Exploring links between multiplication problem types, learners' setting up of models and use of strategies within a small-scale intervention.

Student: Emmanuel Dlamini

Student Number: 513902

Supervisor: Prof. Hamsa Venkat

A research report submitted to the WITS SCHOOL OF EDUCATION, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Education (Primary Mathematics).

Johannesburg, 2014

Declaration

I, Emmanuel Dlamini, hereby declare that this research report is my own work. It is being submitted for the Degree of Master of Education (Primary Mathematics) at the University of the Witwatersrand, Johannesburg. This report has not been submitted for any other degree or examination at any other university.

____________________________ ____________________

Emmanuel Dlamini Date

Acknowledgements

It is my pleasure to dedicate special thanks to my research supervisor, Hamsa Venkat for her patience, support and rigorous guidance in my research work. I appreciate all the time and effort that she invested in my research work. I have since grown in the area of research and I believe that what I have achieved through her guidance will impact my research skills greatly. I would also like to thank the following:

- The learners who consented to participate in my study
- My beloved wife Saziso and sons who always encouraged me and gave me moral and emotional support.

Abstract

This study explored models and strategies used by one class of Grade 6 learners across a range of multiplication problem types under a realistic mathematics approach (RME) in a former Model C primary school in Johannesburg East. Forty learners participated in six intervention lessons over a period of six weeks. Learners were presented with mathematical word problem-solving tasks. Evidence of model use, type of models generated, and how the models and the corresponding strategies were used to solve multiplication problems were assessed. The main focus was to discover the models and strategies that learners were using prior to the intervention lessons (assessed using a pre-test), to identify the shifts with respect to these models and strategies during the course of the intervention lessons and finally to identify the kind of models and strategies that learners were using by the end of the small scale intervention (assessed with a post-test which was a repear sitting of the pre-test). Results indicated that learners were using a limited number of models and strategies at the beginning. As a result of the intervention lessons, learners began to use a broader range of models within their problem-solving. Post-test results indicated that a broader range of models and substantial shifts away from use of the column model were associated with increased success in learners' multiplication problem-solving performance at the end of the intervention lessons.

Table of Contents

Content **Page**

CHAPTER ONE: INTRODUCTION

1.1 Background to the study

The poor performance of South African Grade 6 learners in the Annual National Assessment (ANA) in mathematics is a cause for concern. The recent announcement of the 2013, national mean results by the minister of education (39 %) at Grade 6 reflect a historically poor pattern of performance in mathematics. This figure follows an even lower mean result (27 %) of 2012. My analysis of how Grade 6 learners at one former Model C School (schools serving formerly white suburbs in apartheid era) in Johannesburg performed in the last three years on the four basic operations of addition, subtraction, multiplication and division on ANA questions revealed the following results (percentages indicate proportion of learners getting these items correct).

Year	Addition (+)	Subtraction (-)	Multiplication (x)	Division (÷)
2010	83 %	75 %	42 %	35 %
2011	92 %	67 %	38 %	40 %
2012	75 %	64 %	36 %	28 %
Average (3 yrs.)	83.3 %	68.7 %	38.6 %	34.3 %

Table 1.1: Performance of grade 6 learners in the last 3 years.

This initial analysis illustrates that multiplication and division pose particular difficulties for learners. Learner difficulties with regard to multiplication and division have also been alluded to in a wide range of writing (Davydov, 1991; Greer, 1992; Nunes & Bryant, 1996 and Anghileri, 2000). Hence, it is not unique to South African learners. In this study, I focus specifically on multiplication. In the sections that follow, I detail my reasons for choosing to focus on multiplication and outline literature that pointed towards a focus on multiplication problem types as well as children's models and strategies. I also explain my decision to use Realistic Mathematics Education (Gravemeijer, 1994) as a broad theoretical framework for my study and outline the research design – which is based on an intervention drawing from Askew's (2012) work and incorporating a pre- and post-test. Validity and reliability issues and details of how I dealt with ethical concerns are also dealt with.

1.2 Rationale

1.2.1 Nature of multiplication

The general model of multiplication **n × a** means **n** groups of **a.** Given **n × a = b**, then **n** is called the multiplier, **a** is called the multiplicand and **b** is called the product. Hence, 4 × 8 represents four groups of eight or 8 + 8 + 8 + 8. It follows from this that while 4 × 8 and 8 × 4 are equal expressions in value, they can be viewed as different in representation. Drake & Barlow (2007) argue that learners who are able to recognize that 4 × 8 is modelled differently from 8 × 4 will have a better understanding of multiplication than those learners who are not able to differentiate between the two expressions.

1.2.2 Context

My study is focused on the learning of multiplication in the context of an intervention focused on developing learners' ability to model multiplication situations. Verschaffel & De Corte (1997) and Van den Heuvel-Panhuizen (2003) argue that learner growth in comprehending mathematics lies in the production and use of models of situations. Models are described as "organizing or mathematizing – with the objective of structuring situations in terms of mathematical relations." (Gravemeijer, 2002, p. 142). Hence, it is the use of realistic contexts that gives learners opportunities to imagine problem situations (given situation should be 'real' in their minds) and produce mathematical models of them. It is important that I outline the general performance of South African learners in comparative studies and how performance in mathematics has been viewed locally and internationally.

South African society has been, and is still experiencing changes following the move to democracy. Hence, changes in education, especially in mathematics and science are inevitable. This is so simply because these subjects are regarded as core subjects in terms of developing the human resources to drive the economy. It has been well documented elsewhere that the teaching and learning of mathematics in South Africa leaves a lot be desired. This is evident in the Trends in International Mathematics and Science Studies (TIMSS) that were conducted over the last two decades (1995, 1999 & 2004), where there has been no improvement in sight, according to Mji & Makgato (2006). Learners' performance in numeracy/maths has always been disappointingly low in comparative studies, ranging between 30% and 40% (Mji & Mkgato, 2006). Recently, the Business Day Financial

Mail (Sunday 1 Dec. 2013) cited the World Economic Forum (WEF)'s annual report that ranked South Africa last in a ranking of 62 countries in the quality of maths and science education. Hence, both national and international studies have highlighted the poor quality of South African mathematics results as well as poor mathematics skills of South African learners (Bansilal, James & Naidoo, 2010).

A current trend in Mathematics the world over, including South Africa, is to put more emphasis on the relation or link between mathematics and the environment or everyday world around us (Verhage & De Lange, 2006). This implies therefore, that mathematics has to be learnt in meaningful ways. By this, it is meant that mathematics must be derived from real situations and also be useful to the reality, (Verhage & De Lange, 1996). With this background, an approach based on realistic situations was deemed appropriate for the purposes of teaching and learning of multiplication. RME, which was introduced in the Netherlands during the eighties, emphasizes learning by doing (human activity) where learners construct mathematics and discover their own strategies in the process. Use of models and situated forms of learning plays an important role in structuring and supporting learners' solution strategies. Other key principles of this approach are that it is learner centred, allowing for free production or self-produced models and it also stimulates learners' reasoning. (De Lange, 1987; Gravemeijer, 1994).

Research has shown that word problems can be a valuable resource in providing learners with contexts in which to learn multiplication. (Greer, 1992; Askew, 2012; Carpenter, Fennema, Franke, Levi & Empson, 1999). According to Greer (1992), as cited in Drake & Barlow (2007), there are classes of multiplication problems. Their general model only accounts for two of these problem types. These are repeated addition problems and rate problems. A further problem type identified by Askew (2012) involves scaling problems.

In a study that used problem writing (on multiplication) as an assessment tool, Drake & Barlow (2007) concluded that gaps in learners' understanding of multiplication, especially with regard to the role of the multiplicand and the multiplier and the actual meaning of the product were readily exposed through problem writing. Hence, problem writing could be used by the teacher as a diagnostic tool that would inform and guide instructional decisions as well as specific remediation approaches. Askew (2012) argues that word problems act as a vehicle for learning mathematics. In his view, a teaching style that begins with presentation of a problem to the learners has a profound effect on building mathematical understanding.

Hence, it is this latter approach that I intend to use in this study, where learners are expected to use the given context to construct informal models leading to strategies which in turn lead to solutions.

Models can therefore, support the development of mathematical meaning for the learner and subsequently support or influence the selection of an appropriate strategy. Vest (1985) argues that as much as textbooks include some models for multiplication, the underlying relationships between the models and the strategy (which may enhance understanding) is rarely explored for the benefit of the learners. In this study, a strategy is regarded as the calculation process used to obtain the answer to a given problem, whereas the model is the initial representation of a problem situation. This draws me back to my pre-analysis of learners' work that I have briefly discussed above. What was evident in the learners' test scripts was the dominant use of the column multiplication model accompanied by attempts at traditional multiplication algorithm based strategies. There was strong evidence that most learners resorted to using this model, yet their strategies for dealing with partial products revealed weaknesses in lining up the digits correctly suggesting a lack of incorporation of place value understanding. This also suggests that the traditional approach (use of the column model and its associated algorithm strategy) was frequently used in the school at the expense of alternative models and strategies that are available for multiplication.

In the following section, I expand on what I mean by models and strategies in this study and the ways in which RME underlies my focus on models and strategies before moving into the specifics of problem types, models and strategies for multiplication. My research questions follow my discussion of the literature and theory.

1.3 Models and modeling situations in mathematics

Models are designed to meet specific purposes in given situations. In mathematics, models may be used in situated forms of learning, especially in problem solving (Greeno, 1999; Lesh & Lehrer, 2003). According to Gravemeijer & Stephan (2002), a mathematical model is described in terms of 'transition' and 'fit'. Modeling is primarily seen as organizing activity in which situations are structured in terms of mathematical relationships. They argue that the design of models begins through the informal activities of learners as they look for models that can support their solution strategies. There is subsequent transition from being a 'model of' a situation to becoming a 'model for' mathematical reasoning and calculating leading into strategies. It is this setting up of models and the resultant strategies in multiplication that serve as the key foci in the classroom that I intend to investigate.

According to Carpenter, Fennema, Franke, Levi & Empson (1999) many students drift away from the intuitive modeling skills they often demonstrated at a young age: "If older children simply applied some of the intuitive, analytic modeling skills exhibited by young children to analyse problem situations, it appears that they will avoid some of their most glaring errors" (Carpenter et al, 1999, p. 55). What is crucial therefore is to help students at the elementary level to retain and develop their ability to model situations.

1.4 Critical questions

In order to understand the kinds of models and strategies being used by learners across a range of multiplication problem types, this study will focus on the following aspects:

1. What models and strategies for solving multiplication word problems are Grade 6 learners using prior to a small scale intervention?

2. What models and strategies are advocated across six intervention lessons?

3. What models and strategies for solving multiplication word problems are Grade 6 learners using post a small scale intervention?

1.5 Methodology

The approaches described in the literature and the approaches advocated within RME depend on beginning in the context of problem situations. This brings me to a focus on word problems that involve multiplication. Carpenter et al (1999), claim that word problems lay the foundation for students to learn formal multiplication and division concepts. Askew (2012), drawing from Carpenter et al (1999), argues that there are three main root situations for multiplication, shown in table 1.2 in the next page.

My intervention for this study was based on the work of Askew (2012) and used word problems as a means of supporting informal reasoning about problem situations rather than providing pre-given models and strategies. Askew (2012) provides guidance about how to

Multiplication as repeated	**Multiplication as rate**	**Multiplication as scaling**

addition		
Situations where several groups all the same in size need to be added together.	Situations where there is an implicit ratio, where, explicitly or implicitly, there is a "per" in the context.	Situations where a continuous quantity is increased in size by a scaling factor.

Table 1.2: The three root situations for multiplication (Askew 2012).

discuss alternative approaches and ways to guide learners towards using increasingly efficient models that can be used to select strategies for calculating answers.

I made use of six intervention lessons with 40 learners in a grade 6 class that I teach in a former Model C School in Johannesburg North East. Each intervention lesson was comprised of the three stages suggested by Askew (2012). In the first phase learners had to work in pairs to solve the first of three examples (one at a time). In between each example, there was a follow up discussion in order to capture and compare models and strategies used by learners to solve each problem; in the second phase, learners were expected to discover the underlying structure of the questions and also classify problem types; finally, each learner was expected to do a follow up task involving problems with a similar structure to the initial examples.

A tentative schedule for the six lesson intervention involving a gradual introduction and then mixing across problem types was followed (see Table 1.3). The intervention took the form of 6 intervention lessons of 1.5 hours each (9 hours total), spread over a period of six weeks. The intervention lessons were taught following a regular pattern of one lesson of 1.5 hours in a week. Within these lessons, class and homework tasks were administered that provided data that was analysed in relation to the models and strategies that learners used in the context of different problem types.

I incorporated the use of a pre- and post-test around the intervention lessons. I administered a pre-test as the initial assessment that enabled me to establish the kind of models and strategies that the learners were using before the intervention lessons began. The pre-test was

PLANNED INTERVENTION SCHEDULE GRADE 6: TERM 3.			HOURS: 6 (1.5 hours/lesson)		
Lesson 1 (Week-1)	Lesson 2 (Week-2)	Lesson 3 (Week-3)	Lesson 4 (Week-4)	Lesson 5 (Week-5)	Lesson 6 (Week-6)
Multiplication as repeated addition.	Multiplication as rate.	Multiplication as scaling.	Multiplication as repeated addition & multiplication as rate.	Multiplication as repeated addition & multiplication as scaling.	Multiplication as rate & multiplication as scaling.

Table 1.3: Six lesson intervention schedule.

a pen and paper test consisting of 10 multiplication problems and 4 'buffer' items. The 10 problems included some questions from the Annual National Assessment (ANA) papers.

The main data sources were the pre-test responses. These allowed me to examine the kind of models and strategies that learners in the study sample were using prior to the implementation of the intervention lessons. Learner work that was inclusive of classwork and homework gave me a window for tracking through shifts with regard to previous models and strategies and newly emergent models that learners were able to produce during the course of intervention lessons. Finally, the post-test responses reflected the kind of models and strategies that learners were able to produce after participating in the intervention lessons. Analysis of data was based on situations, models and strategies that were outlined in the literature section. It took the form of coding both models and strategies, organizing them into categories which were then analysed in order to determine the relationships and patterns produced. This also entailed observation of shifts and capturing emerging models and strategies during intervention lessons as well as during marking of learner work.

1.6 Implications and limitations of the study

Having outlined the focus of my study on models and strategies in multiplication as well as the methodology of how the study was carried out and analysed, I need to also examine the implications of the study. In order to examine the production or construction of models in multiplication problems, it is crucial that we examine links that a learner makes with model(s). The underlying idea is that the mathematical representations (models) reflect the internal mental representations in learners (Gravemeijer, 2002). Hence, the assumption is that learning is likely to be supported by the relationships they are able to construct. These models enabled learners to make sense of their own imagination or their own worlds with regard to their internal representation.

Learners' understanding could be interpreted from the models that they produced and the corresponding strategies that they used linked to these models. I was therefore able to understand to some degree the learners' understanding of concepts. Furthermore, in order to assess understanding I was able to ask some of the learners to demonstrate on the chalk board the connections that they had made through reasoning by way of explanations that the learners gave. These explanations were either in written format in learners' assessment tasks or given verbally during intervention lessons. Chi, De Leeuw Chiu & La Vancher (1994) argue that eliciting self-explanations from the learners enhances their learning as well as their understanding.

However, there are also limitations to the study. According to Hiebert & Carpenter (1992), understanding cannot be inferred solely on one particular response on a single task. This is because sometimes a task can be answered correctly without the learner's understanding. Hence, it would need a wide variety of well-planned tasks to build up evidence that points to understanding of a concept. Askew (2012) also notes the significance of an extended period of time in order to successfully introduce models and realise their power in mathematics learning. Hence, six weeks was quite a short period for this purpose. Finally, this study is not straightforwardly generalizable due to its location in a specific context. Regardless of these limitations, there are opportunities for further investigation.

In this study I have focused on Grade 6 learners' production of models and strategies within a short space of time (about six weeks). A longitudinal study across the Intermediate Phase years (Grade 3 – 6), coupled with the use of a control group could achieve more extensive and comprehensive results with regard to models and strategies in multiplication. The extended period would allow learners to be repeatedly exposed to the use of models to an extent that it would become part of the pedagogical content of almost the entire mathematics curricula.

1.7 Structure of research report

This current chapter outlines the introduction of the research report. It lays out the background, the rationale and context for the study as well as the research questions that have been the focus of this study.

The second chapter presents a range of literature that I reviewed prior to carrying out the study. It focuses on literature on models and strategies involving multiplication at the intermediate phase.

The third chapter deals with the theoretical framework of the study, the Realistic Mathematics Education (RME) in particular.

Chapter four outlines the method used in the study. It also outlines the research design and the sampling procedure for the study is described, including the justification why it was carried out the way it was done.

In chapter five, a detailed discussion of the findings with respect to the research questions of the study is presented.

Chapter six presents the conclusion of the research study together with its limitations followed by a short discussion of possible areas of future research.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

The purpose of this section is to introduce a range of literature on multiplication as a basis for a broader literature review in my study. The nature of multiplication and factors that impede successful performance by learners are described. Secondly, a literature-based typology of models and strategies for multiplication is presented, which I use as an analytical framework within the study while remaining open in the analysis to any further models and strategies that children in my sample use.

Unlike addition and subtraction in which the key facts are the number bonds of ten and the patterns associated with the addition and subtraction of ten, multiplication does pose some challenges for learners in that it demands a different kind of thinking (Anghileri, 2007; Barmby et al, 2009). Hence, it is important in my review of literature to first outline the nature of multiplication processes before I move on to the models and associated strategies that can support learners in developing multiplicative thinking in the intermediate phase. It is imperative that I also outline problem situations related to multiplication since my study involves an intervention that uses word problems.

2.2 Nature of multiplication

Anghileri (2007) states that multiplication is a binary operation with two different inputs (e.g. 3 x 4 = 12). One input represents the size of a set and the other represents the number of replications. Hence, the two numbers reflect different types of quantity. Nunes & Bryant (1996) as cited in Barmby, Harries, Higgins & Suggate (2008) emphasize that multiplication and division reflect a qualitative change in the student's thinking as opposed to the commonly held view that these two operations simply follow addition and subtraction. Multiplication involves a replication of sets as opposed to the joining of sets that we see in addition and subtraction.

There are some important aspects of multiplication that need to be understood by learners at primary level with regard to whole numbers. These are:

- Replication
- Binary operation
- Commutativity

- Distributivity
- Associativity

Anghileri (2007) and Cathcart, Pothier, Vance & Bezuk (2000), note that the commutativity property states that for all numbers **m** and **n** in the system of whole numbers, **m** x **n** = **n** x **m.** This implies that if a learner understands that 4 times 6 is 24, then they should also know that 6 times 4 must be 24. That is, numbers may be multiplied in any order. The two writings argue that by applying the commutativity property, we can almost halve the number of facts to be learned in multiplication. Also the associative property enables us to join more than two elements or numbers. For **p**, **q** and **r** in a system of whole numbers (**p** x **q**) x **r** = **p** x (**q** x **r**). When three or more numbers are being multiplied they may be grouped in any order. Hence, learners can combine these properties (commutativity and associativity) to conclude that numbers may be multiplied in any order. In this case, the associative property may allow one to combine easier numbers first. The distributive property is the basis of the common long multiplication method. For all real numbers, **a, b** and **c**, **a** × **(b + c) = a** × **b + a** × **c.** Anghileri (2000) argues that whilst addition and subtraction can be viewed as the joining of sets (with each input representing the same kind of element), multiplication basically involves replication. Hence, addition and subtraction are seen as unary operations whilst multiplication with the two distinct elements (the size of a set and the number of replications) is seen as a binary operation (Anghileri, 2000).

Having discussed the three properties above, it is important to acknowledge that earlier research from small scale studies in English and Welsh primary schools carried out by Dickson, Brown & Gibson (1984) suggest that only a few upper primary school learners could use commutativity and distributivity properties when solving multiplication problems. These studies also show that teaching approaches to multiplication and division differed from those for teaching addition and subtraction. When teaching addition and subtraction, models based on number squares and number lines were deemed to be more effective. However, for multiplication and division, a range of different approaches were found to be appropriate involving the learning of tables, skip counting and algorithms. These strategies were built upon the early foundations of multiplication, namely repeated addition and description of an array when dealing with single digit numbers (Barmby et al, 2009).

2.3 Problem situations related to multiplication

Askew (2012) classifies situations where a certain number of groups of the same size are to be added as **repeated addition**. Situations where a continuous quantity is increased in size by a scale factor are categorised as **multiplication by scaling**. Finally, where there is an implicit ratio or there is the word ‘per’ in context are categorised as **multiplication as rate.**

2.3.1The repeated addition structure for multiplication

The repeated addition structure is an extension of the addition structure. According to Haylock (2010) the common language associated with the repeated addition structure of multiplication is “so many lots (set) of, how many (how much) altogether. For example if there are 8 sets of 5 counters, the repeated addition sum 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 becomes the multiplication (8 x 5) (see figure 2.1).

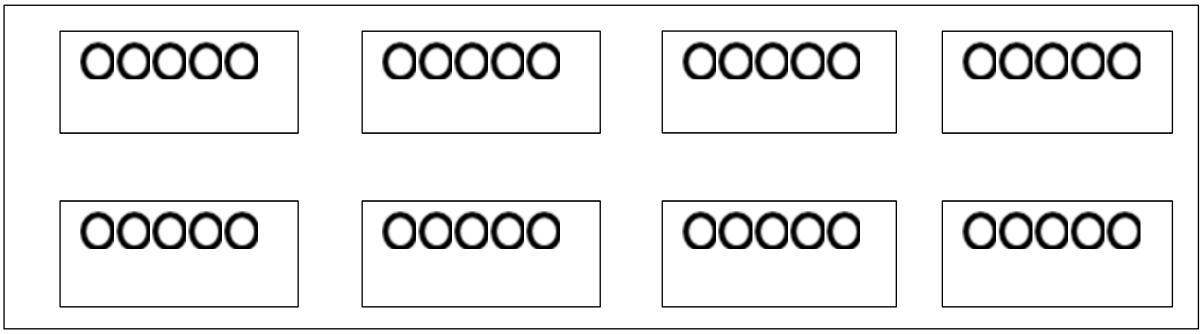
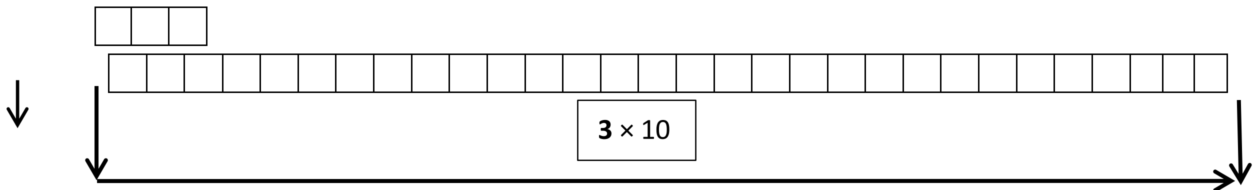


Fig. 2.1: Multiplication as repeated addition (8 x 5).

2.3.2 Scaling structure for multiplication

For the scaling structure, addition means increasing a given quantity by a certain amount. With multiplication we increase a quantity by a scale factor. Hence, multiplying by a particular number, say 10, can be interpreted as scaling quantity by 10. For example,

3 units



Scaled by factor 10

Fig. 2.2: Multiplication as Scaling

Terms such as doubling, scale factor, trebling, so many times as much as, are used in multiplication as scaling. The scaling structure portrays the characteristics of a number line, a model that may be used flexibly in multiplication, especially with low range numbers. It could be more suited to scaling problems as is indicated in the above example.

2.3.3 Multiplication as Rate

When using multiplication as rate, the key words are "each" and "per." There are important situations where repeated addition is presented in the context of cost per unit of measurement. For example, Peter put 15 stamps on each (per) page of his stamp collection book. He filled in 7 pages. How many stamps did he put in his collection book? (In this case one would either multiply 15 by 7 or repeatedly add 15).

2.4 Modelling and problem-solving in multiplication

The use of abstractions of mathematics to solve problems in the real world is regarded as mathematical modeling (Haylock, 2010). This process is illustrated in figure 2.3 below.

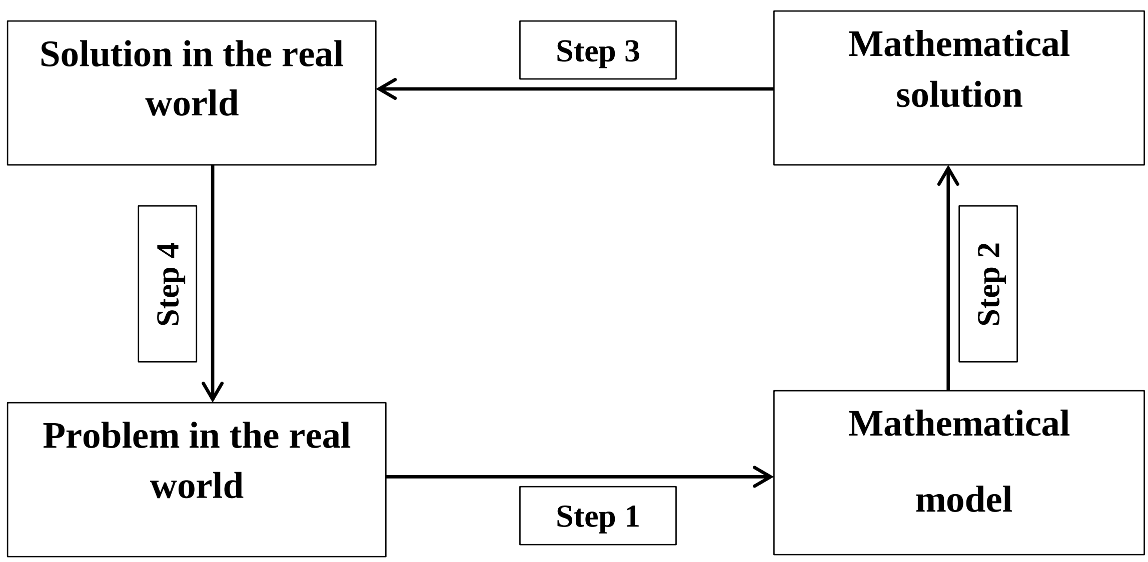


Fig. 2.3: Mathematical modeling (Haylock, 2010)

At the initial stage of the process, a problem in the real world is translated from a realistic situation to the setting up of a mathematical model. According to Realistic Mathematics Education, (described in chapter 3) which is based on the idea of 'mathematizing', this transition from a realistic situation to the setting up of a mathematical model is termed

“Horizontal mathematizing” (Askew, 2012). Using Haylock’s (2010) diagram, the second stage represents vertical mathematization. This is when learners begin to engage with the mathematical solution of the problem, moving from model to strategies that they have produced. At this stage we expect learners to be manipulating the mathematical symbols either mentally or through written strategies in order to find a solution for the given problem. The third step represents interpretation of this mathematical solution in relation to the real world. The final step could be to check the result by considering the real situation to see if it makes sense. However, it is important to note that in this study my focus is on the first two steps of the modelling process (step 1 and step 2), as my interest is in learners’ mathematical solutions.

Haylock (2010) argues that the idea of modeling can be applied to word problems being translated into number statements and Askew (2012) insists that problem solving provides a starting point that can lead to mathematical understanding. He emphasizes that it is crucial to start with a problem-solving context that is meaningful to learners so that they will buy into the problem and begin to generate models and strategies towards a solution.

Askew (2012) argues that the Realistic Mathematics Education approach that makes use of models provides a framework for thinking about how, as teachers, we can help learners to make models to aid their thinking that could lead to effective strategies of dealing with multiplication problems. He argues that learners can progress from the use of a physical image or representation (model) to becoming a mental tool for thinking which can lead to a solution strategy of a given problem. According to RME, models evolve over time from learners’ informal activities, through the process of mathematization, to re-inventing formal ways of reasoning. This implies that the learner gradually gains understanding with the mathematical relations up to a point where the learner begins to internalize (where the learner no longer needs the support of the model for more formal mathematical reasoning) because he/she has appropriated the concept. This, therefore, explains the transition from the informal situated activity (‘model-of’) to the formal reasoning (‘model-for’). An example from Askew (2012) on pages 17 and 18 of this report exemplifies this transition where initially the learner makes use of a grid model to find the product by counting the squares but later on uses her knowledge of multiplying by 10 to calculate the partial product.

2.5 Models and associated strategies for multiplication

In the section that follows I discuss the key models for multiplication. Within each sub-section dealing with a particular model, I review the strategies that literature associates with that particular model.

2.5.1 Array model and associated strategies

Array models provide visual representation of grouped count situations. Learners may progress from multiplication with single digit numbers to multiplication with 2 digit numbers. This model lends itself in turn to grouped counting strategies for calculating totals working either with a column or a row of a group.

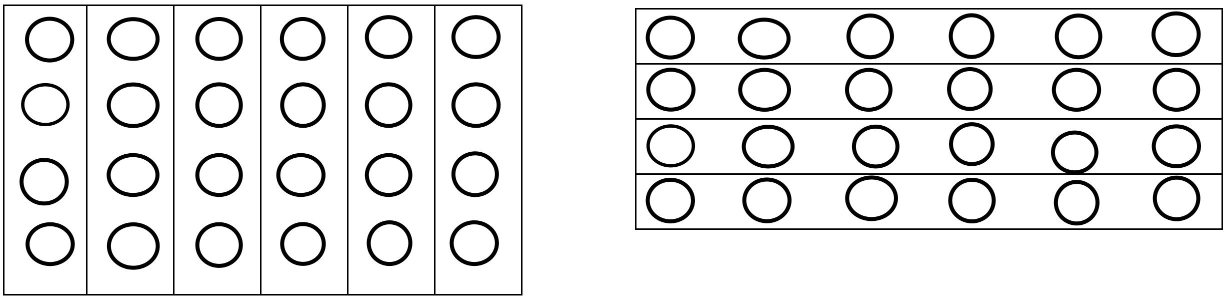


Fig. 2.4: View each column as a group View each row as a group

6 groups of 4 (6 × 4) 4 groups of 6 (4 × 6)

Beckman (2011) argues that viewing multiplication problems in both of these ways is crucial in the sense that the array model allows us to see that commutativity holds, and then decide which version to use in our calculation strategy. In the example above, one would do 4 × 6 rather than 6 × 4 if one is fluent with both the 4 and 6 times tables, as it can be more efficient for children.

Barmby et al (2008), carried out a classroom study with 20 Grade level 4 and 14 Grade level 6 learners in North East of England that was focused on examining how the array representation could support learners' reasoning in multiplication. Four categories of strategies within this model were compared: counting, distributive, rearranging and completing. Counting strategies are described as involving saying out the numbers in order as units (1s), or groups (2s, 3s, etc.). The distributive property is a property of multiplication that can be used to simplify multiplication calculations. For example, 9 groups of 3 can be calculated as 5 groups of 3 and 4 groups of 3. This strategy is useful in that it allows for multiplication situations involving larger numbers to be broken down into separate smaller multiplication parts that can be combined through addition. Awareness and use of this strategy makes multiplication situations involving relatively large numbers possible to solve through mental calculation. The rearranging strategy is used when we multiply using the array model. This is when parts of the array are moved (rearranged) to fill up gaps in the array to make the calculation easier. In this case learners can group the items into groups of round numbers. For example, in 14 x 12 (Barmby et al, 2009) there are 14 counters in each of the 12 rows. So in this example 14 x 12 = 16 x 10 + 8 by moving the 11 and 12th rows to fill two additional columns of 10 with an 8 left over at the end. Learners also completed as many 25s by moving parts of the array and then added on what was left to complete the calculation

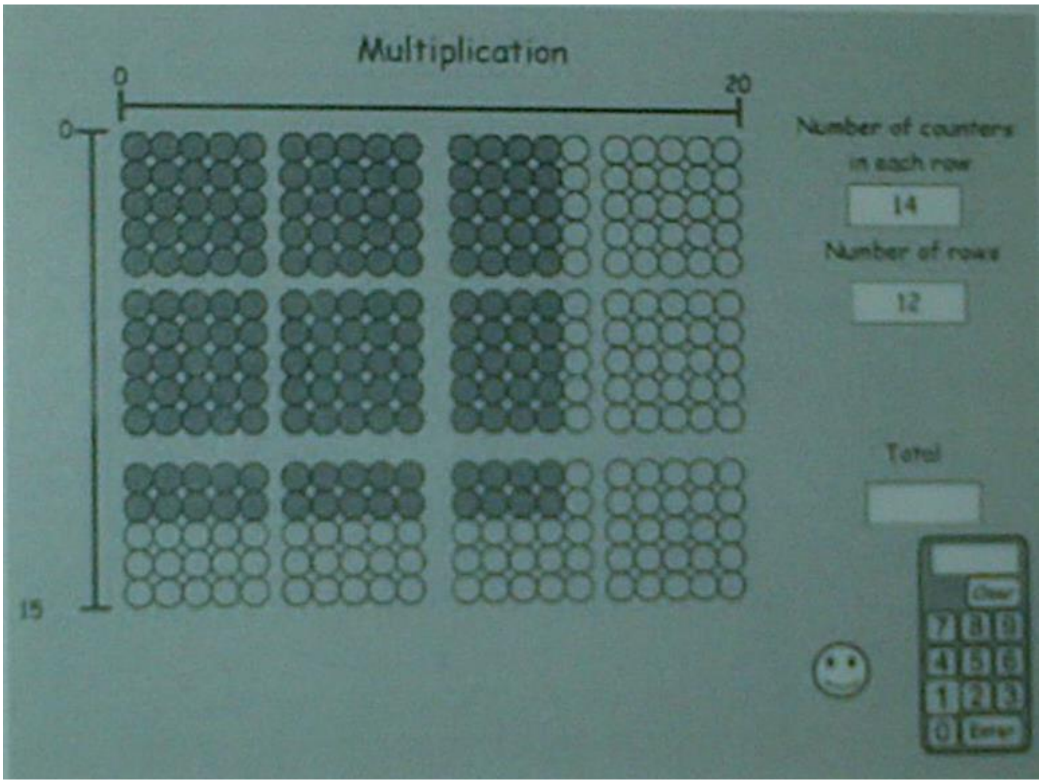


Fig. 2.5: Moving parts of the array to complete the calculation (Barmby et al 2009, p. 231).

of 14 x 12. This arrangement within the array makes mental counting more efficient for the learners. The completing strategy also makes use of round numbers in order to make calculation easier. For example, $9 \times 17 = (10 \times 17) - (1 \times 17)$. In the completing strategy, numbers are rounded and then compensated for (Caliandro, 2000). Barmby et al's (2008) analysis of the performance of intermediate phase learners based on the above strategies is illustrated in table 2.1.

Strategy	Year 4 learners	Year 6 learners
Counting	90%	14%
Distributive	70%	100%
Rearranging	0%	100%
Completing	0%	14%

Table 2.1: Proportion of strategies used by year 4 and 6 learners.

The table indicates that at the beginning of the intermediate phase, the students are largely dependent on the counting strategy. However, a good number start to graduate to the more efficient distributive strategy in later years. This information is important to my study as it gives me some indications of what to expect with regard to the strategies that may be seen in the context of array models. Quantitative comparisons of this nature make me aware of which strategies Grade 6 learners may be likely to use (distributive and rearranging). It will not be surprising if I discover that the strategy of completing is hardly used because this has been revealed by the above comparisons. Although Barmby et al (2008) did not provide explanations for the absence of this strategy, some reasons have been highlighted in other literature, for example Caliandro (2000), who noted that learners frequently struggled with compensating correctly when using the completing strategy.

In a study where learners were working with arrays to explore multiplication, Askew (2012) was able to show how learners can shift from using a representation as a "model of" a situation to being "a model for" calculating the answer and consequently being a tool for thinking and analysing with. Figure 2.5 below shows the use of the array as a model of multiplying a single digit by a two-digit number, (9 x 26) using a squared paper.

In this case, if the learner had counted individual squares in order to find the product of 6 x 26, then the learner would have used it as a "model of," whereas if the learner used her knowledge of multiplying by 10 to work out the partial products of 9 x 20 and 9 x 6, then she would have used it as a "model for" (Askew, 2012).

The above argument is justified by the figure 2.6 below in which the learner did not count squares but used her knowledge of multiplying by 10 to calculate the partial products. Hence, this is a "model for."

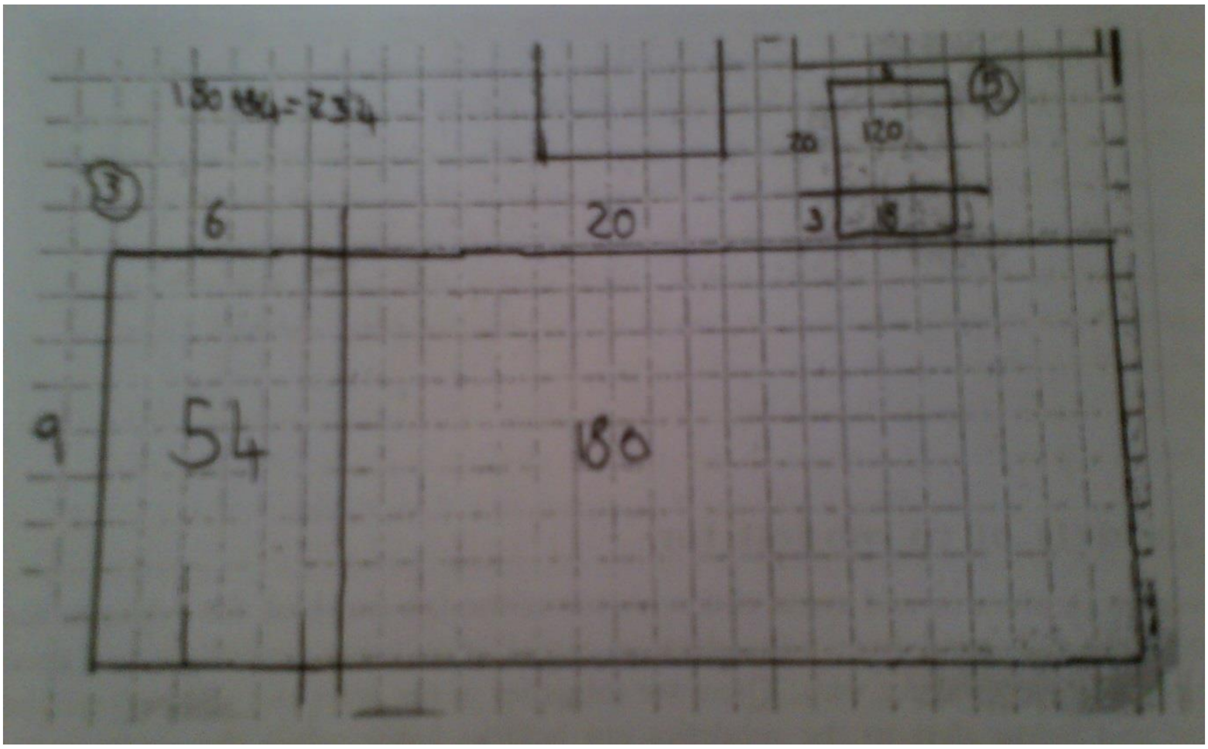


Fig. 2.6: Array as a model of multiplication (Askew 2012, p. 120)

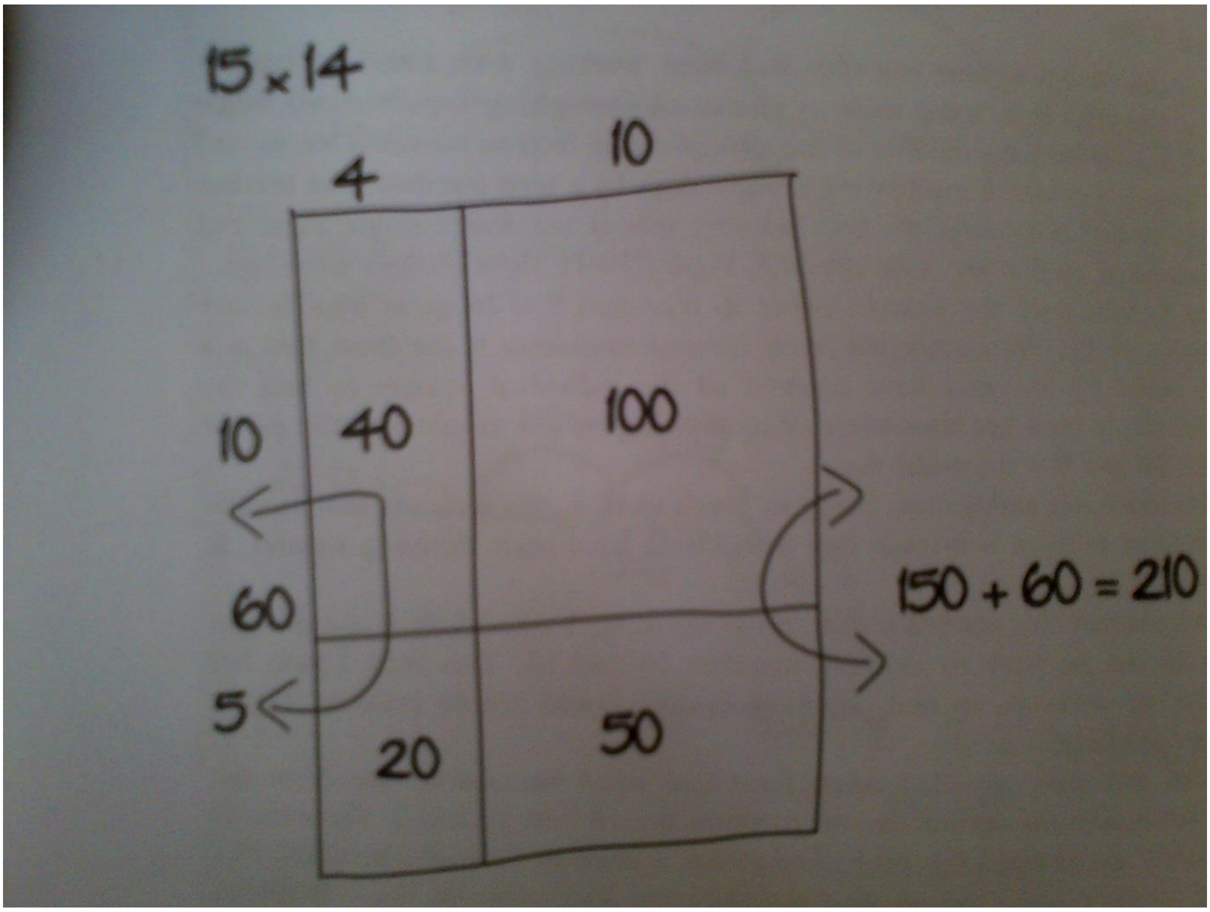


Fig. 2.7: Use of knowledge of multiplying by 10 –"model for" (Askew, 2012, p. 120).

Askew (2012) points out that in this transition from a "model of" to "model for" there is evidence that there is reasoning by the learner as she is thinking through the solution. He also warns though, that introduction of models to learners cannot be done over a short period of time, yet their impact as mathematical tools, if appropriated and internalized, is immense. Hence, repeated exposure is critical for learners to take models on board as part of their daily activities in a mathematics classroom

2.5.2 Area model and associated strategies

Anghileri (2007) emphasizes the importance of modelling using arrays and area models. Subtotals are organized so that they reflect a good understanding of place value. For example, 23 lots of 4 is the same as 20 lots of 4 added to 3 lots of 4 and this can be demonstrated using squared paper marked in the form of the area model shown in figure 2.7 in the next page.

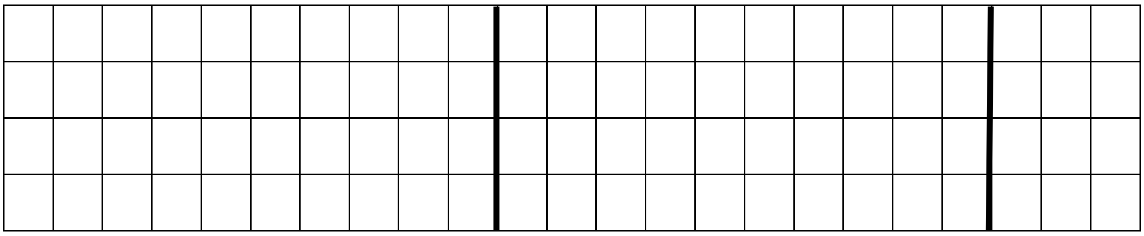


Fig. 2.8: Squared paper marked in the form of the area model (Anghileri 2007, p.77)

This model leads into a strategy based on place value quantity decomposition of the numbers being multiplied. An area model is a truncated version of the grid model, and can be very effective for dealing with two-digit and three-digit numbers. However, the grid model should be introduced at a lower level of multiplication (two-digit by one-digit) before it can be extended for multiplying bigger numbers (Anghileri, 2007).

	3
20	60
4	12

	30	**2**
20	600	40
4	120	8

24 × 3 = 60 + 12 = 72

24 x 32 = 600 + 40 +120 + 8 = 768

Fig. 2.9: Grid model for multiplying 2-and 3-digit numbers (Anghileri, 2007, p. 77).

Haylock (2010), argues that the use of area to interpret multiplication makes it easier for learners to understand multiplication of two-digit numbers. This method is based on splitting both numbers that are being multiplied into tens and units. Using an example taken from Hylock (2010), in simplifying 26 × 34 the two numbers become 20 + 6 and 30 + 4.

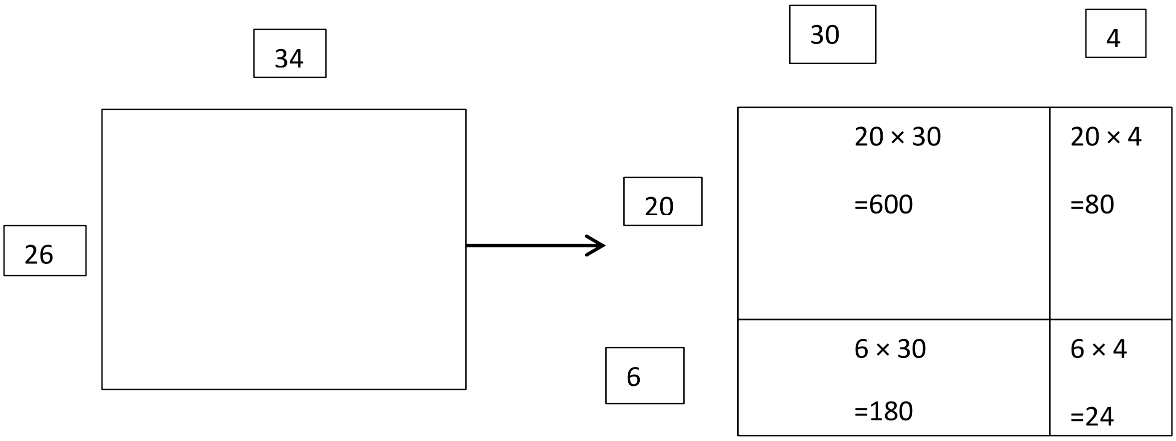


Fig. 2.10: Using area to interpret 26 × 34 (Haylock, 2010).

In the above example a more efficient representation uses the idea of a rectangle. The areas of the four rectangles are then calculated and added together. Once learners have understood the procedure and they have become fluent with it, they may no longer need to draw the rectangles. Learners will simply set out the calculation in the form of a grid shown below:

	30		4	
20	600		80	
6	180		24	
	780	+	104	= 884

In the example above, the grid model has become a “tool for” thinking with. This method enables the learners to handle the computation in a more efficient manner because they deal with friendlier numbers due to the splitting up of both numbers that are being multiplied. The area model is not only restricted to the multiplication of 2-digit numbers only, but can be extended to 3-digit numbers as well. See figure 2.11 below.

	300	40	8
20	300 × 20 = 600	20 × 40 = 800	20 × 8 = 160
5	300 × 5 = 1500	40 × 5 =200	5 × 8 =40

Fig. 2.11: Using the area model to multiply 2-digit by 3-digit numbers.

2.5.3 Column model and associated strategies

Traditional column models were in broad evidence in my initial analysis of Grade 6 learner work in my school, so whilst research cautions against early introduction of this formal model, I note that it is likely to be a model that figures in my empirical dataset, associated with a range of, often incorrect partial product calculation strategies. According to Ma (1999), research that dealt with learners' mistakes in multidigit multiplication revealed that most of these learners forgot to move the numbers over on subsequent lines. This reflects that while learners had no problem selecting the column model they struggled with making sense of, or recalling, the strategies associated with this model. This problem was viewed as "a problem of mathematical learning rather than a careless oversight" (Ma, 1999, p.29). Below is Ma's example of what the children were doing:

123		123	
× 645		× 645	
615	instead of	615	
492		492	
738		738	
1845		79335	(Ma, 1999, p.28).

There were two views here. Most teachers (70%), thought that the problem was associated with the way in which numbers are lined up (partial products). A few of these teachers (30%), felt that students did not understand the actual meanings of the quantities represented by these partial products. Cathcart et al (2000) argue that computations like these tend to depend too much on rules. They argue that to use mental computation effectively, learners need to develop their own strategies and in the process have a good number sense. Knowledge of some basic number facts and a good understanding of the place value numeration system are key.

Column representation has been the traditional way of presenting long multiplication for most of the learners in the sample when two or more digits are being multiplied. This method might be set out as shown below for calculating 26 x 34.

$$\begin{array}{r l}
26 & \\
\underline{\times \quad 34} & \\
780 & \longleftarrow \text{ this is } 26 \times 30 \\
\underline{\quad 104} & \longleftarrow \text{ this is } 26 \times 4 \\
\underline{\quad 884} & \longleftarrow \text{ this is } 26 \times 34
\end{array}$$

(Haylock, 2010, p. 153)

The above example is a left-to-right (L- R) column multiplication (see analysis in chapter 5).

$$\begin{array}{r l}
26 & \\
\underline{\times \quad 34} & \\
104 & \longleftarrow \text{ this is } 26 \times 4 \\
\underline{\quad 780} & \longleftarrow \text{ this is } 26 \times 30 \\
\underline{\quad 884} & \longleftarrow \text{ this is } 26 \times 34
\end{array}$$

(Ma, 1999, p. 28)

The above example is a right-to-left (R – L) column multiplication (see chapter 5).

The two examples reflect the two marginally different approaches to long multiplication. It is important to note that within this study, I view the column representation as a model for multiplication with learners' solution procedures viewed as strategies. Whilst Askew (2012) does not deal with this model/strategy, I include it given the prevalence of this combination in learners' multiplication solutions at my school that I noted in the opening sections.

The long multiplication method is based on the distributive law for multiplication. One of the numbers is broken down into tens and units, and the multiplication by the other number is distributed across these. That is 26 × 34 can be viewed as 26 × (30 + 4), which gives 26 × 30 + 26 × 4 = 780 + 104. Haylock (2010) and Cathcart et al (2000) acknowledge that long multiplication is difficult; hence, there is some potential for errors in the process of calculation.

A good understanding of place value and some mathematical properties of whole numbers can help learners with computational procedures. The commutative law and distributive property of multiplication over addition can support effective computation. For example, multiplication sums below show that if the commutative law is applied, it may be easier to carry out the computational procedures (Cathcart et al, 2000).

40		27
× 27	→	× 40
280		1080
800		
1080		

2.5.4 Rounding and compensating and associated strategies

When learners are able to make use of number facts and relationships with which they are confident, they are likely to find and develop approaches that make sense to them (Caliandro, 2000; Haylock, 2010 ; Steffe, 1994). There is great value in motivating learners to develop their own approaches. Using a combination of additions and subtractions, the distributive law facilitates the breakdown of a number in a multiplication calculation in any way that is easiest to handle (Haylock, 2010). In Caliandro (2000)'s study, learners used the phrase "making numbers friendlier" (Caliandro, 2000, p.423). If learners could breakdown one of the numbers in the multiplication sum into round numbers, they were able to get away with multiplying only by easy numbers. This is illustrated by the two examples below.

Example 1: 26 x 34 could be broken down into (10 x 34) + (10 x 34) + (2 x 34) + (2 x 34) + (2 x 34) = 340 + 340 + 68 + 68 + 68 = 884.

Example 2: The second approach is to think of 34 as 10 + 10 + 10 + 5 – 1 which gives (26 x 10) + (26 x 10) + (26 x 10) + (26 x 5) – (26 x 1) = 260 + 260 + 260 + 130 – 26

= 910 – 26 = 884

In a study conducted by Askew (2012) on learners' use of models when working with arrays to explore multiplication, one learner was evaluating 9 x 16. He modeled carrying out the calculation by setting up the array for 10 x 16. He then "sliced off" the strip of 1 x 16 to evaluate 9 x 16 (Askew, 2012). There is, therefore, a link to the earlier references of the 'completing' model in Barmby et al (2009) and Askew's (2012) writing on 'model of'/ 'model for'. While Barmby et al (2009) discuss this strategy in the context of the array model, Askew (2012) deals with this strategy in the context of a grid model.

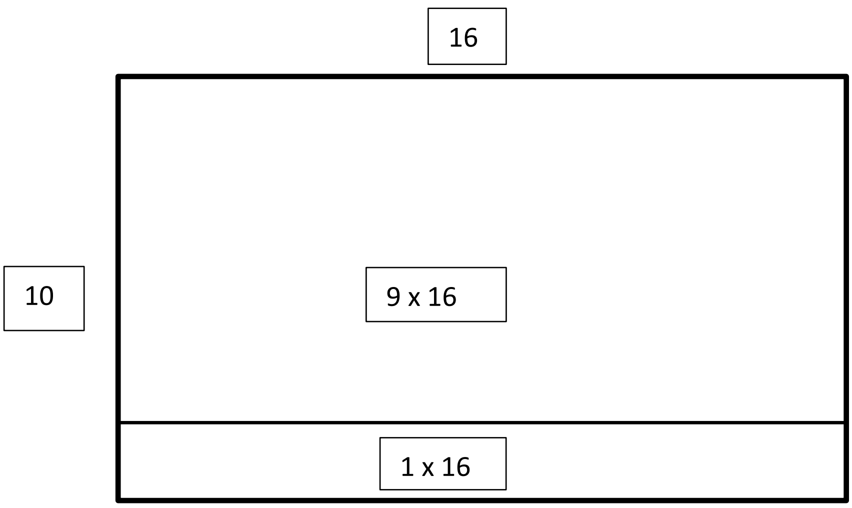


Fig. 2.12: Completing model in context of a grid model (Askew 2012, p. 121)

Therefore, 10 x 16 = 160 and 160 – (1 x 16) = 144. Hence, 9 x 16 = 144.

In this activity Askew (2012) insists that the details of such a representation indicate that the learner is treating it as a model for thinking through a solution. It shows that awkward numbers can be made easier to work with by rounding and then compensating for the difference later.

2.5.5 Strategies based on recalled/ derived facts related to multiplication

Cathcart et al (2000), outlined a couple of thinking strategies that may be used by learners when learning multiplication and division basic facts. There are prerequisite skills for being efficient at mental strategies for multiplication. Knowledge of times tables up to 10 x 10 and being able to recall them instantly is crucial. Underlying fluencies include skip counting, splitting the product into known parts, repeated addition and multiples of 2, 5 and 10.

Skip counting is achieved by counting by the second factor the number of times that is indicated by the first factor. For example, given 4 x 6 is found by skip counting in the following manner: "6, 12, 18, and 24". Skip counting is based on or it builds on meaning of multiplication, since counting by six four times corresponds with "four groups of six". Hence, learners can use skip counting by six to generate the entire six times table.

Repeated addition is achieved when, for example, 3 groups of 7 is expressed as 7 + 7 + 7 = 21 (i.e. seven plus seven plus seven). Large numbers may be written in column form and sometimes learners may begin to use a combination of addition and multiplication (Calliandro, 2000).

Commutative property: this property states that for all numbers **m** and **n** in the system of whole numbers **m** x **n** = **n** x **m** for multiplication. Coming to terms with the commutative property, it is believed, relieves learners of cognitive challenges that may lead to stress (George, 2000). Hence, it is argued that applying the commutative property winds down the multiplication facts to be learned from around 100 to around 55. French (2005) and George (2000) believe that a learner may only come to understand this after going through quite a number of activities that will enable the learner to construct an understanding of the commutative property.

Associative property: only two numbers can be joined at a given time because multiplication is a binary operation (Anghileri, 2007). Hence, more than two sets may be joined in sequence (**m** x **n**) x **p** = **m** x (**n** x **p**). That is, when multiplying more than two numbers, one can group them. Factors can sometimes be used together with the associative law to evaluate multiplication sums. The strategy of factors is particularly effective when dealing with multiples of 2, 5 and 10 because they are easier to handle. For example;

a) 26 x 15 = (13 x 2) x 15

= 13 x (2 x 15) (associative law)

= 13 x 30

= 390

b) 25 x 32 = 25 x (4 x 8)

= (25 x 4) x 8 (associative law)

= 100 x 8

= 800

The above examples could be done mentally or they could be done using paper and pencil, depending on the learner's number sense and confidence with multiplication tables.

Understanding the **distributive property** is mainly dependent on extension facts that are encountered by learners in the junior grades. As pointed out by Cathcart et al (2000), knowledge of some basic facts is key before learners can begin to tackle multiplication of multidigit numbers, for example, basic facts in place value position other than units' position, where the thinking may run as shown below:

Learners know that $\begin{array}{r} 7 \\ \times\ 4 \\ \hline 28 \end{array}$, so it is an easy extension to say $\begin{array}{r} 70 \\ \times\ 4 \\ \hline 280 \end{array}$ or $\begin{array}{r} 70 \\ \times\ 40 \\ \hline 2800 \end{array}$

It follows therefore, that for a sum like 9 x 47, it can be reorganized either mentally or on paper to 9 x (40 + 7) = (9 x 40) + (9 x 7). There are three important points to note here. It is relatively easy to determine the product of 9 and 7 from learners' knowledge of 9 and/ or 7 times tables, the extension fact 9 x 40 is also easy to determine mentally (9 x 4 x 10), and finally the expanded notation is also included with the use of the distribution property (47 = 40 + 7).

Anghileri (2007) suggests the use of web diagrams to emphasise the ways in which 'known' or 'recalled facts' related to multiplication can be used with strategic thinking to derive new facts (figure 2.13).

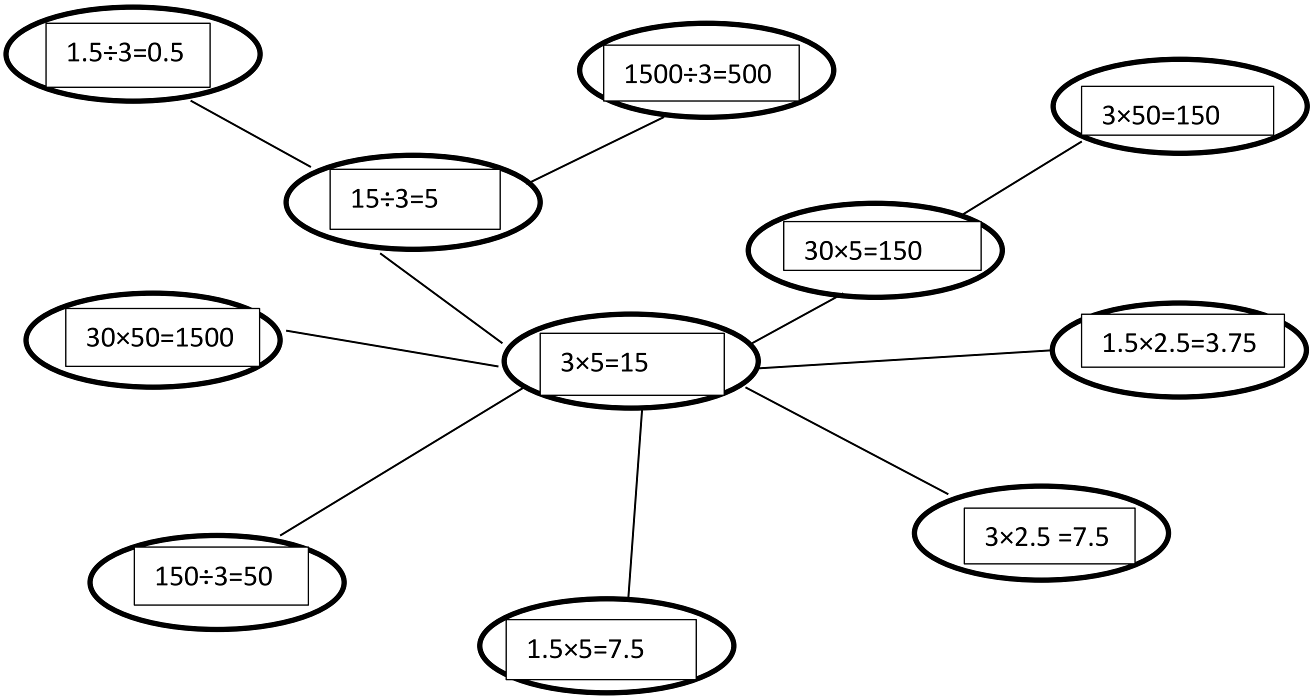


Fig. 2.13: Web diagram of multiplication and division connected facts (Adapted from Anghileri, 2007, p.73).

Doubling, halving and other multiple based strategies come into play through these connections. Repeated doubling is also cited by Anghileri (2007) as well as Caliandro (2000) as an effective method in informal multiplication. It enables one to multiply "… any number by 4 or 8, and even by 16 or 32." (Anghileri, 2007, p. 73). One might find this method quite restrictive in the sense that it can only be used when a number is being multiplied by specific friendly numbers (2, 4, 8, 16, 32, etc.). However, to work out, for example, 5 times 24, one can say 2 × 24 = 48, 4 × 24 = 96, and so 5 × 24 = 96 + 24 =120. Hence, doubling can be used in combination with adding. Doubling and halving strategies also present an easier way of multiplying when multiplying a given number by say 5 or 50. For example, students will find it easier to work out 86 x 5 by multiplying 86 by 10 and then dividing by 2 (or halving the result), rather than multiplying 80 by 5 and 6 by 5 and add the results. Anghileri (2007) points out that "Assessment questions are sometimes based on these friendly numbers and recognition of their factors can provide an effective strategy for calculating" (Anghileri, 2007, p. 75).

2.5.6 Doubling – as an approach to multiplication.

Anghileri (2007), as I have mentioned above, argues that doubling provides an effective strategy when learners have an understanding on the effects on multiplication and division. The advantage of doubling is that bigger products can be obtained through doubling either the multiplier or the multiplicand. This is also echoed by Haylock (2010) who illustrates this approach by using the following example (26 × 23):

26 × 1 = 26

26 × 2 = 52

26 × 4 = 104

26 × 8 = 208

26 × 16 = 416

Hence, 26 × 23 = 416 + 104 + 52 + 26 = 598

In this case repeated doubling is applied on 26 and then the results for the multipliers that add up to 23 are selected (16 × 26, 4 × 26, 2 × 26 and 1 × 26).

In a study conducted by Caliandro (2000), most learners used a combination of addition and multiplication. See example below:

Question: How much money would you have if you receive 75c a day for 15 days?

Learners bracketed every two 75s except the last three. Hence, 75c + 75c = 150c , 150c × 6 = 900c and 900c + 75c + 75c +75c = 1125c.

[75 75]	[75 75]	75	150	900
[75 75]	[75 75]	+ 75	× 6	75
[75 75]	[75 75]	150	900	75
[75 75]	75			+ 75
	75			1125
	75			

Fig. 2.14: Doubling by adding pairs (Caliandro, 2000, p. 421).

In this study, a common procedure for most learners was to add pairs and then pairs of pairs until all the numbers had been added. This procedure is illustrated by the example below.

Question: For a school fair, the students bought 14 cartons of soda. There were 24 bottles in each carton. How many bottles of soda do they have?

Factoring one of the numbers (24 x 14 = 24 x 7 x 2):

$$\begin{array}{r} 24 \\ \underline{\times\ \ 7} \\ \underline{168} \end{array} \qquad \begin{array}{r} 168 \\ \underline{\times\ \ \ 2} \\ \underline{336} \end{array}$$

In this example, the learner split 14 into two 7s because (according to his explanation in the class discussion) he could not work out the product of 24 x 14. Hence, after getting the product of 24 x 7, he then doubled it to get 336. In the same study, one learner was able to use a combination of doubling and halving.

Question: Find the price of postage for forty two letters at 32c each

42 x 32

↓

84 x 16

↓

168 x 8 → $\begin{array}{r} 168 \\ \underline{\times\ \ 8} \\ \underline{1344} \end{array}$

A combination of doubling and halving (Caliandro, 2000).

In the above informal ways of doubling, I have tried to provide more detail in order to clearly illustrate the different approaches. What appears to be significant in each of the examples that have been given is that by breaking down one of the numbers in the multiplication into smaller and/or friendlier numbers, learners are able to build on their ever growing confidence with number, to develop their own strategies and share different approaches to a given multiplication problem. Hence, learners' flexibility with numbers cannot be overemphasized, especially in the early years of learners' schooling.

The literature overall therefore, points to the importance of allowing children to informally devise their own models of different problem situations, and to explore strategies supported by these informal models. The prevalence of column models in my data suggests that Grade 6 learners at my school are not making sense of given multiplication problem situations and translating these into informal models. Strategies are then based on erroneously remembered algorithms that do not appear to make sense. Caliandro (2000) argues that: "Children can invent their own methods for solving multidigit multiplication and division problems without learning the conventional algorithms that we normally teach them" (Caliandro, 2000, p. 420). It is believed that algorithms lead to correct answers, yet they stifle students' thinking processes. This implies that attempts to instil conventional algorithms for students to adopt are unlikely to achieve desired results because it does not allow them to apply their own thinking. Anghileri (2007) also highlights this fact by insisting that students who rely on their own methods as opposed to those that are introduced by the teacher are likely to perform better than those who rely on standard procedures. The significance of knowledge of multiplication tables cannot be overemphasized within this.

French (2005) argues that knowing multiplication tables by heart is critical for learners' success in multiplication. This is also echoed by Askew and William (1995) as cited in French (2005). They insist that children should gradually get to know number facts and multiplication tables by heart. French (2005) argues that multiplication strategies are developed from multiplication table facts.

It is also important for learners to know and understand that any number can be split up in a variety of ways (distributive law), as shown below in 3 different ways of calculating 4 x 58:

$4 \times 58 = 4 \times 50 + 4 \times 8 = 200 + 32 = 232$

$4 \times 58 = 4 \times 60 - 4 \times 2 = 240 - 8 = 232$

$4 \times 58 = 4 \times 55 + 4 \times 3 = 220 + 12 = 232.$

I believe that the above argument holds and can lead to deeper understanding as learners develop facility with larger numbers through discovery as they are solving multidigit multiplication problems. According to Caliandro (2000)'s study on multiplication, learners often multiplied multidigit numbers by writing a column of equal addends. However, later in the year, most of them were able to deal with difficult and complex problems as they developed facility with large numbers.

Hence, this was an exciting discovery for the learners because it was deemed to be learners' own invention. This approach made learners discover that the initial methods that they were using (writing long columns of equal addends), were undesirable to them. Hence, learners looked for easier and faster ways of multiplying (solving problems) multidigit numbers. This process could happen because they were given the opportunity to think and explore, constructing a deeper understanding of multiplication. This was mainly on relationships between multiplication and addition and between doubling and halving the numbers that are being multiplied.

According to Caliandro (2000), learners began to check for reasonableness of the solutions that they got. This observation reflects an important step in developing facility and efficiency with multiplication of large numbers because if a learner is able to check for reasonableness of their solutions, it means that they are more likely to self-correct if their solution is incorrect.

From the above argument we begin to realise that mathematical knowledge cannot be transferred from the teacher to students in the classroom. It is "…something that learners themselves construct by seeking out meanings and making mental connections in an active manner." (Anghileri, 1995, p.3). Students therefore need to actively participate in constructing mathematical knowledge by seeking translation and fit with situations in their construction of models.

CHAPTER 3: THEORETICAL FRAMEWORK

3.1 Introduction

In this section I outline my reasons for selecting the theory of Realistic Mathematics Education (RME) as the theoretical framework in a study that focuses on exploring links between multiplication problem types, learners' setting up of models and use of strategies in multiplication. Askew (2012) points out that there is a general recognition that models and images play a significant role in facilitating the learning of mathematics by the learners.

The Dutch Realistic Mathematics Education theory, with its roots in Hans Freudenthal's (1973; 1991) interpretation of the subject of mathematics, provides a platform for thinking about how learners can be helped in exploring the construction of models, their use as "models of" and the progression to being "models for" in the process of solving multiplication problems. It is my aim, therefore to justify the reasons for selecting RME as a theoretical framework that was deemed suitable for facilitating the intervention lessons as well as analysing data in my study.

3.2 Models and modelling situations in mathematics

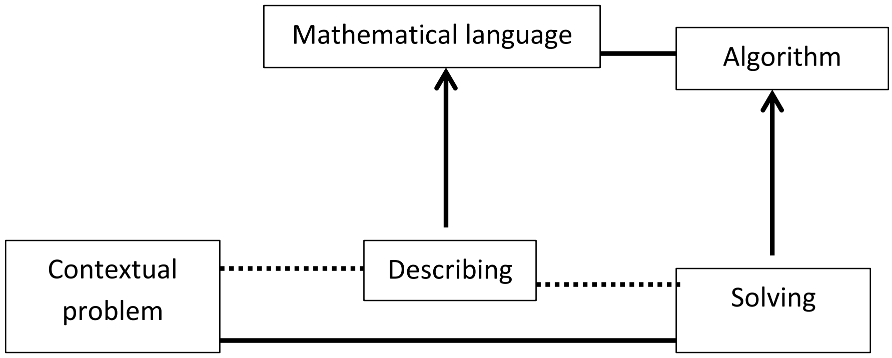
Models are designed to meet specific purposes in given situations. In mathematics, models may be seen as bridges that help learners in transition to understand new concepts (Gravemeijer and Stephan, 2002). Models are therefore embedded in task settings and are derived from learners' informal activities. Gravemeijer (1999) argues that models serve as a means for instructional design in mathematics. A model should help learners in mathematical construction, starting from learners' perspective (bottom-up), as opposed to the use of traditional models that transmit ready-made mathematics (top-down). Modelling is primarily seen as organizing activity in which situations are structured in terms of mathematical relationships. They argue that the design of models begins through the informal activities of the learners as they look for models that can support their solution strategies.. It is this setting up of models and the resultant strategies in multiplication that can serve as tools in the classroom that I intend to investigate.

As noted already, Fennema, Franke, Levi & Empson (1999) note moves away from the intuitive modelling skills demonstrated at a young age: "If older children simply applied some of the intuitive, analytic modeming skills exhibited by young children to analyse problem situations, it appears that they will avoid some of their most glaring errors." (Carpenter et al, 1999, p. 55). What is crucial therefore is to help learners at the elementary level to retain and develop their ability to use modelling when solving mathematical problems.

3.3 Realistic Mathematics Education

RME has its roots in the Freudenthal Institute. The Realistic Mathematics Education approach views mathematics as a human activity (Gravemeijer, 1994). The theoretical underpinnings of Realistic Mathematics Education focus mainly on mathematizing or organizing subject matter that is taken from reality or realistic contexts as opposed to the traditional approach of presenting mathematics as a ready-made system (Gravemeijer, 1994). Through the process of mathematization, learners are expected to learn by way of reinventing mathematical insights, relationships and procedures. As they engage with mathematically authentic contexts, they undergo what RME refers to as a process of horizontal and vertical mathematization. According to Gravemeijer (1994), horizontal mathematization is when learners make use of their informal representation to describe and sometimes solve a given contextual problem. Vertical mathematization is when the learner's informal presentation leads them to solve a given problem using what he/she deems a suitable algorithm. Hence, the Realistic Mathematics Education approach is characterised by problem situations that are experientially real and learners are also expected to be supported and guided in developing their own informal mathematical models and strategies into more formal ones. It involves working with the problem at different levels as shown in the diagram in the next page. Hence, in this study, the way a learner represents a given problem situation will be regarded as a model (horizontal mathematization) and the way a learner carries out the process of calculation, sometimes using an algorithm, will be regarded as a strategy (vertical mathematization) (Gravemeijer, 1994).

In this study, it is assumed therefore, that the RME framework, through the use of contextual problems, will give learners an opportunity to set up models (horizontal mathematization) that can be used to organize and solve problems located in real life situations into the world of symbols, that is, going from the world of life into the world of symbols (Van den Heuvel-Panhuizen, 2003). Figure 3.1 summarizes this framework:



(Gravemeijer, 1994.)

Figure 3.1: Horizontal mathematization (………)

Vertical mathematization (↑)

It will also facilitate the use of strategies (vertical mathematization) through the application of horizontal and vertical mathematization (i.e. moving from a model to selecting a strategy in order to solve a given contextual problem). This involves discovering connections between concepts and strategies and finding possible short-cuts (reorganization within the problem) within the world of symbols. It is also anticipated that different models and strategies will emerge in the process as learners engage with contextual problems.

3.4 Instructional design principles of the Realistic Mathematics Education

According to Gravemeijer & Stephan (2002) and Gravemeijer (1994; 1999), there are three key instructional design principles of the Realistic Mathematics Education (RME). These are guided re-invention, didactical phenomenology and emergent models.

3.4.1 Guided re-invention

According to Freudenthal's (1977) view on mathematics, the subject must not be seen as a subject matter that has to be transmitted, but it must be a human activity whereby learners are given the "guided" opportunity to "re-invent" mathematics by doing it (Gravemeijer, 2004; Van den Heuvel-Panhuizen, 2001). Hence, mathematics education is expected to focus on the actual activity, which according to Freudenthal is the process of "mathematization." It is, therefore, essential that the subject matter should be experientially real for the learner. Van den Heuvel-Panhuizen notes also that situations drawn from the formal world of mathematics can also provide a suitable context as a problem. What is essential is that the problem should be real in the learner's mind.

3.4.2 The Principle of Didactical Phenomenology

According to Freudenthal (1978; 1983), as cited in Van den Heuvel-Panhuizen (2003), problem contexts, in conjunction with mathematization, need to lead learners to discover mathematical structures and concepts. Hence, mathematization by learners has to start from phenomena that are meaningful to learners. "Formal mathematics is not something 'out there' with which the student has to connect. Instead, formal mathematics is seen as something that grows out of the student's activity" (Gravemeijer, 1999, p.160). This can be interpreted as meaning that real world or imagined situations are inseparable from formal mathematics. According to the Realistic Mathematics Education theory, concept formation may be rooted in learners' mathematization of their informal mathematical activities that develop into formal mathematics.

Model formation plays a key role in Realistic Mathematics Education because modelling is viewed as way of organizing within which symbols emerge as well as the model itself. Models facilitate and elicit progress in the process of mathematization (Treffers and Goffree 1985; Gravemeijer 1994; Van den Heuvel-Panhuizen 2002). Models, therefore, can act as a link between the informal understanding within the real world and that of formal mathematics. Models can also give access to formal mathematical concepts because they can function as windows that learners look through. According to Gravemeijer (1999) and Streefland (1985), as cited in Van den Heuvel-Panhuizen (2003), models also have the power to raise the level of understanding between informal and formal level. At the initial stage of learning, the model is organized in a manner that is close to the context or situation at hand. However, the context specific model gradually becomes applicable to related problem situations (generalization). In such situations, the level of understanding would have increased. Hence, this could be associated with mathematical growth in learners. This is explained more clearly under emergent models.

3.4.3 Emergent models

According to Gravemeijer & Stephan (2002), the concept of "emergent models" characterises the way in which models emerge within the RME instructional design. These are models developed by leaners in the process of solving problems (model of the learner's situated informal strategies). Secondly, it characterises the way by which the models gradually evolve over time to become models for more formal mathematical reasoning. Hence, learners are expected, through the process of mathematization, to re-invent formal ways of reasoning as a result of their informal activities.

It is further argued that the sequence passes through four levels of activity during mathematization process in which there is repeated movement (upwards and downwards) between the levels as learners attempt to refine their models and strategies. The first level is that of an activity in a task setting, the second involves the referential activity whereby the learner acts on the task setting through acting with models. Thirdly, the learner gradually gains familiarity with the mathematical relations and finally the learner begins to generalize where the learner no longer requires the support of a model for more formal mathematical reasoning (Gravemeijer & Stephan 2002).

3.5 Traditional versus RME approach

The concept of mathematics as a human activity is embedded in the practice of education and teaching as opposed to the transmission of mathematical concepts as a pre-formed system (Gravemeijer & Terwel, 2001). They argue that in the traditional approach, the result of mathematical activities of others is used as a point of departure for instruction. This is what Freudenthal (1973b) as cited in Gravemeijer & Terwel (2001) described as "anti-didactical inversion." This contradicts the Realistic Mathematics Education approach by teaching the result of an activity instead of beginning with the activity itself. According to Treffers (1987), as cited in Van den Heuvel-Panhuizen (2001), five principles formulate the RME curricula that guide the process of progressive mathematization.

- The dominating place of context problems – of which solving everyday life problems (contextual problems) is an essential part of learning mathematics.
- The broad attention paid to the development of models – production of models (images or representations) aids learners' thinking which can lead to solution strategies.
- The contributions of learners by means of own productions and constructions – learners' own production of models through the process of mathematization.
- The interactive character of the learning process- learning as a social activity

- The intertwinement of learning strands – discovering connections between concepts and strategies.

These principles were further developed into a framework for an instruction theory that reflects both the learning aspect and the instruction aspect model. In this study, the intervention lessons were guided by the principles of Realistic Mathematics Education (see the table 3.2).

General RME theory	
What	**How**
• Meaningful human activity • Horizontal and vertical mathematization • Low level skills and high level skills	**Teaching**: - Context or reality principle. Intertwinement principle Guidance principle **Learning:-** Activity principle Level principle Interaction principle

Table 3.2: General RME instruction theory (Van den Heuvel – Panhuizen, 2001, p.35)

The focal points of the Realistic Mathematics Education approach were emphasized during intervention lessons. The multiplication problems used were connected to the real world, or were expected to offer learners situations that they could imagine (making a given problem real in their minds). According to Van den Heuvel–Panhuizen (2001), it is anticipated that context problems function as a source of learning. Learners can develop tools and strategies closely related to the context.

3.6 Whole class discussion and group work

The role of class discussion and group discussion as applied to interpretation and solution methods has been highlighted by Gravemeijer & Terwel (2001). They posit that discussions will enhance the shifts in solution methods and ways of description (for both models and strategies). These shifts could be due to reflections upon models and strategies and underlying understandings with regard to the given question. In the context of RME it is anticipated that each problem will give rise to different solution methods. Hence, during whole class discussion and group discussion, where solutions are compared, learners are supposed to recognise advantages and disadvantages of such solutions with a view of adopting the approach that seems more formal. Realistic Mathematics Education advocates that both weaker and stronger learners would benefit from these whole class and group discussions.

CHAPTER 4: METHODOLOGY

4.1 Introduction

The quest for improving educational practices has made educational research a valuable source of information. There is renewed interest at all levels of education for decisions to be dependent upon generated evidence (McMillan & Schumacher, 2010). The knowledge that is generated from such research has a direct impact on educational practice in the context of practising educators. According to Opie (2003, p.3), educational research is “the collection and analysis of information on the world of education so as to understand and explain it better.” McMillan & Schumacher (2010, p. 8), define research as “the systematic process of collecting and logically analysing data (i.e. evidence-based) for some purpose.” From these definitions, the main focus is to grow one’s knowledge, and hopefully that of colleagues in education through the production of insights in the area of study. It is a systematic way of inquiry that aims at solving problems or answering questions and expanding knowledge (Bell, 1993). In order to understand the kind of models and strategies being used by learners across a range of multiplication problem types, I chose to use a qualitative research approach.

4.2 Research design

The research design outlines the ways and means for carrying out research. According to McMillan & Schumacher (2010), the design “specifies a plan for generating evidence that will be used to answer research questions” (p.20). Hence, it maps out the entire study in line with the purpose as well as the type of research to be undertaken.

Henning (2004) states that the decision to work with qualitative data is basically linked to the type of inquiry that a researcher intends to conduct. In the process of gathering data during my intervention lessons, there were no externally imposed constraints or control of classroom settings. I acted as both teacher and observer during lessons, listening to interpretations and descriptions of constructed models and strategies. I spent a considerable amount of time looking at learners’ work in their workbooks. This kind of interaction with both participants and their workbooks afforded me the opportunity to stay close to the data. In this section I discuss the procedures for conducting the study, how the data was collected and analysed in order to generate the findings that I used to answer my research questions. I unveil my primary data sources for the study, the sample, ethical issues as well as aspects of validity and reliability (the truthfulness of findings and conclusions)

There are certain limitations to the method of inquiry that I selected for my study. In light of such limitations, an attempt to make meaning through deeper understanding and interpretation of multiplication models and strategies presented by Grade 6 learners were largely dependent upon my ability to see the bigger picture of the study, as well as the ability to convert the raw empirical data into "thick description" (Geertz, 1973; 1993 as cited in Henning et al, 2004). Bearing in mind that I (the researcher) am also human, I am aware that observation is fallible and in many instances it could be coupled with errors as well. Fisher (1999), as cited in Opie (2004), mourns the problem of the large volume of data to be analysed, as well as its complexity with regard to analytical vigour. This is because the exercise of analysing in this qualitative mode is easily open to subjectivity of the researcher. This above view brings me to another disadvantage closely linked to it. Henning et al (2004) argue that qualitative research may be superficial, especially when information is just presented according to themes or categories as if the data does convey meaning, with "thin interpretation" from the researcher.

The above are just some of the challenges that I faced in my endeavour to achieve "thick description", which calls for coherence and more depth in terms of evidence, facts and content of inquiry. By presenting ample evidence from multiple sources complemented by authoritative support in the form of a strong theory and literature base, I strove for depth and quality in my descriptions and interpretations.

4.2.1 The Sample

Selecting research participants (sampling) has a bearing on the study. The sample is the group of individuals or participants from whom the data are collected (McMillan & Schumacher 2010). Opie (2004) argues that how a sample is defined is a prerogative of the one who is carrying out the research. Hence, it should be defined in keeping with the goals of the nonprobability sampling approach that is commonly used in educational research, the convenience sample. This is a group or subjects who happen to be accessible for a given study (Creswell 2012; McMillan & Schumacher 2010). The grade 6 class of 42 learners at the school where I am teaching was a convenience group because I teach them mathematics every day of the week. However, two of these learners did not take part in the study because they decided that they will not participate in the research.

There are advantages to the use of a convenience sample. First and foremost is its accessibility, since this is a class that I teach daily. Secondly, my primary purpose of the research was to better understand the ways in which learners set up models and their use of strategies when dealing with multiplication problems within the Realistic Mathematics Education framework approach, which was facilitated by this group's prior familiarity with me as their teacher. However, there are also disadvantages with the use of a convenience sample. In particular, it is likely that the results will be limited to the type of participants in the sample, rather than being generalizable.

4.2.2 Setting

The sample that I used in my study was made up of forty sixth-grade learners (boys and girls) with a weak mathematical background. These are learners I taught five times a week for only one subject the whole year (2013). I only saw them for their mathematics periods every day. All these learners would have been introduced to the basic algorithms for addition, subtraction and multiplication of 2-digit by 2-digt numbers and multiplication of 3-digit by 2-digit numbers in the fourth- and fifth-grades (see table 1 and 2 below). According to the curriculum statement, learners should be able to multiply 3-digit by 2-digit numbers at grade level 5. They are also expected to use approaches in which they break up numbers as well as the traditional column method. Hence, they are therefore expected to later on progress to multiplication of 4-digit by 3-digit numbers using a variety of methods. They are also expected to do context free calculations on multiplication as well as be able to solve problems in contexts (DoE; 2011).

TERM I		**TERM 2**		**TERM 3**		**TERM 4**	
Topic	**Time**	**Topic**	**Time**	**Topic**	**Time**	**Topic**	**Time**
Whole numbers: Multiplication (2-digit by 2-digit)	6 hours	Whole numbers: Multiplication (3-digit by 2-digit)	7 hours	Whole numbers: Multiplication (3-digit by 2-digit)	7 hours		
Mental mathematics (10 minutes daily)	8 hours	Mental mathematics (10 minutes daily)	7 hours	Mental mathematics (10 minutes daily)	8 hours	Mental mathematics (10 minutes daily)	7 hours

Table: 4.1 Curriculum Assessment Policy Statements (CAPS) for **Grade 5**, (p. 122)

TERM I		TERM 2		TERM 3		TERM 4	
Topic	**Time**	**Topic**	**Time**	**Topic**	**Time**	**Topic**	**Time**
		Whole numbers: Multiplication (4-digit by 2-digit)	7 hours			Whole numbers: Multiplication (3-digit by 2-digit)	5 hours
Mental mathematics (10 minutes daily)	8 hours	Mental mathematics (10 minutes daily)	7 hours	Mental mathematics (10 minutes daily)	8 hours	Mental mathematics (10 minutes daily)	7 hours

Table: 4.2 Curriculum Assessment Policy Statements (CAPS) for **Grade 6**, (p. 212)

In this group of learners, a good number were still struggling to carry out 2-digit by 1-digit multiplication. This was underlain by lack of fluency in their times tables and they also demonstrated a lack of confidence in number operations including competence with multiplication. Orton & Wain (1994) argue that such learners are not incapable, but it is due to the fact that they are not ready to deal with such problems. It is, therefore, fitting for them to explore simpler products. Hence, in all my intervention lesson tasks I incorporated Grade level 5 tasks and a few questions from Grade level 6 in order to try and cater for all the learners.

Superfluous information

In order to help the learners become better problem solvers in multiplication, I included superfluous numerical information, incorporating the need for selection of appropriate quantities. According to Silver & Thomson (1984), for learners to become good problem solvers, they need to identify critical elements and also be able to select appropriate quantities from irrelevant information. Askew (2012) incorporates superfluous information into the multiplication problem exemplar problems.

4.2.3 PRE- / POST-TEST

The task (pre-test and post-test), was made up of six word problems on multiplication, adapted from Askew (2012). Focus questions were questions 1, 2, 4, 6, 7and 9. The rest of the questions (3, 5, 8 and 10) were buffer questions, used in order to explore whether learners were able to select multiplication as the appropriate operation in certain situations.

PRE - / POST-TEST (GRADE 6).

INSTRUCTIONS:

(i) **Show all your working out in the spaces provided. You can work out the problems in any way you want – diagram, picture, calculation sum or give an explanation.**

(ii) **In some questions you might not need to use all the numbers. You will have to read the question carefully and decide which numbers you need to use.**

1. Coco ordered 8 boxes of neck ties. Each tie had 25 spots on it. Each box contained 16 ties. Coco unpacked the ties and tried them all on. How many ties did Coco try on?

2. Every time Marge left a kiwi fruit in her cupboard for a month, the kiwi fruit turned into 8 worms. One month, Marge left 14 kiwi fruit in her cupboard. How many worms did she find?

3. 149 sea snails crawled into a cave. Inside, they found another 379 sea snails. How many sea snails were inside the cave?

4. Jolene's sunflower was 28 cm tall when she first measured it. 8 days later, it was 6 times as tall. How tall was the sunflower then?

5. 2030 × 24

6. The Rea Vaya bus has 27 rows of seats for people to ride in. Each row has 6 seats. You have to be over 120 cm tall to ride the Rea Vaya bus. How many people can ride on the Rea Vaya bus when it is full?

7. Charlie wanted some tea bags. Tea bags come in packets of 36. Charlie bought 9 packets of tea bags and 0.5 l of milk. How many tea bags did Charlie buy?

8. Fiori, the god of nature, holds 12 flowers in each of his hands. Altogether, Fiori holds 264 flowers. How many hands does Fiori have?

9. A black mamba weighs 895 g. A python is 28 times as heavy as a black mamba. How heavy is python

10. 934 x 14

4.2.4 The intervention

The study took the form of qualitative research. This is so because of the type of inquiry that I intended to engage in, of identifying models and strategies as well as patterns and relationships (Henning, Van Rensburg & Smit, 2004). In this research, there is no control of behaviour or of situations as it were. It thrives on rich descriptions that capture what the researcher has observed. In this type of research, gathering of data and its analysis precedes conclusions and generation of generalizations (McMillan & Schumacher, 2010). Whilst my study was biased towards a qualitative approach, this did not rule out some quantitative analyses in instances when figures were used to justify certain observations relating to, for example, the prevalence of some models and strategies over others.

As noted already, my research approach was largely based on an intervention drawn from Askew's (2012) 'Big Book' of word problems related to working with the four operations – detailed further below. Askew advocates for teachers to support learners to make sense of the problem through asking clarifying questions, letting them draw pictures or set up models. He argues that this kind of approach helps them understand other learners' approaches and advances their own approaches. Learners were expected to bring meaning to the way the problem is worded from the perspective that this was going to help them build up an understanding of classes of problems. According to Askew (2012), there are three stages to intervention lessons:

Phase: 1. (3 problems in the Big Book)

Learners work in pairs to solve three problems (one at a time). All these problems are focused on one operation within one of the root situations. In between each problem there is a follow up discussion in order to capture the types of models and strategies used for solving each problem. Askew suggests that the teacher should focus on supporting learners' reasoning through asking "clarifying questions, setting up models pictures or diagrams." (p. 7).

Phase: 2. Linking Problems.

In this stage all three problems are compared to find out if there are any relationships or differences. Learners are expected to discover the underlying structures of the problems so that they can classify problem types according to root situations through the guidance of the teacher.

Phase: 3. Follow up problems.

Each learner is expected to do a follow up task with more or less the same structure as the introductory problems that would have been discussed earlier. The aim is to consolidate the concepts that were introduced in the lesson. Most problems include superfluous numerical information in order to emphasise that learners need to make sense of the situation in their modelling activity, rather than 'grab' at the numbers seen. The teaching that Askew (2012) advocates can be interpreted as aligned with the Realistic Mathematics Education approach in that he suggests following learners' emergent models.

In my intervention, aligned with this advice and with Realistic Mathematics Education theory, I did not try to impose models or strategies but I supported and helped learners to reflect and compare in order for them to choose their own approaches that they find efficient. Different work samples for class discussion were selected and these were placed on individual chart paper (A3) or presented by a learner on the chalkboard. This approach enabled us to discuss each strategy on its own, and then place different strategies side by side for comparing and contrasting. As the researcher I did not impose an approach for learners but I gave them an opportunity to evaluate emerging approaches with my support.

The schedule for the six lesson intervention involving a gradual introduction and then mixing across problem types is shown in the table on page 6 of this report. The word problems adapted from Askew's (2012) Big Book included multiplication as repeated addition, scaling and rate situations – as discussed in the literature review (see Appendix). I also included buffer questions that randomly separated the research questions.

At the end of the intervention program, the same test (as the pre-test) was administered to the learners. However, the buffer items were changed as well as the order of the question so as to avoid direct repetition and increase validity of findings. By looking across the pre-test responses, learner work on class and homework tasks during the intervention, and post-test responses, I aimed to understand the shifts, if any, in learners' use of models and strategies across different problem types, as well as noting any changes in 'success' overall with multiplication problems.

4.3 Summary of data sources

The primary data sources for the study were the pre-test responses, learner work from intervention lessons and homework tasks, and post-test responses. Learners were issued with new exercise books and worksheets which were specifically used for the purposes of gathering data. It was necessary to also keep track of the success-rate across the whole class in order to understand the more/less effective approaches as well. Data was analysed based on the situations, models and strategies discussed in literature. Questioning during lessons, after marking activities or during revision was used for purposes of collecting more in-depth qualitative data. These were used to probe learners' modelling processes as well as their use of models and strategies in multiplication. They were intended to access further learners' thinking with respect to their multiplication approaches and strategies. Interactional process records were kept in a post-lesson reflective journal.

4.4 Reliability and validity

Trustworthiness and confidence in qualitative research depends upon the quality of communication and interpretation of evidence that is gathered to support arguments that are also backed by related literature. Hence, the reader will have a heightened sense of trust in a research report if the concepts of validity and reliability have been carefully incorporated during the entire course of the study.

Validity is normally judged on the basis of available evidence and the precise interpretation of it. That is, good craftsmanship during the entire research process. Henning (2004) insists that in order to achieve validity one must check for bias, scrutinize both for procedures and decisions in a critical manner and make use of theory from other studies and research that is credible. On the other hand reliability is based on the extent to which the measurement is consistent on different forms of settings. Hence it is more or less about the extent to which a strategy or technique is prone to producing similar results. Reliability, therefore, is essential for purposes of enhancing validity.

4.5 Validity

In this study, analysis of learner documents, observation of participants during intervention lessons were expected to enhance validity, together with field notes and recordings that were documented during the course of the research study. Participants' explanations and interpretations of meanings were captured during intervention lessons. Self-scrutiny by the researcher also took human subjectivity into account. Atkinson (1983), as cited in Henning et al (2004) posits that data itself is not valid or invalid, but it is the inference that is drawn from it that determines its validity. It is critical that my reporting captures as much detail as I can from the participants' perspectives. Continued consultation of my supervisor as well as my peers with regard to emerging findings as well as clarification of any assumption with participants assisted me to keep on track.

Validity in qualitative research seeks to address mainly the "degree of congruence between the explanation of the phenomena and the realities of the world" (McMillan & Schumacher 2010, P. 330). According to Opie (2004), the quality of communication between the researcher and the reader rests upon the reader's sense of trust and confidence in what is presented in the report. Hence, validity is dependent on the entire process of carrying out the research, that is, precision in the way in which data is collected and the way in which it is analysed. Wellington (2000), put forward three ingredients with regard to validity; instrument (method or tool), what the researcher intends to measure (intention) and the findings (results of the instrument). This implies, therefore, that all these aspects need to be put into perspective when making judgements on validity. Hence, it is therefore rational to say that validity represents the relationship that exists between the claim and the findings of a data – gathering process (Opie, 2004). Cohen & Manion, (1994) posit that in educational research validity has two distinctions, internal and external validity. Internal validity is concerned with the degrees to which the account is reasonable and self-consistent, whilst external validity pertains to the extent to which findings can be applied to other situations (Opie, 2004). It is further argued that the latter can arise by degrees depending on the claim.

In my study I used different techniques, like learner observations during intervention lessons, listening to learner feedback in class discussions and an extensive use of learner documents (exercise books and worksheets). Hence, I resorted to data triangulation across these techniques. According to McMillan & Schumacher (2010, 331) triangulation refers to "obtaining convergent data using cross validation." Such an approach may reveal different insights and therefore lead to an increase in credibility of findings. In addition to this approach, validity may be enhanced by the fact that, in my research I did not only use my own data in isolation but I made an effort to link it with the findings of seasoned researchers, coupled with the back-up of a very experienced supervisor.

4.6 Reliability

Reliability is an indication of quality work in qualitative research because it enhances the credibility of the study. There are two prominent features of reliability that stand out; these are consistency in terms of results across a range of similar settings and repetition (Wellington, 2000; Bell, 1999). According to Opie (2004), in qualitative research reliability should encompass the entire process of data gathering inclusive of the findings. However, in research studies that take place in the classroom, it may not be easy to replicate results. Hence, it becomes difficult to apply the concept of reliability in order to judge a classroom based research study. As much as it may be difficult, Opie (2004) argues that the process of data gathering itself could be subjected to reliability judgement. Over and above this view, there are three ways of achieving reliability; "test-retest", using an instrument on the same subjects and then comparing the results. Secondly, the "equivalent forms" procedure makes use of two equivalent versions of a data-gathering instrument that is applied on the same subjects and “split-half” makes use of a single instrument in which the results are then split into two halves which are then compared.

In light of the above techniques, I cannot claim that the study was reliable since it was classroom-based and I did not get an opportunity to replicate the results. In addition to the concepts of validity and reliability, self-examination is also one critical aspect that can add credibility to my research. According to McMillan & Schumer (2010): “The researcher’s very act of posing difficult questions to him or herself assumes that he or she cannot be neutral, objective or detached.” Hence, the element of subjectivity was taken into account during the entire process of carrying out the research. In order to enhance reflexivity, I also kept a notebook where I wrote continuous records of events, consultation meetings or discussions with both supervisor and peers. However, potential limitations with regard to attrition did exist. Attrition occurs in a study when participants drop out as a result of some various reasons like lack of interest, absenteeism due to illness or learner transfer. Since the study took some weeks to complete, I anticipated that there were some individuals, especially (low-achievers) those who did not show enthusiasm for the subject, who might decide to pull out of the study.

There were 40 learners who sat for the pre-test. However, as the intervention lessons progressed, numbers fluctuated between 34 and 40 due to absenteeism. Finally, 34 learners sat for the post-test. Matched comparisons of pre- and post-test results analysis were therefore based on these 34 learners.

4.7 Ethical issues

Well before I began my research, I made applications to the university ethical clearance committee (Human Research Ethical Committee), the Department of Education, the School Governing Body and school administration. As soon as I obtained ethical clearance, I fully informed the parents and the learners (participants), both in writing and verbally, about the research study. They needed to be aware that their privacy was protected and also to know how the research was going to be carried out (Henning et al 2004). Both parents and learners gave their consent by first reading through written letters and information sheets before they could sign to acknowledge their consent.

The purpose of the study was clearly outlined to all the parents and the participants involved in the study. They were assured that their privacy and sensitivity will be highly protected

(Henning et al, 2004). Anonymity was guaranteed through the use of codes in place of their names. It was also my responsibility to make it known to the participants that it is their right to decide whether to participate or not in this research. They were also informed of the fact that they could also withdraw from the study at any time if they felt that they were not interested anymore during the course of the study. I clearly explained to participants that their documents (exercise books) will be kept in a safe place and they will remain confidential until they are destroyed within a period of five years. They were also informed as to how the research study was going to benefit them. For purposes of reporting, pseudonyms were used to maintain the participants' anonymity. Finally, the Department of Education as well as the school gave consent before I embarked on the study.

4.8 Data analysis

Analysis of qualitative data normally takes the form of organizing information into categories whereby relationships and patterns are analysed. A process whereby one starts with specific data, employees a systematic process of coding, synthesizes and makes meaning through interpretation of patterns and relationships is termed inductive analysis (McMillan & Schumacher, 2010).

A qualitative approach based on the typologies of models and strategies that have been identified from the literature was used to analyse learner responses on multiplication lessons data as well as pre-test and post-test. Since this is an exploratory study framed by Realistic Mathematics Education, I was cognisant of the fact that the literature that has been dealt with earlier on in this report only provided initial insights into models and strategies that could have been used by learners. However, Realistic Mathematics Education is based on the idea of emergent models (bottom-up) rather than pre-given (top-down) ones. Hence, I was open to alternative models and strategies that may have emerged within the data, given the RME emphasis on modeling activity, indicating that a combination of more inductive and more typological approaches were used.

In my analysis, learners' work was initially marked and then categorized according to models produced and corresponding strategies within the models. Shifts within both models and strategies were observed and captured and any new models and strategies produced were also noted. Data analysis was always an on-going process even during the intervention lessons. In the process of supporting learners, giving guidance through asking probing questions, I also afforded some learners an opportunity to present their work on the chalkboard so that it could be reflected upon by other learners during a whole class discussion.

CHAPTER 5: FINDINGS

5.1 Introduction

As I mentioned at the beginning of this report, the initial motivation for my study was to explore models and strategies being used by learners across a range of multiplication problem types. In this chapter, I present findings of my study in relation to models and strategies based on the results of the pre-test, the six intervention lessons, and the post-test results that marked the end of the intervention exercise. In the pre-test, I intended to establish the kind of models and strategies for solving multiplication problems that the Grade six learners were using prior to the small scale intervention. I examined in detail both models and strategies with an interest in connections that could be made between them. During intervention lessons I focused on monitoring the kind of developments and shifts with regard to models and strategies used by the learners. I drew on the literature to try and establish whether the models produced did support learners' reasoning in the context of multiplication.

Data was gathered from 40 Grade six learners' lesson activities (inclusive of classwork and homework) that were recorded in exercise books as well as from my personal field notes that I jotted down summarizing my interactions with the learners and their written activities. Statistical graphs in the form of pie-charts, tables and bar graphs were used to analyse the results of the study. Both models and strategies were coded and thereafter organised into categories so as to examine patterns and shifts with regard to models and strategies produced during the course of intervention lessons. Within the analysis related to each sub-set of data, and given RME's emphasis on modelling, I categorized learners' written responses to tasks initially according to the model used and whether the answer was correct or incorrect.

The summary of learner performance across the pre-test and post-test is presented at the start of this chapter using a table. This is followed by summary tables showing detailed analyses of models and strategies used across both the pre-test and post-test. For intervention lessons 1 up to 6, composite summary tables are presented. Excerpts drawn from fieldnotes from the six intervention lessons have been included to draw attention to key shifts at the level of models and strategies as well as horizontal mathematization in relation to selection of appropriate numbers given that this was a difficulty that was pronounced in some of the lessons.

5.2 Learner performance across pre-test and post-test multiplication situation problems

Question	Problem type	Pre-test (correct/34)	% Correct	Post-test (correct/34)	% Correct	% Increase
1	Rep. Add.	27	79.4 %	31	91.2 %	11.8 %
2	Rate	26	76.5 %	34	100 %	23.5 %
4	Scaling	23	67.6 %	32	94.1 %	26.5 %
6	Rate	23	67.6 %	31	91.2 %	23.6 %
7	Rep. Add.	19	55.9 %	26	76.5 %	20.6 %
9	Scaling	15	44.1 %	16	47.1 %	3.0 %

Table: 5.1: Summary of performance in multiplication word problems across pre-test and post-test.

Table 5.1 shows shifts in learner performance between the pre-test and the post-test assessment across the six weeks intervention schedule. The overall results suggest that learners indeed benefitted from the intervention lessons, with substantial gains on all but items apart from item 9. Hence, one can conclude that an intervention focused on the use of models served as a vehicle for building mathematical understandings of multiplication. The overview performance gain pointed to the usefulness of looking at the detail of models and strategies – in the pre- and post-tests and the intervention lessons – in order to understand how these gains were produced.

5.3 Models used in the pre-test and post-test

The pre-test results (Table: 5.2) indicate that the column model is predominantly favoured by most of the learners in the sample (averaging 62.4 % across the six items). The use of an area model and doubling averaged 19 % and 11 % respectively. Out of a total of 71 incorrect answers in the pre-test, 49 (69 %) of them are associated with the column model. Yet, only 8 errors (8.2 %) are associated with the area model and 4 errors (4.1 %) with the doubling model. This suggests that, as much as learners favour the column model, it did not give the most favourable results in comparison to the doubling and area models.

Table 5.2: Pre-test summary analysis of models

MODEL	CORRECT	INCORRECT	TOTAL
Qn. = 1 (R – Addition)			**(n = 34)**
Column	16 (72.7 %)	6 (27.3 %) *(2)	22 (64.7 %)
Area Mo.	4 (80 %)	1 (20 %) *(2)	5 (14.7 %)
Doubling	7 (100 %)	0	7 (20.6 %)
	0		
Qn. = 2 (Rate)			**(n = 34)**
Column	15 (75 %)	5 (25 %)	20 (58.8 %)
Area Mo.	5 (71.4 %)	2 (28.6%)	7 (20.6 %)
Doubling	5 (100%)	0	5 (14.7 %)
No working	1 (50%)	1 (50%)	2 (5.9 %)
Qn. = 4 (Scaling)			**(n = 34**)
Column	16 (64 %)	9 (36 %)	25 (73.5 %)
Area Mo.	5 (83.3 %)	1 (16.7 %)	6 (17.6 %)
Doubling	2 (66.7 %)	1 (33.3 %)	3 (8.8 %)
Qn.= 6 (Rate)			**(n = 34)**
Column	14 (77.8 %)	4 (22.2 %) *(2)	18 (52.9 %)
Area	5 (100 %)	0	5 (14.7 %)
Doubling	1 (75 %)	3 (25 %)	4 (11.8 %)
L –Division	0	3 (100 %)	3 (8.8 %)
No working	3 (75 %)	1 (25 %)	4 (11.8 %)
Qn. = 7 (R – Addition)			**(n = 34)**
Column	11 (52.4 %)	10 (47.6 %) *(1)	21 (61.8 %)
Area Mo.	3 (100 %)	0	3 (8.8 %)
Doubling	3 (100 %)	0	3 (8.8 %)
L – Division	0	1 (100 %)	1 (2.9 %)
No working	2 (33.3 %)	4 (66.7 %)	6 (17.7 %)
Qn. 9 (Scaling)			**(n = 34)**
Column	7 (31.8 %)	15 (68.2 %)	22 (64.7 %)
Area Mo.	8 (66.7 %)	4 (33.3 %)	12 (35.3 %)

***(n):** Indicates number of responses where learners translated from problem to this model, but with incorrect selection of figures

In question 1, most of the learners who obtained incorrect answers had used wrong figures (25 x 16) instead of (8 x 16). This could be an indication that the problem is not at the level of model type but it is in translating from a given situation (problem) to model. Two learners just added all the three figures (25 + 16 + 8) indicating that they did not understand the structure of the situation. In question 4 the area model was largely successfully used (83 %) but not used widely. In questions 6 and 7 all the learners who employed the area model were successful, achieving 100% in each question. However, the column model achieved the lowest percentage in question 7 (52 %).

In addition to the observation of overall low performance, these results give a direct answer to the first research question, (what models and strategies for solving multiplication word problems are Grade 6 learners using prior to a small scale intervention?). The data in table 5.2 clearly indicates the dominance of the column model, yet learners struggle at carrying out the algorithm-based strategies associated with this correctly. Also observed is the way learners used the two models (namely the column model and the area model) with regard to the number range. The column model remained dominant in the lower number ranges, whilst the area model gradually gained dominance in higher number ranges.

Post-test summary of the use of models (Table 5.3) reflects the use of new models (distributive, number line and completion) that were discussed during the intervention lessons as well as a pronounced shift in choices of models, especially the column model, where numbers reduced drastically (see Table 5.4). On average, this change shows that roughly 15 learners have moved on to adopt or explore models different from the column model across the items. However, the last question reflects an interesting result because the 4 learners that used the column model in order to answer the question did not succeed to answer it correctly. This is an indication that even those learners who used this model successfully last time have migrated to make use of other approaches. In the main though, the few learners that remained using the column model demonstrated a good understanding of both the model and associated strategies. It is interesting to note that out of a total of 204 responses (34 × 6 questions) in the pre-test, 178 (87 %) of these used the column model. After the pre-test, only 34 (17 %) remained using the column model. This indicates a substantial voluntary shift within the six week period of intervention activities.

Table 5. 3: Post-test summary analysis of models

MODEL	CORRECT	INCORRECT	TOTAL
Qn. = 1 (R– Addition)			**(n = 34)**
Column	3 (75 %)	1 (25 %)	4 (11.8 %)
Area Mod.	13 (100 %)	0	13 (38.2 %)
Doubling	4 (100 %)	0	4 (11.8 %)
Distributive	8 (80 %)	2 (20 %)	10 (29.4 %)
N-line	1 (100 %)	0	1 (2.9 %)
Completion	2 (100 %)	0	2 (5.9 %)
Qn. = 2 (Rate)			**(n = 34)**
Column	5 (100 %)	0	5 (14. 7 %)
Area Mod.	6 (100 %)	0	6 (17.6 %)
Doubling	7 (100 %)	0	7 (20.6 %)
Distributive	8 (100 %)	0	8 (23.5 %)
N – Line	5 (100 %)	0	5 (14.7 %)
Completion	3 (100 %)	0	3 (8.8 %)
Qn. = 4 (Scaling)			**(n =34)**
Column	4 (66.7 %)	2 (33.3 %)	6 (17.6 %)
Area Mod.	9 (100 %)	0	9 (26.5 %)
Doubling	6 (100 %)	0	6 (17.6 %)
Distributive	8 (100 %)	0	8 (23.5 %)
N - Line	1 (100 %)	0	1 (2.9 %)
Completion	4 (100 %)	0	4 (11.8 %)
Qn. = 6 (Rate)			**(n =34)**
Column	8 (88.9 %)	1 (11.1 %)	9 (26.5 %)
Area Mod.	6 (100 %)	0	6 (17.6 %)
Doubling	4 (80 %)	1 (20 %)	5 (14.7 %)
Distributive	9 (100 %)	0	9 926.9 %)
N - Line	1 (50 %)	1 (50 %)	2 (5.9 %)
Completion	3 (100 %)	0	3 (8.8 %)
Qn. =7(R-Addition)			**(n =34)**
Column	4 (66.7 %)	2 (33.3 %)	6 (17.6 %)
Area Mod.	4 (66.7 %)	2 (33.3 %)	6 (17.6 %)
Doubling	5 (62.5 %)	3 (37.5 %)	8 (23.5 %)
Distributive	8 (100 %)	0	8 (23.5%)
N - Line	0	0	0
Completion	5 (83.3 %)	1 (29.4 %)	6 (17.6 %)
Qn. = 9 (Scaling)			**(n = 34)**
Column	0	4 (100 %)	4 (11.8 %)
Area Mod.	10 (50 %)	10 (50 %)	20 (58.8 %)
Doubling	1 (100 %)	0	1 (2.9 %)
Distributive	4 (57.1 %)	3 (42.9 %)	7 (20.6 %)
N – Line	0	1 (100 %)	1 (2.9 %)
Completion	1 (100 %)	0	1 (2.9 %)

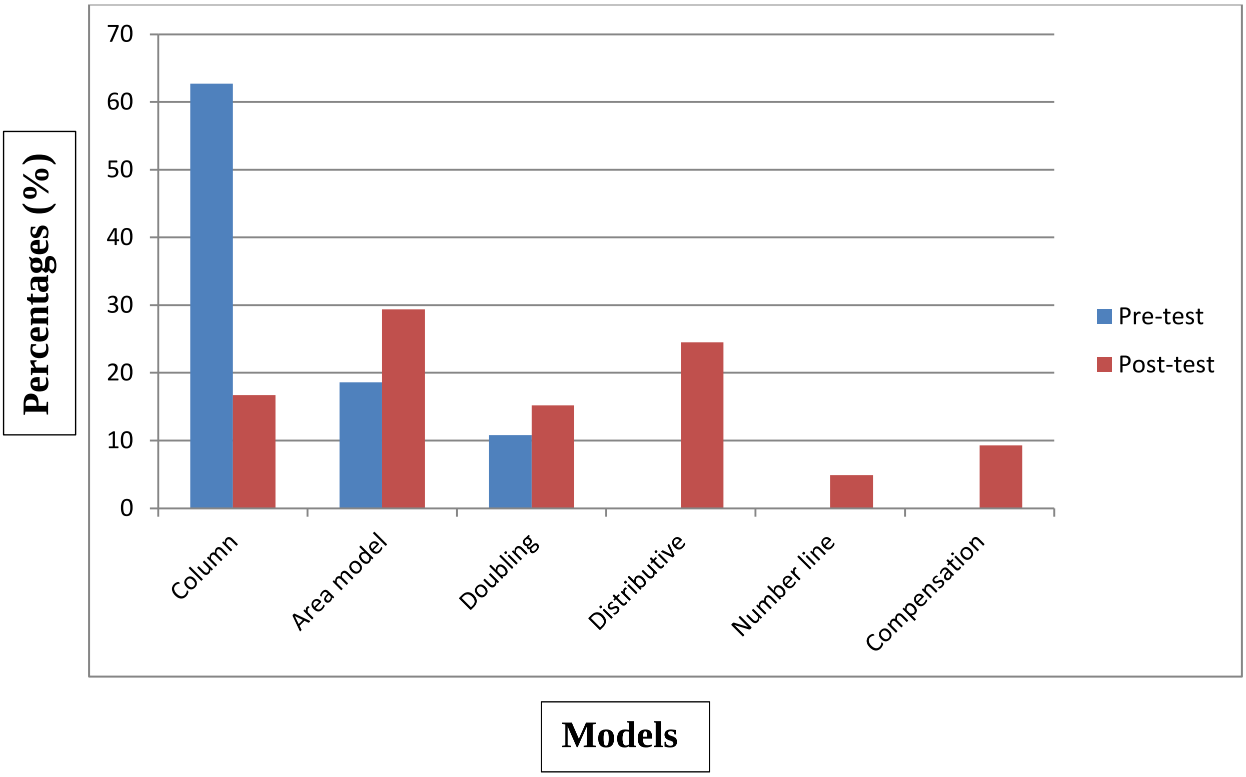
Table 5.4: Comparison of the use of models in the pre- and post-test

Question	Model	Pre-test (learners using the model)	Number correct	Post-test (learners using the model)	Number correct
1	**Column**	**22**	**16**	**4**	**3**
R-Add.	Area Model	5	4	13	13
	Doubling	7	7	4	4
	Distributive	-	-	10	8
	Number line	-	-	1	1
	Completion	-	-	2	2
2	**Column**	**20**	**15**	**5**	**5**
Rate	Area Model	7	5	6	6
	Doubling	5	5	7	7
	Distributive	-	-	8	8
	Number line	-	-	5	5
	Completion	-	-	3	3
4	**Column**	**25**	**16**	**6**	**4**
Scaling	Area Model	6	5	9	9
	Doubling	3	2	6	6
	Distributive	-	-	8	8
	Number line	-	-	1	1
	Completion	-	-	4	4
6	**Column**	**18**	**14**	**9**	**8**
Rate	Area Model	5	5	6	6
	Doubling	4	1	5	4
	Distributive	-	-	9	9
	Number line	-	-	2	1
	Completion	-	-	3	3
7	**Column**	**21**	**11**	**6**	**4**
R. Add.	Area Model	3	3	6	4
	Doubling	3	3	8	5
	Distributive	-	-	8	8
	Number line	-	-	0	0
	Completion	-	-	6	5
9	**Column**	**22**	**7**	**4**	**0**
Scaling	Area Model	12	8	20	10
	Doubling	0	0	0	0
	Distributive	-	-	7	4
	Number line	-	-	1	0
	Completion	-	-	1	1

Shifts in use of the area model portray a different picture from that seen with the column model. Numbers do not show a drastic shift. The pre-test shows that 38 (18.6 %) out of 204 responses made use of the area model and the success rate stood at 30 (79 %). However, after the pre-test 60 (29 %) responses made use of the area model and the success rate improved a little (83 %). It needs to be noted that in questions 1 and 9, the change in numbers was more pronounced (8 more learners) per question. Question 9 elicited the area model more than any other model in the post-test (20 learners) but the success rate was relatively low (50 %). This result indicated lower levels of competence in multiplication of 3-digit by 2-digit numbers than with smaller numbers, although learners' performance using the area model was better than the performance that was demonstrated in the column model. This is an indication that struggle with multiplication of 2-digit by 3-digit numbers is at the heart of the problem here, beyond difficulties dealing with the word problem.

Moving on to doubling, again the change was not as pronounced as it was in the column model. There were 22 (11 %) doubling responses out of a total of 204. Its success rate was high (82 %) in the pre-test and it continued to increase in the post-test. 30 (15 %) responses were based on doubling and the success rate rose to 87 % in this model. Doubling was not used in question 9 during the pre-test. However, only one learner used this model in the post-test and obtained a correct result. The overall picture of the pre-test and post-test results is shown on the bar graph below, using mean figures across items for choice of model.

Fig. 5.1 Comparison of models used in the pre-test and the post-test



It would appear that column model selection has been greatly reduced during the intervention lessons. Over 40 % of learners have moved on to explore other models. As a result, the area model was used more widely by learners, moving from being seen in 18 % to 29 % of responses, and doubling moving from use in 11 % to 15 % of responses. New models also emerged in the post-test following the intervention. These are the distributive (24 %), number line (5 %) and completing (9 %) model. The relative popularity of the distributive model can partially be attributed to Annual National Assessment (ANA) questions that require learners to use this model in the exemplar papers that are distributed to schools early in the year and the final examination papers. Hence, it took one learner to present this approach and a good number of the learners began to adopt it immediately because they had been introduced to it before when we were revising exemplar papers in preparation for the final ANA examinations. This was in spite of not choosing this model in the pre-test. The number line and the completing models can be viewed as new because learners have not used them before as tools for multiplication.

5.4 Strategies used in the pre-test and post-test

5.4.1 Pre-test

Two strategies were used within the column model. There are learners who began multiplying from the right-to-left (R – L) and others who began multiplying from the left-to-right (L – R) - see Table 5.5 (a)

Table 5.5 (a): Strategies used in the pre-test

		R. Add.		Rate		Scaling		Rate		R. Add.		Scaling	
		Qn. 1 (16x8)		Qn. 2 (14 x 8)		Qn. 4 (28 x 6)		Qn. 6 (17 x 6)		Qn. 7 (36 x 9)		Qn. 9 (895 x 28)	
Model	**Strategy**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**
Column	R - L	16	4	15	4	16	6	14	3	11	8	5	13
Column	L - R	0	2	0	1	0	3	0	1	0	2	2	2
	Total	16	6	15	5	16	9	14	4	11	10	7	15
		22 (64.7 %)		20 (58.8 %)		25 (73.5 %)		18 (52.9 %)		21 (61.8 %)		22 (64.7 %)	
Area M.	Mental	4	1	5	2	5	1	5	0	3	0	8	4
	Total	5 (14.7 %)		7 (20.6 %)		6 (17.6 %)		5 (14.7 %)		3 (8.8 %)		12 (35.3 %)	
Doublin	Add. prs	7	0	5	0	2	1	1	3	3	0	0	0
	Total	7 (20.6 %)		5 (14.7 %)		3 (8.8 %)		4 (11.8 %)		3 (8.8%)		0 (0 %)	

Question 1: The right-to-left column addition strategy was much more widely used than the left-to-right. Area model is associated with mental multiplication strategies (indicated by no written working seen for partial products). Although few learners use this model, correct answers are widespread here (80 % correct compared with 72.7 % for the column model). The doubling model and mental strategy of adding pairs of numbers was successful for all the 7 learners who used the model. Both numbers in this question are multiples of 2, making doubling an attractive strategy here.

Question 2: The column model is still favoured here with the R-L strategy. However, its success rate is a little higher (75 %) compared to that of area model (71.4 %). Doubling also achieved 100 % correct responses in this question.

Question 4: Column model R-L strategy remained dominant but with a low success rate (64 %), whilst the area model had a higher success rate. This time doubling decreased to 67 %.

Question 6: The R-L strategy was dominant, with only 1 learner using the L-R strategy within the column model. It is interesting to note that the two learners who have been working from left-to-right have not been successful with this strategy.

Question 7: In this question learners struggled with the R-L strategy and the success rate fell below the halfway mark (38 %). This kind of performance points to lack of fluency in the learners' times tables. However, the three learners who opted for the area model were all successful. The same applies for doubling.

Question 9: This question exposed the learners' weakness in their number facts and times tables because it became clear that vertical mathematization was a challenge as the number range increases. This was evident because of the errors that became more prevalent in learner responses to the questions. Learners' performance in this question persuaded me to focus on modeling in the context of 2-digit by 1-digit and 2-digit by 2-digit multiplications in my intervention lessons.

5.4.2 Post-test

34 learners took the post-test due to learners being absent from school on the day. For the column model, the left-to-right strategy has fallen away with none of the three learners who used it in the pre-test still using it (see table 5.5b)

Table 5.5 (b): Strategies used in the post-test

		R. Add.		Rate		Scaling		Rate		R. Add.		Scaling	
		Qn. 1 (16x8)		Qn. 2 (14 x 8)		Qn. 4 (28 x 6)		Qn. 6 (17 x 6)		Qn. 7 (36 x 9)		Qn. 9 (895 x 28)	
Model	**Strategy**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**	**Corr.**	**Incorr.**
Column	R - L	3	1	5	0	4	2	8	1	4	2	0	4
	Total	4 (11.8 %)		5 (14.7 %)		6 (17.6 %)		9 (26.5 %)		6 (17.6 %)		4 (11.8 %)	
Area M.	Mental	13	0	6	0	9	0	6	0	4	2	10	10
	Total	13 (38.2 %)		6 (17.6 %)		9 (26.5 %)		6 (17.6 %)		6 (17.6 %)		20 (58.8 %)	
Doublin	Add. prs	4	0	7	0	6	0	4	1	5	3	1	0
	Total	4 (14.7 %)		7 (20.6 %)		6 (17.6 %)		5 (14.7 %)		8 (23.5 %)		1 (2.9 %)	
Distrbut.	mental	8	2	8	0	8	0	9	0	8	0	4	3
	Total	10 (29.4 %)		8 (23.5 %)		8 (23.5 %)		9 (26.5 %)		8 (23.5 %)		7 (20.6 %)	
N-line	proportion	1	0	5	0	1	0	1	1	0	0	0	1
	Total	1 (2.9 %)		5 (14.7 %)		1 (2.9 %)		2 (5.9 %)		0 (0 %)		1 (2.9 %)	
Complet	Rounding	2	0	3	0	4	0	3	0	5	1	1	0
	Total	2 (5.9 %)		3 (8.8 %)		4 (11.8 %)		3 (8.8 %)		6 (17.6 %)		2 (2.9 %)	

Question 1: The success rate (31 out of 34) in this question indicates a great deal of improvement in the way the learners dealt with horizontal and vertical mathematization. Only one out of four learners who used the column model had an incorrect answer in number 1 (16 x 8 = 129). This result could have been an error or a case of poor times tables fluency. The other two learners had incorrect responses within the distributive model. One error was based on selection of inappropriate numbers - she used 25 x 16 instead of 16 x 8. However, the strategy was carried out correctly despite the incorrect model. The third learner got the model wrong as well [16 x 8 = (10 + 6) x (16 x 10)].

Question 2: Again question 2 responses were all correct for all the six models used. Pre-test results show that there were 7 incorrect responses. Hence, the post-test results reflect that these learners have overcome difficulties or errors from the pre-test.

Question 4: Two learners using the column model selected inappropriate numbers (28 x 8 instead of 28 x 6). However, strategies were correctly carried out. This is a sign that these learners did not struggle at the strategy level this time but struggled at the horizontal modeling level with translating from a given situation (problem) to model.

Question 6: In this question, one learner using the column model might have erred or she is not fluent with her times tables (27 x 6 = 165). The second learner used the doubling model but selected incorrect figures (120 x 6 instead of 27 x 6). The third learner used a double number line model. Four pairs (1:6, 7:42, 10:60 and 20:120) are marked on the number line. He then decided to add 120 + 60 +42 instead of 120 +42.

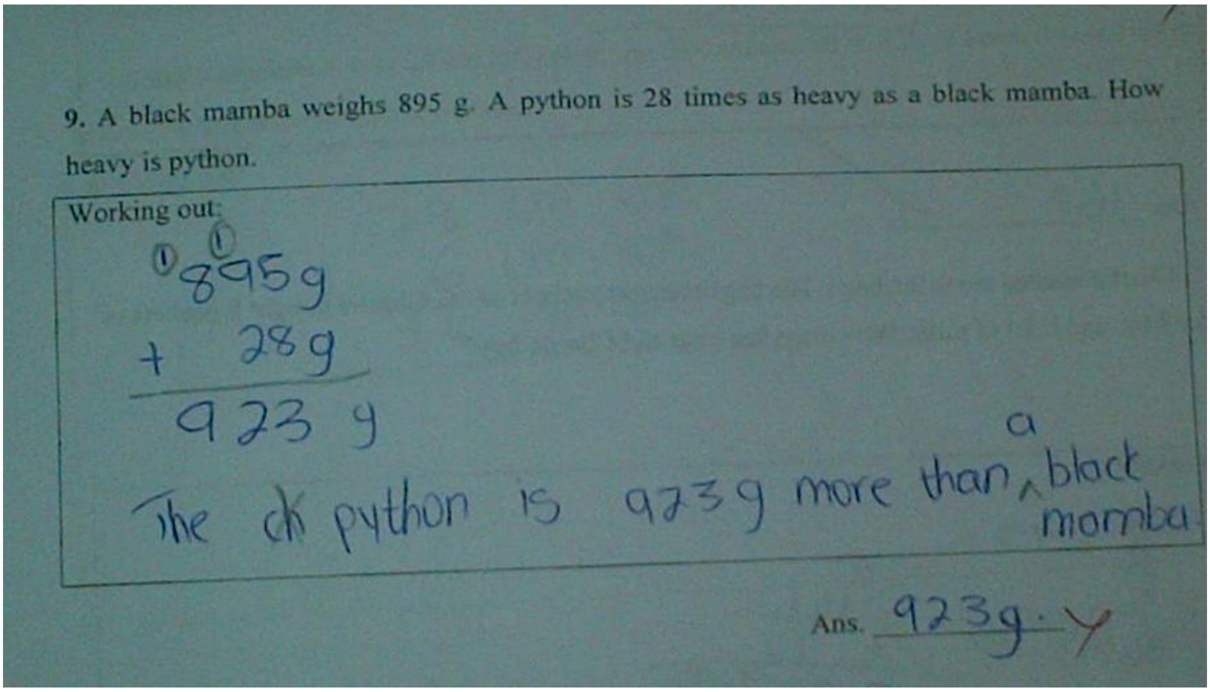
Question 7: This question saw a number of errors at the level of vertical mathematization. Two of the learners demonstrated lack of knowledge of their times tables. They used the column model (36 x 9 = 323) to obtain incorrect answers. Two other learners used the area model and they got the partial products wrong (30 x 9 = 290 and 6 x 9 = 46). The other four learners who used the doubling model added incorrectly at different stages.

Question 9: This question proved difficult for most learners. None of those learners who used the column model (4) got the answer correct due to errors in multiplication and addition of partial sums. Patterns of general success, regardless of strategy are disrupted on questions 7 and 9. Here, area, doubling and completing models are much more successful. More than half the learners opted for the area model but struggled to work out the correct partial products, suggesting lack of fluency with multiplication involving multiples of 10.

Overall, key horizontal mathematization problems seem to be at the level of selecting inappropriate information. Key problems at vertical mathematization level seem to relate to partial products.

Question 9, which is a 2-digit by 3-digit multiplication, reflects a low success-rate compared to the rest of the questions in both the pre- and post-test. Hence, this observation motivated me to also analyse question 10, which is a buffer question and also a 2-digit by 3-digit multiplication just like question 9 that is related to curriculum specification for Grades 5 and 6.The difference between the two is very small (33% and 29%), an indication that struggle with multiplication of 2-digit by 3-digit numbers is at the heart of the problem here, beyond difficulties dealing with the word problem.

Question = 9, (895 x 28)	**Correct**	**Incorrect**	**(Scaling)**
Column	10 (33.3 %)	20 (66.7 %)	30 (71.4 %)
Area Mo.	8 (66.7 %)	4 (33.3 %)	12 (28.6 %)
Question 10, (934 x 14)			**(Buffer)**
Column	7 (29.2 %)	17 (70.8 %)	24 (57.1 %)
Area Mo.	13 (72.2 %)	5 (27.8 %)	18 (42.9 %)



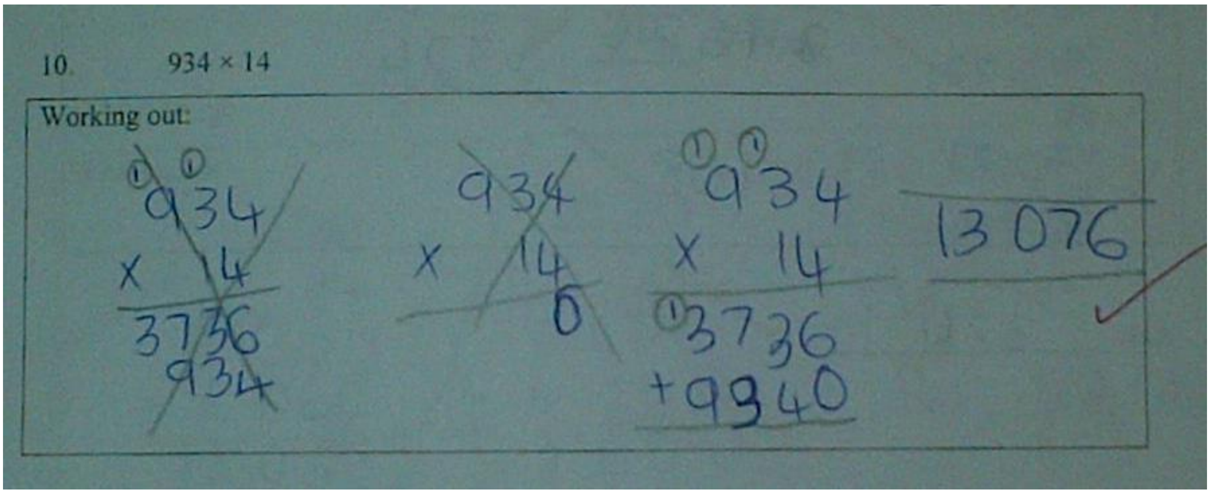


Fig 5.2: Comparison Student no. 33, questions 9 and 10 (pre-test)

A closer look at the responses to the two questions (9 and 10) reveals that a good number of learners (learner numbers 4, 5, 6, 13, 14, 25 and 33) struggled with the translation from the problem to model (horizontal mathematization), see Fig. 5.2 above from learner 33. These are relatively similar problems yet the learner could not apply the appropriate operation in question 9 because she failed to interpret it correctly. However, in question 10, where the operation is a given, the same learner was able to reorganize and apply vertical mathematization using the column model, resulting in a correct response. About 7 (20.6 %) learners experienced a similar problem to question 9, where they could not translate the problem into an appropriate model. Some of the learners struggled with the vertical mathematization in relation to calculation of partial products within the column and the area models. About 13.2 % of the learners struggled due to inaccurate calculations (errors).

However, in these two questions, the use of the area model became more pronounced compared to the earlier questions. In addition to this, its success-rate (area model) in these questions was far higher than the column model (67% and 72% for area model and 33 % and 29 % for column model). The differences in success rates between the area and column models are not so marked in lower number ranges, suggesting that the area model becomes more useful in the higher number ranges specified in Grades 5 - 6. On the other hand, it looks like the column model is associated with increasing error as the number range increases.

From the above discussion of models and strategies used by learners, two things stand out. The low success rate reflected within these models and strategies might be an indication that the learners in the sample have knowledge gaps in the number facts and their multiplication times tables. The selection of inappropriate figures in questions where they had to select numbers appropriate for a given situation also suggests some problems, not at the level of model type, but in translating from a given situation (problem) to model. It may be inferred that some learners struggle with interpreting situation structure correctly. In questions where there were just two numbers to work with, learners displayed their inability (struggled with strategies) to calculate accurately which could be a sign that they lack a good foundation in their times tables. According to the DoE curriculum, and also in defence of teachers in the foundation phase and in the intermediate phase, it is certain that multiplication has been taught, but the teaching has not led to the mastery of multiplication algorithms. Learners, therefore, need assistance towards an adequate understanding of multiplication.

It is this dilemma that has brought the teaching and learning of multiplication into sharp focus. In addition to this, it is quite frustrating for us as teachers to see learners at this level (Grade 6) struggling with times tables and, for example, when simplifying 27 x 9 to see 27 written 9 times as repeated addition. From the literature reviewed, arguments for the importance of encouraging learners to invent their own methods make a lot of sense for me and they correspond to Realistic Mathematics Education approach which is based on the idea of "mathematizing", a "process of moving from a realistic situation to the setting up of a mathematical model" (Askew, 2012, p. 105).

The use of inappropriate figures (mathematizing from problem situation to model) was a concern for me. Elsewhere, language has been cited as playing a significant role in learners' low attainment in mathematics (Setati, 2002). Daniels & Aghileri (1995) cite speech and written language as the critical tools in the learning of mathematics. It is therefore crucial that learners understand the problem before they could construct a model, let alone a strategy for solving it. Although learners in my study use English as their language of learning and teaching (LOLT), they are not comfortable with the use of it when dealing with word problems, hence some of them struggled with translating from a given problem to a (struggle with trying to make sense of the problem) model.

5.5 Summarizing shifts in models and strategies

In the section that follows I closely analyse the shifts that took place after the small scale intervention. I make use of tables and bar graphs in order to bring these to the attention of the reader.

5.5.1 The column model

The results of the post-test reflect a significant shift with regard to the number of learners who initially used the column model before intervention lessons. Data shows that learners struggled mainly with two factors. First was the translation from a situation (problem) to a model. This was evident in questions with superfluous information as well as those that did not have superfluous information. Of the 49 incorrect responses (pre-test) within the column model, 27 (55 %) were associated with the problem of translation from a problem to a model. Inappropriate figures were used as well as incorrect operations. Secondly, there were problems with calculation strategies coupled with learners' weakness in their times tables. This led to increased errors in their calculations.

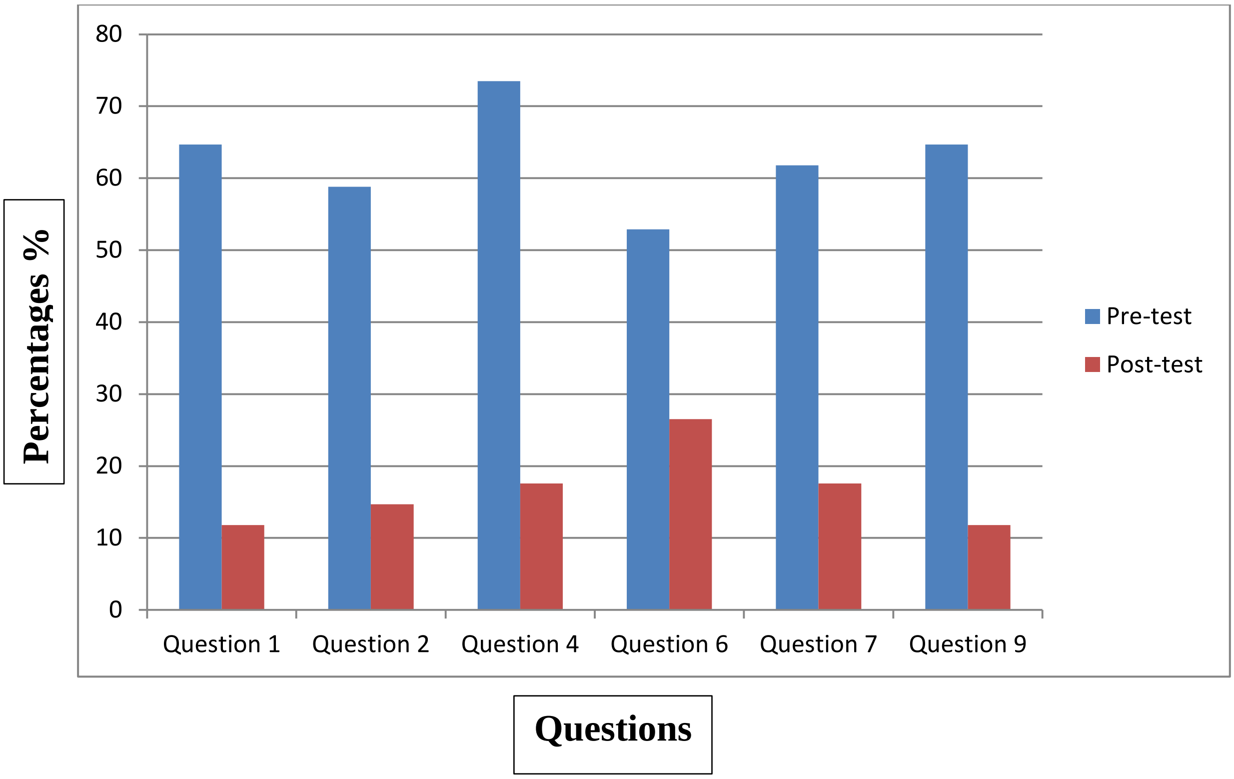
Within the column model, the post-test shows that out of the 10 incorrect responses across the 6 items, only one response was associated with translation from a given situation (problem) to a model (question 6). The other nine incorrect responses indicate that most learners were no longer struggling with the translation to models or the use of inappropriate figures, but there were still struggling with vertical mathematization. For example, table 5.6 shows some of the challenges that learners were facing by the end of the intervention lessons, which are mainly related to strategies.

Question	**1.** (16 x 8**)**	**2 (**14 x 8**)**	**4. (**28 x 6**)**	**6.(**27 x 6)	**7. (**36 x 9)	**9. (**895 x 28**)**
Incorrect strategy	16x8 = 129	0	28 x 6 = 88	0	36 x 9 = 323	895 x 28 = 923
Incorrect Translation				120 x 6 = 720		

Table 5.6 Challenges remaining in post-test in the context of use of column model

Hence, the above examples show that learners were no longer struggling seriously with the translation of a problem to a model but still had problems with vertical mathematization. On average, this change shows that roughly 15 learners have moved on to adopt or explore models different from the column model across items and root situations. However question 9 results indicate that even those learners who used this model successfully previously migrated to make use of other approaches. This is confirmation of the point I raised earlier about

Fig. 5.3 Comparison of the use of the column model in pre- and post-test



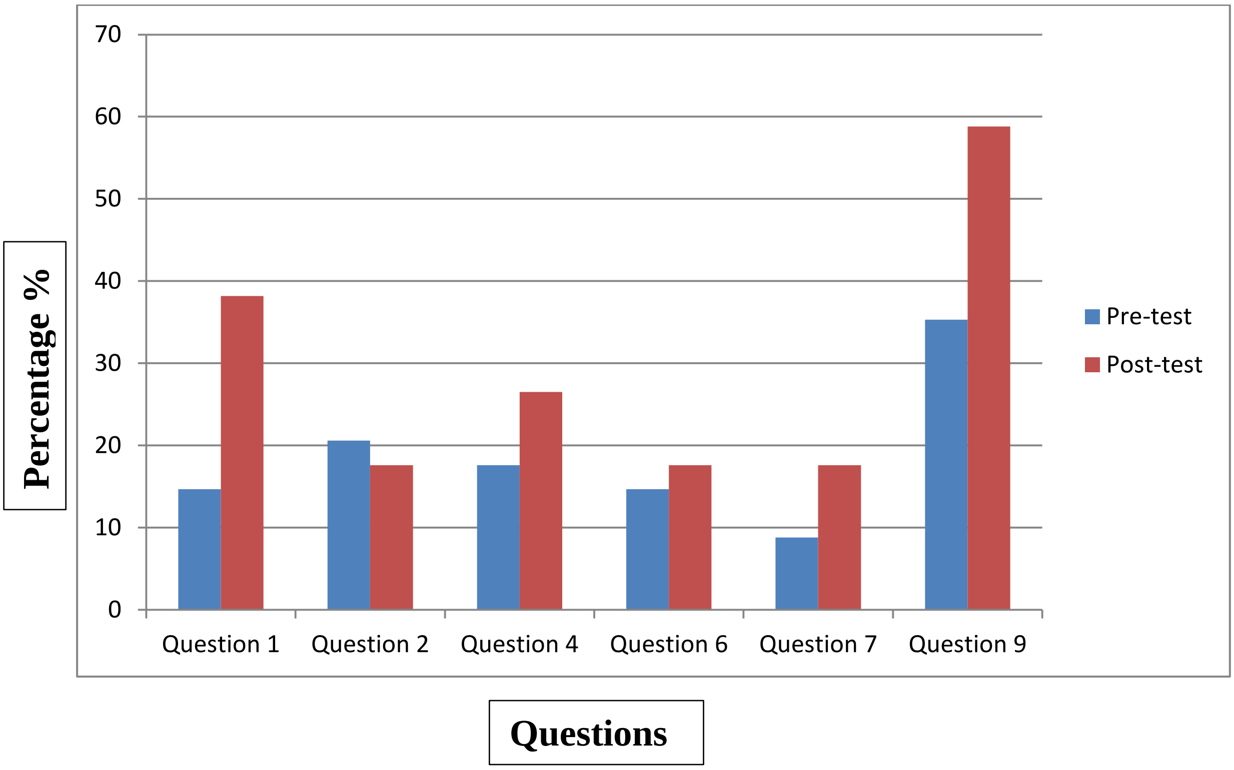
learners' competence with regard to multiplication, poor performance in their times tables and the general lack of confidence in manipulating numbers. It follows therefore, that what is prescribed in the Grade 5 curriculum statement, that learners should be able to multiply 3-digit by 2-digit numbers, has not been achieved near the end of Grade 6 by some learners.

In spite of this, the shift in the column model is very significant. This pattern indicates that learners have now adopted other models which they could not use before due to lack of exposure to them. Results also highlight learners' exposure to the column model as the 'standard' for multiplication in the lower grades. I say this because most of them were using the column model at the beginning of the year. The pattern that is reflected in the comparison of results of the column model is slightly different in the area model.

5.5.2 The area model

Within the area model, a slight increase was realised in numbers of learners preferring this model. Its success rate was high in questions 1, 2, 4 and 6. A few learners experienced challenges with making sense of some of the problems and calculating partial products in the pre-test. However, their performance both in the level of number selection and accuracy in calculating the partial products was greatly improved (Fig. 5.4)

Fig. 5.4 Comparison of the use of the area model in pre- and post-test

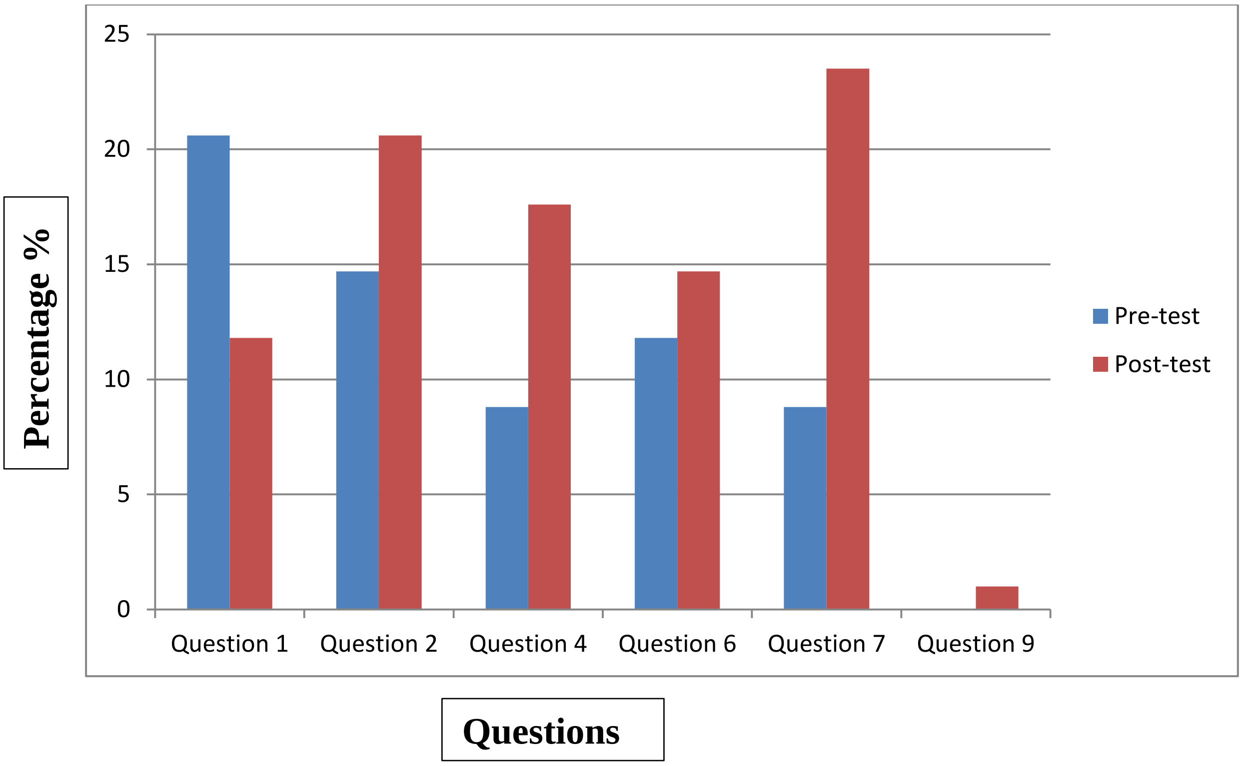


The increase in numbers in question 9 leads to a decrease in the percentage of learners who produced correct responses (from 67.7 % down to 50 %) using the area model. Again, this pattern demonstrates the learners' low level of competence in multiplication of 3-digit by 2-digit numbers, although learners' performance in the area model was slightly better than the performance that was demonstrated in the column model.

5.5.3 Doubling model

Moving on to doubling, the average number of learners per question was around 3 in the pre-test whereas in the post-test it only went up by 1, to 4 learners per question. In questions 2, 4, 6 and 7, the number of learners who used doubling increased. It is only in question 1 where the number of leaners using doubling in the post-test came down from 7 (20.6 %) learners to 4 (11.8 %). Doubling was not used in question 9 during the pre-test. However, only one learner used this model in the post-test and obtained a correct result. The overall picture indicates that there were big shifts in questions 4 and 7 and doubling was the least used model in the pre-test. Its success is reflected by the consistency of the numbers in both the pre- and the post-test.

Fig. 5.5 Comparison of the use of the doubling model in pre- and post-test



Overall, key horizontal mathematization problems seem to be at the level of inappropriate information or incorrect selection of numbers due to failure to make sense of the problem or given situation. On the other hand, key problems at vertical mathematization seem to relate to partial products in both the column and the area model. Data obtained from questions 9 and 10 points to errors at the level of both single digit multiplication as well as multiplication by multiples of 10. However, my observation is that learners' selection of appropriate numbers improved during the intervention lessons compared to vertical mathematization. An increase in number range seemed to favour the area model more than the other models.This is indicated by the post-test results. In order to understand how and when these shifts were produced, I looked across the dataset relating to six intervention lessons.

5.6 Intervention lessons

Drawing on the literature on the use of problems as vehicles for learning multiplication, Askew (2012, p.105), posits that "this requires a style of teaching that begins with engaging the children with the problem". This proposal aligns with the concept of mathematization and Caliandro's (2000) assertion that learners are capable of inventing their own methods for solving multidigit multiplication problems. That is, learners themselves are the constructors of models and strategies to solve multiplication problems. The important part was to draw upon learners' solutions that offer opportunities for teaching points during the whole-class discussions. Also comparing some of the models and strategies coupled with learners' own explanations, I hoped to prompt them to come to a better understanding of multiplication and to also adjust their approaches so as to adopt the more efficient ones according to their individual ability and their level of understanding.

The teacher's role

Teaching using the Realistic Mathematics Education framework has a whole range of implications. It involves "sacrifice of overt control" (Mason & Pimm, 1986) of the class (an approach that is common in our South African classrooms), allowing learners to wrestle with creation of appropriate models, invent corresponding strategies and explain their own solutions to others. It was not easy to stand back and let learners grapple with problems on their own. Neither was it easy for me to forfeit the role of evaluating learners' contributions and instead passing this responsibility on to the whole class. I was more concerned with keeping learners on-task and drawing on various solutions offered by learners so as to intervene when necessary in order to guide the learners. Below is a section of my lesson plan that formed the structure of my first intervention lesson.

Table 5.8: Section of a lesson plan

OBJECTIVES	SKILLS
❖ Learners derive models and schematic representations of word problems.	❖ Learners to imagine problem situations that will help in a solution.
❖ Use representations to solve word problems on multiplication. (repeated addition)	❖ Use/ select appropriate numbers in the question
	❖ Translate linguistic information to Mathematical model or representation.
CONTENT KNOWLEDGE	**RESOURCES**
❖ Basic number facts and relationships	❖ Big Book (Askew, 2012).
❖ Times tables	
❖ Basic knowledge of 2-digit by 2-digit multiplication	
❖ Solution of simple word problems using given situations.	

LEARNER ACTIVITY
❖ Pair-work – learners discuss and solve introductory problems in the book.
❖ Guided discussion and learner – demonstration on the chalk-board.
❖ Learners use A-3 papers to display their solutions in the form of charts and design and begin To compare their solutions.
❖ Learners provide explanations through discussions and sharing ideas
❖ Summary of strategies and models used.
TEACHER ACTIVITY
❖ Provide/ outline expectations to learners.
❖ Encourage learners to solve the 3 Big Book problems (one at a time) by inventing or creating representations and explanations for their solutions.
❖ Engage in discussions and select some examples that learners can use to demonstrate and Clarify some aspects of the given solutions (focus on interpretation and strategies used).
❖ Support learners' explanations through probing questions/setting up models, diagrams etc.
❖ Compare problems during discussions and try to link them
❖ Supervise and support individual learners work

A worksheet of seven questions was prepared. The objective was to let learners work in pairs when doing the first question of the examples. I anticipated that this approach would lead to learners talking among themselves (mathematical talk), leading to construction of diagrams – moving from a realistic situation to the setting up of models (horizontal mathematization). As they imagine and select a way of representation their thinking is likely to be aligned with the produced model. Hence, I challenged the learners to begin with diagrams and then use these images to solve the problems. However, as the lesson progressed I felt that the significance of these representations was not recognised by some learners because in most cases one would not find a link between the image and the solution of a given problem. This is a challenge that is also alluded to by Askew (2012), who warns that sometimes learners may not 'see through" these images to engage with the mathematics.

Before engaging with the next question we had a whole class discussion, which became, for me, a very important part of the lesson. As I had been moving around the class, observing various solutions from different groups, I was in a position to bring to the attention of the whole class, those solutions that were deemed to be of value and that captured key teaching points about the ways in which learners interpreted the question and the strategies that they used for solving that particular problem. Some of these solutions did provoke rich dialogue among the learners, providing an opportunity for them to deal with some errors, misconceptions, etc. and my role was to support and guide learners in their efforts to set up models or diagrams that they used as tools to explore multiplication problems. Data sources were the learner workbooks, the researcher's field notes as well as lesson plans. Key illustrative incidents relating to models and strategies discussions are drawn on within this discussion. The results of the first intervention lesson are discussed below.

5.7 Lesson 1: Results and discussion (multiplication as repeated addition)

Lesson 1 revealed various aspects of learners' performance. The most prevalent was incorrect selection of numbers to be used in a given situation (horizontal mathematization). On average 73 (26.7 %) responses out of a total of 273 responses based on incorrect selection of numbers. However, in instances where incorrect figures were selected, the strategies used were applied correctly in some cases. Learners' struggle with vertical mathematization contributed 13.2 % (36 out of 273 responses). This problem was evident right across all the models produced, seen in calculation errors. Below are typical examples of the kind of errors that emanated from the first lesson.

Question 1: An order of 45 boxes of fire crackers arrived. There were 36 fire crackers in each box. Coco threw each fire cracker on the floor to see if it worked. How many fire crackers did Coco throw on the floor?

The success rate for this question was fairly low (51.3 %). **In Fig. 5.6** below the learner has used the area model correctly to work out the partial products of 45 x 36. However, it is disappointing to note that she is let down by the incorrect lining up of partial products when adding them (1200 + 240 = 3600 and 150 + 30 = 450). Out of 5 incorrect responses in the area model, 3 responses reflected the errors in Fig. 5.6. Errors were quite prevalent in questions where the column model was used, mainly knowledge of times tables was a problem for a good number of learners.

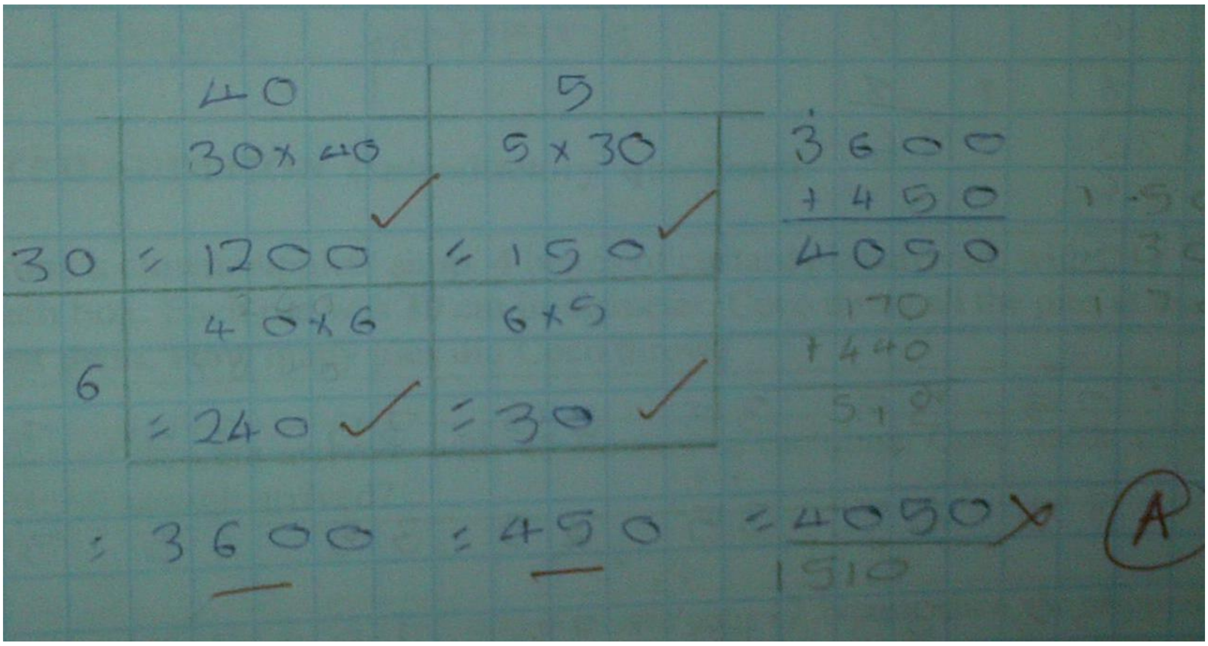


Fig. 5.6 Errors in the area model

Where there was more than two numbers given, some learners struggled with choosing the appropriate figures (Fig. 5.7). A good example is question 2 shown below.

Question 2: Erasers come packed in boxes of 28. Coco was carrying 20 boxes of erasers when she sneezed and dropped 14 boxes. All the erasers in the 14 boxes fell out. How many erasers bounced across the floor?

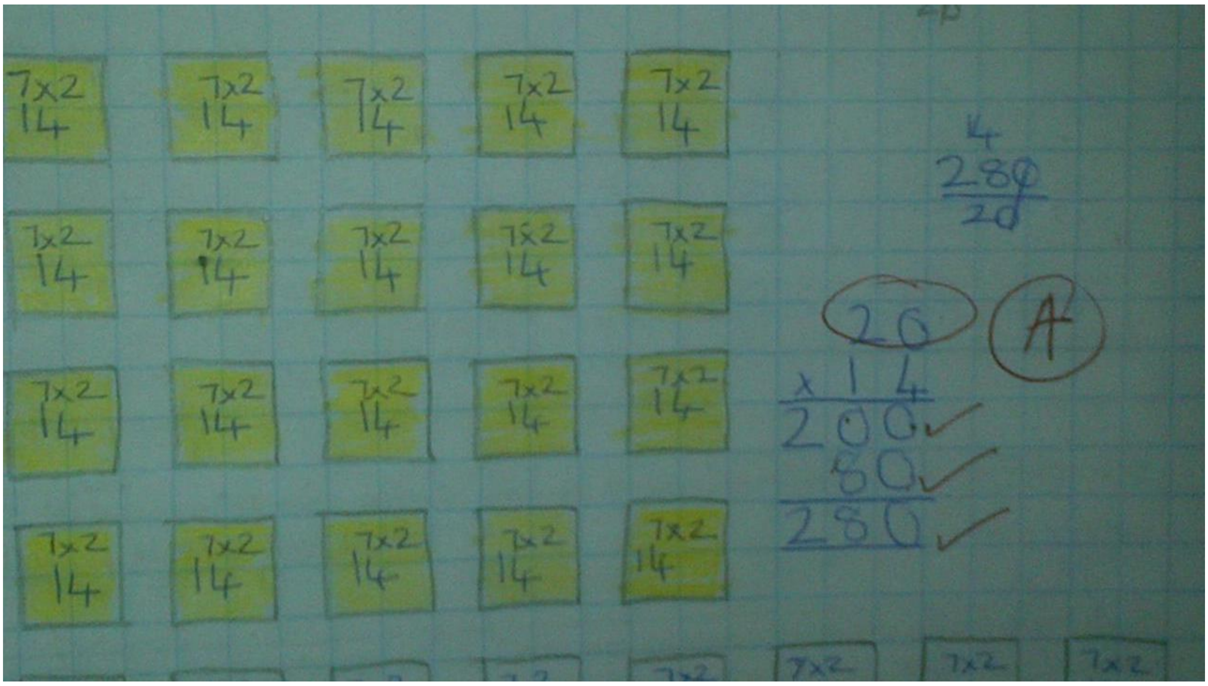


Fig. 5.7

The selection of inappropriate figures (horizontal mathematization) was quite pronounced, especially with questions 2, 5, 6 and 7. Hence, learners resorted to ‘number grabbing’ rather than making sense of the contextual problem (Fig. 5.7). There could be various reasons for learners to perform the way they have done. It could be that learners are struggling with language, learners’ not paying attention to detail or some of them could be rushing through questions without understanding fully what the question requires them to do. However, in most of learners’ solutions (pre-test) problems occurred in both horizontal and vertical mathematization. With horizontal mathematization, the problem was at the level of selection of inappropriate numbers, whilst with vertical mathematization, the problem was at the level of errors (inaccurate calculations). It should be noted that as a result of intervention lessons,the problem of selecting inappropriate numbers was grealty reduced. Out of the 10 incorrect responses in the pre-test, only one was associated with this problem. However, errors with regard to inaccurate calculations were still prevalent in the post-test (see Table 5,6)

As I was marking learner work from lesson 1, there were presentations that reflected learner thinking beyond just dealing with whole numbers. One example was demonstrated by learner 2 in the process of working out question 3 (150g x 48).

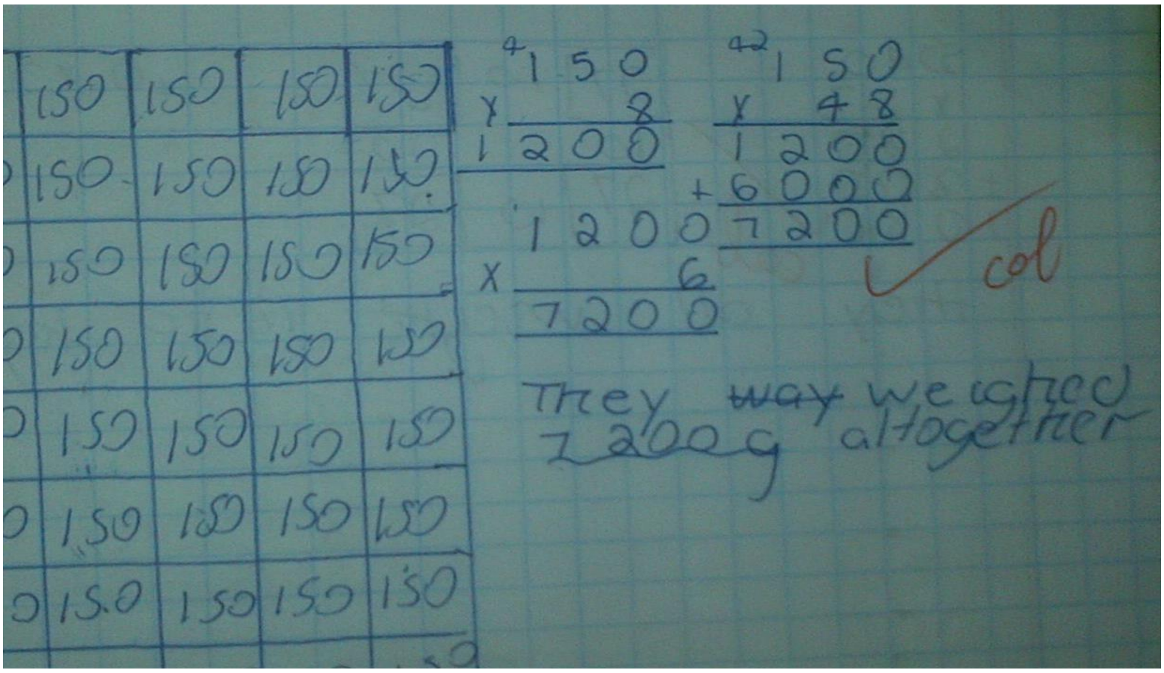


Fig. 5.8 Use of representation (model) to break down the number into factors

Table 5.6 Summary Table of models for lesson 1 (Repeated Addition)

MODEL	CORRECT	INCORRECT	TOTAL
Question = 1			
Column	14 (56%)	11 (44 %)	25 (64.1 %)
Area Mo.	5 (50%)	5 (50 %)	10 (25.6 %)
Doubling	1 (50%)	1 (50 %)	2 (5.1 %)
No working	0	2	2 (5.1 %)
Total	20 (51%)	19 (49 %)	(n = 39)
Question = 2			
Column	16 (*7) (59 %)	11 (41 %)	27 (69.2 %)
Area Mo.	4 (57 %)	3 (43 %)	7 (17.9 %)
Doubling	0 (0 %)	1 (100 %)	1 (2.6 %)
No working	1	3	4 (10.3 %)
Total	21 (53.8 %)	18 (46.2 %)	(n = 39)
Question = 3			
Column	20 (60.6 %)	13 (39.4 %)	33 (84.6 %)
Area Mo.	2 (66. 7 %)	1 (33.3 %)	3 (7.7 %)
No working	0	3	3 (7.7 %)
Total	22 (56.4 %)	17 (43.6 %)	(n = 39)
Question = 4			
Column	14 (63.6 %)	8 (36.4 %)	22 (56.4 %)
Area Mo.	9 (90 %)	1 (10 %)	10 (25.6 %)
Doubling	1 (100 %)	0	1 (2.6 %)
No working	2	2	6 (15.4 %)
Total	26 (66.7 %)	11 (29.7 %)	(n = 39)
Question 5			
Column	6 (*3) (26 %)	17 (74 %)	23 (59 %)
Area Mo.	11 (*8) (91.7 %)	1 (8.3 %)	12 (30.8 %)
Doubling	0 (0 %)	2 (100 %)	2 (5.1 %)
No working	0 (0 %)	2 (100 %)	2 (5.1 %)
Total	6 (15.4 %)	33 (84.6 %)	(39 = 39)
Question = 6			
Column	11 (*2) (50 %)	11 (50 %)	22 (56.4 %)
Area Mo.	8 (*3) (61.5 %)	5 (38.5 %)	13 (33.3 %)
Doubling	1 (50 %)	1 (50 %)	2 (5.1 %)
No working	0	2 (100 %)	2 (5.1 %)
Total	15 (38.5 %)	24 (61.5 %)	(n = 39)
Question = 7			
Column	12 (*3) (54.5 %)	10 (45.5 %)	22 (56.4 %)
Area Mod.	4 (*1) (33.3 %)	8 (66.7 %)	12 (30.8%)
Doubling	2 (100%)	0 (0 %)	2 (5.1 %)
No working	0	3 (100 %)	3 (7.7 %)
Total	14 (35.9 %)	25 (64.1 %)	(n = 39)

The learner made use of the resulting diagram (model) to break down 48 into its factors by counting the unit squares on each strip. In his explanation at the beginning of lesson 2, he says he realised that each vertical strip had a total weight equivalent to (150g x 8) 1200g. Since there were 6 strips, then he needed to multiply the answer to one strip by 6 (1200g x 6 = 7200). After a whole class discussion on this question learners came to realise that the sum (150 x 48) can also be expressed differently (150 x 8 x 6 or 15 x 10 x 48 or 15 x 10 x 8 x 6). They became aware that sometimes breaking down the numbers makes your work a little bit easier compared to tackling big whole numbers as it were.

Table 5.7 Detailed analyses of models and strategies for lesson 1 (R. Add)

	STRATEGY	Qn. 1	Qn. 2	Qn. 3	Qn. 4	Qn. 5	Qn. 6	Qn.7	Total	%
COLM.	R-L (correct)	11	15 (*7)	19 (*1)	11	6 (*2)	9 (*1)	10 (*2)	81	(46.6%)
COLM.	R-L (incorrect)	10	10	12	8	15	11	9	75	(43.1%)
COLM.	L-R (correct)	3	1	1	3	0	2 (*1)	2 (*1)	12	(6.9 %)
COLM.	L-R (incorrect)	1	1	1	0	2	0	1	6	(3.4%)
	Total (39 Lnr)	25 (64.1%)	27 (69.2%)	33 (84.6%)	22 (56.4%)	23 (59 %)	22 (56.4%)	22 (56.4%)	Aver. (63.7%)	
AREA MO.	Mental (correct)	5	4 (*2)	2	9	11	8 (*3)	4	43	(64.2%)
AREA MO.	Mental (incorrect)	5	3	1	1	1	5	8	24	(35.8%)
	Total (39 Lnr)	10 (25.6%)	7 (17.9%)	3 (7.7%)	10 (25.6%)	12 (30.8%)	13 (33.3%)	12 (30.8%)	Aver. (24.5%)	
DBLNG	Correct	1	0	0	1	0	1 (*1)	2	5	(50%)
DBLNG	Incorrect	1	1	0	0	2	1	0	5	(50%)
	Total (39 Lnr)	2 (5.1 %)	1 (2.6%)	0 (0 %)	1 (2.6%)	2 (5.1 %)	2 (5.1 %)	2 (5.1 %)	Aver. (3.7%)	

(*n): These are correct responses in terms of strategies but they are wrong due to the use of inappropriate (horizontal mathematization) figures.

Comparing the use of models in lesson 1, with the use of models in the pre-test reveals slight shifts especially in the area model (from 18.6 % to 24.5 %) and doubling model (from 10.8 % to 3.7 %), with use of the column model remaining largely unchanged (62.8 % to 63.6 %).

5.8 Lesson 2: Results and discussion (multiplication as rate)

In the first lesson, there was evidence to show that some leaners (2) had also focused on the actual numbers involved. Problems here indicated a lack of fluency with basic times tables and multiples of 2, 5 and 10. I felt that it was therefore necessary tobuild this fluency before proceeding with lesson 2 problems. This was my concern because I felt that learners' manipulation of number (number skills) could be harnessed in order to make multiplication easier. In relation to this I was also aware that I had to conduct my lessons in what Gravemeijer (2002, p. 146) calls a "bottom-up manner," whereby learners actively construct mathematical meanings through the process of "mathematization."

My focus was on developing meaning through models whereby the starting point of solving problems lies in these informal mathematical activities of constructing models by the learners. These in turn would support and enhance learners' informal solution strategies. Hence, before I introduced learners to the three examples of lesson two, I dedicated some time on the example that I discussed above (Fig 5.8) and moved on to recitation of times tables of the friendly or round numbers (2, 5, 10, 100, etc.). Anghileri (2007) suggests that "known" or "recalled facts" related to multiplication can be used with strategic thinking to derive new facts. I was, therefore, hoping that this practice of early multiplication would be useful within multiple-based strategies that could come into play through breaking down of numbers into friendly numbers. Below (Table 5.8) are the results of models and strategies used in lesson 2. Two new models emerged during the second lesson in interaction, namely the number line and the distributive model.

Table 5.8: Analysis of models and strategies for lesson 2 (Rate problems), (n = 38).

Model	STRATEGY	Qn. 1	Qn. 2	Qn. 3	Qn. 4	Qn. 5	Qn. 6	Total	%
	R-L (correct)	10	11	10 (*1)	6 (*1)	5	6 (*1)	48	59.2 %
COLM.	R-L (incorrect)	2	1	5	7	8	8	31	38.3 %
COLM.	L-R (correct)	0	0	0	1	1	0	2	2.5 %
COLM.	L-R (incorrect)	0	0	0	0	0	0	0	0
	Total (38 Lnrs)	12 31.6 %	12 31.6 %	15 40 %	14 36.8 %	14 36.8 %	14 36.8 %	81 Avr. = 35.6%	
AREA MO.	Mental (correct)	5	4	6	4	5	6	30	81.1 %
AREA MO.	Mental (incorrect)	0	2	1	3	1	0	7	18.9 %
	Total (38 Lnrs)	5 13 %	6 15.8 %	7 18.4 %	7 18.4 %	6 15.5 %	6 15.5 %	37 Avr = 16.2 %	
DBLNG	Add pairs (correct)	9	6	8	6 (*2)	3	7	39	67.2 %
DBLNG	Add pairs (incorrect)	1	5	0	3	7	3	19	32.8%
	Total (38 Lnrs)	10 26.3 %	11 28.9 %	8 21.1 %	9 23.7 %	10 26.3 %	10 26.3 %	58 Avr. = 25.4 %	
DSTRB	(Correct)	6	4	4	4 (*3)	5	6	29	90.6 %
DSTRB	(Correct)	0	0	0	2	1	0	3	9.4 %
	Total (38 Lnrs)	6 15.8 %	4 10.1 %	4 10.1 %	6 15.8 %	6 15.8 %	6 15.8 %	32 Avr. = 14 %	
N-Lne Dble	(Correct)	3	4	3	2 (*1)	0	0	12	63.2 %
N-Lne Dble	(Incorrect)	1	1	1	0	2	2	7	36.8 %
	Total (38 Lnrs)	4 10.1 %	5 13.2 %	4 10.1 %	2 5.3 %	2 5.3 %	2 5.3 %	19 Avr = 8.3 %	

Number 1: Putting a pear into the magic cupboard for a month turns it into 7 flies**.** Marge picked 18 pears and put them in her cupboard for a month. How many flies did she find a month later?

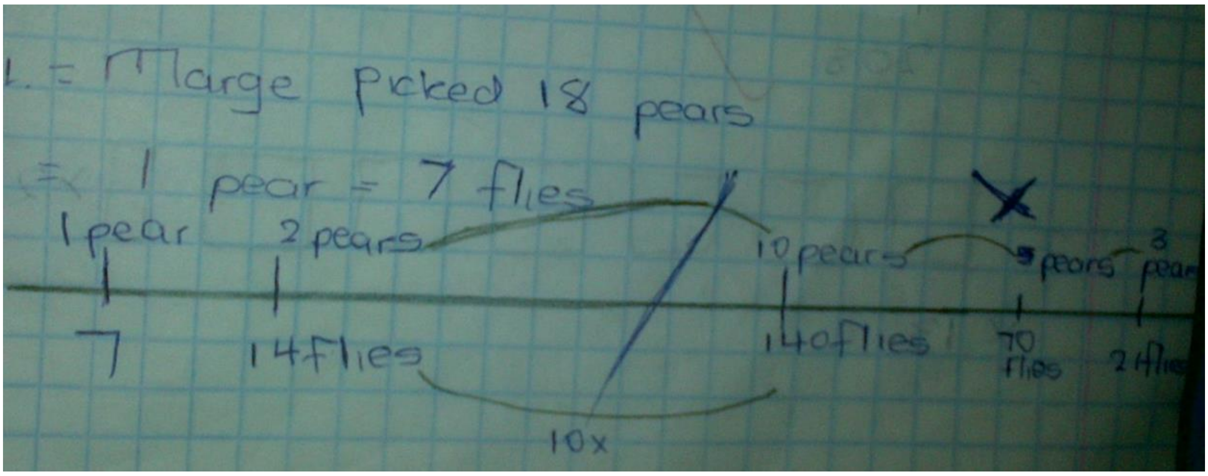


Fig. 5.9 Learner no.36, lesson 2 (number line model).

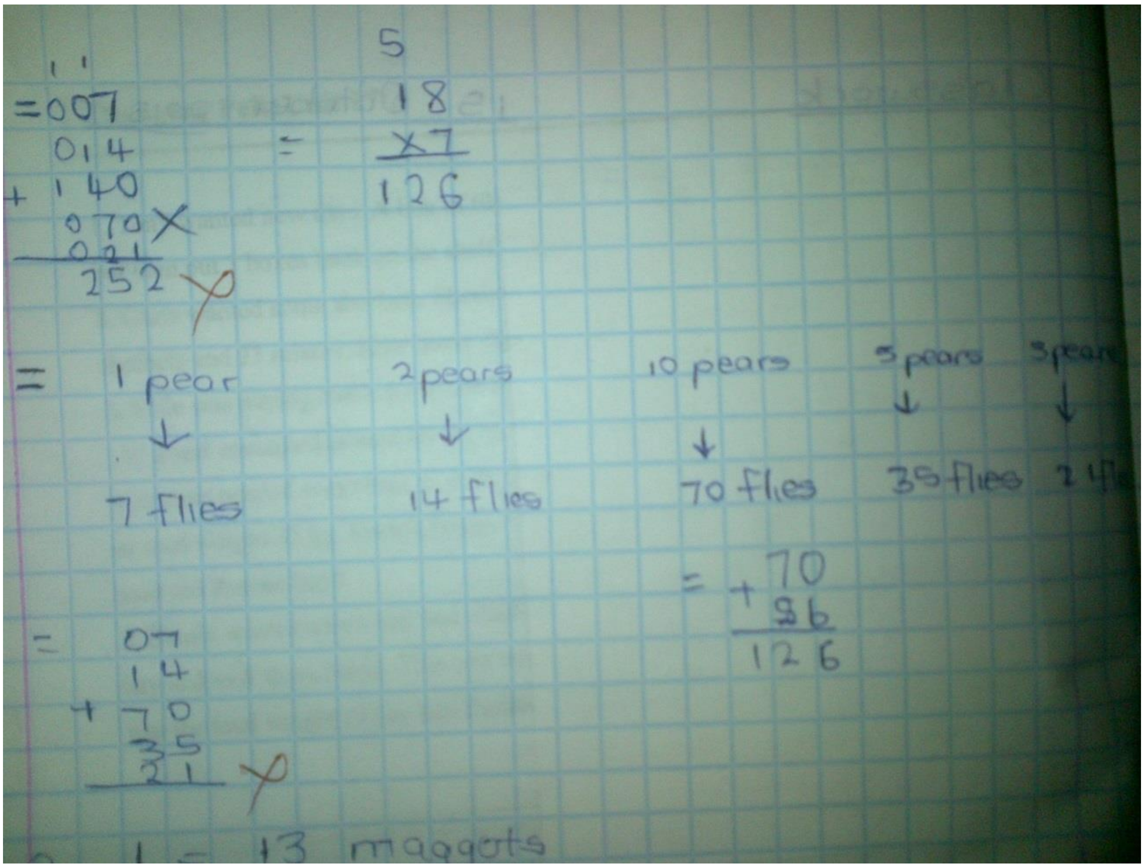


Fig. 5.10 Learner no.36, lesson 2 (example 1) – number line model.

The above example was used for discussion. From the example, an informal number line emerged (horizontal mathematization), even though the numbers were not placed in order. The first two pairings (1; 7) and (2; 14) reflected the corresponding number of flies per pear. In a bid to take advantage of a friendly number, the learner attempted to note down the number of flies for 10 pears which he wrote down as 14 x 10 = 140 (first model), instead of 7 x 10 = 70. However, in the second model, the learner realized his mistake and he corrected it. The learner might have used the pairings (1;7) and (2;14) to obtain 21 (7 + 14) or it could have been a prduct of 3 and 7. Looking at the last 3 pairings, it is clear that the learner was focusing on breaking down 18 into 10 + 5 +3 and added the corresponding number of flies to obtain the total number of flies (70 + 56 = 126). Although he added correctly during the discussion (right), he had erred in the first attempt where he added 07 + 14 + 70 + 35 + 21 (left side). This means that he was adding flies from 21 pears instead of 18. During this discussion, the number line was introduced as a model for multiplication, and a few learners (4) adopted it during the class and homework activities. This example confirms Anghileri's (2007) assertion in the literature that says recalled facts can be used with strategic thinking to derive new facts.

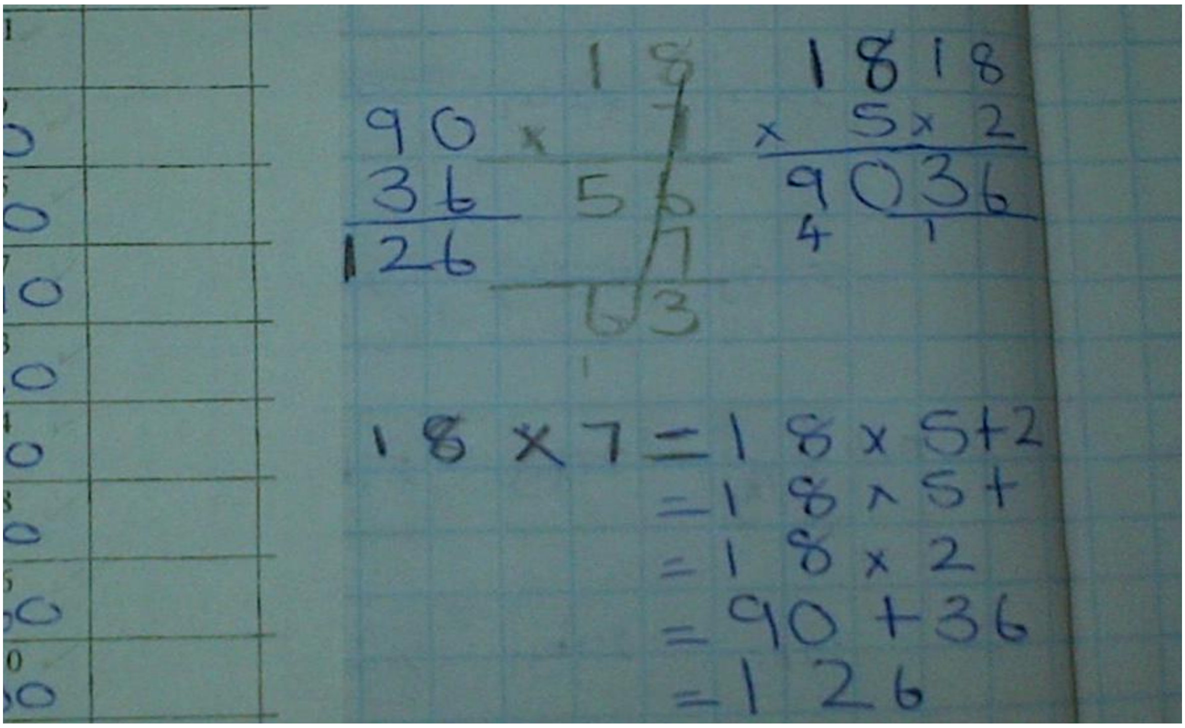


Fig. 5.11 Learner no. 27 (Distributive model)

Learner 27 had a different approach for the same question (Fig. 5.11). However, it is the way the sum was arranged or presented that provoked a lot of debate. The above example also shows two stages, an indication that the learner was not really happy with her initial layout (horizontal mathematization) of her working. The answer is supposed to be written as 90 + 36 but it is written as 9036, yet on the side a correct working is shown, an indication that the learner had made sense of her emergent model but needed to modify her way of laying out or presenting the solution strategy (vertical mathematization). Whilst discussing this example, several learners began to appreciate it more because it became familiar as it was discussed and modified with regard to how it can be better laid out. We had used the distributive model during the Annual National Assessment (ANA) revision period before the learners wrote the ANA examinations. Although I had not formally taught this model, it was included in the exemplar papers that had been delivered to schools earlier on before the ANA examinations. Hence, a few learners made use of it during class and homework activities immediately after it was introduced during class discussions.

5.9 Lesson 3: Results and discussion (multiplication as scaling)

In lesson 3, the number of learners using the column model continued to drop further down, from an average of 12 in lesson two, to an average of 7 in lesson three. In addition to this, the learners who were using the strategy of beginning from left to right when multiplying no longer used this approach or they have adopted other models altogether. The learners using the area model had been consistent between the two lessons (2 and 3). However, those using the doubling model dropped once again from an average of 10 in the second lesson to an average of 6 in lesson 3. These changes from one model to the other are likely to be influenced by the fact that learners are exploring emergent models in order to find the one that is more efficient and they understand better.

There was a sharp rise in the use of distributive model which could be associated with the fact that learners' familiarity with the model from the ANA papers. Number line model use remained steady, with a maximum of three learners using it at the end of the third lesson. One new model, the completing model, was used by one learner (number 21) in the third lesson. See summary below.

Table 5.9: Analysis of models and strategies for lesson 3 (Scaling problems)

Model	Strategy	Qn.1	Qn.2	Qn.3	Qn.4	Qn.5	Qn.6	Qn.7	Total	%
COLM.	R-L (correct)	7	10	7	3	4	6	9	46	70.8 %
COLM.	R-L (incorrect)	0	2	0	3	5	3	6	19	29.2 %
	Total (40 Lnrs)	7 17.5 %	12 30 %	7 17.5 %	6 15 %	9 22.5 %	9 22.5 %	%	65/280 Avr. = 23.2 %	
AREA MO.	Mental (correct)	6	4	6	5	4	7	5	37	78.7 %
AREA MO.	Mental (incorrect)	0	0	0	3	1	4	2	10	21.3 %
	Total (40 Lnrs)	6 15 %	4 10 %	6 15 %	8 20 %	5 12.5 %	11 11.5 %	%	47/280 Avr = 16.8 %	
DBLNG	Add pairs (correct)	9	6	9	5	2	0	3	34	85 %
DBLNG	Add pairs (incorrect)	0	0	0	3	1	1	1	6	15 %
	Total (40 Lnrs)	9 22.5 %	6 15 %	9 22.5 %	8 20 %	3 7.5 %	1 2.5 %	%	40/280 Avr. = 14.3 %	
DSTRB	(Correct)	9	11	9	5	8	9	9	60	63.2 %
DSTRB	(Correct)	6	2	6	7	6	5	3	35	36.8 %
	Total (40 Lnrs)	15 37.5 %	13 32.5 %	15 37.5 %	12 30 %	14 35 %	14 35 %	%	95/280 Avr. =33.9 %	
N-Lne Dble	(Correct)	3	3	3	1	1	0	0	11	91.7 %
N-Lne Dble	(Incorrect)	0	0	0	0	0	0	1	1	8.3 %
	Total (40 Lnrs)	3 7.5 %	3 7.5 %	3 7.5 %	1 2.5 %	1 2.5 %	0 0 %	%	12/280 Avr =4.3 %	
Comp	(Correct)	-	1	-	0	2	0	1	4	80 %
Comp	(Incorrect)	-	0	-	0	1	0	0	1	20 %
	Total (40 Lnrs)	0 0 %	1 2.5 %	0 0 %	0 0 %	3 2.5 %	0 0 %	%	5/280 Avr =1.8 %	

.

Lesson 3, example 1: Putting a pear into the magic cupboard for a month turns it into 7 flies. Marge picked 18 pears and put them in her cupboard. How many flies did she find a month later?

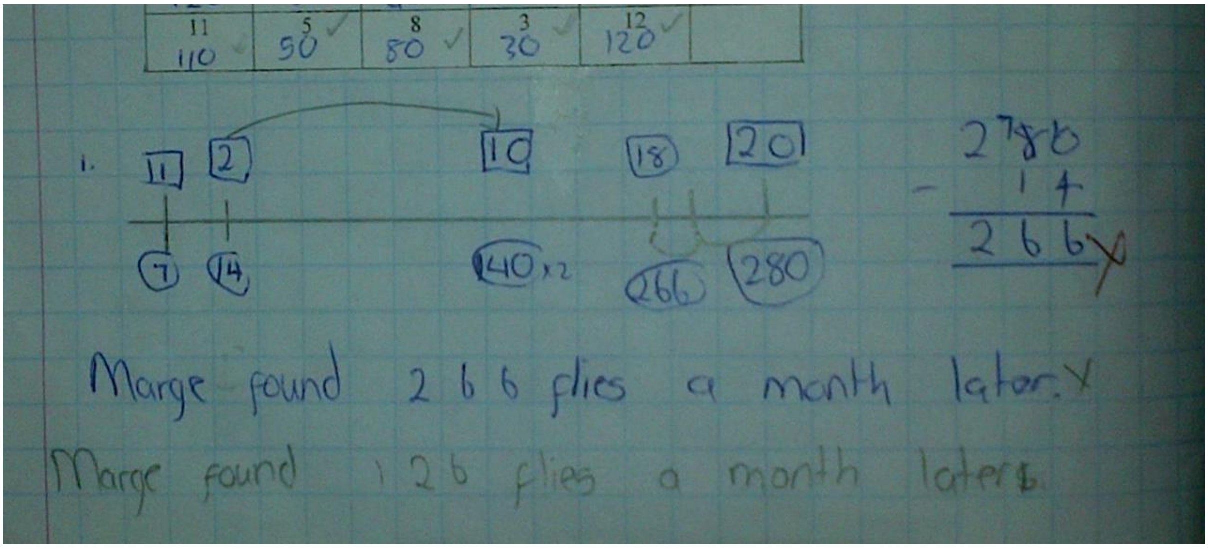


Fig. 5.12: Number line and the completion models

In this example the learner was focused on using round or friendly numbers. She began with first two pairs, (1;7) and (2;14) and then skipped some of the figures to get 10 pears. However, the big jump started from 2 instead of 1. As a result she multiplied 14 by 10 instead of 7 by 10. She further doubled 140 to obtain 280 from 20 pears. Hence, the initial mistake led to an incorrect answer at the end (266). Two models are involved in this example, the number line and the completing model (20 x 7) – (20 x 2). This shows that error is at the level of the strategy (vertical mathematization) as opposed to level of the model. I specifically chose this example because it was used for a class discussion at the beginning of the third lesson, which led to the introduction of the completing model. I had missed it during the second lesson, only to pick it up during marking. Following this discussion two learners tried to make use of this model.

Question 1, lesson 3: Jen woke to find some visitors in her room. Starjen looks a bit like Jen, but was 38 times as old as Jen. Jen is 17 years old. How old is Starjen?

The two numbers, 38 and 17 can easily be rounded to the nearest friendly numbers (40 and 20). Learner 35’s choice of 20 was appropriate since it is associated with doubling. However, his calculation of 38 x 3 = 115 was inaccurate. Such errors kept cropping up in several learners’ work, indicating a lack of fluency with more basic multiplication facts.

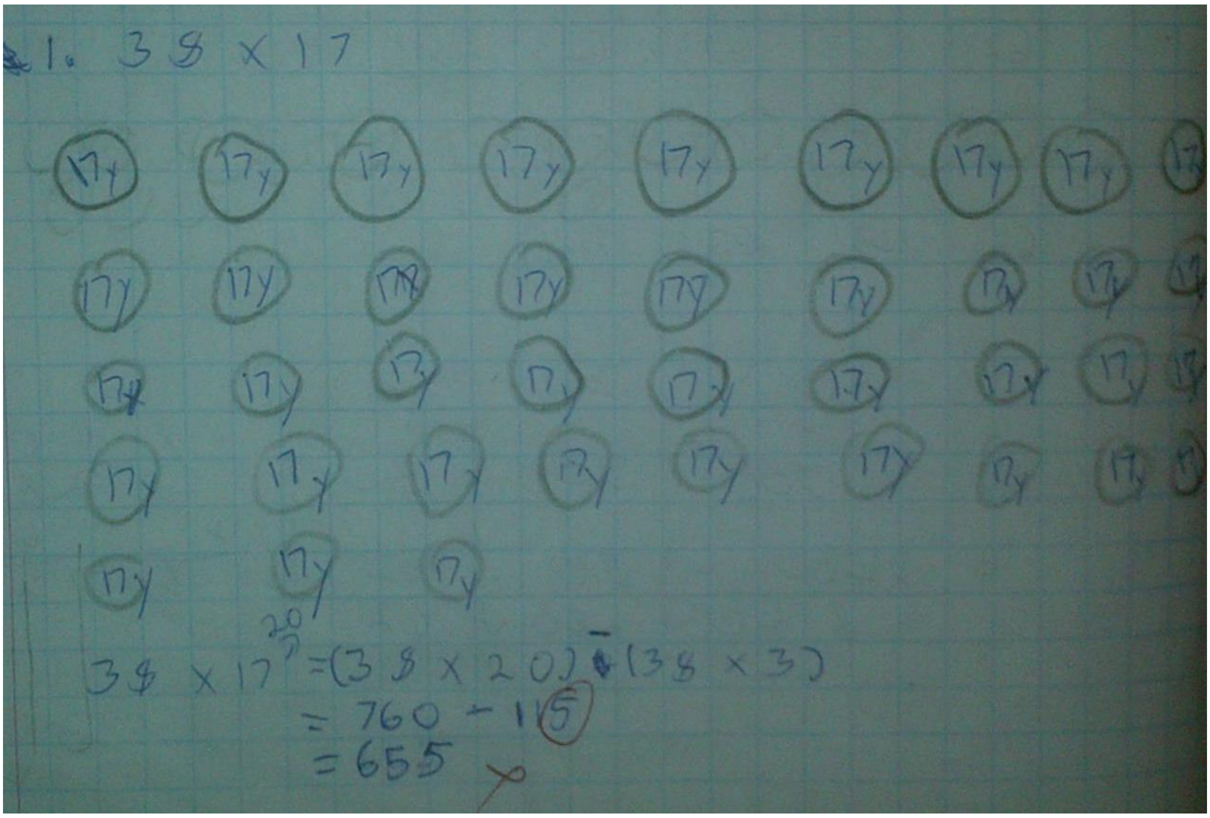


Fig. 5.13: Learner 35 (Completing model)

The summary of the three intervention lessons in terms of learner's success rate indicate that the learners performed better on multiplication as lessons progressed and performance within each model improved across the lessons (Table 5.10).

Table 5.10: Comparison of the use of models and their success rate in the first 3 lessons

	Lesson 1	**Success**	**Lesson 2**	**Success**	**Lesson 3**	**Success**
Model	**R. Ad.**		**Rate**		**Scaling**	
Column	46.9 %	**60.9 %**	35.6 %	**61.7 %**	23.2 %	**70.8 %**
Area model	25.7 %	**65.7 %**	16.1 %	**81.1 %**	16.8 %	**78.7 %**
Doubling	3.7 %	**50 %**	25.4 %	**67.2 %**	14.3 %	**85 %**
Distributive	—	—	13.9 %	**90.6 %**	33.9 %	**63.2 %**
N- Line	—	—	08.2 %	**63.2 %**	04.3 %	**91.7 %**
Completing	—	—	—	—	01.8 %	**80 %**
		58.9 %		**72.8 %**		**77.9 %**

These figures cannot give a true reflection of how the learners performed per model that they used because in some models like the number line and the completing model, the numbers of learners using that particular model is so small that it becomes difficult to compare.

Table 5.11: Planned schedule for lessons 3,4 and 6

PLANNED INTERVENTION SCHEDULE (lesson 4, 5 and 6).		
Lesson 4 (week 4)	Lesson 5 (week 5)	Lesson 6 (week 6)
Multiplication as repeated addition & multiplication as rate.	Multiplication as repeated addition & multiplication as scaling.	Multiplication as rate & multiplication as scaling.

5.10 Lesson 4: Results and discussion (repeated addition and rate)

The general performance of learners in lesson 4 was surprisingly poor (roughly 51 %). This unimpressive performance might be associated with the fact that most of the problems included superfluous information and they all involved 2-digit by 2-digit multiplication. A comparison of questions 1 (with superfluous information) and 2 illustrates that superfluous information might have been a source of learners' underperfomance. Only 15 learners out of 39 were successful in question 1, whilst 32 learners were successful in question 2.The performance patterns revealed that learners' solutions displayed that they struggled more with multiplication as repeated addition questions compared to those of multiplication as rate. At this stage, I observed that more learners were beginning to rely on the distributive model, yet some of them were making errors on calculation strategies.

Table 5.13: Summary of lesson 4 results (R. addition and rate problems).

	STRATEGY	Qn. 1 R. Add.	Qn. 2 Rate	Qn.3 R. Add.	Qn. 4 Rate	Qn. 5 R. Add.	Qn. 6 R. Add.	Qn.7 Rate	Total	%
COLM.	R-L (correct)	4	7	8	5	7	6	7	44	42.3 %
COLM.	R-L (incorrect)	12	1	9	12	9	9	8	60	57.7 %
	Total (39 Lnrs)	16 41.1 %	8 20.5 %	17 43.6 %	17 43.6 %	16 41.1 %	15 38.5 %	15 38.5 %	104/273 Avr. = 38.1 %	
AREA MO.	Mental (correct)	5	8	3	1	6	6	10	39	67.2 %
AREA MO.	Mental (incorrect)	2	0	4	6	3	1	3	19	32.8 %
	Total (39 Lnrs)	7 17.9 %	8 20.5 %	7 17.9 %	7 17.9 %	9 23.1 %	7 17.9 %	13 33.3 %	58/273 Avr =21.2 %	
DBLNG	Add pairs (correct)	1	2	1	0	1	0	0	5	26.3 %
DBLNG	Add pairs (incorrect)	1	2	1	2	1	6	1	14	73.7 %
	Total (39 Lnrs)	2 5.1 %	4 10.3 %	2 5.1 %	2 5.1 %	2 5.1 %	6 15.4 %	1 2. 6 %	19/273 Avr. = 7%	
DSTRB	(Correct)	2	7	4	4	1	4	2	24	43.6 %
DSTRB	(Incorrect)	6	2	6	3	6	5	3	31	56.4 %
	Total (39 Lnrs)	8 20.5 %	9 23.1 %	9 23.1 %	7 17.9 %	7 17.9 %	9 23.1 %	5 12.8 %	55/273 Avr. = 20.1%	
N-Lne Dble	(Correct)	2	5	1	1	1	0	0	10	55.6 %
N-Lne Dble	(Incorrect)	2	1	0	3	1	1	0	8	44.4 %
	Total (39 Lnrs)	4 10.3 %	6 15.4 %	1 2.6 %	4 10.3 %	2 5.1 %	1 2.6 %	0 %	18/273 Avr = 6.6 %	
Comp	(Correct)	1	3	1	1	2	0	4	12	66.7 %
Comp	(Incorrect)	2	1	0	1	1	1	0	6	33.3 %
	Total (39 Lnrs)	3 7.7 %	4 10.3 %	1 2.6 %	2 5.1 %	3 7.7 %	1 2.6 %	4 10.3 %	18/273 Avr = 6.6 %	

Table 5.12: Performance on the two classes of multiplication (rep. addition and rate).

MODEL	RATE	REPEATED ADDITION
Column	50 %	50 %
Area	60.9 %	39.1 %
Doubling	63.9 %	36.1 %
Distributive	60.7 %	39.3 %
Number line	66.7 %	33.3 %
Completing	72.7 %	27.3 %
Average	62.5 %	37.5 %

The percentages for the use of each model on the two classes of multiplication take into account the fact that there were less rate questions compared to those of repeated addition (4 multiplications as repeated addition and 3 multiplications as rate questions).

An important concern for me was the errors associated with the distributive model.

Lesson 4, Question 7: The blood-chiller has 33 rows of seats for people to ride in. Each row has 28 seats. You have to be over 146 cm tall to ride the Blood-chiller. How many people can ride on the Blood-chiller when it is full?

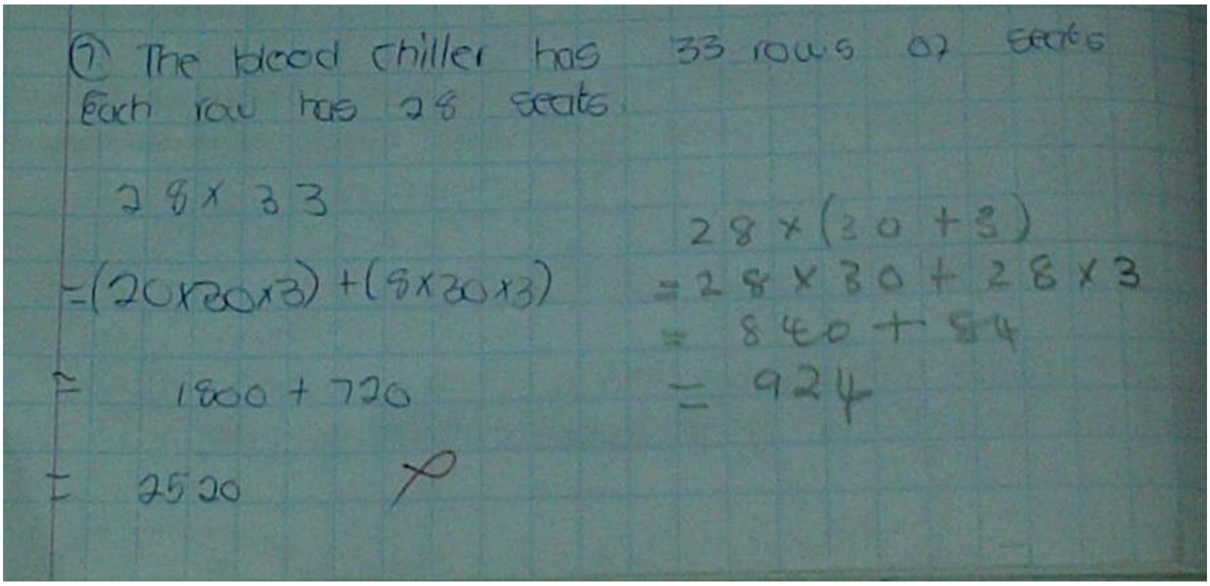


Fig. 5.14: Learner 37 (lesson 4, question 7).

The above example demonstrates that the learner has not clearly grasped the distributive property. Hence, the distributive model does pose challenges for learners, especially when learners split both numbers incorrectly and begin to use inappropriate signs. However, there were a few learners who managed to work out answers accurately.

The above observation takes me back to what was alluded to in the litaerature. Askew (2012), suggests that only a few models need to be introduced to learners at a given time. “We are better off introducing children to a small number of models and working intensively with these over time.” (Askew, 2012, p. 109). The point to note here is that learners need time to gain deeper understanding with respect to the introduced modules.

5.11: Lesson 5: Results and discussion (Scaling and Repeated addition)

Lesson 5 was a combination of multiplication as repeated addition and multiplication as scaling. By splitting the correct responses into two categories of scaling and repeated addition, it is clear that multiplication as scaling problems were better responded to than the repeated addition questions. The doubling and number line models were the only models that produced better responses with respect to multiplication as repeated addition (√). The rest of the models, column, area model, distributive and the completing model show that learners performed better in the multiplication by scaling problems.

Table 5.14: Comparison of usage of models and learners’ performance in lesson 5

MODEL	% USE OF THE MODEL	SCALING	R. - ADDITION	TOTAL % OF CORRECT RESP.
Column	22.8 %	40.7 %	28.8 %	69.5 %
Area model	28.6 %	51.4 %	28.4 %	79.8 %
Doubling	10.0 %	50.0 %	42.3 % √	92.3 %
Distributive	25.9 %	50.7 %	31.3 %	82.0 %
N- Line	04.2 %	36.4 %	63.6 % √	100 %
Completing	08.5 %	68.2 %	22.7 %	90.9 %

.

Table 5.15: Analyses of models and strategies for Lesson 5 (Scaling and R. Addition).

Model	STRATEGY	Qn. 1o Scaling	Qn. 2 R. Add.	Qn. 3 Scaling	Qn. 4 R. Add.	Qn. 5 Scaling	Qn. 6 R. Add.	Qn.7 Scaling	Total	%
COLM.	R-L (correct)	4	4	7	6	6	7	7	41	69.5 %
COLM.	R-L (incorrect)	3	3	1	1	2	7	1	18	30.5 %
	Total (37 Lnrs)	7 18.9 %	7 18.9 %	8 21.6 %	7 18.9 %	8 21.6 %	14 37.8 %	8 21.6 %	59/259 Avr. = 22.8 %	
AREA MO.	Mental (correct)	6	11	11	9	11	1	10	59	79.7 %
AREA MO.	Mental (incorrect)	3	4	1	0	2	4	1	15	20.3 %
	Total (37 Lnrs)	9 24.3 %	15 40.5 %	12 32.4 %	9 24.3 %	13 35.1 %	5 13. 5 %	11 29.7 %	74/259 Avr = 28.6 %	
DBLNG	Add pairs (correct)	5	2	1	6	3	3	4	24	92.3 %
DBLNG	Add pairs (incorrect)	0	1	0	1	0	0	0	2	7.7 %
	Total (37 Lnrs)	5 13.5 %	3 8.1 %	1 2.7 %	7 18.9 %	3 8.1 %	3 8.1 %	4 10.8 %	26/259 Avr.= 10 %	
DSTRB	(Correct)	10	4	8	11	10	6	6	55	82.1 %
DSTRB	(Correct)	1	3	1	1	0	4	2	12	17.9 %
	Total (37 Lnrs)	11 29.7 %	7 18.9 %	9 24.3 %	12 32.4 %	10 27 %	10 27 %	8 21.6 %	67/259 Avr. = 25.9 %	
N-Lne Dble	(Correct)	1	3	3	2	0	2	0	11	100 %
N-Lne Dble	(Incorrect)	0	0	0	0	0	0	0	0	0 %
	Total (37 Lnrs)	1 2.7 %	3 8.1 %	3 8.1 %	2 5.4 %	0 %	2 5.4 %	0 %	11/259 Avr = 4.2 %	
Comp	(Correct)	4	2	3	0	2	3	6	20	90.9 %
Comp	(Incorrect)	1	0	1	0	0	0	0	2	9.1 %
	Total (37 Lnrs)	5 13.5 %	2 5.4 %	4 10.8 %	0 %	2 5.4 %	3 8.1 %	6 16.2 %	22/259 Avr = 8.5 %	

Most learners made use of the area model and the distributive models. The column model is no longer regarded as the standard model of multiplication as it was at the beginning of the intervention lessons. Learners selected from a wide range of choices unlike before. However, very few learners are using the number line and the completing models at this stage. See summary of lesson 5 (Table 5.15).

My observation in lesson 5 was that most of the incorrect responses that we dealt with in class discussion pointed to the fact that learners made calculation errors within strategies (vertical mathematization) and they were prone to making errors on partial products, indicating a lack of fluency. Below are examples that illustrate my observations.

Question 1, lesson 5: Elizabeth counted that there were 48 strawberries on her strawberry plant. 10 days later,there were 12 times as many strawberries on the plant. How many strawberries was that?

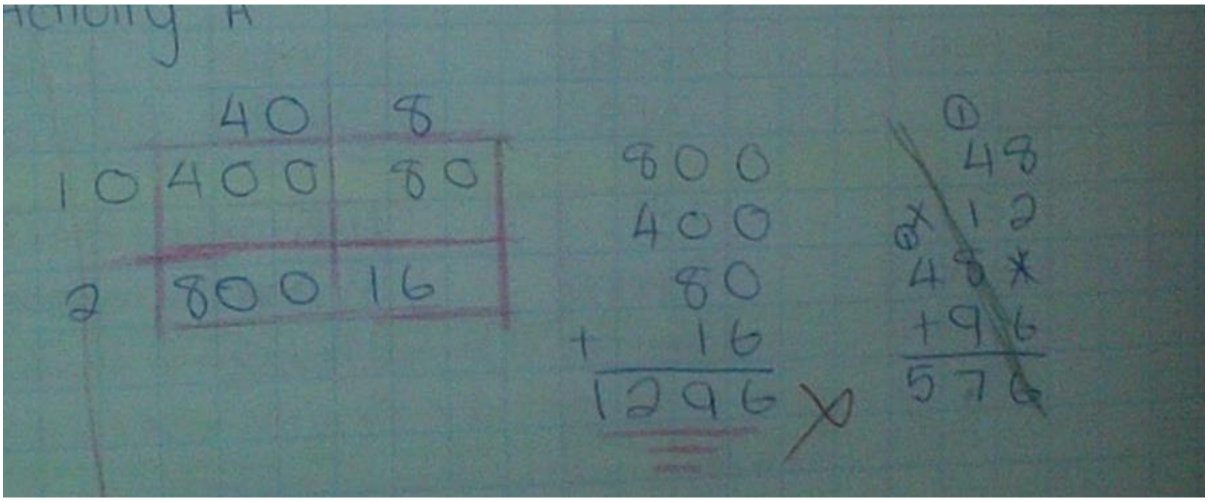


Fig.5.15 Learner 19, (errors in strategies)

I mentioned earlier on that the area model is associated with mental multiplication strategies (indicated by the absence of written working seen in partial products) when most learners are not yet fluent in their number sense. I realised that there was a need for me to insist on written working as one of the measures of trying to improve accuracy at the level of strategy. They are likely to have corrected these answers immediately. Inaccurate strategies were not associated with particular models but were widespread across all the models. See **Fig. 5.16** and **5.17** for learners 31 and 29 who obtained incorrect answers due to number sense related errors.

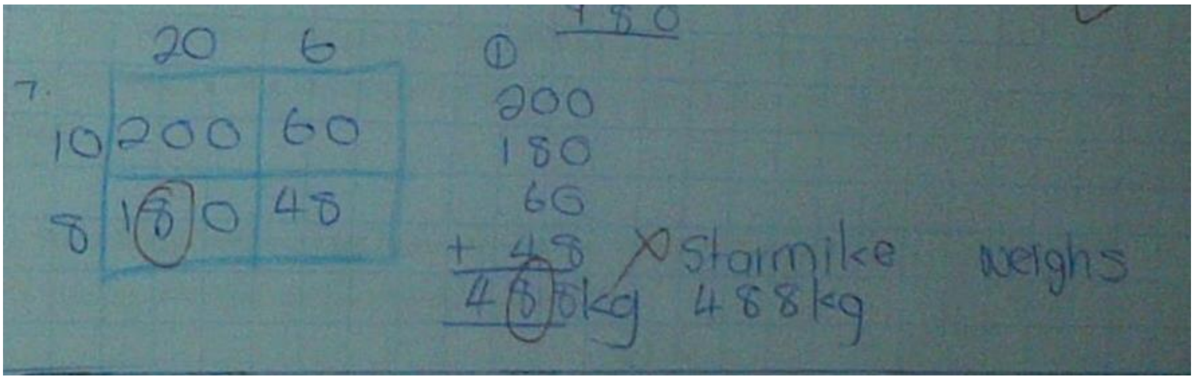


Fig. 5.16 Learner 31 (errors in strategies)

Question 7, lesson 5: Mike woke to find some visitors in his room. Starmike looked a bit like Mike, but was 18 times as heavy as Mike. Mike weighs 26 kg. How heavy is Starmike?

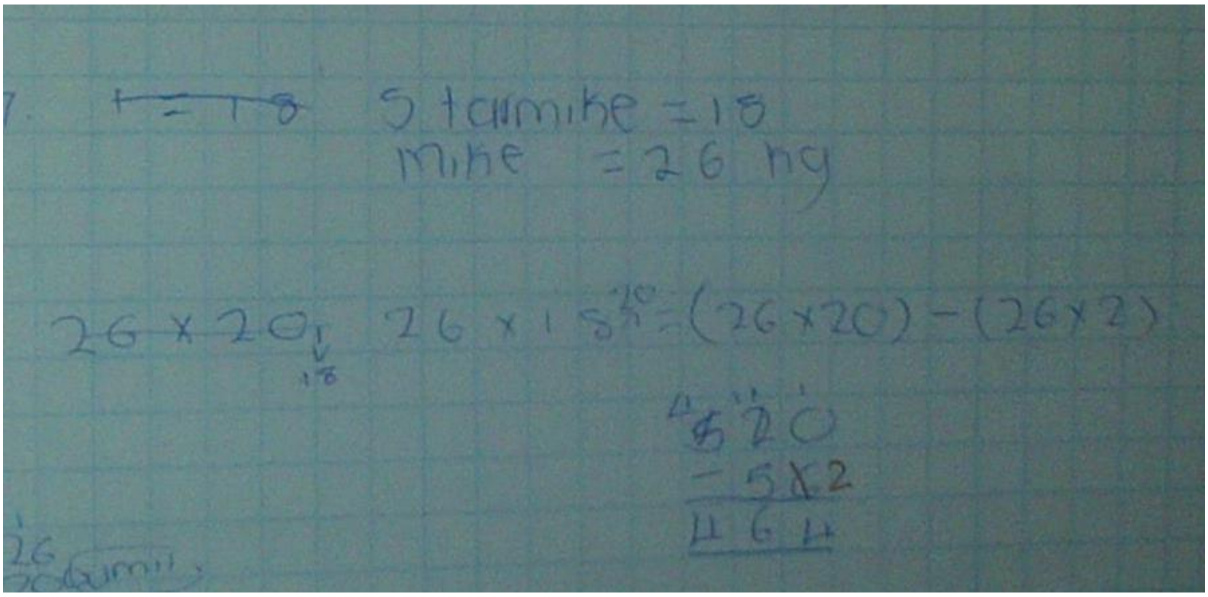


Fig. 5.17: Learner 29 (errors in strategies)

In Fig. 5.17, learner 29 used the completion model to work out 26 x 18. The transition from the problem to model was appropriately done. However, it is at the level of carrying out the strategy where the leaner falters (26 x 2 = 56) and she fails to realize her mistake. These are just a few of the errors that provide us with a glimpse of the bigger picture of where learners are struggling (errors at the level of strategy) mostly.

5.12 Lesson 6: Results and discussion (Rate and scaling)

Lesson 6 comprises of multiplication as rate and multiplication as scaling problems. This was the last lesson of the intervention program. The lesson came at a time when learners had written and completed their end of year examinations. Hence, absenteeism was a concern at this time of the year. Seven learners were absent on the day of the lesson, despite the fact that learners had been strongly advised to attend the entire intervention lessons. Analysis of the results indicate that the area model continues to attract more learners (31.6 %) compared to other models across both multiplication as rate and multiplication as scaling problems. Its success rate (93.2 %) is also higher than that of other models. However, it is also important to note that the success rate in the column model has also improved greatly (92.2%). This improvement may be associated with better understanding on the part of the learners due to repeated practice during intervention lessons. The number line, doubling and the completing models have continuously been used by very few learners. Hence, the percentages in these models cannot be used in making valid judgements with regard to their use or success rate.

In lesson 6, whole class discussions were focused more on working accurately on strategies. See Fig. 5.15 above. I emphasized written working as opposed to mental strategies that led to error in earlier lessons. The use of the column model dropped slightly from lesson 4 up to lesson 6. Two models, namely distributive and the area model gained more leaners, especially the area model (from 21.2 % to 27,1 %). The same can be said about the success rate in these three models. The column and the distributive model achieved higher success rates compared to the area model by the end of lesson 6. However, the area model has proved to be more consistent and reliable for those learners using it as is indicated by the average of the success rate. This consistency of the area model was also reflected by the first 3 lessons based on multiplication as repeated addition, multiplication as rate and multiplication as scaling.

Lesson 6, no. 3: Clark's leek was 18 cm long when she first measured it. 10 days later it was 15 times as long. How long was the leek then?

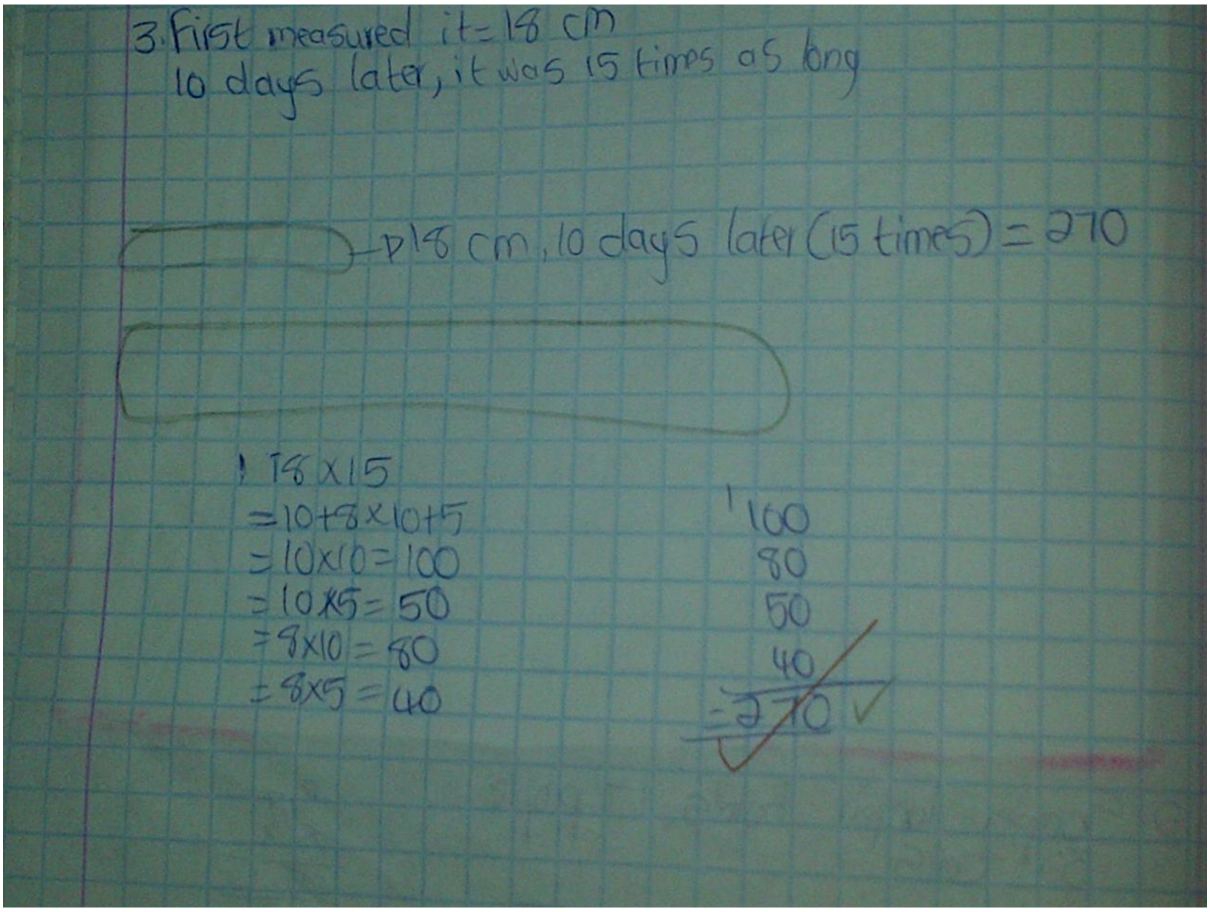


Fig. 5.18: Learner 9, (lesson 6 no. 3)

Taking a closer look at the column model, it can be seen that almost half the class used this model as a standard model of multiplication at the beginning of the intervention lessons. However, by the end of lesson three, a good number of learners had since moved on to adopt other models. The gradual rise in the success rate (column model) during lessons may be assigned to a wide range of factors. Either those learners who struggled with the column model have decided to abandon it and they have adopted other models or they have developed a better understanding due to repeated practice of multiplication. See models and strategies for lesson 6 (Table 5:16).

Table 5.16: Analysis of models and strategies for lesson 6 (rate and scaling)

	STRATEGY	Qn. 1	Qn. 2	Qn. 3	Qn. 4	Qn. 5	Qn. 6	Qn.7	Total	%
COLM.	R-L (correct)	6	12	7	7	6	10	11	59	92.2 %
COLM.	R-L (incorrect)	2	0	0	1	0	1	1	5	7.8 %
	Total (33 Lnrs)	8 24.2 %	12 36.4 %	7 21.2 %	8 24.2 %	6 18.2 %	11 33.3 %	12 36.4%	64/231 Avr. = 27.7 %	
AREA MO.	Mental (correct)	11	6	13	9	10	9	10	68	93.2 %
AREA MO.	Mental (incorrect)	0	0	0	2	2	0	1	5	6.8 %
	Total (33 Lnrs)	11 33.3 %	6 18.2 %	13 39.4 %	11 33.3 %	12 36.4 %	9 27.3 %	11 33.3%	73 Avr. = 31.6 %	
DBLNG	Add pairs (correct)	3	3	3	4	2	3	1	19	82.6 %
DBLNG	Add pairs (incorrect)	0	1	0	0	1	1	1	4	17.4 %
	Total (33 Lnrs)	3 9.1 %	4 12,1 %	3 9.1 %	4 12.1 %	3 9.1 %	4 12.1 %	2 6.1 %	23/231 Avr. = 10 %	
DSTRB	(Correct)	8	8	3	5	9	5	6	44	89.8 %
DSTRB	(Correct)	0	1	1	2	0	1	0	5	10.2 %
	Total (33 Lnrs)	8 24.2 %	9 27.3 %	4 12.1 %	7 21.2 %	9 27.3 %	6 18.2 %	6 18.2 %	49/231 Avr. = 21.2 %	
N-Lne Dble	(Correct)	0	0	2	0	0	0	0	2	50 %
N-Lne Dble	(Incorrect)	0	1	1	0	0	0	0	2	50 %
	Total (33 Lnrs)	0 0 %	1 3 %	3 9.1 %	0 0 %	0 %	0 0 %	0 0 %	4/231 Avr = 1.7 %	
Comp	(Correct)	2	1	3	3	2	2	2	15	83.3 %
Comp	(Incorrect)	1	0	0	0	1	1	0	3	16.7 %
	Total (33 Lnrs)	3 9.1 %	1 3 %	3 9.1 %	3 9.1 %	3 9.1 %	3 9.1 %	2 6.1 %	18/231 Avr. = 7.8 %	

The summary of the three intervention lessons in terms of learner's success rate indicate that the learners performed better on multiplication as lessons progressed and performance within each model improved across the lessons the three lessons.

Table 5.17: Comparison of the use of models and their success rate (lessons 4, 5 and 6).

	Lesson 4	**Success**	**Lesson 5**	**Success**	**Lesson 6**	**Success**		**Success**
Model	**Rat/ R.A**		**Scal/ R. A.**		**Rat/ Scal**		**Aver.**	
Column	38.1 %	**42.3 %**	22.8 %	**69.5 %**	27.7 %	**92.2 %**	29.5 %	**68 %**
Area model	21.2 %	**67.2 %**	28.6 %	**79.7 %**	31.6 %	**93.2 %**	27.1 %	**80.1 %**
Doubling	07.0 %	**26.3 %**	10.0 %	**92.3 %**	10.0 %	**82.6 %**	09.0 %	**67.1 %**
Distributive	20.1 %	**43.6 %**	25.9 %	**82.1 %**	21.2 %	**89.9 %**	22.4 %	**71.9 %**
N- Line	06.6 %	**55.6 %**	04.2 %	**100 %**	01.7 %	**50 %**	04.2 %	**68.5 %**
Completing	06.6 %	**66.7 %**	08.5 %	**90.9 %**	07.8 %	**83.3 %**	07.6 %	**80.3 %**

By the end of lesson six, 28 % of the learners were using the column model and the success rate had improved greatly (92 %). The same cannot be said about the area model. The first 3 lessons saw a slight drop in the use of this model, yet the success rate rose from about 65 % to 78 % at the end of the third lesson. However, its use began to shift during the last three lessons (4, 5 and 6) from around 20 % to roughly 32 % after the last lesson. Coupled with this rise in numbers, the success rate also increased greatly by almost 26 % (67.2 % to 93.2 %). This phenomenon is in contrast with that which was observed in the column model. Hence, I am convinced that this model was better understood by most learners who decided to use it in these multiplication problems.

The distributive model became popular model for learners. Although the model did not have a good success rate during the first phase of the intervention lessons, learners displayed a better understanding of it during the last three lessons. Considering that this model only emerged in the second lesson, even though learners were familiar with it, its subsequent popularity indicates that learners gained a better understanding of the model during the intervention lessons. The other models, doubling, completion and the number line were used sparingly by learners. Not much can be read from them in terms of their efficiency or success rate. However, the doubling model maybe has proved to be less efficient because its use experienced a rise during the second and the third lesson and it later on subsided to lower figures again in the last 3 lessons (3,4 and 6). Its use may well be related to numbers that lend themselves to doubling strategies (e.g. even numbers, powers of 2).

Chapter Six: Conclusion

6.1 Introduction

In this chapter I highlight the key findings of my study with respect to the use of models and strategies employed by Grade 6 learners in solving multiplication word problems. In addition to findings, I also highlight what I perceive to be the limitations of the study and offer some recommendations for future research.

The initial motivation for my study was the poor performance in multiplication of Grade 6 learners over the years at the school where I teach. Learner difficulties in multiplication that have been alluded to in a wide range of literature also prompted me to explore models and strategies used by learners across a range of multiplication problem types. Realistic Mathematics Education (RME) was used as the theoretical framework for the study because there is a general recognition that models and images play a significant role in facilitating the learning of mathematics (Askew 2012). Hence, I saw RME as a theory that provides a platform for thinking about how learners can be helped to explore the construction of models and their use in contextual word problems. The relation or link between mathematics and the environment or everyday world meant that mathematics was derived from real situations using the RME approach. Hence, the use of models and situated forms of learning played an important role in structuring and supporting learners' solution strategies.

The main focus was three fold; to discover the models and strategies that the learners were using prior to the intervention lessons, to identify the shifts with respect to these models and strategies during the course of the intervention lessons and finally to identify the kind of models and strategies that learners would have adopted by the end of the small scale intervention.

Data gathered and analysed within the course of my study revealed interesting findings with regard to multiplication in the context of the RME approach. Initially, the pre-test revealed that the Grade 6 learners in the sample used quite a limited number of models and strategies. However, as a result of intervention lessons involving the adoption of RME approaches that make use of modelling in teaching and learning of multiplication, learners experienced increased success and used a broader range of models within their problem-solving.

Returning to the research questions that I posed at the beginning of this study, I categorized learners' written responses to tasks initially according to the model, the strategy used within the model and whether the answer was correct or incorrect. Both models and strategies were coded and thereafter organised into categories so as to examine patterns and shifts with regard to models and strategies produced during the course of intervention lessons. Findings from the study show that key horizontal mathematization problems were at the level of inappropriate information or incorrect selection of numbers due to failure to make sense of the problem or given situation. Key problems at vertical mathematization related to partial products in both the column and the area model. Data obtained points to errors at the level of both single digit multiplication as well as multiplication by multiples of 10 and 100. However, overall there were interesting shifts with regard to the use of models and strategies. In order to understand how and when these shifts were produced, I looked across the dataset relating to six intervention lessons.

My observation is that learners' selection of appropriate information (numbers) gradually improved during the intervention lessons, and more substantially compared to vertical mathematization, but there were broad improvements at the level of both models and strategies. Overall results suggest that learners indeed benefitted from the intervention lessons, with substantial gains on all items apart from item 9 (Chapter 5). The summary of the six intervention lessons in terms of learner's success rate indicate that the learners performed better on multiplication as lessons progressed and performance within each model improved across the intervention lessons. An increase in number range seemed to favour the area model more than the other models.

Learner selection and use of models also revealed major shifts from the initially dominant column model to other models and newly emerging ones. Only three models were in use in the pre-test. Namely, column, area and the doubling model. The column model was widely used by more than 60 % of the learners across individual items. As a result of the intervention lessons, learners began to explore alternative models, leading to shifts coupled with more efficient strategies which led to an improved learner performance across individual tasks. Three more models were in use in the post-test, namely completion, number line and distributive model. Few patterns emerged relating to models and strategies in relation to problem types (the three main root situations for multiplication), except the fact that learners seemed to perform better in multiplication as scaling problems compared to multiplication as repeated addition and multiplication as rate.

It would appear that opportunity to engage in modelling activities using word problems served as a vehicle for learners' construction of mathematical skills and knowledge. I also noted that discussions and interactions in the small groups and in whole class discussion when different opinions are engaged often drive the modelling process. In the excerpts that I presented (Chapter 5), the learners often demanded some explanations from each other whilst I, the researcher was only involved in a moderating capacity. I motivated them to ask questions so that they could explore and justify their solutions. I tried to spend less time asking questions and instead of asking directive questions, I focussed on guiding questions that capitalised on learners' emergent ideas on models and strategies. As a result of this approach, learners began to adapt gradually. They began to spontaneously talk to each other about their ideas. Over the course of the study I felt that the intervention lessons were helping the learners construct mathematical ideas that were effective in their modelling process. They began to grow confidence, building on each other's ideas in more unique ways. The divergent ways of thinking that emanated from the class sessions towards the end of the intervention schedule suggests that learners were beginning to develop some sense of ownership of some of the modelling strategies they employed in solving multiplication problems.

Sometimes learners' interpretation of the context posed difficulties. In traditional approaches to teaching and learning of mathematics learners are rarely presented with problems that ask for interpretation of their context, or asked to model and explain the context. Hence, exposure to such an approach during the course of intervention lessons saw a number of learners getting things wrong at the level of models (mathematizing from problem situation to model), even when the vertical mathematization was in order. This suggests that learners develop divergent ways of thinking given a task. However, it is the inability to interpret the context correctly and the un-mastered skills that often prevent their progress. However, practice with modeling has produced substantial improvements at the level of modeling, feeding into overall improvements in performance, suggesting that it is worth persevering through the early difficulties and incorrect answers.

6.2 Limitations

This study has its limitations. It was limited in terms of both time and sample size. Hence, it cannot be generalized due to its location in a specific context. In spite of this, the study provides enough evidence to signal or flag the role that modelling can play in organizing contextual problems (horizontal mathematization) in conjunction with RME approaches.

Factors like the low external validity of qualitative findings as well as the subjectivity of the researcher are worth mentioning in this study. My study was exploratory and small scale, and therefore did not include a comparable control group – which further constrains the claims that can be made.

Another limitation of my study is that understanding cannot be inferred on one particular small scale study. Hence, it would need a wide variety of well-planned longitudinal studies in order to build up evidence that points to trajectories of understanding of concepts, via the use of models and strategies.

6.3 Recommendations for future research

Despite this, the study provides strong motivation for further research as to how modelling perspectives may be integrated into multiplication teaching and learning in class, as well as with other topics. "When it is used mathematics stops being a mere transmission of resolution techniques and becomes a tool in another area of knowledge" (Lesh et al, 2010). Although the use of modelling is not a panacea to solve all word problems in the teaching and learning of mathematics, this short study has provided a window to the effect that the use of models can have on performance and processes in solving word problems in multiplication.

My results suggest that it would be interesting to set up a longitudinal study across the intermediate phase years incorporating the use of a control group. Such a study could reveal more extensive results on transitions over time with regard to models and strategies in multiplication. I am persuaded to think of such a study due to what was alluded to in the literature. "We are better off introducing children to a small number of models and working intensively with these over time." (Askew, 2012, p. 109). The point to note here is that learners need time to gain deeper understanding with respect to the introduced models. The extended period would allow learners to be repeatedly exposed to the use of models to an extent that they (models) will be intensively entrenched into the mathematics curricula for learners' benefit.

Bibliograp

Anghileri, J. (1995). *Children's Mathematical Thinking in the Primary Years.* London: Biddles Ltd, Guildford and King's Lynn.

Anghileri, J. (2007). Developing Multiplicative Thinking. In J. Anghileri, *Developing Number Sense Progression in the middle years* (pp. 71 - 91). London: Conhnuum.

Askew, M. (2012). *Beam Education.* London: Cromwell Press.

Askew, M. (2012). Tools. In M. Askew, *Transforming Primary Mathematics* (pp. 109 - 127). New York: Routledge.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2008). The Array Representation and Primary Children's Understanding and Reasoning in multiplication. *Education Studies in Mathematics, 70*, 217-241.

Beckmann, S. (2011). *Mathematics for Elementary Techers with Activity Manual* (3rd ed.). Boston: Pearson Publishers.

Caliandro, C. K. (2000). Children's Iinvestigations for Multidigit Multiplication and Division. In C. K. Caliandro, & A. Reynolds (Ed.), *Research into Practice* (pp. 420 - 426). New York.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's Mathematics: Cognitively Guided Instruction.* Portsmouth: NH: Heinemann.

Cathcart, G. W., Pothier, M. Y., Vance, H. J., & Bezuk, S. N. (2000). *Learning Mathematics in Elementary and Middle schools.* New Jersey: Marrill: Upper Saddle River.

Davydov, V. V. (1991). Psychological analysis of Multiplication. (L. P. Steffe, Ed.) *Psychological abilities of primary school children in learning mathematics. Soviet studies in mathematics education series, 6*, 1-85.

DBE. (2011a). Curriculum and Assessment Policy Statement (CAPS): Intermediate Phase Mathematics, Grade 4 - 6. Pretoria: DBE.

Drake, J. M., & Barlow, A. T. (2007). Assessing Students' Levels of Understanding Multiplication through Problem Writing. *Teaching Children Mathematics, 14 (5)*, 272-277.

Gravemeijer, K. (1994). *Developing Realistic Mathematics Education.* The Netherlands, Utrecht: Freudenthal Institute.

Gravemeijer, K., & Doorman, M. (1999). Context problems in Realistic Mathematics Education: A calculus Course as an example. *Educational Studies in Mathematics, 39*(1-3), 111-129.

Gravemeijer, K., & Stephan, M. (2002). Emergent Models As An Instructional Design Heuristic. In K. Gravemeijer, R. Lehrer, B. Van Oers, & L. Verschaffel, *Symbolizing, Modeling And Tool Use In Mathematics Education* (Vol. 30, pp. 145-169). London: Kluwer Academic Publishers.

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of curriculum studies, 32*(6), 777 - 796.

Hatch, J. (2002). Analysing qualitative data. In *Doing Qualitative research in education settings* (pp. 147-191). Albany: Suny Press.

Haylock, D. (2010). *Mathematics Explained for Primary Teachers* (4th ed.). Los Angeles; London: SAGE.

Henning, E., Van Rensburg, W., & Smit, B. (2004). *Finding your way in qualitative research.* Pretoria: Van Schaik Publishers.

Lesh, R. (2010). Modeling Student's Mathematical Modeling Competencies. New York: Springer Science and Business Media.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics.* New Jersey: Lawrence Erlbaum Associates, Inc.

McMillan, J. H., & Schumacher, S. (2010). *Research in Education* (7th ed.). New Jersey: Pearson.

Nesher, P. (1987). Towards an instructional theory: The role of students' misconceptions. *For the learning of mathematics , 7*(3), 33-39.

Opie, C. (Ed.). (2004). *Doing Educational Research.* London: SAGE .

Sepeng, P., & Webb, P. (2012). Exploring mathematical discussion in word problem solving. *Pythagoras, 33*(1), 8 pages.

Silver, E. A., & Thompson, A. G. (1984). Research Perspectives on Problem Solving in Elementary School Mathematics. *The Elementary School Journal, 84*(5), 529 - 545.

van den Heuvel-Panhuizen, M. (2001). Realistic Mathematics Education as Work in Progress. In F. L. Lin (Ed.), *The Netherlands and Taiwan conference*, (pp. 1 - 43). Taipei.

Verhage, H., & de Lange, J. (1996). Mathematics Education and Assessment. *Amesa conference*, (pp. 1 - 11). Netherlands.

Verschaffel, L., & De Corte, E. (1997). Teaching Realistic Mathematical Modeling in the Elementary School: A Teaching Experiment with Fifth Graders. *Journal for research in mathematics education, 28*(5), 577-601.

Vest, F. (1985, September). Episodes with Several Models of Multiplication. *Mathematics in School Publication Info, 14*(4), 24-26.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education , 27*(4), 458-477.

Zbiek, R., & Conner, A. (2006, September). Beyond Motivation: Exploring Mathematical Modelings as a Context for Deeping Students' Understanding of Curricular Mathematics. *Educational Studies in Mathematics, 63*(1), 89-112.