

The chromatic polynomials of the q -analogue of certain graph operations

by

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.



(Signature of candidate)

21 day of September 20 18 in University of Witwatersrand.

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Abstract

In this dissertation, we start by giving basic definitions in graph theory and some classes of graphs which are relevant in this work. We then introduce the chromatic polynomial of a graph which is the main focus of this dissertation. In addition, we discuss some properties of a graph, and the chromatic polynomials which are relevant to this work. In particular, explicit expression of chromatic polynomials of certain classes of graphs are presented. Certain graph operations and the chromatic polynomials of the resultant graphs are discussed. Finally, we introduce the 2-vertex join of a graph and we study its chromatic polynomial, in particular we give the explicit expression of the flow polynomial of the 2-vertex join of a path P_n .

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Chapter 1

Introduction

1.1 Literature review

The four color problem is considered as the origin of graph theory. This problem has been fascinating for over a century and a half to mathematicians and non-mathematicians alike. There is a huge volume of published literature on the history of the conjecture and attempted proofs. Until now, there are still outstanding issues on the four color problem which are not yet resolved, as some mathematicians still do not agree that it has been proved correctly. Several methods have been introduced over the years, trying to solve the four color problem, with the hope of finding a solution of this well-known problem.

The four color problem was introduced by Francis Guthrie in 1852 while trying to color the map of counties of England. In particular, Francis observed that the four colors seemed sufficient to color most maps, but he became curious and wondered if this condition was true for all maps. Francis posed this question as a conjecture and tried to prove it. His attempts to prove the conjecture did not give any promising answers. At this point, Francis passed the conjecture to his brother, Frederick Guthrie, to try and prove it. Just like his brother, Frederick failed to prove the conjecture. Frederick Guthrie then asked his supervisor, Professor Augustus De Morgan, for help to solve

the problem.

Professor Augustus De Morgan was the first professor of mathematics at the new University College London then. He studied the four color problem and declared that the problem was new and interestingly difficult. De Morgan then posed the four color problem to the mathematical community. In 1852, the four color conjecture appeared for the first time in a letter written by Professor Augustus De Morgan to his friend William Rowan Hamilton, a famous Irish mathematician. This letter led to several attempts in proving the conjecture, but no solution was found, see [20].

In 1878, Arthur Cayley, was the first mathematician to publish the problem as a puzzle for the public at a meeting of the London Mathematical Society. Cayley asked at the meeting if anyone has a solution for Professor Augustus De Morgan's original question. This revived interest in the original question and attracted mathematicians to look for a solution, but there was no significant breakthrough which came out of it. In an attempt to prove the conjecture, Cayley published a short paper titled, *On the coloring of maps*, in American Journal of Mathematics Pure and Applied. In this paper, Cayley discussed some of the hurdles encountered in attempting to solve the four color problem and suggested some possible way of approaching the problem. This approach led to the application of mathematical induction on the four color problem [5].

In 1879, Alfred Bray Kempe, a student of Cayley, a lawyer by training and a member of the London Mathematical Society, revived interest when he developed a procedure called the Method of *Kempe Chains*, to find a proof of the four color theorem. In July 1879, Kempe announced through the British Journal Nature to the mathematical world that he had solved the four color problem. Kempe's complete proof was however published in the 1879, Volume 2 of America Journal of Mathematics, see [9]. Kempe also published two more proofs which stood for ten years. He was honored for this work and elected as Fellow of Royal Society in 1881.

In 1880, Peter G. Tait, a mathematical physicist, established an equivalent formu-

lation of the four color problem in regards to the three-edge coloring, see [15]. This approach was believed to have offered a solution to the four color problem although a similar proof had appeared earlier in Kempe's work.

In 1889, Percy J. Heawood, a lecturer at Durham England, saw a defect in Kempe's proof and revived interest in the four color conjecture by publishing a paper titled *Map coloring theorem*, see [2]. In this paper, Heawood, discussed that a map can be colored with five colors. In 1891, Julius Peterson, declared Heawood's work incomplete and went ahead to give a proof for the four-color conjecture, see [4, 7]. Henceforth, there was a lot of interest in the four color conjecture. For example, Lewis Carroll, author of the famous children's story *Alice in Wonderland* developed a game for two players where a player can design a map in four colors for his opponent. In 1889, Frederick Temple, published his own solution of the four color problem in the Journal of Education.

Notably, of much interest to this work, in 1912, an American mathematician famously credited for the ergodic theorem, George Birkhoff attempted to prove the four color conjecture and introduced a polynomial, now known as the chromatic polynomial. Birkhoff's contribution helped Franklin in 1922 to come up with a proof for the four color conjecture. Franklin concluded that the conjecture is true for maps with at most twenty five regions, see [3]. This development allowed both mathematicians and non mathematicians to make huge progress on the proof of the four color conjecture. In 1932, a doctoral student of George Birkhoff, Hassler Whitney, translated some results on the chromatic polynomials from maps to graphs. Hassler introduced a *skein relation* for the chromatic polynomial, see [18, 17]. Sadly, the results of Whitney could not be used to prove the four color conjecture. In 1936, Heinrich Heesch, from the University of Hannover, was the first mathematician after Kempe to develop a modern computer based proof method, as a way forward to prove the four color conjecture, see [1]. Mathematicians made errors and flaws for many years. In 1976, Appel and Haken, developed a proof for the four color conjecture. This method is based on

reducibility using Kempe chains. This proof is still standing as correct until someone prove that it is not correct. This proof is very long, uses a computer programme and takes up a whole book, see [1].

Although, chromatic polynomials were initially introduced by Birkhoff in 1912, there was not much interest in the topic. In 1968, C. Read, wrote a survey paper on the chromatic polynomial of graphs and posed a few questions, see [11]. This survey paper enjoys the status of being the activator of interest in this now very active research area of graph theory, the chromatic polynomial. From then till now, researchers have explored different areas of the chromatic polynomial. Just to mention a few, some areas explored are coefficients of chromatic polynomials, roots of chromatic polynomials, chromatic equivalence, the chromatic factorization and certificates etc. Chromatic polynomials have also found applications in other sciences. For example in physics, there is a connection to the Potts model, see [6].

It has been determined that computing the chromatic polynomial of a graph is equivalent to determining the smallest number of colors required to color that graph which is known to be NP-complete, see [8, 13]. As a result, any process which can reveal information about the chromatic polynomial of a graph, without requiring computation is of interest, for example finding explicit expression of the chromatic polynomial for classes of graphs which is our objective in this dissertation.

1.2 Overview of dissertation

In Chapter 1, we give some definitions and properties of graphs which are relevant to this dissertation.

In Chapter 2, we discuss some basic properties of the chromatic polynomial of a graph and different methods for computing the chromatic polynomial of a graph. We then demonstrate how to compute chromatic polynomials for some special classes of graphs. Finally, we introduce different forms of representing the chromatic polynomial

of a graph.

In Chapter 3, we create new graphs through some graph operations and give the chromatic polynomial of the created graphs in terms of the chromatic polynomial of the original graphs, respectively.

In Chapter 4, we study some q -analogues of some graph operations and we investigate if we can find the chromatic polynomial of the q -analogue graph in terms of the original graph.

1.3 Useful definitions

In this section, we discuss some useful definitions and notions in graph theory that will be relevant in this dissertation. We will give a few examples to clarify some of the definitions. We refer the reader to [19], for basic concepts in graph theory unless stated otherwise.

Definition 1.3.1. A graph $G = (V(G), E(G))$ is a non-empty finite set $V(G)$ of vertices and a set $E(G)$ of unordered pairs of vertices from $V(G)$ called edges. That is if the vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$ then the set of edges, $E(G) = \{(w_i, w_j) | i, j \in \{1, 2, \dots, n\}\}$. The order of the graph G is the number of vertices of G , $|V(G)|$. The size of the graph G is the number of edges of G , $|E(G)|$.

In this dissertation, a graph G will be defined interchangeably using just G or $G = (V(G), E(G))$ when necessary.

Definition 1.3.2. Let $G = (V(G), E(G))$ be a graph. The graph $H = (V(H), E(H))$ is a subgraph of the graph G if $E(H) \subseteq E(G)$ and $V(H) \subseteq V(G)$.

For all definitions in this section, we consider a graph $G = (V(G), E(G))$ with vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$ and edge set $E(G) = \{(w_i, w_j) | i, j \in \{1, 2, \dots, n\}\}$.

Definition 1.3.3. Let $e = (w_i, w_j)$ be an edge of a graph G . Then the vertices w_i and w_j are said to be adjacent in G . In addition, the edge e is said to be incident to the vertices w_i and w_j .

Definition 1.3.4. The degree of a vertex $w_i \in V(G)$ is the number of times w_i appears in the unordered pairs in $E(G)$.

In the literature, the vertex degree is also known as the local degree or valency. We denote the degree of vertex w_i by $d(w_i)$ and $d(w_i) \geq 0 \in \mathbb{Z}^+$. If we arrange all the degrees of vertices in G , in descending order, we say that we have a degree sequence of G . The maximum degree in G is denoted by $\Delta(G) = \max\{deg(w_i)|w_i \in V(G)\}$ and the minimum degree of G is denoted by $\delta(G) = \min\{deg(w_i)|w_i \in V(G)\}$.

Definition 1.3.5. Let $e = (w_i, w_j)$ be an edge of G . If there exists another edge in $E(G)$ say, edge $e_1 = (w_i, w_j)$, then edges e and e_1 are said to be parallel edges. A set of parallel edges is called a parallel class. If a graph G has a parallel class, then G is called a multigraph.

Definition 1.3.6. An edge $e = (w_i, w_i) \in E(G)$ for some $i \in \{1, 2, \dots, n\}$ is called a loop.

Definition 1.3.7. A graph $G = (V(G), E(G))$ without any parallel class or any loop is called a simple graph. The simplification of a multigraph G is identifying each parallel class as a single edge. The graph obtained by simplification of a multigraph is called the simple graph of G and we denote it by \tilde{G} .

We clarify some of the defined concepts using a graph G shown in Figure 1.1. The vertex set of G , $V(G) = \{w_1, w_2, \dots, w_5\}$ and the edge set of G , $E(G) = \{e_1, e_2, \dots, e_7\}$.

Alternatively, the edge set

$E(G) = \{(w_1, w_1), (w_1, w_2), (w_2, w_3), (w_3, w_4), (w_2, w_5), (w_1, w_4), (w_2, w_3)\}$. The degree of vertex w_2 is 4, the edges e_3 and e_7 are parallel, the edge e_1 is a loop and the edge e_5 is a bridge.

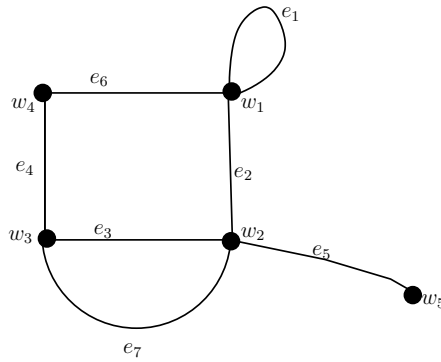


Figure 1.1: Graph G

Definition 1.3.8. Let $G = (V(G), E(G))$ be a graph. An alternating sequence of adjacent vertices and edges, $w_1, e_1, w_2, e_2, w_3, e_3, \dots, e_{n-1}, w_n$ is called a walk in the graph G . Thus the walk begins at vertex w_1 and ends at vertex w_n , and vertices w_{i-1} and w_i are incident with the edge e_{i-1} where $i \in \{2, 3, \dots, n\}$.

The length of a walk is the number of edges contained in the walk. A walk in a graph G is closed if $w_1 = w_n$.

Definition 1.3.9. Let $G = (V(G), E(G))$ be a graph. A trail in G is a walk with the property that no edge is repeated. A closed trail is called a cycle.

Definition 1.3.10. A path in G is a walk with the property that no edge and no vertex is repeated.

Definition 1.3.11. The distance between two vertices w_i and w_j of a graph G is the length of the shortest path between the two vertices.

We use the graph G shown in the diagram of Figure 1.2 to illustrate four concepts walk, cycle, path and trail. In a graph G

a sequence $w_1, e_1, w_2, e_2, w_3, e_3, w_3, e_4, w_4, e_5, w_6, e_6, w_7, e_7, w_5$ is a walk,

a sequence $w_1, e_1, w_2, e_2, w_3, e_3, w_4, e_4, w_5, e_5, w_6, e_6, w_7$ is a path,

a sequence $w_5, e_5, w_6, e_6, w_7, e_7, w_5$ is a cycle, and

a sequence $w_1, e_8, w_7, e_7, w_5, w_4$ is a trail.

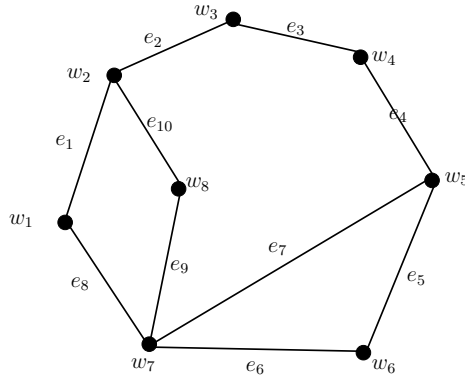


Figure 1.2: Graph G

Definition 1.3.12. Let $G = (V(G), E(G))$ be a graph. G is connected if there exists a walk between every pair of vertices w_1 and w_2 in G , otherwise the graph is disconnected.

Definition 1.3.13. Let $G = (V(G), E(G))$ be a graph such that $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_m)$, $V(G) = V(G_1) \cup V(G_2) \cup \dots \cup V(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ and $V(G_i) \cap V(G_j) = \emptyset$ for $0 \leq i \leq m$. If the graph $G_i = (V(G_i), E(G_i))$ is connected, then G_i is called a component of the graph G . We denote the number of components of G by $c(G)$.

Definition 1.3.14. Let $G = (V(G), E(G))$ be a connected graph and an edge $e \in E(G)$. An edge e is called a bridge in the graph G if its removal from $E(G)$ disconnects the graph G .

The graph in Figure 1.3 is an example of a disconnected graph with two components. The graph $G = G_1 \cup G_2$; the subgraphs G_1 and G_2 are the components of G . However if you consider G_1 as a graph on its own, then G_1 is a connected graph.

Definition 1.3.15. Isomorphism between two simple graphs G and H is a bijection between the vertex sets, $\psi : V(G) \rightarrow V(H)$, such that vertex w_1 is adjacent to vertex w_2 in the graph G iff $\psi(w_1)$ is adjacent to $\psi(w_2)$ in the graph H . Furthermore, there

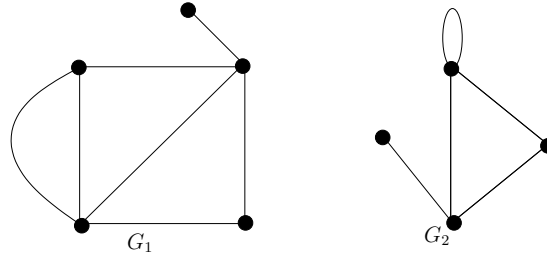


Figure 1.3: Graph $G = G_1 \cup G_2$

is an edge bijection $\phi : E(G) \rightarrow E(H)$ such that $(w_1, w_2) \rightarrow (\phi(w_1), \phi(w_2))$. G is isomorphic to H is denoted by $G \cong H$.

The following proposition is a direct consequence of Definition 1.3.15.

Proposition 1.3.16. *Let G and H be isomorphic graphs. Then*

(i) $|V(G)| = |V(H)|$.

(ii) $|E(G)| = |E(H)|$.

(iii) *the degree sequences of G and H are equal.*

1.4 Classes of graphs

In this section, we define some well known classes of graphs and state a few properties for some of the classes of graphs which are relevant to this dissertation.

Definition 1.4.1. A graph G of order n is a null graph if the edge set, $E(G)$, is empty. The null graph on n vertices is denoted by N_n .

Definition 1.4.2. A graph G is a tree if it is connected and does not contain any cycle. A tree on n vertices is denoted by t_n .

Remark 1.4.3. *There are different types of non isomorphic trees on n vertices. A path P_n in Definition 1.3.10, is a tree on n vertices with degree sequence $1, 1, \underbrace{2, 2, \dots, 2}_{n-2}$*

and a star on n vertices, denoted by S_n , is a tree with degree sequence $\underbrace{1, 1, 1, \dots, 1, 1}_{n-1}, n - 1$. Thus, based on degree sequences, it is clear that S_n and P_n are non isomorphic.

The following proposition states some well known properties of a tree which are of interest in this dissertation.

Proposition 1.4.4. *Let t_n be a tree on n vertices. Then*

- (i) $|E(t_n)| = n - 1$.
- (ii) every edge of t_n is a bridge.
- (iii) t_n is a simple graph.

Definition 1.4.5. A vertex of degree 1 in a tree is called a leaf.

From the degree sequence of a path, P_n , we conclude that P_n has two leaves and from the degree sequence of a star, S_n , we conclude that S_n star has $n - 1$ leaves.

Definition 1.4.6. A complete graph, denoted by K_n , is a graph with vertex set

$$V(K_n) = \{w_1, w_2, \dots, w_n\}$$

and edge set

$$E(G) = \{(w_i, w_j) \forall i, j \in \{1, 2, \dots, n\}\}.$$

The following proposition states some well known properties of a complete graph which are of interest in this dissertation.

Proposition 1.4.7. *Let K_n be a complete graph on n vertices. Then*

- (i) $|E(K_n)| = \binom{n}{2}$.
- (ii) every vertex of K_n is of degree $n - 1$.
- (iii) any pair of vertices is adjacent in K_n .

(iv) K_n is a simple graph.

Definition 1.4.8. A cycle graph, denoted by C_n , is a graph with vertex set

$$V(C_n) = \{w_1, w_2, \dots, w_n\}$$
 and edge set

$$E(C_n) = \{(w_1, w_2), (w_2, w_3), \dots, (w_{n-2}, w_{n-1}), (w_{n-1}, w_1)\}.$$

The following proposition states some well known properties of a cycle graph which are of interest in this dissertation.

Proposition 1.4.9. *Let C_n be a cycle graph on n vertices. Then*

1. $|E(C_n)| = n$.
2. every vertex of C_n is of degree 2.
3. C_n is a simple graph for $n > 2$.

Definition 1.4.10. A graph G is bipartite if the vertex set $V(G)$ can be partitioned into two sets X and Y , that is $V(G) = X \cup Y$ and $X \cap Y = \emptyset$, such that for each edge $e \in E(G)$, one endpoint is in X and other endpoint is in Y .

Definition 1.4.11. A graph G is said to be k -regular if every vertex has degree k . Thus G has a degree sequence k, k, \dots, k .

By Proposition 1.4.7, it is clear that a complete graph, K_n , is an $(n-1)$ -regular graph and by Proposition 1.4.9, it is clear that a cycle graph, C_n , is a 2-regular graph.

The graphs shown in the Figure 1.4 are examples of some classes of graphs on 5 vertices. In particular, examples of a null graph, tree, path, star, cycle and complete graph.

1.5 Graph operations

In this section, we will discuss some graph operations which are known in the literature and are useful to this dissertation.

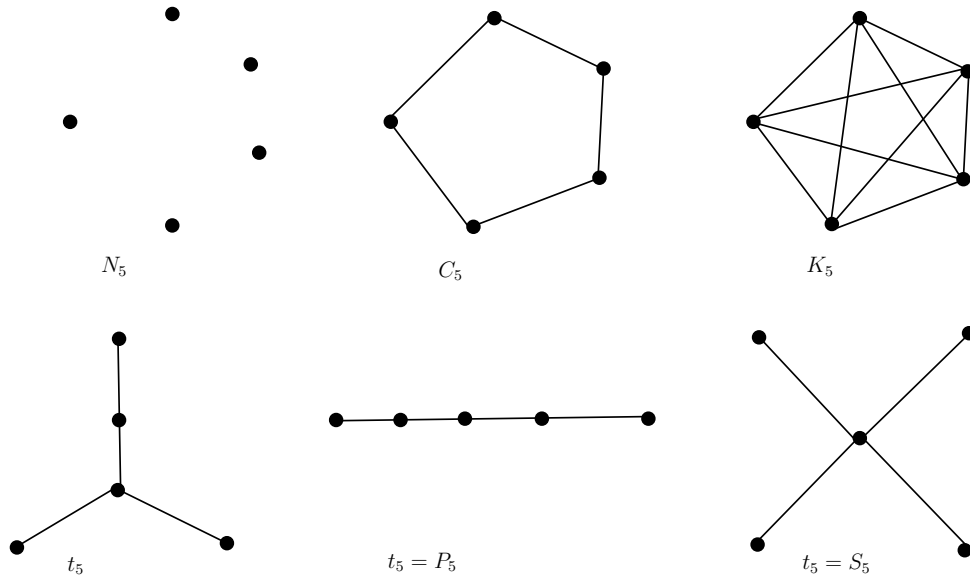


Figure 1.4: Example of some classes of graphs

A graph operation is the process of creating a new graph from one or more given graphs. There are several graph operations which are widely used in graph theory. We start by discussing two of the widely used graph operations, the deletion of an edge and contraction of an edge. Finally, we discuss the vertex join of a graph, the join of two graphs and the K_r -gluing of graphs.

Definition 1.5.1. *We define*

(i) **edge deletion.**

Let G be a graph of order n with edge set $E(G) = \{e_1, e_2, \dots, e_m\}$ and vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$. The graph obtained from G by deleting an edge e_1 is a graph of order n with edge set $\{e_2, \dots, e_m\}$ and vertex set $\{w_1, w_2, \dots, w_n\}$ and is denoted by $G \setminus e_1$.

(ii) **edge contraction.**

Let G be a graph, of the order n with vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\} = \{(w_i, w_j) \mid i, j \in \{1, 2, \dots, n\}\}$. The graph

obtained by contracting an edge $e_1 = (w_1, w_2)$ is a graph of order $n - 1$ with vertex set $\{w_2, w_3, \dots, w_n\}$ and edge set $E(G) = \{(w_1, w_j) | (w_1, w_j) \in E(G)\} \cup \{(w_2, w_j) | (w_1, w_j) \in E(G), j \neq 2\}$ and is denoted by G/e_1 .

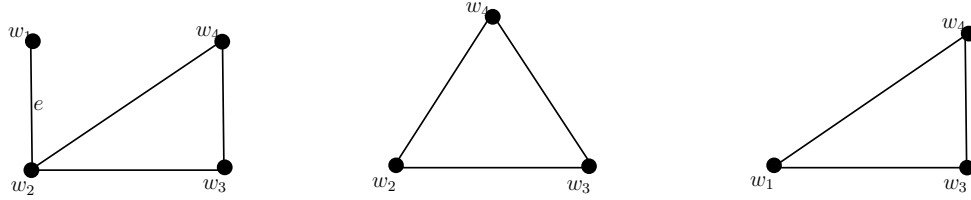


Figure 1.5: Edge deletion and edge contraction

As an example, consider the graph G shown in the diagrams of Figure 1.5 with vertex set $V(G) = \{w_1, w_2, w_3, w_4\}$ and edge set

$$E(G) = \{(w_1, w_2), (w_2, w_3), (w_3, w_4), (w_2, w_4)\}.$$

Let edge $e = (w_2, w_3)$, then the graph $G \setminus e$ have edge set $\{(w_1, w_2), (w_2, w_4), (w_3, w_4)\}$ and vertex set $\{w_1, w_2, w_3, w_4\}$.

Let edge $e = (w_2, w_3)$, then the graph G/e have edge set $\{(w_1, w_2), (w_2, w_4), (w_2, w_4)\}$ and vertex set $\{w_1, w_2, w_4\}$.

(iii) **Join of graphs.**

Let G_1 and G_2 be graphs. The join of G_1 and G_2 , denoted by $G_1 + G_2$, is the graph obtained by connecting every vertex in graph G_1 to all vertices in graph G_2 with an edge. Thus the graph $G_1 + G_2$ will have vertex set $V(G_1) \cup V(G_2)$ and edges set $E(G_1) \cup E(G_2) \cup \{(u, v) | u \in V(G_1) \text{ and } v \in V(G_2)\}$.

As an example graph shown in Figure 1.6 are two graphs G_1 and G_2 and the join $G_1 + G_2$.

(iv) **Vertex join of a graph: (Special case of join of two graphs).**

Let $G = (V(G), E(G))$ be a graph of order n and let $V(G) = \{w_1, w_2, \dots, w_n\}$. A



Figure 1.6: The join of G_1 and G_2 .

vertex join of G , denoted by \widehat{G} is a graph of order $n+1$ obtained by taking the join of G and K_1 . Thus the vertex join of G has vertex set $V(\widehat{G}) = V(G) \cup \{w\}$ where $V(K_1) = \{w\}$ and edge set $E(\widehat{G}) = E(G) \cup \{(w_1, w), (w_2, w), \dots, (w_n, w)\}$.

As an example, graphs shown in Figure 1.7 is a graph G and a graph \widehat{G} , the vertex join of G .

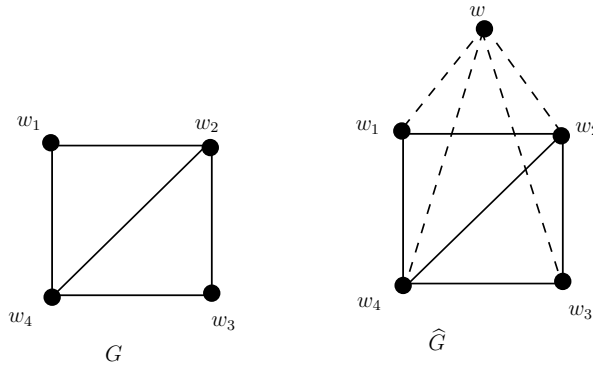


Figure 1.7: Graph G and vertex join \widehat{G} .

(v) **K_r -gluing.**

Let G_1 and G_2 be two graphs, each with a subgraph K_r . The graph obtained by merging K_r in G_1 and K_r in G_2 is called the K_r -gluing, denoted by $G_1 \cup_{K_r} G_2$. Of special interest is a K_1 and K_2 gluing, since they have been studied as graph operations on their own under different names such as 1-sum and 2-sum of graphs.

(vi) **Edge Gluing:**(special case of K_r gluing).

A K_2 -gluing of two graphs, is known as an edge gluing. As an example, the diagrams in Figure 1.8 are two graphs G_1 and G_2 and the corresponding edge gluing of G_1 and G_2 .

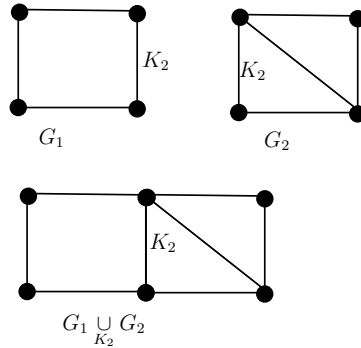


Figure 1.8: The graphs G_1 , G_2 and $G_1 \cup_{K_2} G_2$.

Definition 1.5.2. A wheel W_n on n vertices is a graph obtained by taking the vertex join of the cycle graph C_{n-1} .

Chapter 2

Introduction of the chromatic polynomial

In this chapter, we define a proper coloring of a graph, then we introduce the chromatic polynomial of a graph and discuss some of its basic properties. We then discuss some methods for computing the chromatic polynomial of a graph. In addition, we state chromatic polynomials for some special classes of graphs. We finish this chapter by giving different ways or forms of presenting the chromatic polynomial of a graph. For further details on the theory discussed in this chapter, we refer the reader to [6, 14] unless stated otherwise.

2.1 Chromatic polynomial of a graph

In this section we define a proper coloring of a graph. We then introduce the chromatic polynomial of a graph and give an example of the chromatic polynomial of a graph.

Definition 2.1.1. A proper coloring is the assignment of a color to each vertex of a graph so that no two adjacent vertices are assigned the same color. A k -coloring of a graph is a proper coloring which involves a total of k colors. A graph that has a k -coloring is said to be k -colorable.

Definition 2.1.2. The chromatic polynomial of a graph G count the number of ways of properly coloring the vertices of G using λ colors. In this case, the colorings are counted as distinct, even if they differ only by permutation of colors. We denote the chromatic polynomial of a graph G by $\chi(G; \lambda)$.

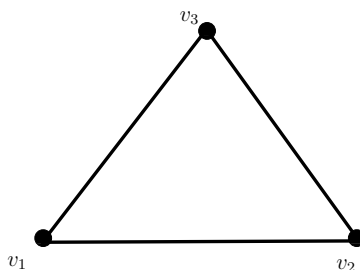


Figure 2.1: Graph G

Example 2.1.3. We clarify the concept of a chromatic polynomial by using the graph G given in the diagram of Figure 2.1. To properly color the graph G shown in Figure 2.1, we note that all pairs of vertices of G are adjacent. Hence we assign different colors to each vertex of G . Assume that we have λ colors, if we color vertex v_1 with any of the λ colors, then vertex v_2 can be colored with any of the remaining $\lambda - 1$ colors. Finally, we color v_3 with the remaining $\lambda - 2$ colors. By Definition 2.1.2, the chromatic polynomial of G is

$$\chi(G; \lambda) = \lambda(\lambda - 1)(\lambda - 2),$$

the number of ways of properly coloring the vertices of G using λ colors.

The chromatic polynomial of a graph G can be defined mathematically. We first need the concept of rank of a graph to define the chromatic polynomial.

Definition 2.1.4. Let G be a graph of order $|V(G)| = n$ with $c = c(G)$ connected components. The rank of the graph G , denoted by $r(G)$, is defined by $r(G) = |V(G)| - c(G) = n - c$.

Recall the definition of a subgraph of G from Chapter 1, Definition 1.3.2.

Definition 2.1.5. Let G be a graph of order n . The chromatic polynomial of the graph G is defined as

$$\chi(G; \lambda) = \lambda^c \sum_{X \subseteq E(G)} (-1)^{|X|} \lambda^{r(G) - r(X)}$$

where c is the number of connected components of G and the sum is over all subgraphs of the graph G .

We demonstrate that the concept of chromatic polynomial, defined mathematically, is the same as the original definition, Definition 2.1.2.

Example 2.1.6. We use the graph G given in the diagram of Figure 2.1. We give in a table form the information required to find the chromatic polynomial of G . Let X be a subgraph of G , then $E(X)$ is the edge set of X and $r(X)$ is the rank of the subgraph X . G has one component and the rank of G , $r(G) = 3 - 1 = 2$.

$E(X)$	$ X $	$r(X) = V(X) - c(X)$
\emptyset	0	0
$\{(v_1, v_2)\}$	1	1
$\{(v_1, v_3)\}$	1	1
$\{(v_2, v_3)\}$	1	1
$\{(v_1, v_2), (v_2, v_3)\}$	2	2
$\{(v_1, v_2), (v_1, v_3)\}$	2	2
$\{(v_1, v_3), (v_2, v_3)\}$	2	2
$\{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}$	3	2

Table 2.1:

Hence the chromatic polynomial of G ,

$$\begin{aligned}
\chi(G; \lambda) &= \lambda^1 \sum_{X \subseteq E(G)} (-1)^{|X|} \lambda^{2-r(X)} \\
&= \lambda [(-1)^0 \lambda^{2-0} + 3(-1)^1 \lambda^{2-1} + 3(-1)^2 \lambda^{2-2} + (-1)^3 \lambda^{2-2}] \\
&= \lambda [\lambda^2 - 3\lambda + 3 - 1] \\
&= \lambda^3 - 3\lambda^2 + 2\lambda \\
&= \lambda(\lambda - 1)(\lambda - 2),
\end{aligned}$$

verifying the result we found in Example 2.1.3.

The following theorem summarizes some of the well known properties of the chromatic polynomial, see [12].

Theorem 2.1.7. *Let G be a graph of order n and size m and let $\chi(G; \lambda)$ be the chromatic polynomial of G . Then*

- (i) *the degree of $\chi(G; \lambda)$ is n .*
- (ii) *the coefficient of λ^n is 1.*
- (iii) *the coefficient of λ^{n-1} is m .*
- (iv) *the coefficients of $\chi(G; \lambda)$ alternate in signs.*
- (v) *there is no constant term in $\chi(G; \lambda)$.*

2.2 Deletion-contraction method for computing the chromatic polynomial

In this section, we discuss the deletion-contraction algorithm for computing the chromatic polynomial of a graph G . This algorithm is based on reducing the graph by deletion-contraction to graphs which are easy to find the chromatic polynomial or whose chromatic polynomial is already known. This algorithm makes it easier to

compute the chromatic polynomial of a graph, than using the definition of the chromatic polynomial, see [12].

We state the algorithm in form of a theorem.

Theorem 2.2.1. *Let G be a graph of order n and $e \in E(G)$. Let $G \setminus e$ and G/e be the graphs obtained by deleting and contracting an edge e of G , respectively. Then if e is neither a loop nor a bridge of G , the chromatic polynomial of G ,*

$$\chi(G; \lambda) = \chi(G \setminus e; \lambda) - \chi(G/e; \lambda).$$

Corollary 2.2.2. *Let G be a graph of order n and $e \in E(G)$. If e is a bridge of G , then $\chi(G; \lambda) = (\lambda - 1)\chi(G/e; \lambda)$.*

Rearranging Theorem 2.2.1 and substituting $G = (H + e)$, $G/e = (H + e)/e$ and $H = G \setminus e$ we get the following corollary.

Corollary 2.2.3. *Let H be a graph of order n , $H + e$ a graph of order n obtained from H by adding an edge e . Then the chromatic polynomial of the graph H ,*

$$\chi(H; \lambda) = \chi((H + e); \lambda) + \chi((H + e)/e; \lambda).$$

Corollary 2.2.3 outlines a formula for computing the chromatic polynomial of a graph by adding edges, other than deletion and contracting of an edge.

The following theorem states some useful results which can be used to ease computation of the chromatic polynomial of any graph.

Theorem 2.2.4. *Let G be a graph.*

(i) *If G has a loop, then $\chi(G; \lambda) = 0$.*

(ii) *If G is a loopless multigraph and \tilde{G} is the simple graph of G , then*

$$\chi(G; \lambda) = \chi(\tilde{G}; \lambda).$$

2.3 Chromatic polynomial of some classes of graphs

In this section we will state explicit expressions of the chromatic polynomial for some classes of graphs defined in Section 1.4; we refer to [6, 12].

Theorem 2.3.1. *Let N_n be the null graph of order n . Then the chromatic polynomial of N_n ,*

$$\chi(N_n; \lambda) = \lambda^n.$$

Proof. It is clear that each of the n vertices can be separately colored by using any of the λ colors, since there are no edges. Hence the result. \square

Theorem 2.3.2. *Let t_n be a tree of order n . Then the chromatic polynomial of t_n ,*

$$\chi(t_n; \lambda) = \lambda(\lambda - 1)^{n-1}.$$

Proof. We prove by induction on the number of vertices n . Consider a tree of order $n = 1$. Then $t_1 = N_1$. By Theorem 2.3.1,

$$\begin{aligned}\chi(t_1; \lambda) &= \chi(N_1; \lambda) = \lambda \\ &= \lambda(\lambda - 1)^{1-1}.\end{aligned}$$

Therefore the statement is true for $n = 1$.

Assume that the statement is true for a tree of order some $n = k$, that is $\chi(t_k; \lambda) = \lambda(\lambda - 1)^{k-1}$.

Now consider a tree of order $n = k + 1$. By Proposition 1.4.4, part (ii), every edge of a tree is a bridge. Thus by Corollary 2.2.2, we get

$$\begin{aligned}\chi(t_{k+1}; \lambda) &= (\lambda - 1)\chi(t_k; \lambda) \\ &= (\lambda - 1) [\lambda(\lambda - 1)^{k-1}] \\ &= \lambda(\lambda - 1)^k.\end{aligned}$$

Therefore the statement is true for a tree of order $n = k + 1$. Thus by principle of induction, the theorem is true. \square

Theorem 2.3.3. *Let C_n be the cycle graph of order $n \geq 3$. Then the chromatic polynomial of C_n ,*

$$\chi(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

Proof. We prove by induction on the number of vertices n . Consider a cycle graph of order $n = 3$. Then we need to show that $\chi(C_3; \lambda) = (\lambda - 1)^3 + (-1)^3(\lambda - 1)$.

We start with the LHS of the statement,

$$\begin{aligned} LHS &= (\lambda - 1)^3 + (-1)^3(\lambda - 1) \\ &= (\lambda - 1) [(\lambda - 1)^2 - 1] \\ &= (\lambda - 1) [(\lambda^2 - 2\lambda + 1) - 1] \\ &= \lambda(\lambda - 1)(\lambda - 2). \end{aligned}$$

Now we consider the RHS of statement. Let e be any edge of C_n . Thus by Theorem 2.2.1, we get

$$\begin{aligned} RHS &= \chi(C_3; \lambda) \\ &= \chi(C_3 \setminus e; \lambda) - \chi(C_3 / e; \lambda) \\ &= \chi(P_3 \setminus e; \lambda) - \chi(P_2; \lambda) \\ &= \lambda(\lambda - 1)^{3-1} - \lambda(\lambda - 1)^{2-1} \text{ by Theorem 2.3.2} \\ &= \lambda(\lambda - 1) [\lambda - 1 - 1] \\ &= \lambda(\lambda - 1)(\lambda - 2) \\ &= LHS \end{aligned}$$

Hence $\chi(C_3; \lambda) = (\lambda - 1)^3 + (-1)^3(\lambda - 1)$. Therefore the statement is true for $n = 3$.

Assume that the statement is true for a cycle graph of order $n = k$, that is

$$\chi(C_k; \lambda) = (\lambda - 1)^k + (-1)^k(\lambda - 1).$$

Now consider a cycle graph of order $n = k + 1$. It is observed that if we delete any single edge of C_n we get a path P_n and if we contract any single edge of C_n we get a cycle graph, C_{n-1} . Thus by Theorem 2.2.1, we get

$$\begin{aligned}
\chi(C_{k+1}; \lambda) &= \chi(C_{k+1} \setminus e; \lambda) - \chi(C_{k+1}/e; \lambda) \\
&= \chi(P_{k+1} \setminus e; \lambda) - \chi(C_k; \lambda) \\
&= \lambda(\lambda - 1)^k - [(\lambda - 1)^k + (-1)^k(\lambda - 1)] \\
&\quad \text{by Theorem 2.3.2 and induction assumption} \\
&= (\lambda - 1)^k(\lambda - 1) - (-1)^k(\lambda - 1) \\
&= (\lambda - 1)^{k+1} + (-1)^{k+1}(\lambda - 1).
\end{aligned}$$

Therefore the statement is true for a cycle graph of order $n = k + 1$. Thus by principle of induction, the theorem is true. \square

Remark 2.3.4. Note that Theorem 2.3.3 is true for $n = 2$, since $\chi(C_2; \lambda) = \chi(P_2; \lambda) = \lambda(\lambda - 1) = (\lambda - 1)^2 + (-1)^2(\lambda - 1)$. However it is clear that C_2 is a multigraph, a pair of parallel edges and in the literature most generalizations for cycle graphs considers $n \geq 3$, because C_n is a simple graph for $n \geq 3$.

Theorem 2.3.5. Let K_n be the complete graph of order n . Then the chromatic polynomial of K_n ,

$$\chi(K_n; \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1).$$

Proof. From Definition 2.1.1, no two adjacent vertices are assigned the same color. By Proposition 1.4.7, any two vertices are adjacent in K_n . Thus if we have λ colors we can color the first vertex with any of the λ colors, the second vertex can be colored with the remaining $\lambda - 1$, the third vertex with the remaining $\lambda - 2$, \cdots and the n^{th} vertex will be colored with the remaining $\lambda - (n - 1)$. Hence the result. \square

Remark 2.3.6. In the literature $\lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$ is also known as the falling factorial and is denoted by λ_n .

An explicit expression of the chromatic polynomial of a wheel will be stated in the next Chapter, because a wheel is a result of a graph operation, the vertex join of a cycle graph, see Definition 1.5.2.

2.4 Some forms of presenting the chromatic polynomial

In this section we will give three different forms of presenting the chromatic polynomial of a graph. Namely null graph form, complete graph form and tree form. Finally we will try to experiment, the possibility of presenting the chromatic polynomial of a graph in the cycle graph form. For further details about this section, we refer the reader to [10, 12], unless otherwise stated.

Computing the chromatic polynomial of a graph using repeated application of deletion and contraction or the reverse process of adding edges, can decompose the chromatic polynomial of the graph into chromatic polynomials of null graphs, complete graphs or trees. Thus we can choose these three ways of presenting the chromatic polynomial of a graph.

In this section, we consider the graph shown in the diagram of Figure 2.2 to demonstrate the three different forms of presenting the chromatic polynomial.

Note that to ease notation, the graphs shown in the diagrams of Figure 2.3, Figure 2.4 and Figure 2.5, represent the chromatic polynomial of that graph. In case we have a multigraph in the process, we represent the multigraph by its simplification graph, since by Proposition 2.2.4 a multigraph and its simplification have the same chromatic polynomial.

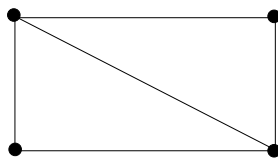


Figure 2.2: Graph G

2.4.1 Null graph form

This is the mostly used form of presenting the chromatic polynomial of a graph. The chromatic polynomial of any given graph G can be expressed in the form

$$\chi(G; \lambda) = \sum_{i=1}^n a_i \lambda^i$$

where a_i is a constant. In addition, Theorem 2.3.1 states that

$$\chi(N_n; \lambda) = \lambda^n.$$

Thus implying that $\chi(G; \lambda)$ is a sum of chromatic polynomials of some null graphs. Alternatively, the chromatic polynomial of a graph G can be computed using the deletion and contraction algorithm repeatedly, until we get a sum of chromatic polynomials of null graphs. We illustrate this procedure with the following example.

Example 2.4.1. Let G be the graph given in Figure 2.2. We compute the chromatic polynomial of G expressed in the null graph form, using the deletion-contraction as demonstrated in Figure 2.3.

The chromatic polynomial of a null graph is given in Theorem 2.3.1. Hence substituting this in place of the null graphs in the diagrams of Figure 2.3, we get $\chi(G; \lambda) = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda$.

2.4.2 Complete graph form

We compute the chromatic polynomial of a graph G expressed in terms of chromatic polynomials of complete graphs. We apply the procedure outlined in Corollary 2.2.3,

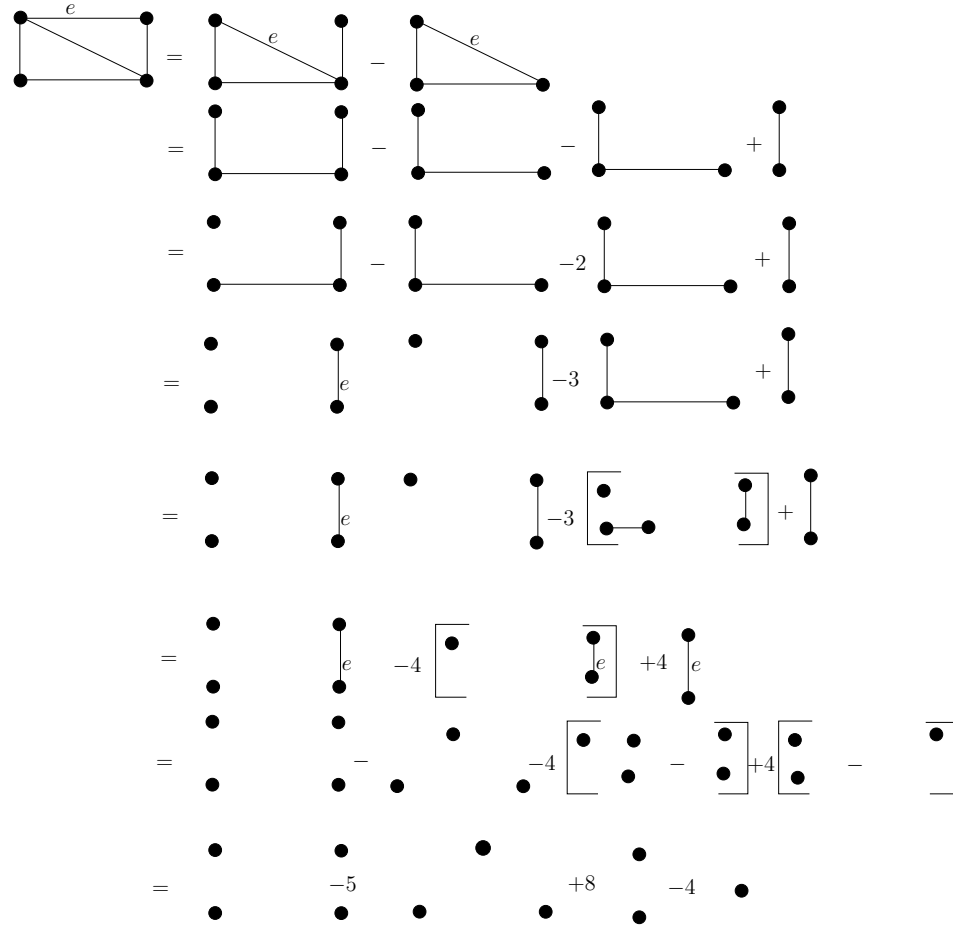


Figure 2.3: Null graph form of $\chi(G; \lambda)$

adding edges to build complete graphs. However, where necessary we apply the deletion and contraction process. We illustrate this procedure with the following example.

Example 2.4.2. Let G be the graph shown in Figure 2.2. We compute the chromatic polynomial of G expressed in the complete graph form, as demonstrated in Figure 2.4.

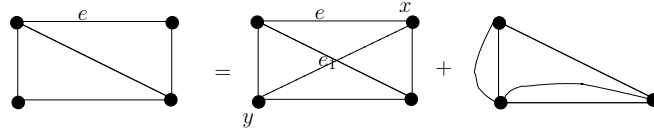


Figure 2.4: Complete graph form of $\chi(G; \lambda)$

Recall that Theorem 2.3.5 states that $\chi(K_n; \lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) = (\lambda)_n$. Hence substituting $\chi(K_m; \lambda)$ in the place of the complete graph of order m shown in the Figure 2.4, we get

$$\begin{aligned} \chi(G; \lambda) &= \chi(K_4; \lambda) + \chi(K_3; \lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) \\ &= \lambda_4 + \lambda_3. \end{aligned}$$

2.4.3 Tree form

We compute the chromatic polynomial of a graph G expressed in terms of chromatic polynomials of trees. We use the deletion and contraction method given in Proposition 2.2.1 until we get to some chromatic polynomials of trees. We illustrate this procedure with the following example.

Example 2.4.3. Let G be the graph shown in Figure 2.2, we compute the chromatic polynomial of G as demonstrated in Figure 2.5.

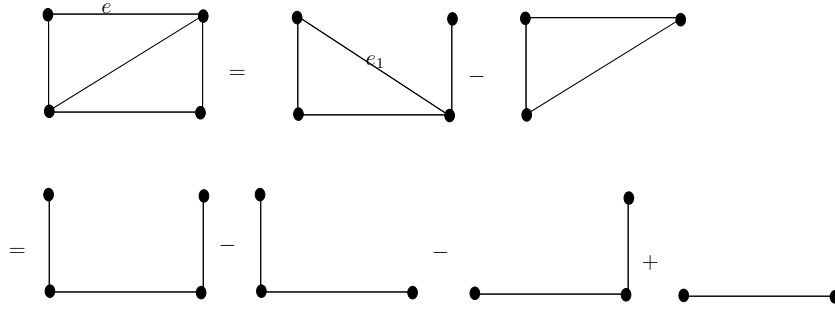


Figure 2.5: Tree form of $\chi(G; \lambda)$

Recall that Theorem 2.3.2 states that $\chi(t_n; \lambda) = \lambda(\lambda - 1)^{n-1}$. Hence substituting $\chi(t_m; \lambda)$ in the place of a tree of order m shown in the Figure 2.5, we get

$$\begin{aligned} \chi(G; \lambda) &= \chi(t_4; \lambda) - 2\chi(t_3; \lambda) + \chi(t_2; \lambda) \\ &= \lambda(\lambda - 1)^3 - 2\lambda(\lambda - 1)^2 + \lambda(\lambda - 1). \end{aligned}$$

We now verify that all the three forms of the chromatic polynomial for the graph G shown in the Figure 2.2 is just one polynomial. We expand the complete graph form and the tree form to the null graph form.

The chromatic polynomial of G in complete graph form,

$$\begin{aligned} \chi(G; \lambda) &= \chi(K_4; \lambda) + \chi(K_3; \lambda) \\ &= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2) \\ &= \lambda(\lambda - 1)(\lambda - 2)^2 \\ &= (\lambda^2 - \lambda)(\lambda^2 - 4\lambda + 4) \\ &= (\lambda^4 - 4\lambda^3 + 4\lambda^2) - (\lambda^3 - 4\lambda^2 + 4\lambda) \\ &= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda, \end{aligned}$$

which is equal to the chromatic polynomial of G in null graph form.

Similarly, the chromatic polynomial of G in tree form,

$$\begin{aligned}
\chi(G; \lambda) &= \chi(t_4; \lambda) - 2\chi(t_3; \lambda) + \chi(t_2; \lambda) \\
&= \lambda(\lambda - 1)^3 - 2\lambda(\lambda - 1)^2 + \lambda(\lambda - 1) \\
&= (\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda) - (2\lambda^3 - 4\lambda^2 + 2\lambda) + (\lambda^2 - \lambda) \\
&= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda,
\end{aligned}$$

which is equal to the chromatic polynomial of G in null graph form.

2.5 The chromatic polynomial in a cycle form

In this section we experiment with a similar idea, presenting the chromatic polynomial of a graph G in terms of the chromatic polynomials of some cycle graphs. To get the chromatic polynomials of cycle graphs, we combine the deletion and contraction process

$$\chi(G; \lambda) = \chi(G \setminus e; \lambda) - \chi(G/e; \lambda)$$

and its reverse process outlined in Corollary 2.2.3,

$$\chi(G; \lambda) = \chi((G + e); \lambda) - \chi((G + e)/e; \lambda)$$

adding edges where necessary. We illustrate this procedure with the following example.

Example 2.5.1. Let G be the graph given in Figure 2.2, we compute the chromatic polynomial of G as demonstrated in Figure 2.6.

Recall that Theorem 2.3.3 states that

$$\chi(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

Hence substituting $\chi(C_m; \lambda)$ in the place of the cycle graph of order m shown in Figure 2.6, we get

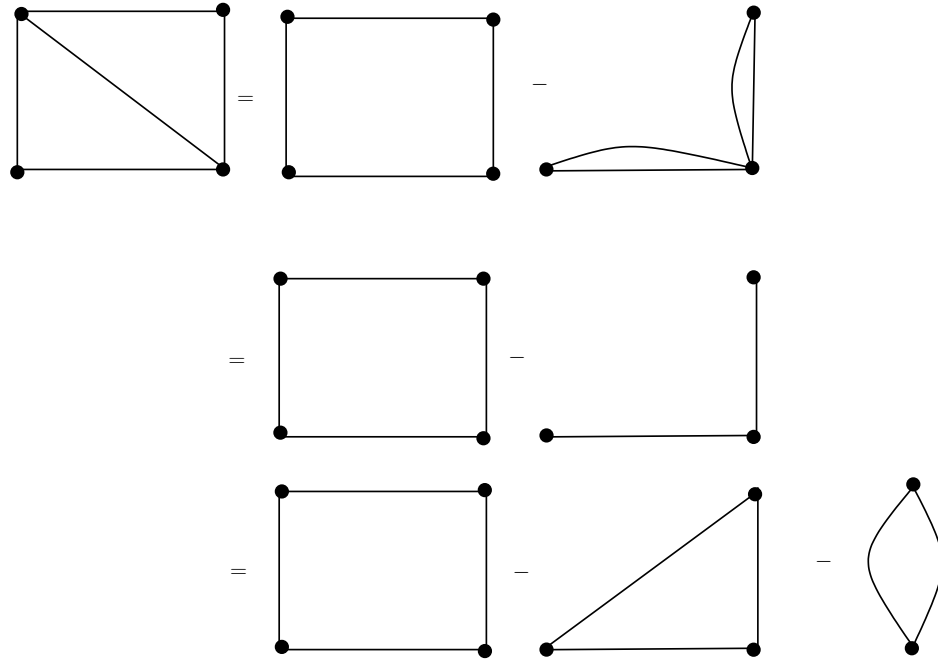


Figure 2.6: Cycle graph form of $\chi(G; \lambda)$

$$\begin{aligned}
 \chi(G; \lambda) &= \chi(C_4; \lambda) - \chi(C_3; \lambda) - \chi(C_2; \lambda). \\
 &= [(\lambda - 1)^4 + (-1)^4(\lambda - 1)] - [(\lambda - 1)^3 + (-1)^3(\lambda - 1)] - \lambda(\lambda - 1).
 \end{aligned}$$

We now verify that the cycle graph form of the chromatic polynomial for the graph G shown in Figure 2.2 is equal to the three other forms, that is we expand the cycle graph form to the null graph form.

The chromatic polynomial of G in complete graph form,

$$\begin{aligned}
\chi(G; \lambda) &= \chi(C_4; \lambda) - \chi(C_3; \lambda) - \chi(C_2; \lambda) \\
&= [(\lambda - 1)^4 + (-1)^4(\lambda - 1)] - [(\lambda - 1)^3 + (-1)^3(\lambda - 1)] - [\lambda(\lambda - 1)] \\
&= (\lambda - 1)^4 - (\lambda - 1)^3 + 2(\lambda - 1) - \lambda(\lambda - 1) \\
&= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 - \lambda^3 + 3\lambda^2 - 3\lambda + 1 + 2\lambda - 2 - \lambda^2 + \lambda \\
&= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda
\end{aligned}$$

which is equal to the chromatic polynomial of G in null graph form.

We give the chromatic polynomial of a tree t_n in terms of chromatic polynomials of cycle graphs. It is well known in the literature, that all non-isomorphic trees of the same order, have the same chromatic polynomial, see [12].

Proposition 2.5.2. *Let t_n be a tree of order n . Then, the chromatic polynomial in cycle form,*

$$\chi(t_n; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)$$

where C_n is a cycle graph of order n .

Proof. Recall from Theorem 2.3.3 that the chromatic polynomial of a cycle graph, C_k ,

$$\chi(C_k; \lambda) = (\lambda - 1)^k + (-1)^k(\lambda - 1).$$

Thus

$$\begin{aligned}
\chi(C_n; \lambda) + \chi(C_{n-1}; \lambda) &= [(\lambda - 1)^n + (-1)^n(\lambda - 1)] + [(\lambda - 1)^{n-1} + (-1)^{n-1}(\lambda - 1)] \\
&= (\lambda - 1)^n + (\lambda - 1)^{n-1} + (-1)^n(\lambda - 1) + (-1)^{n-1}(\lambda - 1) \\
&= (\lambda - 1)^{n-1}(\lambda - 1 + 1) + (-1)^{n-1}(\lambda - 1)((-1) + 1) \\
&= \lambda(\lambda - 1)^{n-1}.
\end{aligned}$$

But by Theorem 2.3.2, $\chi(t_n; \lambda) = \lambda(\lambda - 1)^{n-1}$. □

Chapter 3

Graph operations and chromatic polynomials

In this chapter, we discuss chromatic polynomials of graphs obtained after performing graph operations on given graphs. A graph operation is a process which starts with one or more graphs to create a new graphs, see details in Chapter 1, Section 1.5. In particular, we are interested in graph operations, such that the chromatic polynomial of the new graph can be expressed in terms of the chromatic polynomial of the original graph. We discuss two such graph operations, the vertex join of a graph and the K_r -gluing of two graphs. For further details about this chapter, we refer the reader to [6].

3.1 Vertex-join of a graph and its chromatic polynomial

In this section, we discuss the chromatic polynomial of the vertex join of a graph. We recall the vertex-join of a graph; see Definition 1.5.1, part (iv). A vertex-join of

a graph $G = (V(G), E(G))$ of order n with vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$, is a graph \widehat{G} of order $n + 1$ with vertex set

$$V(\widehat{G}) = V(G) \cup \{w\}$$

and edge set

$$E(\widehat{G}) = E(G) \cup \{(w_1, w), (w_2, w), \dots, (w_n, w)\}.$$

Before stating the theorem on the relationship between the chromatic polynomial of the vertex-join \widehat{G} and the chromatic polynomial of G , we state and prove the following two lemmas which will be used in the proof of the theorem.

Lemma 3.1.1. *Let G be a graph and e an edge of G . Then the following two process are equal.*

(i) *Form vertex join of G , \widehat{G} , then delete an edge e in \widehat{G} and*

(ii) *Delete an edge e in G , then form the vertex join of $G \setminus e$.*

Proof. Let G be a graph, with edge set $E(G) = \{e_1, e_2, \dots, e_m\}$ and vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$. Hence $V(\widehat{G}) = V(G) \cup \{w\}$ and

$E(\widehat{G}) = E(G) \cup \{(w_1, w), (w_2, w), \dots, (w_n, w)\}$ by Definition 1.5.1 part (iv).

Without loss of generality, let $e_1 \in E(G)$. Consider a subgraph $(\widehat{G} \setminus e_1)$ of \widehat{G} .

Then the graph $(\widehat{G} \setminus e_1)$ has edge set

$$E((\widehat{G} \setminus e_1)) = \{e_2, \dots, e_m\} \cup \{(w_1, w), (w_2, w), \dots, (w_n, w)\}.$$

The vertex set, $V((\widehat{G} \setminus e_1)) = V(\widehat{G}) = \{w_1, w_2, \dots, w_n\} \cup \{w\}$ since deletion of an edge does not affect the number of vertices.

Now, consider the vertex join of the subgraph $(G \setminus e_1)$ denoted by $(\widehat{G \setminus e_1})$. Then the graph $(\widehat{G \setminus e_1})$

has edge set $E(\widehat{G \setminus e_1}) = \{e_2, \dots, e_m\} \cup \{(w_1, w), (w_2, w), \dots, (w_n, w)\}$.

The vertex set, $V(\widehat{G \setminus e_1}) = V(\widehat{G}) = \{w_1, w_2, \dots, w_n\} \cup \{w\}$ since deletion of an edge does not affect the number of vertices.

Hence $(\widehat{G} \setminus e_1) = (\widehat{G \setminus e_1})$. □

We clarify Lemma 3.1.1 with the following example.

Example 3.1.2. Let the graph G and its vertex join \widehat{G} be the graphs in Figure 3.1.

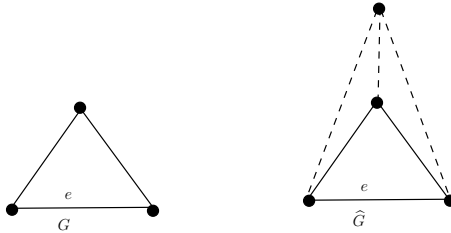


Figure 3.1: Graph G and graph \widehat{G}

The process of first forming a vertex join of G , \widehat{G} , then deleting an edge e in \widehat{G} , is illustrated in the diagrams of Figure 3.2.

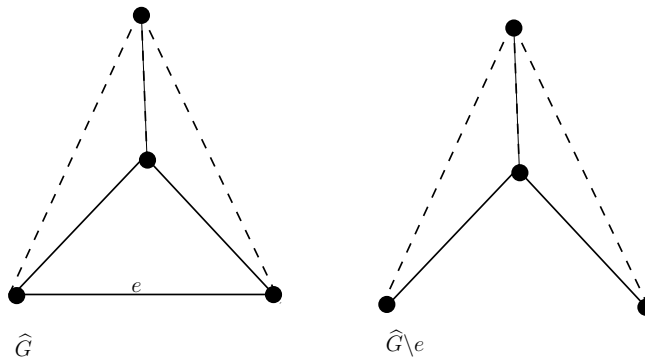


Figure 3.2: Graph \widehat{G} and graph $\widehat{G} \setminus e$

The process of first deleting an edge e in G , then forming the vertex join of $G \setminus e$ is illustrated in the diagrams of Figure 3.3.

Recall Definition 1.3.5 that a set of parallel edges is called a parallel class. Thus multigraphs are isomorphic up to parallel class means that they have the same simple graph.

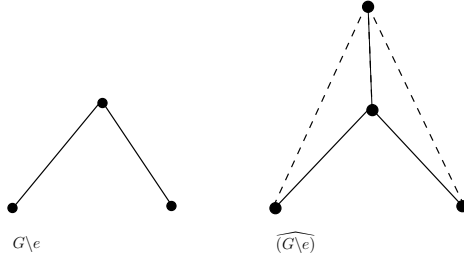


Figure 3.3: Graph $G \setminus e$ and graph $\widehat{(G \setminus e)}$

Lemma 3.1.3. *Let G be a graph with $e \in E(G)$ and let \widehat{G} be the vertex join of G . Then the graph obtained by contracting e in \widehat{G} is isomorphic up to parallel class to the vertex join of the graph G/e .*

Proof. Let G be a graph with edge set $E(G) = \{e_1, e_2, \dots, e_m\}$ and vertex set $V(G) = \{w_1, w_2, \dots, w_n\}$. Then $V(\widehat{G}) = V(G) \cup \{w\}$ and $E(\widehat{G}) = E(G) \cup \{(w_1, w)(w_2, w), \dots, (w_n, w)\}$ by Definition 1.5.1 part (iv).

Without loss of generality, let $e_1 = (w_1, w_2) \in E(G)$ and w_{1_2} be the new vertex resulting from identifying vertex w_1 and w_2 in the contraction of e_1 in the graphs G and \widehat{G} .

Consider the graph \widehat{G}/e_1 . In the process of contraction in \widehat{G} , we have replaced two vertices, w_1 and w_2 by a single vertex w_{1_2} . Hence the vertex set

$$\begin{aligned} V(\widehat{G}/e_1) &= \{V(\widehat{G}) - \{w_1, w_2\} \cup \{w_{1_2}\} \\ &= V(G) \cup \{w\} - \{w_1, w_2\} \cup \{w_{1_2}\} \\ &= \{w_3, \dots, w_n, w, w_{1_2}\}. \end{aligned}$$

For the edge set of \widehat{G}/e_1 , we observe that in the process of contraction all the vertices which were adjacent to w_1 and w_2 in \widehat{G} are now adjacent to vertex w_{1_2} in \widehat{G}/e_1 since

$w_1 = w_2 = w_{1_2}$. Hence the edge set

$$\begin{aligned}
E(\widehat{G}/e_1) &= \{(w_i, w_j) \mid (w_i, w_j) \in E(G) \text{ and } i, j \notin \{1, 2\}\} \\
&= \cup\{(w_k, w) \mid k \in \{3, \dots, n\}\} \\
&= \cup\{(w_{1_2}, w_t) \mid \forall t \text{ such that } (w_1, w_t), (w_2, w_t) \in E(G)\} \\
&= \cup\{(w_{1_2}, w), (w_{1_2}, w)\} \text{ since } (w_1, w), (w_2, w) \in E(\widehat{G}).
\end{aligned}$$

On the other hand consider the graph G/e_1 where $e_1 = (w_1, w_2) \in E(G)$. In the process of contraction in G , we have replaced two vertices, w_1 and w_2 , by a single vertex w_{1_2} . Hence the vertex set

$$V(G/e_1) = V(G) - \{w_1, w_2\} \cup \{w_{1_2}\} = \{w_3, \dots, w_n, w_{1_2}\}$$

For the edge set of G/e_1 , we observe that in the process of contraction all the vertices which were adjacent to w_1 and w_2 in G are now adjacent to vertex w_{1_2} in G/e_1 since $w_1 = w_2 = w_{1_2}$. Hence the edge set

$$\begin{aligned}
E(G/e_1) &= \{(w_i, w_j) \mid (w_i, w_j) \in E(G) \text{ and } i, j \notin \{1, 2\}\} \\
&= \cup\{(w_{1_2}, w_t) \mid \forall t \text{ such that } (w_1, w_t), (w_2, w_t) \in E(G)\}.
\end{aligned}$$

Now consider the vertex join of the graph G/e_1 . By Definition 1.5.1, part (iv) we get the vertex set of the vertex join of the graph G/e_1 ,

$$\begin{aligned}
V(\widehat{(G/e_1)}) &= V(G/e_1) \cup \{w\} \\
&= V(G) - \{w_1, w_2\} \cup \{w_{1_2}\} \cup \{w\} \\
&= \{w_3, \dots, w_n, w_{1_2}, w\}.
\end{aligned}$$

Then the edge set of the vertex join of the graph G/e_1 ,

$$\begin{aligned}
E(\widehat{(G/e_1)}) &= E(G/e_1) \cup \{(v, w) \mid \in V(G/e_1)\} \\
&= \{(w_i, w_j) \mid (w_i, w_j) \in E(G) \text{ and } i, j \notin \{1, 2\}\} \\
&= \cup \{(w_{1_2}, w_t) \mid \forall t \text{ such that } (w_1, w_t), (w_2, w_t) \in E(G)\} \\
&= [\text{these are all edges in } E(G/e_1)] \\
&= \cup \{(w_k, w) \mid \in \{3, \dots, n\}\} \cup \{(w_{1_2}, w)\} \\
&= [\text{these are edges of the form } \{(v, w) \mid \in V(G/e_1)\}].
\end{aligned}$$

Hence $V(\widehat{G/e_1}) = V(\widehat{(G/e_1)})$ and $E(\widehat{G/e_1}) = E(\widehat{(G/e_1)}) \cup \{(w_{1_2}, w)\}$. Thus the edge (w_{1_2}, w) is a parallel class in $\widehat{G/e_1}$. Therefore $\widehat{G/e_1}$ is isomorphic to $\widehat{(G/e_1)}$ up to parallel class. \square

We now give the chromatic polynomial of a vertex join of a graph G in the following theorem.

Theorem 3.1.4. *Let G be a graph and \widehat{G} be the vertex join of G . Then the chromatic polynomial of \widehat{G} ,*

$$\chi(\widehat{G}; \lambda) = \lambda \chi(G; \lambda - 1).$$

Proof. We will prove this theorem by induction on the size of a graph G of order n . Let G be a graph of size 0 of order n , then G is the null graph on n vertices. Then by applying Definition 1.5.1, part (iv) and Remark 1.4.3, we get that the vertex join of G , \widehat{G} is a star graph on $n + 1$ vertices, S_{n+1} . In addition, S_{n+1} is a tree t_{n+1} . Hence

$$\begin{aligned}
\chi(\widehat{G}; \lambda) &= \chi(t_{n+1}; \lambda) \\
&= \lambda(\lambda - 1)^n \text{ by Theorem 2.3.2} \tag{3.1}
\end{aligned}$$

By Theorem 2.3.1 we have $\chi(G; \lambda) = \lambda^n$ and this implies $\chi(G; \lambda - 1) = (\lambda - 1)^n$. Substituting this in Equation 3.1, we get

$$\chi(\widehat{G}; \lambda) = \lambda(\lambda - 1)^n = \lambda \chi(G; \lambda - 1).$$

Therefore the theorem is true for a graph G of size 0 and order n .

Now we assume that the theorem is true for a graph G of size less or equal to k and order less or equal to n . Thus if $|E(G)| \leq k$,

$$\chi(\widehat{G}; \lambda) = \lambda\chi(G; \lambda - 1).$$

Now we consider a graph G of size $k + 1$ and order n , i.e. $|E(G)| = k + 1$. Let e be an edge of G . Then there are two cases:

- (i) If e is a loop of G , then the theorem is true because $\chi(G; \lambda) = \chi(\widehat{G}; \lambda) = 0$ by Theorem 2.2.4, part (ii). Therefore $\chi(G; \lambda - 1) = 0$. Hence $\chi(\widehat{G}; \lambda) = 0 = \lambda\chi(G; \lambda - 1)$.
- (ii) If e is an edge of graph G and e is not a loop. Then by Lemma 3.1.1, $(\widehat{G}) \setminus e = \widehat{(G \setminus e)}$.

$$\chi(\widehat{G}; \lambda) = \chi(\widehat{G} \setminus e; \lambda) - \chi(\widehat{G}/e; \lambda) \text{ by Theorem 2.2.1.} \quad (3.2)$$

By Lemma 3.1.1, $(\widehat{G}) \setminus e = \widehat{(G \setminus e)}$. But $G \setminus e$ has size k and order n . Thus by the induction hypothesis

$$\begin{aligned} \chi(\widehat{G} \setminus e; \lambda) &= \chi(\widehat{(G \setminus e)}; \lambda) \\ &= \lambda\chi((G \setminus e); \lambda - 1). \end{aligned} \quad (3.3)$$

By Lemma 3.1.3, \widehat{G}/e is isomorphic to $\widehat{(G/e)}$, up to parallel class, that is they have the same simple graph. But the graph (G/e) has size k and order $n - 1$.

Thus by the induction hypothesis

$$\begin{aligned} \chi(\widehat{G}/e; \lambda) &= \chi(\widehat{(G/e)}) \\ &= \lambda\chi(G/e; \lambda - 1). \end{aligned} \quad (3.4)$$

Substituting Equation 3.3 and Equation 3.4 into Equation 3.2 we get,

$$\begin{aligned}
\chi(\widehat{G}; \lambda) &= \chi(\widehat{G} \setminus e; \lambda) - \chi(\widehat{G}/e; \lambda) \\
&= \lambda\chi(G \setminus e; \lambda - 1) - \lambda\chi((G/e); \lambda - 1) \\
&= \lambda[\chi(G \setminus e; \lambda - 1) - \chi(G/e; \lambda - 1)] \\
&= \lambda\chi(G; \lambda - 1).
\end{aligned}$$

Thus the theorem is true for a graph G of size $k + 1$. Therefore, by principle of mathematical induction, the theorem holds. \square

As an application of Theorem 3.1.4, we state and prove a formula for the chromatic polynomial of a wheel graph.

Theorem 3.1.5. *Let W_{n+1} be the wheel graph of order $n + 1$. Then*

$$\chi(W_{n+1}; \lambda) = \lambda[(\lambda - 2)^n + (-1)^n(\lambda - 2)].$$

Proof. Recall from Definition 1.5.2 that a wheel graph, W_{n+1} , is the vertex join of the cycle graph C_n . Hence by Theorem 3.1.4 we get,

$$\begin{aligned}
\chi(W_{n+1}; \lambda) &= \lambda\chi(C_n; \lambda - 1) \\
&= \lambda[((\lambda - 1) - 1)^n + (-1)^n((\lambda - 1) - 1)] \text{ by Theorem 2.3.3} \\
&= \lambda[(\lambda - 2)^n + (-1)^n(\lambda - 2)].
\end{aligned}$$

\square

In the following example, we verify Theorem 3.1.4. We calculate the chromatic polynomial of the vertex-join \widehat{G} of a graph G using deletion and contraction method and compare it with the chromatic polynomial of \widehat{G} given by the theorem.

Example 3.1.6. In this example we let G and \widehat{G} be the graphs given in the diagrams of Figure 3.4.

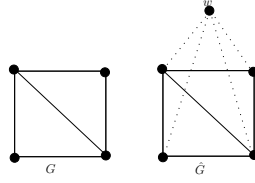


Figure 3.4: Graph G and graph \widehat{G}

Recall from Chapter 2, Example 2.4.1 that the chromatic polynomial of G

$$\chi(G; \lambda) = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda. \quad (3.5)$$

Hence by applying Theorem 3.1.4, we get

$$\begin{aligned} \chi(\widehat{G}; \lambda) &= \lambda\chi(G; \lambda - 1) \\ &= \lambda [(\lambda - 1)^4 - 5(\lambda - 1)^3 + 8(\lambda - 1)^2 - 4(\lambda - 1)] \\ &= \lambda [\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda^1 + 1] \\ &= -5\lambda [\lambda^3 - 3\lambda^2 + 3\lambda^1 - 1] \\ &= +8\lambda [\lambda^2 - 2\lambda^1 + 1] - 4\lambda + 4 \\ &= \lambda [\lambda^4 - 9\lambda^3 + 29\lambda^2 - 39\lambda + 18] \end{aligned} \quad (3.6)$$

We now compute the chromatic polynomial of \widehat{G} using the deletion and contraction method. The graphs shown in the Figure 3.5 represents the chromatic polynomial of that graph.

The chromatic polynomial of the graph G is given in Equation 3.5, and the chromatic polynomial of K_4 is given in Theorem 2.3.5. Hence substituting this in place of the graphs shown in the Figure 3.5, we get

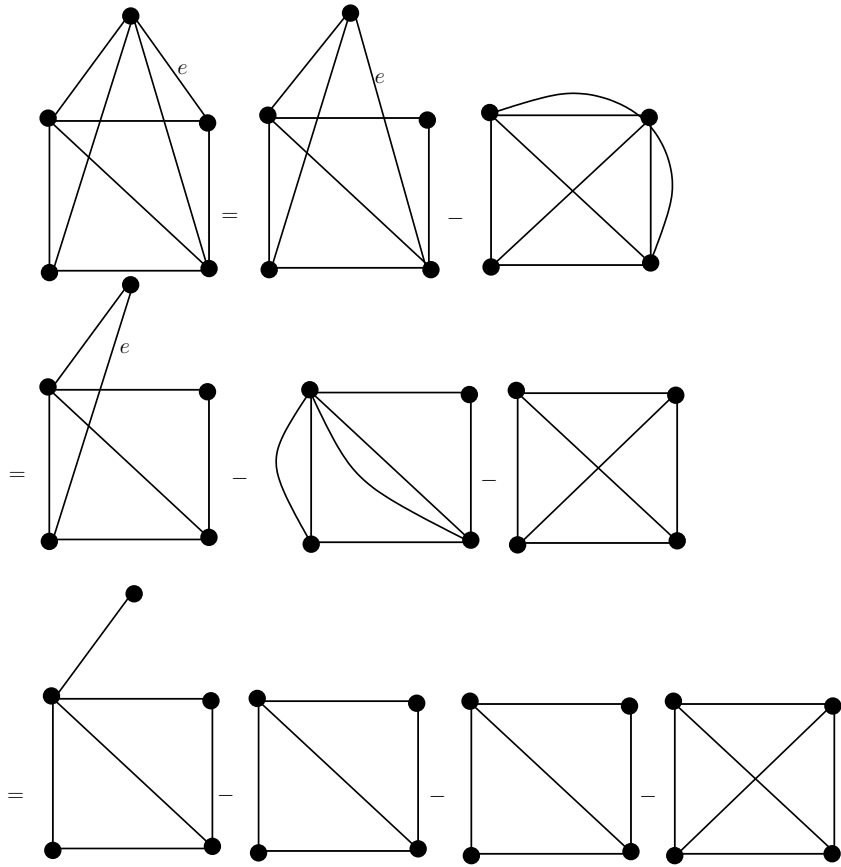


Figure 3.5:

$$\begin{aligned}
\chi(\widehat{G}; \lambda) &= (\lambda - 1)\chi(G; \lambda) - \chi(G; \lambda) - \chi(G; \lambda) - \chi(K_4; \lambda) \\
&= (\lambda - 3)\chi(G; \lambda) - \chi(K_4; \lambda) \\
&= (\lambda - 3) [\lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda] \\
&\quad - \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) \\
&= \lambda^5 - 5\lambda^4 + 8\lambda^3 - 4\lambda^2 - 3\lambda^4 + 15\lambda^3 - 24\lambda^2 \\
&\quad + 12\lambda - \lambda^4 + 6\lambda^3 - 11\lambda^2 + 6\lambda \\
&= \lambda^5 - 9\lambda^4 + 29\lambda^3 - 39\lambda^2 + 18\lambda \\
&= \lambda\chi(G; \lambda - 1) \text{ given in Equation 3.6,}
\end{aligned}$$

verifying Theorem 3.1.4.

3.2 K_r gluing of two graphs and its chromatic polynomial

In this section, we discuss the chromatic polynomial of the graph obtained by a K_r gluing of two graphs. Then we give an example to verify the given theorem.

We recall the operation of K_r gluing of two graphs Definition 1.5.1, part (v). The graph $G_1 \cup_{K_r} G_2$ obtained by merging K_r in G_1 and K_r in G_2 is called the K_r -gluing.

We state the following theorem without proof, we refer the reader for further details and proof to [6].

Theorem 3.2.1. *Let G be the K_r gluing of graphs G_1 and G_2 . The chromatic polynomial of G ,*

$$\chi(G; \lambda) = \frac{\chi(G_1; \lambda)\chi(G_2; \lambda)}{\chi(K_r; \lambda)}$$

To verify the given Theorem 3.2.1 by computing the chromatic polynomial of G using deletion and contraction and compare it with the chromatic polynomial found using the theorem.

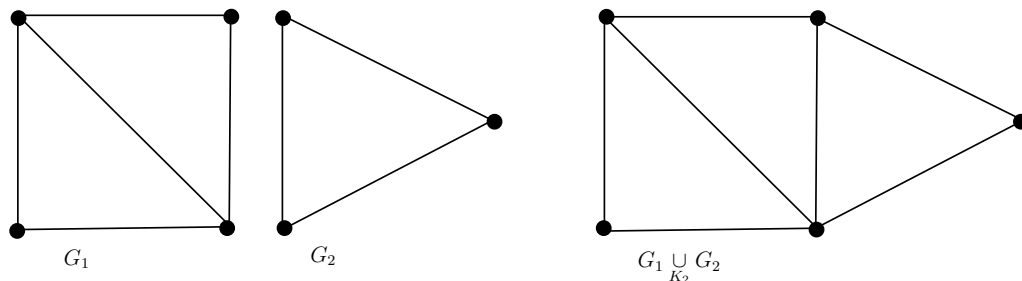


Figure 3.6: Graph $G_1 \cup_{K_2} G_2$

Example 3.2.2. Let G be the graph $G_1 \cup_{K_2} G_2$ given in Figure 3.6.

We compute the chromatic polynomial of G using Theorem 3.2.1.

$$\begin{aligned}
 \chi(G; \lambda) &= \frac{\chi(G_1; \lambda)\chi(G_2; \lambda)}{\chi(K_2; \lambda)} \\
 &= \frac{\chi(G_1; \lambda)\chi(K_3; \lambda)}{\lambda(\lambda - 1)} \\
 &= \frac{\chi(G_1; \lambda)[\lambda(\lambda - 1)(\lambda - 2)]}{\lambda(\lambda - 1)} \text{ by Theorem 2.3.5} \\
 &= (\lambda - 2)\chi(G_1; \lambda).
 \end{aligned}$$

Now we compute the chromatic polynomial of G using the deletion and contraction method given in Theorem 2.2.1.

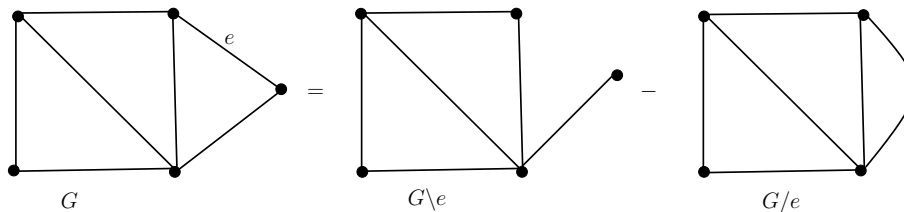


Figure 3.7: Chromatic polynomial of $G_1 \cup_{K_2} G_2$

The graphs in Figure 3.7 represents the chromatic polynomial of that graph. Hence

by applying Corollary 2.2.2 and Theorem 2.2.4, part(ii),

$$\begin{aligned}\chi(G; \lambda) &= (\lambda - 1)\chi(G_1; \lambda) - \chi(G_1; \lambda) \\ &= (\lambda - 2)\chi(G_1; \lambda),\end{aligned}$$

verifying the Theorem.

Remark 3.2.3. It is interesting to note that the graph G given in Figure 2.2, used in examples of Chapter 2 can be viewed as a K_2 gluing of K_3 and K_3 , see Figure 3.8.

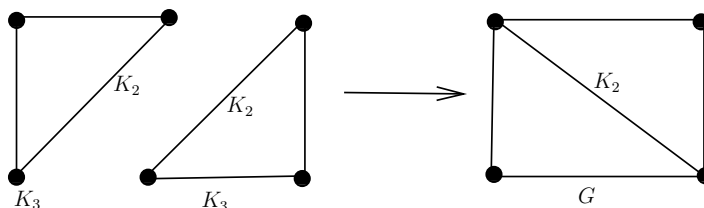


Figure 3.8: Graph $K_3 \cup_{K_2} K_3$

By application of Theorem 3.2.1, we get the chromatic polynomial of

$$\begin{aligned}\chi(G; \lambda) &= \frac{\chi(K_3; \lambda)\chi(K_3; \lambda)}{\chi(K_2; \lambda)} \\ &= \frac{\lambda(\lambda - 1)(\lambda - 2)\lambda(\lambda - 1)(\lambda - 2)}{\lambda(\lambda - 1)} \text{ by Theorem 2.3.5} \\ &= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 2) \\ &= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda.\end{aligned}$$

Verifying the results in Chapter 2, Example 2.4.1.

Theorem 3.2.4. Let G be the union of two graphs G_1 and G_2 . The chromatic polynomial of G ,

$$\chi(G; \lambda) = \chi(G_1; \lambda)\chi(G_2; \lambda).$$

The following Example illustrate the application of Theorem 3.2.4.

Example 3.2.5. Let the graphs G_1 and G_2 be the graphs given in Figure 3.9 and let G be the union of G_1 and G_2 .

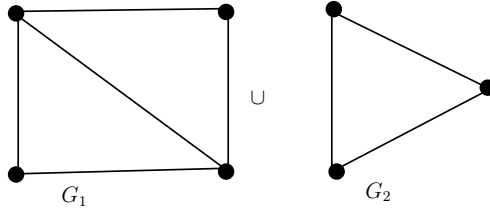


Figure 3.9: Graph $G_1 \cup G_2$

The chromatic polynomial of G , by Theorem 3.2.4,

$$\begin{aligned}
 \chi(G; \lambda) &= \chi(G_1; \lambda)\chi(G_2; \lambda) \\
 &= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 2)\lambda(\lambda - 1)(\lambda - 2) \text{ by Remark 3.2.3 and Theorem 2.3.5} \\
 &= \lambda^2(\lambda - 1)^2(\lambda - 2)^3.
 \end{aligned}$$

Chapter 4

Chromatic polynomials of the q -vertex join

In this chapter, we define the q -vertex join of a graph. We investigate the 2-vertex join of a graph G to test if there is a possibility of generalizing the chromatic polynomial of the q -vertex join of a graph in terms of the original graph, since there is a known result for 1-vertex join as discussed it in chapter 2. We then discuss the flow polynomials of a graph G , which is the chromatic polynomial of the dual graph of G . We find the flow polynomial of the q -analogue of a 1-fan.

4.1 q -vertex join

Let G be a graph with edge set $\{e_1, e_2, \dots, e_m\}$ and vertex set $\{w_1, w_2, \dots, w_n\}$. Recall the definition of vertex join of a graph from Chapter 1, Definition 1.5.1, part (iv). A vertex join of G , denoted \widehat{G} , is a graph with vertex set $V(\widehat{G}) = V(G) \cup \{w\}$ and edge set $E(\widehat{G}) = E(G) \cup \{(w_i, w) | i = 1, \dots, n\}$. We now extend this definition to q -vertex join of G . The graph obtained by replacing the edges of the form (w_i, w) in the vertex join of the graph G by a path p_{q+1} is called the q -vertex join of the graph G and is denoted by \widehat{G}^q . Note that the vertex join of a graph G is just the 1-vertex

join of G .

The graphs shown in the Figure 4.1 are examples of a graph G , graph \widehat{G} and graph \widehat{G}^3 .

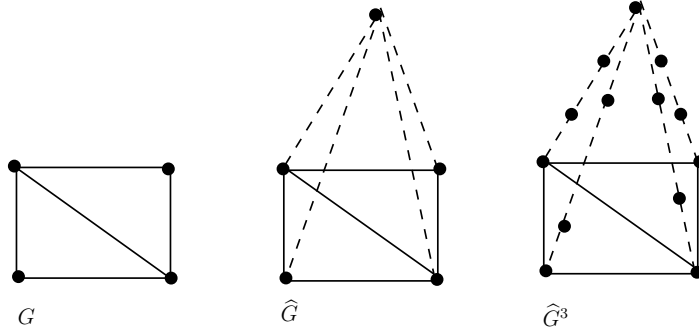


Figure 4.1:

To begin the investigation on the chromatic polynomial of the q -vertex join of G we use the null graph N_n since its chromatic polynomial is easy to compute. Then we do a similar investigation on the path P_n since its chromatic polynomial is also easy to compute.

4.2 Chromatic polynomial of the q -vertex join of a null graph

In this section, we compute the chromatic polynomial of the q -vertex join of a Null graph N_n .

The graphs shown in Figure 4.2 are examples of a graph N_5 , graph \widehat{N}_5 and graph \widehat{N}_5^3 .

Recall from Theorem 2.3.1 that the chromatic polynomial of the null graph of order n , $\chi(N_n; \lambda) = \lambda^n$. In addition, note that the vertex join of the null graph N_n is a star graph S_{n+1} defined in Chapter 1, Remark 1.4.3. Thus, the vertex join of the null

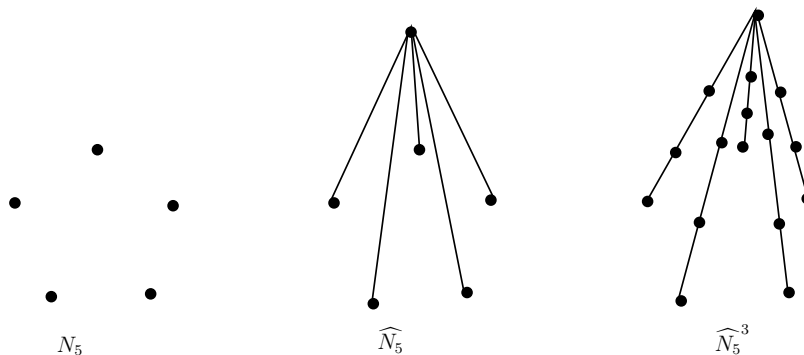


Figure 4.2:

graph N_n is a tree of order $n + 1$. Hence the chromatic polynomial of the vertex join of the null graph, $\chi(\widehat{N}_n; \lambda) = \lambda(\lambda - 1)^n$. It is clear from the definition that if we take the q -vertex join of the null graph, we have a tree of order $nq + 1$. Hence, we state the following proposition without proof.

Proposition 4.2.1. *Let \widehat{N}_n^q be the q -vertex join of the null graph, N_n . Then the chromatic polynomial*

$$\chi(\widehat{N}_n^q; \lambda) = \lambda(\lambda - 1)^{nq}.$$

By inspection of the chromatic polynomial of the the q -vertex join of the null graph, there is no clear relationship with the chromatic polynomial of the original graph N_n . Hence we investigate further; we take the q -vertex join of a path and compute its chromatic polynomial.

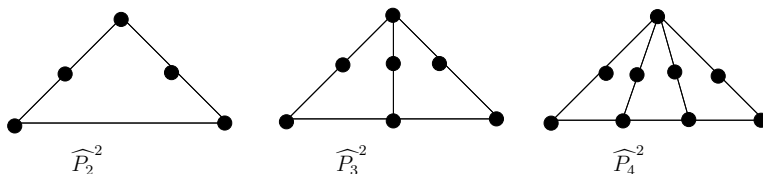


Figure 4.3: 2-vertex joins of paths

The graphs shown in Figure 4.3 are examples of 2-vertex joins of certain paths, namely

\widehat{P}_2^2 , \widehat{P}_3^2 and \widehat{P}_4^2 .

4.3 Chromatic polynomial of the 2-vertex join of a path

In this section, we compute and study the chromatic polynomial of the 2-vertex join of a path.

We begin by computing the chromatic polynomial of the 2-vertex join of a path P_2 . To ease notation, each graph in Figure 4.4, represent the chromatic polynomial of that graph.

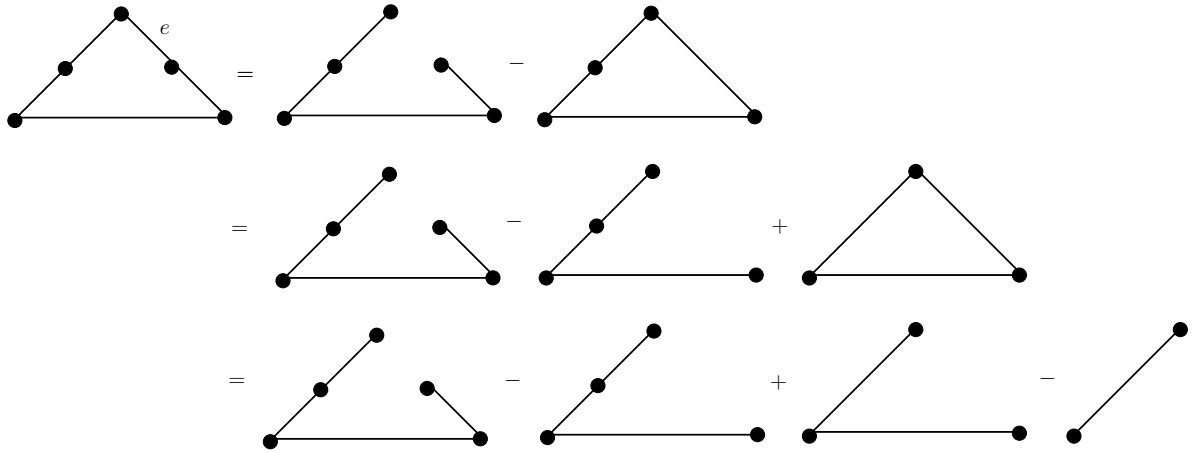


Figure 4.4:

Hence if we replace each graph by its chromatic polynomial, we get

$$\begin{aligned}
 \chi(\widehat{P}_2^2; \lambda) &= \lambda(\lambda - 1)^4 - \lambda(\lambda - 1)^3 + \lambda(\lambda - 1)^2 - \lambda(\lambda - 1) \\
 &= \lambda(\lambda - 5\lambda^3 + 10\lambda^2 - 10\lambda + 4) \\
 &= (\lambda - 2)(\lambda - 1)\lambda(\lambda^2 - 2\lambda + 2).
 \end{aligned}$$

We now compute the chromatic polynomial of the 2-vertex join of a path P_3 given in

Figure 4.5.

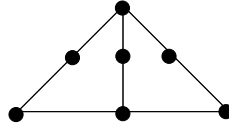


Figure 4.5:

We use mathematica to get

$$\begin{aligned}
 \chi(\widehat{P_3}^2; \lambda) &= -\lambda^2(\lambda - 1)^5 + 3\lambda^4 - 4\lambda(\lambda - 1)^3 + 3\lambda(\lambda - 1)^2 - \lambda(\lambda - 1)^6 - \lambda^2(\lambda - 1)^5 \\
 &= \lambda^2(\lambda - 1)^6 - \lambda^2(\lambda - 1)^5 + 3\lambda(\lambda - 1)^4 - 4\lambda(\lambda - 1)^3 + 3\lambda(\lambda - 1)^2 - (\lambda - 1)\lambda \\
 &= (\lambda - 2)(\lambda - 1)\lambda(\lambda^2 - 2\lambda + 2)(\lambda^3 - 2\lambda^2 - \lambda + 3).
 \end{aligned}$$

Finally, we compute the chromatic polynomial of the 2-vertex join of a path P_4 given in Figure 4.6.

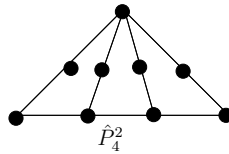


Figure 4.6:

We use mathematica to get

$$\begin{aligned}
 \chi(\widehat{P_4}^2; \lambda) &= \lambda^3(\lambda - 1)^8 - 3\lambda^3(\lambda - 1)^5 - 2\lambda^2(\lambda - 1)^7 + 2\lambda^2(\lambda - 1)^6 \\
 &+ \lambda^2(\lambda - 1)^6 - 3\lambda^2(\lambda - 1)^5 + 3\lambda^2(\lambda - 1)^4 + \lambda^2(\lambda - 1)^4 - 2\lambda^2(\lambda - 1)^3 \\
 &+ (\lambda - 2)(\lambda^2 - 2\lambda + 2 + \lambda + 3)\lambda(\lambda - 1) + 2\lambda(\lambda - 1)^6 - 5\lambda(\lambda - 1)^5 \\
 &+ 5\lambda(\lambda - 1)^4 - 4\lambda(\lambda - 1)^3 + 2\lambda(\lambda - 1)^2 \\
 &= (\lambda - 2)(\lambda - 1)\lambda(\lambda^8 - 5\lambda^7 + 9\lambda^6 - 2\lambda^5 - 17\lambda^4 + 25\lambda^3 - 3\lambda^2 - 20\lambda + 15).
 \end{aligned}$$

We observe that after factorizing the chromatic polynomials of \widehat{P}_2^2 , \widehat{P}_3^2 and \widehat{P}_4^2 , there is no clear indication how we can obtain these chromatic polynomials from the chromatic polynomials of the original graphs P_2 , P_3 and P_4 .

Although we are not able to find the expression of the chromatic polynomial of the q -vertex join of G in terms of the chromatic polynomial of G , we are able to find some results on the q -vertex join of G of a related polynomial, known as the Flow polynomial.

4.4 Flow polynomial

In this section, we begin by defining and constructing a dual graph of a planar graph. Then we discuss the relationship between the chromatic polynomial and the flow polynomial of a planar graph. Finally, we give the flow polynomial of the q -vertex join of a path.

4.4.1 Construction of a dual graph

Definition 4.4.1. Let G be a graph. If there exists a drawing of the graph G in the plane such that no two edges intersect in a point other than a vertex of the graph, then G is called a planar graph. A plane graph is planar graph which has been drawn without any edges crossing.

The graphs in Figure 4.7 are examples of a planar graph and a plane graph.

Recall from Definition 1.3.9 that a closed trail is called a cycle of a graph. If G is a plane graph, then each closed trail (cycle) divides the plane into regions. These regions are called faces of the graph G . Note that the outside unbounded region is also a face, usually called the exterior face. Each edge of a cycle in a plane graph is said be adjacent to the faces it splits in the plane.

Note that f_1, f_2, \dots, f_7 in Figure 4.8 are the faces of the given plane graph. The edge e in the graph is adjacent to faces f_2 and f_3 .

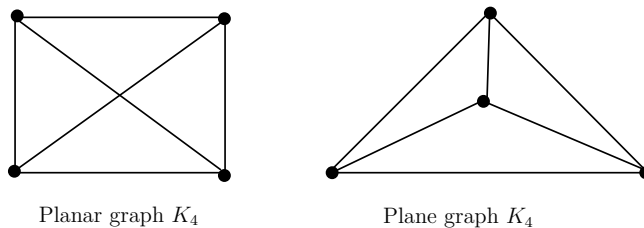


Figure 4.7:

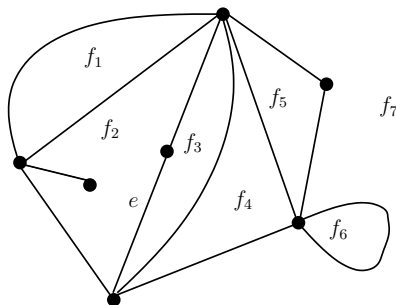


Figure 4.8:

Let G be a plane graph with a set of n faces $\{f_1, f_2, \dots, f_n\}$ and a set of cycles $\{c_1, c_2, \dots, c_m\}$. We define the dual graph of G to be the graph with vertex set $\{f_1, f_2, \dots, f_n\}$ and edge set $\{(f_i, f_j) \mid f_i, f_j \text{ is adjacent to an edge } e_k \forall e_k \in E(c_m)\}$.

Example 4.4.2. The graphs in Figure 4.9 demonstrates the construction of the dual graph G^* of a graph G .

Definition 4.4.3. A vertex is said to be conservative if its weight is zero.

Definition 4.4.4. A vertex is said to be conservative if its weight is zero. A flow on a graph G is a labeling of the edges with values in an Abelian group, such that each vertex $v \in V(G)$ is conservative. A nowhere-zero flow is a flow such that none of the edges are labeled zero.

Recall from Definition 2.1.2 that the chromatic polynomial, $\chi(G; \lambda)$, of a graph G

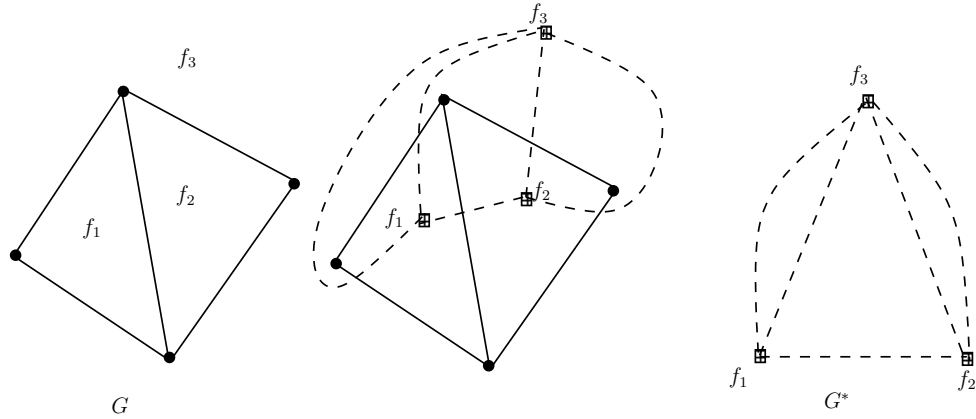


Figure 4.9:

count the number of ways of properly coloring the vertices of G using λ colors. In a similar way we now define the flow polynomial of a graph G .

Definition 4.4.5. The flow polynomial of a graph G gives the number of nowhere-zero λ flows in G independently of the selected angle and is denoted by $F(G; \lambda)$.

The following alternative definition of the flow polynomial is given by Tutte in [16].

Definition 4.4.6. Let G be a graph. The flow polynomial of G ,

$$F(G; \lambda) = (-1)^{|E(G)|} \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{V(G:S)},$$

where $G : S$ denotes the spanning subgraph of G with edge set S .

It is widely known in the literature that the flow polynomial and the chromatic polynomial of a graph have a special connection. We now state without proof the relationship between the flow polynomial and the chromatic polynomial in the following theorem. For further details we refer the reader to [6, 21].

Theorem 4.4.7. Let G be a planar graph, $F(G; \lambda)$ the flow polynomial of G and $\chi(G^*; \lambda)$ the chromatic polynomial of G^* . Then

$$F(G; \lambda) = \frac{1}{\lambda^{k(G)}} \chi(G^*; \lambda)$$

where $k(G)$ is the number of connected components of G .

4.5 Flow polynomial of the 2-vertex join of a path P_n

In Section 4.3, we tried to find the formula for the chromatic polynomial of the 2-vertex join of a path. In this section we study and generalize the dual graph of the 2-vertex join of a path P_n . Finally, we find a general formula for the flow polynomial of the 2-vertex join of a path.

Recall from Definition 1.3.7 that the graph obtained by identifying each parallel class as a single edge in a multigraph G is called the simple graph of G . That is if G' is a multigraph and \tilde{G} its simple graph, we say G' is isomorphic to \tilde{G} up to parallel class. We begin by studying the dual graphs of certain 2-vertex joins of paths. The graphs in Figure 4.10 gives the 2-vertex join of a path \widehat{P}_n^2 , the dual of the 2-vertex join of a path $(\widehat{P}_n^2)^*$, and the simple graph of the dual of the 2-vertex join of a path $(\widehat{P}_n^2)^*$.

An observation in Figure 4.10 lead to the following theorem.

Theorem 4.5.1. *Let \widehat{P}_n^2 be a 2-vertex join of a path P_n . Then the dual graph $(\widehat{P}_n^2)^*$ is isomorphic to the vertex join of a path P_{n-1} , \widehat{P}_{n-1}^2 , up to parallel class.*

Proof. Our proof is by induction on $n \geq 2$.

We start with $n = 2$. The diagrams in Figure 4.10, shows that up to parallel class the dual graph $(\widehat{P}_2^2)^*$ is isomorphic to P_2 . But by Definition 1.5.1 of vertex join we get that $\widehat{P}_1 \cong P_2$. Hence the result is true for $n = 2$.

Assume that the theorem is true for some $n = k$, that is the dual graph of \widehat{P}_k^2 , $(\widehat{P}_k^2)^* \cong \widehat{P}_{k-1}^2$ up to parallel class.

Now we consider the case when $n = k + 1$, i.e the 2-vertex join of P_{k+1} .

We prove by constructing the 2-vertex join of P_{k+1} from the 2-vertex join of P_k .

In general, consider the diagrams in Figure 4.11 such that the graph \widehat{P}_5^2 is being constructed from \widehat{P}_4^2 , by adding one face f_5 , two vertices and three edges.

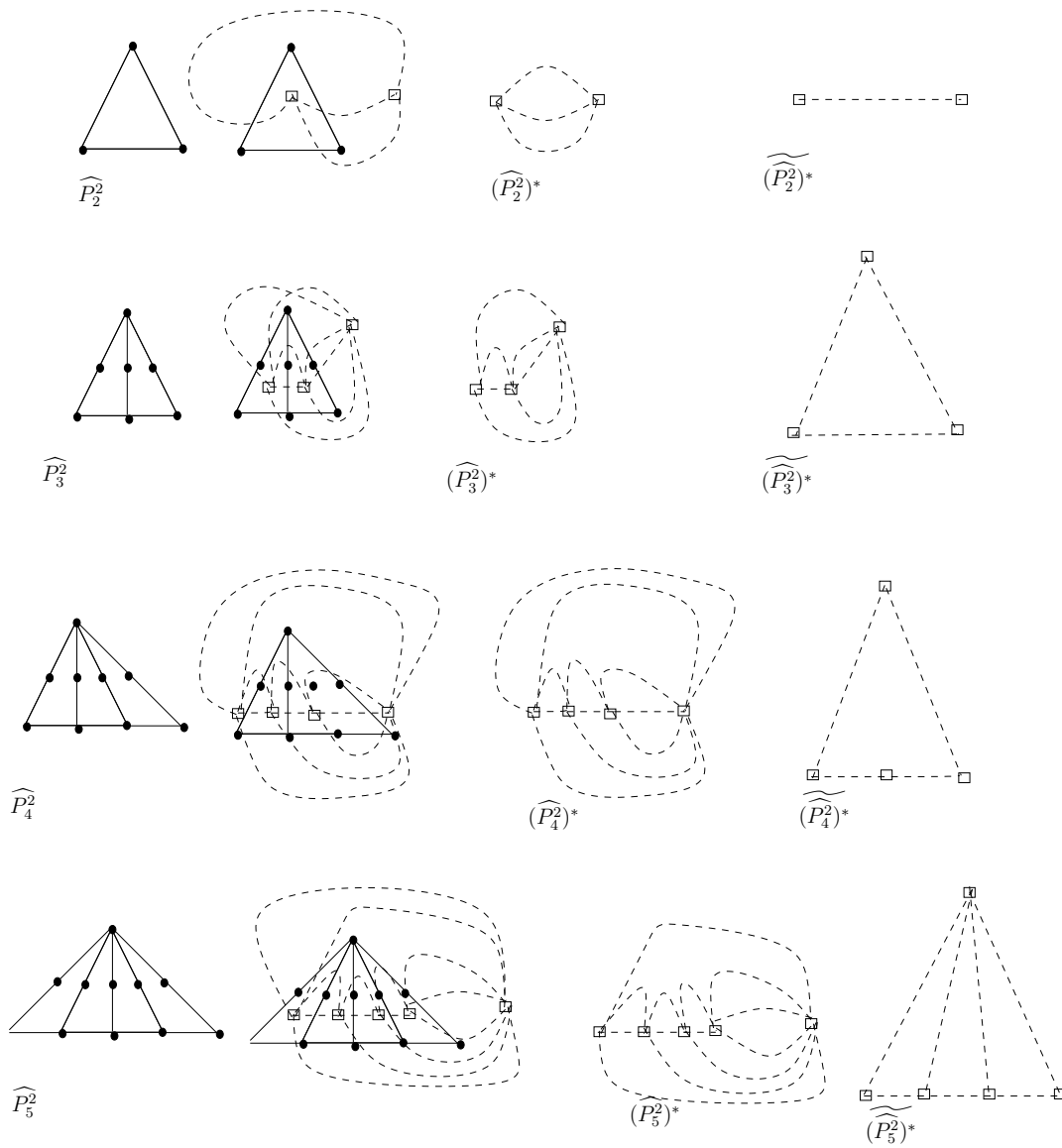


Figure 4.10:

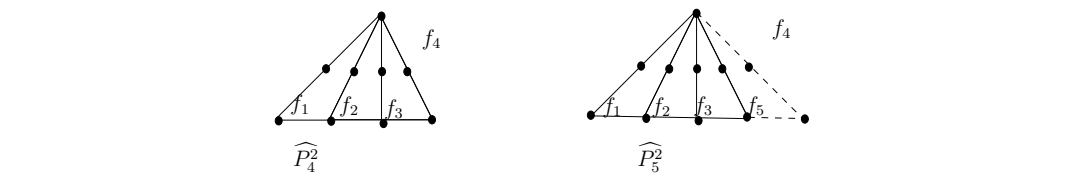


Figure 4.11:

Let $G = \widehat{P}_4^2$ and $H = \widehat{P}_5^2$. Then the vertex set $V(G^*) = \{f_1, f_2, f_3, f_4\}$ and edge set

$$\begin{aligned} E(G^*) &= \{(f_1, f_4), (f_1, f_4), (f_1, f_4), (f_1, f_2), (f_1, f_2)\} \\ &= \cup\{(f_2, f_3), (f_2, f_3), (f_2, f_4), (f_3, f_4), (f_3, f_4), (f_3, f_4)\}. \end{aligned}$$

Hence the simple graph of G^* will have edge set

$$E(\widetilde{G}^*) = \{(f_1, f_4), (f_1, f_2), (f_2, f_3), (f_2, f_4), (f_3, f_4)\}.$$

Thus our induction assumption $\{(f_1, f_2), (f_2, f_3)\}$ is an edge set of a path P_2 and f_4 is the join vertex.

It is clear that in the construction of the dual graph of H we add three parallel edges, (f_5, f_4) , two parallel edges (f_5, f_3) and removed two parallel edges (f_4, f_3) leaving only one (f_4, f_3) to the edge set of the dual of G^* . Hence $E(\widetilde{H}^*) = E(\widetilde{G}^*) \cup \{(f_5, f_4), (f_5, f_3)\}$. Thus

$$\begin{aligned} E(\widetilde{H}^*) &= E(\widetilde{G}^*) \cup \{(f_5, f_4), (f_5, f_3)\} \\ &= \{(f_1, f_4), (f_1, f_2), (f_2, f_3), (f_2, f_4), (f_3, f_4), (f_5, f_3), (f_5, f_4)\}. \end{aligned}$$

Thus by the induction assumption $\{(f_1, f_2), (f_2, f_3), (f_5, f_3)\}$ is an edge set of a path P_3 and f_4 is the join vertex. Hence the theorem is true for all n by principle of mathematical induction. \square

Theorem 4.5.2. *Let $G = \widehat{P}_n^2$ be a 2-vertex join of a path P_n . Then the flow polynomial of G ,*

$$F(G; \lambda) = (\lambda - 1)(\lambda - 2)^{n-2}.$$

Proof. By Theorem 4.4.7

$$F(G; \lambda) = \frac{1}{\lambda^{k(G)}} \chi(G^*; \lambda). \quad (4.1)$$

G is connected and hence has one component, thus $k(G) = 1$. Further, by Theorem 4.5.1 we know that $G^* \cong \widehat{P}_{n-1}$ up to parallel class. Applying Theorem 2.2.4 we

get $\chi(G^*; \lambda) = \chi(\widehat{P_{n-1}}; \lambda)$. Substituting these in Equation 4.2, we get

$$\begin{aligned} F(G; \lambda) &= \frac{1}{\lambda^k(G)} \chi(G^*; \lambda) \\ &= \frac{1}{\lambda} \chi(\widehat{P_{n-1}}; \lambda) \\ &= \frac{1}{\lambda} \lambda \chi(P_{n-1}; \lambda - 1) \text{ by Theorem 3.1.4} \\ &= \chi(P_{n-1}; \lambda - 1) \\ &= (\lambda - 1)((\lambda - 1) - 1)^{n-2} \text{ by Theorem 2.3.2} \\ &= (\lambda - 1)(\lambda - 2)^{n-2}. \end{aligned}$$

□

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