University of the Witwatersrand

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Master's Dissertation

Enumerations and bijections of Dyck paths

Author: Derrick Mohlala (1375363)
Supervisor: Prof. Margaret Archibald
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in the
School of Mathematics

Declaration

I, Derrick Mohlala (1375363), declare that this Master's Dissertation titled, "Enumerations and Bijections of Dyck Paths" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
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ABSTRACT

A Dyck path is a non-negative lattice path with the steps $u = (1, 1)$ and $d = (1, -1)$ such that the path starts at the origin and ends on the x-axis. In this research we consider some bijections that Dyck paths have with certain Catalan objects: bargraphs, d-ary trees, Motzkin paths and other Dyck paths. We apply the bijections to derive relationships that arise between the statistics of the Dyck paths and the Catalan objects, and subsequently show the enumerations of Dyck paths with regard to these statistics. The statistics that we consider include: the semiperimeter minus the number of peaks of the corresponding bargraph, the semilength and size k of the downward step $d = (1, -k)$ of the k-Dyck path, the semilength, the size k of the downward step $d = (1, -k)$ and the lower bound $y = -t$ of the k_t-Dyck path, the number of hills, odd rises, even rises, returns and semilength of the Dyck path, and lastly the number of centred, left and right tunnels, centred multitunnels and semilength of the Dyck path. Finally, we apply several techniques of the symbolic method to derive the enumeration of cornerless Motzkin paths, bargraphs and k_t-Dyck paths.

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- Last but not least, I would like to thank my parents for giving me the freedom to explore my love for Mathematics.

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Chapter 1

Introduction

A lattice path is a walk with some defined set of steps on the Euclidean space $\mathbb{R}^n$. Lattice paths have been studied since the late 19th century and have since been applied in mathematics to encode many combinatorial objects; in computer science to represent data structures such as search trees; and in statistics to model random walks and other complex statistical problems. One of the classic applications of lattice paths was to the ballot problem, which was formulated in 1887 by Joseph Bertrand in [2], stated as follows.

Ballot problem: Suppose there are two candidates A and B running for office and there are $2n$ voters with n voting for A and n voting for B. How many ways can the ballots be counted, such that A is always ahead of or tied with B?

For example, say we have 6 voters. Then the number of ways to count the ballots such that A is always ahead of or tied with B is 5, as shown below:

- $AAABBB$
- $AABABB$
- $AABBAB$
- $ABAABB$
- $ABABAB$

There have been several solutions to the ballot problem since its formulation. The original solution was published in 1887 by Joseph Bertrand himself in [2] using an inductive proof and later in the same year, Désiré André published a direct proof in [1].
To show the solution to the ballot problem, we can formulate it as a lattice path by assigning the right step $(1,0)$ when A is counted and the up step $(0,1)$ when B is counted. It is then clear that a good path will start at $(0,0)$ and end on (n,n) whilst staying on or below the line $y=x$ but a bad path will go above the line.

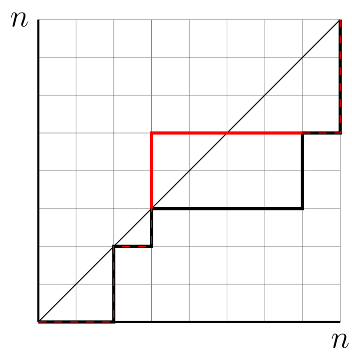


Figure 1.1: A good ballot path (black) and bad ballot path (red).

The total number of possible paths in the plane that have n up steps and n right steps is $\binom{2n}{n}$. Now, we need to remove all the bad paths from the total paths to remain with only the good paths. To achieve this, we simply switch the assignment of every step in the bad path after the first step that goes above $y=x$ by letting the up steps become right steps and vice versa. The resulting path will always end on $(n-1,n+1)$. See Figure 1.2 below.

This is intuitive since we can only go above the line $y=x$ if we have one more up step than right steps, therefore by switching the assignment, this gives us an extra up step and one less right step. In Figure 1.2, the path crosses $y=x$ when there are 3 right steps and 4 up steps. The remainder of the path has 5 right steps and 4 up steps. When we switch the assignment in the remaining path, we get 4 right steps and 5 up steps. Therefore, the new path has in total 7 right steps and 9 up steps.

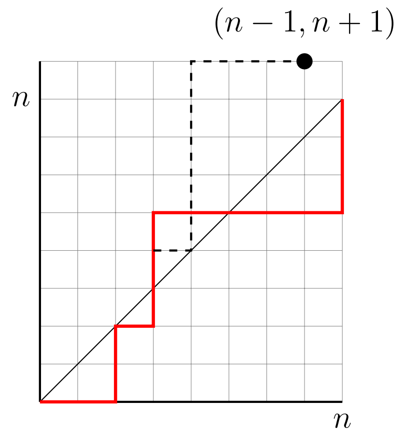


Figure 1.2: A bad ballot path.

Therefore, we can firmly establish that there is a one-to-one correspondence between bad paths and paths that have $(n-1)$ right steps and $(n+1)$ up steps. Now, the number of good ballot paths can be solved for as follows.

$$\begin{aligned}
\#\text{ Good ballot paths} &= \#\text{ Total ballot paths} - \#\text{ Bad ballot paths} \\
&= \binom{2n}{n} - \binom{2n}{n-1} \\
&= \frac{2n!}{n!n!} - \frac{2n!}{(n-1)!(n+1)!} \\
&= \frac{2n!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{2n!}{n!(n-1)!}\left(\frac{1}{n(n+1)}\right) \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

The solution to the ballot problem is therefore the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.1}$$

The good paths that we considered in the ballot problem are referred to as Dyck paths, named in honour of the German mathematician Walther von Dyck. We usually represent Dyck paths along the x-axis of the Cartesian plane by applying the following transformation. Let every right step $(1, 0)$ in the good path become the up step $u = (1, 1)$ and every up step $(0, 1)$ become the down step $d = (1, -1)$.

The good path in Figure 1.1 corresponds to the Dyck path *uudduduuuudduddd* given in Figure 1.3.

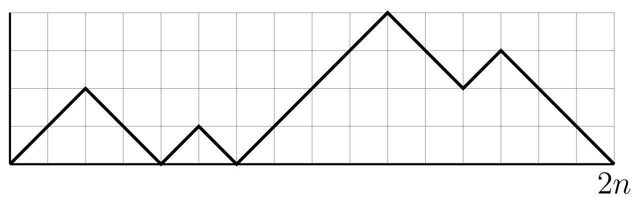


Figure 1.3: A good ballot path as a Dyck path.

We can interpret the n-th Catalan number as the number of Dyck paths that end on $(2n, 0)$. Each enumeration that arises from the statistics of Dyck paths provides us with a new and different interpretation of Catalan numbers and consequently other related Catalan objects. There are various statistics of Dyck paths that have been studied over the years. We highlight some important ones below.

Emeric Deutsch showed in [8] by generating functions, the enumeration of Dyck paths according to the statistics: the number of peaks, number of low and number of high peaks, number of low peaks, number of high peaks, number of peaks and return steps, number of return steps, number of peaks and height of first peak, height of first peak, number of low valleys and high valleys, number of return descents of odd and even length, number of low valleys, number of high valleys, and number of peaks before and after the first return step.

Sergi Elizalde and Emeric Deutsch showed in [10] by a bijective proof, the enumeration of Dyck paths according to the statistics: the number of centred, left, and right tunnels, and number of centred multitunnels and the enumeration according to the statistics: the number of centred, left, and right tunnels, centred multitunnels, and the semilength.

Toufik Mansour showed in [16] by generating functions the enumeration of Dyck paths according to the statistic: the number of $uu \cdots udu's$. Sun Yidong proved in [23] by generating functions the enumeration of Dyck paths according to the statistic: number of $udu's$.

Generalised Dyck paths have also been studied extensively in the literature. Philippe Duchon proved in [9] the enumeration of generalised Dyck words, which have a one-to-one correspondence with Dyck paths. Dyck words are a sequence of words made up of n "(" words and n ")" words such that at no point from the start of the sequence are there more "(" words than ")" words. For example, "(()(()))" is a Dyck word. Anna De Mier and Marc Noy also proved in [7] the enumeration of generalised Dyck paths by proving that the solution to the *tennis ball problem* is generalised Dyck paths. The *tennis ball problem* is stated as follows: Imagine there is a game consisting of n turns and there are $2n$ balls numbered $1, 2, 3, \cdots, 2n$. In the first turn balls 1 and 2 are put into a basket and one of them is removed. In the second turn balls 3 and 4 are put into the basket and one of the three remaining balls is removed. Next balls 5 and 6 go in and one of the four remaining balls is removed. At the end there are exactly n balls outside the basket. The question is how many different sets of balls can we have at the end outside the basket. Many other authors have shown various enumerations of Dyck paths according to several other statistics, which are out of the scope of this research.

The objective of this dissertation is to study the enumerations of different classes of Dyck paths and the bijections they share with other Catalan objects and themselves. We will do so using bijective proofs, generating functions and the symbolic method. We mainly reference the symbolic method and generating functions as described by Robert Sedgewick and Philippe Flajolet in their book [19], *"An introduction to the analysis of algorithms"*.

We begin in Chapter 2 by giving formal definitions of Dyck paths and bargraphs and the statistics of each that we use in subsequent chapters. These definitions are mainly from Sergi Elizalde and Emeric Deutsch's paper [12], *"A bijection between bargraphs and Dyck paths"*. We also provide formal definitions for generating functions and show the different methods of extracting coefficients from generating functions that we will rely on in subsequent chapters.

In Chapter 3 we study the bijection between Dyck paths and bargraphs given by Sergi Elizalde and Emeric Deutsch in [12]. We take inspiration from this bijection that relies on the sequence of heights of the steps of the Dyck path and the resulting column heights in the bargraph to come up with our own original bijection and proof that uses the consecutive peaks and valleys of the Dyck path and the horizontal steps of the bargraph. We then show the relationship between the semiperimeter and number of peaks of a bargraph and the semilength of a Dyck path through another original proof that follows from our bijection. We then show that Catalan numbers can be interpreted as a set of bargraphs where the semiperimeter minus the number of peaks is n, a result that was proven by Sergi Elizalde and Emeric Deutsch in [12]. Lastly, we show the derivation of the generating function of bargraphs that was proven by Mireille Bousquet-Mêlou and Andrew Rechnitzer in [5] using the wasp-waist decomposition.

In Chapter 4 we study the bijection between cornerless Motzkin paths and bargraphs and subsequently derive the generating function of bargraphs using the bijection and the decomposition of cornerless Motzkin paths. Both these results were proved by Sergi Elizalde and Emeric Deutsch in [11].

In Chapter 5 we study two generalisations of Dyck paths, namely k-Dyck paths which have the down step $d = (1, -k)$, and k_t-Dyck paths which have the down step $d = (1, -k)$ and are bounded below by $y = -t$ where $t \geq 0$.

There are several papers on these generalisations, however, for this chapter we mainly study the proofs provided by Sarah Jane Selkirk in [20]. We first show, in our own words, a bijection between $(k+1)$-ary trees and k-Dyck paths using tree traversal. As a consequence of the bijection it follows that the enumeration of k-Dyck paths is given by generalised Catalan numbers,

$$\frac{1}{kn+1}\binom{(k+1)n}{n}.$$

We establish another bijection between tuples of k-Dyck paths and k_t-Dyck paths. We interpret this bijection in our own way by using geometric methods. It follows from this bijection that the enumeration of k_t-Dyck paths is given by

$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

Lastly, we explain the derivation of the generating function of k_t-Dyck paths directly using the symbolic method, as shown by Selkirk in [20]. Through our explanation, we fix a few errors with the k_t-Dyck path decomposition illustration and the final result of the proof given in Selkirk. In particular, we show that the right-most $\mathcal{K}_t$ path in the decomposition is not bounded below by $y=-i$ and that it can go down to the level $y=-t$, where $i \leq t$. And we show that k_t-Dyck paths are enumerated by the generalised binomial series $\mathcal{B}_{k+1}(x^{k+1})^{t+1}$ instead of $\mathcal{B}_{k+1}(x^{k+1})^{t}$ as shown in [20].

Finally, in Chapter 6 we map Dyck paths back onto themselves using tunnel statistics. The original proof of this bijection and its generalisation are given by Sergi Elizalde and Emeric Deutsch in [10]. We expand the proof, by adding more details. We then apply the bijection to derive the generating function of Dyck paths with regard to the number of hills, odd rises, even rises, returns and the semilength. We add details to this derivation as well.

Chapter 2

Lattice paths and generating functions

The central topic of investigation in this dissertation is Dyck paths. We begin by formalising the definition of the family of lattice paths that Dyck paths belong to and define the statistics of Dyck paths that play a vital role throughout this research.

2.1 Lattice paths

There are several ways to define lattice paths, however out of necessity we restrict our definition to those whose lattice is the $\mathbb{Z}^2$ plane.

Definition 2.1.1. A *non-negative lattice path* with steps $S \in \mathbb{Z}^2$ is a sequence of vectors $s_1, s_2, ..., s_n \in \mathbb{Z}^2$, where $n \in \mathbb{N}$, that start at the origin, stay on or above the x-axis and $S_i - S_{i-1} \in S$ (See [6, Chapter 8]).

2.1.1 Bargraphs

Definition 2.1.2. A *bargraph* is a sequence of steps $U = (0, 1)$, $H = (1, 0)$ and $D = (0, -1)$ such that the steps start at the origin and end on the x-axis. The steps stay on or above the x-axis, and there is no pair of consecutive steps DU or UD (see [12]).

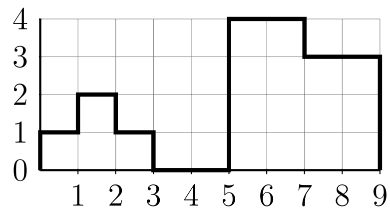


Figure 2.1: Bargraph $UHUHDHDHHUUUUHHDHHDDD$.

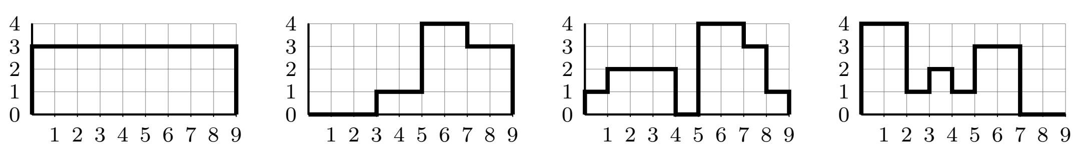
Definition 2.1.3. The *semiperimeter* $\mathrm{sp}(B)$ of a bargraph B is the number of U steps plus the number of H steps (see [12]).

In Figure 2.1, the semiperimeter is given by nine H steps plus six U steps giving $\mathrm{sp}(B) = 15$.

Definition 2.1.4. A *peak* of a bargraph is the occurrence of the sequence of steps $UH^\ell D$ for some $\ell \in \mathbb{N}$. A *valley* of a bargraph is the occurrence of the sequence of steps $DH^\ell U$ for some $\ell \in \mathbb{N}$ (see [12]). We denote the number of peaks and valleys by $\mathrm{pk}(B)$ and $\mathrm{vl}(B)$ respectively.

In Figure 2.1, we have two peaks: UHD between $x = 1$ and $x = 2$ and $UHHD$ between $x = 5$ and $x = 7$, and one valley $DHHU$ between $x = 3$ and $x = 5$.

More examples:



(a) $\mathrm{sp}(B) = 12$, $\mathrm{pk}(B) = 1$, $\mathrm{vl}(B) = 0$.

(b) $\mathrm{sp}(B) = 13$, $\mathrm{pk}(B) = 1$, $\mathrm{vl}(B) = 0$.

(c) $\mathrm{sp}(B) = 15$, $\mathrm{pk}(B) = 2$, $\mathrm{vl}(B) = 1$.

(d) $\mathrm{sp}(B) = 16$, $\mathrm{pk}(B) = 3$, $\mathrm{vl}(B) = 2$.

Figure 2.2: Bargraphs enumerating semiperimeter, peaks and valleys.

2.1.2 Dyck paths

Definition 2.1.5. A *Dyck path* is a non-negative lattice path with steps $u = (1, 1)$ and $d = (1, -1)$ that start at the origin and end on the x-axis (see [8]).

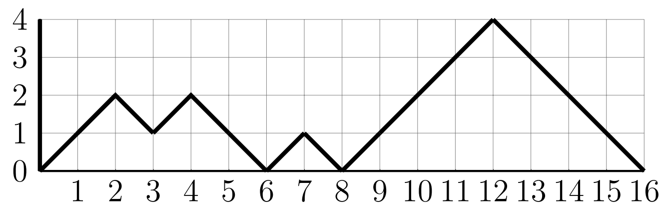


Figure 2.3: Dyck path *uududdduduuuudddd*.

Definition 2.1.6. The *semilength* $\mathrm{sl}(P)$ of a Dyck path is half of the number of steps in the path, and we denote D_n to be the set of Dyck paths of semilength n (see [8]).

In Figure 2.3 the number of u steps is 8 and d steps is 8 then, $2n = 16$ which gives the semilength $\mathrm{sl}(P) = 8$.

Definition 2.1.7. A *peak* of a Dyck path is a pair of consecutive steps ud and a *valley* is the pair of consecutive steps du (see [8]).

In Figure 2.3 there are four peaks at $x = 2, 4, 7$ and 12 and three valleys at $x = 3, 6$ and 8.

Definition 2.1.8. A *return* in a Dyck path is a d step ending on the x-axis (see [12]).

In Figure 2.3 there are 3 returns at the steps ending on $x = 6, 8, 16$.

Remark. A Dyck path must have at least one return, since by definition it has to end on the x-axis.

Definition 2.1.9. The *height* of a Dyck path is the maximum y-coordinate that the path reaches (see [12]).

In Figure 2.3 the maximum y-coordinate the path reaches is 4 at $x = 12$.

Definition 2.1.10. The *height* of a step in a Dyck path is the maximum y-coordinate that the step reaches.

In Figure 2.3 the maximum y-coordinate the step ending at $x = 11$ reaches is 3.

Definition 2.1.11. A *peak height* in a Dyck path is the maximum y-coordinate that the peak reaches.

In Figure 2.3 the peak height of the peak at $x = 4$ is 2.

Definition 2.1.12. A *valley height* in a Dyck path is the y-coordinate that the d step of the valley starts on.

In Figure 2.3, the valley height of the valley du at $x = 6$ is 1 since the d step starts at $y = 1$.

Definition 2.1.13. A *hill* in a Dyck path is a peak with a peak height of 1 (see [10]).

In Figure 2.3 the peak at $x = 7$ is a hill.

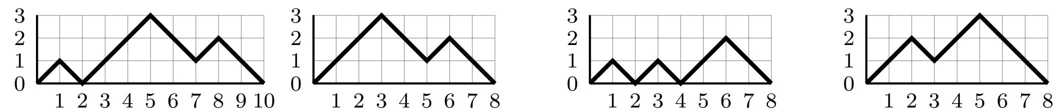
Definition 2.1.14. An *odd rise* in a Dyck path is a step u that ends on an odd numbered x-axis coordinate. Similarly, an *even rise* is a step u that ends on an even numbered x-axis coordinate (see [10]).

In Figure 2.3 there are 4 odd rises and 4 even rises.

Definition 2.1.15. A *tunnel* of a Dyck path is a horizontal line segment between two points such that the line segment touches only two unique horizontal points on the Dyck path (see [10]).

In Figure 2.3 there are 8 tunnels: between (1,1) and (3,1); (3,1) and (5,1); (0,0) and (6,0); (6,0) and (8,0); (11,3) and (13,3); (10,2) and (14,2); (9,1) and (15,1); and (8,0) and (16,0).

More examples:



(a)
sl(P) = 5,
height = 3,
#returns = 2,
#peaks = 3,
#valleys = 2,
#tunnels = 5,
#odd rises = 3,
#even rises = 2.

(b)
sl(P) = 4,
height = 3,
#returns = 1,
#peaks = 2,
#valleys = 1,
#tunnels = 4,
#odd rises = 2,
#even rises = 2.

(c)
sl(P) = 4,
height = 2,
#returns = 3,
#peaks = 3,
#valleys = 2,
#tunnels = 4,
#odd rises = 3,
#even rises = 1.

(d)
sl(P) = 4,
height = 3,
#returns = 1,
#peaks = 2,
#valleys = 1,
#tunnels = 4,
#odd rises = 2,
#even rises = 2.

Figure 2.4: Dyck paths enumerating semilength, height, the number of returns, peaks, valleys, tunnels and odd and even rises.

2.2 Generating functions

Throughout this research we will investigate different enumerations of Dyck paths and the main tool that we will use for that purpose is generating functions. Generating functions allow us to interpret combinatorial objects as sequences whose coefficients give the enumeration of the parameter represented by that term.

There are many types of generating functions such as ordinary generating functions, exponential generating functions, Lambert series, Bell series, Dirichlet series, etc, however we will only focus on ordinary generating functions and exponential generating functions as they are sufficient for the purpose of our research. We mainly use the definitions of generating functions given by Robert Sedgewick and Philippe Flajolet in [19] as inspiration for this section.

2.2.1 Ordinary generating functions

Definition 2.2.1. A *power series* is an infinite series in the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z^1 + a_2 z^2 + \cdots ,$$

where a_n is the coefficient of the $(n+1)$-th term z^n.

Definition 2.2.2. An *ordinary generating function* (OGF) of a sequence $\{g_n\}_{n=0}^{\infty}$ is the power series

$$G(z) = \sum_{n=0}^{\infty} g_n z^n$$

where z^n represents the n-th occurrence of a feature of the combinatorial object represented by the function and the coefficient g_n represents the number of such occurrences.

Remark. We denote the coefficient of the term z^n by $[z^n]G(z)$.

Ordinary generating functions have the following properties as shown in [19, Section 3.1]:

1. $F(z) + G(z) = \sum_{n=0}^{\infty} (f_n + g_n) z^n$, *addition.*
2. $(1 - z)G(z) = g_0 + \sum_{n=1}^{\infty} (g_n - g_{n-1}) z^n$, *difference.*
3. $z^m G(z) = \sum_{n=m}^{\infty} g_{n-m} z^n$ where $m \geq 0$, *right shift.*
4. $\dfrac{G(z) - g_0 - g_1 z^1 - \cdots - g_{m-1} z^{m-1}}{z^m} = \sum_{n=0}^{\infty} g_{n+m} z^n$ where $m \geq 0$, *left shift.*
5. $G(cz) = \sum_{n=0}^{\infty} c^n g_n z^n$, *scaling.*

6. $G'(z) = \sum_{n=0}^{\infty}(n+1)g_{n+1}z^n$, *index multiply.*

7. $zG'(z) = \sum_{n=0}^{\infty} ng_n z^n$, *index multiply.*

8. $\int_0^z G(t)dt = \sum_{n=1}^{\infty} \frac{1}{n} g_{n-1} z^n$, *index divide.*

9. $F(z)G(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k g_{n-k} \right) z^n$, *convolution.*

10. $\frac{A(z)}{1-z} = \sum_{n=0}^{\infty} \left(\sum_{0 \le k \le n} a_k \right) z^n$, *partial sum.*

Example 2.2.1. The ordinary generating function for the sequence $\{1, 1, 1, \dots\}$, i.e. $\{a_n\}_{n=0}^{\infty}$ where $a_n = 1 \; \forall \; n$ is given by

$$A(z) = \sum_{n=0}^{\infty} 1 \cdot z^n = \frac{1}{1-z}.$$

We can use the above properties of ordinary generating functions to obtain generating functions of other sequences that can be derived by applying the appropriate transformations to the sequence in the above example. Say we have the sequence $\{b_n\}_{n=0}^{\infty} = 0, 1, 2, 3, \dots$, then what is its generating function?

We notice that if we multiply the sequence $\{1, 1, 1, \dots\}$ by its index, i.e., $\{0 \times 1, 1 \times 1, 2 \times 1, \dots\}$ then we obtain $\{0, 1, 2, 3, \dots\}$. By property 6, the generating function of the sequence $\{b_n\}_{n=0}^{\infty}$ is given by

$$zA'(z) = \frac{z}{(1-z)^2}.$$

Solving Recurrences

We can use generating functions to solve a recurrence equation of a sequence in the following way:

1. Multiply both sides of the recurrence equation by z^n and sum over all possible values of n to obtain the respective generating functions.
2. Obtain the functional equation for the recurrence.
3. Solve the functional equation.
4. Extract the required coefficients.

Example 2.2.2. Given the recurrence equation $a_n = a_{n-1} + 3$ with $n \geq 1$ and $a_0 = 0$,

1. Multiply both sides of the recurrence equation by z^n and sum over all possible values of n to obtain the respective generating functions

$$\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_{n-1} z^n + \sum_{n=1}^{\infty} 3z^n. \tag{2.1}$$

2. Obtain the functional equation for Equation (2.1).

   Let $A(z) = \sum_{n=1}^{\infty} a_n z^n$ then,

$$A(z) = zA(z) + \frac{3z}{1-z}. \tag{2.2}$$

3. Solve the functional equation,

$$A(z) = \frac{3z}{(1-z)^2}. \tag{2.3}$$

4. Extract the required coefficient,

$$[z^n]A(z) = 3\binom{n}{1} = 3n. \tag{2.4}$$

For instance $[z^2]A(z) = 3\binom{2}{1} = 6$ and $[z^3]A(z) = 3\binom{3}{1} = 9$ and this satisfies the recurrence equation $a_3 = a_2 + 3$ as required.

2.2.2 Exponential generating functions

Definition 2.2.3. The *exponential generating function* (EGF) of a sequence $\{h_n\}_{n=0}^{\infty}$ is the power series

$$H(z) = \sum_{n=0}^{\infty} h_n \frac{z^n}{n!},$$

and the coefficient of the term z^n is denoted $n![z^n]H(z)$.

The following properties hold for exponential generating functions as shown in [19, Section 3.2]:

1. $H(z) + F(z) = \sum_{n=0}^{\infty} (h_n + f_n) \frac{z^n}{n!}$, *addition.*

2. $H'(z) - H(z) = \sum_{n=0}^{\infty} (h_{n+1} - h_n) \frac{z^n}{n!}$, *difference.*

3. $\int_0^z H(t)dt = \sum_{n=1}^{\infty} h_{n-1} \frac{z^n}{n!}$, *right shift.*

4. $H'(z) = \sum_{n=0}^{\infty} h_{n+1} \frac{z^n}{n!}$, *left shift.*

5. $\frac{H(z) - H(0)}{z} = \sum_{n=1}^{\infty} \frac{h_{n+1}}{n+1} \frac{z^n}{n!}$, *index divide.*

6. $zH(z) = \sum_{n=0}^{\infty} n h_{n-1} \frac{z^n}{n!}$, *index multiply.*

7. $H(z)F(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} h_k f_{n-k} \right) \frac{z^n}{n!}$, *binomial convolution.*

8. $e^z H(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} h_k \right) \frac{z^n}{n!}$, *binomial sum.*

Example 2.2.3. The exponential generating function for the sequence $\{1, 1, 1, \dots\}$ is given by

$$A(z) = \sum_{n=0}^{\infty} \frac{1 \cdot z^n}{n!} = e^z,$$

and the sequence $\{b_n\}_{n=0}^{\infty} = \{0, 1, 2, 3, \dots\}$, can be obtained by index multiplication of $A(z) = e^z$. So by property 6 the generating function of the sequence $\{b_n\}_{n=0}^{\infty}$ is given by

$$zA(z) = ze^z.$$

Remark. We can solve recurrences with either ordinary or exponential generating functions. Both ordinary and exponential generating functions can be used on the same sequences. The factorial in the exponential generating function does however suggest that it is better suited for counting problems involving permutations of different objects, since these they often involve factorials.

For example, if we are counting the number of ways to seat n people at a round table, then we can do that in $(n-1)!$ ways. If we use the exponential generating function to represent this solution, it simplifies to $\sum_{n\geq 1} \frac{1}{n} z^n$. The equivalent OGF is $\sum_{n\geq 1} (n-1)! z^n$.

Another difference between ordinary and exponential generating functions is that exponential generating functions excel on counting problems with labelled objects. This is explored further by Sedgewick and Flajolet in [19, Section 3.9] with respect to the symbolic method.

2.2.3 Extracting coefficients

In order to be able to find enumerations of combinatorial objects represented by generating functions, we need to be able to extract the coefficients from those generating functions. The straightforward approach to extract coefficients from a generating function is to expand its closed form and obtain the necessary coefficients, however this can become tedious with more complicated generating functions. In this section, we explore different methods of extracting coefficients from generating functions that we will apply throughout this research.

Method 1:

The Taylor expansion formula

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots \tag{2.5}$$

provides a simple method of extracting coefficients from generating functions that are easily differentiable.

Example 2.2.4. Given the generating function $G(z) = \frac{1}{(1-z)^4}$, the n-th coefficient $[z^n]G(z)$ is found by applying the Taylor expansion formula. We find the derivatives of $G(z)$ below:

First derivative: $G'(z) = \frac{4}{(1-z)^5}$

Second derivative: $G''(z) = \frac{4 \cdot 5}{(1-z)^6}$

Third derivative: $G'''(z) = \frac{4 \cdot 5 \cdot 6}{(1-z)^7}$

n-th derivative: $G^{(n)}(z) = \frac{(n+3)!}{6(1-z)^{n+4}}$

Let $z = 0$ to obtain $[z^n]G(z) = \frac{G^{(n)}(0)}{n!} = \frac{(n+3)!}{6n!}$.

Method 2:

The Lagrange inversion theorem is a very powerful tool for solving functional equations involving generating functions. We mainly encounter these types of functional equations when using the symbolic method to derive generating functions. The theorem is given by Sedgewick and Flajolet in [19, Section 6.12].

Theorem 2.2.1. (Lagrange inversion theorem) If a generating function $G(z) = \sum_{n=0}^{\infty} g_n z^n$ satisfies the functional equation $z = f(G(z))$, where $f(z)$ satisfies $f(0) = 0$ and $f'(0) \neq 0$ then

$$g_n \equiv [z^n]G(z) = \frac{1}{n}[u^{n-1}]\left(\frac{u}{f(u)}\right)^n$$

and

$$[z^n](G(z))^m = \frac{m}{n}[u^{n-m}]\left(\frac{u}{f(u)}\right)^n$$

and

$$[z^n]t(G(z)) = \frac{1}{n}[u^{n-1}]t'(u)\left(\frac{u}{f(u)}\right)^n.$$

Example 2.2.5. Given the functional equation $z = \frac{A(z)}{1-A(z)}$, we extract $[z^n]A(z)$ by applying the Lagrange inversion theorem as follows:

Let $f(u) = \frac{u}{1-u}$ then

$$\begin{aligned}
[z^n]A(z) &= \frac{1}{n}[u^{n-1}]\left(\frac{u}{u/(1-u)}\right)^n \\
&= \frac{1}{n}[u^{n-1}](1-u)^n \\
&= \frac{(-1)^{n-1}}{n}\binom{n}{n-1} \\
&= (-1)^{n-1}.
\end{aligned}$$

Chapter 3

Bargraphs and Dyck paths

We derive a bijection between bargraphs and Dyck paths through a new and different proof from the original that is given by Elizalde and Deutsch in [12]. The proof provided by Elizalde and Deutsch relies on the sequence of heights of the steps of the Dyck path and the resulting column heights in the bargraph. We prove a bijection using the peaks and valleys of the Dyck path and the corresponding horizontal steps in the bargraph. We then use the bijection to find the relationship between the statistics semiperimeter and the number of peaks of bargraphs and the semilength of Dyck paths. This gives us a interpretation of Catalan numbers in terms of these statistics of bargraphs.

3.1 Bijection

The definition of the semiperimeter of a bargraph, semilength, peak, height, peak and valley height, valley of a Dyck path are given in Section 2.1.1 and we will use these in establishing the bijections.

The concept of consecutive peaks and valleys plays a pivotal role in establishing our bijection. We define it below.

Definition 3.1.1. *Consecutive peaks* in a Dyck path are one or more connected peaks with the same peak height. We say peaks are connected if there are no other steps between them in the path.

Definition 3.1.2. *Consecutive valleys* in a Dyck path are one or more connected valleys with the same valley height. We say valleys are connected if there are no other steps between them in the path.

When there are more connected peaks than connected valleys at some peak height, then we only consider the consecutive peaks and vice versa.

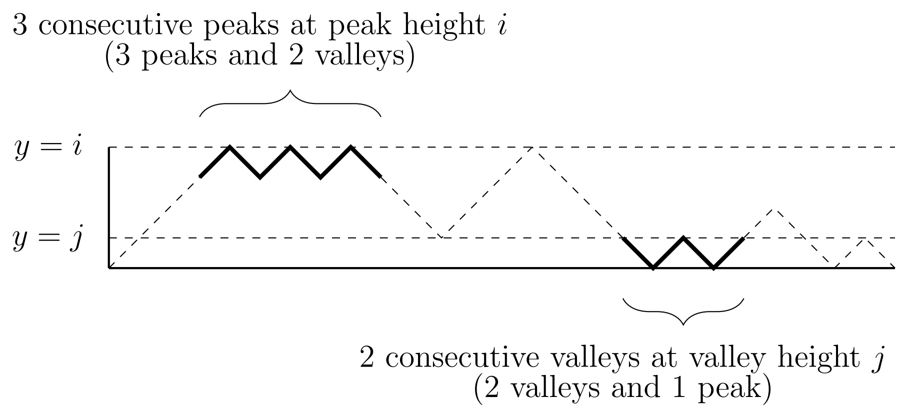


Figure 3.1: Consecutive peaks and valleys.

Consider the Dyck path in Figure 3.2. There is one consecutive peak $x = 2$ with peak height of 2. Then there are two consecutive peaks $x = 10$ and $x = 12$ that are connected, with peak height of 2.

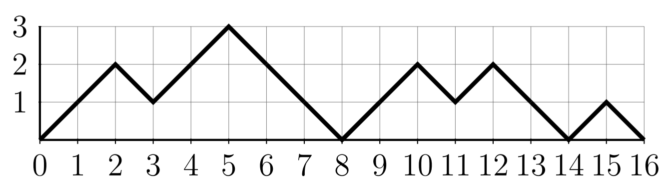


Figure 3.2: Dyck path *uuduudddduududdud*.

The bijection between bargraphs and Dyck paths in [12] is described as follows:

Let ϕ be a bijection from Dyck paths to bargraphs where the semilength $\mathrm{sl}(P)$ of the Dyck path becomes the semiperimeter minus the number of peaks in the bargraph.

For a sequence of heights of the steps of P, to construct the bargraph $\phi(P)$ from this sequence, we turn each maximal block of c consecutive letters h into $\lfloor \frac{c}{2} \rfloor$ columns of height h. We think of these maximal blocks as sections in the sequence of heights where heights are equal, i.e., a sequence 111122233 has 3 blocks of $c = 4$ consecutive 1's, $c = 3$ consecutive 2's and $c = 2$ consecutive 3's.

Example 3.1.1. Consider the Dyck path *uduuudduddududd* in Figure 3.3.

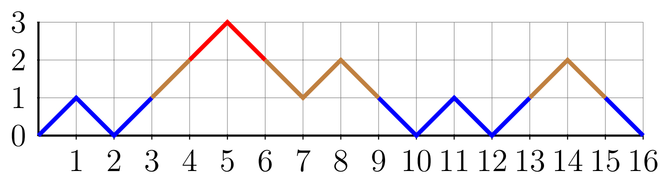


Figure 3.3: Dyck path with a sequence of step heights 1112332221111221.

The first block of consecutive heights is 1 with $c = 3$ therefore $\lfloor \frac{3}{2} \rfloor = 1$ column of height 1. The second block of consecutive heights is 2 with $c = 1$ therefore $\lfloor \frac{1}{2} \rfloor = 0$. The third block of consecutive heights is 33 with $c = 2$ therefore $\lfloor \frac{2}{2} \rfloor = 1$ column of height 3. The fourth block of consecutive heights is 222 with $c = 3$ therefore $\lfloor \frac{3}{2} \rfloor = 1$ column of height 1. The fifth block of consecutive heights is 1111 with $c = 4$ therefore $\lfloor \frac{4}{2} \rfloor = 2$ columns of height 1. The fourth block of consecutive heights is 22 with $c = 2$ therefore $\lfloor \frac{2}{2} \rfloor = 1$ column of height 1. The last block of consecutive heights is 1 with $c = 1$ therefore $\lfloor \frac{1}{2} \rfloor = 0$ column of height 1. This gives the bargraph 132112 illustrated in Figure 3.4.

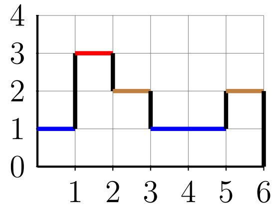


Figure 3.4: Corresponding bargraph to Figure 3.3 with a sequence of column heights 132112.

To get back the Dyck Path from the bargraph, we describe the inverse mapping as follows. Given a bargraph described by a sequence of column heights $h_1^{a_1}h_2^{a_2}...h_r^{a_r}$. By inserting the terms $a_j = 0$ if necessary, we can assume $h_1 = h_r = 1$ and $|h_{i+1} - h_i| = 1$ for all i, that is, the difference between consecutive heights must be 1 else insert a necessary height to make this true but set its $a_i = 0$. Define $h_0 = h_{r+1} = 0$, this ensures a value to compare the first and last height to.

To construct the Dyck path, for $i = 1, 2, ..., r$ we replace $h_i^{a_i}$ with

$$\begin{cases} (ud)^{a_i} & \text{if } h_{i-1} < h_i > h_{i+1}, \\ (ud)^{a_i}u & \text{if } h_{i-1} < h_i < h_{i+1}, \\ (du)^{a_i}d & \text{if } h_{i-1} > h_i > h_{i+1}, \\ (du)^{a_i} & \text{if } h_{i-1} > h_i < h_{i+1}. \end{cases}$$

For the bargraph above defined by a sequence of column heights 132112, we write these as $1^1 2^1 3^1 2^1 1^2 2^1 1^0$ which becomes $(ud)^1u(ud)^0u(ud)^1(du)^1d(du)^2(ud)^1(du)^0d$ giving the Dyck path $uduuudduddududuudd$.

We establish our bijection between bargraphs and Dyck paths in this manner:

Proposition 3.1.1. There exists a bijection between bargraphs and Dyck paths in terms of the consecutive peaks and valleys in the Dyck path and the horizontal steps in the bargraph.

Proof. Given a Dyck path, we read the path from left to right and count the number of consecutive peaks and valleys at each peak or valley height. If the number of consecutive peaks is more than the number of valleys then we let each peak ud become the horizontal step $H = (1, 0)$ and if the number of consecutive valleys is more than the number of peaks then we let each valley du become the horizontal step $H = (1, 0)$, otherwise if there is an equal number of consecutive peaks and valleys then let each of either the left-most peaks or valleys become the horizontal step.

We draw the horizontal steps H at the respective peak heights and valley heights and join the path with the steps $U = (1, 0)$ and $D = (0, -1)$ as required to obtain the corresponding bargraph.

To get back the Dyck path, read the bargraph from left to right and let every horizontal step H we encounter become a peak with a peak height of H if the preceding step is U or H, otherwise, let it become a valley with a valley height of H. We join up the path with the steps $u = (1, 1)$ and $d = (1, -1)$ as required and acquire the original Dyck path. □

Example 3.1.2. Consider the Dyck path $uuduudddduududddud$ given in Figure 3.2.

⇒ We read the path from left to right and find one peak and one valley at peak height 2 therefore we draw a horizontal step H at height 2. We then find one peak at peak height 3 therefore we draw a horizontal step H at height 3. We then find one valley at valley height 1 therefore we draw a horizontal step H at height 1. We further find two consecutive peaks at peak height 2 therefore we draw two horizontal steps H at height 2. Finally, we find one peak and one valley at peak height 1 and draw a final horizontal step H at height one.

We join these horizontal steps with the steps $U = (1, 0)$ and $D = (0, -1)$ to obtain the bargraph $UUHUHDDHUHHDHD$ in Figure 3.5.

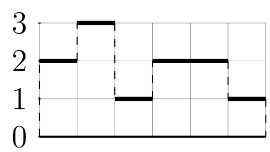


Figure 3.5: Bargraph $UUHUHDDHUHHDHD$.

$\Leftarrow$ Read the bargraph from left to right. We let the first and second H steps become peaks with peak heights 2 and 3 respectively, since the steps before them are U steps. The third H step becomes a valley with valley height 1 since the step before it is D. The fourth and fifth H steps both become peaks with peak heights 2 and the sixth and last H step becomes a valley with since the step before it is D. We join up the rest of the path with the steps $u = (1, 1)$ and $d = (1, -1)$ as required to obtain the Dyck path *uuduudddduududdud* given in Figure 3.6.

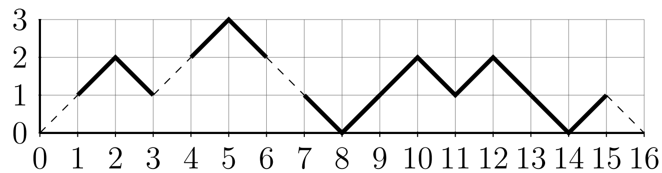


Figure 3.6: Corresponding Dyck path to Figure 3.5.

∎

We now show the relationship between the semiperimeter and number of peaks of a bargraph and the semilength of a Dyck path. This relationship is given by the following theorem that follows directly from the bijection. This theorem is originally given in [12].

Theorem 3.1.1. Given a Dyck path P and its corresponding bargraph B then $sl(P) = sp(B) - pk(B)$.

Proof. Recall by Definition 2.1.3 that $sp(B) = \#U + \#H$ where $\#U$ and $\#H$ are the number of upward steps and horizontal steps of the bargraph respectively.

We solve for $\#H$ in B by considering the semilength of P as follows:

Every Dyck path P is made up of a sequence of consecutive peaks and consecutive valleys, and all the other steps that do not form part of the consecutive peaks and consecutive valleys. The total set of consecutive peaks or consecutive valleys, denoted $\mathcal{C}$, can be broken down into these disjoint subsets:

i. $\mathcal{C}_{p>v}$ contains all consecutive peaks where the number of connected peaks is more than the number of connected valleys at the same peak height,

ii. $\mathcal{C}_{p<v}$, contains all consecutive valleys where the number of connected valleys is more than the number of connected peaks at the same valley height,

iii. $\mathcal{C}_{p=v}$, contains all consecutive valleys or peaks where the number of connected valleys is equal to the number of connected peaks at the same peak height.

For example, in Figure 3.2, the subset $\mathcal{C}_{p>v}$ contains the peak $x = 5$ at peak height 3 and the two connected peaks $x = 10$ and $x = 12$ at peak height 2. The subset $\mathcal{C}_{p<v}$ contains the valley at $x = 8$ at valley height 1 and the subset $\mathcal{C}_{p=v}$ contains the peak at $x = 2$ and the peak $x = 15$ at peak height 1.

By definition of the bijection, all the H steps in B are created from these three subsets of $\mathcal{C} = \mathcal{C}_{p>v} + \mathcal{C}_{p<v} + \mathcal{C}_{p=v}$ in P. Now, since a single H step is created by either a peak or valley which is made up of two steps then

$$\#H = |\mathcal{C}| = |\mathcal{C}_{p>v}| + |\mathcal{C}_{p<v}| + |\mathcal{C}_{p=v}|. \tag{3.1}$$

The path in Figure 3.2 has $|\mathcal{C}| = |\mathcal{C}_{p>v}| + |\mathcal{C}_{p<v}| + |\mathcal{C}_{p=v}| = 3 + 1 + 2 = 6$, which is the number of H steps in the corresponding bargraph in Figure 3.5.

Notice however that for the subset $\mathcal{C}_{p=v}$, there will always be a single unused step, e.g., say for some peak height we have one peak and one valley udu then we use the peak ud to create a H step and leave the remaining u step unused. We take a set $\mathcal{L}_{p=v}$ of all the unused steps from each case $\mathcal{C}_{p=v}$ together with the set $\mathcal{O}$ of all the other steps that are not in $\mathcal{C}$ then we have the rest of the steps in P. It follows that,

$$\begin{aligned} 2sl(P) &= 2(|\mathcal{C}_{p>v}| + |\mathcal{C}_{p<v}| + |\mathcal{C}_{p=v}|) + (|\mathcal{L}_{p=v}| + |\mathcal{O}|) \\ sl(P) &= |\mathcal{C}_{p>v}| + |\mathcal{C}_{p<v}| + |\mathcal{C}_{p=v}| + \left(\frac{|\mathcal{L}_{p=v}| + |\mathcal{O}|}{2}\right) \\ &= \#H + \left(\frac{|\mathcal{L}_{p=v}| + |\mathcal{O}|}{2}\right) \quad \text{(By Equation 3.1).} \end{aligned} \tag{3.2}$$

The path in Figure 3.2 has $|\mathcal{L}_{p=v}| = 2$ since $|\mathcal{C}_{p=v}| = 2$. These 2 steps in $\mathcal{L}_{p=v}$ are the steps ending on $x = 4$ and $x = 16$. The steps in $\mathcal{O}$ are the steps ending on $x = 1$ and $x = 7$, giving that $|\mathcal{O}| = 2$.

We solve for $\#U$ in B by considering the number of peaks and valleys in B as follows:

By definition of a bargraph, the number of upward steps U is always equal to the number of downward steps D therefore

$$\#U = \frac{\#U + \#D}{2}. \tag{3.3}$$

By definition of the bijection, the resulting bargraph will either have an upward step or downward step whenever we encounter a different step height in P. This will occur before $\mathcal{C}$, the consecutive peaks or valleys at a peak height, and $\mathcal{O}$, all the other steps that are not part of any peaks or valleys in P. There will be a final downward step after all the cases giving that

$$\#U + \#D = |\mathcal{O}| + Occ(\mathcal{C}_{p>v}) + Occ(\mathcal{C}_{p<v}) + Occ(\mathcal{C}_{p=v}) + 1, \tag{3.4}$$

where

i. $Occ(\mathcal{C}_{p>v})$ is the number of occurrences of consecutive peaks at a peak height, where the number of connected peaks is more than the number of connected valleys,

ii. $Occ(\mathcal{C}_{p<v})$ is the number of occurrences of consecutive valleys at the valley height, where the number of connected valleys is more than the number of connected peaks,

iii. $Occ(\mathcal{C}_{p=v})$ is the number of occurrences of consecutive peaks at the peak height, where the number of connected peaks is equal to the number of connected valleys.

The path in Figure 3.2 has 2 occurrences of $\mathcal{C}_{p=v}$ at peak heights 2 and 1 with the single consecutive peaks $x = 2$ and $x = 15$. There are 2 occurrences of $\mathcal{C}_{p>v}$ at peak heights 3 and 2 with the single consecutive peak $x = 5$ and two consecutive peaks $x = 10$ and $x = 12$. Lastly, there is 1 occurrence of $\mathcal{C}_{p<v}$ at height 1 with the single consecutive valley $x = 8$. Recall that $|\mathcal{O}| = 2$,

therefore, using Equation (3.4) we get that $\#U+\#D = 2+2+1+2+1 = 8$, as shown by the bargraph in Figure 3.5.

Notice that all the peaks and valleys in B come from the subsets $\mathcal{C}_{p>v}$ or $\mathcal{C}_{p<v}$. This is because in the case of $\mathcal{C}_{p>v}$ and $\mathcal{C}_{p<v}$ the first and last step at the peak and valley height will be the same, thus the height change before and after $\mathcal{C}_{p>v}$ or $\mathcal{C}_{p<v}$ will always be in the opposite direction of each other therefore creating a peak or valley in the bargraph. However, in the case when $\mathcal{C}_{p=v}$, the first step and the last step at the peak or valley height can never be the same, thus the height change before and after $\mathcal{C}_{p=v}$ will be in the same direction therefore it will always create the sequence of steps $DH^{\ell}D$ or $UH^{\ell}U$. Since peaks and valleys in B alternate, always starting with peaks, the number of peaks is always one more than the number of valleys. That is, $pk(B) = vl(B) + 1$, giving that

$$\begin{aligned} Occ(\mathcal{C}_{p>v}) + Occ(\mathcal{C}_{p<v}) &= pk(B) + vl(B) \\ &= pk(B) + (pk(B) - 1) \\ &= 2pk(B) - 1. \end{aligned} \tag{3.5}$$

The number of unused steps $|\mathcal{L}_{p=v}|$ from each subset $\mathcal{C}_{p=v}$ is exactly equal to $Occ(\mathcal{C}_{p=v})$, then

$$|\mathcal{L}_{p=v}| = Occ(\mathcal{C}_{p=v}).$$

We substitute Equation (3.5) into Equation (3.4) and obtain that

$$\begin{aligned} \#U + \#D &= |\mathcal{O}| + Occ(\mathcal{C}_{p>v}) + Occ(\mathcal{C}_{p<v}) + Occ(\mathcal{C}_{p=v}) + 1 \\ &= |\mathcal{O}| + Occ(\mathcal{C}_{p=v}) + 2pk(B) \\ &= |\mathcal{O}| + |\mathcal{L}_{p=v}| + 2pk(B). \end{aligned} \tag{3.6}$$

The semiperimeter of a bargraph is therefore

$$\begin{aligned}
sp(B) &= \#U + \#H \\
&= \#U + sl(P) - \frac{|\mathcal{L}_{p=v}| + |\mathcal{O}|}{2} && \text{(by Equation (3.2))} \\
&= \frac{\#U + \#D}{2} + sl(P) - \frac{|\mathcal{L}_{p=v}| + |\mathcal{O}|}{2} && \text{(by Equation (3.3))} \\
&= \frac{|\mathcal{O}| + |\mathcal{L}_{p=v}| + 2pk(B)}{2} + sl(P) - \frac{|\mathcal{L}_{p=v}| + |\mathcal{O}|}{2} \\
& && \text{(by Equation (3.6))} \\
&= pk(B) + sl(P).
\end{aligned}$$

□

3.2 The generating function

We have proved the bijection between bargraphs and Dyck paths and consequently showed the relationship between some statistics. Dyck paths are enumerated by their semilength and thus, using the statistics: semiperimeter and number of peaks of bargraphs, we show that Catalan numbers can be interpreted with regard to these statistics.

The following corollary follows directly from Theorem 3.1 and was originally proven in [12].

Corollary 3.2.1. Let $\mathcal{B}$ be a set of bargraphs, then the Catalan number C_n is the cardinality of a set of bargraphs whose semiperimeter minus the number of peaks is n, that is,

$$C_n = |B \in \mathcal{B} : sp(B) - pk(B) = n|. \tag{3.7}$$

Proof. Let z represent the semiperimeter and w represent the number of peaks of B, then the generating function G of bargraphs in terms of semiperimeter and peaks is given by

$$G(w, z) = \sum_{B \in \mathcal{B}} w^{pk(B)} z^{sp(B)}. \tag{3.8}$$

It was shown by Blecher, Brennan and Knopfmacher in [4] and originally by Bousquet-Mélou and Rechnitzer in [5] that by the wasp-waist decomposition, we can factor any bargraph into the following:

1. A single block (semiperimeter 2 and one peak), or
2. A single block (semiperimeter 2) attached to a bargraph, or
3. A bargraph raised by one step up (semiperimeter 1), or
4. A bargraph raised by one step up (semiperimeter 1) and attached to a single block (semiperimeter 1), or
5. A bargraph raised by one step up (semiperimeter 1), connected to a single block (semiperimeter 1) then connected to another bargraph.

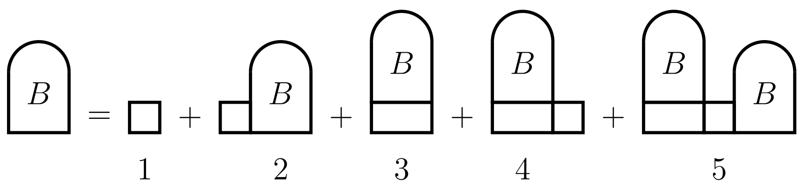


Figure 3.7: Wasp-waist decomposition of bargraphs.

Let $G(w, z)$ be the generating function of a bargraph that is enumerated by the number of peaks w and semiperimeter z, then its functional equation is given by

$$G(w,z) = \underbrace{z^2 w}_{1} + \underbrace{zG(w,z)}_{2} + \underbrace{zG(w,z)}_{3} + \underbrace{z^2 G(w,z)}_{4} + \underbrace{zG(w,z)\left(G(w,z) - \frac{z^2 w}{1-z} + \frac{z^2}{1-z}\right)}_{5}. \tag{3.9}$$

In Case 5, we remove any subsequent bargraph consisting of a sequence of columns of height one, since all the possible blocks adjacent to the middle single block would not form any peak. We then add back the columns without counting the peak.

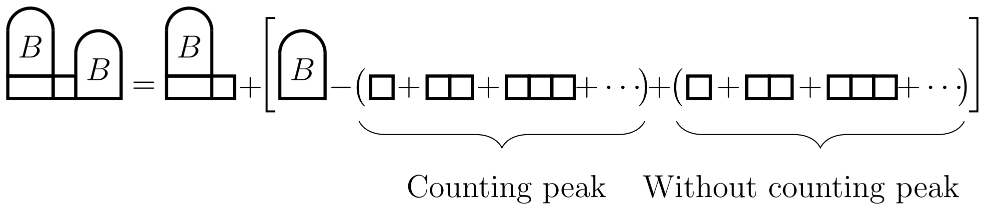


Figure 3.8: Case 5 decomposition.

Therefore, we simplify Equation (3.9) to become

$$G(w,z) = z^2w + 2zG(w,z) + z^2G(w,z) + zG(w,z)\left(G(w,z) - \frac{z^2w}{1-z} + \frac{z^2}{1-z}\right),$$

which we rearrange to get

$$z^2(1-z)w - (1+z^2-3z+z^3w)G(w,z) + z(1-z)G(w,z)^2 = 0. \quad (3.10)$$

To obtain the generating function $H(w,z)$ of bargraphs in terms of semiperimeter minus peaks, let

$$H(w,z) := G\left(\frac{1}{w}, wz\right) = \sum_{B\in\mathcal{B}} w^{sp(B)-pk(B)} z^{sp(B)},$$

then Equation (3.10) becomes

$$wz^2(1-wz) - (1-3wz+wz^2+w^2z^3)H(w,z) + wz(1-wz)H(w,z)^2 = 0.$$

Set $z=1$, which yields that

$$w(1-w) - (1-3w+w^2+w^2)H(w,1) + w(1-w)H(w,1)^2 = 0$$

and simplify to obtain

$$wH(w,1)^2 - (1-2w)H(w,1) + w = 0.$$

Solving for $H(w,1)$ gives

$$H(w,1) = \frac{1-\sqrt{1-4w}}{2w} - 1 = \sum_{n\geq 1} C_n z^n. \quad (3.11)$$

□

It is clear that Catalan numbers and consequently the enumeration of Dyck paths of semilength n can be interpreted as the number of bargraphs whose semiperimeter minus the number of peaks is exactly the semilength of those Dyck paths.

Chapter 4

Bargraphs and cornerless Motzkin paths

We derive the generating function of bargraphs using a bijection between cornerless Motzkin paths and bargraphs. This bijection and the decomposition of cornerless Motzkin paths are originally proved by Elizalde and Deutsch in [11]. We have already derived the generating function of bargraphs using the wasp-waist decomposition, however we will study a much easier proof using the decomposition of Motzkin paths and the bijection they share with bargraphs shown in [11].

4.1 Definitions

We begin by giving the definition of the Motzkin path family and then define cornerless Motzkin paths that belong to that family.

Definition 4.1.1. A *Motzkin path* is a non-negative lattice path with steps $u = (1, 1)$, $h = (1, 0)$ and $d = (1, -1)$ that starts at the origin and ends on the x-axis (see [11]).

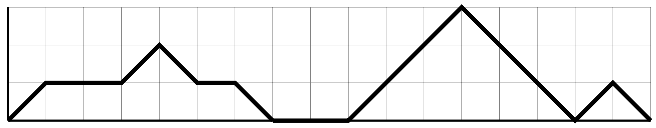


Figure 4.1: A Motzkin path *uhhudhdhhuuuddd*.

Definition 4.1.2. A *cornerless Motzkin path* is a non-negative lattice path with steps $u = (1, 1)$, $h = (1, 0)$ and $d = (1, -1)$ that start at the origin and end on the x-axis with no consecutive ud or du steps (see [11]).

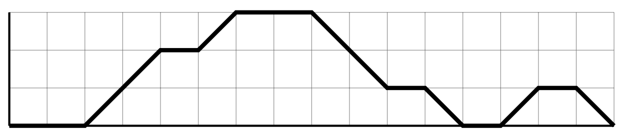


Figure 4.2: A cornerless Motzkin path *hhuuhuhhddhdhuhd*.

4.2 Bijection

We now show the bijection between cornerless Motzkin paths and bargraphs.

Proposition 4.2.1. Let M_n be the set of cornerless Motzkin paths where the number of up steps and horizontal steps are n and let B_n be the set of bargraphs of semiperimeter n. Then there exists a bijection $\Delta : M_{n-1} \leftrightarrow B_n$ between cornerless Motzkin paths and bargraphs.

Proof. We define the mapping from cornerless Motzkin paths to bargraphs as follows:

- Replace the steps $u = (1, 1)$ and $d = (1, -1)$ with $u' = (0, 1)$ and $d' = (0, -1)$ respectively, and leave the $h = (1, 0)$ steps as is.
- Raise the path by adding an initial $u = (0, 1)$ and consequently a last return $d = (0, -1)$.

The mapping from bargraphs to cornerless Motzkin paths is defined as follows:

- Replace the steps $u' = (0,1)$ and $d' = (0,-1)$ with $u = (1,1)$ and $d = (1,-1)$ respectively, and leave the $h = (1,0)$ steps as is.
- Lower the path by removing the initial $u = (0,1)$ and consequently the last return $d = (0,-1)$.

□

Example 4.2.1. Consider the cornerless Motzkin path M_{12} in Figure 4.2 with $n = 12$. Its corresponding bargraph in Figure 4.3 has semiperimeter 13.

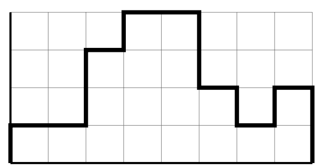


Figure 4.3: A bargraph with semiperimeter 13.

4.3 The generating function

The functional equation of cornerless Motzkin paths can be found by characterising them with a decomposition such that every cornerless Motzkin path can be broken down into either

1. An empty path,
2. A path that starts with the horizontal step h,
3. An elevated non-empty path,
4. An elevated non-empty path that ends with the horizontal step h followed by a possibly empty path.

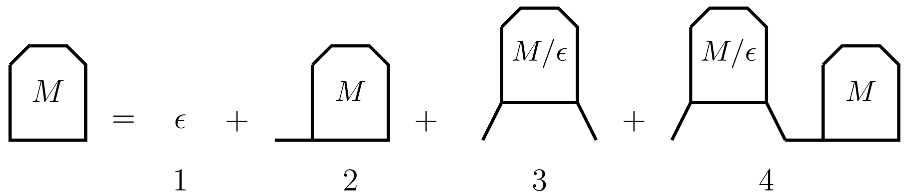


Figure 4.4: Decomposition of cornerless Motzkin paths.

Let x denote the number of horizontal steps and y denote the number of upward steps, then the functional equation of cornerless Motzkin paths is given by

$$M(x,y) = 1 + xM(x,y) + y(M(x,y) - 1) + y(M(x,y) - 1)xM(x,y). \quad (4.1)$$

From the bijection above, we have established that $B(x,y) = y(M(x,y)-1)$. That is, an extra up step is added to the cornerless Motzkin path when we raise it.

We substitute $M(x,y) = B(x,y)/y + 1$ into Equation (4.1) to get that

$$\begin{aligned} B(x,y)/y = {} & x(B(x,y)/y + 1) + y(B(x,y)/y) \\ & + y(B(x,y)/y)x(B(x,y)/y + 1). \end{aligned} \quad (4.2)$$

We simplify Equation (4.2) to obtain that

$$B(x,y) = xB(x,y) + xy + yB(x,y) + xB(x,y)^2 + xyB(x,y). \quad (4.3)$$

We then rearrange Equation (4.3) to find that

$$0 = xB(x,y)^2 - (1 - x - y - xy)B(x,y) + xy. \quad (4.4)$$

We solve Equation (4.4) using the quadratic formula to obtain the generating function of bargraphs in terms of the number of up steps and horizontal steps which gives

$$B(x,y) = \frac{(1 - x - y - xy) - \sqrt{(1 - x - y - xy)^2 - 4x^2y}}{2x}.$$

Let $y = x$ then we get the generating function of bargraphs in terms of semiperimeter is given by

$$B(x) = \frac{(1 - 2x - x^2) - \sqrt{1 - 4x + 2x^2 + x^3}}{2x}.$$

By the Taylor expansion formula, we find that

$$B(x) = x^2 + 2x^3 + 5x^4 + 13x^5 + 35x^6 + 97x^7 + O(x^8).$$

We can deduce that there are 5 bargraphs with semiperimeter 4.

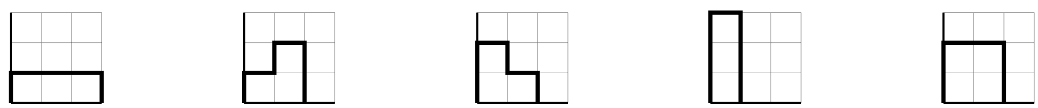


Figure 4.5: Bargraphs with semiperimeter = 4.

Chapter 5

Generalisations of Dyck paths

In this chapter, we show that the enumeration of generalised Dyck paths can be found using generalised Catalan numbers. This was proved by Selkirk in [20] and several other papers, however we will mainly focus on the methods used in [20]. We establish a bijection between d-ary trees and k-Dyck paths and then another bijection between k-Dyck paths and k_t-Dyck paths, and derive the enumerations from each bijection. Lastly, we show the derivation of the enumeration of k_t-Dyck paths directly using the symbolic method.

5.1 Definitions

Definition 5.1.1. A k-Dyck path is a non-negative lattice path with the steps $u = (1, 1)$ and $d = (1, -k)$ that starts at the origin $(0, 0)$ and ends on the x-axis at $((k + 1)n, 0)$ where $n \in \{0, 1, 2, 3, ...\}$ and $k \in \mathbb{N}$.

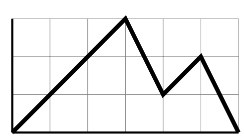
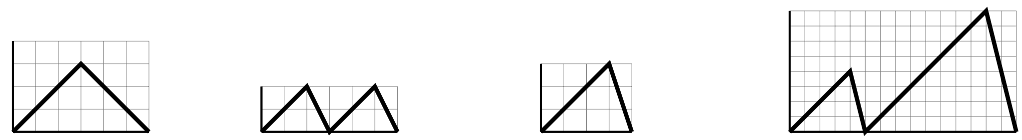


Figure 5.1: A 2-Dyck path $uuudud$, with $n = 2$.

More examples:



(a) k=1. (b) k=2. (c) k=3. (d) $k = 4$.

Figure 5.2: k-Dyck paths.

Definition 5.1.2. A k_t-Dyck path is a lattice path with the steps $u = (1, 1)$ and $d = (1, -k)$ that starts at the origin $(0, 0)$ and ends on the x-axis at $((k + 1)n, 0)$ but does not go below $y = -t$, where $n \in \{0, 1, 2, 3, ...\}$, $k \in \mathbb{N}$ and $0 \leq t \leq k$.

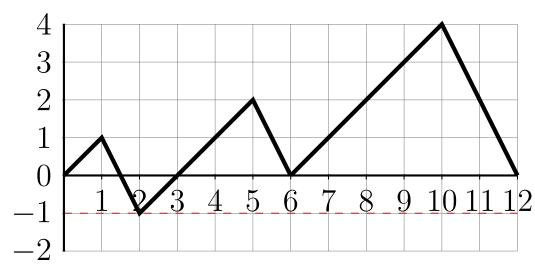
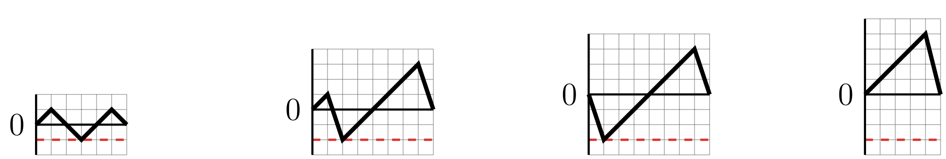


Figure 5.3: k_t-Dyck path with $t = 1, k = 2$ and $n = 4$.

More examples:



(a) k=1, t=1. (b) k=3, t=2. (c) k=3, t=3. (d) $k = 4$, $t = 3$.

Figure 5.4: k_t-Dyck paths.

Definition 5.1.3. A *rooted tree* is a rooted connected acyclic graph made up of internal (circles in Figure 5.5) and external (rectangles in Figure 5.5) nodes.

The internal node at the top of the tree is called the root node. Each internal node is connected to adjacent internal or external nodes by edges. Note that an internal node is always a parent node and a child node except if it is the root node.

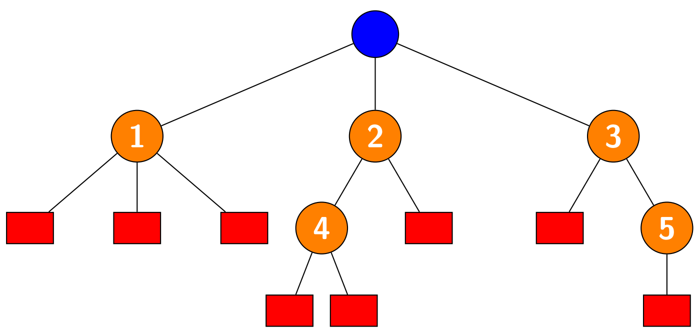


Figure 5.5: A rooted tree with 14 nodes, 6 parent nodes and 13 child nodes.

Remark. Internal nodes 1,2,3,4 and 5 are both parent and child nodes. e.g., node 2 is a child node to the root node and it also a parent to two child nodes, one of which is the internal node 4 in orange and the other the adjacent external node in red.

Definition 5.1.4. A d-ary tree is a rooted tree where each internal node has exactly d child nodes (see [20]).

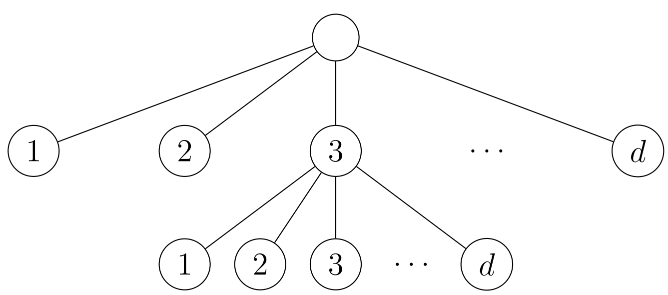


Figure 5.6: A d-ary tree with $2d + 1$ nodes.

We let $\mathcal{D}$ denote the class of d-ary trees then we can think of d-ary trees as an empty tree or a d-ary tree where the root tree has d child nodes with each child node being a d-ary tree itself, as illustrated below:

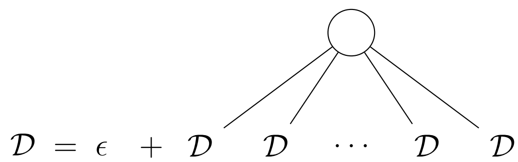


Figure 5.7: Representation of d-ary trees.

Lemma 5.1.1. The generating function of d-ary trees is defined as,

$$D(z) = \sum_{n \geq 0} d_n z^n,$$

where $d_n = \dfrac{1}{(d-1)n+1}\dbinom{dn}{n}$ is the number of d-ary trees with n internal nodes.

Proof. Using the representation of d-ary trees in Figure 5.7 above, let z denote an internal node, then the functional equation of d-ary trees is given by $D(z) = 1 + zD(z)^d$. We apply Theorem 2.2.1 to extract the n-th coefficient from the generating function.

Recall Theorem 2.2.1 (Lagrange inversion theorem):

$$g_n \equiv [z^n]G(z) = \frac{1}{n}[u^{n-1}]\left(\frac{u}{f(u)}\right)^n.$$

We apply the theorem to the d-ary trees functional equation as follows:

Let $G(z) = D(z) - 1$ then $z = \dfrac{G(z)}{(G(z)+1)^d}$ and $f(z) = \dfrac{z}{(z+1)^d}$ satisfies $f(0) = 0$ and $f'(0) \neq 0$.

Then for $n \geq 1$,

$$\begin{aligned} g_n = d_n \equiv [z^n]G(z) = [z^n]D(z) - 1 &= \frac{1}{n}[u^{n-1}]\left(\frac{u}{f(u)}\right)^n \\ &= \frac{1}{n}[u^{n-1}]\left(\frac{u(u+1)^d}{u}\right)^n \\ &= \frac{1}{n}[u^{n-1}]\left((u+1)^d\right)^n \\ &= \frac{1}{n}[u^{n-1}]\,(u+1)^{dn} \\ &= \frac{1}{n}\binom{dn}{n-1} \\ &= \frac{(dn)!}{n(n-1)!(dn-n+1)!} \\ &= \frac{1}{dn-n+1}\binom{dn}{n} \\ &= \frac{1}{(d-1)n+1}\binom{dn}{n}. \end{aligned}$$

□

5.2 d-ary trees and k-Dyck paths

We show that k-Dyck paths are enumerated by the generalised Catalan numbers (see [20])

$$C_{n,k} = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

5.2.1 Bijection

We establish a bijection between $(k+1)$-ary trees and k-Dyck paths, and then use that bijection to show enumeration of k-Dyck paths.

Proposition 5.2.1. There exists a bijection between $(k+1)$-ary trees and k-Dyck paths.

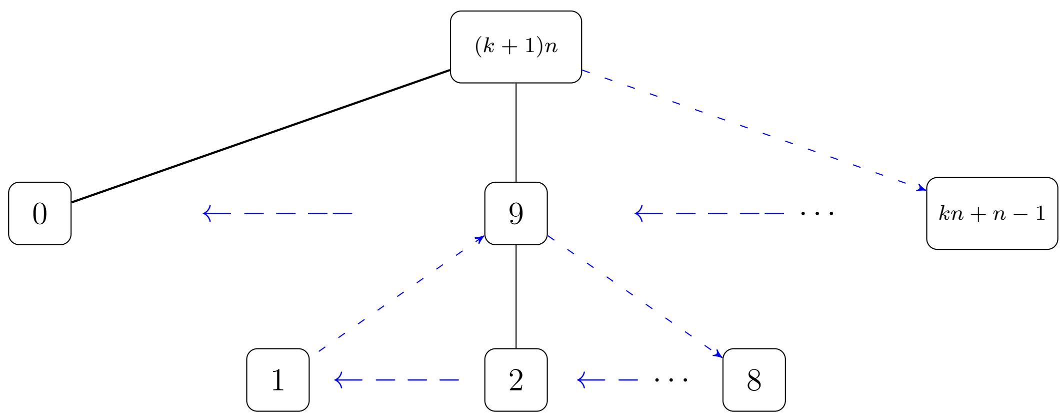


Figure 5.8: Post order traversal of a tree.

Proof. The mapping from $(k+1)$-ary trees to k-Dyck paths is described as follows:

We apply post order traversal on the tree, that is, label the $(k+1)$-ary tree nodes from right to left in descending order starting at the root node with $(k+1)n$ until the left-most external node with 0. The labelling is such that when we arrive at an internal node, we label its children from right to left in descending order, continuing from the numbering of that parent node. When we are done labelling all the children, we travel up to the parent node and continue labelling the rest of the tree in the same manner.

To plot our corresponding k-Dyck path, we assign the upward step $u = (1, 1)$ to the external nodes and the downward step $d = (1, -k)$ to the internal nodes. We plot the path starting at the origin $(0, 0)$ with the step ending on the x-coordinate corresponding to the label on each node until the root node $(k + 1)n$. It is clear that the node labelled 0 will be ignored.

The mapping from k-Dyck paths back to $(k + 1)$-ary trees is described as follows:

We first label each upward step $u = (1, 1)$ and downward $d = (1, -k)$ as an external node and internal node, respectively. To obtain the corresponding tree, we travel from right to left of the Dyck path, that is, starting at the final return until the initial step. We draw the tree from right to left starting with the root node that corresponds to the final step of the Dyck path and then inserting an internal node or an external node accordingly as we travel on the path. Our resulting tree will be missing one external node from its left-most internal node, so we append an extra external node to that internal node.

□

Recall from Lemma 5.1.1 that the number of k-ary trees are enumerated by

$$\frac{1}{(k-1)n+1}\binom{kn}{n},$$

therefore as a direct consequence of the bijection between $(k + 1)$-ary trees and k-Dyck paths we get the theorem below.

Theorem 5.2.1. The number of k-Dyck paths is enumerated by

$$\frac{1}{kn+1}\binom{(k+1)n}{n}.$$

Proof. This follows directly from the enumeration of $(k + 1)$-ary trees and the bijection. □

Example 5.2.1. Consider the labelled 2-ary (binary) tree with $n = 4$ internal nodes given in Figure 5.9.

$\Rightarrow$ We label the tree in descending order and assign each internal node to a downward step $d = (1, -1)$ and each external node to an upward step $u = (1, 1)$ to obtain the corresponding 1-Dyck path.

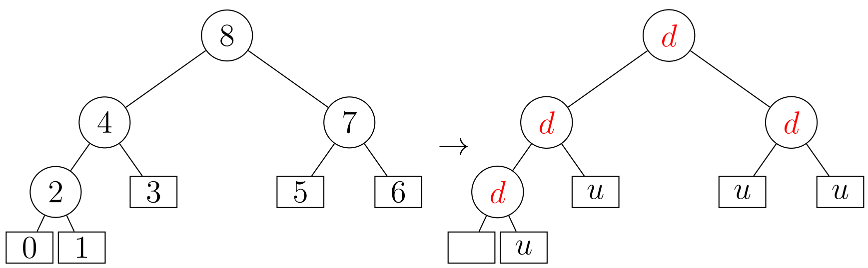


Figure 5.9: A labelled 2-ary tree with $n = 4$.

This gives the Dyck path in Figure 5.10.

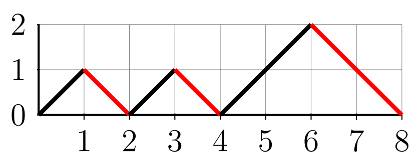


Figure 5.10: The corresponding 1-Dyck path with $n = 4$.

$\Leftarrow$ To get back the 2-ary tree, let every upward step $u = (1, 1)$ be an external node and every downward step $d = (1, -1)$ be an internal node. Read the Dyck path from right to left and plot the tree from right to left starting at the root until the last external node and append the left-most external most to complete the tree.

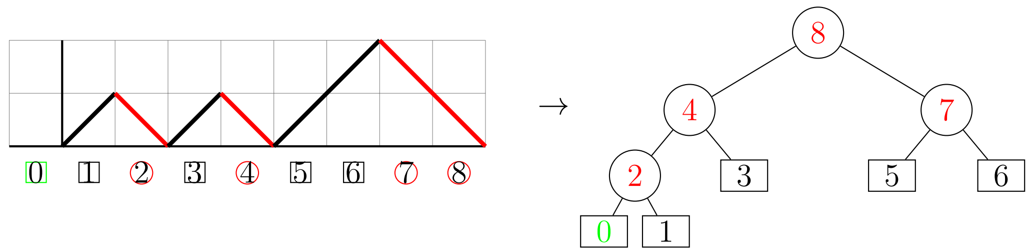


Figure 5.11: The corresponding 2-ary tree, with $n = 4$.

Example 5.2.2. Consider the 4-ary tree given in Figure 5.12 below.

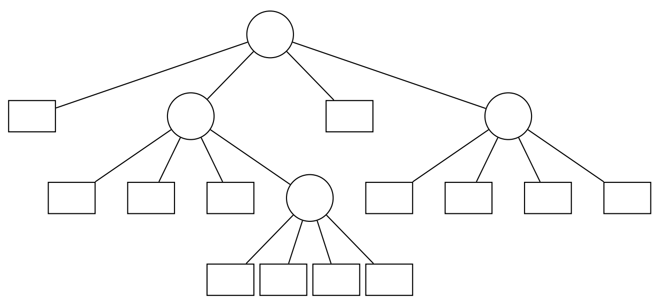


Figure 5.12: A 4-ary tree with $n = 4$.

$\Rightarrow$ We traverse the tree, labelling each node in descending order until 0, and then we let the upward steps $u = (1, 1)$ be the external nodes and the downward steps $d = (1, -3)$ be the internal nodes.

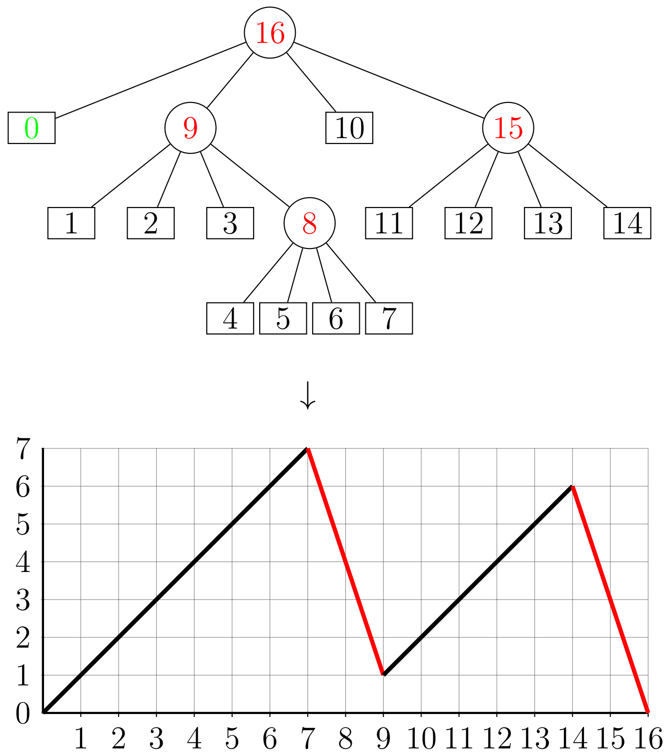


Figure 5.13: The corresponding 3-Dyck path.

$\Leftarrow$ To get back the original tree. Let every upward step $u = (1, 1)$ be an external node and every downward step $d = (1, -1)$ be an internal node. The last downward step ending on $x = 16$ will be the root node which has the child node from the downward step ending on $x = 15$. This child node which is also an internal node will have adjacent external nodes from the steps ending on $x = 14, 13, 12, 11$. The step ending on $x = 10$ will become an external node of the root node. The step ending on $x = 9$ is adjacent to the root node and has a child node from the step ending on $x = 8$. This child node which is also an internal node will have adjacent external nodes from the steps ending on $x = 7, 6, 5, 4$. The steps ending on $x = 3, 2, 1$ become the external nodes adjacent to the internal node from the step ending on $x = 9$. The root node will have 3 child nodes, therefore we add an extra external node to complete the 4-ary tree. ■

Example 5.2.3. Consider the 2-Dyck path given in Figure 5.14 below.

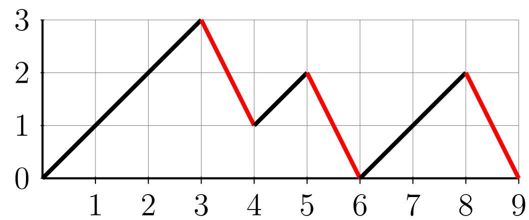


Figure 5.14: A 2-Dyck path, with $n = 3$.

$\Leftarrow$ We begin by labelling each of the upward steps $u = (1, 1)$ and downward steps $d = (1, -2)$ as external nodes and internal nodes, respectively. We add an external node labelled 0 at the beginning. Then we plot the 3-ary tree as starting at the root as follows:

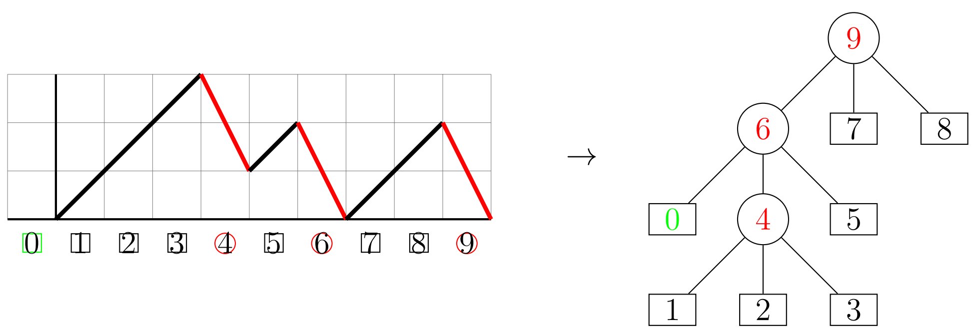


Figure 5.15: The corresponding 3-ary tree.

■

5.3 k-Dyck paths and k_t-Dyck paths

We have derived the enumeration of k-Dyck paths in the previous section. Now we extend this enumeration to k_t-Dyck paths by establishing a bijection between these two sets of Dyck paths to show that k_t-Dyck paths are enumerated by the generalised Catalan numbers (see [20])

$$C_{n,k_t} = \frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

Definition 5.3.1. A generalised binomial series is defined by Knuth in [14, Section 5.4] as

$$\mathcal{B}_m(x) := \sum_{n\geq 0} \frac{1}{mn+1}\binom{mn+1}{n}x^n.$$

Lemma 5.3.1. The enumeration of k-Dyck paths is given by

$$\frac{1}{kn+1}\binom{(k+1)n}{n},$$

the n-th coefficient of the generalised binomial series $\mathcal{B}_{k+1}(x)$.

Proof. Given

$$\mathcal{B}_m(x) := \sum_{n\geq 0} \frac{1}{mn+1}\binom{mn+1}{n}x^n,$$

let $m = k+1$ then

$$\begin{aligned}
\mathcal{B}_{k+1}(x) &= \sum_{n\geq 0} \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}x^n \\
&= \sum_{n\geq 0} \frac{1}{(k+1)n+1}\frac{((k+1)n+1)!}{n!((k+1)n+1-n)!}x^n \\
&= \sum_{n\geq 0} \frac{1}{(k+1)n+1}\frac{((k+1)n+1)!}{n!(kn+1)!}x^n \\
&= \sum_{n\geq 0} \frac{((k+1)n)!}{n!(kn+1)!}x^n \\
&= \sum_{n\geq 0} \frac{((k+1)n)!}{n!(kn)!(kn+1)}x^n \\
&= \sum_{n\geq 0} \frac{1}{kn+1}\binom{(k+1)n}{n}x^n.
\end{aligned}$$

□

Definition 5.3.2. The generalised binomial series given in Definition 5.3.1 satisfies the following identity with regard to its tuple of objects (see [14, Section 5.4]):

$$(\mathcal{B}_m(x))^r := \sum_{n\geq 0} \frac{r}{mn+r}\binom{mn+r}{n}x^n.$$

5.3.1 Bijection

We establish a bijection between k_t-Dyck paths and an ordered tuple of $(t+1)$ k-Dyck paths and then use the generalised binomial series identity in Definition 5.3.2 above to show that the generating function of k_t-Dyck paths is given by

$$(\mathcal{B}_{k+1}(x))^{t+1} = \sum_{n\geq 0} \frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}x^n.$$

Proposition 5.3.1. There exists a bijection between an ordered tuple of $(t+1)$ k-Dyck paths and k_t-Dyck paths.

Proof. We define the mapping from k_t-Dyck paths to k-Dyck paths as follows:

Consider a k_t-Dyck path of length $(k+1)n$ where n is the number of downward steps $d = (1, -k)$. Find a point that is the furthest right at the lowest level $y = -t + i$, where $0 \leq i \leq t$, that the path touches and label it m_i. Repeat the process until you reach the last level $y = -t+t = 0$ on the x-axis at $x = (k+1)n$.

We read the path from left to right. To obtain the $(t+1)$ tuple of k-Dyck paths, we look for a labelled m_i point at each level starting at $y = -t$ and if there exists such a m_i for $0 \leq i \leq t$ then we slice the k_t-Dyck path at the next point to the right that lies on the x-axis to obtain a k_t-Dyck subpath p_i, where p_i is in the tuple, and the remaining Dyck path. Otherwise, if there exists no such labelled point at level $y = -t$, then we obtain an empty k-Dyck path in the tuple. We repeat this process for each subsequent labelled $m_{i'}$ point, where $i' > i$, in the remaining Dyck path, with the exception that if the $m_{i'}$ point lies in p_i then we get an empty k-Dyck path in the tuple and if it is in the remaining Dyck path then we slice that path at the next point to the right that lies on the x-axis and add another path in the tuple. We recursively repeat this process until we arrive at the last point on the x-axis at $(k+1)n$.

We now have $(t+1)$ subpaths of the k_t-Dyck path. We simply move each down step $d = (1, -k)$ to the right and up by $(t-i)$ steps for each subpath p_i where $0 \leq i \leq t$ and connect the rest of the path with $u = (1, 1)$ steps to get an ordered tuple $(q_0, q_1, .., q_t)$ of $(t+1)$ k-Dyck paths.

The mapping from tuples of k-Dyck paths to k_t-Dyck paths is defined as follows:

For each k-Dyck path q_i where $0 \leq i \leq t$ in the tuple, shift each step $d = (1, -k)$ to the left and down by $(t-i)$ steps to get the subpaths p_i of the k_t-Dyck path and then join all the subpaths to get the corresponding k_t-Dyck path.

□

Example 5.3.1. Consider the 3_3-Dyck path given in Figure 5.16. The furthest right point at the lowest level $y = -3$ is located at $x = 9$ which we label m_0 and the furthest right point at level $y = -2$ is located at $x = 14$ which we label m_1 and at level $y = -1$ located at $x = 15$ which we label m_2 and lastly at level $y = 0$ located at $x = 16$ labelled m_3.

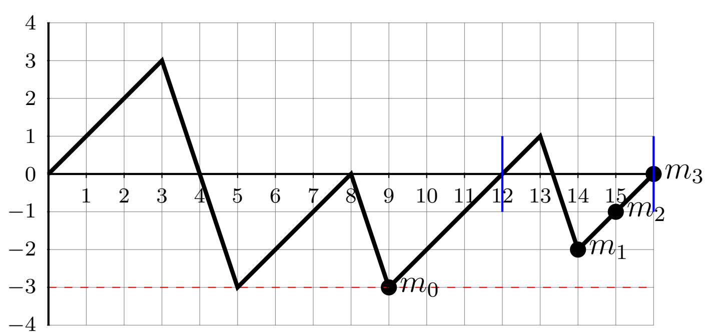


Figure 5.16: A 3_3-Dyck path.

We break this path into 2 subpaths p_0 and p_1 by slicing at $x = 12$ and $x = 16$, the next point on the x-axis from m_0 and m_1 respectively. Since m_2 and m_3 lie on p_1, we will get 2 empty k_t-Dyck paths.

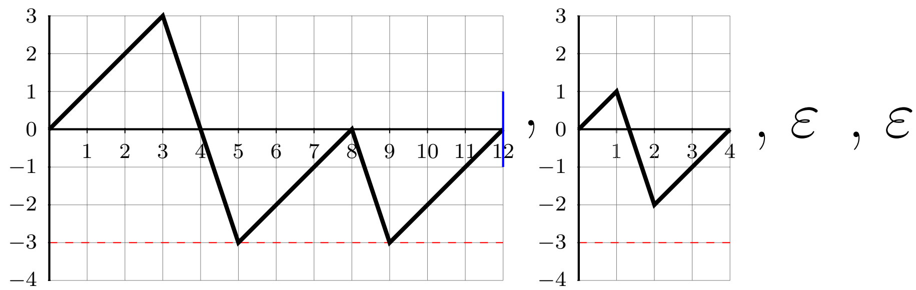


Figure 5.17: Four subpaths p_0, p_1, p_2 and p_3 of the path in Figure 5.16.

We then move every step $d = (1, -k)$ in subpath p_0 right and up by $3 - 0 = 3$ and every step in subpath p_1 right and up by $3 - 1 = 2$ and use $u = (1, 1)$ to complete the subpaths and obtain the respective k-Dyck paths q_0,q_1,q_3,q_4.

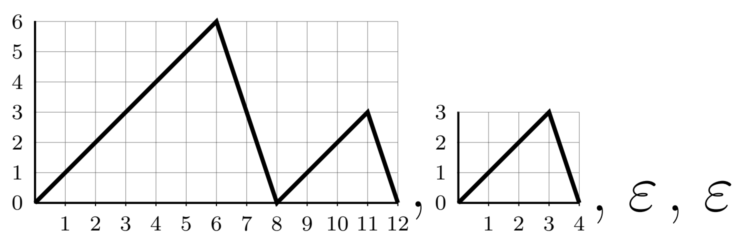


Figure 5.18: An ordered tuple of four 3-Dyck paths q_0,q_1,q_2,q_3 corresponding to Figure 5.16.

To get back the original k_t-Dyck path, we move every step $d = (1, -k)$ in q_0 left and down by 3 and every step in q_1 left and down by 2 and join all the resulting subpaths to obtain the respective k_t-Dyck path. ■

Example 5.3.2. Consider the 4_3-Dyck path given in Figure 5.19. There is no furthest right point m_0 at $y = -3$. The furthest right point m_1 at $y = -2$ is located at $x = 3$. The furthest right point m_2 at $y = -1$ is located at $x = 4$. Lastly, the furthest right point m_3 at $y = 0$ is located at $x = 10$.

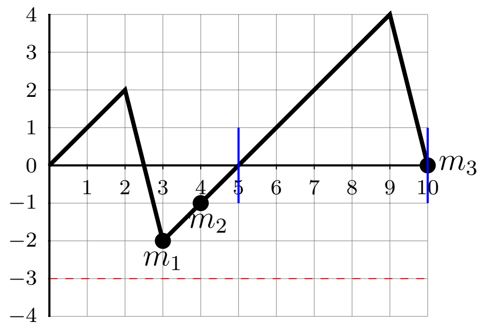


Figure 5.19: A 4_3-Dyck path.

Since there is no labelled point at $y = -3$, we get an empty 4_3 subpath. There is a labelled point at $y = -2$ and $y = 0$, then we slice the 4_3 path at $x = 5$ and $x = 10$ respectively. The point m_2 lies on the subpath p_1 therefore we get an empty 4_3 subpath.

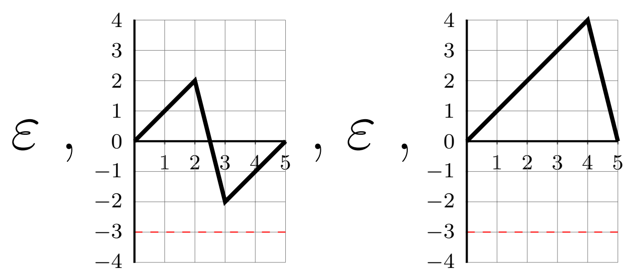


Figure 5.20: Four subpaths p_0, p_1, p_2 and p_3 of the path in Figure 5.19.

We then move the step $d = (1, -4)$ in p_1 right and up by 2 and the step $d = (1, -4)$ in p_3 right and up by 0, i.e, we don't move it, to obtain the ordered tuple:

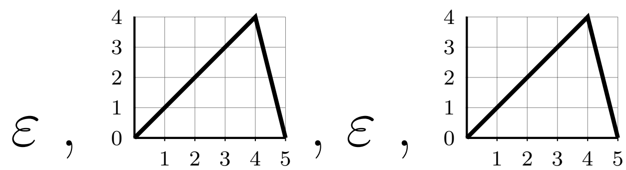


Figure 5.21: An ordered tuple of four 4-Dyck paths q_1,q_2,q_3,q_4 corresponding to Figure 5.19.

■

Other examples:

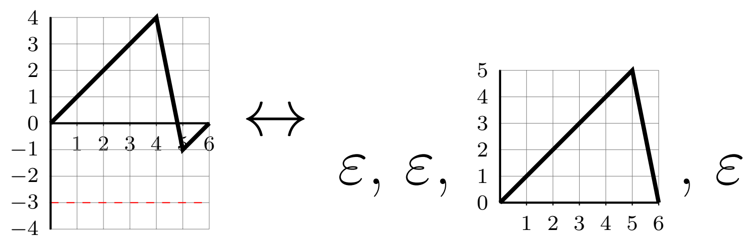


Figure 5.22: $k = 5$ and $t = 3$.

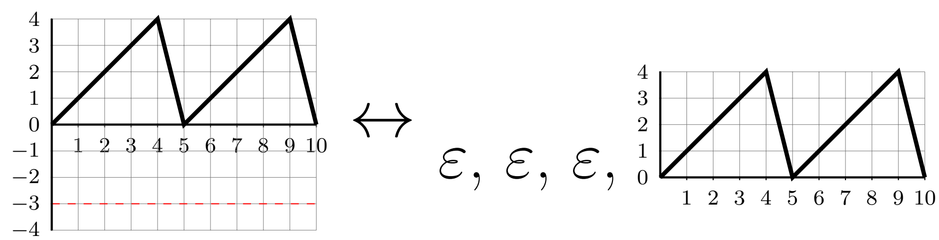


Figure 5.23: $k = 4$ and $t = 3$.

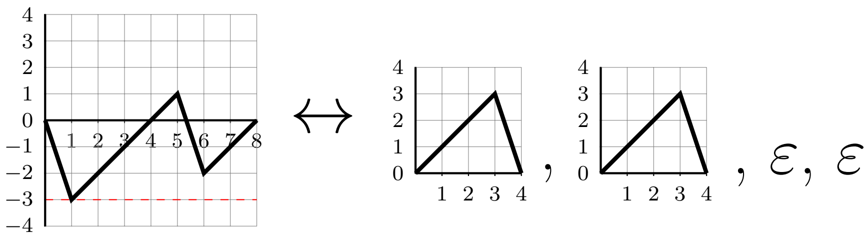


Figure 5.24: $k = 3$ and $t = 3$.

5.4 Generating functions

We have shown the enumeration of k_t-Dyck paths using a bijection. We now prove this enumeration directly using the symbolic method. This proof is given in [20].

5.4.1 k_t-Dyck paths

Theorem 5.4.1. Let $\mathcal{K}_t$ be the class of k_t-Dyck paths. For $k \geq 2$ and $0 \leq t \leq k$, the decomposition of k_t-Dyck paths with regard to the first return is given by

$$\mathcal{K}_t = \epsilon + \sum_{i=0}^{t} u^{k-i} \mathcal{K}_{k-i-1} d u^i \mathcal{K}_t$$

where $\mathcal{K}_{-1}$ is defined to be ϵ and u is an up step $(1,1)$ and d is a down step $(1,-k)$.

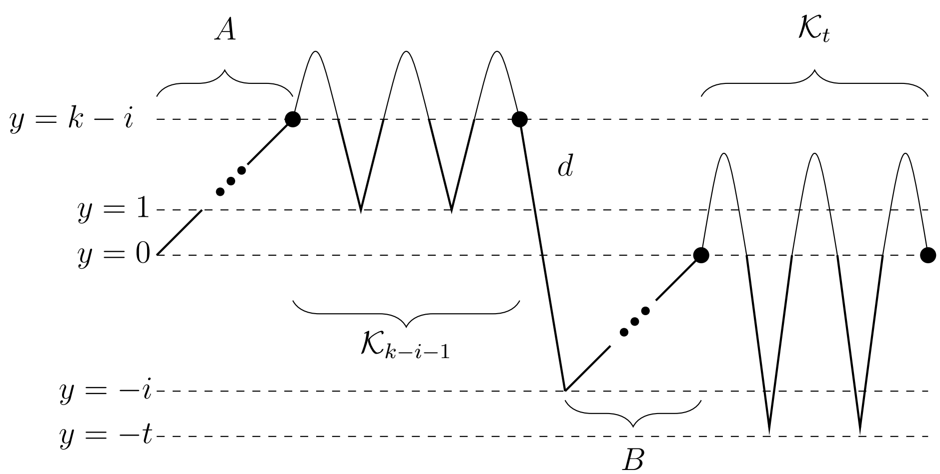


Figure 5.25: $\mathcal{K}_t$-decomposition.

Proof. Consider a chain A of length j of up steps at the beginning of the Dyck path that ends at $y = j = k - i$ and a chain B of length i of up steps below the x-axis after the down step d that starts at $y = -i$ and ends at $y = 0$.

By definition, a k_t-Dyck path must stay on and above $y = -t$, therefore the first return can either come from below or above the x-axis. When the first return comes from below the x-axis, there is the chain B with at most t up steps that ends on the x-axis. For example, when this chain has exactly t up steps, then it starts at $y = -t$ and ends at $y = 0$. When this chain has 0 up steps, then we have that the first return comes from above the x-axis. Since i is the length of the chain B, then it is clear that $0 \leq i \leq t \leq k$.

Preceding the chain B is the down step $d = (1, -k)$. When the length of the chain B is at least equal to 1 then the down step d crosses the x-axis from above (without creating a return) and connects with the chain below the x-axis leading to a first return that comes from below the x-axis. In the case when the length of the chain is 0 then the down step d will end on the x-axis and thus be the first return.

At the beginning of every k_t-Dyck path, there is the chain A of j initial up steps that are numerically equal to the levels of the down step d that are above the x-axis. Succeeding the chain A is a k_{j-1}-Dyck path. The k_{j-1}-Dyck path stays strictly above the x-axis, and therefore does not create any returns. When $j = 0$ then, the k_{-1}-Dyck path is empty and the Dyck path starts with the down step and A is empty. When $j = 1$ there is a single initial up step (A has length 1), which is then succeeded by a k_0-Dyck path (possibly empty) that stays above $y = 1$. When $j = k$ there is a k_{k-1}-Dyck path succeeding the initial j up steps. Clearly, no returns are created from the k_{j-1}-Dyck path.

The rest of the Dyck path after the first return is a k_t-Dyck path. Note that when $t < k$ then the k_t-Dyck path has to begin with an up step, otherwise the initial down step will go below $-t$.

In summary, in a k_t-Dyck path with a chain B of length i, the down step d will be $k - i$ above the x-axis and consequently chain A will have length $j = k - i$ followed by a k_{k-i-1}-Dyck path. We obtain the decomposition

$$\mathcal{K}_t = \epsilon + \sum_{i=0}^{t} A\mathcal{K}_{k-i-1} dB\mathcal{K}_t.$$

Since both chain A and chain B are made of up steps, we simply replace them with u to get that

$$\mathcal{K}_t = \epsilon + \sum_{i=0}^{t} u^{k-i}\mathcal{K}_{k-i-1} du^i \mathcal{K}_t$$

□

Now that we have the decomposition of k_t-Dyck paths, we can proceed to derive the generating function of k_t-Dyck paths.

Theorem 5.4.2. The generating function of k_t-Dyck paths is given by

$$\left(\mathcal{B}_{k+1}(x^{k+1})\right)^{t+1}.$$

Proof. The functional equation of k_t-Dyck paths can be deduced from Theorem 5.4.1 by letting x represent both down and up steps, then

$$K_t(x) = 1 + x^{k+1} K_t(x) \sum_{i=0}^{t} K_{k-i-1}(x). \tag{5.1}$$

We solve for $K_t(x)$ as follows:

First, we divide Equation (5.1) by $\frac{1}{K_t(x)}$ and rearrange to find that

$$\frac{1}{K_t(x)} = 1 - x^{k+1} \sum_{i=0}^{t} K_{k-i-1}(x).$$

We then subtract two consecutive values of t and get that

$$\frac{1}{K_t(x)} - \frac{1}{K_{t+1}(x)} = \left(1 - x^{k+1}\sum_{i=0}^{t} K_{k-i-1}(x)\right) - \left(1 - x^{k+1}\sum_{i=0}^{t+1} K_{k-i-1}(x)\right)$$

$$= x^{k+1} K_{k-t-2}(x). \tag{5.2}$$

We want to prove that $K_t(x) = \left(\mathcal{B}_{k+1}(x^{k+1})\right)^{t+1}$ using an ascending-descending induction proof, therefore we first need to get both the ascending and descending forms of $K_t(x)$.

To get the ascending form of $K_t(x)$, let $t = -1$ then by definition $K_{-1}(x) = 1$ and Equation (5.2) simplifies to

$$1 - \frac{1}{K_0(x)} = x^{k+1} K_{k-1}(x),$$

which implies that

$$K_0(x) = \left(1 - x^{k+1} K_{k-1}(x)\right)^{-1}.$$

We want to show that for all $0 \leq j \leq k-1$

$$K_j(x) = \left(1 - x^{k+1} K_{k-1}(x)\right)^{-j-1}. \tag{5.3}$$

For the descending form of $K_t(x)$, we let $t = k-1$ then Equation (5.2) becomes

$$\frac{1}{K_{k-1}(x)} - \frac{1}{K_k(x)} = x^{k+1} K_{-1}(x), \tag{5.4}$$

and using the fact that $K_{-1}(x) = 1$, we simplify Equation (5.4) as follows:

$$\begin{aligned} K_k(x) - K_{k-1}(x) &= x^{k+1} K_{k-1}(x) K_k(x); \\ \frac{K_{k-1}(x)}{K_k(x)} &= 1 - x^{k+1} K_{k-1}(x), \end{aligned}$$

and finally we find that

$$K_k(x) = K_{k-1}(x) \left(1 - x^{k+1} K_{k-1}(x)\right)^{-1}.$$

We want to show that for all $0 \leq j \leq k-1$

$$K_{k-j}(x) = K_{k-1}(x) \left(1 - x^{k+1} K_{k-1}(x)\right)^{j-1}. \tag{5.5}$$

Base case: We proved above that K_0 is true in Equation (5.3), and that K_k is true in Equation (5.5), then the case for $j = 0$ is satisfied for both the ascending and descending induction.

Ascending induction: We show that assuming $K_j(x) = \left(1 - x^{k+1} K_{k-1}(x)\right)^{-j-1}$ is true for all $1 \leq j \leq \ell$ then it is true for $j = \ell + 1$.

Assume $K_{k-j}(x) = K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{j-1}$ is true for all $0 \le j \le \ell{+}2$.

From Equation (5.2) we have that

$$\frac{1}{K_{\ell+1}(x)} = \frac{1}{K_\ell(x)} - x^{k+1}K_{k-\ell-2}(x). \tag{5.6}$$

Substitute Equation (5.3) and Equation (5.5) (with $j = \ell + 2$) into Equation (5.6) to get

$$\begin{aligned}\frac{1}{K_{\ell+1}(x)} &= \frac{1}{K_\ell(x)} - x^{k+1}K_{k-\ell-2}(x)\\ &= \left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell+1} - x^{k+1}K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell+1}\\ &= \left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell+1}\left(1 - x^{k+1}K_{k-1}(x)\right)\\ &= \left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell+2}.\end{aligned}$$

Therefore

$$K_{\ell+1}(x) = \left(1 - x^{k+1}K_{k-1}(x)\right)^{-\ell-2}.$$

Descending induction: We show that assuming $K_{k-j}(x) = K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{j-1}$ is true for all $1 \le j \le \ell - 1$ then it is true for $j = \ell$.

From Equation (5.2) we have that

$$\frac{1}{K_{k-\ell}(x)} = \frac{1}{K_{k-\ell+1}(x)} + x^{k+1}K_{k-(k-\ell)-2}(x) = \frac{1}{K_{k-\ell+1}(x)} + x^{k+1}K_{\ell-2}(x). \tag{5.7}$$

Substitute Equation (5.3) (with $j = \ell - 2$) and Equation (5.5) (with $j = \ell - 1$) into Equation (5.7) to get

$$\begin{aligned}
\frac{1}{K_{k-\ell}(x)} &= \frac{1}{K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell-2}} + x^{k+1}\left(1 - x^{k+1}K_{k-1}(x)\right)^{-\ell+1} \\
&= \frac{1 + \left(K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell-2}\right)\left(x^{k+1}\left(1 - x^{k+1}K_{k-1}(x)\right)^{-\ell+1}\right)}{K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell-2}} \\
&= \frac{1 + x^{k+1}K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{-1}}{K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell-2}} \\
&= \frac{\left(1 - x^{k+1}K_{k-1}(x)\right) + x^{k+1}K_{k-1}(x)}{K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell-1}} \\
&= \frac{1}{K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell-1}}.
\end{aligned}$$

Therefore

$$K_{k-\ell}(x) = K_{k-1}(x)\left(1 - x^{k+1}K_{k-1}(x)\right)^{\ell-1}.$$

The ascending and descending induction proof is complete. That is, we have shown that the ascending identity holds when the descending identity is true and vice versa. Therefore, any gaps left by the ascending identity can be filled by the descending identity so that both identities hold, and we can conclude that for $0 \leq j \leq \ell$

$$K_j(x) = \left(1 - x^{k+1}K_{k-1}(x)\right)^{-j-1}, \tag{5.8}$$

and we can immediately use Equation (5.8) to find that

$$K_{k-1}(x) = \left(1 - x^{k+1}K_{k-1}(x)\right)^{-k}. \tag{5.9}$$

Let $z = x^{k+1}K_{k-1}(x)$ then Equation (5.9) becomes

$$x^{k+1} = z(1-z)^k$$

and Equation (5.8) becomes

$$K_j(x) = (1-z)^{-j-1}.$$

We now apply the Cauchy integral formula to solve for $K_j(x)$ as follows:

Theorem 5.4.3. (Cauchy's integral formula) Let $f(z)$ be analytic in a region Ω containing 0 and λ be a simple loop around 0 in Ω that is positively oriented. Then the coefficient $[x^n]f(x)$ admits the integral representation (see [18, Chapter IV])

$$[x^n]f(z) = \frac{1}{2\pi i}\int_\lambda f(x)\frac{dx}{x^{n+1}}.$$

Our aim is to find the n-th coefficient of the binomial term $z(1-z)^k$ in $(1-z)^{-j-1}$.

Let $x = z(1-z)^k$ then $dx = (1-z)^{k-1}(1-z(k+1))dz$ and

$$\begin{aligned}
[x^n]\frac{1}{(1-z)^{j+1}} &= \frac{1}{2\pi i}\oint \frac{\frac{1}{(1-z)^{j+1}}}{x^{n+1}}dx \\
&= \frac{1}{2\pi i}\oint \frac{\frac{1}{(1-z)^{j+1}}}{(z(1-z)^k)^{n+1}}\left((1-z)^{k-1}(1-z(k+1))dz\right) \\
&= \frac{1}{2\pi i}\oint \frac{(1-z)^{k-1}(1-z(k+1))}{z^{n+1}(1-z)^{((k+1)-1)n+k}(1-z)^{j+1}}dz \\
&= \frac{1}{2\pi i}\oint \frac{(1-z(k+1))}{z^{n+1}(1-z)^{((k+1)-1)n+(j+1)+1}}dz.
\end{aligned}$$

Take coefficients

$$\begin{aligned}
[x^n]\frac{1}{(1-z)^{j+1}} &= [z^n]\frac{1-z(k+1)}{(1-z)^{((k+1)-1)n+(j+1)+1}} \\
&= [z^n]\frac{1}{(1-z)^{((k+1)-1)n+(j+1)+1}} - [z^{n-1}]\frac{(k+1)}{(1-z)^{((k+1)-1)n+(j+1)+1}} \\
&= \binom{(k+1)n+(j+1)}{n} - (k+1)\binom{(k+1)n+(j+1)-1}{n-1} \\
&= \binom{(k+1)n+(j+1)-1}{n-1}\left(\frac{(k+1)n+(j+1)}{n} - (k+1)\right).
\end{aligned}$$

Expand and simplify binomial coefficients

$$\begin{aligned}
[x^n]\frac{1}{(1-z)^{j+1}} &= \frac{((k+1)n+(j+1)-1)!((k+1)n+(j+1)-(k+1)n)}{n(n-1)!((k+1)n+(j+1)-n)!} \\
&= \frac{((k+1)n+(j+1)-1)!(j+1)}{n!((k+1)n+(j+1)-n)!} \\
&= \frac{j+1}{(k+1)n+(j+1)}\binom{(k+1)n+(j+1)}{n} \\
&= \left(\mathcal{B}_{k+1}\left(z(1-z)^k\right)\right)^{j+1}.
\end{aligned}$$

Replace $z(1-z)^k$ with x^{k+1} and j with t and we get that

$$K_t(x) = \left(\mathcal{B}_{k+1}(x^{k+1})\right)^{t+1}.$$

□

Chapter 6

Mapping Dyck paths onto themselves

In this chapter, we use the tunnel statistic to map a set of Dyck paths onto itself. This bijection was originally proved by Elizalde and Deutsch in [10]. We also show the relationships between statistics of Dyck paths that arise from the bijection and finally as a consequence, the generating function of Dyck paths in terms of the number of hills, odd rises, even rises, returns and semilength.

6.1 Definitions

Recall the definition of a return (Definition 2.1.8), a hill (Definition 2.1.13), an odd and even rise (Definition 2.1.14) and a tunnel (Definition 2.1.15) given in Section 2.1.

We extend the definition of a tunnel to include different types of tunnels that we will later use in this chapter.

Definition 6.1.1. A *left tunnel* of a Dyck path is a tunnel whose midpoint is to the left of the midpoint of the Dyck path.

Similarly, a *right tunnel* of a Dyck path is a tunnel whose midpoint is to the right of the midpoint of the Dyck path.

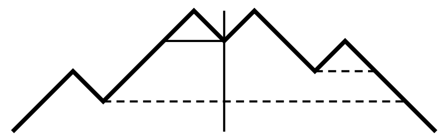


Figure 6.1: One left tunnel (solid line) and two right tunnels (dashed line).

Definition 6.1.2. A *centred tunnel* of a Dyck path is a tunnel whose midpoint is the midpoint of the Dyck path.

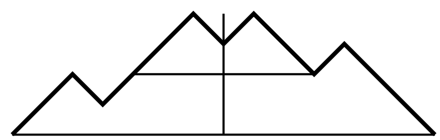


Figure 6.2: Two centred tunnels.

Definition 6.1.3. An *arch* of Dyck path is a tunnel that is on the x-axis. Clearly, every arch creates a return in the path.

Definition 6.1.4. A *multitunnel* of a Dyck path is the concatenation of two or more adjacent tunnels.

Definition 6.1.5. A *centred multitunnel* of a Dyck path is a multitunnel whose midpoint is the midpoint of the Dyck path.

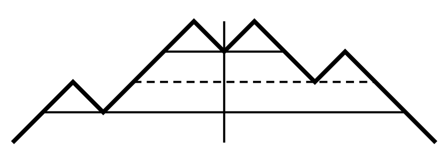


Figure 6.3: Three multitunnels and two centred multitunnels (solid line).

6.2 The bijection

Proposition 6.2.1. Let $\mathcal{D}_n$ be the set of Dyck paths with the semilength n. There exists a bijection Φ that maps the set of Dyck paths $\mathcal{D}_n$ back to itself.

Proof. Given a Dyck path $D \in \mathcal{D}_n$, we define the mapping $\Phi(D)$ as follows:

Let $\sigma \in S_{2n}$ be the permutation of "zigzag" ordered elements defined by

$$\sigma_i = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ 2n+1-\frac{i}{2} & \text{if } i \text{ is even,} \end{cases}$$

which is equivalent to

$$\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n \\ 1 & 2n & 2 & 2n-1 & 3 & 2n-2 & \cdots & n-1 & n+2 & n & n+1 \end{smallmatrix}\right).$$

Consider all the tunnels formed in the Dyck path, beginning with the step that starts at $x = i-1$ and ends at $x = i$ for $1 \leq i \leq 2n$. Each tunnel has an opening step $u = (1, 1)$ that ends at $x = i$ at the beginning of the tunnel and a matching closing step $d = (1, -1)$ that ends at $x = t(i)$ at the end of the tunnel.

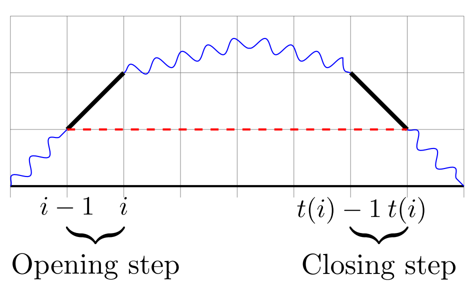


Figure 6.4: Opening and closing steps of a tunnel.

The permutation σ tells us the order in which we will read the path, and each σ_i represents the opening step of a tunnel at $x = i$ or a closing step of a tunnel at $x = t(i)$. By definition of the permutation σ, this order is zigzag starting at the beginning of the path then going to the end of the path and back to the subsequent step at the beginning of the path and so on.

If σ_i represents the closing step of a tunnel in the path, then find the corresponding point $x = i$ at the matching opening step of that tunnel. If we have not read the corresponding point before then assign the upward step $u = (1, 1)$ in the corresponding Dyck path, otherwise if we have read it already then assign the downward step $d = (1, -1)$. And if σ_i represents the opening step of a tunnel in the path, then we find the corresponding point $x = t(i)$ at the matching closing step of that tunnel. Similarly, if we have not read the corresponding point before then assign the upward step $u = (1, 1)$, otherwise if we have read it already then assign the downward step $d = (1, -1)$.

Note that $\Phi(D)$ is indeed a Dyck path. The path never goes below the x-axis since at no point is the number of downward steps more than upward steps, since a downward step is only assigned after a matching step has already been read and that step would have produced an upward step before. The path ends on $(2n, 0)$ since each pair of matching steps produce an upward and downward step in $\Phi(D)$ and we always end on a downward step.

We define the inverse mapping $\Phi^{-1}(D)$ as follows:

Given a Dyck path $\Phi(D)$ and permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n \\ 1 & 2n & 2 & 2n-1 & 3 & 2n-2 & \cdots & n-1 & n+2 & n & n+1 \end{pmatrix}.$$

We define a word $W = w_1 w_2 w_3 \cdots w_{2n}$ as follows:

Let o and c denote an opening and closing of a tunnel, respectively. Let $w_{\sigma_i} = o$ if there is a step $u = (1, 1)$ between $x = i - 1$ and $x = i$ for $1 \leq i \leq 2n$ in the given Dyck path, otherwise let $w_{\sigma_i} = c$. That is,

$$w_{\sigma_i} = \begin{cases} o & \text{if } \exists\ u = (1,1) \text{ at step ending on } x = i, \\ c & \text{if } \exists\ d = (1,-1) \text{ at step ending on } x = i. \end{cases}$$

To obtain the corresponding Dyck path, we split W in half and match each opening to a closing in zigzag order. That is, starting in the first half (left to right) of W then going to the other half (right to left) and back. If $w_{\sigma_i} = o$, then label it with the smallest i that has not been used yet. If $w_{\sigma_i} = c$ then match the label to the smallest labelled unmatched o in its half, otherwise if there is no such o then match it with the smallest labelled unmatched o in the other half. We read the labelled word from left to right and whenever we encounter a new label then we assign it to $u = (1, 1)$ otherwise $d = (1, -1)$ to obtain the corresponding Dyck path.

Remark. In the original paper [10] they match each c to the largest labelled unmatched o in its half, and if such an o does not exist then match it to the smallest labelled unmatched o in the other half. Note however, this only changes the labelling but overall the down steps and up steps will be in the same position even if you match each c to the smallest o in its half else the other half. This is because if you match a c to any (largest or smallest) labelled o in its half then it becomes a down step either way.

□

Example 6.2.1. Consider the Dyck path *uuddud* with 6 steps given in Figure 6.5.

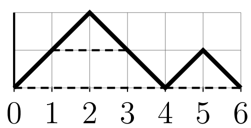


Figure 6.5: A Dyck path *uuddud* with 3 tunnels.

⇒ We define the permutation

$$\sigma = \left(\begin{smallmatrix}1&2&3&4&5&6\\1&6&2&5&3&4\end{smallmatrix}\right).$$

We read the path in the zigzag order given by the permutation σ and consider the tunnels that form with each $x = \sigma_i$. The tunnel that opens at the step ending on $x = 1$ closes at $x = 4$, which has not been read before, therefore assign the step $\boldsymbol{u}$. The tunnel that closes at $x = 6$ opens at the step ending on $x = 5$, which has not been read before, therefore assign the step $\boldsymbol{u}$. The tunnel that opens at the step ending on $x = 2$ closes at $x = 3$, which has not been read before, therefore assign the step $\boldsymbol{u}$. The tunnel that opens at the step ending on $x = 5$ closes at $x = 6$, which has been read before, therefore assign the step $\boldsymbol{d}$. The tunnel that closes at $x = 3$ opens at the step ending on $x = 2$, which has been read before, therefore assign the step $\boldsymbol{d}$. Lastly, the tunnel that closes at $x = 4$ opens at the step ending on $x = 1$, which has been read before, therefore assign the step $\boldsymbol{d}$. We obtain the Dyck path *uuuddd*.

⇐ To get back the original Dyck path. Consider the Dyck path *uuuddd* given in Figure 6.6

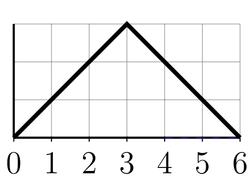


Figure 6.6: A Dyck path *uuuddd*.

and the permutation

$$\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 5 & 3 & 4 \end{smallmatrix}\right).$$

We form the word $W = w_1 w_2 w_3 \cdots w_6$ as follows.

$$\begin{aligned}
w_1 &= w_{\sigma_1} = o \text{ since the step ending on } x = 1 \text{ is } u,\\
w_2 &= w_{\sigma_3} = o \text{ since the step ending on } x = 3 \text{ is } u,\\
w_3 &= w_{\sigma_5} = c \text{ since the step ending on } x = 5 \text{ is } d,\\
w_4 &= w_{\sigma_6} = c \text{ since the step ending on } x = 6 \text{ is } d,\\
w_5 &= w_{\sigma_4} = c \text{ since the step ending on } x = 4 \text{ is } d,\\
w_6 &= w_{\sigma_2} = o \text{ since the step ending on } x = 2 \text{ is } u.
\end{aligned}$$

We obtain $W = oocccco$, which we split in half to get $W = ooc|cco$. We read W in zigzag order, matching every opening with its respective closing. The first o from the left is labelled o_1 and the first o from the right is labelled o_2. The second o from the left is labelled o_3. The first c from the right is labelled c_2 to match the unmatched o_2 in that half. The first c from the left is labelled c_1 to match the smallest unmatched o in that half. The second c from the right is labelled c_3 since there is no unmatched o in the second half, we therefore match it with the smallest unmatched o in the first half o_3. We obtain $o_1 o_3 c_1 c_3 c_2 o_2$ and assign every new label the step u else d and get the Dyck path *uuddud*. ■

Example 6.2.2. Consider the Dyck path *uududuuudduddd* with 14 steps given in Figure 6.7.

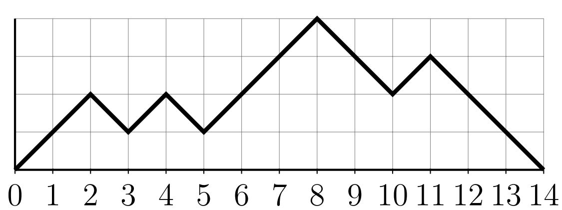


Figure 6.7: A Dyck path *uududuuudduddd* with 14 steps.

⇒ Define the permutation

$$\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 14 & 2 & 13 & 3 & 12 & 4 & 11 & 5 & 10 & 6 & 9 & 7 & 8 \end{smallmatrix}\right).$$

We read the path in the zigzag order given by σ and find the matching step for each tunnel. If we encounter a matching step that we have read before, we assign it the downward step, otherwise the upward step.

σ_i	Matches to	Read before?	Assign
Opening step that ends at 1	Closing step that ends at 14	No	u
Closing step that ends at 14	Opening step that ends at 1	Yes	d
Opening step that ends at 2	Closing step that ends at 3	No	u
Closing step that ends at 13	Opening step that ends at 6	No	u
Closing step that ends at 3	Opening step that ends at 2	Yes	d
Closing step that ends at 12	Opening step that ends at 11	No	u
Opening step that ends at 4	Closing step that ends at 5	No	u
Opening step that ends at 11	Closing step that ends at 12	Yes	d
Closing step that ends at 5	Opening step that ends at 4	Yes	d
Closing step that ends at 10	Opening step that ends at 7	No	u
Opening step that ends at 6	Closing step that ends at 13	Yes	d
Closing step that ends at 9	Opening step that ends at 8	No	u
Opening step that ends at 7	Closing step that ends at 10	Yes	d
Opening step that ends at 8	Closing step that ends at 9	Yes	d

We obtain the Dyck path *uduuduuddududd*.

⇐ To get back the original Dyck path. Consider the Dyck path *uduuduuddududd* given in Figure 6.8

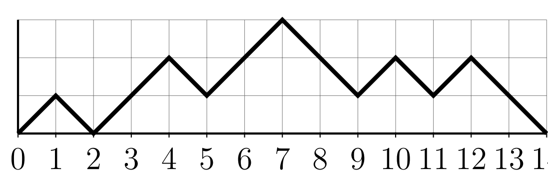


Figure 6.8: A Dyck path *uduuduuddududd* with $n = 7$.

and the permutation

$$\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 14 & 2 & 13 & 3 & 12 & 4 & 11 & 5 & 10 & 6 & 9 & 7 & 8 \end{smallmatrix}\right).$$

We form the word $W = w_1 w_2 w_3 \cdots w_{14}$ as follows,

$$\begin{aligned}
w_1 &= w_{\sigma_1} = o \text{ since the step ending on } x = 1 \text{ is } u,\\
w_2 &= w_{\sigma_3} = o \text{ since the step ending on } x = 3 \text{ is } u,\\
w_3 &= w_{\sigma_5} = c \text{ since the step ending on } x = 5 \text{ is } d,\\
w_4 &= w_{\sigma_7} = o \text{ since the step ending on } x = 7 \text{ is } u,\\
w_5 &= w_{\sigma_9} = c \text{ since the step ending on } x = 9 \text{ is } d,\\
w_6 &= w_{\sigma_{11}} = c \text{ since the step ending on } x = 11 \text{ is } d,\\
w_7 &= w_{\sigma_{13}} = c \text{ since the step ending on } x = 13 \text{ is } d,\\
w_8 &= w_{\sigma_{14}} = c \text{ since the step ending on } x = 14 \text{ is } d,\\
w_9 &= w_{\sigma_{12}} = o \text{ since the step ending on } x = 12 \text{ is } u,\\
w_{10} &= w_{\sigma_{10}} = o \text{ since the step ending on } x = 10 \text{ is } u,\\
w_{11} &= w_{\sigma_8} = c \text{ since the step ending on } x = 8 \text{ is } d,\\
w_{12} &= w_{\sigma_6} = o \text{ since the step ending on } x = 6 \text{ is } u,\\
w_{13} &= w_{\sigma_4} = o \text{ since the step ending on } x = 4 \text{ is } u,\\
w_{14} &= w_{\sigma_2} = c \text{ since the step ending on } x = 2 \text{ is } d.
\end{aligned}$$

We obtain $W = oococcccoocooc$, which we split in half and get $W = oococcc|coocooc$. We then read W in zigzag order and match each opening with a respective closing to obtain $o_1 o_2 c_2 o_5 c_5 c_4 c_6 c_7 o_7 o_6 c_3 o_4 o_3 c_1$ and assign every new label the step u otherwise d and get the Dyck path $uududuuudduddd$. ■

6.2.1 Generalisation of the bijection

We define a generalisation Φ_r, also given in [10], of the bijection Φ that depends on a non-negative integer parameter r where $r \leq n$. The major difference between Φ_r and Φ is how we define the permutation $\sigma \in S_{2n}$. For Φ_r we let

$$\sigma_i^{(r)} = \begin{cases} i & \text{if } i \leq 2r, \\ \frac{i+1}{2} + r & \text{if } i > 2r \text{ and } i \text{ is odd,} \\ 2n + 1 - \frac{i}{2} + r & \text{if } i > 2r \text{ and } i \text{ is even.} \end{cases}$$

Note that the bijection Φ is a special case of Φ_r where $r = 0$.

Given $\sigma_i^{(r)}$ and a Dyck path $D \in D_n$, we define $\Phi_r(D)$ as follows:

Copy the first $2r$ steps of D, then read the rest of the path in the zigzag order given by $\sigma_i^{(r)}$. If we encounter a corresponding step in D that has been read before then we assign it the downward step $d = (1, -1)$, otherwise if it has not been read before then assign it the upward step $u = (1, 1)$.

Given $\sigma_i^{(r)}$ and a Dyck path $\Phi_r(D)$, the inverse mapping $\Phi_r^{-1}(D)$ is defined as follows:

Define the word $W = w_1 w_2 w_3 \cdots w_{2n}$ as

$$w_{\sigma_i} = \begin{cases} o & \text{if } \exists\ u = (1, 1) \text{ at step ending on } x = i, \\ c & \text{if } \exists\ d = (1, -1) \text{ at step ending on } x = i. \end{cases}$$

We split W at $2r$ and at $n + r$ to obtain $w_1 \cdots w_{2r} | w_{2r+1} \cdots w_{n+r} | w_{n+r+1} \cdots w_{2n}$. Read $w_1 w_2 \cdots w_{2r}$ from left to right and label each o with the smallest unused i, where $1 \leq i \leq 2r$, and match any c to the smallest unmatched o. We are now left with two unlabelled halves

$$w_{2r+1} \cdots w_{n+r} | w_{n+r+1} \cdots w_{2n},$$

which we read in zigzag order. If $w_{\sigma_i} = o$, then label it with smallest i that has not been used yet. If $w_{\sigma_i} = c$ then match the label to the smallest labelled unmatched o in $W_1 = w_1 w_2 \cdots w_{2r}$ and if such an o does not exist then match the label to the smallest unmatched o in its half otherwise if there is no such o in both then match it with the smallest labelled unmatched o in the other half. We read the labelled word from left to right and whenever we encounter a new label we assign it to $u = (1, 1)$ otherwise $d = (1, -1)$ to obtain the corresponding Dyck path.

Example 6.2.3. Consider the Dyck path D given in Figure 6.9.

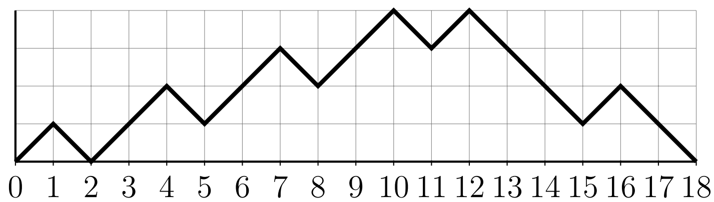


Figure 6.9: The Dyck path *uduuduuduududdddudd* with $n = 9$ steps.

$\Rightarrow$ Let $r = 3$ then $\sigma_i^{(3)}$ is given by

$$\sigma^{(3)} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 18 & 8 & 17 & 9 & 16 & 10 & 15 & 11 & 14 & 12 & 13 \end{smallmatrix}\right).$$

We copy the first $2r = 6$ steps and then read the path in zigzag order from step 7. The 7th step matches with the 8th step which has not been read before, therefore we draw an upward step. The 18th step matches with the 3rd step which has been read before when we copied the first $2r$ steps, therefore we draw a downward step. The 8th step matches with the 7th step which has been read before, therefore we draw a downward step. The 17th step matches with the 16th step which has not been read, therefore we draw an upward step. We continue on and on until the 13th step which matches with the 12th step which we would have read before, therefore we draw a downward step.

This gives the Dyck path *uduuduudduududdddud*.

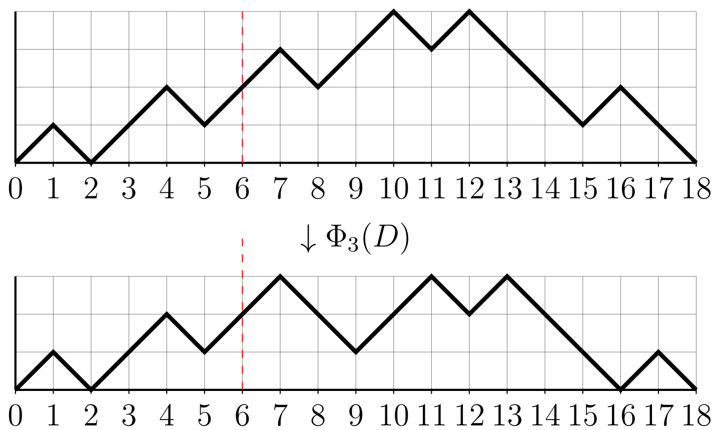


Figure 6.10: The corresponding Dyck path *uduuduuddduududddud* from $\Phi_3(D)$.

$\Leftarrow$ Given the Dyck Path $\Phi_3(D)$ and the permutation

$$\sigma^{(3)} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 18 & 8 & 17 & 9 & 16 & 10 & 15 & 11 & 14 & 12 & 13 \end{smallmatrix}\right),$$

we define the word $W = ocoocoococoocccccoc$ and then split it at $2r = 6$ and $n + r = 12$ to get $W = ocooco|ocooco|ccccoc$. We label $ocooco|$ from left to right, matching every c with an unmatched o to get $W_1 = o_1c_1o_2o_3c_2o_4|$. We now label $|ocooco|ccccoc$ in zigzag order as follows. The first o in the first half is labelled o_5, the smallest unused i. The first c in the other half is labelled c_3 to match the smallest unmatched o in W_1. The second c in the first half is labelled c_4 to match the smallest unmatched o in W_1. We label the second last o in the other half o_6 and the third o in the first half is labelled o_7 and so on until we obtain $o_1c_1o_2o_3c_2o_4|o_5c_4o_7o_8c_7o_9|c_9c_8c_5c_6o_6c_3$. We read from left to right and assign the upward step to a new label, otherwise assign a downward step if we have read it before to obtain the Dyck path *uduuduuduududdddudd*. ■

6.3 The generating function

The following properties follow directly from the above bijection Φ.

Theorem 6.3.1. Let D be the Dyck path and $D' = \Phi(D)$. Let $ct(D)$ be the number of centred tunnels, $rt(D)$ be the number of right tunnels, $lt(D)$ be the number of left tunnels, $cmt(D)$ be the number of centred multitunels, $er(D)$ be the number of even rises, $or(D)$ the number of odd rises, $ret(D)$ be the number of returns and $h(D)$ be the number of hills, then

i. $ct(D) = h(D')$

ii. $rt(D) = er(D')$

iii. $lt(D) + ct(D) = or(D')$

iv. $cmt(D) = ret(D')$.

Proof. Let $\mathcal{D}$ be the set of Dyck paths, then we begin by defining a decomposition $D = ABC$ of a Dyck path D, where $B \in \mathcal{D}$ and A and C have the same length.

By definition of Φ, we read the path D in zigzag order, therefore we will read A and C first before we reach the middle section B of the path. Therefore, it follows by this reasoning that $D' = \Phi(D) = \Phi(AC)\Phi(B)$.

(i) $\Rightarrow$ Consider a centred tunnel given by the decomposition $D = AuBdC$. This is indeed a centred tunnel since A and C are the same length and u raises B up by one step, therefore creating a centred tunnel. Now we apply the bijection to the decomposition.

$$D' = \Phi(D) = \Phi(AuBdC) = \Phi(AC)\Phi(uBd) = \Phi(AC)\Phi(ud)\Phi(B).$$

Notice that $\Phi(ud) = ud$ since when we read the path in zigzag order, the u step produces a u step in D' since we have not read its matching step yet and the d step produces a d step in D' since we have already read the u step. Then we have that

$$D' = \Phi(AC)ud\Phi(B),$$

therefore we have a hill ud in D'.

$\Leftarrow$ Consider a hill given by the decomposition $D' = XudY$ where $X, Y \in \mathcal{D}$. This hill can be obtained from a centred tunnel in $D = Z_1 u \Phi^{-1}(Y) d Z_2$, where Z_1 and Z_2 have the same length and are the first and second halves of $\Phi^{-1}(X)$ as follows:

$$\begin{aligned}\Phi(D) &= \Phi(Z_1 u \Phi^{-1}(Y) d Z_2)\\ &= \Phi(Z_1 Z_2)\Phi(u\Phi^{-1}(Y)d)\\ &= \Phi(\Phi^{-1}(X)) ud \Phi(\Phi^{-1}(Y))\\ &= XudY.\end{aligned}$$

(ii) $\Rightarrow$ Consider a right tunnel given by the decomposition $D = XuBdY$, where $X, Y \in D$ and $\text{length}(X) > \text{length}(Y)$. This is indeed a right tunnel, since the centre of the tunnel between u and d will lie to the right of the midpoint of D.

Now since $\text{length}(X) > \text{length}(Y)$, when we read the path D in zigzag order, we will read the d step first before the u step in the decomposition. Then the d step will produce a u step in D' and that u step will be an even rise since the zigzag order of reading dictates that the parity of all the steps in the second half of D will always be even in D'.

$\Leftarrow$ Consider an even rise in D'. We know the even rise comes from the right half of D from a step, say b, that is read before its matching step. Therefore, we may conclude that b opens a tunnel in D which is closed by a matching step. That tunnel is a right tunnel since b is read first.

(iii) By definition a tunnel connects two steps in the path therefore the number of tunnels in the path equals the semilength, that is

$$lt(D) + ct(D) + rt(D) = n. \tag{6.1}$$

By definition of a Dyck path, the number of up steps is equals to the semilength. Therefore, the number of even and odd rises of a path equals the semilength, that is

$$or(D') + er(D') = n. \tag{6.2}$$

We put together Equations (6.1) and (6.2) and obtain that

$$\begin{aligned} lt(D) + ct(D) + rt(D) &= or(D') + er(D') \\ lt(D) + ct(D) + rt(D) &= or(D') + rt(D) \quad \text{by (ii)} \\ lt(D) + ct(D) &= or(D'). \end{aligned}$$

(iv) $\Rightarrow$ Consider a centred multitunnel given by the decomposition $D = ABC$, where $B \in \mathcal{D}$ and A and C have the same length. This is indeed a centred multitunnel since any path above a centred multitunnel is a Dyck path and A and C have the same length therefore, from our decomposition, the centred multitunnel is between A and C and creates the Dyck path B.

We apply the bijection to the decomposition

$$D' = \Phi(D) = \Phi(ABC) = \Phi(AC)\Phi(B).$$

Now by definition of the bijection, the first step of $\Phi(B)$ is an up step which opens an arch in D' and we know for any $D' \in \mathcal{D}$ that the number of arches is equal to $ret(D')$.

$\Leftarrow$ It is clear that the arch created by $\Phi(B)$ in the decomposition $\Phi(D) = \Phi(AC)\Phi(B)$ can be created by a centred multitunnel $D = ABC$ in D. In other words, every arch in D' and therefore every $ret(D')$ maps back to a centred multitunnel in D.

□

The above theorem allows us to derive the generating function of Dyck paths in terms of tunnels as follows.

Theorem 6.3.2. Let $\mathcal{D}$ be a set of Dyck paths then the generating function of Dyck paths in terms of the number of hills, odd rises, even rises, returns and semilength is given by

$$\sum_{D \in \mathcal{D}} t^{h(D)} u^{or(D)} v^{er(D)} w^{ret(D)} z^{|D|} = \frac{2}{2 - w + (v + u - 2tu)wz + w\sqrt{1 - 2(v+u)z + (v-u)^2 z^2}}.$$

Proof. Let $\tilde{G}$ be the generating function of Dyck paths in terms of the number of centred, left and right tunnels, centred multitunnels and semilength then $\tilde{G}$ is defined as

$$\tilde{G}(m,n,o,p,q) := \sum_{D\in\mathcal{D}} m^{ct(D)} n^{lt(D)} o^{rt(D)} p^{cmt(D)} q^{|D|}.$$

Replace m with mn so that n also counts the central tunnels then

$$\begin{aligned}\tilde{G}(mn,n,o,p,q) &= \sum_{D\in\mathcal{D}} (mn)^{ct(D)} n^{lt(D)} o^{rt(D)} p^{cmt(D)} q^{|D|} \\ &= \sum_{D\in\mathcal{D}} m^{ct(D)} n^{ct(D)} n^{lt(D)} o^{rt(D)} p^{cmt(D)} q^{|D|}. \qquad (6.3)\end{aligned}$$

We apply Theorem 6.3.1 to Equation (6.3) so that the centred tunnels become hills, right tunnels become even rises, left tunnels plus centred tunnels become odd rises, centred multitunnels become returns. This gives the generating function of Dyck paths in terms of the number of hills, odd rises, even rises, returns and semilength. That is,

$$\begin{aligned}\tilde{G}(m,n,o,p,q) &= \sum_{D\in\mathcal{D}} m^{h(D)} n^{lt(D)+h(D)} o^{er(D)} p^{ret(D)} q^{|D|} \\ &= \sum_{D\in\mathcal{D}} m^{h(D)} n^{or(D)} o^{er(D)} p^{ret(D)} q^{|D|}.\end{aligned}$$

Let $t = m, u = n, v = o, w = p$ and $z = q$ then

$$G(t,u,v,w,z) = \sum_{D\in\mathcal{D}} t^{h(D)} u^{or(D)} v^{er(D)} w^{ret(D)} z^{|D|}.$$

Now consider the decomposition $D = \epsilon + uAdB$ of a Dyck path D, where $A, B \in \mathcal{D}$.

The number of hills of D is given by

$$h(uAdB) = \begin{cases} h(B)+1 & \text{if } A \text{ is empty,} \\ h(B) & \text{otherwise.} \end{cases}$$

This is because if A is non-empty then we only consider the hills of B since A is raised up by one step however, if A is empty then the ud steps in the decomposition udB form an extra hill.

The number of even rises of D is given by

$$er(uAdB) = or(A) + er(B).$$

This is because the A is shifted one step to the right in the decomposition by the u step, therefore all the odd rises of A become even rises. The even rises of B remain even rises since it is always shifted by an even number of steps to the right by uAd.

The number of odd rises of D is given by

$$or(uAdB) = er(A) + or(B) + 1.$$

This time all the odd rises of A become even rises as explained above. We add an extra odd rise from the u step in the decomposition.
The number of returns of D is given by

$$ret(uAdB) = ret(B) + 1.$$

This is because A is raised up by one step, therefore by definition cannot have a return. Therefore, we consider only the returns of B plus the extra return from the d step in the decomposition.

We apply the symbolic method as described by Sedgewick and Flajolet in [19] to obtain that the functional equation of G is

$$\begin{aligned} G(t,u,v,w,z) = \underbrace{1}_{1} + \underbrace{uwzG(t,u,v,w,z)G(1,v,u,1,z)}_{2} + \underbrace{tuwzG(t,u,v,w,z)}_{3} \\ - \underbrace{uwzG(t,u,v,w,z)}_{4}. \end{aligned} \tag{6.4}$$

Let us break down what each of the terms in the above equation represent.

- Case 1 represents the empty Dyck path.
- Case 2 represents the case when A or B can be empty or non-empty. The term $G(t,u,v,w,z)$ represents the Dyck path B in the decomposition. The term $G(1,v,u,1,z)$ represents the Dyck path A in the

decomposition, and we set $t = 1$ and $w = 1$ since A does not have any hills and returns. We also swap the order of the variables u and v in $G(1, v, u, 1, z)$ since the even rises of A contribute towards the odd rises of $D = uAdB$.

- Case 3 represents the case when A is empty. That is, we have the decomposition $D = udB$. Recall, when A is empty, there is an extra hill.

- Case 4 removes case 2 when A is empty to avoid double counting.

To solve for $G(t, u, v, w, z)$, we begin by simplifying Equation (6.4) to be

$$G(t, u, v, w, z) = 1 + uwz\left(G(1, v, u, 1, z) - 1 + t\right) G(t, u, v, w, z). \tag{6.5}$$

We then rearrange Equation (6.5) to get that

$$G(t, u, v, w, z) = \frac{1}{1 - uwzG(1, v, u, 1, z) - 1 + t}. \tag{6.6}$$

Let $H_1 := G(1, v, u, 1, z)$ and $G_1 := G(1, u, v, 1, z)$. Substitute $t = 1$ and $w = 1$ into Equation (6.5) then we get that

$$G_1 = 1 + uzH_1G_1, \tag{6.7}$$

and then we interchange u and v in Equation (6.5) and get that

$$H_1 = 1 + vzG_1H_1. \tag{6.8}$$

We solve Equations (6.7) and (6.8) simultaneously and use the quadratic formula to obtain that

$$H_1 = \frac{1 + (u - v)z - \sqrt{1 - 2(v + u)z + (v - u)^2z^2}}{2uz}. \tag{6.9}$$

We substitute Equation (6.9) into Equation (6.6) and obtain that the generating function of Dyck paths in terms of the number of hills, odd rises, even rises, returns and semilength is given by

$$\begin{aligned}G(t,u,v,w,z) &= \frac{1}{1-uwz(G(1,v,u,1,z)-1+t)} \\ &= \frac{1}{1-uwz(H_1-1+t)} \\ &= \frac{2}{2-w+(v+u-2tu)wz+w\sqrt{1-2(v+u)z+(v-u)^2z^2}}.\end{aligned}$$

□

The following corollary follows directly from the above theorem.

Corollary 6.3.3. Let $\mathcal{D}$ be the set of Dyck paths, then the generating function of Dyck paths in terms of the number of centred, left and right tunnels, centred multitunnels and semilength is given by

$$\sum_{D\in\mathcal{D}} m^{ct(D)}n^{lt(D)}o^{rt(D)}p^{cmt(D)}q^{|D|} = \frac{2}{2-p+(o+n-2m)pq+p\sqrt{1-2(o+n)q+(o-n)^2q^2}}.$$

Proof. From Theorem 6.3.2 we have that the generating function of Dyck paths in terms of the number of hills, odd rises, even rises, returns and semilength is

$$G(t,u,v,w,z) = \frac{2}{2-w+(v+u-2tu)wz+w\sqrt{1-2(v+u)z+(v-u)^2z^2}}.$$

We simply reverse the transformation in the proof of Theorem 6.3.2 of $G(t,u,v,w,z)$ back to $\tilde{G}(m,n,o,p,q)$ by letting $m=t$, $n=u$, $o=v$, $p=w$ and $q=z$ and lastly $m=mn$ to obtain that

$$\tilde{G}(m,n,o,p,q) = \frac{2}{2-p+(o+n-2m)pq+j\sqrt{1-2(o+n)q+(o-n)^2q^2}}.$$

□

Chapter 7

Conclusion

In this research we have studied various papers about the bijections that Dyck paths share with bargraphs, d-ary trees and other Dyck paths. We also studied the bijection bargraphs have with Motzkin paths. As a direct consequence of the established bijections, the authors were able to enumerate the Dyck paths and bargraphs with regard to their various statistics.

We began in Chapter 3 by studying the paper by Elizalde and Deutsch [12] that proves the bijection between Dyck paths and bargraphs. The proof in this paper mainly relies on the sequence of heights of the steps of the Dyck path and the resulting column heights in the bargraph. We took inspiration from this proof to establish an original bijection between Dyck paths and bargraphs. Our bijection is established using the consecutive peaks and valleys in the Dyck path and resulting horizontal steps that form in the corresponding bargraph. The authors proved in [12] the relationship between the semilength of a Dyck path and the semiperimeter minus the number of peaks of bargraphs, that is, $sl(P) = sp(B) - pk(B)$. Using the bijection in this dissertation, we provided an alternative proof for this relationship. Lastly, we used this relationship between these statistics of Dyck paths and bargraphs and showed, through a generating function, that Catalan numbers and consequently Dyck paths of semilength n can be interpreted as the set of bargraphs whose semiperimeter minus the number of peaks is n. This final result is also originally proved in [12]. We extended the proof by adding more details to it, such as the wasp-waist decomposition of bargraphs.

In Chapter 4 we focused on a different type of path in the lattice path family called the cornerless Motzkin path. We studied the paper by Elizalde and Deutsch [11] that shows the bijection between cornerless Motzkin paths and bargraphs. We added more details, such as explaining the decomposition of cornerless Motzkin paths, to the proof of the derivation of the generating function of bargarphs in terms of the semiperimeter. This generating function is given by

$$B(x, y) = \frac{(1 - 2x - x^2) - \sqrt{1 - 4x + 2x^2 + x^4}}{2x}.$$

We offered this method of deriving the generating function of bargraphs as an easier alternative to the wasp-waist decomposition method that we used in Chapter 3 to derive the same result.

Selkirk [20] gave two generalisations of Dyck paths, which we elaborate on in Chapter 5. The first generalisation is k-Dyck paths, where each down step $d = (1, -1)$ in a Dyck path is replaced with the down step $d' = (1, -k)$, where $k \in \mathbb{N}$. We expressed the bijection between the k-Dyck paths and $(k+1)$-ary trees in our own words and added more details to the proof, which involves explaining the tree traversal method and showing that k-Dyck paths are enumerated by the generalised Catalan numbers

$$\frac{1}{kn + 1}\binom{(k + 1)n}{n}.$$

The second generalisation is k_t-Dyck paths, where we bound a k-Dyck path below at some level $y = -t$, where $0 \leq t \leq k$. We study the original bijection between k-Dyck paths and k_t-Dyck paths and interpret it in a more geometric way. Our interpretation relies less on the algebraic calculations involved in the original proof and more on the reading of the path. From this bijection, it follows that k_t-Dyck paths are enumerated by another generalisation of Catalan numbers

$$\frac{t + 1}{(k + 1)n + t + 1}\binom{(k + 1)n + t + 1}{n}.$$

Lastly, we explained the direct proof in [20] that uses the symbolic method to show that the generating function of k_t-Dyck paths is the generalised binomial series $\mathcal{B}_{k+1}(x^{k+1})^{t+1}$. In this proof, we elaborated on the decomposition of k_t-Dyck paths illustrated in [20] by showing that the right-most $\mathcal{K}_t$ path in the decomposition is not bounded below by $y = -i$ and that it can go down to the level $y = -t$, where $i \leq t$. We also fixed a typo in [20] that k_t-Dyck paths are enumerated by $\mathcal{B}_{k+1}(x^{k+1})^t$. We instead show that they are actually enumerated by $\mathcal{B}_{k+1}(x^{k+1})^{t+1}$.

Finally, in Chapter 6 we studied the paper by Elizalde and Deutsch [10] that shows how to map Dyck paths back onto themselves through a bijection that uses the tunnel statistic. We explained this bijection in our own words by showing how to read the path in zigzag order using the tunnel statistic and obtain the resulting Dyck path. We similarly explain the generalisation of this bijection that considers mapping only a specific section of the Dyck path onto another Dyck path. It follows from the bijection that there exists a relationship between the statistics: the number of centred tunnels $ct(D)$ and number of hills $h(D')$, the number of right tunnels $rt(D)$ and number of even rises $er(D')$, the number of left tunnels $lt(D)$ plus centred tunnels $ct(D)$ and number of odd rises $or(D')$ and lastly, the number of centred multitunnels $cmt(D)$ and number of returns $ret(D')$, between the original Dyck path D and the corresponding Dyck path D'. We then added more details, particularly to the decomposition in [10] of Dyck paths provided with regard to the above-mentioned statistics. It followed that the generating function of Dyck paths in terms of the number of hills, odd rises, even rises, returns and semilength is given by

$$\sum_{D\in\mathcal{D}} t^{h(D)}u^{or(D)}v^{er(D)}w^{rt(D)}z^{|D|} = \frac{2}{2 - w + (v+u-2tu)wz + w\sqrt{1-2(v+u)z+(v-u)^2z^2}}$$

and that the generating function of Dyck paths in terms of the number of centred, left and right tunnels, centred multitunnels and semilength is given by

$$\sum_{D\in\mathcal{D}} m^{ct(D)}n^{lt(D)}o^{rt(D)}p^{cmt(D)}q^{|D|} = \frac{2}{2 - p + (o+n-2m)pq + p\sqrt{1-2(o+n)q+(o-n)^2q^2}}.$$

In summary, we have studied the bijections between Dyck paths and bargraphs, cornerless Motzkin paths and bargraphs, k-Dyck paths and d-ary trees, k-Dyck paths and k_t-Dyck paths, and lastly Dyck paths and themselves. We have included the enumeration of Dyck paths with regard to the number of centred, left and right tunnels, centred multitunnels and semilength, and that generalised Dyck paths are enumerated by generalised Catalan numbers.

We have the following questions and comments:

- There are many other interesting statistics like the number of $udu's$, as shown by Mansour in [16], and the number of low peaks and number of high peaks, as shown by Deutsch in [8], that we can apply the established bijections to in order to investigate the relationships they share with the statistics of the corresponding combinatorial objects in this paper.

- We have shown that $(k+1)$-ary trees have a bijection with k-Dyck paths and that tuples of k-Dyck paths have a bijection with k_t-Dyck paths. We plan to conduct further research to establish a bijection between $(k+1)$-ary trees and k_t-Dyck paths.

- An even more interesting topic of investigation is the enumeration of all types of generalised Dyck paths. For example, k_t^r-Dyck path, which are k_t-Dyck paths that also bounded above at some level $y = r$ where $r > 0$. We plan to derive this enumeration in further studies.

- One of the open problems with regard to Dyck paths that was investigated by Ferrari in [13] is with respect to proving that the sequence of the number of Dyck paths of semilength n having area $n + 2k$ is unimodal, for all n and $k = 0, 1, 2, 3...$ We plan to derive the enumeration of such paths in further studies.

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