Difference Equations And Their Symmetries

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Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Abstract

The aim of the dissertation is to extend on the work done by Hydon in [17]. We only consider second order ordinary difference equations and calculate their symmetry generators, first integrals and reduce their order, that is, find a general solution. We investigate the association between a symmetry generator and a first integral. Furthermore, we investigate when a reduced equation may be further reduced and lead to a double reduction. The examples considered are obtained from [17].

To my beloved family.

With God on our side, the best is yet to come.

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"Trust in the Lord with all your heart and lean not on your own understanding, in all your ways acknowledge him, and he will make your paths straight." - Proverbs 3:5-6.
Thank you Lord God almighty, without you this dissertation would not exist.

A famous Zulu proverb states that "umuntu ngumuntu ngabantu", meaning a person is a person because of other people. The following people have shaped me to the person I am today.

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Contents

Chapter 1

Introduction

1.1 Historical Background

S. Elaydi in [13] states that difference equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)^{st}$ generation, $x(n+1)$ is a function of the n^{th} generation $x(n)$ and the relation can be conveyed by the difference equation

$$x(n+1) = f(x(n)).$$

It is worth noting that the term"difference equation" is frequently used to refer to any recurrence relation. A recurrence relation can be defined as an equation that recursively defines a sequence, once one or more initial conditions are provided. Each further term of the sequence is defined as a function of the preceding term as shown in the above example.

The philosophy of difference equations and their applications have cemented a central position in applicable analysis. According to [27], in the last 12 years, the subject of difference equations has produced hundreds of research articles, several monographs, many international conferences, and numerous special sessions.
Difference equations have been studied by many well-known mathematicians including Leonardo Fibonacci (1170-1250), Abraham de Moivre (1667-1754), and Gottfried Wilhelm von Leibnz (1646-1716). The influence of these mathematicians on the history and developments of difference equations is a rather interesting one. We shall follow its development carefully as discussed in [19].

As early as 2000 B.C, the idea of computing by recursion was recognised by the Babylonians in efforts to extract roots. The idea was extended in a more explicit form around 450 B.C in the Pythagoreans' study of figurative numbers. As stated in [19], the Pythagoreans further used a system of difference equations to generate large solutions of Pell's equation and thereby approximations of $\sqrt{2}$. About 250 B.C, Archimedes employed the difference equations

$$\begin{aligned} P_{2n} &= 2p_nP_n/(p_n+P_n), \\ p_{2n} &= \sqrt{p_nP_{2n}}, \end{aligned}$$

in attempts to calculate the circumference of a circle.

The earliest known example of a difference equation in two indices, namely the equation $b_{n+1,r} = b_{n,r} + b_{n,r-1}$ for the binomial coefficients, can be sketched back to Chia Hsien (1010-1070) and Omar Khayyan (1048-1131) [19]. It was not until 1202 that Leonardo Fibonacci formulated his famous rabbit problem that led to the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, known as the Fibonacci sequence and has the corresponding difference equation $F_n = F_{n-2} + F_{n-1}$ with initial values $F_0 = 1$ and $F_1 = 1$. In 1572 an Italian mathematician, Rafael Bombelli (1526-1572), studied the equation $y_n = 2 + \frac{1}{y_{n-1}}$, which is similar to the equation $z_n = 1 + \frac{1}{z_{n-1}}$ satisfied by ratios of Fibonacci numbers, in order to approximate $\sqrt{2}$.

Also from [19] it is noted that when Francesco Maurolico (1494-1575) developed the concept of mathematical induction in the sixteenth century, there was a significant advancement in the method of recursion and hence the concept of difference equations. The introduction of the concept of calculus of finite differences by Sir Thomas Harriet $(1560-1621)$ was crucial as it was used by Newton, Euler, Lagrange, Gauss, and many others to study interpolation theory. It was in 1669 that Newton used an important class of non-linear difference equations to study solutions of $y^3-2y-5=0$ and later in computations for Kepler's equation.

The authors of [19] mention that the basic theory of linear difference equations was developed in the eighteenth century by de Moivre, Euler, Lagrange, Laplace, and others. In particular, Laplace exploited the generating functions which were first used by de Moivre to solve the Fibonacci equation. The concept of using difference equations to approximate solutions of differential equations originated in 1769 with

Euler's polygon method. In the 1880's, Jules Henri Poincare (1854-1912) laid the foundation for the study of the asymptotic properties of solutions of linear difference equations. The urgent need for numerical approximations during World War I significantly stimulated research into the study of the properties of difference equations. Furthermore, the development of the digital computer brought an influx of publications in the area of difference equations. Then in 1952, the application of linear difference equations to the computation of special functions originated with Miller's algorithm for Bessel functions.

During the 1950's, as indicated in [19], an important application of difference equations was presented. Numerous ecologists used simple non-linear difference equations, including the logistic equation which is a model of population growth first published by Pierre Verhulst, to study the change in populations from one year (or season) to the next. Soon after this, momentous discoveries were made about the logistic and related equations by York, Sarkovskii, Feigenbaum, and others. The success of these studies attracted the attention of researchers who attempted to apply the results to fields of economics, medicine, engineering and physics.

A new area in the study of difference equations is that of imposing boundary conditions. Such can be seen in [2] where Atkinson sets up boundary problems on a three term recurrence relation given by

$$c_n y_{n+1} + (a_n \lambda + b_n) y_n + c_{n-1} y_{n-1} = 0. \tag{1.1}$$

In their work, S. Currie and A. Love considered a variation of the above recurrence relation, that is, they studied a weighted second-order difference equation of the form

$$c_n y_{n+1} - b_n y_n + c_{n-1} y_{n-1} = -c_n \lambda y_n, \tag{1.2}$$

where a_n in equation (1.1) is replaced by c_n. In a series of five publications, [6], [7], [8], [9] and [10], S. Currie and A. Love studied the effect of applying two Crum-type transformations to the above recurrence relation with the several combinations of Dirichlet, non-Dirichlet, and affine λ-dependent boundary conditions at the end points [9], [10]. In [6], [7] and [8], they considered general λ-dependent boundary conditions at both the initial and terminal ends. The brief history provided clearly shows that the area of difference equations is broad and that there is a scope to discover new theories.

Quispel and Sahadevan assert that, “at present, the state of the theory of difference equations is somewhat similar to the state of the theory of differential equations one hundred years ago” (cf [24], p 64). However, difference equations are important because of their many applications in mathematical modelling. They have applications in biology, digital signal processing and economics, just to name a few. We consider applications provided in [28]. In biology, for example, the Fibonacci numbers were once used as a model for the growth of a rabbit population. The logistic map, mathematically written

$$x_{n+1} = rx_n(1 - x_n),$$

is used to model population growth, or as a starting point to more detailed models. Another well-known use of difference equations in biology is the modelling of the interaction of two or more populations. An example would be the Nicholson-Bailey model for a host-parasite interaction given by

$$\begin{aligned} N_{t+1} &= \lambda N_t e^{-aP_t}, \\ P_{t+1} &= N_t\left(1 - e^{-aP_t}\right), \end{aligned}$$

where N_t represents the hosts and P_t the parasites at time t. In digital signal processing, recurrence relations can model feedback in a system, where outputs at one time become inputs for future time. Linear recurrence relations are used extensively in both theoretical and empirical economics. In particular, in macroeconomics where a model may be developed for broad sectors of the economy e.g. the financial, goods or labour sectors. Next we provide a brief history and review the main publications in the topic of difference equations.

More than 100 years ago, Sophus Lie (1842-1899) who was mostly responsible for the theory of continuous symmetries, presented symmetry-based techniques for solving ordinary differential equations (ODEs). His techniques enabled individuals to find Lie groups of symmetries of a certain ODE. His methods could also be used to solve ODEs provided that a large symmetry group could be established. In particular, the similarity method is a tool that is still used to solve differential equations (for example, see [3]). However, it is stated in [15] that Lie's methods for determining and using symmetries were greatly ignored until recently. Most of the theory provided in this dissertation is extracted from the work done by P. Hydon in [17] where he provides a symmetry-based approach to solving a given ordinary difference equation (OΔE). In [16], he introduces a technique for obtaining the conservation laws of a given partial difference equation (PΔE) with two independent variables. Differential equations and difference equations share common ground. Understanding symmetry methods for differential equations is key to understanding the material provided in this dissertation. An introduction to symmetry methods for ODEs can be found in [4], [16], [18] or [26]. In the present study, particular focus will be placed on the theory of symmetry methods for difference equations.

In 1987, Maeda [22] applied the similarity method, which is an extension of Lie's method, to ordinary difference equations and showed, using two examples, that the similarity method is useful for obtaining expressions for difference equations. He indicated that autonomous systems of first-order OΔEs can be simplified or solved using this method. In his 1993 publication [14], Gaeta discussed the relation between Lie-point symmetries of ODEs and of the maps obtained by discretizing it over a time step λ. Gaeda emphasises that this subject was also studied by Dorodnitsyn [11] where he gave a general formulation, discussion of the problem and corresponding general results. Dorodnitsyn teamed up with Kozlov and Winternitz in [12] where they used the same technique as in [14] to descretize the second-order ODEs.

Quispel and Sahadevan [24] generalised the Lie method to ordinary difference equations. By explicitly solving a first-order difference equation they proved that the order of ordinary difference equations can be reduced by one. Furthermore, they provided a sufficient condition for reduction of the order by two which they showed by solving a second-order difference equation. The examples they provided were autonomous ordinary difference equations, however, they stressed that their method could be applied to nonautonomous equations. In [21], Levi presented a constructive general formalism for calculating symmetries of difference equations and by doing so extended on Maeda's ideas. Levi also reviewed the use of Lie groups to study difference equations in [20]. He showed that the mismatch between continuous symmetries and discrete equations could be showed by using generalized symmetries acting on solutions of difference equations, but leaving the lattice invariant.

Recently, P. Hydon [17] described new methods for finding and using symmetries of a given OΔE by studying the local structure of the set of solutions. That is, he presented a systematic method for finding Lie symmetries of a given OΔE.

The remainder of this dissertation is set out as follows.

In Chapter 2 we give important definitions and present the different methods used to solve ordinary difference equations. In Chapter 3, we use the symmetry-finding algorithm to find the Lie symmetry generators of a second order difference equation, denoted by OΔE.

Chapter 4 is divided into two sections. Section 4.1, illustrates how to find the general solution and in Section 4.2, we find the first integral of an OΔE. In Chapter 5, we consider two different examples. The first being a non-linear example, while the second is linear. In both cases we find their symmetry generators, first integrals and reduce their orders.

Chapter 2

Mathematical Preliminaries

There are two notations used to represent difference equations, both used in the previous chapter. For example, the difference equations for the Fibonacci sequence can be denoted as $F(n) = F(n-2) + F(n-1)$ or $F_n = F_{n-2} + F_{n-1}$. In this dissertation we will use both notations.

We dedicate this chapter to providing definitions and other mathematical preliminaries that will be used in the coming chapters. Furthermore, we define basic concepts that were mentioned in the introduction. We will also provide examples to further explain certain definitions. However, since the theory of symmetries and first order integrals of difference equations is broad, we divide this chapter into three sections. Section 2.1 will provide basic definitions about difference equations and Section 2.2 will introduce a method of solving difference equations using powers of matrices. Section 2.3 will provide more complex definitions focusing on the theory provided in [17]. We start by giving definitions of the basic concepts of difference equations.

2.1 Basic Definitions

We begin this section by giving the definition of the order of a difference equation.

Definition 2.1.1. *(cf [5], p 1)*
The order of a recurrence relation is the difference between the greatest and lowest subscripts of the terms of the sequence in the equation.

For example, $u_{n+3} + u_{n+1} = -u_n$ is a recurrence relation of order 3. The order is important as it allows us to name/classify difference equations. For example the difference equation

$$u_{n+k} = g(u_{n+k-1}, u_{n+k-2}, \dots, u_n),$$

for a given function g, is called a difference equation of order k.

Elaydi (cf [13], p 2) defines and classifies difference equations as autonomous or nonautonomous. An autonomous difference equation is given by

$$x(n+1) = f(x(n)).$$

If the above function f is replaced by a function g of two variables, that is, $g : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{Z}^+$ is the set of nonnegative integers and $\mathbb{R}$ is the set of real numbers, then we have a nonautonomous equation given by

$$x(n+1) = g(n, x(n)).$$

According to Elaydi, " the study of nonautonomous equations is much more complicated and does not lend itself to the discrete dynamical system theory of first order equations," see [13]. Next we consider the linear difference equation and provide the definitions of a linear homogeneous and nonhomogeneous first-order equations.

Definition 2.1.2. *(cf [13], p 2)*
A typical linear homogeneous first-order equation is given by

$$x(n+1) = a(n)x(n), \qquad x(n_0) = x_0, \qquad n \geq n_0 \geq 0,$$

and the associated nonhomogeneous equation is given by

$$y(n+1) = a(n)y(n) + g(n), \qquad y(n_0) = y_0, \qquad n \geq n_0 \geq 0,$$

where in both equations it is assumed that $a(n) \neq 0$, and $a(n)$ and $g(n)$ are real-valued functions defined for $n \geq n_0 \geq 0$.

This brings us to the end of this section. The next section explains how to solve difference equations using powers of matrices. We also introduce the Cayley Hamilton theorem.

2.2 Solving difference equations using powers of matrices

One method of solving difference equations is using powers of matrices. However, this method is better illustrated by example. Suppose we have a difference equation

$$a_{n+1} = ca_n + da_{n-1}.$$

We can solve for a_n explicitly in terms of n by considering the equations

$$\begin{aligned} a_{n+1} &= ca_n + da_{n-1}, \\ a_n &= a_n, \end{aligned}$$

and writing them in matrix form as follows

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} c & d \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$

If we let $A = \begin{pmatrix} c & d \\ 1 & 0 \end{pmatrix}$, we have the following

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$

In the same way

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}.$$

Therefore

$$\begin{aligned} \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} &= A.A \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} \\ &= A^2 \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}. \end{aligned}$$

After sufficient iterations, we finally have

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}.$$

So, if we have an expression for A^n then we can express a_n in terms of a_1 and a_0 (both of which are usually given as initial conditions). For the Fibonacci numbers,

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha^{n+1} - \beta^{n+1} \\ \alpha^n - \beta n \end{pmatrix}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Therefore, $a_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$. However $\beta^n \to 0$ since $\beta \approx -0.618$ and hence $a_n \to \frac{1}{\sqrt{5}}\alpha^n$.

For a given matrix A, we can find the characteristic equation $c_A(\lambda)$ where

$$c_A(\lambda) = \det(A - \lambda I_n).$$

The values of λ are the eigenvalues and have corresponding eigenvectors X $(\neq \vec{0})$ i.e. $AX = \lambda X$. Generally, distinct eigenvalues have distinct eigenvectors, but not always. Let T be the matrix of eigenvectors corresponding to the eigenvalues λ i.e. the $X's$ are the columns of T. Then T is invertible (if eigenvectors are independent) and

$$T^{-1}AT = D \tag{2.1}$$

where D is a diagonal matrix with the eigenvalues appearing on the main diagonal. From (2.1)

$$\begin{aligned} A &= TDT^{-1}, \\ A^2 &= (TDT^{-1})(TDT^{-1}), \\ &= TD^2T^{-1}, \\ &\vdots \\ A^n &= TD^nT^{-1}. \end{aligned}$$

If T is not invertible, i.e. repeated eigenvectors, then we need to use the Cayley-Hamilton theorem. The Cayley-Hamilton theorem is used in applications in which diagonalization fails because there are insufficient independent independent eigenvectors. Below we state the Cayley-Halimton theorem provided in [25].

Theorem 2.2.1. *([25], Chapter 2)*
Every square matrix satisfies its own characteristic equation.

Now suppose A is a matrix of constant coefficients and that

$$\dot{X} = AX$$

where dots denote differentiation with respect to t. Then by differentiation,

$$\ddot{X} = A\dot{X} = A(AX) = A^2X.$$

Repeating this process

$$\frac{d^n}{dt^n}X = A^n X \quad \forall n$$

and by taking linear combinations of these derivatives we obtain a "polynomial of derivatives"

$$P\left(\frac{d^n}{dt^n}\right)X = P(A)X$$

where the degree of the derivatives corresponds to the power of the matrix, and this holds for any polynomial P. In particular, consider the characteristic equation and let

$$P(\lambda) = c_A(\lambda).$$

Then

$$P(A) = c_A(A) = 0$$

by the Cayley-Hamilton theorem. Thus,

$$c_A\left(\frac{d}{dt}\right)X = c_A(A)X = 0X = \vec{0}.$$

This equation can be solved by using D-operator methods with the arbitrary constants now being arbitrary constant vectors.

For example, consider the difference equation

$$A(n+1) = A(n). \tag{2.2}$$

We can generate the following system of equations

$$\begin{aligned} A(n+1) &= A(n), \\ A(n) &= A(n), \end{aligned}$$

which can be written in matrix form as

$$\begin{pmatrix} A(n+1) \\ A(n) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A(n) \\ A(n-1) \end{pmatrix},$$

where $A(1) = A(0) = c$ (c a constant).

Set $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. To find the eigenvalues of B we solve $c_B(\lambda) = 0$. Then,

$$c_B(\lambda) = \det\begin{pmatrix} 1-\lambda & 0 \\ 1 & -\lambda \end{pmatrix}$$

$$
\begin{aligned}
&= -\lambda(1-\lambda) - 0 \\
&= \lambda(\lambda - 1).
\end{aligned}
$$

That is, the eigenvalues are $\lambda_1 = 0$ or $\lambda_2 = 1$.
When $\lambda = 0$, we have

$$
\begin{aligned}
(B - \lambda I_2 : 0) &\sim \begin{pmatrix} 1 & 0 & : & 0 \\ 1 & 0 & : & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & : & 0 \\ 0 & 0 & : & 0 \end{pmatrix}.
\end{aligned}
$$

Then, $x = 0$ and y can take any value. Take $y = 1$, then the eigenvector associated with $\lambda_1 = 0$ is $v_1 = (0, 1)$. Check: $B\underline{v}_1 = \lambda_1 \underline{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
When $\lambda = 1$, we have

$$
(B - \lambda I_2 : 0) \sim \begin{pmatrix} 0 & 0 & : & 0 \\ 1 & -1 & : & 0 \end{pmatrix}.
$$

Then $x = y$, that is the eigenvector corresponding to $\lambda_2 = 1$ is $v_2 = (1, 1)$.
Define $T = (\underline{v}_1, \underline{v}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then,

$$
\begin{aligned}
T^{-1} &= \frac{1}{1(0) - (1)(1)} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}
$$

Now let's define D, the diagonal matrix, as

$$
\begin{aligned}
D &= T^{-1}BT \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{so} \\
D^n &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Then,

$$\begin{aligned}
B^n &= TD^nT^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore, using the generalized work above we have

$$\begin{aligned}
\begin{pmatrix} A(n+1) \\ A(n) \end{pmatrix} &= B \begin{pmatrix} A(n) \\ A(n-1) \end{pmatrix} \\
&= B^n \begin{pmatrix} A(1) \\ A(0) \end{pmatrix}, \text{by the generalised work above.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\begin{pmatrix} A(n+1) \\ A(n) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ c \end{pmatrix} \\
&= \begin{pmatrix} c \\ c \end{pmatrix},
\end{aligned}$$

which implies $A(n) = c$.

Note that for this example we did not have to use the Cayley Hamilton theorem. For more complicated difference equations we will use the R-Solve command in *Mathematica*. The next Section provides an easier way of solving second order difference equations.

2.3 Using the auxiliary equation to find the general solution

Most of the difference equations we will have to solve are second order difference equations. In this section, we provide the general solution of second order difference equations. To solve first order difference equations, it is possible to use an iterative technique to find the general solution. However, an iterative technique does not work for second order difference equations.

Consider the second order linear difference equation

$$u_n = pu_{n-1} + qu_{n-2} \tag{2.3}$$

where p, q are constants and $n \geq 2$. To find a general solution of (2.3), we try

$$u_n = Am^n$$

where m and A are constants. Now substituting

$$\begin{aligned} u_n &= Am^n, \\ u_{n-1} &= Am^{n-1}, \\ u_{n-2} &= Am^{n-2}, \end{aligned}$$

into (2.3) gives

$$\begin{aligned} pAm^{n-1} + qAm^{n-2} &= Am^n, \\ Am^{n-2}\left(m^2 - pm - q\right) &= 0. \end{aligned}$$

If $m = 0$ or $A = 0$, then (2.3) has a trivial solution and $u_n = 0$. If $m \neq 0$ and $A \neq 0$, then

$$m^2 - pm - q = 0. \tag{2.4}$$

Equation (2.4) is called the auxiliary equation of (2.3), and has the solution

$$\begin{aligned} m_1 &= \frac{p + \sqrt{p^2 + 4q}}{2} \\ \text{or} \quad m_2 &= \frac{p - \sqrt{p^2 + 4q}}{2} \end{aligned}$$

where m_1 and m_2 can be real or complex. Suppose that $m_1 \neq m_2$, then both

$$u_n = Am_1^n, \tag{2.5}$$

$$\text{and} \quad u_n = Bm_2^n \tag{2.6}$$

are solutions of (2.3), where A and B are constants. Next we show that the linear combination of (2.5) and (2.6) is also a solution. Since both Am_1^n and Bm_2^n satisfy (2.3) giving

$$\begin{aligned} Am_1^n &= Am_1^{n-1}p + Am_2^{n-2}q, \\ Bm_2^n &= Bm_2^{n-1}p + Bm_2^{n-2}q. \end{aligned}$$

This implies,

$$Am_1^n + Bm_2^n = p(Am_1^{n-1} + Bm_2^{n-1}) + q(Am_1^{n-2} + Bm_2^{n-2})$$

and hence $Am_1^n + Bm_2^n$ is also a solution of (2.3) and can be shown to be the general solution. In summary, the general solution of $u_n = pu_{n-1} + qu_{n-2}$ is

$$u_n = Am_1^n + Bm_2^n, \quad m_1 \neq m_2$$

where A, B are constants and m_1, m_2 are the solutions of the auxiliary equation $m^2 - pm - q = 0$.

Now we consider the case when $m_1 = m_2$. This implies that

$$\begin{aligned} u_n &= Am_1^n + Bm_1^n \\ &= m_1^n(A + B) \\ &= m_1^n C \end{aligned}$$

where $C = A + B$ is a constant. It can be shown that another solution to (2.3) is

$$u_n = Dnm_1^n. \tag{2.7}$$

If $u_n = Dnm_1^n$ then

$$\begin{aligned} u_{n-1} &= D(n-1)m_1^{n-1}, \\ \text{and} \quad u_{n-2} &= D(n-2)m_1^{n-2}. \end{aligned}$$

If (2.7) is a solution of (2.3), then

$$u_n - pu_{n-1} - qu_{n-2} = 0.$$

That is

$$Dnm_1^n - pD(n-1)m_1^{n-1} - qD(n-2)m_1^{n-2} = 0,$$

$$\begin{aligned}
Dm_1^{n-2}\left[nm_1^2-(n-1)pm_1-(n-2)q\right] &= 0, \\
Dm_1^{n-2}\left[n\left(m_1^2-pm_1-q\right)+pm_1+2q\right] &= 0, \\
Dm_1^{n-2}(pm_1+2q) &= 0, \quad \text{by (2.4)}.
\end{aligned}$$

Since the auxiliary equation has equal roots then $p^2+4q=0$ and

$$m_1 = \frac{p}{2}. \tag{2.8}$$

Thus by (2.8),

$$\begin{aligned}
u_n - pu_{n-1} - qu_{n-2} &= Dm_1^{n-2}(pm_1+2q), \\
&= Dm_1^{n-2}\left(p\cdot\frac{p}{2}+2q\right), \\
&= 2Dm_1^{n-2}\left(p^2+4q\right), \\
&= 0, \quad \text{since } p^2+4q=0.
\end{aligned}$$

Thus $u_n = Dnm_1^n$ is a solution and hence $Dnm_1^n + Cm_1^n$ will also be a solution. In summary, when $p^2+4q=0$ the general solution of (2.3) is

$$u_n = Dnm_1^n + Cm_1^n \tag{2.9}$$

where C and D are arbitrary constants.

2.4 Other Definitions

In this section we aim to provide more advanced definitions which are needed to understand the work provided in the next chapters. The material provided in this section is obtained from [17] and therefore the reader can consult this paper for more clarification. Consider the N^{th}-order OΔE

$$u_{n+N} = w(n, u_n, u_{n+1}, \dots, u_{n+N-1}) \tag{2.10}$$

where w is a smooth function such that $\dfrac{\partial w}{\partial u_n} \neq 0$ and integer n is an independent variable. The general solution of (2.10) can be written in the form

$$u_n = F(n, c_1, \dots, c_N) \tag{2.11}$$

and depends on N arbitrary independent constants c_i. In Section 4.1 we show how to find the general solution of (2.10) for $N = 2$.
Next we introduce an operator, $\mathcal{S}$, which will be useful for the theory provided.

Definition 2.4.1. *([17], p 3)*
Throughout this dissertation, we define $\mathcal{S}$ to be the shift operator acting on n as follows:

$$\mathcal{S} : n \mapsto n + 1. \tag{2.12}$$

That is, if $u_n = F(n, c_1, \dots, c_N)$ then,

$$\begin{aligned} \mathcal{S}(u_n) &= \mathcal{S}(F(n, c_1, \dots, c_N)) \\ &= F(\mathcal{S}(n), c_1, \dots, c_N) \\ &= F(n + 1, c_1, \dots, c_N) \\ &= u_{n+1} \end{aligned}$$

where the c_i's are independent of n. In the same way,

$$\mathcal{S}(u_{n+k}) = u_{n+k+1}, \quad k = 0, \dots, N - 2.$$

Therefore, $\mathcal{S}$ is an operator on n and hence on u_{n+k} as defined in (2.12). Below we provide the definition of a first integral.

Definition 2.4.2. *([17], p 4)*
If ϕ is a first integral, then it is constant on the solutions of the OΔE and hence satisfies

$$\begin{aligned} \mathcal{S}(\phi(n, u_n, \ldots, u_{n+N-1})) &= \phi(n, u_n, \ldots, u_{n+N-1})), \\ \phi(n+1, u_{n+1}, \ldots, \omega(n, u_n, \ldots, u_{n+N-1})) &= \phi(n, u_n, \ldots, u_{n+N-1})), \end{aligned} \tag{2.13}$$

where $\mathcal{S}$ is the shift operator defined in (2.12).

Below we provide the definition of a symmetry generator, X, of an ordinary difference equation.

Definition 2.4.3. *([17], p 6)*
A symmetry generator, X, of (2.10) is given by

$$\begin{aligned} X = Q(n, u_n, \ldots, u_{n+N-1})\frac{\partial}{\partial u_n} + (\mathcal{S}Q(n, u_n, \ldots, u_{n+N-1}))\frac{\partial}{\partial u_{n+1}} + \\ \cdots + \left(\mathcal{S}^{N-1}Q(n, u_n, \ldots, u_{n+N-1})\right)\frac{\partial}{\partial u_{n+N-1}}, \end{aligned}$$

and satisfies the symmetry condition

$$S^N Q(n, u_n, \ldots, u_{n+N-1}) - X\omega = 0 \tag{2.14}$$

where $Q = Q(n, u_n, \ldots, u_{n+N-1})$ is a function called the characteristic of the one-parameter group.

Throughout this dissertation, we only deal with second order OΔE's. From equation (2.14) the symmetry condition for a second order OΔE, that is $N = 2$, is

$$\begin{aligned} \mathcal{S}^2 Q(n, u_n, u_{n+1}) - X\omega &= 0, \\ Q(n+2, \omega, \mathcal{S}\omega) - X\omega &= 0, \end{aligned} \tag{2.15}$$

where

$$X = Q(n, u_n, u_{n+1})\frac{\partial}{\partial u_n} + Q(n+1, u_{n+1}, \omega(n, u_n, u_{n+1}))\frac{\partial}{\partial u_{n+1}}. \tag{2.16}$$

The symmetry generators play an important role in reducing the order of a difference equation and finding a general solution.

Chapter 3

Constructing the Symmetry Generator

The aim of this chapter is to discuss the method of calculating the symmetries of OΔE's, particularly second order difference equations considered in [17]. This is done in Section 3.2. Section 3.1 provides an application of the implicit function theorem which will aid us in better understanding the theory provided in Section 3.2.

3.1 Implicit Function Theorem

This section aims to illustrate how the implicit function theorem is used to determine the differential operator needed to solve the symmetry condition

$$\mathcal{S}^2 Q(n, u_n, u_{n+1}) - X\omega = 0.$$

Using the information provided in ([29], p 2), the implicit function theorem can be used to show that

$$\frac{\partial u_{n+1}}{\partial u_n} = -\frac{\frac{\partial \omega}{\partial u_n}}{\frac{\partial \omega}{\partial u_{n+1}}} \tag{3.1}$$

where $u_{n+2} = \omega(n, u_n, u_{n+1})$. As stated in ([29]), consider a function given by

$$y = f(x_1, x_2, \ldots, x_n)$$

where y is a function with n variables. The equation can be implicitly written as

$$\phi(x_1, x_2, \ldots, x_n, y) = f(x_1, x_2, \ldots, x_n) - y = 0 \tag{3.2}$$

where ϕ is now a function with $n+1$ variables. Then assume that ϕ is continuously differentiable and that

$$\frac{\partial \phi}{\partial x_k} = \frac{\partial f}{\partial x_k} \neq 0, \quad j = 1, 2, \ldots, n.$$

Note that the implicit function theorem is stated in ([29]) and that ϕ satisfies the theorem. Since the implicit function theorem holds, we can solve (3.2) for x_k as a function of y and the other x's. That is,

$$x_k = \psi_k(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, y).$$

Thus it holds that

$$\phi(y, x_1, x_2, \ldots, x_{k-1}, \psi_k(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, y), x_{k+1}, \ldots, x_n) \equiv 0$$

or that

$$y \equiv f(x_1, x_2, \ldots, x_{k-1}, \psi_k(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, y), x_{k+1}, \ldots, x_n). \tag{3.3}$$

Differentiating (3.3) with respect to x_j gives

$$0 = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial x_j}$$

and rearranging gives us

$$\frac{\partial x_k}{\partial x_j} = -\frac{\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial x_k}}. \tag{3.4}$$

By replacing $y = u_{n+2}$, $f = \omega$, $x_k = u_{n+1}$ and $x_j = u_n$, we obtain that (3.1) is true.

3.2 Symmetry Generator

In this Section we provide the procedure for calculating the symmetry generator of a second order OΔE. Consider the second order OΔE

$$u_{n+2} = \frac{u_n u_{n+1}}{2u_n - u_{n+1}} = \omega(n, u_n, u_{n+1}) \tag{3.5}$$

which is provided as an example in ([17], p 7). Finding the symmetry generator of (3.5), requires us to determine the *characteristic* of the one-parameter group, Q. Suppose that we seek characteristics of the form $Q = Q(n, u_n)$. To do this, we use the symmetry condition given by equation (2.15), and solve for $Q = Q(n, u_n)$. We call this method the *symmetry-finding algorithm.*

Recall from (2.15), the symmetry condition for a second order OΔE is

$$Q(n+2, \omega, \mathcal{S}\omega) - X\omega = 0.$$

Since $Q = Q(n, u_n)$, the symmetry condition becomes

$$\begin{aligned} Q(n+2, u_{n+2}) - X\omega &= 0, \\ Q(n+2, \omega) - X\omega &= 0. \end{aligned} \tag{3.6}$$

We start by simplifying (3.6) which is a tedious task. We get

$$\begin{aligned} X\omega &= Q(n, u_n)\frac{\partial \omega}{\partial u_n} + (\mathcal{S}Q(n, u_n))\frac{\partial \omega}{\partial u_{n+1}} \\ &= Q(n, u_n)\frac{\partial \omega}{\partial u_n} + Q(n+1, u_{n+1})\frac{\partial \omega}{\partial u_{n+1}} \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \frac{\partial \omega}{\partial u_n} &= \frac{(u_{n+1} + u_n \cdot 0)(2u_n - u_{n+1}) - u_n u_{n+1}(2)}{(2u_n - u_{n+1})^2} \\ &= \frac{u_{n+1}(2u_n - u_{n+1}) - 2u_n u_{n+1}}{(2u_n - u_{n+1})^2} \\ &= \frac{-u_{n+1}^2}{(2u_n - u_{n+1})^2} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \frac{\partial \omega}{\partial u_{n+1}} &= \frac{u_n(2u_n - u_{n+1}) - u_n u_{n+1}(-1)}{(2u_n - u_{n+1})^2} \\ &= \frac{u_n(2u_n - u_{n+1}) + u_n u_{n+1}}{(2u_n - u_{n+1})^2} \\ &= \frac{2u_n^2}{(2u_n - u_{n+1})^2}. \end{aligned} \tag{3.9}$$

Substituting equations (3.8) and (3.9) into (3.7) gives

$$X\omega = -Q(n,u_n)\frac{u_{n+1}^2}{(2u_n - u_{n+1})^2} + Q(n+1,u_{n+1})\frac{2u_n^2}{(2u_n - u_{n+1})^2}. \tag{3.10}$$

Substitute (3.10) into (3.6) to obtain

$$Q(n+2,\omega) - Q(n+1,u_{n+1})\frac{2u_n^2}{(2u_n - u_{n+1})^2} + Q(n,u_n)\frac{u_{n+1}^2}{(2u_n - u_{n+1})^2} = 0. \tag{3.11}$$

The above equation is difficult to solve because there are three separate pairs of arguments associated with the characteristic Q namely, $Q(n+2,\omega)$, $Q(n+1,u_{n+1})$ and $Q(n,u_n)$. However, the solution can be achieved in a series of steps. Firstly, differentiate (3.11) with respect to u_n, keep w fixed and consider u_{n+1} to be a function of n, u_n and ω. That is, $u_{n+1} = u_{n+1}(n,u_n,\omega)$. Using the implicit function theorem (equation (3.1)) and differentiating u_{n+1} with respect to u_n yields

$$\begin{aligned}\frac{\partial u_{n+1}}{\partial u_n} &= \frac{\partial u_{n+1}(n,u_n,\omega)}{\partial u_n}\\ &= -\frac{\frac{\partial\omega}{\partial u_n}}{\frac{\partial\omega}{\partial u_{n+1}}}\\ &= \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}\cdot\frac{(2u_n - u_{n+1})^2}{2u_n^2}\\ &= \frac{u_{n+1}^2}{2u_n^2}.\end{aligned}$$

Secondly, apply the *differential operator*, given by

$$\frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}} = \frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{2u_n^2}\frac{\partial}{\partial u_{n+1}},$$

to (3.11) and obtain

$$\left[\frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{2u_n^2}\frac{\partial}{\partial u_{n+1}}\right]\left[Q(n+2,\omega) - \frac{2u_n^2}{(2u_n - u_{n+1})^2}Q(n+1,u_{n+1}) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q(n,u_n)\right] = 0. \tag{3.12}$$

Now note that

$$\left.\begin{aligned}\frac{\partial}{\partial u_n}Q(n+1,u_{n+1}) &= 0,\\ \frac{\partial}{\partial u_{n+1}}Q(n,u_n) &= 0,\\ \frac{\partial}{\partial u_{n+1}}Q(n+1,u_{n+1}) &= Q'(n+1,u_{n+1}),\\ \text{and}\quad \frac{\partial}{\partial u_n}Q(n,u_n) &= Q'(n,u_n),\end{aligned}\right\} \tag{3.13}$$

where a dash represents a derivative with respect to a continuous variable, that is u_{n+1} and u_n in the equations above.

To simplify this computation we will first apply the differential operator on the $Q(n+1, u_{n+1})$ term.

$$\left[\frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{2u_n^2}\frac{\partial}{\partial u_{n+1}}\right]\left[-\frac{2u_n^2}{(2u_n - u_{n+1})^2}Q(n+1, u_{n+1})\right]$$
$$= -\left[\frac{4u_n(2u_n - u_{n+1})^2 - 8u_n^2(2u_n - u_{n+1})}{(2u_n - u_{n+1})^4}Q(n+1, u_{n+1}) + \right.$$
$$\left.\frac{u_{n+1}^2}{2u_n^2}\left(\frac{4u_n^2(2u_n - u_{n+1})}{(2u_n - u_{n+1})^4}Q(n+1, u_{n+1}) \ + \frac{2u_n^2}{(2u_n - u_{n+1})^2}Q'(n+1, u_{n+1})\right)\right]$$
$$= -\left[\frac{8u_n^2 - 4u_n u_{n+1} - 8u_n^2}{(2u_n - u_{n+1})^3}Q(n+1, u_{n+1}) + \right.$$
$$\left.\frac{2u_{n+1}^2}{(2u_n - u_{n+1})^3}Q(n+1, u_{n+1}) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n+1, u_{n+1})\right]$$
$$= -\left[-\frac{4u_n u_{n+1}}{(2u_n - u_{n+1})^3}Q(n+1, u_{n+1}) + \frac{2u_{n+1}^2}{(2u_n - u_{n+1})^3}Q(n+1, u_{n+1}\ ) + \right.$$
$$\left.\frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n+1, u_{n+1})\right]$$
$$= \frac{2u_{n+1}}{(2u_n - u_{n+1})^2}Q(n+1, u_{n+1}) \ - \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n+1, u_{n+1}). \tag{3.14}$$

Next we apply the differential operator on the $Q(n, u_n)$ term.

$$\left[\frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{2u_n^2}\frac{\partial}{\partial u_{n+1}}\right]\left[\frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q(n, u_n)\right]$$
$$= -\frac{4u_{n+1}^2(2u_n - u_{n+1})}{(2u_n - u_{n+1})^4}Q(n, u_n) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n, u_n) +$$
$$\frac{u_{n+1}^2}{2u_n^2}\left[\frac{2u_{n+1}(2u_n - u_{n+1})^2 + 2u_{n+1}^2(2u_n - u_{n+1})}{(2u_n - u_{n+1})^4}Q(n, u_n)\right]$$
$$= -\frac{4u_{n+1}^2}{(2u_n - u_{n+1})^3}Q(n, u_n) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n, u_n) +$$
$$\frac{u_{n+1}^2}{2u_n^2}\left[\frac{4u_n u_{n+1} - 2u_{n+1}^2 + 2u_{n+1}^2}{(2u_n - u_{n+1})^3}Q(n, u_n)\right]$$
$$= \left[\frac{-4u_{n+1}^2}{2u_n}\frac{(2u_n - u_{n+1})}{(2u_n - u_{n+1})^3}\right]Q(n, u_n) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n, u_n)$$
$$= -\frac{2u_{n+1}^2}{u_n(2u_n - u_{n+1})^2}Q(n, u_n) + \frac{u_{n+1}^2}{(2u_n - u_{n+1})^2}Q'(n, u_n). \tag{3.15}$$

Finally the differential operator acting on $Q(n+2, \omega)$ gives

$$\left[\frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{2u_n^2}\frac{\partial}{\partial u_{n+1}}\right]Q(n+2, w) = 0. \tag{3.16}$$

Now substitute (3.14), (3.15) and (3.16) into (3.12) to obtain

$$\begin{aligned}&\frac{2u_{n+1}}{(2u_n-u_{n+1})^2}Q(n+1,u_{n+1})-\frac{u_{n+1}^2}{(2u_n-u_{n+1})^2}Q'(n+1,u_{n+1})\\&-\frac{2u_{n+1}^2}{u_n(2u_n-u_{n+1})^2}Q(n,u_n)+\frac{u_{n+1}^2}{(2u_n-u_{n+1})^2}Q'(n,u_n)=0.\end{aligned} \tag{3.17}$$

Multiplying both sides of (3.17) by $\frac{(2u_n-u_{n+1})^2}{u_{n+1}^2}$, gives

$$-Q'(n+1,u_{n+1})+\frac{2}{u_{n+1}}Q(n+1,u_{n+1})+Q'(n,u_n)-\frac{2}{u_n}Q(n,u_n)=0. \tag{3.18}$$

To solve (3.18), we differentiate it with respect to u_n, keeping u_{n+1} fixed. This gives us the ODE

$$\frac{d}{du_n}\left(Q'(n,u_n)-\frac{2}{u_n}Q(n,u_n)\right)=0.$$

Integrating both sides with respect to u_n gives

$$Q'(n,u_n)-\frac{2}{u_n}Q(n,u_n)=A(n) \tag{3.19}$$

which is a first order linear differential equation. The integrating factor of this differential equation is

$$\begin{aligned}\exp\left(-\int\frac{2}{u_n}\,\mathrm{d}u_n\right) &= \exp(-2\ln(u_n))\\ &= \frac{1}{u_n^2}.\end{aligned}$$

Multiplying both sides of (3.19) by the integrating factor and taking integrals gives

$$\begin{aligned}\int\left(\frac{1}{u_n^2}Q'(n,u_n)-\frac{2}{u_n^3}Q(n,u_n)\right)\mathrm{d}u_n &= \int\left(A(n)\frac{1}{u_n^2}\right)\mathrm{d}u_n,\\ \int\frac{\mathrm{d}}{\mathrm{d}u_n}\left(\frac{1}{u_n^2}Q(n,u_n)\right)\mathrm{d}u_n &= -A(n)u_n^{-1}+B(n),\\ \text{i.e.}\quad \frac{1}{u_n^2}Q(n,u_n) &= -A(n)u_n^{-1}+B(n).\end{aligned}$$

Hence,

$$Q(n,u_n)=-A(n)u_n+B(n)u_n^2. \tag{3.20}$$

From (3.20) we obtain the following equations

$$Q(n+1,u_{n+1}) = -A(n+1)u_{n+1}+B(n+1)u_{n+1}, \tag{3.21}$$

$$
\begin{aligned}
Q'(n,u_n) &= -A(n)+2B(n)u_n, &(3.22)\\
Q'(n+1,u_{n+1}) &= -A(n+1)+2B(n+1)u_{n+1}. &(3.23)
\end{aligned}
$$

Next we substituting equations (3.20), (3.21), (3.22) and (3.23) back into (3.18) and obtain

$$
\begin{aligned}
&-\left[-A(n+1)+2B(n+1)u_{n+1}\right]+\frac{2}{u_{n+1}}\left[-A(n+1)u_{n+1}+B(n+1)u_{n+1}^2\right]\\
&-A(n)+2B(n)u_n-\frac{2}{u_n}\left[-A(n)u_n+B(n)u_n^2\right]=0.
\end{aligned}
$$

Simplifying further we get

$$
\begin{aligned}
&A(n+1)-2B(n+1)u_{n+1}-2A(n+1)+2B(n+1)u_{n+1}-A(n)\\
&+2B(n)u_n+2A(n)-2B(n)=0
\end{aligned}
$$

and hence

$$
\begin{aligned}
A(n)-A(n+1) &= 0,\\
A(n+1) &= A(n). &(2.2)
\end{aligned}
$$

Recall that equation (2.2) was considered in Section 2.2 as an example. As calculated in Section 2.2, the general solution of equation (2.2) is

$$
A(n)=-c_1, \tag{3.24}
$$

where c_1 is a constant. Substituting (3.24) into (3.20) yields

$$
Q(n,u_n)=c_1u_n+B(n)u_n^2, \tag{3.25}
$$

where c_1 is a constant. It follows that

$$
\begin{aligned}
Q(n+1,u_{n+1}) &= c_1u_{n+1}+B(n+1)u_{n+1}^2, &(3.26)\\
Q(n+2,w) &= Q(n+2,u_{n+2})\\
&= c_1u_{n+2}+B(n+2)u_{n+2}^2\\
&= \frac{c_1u_nu_{n+1}}{2u_n-u_{n+1}}+B(n+2)\frac{u_n^2u_{n+1}^2}{(2u_n-u_{n+1})^2}. &(3.27)
\end{aligned}
$$

To determine $B(n)$, substitute equations (3.25), (3.26) and (3.27) into (3.11) and obtain

$$
\frac{c_1u_nu_{n+1}}{2u_n-u_{n+1}}+B(n+2)\frac{u_n^2u_{n+1}^2}{(2u_n-u_{n+1})^2}-\frac{2u_n^2}{(2u_n-u_{n+1})^2}\left[c_1u_{n+1}+B(n+1)u_{n+1}^2\right]
$$

$$+\frac{u_{n+1}^2}{(2u_n-u_{n+1})^2}\left[c_1u_n+B(n)u_n^2\right]=0.$$

Expanding the terms in the above equation gives

$$\frac{c_1u_nu_{n+1}}{2u_n-u_{n+1}}+B(n+2)\frac{u_n^2u_{n+1}^2}{(2u_n-u_{n+1})^2}-\frac{2c_1u_n^2u_{n+1}}{(2u_n-u_{n+1})^2}$$
$$-\frac{2u_n^2u_{n+1}^2}{(2u_n-u_{n+1})^2}B(n+1)+\frac{c_1u_nu_{n+1}^2}{(2u_n-u_{n+1})^2}+\frac{u_n^2u_{n+1}^2}{(2u_n-u_{n+1})^2}B(n)=0. \quad (3.28)$$

Multiplying both sides of equation (3.28) by $(2u_n-u_{n+1})^2$ yields

$$c_1u_nu_{n+1}(2u_n-u_{n+1})+B(n+2)u_n^2u_{n+1}^2-2c_1u_n^2u_{n+1}$$
$$-2B(n+1)u_n^2u_{n+1}^2+c_1u_nu_{n+1}^2+B(n)u_n^2u_{n+1}^2=0. \quad (3.29)$$

Simplifying (3.29) further, gives

$$B(n+2)u_n^2u_{n+1}^2-2B(n+1)u_n^2u_{n+1}^2+B(n)u_n^2u_{n+1}^2=0$$

and dividing all the terms in the above equation by $u_n^2u_{n+1}$ we finally obtain

$$B(n+2)-2B(n+1)+B(n)=0. \quad (3.30)$$

Equation (3.30) can be solved using the auxiliary equation theory provided in Section 2.3. The auxiliary equation of equation (3.30) is $m^2-2m+1=0$ so that $m=1$. Since the auxiliary equation has equal roots, the general solution of (3.30) is

$$\begin{aligned} B(n) &= c_2n(1)^n+c_3(1)^n \\ &= c_2n+c_3 \end{aligned} \quad (3.31)$$

where c_2 and c_3 are constants. Hence we have that the characteristic of the one parameter group, $Q(n,u_n)$, is

$$Q(n,u_n)=c_1u_n+(c_2n+c_3)u_n^2. \quad (3.32)$$

Setting one of the constants in equation (3.32) equal to one and the other two constants equal zero, we obtain a three-dimensional Lie algebra of symmetry generators, whose characteristics are linear combinations of

$$c_1=1, c_2=c_3=0 \Rightarrow Q_1=u_n,$$
$$c_2=1, c_1=c_3=0 \Rightarrow Q_2=nu_n^2,$$
$$c_3=1, c_1=c_2=0 \Rightarrow Q_3=u_n^2.$$

Hence the Lie symmetry generators of (3.5) are

$$
\begin{aligned}
X_1 &= Q_1\frac{\partial}{\partial u_n} = u_n\frac{\partial}{\partial u_n},\\
X_2 &= Q_2\frac{\partial}{\partial u_n} = nu_n^2\frac{\partial}{\partial u_n},\\
X_3 &= Q_3\frac{\partial}{\partial u_n} = u_n^2\frac{\partial}{\partial u_n}.
\end{aligned}
$$

Thus we have obtained the desired result. Note that for each

$$X_i = Q_i\frac{\partial}{\partial u_n},$$

where $i = 1, 2, 3$, the symmetry condition holds. That is

$$Q(n+2, \omega) - X_i\omega = 0$$

holds.

3.3 Conclusion

In this chapter we calculate the differential operator and hence solve the symmetry condition $Q(n+2, \omega) - X\omega = 0$. We also used the symmetry-finding algorithm to calculate the characteristic, $Q = Q(n, u_n)$, and hence the Lie symmetry generators of a second order OΔE.

Chapter 4

Calculating the general solution and first integral

This chapter is divided into two sections. Section 4.1 shows how we use Lie symmetries to reduce the order, that is, find the general solution of an OΔE. In Section 4.2 we show how to calculate a first integral, ϕ, of an OΔE.

4.1 Using symmetries to obtain the general solution of an OΔE

In this chapter, we present the theory and an example of how to use the symmetry generator of an OΔE to reduce its order, that is, find its general solution. Again we consider the theory and example provided by Hydon in [17] and are only interested in second-order OΔEs. We begin this section by providing the definition of a commutator given below.

Definition 4.1.1. *([26])*
The commutator of two symmetry generators X_N and X_M is denoted $[X_N, X_M]$ and defined by

$$\begin{aligned} [X_N, X_M] &= X_N X_M - X_M X_N \\ &= -[X_M, X_N]. \end{aligned} \tag{4.1}$$

Also note that for commutators like (4.1), the Jacobian identity

$$[X_N, [X_M, X_P]] + [X_M, [X_P, X_N]] + [X_P, [X_M, X_P]] = 0$$

always holds simply because of the definition of a commutator.

The *invariant* of an OΔE plays an important role in reducing its order. Given a symmetry generator for a second-order OΔE,

$$X = Q(n, u_n, u_{n+1})\frac{\partial}{\partial u_n} + Q(n+1, u_{n+1}, \omega(n, u_n, u_{n+1}))\frac{\partial}{\partial u_{n+1}},$$

a first order invariant

$$v_n = v(n, u_n, u_{n+1}), \tag{4.2}$$

is a function satisfying

$$X v_n = 0. \tag{4.3}$$

To determine the invariant, we use the *method of characteristics.* Note that the invariant satisfies

$$\left[Q\frac{\partial}{\partial u_n} + \mathcal{S}Q\frac{\partial}{\partial u_{n+1}}\right] v_n \quad = \quad 0.$$

Using the method of characteristics we obtain

$$\frac{\mathrm{d}u_n}{Q} = \frac{\mathrm{d}u_{n+1}}{\mathcal{S}Q} = \frac{\mathrm{d}v_n}{0}.$$

Note: $\frac{\mathrm{d}v_n}{0}$ implies that v_n is a dependent variable and $v_n = \gamma$ where $\gamma = f(\alpha)$ and α is an independent variable. To simplify calculations, we choose $f(\alpha) = \alpha$. We obtain α by solving

$$\frac{\mathrm{d}u_n}{Q} = \frac{\mathrm{d}u_{n+1}}{\mathcal{S}Q}$$

where α is the integrating constant.

We make the assumption that (4.2) can be inverted to obtain

$$u_{n+1} = \omega(n, u_n, v_n) \tag{4.4}$$

for some function ω. Solving (4.4) requires finding a canonical coordinate

$$s_n = s(n, u_n) \tag{4.5}$$

which satisfies

$$X s_n = 1. \tag{4.6}$$

According to ([17], p 13), the most obvious choice of canonical coordinate is

$$s(n, u_n) = \int \frac{\mathrm{d}u_n}{Q(n, u_n, \omega(n, u_n, f(n; c_1)))} \tag{4.7}$$

with a general solution of the form

$$s_n = c_2 + \sum_{k=n_0}^{n-1} g(k, f(k; c_1)),$$

for some function g and any integer n_0. Alternatively, if we cannot determine the canonical coordinate $s(n, u_n)$, we use the RSolve command in *Mathematica* to solve (4.4).

We consider an example provided in ([17], p 13). We will calculate the invariant and reduce the order of the nonlinear OΔE,

$$u_{n+2} = \frac{2u_{n+1}^3}{u_n^2} - u_{n+1}. \tag{4.8}$$

Using the approach provided in Chapter 3, the set of characteristics of the one parameter group, $Q = Q(n, u_n, u_{n+1})$, are

$$Q_1 = \frac{u_{n+1}}{u_n}, \qquad Q_2 = u_n. \tag{4.9}$$

Define:

$$X_1 = Q_1 \frac{\partial}{\partial u_n} = \frac{u_{n+1}}{u_n} \frac{\partial}{\partial u_n}, \tag{4.10}$$

and

$$X_2 = Q_2 \frac{\partial}{\partial u_n} = u_n \frac{\partial}{\partial u_n}. \tag{4.11}$$

Then

$$\begin{aligned}
[X_1, X_2] &= X_1 X_2 - X_2 X_1 \\
&= \left(\frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_n}\right)\left(u_n \frac{\partial}{\partial u_n}\right) - \left(u_n \frac{\partial}{\partial u_n}\right)\left(\frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_n}\right) \\
&= \frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_n} - u_n\left(-\frac{u_{n+1}}{u_n^2}\right)\frac{\partial}{\partial u_n} \\
&= 2\left(\frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_n}\right) \\
&= 2X_1.
\end{aligned}$$

Since the commutator $[X_1, X_2]$ is linear in X_1, the order of the OΔE will be reduced by first using X_1. As a result, the reduced OΔE will inherit the symmetries generated by X_2. Now suppose that $v_n = v(n, u_n, u_{n+1})$ is the invariant generated by X_1. Now using the condition given by equation (4.3) we have

$$\begin{aligned}
X_1 v_n &= \left(Q_1 \frac{\partial}{\partial u_n} + SQ_1 \frac{\partial}{\partial u_{n+1}}\right) v_n \\
&= \left(\frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_n} + \frac{u_{n+2}}{u_{n+1}}\frac{\partial}{\partial u_{n+1}}\right) v_n \\
&= \left[\frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_n} + \frac{1}{u_{n+1}}\left(\frac{2u_{n+1}^3}{u_n^2} - u_{n+1}\right)\frac{\partial}{\partial u_{n+1}}\right] v_n \\
&= \left[\frac{u_{n+1}}{u_n}\frac{\partial}{\partial u_n} + \left(\frac{2u_{n+1}^2}{u_n^2} - 1\right)\frac{\partial}{\partial u_{n+1}}\right] v_n \\
&= 0.
\end{aligned}$$

Next we use the method of characteristic to solve for v_n and hence

$$\frac{du_n}{\frac{u_{n+1}}{u_n}} = \frac{du_{n+1}}{\frac{2u_{n+1}^2}{u_n^2} - 1} = \frac{dv_n}{0}.$$

We first determine the independent variable α by solving

$$\frac{du_n}{\frac{u_{n+1}}{u_n}} = \frac{du_{n+1}}{\frac{2u_{n+1}^2}{u_n^2} - 1}.$$

Rearranging we get

$$\begin{aligned}
\frac{du_{n+1}}{du_n} &= \left(\frac{2u_{n+1}^2 - u_n^2}{u_n^2}\right)\left(\frac{u_n}{u_{n+1}}\right) \\
&= \frac{2u_{n+1}^2 - u_n^2}{u_n u_{n+1}}. \qquad (4.12)
\end{aligned}$$

Observe that (4.12) is a first order homogeneous equation. If we let $u_{n+1} = y$ and $u_n = x$, where $y = kx$ and k is a new variable, (4.12) becomes

$$\begin{aligned}
x\frac{\mathrm{d}k}{\mathrm{d}x} + k &= \frac{2(kx)^2 - x^2}{kx^2}, \\
x\frac{\mathrm{d}k}{\mathrm{d}x} &= \frac{2k^2 - k^2 - 1}{k}, \\
\frac{k}{k^2 - 1}\mathrm{d}k &= \frac{\mathrm{d}x}{x}, \\
\left(\frac{\frac{1}{2}}{k-1} + \frac{\frac{1}{2}}{k+1}\right)\mathrm{d}k &= \frac{\mathrm{d}x}{x}, \\
\left(\frac{1}{k-1} + \frac{1}{k+1}\right)\mathrm{d}k &= 2\frac{\mathrm{d}x}{x}.
\end{aligned}$$

Taking integrals on both sides gives

$$\begin{aligned}
\ln(k-1)+\ln(k+1) &= 2\ln x + A, \quad \text{where A is a constant} \\
\ln(k-1)(k+1) &= \ln x^2 + A.
\end{aligned}$$

Now multiply both sides by the natural exponent, e, to obtain

$$(k-1)(k+1) = x^2 e^A.$$

Simplify the right hand side and let the integration constant $\alpha = e^A$, so that the equation becomes

$$k^2 - 1 = \alpha x^2. \tag{4.13}$$

Recall that $k = \frac{y}{x}$, then equation (4.13) becomes

$$\begin{aligned}
\frac{y^2}{x^2} - 1 &= \alpha x^2 \\
y^2 &= \alpha x^4 + x^2 \\
\alpha &= \frac{y^2 - x^2}{x^4} \\
&= \frac{u_{n+1}^2 - u_n^2}{u_n^4},
\end{aligned}$$

which is the independent variable.
Next we consider $\frac{\mathrm{d}v_n}{0}$. By the method of characteristics, this implies that the dependent variable $v_n = \gamma$ where $\gamma = f(\alpha) = \alpha$. Then,

$$v_n = \frac{u_{n+1}^2 - u_n^2}{u_n^4} = \alpha. \tag{4.14}$$

Then using equation (4.14), we obtain

$$\begin{aligned}
\mathcal{S}(v_n) &= v_{n+1} \\
&= \frac{u_{n+2}^2 - u_{n+1}^2}{u_{n+1}^4} \\
&= \frac{\left(\frac{2u_{n+1}^3 - u_n^2 u_{n+1}}{u_n^2}\right)^2 - u_{n+1}^2}{u_{n+1}^4} \\
&= \left(\frac{4u_{n+1}^6 - 4u_n^2 u_{n+1}^4 + u_n^4 u_{n+1}^2 - u_n^4 u_{n+1}^2}{u_n^4}\right)\frac{1}{u_{n+1}^4} \\
&= \frac{4u_{n+1}^4}{u_{n+1}^4}\left(\frac{u_{n+1}^2 - u_n^2}{u_n^4}\right) \\
v_{n+1} &= 4v_n.
\end{aligned} \tag{4.15}$$

The general solution of (4.15) is calculated by iteration as follows

$$\begin{aligned}
v_{n+1} &= 4v_n \\
&= 4(4v_{n-1}) = 4^2 v_{n-1} \\
&= 4^2(4v_{n-2}) = 4^3 v_{n-2} \\
&= \vdots \\
&= 4^n v_1 \\
&= 4^{n+1} v_0 \\
v_{n+1} &= 4^{n+1} c_1
\end{aligned} \tag{4.16}$$

where $c_1 = v_0$ is a constant. Then from (4.16) we obtain the general solution

$$v_n = 4^n c_1. \tag{4.17}$$

From (4.14) and (4.17) we obtain

$$\begin{aligned}
v_n &= \frac{u_{n+1}^2 - u_n^2}{u_n^4}, \\
4^n c_1 &= \frac{u_{n+1}^2 - u_n^2}{u_n^4}, \\
u_{n+1}^2 - u_n^2 &= c_1 4^n u_n^4, \\
u_{n+1}^2 &= u_n^2 \left(c_1 4^n u_n^2 + 1\right), \\
u_{n+1} &= \pm u_n \sqrt{1 + c_1 4^n u_n^2}.
\end{aligned} \tag{4.18}$$

We consider the positive square root so that we may not have any inconsistencies with equation (4.8). Thus,

$$u_{n+1} = u_n \sqrt{1 + c_1 4^n u_n^2}. \tag{4.19}$$

At this point we can use the RSolve command in *Mathematica* to find the general solution of equation (4.19) which would lead to the general solution of (4.8). However, since

$$\frac{u_{n+1}}{u_n} = \sqrt{1 + c_1 4^n u_n^2}$$

and from (4.9) we get

$$Q_1 = \sqrt{1 + c_1 4^n u_n^2}, \tag{4.20}$$

we can determine $s(n, u_n)$ using equation (4.7). We can then use $s_n = s(n, u_n)$ to find the general solution of (4.8).

Substituting (4.20) into (4.7), we obtain the canonical coordinate given by

$$s_n = \int \frac{\mathrm{d}u_n}{\sqrt{1 + c_1 4^n u_n^2}}. \tag{4.21}$$

By ([23], p 419), for $a \geq 0$

$$\begin{aligned} \int \frac{\mathrm{d}x}{\sqrt{a^2 + x^2}} &= \ln\left|x + \sqrt{a^2 + x^2}\right| \\ &= \sinh^{-1}\left(\frac{x}{a}\right). \end{aligned} \tag{4.22}$$

We use (4.22) to solve (4.21). Let $x = \sqrt{c_1}2^n u_n$ in (4.21) where $a = 1$. Then $\mathrm{d}x = \sqrt{c_1}2^n \mathrm{d}u_n$ and (4.21) becomes

$$\begin{aligned} s_n &= \int \frac{\mathrm{d}u_n}{\sqrt{1 + c_1 4^n u_n^2}} \\ &= \frac{1}{\sqrt{c_1}2^n} \int \frac{\mathrm{d}x}{\sqrt{1 + x^2}} \\ &= \frac{1}{\sqrt{c_1}2^n} \ln\left|x + \sqrt{1 + x^2}\right| \\ &= \frac{1}{\sqrt{c_1}2^n} \sinh^{-1}\left(\frac{x}{1}\right) \\ &= \frac{1}{\sqrt{c_1}2^n} \sinh^{-1}\left(\sqrt{c_1}2^n u_n\right). \end{aligned} \tag{4.23}$$

Then (4.18) is equivalent to

$$s_{n+1} = s_n$$

if we assume $g(n, v_n) = 0$, as a special case, for ease of computation. Then s_n has the general solution

$$s_n = c_2 \tag{4.24}$$

where c_2 is a constant. Equating (4.23) and (4.24) gives

$$c_2 = \frac{1}{\sqrt{c_1}2^n} \sinh^{-1}\left(\sqrt{c_1}2^n u_n\right) \tag{4.25}$$

and solving for u_n we have

$$c_2\sqrt{c_1}2^n = \sinh^{-1}\left(\sqrt{c_1}2^n u_n\right),$$

$$\begin{aligned}
\sinh\left(c_2\sqrt{c_1}2^n\right) &= \sinh\left(\sinh^{-1}\left(\sqrt{c_1}2^n u_n\right)\right), \\
\sinh\left(c_2\sqrt{c_1}2^n\right) &= \sqrt{c_1}2^n u_n, \\
u_n &= \frac{1}{\sqrt{c_1}2^n}\sinh\left(c_2\sqrt{c_1}2^n\right). \qquad (4.26)
\end{aligned}$$

Hence the general solution of (4.8) is

$$u_n = \frac{1}{\sqrt{c_1}2^n}\sinh\left(c_2\sqrt{c_1}2^n\right).$$

Note that the order of the difference equation (4.8) has been reduced by two. We say that we have calculated the *general solution* of the OΔE.

4.2 A direct construction of first integrals

In [1], Anco and Bluman show how to find the integrating factors and corresponding first integrals for any system of ordinary differential equations. In [17], Hydon shows how to construct the first integrals of OΔEs directly. For this method, the symmetries of the OΔE need not be known. The theory provided below is obtained from [17]. Again, we will only consider second-order OΔE's.

Recall the definition of a first integral given by equation (2.13) is $\mathcal{S}\phi = \phi$. Now add the condition that

$$\mathcal{S}\phi = \phi, \qquad \frac{\partial \phi}{\partial u_{n+N-1}} \neq 0. \tag{4.27}$$

Note that for a second-order OΔE, (4.27) becomes

$$\begin{aligned} \phi(n+1, u_{n+1}, u_{n+2}) &= \phi(n, u_n, u_{n+1}), \\ \phi(n+1, u_{n+1}, w(n, u_n, u_{n+1})) &= \phi(n, u_n, u_{n+1}), \qquad \frac{\partial \phi}{\partial u_{n+1}} \neq 0. \end{aligned} \tag{4.28}$$

Now let

$$P_1(n, u_n, u_{n+1}) = \frac{\partial \phi}{\partial u_n}(n, u_n, u_{n+1}) \tag{4.29}$$

and

$$P_2(n, u_n, u_{n+1}) = \frac{\partial \phi}{\partial u_{n+1}}(n, u_n, u_{n+1}). \tag{4.30}$$

Next we differentiate (4.28) with respect to u_n and obtain

$$\begin{aligned} \frac{\partial \phi(n, u_n, u_{n+1})}{\partial u_n} &= \frac{\partial \phi(n+1, u_{n+1}, \omega(n, u_n, u_{n+1})}{\partial u_n}, \\ \frac{\partial \phi}{\partial u_n} &= \frac{\partial \phi}{\partial \omega}\frac{\partial \omega}{\partial u_n}, \\ P_1 &= \frac{\partial \phi(n+1, u_{n+1}, u_{n+2})}{\partial u_{n+2}}\frac{\partial \omega}{\partial u_n} \\ &= \mathcal{S}\left(\frac{\partial \phi(n, u_n, u_{n+1})}{\partial u_{n+1}}\right)\frac{\partial \omega}{\partial u_n} \\ &= \mathcal{S}P_2\frac{\partial \omega}{\partial u_n}. \end{aligned} \tag{4.31}$$

Differentiating (4.28) with respect to u_{n+1} we get

$$\frac{\partial \phi(n, u_n, u_{n+1})}{\partial u_{n+1}} = \frac{\partial \phi(n+1, u_{n+1}, \omega(n, u_n, u_{n+1})}{\partial u_{n+1}},$$

$$
\begin{aligned}
P_2 &= \frac{\partial \phi}{\partial u_{n+1}} + \frac{\partial \phi}{\partial \omega}\frac{\partial \omega}{\partial u_{n+1}} \\
&= \frac{\partial \phi(n+1, u_{n+1}, u_{n+2})}{\partial u_{n+1}} + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S} P_2 \\
&= \mathcal{S}\left(\frac{\partial \phi(n, u_n, u_{n+1})}{\partial u_n}\right) + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S} P_2 \\
&= \mathcal{S} P_1 + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S} P_2. \qquad (4.32)
\end{aligned}
$$

Substituting (4.31) into (4.32) we observe that P_2 satisfies the second-order linear functional equation, or *first integral condition*,

$$
\begin{aligned}
\mathcal{S}\left(\frac{\partial \omega}{\partial u_n} \mathcal{S} P_2\right) + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S} P_2 - P_2 &= 0, \\
\mathcal{S}\left(\frac{\partial \omega}{\partial u_n}\right) \mathcal{S}^2 P_2 + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S} P_2 - P_2 &= 0. \qquad (4.33)
\end{aligned}
$$

At this stage we can solve (4.33) for P_2. To do this, we use the symmetry-finding algorithm described in Chapter 3. Given a solution for P_2, we can solve for P_1 by using equation (4.31). After solving for P_2 and constructing P_1, we check that the *integrability condition*

$$
\frac{\partial P_1}{\partial u_{n+1}} = \frac{\partial P_2}{\partial u_n} \qquad (4.34)
$$

is satisfied. If (4.34) is satisfied then we have an exact differential equation and there exists a function $\phi = \phi(n, u_n, u_{n+1})$ such that $\frac{\partial \phi}{\partial u_n} = P_1(n, u_n, u_{n+1})$ and $\frac{\partial \phi}{\partial u_{n+1}} = P_2(n, u_n, u_{n+1})$ where

$$
\phi = \int (P_1 \mathrm{d}u_n + P_2 \mathrm{d}u_{n+1}) + G(n).
$$

Hence the first integral takes the form

$$
\phi = \int (P_1 \mathrm{d}u_n + P_2 \mathrm{d}u_{n+1}) + G(n). \qquad (4.35)
$$

To solve for $G(n)$, we substitute (4.35) into (4.28) and solve for the resulting first order OΔE.

We go over the example on page 17 of ([17]) to illustrate the method explained above. Consider the second order OΔE,

$$
\omega = u_{n+2} = \frac{n}{n+1} u_n + \frac{1}{u_{n+1}}. \qquad (4.36)
$$

Then differentiating with respect to u_n and u_{n+1} respectively yields

$$\frac{\partial \omega}{\partial u_n} = \frac{n}{n+1}$$

and

$$\frac{\partial \omega}{\partial u_{n+1}} = -\frac{1}{u_{n+1}^2}.$$

Suppose that $P_2 = P_2(n, u_n)$, then (4.33) can be rewritten to give

$$\begin{aligned} \mathcal{S}\left(\frac{n}{n+1}\right)\mathcal{S}^2 P_2(n,u_n) - \frac{1}{u_{n+1}^2}\mathcal{S}P_2(n,u_n) - P_2(n,u_n) &= 0, \\ \left(\frac{n+1}{n+2}\right)P_2(n+2,u_{n+2}) - \frac{1}{u_{n+1}^2}P_2(n+1,u_{n+1}) - P_2(n,u_n) &= 0. \qquad (4.37) \end{aligned}$$

To solve for $P_2(n, u_n)$ in (4.37), make use of the symmetry-finding algorithm from Chapter 3. We first differentiate u_{n+1} with respect to u_n to obtain the differential operator

$$\begin{aligned} \frac{\partial u_{n+1}}{\partial u_n} &= \frac{\partial u_{n+1}(n,u_n,w)}{\partial u_n} \\ &= -\frac{\frac{\partial w}{\partial u_n}}{\frac{\partial w}{\partial u_{n+1}}} \\ &= \frac{-n}{n+1}\cdot\frac{u_{n+1}^2}{-1} \\ &= \frac{n u_{n+1}^2}{n+1}. \end{aligned}$$

Hence the differential operator is

$$\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}} \qquad (4.38)$$

which we apply to (4.37). That is, we have

$$\left[\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right]\left[\left(\frac{n+1}{n+2}\right)P_2(n+2,u_{n+2}) - \frac{1}{u_{n+1}^2}P_2(n+1,u_{n+1}) - P_2(n,u_n)\right] = 0. \qquad (4.39)$$

Applying the differential operator on $P_2(n+2, u_{n+2})$ yields a zero. That is

$$\left[\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right]\left[\left(\frac{n+1}{n+2}\right)P_2(n+2,u_{n+2})\right] = 0. \qquad (4.40)$$

For ease of computation, we consider the terms separately. We start by applying the differential operator on the $P_2(n+1, u_{n+1})$.

$$\begin{aligned}
&\left[\frac{\partial}{\partial u_n} + \frac{nu_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right]\left[-\frac{1}{u_{n+1}^2}P_2(n+1,u_{n+1})\right] \\
&= 0 - \frac{nu_{n+1}^2}{n+1}\left[\frac{-2}{u_{n+1}^3}P_2(n+1,u_{n+1}) + \frac{1}{u_{n+1}^2}P_2'(n+1,u_{n+1})\right] \\
&= \frac{n}{n+1}\left[\frac{2}{u_{n+1}}P_2(n+1,u_{n+1}) - P_2'(n+1,u_{n+1})\right]. \qquad (4.41)
\end{aligned}$$

Applying the differential operator on $P_2(n, u_n)$ term yields

$$\begin{aligned}
&\left[\frac{\partial}{\partial u_n} + \frac{nu_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right](-P_2(n,u_n)) \\
&= -P_2'(n,u_n) + 0 \\
&= -P_2'(n,u_n). \qquad (4.42)
\end{aligned}$$

Substitute (4.40), (4.41) and (4.42) into (4.39) to get

$$\frac{n}{n+1}\frac{2}{u_{n+1}}P_2(n+1,u_{n+1}) - \frac{n}{n+1}P_2'(n+1,u_{n+1}) - P_2'(n,u_n) = 0. \qquad (4.43)$$

Next we differentiate (4.43) with respect to u_n keeping u_{n+1} constant to obtain

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}u_n}\left(P_2'(n,u_n)\right) &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}u_n}\left(P_2'(n,u_n)\right) &= 0, \\
P_2'(n,u_n) &= B(n), \\
P_2(n,u_n) &= B(n)u_n + c, \\
P_2(n,u_n) &= B(n)u_n, \qquad (4.44)
\end{aligned}$$

if we take $c = 0$. Now substitute (4.44) into (4.43) to find a difference equation in $B(n)$.

$$\begin{aligned}
\frac{n}{n+1}\frac{2}{u_{n+1}}B(n+1)u_{n+1} - \frac{n}{n+1}B(n+1) - B(n) &= 0, \\
\frac{n}{n+1}B(n+1) - B(n) &= 0, \\
B(n+1) &= \frac{n+1}{n}B(n). \qquad (4.45)
\end{aligned}$$

To solve for $B(n)$, we use an iterative method as shown below.

$$B(n+1) = \frac{n+1}{n}B(n)$$

$$
\begin{aligned}
&= \left(\frac{n+1}{n}\right)\left(\frac{n}{n-1}\right)B(n-1) \\
&= \left(\frac{n+1}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{n-1}{n-2}\right)B(n-2) \\
&\vdots \\
&= \left(\frac{n+1}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{n-1}{n-2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{2}{1}\right)B(1) \\
&= (n+1)B(1). \qquad (4.46)
\end{aligned}
$$

Now choose $B(1) = 1$, to obtain $B(n+1) = n+1$ and hence

$$
B(n) = n. \qquad (4.47)
$$

Now substitute (4.47) into (4.44) to get

$$
P_2(n, u_n) = nu_n. \qquad (4.48)
$$

From (4.31),

$$
\begin{aligned}
P_1(n+1, u_{n+1}) &= \mathcal{S}P_2\frac{\partial\omega}{\partial u_n} \\
&= \mathcal{S}(nu_n)\left(\frac{n}{n+1}\right) \\
&= (n+1)u_{n+1}\left(\frac{n}{n+1}\right) \\
&= nu_{n+1}. \qquad (4.49)
\end{aligned}
$$

Now check that the integrability condition is satisfied:

$$
\frac{\partial P_1}{\partial u_{n+1}} = n = \frac{\partial P_2}{\partial u_n}.
$$

Since the integrability condition holds we can calculate the first integral ϕ. From (4.48) and (4.49) we have

$$
\begin{aligned}
\phi &= \int (P_1 \mathrm{d}u_n + P_2 \mathrm{d}u_{n+1}) + G(n) \\
&= \int ((nu_{n+1})\mathrm{d}u_n + (nu_n)\mathrm{d}u_{n+1}) + G(n) \\
&= nu_n u_{n+1} + G(n). \qquad (4.50)
\end{aligned}
$$

To find $G(n)$ we substitute equation (4.50) into (4.28). We obtain

$$
\begin{aligned}
\mathcal{S}\phi &= \phi, \\
\mathcal{S}(nu_n u_{n+1} + G(n)) &= nu_n u_{n+1} + G(n),
\end{aligned}
$$

$$\begin{aligned}
(n+1)u_{n+1}u_{n+2} + G(n+1) &= nu_n u_{n+1} + G(n), \\
(n+1)u_{n+1}\left(\frac{n}{n+1}u_n + \frac{1}{u_{n+1}}\right) + G(n+1) &= nu_n u_{n+1} + G(n), \\
nu_n u_{n+1} + (n+1) + G(n+1) &= nu_n u_{n+1} + G(n), \\
G(n+1) - G(n) + n + 1 &= 0. \qquad (4.51)
\end{aligned}$$

The general solution of (4.51) is

$$G(n) = -\frac{n(n+1)}{2} + c_1,$$

where c_1 is a constant. Choose $c_1 = 0$, then

$$G(n) = -\frac{n(n+1)}{2}. \qquad (4.52)$$

Finally substitute (4.52) into (4.50) to obtain the first integral

$$\phi = nu_n u_{n+1} - \frac{n(n+1)}{2}. \qquad (4.53)$$

4.3 Conclusion

In Section 4.1 we demonstrated how to reduce the order of an OΔE. We introduced the commutator and the invariant, $v_n = v(n, u_n, u_{n+1})$, which satisfies $Xv_n = 0$. Furthermore, we illustrated how the general solution of the canonical coordinate can be used, whenever the characteristic is $Q = Q(n, {}_n, u_{n+1})$, to reduce the order of the OΔE.

In Section 4.2, we calculate the first integral. The approach used required us to determine $P_2 = P_2(n, u_n, u_{n+1})$ and $P_1(n, u_n, u_{n+1}) = \mathcal{S}P_2\frac{\partial \omega}{\partial u_n}$. To calculate P_2 we used equation (4.33) as well as the symmetry-finding algorithm. After solving for P_2 and constructing P_1, we calculated the first integral which takes the form

$$\phi = \int (P_1 \mathrm{d}u_n + P_2 \mathrm{d}u_{n+1}) + G(n).$$

Chapter 5

Application

The aim of this chapter is to consider two examples and find their symmetries, first integrals and general solution. We also discuss the notion of 'double reduction' and 'association'.

5.1 Problem 1

Throughout this Section we consider the OΔE given by equation (4.36), that is

$$\omega = u_{n+2} = \frac{n}{n+1}u_n + \frac{1}{u_{n+1}},$$

which is clearly a non-linear equation. Recall that we calculated the first integral of (4.36) in Section 4.2 and now in Sub-section 5.1.1 we will find its symmetry.

5.1.1 Calculating the symmetry generator

Again, we use the symmetry-finding algorithm used in Chapter 3. Suppose that the characteristics of the one-parameter group satisfy $Q = Q(n, u_n)$. Then

$$\frac{\partial \omega}{\partial u_n} = \frac{n}{n+1} \tag{5.1}$$

and

$$\frac{\partial \omega}{\partial u_{n+1}} = -\frac{1}{u_{n+1}^2}. \tag{5.2}$$

Using (5.1) and (5.2), the symmetry condition is

$$Q(n+2, \omega) - X\omega \;\; = \;\; 0,$$

$$
\begin{aligned}
Q(n+2,\omega) - Q(n,u_n)\frac{\partial \omega}{\partial u_n} - Q(n+1,u_{n+1})\frac{\partial \omega}{\partial u_{n+1}} &= 0, \\
Q(n+2,\omega) + Q(n+1,u_{n+1})\frac{1}{u_{n+1}^2} - Q(n,u_n)\left(\frac{n}{n+1}\right) &= 0.
\end{aligned} \tag{5.3}
$$

Next we calculate the differential operator by first differentiating u_{n+1} with respect to u_n. That is,

$$
\begin{aligned}
\frac{\partial u_{n+1}}{\partial u_n} &= \frac{\partial u_{n+1}(n,u_n,w)}{\partial u_n} \\
&= -\frac{\frac{\partial w}{\partial u_n}}{\frac{\partial w}{\partial u_{n+1}}} \\
&= \frac{n}{n+1} \times u_{n+1}^2 \\
&= \frac{n u_{n+1}^2}{n+1}
\end{aligned} \tag{5.4}
$$

and hence the differential operator is

$$
\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}. \tag{5.5}
$$

Applying the differential operator to equation (5.3) gives

$$
\left[\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right]\left[Q(n+2,\omega) + \frac{1}{u_{n+1}^2}Q(n+1,u_{n+1}) - \frac{n}{n+1}Q(n,u_n)\right] = 0. \tag{5.6}
$$

Note that we will make use of the equations given in (3.13). Then

$$
\begin{aligned}
&\left[\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right]\left[\frac{1}{u_{n+1}^2}Q(n+1,u_{n+1})\right] \\
&= 0 + \frac{n u_{n+1}^2}{n+1}\left[-\frac{2}{u_{n+1}^3}Q(n+1,u_{n+1}) + \frac{1}{u_{n+1}^2}Q'(n+1,u_{n+1})\right] \\
&= -\frac{2n}{(n+1)u_{n+1}}Q(n+1,u_{n+1}) + \frac{n}{n+1}Q'(n+1,u_{n+1})
\end{aligned} \tag{5.7}
$$

and

$$
\begin{aligned}
&\left[\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right]\left[-\frac{n}{n+1}Q(n,u_n)\right] \\
&= -\frac{n}{n+1}Q'(n,u_n).
\end{aligned} \tag{5.8}
$$

Finally, applying the differential operator to $Q(n+2,\omega)$ gives zero. That is,

$$
\left[\frac{\partial}{\partial u_n} + \frac{n u_{n+1}^2}{n+1}\frac{\partial}{\partial u_{n+1}}\right][Q(n+2,\omega)] = 0. \tag{5.9}
$$

By combining (5.7), (5.8) and (5.9), equation (5.6) becomes

$$-\frac{2n}{(n+1)u_{n+1}}Q(n+1,u_{n+1})+\frac{n}{n+1}Q'(n+1,u_{n+1})-\frac{n}{n+1}Q'(n,u_n)=0. \quad (5.10)$$

To solve (5.10), we differentiate it with respect to u_n keeping u_{n+1} fixed. As a result we obtain the following ODE

$$-\frac{\mathrm{d}}{\mathrm{d}u_n}\left(\frac{n}{n+1}Q'(n,u_n)\right) \;=\; 0, \quad (5.11)$$

where $\frac{n}{n+1}$ may be omitted, here and subsequent steps. Solving (5.11) we obtain

$$\begin{aligned}\frac{n}{n+1}Q'(n,u_n) &= A(n),\\ Q'(n,u_n) &= \left(\frac{n+1}{n}\right)A(n),\\ Q(n,u_n) &= \left(\frac{n+1}{n}\right)A(n)u_n+B(n). \quad (5.12)\end{aligned}$$

Suppose that $B(n)=0$ for ease of computation. Then

$$Q(n,u_n)=\left(\frac{n+1}{n}\right)A(n)u_n \quad (5.13)$$

and hence

$$Q(n+1,u_{n+1})=\left(\frac{n+2}{n+1}\right)A(n+1)u_{n+1}. \quad (5.14)$$

Next we substitute (5.13) and (5.14) into (5.10) to yield

$$\left(-\frac{2n}{(n+1)u_{n+1}}\right)\left(\frac{n+2}{n+1}\right)A(n+1)u_{n+1}+\left(\frac{n}{n+1}\right)\left(\frac{n+2}{n+1}\right)A(n+1)$$
$$-\left(\frac{n}{n+1}\right)\left(\frac{n+1}{n}\right)A(n)=0 \quad (5.15)$$

and simplifying we obtain

$$\begin{aligned}\left(\frac{-2n(n+2)}{(n+1)^2}\right)A(n+1)+\left(\frac{n(n+2)}{(n+1)^2}\right)A(n+1)-A(n) &= 0,\\ \left[\frac{-2n(n+2)+n(n+2)}{(n+1)^2}\right]A(n+1) &= A(n)\\ \left[\frac{-n(n+2)}{(n+1)^2}\right]A(n+1) &= A(n). \quad (5.16)\end{aligned}$$

Using Mathematica to solve equation (5.16) for $A(n)$, we get

$$A(n)=\left(\frac{n}{n+1}\right)2c(-1)^{n-1} \quad (5.17)$$

where c is a constant. Substituting (5.17) into (5.13) gives

$$\begin{aligned} Q(n,u_n) &= \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right)2c(-1)^{n-1}u_n \\ &= 2c(-1)^{n-1}u_n. \end{aligned} \tag{5.18}$$

Hence we obtain the symmetry generator, namely

$$X = 2c(-1)^{n-1}u_n\frac{\partial}{\partial u_n}. \tag{5.19}$$

5.1.2 Calculating the first integral

Recall that the first integral of (4.36) was calculated in Section 4.2 and was found to be

$$\phi = nu_nu_{n+1} - \frac{n(n+1)}{2},$$

which is given by equation (4.53).
Now we apply X, the symmetry generator given by (5.19), to the first integral, ϕ.

$$\begin{aligned} X\phi &= Q(n,u_n)\frac{\partial\phi}{\partial u_n} + Q(n+1,u_{n+1})\frac{\partial\phi}{\partial u_{n+1}} \\ &= 2c(-1)^{n-1}u_n\frac{\partial\phi}{\partial u_n} + 2c(-1)^{n}u_{n+1}\frac{\partial\phi}{\partial u_{n+1}} \\ &= 2c(-1)^{n-1}u_n(nu_{n+1}) - 2c(-1)^{n-1}u_{n+1}(nu_n) \\ &= 2c(-1)^{n-1}(nu_nnu_{n+1} - nu_nnu_{n+1}) \\ &= 0. \end{aligned}$$

Observe that ϕ is invariant under X. We say X and ϕ are *associated*.

5.1.3 General Solution

Recall in Subsection 5.1.1 we calculated the symmetry generator, X, to be

$$X = 2c(-1)^{n-1}u_n\frac{\partial}{\partial u_n}$$

given by equation (5.19). Suppose $v_n = v(n,u_n,u_{n+1})$ is an invariant of X. Then

$$\begin{aligned} Xv_n &= \left(Q(n,u_n)\frac{\partial}{\partial u_n} + \mathcal{S}Q(n,u_n)\frac{\partial}{\partial u_{n+1}}\right)v_n \\ &= \left(2c(-1)^{n-1}u_n\frac{\partial}{\partial u_n} + 2c(-1)^{n}u_{n+1}\frac{\partial}{\partial u_{n+1}}\right)v_n \end{aligned}$$

$$= \quad 0. \tag{5.20}$$

Now we use the method of characteristics to solve for v_n and construct the equation,

$$\frac{\mathrm{d}u_n}{2c(-1)^{n-1}u_n} = \frac{\mathrm{d}u_{n+1}}{2c(-1)^n u_{n+1}} = \frac{\mathrm{d}v_n}{0}. \tag{5.21}$$

To obtain the independent variable, α, we solve

$$\begin{aligned}
\frac{\mathrm{d}u_{n+1}}{2c(-1)^n u_{n+1}} &= \frac{\mathrm{d}u_n}{2c(-1)^{n-1}u_n}, \\
\frac{\mathrm{d}u_{n+1}}{u_{n+1}} &= -\frac{\mathrm{d}u_n}{u_n}, \\
\ln u_{n+1} &= -\ln(u_n) + A, \\
A &= \ln(u_{n+1}) + \ln(u_n), \\
\exp(A) &= \exp(\ln(u_{n+1}))\exp(\ln(u_n)), \\
&= u_n u_{n+1}.
\end{aligned}$$

Let $\exp(A) = \alpha$, then the independent variable is

$$\alpha = u_n u_{n+1}. \tag{5.22}$$

Also $\frac{\mathrm{d}v_n}{0}$ implies that $v_n = \gamma$ where $\gamma = \alpha$. Therefore by equation (5.22), the dependent variable, v_n, is

$$v_n = u_n u_{n+1}. \tag{5.23}$$

Applying the shift operator, $\mathcal{S}$, on v_n gives

$$\begin{aligned}
\mathcal{S}(v_n) &= v_{n+1} \\
&= u_{n+1}u_{n+2} \\
&= u_{n+1}\left(\frac{n}{n+1}u_n + \frac{1}{u_{n+1}}\right) \qquad \text{from equation (4.36)} \\
&= \frac{n}{n+1}u_n u_{n+1} + 1 \\
&= \frac{n}{n+1}v_n + 1.
\end{aligned}$$

That is, we get the first order difference equation

$$v_{n+1} = \frac{n}{n+1}v_n + 1. \tag{5.24}$$

By using the RSolve Mathematica command, we obtain that the general solution of equation (5.24) is

$$v_n = \frac{n+1}{2} + \frac{c}{n} \tag{5.25}$$

where c is a constant. Then equating (5.23) and (5.25) we obtain

$$\begin{aligned} v_n &= u_n u_{n+1}, \\ \frac{n+1}{2} + \frac{c}{n} &= u_n u_{n+1}, \\ u_{n+1} &= \frac{n+1}{2u_n} + \frac{c}{u_n}. \end{aligned} \tag{5.26}$$

Note that equation (4.36) has been reduced by one order into equation (5.26). It can be further reduced by using the RSolve command in *Mathematica*.

Solving (5.26) for c, gives

$$\begin{aligned} c &= n u_n u_{n+1} - \frac{n(n+1)}{2} \\ &= \phi. \end{aligned} \tag{5.27}$$

Note that the first integral ϕ, given by equation (4.53), and the reduction are the same. This suggests a relationship between ϕ and X. In fact, the association is that ϕ is *invariant* under X. Thus, the reduced equation (5.27), $\phi = c$, may be further reduced using X again. This is referred to as *double reduction* in differential equations. It would appear therefore, that the method of double reduction can also be applied to difference equations.

5.2 Problem 2

We now consider the linear difference equation $u_{n+2} = 2u_{n+1} - u_n$ obtained from ([17], p 11). Again we calculate the symmetries, first integrals and then reduce the order of the difference equation.
We start by calculating the symmetry.

5.2.1 Calculating the symmetry generator

Given that

$$\omega = u_{n+2} = 2u_{n+1} - u_n, \tag{5.28}$$

then it follows that differentiating with respect to u_n and u_{n+1} respectively gives

$$\begin{aligned} \frac{\partial \omega}{\partial u_n} &= -1, &(5.29)\\ \frac{\partial \omega}{\partial u_{n+1}} &= 2. &(5.30) \end{aligned}$$

Suppose that $Q = Q(n, u_n)$, then the symmetry condition becomes

$$\begin{aligned} Q(n+2,\omega) - Q(n,u_n)\frac{\partial \omega}{\partial u_n} - Q(n+1,u_{n+1})\frac{\partial \omega}{\partial u_{n+1}} &= 0,\\ Q(n+2,\omega) - Q(n,u_n)(-1) - Q(n+1,u_{n+1})(2) &= 0,\\ Q(n+2,\omega) - 2Q(n+1,u_{n+1}) + Q(n,u_n) &= 0. &(5.31) \end{aligned}$$

To solve equation (5.31) for $Q = Q(n, u_n)$, we use the differential operator method. Differentiating u_{n+1} with respect to u_n yields

$$\begin{aligned} \frac{\partial u_{n+1}}{\partial u_n} &= -\frac{\frac{\partial \omega}{\partial u_n}}{\frac{\partial \omega}{\partial u_{n+1}}},\\ &= \frac{1}{2} \end{aligned}$$

and hence the differential operator is

$$\frac{\partial}{\partial u_n} + \frac{1}{2}\frac{\partial}{\partial u_{n+1}}. \tag{5.32}$$

Applying (5.32) to (5.31) gives

$$\begin{aligned} \left[\frac{\partial}{\partial u_n} + \frac{1}{2}\frac{\partial}{\partial u_{n+1}}\right][Q(n+2,\omega) - 2Q(n+1,u_{n+1}) + Q(n,u_n)] &= 0,\\ Q'(n,u_n) - Q'(n+1,u_{n+1}) &= 0. &(5.33) \end{aligned}$$

We differentiate equation (5.33), with respect to u_n keeping u_{n+1} fixed, to solve for $Q(n, u_n)$. Therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u_n}\left[Q'(n,u_n)\right] &= 0, \\
Q'(n,u_n) &= A(n), && (5.34) \\
Q(n,u_n) &= A(n)u_n + B(n), && (5.35)
\end{aligned}$$

and hence we have

$$\begin{aligned}
Q(n+1,u_{n+1}) &= A(n+1)u_{n+1} + B(n+1), \\
Q'(n+1,u_{n+1}) &= A(n+1). && (5.36)
\end{aligned}$$

Next we solve for $A(n)$ by substituting equations (5.34) and (5.36) into (5.33). This gives

$$\begin{aligned}
A(n) - A(n+1) &= 0, \\
A(n+1) &= A(n), \\
A(n) &= a, && (5.37)
\end{aligned}$$

where a is a constant. Substituting $A(n) = a$ into equation (5.35) yields

$$\begin{aligned}
Q(n,u_n) &= au_n + B(n), && (5.38) \\
Q(n+1,u_{n+1}) &= au_{n+1} + B(n+1), && (5.39) \\
Q(n+2,u_{n+2}) &= au_{n+2} + B(n+2). && (5.40)
\end{aligned}$$

To solve for $B(n)$, substitute (5.38), (5.39) and (5.40) into (5.31). By doing this we get

$$\begin{aligned}
au_{n+2} + B(n+2) - 2\left[au_{n+1} + B(n+1)\right] + au_n + B(n) &= 0, \\
a[2u_{n+1} - u_n] + B(n+2) - 2au_{n+1} - 2B(n+1) + au_n + B(n) &= 0, \\
2au_{n+1} - au_n + B(n+2) - 2au_{n+1} - 2B(n+1) + au_n + B(n) &= 0, \\
B(n+2) - 2B(n+1) + B(n) &= 0. && (5.41)
\end{aligned}$$

Recall that equation (5.41) was solved in Chapter 3 using auxiliary equations (refer to equation (3.30)) and has general solution

$$B(n) = bn + c \qquad (5.42)$$

where b and c are arbitrary constants. Finally we substitute (5.42) into (5.38) and obtain the characteristic

$$Q(n, u_n) = au_n + bn + c. \tag{5.43}$$

Now we let one of the constants equal one and the other two constants equal zero to obtain a three-dimensional Lie algebra of symmetry generators, whose characteristics are linear combinations of

$$\begin{aligned} a &= 1,\ b = c = 0 \Rightarrow Q_1 = u_n, \\ b &= 1,\ a = c = 0 \Rightarrow Q_2 = n, \\ c &= 1,\ a = b = 0 \Rightarrow Q_3 = 1. \end{aligned}$$

Therefore the Lie symmetry generators are

$$X_1 = Q_1\frac{\partial}{\partial u_n} = u_n\frac{\partial}{\partial u_n}, \tag{5.44}$$

$$X_2 = Q_2\frac{\partial}{\partial u_n} = n\frac{\partial}{\partial u_n}, \tag{5.45}$$

$$X_3 = Q_3\frac{\partial}{\partial u_n} = \frac{\partial}{\partial u_n}. \tag{5.46}$$

In the next Subsection we calculate a first integral of equation (5.28).

5.2.2 Calculating the first integral

Suppose that $P_2 = P_2(n, u_n)$ and recall that the *first integral condition* is

$$\mathcal{S}\left(\frac{\partial \omega}{\partial u_n}\right)\mathcal{S}^2 P_2(n, u_n) + \frac{\partial \omega}{\partial u_{n+1}}\mathcal{S}P_2(n, u_n) - P_2(n, u_n) = 0,$$

given by equation (4.33).
Substituting (5.29) and (5.30) into (4.33) gives

$$\begin{aligned} \mathcal{S}(-1)P_2(n+2, u_{n+2}) + 2P_2(n+1, u_{n+1}) - P_2(n, u_n) &= 0, \\ -P_2(n+2, u_{n+2}) + 2P_2(n+1, u_{n+1}) - P_2(n, u_n) &= 0, \\ P_2(n+2, \omega) - 2P_2(n+1, u_{n+1}) + P_2(n, u_n) &= 0. \end{aligned} \tag{5.47}$$

Note that (5.47) is the same as (5.31). Hence, by the same approach used to solve equation (5.31), we have that the solution to (5.47) is

$$P_2(n, u_n) = ku_n + pn + q \tag{5.48}$$

where k, p and q are constants. Then by equation (4.31), we have that

$$\begin{aligned} P_1(n+1, u_{n+1}) &= \mathcal{S} P_2(n, u_n) \frac{\partial \omega}{\partial u_n} \\ &= \mathcal{S}(k u_n + pn + q) \frac{\partial \omega}{\partial u_n} \\ &= [k u_{n+1} + p(n+1) + q](-1) \\ &= -k u_{n+1} - pn - p - q. \end{aligned} \tag{5.49}$$

Recall the first integral is given by equation (4.35), that is,

$$\phi = \int (P_1 \mathrm{d}u_n + P_2 \mathrm{d}u_{n+1}) + G(n).$$

Substituting (5.48) and (5.49) into (4.35) we obtain

$$\begin{aligned} \phi &= \int [(-k u_{n+1} - pn - p - q)\, \mathrm{d}u_n + (k u_n + pn + q)\, \mathrm{d}u_{n+1}] + G(n) \\ &= -k u_{n+1} u_n - pn u_n - (p+q) u_n + k u_n u_{n+1} + pn u_{n+1} + q u_{n+1} + G(n) \\ &= -pn u_n + pn u_{n+1} + q u_{n+1} - (p+q) u_n + G(n) \\ &= pn(u_{n+1} - u_n) + q(u_{n+1} - u_n) - p u_n + G(n). \end{aligned} \tag{5.50}$$

To solve for $G(n)$, we substitute (5.50) into the definition of the first integral given by (2.13). Then we have

$$\begin{aligned} \mathcal{S}\phi &= p(n+1)(u_{n+2} - u_{n+1}) + q(u_{n+2} - u_{n+1}) - p u_{n+1} + G(n+1) \\ &= p(n+1)(u_{n+1} - u_n) + q(u_{n+1} - u_n) - p u_{n+1} + G(n+1), \quad \text{by (5.28)} \\ &= pn(u_{n+1} - u_n) + p(u_{n+1} - u_n) + q(u_{n+1} - u_n) - p u_{n+1} + G(n+1) \\ &= pn(u_{n+1} - u_n) + q(u_{n+1} - u_n) - p u_n + G(n+1). \end{aligned} \tag{5.51}$$

To satisfy equation (2.13), equate RHS's of (5.50) and (5.51). After equating and cancelling like terms we obtain

$$\begin{aligned} G(n+1) &= G(n), \\ G(n) &= r, \end{aligned} \tag{5.52}$$

where r is a constant. Substituting (5.52) into (5.50) we get that the first integral is

$$\begin{aligned} \phi &= pn(u_{n+1} - u_n) + q(u_{n+1} - u_n) - p u_n + r \\ &= (pn + q) u_{n+1} - (pn + p + q) u_n + r. \end{aligned} \tag{5.53}$$

Then differentiating ϕ with u_{n+1} and u_n respectively, gives

$$\frac{\partial \phi}{\partial u_{n+1}} = pn + q, \tag{5.54}$$

$$\frac{\partial \phi}{\partial u_{n+1}} = -pn - p - q. \tag{5.55}$$

Next we check if ϕ is associated with the symmetry generators given by equations (5.44), (5.45) and (5.46). That is, when does the condition $X\phi = 0$ hold?

Part I

For equation (5.44), that is, when the symmetry generator is

$$X_1 = u_n \frac{\partial}{\partial u_n},$$

we have

$$\begin{aligned} X_1\phi &= u_n \frac{\partial \phi}{\partial u_n} + u_{n+1}\frac{\partial \phi}{\partial u_{n+1}} \\ &= u_n\left(-pn - q - p\right) + u_{n+1}\left(pn + q\right), \quad \text{by (5.54) and (5.55)} \\ &= -u_n\left(pn + q + p\right) + u_{n+1}\left(pn + q\right). \end{aligned}$$

Thus ϕ is associated with X_1, that is $X_1\phi = 0$, if the following two equations are satisfied:

$$pn + q + p = 0 \quad \text{and} \quad pn + q = 0.$$

Solving the above equations simultaneously gives $p = 0$ and $q = 0$. Hence, for ϕ to be associated with X_1, then

$$\phi = r, \tag{5.56}$$

which is trivial.

Part II

Next consider the symmetry generator

$$X_2 = n\frac{\partial}{\partial u_n},$$

given by equation (5.45). Then

$$X_2\phi = n\frac{\partial \phi}{\partial u_n} + (n+1)\frac{\partial \phi}{\partial u_{n+1}}$$

$$
\begin{aligned}
&= n(-pn-q-p)+(n+1)(pn+q) \quad \text{by (5.53)} \\
&= q.
\end{aligned}
$$

Hence ϕ is associated with X_2 if $q=0$, that is, if

$$\phi = pnu_{n+1} - (pn+p)u_n + r. \tag{5.57}$$

Part III

Finally consider equation (5.46), which is the symmetry generator given by

$$X_3 = \frac{\partial}{\partial u_n}.$$

Then,

$$
\begin{aligned}
X_3\phi &= \frac{\partial \phi}{\partial u_n} + \frac{\partial \phi}{\partial u_{n+1}} \\
&= -pn-q-p+pn+q, \quad \text{by (5.54) and (5.55)} \\
&= -p.
\end{aligned}
$$

Here ϕ is associated with X_3 if $p=0$. Therefore, the ϕ associated with X_3 is

$$\phi = q(u_{n+1}-u_n)+r.$$

5.2.3 General Solution

This subsection aims to find the general solution of (5.28), that is, reduce the order. We find the commutators of the symmetries so that we may determine which symmetry generator to use first to find the general solution of the difference equation.

Part I

Below we find the commutator of X_1 and X_2, given by equations (5.44) and (5.45) respectively,

$$
\begin{aligned}
[X_1, X_2] &= X_1X_2 - X_2X_1 \\
&= \left(u_n\frac{\partial}{\partial u_n}\right)\left(n\frac{\partial}{\partial u_n}\right) - \left(n\frac{\partial}{\partial u_n}\right)\left(u_n\frac{\partial}{\partial u_n}\right) \\
&= -n\left(\frac{\partial}{\partial u_n}\right) \\
&= -X_2.
\end{aligned}
$$

Hence $[X_1, X_2]$ is linear in X_2 and thus (5.28) will be reduced using X_2 first. Suppose that $v_n = v(n, u_n, u_{n+1})$ is the invariant of X_2. Then

$$\begin{aligned} X_2 v_n &= \left[n\frac{\partial v_n}{\partial u_n} + (n+1)\frac{\partial v_n}{\partial u_{n+1}} \right] \\ &= 0. \end{aligned}$$

Using the method of characteristics we have

$$\frac{\mathrm{d}u_n}{n} = \frac{\mathrm{d}u_{n+1}}{n+1}$$

and integrating gives

$$\begin{aligned} \alpha + (n+1)u_n &= nu_{n+1}, \\ \alpha &= nu_{n+1} - (n+1)u_n, \end{aligned} \tag{5.58}$$

where the integration constant, α, is the independent variable. Also $\frac{\mathrm{d}v_n}{0}$ implies that the dependent variable v_n is

$$v_n = \gamma \tag{5.59}$$

where γ satisfies $\gamma = \alpha$. Hence substituting (5.58) into (5.59) we get

$$v_n = nu_{n+1} - (n+1)u_n. \tag{5.60}$$

Applying the shift operator S on v_n yields

$$\begin{aligned} \mathcal{S}(v_n) &= v_{n+1} \\ &= (n+1)u_{n+2} - (n+2)u_{n+1} \quad \text{by (5.60)} \\ &= (n+1)\left(2u_{n+1} - u_n\right) - (n+2)u_{n+1} \quad \text{by (5.28)} \\ &= 2nu_{n+1} - nu_n + 2u_{n+1} - u_n - nu_{n+1} - 2u_{n+1} \\ &= nu_{n+1} - (n+1)u_n \\ &= v_n, \end{aligned}$$

that is,

$$v_{n+1} = v_n. \tag{5.61}$$

Solving (5.61), we get the general solution

$$v_n = c_1, \tag{5.62}$$

where c_1 is a constant. Equating the RHS of equations (5.60) and (5.62) we have

$$\begin{aligned} c_1 &= nu_{n+1} - (n+1)u_n, \\ u_{n+1} &= \frac{c_1}{n} + \left(1 + \frac{1}{n}\right) u_n. \end{aligned} \tag{5.63}$$

Using *Mathematica*, we obtain

$$u_n = nc_2 + c_1(n-1) \tag{5.64}$$

where c_2 is an arbitrary constant. Equation (5.64) is the general solution of (5.28).

Note that solving for c_1 in equation (5.63) yields

$$c_1 = nu_{n+1} - (n+1)u_n.$$

Therefore, ϕ (given by equation (5.57)) and the reduction are the same if $p = 1$ and $r = 0$. That is $\phi = c_1$. This equation may be further reduced using X_2 again.

Part II

We can also find another general solution of (5.28) by using a different symmetry generator from X_2. To determine what symmetry to reduce the order by, we find the commutator of X_1 and X_3 which are given by equations (5.44) and (5.46) respectively.

$$\begin{aligned} [X_1, X_3] &= X_1X_3 - X_3X_1 \\ &= \left(u_n \frac{\partial}{\partial u_n}\right)\left(\frac{\partial}{\partial u_n}\right) - \left(\frac{\partial}{\partial u_n}\right)\left(u_n \frac{\partial}{\partial u_n}\right) \\ &= -\left(\frac{\partial}{\partial u_n}\right) \\ &= -X_1, \end{aligned}$$

which is linear in X_1 and hence (5.28) will be reduced using X_1 first. Again suppose that $v_n = v(n, u_n, u_{n+1})$ is invariant of X_1. Then

$$\begin{aligned} X_1 v_n &= \left[u_n \frac{\partial v_n}{\partial u_n} + u_{n+1} \frac{\partial v_n}{\partial u_{n+1}}\right] \\ &= 0. \end{aligned}$$

Using the method of characteristics, we have that

$$\frac{\mathrm{d}u_n}{u_n} = \frac{\mathrm{d}u_{n+1}}{u_{n+1}}. \tag{5.65}$$

To solve (5.65), we take the integral on both sides and obtain

$$\begin{aligned}
\ln(\alpha)+\ln(u_n) &= \ln(u_{n+1}), \\
\ln(\alpha) &= \ln(u_{n+1})-\ln(u_n), \\
\exp(\ln(\alpha)) &= \exp\left(\ln(u_{n+1}-\ln(u_n))\right), \\
\alpha &= \frac{u_{n+1}}{u_n}, \qquad (5.66)
\end{aligned}$$

where α is the independent variable. Also by the method of characteristics, $\frac{\mathrm{d}v_n}{0}$ implies that the dependent variable, v_n, satisfies $v_n = \gamma$ where $\gamma = \alpha$. Hence,

$$\begin{aligned}
v_n &= \alpha \\
&= \frac{u_{n+1}}{u_n}. \qquad (5.67)
\end{aligned}$$

Therefore applying the shift operator on v_n gives

$$\begin{aligned}
\mathcal{S}(v_n) &= v_{n+1} \\
&= \frac{u_{n+2}}{u_{n+1}} \\
&= \frac{2u_{n+1}-u_n}{u_{n+1}} \quad \text{by (5.28)} \\
&= 2-\frac{u_n}{u_{n+1}} \\
&= 2-\frac{1}{v_n}.
\end{aligned}$$

That is,

$$v_{n+1} = 2-\frac{1}{v_n}, \qquad (5.68)$$

and using the R-Solve command in *Mathematica* to solve for v_n we get

$$v_n = \frac{1+2c_1+nc_1}{1+c_1+nc_1}, \qquad (5.69)$$

where c_1 is a constant. Equating (5.67) and (5.69) results in

$$\begin{aligned}
\frac{u_{n+1}}{u_n} &= \frac{1+2c_1+nc_1}{1+c_1+nc_1}, \\
u_{n+1} &= \left(\frac{1+2c_1+nc_1}{1+c_1+nc_1}\right)u_n. \qquad (5.70)
\end{aligned}$$

To find the general solution of (5.28) we solve equation (5.70) by using *Mathematica*. We then have a general solution of (5.28) which is

$$u_n = \frac{(1+c_1+nc_1)\,c_2}{1+c_1}, \qquad (5.71)$$

where c_2 is a constant. Also, note that equation (5.71) is equivalent to equation (5.64), which is expected for the general solution.

Part III

Finally we consider the commutator of (5.45) and (5.46), that is, X_2 and X_3.

$$\begin{aligned}[X_2, X_3] &= X_2X_3 - X_3X_2 \\ &= \left(n\frac{\partial}{\partial u_n}\right)\left(\frac{\partial}{\partial u_n}\right) - \left(\frac{\partial}{\partial u_n}\right)\left(n\frac{\partial}{\partial u_n}\right) \\ &= 0.\end{aligned}$$

Since the commutator is 0, we can first reduce the OΔE with either X_2 or X_3. However, since we have already reduced (5.28) with X_2, we will use X_3. As before, suppose $v_n = v(n, u_n, u_{n+1})$ is invariant of X_3. Then

$$\begin{aligned}X_3 v_n &= \left[\frac{\partial v_n}{\partial u_n} + \frac{\partial v_n}{\partial u_{n+1}}\right] \\ &= 0.\end{aligned}$$

Applying the method of characteristics, we have

$$\frac{\mathrm{d}u_n}{1} = \frac{\mathrm{d}u_{n+1}}{1} = \frac{\mathrm{d}v_n}{0}.$$

From

$$\mathrm{d}u_n = \mathrm{d}u_{n+1},$$

we get

$$\begin{aligned}\alpha + u_n &= u_{n+1}, \\ \alpha &= u_{n+1} - u_n.\end{aligned}$$

From $\frac{\mathrm{d}v_n}{0}$ we obtain $v_n = \gamma$ where $\gamma = \alpha$. Hence

$$v_n = u_{n+1} - u_n. \tag{5.72}$$

Applying the shift factor, $\mathcal{S}$, on (5.72) gives

$$\begin{aligned}\mathcal{S}(v_n) &= v_{n+1} \\ &= u_{n+2} - u_{n+1} \\ &= 2u_{n+1} - u_n - u_{n+1} \\ &= u_{n+1} - u_n \\ &= v_n. \end{aligned} \tag{5.73}$$

Solving (5.73) we get

$$v_n = c_1. \tag{5.74}$$

Equating (5.72) and (5.74) gives

$$\begin{aligned} c_1 &= u_{n+1} - u_n, \\ u_{n+1} &= u_n + c_1. \end{aligned} \tag{5.75}$$

Using *Mathematica*, we solve (5.75) and find

$$u_n = nc_1 + c_2, \tag{5.76}$$

which is the general solution for (5.28), again equivalent to equation (5.64). Note from (5.75)

$$c_1 = u_{n+1} - u_n,$$

which is the same as ϕ (given by equation (5.56)) if $q = 1$ and $r = 0$.

Conclusion

In this dissertation we have shown how to calculate the symmetry generator, find the first integral and reduce the order of a second-order ordinary difference equation. In Chapter 3, we provided the method for calculating the characteristic and hence the Lie symmetry generators. In Chapter 4 we demonstrated how to use the Lie symmetry generators to find the general solution of an OΔE. We also showed how to directly construct the first integral of an OΔE. The method used to find the first integral did not require the symmetry generators of the difference equation to be known.

In Chapter 5, using difference equations obtained from [17], we have showed that if the symmetry generator, X, and first integral, ϕ, are associated then they satisfy $X\phi = 0$. Furthermore, we showed that if ϕ and the reduction of the difference equation are the same, then ϕ is invariant under X. Hence the reduced equation may be further reduced using X again, which is called double reduction. Some future problems arising from the methods discussed in this dissertation would be a possible application to partial difference equations (PΔE).

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