{q_s} \end{array}\right\} \text{which alternately strictly increase or strictly decrease.}$$

In the event that a composition begins with equalities, the sequence above is chosen so that the first block has one part only.

This definition is a generalization of that used in [2] to define the blocks of the concave compositions.

Examples 7.1 *The composition 22212 is split (2)(2)(21)(2) and not (2)(2)(2)(1)(2). The composition 133222 is split (13)(32)(2)(2).*

With blocks as defined in this section, an occurrence of any of the patterns 121, 231 or 132 (strict peaks) or alternatively 212, 213 or 312 (strict valleys) or 11 (equalities) marks the beginning of a new block.

Thus the generating function for the total number of alternating blocks is

$$\begin{aligned}
G_3 &= \left\{\left[\frac{\partial C_{\mathbb{N}}^{\nu_1}}{\partial q} + \frac{2\partial C_{\mathbb{N}}^{\nu_2}}{\partial q}\right] + \left[\frac{\partial C_{\mathbb{N}}^{\tau_1}}{\partial q} + \frac{2\partial C_{\mathbb{N}}^{\tau_2}}{\partial q}\right] + \left[\frac{\partial C_{\mathbb{N}}^{\nu_4}}{\partial q}\right]\right\}\Bigg|_{\substack{\ell=3\\ q=1}} + \left(\frac{1-x}{1-2x}\right) \\
&= \left(\frac{1-x}{1-2x}\right)^2 \left[\frac{1-2x+3x^4+x^5+x^6}{(1-x)(1-x^2)(1-x^3)}\right] \\
&= -\frac{1}{4} + \frac{2}{3(1-x)} + \frac{1(1-x)}{3(1-x^2)} - \frac{5}{294(1-2x)} + \frac{5}{28(1-2x)^2} \\
&\quad + \frac{1(13-2x-11x^2)}{147(1-x^3)}.
\end{aligned}$$

Extracting coefficients of x^n from this expression, we obtain

$$[x^n] = \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{588}\big(x(n) + A(n)\big) & n \neq 0 \text{ as below} \end{cases}$$

where

$$x(n) = \begin{cases} 640 & \text{if } n \equiv 0 \mod 6 \, (n \neq 0) \\ 188 & \text{if } n \equiv 1 \mod 6 \\ 544 & \text{if } n \equiv 2 \mod 6 \\ 248 & \text{if } n \equiv 3 \mod 6 \\ 580 & \text{if } n \equiv 4 \mod 6 \\ 152 & \text{if } n \equiv 5 \mod 6 \end{cases} \qquad \text{and } A(n) = 95 \times 2^n + 105 \times 2^n n.$$

Since there are 2^{n-1} compositions of the integer n, the expected number of blocks as $n \to \infty$ is $\frac{95}{294} + \frac{5}{14}n$. Since the expected number of parts is $\frac{n}{2}$, the expected number of parts per block is $\frac{7}{5}$.

7.6 Summary of Results

Above we have proved:

Theorem 1. *For compositions of n split up into blocks of alternately increasing/decreasing partitions, the asymptotic expectation for the number of blocks with corresponding number of parts is given in the following table:*

Definition of type of decomposition	*Asymptotically expected number of blocks*	*Corresponding number of parts per block*
Section 3 (Weakly increasing/weakly decreasing)	$(0.273362\ldots)n + 0(1)$	$1.82098\ldots$
Section 4 (Weakly increasing/strictly decreasing)	$\frac{2}{7}n + \frac{1}{49}$	$1\frac{3}{4}$
Section 5 (Strictly increasing/strictly decreasing)	$\frac{5}{14}n + \frac{95}{294}$	$1\frac{2}{5}$

7.7 Bibliography for Paper B

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8 Some background for Paper C

What is necessary for this paper is to understand the notions of compositions and partitions of the integer n. These have already been repeatedly defined for both papers A and B and also more simply in the background section for these two papers.

A concept that is required specifically for this paper (as well as for paper B) is the notion of a subword pattern of length 3. This is formally defined in the paper that follows. Here we try to communicate less formally an understanding of this notion:

Given a word consisting of three positive integers, we replace all occurrences of the smallest integer by 1, the second smallest (if it exists) by 2, and the next smallest by 3.

Using this process, 833 would be an occurrence of the *pattern* 211 and 423 would be an occurrence of the *pattern* 312 (called in Paper C, a *strict valley pattern*).

9 Paper C

9.1 Introduction

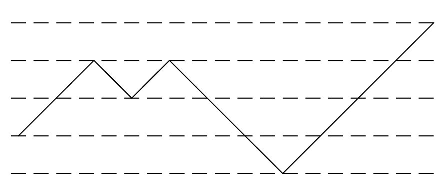
Compositions and partitions of integers were introduced by MacMahon [6] in 1917. A survey of partition theory is found in [1] and a survey of composition theory in [5]. Partitions have not generally been studied in terms of patterns. As in [5], composition generating functions often have occurrences of particular patterns as one variable. So compositions and partitions have usually been studied in separate theoretical frameworks. Recently some work has been done on expressing composition generating functions in terms of partition generating functions. For example in [2] a generating function for a certain class of compositions, “concave” compositions having one “valley” pattern (and no peaks) was developed. In Section 2.5, Chapter 2 of [8], a similar idea was done for another type of “unimodal” compositions having a single “peak” pattern (and no valleys).

Here, we consider arbitrary compositions and develop a generating function in terms of the size of the composition, the number of parts and the total number of peaks and valleys. In other words any composition may be viewed as an alternating sequence of “increasing” or “decreasing” partition blocks. This depends on precisely how “increasing” and ”decreasing” are defined. The generating function which we obtain, accounts for how many partition blocks each composition of n with m parts splits into.

In Section 2 of this paper, we specify increasing partition blocks as those which are non-decreasing, in other words weakly increasing, and decreasing partition blocks as those which are strictly decreasing. With these specifications an arbitrary composition is split into an alternating sequence of increasing/decreasing partitions precisely at the part of the composition in which the midpoint of any of the following patterns occur:

TYPE	PATTERN
Strict peaks	121, 231, 132
Weak peak	221
Strict valleys	212, 213,312
Weak valley	211

As an illustration, we may perhaps represent an arbitrary composition as a zigzag graph where every left to right increasing block represents a non-decreasing partition with a line going up and where every left to right decreasing block represents a strictly decreasing partition with a line going down. In the diagram below, there are 2 peaks, 2 valleys and 5 partition blocks.



The joining point between these is at any of the peak or valley patterns above. Hence the number of such partition blocks = number of peaks + number of valleys + 1. We now turn to the precise definitions required to develop the generating functions.

9.2 Definitions

We need the following definitions:

A composition $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ of a positive integer n is an ordered collection of one or more positive integers whose sum is n. Each summand σ_i is called a part of the composition.

A partition of a positive integer n is either a non-increasing or a non-decreasing composition of n.

For example, for the partition of n with k parts

$$n = a_1 + a_2 + \cdots + a_k$$

either

$$a_1 \geq a_2 \geq \cdots \geq a_k \geq 1 \quad \text{or} \quad 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k.$$

As per the definition found in [7], let $[n] = \{1, 2, \cdots, n\}$ and let $[n]^l$ denote the set of words of length l in the alphabet $[n]$. For any word σ in $[n]^m$, let $red(\sigma)$ denote the member of $[n]^m$ obtained by replacing the smallest letter of σ by 1, replacing all letters corresponding to the second smallest element of σ by 2 and so on.

For example: if $\sigma = 42244 \in [4]^5$ then $red(\sigma) = 21122$.

We call $\{red(\sigma) : \sigma \in [n]^m, 1 \leq m \leq l\}$ the set of subword patterns in $[n]^l$. We also say that there is an occurrence of the pattern $\tau = \sigma_1, \sigma_2, \cdots, \sigma_m$ at index i in the composition or word $\alpha = \alpha_1\alpha_2\cdots\alpha_s$ if $red(\alpha_1, \alpha_{i+1}, \cdots, \alpha_{i+m-1}) = \tau$, where $i \leq s + 1 - m$.

Consider arbitrary compositions of n. We split them into alternating blocks of weakly increasing (non-decreasing) or strictly decreasing partitions each of maximum size. The first block is either increasing or decreasing according to which is the larger of the two. Thereafter, we alternate the blocks.

For example, consider the composition of 25: 3 3 4 2 1 2 5 4 1.

It can be split correctly as (3 3 4)(2 1)(2 5)(4 1) starting with a weakly increasing partition and incorrectly as (3)(3 4)(2 1)(2 5)(4 1) starting with a strictly decreasing block. The former is chosen as it has a larger initial block.

We define peaks and valleys as in Section 1.

Counting peaks and valleys in compositions (represented as words) is an extension of the concept of counting the comparable patterns on k-ary words or other finite discrete structures. For example, there is the paper [9] which considers the problem on k-ary words. See also the recent book by Mansour [10] (and the references contained therein) for results concerning the problem on finite set partitions, represented canonically as restricted growth functions.

We introduce the generating function

$$F(x, y, q) := \sum_{n \geq 0} \sum_{m=0}^{n} x^n \, y^m \, q^r, \tag{9.1}$$

where x marks the size of the composition, y the number of parts and q the number of occurrences of any of the above patterns stated in Section 1.

We also define $F_a := F_a(x, y, q)$ to be the generating function for all compositions of n where the first part is a. Hence, considering all possible starting value for a, we have

$$F(x, y, q) = 1 + \sum_{a \geq 1} F_a \tag{9.2}$$

which includes the empty composition.

We extend this notation to $F_{a_1 a_2 \cdots a_s}$ as the generating function for the compositions starting with $a_1 a_2 \cdots a_s$.

Consider the pair of letters aj. We have two cases depending on the value of j.

Firstly, if $j < a$, we define L_a to be the generating function for all composition for which there is a strict descent following the initial letter a.

Secondly, if $j \geq a$, we define M_a to be the generating function describing a weak ascent following a. Thus

$$L_a = \sum_{j<a} F_{aj} \tag{9.3}$$

and

$$M_a = \sum_{j \geq a} F_{aj}. \tag{9.4}$$

Putting all the possible cases together, we have the generating function for all compositions starting with the letter a:

$$F_a = x^a y + L_a + M_a, \tag{9.5}$$

where the first term is for a one part composition.

9.3 Generating functions

In this section, we find the generating function F_{aj} for all words beginning with aj where $j < a$ and F_{aj} for all words beginning with aj where $j \geq a$.

Let $j < a$,

$$\begin{aligned} F_{aj} &= x^{a+j}y^2 + \sum_{i<j} F_{aji} + \sum_{i\geq j} F_{aji} \\ &= x^{a+j}y^2 + x^a y(L_j + qM_j), \end{aligned} \tag{9.6}$$

and similarly let $j \geq a$,

$$F_{aj} = x^{a+j}y^2 + x^a y(qL_j + M_j). \tag{9.7}$$

We substitute (9.6) into (9.3) and (9.7) into (9.4) to obtain

$$L_a = \sum_{j=1}^{a-1} x^{a+j}y^2 + x^a y \sum_{j=1}^{a-1} (L_j + qM_j) \tag{9.8}$$

and

$$M_a = \sum_{j=a}^{\infty} x^{a+j}y^2 + x^a y \sum_{j=a}^{\infty} (qL_j + M_j)\,. \tag{9.9}$$

We now define the following generating functions

$$F(t) = F(x, y, q; t) = \sum_{a\geq 1} F_a\, t^a, \tag{9.10}$$

$$L(t) = \sum_{a\geq 1} L_a\, t^a, \tag{9.11}$$

and

$$M(t) = \sum_{a\geq 1} M_a\, t^a. \tag{9.12}$$

Clearly, by (9.5) we have

$$F(x,y,q) = 1 + \frac{xy}{1-x} + L(1) + M(1). \tag{9.13}$$

Thus substituting (9.8) into (9.11) and similarly (9.9) into (9.12), we obtain

$$\begin{aligned}
L(t) &= \sum_{a\geq 1}\sum_{j=1}^{a-1} x^{a+j}y^2t^a + \sum_{a\geq 1} x^a y \sum_{j=1}^{a-1}(L_j+qM_j)t^a\\
&= \frac{x^3y^2t^2}{(1-xt)(1-x^2t)} + \sum_{j=1}^{\infty}(L_j+qM_j)y\sum_{a=j+1}^{\infty}(tx)^a\\
&= \frac{x^3y^2t^2}{(1-xt)(1-x^2t)} + \frac{xyt}{1-tx}\,L(tx) + \frac{qxyt}{1-tx}\,M(tx),
\end{aligned} \tag{9.14}$$

and

$$\begin{aligned}
M(t) &= \sum_{a\geq 1}\sum_{j=a}^{\infty}\left(x^{a+j}y^2 + x^a y(qL_j+M_j)\right)t^a\\
&= \sum_{a\geq 1} y^2(tx)^a\frac{x^a}{1-x} + \sum_{j=1}^{\infty} y(qL_j+M_j)\sum_{a=1}^{j}(tx)^a\\
&= \frac{(xy)^2t}{(1-x)(1-tx^2)} + \sum_{j=1}^{\infty}\left(\frac{xyqtL_j}{1-xt} + \frac{xytM_j}{1-xt} - \frac{xyqt(xt)^jL_j}{1-xt}\right.\\
&\qquad \left. - \frac{xyt(xt)^jM_j}{1-xt}\right)\\
&= \frac{(xy)^2t}{(1-x)(1-x^2t)} + \frac{xyqtL(1)}{1-xt} + \frac{xytM(1)}{1-xt} - \frac{xyqtL(xt)}{1-xt}\\
&\qquad - \frac{xytM(xt)}{1-xt}.
\end{aligned} \tag{9.15}$$

If we define

$$A(t) = \frac{x^3y^2t^2}{(1-xt)(1-x^2t)},$$
$$B(t) = \frac{x^2y^2t}{(1-x)(1-x^2t)} + \frac{xyt}{1-xt}(qL(1)+M(1)),$$

then $L(t)$ and $M(t)$ can be expressed using matrices as follows

$$\begin{pmatrix} L(t) \\ M(t) \end{pmatrix} = \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} + \frac{xyt}{1-xt}\begin{pmatrix} 1 & q \\ -q & -1 \end{pmatrix}\begin{pmatrix} L(xt) \\ M(xt) \end{pmatrix}. \tag{9.16}$$

We keep iterating (9.16) and use the fact that $x^j, y^j, t^j \to 0$ as $j \to \infty$ for $|x|, |y|, |t| \leq \rho < 1$. Finally we put $t = 1$ and obtain the following result.

Theorem 9.1 *The generating functions $L(1)$ and $M(1)$ are given by*

$$\begin{aligned} \begin{pmatrix} L(1) \\ M(1) \end{pmatrix} &= \sum_{j\geq 0} \alpha_j \begin{pmatrix} 1 & q \\ -q & -1 \end{pmatrix}^j \begin{pmatrix} \frac{x^{j+2}}{1-x^{j+1}} \\ \frac{x}{1-x} \end{pmatrix} \\ &\quad + \sum_{j\geq 0} \beta_j \begin{pmatrix} 1 & q \\ -q & -1 \end{pmatrix}^j \begin{pmatrix} 0 \\ qL(1)+M(1) \end{pmatrix}. \end{aligned}$$

where $\alpha_j = \frac{x^{\binom{j+2}{2}} y^{j+2}}{(1-x)(1-x^2)\cdots(1-x^j)(1-x^{j+2})}$ *and* $\beta_j = \frac{x^{\binom{j+2}{2}} y^{j+1}}{(1-x)(1-x^2)\cdots(1-x^{j+1})}$.

Using the fact that

$$\begin{pmatrix} 1 & q \\ -q & -1 \end{pmatrix}^{2j} = (1-q^2)^j I \quad \text{and} \quad \begin{pmatrix} 1 & q \\ -q & -1 \end{pmatrix}^{2j+1} = (1-q^2)^j \begin{pmatrix} 1 & q \\ -q & -1 \end{pmatrix},$$

Theorem 9.1 gives

$$\begin{pmatrix} L(1) \\ M(1) \end{pmatrix} = \sum_{j\geq 0} \alpha_{2j}(1-q^2)^j \begin{pmatrix} \frac{x^{2j+2}}{1-x^{2j+1}} \\ \frac{x}{1-x} \end{pmatrix}$$
$$+ \sum_{j\geq 0} \alpha_{2j+1}(1-q^2)^j \begin{pmatrix} \frac{x^{2j+3}}{1-x^{2j+2}} + \frac{qx}{1-x} \\ -\frac{qx^{2j+3}}{1-x^{2j+2}} - \frac{x}{1-x} \end{pmatrix}$$
$$+ \sum_{j\geq 0} \beta_{2j}(1-q^2)^j \begin{pmatrix} 0 \\ qL(1)+M(1) \end{pmatrix}$$
$$+ \sum_{j\geq 0} \beta_{2j+1}(1-q^2)^j \begin{pmatrix} q^2L(1)+qM(1) \\ -qL(1)-M(1) \end{pmatrix}.$$

This is equivalent to

$$\begin{pmatrix} 1-q^2\sum\limits_{j\geq 0}\beta_{2j+1}(1-q^2)^j & -q\sum\limits_{j\geq 0}\beta_{2j+1}(1-q^2)^j \\ q\sum\limits_{j\geq 0}(\beta_{2j+1}-\beta_{2j})(1-q^2)^j & 1+\sum_{j\geq 0}(\beta_{2j+1}-\beta_{2j})(1-q^2)^j \end{pmatrix} \begin{pmatrix} L(1) \\ M(1) \end{pmatrix}$$
$$= \sum_{j\geq 0} \alpha_{2j}(1-q^2)^j \begin{pmatrix} \frac{x^{2j+2}}{1-x^{2j+1}} \\ \frac{x}{1-x} \end{pmatrix} + \sum_{j\geq 0} \alpha_{2j+1}(1-q^2)^j \begin{pmatrix} \frac{x^{2j+3}}{1-x^{2j+2}} + \frac{qx}{1-x} \\ -\frac{qx^{2j+3}}{1-x^{2j+2}} - \frac{x}{1-x} \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1-q^2\sum\limits_{j\geq 0}\beta_{2j+1}(1-q^2)^j & -q\sum\limits_{j\geq 0}\beta_{2j+1}(1-q^2)^j \\ q\sum\limits_{j\geq 0}(\beta_{2j+1}-\beta_{2j})(1-q^2)^j & 1+\sum\limits_{j\geq 0}(\beta_{2j+1}-\beta_{2j})(1-q^2)^j \end{pmatrix}^{-1}$$
$$= \frac{1}{1-\sum\limits_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}}\mathbf{T}$$

with

$$\mathbf{T} = \begin{pmatrix} 1+\sum\limits_{j\geq 0}(\beta_{2j+1}-\beta_{2j})(1-q^2)^j & q\sum\limits_{j\geq 0}\beta_{2j+1}(1-q^2)^j \\ q\sum\limits_{j\geq 0}(\beta_{2j}-\beta_{2j+1})(1-q^2)^j & 1-q^2\sum\limits_{j\geq 0}\beta_{2j+1}(1-q^2)^j \end{pmatrix},$$

we obtain

$$\begin{pmatrix} L(1) \\ M(1) \end{pmatrix} =$$

$$\frac{\mathbf{T}\sum_{j\geq 0}\alpha_{2j}(1-q^2)^j\begin{pmatrix}\frac{x^{2j+2}}{1-x^{2j+1}}\\ \frac{x}{1-x}\end{pmatrix}+\mathbf{T}\sum_{j\geq 0}\alpha_{2j+1}(1-q^2)^j\begin{pmatrix}\frac{x^{2j+3}}{1-x^{2j+2}}+\frac{qx}{1-x}\\ -\frac{qx^{2j+3}}{1-x^{2j+2}}-\frac{x}{1-x}\end{pmatrix}}{1-\sum_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}}. \tag{9.17}$$

We substitute the expressions for $L(1)$ and $M(1)$ into (9.13) to obtain our main result, which is the generating function $F(x, y, q)$ as shown in the next theorem:

Theorem 9.2 *Let*

$$G = 1 + (1-q)\sum_{j\geq 0}(\beta_{2j+1} - \beta_{2j})(1-q^2)^j,$$
$$H = 1 + q(1-q)\sum_{j\geq 0}\beta_{2j+1}(1-q^2)^j.$$

The generating function $F(x, y, q)$ is given by

$$1+\frac{xy}{1-x}+\frac{G\left(\sum_{j\geq 0}(\beta_{2j+1}+\beta_{2j+2})(1-q^2)^j+\frac{qx}{1-x}\sum_{j\geq 0}\alpha_{2j+1}(1-q^2)^j\right)}{1-\sum_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}}$$
$$+\frac{H\left(\frac{x}{1-x}\sum_{j\geq 0}(\alpha_{2j}-\alpha_{2j+1})(1-q^2)^j-q\sum_{j\geq 0}\beta_{2j+2}(1-q^2)^j\right)}{1-\sum_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}},$$

where $\alpha_j = \frac{x^{\binom{j+2}{2}}y^{j+2}}{(1-x)(1-x^2)\cdots(1-x^j)(1-x^{j+2})}$ *and* $\beta_j = \frac{x^{\binom{j+2}{2}}y^{j+1}}{(1-x)(1-x^2)\cdots(1-x^{j+1})}$.

The proof is a matter of simplification of the expression for $L(1) + M(1)$, bearing in mind that $\alpha_j \frac{x^{j+2}}{1-x^{j+1}} = \beta_{j+1}$. This is immediate from the definitions of α_j and β_j.

Applying the above theorem for $q = 1$, we obtain $G = H = 1$ and then

$$\begin{aligned} F(x,y,1) &= 1 + \frac{xy}{1-x} + \frac{1}{1-\beta_0}\left(\beta_1 + \beta_2 + \frac{x}{1-x}\alpha_1\right) \\ &\quad + \frac{1}{1-\beta_0}\left(\frac{x}{1-x}(\alpha_0 - \alpha_1) - \beta_2\right) \\ &= 1 + \frac{xy}{1-x} + \frac{1}{1-\frac{xy}{1-x}}\left(\frac{x^3y^2}{1-x)(1-x^2)} + \frac{x^2y^2}{(1-x)(1-x^2)}\right) \\ &= 1 + \frac{xy}{1-x} + \frac{x^2y^2}{(1-x)(1-x-xy)} = \frac{1-x}{1-x-xy}, \end{aligned}$$

which is well-known, for an example see [5].

We are now in a position to find the generating function for the compositions of n that consist of precisely two alternating blocks, either starting with an ascending or descending block. Since we are only interested in the occurrence of two blocks, only one peak or one valley will occur. Thus we need to find the coefficient of q in the generating function in Theorem 2. Hence

The generating function for compositions of n with two alternating blocks is

$$\frac{x}{1-x}\frac{\sum_{j\geq 0}\beta_{2j+1}\sum_{j\geq 0}\alpha_{2j} + \sum_{j\geq 0}\alpha_{2j+1}\left(1 - \sum_{j\geq 0}\beta_{2j}\right)}{1-\sum_{j\geq 0}(-1)^j\beta_j} - \frac{\sum_{j\geq 0}(\beta_{2j+1} - \beta_{2j})\sum_{j\geq 1}\beta_j + \sum_{j\geq 1}\beta_{2j}}{1-\sum_{j\geq 0}(-1)^j\beta_j}.$$

We leave this proof to the reader, noting that the constant term in $\frac{1}{1-\sum_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}}$ is $\frac{1}{1-\sum_{j\geq 0}(-1)^j\beta_j}$.

9.4 Generating function for the total number of peaks and valleys

In this section, we find the generating function for the total number of peaks and valleys in any composition of n. Let

$$A = \sum_{j\geq 0}(\beta_{2j+1}+\beta_{2j+2})(1-q^2)^j + \frac{qx}{1-x}\sum_{j\geq 0}\alpha_{2j+1}(1-q^2)^j,$$
$$B = \frac{x}{1-x}\sum_{j\geq 0}(\alpha_{2j}-\alpha_{2j+1})(1-q^2)^j - q\sum_{j\geq 0}\beta_{2j+2}(1-q^2)^j.$$

Then Theorem 9.2 gives

$$\frac{d}{dq}F(x,y,q)\mid_{q=1} = \frac{d}{dq}\left(\frac{GA+HB}{1-\sum_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}}\right)_{q=1}.$$

Using the facts that $G\mid_{q=1}= H\mid_{q=1}= 1$, $A\mid_{q=1}= \beta_1+\beta_2+\frac{x}{1-x}\alpha_1$, $B\mid_{q=1}= \frac{x}{1-x}(\alpha_0-\alpha_1)-\beta_2$, $\frac{d}{dq}G\mid_{q=1}= \beta_0-\beta_1$, $\frac{d}{dq}H\mid_{q=1}= -\beta_1$, $\frac{d}{dq}A\mid_{q=1}= -2(\beta_3+\beta_4)+\frac{x}{1-x}(\alpha_1-2\alpha_3)$ and $\frac{d}{dq}B\mid_{q=1}= -\frac{2x}{1-x}(\alpha_2-\alpha_3)-\beta_2+2\beta_4$, we obtain

$$\begin{aligned}&\frac{d}{dq}F(x,y,q)\mid_{q=1}\\ &= \frac{x}{1-x}\frac{\alpha_0(\beta_1+\beta_1\beta_0-2\beta_2)+\alpha_1(1-\beta_0^2)+2\alpha_2(\beta_0-1)}{(1-\beta_0)^2}\\ &- \frac{2\beta_3+\beta_2+\beta_1(2\beta_2-\beta_1)-\beta_0(2\beta_2+2\beta_3+\beta_1^2+\beta_1)+\beta_0^2(\beta_1+\beta_2)}{(1-\beta_0)^2}.\end{aligned} \tag{9.18}$$

Theorem 9.3 *The generating function for the total number of peaks and valleys in any composition of n with m parts is given by*

$$\frac{2x^4y^3}{(1-x^3)(1-x-xy)^2}.$$

Moreover, the total number of peaks and valleys in any compositions of n with m parts is given by

$$2(m-2)\sum_{j=0}^{\lfloor (n-3)/3\rfloor}\binom{n-3-3j}{m-2}.$$

The first part follows immediately from substituting the values of α_j and β_j in (9.18). Direct calculations show

$$\begin{aligned}\frac{2x^4y^3}{(1-x^3)(1-x-xy)^2} &= \sum_{m\geq 2}\frac{2(m-2)x^{m+1}}{(1-x)^{m-1}(1-x^3)}y^m\\ &= \sum_{m\geq 2}\sum_{i,j\geq 0}2(m-2)\binom{m-2+i}{i}x^{3j+i+m+1}y^m,\end{aligned}$$

which implies the total number of peaks and valleys in any compositions of n with m parts is given by $2(m-2)\sum_{j=0}^{\lfloor (n-3)/3\rfloor}\binom{n-3-3j}{m-2}$, as claimed.

Remark: One can easily obtain the asymptotics expression for the mean number of partition blocks per composition of n, as n tends to infinity, to be $\frac{1}{49}+\frac{2n}{7}$. We use the fact that the number of peaks and valleys plus one equals the number of partition blocks. This result matches the result found in [**?**].

Indeed, our techniques obtain more results. For instance, if we denote the generating function for all compositions of n with m parts where the first part is a descent (respectively, non-descent) by $L(x,y,q)$ (respectively, $M(x,y,q)$), then (9.17) gives the following result.

The generating functions $L(x,y,q)\left(1-\sum_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}\right)$ and $M(x,y,q)\left(1-\sum_{j\geq 0}(-1)^j\beta_j(1-q^2)^{\lfloor (j+1)/2\rfloor}\right)$ are given by

$$q\sum_{j\geq 0}\beta_{2j+1}(1-q^2)^j\left(\frac{x}{1-x}\sum_{j\geq 0}(\alpha_{2j}-\alpha_{2j+1})(1-q^2)^j-q\sum_{j\geq 0}\beta_{2j+2}(1-q^2)^j\right)$$
$$+\left(1+\sum_{j\geq 0}(\beta_{2j+1}-\beta_{2j})(1-q^2)^j\right)\times$$
$$\times\left(\sum_{j\geq 0}\beta_{j+1}(1-q^2)^{\lfloor j/2\rfloor}+\frac{qx}{1-x}\sum_{j\geq 0}\alpha_{2j+1}(1-q^2)^j\right),$$

and

$$\left(1-q^2\sum_{j\geq 0}\beta_{2j+1}(1-q^2)^j\right)\times$$
$$\times\left(\frac{x}{1-x}\sum_{j\geq 0}(\alpha_{2j}-\alpha_{2j+1})(1-q^2)^j-q\sum_{j\geq 0}\beta_{2j+2}(1-q^2)^j\right)$$
$$+q\sum_{j\geq 0}(\beta_{2j}-\beta_{2j+1})(1-q^2)^j\times$$
$$\times\left(\sum_{j\geq 0}\beta_{j+1}(1-q^2)^{\lfloor j/2\rfloor}+\frac{qx}{1-x}\sum_{j\geq 0}\alpha_{2j+1}(1-q^2)^j\right),$$

respectively.

9.5 Bibliography for Paper C

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10 Extensions, open questions and further research stemming from papers B and C

The reader may have noticed that in the background Section 7, the preliminary examples quoted there dealt with both alternating and non-alternating concatenations of partitions for an arbitrary composition of n. Yet, in both Papers B and C, we have only dealt with alternating cases.

In a later paper, which is not part of this thesis, [6], this author and collaborators have dealt with a non-alternating concatenation.

Not all cases have been covered and certainly this is an area for further research.

Part of the original motivation for Papers B and C is the development of a generating function for arbitrary compositions so as to account for the number of partitions used, the size of the composition and the number of parts. This was achieved but unfortunately the expressions are very unwieldy. In certain simplified situations where there are precisely two partition blocks, other researchers have developed nice simple generating functions. See, for example, [4, 16]. The question remains open whether or not the theory contained in this paper can be used to obtain a nice expression for the concatenation of three partition blocks in certain precisely defined situations.

11 Some background for Paper D

Stanley's seminal paper [15] set the task of finding generating functions for amongst others, the sets of

(a) TSPPs; and

(b) CSPPs.

These problems were solved by various authors and the effort was described in Bressoud's book [7].

The approach is somewhat indirect and involved an analysis of the theory of alternating sign matrices.

The author set about trying to investigate whether a more direct and simpler approach could be found that would allow these objects, say a TSPP of n, to be viewed as being built up out of the union of simpler basis elements and thereby provide a simpler approach to Stanley's questions.

I didn't succeed in doing this and whether this approach can indeed be successful remains an open problem.

What the author did however manage to explore and unpack was the case of a subclass of both these objects (CSPPs and TSPPs) with self conjugate main-diagonal. In the process of this investigation it became clear that relatively little work had been done on considering the consequence of imposing additional constraints on the general structures.

So what emerged from the investigation was an exploration of the geometrical structure of symmetric plane partitions. The details of this approach are given in Paper D which follows.

Here follows a brief exposition of how this study evolved in the author's mind:

A simple and elegant overview of general partition theory is to be found in the book by Andrews and Ericcson [1].

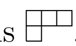
To make this thesis self-contained let me introduce here some of the basic concepts. We start with that of a partition of a positive integer n.

This is a set of positive integers, $n_1 \geq n_2 \geq \cdots \geq n_k$ (each called a part of the partition), such that $n = n_1 + n_2 + \cdots + n_k$. The terminology is extended to allow us to say that $n_1 + n_2 + \cdots + n_k$ is a partition of n.

So for example the partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

There are precisely 5 such partitions. Now we may model a partition using a so-called Young's or Ferrers diagram. This is a left justified array of squares where row i has precisely n_i such squares.

For example the Young's diagram for $2+2$ is  and for $3+1$ is .

The **conjugate** of a given partition is produced by replacing the word "row" in the description of its Ferrers diagram with the word "column". Equivalently, this means the initial Ferrers diagram has been reflected in its main top-left to bottom-right diagonal.

A given partition is **self conjugate** if it is its own conjugate.

Of the partitions of 4 above, only $2+2$ is self conjugate.

Next a plane partition of a positive integer n is an array (or matrix) of positive integers

$$\begin{array}{llll} n_{11} & n_{12} & \cdots & n_{1i_1} \\ n_{21} & n_{12} & \cdots & n_{2i_2} \\ \vdots & & & \\ n_{j1} & n_{j2} & \cdots & n_{ji_j} \end{array} \qquad \text{where } i_1 \geq i_2 \geq \cdots \geq i_j$$

and each row and each column is non-increasing, and where the sum of all integers in the array is n.

This in turn may be modelled by a sequence of Young diagrams, but where the squares are replaced by cubes, and each row of the array is another Young diagram which is aligned with the previous row.

So for example $\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array}$ would be modelled using the following diagram.

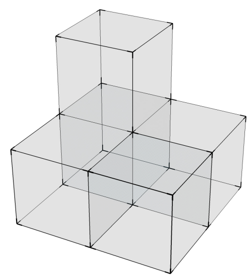


Figure 1: Model of this plane partition.

(in other words, a stack of children's blocks placed in the corner of the room against the planes of the walls and floor).

12 Paper D

12.1 Introduction

Cyclically symmetric and totally symmetric plane partitions (CSPPs and TSPPs, respectively) have been thoroughly investigated (see, e.g., [1],[2],[3], [5] and [7]). However there has been relatively little research done when additional structure is imposed on CSPPs. Here we show that if we require the CSPP to have a self-conjugate diagonal, symmetry (see Definition 5, on next page) and uniqueness of the plane partition are forced.

In Section 4, we show that the generating function (see Theorem 6 below) for all TSPPs with self-conjugate main diagonal is

$$\prod_{i=0}^{\infty} \left(1 + q^{3i^2+3i+1}\right).$$

The method of proof is the embedding of all "smaller" TSPPs in 1-shell TSPPs, concepts which are all defined in this paper.

The "embedding" notion is a (plane partition) generalisation of a procedure for "nesting" self-conjugate ordinary partitions in other self-conjugate ordinary partitions which have a larger number of parts. Because this method is at the heart of what follows, the process for self-conjugate ordinary partitions is used to obtain its well known generating function, i.e.,

$$\prod_{i=0}^{\infty}\left(1+q^{2i+1}\right).$$

In Proposition 5 we specify the generating function for 1-shell TSPPs as

$$1+\sum_{k=1}^{\infty}q^{3k-2}\prod_{i=0}^{k-2}\left(1+q^{6i+3}\right).$$

12.2 Definitions

We will need the following definitions:

[1] An ordinary partition of n is a row of positive integers which is non-increasing from left to right where n is the sum of all entries.

[2] A Ferrers graph is a grid of squares representing an ordinary partition as discussed in [2], page 15.

[3] The conjugate of a partition is that partition obtained when we reflect a Ferrers graph about its main diagonal (in the analogous matrix sense). A partition is self-conjugate if it is its own conjugate.

[4] A plane partition of n is a left justified array of rows of positive integers which are non-increasing along rows and columns and where n is the sum of all entries in the array. (Definitions 4 – 7 are given in [3] and discussed in [8].)

[5] A symmetric plane partition is one in which the array regarded as a matrix is symmetric in the matrix sense.

[6] A cyclically symmetric plane partition (CSSP) is one in which the i-th column is the conjugate (as an ordinary partition) of the i-th row.

[7] A totally symmetric plane partition (TSPP) is a plane partition which is both symmetric and cyclically symmetric.

[8] The main diagonal of a TSPP is the ordinary partition constituted by the main diagonal of the array in the matrix sense.

[9] A TSPP with self-conjugate main diagonal is called a conjugate TSPP.

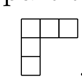
12.3 Preliminary example as a model for later proofs

We first consider self-conjugate ordinary partitions with largest part k (and therefore number of parts also k). Let the generating function for such partitions be $\alpha(k)$. We define $\alpha(0) = 1$.

For $k = 1$, the only partition is 1 and the generating function is $\alpha(1) = q$.

For $k = 2$, we nest all self-conjugate partitions with largest part < 2 in the "arms" of the partition representing the plane partition $\begin{matrix}2\\1\end{matrix}$; in other words, we consider its Ferrers diagram,  , and place any smaller (i.e., having a smaller largest part) self-conjugate partition whose generating function is a term from $\alpha(1)$ below the largest part and to the right of the other parts. The nested part is represented by $\alpha(0) + \alpha(1)$.
So

$$\alpha(2) = q^3\big(\alpha(0) + \alpha(1)\big).$$

For $k = 3$, we nest in the first row and column (the "arms") of the partition represented by the Ferrers diagram for the plane partition $\begin{matrix}3\\1\\1\end{matrix}$, i.e.,  .

The generating function for the "arms" is $q^5 = q^{2\cdot 3-1}$.
Hence $\alpha(3) = q^5\big(\alpha(0) + \alpha(1) + \alpha(2)\big)$.

Repeating this process, we obtain

$$\alpha(k+1) = q^{2k+1}\big(\alpha(0) + \alpha(1) + \cdots + \alpha(k)\big).$$

Applying this formula again to $\alpha(k)$, we obtain

$$\alpha(k+1) = q^{2k+1}(1 + q^{2k-1})\big(\alpha(0) + \alpha(1) + \cdots + \alpha(k-1)\big).$$

Doing this recursively to the highest α value each time, we obtain

$$\begin{aligned}\alpha(k+1) &= q^{2k+1}(1+q^{2k-1})(1+q^{2k-3})\dots(1+q) \\ &= q^{2k+1}\prod_{i=1}^{k}(1+q^{2i-1}),\end{aligned}$$

where the product $\prod_{i=1}^{k}(1+q^{2i-1})$ represents the generating function for all self-conjugate partitions with largest part $\leq k$.

Letting $k \to \infty$, we obtain $\prod_{i=1}^{\infty}(1+q^{2i-1})$, which is the well known generating function for self-conjugate partitions.

12.4 Lemmas, Algorithm and Theorems

We shall use an adaptation of this nesting or embedding procedure to produce a generating function for conjugate TSPPs.

We begin with the following lemma:

Any CSPP with self-conjugate main diagonal and largest part k is of the form

$$\begin{matrix} k & \cdots & k \\ \vdots & \ddots & \vdots \\ k & \cdots & p \end{matrix}$$

where the element p in the (k,k)th position of the array is non-zero.

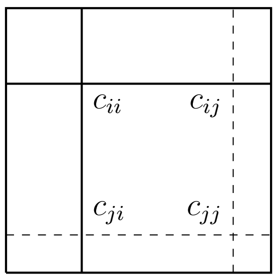
Let P be a CSPP with self-conjugate main diagonal and largest part k. Being self-conjugate means that the main diagonal has k parts and hence its k^{th} part, the $(k,k)^{\text{th}}$ element of the array, is non-zero. The non-increasing condition (in the definition of a plane partition) implies that the last row and last column each have k non-zero parts. Each of these, being the conjugate of the other, have k as first element. So the first and last element of both the first row and column is k. Hence every element of the first row and column is k (by the non-increasing requirement in the plane partition definition).

In the above lemma, we have used a special case of the more general fact that if $\pi = \pi_1\pi_2\dots\pi_k$ and $\pi^c = \pi_1^c\pi_2^c\dots\pi_n^c$ are conjugate partitions, then $\pi_i \geq s \implies \pi_s^c \geq i$. For an explanation of this, see the expression for a partition and its conjugate in [6].

Any CSPP with self-conjugate main diagonal is a TSPP.

Consider an arbitrary CSPP $= (c_{ij})$ with largest part n and self-conjugate main diagonal. We shall show that it is symmetric: i.e., we will show that for all $1 \le i \le n$ and $1 \le j \le n$, $c_{ij} = c_{ji}$.

To obtain a contradiction, suppose that for some $i < j$, $c_{ij} \ne c_{ji}$. Without loss of generality, we may suppose further that j is the largest such value for that particular i. Let $a = \max\{c_{ij}, c_{ji}\}$ and $b = \min\{c_{ij}, c_{ji}\}$.



We will consider two cases.

(i) $b < i$;

(ii) $b \ge i$.

(i) By the defining property for CSPPs, each row is the conjugate of the corresponding column and therefore the ith row has precisely c_{ji} initial elements $\ge j$ and the ith column has precisely c_{ij} initial elements $\ge j$ (statement 1). (See preceding remark.) By the non-increasing property of plane partitions, the main diagonal has precisely a initial elements $\ge j$. Since the main diagonal is also self-conjugate, $c_{jj} = a$. Again relative conjugacy of rows to columns implies that both the jth row and column have precisely a initial entries $\ge j$ (statement 2).

But by statement (1), either row i or column i has only b (initial) elements $\ge j$. Again by the non-increasing property of rows and columns, either row j or column j has at most b elements $\ge j$. This is a contradiction of statement (2) and establishes that $c_{ij} = c_{ji}$ in this case.

(ii) $b \ge i$ implies $c_{ij}, c_{ji} \ge b \ge i$ which, by the previous remark, implies that at least b initial elements in the ith row and column $\ge j$. By the non-increasing property of plane partitions, $c_{ii} \ge j$, and by self-conjugacy of the main diagonal, $c_{jj} \geq i$. Again, by the previous remark and conjugacy of the jth row and jth column, there are at least i initial elements in this row and column $\geq j$. This forces $b = \min\{c_{ij}, c_{ji}\} \geq j$. Hence $a > j$.

Now if $j = n$, then $a > n$ which contradicts the largest element in the partition being n.

Otherwise there are two options. We have either $c_{ir} \neq c_{ri}$ for some $r \geq j+1$, which is impossible since j was chosen as large as possible, or $c_{ir} = c_{ri}$ for all r, $j+1 \leq r \leq n$. So let us assume the latter. Note that if $\max\{c_{is}, c_{si}\} \geq j+1$ for some s, then $c_{is} = c_{si}$. To see this, observe that the conjugacy (see statement 1 above) and the fact $c_{ir} = c_{ri}$ if $r \geq j+1$ together imply that the number of occurrences of each letter t, where $t \geq j+1$, is the same in the ith row as in the ith column, whence $c_{is} = c_{si}$ since $\max\{c_{is}, c_{si}\} \geq j+1$. But then we have $a = \max\{c_{ij}, c_{ji}\} \geq j+1$ with $c_{ij} \neq c_{ji}$, which is impossible. Hence $c_{ij} = c_{ji}$ in this case also.

The converse of this theorem is false, i.e., there exist TSPPs which are not conjugate (see the example immediately following the example of how the algorithm is applied).

Unless otherwise stated, for the rest of the discussion we will assume that when we refer to any TSPP it will be a conjugate TSPP.

We define the <u>shell</u> of a conjugate TSPP as the first row and first column together with 1's in every other position of the square array.
For example, the shell of

$$\begin{matrix} 4 & 4 & 4 & 4 \\ 4 & 3 & 3 & 1 \\ 4 & 3 & 3 & 1 \\ 4 & 1 & 1 & 1 \end{matrix} \quad \text{is} \quad \begin{matrix} 4 & 4 & 4 & 4 \\ 4 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{matrix}.$$

We shall also call any TSPP in which all elements other than those in the first row and column are 1, a shell, without it being necessary to refer it to some other conjugate TSPP.

A shell is a conjugate TSPP.

For conjugate TSPPs P and Q, where P is a shell, Q is said to be smaller than P iff the largest part of $Q <$ largest part of P.

If such Q is smaller than a shell P, we define the embedding of Q in P as the plane partition with array whose (i, j) entry is:

$$\begin{array}{ll} P_{i,j} & \text{if } i = 1 \text{ or } j = 1 \text{ or } Q_{i-1,j-1} \text{ is not defined} \\ P_{i,j} + Q_{i-1,j-1} & \text{otherwise.} \end{array}$$

For example, the embedding of $\begin{array}{cc} 2 & 2 \\ 2 & 1 \end{array}$ in

$$\begin{array}{cccc} 4 & 4 & 4 & 4 \\ 4 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{array} \quad \text{is} \quad \begin{array}{cccc} 4 & 4 & 4 & 4 \\ 4 & 1+2 & 1+2 & 1 \\ 4 & 1+2 & 1+1 & 1 \\ 4 & 1 & 1 & 1 \end{array}.$$

The embedding of a conjugate TSPP Q in a larger TSPP shell P is also a conjugate TSPP and, conversely, any conjugate TSPP with largest part k ($k \geq 2$) can be obtained by embedding a smaller conjugate TSPP in the shell with all first row and column entries $= k$ and all other entries $= 1$.

The first row and column of the embedding inherit the self-conjugate and symmetric property from P.

Any other row or column or indeed the main diagonal of the embedding consists of a nesting of a smaller self-conjugate ordinary partition in the "arms" of a larger self-conjugate partition (as per the process described at the start of this paper) and therefore result in that row or column being self-conjugate. The symmetry is obviously inherited by the embedding from the symmetry of P and Q. Also the non-increasing property of the embedding follows from the non-increasing property of P and Q as well as the fact that Q is smaller than P.

Conversely, we may obtain the conjugate TSPP $P = (\alpha_{i,j})$, where $i, j \in \{1, \ldots, k\}$, by embedding conjugate $Q = (\beta_{i,j})$, $i, j \in \{1, \ldots, k-1\}$, in the shell

$$\begin{array}{cccc} \alpha_{11} & & \ldots & \alpha_{1k} \\ & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \\ \alpha_{k1} & 1 & & 1 \end{array} \quad \text{where } \beta_{i,j} = \alpha_{i+1,j+1} - 1.$$

Q is again a conjugate TSPP since all its rows are obtained by the reverse procedure of the nesting process described in Section 3.

These two lemmas lead to the following corollary.

Any conjugate TSPP is uniquely defined by the diagonal.

The proof of the lemma above provides an algorithm for recovering the unique conjugate TSPP from its main diagonal:

Example Let the TSPP have self-conjugate main diagonal 6 5 3 2 2 1.

This would be obtained (by the procedure already described) by nesting 4 2 1 1 in the arms of 6 1 1 1 1 1.

4 2 1 1 is in turn obtained by nesting 1 in the arms of 4 1 1 1.

The associated embedding procedure described above is to embed

$$1 \quad \text{in} \quad \begin{matrix} 4 & 4 & 4 & 4 \\ 4 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{matrix}$$

to obtain

$$\begin{matrix} 4 & 4 & 4 & 4 \\ 4 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{matrix},$$

and in turn to embed this in

$$\begin{matrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 1 & 1 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

to obtain

$$\begin{matrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 5 & 5 & 5 & 5 & 1 \\ 6 & 5 & 3 & 2 & 2 & 1 \\ 6 & 5 & 2 & 2 & 2 & 1 \\ 6 & 5 & 2 & 2 & 2 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 \end{matrix}.$$

This is the unique conjugate TSPP with the given main diagonal.

Example As the following examples show, an ordinary partition which is not self-conjugate may be the main diagonal of different TSPPs, e.g.,

$$\begin{array}{ccc} 3 & 2 & 1 \\ 2 & 2 & \\ 1 & & \end{array} \quad \text{and} \quad \begin{array}{ccc} 3 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & \end{array}$$

As an aside, we may extend the embedding process to the case of TSPPs which do not necessarily have a self-conjugate main diagonal:

Define the shell of TSPP P as the array with (i, j)th entry

$$\begin{array}{cc} \text{same as } P & \text{if } i = 1 \text{ or } j = 1 \\ 1 & \text{if } i \neq 1 \text{ and } j \neq 1 \text{ and } P_{ij} \geq 1 \\ \text{non-existent} & \text{otherwise} \end{array} .$$

Such a shell is also a TSPP and identical with the previous definition (when the main diagonal is self-conjugate).

For example, the shell of

$$\begin{array}{cccc} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & & \\ 2 & 2 & & \end{array} \quad \text{is} \quad \begin{array}{cccc} 4 & 4 & 2 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 1 & & \\ 2 & 1 & & \end{array} .$$

We say that a TSPP Q is "smaller" than a shell P if for each non-zero element in row 1 of Q, $Q_{i,j} + 1 \leq P_{i,j+1}$ (where $i = 1$). This also coincides with "smaller" in the case of self-conjugate main diagonals.

We may use the same definition as for TSPPs with self-conjugate main diagonal to obtain the embedding of a smaller TSPP Q in a shell P. We can easily show that such an embedding is also a TSPP.

For example the embedding of the smaller TSPP

$$\begin{array}{cc} 2 & 1 \\ 1 & \end{array} \quad \text{in the shell} \quad \begin{array}{cccc} 4 & 4 & 2 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 1 & & \\ 2 & 1 & & \end{array} \quad \text{is} \quad \begin{array}{cccc} 4 & 4 & 2 & 2 \\ 4 & 1+2 & 1+1 & 1 \\ 2 & 1+1 & & \\ 2 & 1 & & \end{array}$$

and the embedding of the smaller

$$\begin{matrix} 2 & 1 \\ 1 & \end{matrix} \quad \text{in the shell} \quad \begin{matrix} 3 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{matrix} \quad \text{is} \quad \begin{matrix} 3 & 3 & 3 \\ 3 & 3 & 2 \\ 3 & 2 & 1 \end{matrix}.$$

It is interesting that the latter example yields a TSPP which does not have a self-conjugate main diagonal. (This example proves the remark following Lemma 2.)

Consistent with the definition of shells above, let us call a TSPP a 1-shell if it has any self-conjugate 1st row/column (as an ordinary partition) and all other entries are 1, e.g.:

$$1, \qquad \begin{matrix} 2 & 1 \\ 1 & \end{matrix}, \qquad \begin{matrix} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{matrix}, \qquad \begin{matrix} 4 & 4 & 2 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 1 & & \\ 2 & 1 & & \end{matrix}.$$

(The justification for this name is that, roughly speaking, such TSPPs have an "exterior" of thickness 1.)

We develop a generating function for all 1-shell TSPPs as follows:

First notice that 1-shell TSPPs are completely determined by their first row for which the generating function is the generating function for all self-conjugate ordinary partitions.

Let the largest part of the 1-shell TSPP be k $(k \geq 2)$. We can regard the first row and first column as copies of the same self-conjugate ordinary partition nested, respectively, in the row or column "arms" of

$$\begin{matrix} k & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & & \\ 1 & & & \end{matrix} \tag{12.1}$$

and the generating function for (1) is

$$q^{k+(k-1)+(k-1)} = q^{3k-2}. \tag{12.2}$$

For example $\begin{matrix} 2 & 1 \end{matrix}$ can be nested in

$$\begin{matrix} 4 & 1 & 1 & 1 \\ 1 & & & \\ 1 & & & \\ 1 & & & \end{matrix} \quad \text{as} \quad \begin{matrix} 4 & 1+2 & 1+1 & 1 \\ 1+2 & & & \\ 1+1 & & & \\ 1 & & & \end{matrix} .$$

Now all the 1's in the i, j position (where $i \neq 1$ and $j \neq 1$) constitute a 3rd copy of the already (doubly nested) self-conjugate partitions in the first row or column.

The generating function for (one copy of) self-conjugate partitions with largest part $\leq k-1$ is $\prod_{i=0}^{k-2} \left(1 + q^{2i+1}\right)$.

Hence for three copies it is

$$\prod_{i=0}^{k-2} \left(1 + (q^3)^{2i+1}\right) = \prod_{i=0}^{k-2} \left(1 + q^{6i+3}\right). \tag{12.3}$$

Let $\gamma(k)$ be the generating function for all 1-shell TSPPs with largest part k and define $\gamma(0) = 1$. Also $\gamma(1) = q$ because 1 is the only such partition of 1. Now the generating function for all 1-shell TSPPs with largest part k ($k \geq 2$) is obtained by combining (2) and (3). Hence

$$\gamma(k) = q^{3k-2} \prod_{i=0}^{k-2} \left(1 + q^{6i+3}\right) \quad (k \geq 1).$$

So we have proved the following proposition:

The generating function for all 1-shell TSPPs is

$$1 + \sum_{k=1}^{\infty} q^{3k-2} \prod_{i=0}^{k-2} \left(1 + q^{6i+3}\right).$$

The first few terms of this generating function are :

$$1 + q + q^4 + 2q^7 + 2q^{10} + 2q^{13} + 3q^{16} + 4q^{19} + 4q^{22} + 5q^{25} + 6q^{28}.$$

We can naturally extend the concept of a 1-shell TSPP to that of a 2-shell TSPP (by embedding a smaller 1-shell TSPP in another 1-shell TSPP). And by an inductive process we may, if we wish, define an n-shell TSPP. Finally, by analogy with the concept of a Durfee square (see, e.g., [2]), we may define a Durfee "cube" for plane partitions as the largest cube which fits entirely inside the plane partition. The length of a side of this cube, n say, is easily seen to coincide with the largest n-shell contained entirely inside the given plane partition.

However the generating function for 2-shell TSPPs is not easy to obtain.

To return to the main topic, we may now use the two lemmas to develop the generating function for all conjugate TSPPs.

Let the generating function for such partitions with largest part k be $\gamma(k)$.
Define $\gamma(0) = 1$.
For $k = 1$, the only TSPP is 1 and so $\gamma(1) = q$.
For $k = 2$, we embed the smaller partitions with generating functions $\gamma(0)$ and $\gamma(1)$ respectively in the shell $\begin{matrix} 2 & 2 \\ 2 & 1 \end{matrix}$.

(This produces $\begin{matrix} 2 & 2 \\ 2 & 1 \end{matrix}$ in the case of the empty partition and $\begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix}$ in the case of $\gamma(1)$).

By Lemma 1, Lemma 2 and the nesting process for all self-conjugate partitions described at the beginning of this paper, this yields all TSPPs with self-conjugate main diagonal and largest part 2.

Hence

$$\gamma(2) = (q^2)^{2.2-1} \cdot q^{1^2}\big(\gamma(0) + \gamma(1)\big).$$

For $k = 3$, we embed the partitions with generating functions $\gamma(0)$, $\gamma(1)$ and $\gamma(2)$ in the shell $\begin{matrix} 3 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{matrix}$, and so the same argument yields

$$\gamma(3) = (q^3)^{2.3-1} \cdot q^{2^2}\big(\gamma(0) + \gamma(1) + \gamma(2)\big).$$

And in general :

$$\begin{aligned} \gamma(k+1) &= (q^{k+1})^{2(k+1)-1} \cdot q^{k^2}\big(\gamma(0) + \gamma(1) + \cdots + \gamma(k)\big) \\ &= q^{3k^2+3k+1}\big(\gamma(0) + \gamma(1) + \cdots + \gamma(k)\big). \end{aligned}$$

As we did in the case of nested self-conjugate ordinary partitions, we apply this formula recursively to itself to obtain

$$\gamma(k+1) = q^{3k^2+3k+1} \prod_{i=0}^{k-1} \left(1 + q^{3i^2+3i+1}\right),$$

where the product

$$\prod_{i=0}^{k-1} \left(1 + q^{3i^2+3i+1}\right)$$

represents the generating function for all conjugate TSPPs with largest part $\leq k$.

By letting $k \to \infty$, we have proved the following:

Theorem 12.1 *The generating function for all conjugate TSPPs is*

$$\prod_{i=0}^{\infty} \left(1 + q^{3i^2+3i+1}\right).$$

The first few terms of this expansion are

$$1 + q + q^7 + q^8 + q^{19} + q^{20} + q^{26} + q^{27} + q^{37} + q^{38} + q^{44} + q^{45} \\ + q^{56} + q^{57} + q^{61} + q^{62} + q^{63} + q^{64} + \cdots .$$

It appears that there are many gaps in this expansion, but in fact $0q^{860}$ is the last term with a 0 coefficient. This is shown in [4].

12.5 Bibliography for Paper D

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13 Extensions and further research

The method of this paper has been to define a kind of "basis" element for conjugate TSPPs. These are the shells which were defined immediately following Lemma 6.5.2. Any conjugate TSPP is then uniquely generated by a "sum" of these shells (as explained in the example following on Corollary 6.5.4).

Before producing this paper, the author had a more ambitious idea, namely could an arbitrary TSPP (not necessarily conjugate) be represented as a certain "sum" of shells? This in fact was the motivation for defining the more general idea of a TSPP n-shell which is contained in the two remarks immediately following the example referred to in the previous paragraph.

The idea was to take an arbitrary TSPP and express it as a unique "sum" of the 1-shells. I have not been able to solve how to carry out that step in this thesis and it remains an open question. If it can be done it may (I believe) allow a generating function to be found for all TSPPs. Of course this has already been done but it might be simpler than the method employing the use of the theory of alternating sign matrices that was so eloquently described in [7], below.

Moreover, the same type of generalisation may also work in obtaining a generating function for CSPPs. This, however, is another open question.

14 Conclusions of the thesis

One of the (understated) aims of this thesis has been to provide an informed lay person with the background required to understand ongoing research in the area of generating functions applied to compositions, integer partitions and plane partitions.

To do this, it has been neccessary to give additional background to each of the four research papers contained in the thesis. This has been done in Sections 4, 5, 7, 9, 12.

Besides this, the author has also attempted in the main body of work to use generating functions to expand understanding of the relationship between certain central areas of current research in the field of integer composition theory as well as those fields summarized by the title of the thesis.

More specifically:

1. As per the concept of the Durfee square for integer partition (which was extensively discussed in Paper A), a generalization to that of a Durfee cube for plane partitions was suggested in Paper D. This analogy was left undeveloped but may be a fruitful area for further research.

2. A generalized framework for embedding integer partition theory in the field of integer compositions was sketched in Papers B and C. This too appears to be amenable to further investigation.

3. Finally, Paper D has investigated adding certain restrictions or conditions to certain classes of plane partition. This has proved fruitful in laying bare some unexpected relationships (e.g. between cyclic and total symmetry) for plane partitions. The author believes that further investigation of the effect of other constraints would yield useful results.

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