COMPARITIVE ANALYSIS OF BORNOLOGY IN THE CATEGORIES OF FRÖLICHER SPACES AND TOPOLOGICAL SPACES.

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A thesis submitted to the Faculty of Science, University of the Witwatersrand, in fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics**.

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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(Signature of candidate)

16th day of FEBRUARY 20 23 at UNIVERSITY OF THE WITWATERSRAND.

Abstract

This thesis seeks to introduce the concept of bornology to the theory of Frölicher spaces. Bornologies are induced from the Frölicher structure, Frölicher topology and the canonical topology on the underlying set of Frölicher space. In each case the bornologies are compared in a general Frölicher space, subspace, product, coproduct and quotient. The Frölicher topology refers to the topology induced from structure functions of the Frölicher space. The bornology induced from the Frölicher structure is induced from the structure functions of Frölicher space. An initial bornology is canonically induced on the underlying set of Frölicher subspace and product, and a final bornology is induced canonically on the underlying set of Frölicher coproduct and quotient. For Frölicher subspace and product the initial bornology is finer than the bornology induced from structure functions. Dually, the final bornology is coarser than the bornology induced from structure functions for Frölicher coproduct and quotient. Relatively-compact and compact bornologies are induced from the Frölicher topology and the canonical topology on the underlying set of Frölicher space. For each of Frölicher subspace, product, coproduct and quotient, that is, the objects in the category of Frölicher spaces under the study of this thesis, there are two topologies - the canonical topology induced from the underlying set and the Frölicher topology. Subsequently there are two relatively-compact and compact bornologies, induced from these topologies, for each of the mentioned objects. The bornological comparison between the relatively-compact bornologies and the bornological comparison between the compact bornologies, for each object, is determined by the comparison of these topologies, that is, the comparison between the Frölicher topology and the canonical topology on the underlying set.

I dedicate this work to my mother Ntombi Jester MAHUDU, and to the memory of my father Sebushi Johanness MAHUDU.

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Contents

Chapter 1

Introduction.

In this thesis we introduce the concept of bornology on the theory of Frölicher spaces. The concept of bornology on a set was introduced by H. Hogbe-Nlend in [51]. Bornologies have been studied and used mainly in functional analysis, on metric spaces, Banach spaces, Schwartz spaces and convenient spaces (see [4], [45], [51], [56] and [67]). They [bornologies] have also been applied to other concepts, for example on metrizable spaces (see [15], [16], [18] and [52]), general topology (see [61]), the theory of locally convex sets (see [51]), bitopological metric spaces (see [74]), etc. The unified theory of differentiability for functions defined on a general normed space was developed using bornologies (see [23]).

The theory of Frölicher spaces is a developing field, thus our introduction of the concept of bornology to the theory of Frölicher spaces is a contribution and an advacement to the theory of Frölicher spaces. There is a series of research that culminated into the theory and concept of Frölicher spaces, as we will mention below. Frölicher spaces are named after Alfred Frölicher, who was the first to define these spaces. Alfred Frölicher referred to these spaces [Frölicher spaces] as smooth spaces (see [45]). The phrase "Frölicher spaces" appeared for the first time in [56] as used by A. Kriegl and P. Michor. However P. Cherenack in [30] was the first person to refer to these smooth spaces defined by Frölicher as Frölicher spaces. This is because Cherenack's paper, [30], was submitted in the year 1996 and published in the year 2000 whereas the work by Kriegl and Michor, [56], was published in 1997. The book titled *Linear Spaces and Differentiation Theory* by Alfred Frölicher and Andreas Kriegl, [45], is the "genesis" and conception of the theory of Frölicher spaces. This book is a culmination of the work of Frölicher in [40], [41], [42] and [43]. Another work that contributed in this theory is the work in [44] by Frölicher and Kriegl. P. Cherenack, alongside his former students, viz A. Batubenge, P. Ntumba and B. Dugmore, had significant contributions to the research and study of Frölicher spaces (see [7], [8], [9], [10], [13], [14] [28], [29], [30], [31], [32], [33], [36] and [37]). A. Batubenge gives a detailed survey of Frölicher spaces in [7]. There has been an interest in applying other theories to the theory of Frölicher spaces and vice versa, and that is: Lie group and Lie algebra (see

[10] and [58]), homotopy theory (see [27] and [36]), symplectic geometry (see [25], [26] and [38]), symplectic reduction (see [62], [66], [71], [72], [75] and [89]), sheaf theory and sheaf cohomology (see [6] and [34]), De Rham cohomology (see [49], [59], [77], [81], [82] and [87]), Alexander-Spanier, Singular and Čech cohomologies (see [88]), Poisson geometry (see [73] and [91]), modern Hamiltonian formalism of mechanics (see [8], [9], [11] and [53]), cosmology and tensor calculus (see [29], [30] and [32]).

A category is a collection of mathematical objects and associated morphisms between these objects such that there is an identity morphism for each object and a composition of morphisms, these are subject to associativity of morphisms and unity in morphisms (see [1], [2], [5] and [64]). Frölicher spaces and smooth maps form a category - the category of Frölicher spaces as proven by Frölicher and Kriegl in [45]. There is an extensive research in the category of Frölicher spaces (see [7], [13], [14], [29], [30] and [86]). Some of the categorical properties of the category of Frölicher spaces are that it is Cartesian-closed, complete, cocomplete and topological over the category of sets (see [29] and [31]). One example of a Frölicher space is the smooth manifold, thus smooth manifolds are objects in the category of Frölicher space (see [8], [29], [45] and [56]). The category of Frölicher spaces is related to other categories of smooth spaces - it is a full subcategory of diffeological spaces in the sense of Souriau (see [46] and [83]), differential spaces in the sense of Sikorski (see [79] and [80]) and Smith spaces (see [82]). Bounded maps and bornological spaces form the category of bornological spaces, which is also topological over the category of sets (see [22]). Another category which is critical in the study of this thesis is the category of topological spaces which consists of topological spaces as objects and continuous maps as morphisms. Some of the categorical properties of the category of topological spaces is that it is Cartesian-closed, has a final (terminal) and and initial objects, has products and for each pair of topological spaces there is an equalizer and coequalizer (see [63] and [78]).

A Frölicher space is a triple (X, C_X, F_X) where X is a non-empty set and (C_X, F_X) is a Frölicher structure (see [7], [10], [29], [32], [36], [37], [45], [86], [88] and [89]). There are two topologies from the Frölicher structure (C_X, F_X), that is, the functional topology, denoted τ_{F_X}, and the curvaceous topology, denoted τ_{C_X}, induced from structure functions and structure curves of X respectively (see [13], [14] and [45]). The curvaceous topology τ_{C_X} is finer than the functional topology τ_{F_X} (see [14]). The basis and subbasis of τ_{F_X} are given by the collections $\{f^{-1}(0, +\infty) \mid f \in F_X\}$ and $\{f^{-1}(0, 1) \mid f \in F_X\}$, respectively (see [36] and [45]). The topology τ_{F_X} is the coarsest topology on X in which all smooth maps are continuous. Due to that, the topology τ_{F_X} is considered as the Frölicher topology (see [14]). Since we have topologies from the Frölicher structure therefore the category of Frölicher spaces is a subcategory of the category of topological spaces. For each of Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient A. Batubenge and M. Tshilombo compared the canonical topology on the underlying set with the Frölicher topologies (see [13] and [14]).

The preliminary work of A. Frölicher and A. Kriegl in [45] included bornologies. They showed that the category of bounded sequences of real numbers embeds into the category of bornological spaces and that there is a bornology from the structure endowed on the set of bounded sequences of real numbers. This thesis determines the induction of bornology from Frölicher space. The comparison of topologies induced from Frölicher space by A. Batubenge and M. Tshilombo, in [13] and [14], inspired the bornological comparison on Frölicher space as studied in this thesis. Following from the work of A. Batubenge and M. Tshilombo, as cited, the hypothesis of this thesis is that bornologies can be induced from the Frölicher structure, the canonical topology from the underlying set of the Frölicher space and from the Frölicher topology. For a general Frölicher space, canonical bornologies are induced from the Frölicher structure and from the Frölicher topology. An initial bornology (see [51]) is induced from the underlying set of Frölicher subspace and Frölicher product since they are initial structures in the category of Frölicher spaces (see [13], [14], [29] and [45]). Similarly the final bornology (see [51]) is induced from the underlying set of Frölicher coproduct and Frölicher quotient as they are final structures in category of Frölicher spaces (see [13], [14], [29] and [45]). There is a relationship between topology and bornology in the sense that a bornology can be induced from a topology (see [47] and [51]). For each of Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient there is canonical topology from the underlying set and the Frölicher topology from the Frölicher structure. Relatively-compact and compact bornologies are induced from the Frölicher topology and from the canonical topology on the underlying set, for each of Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient. There are three phases of bornological comparison in this thesis. The first phase is the bornological comparison between initial bornologies and bornologies from Frölicher structure on Frölicher subspace and Frölicher product, and the final bornology and the bornologies from the Frölicher structure on Frölicher coproduct and Frölicher quotient. The second and third phase of bornological comparison, respectively, is between the relatively-compact and compact bornologies induced from the canonical topology on the underlying set and the Frölicher topology on Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient. We conclude by applying these phases of bornological comparison on Frölicher groups.

The research questions to be investigated include but are not limited to the following:

- Under which conditions can we induce a bornology from the Frölicher topology and from the Frölicher structure?
- What is the bornological relation between the bornologies induced from the Frölicher structure?
- For initial structures in the category of Frölicher spaces, that is Frölicher

subspace and Frölicher product, what is the relationship between the initial bornology and the bornologies induced from the Frölicher structure?

- For final structures in the category of Frölicher space, that is Frölicher coproduct and Frölicher quotient, what is the relationship between the final bornology and the bornologies induced from the Frölicher structure?

- For each of Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient what is the bornological relation between the bornologies induced from the canonical topology on the underlying set and the bornology induced from the Frölicher topology? Does that bornological comparison follow from the comparison of the topologies from which the bornologies are induced from?

The work of this thesis is divided into six chapters. As usual the first chapter is the introduction and the second chapter is on preliminaries. Under preliminaries we will outline some concepts from category theory, general topology, concepts on Frölicher spaces and bornology. Concepts on Frölicher spaces will include the theory of Frölicher spaces and structures, the category of Frölicher spaces, Frölicher topologies, Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient. The third chapter will cover bornologies induced from Frölicher structure. These bornologies are compared for each of the initial structures in the category of Frölicher spaces, that is Frölicher subspace and Frölicher product, they are compared with the initial bornology induced from the underlying set of these structures, and for the final structures, that is Frölicher coproduct and Frölicher quotient, they are compared with the final bornology. The fourth chapter covers on relatively-compact bornologies induced from the Frölicher topology and the canonical topology on the underlying set of Frölicher space, and the fifth chapter covers compact bornologies induced from Frölicher topology and the canonical topology on the underlying set of Frölicher space. In both the fourth and the fifth chapter, for each of Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient, these bornologies are compared. We conclude with the sixth chapter which is on the application of the induction and comparison of bornologies, as in the third, fourth and fifth chapter, on Frölicher groups.

Chapter 2

Preliminaries.

2.1 Concepts from category theory.

We start off with concepts from category theory. We will only discuss concepts that are relevant and useful for this thesis.

2.1.1 Category

Definition 2.1.1 ***Morphism.***
In category theory a morphism is an arrow or mapping between two objects, say between the objects A and B, written $A \longrightarrow B$. Object A is called the domain of the morphism and object B is called the codomain of the morphism. Objects in this case refers to mathematical (abstract) objects such as sets, groups, algebras, topological spaces, etc.

Definition 2.1.2 ***Category.***
Let $\{X_i \mid i \in I\}$ be a collection of objects and $\{f_i \mid i \in I\}$ be a collection of morphisms between these objects. Then $\{X_i \mid i \in I\}$ and $\{f_j \mid j \in J\}$ form a category if the following are satisfied:

1. *Identity: For each object X_i there exists an identity morphism $I_{X_i} : X_i \longrightarrow X_i$.*
2. *Composition: For each pair of morphisms f_{j_1} and f_{j_2}, with $j_1, j_2 \in J$, such that the codomain of f_{j_1}, denoted $cod(f_{j_1})$, and the domain of f_{j_2}, denoted $dom(f_{j_2})$, are such that $cod(f_{j_1}) = dom(f_{j_2})$, then there exists the composition morphism $f_{j_2} \circ f_{j_1} : dom(f_{j_1}) \longrightarrow cod(f_{j_2})$.*

The above conditions are subject to the following conditions:

1. *Associativity: For every morphisms f_{j_1}, f_{j_2} and f_{j_3}, with $j_1, j_2, j_3 \in J$, such that $cod(f_{j_1}) = dom(f_{j_2})$ and $cod(f_{j_2}) = dom(f_{j_3})$ then $f_{j_3} \circ (f_{j_2} \circ f_{j_1}) = (f_{j_3} \circ f_{j_2}) \circ f_{j_1}$.*

2. *Unity: For every morphism $f_j : X_a \longrightarrow X_b$, where $j \in J$ and $a, b \in I$, we have that $f_j \circ I_{X_a} = f_j$ and $I_{X_b} \circ f_j = f_j$ where, $I_{X_a} : X_a \longrightarrow X_a$ and $I_{X_b} : X_b \longrightarrow X_b$ are identity morphisms on X_a and X_b, respectively.*

Examples of categories:

Example 2.1.1 ***The category of sets.***
Let A_1, A_2 and A_3 be sets such that $f : A_1 \longrightarrow A_2$ and $g : A_2 \longrightarrow A_3$ are functions. Since f and g are functions then the composition function $g \circ f : A_1 \longrightarrow A_3$ exists. For any set X there exists an identity function $I_X : X \longrightarrow X$. Therefore sets and functions form a category, called the category of sets.

Example 2.1.2 ***The category of posets.***
Let X be a non-empty set. Then X is a partially-ordered set (poset) if for the relation $\leq$, on X, we have that $\forall\ a, b, c \in X$ the following conditions are satisfied:

1. *$a \leq a$ (reflexivity).*
2. *$a \leq b$ and $b \leq c \Longrightarrow a \leq c$ (transitivity).*
3. *$a \leq b$ and $b \leq a \Longrightarrow a = b$ (antisymmetry).*

Let X and Y be posets, then the morphism $f : X \longrightarrow Y$ is called a monotone function if $\forall\ x_1, x_2 \in X,\ x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$. For any poset X the identity morphism $I_X : X \longrightarrow X$ is a monotone function since $\forall\ x \in X$, $x \leq x \iff I_X(x) \leq I_X(x)$. Let Z be a poset and $g : Y \longrightarrow Z$ be a monotone function. Since $\forall\ x_1, x_2 \in X,\ f(x_1),\ f(x_2) \in Y$ and g is a monotone function therefore $f(x_1) \leq f(x_2) \implies g(f(x_1)) \leq g(f(x_2)) \iff (g \circ f)(x_1) \leq (g \circ f)(x_2)$. Therefore $g \circ f : X \longrightarrow Z$ is a monotone function. Thus, posets and monotone functions form a category - the category of posets.

Example 2.1.3 ***Categories of structured sets.***
Structured sets are sets endowed with a structure and structure preserving functions (see [5]). The structure on a poset is the relation defined on the poset and the monotone functions preserve the relation. Therefore a poset is a structured set. Other examples of structured sets are differential manifolds, groups, graphs, topological spaces, vector spaces, etc. Structured sets and their preserving structure functions form a category. For example:

1. *Posets and monotone functions form the category of posets as we have seen in the previous example.*
2. *Differential manifolds and smooth maps form the category of differential manifolds.*

3. *Groups and group homomorphisms form the category of groups.*

4. *Graphs and graph homomorphisms form the category of graphs.*

5. *Topological spaces and continuous maps form the category of topological spaces.*

6. *Vector spaces and linear maps form the category of vector spaces.*

2.1.2 Initial and final objects.

Definition 2.1.3 ***Initial object.***
Let $\mathcal{C}$ be a category and I be an object in $\mathcal{C}$. The object I is called an initial object if the morphism $I \longrightarrow X$ is unique for every object X in $\mathcal{C}$.

Definition 2.1.4 ***Final object.***
Let $\mathcal{C}$ be a category and F be an object in $\mathcal{C}$. The object F is called a final object if the morphism $X \longrightarrow F$ is unique for every object X in $\mathcal{C}$.

The other phrase used for final objects is terminal objects (see [1], [2] and [5]). Initial and final objects are the algebraic duals of each other.

Examples of initial and final objects:

Example 2.1.4 *Let ϕ denote the empty set, then it is trivial that the morphism $\phi \longrightarrow X$ is unique for any set X. Thus, ϕ is the initial object in the category of sets. Consider any singleton $\{x\}$ in the category of sets. Then it is also trivial that for any set Y, the morphism $Y \longrightarrow \{x\}$ is unique. Giving that any singleton in the category of sets is a final object.*

Example 2.1.5 *A poset is a set endowed with a relation. Thus the category of posets is a subcategory of the category of sets. Therefore the empty set is an initial object in the category of sets. Every singleton $\{u\}$ in the category of sets with the relation $u \leq u$ is a poset. It follows trivially that the morphism $P \longrightarrow \{u\}$ is unique for every poset P. Thus, any singleton is a final object in the category of posets.*
Let $(X, \leq)$ be a poset. Then $\forall\ x, y \in X$ we define a moprhism $x \longrightarrow y$ if and only if $x \leq y$. Then $(X, \leq)$ is a category. Let m be the least element in X, that is, $m \leq x$, $\forall\ x \in X$. Let M be the greatest element in X, that is $x \leq M$, $\forall\ x \in X$. Therefore the morphisms $m \longrightarrow x$ and $x \longrightarrow M$ are unique, $\forall\ x \in X$. Thus, the least element in a poset is an initial object and the greatest element in a poset is a final object. That is a poset has one initial object and one final object.

Remark 2.1.1 *A category can have multiple initial objects and multiple final objects, as seen from the examples above. However, there are categories that do* *not have neither an initial object nor a final object (see [1] and [5]). The category of the posets of the form* $(\mathbb{Z}, \leq)$ *is such a category, it has neither an initial object nor a final object. Interestingly, there are categories with objects that are both initial and final objects. An object that is both an initial and a final object is called a zero object (see [1]).*

2.1.3 Functors.

Definition 2.1.5 ***Functor.***
Let $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be categories. The morphism* $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ *is a functor if:*

1. F *assigns every object* A *in* $\mathcal{C}_1$ *to an object* $F(A)$ *in* $\mathcal{C}_2$*.*
2. F *assigns every morphism* $f : X \longrightarrow Y$ *in* $\mathcal{C}_1$ *to a morphism* $F(f) : F(X) \longrightarrow F(Y)$ *in* $\mathcal{C}_2$ *such that* F *preserves composition and identity morphisms. That is, for every pair of morphisms* f_1 *and* f_2 *in* $\mathcal{C}_1$ *such that* $f_1 \circ f_2$ *is a morphism in* $\mathcal{C}_1$ *then* $F(f_1 \circ f_2) = F(f_1) \circ F(f_2)$*, and for the identity morphisms* I_A *in* $\mathcal{C}_1$ *and* $I_{F(A)}$ *in* $\mathcal{C}_2$*, we have that* $F(I_A) = I_{F(A)}$*.*

Examples of functors:

Example 2.1.6 ***The identity functor.***
For any category $\mathcal{C}$*, there is an identity functor* $id : \mathcal{C} \longrightarrow \mathcal{C}$ *that assigns every morphism* f *in* $\mathcal{C}$ *to itself and asigns every object* c *in* $\mathcal{C}$ *to itself.*

Example 2.1.7 ***The forgetful functor.***
Let $\mathcal{C}$ *be a construct, that is, a category of a structured set (see [1], [2], [5] and [64]). There is a functor* $U : \mathcal{C} \longrightarrow$ ***Set****, called the forgetful functor, that assigns every object in* $\mathcal{C}$ *to it's underlying set and assigns every morphism* f *in* $\mathcal{C}$ *to it's underlying function. For example, the category of topological spaces, denoted* ***Top****, is a construct as topological spaces are structured sets. Then the forgetful functor* U : ***Top*** $\longrightarrow$ ***Set*** *assigns every topological space* (X, τ) *to it's underlying set* X *and assigns every continuous map* $f : (X, \tau) \longrightarrow (Y, \tau)$ *between topological spaces to the function* $U(f) : X \longrightarrow Y$*. That is, the forgetful functor "forgets" the structure in each assignment. The forgetful functor* U : ***Top*** $\longrightarrow$ ***Set*** *"forgets" the topological structure.*

2.1.4 Limits and colimits.

Definition 2.1.6 ***Diagram.***
Let $\mathcal{C}$ *be a category. A diagram in* $\mathcal{C}$ *is a functor* $D : \mathcal{A} \longrightarrow \mathcal{C}$ *for some category* $\mathcal{A}$*. The category* $\mathcal{A}$ *is called the scheme or the index of the diagram* D*.*

Remark 2.1.2 *There is a nuance between a diagram and a functor, which can create confusion between the two. Technically, a diagram is a functor but a functor is not necessarily a diagram. A diagram* $D : \mathcal{A} \longrightarrow \mathcal{C}$ *in the category* $\mathcal{C}$*, with the index category* $\mathcal{A}$*, as defined above can be viewed as indexing a collection of objects and morphisms in* $\mathcal{C}$ *through* $\mathcal{A}$*. This is analogous to an index set in set theory.*

Definition 2.1.7 ***Limit.***
Let $D : \mathcal{A} \longrightarrow \mathcal{C}$ *be a diagram in the category* $\mathcal{C}$*.*

1. *Let* A *and* B *be objects in* $\mathcal{A}$*. The morphism* $N : \mathcal{C} \longrightarrow D(A)$ *is said to be a natural source for the diagram* D *if for the morphism* $M : \mathcal{C} \longrightarrow D(B)$ *we have that for every morphism* $f : A \longrightarrow B$ *in* $\mathcal{A}$*,* $M = D(f) \circ N$*. That is, the triangle*

   $$\begin{array}{ccc} \mathcal{C} & \xrightarrow{N} & D(A) \\ & {}_{M}\searrow & \downarrow{\scriptstyle D(f)} \\ & & D(B) \end{array}$$

   commutes.

2. *Let* A *be an object in* $\mathcal{A}$*. The limit of the diagram* D *is the morphism* $l : \mathcal{L} \longrightarrow D(A)$ *if for every natural source* $N : \mathcal{C} \longrightarrow D(A)$ *there exists a unique morphism* $g : \mathcal{C} \longrightarrow \mathcal{L}$ *such that* $N = L \circ g$*. That is, the triangle*

   $$\begin{array}{ccc} \mathcal{C} & \xrightarrow{g} & \mathcal{L} \\ & {}_{N}\searrow & \downarrow{\scriptstyle l} \\ & & D(A) \end{array}$$

   commutes.

The algebraic dual of limits is colimits.

Definition 2.1.8 ***Colimits.***
Let $D : A \longrightarrow \mathcal{C}$ *be a diagram in the category* $\mathcal{C}$*.*

1. *Let* A *and* B *be objects in* $\mathcal{A}$*. The morphism* $N : D(A) \longrightarrow \mathcal{C}$ *is said to be a natural sink for the diagram* D *if for the morphism* $M : D(B) \longrightarrow \mathcal{C}$ *we have that for every morphism* $f : B \longrightarrow A$ *in* $\mathcal{A}$*,* $M = N \circ D(f)$*. That is, the triangle*

   $$\begin{array}{ccc} D(B) & \xrightarrow{D(f)} & D(A) \\ & {}_{M}\searrow & \downarrow{\scriptstyle N} \\ & & \mathcal{C} \end{array}$$

   commutes.

2. *Let A be an object in $\mathcal{A}$. The colimit of the diagram D is a morphism $c : D(A) \longrightarrow \mathcal{K}$ if for every natural sink $N : D(A) \longrightarrow \mathcal{C}$ there exists a unique morphism $g : \mathcal{K} \longrightarrow \mathcal{C}$ such that $N = g \circ c$. That is, the triangle*

$$\begin{array}{ccc} D(A) & \xrightarrow{c} & \mathcal{K} \\ & {\scriptstyle N}\searrow & \downarrow{\scriptstyle g} \\ & & \mathcal{C} \end{array}$$

commutes.

2.1.5 Cartesian-closed categories and exponential law.

Definition 2.1.9 *Let $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be a functor, and A be an object in the category $\mathcal{C}_1$ and B be an object in the category $\mathcal{C}_2$.*

1. *The morphism $f : B \longrightarrow F(A)$ is called an F-structured morphism with domain B.*

2. *An F-structured morphism $f : B \longrightarrow F(A)$ with domain B is called an F-universal morphism for B given that for each F-structured morphism $\acute{f} : B \longrightarrow F(\acute{A})$ with domain B there exists a unique morphism $\hat{f} : A \longrightarrow \acute{A}$ in the category $\mathcal{C}_1$ such that $\acute{f} = F(\hat{f}) \circ f$. That is the triangle*

$$\begin{array}{ccc} B & \xrightarrow{f} & F(A) \\ & {\scriptstyle \acute{f}}\searrow & \downarrow{\scriptstyle F(\hat{f})} \\ & & F(\acute{A}) \end{array}$$

commutes.

Definition 2.1.10 ***Adjoint and co-adjoint functors.***
Let $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be a functor. Then F is called

1. *An adjoint functor if there exists an F-universal morphism with domain B, for every object B in $\mathcal{C}_2$.*

2. *A co-adjoint functor if there exists an F-universal morphism with codomain B, for every object B in $\mathcal{C}_2$.*

That is, the adjoint functor and the co-adjoint functor are algebraic duals of each other. The other term used for adjoint and co-adjoint functor is left-adjoint and right-adjoint functor, respectively.

Definition 2.1.11 ***Isomorphisms.***
Let $\mathcal{C}$ be a category. If for the morphism $f : A \longrightarrow B$ in $\mathcal{C}$ there exists a morphism $g : B \longrightarrow A$ in $\mathcal{C}$ such that $g \circ f = I_A$ and $f \circ g = I_B$, where $I_A : A \longrightarrow A$ and $I_B : B \longrightarrow B$ are identity morphisms in $\mathcal{C}$.

Definition 2.1.12 ***Isomorphic objects.***
Let $\mathcal{C}$ be a category. If for the objects A and B in $\mathcal{C}$ there exists an isomorphism $f : A \longrightarrow B$ in $\mathcal{C}$, then we say that A and B are isomorphic, written $A \cong B$.

Definition 2.1.13 ***Functor isomorphism.***
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories. If for the functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ there exists a functor $G : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that $G \circ F = id_{\mathcal{C}_1}$ and $F \circ G = id_{\mathcal{C}_2}$, where $id_{\mathcal{C}_1} : \mathcal{C}_1 \longrightarrow \mathcal{C}_1$ and $id_{\mathcal{C}_2} : \mathcal{C}_2 \longrightarrow \mathcal{C}_2$ are identity functors, then F is called a functor isomorphism.

Definition 2.1.14 ***Isomorphic categories.***
The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be isomorphic if there exists a functor isomorphism $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic we write $\mathcal{C}_1 \cong \mathcal{C}_2$.

Definition 2.1.15 ***Exponential objects.***
Let $\mathcal{C}$ be a category with products and let A and B be objects in $\mathcal{C}$. We denote all morphisms from A to B by B^A. For any object C in $\mathcal{C}$, the condition $B^{A \times C} \cong (B^A)^C$ is called the exponential law. The objects A and B are exponentiable or called exponential objects if they satisfy the exponential law.

Definition 2.1.16 ***Cartesian-closed categories.***
Let $\mathcal{C}$ be a category with final objects and finite products. If for each object A in $\mathcal{C}$ the functor $(A \times -) : \mathcal{C} \longrightarrow \mathcal{C}$ is a co-adjoint functor then $\mathcal{C}$ is said to be Cartesian-closed. The functor $(A \times -) : \mathcal{C} \longrightarrow \mathcal{C}$ maps every object B in $\mathcal{C}$ to the product $A \times B$ in $\mathcal{C}$.

Following from the definition of Cartesian-closed categories we have that a category is Cartesian-closed if and only if it has final objects, finite products and exponential objects. We have been consistent with the category of sets and the category of posets thus far in this thesis, and these two categories are Cartesian-closed (see [1] and [5]).

2.2 Concepts from general topology.

2.2.1 Topology.

Definition 2.2.1 ***Topology.***
Let X be a set, $P(X)$ be the power set of X and $\tau \subseteq P(X)$. Then τ is a topology on X if the following conditions are satisfied:

1. *$\phi, X \in \tau$, where ϕ denote the empty set.*
2. *If $U_1, U_2, \cdots, U_n \in \tau$ then $\bigcap_{i=1}^{n} U_i \in \tau$. That is, τ is closed under finite intersection.*

3. *If* $U_\alpha \in \tau$, $\alpha \in I$ *then* $\bigcup_{\alpha \in I} U_\alpha \in \tau$. *That is,* τ *is closed under infinite union.*

The pair (X, τ) *is called a topological space.*

Examples of topologies:

Example 2.2.1 ***Trivial topology.***
Let X *be a set and* ϕ *be the exmpty set. It is trivial that the collection* $\{X, \phi\}$ *is a topology on* X *since* $X \cap \phi = \phi$ *and* $X \cup \phi = X$. *Thus,* $\{X, \phi\}$ *is called the trivial topology.*

Example 2.2.2 ***Discrete topology.***
Let X *be a non-empty set. Consider* $P(X) := \{U \mid U \subseteq X\}$, *that is, the power set of* X. *Since* $X, \phi \subseteq X$ *therefore* $X, \phi \in P(X)$. *Let* $U_1, U_2, \cdots, U_n \in P(X)$, *then* $U_i \subseteq X$, $\forall\ i = 1, 2, \cdots, n$. *Since* $\bigcap_{i=1}^{n} U_i \subseteq X$ *therefore* $\bigcap_{i=1}^{n} U_i \in P(X)$. *That is,* $P(X)$ *is closed under finite intersection.*
Let $U_\alpha \in P(X)$, $\alpha \in I$, *then* $U_\alpha \subseteq X$. *Since* $\bigcup_{\alpha \in I} U_\alpha \subseteq X$ *therefore* $\bigcup_{\alpha \in I} U_\alpha \in P(X)$. *That is,* $P(X)$ *is closed under infinite union. With* $P(X)$ *closed under infinite union and finite intersection and* $\phi, X \in P(X)$ *therefore* $P(X)$ *is a topology on* X, *called the discrete topology.*

Example 2.2.3 ***Finite complement topology.***
Let X *be a non-empty set and let*

$$\tau_f := \{U \subseteq X \mid X - U \ is\ finite\ or\ X - U = X\}.$$

Since $\phi, X \in P(X)$ *then* $\phi, X \subseteq X$ *and we have that* $X - \phi = X$ *and* $X - X = \phi$. *Therefore* $\phi, X \in \tau_f$.
Let $U_1, U_2, \ldots, U_n \in \tau_f$, *therefore* $U_i \subseteq X$ *and* $X - U_i$ *is finite or* $X - U_i = X$, $\forall\ i = 1, 2, \ldots, n$. *Since* $U_i \subseteq X$, $\forall\ i = 1, 2, \ldots, n$, *then* $\bigcap_{i=1}^{n} U_i \subseteq X$. *Also* $X - \bigcap_{i=1}^{n} U_i = \bigcap_{i=1}^{n} (X - U_i)$. *Therefore* $X - \bigcap_{i=1}^{n} U_i$ *is finite or all of* X. *Thus* $\bigcap_{i=1}^{n} U_i \in \tau_f$, *that is,* τ_f *is closed under finite intersection.*
Let $U_\alpha \in \tau_f$, $\alpha \in I$, *then* $U_\alpha \subseteq X$ *such that* $X - U_\alpha$ *is finite or all of* X, $\alpha \in I$. *Therefore* $\bigcup_{\alpha \in I} U_\alpha \subseteq X$ *and* $X - \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} (X - U_\alpha)$. *Thus* $X - \bigcup_{\alpha \in I} U_\alpha$ *is finite or all of* X. *Therefore* $\bigcup_{\alpha \in I} U_\alpha \in \tau_f$. *That is,* τ_f *is closed under infinite union.*
Hence τ_f *is a topology on* X *and is called the finite complement topology.*

Definition 2.2.2 *Let (X, τ_X) and (Y, τ_Y) be topological spaces, then the morphism $f : X \longrightarrow Y$ is continuous if $f^{-1}(U) \in \tau_X$, $\forall\ U \in \tau_Y$.*

Definition 2.2.3 ***Basis of a topology.***
Let X be a non-empty set. A collection β of subsets of X, that is $\beta \subseteq P(X)$, is a basis for a topology on X if:

1. *$\forall\ x \in X\ \exists\ B \in \beta$ such that $x \in B$.*
2. *If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subset B_1 \cap B_2$.*

The topology τ_β generated by the basis β of a topology on X is defined as

$$\tau_\beta = \{U \subseteq X \mid \forall\ x \in U\ \exists\ B \in \beta \text{ such that } x \in B \subset U\}.$$

Examples of basis of a topology:

Example 2.2.4 *Consider the open interval*

$$(a, b) = \{x \mid a < x < b\}$$

in the real line $\mathbb{R}$ and let

$$\beta = \{(a, b) \mid a, b \in \mathbb{R}\}.$$

Let $r \in \mathbb{R}$ be arbitrary, then there exits $a, b \in \mathbb{R}$ such that $a < r < b$. That is, $\forall\ r \in \mathbb{R}\ \exists\ (a, b) \in \beta$ such that $r \in (a, b)$. Now let $(a, b), (z, w) \in \beta$ and $r \in (a, b) \cap (z, w)$. Since $(a, b) \cap (z, w) \neq \phi$ then $(a, b) \cap (z, w) = (x, y) \in \beta$ for some $x, y \in \mathbb{R}$. Then there exists $m, n \in \mathbb{R}$ such that $r \in (m, n) \subset (x, y)$. Therefore β is a basis of a topology on $\mathbb{R}$. The topology generated by β is called the standard topology on $\mathbb{R}$, denoted $\tau_{\mathbb{R}}$. Thus, β is the basis of the standard topology $\tau_{\mathbb{R}}$.

Example 2.2.5 *Consider the half-open interval*

$$[a, b) = \{x \mid a \leq x < b\}$$

in the real line $\mathbb{R}$ and let

$$\tilde{\beta} = \{[a, b) \mid a, b \in \mathbb{R}\}.$$

Let $r \in \mathbb{R}$, then $r \in [r, b)$, $\forall\ b \in \mathbb{R}$. That is $\forall\ r \in \mathbb{R}\ \exists\ B \in \tilde{\beta}$ such that $r \in B$. Let $[a, b), [z, w) \in \tilde{\beta}$ and $r \in [a, b) \cap [z, w)$. Since $[a, b) \cap [z, w) \neq \phi$ then $[a, b) \cap [z, w) = [x, y)$ for some $x, y \in \mathbb{R}$. Then there exists $t \in \mathbb{R}$ such that $t < y$ where $r \in [x, t) \subset [x, y)$. Therefore $\tilde{\beta}$ is a basis of a topology on $\mathbb{R}$. The topology generated by $\tilde{\beta}$ is called the lower limit topology. When $\mathbb{R}$ is endowed with the lower limit topology it is denoted $\mathbb{R}_\ell$. Thus we denote the lower limit topology by $\tau_{\mathbb{R}_\ell}$. Thus $\tilde{\beta}$ is the basis of the lower limit topology $\tau_{\mathbb{R}_\ell}$. The topological space $(\mathbb{R}_\ell, \tau_{\mathbb{R}_\ell})$ is called the Sorgenfrey line.

Example 2.2.6 *Let*

$$K = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\},$$

and let $a, b \in \mathbb{R}$ and $\hat{\beta}$ be the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$, that is,

$$\hat{\beta} = \{(a, b) \mid a, b \in \mathbb{R}\} \cup \{(a, b) - K \mid a, b \in \mathbb{R}\}.$$

Since $\beta := \{(a, b) \mid a, b \in \mathbb{R}\}$ generates the standard topology $\tau_{\mathbb{R}}$ on $\mathbb{R}$ (Example 2.2.4) then $\forall\ r \in \mathbb{R}\ \exists\ (a, b)$ such that $r \in (a, b)$. It follows that for $r \neq \frac{1}{n}$, $n \in \mathbb{Z}^+$ then there exists (a, b) such that $r \in ((a, b) - K)$. Therefore $\forall\ r \in \mathbb{R}\ \exists\ B \in \hat{\beta}$ such that $r \in B$. Let $B_1, B_2 \in \hat{\beta}$, then we have 3 cases:
Case 1: $B_1, B_2 \in \beta := \{(a, b) \mid a, b \in \mathbb{R}\}$.
Since β generates the standard topology $\tau_{\mathbb{R}}$ then $\forall\ x \in B_1 \cap B_2\ \exists\ B_3 \in \beta$ such that $x \in B_3 \subset B_1 \cap B_2$.
Case 2: $B_1, B_2 \in \{(a, b) - K \mid a, b \in \mathbb{R}\}$.
Let $x \in B_1 \cap B_2$ then $B_1 \cap B_2 \neq \phi$ and $x \neq \frac{1}{n}$, $n \in \mathbb{Z}^+$, therefore $B_1 \cap B_2 = (z, w) - K$ for some $z, w \in \mathbb{R}$. Then there exists $m, n \in \mathbb{R}$ such that $x \in ((m, n) - K) \subset ((z, w) - K)$.
Case 3: $B_1 \in \{(a, b) \mid a, b \in \mathbb{R}\}$ and $B_2 \in \{(a, b) - K \mid a, b \in \mathbb{R}\}$.
Let $x \in B_1 \cap B_2$ which implies that $B_1 \cap B_2 \neq \phi$ and $x \neq \frac{1}{n}$, $n \in \mathbb{Z}^+$, then $B_1 \cap B_2 = (z, w) - K$ for some $z, w \in \mathbb{R}$. Therefore there exists $m, n \in (\mathbb{R} - K)$ such that $x \in (m, n) \subset ((z, w) - K)$. Hence $\hat{\beta}$ is a basis of a topology on $\mathbb{R}$. The topology generated by $\hat{\beta}$ is called the K-topology on $\mathbb{R}$. When $\mathbb{R}$ is endowed with the K-topology it is denoted $\mathbb{R}_K$. Thus, we denote the K-topology on $\mathbb{R}$ by $\tau_{\mathbb{R}_K}$. That is, $\hat{\beta}$ is the basis of $\tau_{\mathbb{R}_K}$.

2.2.2 Category of topological spaces.

Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be topological spaces and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps, that is, $f^{-1}(U) \in \tau_X$, $\forall\ U \in \tau_Y$ and $g^{-1}(V) \in \tau_Y$, $\forall\ V \in \tau_Z$. Consider the composition $g \circ f : X \longrightarrow Z$. Let $W \in \tau_Z$, then $g^{-1}(W) \in \tau_Y$ since $g : Y \longrightarrow Z$ is a continuous map, which implies that $f^{-1}(g^{-1}(W)) \in \tau_X$ since $f : X \longrightarrow Y$ is a continuous map. That is, $\forall\ W \in \tau_Z$ we have that $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \tau_X$, therefore $g \circ f : X \longrightarrow Z$ is a continuous map. Let $I_X : X \longrightarrow X$ be an identity map then $\forall\ U \in \tau_X$ we have that $I_X(U) = U \in \tau_X$, therefore $I_X : X \longrightarrow X$ is a continuous map. Thus, the topological spaces and the continuous maps form a category - the category of topological spaces, usually denoted **Top**. That is, in the category of topological spaces the objects are topological spaces and the morphisms are continuous maps. The category of topological spaces is a category of a structured set since a topology is a structured set.

We now look at some properties of the category of topological spaces:

1. An initial object and final objects exist in the category of topological spaces. In the category of topological spaces the empty set is an initial object and any singleton topological space is a final object.

2. The category of topological spaces is complete and cocomplete. That is, limits and colimits exist in the category of topological spaces.

3. There is a canonical forgetful functor, $U : \mathbf{Top} \longrightarrow \mathbf{Set}$, from the category of topological spaces to the category of sets. This forgetful functor assigns each topological space to its underlying set. Thus the category of topological spaces is topological over sets (see [24]).

4. The category of topological spaces is not Cartesian-closed (see [50]).

2.2.3 Compactness.

Definition 2.2.4 ***Open cover.***
Let (X, τ) be a topological space. An open cover of X is a collection $\{O_\alpha \mid \alpha \in I\}$ of open sets (of the topology τ) such that $X = \bigcup_{\alpha \in I} O_\alpha$.

Definition 2.2.5 ***Subcover of an open cover.***
Let (X, τ) be a topological space and the collection $\{O_\alpha \mid \alpha \in I\}$ be an open cover of X, then the subcover of the open cover $\{O_\alpha \mid \alpha \in I\}$ is the collection $\{O_\alpha \mid \alpha \in J\}$ such that $J \subset I$ and $X = \bigcup_{\alpha \in J} O_\alpha$.

Definition 2.2.6 ***Compact space.***
The topological space (X, τ) is said to be a compact space if every open cover of X has a finite subcover. That is, for every open cover $\{O_\alpha \mid \alpha \in I\}$ of X, thus $X = \bigcup_{\alpha \in I} O_\alpha$, there exists a finite set J such that $J \subset I$ and $\{O_\alpha \mid \alpha \in J\}$ covers X, that is $X = \bigcup_{\alpha \in J} O_\alpha$.

Definition 2.2.7 ***Compact set.***
Let (X, τ) be a topological space and $U \subseteq X$, then the set U is said to be compact if for every collection $\{O_\alpha \mid \alpha \in I\}$ of open sets (of the topology τ) such that $U \subseteq \bigcup_{\alpha \in I} O_\alpha$ there exists a finite set J such that $J \subset I$ and that for $\{O_\alpha \mid \alpha \in J\}$ we have that $U \subseteq \bigcup_{\alpha \in J} O_\alpha$.

Theorem 2.2.1 *The finite union of compact sets is compact.*

Proof: Let (X, τ) be a topological space, $A \subseteq X$ and $B \subseteq X$, with A and B being compact. It is sufficient to prove that $A \cup B$ is compact. Since A is compact then the open cover of A, say $\{O_\alpha \mid \alpha \in I\}$, has a finite subcover $\{O_{\alpha_i} \mid i = 1, 2, \cdots, n\}$. Similarly, since B is also compact then every open cover, say $\{U_\lambda \mid \lambda \in J\}$, of B has a finite subcover $\{U_{\lambda_j} \mid j = 1, 2, \cdots, m\}$. Then

$$\{O_\alpha \mid \alpha \in I\} \bigcup \{U_\lambda \mid \lambda \in J\}$$

forms an open cover for $A \cup B$, and

$$\{O_{\alpha_i} \mid i = 1, 2, \cdots, n\} \bigcup \{U_{\lambda_j} \mid j = 1, 2, \cdots, m\}$$

is its finite subcover. Therefore $A \cup B$ is compact. □

Definition 2.2.8 ***Closed set.***
Let (X, τ) be a topological space and $U \subseteq X$. The set U is said to be closed if its complement is open. Thus if $X \setminus U \in \tau$.

Definition 2.2.9 ***Closure of a set.***
Let (X, τ) be a topological space and $U \subseteq X$. The closure of U, denoted $Cl(U)$, is the smallest closed subset of X containing U, and that is the intersection of all closed subsets of X containing U.

Proposition 2.2.1 *Let (X, τ) be a topological space and $U \subseteq X$, then $U \subseteq Cl(U)$.*

Proof: This follows from Definition 2.2.9. □

Theorem 2.2.2 *Let (X, τ) be a topological space and $U \subseteq X$, then U is closed if and only if $U = Cl(U)$.*

Proof: ($\Longrightarrow$) Let U be a closed subset of X. The smallest closed subset of X containing U is U itself. Therefore by Definition 2.2.9 we have that $U = Cl(U)$.
($\Longleftarrow$) Let $U = Cl(U)$. Since the closure of a set is closed (see Definition 2.2.9) then $Cl(U)$ is closed. Therefore U is also closed since $U = Cl(U)$. □

Definition 2.2.10 ***Relatively compact set.***
Let (X, τ) be a topological space and $U \subseteq X$, then the set U is relatively compact if its closure, that is $Cl(U)$, is compact.

Definition 2.2.11 ***Hausdorff space.***
A topological space (X, τ) is a Hausdorff space if $\forall\ x, y \in X$ with $x \neq y$ there exist open sets $U, V \subseteq X$ such that for $x \in U$ and $y \in V$ we have $U \cap V = \phi$.

Theorem 2.2.3 *Let X be a Hausdorff space and $S \subseteq X$, then S is Hausdorff space.*

Proof: Let X be a Hausdorff space and $s_1, s_2 \in S$, with $s_1 \neq s_2$, then $s_1, s_2 \in X$ since $S \subseteq X$. Since X is a Hausdorff space then there exist open sets $U, V \subseteq X$ such that $s_1 \in U$, $s_2 \in V$ and $U \cap V = \phi$. We have that $s_1 \in U \cap S$ and $s_2 \in V \cap S$, and $U \cap S$ and $V \cap S$ are open in S. Then

$$(U \cap S) \cap (V \cap S) = (U \cap V) \cap S = \phi \cap S = \phi.$$

Therefore S is a Hausdorff space. □

Theorem 2.2.4 *Let X and Y be Hausdorff spaces, then $X \times Y$ is a Hausdorff space.*

Proof: Let X and Y be Hausdorff spaces. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$. Since X is a Hausdorff space then there exist open sets $U_1, U_2 \subseteq X$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \phi$. Similarly, there exist open sets $V_1, V_2 \subseteq Y$ such that $y_1 \in V_1$, $y_2 \in V_2$ and $V_1 \cap V_2 = \phi$ since Y is a Hausdorff space. Since $x_1 \neq x_2$ and $y_1 \neq y_2$ then we have that (x_1, y_1) and (x_2, y_2) are distinct in $X \times Y$. Thus $(x_1, y_1) \in U_1 \times V_1$ and $(x_2, y_2) \in U_2 \times V_2$, and $U_1 \times V_1$ and $U_2 \times V_2$ are open in $X \times Y$. Since

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \phi \times \phi = \phi,$$

therefore $X \times Y$ is a Hausdorff space. □

Theorem 2.2.5 *Let X and Y be Hausdorff spaces, then the coproduct $X \coprod Y$ is a Hausdorff space.*

Proof: Let

$$X \coprod Y = \{(x, 1) \mid x \in X\} \cup \{(y, 2) \mid y \in Y\},$$

then we have the canonical projections

$$\varphi_1 : X \longrightarrow X \coprod Y$$

and

$$\varphi_2 : Y \longrightarrow X \coprod Y.$$

Let $x_1, x_2 \in X$ and $(x_1, 1), (x_2, 1) \in X \coprod Y$ such that $(x_1, 1) \neq (x_2, 1)$. But $(x_1, 1) \neq (x_2, 1) \implies x_1 \neq x_2$, and since X is a Hausdorff space then there exist open subsets M and N of X such that $x_1 \in M$, $x_2 \in N$ and $M \cap N = \phi$. With $x_1 \in M$ and $x_2 \in N$ therefore $(x_1, 1) \in (M \times 1)$ and $(x_2, 1) \in (N \times 1)$, and $M \times 1$ and $N \times 1$ are open subsets of $X \coprod Y$ since $\varphi_1^{-1}(M \times 1) = M$ and $\varphi_1^{-1}(N \times 1) = N$, and

$$(M \times 1) \cap (N \times 1) = (M \cap N) \times 1 = \phi \times 1 = \phi.$$

Similarly for $(y_1, 2), (y_2, 2) \in X \coprod Y$ such that $(y_1, 2) \neq (y_2, 2)$ we have that there exist disjoint open subsets, say $(U \times 2)$ and $(V \times 2)$, of $X \coprod Y$ with $(y_1, 2) \in (U \times 2)$ and $(y_2, 2) \in (V \times 2)$, and

$$(U \times 2) \cap (V \times 2) = (U \cap V) \times 2 = \phi \times 2 = \phi.$$

Now we need to account for having two non-equal elements with one coming from $\{(x, 1) \mid x \in X\}$ and the other from $\{(y, 2) \mid y \in Y\}$. Let $x \in X$ and $y \in Y$ then $(x, 1), (y, 2) \in X \coprod Y$ and by definition of $X \coprod Y$ we have that $(x, 1) \neq (y, 2)$. Since

$$\varphi_1^{-1}\left(\{(x, 1) \mid x \in X\}\right) = X$$

and

$$\varphi_2^{-1}(\{(y, 2) \mid y \in Y\}) = Y$$

therefore $\{(x, 1) \mid x \in X\}$ and $\{(y, 2) \mid y \in Y\}$ are open subsets of $X \coprod Y$. By definition

$$\{(x, 1) \mid x \in X\} \cap \{(y, 2) \mid y \in Y\} = \phi.$$

Hence $X \coprod Y$ is a Hausdorff space. □

Theorem 2.2.6 *Let X be a Hausdorff space and $U \subset X$, then U is closed if it is compact.*

Proof: We have to show that $X \setminus U$ is open. Let $x \in X \setminus U$. For any $y \in U$ we have that x and y are distinct. Note that $y \in U \implies y \in X$ since $U \subset X$. Therefore, since X is Hausdorff, $\forall\ y \in U$ there exists open sets A_y and B_y such that $x \in A_y,\ y \in B_y$ and $A_y \cap B_y = \phi$. Having that U is compact it follows that there exists a finite set $I \subseteq U$ such that $U \subseteq \bigcup_{y \in I} B_y$. Since $x \in A_y,\ \forall\ y \in U$, then $x \in \bigcap_{y \in I} A_y$ and $\bigcap_{y \in I} A_y$ is open since the intersection of finite open sets is open. Then it follows that $U \cap \left(\bigcap_{y \in I} A_y\right) = \phi$. That is, $\bigcap_{y \in I} A_y$ is a neighbourhood of x disjoint from U. Therefore $X \setminus U$ is open which implies that U is closed. □

Theorem 2.2.7 *Let X be a Hausdorff space. If $U \subseteq X$ is compact then it is relatively compact.*

Proof: Let X be a Hausdorff space and $U \subseteq X$ be compact. We have to show that the closure of U, that is $Cl(U)$, is compact. Since $U \subseteq X$ and X is a Hausdorff space then by Theorem 2.2.6 we have that U is closed. Now since U is closed then by Theorem 2.2.2 we have that $U = Cl(U)$. This implies that $Cl(U)$ is compact since U is compact. With $Cl(U)$ compact therefore U is relatively compact. □

2.3 Concepts on Frölicher spaces.

2.3.1 Frölicher structures and spaces.

Let X be a non-empty set,

$$\mathbb{R}^X := \{f : X \longrightarrow \mathbb{R} \mid f \text{ } is \text{ } a \text{ } function\}$$

and

$$X^{\mathbb{R}} := \{c : \mathbb{R} \longrightarrow X \mid c \text{ } is \text{ } a \text{ } curve\}.$$

Consider the power sets $P(\mathbb{R}^X)$ and $P(X^{\mathbb{R}})$. Let $U : F_1 \hookrightarrow F_2$ and $V : F_2 \hookrightarrow F_3$ be inclusion maps where $F_1, F_2, F_3 \in P(\mathbb{R}^X)$. Then we have the composition function $V \circ U : F_1 \hookrightarrow F_3$ by transitivity of sets. Also, $\forall$ $F \in P(\mathbb{R}^X)$ we have the identity function $I_F : F \longrightarrow F$. Therefore subsets of $\mathbb{R}^X$ and inclusion maps make $P(\mathbb{R}^X)$ a category. Similarly, $P(X^{\mathbb{R}})$ is also a category.

Let $F \in P(\mathbb{R}^X)$, $C \in P(X^{\mathbb{R}})$ and $\Gamma : P(\mathbb{R}^X) \longrightarrow P(X^{\mathbb{R}})$ and
$\Phi : P(X^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^X)$ be defined by

$$\Gamma F := \{c : \mathbb{R} \longrightarrow X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F\}$$

and

$$\Phi C := \{f : X \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C\},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ is the set of all infinitely differentiable (smooth) maps from $\mathbb{R}$ to $\mathbb{R}$. That is, Γ maps a set of real-valued functions to a set of curves into X, and Φ maps a set of curves into X to a set of real-valued functions on X. Thus, $F \in P(\mathbb{R}^X) \implies \Gamma F \in P(X^{\mathbb{R}})$ and $C \in P(X^{\mathbb{R}}) \implies \Phi C \in P(\mathbb{R}^X)$.

Lemma 2.3.1 *Let X be a non-empty set, $\mathbb{R}^X$ be the set of all real-valued functions on X and $X^{\mathbb{R}}$ be the set of all curves into X. Given $F_0, F_1 \in P(\mathbb{R}^X)$ and $C_0, C_1 \in P(X^{\mathbb{R}})$.*

1. *If $F_0 \subseteq F_1$ then $\Gamma F_1 \subseteq \Gamma F_0$, $F_0 \subseteq \Phi\Gamma F_0$ and $\Gamma F_0 = \Gamma\Phi\Gamma F_0$.*
2. *If $C_0 \subseteq C_1$ then $\Phi C_1 \subseteq \Phi C_0$, $C_0 \subseteq \Gamma\Phi C_0$ and $\Phi C_0 = \Phi\Gamma\Phi C_0$.*

Proof:

1. Let $F_0, F_1 \in P(\mathbb{R}^X)$ such that $F_0 \subseteq F_1$. For $\Gamma : P(\mathbb{R}^X) \longrightarrow P(X^{\mathbb{R}})$ we have that

$$\Gamma F_0 = \{c : \mathbb{R} \longrightarrow X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_0\}$$

and

$$\Gamma F_1 = \{c : \mathbb{R} \longrightarrow X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_1\}.$$

Let $c \in \Gamma F_1$ therefore $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, $\forall\ f \in F_1$. But since $F_0 \subseteq F_1$ then it follows that $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, $\forall\ f \in F_0$, and this implies that $c \in \Gamma F_0$. That is $c \in \Gamma F_1 \implies c \in \Gamma F_0$, therefore $\Gamma F_1 \subseteq \Gamma F_0$.
Let $\Phi : P(X^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^X)$ be defined, and with

$$\Gamma F_0 = \{c : \mathbb{R} \longrightarrow X \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}),\ \forall\ f \in F_0\}$$

we have that

$$\Phi\Gamma F_0 = \{f : X \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}),\ \forall\ c \in \Gamma F_0\}.$$

Let $f \in F_0$ then $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, $\forall\ c \in \Gamma F_0$, which implies that $f \in \Phi\Gamma F_0$. That is $f \in F_0 \implies f \in \Phi\Gamma F_0$, therefore $F_0 \subseteq \Phi\Gamma F_0$. Let $f \in \Gamma\Phi F_0$, then $\forall\ c \in \Gamma F_0$ we have that $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$. But $c \in \Gamma F_0 \implies f \in F_0$, therefore $\Phi\Gamma F_0 \subseteq F_0$. That is $F_0 \subseteq \Phi\Gamma F_0$ and $\Phi\Gamma F_0 \subseteq F_0$. Therefore $F_0 = \Phi\Gamma F_0$ and since $F_0, \Phi\Gamma F_0 \in P(\mathbb{R}^X)$ therefore $\Gamma F_0 = \Gamma\Phi\Gamma F_0$.

2. Let $C_0, C_1 \in P(X^{\mathbb{R}})$ such that $C_0 \subseteq C_1$. With $\Phi : P(X^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^X)$ defined we have that

$$\Phi C_0 = \{f : X \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}),\ \forall\ c \in C_0\}$$

and

$$\Phi C_1 = \{f : X \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}),\ \forall\ c \in C_1\}.$$

Let $f \in \Phi C_1$ then $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, $\forall\ c \in C_1$, which implies that $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, $\forall\ c \in C_0$, since $C_0 \subseteq C_1$, therefore $f \in \Phi C_0$. That is $f \in \Phi C_1 \implies f \in \Phi C_0$, therefore $\Phi C_1 \subseteq \Phi C_0$.
Let $\Gamma : P(\mathbb{R}^X) \longrightarrow P(X^{\mathbb{R}})$ be defined then applying Γ on

$$\Phi C_0 = \{f : X \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}),\ \forall\ c \in C_0\}$$

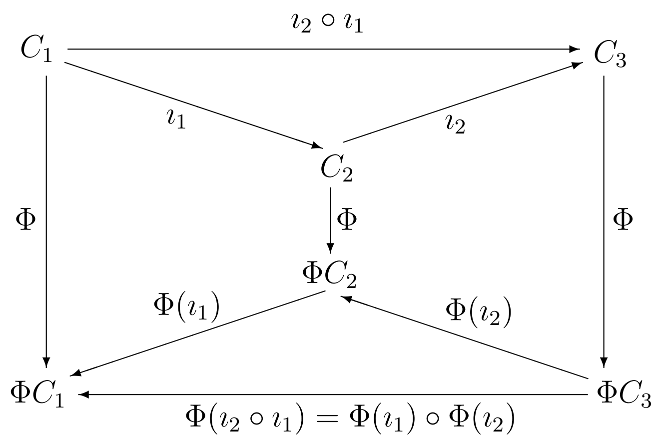
we have that

$$\Gamma\Phi C_0 = \{c : \mathbb{R} \longrightarrow X \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}),\ \forall\ f \in \Phi C_0\}.$$

Let $c \in C_0$, then $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, $\forall\ f \in \Phi C_0$, which implies that $f \in \Gamma\Phi C_0$. That is $c \in C_0 \implies c \in \Gamma\Phi C_0$, therefore $C_0 \subseteq \Gamma\Phi C_0$. Let $c \in \Phi\Gamma C_0$, then $\forall\ f \in \Phi C_0$ we have that $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, which implies that $c \in C_0$, therefore $\Gamma\Phi C_0 \subseteq C_0$. That is, $C_0 \subseteq \Gamma\Phi C_0$ and $\Gamma\Phi C_0 \subseteq C_0$, therefore $C_0 = \Gamma\Phi C_0$ and since $C_0, \Gamma\Phi C_0 \in P(X^{\mathbb{R}})$ therefore $\Phi C_0 = \Phi\Gamma\Phi C_0$. □

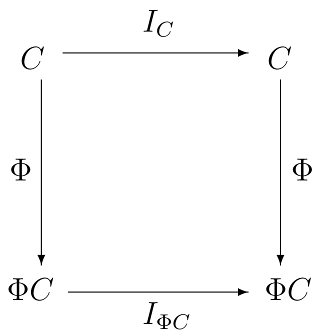
Let $C_1, C_2, C_3 \in P(X^{\mathbb{R}})$ such that $C_1 \subseteq C_2 \subseteq C_3$, then we have that $\Phi : C_i \mapsto \Phi C_i$ where $\Phi C_i \in P(\mathbb{R}^X)$ for $i = 1, 2, 3$. Now since $C_1 \subseteq C_2 \subseteq C_3$ in $P(X^{\mathbb{R}})$, then we have the inclusion maps $\imath_1 : C_1 \hookrightarrow C_2$ and $\imath_2 : C_2 \hookrightarrow C_3$ as morphisms in $P(X^{\mathbb{R}})$. Then it follows that we have the inclusion $\imath_2 \circ \imath_1 : C_1 \hookrightarrow C_3$. But since $C_1 \subseteq C_2 \subseteq C_3$ then by Lemma 2.3.1, $\Phi C_3 \subseteq \Phi C_2 \subseteq \Phi C_1$, that is, we have the inclusion maps $\Phi(\imath_1) : \Phi C_2 \hookrightarrow \Phi C_1$ and $\Phi(\imath_2) : \Phi C_3 \hookrightarrow \Phi C_2$ as morphisms in

$P(\mathbb{R}^X)$. Then it follows that we have the inclusion $\Phi(\imath_1) \circ \Phi(\imath_2) : \Phi C_3 \hookrightarrow \Phi C_1$. Then since $C_1 \subseteq C_3$ then we have that $\Phi C_3 \subseteq \Phi C_1$ by Lemma 2.3.1, that is, the inclusion $\Phi(\imath_2 \circ \imath_1) : \Phi C_3 \hookrightarrow \Phi C_1$ is the corresponding morphism in $P(\mathbb{R}^X)$. Then it follows that $\Phi(\imath_2 \circ \imath_1) = \Phi(\imath_1) \circ \Phi(\imath_2)$. These is illustrated in the diagram below:



The arrows $\imath_1$, $\imath_2$, $\imath_2 \circ \imath_1$, $\Phi(\imath_1)$, $\Phi(\imath_2)$ and $\Phi(\imath_2 \circ \imath_1)$ are inclusion maps.

Let C be any element in $P(X^{\mathbb{R}})$ and consider the identity map $I_C : C \longrightarrow C$. Then $\Phi : C \mapsto \Phi C$ and since $C \subseteq C$ then by Lemma 2.3.1 $\Phi C \subseteq \Phi C$ and thus we have the identity map $I_{\Phi C} : \Phi C \longrightarrow \Phi C$ in $P(\mathbb{R}^X)$. Therefore it follows that $\Phi(I_C) = I_{\Phi C}$. These is illustrated in the diagram below:



From the above diagrams we have that Φ preserves composition and identity. The same process can be applied for $\Gamma : P(\mathbb{R}^X) \longrightarrow P(X^{\mathbb{R}})$. Hence Φ and Γ are functors.

Definition 2.3.1 ***Frölicher structure and Frölicher space.***
Let X be a non-empty set, $\Gamma : P(\mathbb{R}^X) \longrightarrow P(X^{\mathbb{R}})$ and $\Phi : P(X^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^X)$ be defined by

$$\Gamma F := \{c : \mathbb{R} \longrightarrow X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in F\}$$

and

$$\Phi C := \{f : X \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C\},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ is the set of all infinitely differentiable (smooth) maps from $\mathbb{R}$ to $\mathbb{R}$. A Frölicher structure on X is the pair (C_X, F_X) where $C_X \in P(X^{\mathbb{R}})$ and $F_X \in P(\mathbb{R}^X)$ with the duality or compatibility condition that

$$\Gamma F_X = C_X$$

and

$$\Phi C_X = F_X.$$

The triple (X, C_X, F_X) is called a Frölicher space.

That is for a Frölicher space (X, C_X, F_X) we have that

$$C_X = \Gamma F_X = \{c : \mathbb{R} \longrightarrow X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in F_X\}$$

and

$$F_X = \Phi C_X = \{f : X \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C_X\}$$

where $\Gamma : P(\mathbb{R}^X) \longrightarrow P(X^{\mathbb{R}})$ and $\Phi : P(X^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^X)$. Thus, the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{c} & X \\ & \searrow^{f\circ c} & \downarrow f \\ & & \mathbb{R} \end{array}$$

commutes, $\forall\ f \in F_X$ and $c \in C_X$. Frölicher spaces were previously called smooth spaces (see [45])

Lemma 2.3.2 *Let (X, C_X, F_X) be a Frölicher space. Then $c \in C_X$ if and only if $F_X \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})$ and $f \in F_X$ if and only if $f \circ C_X \subset C^\infty(\mathbb{R}, \mathbb{R})$. And $F_X \circ C_X \subset C^\infty(\mathbb{R}, \mathbb{R})$.*

Proof: This follows from the definition of Frölicher structure and Frölicher space. □

Theorem 2.3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function. Then f is smooth on $\mathbb{R}^n$ if and only if $f \circ c : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth for each curve $c : \mathbb{R} \longrightarrow \mathbb{R}^n$. That is, $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each curve $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$.*

Theorem 2.3.1 and its proof are the result of the work by Boman in [21].

Examples of Frölicher spaces:

Example 2.3.1 ***The canonical Frölicher space.***
Let $(\mathbb{R})^{\mathbb{R}^n}$, $n \in \mathbb{N}$, be the set of all real-vlued functions on $\mathbb{R}^n$ and $(\mathbb{R}^n)^{\mathbb{R}}$ be the set of all curves into $\mathbb{R}^n$, then $P((\mathbb{R})^{\mathbb{R}^n})$ and $P((\mathbb{R}^n)^{\mathbb{R}})$ are categories. Let $f \in (\mathbb{R})^{\mathbb{R}^n}$ and $c \in (\mathbb{R}^n)^{\mathbb{R}}$, and let $\Gamma : P((\mathbb{R})^{\mathbb{R}^n}) \longrightarrow P((\mathbb{R}^n)^{\mathbb{R}})$ and $\Phi : P((\mathbb{R}^n)^{\mathbb{R}}) \longrightarrow P((\mathbb{R})^{\mathbb{R}^n})$ be functors defined by

$$\Gamma F = \{c : \mathbb{R} \longrightarrow \mathbb{R}^n \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F\}$$

and

$$\Phi C = \{f : \mathbb{R}^n \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C\}.$$

Let $F = C^\infty(\mathbb{R}^n, \mathbb{R})$. With $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, $\forall\ f \in F = C^\infty(\mathbb{R}^n, \mathbb{R})$, then $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ by Theorem 2.3.1. Similarly let $C = C^\infty(\mathbb{R}, \mathbb{R}^n)$, then with $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, $\forall\ c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$, we have that $f \in C^\infty(\mathbb{R}^n, \mathbb{R}) = F$. That is,

$$\begin{aligned}\Gamma C^\infty(\mathbb{R}^n, \mathbb{R}) &= \{c : \mathbb{R} \longrightarrow \mathbb{R}^n \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in C^\infty(\mathbb{R}^n, \mathbb{R})\} \\ &= C^\infty(\mathbb{R}, \mathbb{R}^n)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\Phi C^\infty(\mathbb{R}, \mathbb{R}^n) &= \{f : \mathbb{R}^n \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C^\infty(\mathbb{R}, \mathbb{R}^n)\} \\ &= C^\infty(\mathbb{R}^n, \mathbb{R}).\end{aligned} \tag{2.2}$$

Therefore $(C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is a Frölicher structure. Hence the triple $(\mathbb{R}^n, C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is a Frölicher space. That is, the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{c} & \mathbb{R}^n \\ & {\scriptstyle f\circ c}\searrow & \downarrow{\scriptstyle f} \\ & & \mathbb{R} \end{array}$$

commutes $\forall\ f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $\forall\ c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$. The Frölicher space $(\mathbb{R}^n, C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is called the canonical Frölicher space.

Example 2.3.2 ***Manifolds.***
Let $\mathbb{R}^M$ and $M^{\mathbb{R}}$ be the set of all real-valued functions on M and a set of all curves into M, respectively, where M is a smooth manifold. Let $f : M \longrightarrow \mathbb{R}$ and $c : \mathbb{R} \longrightarrow M$ be a smooth real-valued function on M and a smooth curve into M, respectively. Then $f \circ c : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth map since M is a smooth manifold. Let $\Gamma : P(\mathbb{R}^M) \longrightarrow P(M^{\mathbb{R}})$ and $\Phi : P(M^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^M)$ be functors defined by

$$\Gamma F = \{c : \mathbb{R} \longrightarrow M \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F\}$$

and

$$\Phi C = \{f : M \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C\}$$

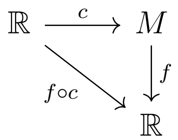
Then since $c \in C^\infty(\mathbb{R}, M)$, $f \in C^\infty(M, \mathbb{R})$ and $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, then we have that

$$\begin{aligned}\Gamma C^\infty(M, \mathbb{R}) &= \{c : \mathbb{R} \longrightarrow M \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in C^\infty(M, \mathbb{R})\} \\ &= C^\infty(\mathbb{R}, M)\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\Phi C^{\infty}(\mathbb{R}, M) &= \{f : M \longrightarrow \mathbb{R} \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\, c \in C^{\infty}(\mathbb{R}, M)\} \\ &= C^{\infty}(M, \mathbb{R}).\end{aligned} \tag{2.4}$$

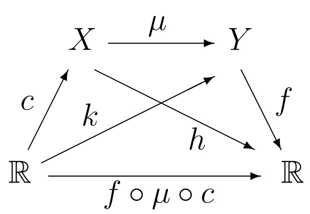
Therefore the pair $(C^{\infty}(\mathbb{R}, M), C^{\infty}(M, \mathbb{R}))$ *is a Frölicher structure, and thus the triple* $(M, C^{\infty}(\mathbb{R}, M), C^{\infty}(M, \mathbb{R}))$ *is a Frölicher space. That is, the diagram*



commutes $\forall\ f \in C^{\infty}(M, \mathbb{R})$ *and* $\forall\ c \in C^{\infty}(\mathbb{R}, M)$*.*

2.3.2 The category of Frölicher spaces

Definition 2.3.2 ***Frölicher smooth map.***
Let $\mu : X \longrightarrow Y$ *be a mapping where* (X, C_X, F_X) *and* (Y, C_Y, F_Y) *are Frölicher spaces. The mapping* $\mu : X \longrightarrow Y$ *is a Frölicher smooth map if* $\{\mu \circ c \mid c \in C_X\} \subseteq C_Y$ *and* $\{f \circ \mu \mid f \in F_Y\} \subseteq F_X$*. That is,* $\mu : X \longrightarrow Y$ *is a Frölicher smooth map if* $f \circ \mu \in C^{\infty}(X, \mathbb{R})$*,* $\mu \circ c \in C^{\infty}(\mathbb{R}, Y)$ *and* $f \circ \mu \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R})$*. Then we have the diagram*



where $h := f \circ \mu$ *and* $k := \mu \circ c$*.*

Let (X, C_X, F_X), (Y, C_Y, F_Y) and (Z, C_Z, F_Z) be Frölicher spaces. Let $\mu : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$ and $\rho : (Y, C_Y, F_Y) \longrightarrow (Z, C_Z, F_Z)$ be Frölicher smooth maps. That is, $\{\mu \circ c \mid c \in C_X\} \subseteq C_Y$, $\{f \circ \mu \mid f \in F_Y\} \subseteq F_X$, $\{\tilde{f} \circ \rho \mid \tilde{f} \in F_Z\} \subseteq F_Y$ and $\{\rho \circ \tilde{c} \mid \tilde{c} \in C_Y\} \subseteq C_Z$. Since the codomain of μ and the domain of ρ are the same therefore we have $\rho \circ \mu : (X, C_X, F_X) \longrightarrow (Z, C_Z, F_Z)$. Let $d \in C_X$ and $g \in F_Z$. We have that $g \circ \rho \in F_Y$ since $g \in F_Z$ and ρ is Frölicher smooth. Since μ is Frölicher smooth then it follows that $g \circ (\rho \circ \mu) = (g \circ \rho) \circ \mu \in F_X$. Similarly $\mu \circ d \in C_Y$ and $(\rho \circ \mu) \circ d = \rho \circ (\mu \circ d) \in C_Z$ since $d \in C_X$ and that ρ and μ are Frölicher smooth. That is, we have that $\{g \circ (\rho \circ \mu) \mid g \in F_Z\} \subset F_X$ and $\{(\rho \circ \mu) \circ d \mid d \in C_X\} \subset C_Z$, therefore $\rho \circ \mu$ is Frölicher smooth.

Let (W, C_W, F_W) be a Frölicher space and $I : (W, C_W, F_W) \longrightarrow (W, C_W, F_W)$ be an identity map. Let $f \in F_W$ and $c \in C_W$, then $f \circ I = f$ and $I \circ c = c$. That is $\{f \circ I \mid f \in F_W\} = F_W$ and $\{I \circ c \mid c \in C_W\} = C_W$. Therefore $I : (W, C_W, F_W) \longrightarrow (W, C_W, F_W)$ is a Frölicher smooth map. That is, for any Frölicher space (W, C_W, F_W) the identity morphism $I : (W, C_W, F_W) \longrightarrow (W, C_W, F_W)$ is Frölicher smooth.

Thus Frölicher spaces and Frölicher smooth maps form a category, that is the category of Frölicher spaces, which consists of Frölicher spaces as objects and Frölicher smooth maps as morphisms (see [7] [8], [10], [13], [14], [29], [30], [32], [36], [86], [88] and [89]).

The category of Frölicher spaces possesses some interesting properties (see [7], [13], [30], [45] and [56]). Here are some of the properties:

1. The category of Frölicher spaces is complete and cocomplete. That is the category of Frölicher spaces has limits and colimits (see definitions 2.1.7 and 2.1.8).

2. The category of Frölicher spaces is Cartesian-closed (this follows from [60], see also [13]).
   Consider the set $C^\infty(Y, Z)$ of all smooth maps from the Frölicher space $Y := (Y, C_Y, F_Y)$ to the Frölicher space $Z := (Z, C_Z, F_Z)$. For any Frölicher spaces $X := (X, C_X, F_X)$, $Y := (Y, C_Y, F_Y)$ and $Z := (Z, C_Z, F_Z)$, the set $C^\infty(Y, Z)$ satisfies the isomorphism

   $$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z)).$$

   If $X = \mathbb{R}$, that is $C^\infty(\mathbb{R} \times Y, Z) \cong C^\infty(\mathbb{R}, C^\infty(Y, Z))$, then we construct the set $C_{Y,Z}$ of curves $c : \mathbb{R} \longrightarrow C^\infty(Y, Z)$ by requiring that the map $\hat{c} : \mathbb{R} \times Y \longrightarrow Z$ defined by $\hat{c}(t, y) := c(t)(y)$ is smooth. Let $F_{Y,Z}$ be a set of real-valued functions $\bar{f} : C^\infty(Y, Z) \longrightarrow \mathbb{R}$ such that $\bar{f} \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in C_{Y,Z}$. Let $\mathbb{R}^{C^\infty(Y,Z)}$ and $(C^\infty(Y, Z))^{\mathbb{R}}$ be the set of all real-valued functions on $C^\infty(Y, Z)$ and the set of all curves into $C^\infty(Y, Z)$, respectively. Let $\Gamma : P(\mathbb{R}^{C^\infty(Y,Z)}) \longrightarrow P((C^\infty(Y, Z))^{\mathbb{R}})$ and $\Phi : P((C^\infty(Y, Z))^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^{C^\infty(Y,Z)})$ be functors defined by

   $$\Gamma F := \{c : \mathbb{R} \longrightarrow C^\infty(Y, Z) \mid \bar{f} \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ \bar{f} \in F\}$$

   and

   $$\Phi C := \{\bar{f} : C^\infty(Y, Z) \longrightarrow \mathbb{R} \mid \bar{f} \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C\}.$$

   Now let $F = F_{Y,Z}$ and $C = C_{Y,Z}$, then since $\bar{f} \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in C_{Y,Z} = C$, with $\bar{f} \in F_{Y,Z}$, therefore $\Phi(C_{Y,Z}) = F_{Y,Z}$. Then it follows that

   $$\Gamma F_{Y,Z} = \{c : \mathbb{R} \longrightarrow C^\infty(Y, Z) \mid \bar{f} \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ \bar{f} \in F_{Y,Z}\} = C_{Y,Z}$$

That is. the duality or compatibility condition holds. Therefore $(C_{Y,Z}, F_{Y,Z})$ is a Frölicher structure on $C^\infty(Y, Z)$. Thus the triple $(C^\infty(Y, Z), C_{Y,Z}, F_{Y,Z})$ is a Frölicher space. Then the canonical isomorphism

$$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z))$$

is an exponential law for the category of Frölicher spaces and it holds for every Frölicher space $X := (X, C_X, F_X)$, $Y := (Y, C_Y, F_Y)$ and $Z := (Z, C_Z, F_Z)$. Therefore the category of Frölicher spaces is Cartesian-closed.

3. The category of Frölicher spaces is topological over the category of sets (see [13], [30] and [56]). That is, the faithful (forgetful or underlying) functor from the category of Frölicher spaces to the category of sets is faithful (see Subsection 2.1.3).

4. The category of Frölicher spaces is a subcategory of the category of topological spaces since Frölicher spaces are topological spaces.

5. Subspace, product, coproduct and quotient exist in the category of Frölicher spaces (see [5], [6], [7], [9], [10], [12], [17], [18] and [19]).

2.3.3 Frölicher topologies.

Let (X, C_X, F_X) be a Frölicher space and consider the collection

$$\tau_{F_X} = \{U \subseteq X \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_X\},$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Since $\phi, X \in P(X)$, with $P(X)$ the power set of X, then $\phi, X \subseteq X$. For every real-valued function $f \in F_X$ we have that $X = f^{-1}(\mathbb{R})$ and $\phi = f^{-1}(\phi)$. Since $\mathbb{R}, \phi \in \tau_{\mathbb{R}}$, as $\tau_{\mathbb{R}}$ is a topology on $\mathbb{R}$, therefore $X, \phi \in \tau_{F_X}$.
Let $U_1, U_2, \ldots, U_n \in \tau_{F_X}$, therefore $U_i \subseteq X$ and $\forall\ f \in F_X$ we have that $U_i = f^{-1}(V_i)$ where $V_i \in \tau_{\mathbb{R}}$, $\forall\ i = 1, 2, \ldots, n$. Then $\bigcap_{i=1}^{n} U_i \subseteq X$ since $U_i \subseteq X$, $\forall\ i = 1, 2, \ldots, n$, and

$$\bigcap_{i=1}^{n} U_i = \bigcap_{i=1}^{n} f^{-1}(V_i) = f^{-1}\left(\bigcap_{i=1}^{n} V_i\right),\ \forall\ f \in F_X.$$

But $\bigcap_{i=1}^{n} V_i \in \tau_{\mathbb{R}}$ since $V_i \in \tau_{\mathbb{R}}$, $\forall\ i = 1, 2, \ldots, n$, (as $\tau_{\mathbb{R}}$ is closed under finite intersection) therefore $\bigcap_{i=1} U_i \in \tau_{F_X}$. That is, τ_{F_X} is closed under finite intersection.

Now let $A_\alpha \in \tau_{F_X}$, $\alpha \in I$. Therefore $A_\alpha \subseteq X$ such that $A_\alpha = f^{-1}(V_\alpha)$, $\forall\, f \in F_X$, where $V_\alpha \in \tau_{\mathbb{R}}$, $\alpha \in I$. Then $\bigcup\limits_{\alpha\in I} A_\alpha \subseteq X$ since $A_\alpha \subseteq X$, $\alpha \in I$. Also

$$\bigcup_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I} f^{-1}(V_\alpha) = f^{-1}\left(\bigcup_{\alpha\in I} V_\alpha\right),\ \forall\, f \in F_X$$

and since $\tau_{\mathbb{R}}$ is closed under infinite union then $\bigcup\limits_{\alpha\in I} V_\alpha \in \tau_{\mathbb{R}}$. Therefore $\bigcup\limits_{\alpha\in I} A_\alpha \in \tau_{F_X}$. That is, τ_{F_X} is closed under infinite union

Having that τ_{F_X} is closed under finite intersection and infinite union and that $\phi, X \in \tau_{F_X}$, hence τ_{F_X} is a topology on X.

Now consider the collection

$$\tau_{C_X} = \{U \subseteq X \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\, c \in C_X\}.$$

We have that $\phi, X \subseteq X$ since $\phi, X \in P(X)$. Since $c \in C_X$ is a curve into X then $c^{-1}(X) = \mathbb{R}$, $\forall\, c \in C_X$. Also, $\forall\, c \in C_X$, we have that $c^{-1}(\phi) = \phi$. Since $\tau_{\mathbb{R}}$ is a standard topology on $\mathbb{R}$ therefore $\phi, \mathbb{R} \in \tau_{\mathbb{R}}$. Then it follows that $\phi, X \in \tau_{C_X}$. Let $R_1, R_2, \ldots, R_k \in \tau_{C_X}$, then $R_j \subseteq X$ and $\forall\, c \in C_X$ we have that $c^{-1}(R_j) \in \tau_{\mathbb{R}}$, $\forall\, j = 1, 2, \ldots, k$. And $\bigcap\limits_{j=1}^{k} R_j \subseteq X$ since $R_j \subseteq X$, $\forall\, j = 1, 2, \ldots, k$. But, $\forall\, c \in C_X$, $c^{-1}\left(\bigcap\limits_{j=1}^{k} R_j\right) = \bigcap\limits_{j=1}^{k} c^{-1}(R_j) \in \tau_{\mathbb{R}}$ since $\forall\, c \in C_X$ we have that $c^{-1}(R_j) \in \tau_{\mathbb{R}}$, $\forall\, j = 1, 2, \ldots, k$. Therefore $\bigcap\limits_{j=1}^{k} R_j \in \tau_{C_X}$. That is, τ_{C_X} is closed under finite intersection.

Let $Q_q \in \tau_{C_X}$, $q \in I$, then $Q_q \subseteq X$ and $c^{-1}(Q_q) \in \tau_{\mathbb{R}}$, $\forall\, c \in C_X$. Also $c^{-1}\left(\bigcup\limits_{q\in I} Q_q\right) = \bigcup\limits_{q\in I} c^{-1}(Q_q) \in \tau_{\mathbb{R}}$, $\forall\, c \in C_X$, since $\forall\, c \in C_X$ we have that $c^{-1}(Q_q) \in \tau_{\mathbb{R}}$, $q \in I$. Therefore $\bigcup\limits_{q\in I} Q_q \in \tau_{C_X}$ since $\bigcup\limits_{q\in I} Q_q \subseteq X$. That is, τ_{C_X} is closed under infinite union.

Thus, τ_{C_X} is a topology on X since it is closed under finite intersection and infinite union and $\phi, X \in \tau_{C_X}$.

The topology τ_{F_X} is induced from the structure functions of F_X and the topology τ_{C_X} is induced from the structure curves of C_X. Thus these topologies are induced from the Frölicher space and for that we call them Frölicher topologies (see [14]).

Remark 2.3.1 *All structure functions and structure curves, of which are Frölicher smooth, are continuous in* τ_{F_X} *and* τ_{C_X} *respectively.*

Remark 2.3.2 *The basis and subbasis of* τ_{F_X} *are given by the collections* $\{f^{-1}(0,+\infty) \mid f \in F_X\}$ *and* $\{f^{-1}(0,1) \mid f \in F_X\}$*, respectively (see [36] and [45]).*

Lemma 2.3.3 *Let* (X, C_X, F_X) *be a Frölicher space and* τ_{F_X} *and* τ_{C_X} *be the Frölicher topologies. Then* $\tau_{F_X} \subset \tau_{C_X}$*.*

Proof: Let $A \in \tau_{F_X}$, then $A \subseteq X$ and $\forall\ f \in F_X$, $A = f^{-1}(V)$, where $V \in \tau_{\mathbb{R}}$ and $\tau_{\mathbb{R}}$ is a standard topology on $\mathbb{R}$. Let $c \in C_X$ be arbitrary, then $\forall\ f \in F_X$ we have that

$$c^{-1}(A) = c^{-1}(f^{-1}(V)) = (f \circ c)^{-1}(V).$$

Since $f \in F_X$ and $c \in C_X$ then $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Therefore $f \circ c : \mathbb{R} \longrightarrow \mathbb{R}$ is Frölicher smooth since $(\mathbb{R}, C^\infty(\mathbb{R}, \mathbb{R}), C^\infty(\mathbb{R}, \mathbb{R}))$ is a Frölicher space. This implies that $f \circ c : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Therefore $(f \circ c)^{-1}(V) \in \tau_{\mathbb{R}}$ which implies that $c^{-1}(A) \in \tau_{\mathbb{R}}$, $\forall\ c \in C_X$. It follows that $A \in \tau_{C_X}$. That is $A \in \tau_{F_X} \implies A \in \tau_{F_X}$, hence $\tau_{F_X} \subset \tau_{C_X}$. □

2.3.4 Frölicher subspace.

Let (X, C_X, F_X) be a Frölicher space, S be a non-empty set such that $S \subseteq X$ and $\imath : S \hookrightarrow X$ be the canonical inclusion. Let $\mathbb{R}^S$ be the set of all real-valued functions on S and $S^{\mathbb{R}}$ be the set of all curves into S. With $P(\mathbb{R}^S)$ and $P(S^{\mathbb{R}})$ being power sets let $G \in P(\mathbb{R}^S)$, $D \in P(S^{\mathbb{R}})$ and $\Gamma : P(\mathbb{R}^S) \longrightarrow P(S^{\mathbb{R}})$ and $\Phi : P(S^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^S)$ be functors defined by

$$\Gamma G := \{d : \mathbb{R} \longrightarrow S \mid g \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ g \in G\}$$

and

$$\Phi D := \{g : S \longrightarrow \mathbb{R} \mid g \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ d \in D\}.$$

Let

$$F^* := \{f \circ \imath \mid f \in F_X \text{ and } \imath : S \hookrightarrow X\}.$$

The function $f \circ \imath$ with $f \in F_X$ is a restriction of f on S and we denote that by $f|_S$, that is, $f|_S = f \circ \imath$. Let $d \in S^{\mathbb{R}}$ then since $c(\mathbb{R}) \subseteq X$, $\forall\ c \in C_X$, therefore there exists $c \in C_X$ such that $c = \imath \circ d$. Thus

$$f|_S \circ d = (f \circ \imath) \circ d = f \circ (\imath \circ d) = f \circ c,$$

for some $c \in C_X$. We have that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_X$ and $\forall\ c \in C_X$, therefore $f|_S \circ d \in C^\infty(\mathbb{R}, \mathbb{R})$. By definition of Γ we have that

$$\Gamma F^* = \{d : \mathbb{R} \longrightarrow S \mid f|_S \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f|_S \in F^*\} := C_S.$$

Then by definition of Φ it follows that

$$\Phi\Gamma F^* = \{g : S \longrightarrow \mathbb{R} \mid g \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ d \in \Gamma F^* = C_S\} := F_S.$$

Since $f|_S \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ d \in \Gamma F^* = C_S$, therefore $f|_S \in \Phi\Gamma F^* = F_S$ which implies that

$$F^* \subset \Phi\Gamma F^* = F_S \tag{2.5}$$

Then by Lemma 2.3.1 we have that $\Gamma F^* = \Gamma\Phi\Gamma F^*$. Therefore

$$\Gamma F_S = \Gamma\Phi\Gamma F^* = \Gamma F^* = C_S \tag{2.6}$$

and

$$\Phi C_S = \Phi\Gamma F^* = F_S \tag{2.7}$$

Thus, (2.6) and (2.7) gives the compatibility condition, therefore

$$(\Gamma F^*, \Phi\Gamma F^*) = (C_S, F_S)$$

is a Frölicher structure. Therefore

$$(S, \Gamma F^*, \Phi\Gamma F^*) = (S, C_S, F_S)$$

is a Frölicher space. Since $S \subseteq X$ then (S, C_S, F_S) is a Frölicher subspace of the Frölicher space (X, C_X, F_X).

Let

$$C^* := \{\imath \circ d \mid d \in C_S \text{ and } \imath : S \hookrightarrow X\}$$

then

$$C^* \subseteq C_X \tag{2.8}$$

since $f|_S \circ d = f \circ (\imath \circ d) \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_X$, which implies that $\imath \circ d \in C_X$. Therefore by (2.5) and (2.8) we have that $\imath : S \hookrightarrow X$ is Frölicher smooth. That is, the morphism $(S, C_S, F_S) \hookrightarrow (X, C_X, F_X)$ is a morphism in the category of Frölicher spaces.

Since the Frölicher subspace (S, C_S, F_S) is a Frölicher space then we have Frölicher topologies induced from the Frölicher structure (C_S, F_S). That is the topology

$$\tau_{C_S} := \{U \subseteq S \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_S\}$$

induced from structure curves, and the topology

$$\tau_{F_S} := \{U \subseteq S \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_S\}$$

induced from structure functions, where $\tau_{\mathbb{R}}$ denotes the standard topology on $\mathbb{R}$. By Lemma 2.3.3 we have that $\tau_{F_S} \subset \tau_{C_S}$. We also have canonical topologies induced on S, that is, the subspace topologies

$$\tau_{F_X}(S) := \{S \cap U \mid U \in \tau_{F_X}\}$$

and

$$\tau_{C_X}(S) := \{S \cap U \mid U \in \tau_{C_X}\}.$$

The subspace topologies make the inclusion $\imath : S \hookrightarrow X$ continuous.

Lemma 2.3.4 *Let (S, C_S, F_S) be a Frölicher subspace of the Frölicher space (X, C_X, F_X), then $\tau_{F_X}(S) \subset \tau_{C_X}(S)$.*

Proof: Let $S \cap U \in \tau_{F_X}(S)$, then $U \in \tau_{F_X}$. But by Lemma 2.3.3 $\tau_{F_X} \subset \tau_{C_X}$, therefore $U \in \tau_{C_X}$, and hence $S \cap U \in \tau_{C_X}(S)$. Thus $\tau_{F_X}(S) \subset \tau_{C_X}(S)$. $\square$

Lemma 2.3.5 *Let (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X), then $\tau_{F_X}(S) \subset \tau_{F_S} \subset \tau_{C_S}$.*

Proof: By Lemma 2.3.3 we have that $\tau_{F_S} \subset \tau_{F_S}$ since the Frölicher subspace (S, C_S, F_S) is a Frölicher space. It is therefore sufficient to show that $\tau_{F_X}(S) \subset \tau_{F_S}$. We have to show that if $U \in \tau_{F_X}(S)$ then $U \in \tau_{F_S}$. Let $U \in \tau_{F_X}(S)$, then $U = S \cap V$ where $V \in \tau_{F_X}$. Let $V = \bigcup_{\alpha \in I} f_\alpha^{-1}(0, \infty)$, $f_\alpha \in F_X$. Note that $f_\alpha^{-1}(0, \infty)$, $\alpha \in I$, is in the basis of τ_{F_X}, therefore $V = \bigcup_{\alpha \in I} f_\alpha^{-1}(0, \infty) \in \tau_{F_X}$. Then

$$\begin{aligned} U &= S \cap V \\ &= S \cap \left(\bigcup_{\alpha \in I} f_\alpha^{-1}(0, \infty) \right) \\ &= \bigcup_{\alpha \in I} (S \cap f_\alpha^{-1}(0, \infty)) \\ &= \bigcup_{\alpha \in I} (\imath^{-1}(f_\alpha^{-1}(0, \infty))) \\ &= \bigcup_{\alpha \in I} ((f_\alpha \circ \imath)^{-1}(0, \infty)) \end{aligned} \tag{2.9}$$

Since $f_\alpha \in F_X$ therefore $f_\alpha \circ \imath = f_\alpha|_S \in F_S$, and thus $(f_\alpha \circ \imath)^{-1}(0, \infty)$, $\alpha \in I$, is an element of the basis of τ_{F_S}. Hence $U = S \cap V = \bigcup_{\alpha \in I} ((f_\alpha \circ \imath)^{-1}(0, \infty)) \in \tau_{F_S}$, thus $\tau_{F_X}(S) \subset \tau_{F_S}$. $\square$

Proposition 2.3.1 *Let (S, C_S, F_S) be a Frölicher subspace of the Frölicher space (X, C_X, F_X). Then the following hold:*

1. *If $S \in \tau_{F_X}$, then $\tau_{F_S} = \tau_{F_X}(S)$.*
2. *If $S \in \tau_{C_X}$, then $\tau_{C_S} = \tau_{C_X}(S)$.*

Proof:

1. Let $S \in \tau_{F_X}$, then $S \subseteq X$. By Lemma 2.3.5 we have that $\tau_{F_X}(S) \subset \tau_{F_S}$. Then it is sufficient to prove that $\tau_{F_S} \subset \tau_{F_X}(S)$. Let $U \in \tau_{F_S}$ then $U = \bigcup_{\alpha \in I}((f_\alpha \circ \imath)^{-1}(0,\infty))$ for some $f_\alpha \in F_X$, $\alpha \in I$, where $\imath : S \hookrightarrow X$. Then it follows that
$$\begin{aligned} U &= \imath^{-1}\imath(U) \\ &= \imath^{-1}\imath\left(\bigcup_{\alpha \in I}((f_\alpha \circ \imath)^{-1}(0,\infty))\right) \\ &= \imath^{-1}\left(\bigcup_{\alpha \in I}(\imath((f_\alpha \circ \imath)^{-1}(0,\infty)))\right) \\ &= \imath^{-1}\left(\bigcup_{\alpha \in I}(\imath(\imath^{-1}(f_\alpha^{-1}(0,\infty))))\right) \\ &= \imath^{-1}\left(\bigcup_{\alpha \in I} f_\alpha^{-1}(0,\infty)\right) \\ &= \bigcup_{\alpha \in I}\imath^{-1}(f_\alpha^{-1}(0,\infty)) \\ &= \bigcup_{\alpha \in I}(S \cap f_\alpha^{-1}(0,\infty)) \end{aligned} \tag{2.10}$$
Since $f_\alpha^{-1}(0,\infty)$ is an element of the basis of τ_{F_X} then $U = \bigcup_{\alpha \in I}(S \cap f_\alpha^{-1}(0,\infty)) \in \tau_{F_X}(S)$. That is, $U \in \tau_{F_S} \implies U \in \tau_{F_X}(S)$. Therefore $\tau_{F_S} \subset \tau_{F_X}(S)$, and since $\tau_{F_X}(S) \subset \tau_{F_S}$, hence $\tau_{F_S} = \tau_{F_X}(S)$.

2. Let $S \in \tau_{C_X}$ and let $U \in \tau_{C_S}$. That is, $S \subseteq X$, $c^{-1}(S) \in \tau_{\mathbb{R}}$, $\forall\, c \in C_X$, and $U \subseteq S$, $d^{-1}(U) \in \tau_{\mathbb{R}}$, $\forall\, d \in C_S$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Then it follows that
$$\begin{aligned} d^{-1}(U) &= d^{-1}(\imath^{-1}(\imath(U))) \\ &= (\imath \circ d)^{-1}(\imath(U)) \\ &= (\imath \circ d)^{-1}(U). \end{aligned} \tag{2.11}$$

Then since $\imath \circ d \in C_X$ and $d^{-1}(U) \in \tau_{\mathbb{R}}$ therefore $(\imath \circ d)^{-1}(U) \in \tau_{\mathbb{R}}$. Thus $U \in \tau_{C_X}$ as $U \subseteq X$. We also have that

$$\begin{aligned} U &= \imath^{-1}(\imath(U)) \\ &= S \cap \imath(U) \\ &= S \cap U \end{aligned} \tag{2.12}$$

Then $U = S \cap U \in \tau_{C_X}(S)$ since $U \in \tau_{C_X}$. That is, $U \in \tau_{C_S} \implies U \in \tau_{C_X}(S)$, therefore $\tau_{C_S} \subset \tau_{C_X}(S)$.
Let $V \in \tau_{C_X}(S)$ then $V = S \cap W$ where $W \in \tau_{C_X}$, that is $c^{-1}(W) \in \tau_{\mathbb{R}}$, $\forall\ c \in C_X$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Also, $S \in \tau_{C_X}$ therefore $c^{-1}(U) \in \tau_{\mathbb{R}}$, $\forall\ c \in C_X$. Then we have that

$$c^{-1}(V) = c^{-1}(S \cap W) = c^{-1}(S) \cap c^{-1}(W) \in \tau_{\mathbb{R}}$$

since $\tau_{\mathbb{R}}$ is closed under finite intersection as it is a topology. Therefore $V \in \tau_{C_X}$. This implies that $(\imath \circ d)^{-1}(V) \in \tau_{\mathbb{R}}$ since $\imath \circ d \in C_X$. But

$$\begin{aligned} (\imath \circ d)^{-1}(V) &= d^{-1}(\imath^{-1}(V)) \\ &= d^{-1}(V \cap S) \\ &= d^{-1}(V) \cap d^{-1}(S) \in \tau_{\mathbb{R}} \end{aligned} \tag{2.13}$$

And $d^{-1}(V) \cap d^{-1}(S) \in \tau_{\mathbb{R}} \implies d^{-1}(V) \in \tau_{\mathbb{R}}$ therefore $W \in \tau_{C_S}$. That is, $V \in \tau_{C_X}(S) \implies V \in \tau_{C_S}$ therefore $\tau_{C_X}(S) \subset \tau_{C_S}$. Thus we have that $\tau_{C_S} \subset \tau_{C_X}(S)$ and $\tau_{C_X}(S) \subset \tau_{C_S}$ hence $\tau_{C_S} = \tau_{C_X}(S)$.

□

2.3.5 Frölicher product.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \dots, n\}$ be a family of Frölicher spaces and let $\mathfrak{X} := \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets. Let $\mathbb{R}^{\mathfrak{X}}$ be the set of all real-valued functions on $\mathfrak{X}$ and $\mathfrak{X}^{\mathbb{R}}$ be the set of all curves into $\mathfrak{X}$. With $P(\mathbb{R}^{\mathfrak{X}})$ and $P(\mathfrak{X}^{\mathbb{R}})$ being power sets let $F \in P(\mathbb{R}^{\mathfrak{X}})$, $C \in P(\mathfrak{X}^{\mathbb{R}})$ and $\Gamma : P(\mathbb{R}^{\mathfrak{X}}) \longrightarrow P(\mathfrak{X}^{\mathbb{R}})$ and $\Phi : P(\mathfrak{X}^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^{\mathfrak{X}})$ be functors defined by:

$$\Gamma F := \{c : \mathbb{R} \longrightarrow \mathfrak{X} \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F\}$$

$$\Phi C := \{f : \mathfrak{X} \longrightarrow \mathbb{R} \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C\}.$$

Let $p_i : \mathfrak{X} \longrightarrow X_i$, $\forall\ i = 1, 2, \dots, n$, be a canonical projection and let

$$K := \{f_i \circ p_i \mid f_i \in F_{X_i},\ i = 1, 2, \dots, n\}.$$

Let $c_i \in C_{X_i}$ then $c_i(\mathbb{R}) \subseteq X_i$ and $p_i^{-1}(c_i(\mathbb{R})) \neq \phi$ since $p_i : \mathfrak{X} \longrightarrow X_i$ is a surjective function $\forall\ i = 1, 2, \dots, n$. Then there exists $d \in \mathfrak{X}^{\mathbb{R}}$ such that $d(\mathfrak{X}) = p_i^{-1}(c_i(\mathbb{R}))$.

Therefore $\forall\ c_i \in C_{X_i}\ \exists\ d \in \mathfrak{X}^{\mathbb{R}}$ such that $c_i = p_i \circ d$, $\forall\ i = 1, 2, \dots, n$. Thus $\forall\ i = 1, 2, \dots, n$ we have that

$$(f_i \circ p_i) \circ d = f_i \circ (p_i \circ d) = f_i \circ c_i \in C^\infty(\mathbb{R}, \mathbb{R})$$

since $f_i \in F_{X_i}$, $c_i \in C_{X_i}$ and (X_i, C_{X_i}, F_{X_i}) is a Frölicher space. By definition of Γ we have that

$$\Gamma K := \{d : \mathbb{R} \longrightarrow \mathfrak{X} \mid (f_i \circ c_i) \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ (f_i \circ p_i) \in K\} := C_{\mathfrak{X}}.$$

Then by definition of Φ it follows that

$$\Phi\Gamma K := \{g : \mathfrak{X} \longrightarrow \mathbb{R} \mid g \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ d \in \Gamma F_{\mathfrak{X}}\} := F_{\mathfrak{X}}.$$

Now since $(f_i \circ p_i) \circ d \in C^\infty(\mathbb{R}, \mathbb{R})$ then we have that $K \subset \Phi\Gamma K := F_{\mathfrak{X}}$. Then by Lemma 2.3.1 we have that $\Gamma K = \Gamma\Phi\Gamma K$. Therefore

$$\Gamma F_{\mathfrak{X}} = \Gamma\Phi\Gamma K = \Gamma K = C_{\mathfrak{X}} \tag{2.14}$$

and

$$\Phi C_{\mathfrak{X}} = \Phi\Gamma K = F_{\mathfrak{X}} \tag{2.15}$$

Thus (2.14) and (2.15) gives the compatibility/duality condition therefore

$$(\Gamma K, \Phi\Gamma K) = (C_{\mathfrak{X}}, F_{\mathfrak{X}})$$

is a Frölicher structure. Therefore the triple $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is a Frölicher space. Since $\mathfrak{X} = \prod_{i=1}^{n} X_i$ is a Cartesian product in the category of sets then we call $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ the Frölicher product.

With $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ a Frölicher product, $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \dots, n\}$ a family of Frölicher spaces and $p_i : \mathfrak{X} \longrightarrow X_i$ a canonical projection $\forall\ i = 1, 2, \dots, n$ then $\forall\ i = 1, 2, \dots, n$ we have that $p_i \circ d \in C_{X_i}\ \forall\ d \in C_{\mathfrak{X}}$, since $f_i \circ (p_i \circ d) \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f_i \in F_{X_i}$. This implies that $\forall\ i = 1, 2, \dots, n$ we have that

$$K^* := \{p_i \circ d \mid d \in C_{\mathfrak{X}}\} \subset C_{X_i}.$$

Recall that we also have that

$$K := \{f_i \circ p_i \mid f_i \in F_{X_i},\ \forall\ i = 1, 2, \dots, n\} \subset F_{\mathfrak{X}}.$$

Therefore with $K^* \subset C_{X_i}$ and $K \subset F_{\mathfrak{X}}$ then we have that the projection $p_i : \mathfrak{X} \longrightarrow X_i$ is Frölicher smooth $\forall\ i = 1, 2, \dots, n$.

Since the Frölicher product $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is a Frölicher space then there are Frölicher topologies induced from the Frölicher structure $(C_{\mathfrak{X}}, F_{\mathfrak{X}})$. These topologies are denoted $\tau_{C_{\mathfrak{X}}}$ and $\tau_{F_{\mathfrak{X}}}$, that is the topology induced from structure curves of $C_{\mathfrak{X}}$ and structure functions of $F_{\mathfrak{X}}$, respectively. By definition we have that

$$\tau_{F_{\mathfrak{X}}} = \{U \subseteq \mathfrak{X} \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_{\mathfrak{X}}\}$$

and

$$\tau_{C_{\mathfrak{X}}} = \{U \subseteq \mathfrak{X} \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_{\mathfrak{X}}\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. By Lemma 2.3.3 we have that

$$\tau_{F_{\mathfrak{X}}} \subset \tau_{C_{\mathfrak{X}}}$$

since $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is a Frölicher space.

We also have the canonical product topology induced on $\mathfrak{X} = \prod_{i=1}^{n} X_i$, which we denote by τ_{π}, and is generated by the basis

$$\hat{\beta} = \left\{ \bigcap_{i=1}^{n} p_i{}^{-1}(U_i) \mid U_i \in \tau_{F_{X_i}}, \forall\ i = 1, 2, \ldots, n \right\}$$

(see Lemma 2.6.1 in [89]) where $p_i : \mathfrak{X} \longrightarrow X_i$ is a canonical projection, $\forall\ i = 1, 2, \ldots, n$. The product topology τ_{π} is canonical on $\mathfrak{X}$ and makes the projection $p_i : \mathfrak{X} \longrightarrow X_i$ continuous $\forall\ i = 1, 2, \ldots, n$.

Lemma 2.3.6 *Let $p_i : \mathfrak{X} \longrightarrow X_i$ be the canonical projection, $\forall\ i = 1, 2, \ldots, n$, therefore $\bigcap_{i=1}^{n} p_i^{-1}(U_i) = \prod_{i=1}^{n} U_i$, $\forall\ U_i \subseteq X_i$, $\forall\ i = 1, 2, \ldots, n$.*

Proof: Let $z = (z_1, z_2, \ldots, z_n)$ and $p_i : \mathfrak{X} \longrightarrow X_i$ be the canonical projection, $\forall\ i = 1, 2, \ldots, n$, such that $z \in \bigcap_{i=1}^{n} p_i^{-1}(U_i)$. This implies that $z \in p_i^{-1}(U_i)$, $\forall\ i = 1, 2, \ldots, n$. Therefore $p_i(z) = p_i(z_1, z_2, \ldots, z_n) = z_i \in U_i,\ \forall\ i = 1, 2, \ldots, n$. This implies that $z \in \prod_{i=1}^{n} U_i$. That is $z \in \bigcap_{i=1}^{n} p_i^{-1}(U_i) \implies z \in \prod_{i=1}^{n} U_i$ therefore $\bigcap p_i^{-1}(U_i) \subset \prod_{i=1}^{n} U_i$. Similarly, let $z = (z_1, z_2, \ldots, z_n) \in \prod_{i=1}^{n} U_i$, therefore $p_i(z) = z_i \in U_i,\ \forall\ i = 1, 2, \ldots, n$. Which implies that $\forall\ i = 1, 2, \ldots, n$ we have that $z \in p_i^{-1}(U_i)$, therefore $z \in \bigcap_{i=1}^{n} p_i^{-1}(U_i)$. That is $z \in \prod_{i=1}^{n} U_i \implies z \in \bigcap_{i=1}^{n} p_i^{-1}(U_i)$, therefore $\prod_{i=1}^{n} U_i \subset \bigcap_{i=1}^{n} p_i^{-1}(U_i)$. Hence $\prod_{i=1}^{n} U_i = \bigcap_{i=1}^{n} p_i^{-1}(U_i)$. $\square$

Lemma 2.3.7 *Let $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product, $\tau_{F_{\mathfrak{X}}}$ be the Frölicher topology induced from the structure functions of $F_{\mathfrak{X}}$ and τ_π be the canonical product topology on $\mathfrak{X}$. Then $\tau_\pi = \tau_{F_{\mathfrak{X}}} \subset \tau_{C_{\mathfrak{X}}}$.*

Proof: By Lemma 2.3.3 we have that $\tau_{F_{\mathfrak{X}}} \subset \tau_{C_{\mathfrak{X}}}$ since the Frölicher product $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is a Frölicher space. It is then sufficient to show that $\tau_\pi = \tau_{F_{\mathfrak{X}}}$. That is we have to show that $\tau_\pi \subset \tau_{F_{\mathfrak{X}}}$ and that $\tau_{F_{\mathfrak{X}}} \subset \tau_\pi$. We first show that $\tau_\pi \subset \tau_{F_{\mathfrak{X}}}$. Let $\prod_{i=1}^{n} U_i \in \tau_\pi$ but since $\prod_{i=1}^{n} U_i = \bigcap_{i=1}^{n} p_i^{-1}(U_i)$ by Lemma 2.3.6 therefore $U_i \in \tau_{F_{X_i}}$, $\forall\ i = 1, 2, \ldots, n$. Then $\forall\ i = 1, 2, \ldots, n$ we have that $U_i \subseteq X_i$ such that $U_i = f_i^{-1}(V)$, $\forall\ f_i \in F_{X_i}$, and $V \in \tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Thus $\prod_{i=1}^{n} U_i \subseteq \prod_{i=1}^{n} X_i = \mathfrak{X}$ since $U_i \subseteq X_i$, $\forall\ i = 1, 2, \ldots, n$. Then $\forall\ (u_1, u_2, \ldots, u_n) \in \prod_{i=1}^{n} U_i\ \exists\ W \in \tau_{\mathbb{R}}$ such that $g(u_1, u_2, ..., u_n) \in W$, with $g \in F_{\mathfrak{X}}$. Therefore

$$
\begin{aligned}
g^{-1}(W) &= \{(u_1, u_2, ..., u_n) \in \mathfrak{X} \mid g(u_1, u_2, ..., u_n) \in W\} \\
&= \left\{(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} U_i \mid g(u_1, u_2, ..., u_n) \in W\right\} \\
&= \{(u_1, u_2, ..., u_n) \mid u_i \in U_i\} \\
&= \prod_{i=1}^{n} U_i
\end{aligned}
\tag{2.16}
$$

If $f_i(x_i) \in V$, $\forall\ f_i \in F_{X_i}$, for some $V \in \tau_{\mathbb{R}}$, then by pulling-back we have that

$$g(x_1, x_2, ..., x_n) = (f_i \circ p_i)(x_1, x_2, ..., x_n) \in V$$

where $p_i : \mathfrak{X} \longrightarrow X_i$ is a canonical projection. Furthermore, since $U_i = f_i^{-1}(V)$, $\forall\ f_i \in F_{X_i}$, for some $V \in \tau_{\mathbb{R}}$, therefore

$$
\begin{aligned}
\prod_{i=1}^{n} U_i &= \prod_{i=1}^{n} f_i^{-1}(V) \\
&= \prod_{i=1}^{n} \{x_i \in X_i \mid f_i(x_i) \in V\} \\
&= \{(x_1, x_2, ..., x_n) \in \mathfrak{X} \mid g(x_1, x_2, ..., x_n) \in V\} \\
&= g^{-1}(V)
\end{aligned}
\tag{2.17}
$$

Thus $\prod_{i=1}^{n} U_i \in \tau_{F_{\mathfrak{X}}}$. That is, $\prod_{i=1}^{n} U_i \in \tau_\pi \implies \prod_{i=1}^{n} U_i \in \tau_{F_{\mathfrak{X}}}$, therefore $\tau_\pi \subset \tau_{F_{\mathfrak{X}}}$.

Now we have to show that $\tau_{F_{\mathfrak{X}}} \subset \tau_{\pi}$. Let $U \in \tau_{F_{\mathfrak{X}}}$, then $U \subseteq \mathfrak{X}$, and therefore $U = \prod_{i=1}^{n} Y_i$ where $Y_i \subseteq X_i$. Then $\prod_{i=1}^{n} Y_i = U \in \tau_{F_{\mathfrak{X}}}$, then we have that $\prod_{i=1}^{n} Y_i = U = g^{-1}(T)$, $\forall\, g \in F_{\mathfrak{X}}$, where $T \in \tau_{\mathbb{R}}$. Therefore

$$\begin{aligned}
\prod_{i=1}^{n} Y_i &= U \\
&= g^{-1}(T) \\
&= (f_i \circ p_i)^{-1}(T) \\
&= {p_i}^{-1}({f_i}^{-1}(T)) \\
&= \left\{(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} X_i = \mathfrak{X} \mid p_i(x_1, x_2, ..., x_n) \in {f_i}^{-1}(T),\ T \in \tau_{\mathbb{R}}\right\} \\
&= \left\{(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} X_i = \mathfrak{X} \mid x_i \in {f_i}^{-1}(T),\ T \in \tau_{\mathbb{R}}\right\} \\
&= \prod_{i=1}^{n} {f_i}^{-1}(T)
\end{aligned} \tag{2.18}$$

Thus $Y_i = {f_i}^{-1}(T)$ where $T \in \tau_{\mathbb{R}}$, therefore $Y_i \in \tau_{F_{X_i}}$. By Lemma 2.3.6 this implies that $\prod_{i=1}^{n} Y_i \in \tau_{\pi}$. That is $U = \prod_{i=1}^{n} Y_i \in \tau_{F_{\mathfrak{X}}} \implies U = \prod_{i=1}^{n} Y_i \in \tau_{\pi}$, therefore $\tau_{F_{\mathfrak{X}}} \subset \tau_{\pi}$. Since $\tau_{\pi} \subset \tau_{F_{\mathfrak{X}}}$ and $\tau_{F_{\mathfrak{X}}} \subset \tau_{\pi}$, hence $\tau_{\pi} = \tau_{F_{\mathfrak{X}}}$. $\square$

2.3.6 Frölicher coproduct.

Let $\{(X_j, C_{X_j}, F_{X_j}) \mid j \in J\}$ be a family of Frölicher spaces and let $\mathcal{N} := \coprod_{j \in J} X_j$ be a coproduct in the category of sets. Let $\mathbb{R}^{\mathcal{N}}$ be the set of all real-valued functions on $\mathcal{N}$ and $\mathcal{N}^{\mathbb{R}}$ be the set of all curves into $\mathcal{N}$, with the power sets $P(\mathbb{R}^{\mathcal{N}})$ and $P(\mathcal{N}^{\mathbb{R}})$. Let $F \in P(\mathbb{R}^{\mathcal{N}})$, $C \in P(\mathcal{N}^{\mathbb{R}})$ and $\Gamma : P(\mathbb{R}^{\mathcal{N}}) \longrightarrow P(\mathcal{N}^{\mathbb{R}})$ and $\Phi : P(\mathcal{N}^{\mathbb{R}}) \longrightarrow P(\mathbb{R}^{\mathcal{N}})$ be functors defined by:

$$\Gamma F := \{d : \mathbb{R} \longrightarrow \mathcal{N} \mid g \circ d \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\, g \in F\}$$

$$\Phi C := \{g : \mathcal{N} \longrightarrow \mathbb{R} \mid g \circ d \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\, d \in C\}.$$

Consider the canonical injection $\varphi_j : X_j \longrightarrow \mathcal{N}$, $j \in J$, and let

$$C_{\coprod} := \{\varphi_j \circ c_j \mid c_j \in C_{X_j},\ j \in J\} \subset \mathcal{N}^{\mathbb{R}}.$$

By definition of Φ we have that

$$\Phi C_{\coprod} = \{g : \mathcal{N} \longrightarrow \mathbb{R} \mid g \circ (\varphi_j \circ c_j) \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ (\varphi_j \circ c_j) \in C_{\coprod},\ j \in J\} := F_{\mathcal{N}}.$$

By definition of Γ it follows that

$$\Gamma\Phi C_{\coprod} = \{d : \mathbb{R} \longrightarrow \mathcal{N} \mid g \circ d \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ g \in \Phi C_{\coprod}\} := C_{\mathcal{N}}.$$

Let $(\varphi_j \circ c_j) \in C_{\coprod}$, $j \in J$, then $g \circ (\varphi_j \circ c_j) \in C^{\infty}(\mathbb{R}, \mathbb{R})$, $\forall\ g \in \Phi C_{\coprod}$, which implies that $(\varphi_j \circ c_j) \in \Gamma\Phi C_{\coprod} = C_{\mathcal{N}}$. Thus $(\varphi_j \circ c_j) \in C_{\coprod} \implies (\varphi_j \circ c_j) \in \Gamma\Phi C_{\coprod} := C_{\mathcal{N}}$, therefore

$$C_{\coprod} \subset \Gamma\Phi C_{\coprod} := C_{\mathcal{N}} \tag{2.19}$$

Then by Lemma 2.3.1 we have that

$$\Phi C_{\coprod} = \Phi\Gamma\Phi C_{\coprod} \tag{2.20}$$

Therefore

$$\Phi C_{\coprod} = \Phi\Gamma\Phi C_{\coprod} = \Phi C_{\mathcal{N}} := F_{\mathcal{N}} \tag{2.21}$$

and

$$\Gamma F_{\mathcal{N}} = \Gamma\Phi C_{\coprod} := C_{\mathcal{N}} \tag{2.22}$$

Thus (2.21) and (2.22) gives the compatibility/duality condition, therefore

$$(\Gamma\Phi C_{\coprod}, \Phi C_{\coprod}) = (C_{\mathcal{N}}, F_{\mathcal{N}})$$

is a Frölicher structure. Therefore the triple $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ is a Frölicher space. Since $\mathcal{N} := \coprod_{j \in J} X_j$ is a coproduct in the category of sets then we call $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ the Frölicher coproduct.

Let

$$F_{\coprod} := \{g \circ \varphi_j \mid g \in F_{\mathcal{N}},\ j \in J\}.$$

Then since $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ is a Frölicher space and $(\varphi_j \circ c_j) \in C_{\mathcal{N}}$ by (2.19) then $\forall\ g \in F_{\mathcal{N}}$ we have that $g \circ (\varphi_j \circ c_j) = (g \circ \varphi_j) \circ c_j$, $\forall\ c_j \in C_{X_j}$, therefore $g \circ \varphi_j \in F_{X_j}$, $j \in J$. That is, $g \circ \varphi_j \in F_{\coprod} \implies g \circ \varphi_j \in F_{X_j}$ therefore for $j \in J$ we have that

$$F_{\coprod} \subset F_{X_j}. \tag{2.23}$$

Then by (2.19) and (2.23) the canonical injection $\varphi_j : X_j \longrightarrow \mathcal{N}$ is Frölicher smooth, $j \in J$.

Remark 2.3.3 *The morphism $\varphi_j : X_j \longrightarrow \mathcal{N}$ is unique only for the coproduct $\mathcal{N} := \coprod_{j \in J} X_j$. That is the morphism $\mathbb{F} \longrightarrow \mathcal{N}$ is not unique for any Frölicher space $(\mathbb{F}, C_{\mathbb{F}}, F_{\mathbb{F}})$. For example, let $\{(Y_j, C_{Y_j}, F_{Y_j}) \mid j \in J\}$ be a family of Frölicher spaces and suppose that Y_j is a group under the binary operation $*$, for $j \in J$. That is Y_j is closed and associative under the operation $*$, and there is an inverse and an identity in Y_j under the operation $*$. Thus for $j \in J$*

1. *$a * b \in Y_j, \ \forall\ a, b \in Y_j$.*
2. *$a * (b * c) = (a * b) * c, \ \forall\ a, b, c \in Y_j$.*
3. *$\forall\ a \in Y_j\ \exists\ e \in Y_j$ such that $a * e = e * a = a$. In this case e is called an identity in Y_j.*
4. *$\forall\ a \in Y_j\ \exists\ a^{-1} \in Y_j$ such that $a^{-1} * a = a * a^{-1} = e$ where e is an identity in Y_j. In this case we say that a^{-1} is the inverse of a.*

Consider the Frölicher product $(\mathfrak{Y}, C_{\mathfrak{Y}}, F_{\mathfrak{Y}})$ where $\mathfrak{Y} := Y_m \times Y_m$ and the morphism $\psi : \mathfrak{Y} \longrightarrow \coprod_{j \in J} Y_j$. Let $x, y \in Y_m$ then ψ can be defined as $\psi(x, y) = (x, m)$, $\psi(x, y) = (y, m)$ and $\psi(x, y) = (x * y, m)$. Therefore ψ is not unique.

Since the Frölicher coproduct $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ is a Frölicher space, as such, we can induce Frölicher topologies from the Frölicher structure $(C_{\mathcal{N}}, F_{\mathcal{N}})$. These topologies are denoted $\tau_{C_{\mathcal{N}}}$ and $\tau_{F_{\mathcal{N}}}$ induced from the structure curves of $C_{\mathcal{N}}$ and the structure functions of $F_{\mathcal{N}}$, respectively. They are defined as

$$\tau_{C_{\mathcal{N}}} = \{U \subseteq \mathcal{N} \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_{\mathcal{N}}\}$$

and

$$\tau_{F_{\mathcal{N}}} = \{U \subseteq \mathcal{N} \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_{\mathcal{N}}\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Lemma 2.3.3 gives that

$$\tau_{F_{\mathcal{N}}} \subset \tau_{C_{\mathcal{N}}}$$

since $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ is a Frölicher space.

We also have a canonical topology on $\mathcal{N} := \coprod_{j \in J} X_j$. That is the coproduct topology (see [3] and [70]). The coproduct topology on $\mathcal{N}$ is given by the collection

$$\tau_{\amalg} := \{U \subseteq \mathcal{N} \mid {\varphi_j}^{-1}(U) \in \tau_{F_{X_j}},\ j \in J\}.$$

Lemma 2.3.8 *The coproduct topology $\tau_{\amalg}$ is the finest topology in which all canonical injections $\varphi_j : X_j \longrightarrow \mathcal{N}$ are continuous, $j \in J$, and $\tau_{C_{\mathcal{N}}} \subset \tau_{\amalg}$.*

Proof: Let τ be an arbitrary topology on $\mathcal{N} := \coprod_{j \in J} X_j$ for which $\varphi_j : X_j \longrightarrow \mathcal{N}$ is continuous, $j \in J$. That is, $\forall\ V \in \tau$ we have that ${\varphi_j}^{-1}(V) = U_j \in \tau_{F_{X_j}}$, thus ${\varphi_j}^{-1}(V)$ is open in $\tau_{F_{X_j}}$, $j \in J$. Then by applying φ_j on both sides of $U_j = {\varphi_j}^{-1}(V)$ we have that

$$\varphi_j(U_j) = \varphi_j({\varphi_j}^{-1}(V)) = V \cap \varphi_j(U_j).$$

Since $U_j \in \tau_{F_{X_j}}$ then $U_j \subseteq X_j$ implying that $\varphi_j(U_j) \subseteq \mathcal{N}$ since $\varphi_j : X_j \to \mathcal{N}$. But ${\varphi_j}^{-1}(\varphi_j(U_j)) = U_j \in \tau_{F_{X_j}}$, therefore $\varphi_j(U_j) \in \tau_{\amalg}$. Thus $\tau_{\amalg} \ni \varphi_j(U_j) = V \cap \varphi_j(U_j)$. Since $\tau_{\amalg}$ is a topology and thus is closed under finite intersection therefore $V \in \tau_{\amalg}$. That is $V \in \tau \implies V \in \tau_{\amalg}$, therefore $\tau \subset \tau_{\amalg}$. Since τ is arbitrary therefore $\tau_{\amalg}$ is the finest topology in which φ_j is continuous. With τ being arbitrary then we let $\tau = \tau_{C_{\mathcal{N}}}$, then we have that $\tau_{C_{\mathcal{N}}} \subset \tau_{\amalg}$. □

Lemma 2.3.9 *Let the Frölicher topologies $\tau_{F_{\mathcal{N}}}$ and $\tau_{C_{\mathcal{N}}}$ and the canonical coproduct topology $\tau_{\amalg}$ be defined as usual, then $\tau_{F_{\mathcal{N}}} = \tau_{C_{\mathcal{N}}} = \tau_{\amalg}$.*

Proof: By Lemma 2.3.8 we have that $\tau_{F_{\mathcal{N}}} \subset \tau_{\amalg}$ since $\tau_{F_{\mathcal{N}}} \subset \tau_{C_{\mathcal{N}}} \subset \tau_{\amalg}$. It is sufficient to show that $\tau_{\amalg} \subset \tau_{F_{\mathcal{N}}}$. Let $U \in \tau_{\amalg}$, that is $U \subseteq \mathcal{N}$ and ${\varphi_j}^{-1}(U) \in \tau_{F_{X_j}}$, $j \in J$. Then ${\varphi_j}^{-1}(U) \in \tau_{F_{X_j}}$ implies that ${\varphi_j}^{-1}(U) \subseteq X_j$ and ${\varphi_j}^{-1}(U) = {f_j}^{-1}(V)$, $\forall\ f_j \in F_{X_j}$, $j \in J$ and $V \in \tau_{\mathbb{R}}$. Then it follows that ${\varphi_j}^{-1}(U) = {f_j}^{-1}(V)$, which implies that $U = \varphi_j({f_j}^{-1}(V))$. Then $\forall\ g \in F_{\mathcal{N}}$ we have that $f_i = g \circ \varphi_i$, therefore

$$\begin{aligned} U &= \varphi_j({f_j}^{-1}(V)) \\ &= \varphi_j((g \circ \varphi_j)^{-1}(V)) \\ &= \varphi_j(({\varphi_j}^{-1} \circ g^{-1})(V)) \\ &= \varphi_j({\varphi_j}^{-1}(g^{-1}(V))) \\ &= g^{-1}(V) \end{aligned} \tag{2.24}$$

Since $U \subseteq \mathcal{N}$ therefore we have that $U \in \tau_{F_{\mathcal{N}}}$. That is $U \in \tau_{\amalg} \implies U \in \tau_{F_{\mathcal{N}}}$, hence $\tau_{\amalg} \subset \tau_{F_{\mathcal{N}}}$. Now since $\tau_{F_{\mathcal{N}}} = \tau_{\amalg}$ and $\tau_{C_{\mathcal{N}}} \subset \tau_{\amalg}$ then $\tau_{C_{\mathcal{N}}} \subset \tau_{F_{\mathcal{N}}}$. With $\tau_{F_{\mathcal{N}}} \subset \tau_{C_{\mathcal{N}}}$ (by Lemma 2.3.3) therefore $\tau_{F_{\mathcal{N}}} = \tau_{C_{\mathcal{N}}} = \tau_{\amalg}$. □

2.3.7 Frölicher quotient.

Let (X, C_X, F_X) and (Y, C_Y, F_Y) be Frölicher spaces and $h : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$ be a morphism in the category of Frölicher spaces and $\sim$ be a relation on X defined by $x_1 \sim x_2 \iff h(x_1) = h(x_2)$, $\forall\ x_1, x_2 \in X$. The relation $\sim$ is reflexive since $x \sim x \iff h(x) = h(x)$, $\forall\ x \in X$, is trivial. Also $\forall\ x_1, x_2 \in X$ we have that $h(x_1) = h(x_2) \iff h(x_2) = h(x_1)$ therefore $x_1 \sim x_2 \iff x_2 \sim x_1$, we say that $\sim$ is symmetric. Let $x_1 \sim x_2$ and $x_2 \sim x_3$ $\forall\ x_1, x_2, x_3 \in X$ therefore $h(x_1) = h(x_2)$ and $h(x_2) = h(x_3)$ which implies that $h(x_1) = h(x_3) \iff x_1 \sim x_3$, thus $\sim$ is transitive. That is $\sim$ is reflexive, symmetric and transitive therefore it is an equivalence relation on X.

Definition 2.3.3 ***Kernel equivalence.***
Let $h : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$ be a morphism in the category of Frölicher spaces, the equivalence relation $\sim$ defined by $x_1 \sim x_2 \iff h(x_1) = h(x_2)$, $\forall\ x_1, x_2 \in X$, is called the kernel equivalence of h.

Let (X, C_X, F_X) be a Frölicher space, $\sim$ be the kernel equivalence of h where h is a morphism in the category of Frölicher spaces with the domain (X, C_X, F_X). Let $Q = X/_\sim$ be a quotient set and $q : X \longrightarrow Q$ be a canonical projection. Let $\mathbb{R}^Q$ be the set of all real-valued functions on Q and $Q^\mathbb{R}$ be the set of all curves into $\mathbb{R}$. Let $P(\mathbb{R}^Q)$ and $P(Q^\mathbb{R})$ be power sets. Let $F \in P(\mathbb{R}^Q)$, $C \in P(Q^\mathbb{R})$, and $\Gamma : P(\mathbb{R}^Q) \longrightarrow P(Q^\mathbb{R})$ and $\Phi : P(Q^\mathbb{R}) \longrightarrow P(\mathbb{R}^Q)$ be functors defined by

$$\Gamma F := \{d : \mathbb{R} \longrightarrow Q \mid g \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ g \in F\}$$

$$\Phi C := \{g : Q \longrightarrow \mathbb{R} \mid g \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ d \in C\}.$$

Let

$$\mathcal{H} := \{q \circ c \mid c \in C_X\}$$

then by definition of Φ we have that

$$\Phi\mathcal{H} = \{g : Q \longrightarrow \mathbb{R} \mid g \circ (q \circ c) \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ (q \circ c) \in \mathcal{H}\} := F_Q.$$

By definition of Γ it follows that

$$\Gamma\Phi\mathcal{H} = \{d : \mathbb{R} \longrightarrow Q \mid g \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ g \in \Phi\mathcal{H}\} = C_Q.$$

Let $d \in \mathcal{H}$ therefore $d = q \circ c$ where $q : X \longrightarrow Q$ is a canonical projection and $c \in C_X$. Then $g \circ d = g \circ (q \circ c) \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ g \in \Phi\mathcal{H}$, which implies that $d \in \Gamma\Phi\mathcal{H} := C_Q$. That is $d \in \mathcal{H} \implies d \in \Gamma\Phi\mathcal{H} := C_Q$, therefore

$$\mathcal{C} \subset \Gamma\Phi\mathcal{H} := C_Q \tag{2.25}$$

Then by Lemma 2.3.1 we have that

$$\Phi\mathcal{H} = \Phi\Gamma\Phi\mathcal{H} \tag{2.26}$$

Therefore

$$\Phi C_Q = \Phi\Gamma\Phi\mathcal{H} = \Phi\mathcal{H} := F_Q \tag{2.27}$$

and

$$\Gamma F_Q = \Gamma\Phi\mathcal{H} := C_Q \tag{2.28}$$

That is (2.27) and (2.28) gives the compatibility/duality condition, therefore

$$(\Gamma\Phi\mathcal{H}, \Phi\mathcal{H}) = (C_Q, F_Q)$$

is a Frölicher structure. Thus the triple (Q, C_Q, F_Q) is a Frölicher space. We call (Q, C_Q, F_Q) the Frölicher quotient since $Q = X/_\sim$ is a quotient set. And the set $\mathcal{H}$ is the generating set of the Frölicher quotient.

Let

$$\mathcal{F} := \{g \circ q \mid g \in F_Q\}.$$

With $g : Q \longrightarrow \mathbb{R}$ a function on Q and the canonical projection $q : X \longrightarrow Q$ a surjective function on X then $g \circ q : X \longrightarrow \mathbb{R}$ is a function on X and

$\forall\ g \in F_Q = \Phi\mathcal{H}$ we have that $g \circ (q \circ c) = (g \circ q) \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C_X$, therefore $g \circ q \in F_X$. Thus

$$\mathcal{F} \subset F_X \tag{2.29}$$

By (2.25) and (2.29) we have that the canonical projection $q : X \longrightarrow Q$ is Frölicher smooth.

Remark 2.3.4 *The morphism $\mathbb{F} \longrightarrow Q$ is not unique for every Frölicher space $(\mathbb{F}, C_{\mathbb{F}}, F_{\mathbb{F}})$. That is, $q : X \longrightarrow Q$ is unique only for X and the quotient set $Q = X/_\sim$. For example, Let (X, C_X, F_X) and (Y, C_Y, F_Y) be Frölicher spaces such that $Y \subset X$. With the Cartesian product $\mathfrak{X} = X \times Y$ then $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is a Frölicher product. Consider the morphism $\pi^* : \mathfrak{X} \longrightarrow Q$. Let $f \in F_Q$, then we have that $f \circ \pi^* : \mathfrak{X} \longrightarrow \mathbb{R}$. Since $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is a Frölicher space then $(f \circ \pi^*) \circ d \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ d \in C_{\mathfrak{X}}$, therefore $f \circ \pi^* \in F_{\mathfrak{X}}$. Similarly, we have that $\pi^* \circ d : \mathbb{R} \longrightarrow Q$ for $d \in C_{\mathfrak{X}}$. Then since (Q, C_Q, F_Q) is a Frölicher space, $f \circ (\pi^* \circ d) \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_Q$, therefore $\pi^* \circ d \in C_Q$. Hence the morphism $\pi^* : \mathfrak{X} \longrightarrow Q$ is Frölicher smooth. But $\pi^* : \mathfrak{X} \longrightarrow Q$ can be defined as $\pi^*(x, y) = [x]$ or $\pi^*(x, y) = [y]$, $[x] \neq [y]$, for $(x, y) \in \mathfrak{X}$. Therefore $\pi^* : \mathfrak{X} \longrightarrow Q$ is not unique.*

Remark 2.3.5 *For any $g \in F_X$ there exists a unique map $f \in F_Q$ such that $g = f \circ q$, where $q : X \longrightarrow Q$ is the canonical projection.*

Since the Frölicher quotient (Q, C_Q, F_Q) is a Frölicher space then we have Frölicher topologies induced from the Frölicher structure (C_Q, F_Q). The Frölicher topologies are denoted τ_{C_Q} and τ_{F_Q} and are induced respectively by structure curves of C_Q and structure functions of F_Q. By definition

$$\tau_{C_Q} = \{U \subseteq Q \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_Q\}$$

and

$$\tau_{F_Q} = \{U \subseteq Q \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_Q\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. By Lemma 2.3.3 we have that $\tau_{F_Q} \subset \tau_{C_Q}$.

We also have the canonical topology on Q, that is, the quotient topology

$$\tau_\sim = \{V \subseteq Q \mid q^{-1}(V) \in \tau_{F_X}\}$$

(see [3] and [70]). The quotient topology makes the canonical projection $q : X \longrightarrow Q$ continuous. That is the quotient topology is the largest, that is, the finest topology on Q for which $q : X \longrightarrow Q$ is continuous.

Lemma 2.3.10 *Let (Q, C_Q, F_Q) be the Frölicher quotient, τ_{F_Q} be a Frölicher topology induced from the structure functions of F_Q and $\tau_\sim$ be the canonical quotient topology on Q, then $\tau_{F_Q} = \tau_\sim$.*

Proof: We have to show that $\tau_{F_Q} \subset \tau_\sim$ and $\tau_\sim \subset \tau_{F_Q}$. Let $U \in \tau_{F_Q}$, then $U \subseteq Q$ such that $U = f^{-1}(V)$, $\forall\ f \in F_Q$, $V \in \tau_\mathbb{R}$, where $\tau_\mathbb{R}$ is the standard topology on $\mathbb{R}$. Recall that $\forall\ g \in F_X\ \exists\ f \in F_Q$ such that $g = f \circ q$ where $q : X \longrightarrow Q$ is the canonical projection (see Remark 2.3.5). Therefore

$$\begin{aligned} q^{-1}(U) &= q^{-1}(f^{-1}(V)) \\ &= (q^{-1} \circ f^{-1})(V) \\ &= (f \circ q)^{-1}(V) \\ &= g^{-1}(V) \end{aligned} \tag{2.30}$$

Since $q^{-1}(U) \subseteq X$, therefore $q^{-1}(U) \in \tau_{F_X}$. Thus $U \in \tau_\sim$, and hence $\tau_{F_Q} \subset \tau_\sim$. Let $W \in \tau_\sim$, that implies that $W \subseteq Q$ and $q^{-1}(W) \in \tau_{F_X}$ where $q : X \longrightarrow Q$ is the canonical projection. That is $q^{-1}(W) \subseteq X$ and $q^{-1}(W) = g^{-1}(V)$, $\forall\ g \in F_X$, $V \in \tau_\mathbb{R}$, where $\tau_\mathbb{R}$ is the standard topology on $\mathbb{R}$. Since $q^{-1}(W) = g^{-1}(V)$, then we have that $W = (q \circ g^{-1})(V)$. By Remark 2.3.5 there exists a unique map $f \in F_Q$ such that $g = f \circ q$. Therefore

$$\begin{aligned} W &= (q \circ g^{-1})(V) \\ &= q(g^{-1}(V)) \\ &= q((f \circ q)^{-1}(V)) \\ &= q((q^{-1} \circ f^{-1})(V)) \\ &= f^{-1}(V) \end{aligned} \tag{2.31}$$

That is $W \subseteq Q$ and $\forall\ f \in F_Q\ \exists\ V \in \tau_\mathbb{R}$ such that $W = f^{-1}(V)$, therefore $W \in \tau_{F_Q}$. That is, $W \in \tau_\sim \implies W \in \tau_{F_Q}$, hence $\tau_\sim \subset \tau_{F_Q}$. □

Lemma 2.3.11 *Let (Q, C_Q, F_Q) be the Frölicher quotient, τ_{C_Q} be the Frölicher quotient induced from the structure curves of C_Q and $\tau_\sim$ be the canonical topology on Q, then $\tau_{C_Q} = \tau_\sim$.*

Proof: We have to show that $\tau_{C_Q} \subset \tau_\sim$ and that $\tau_\sim \subset \tau_{C_Q}$. Let $U \in \tau_{C_Q}$, then $U \subseteq Q$ and $c^{-1}(U) \in \tau_\mathbb{R}$, $\forall\ c \in C_Q$, where $\tau_\mathbb{R}$ is the standard topology on $\mathbb{R}$. Since $\forall\ g \in F_X\ \exists\ f \in F_Q$ such that $g = f \circ q$ then $f^{-1} = q \circ g^{-1}$. Therefore

$$\begin{aligned} (g \circ q^{-1})(U) &= (q \circ g^{-1})^{-1}(U) \\ &= (f^{-1})^{-1}(U) \\ &= f(U) \\ &= V \end{aligned} \tag{2.32}$$

for some $V \in \tau_\mathbb{R}$. Thus $q^{-1}(U) = g^{-1}(V)$, $\forall\ g \in F_X$. Therefore $q^{-1}(U) \in \tau_{F_X}$, which implies that $U \in \tau_\sim$, hence $\tau_{C_Q} \subset \tau_\sim$.
Let $W \in \tau_\sim$, that is $W \subseteq Q$ and $q^{-1}(W) \in \tau_{F_X}$ where $q : X \longrightarrow Q$ is the canonical projection. But $q^{-1}(W) \in \tau_{F_X}$ implies that $q^{-1}(W) \subseteq X$ such that $q^{-1}(W) = g^{-1}(V)$, $\forall\ g \in F_X$, $V \in \tau_\mathbb{R}$, where $\tau_\mathbb{R}$ is the standard topology on $\mathbb{R}$.

But $q^{-1}(W) = g^{-1}(V) \implies W = q(g^{-1}(V))$. By Remark 2.3.5 there exists a unique map $f \in F_Q$ such that $g = f \circ q$, thus

$$g^{-1}(V) = (f \circ q)^{-1}(V) = (q^{-1} \circ f^{-1})(V).$$

Therefore

$$\begin{aligned} W &= q(g^{-1}(V)) \\ &= q((q^{-1} \circ f^{-1})(V)) \\ &= q(q^{-1}(f^{-1}(V))) \\ &= f^{-1}(V) \end{aligned} \tag{2.33}$$

Then $\forall\ c \in C_Q$ we have that

$$\begin{aligned} c^{-1}(W) &= c^{-1}(f^{-1}(V)) \\ &= (c^{-1} \circ f^{-1})(V) \\ &= (f \circ c)^{-1}(V) \end{aligned} \tag{2.34}$$

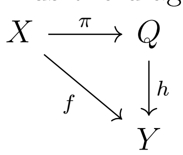
But $f \circ c : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous since $f \in F_Q$, $c \in C_Q$ and $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Therefore $(f \circ c)^{-1}(V) \in \tau_{\mathbb{R}}$ implying that $c^{-1}(W) \in \tau_{\mathbb{R}}$, $\forall\ c \in C_Q$, thus $W \in \tau_{C_Q}$. That is, $W \in \tau_\sim \implies W \in \tau_{C_Q}$, hence $\tau_\sim \subset \tau_{C_Q}$. □

Lemma 2.3.12 *Let (Q, C_Q, F_Q) be the Frölicher quotient, τ_{F_Q} be the Frölicher topology induced from the structure functions of F_Q, τ_{C_Q} be the Frölicher topology induced from structure curves of C_Q and $\tau_\sim$ be the quotient topology on Q, then $\tau_{F_Q} = \tau_{C_Q} = \tau_\sim$.*

Proof: From Lemma 2.3.10 we have that $\tau_{F_Q} = \tau_\sim$ and from Lemma 2.3.11 we have that $\tau_{C_Q} = \tau_\sim$. Then it follows that $\tau_{F_Q} = \tau_{C_Q} = \tau_\sim$. □

Theorem 2.3.2 *Let X be a topological space, Y be a Hausdorff space, $f : X \longrightarrow Y$ a surjection and $\sim$ an equivalence relation on X defined by $x \sim y$ if and only if $f(x) = f(y)$, then the quotient set $Q := X/_\sim$ is a Hausdorff space.*

Proof: Since $f : X \longrightarrow Y$ is a surjection then there exists a unique function $h : Q \longrightarrow Y$ defined by $h : [x] \mapsto f(x)$ such that $f : X \longrightarrow Y$ factors uniquely via $\pi : X \longrightarrow Q$, that is $f = h \circ \pi$ (see Proposition 2.4 and Theorem 3.1 in [90]). Thus the diagram



commutes. Let $[x], [y] \in Q := X/_\sim$ such that $[x] \neq [y]$. By the definition of the equivalence relation $\sim$ we have that $[x] \neq [y] \iff x \nsim y \iff f(x) \neq f(y)$. Since Y is a Hausdorff space with $f(x), f(y) \in Y$ and $f(x) \neq f(y)$ then there exists open sets W and Z in Y such that $f(x) \in W$, $f(y) \in Z$ and $W \cap Z = \phi$. By definition of $h : Q \longrightarrow Y$ we have that $h([x]) = f(x)$ and $h([y]) = f(y)$, therefore $[x] = (h^{-1} \circ f)(x)$ and $[y] = (h^{-1} \circ f)(y)$. Since $f(x) \in W$ and $f(y) \in Z$ therefore $[x] \in h^{-1}(W)$ and $[y] \in h^{-1}(Z)$, and $h^{-1}(W)$ and $h^{-1}(Z)$ are open sets since the pre-image of an open set is open. Also

$$h^{-1}(W) \cap h^{-1}(Z) = h^{-1}(W \cap Z) = h^{-1}(\phi) = \phi.$$

Therefore Q is a Hausdorff space. □

Lemma 2.3.13 *Let $f : X \longrightarrow Y$ be an injection, $\sim$ be the kernel equivalence of f, that is, $x \sim y$ if and only if $f(x) = f(y)$, $\forall\ x, y \in X$ and $Q := X/_\sim$ be a quotient set. The canonical projection $\pi : X \longrightarrow Q$ is an injection.*

Proof: Let $x, y \in X$. Then $\pi(x) = \pi(y) \iff [x] = [y] \iff x \sim y \iff f(x) = f(y)$. And $f(x) = f(y) \implies x = y$ since $f : X \longrightarrow Y$ is an injection. Therefore $\pi(x) = \pi(y) \implies x = y$. Hence $\pi : X \longrightarrow Q$ is an injection. □

Theorem 2.3.3 *Let X be a Hausdorff space, $f : X \longrightarrow Y$ be an injective, $\sim$ be an equivalence relation on X defined by $x \sim y$ if and only if $f(x) = f(y)$, $\pi : X \longrightarrow Q$ be the canonical projection, τ_{F_X} be the functional topology on X and $\tau_\sim$ be the quotient topology. The quotient set $Q := X/_\sim$ is Hausdorff.*

Proof: Let $[x]$ and $[y]$ be equivalence classes such that $[x], [y] \in Q$ and $[x] \neq [y]$. That is $x \nsim y$, which implies that $x \neq y$, and we have that $x, y \in X$ since $[x], [y] \in Q$. Therefore there exists $U, V \in \tau_{F_X}$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Having the canonical projection $\pi : X \longrightarrow Q$, which is defined by $\pi(x) = [x]$, $\forall\ x \in X$, therefore $\pi(U)$, $\pi(V) \subset Q$. Having that $f : X \longrightarrow Y$ is an injection implies that $\pi : X \longrightarrow Q$ is also an injection, by Lemma 2.3.13. Therefore we have that $\pi^{-1}(\pi(U)) = U \in \tau_{F_X}$ and $\pi^{-1}(\pi(V)) = V \in \tau_{F_X}$ since $\pi : X \longrightarrow Q$ is an injection, which implies that $\pi(U)$, $\pi(V) \in \tau_\sim$. But $x \in U \implies [x] \in \pi(U)$ and $y \in V \implies [y] \in \pi(V)$. Since $U \cap V = \phi$ therefore

$$\pi(U) \cap \pi(V) = \pi(U \cap V) = \pi(\phi) = \phi.$$

Hence $Q := X/_\sim$ is Hausdorff. □

2.4 Concepts on bornology.

Definition 2.4.1 ***Bornology and bounded sets.***
Let X be a non-empty set. A bornology β on X is a collection of subsets of X, that is $\beta \subseteq P(X)$, with $P(X)$ the power set of X, such that the following are satisfied:

1. $X = \bigcup_{U \in \beta} U$, *that is,* β *covers* X.

2. *If* $U_1, U_2, \ldots, U_n \in \beta$ *then* $\bigcup_{i=1}^{n} U_i \in \beta$. *That is,* β *is closed under finite union.*

3. *If* $A \subseteq B$ *and* $B \in \beta$ *then* $A \in \beta$. *That is,* β *is closed under inclusion.*

The elements of the bornology β *are called bounded sets.*

Definition 2.4.2 ***Bornological set.***
Let X *be a non-empty set and* β *be a bornology on* X*, then the pair* (X, β) *is called the bornological set.*

Remark 2.4.1 *Let* β *be a bornology as defined in the definition above. Then* β *covers* X*, that is* $X = \bigcup_{U \in \beta} U$. *Therefore there exists* $U \in \beta$ *such that* $\{x\} \subset U$*,* $\forall\ x \in X$. *Therefore* $\{x\} \in \beta$*,* $\forall\ x \in X$*, since* β *is closed under inclusion. Since* $\beta \subseteq P(X)$*, it then follows trivially that* $\{x\} \in \beta$*,* $\forall\ x \in X$*, implies that* $X = \bigcup_{U \in \beta} U$. *Therefore the condition that* β *covers* X *and the condition that* $\{x\} \in \beta$*,* $\forall\ x \in X$*, are equivalent (see [84]).*

If we have that β is a bornology and is closed under finite non-disjoint union then β will be called a prebornology (see [55]). Herewith is the definition of a prebornology:

Definition 2.4.3 ***Prebornology.***
Let X *be a non-empty set. A prebornology* β *on* X *is a collection of subsets of* X*, that is* $\beta \subseteq P(X)$*, with* $P(X)$ *the power set of* X*, such that the following are satisfied:*

1. β *covers* X*, that is,* $X = \bigcup_{B \in \beta} B$.

2. *If* $A \subseteq B$ *and* $B \in \beta$ *then* $A \in \beta$. *That is,* β *is closed under inclusion.*

3. *If* $U_1, U_2, \ldots, U_n \in \beta$ *and* $\bigcap_{i=1}^{n} U_i \neq \phi$ *then* $\bigcup_{i=1}^{n} U_i \in \beta$. *That is,* β *is closed under finite non-disjoint union.*

Definition 2.4.4 ***Base of a bornology.***
Let (X, β) *be a bornological set then* β_0 *is a base of the bornology* β *if* $\beta_0 \subset \beta$ *and* $\forall\ U \in \beta\ \exists\ V \in \beta_0$ *such that* $U \subset V$. *That is,* β_0 *is a base of the bornology of* β *if* β_0 *is a subset of* β *and every element of* β *is contained in an element of* β_0.

Lemma 2.4.1 *Let X be a non-empty set and β_0 be a collection of subsets of X. Then β_0 is a base of a bornology on X if and only if β_0 covers X and $\forall\ U_1, U_2, \ldots, U_n \in \beta_0\ \exists\ V \in \beta_0$ such that $\bigcup_{i=1}^{n} U_i \subset V$.*

Proof: ($\Longrightarrow$) Let β be a bornology on X and β_0 be its base. That is, $\beta_0 \subset \beta$ and $\forall\ U \in \beta\ \exists\ V \in \beta_0$ such that $U \subset V$. Let $U_1, U_2, \ldots, U_n \in \beta_0$, therefore $U_1, U_2, \ldots, U_n \in \beta$ since $\beta_0 \subset \beta$. This implies that $\bigcup_{i=1}^{n} U_i \in \beta$ since β is closed under finite union as it is a bornology. Then there exists $V \in \beta_0$ such that $\bigcup_{i=1}^{n} U_i \subset V$ since β_0 is a base of β. Therefore $\forall\ U_1, U_2, \ldots, U_n \in \beta_0\ \exists\ V \in \beta_0$ such that $\bigcup_{i=1}^{n} U_i \subset V$. Since β is a bornology on X then β covers X, that is $X = \bigcup_{U \in \beta} U$. Therefore $\bigcup_{V \in \beta_0} V \subset \bigcup_{U \in \beta} U = X$ since $\beta_0 \subset \beta$. We also have that $X = \bigcup_{U \in \beta} U \subset \bigcup_{V \in \beta_0} V$ since $\forall\ U \in \beta\ \exists\ V \in \beta_0$ such that $U \subset V$. That is we have that $X \subset \bigcup_{V \in \beta_0} V$ and $\bigcup_{V \in \beta_0} V \subset X$, therefore $X = \bigcup_{V \in \beta_0} V$ and thus β_0 covers X.
($\Longleftarrow$) Let β_0 be the collection of subsets of X such that β_0 covers X and that $\forall\ U_1, U_2, \ldots, U_n \in \beta_0\ \exists\ V \in \beta_0$ such that $\bigcup_{i=1}^{n} U_i \subset V$. Therefore there exists a bornology β on X such that $\beta_0 \subset \beta$ since bornologies are closed under finite union and inclusion. Let $U_1, U_2, \ldots, U_n \in \beta_0$ then $U_1, U_2, \ldots, U_n \in \beta$ since $\beta_0 \subset \beta$, therefore $\bigcup_{i=1}^{n} U_i \in \beta$ since β is closed under finite union. Now since both β_0 and β cover X, that is $X = \bigcup_{V \in \beta_0} V = \bigcup_{U \in \beta} U$, and we also have that $\beta_0 \subset \beta$ therefore $\forall\ B \in \beta\ \exists\ U_1, U_2, \ldots, U_n \in \beta_0$ such that $B = \bigcup_{i=1}^{n} U_i$. It follows that $\forall\ U \in \beta\ \exists\ V \in \beta_0$ such that $U \subset V$. $\square$

Remark 2.4.2 *Let X be a non-empty set and β_0 be the collection of subsets of X. The collection*

$$\beta := \{U \subseteq X \mid U \subset V \in \beta_0\}$$

is such that it is a bornology on X and β_0 is its base (see [51]).

Definition 2.4.5 ***Bounded maps.***
Let (X, β_X) and (Y, β_Y) be bornological sets. We say that the mapping $\mu : (X, \beta_X) \longrightarrow (Y, \beta_Y)$ is bounded if $\mu(U) \in \beta_Y,\ \forall\ U \in \beta_X$.

Definition 2.4.6 ***Finer/coarser bornology.***
Let (X, β_1) and (X, β_2) be bornological sets. If the identity mapping $id : (X, \beta_1) \longrightarrow (X, \beta_2)$ is bounded then we say that the bornology β_1 is coarser than the bornology β_2 or the bornology β_2 is finer than the bornology β_1. This is equivalent to $\beta_1 \subset \beta_2$.

Definition 2.4.7 ***Bornological isomorphism.***
Let (X, β_X) and (Y, β_Y) be bornological sets then the mapping $\mu : (X, \beta_X) \longrightarrow (Y, \beta_Y)$ is called a bornological isomorphism if it is bounded and its inverse is also bounded.

Examples of bornologies:

Example 2.4.1 *Let X be any non-empty set and consider the power set*

$$P(X) := \{U \mid U \subseteq X\}$$

of X. By definition of the power set $P(X)$ we have that $X = \bigcup_{U \in P(X)} U$. That is, $P(X)$ covers X.
Let $U_1, U_2, \ldots, U_n \in P(X)$ then $U_i \subseteq X,\ \forall\ i = 1, 2, \ldots, n$, which implies that $\bigcup_{i=1}^{n} U_i \subseteq X \implies \bigcup_{i=1}^{n} U_i \in P(X)$. Therefore $P(X)$ is closed under finite union.
Let $V \subset U \in P(X)$ then $V \subset U \subseteq X \implies V \subset X \implies V \in P(X)$. Therefore $P(X)$ is closed under inclusion.
We have that $P(X)$ is closed under finite union and inclusion and that it covers X, therefore it is a bornology on X.

Example 2.4.2 *Let X be any set and let*

$$F(X) := \{U \subseteq X \mid U \text{ is finite}\}.$$

Let $U_1, U_2, \ldots, U_n \in F(X)$ then $U_i \subseteq X$ is finite $\forall\ i = 1, 2, \ldots, n$. Therefore $\bigcup_{i=1}^{n} U_i \subseteq X$ is finite which implies that $\bigcup_{i=1}^{n} U_i \in F(X)$. Therefore $F(X)$ is closed under finite union.
Let $U \in F(X)$ and $V \subset U$. Since $U \in F(X)$ then $U \subseteq X$ is finite, therefore $V \subset U \implies V \subset X$ which implies that V is finite, therefore $V \in F(X)$. That is, $V \subset U \in F(X) \implies V \in F(X)$, thus $F(X)$ is closed under inclusion.
The singleton $\{x\} \subset X$ is finite $\forall\ x \in X$, therefore $\{x\} \in F(X),\ \forall\ x \in X$. Thus $X = \bigcup_{U \in F(X)} U$, that is $F(X)$ covers X.
Hence $F(X)$ is a bornology on X since it covers X and it is closed under inclusion and finite union.

Remark 2.4.3 *With X a non-empty set, the power set $P(X)$ is the largest bornology on X and the set $F(X) := \{U \subseteq X \mid U \text{ is finite}\}$ is the smallest bornology on X (see [17]).*

Definition 2.4.8 *Let X be a vector space over the field K.*

1. *Let $\beta \subset P(X)$ such that β is closed under vector addition, scalar multiplication and the formation of balanced hulls (that is, the sum of two bounded sets is bounded) then β is called a vector bornology on X.*
2. *If $\forall\ \lambda \in K$, $|\lambda| \leq 1$, we have that $\lambda U \subset U$ for $U \subseteq X$ then we say that U is circled (see [65]).*
3. *We say that $A \subseteq X$ absorbs $B \subseteq X$ if $\exists\ r \in \mathbb{R}$, $r > 0$, such that $\lambda A \subset B$ whenever $r \leq |\lambda|$. If $A \subseteq X$ such that it absorbs every singleton of X then we say that A is absorbent (see [65]).*

Example 2.4.3 *Let Y be a topological vector space and let*

$$\beta_Y := \{U \subseteq Y \mid \forall \text{ neighbourhood } V \text{ of } 0\ \exists\ r > 0 \text{ such that } U \subseteq rV\}.$$

Let β_0 be a base of circled neighbourhoods of 0 in X then $\forall\ A \subseteq X$, A is bounded if and only if $\forall\ B \in \beta_0\ \exists\ r > 0$ such that $A \subseteq rB$. Therefore β_Y is a covering of Y since every neighbourhood of 0 is absorbent. Let $U_1, U_2 \in \beta_Y$ and $B \in \beta_0$ then there exists $D \in \beta_0$ such that $D + D \subset B$. Since $U_1, U_2 \in \beta_Y$ then there exists $r_1, r_2 > 0$ such that $U_1 \subset r_1 D$ and $U_2 \subset r_2 D$. By letting $r = max\{r_1, r_2\}$ then we have that

$$(U_1 + U_2) \subset (r_1 D + r_2 D) \subset (rD + rD) \subset r(D + D) \subset rB.$$

That is, β_Y is closed under vector addition. We also have that β_Y is closed under the formation of circled hulls since β_0 is closed under the formation of circled hulls. Therefore β_Y is a vector bornology on Y.

Definition 2.4.9 *Let $D \subseteq \mathbb{R}$ then we say that:*

1. *D is bounded below if $\forall\ d \in D\ \exists\ l \in \mathbb{R}$ such that $l \leq d$. We call l the lower bound of D.*
2. *D is bounded above if $\forall\ d \in D\ \exists\ L \in \mathbb{R}$ such that $d \leq L$. We call L the upper bound of D.*
3. *D is bounded if it is bounded below and bounded above. That is $\forall\ d \in D\ \exists\ l, L \in \mathbb{R}$ such that $l \leq d \leq L$*

Example 2.4.4 *Consider the collection*

$$\beta_{\mathbb{R}} := \{S \subseteq \mathbb{R} \mid S \text{ is bounded}\}$$

of subsets of $\mathbb{R}$.
Let $A, B \in \beta_{\mathbb{R}}$, *then* $A \subseteq \mathbb{R}$ *and* $B \subseteq \mathbb{R}$ *such that* A *and* B *are bounded. That is* $\forall\ a \in A\ \exists\ m, n \in \mathbb{R}$ *such that* $m \leq a \leq n$, *and* $\forall\ b \in B\ \exists\ p, q \in \mathbb{R}$ *such that* $p \leq b \leq q$. *Since* $A \subseteq \mathbb{R}$ *and* $B \subseteq \mathbb{R}$ *then* $A \cup B \subseteq \mathbb{R}$. *Let* $k \in A \cup B$ *then* $k \in A$ *or* $k \in B$, *and* $k \in A \implies m \leq k \leq n$ *and* $k \in B \implies p \leq k \leq q$. *Therefore* $A \cup B \subseteq \mathbb{R}$ *is bounded. Thus* $A \cup B \in \beta_{\mathbb{R}}$. *That is* $\beta_{\mathbb{R}}$ *is closed under finite union.*
Let $U \subseteq V$ *and* $V \in \beta_{\mathbb{R}}$, *then* $V \subseteq \mathbb{R}$ *is bounded, that is* $\forall\ v \in V\ \exists\ l, L \in \mathbb{R}$ *such that* $l \leq v \leq L$. *Since* $V \subseteq \mathbb{R}$ *and* $U \subseteq V$ *then* $U \subseteq \mathbb{R}$. *Let* $u \in U$, *then* $u \in V$ *since* $U \subseteq V$, *therefore* $l \leq u \leq L,\ \forall\ u \in U$. *That is* $u \in U \implies l \leq u \leq L$, *therefore* $U \subseteq \mathbb{R}$ *is bounded. Thus,* $U \in \beta_{\mathbb{R}}$. *That is,* $U \subseteq V \in \beta_{\mathbb{R}} \implies U \in \beta_{\mathbb{R}}$, *thus* $\beta_{\mathbb{R}}$ *is closed under inclusion.*
Let $r \in \mathbb{R}$ *and consider the singleton* $\{r\}$. *Since* $\mathbb{R}$ *is a partially ordered set then* $\forall\ r \in \mathbb{R}\ \exists\ l, L \in \mathbb{R}$ *such that* $l \leq r \leq L$. *Thus* $\{r\} \subseteq \mathbb{R}$ *is bounded* $\forall\ r \in \mathbb{R}$. *Therefore* $\{r\} \in \beta_{\mathbb{R}},\ \forall\ r \in \mathbb{R}$, *which implies that* $\mathbb{R} = \bigcup_{U \in \beta_{\mathbb{R}}} U$, *that is* $\beta_{\mathbb{R}}$ *covers* $\mathbb{R}$.
That is, we have that $\beta_{\mathbb{R}}$ *is closed under finite union and inclusion, and it covers* $\mathbb{R}$, *therefore* $\beta_{\mathbb{R}}$ *is a bornology on* $\mathbb{R}$. *The bornology* $\beta_{\mathbb{R}}$ *is the standard bornology on* $\mathbb{R}$.

The category of bornological sets.

Let (X, β_X), (Y, β_Y) and (Z, β_Z) be bornological sets and $f : (X, \beta_X) \longrightarrow (Y, \beta_Y)$ and $g : (Y, \beta_Y) \longrightarrow (Z, \beta_Z)$ be bounded maps. Since the codomain of f and the domain of g are equal then $g \circ f : (X, \beta_X) \longrightarrow (Z, \beta_Z)$. We have that $f(U) \in \beta_Y,\ \forall\ U \in \beta_X$, and $g(V) \in \beta_Z,\ \forall\ V \in \beta_Y$, since f and g are bounded maps. Let $W \in \beta_X$ therefore $(g \circ f)(W) = g(f(W)) \in \beta_Z$ since $f(W) \in \beta_Y$ and g is bounded. That is $\forall\ W \in \beta_X$ we have that $(g \circ f)(W) \in \beta_Z$ therefore $g \circ f$ is bounded. Every identity $id : (X, \beta_X) \longrightarrow (X, \beta_X)$ is bounded for every bornological set (X, β_X) since $id(U) = U \in \beta_X,\ \forall\ U \in \beta_X$. Therefore bornological sets and bounded maps between the bornological sets form a category - the category of bornological sets. Here are some properties of the category of bornological sets:

- Limits and colimits exist in the category of bornological sets and the forgetful functor from the category of bornological sets to the category of sets gives that there are initial and final structures in the category of bornological sets, (see [45]).
- The category of bornological sets is Cartesian closed, (see [20]).

Chapter 3

Bornologies induced from Frölicher space.

3.1 Bornology induced from structure functions.

3.1.1 Structure.

A. Frölicher and A. Kriegl in [45] showed that l^∞, that is the set of bounded sequences of real numbers, embeds into the category of bornological spaces. The l^∞-structure $(\mathcal{C}, \mathcal{F})$ of the l^∞-space $(X, \mathcal{C}, \mathcal{F})$ consists of a set $\mathcal{C}$ of structure curves $c : \mathbb{N} \longrightarrow X$ and a set $\mathcal{F}$ of structure functions $f : X \longrightarrow \mathbb{R}$, such that $f \circ c \in C^\infty(\mathbb{N}, \mathbb{R})$, that is $\mathcal{C}$ and $\mathcal{F}$ determines each other (see [45]). Then given an l^∞-space $(X, \mathcal{C}, \mathcal{F})$ the subset $U \subseteq X$ is bounded if and only if $f(U) \subseteq \mathbb{R}$ is bounded, $\forall\ f \in \mathcal{F}$. This is equivalent to $U \subseteq X$ is bounded if and only if every sequence $c : \mathbb{N} \longrightarrow X$, with $c(\mathbb{N}) \subseteq U$, belongs to $\mathcal{C}$ (Propostion 1.2.4 in [45]). The l^∞-structure $(\mathcal{C}, \mathcal{F})$ becomes a Frölicher structure by replacing $\mathbb{N}$ with $\mathbb{R}$ in $\mathcal{C}$, that is, by replacing $\mathbb{N}$ by $\mathbb{R}$ as the domain of the structure curves in $\mathcal{C}$. Based on this it is assumed that there is a bornology from the Frölicher structure. This assumption comes about and will be determined based on the structural similarities between the l^∞-structure and the Frölicher structure.

Lemma 3.1.1 *Let (X, C_X, F_X) be a Frölicher space then the collection*

$$\beta_{F_X} := \{U \subseteq X \mid f(U) \subset \mathbb{R} \text{ is bounded}, \ \forall\ f \in F_X\}$$

is a bornology on X.

Proof: We have to show that β_{F_X} covers X, it is closed under finite union and it is closed under inclusion.
Consider any singleton $\{x\} \subset X$. Since $f \in F_X$ is such that $f : X \longrightarrow \mathbb{R}$ is a real-valued function, then the image of any singleton $\{x\} \subset X$ under f is such

that $f(\{x\}) = \{f(x)\} \subset \mathbb{R}$. Since $f(x) \in \mathbb{R}$ and $\beta_{\mathbb{R}}$, that is the standard bornology on $\mathbb{R}$ (see Example 2.4.4), covers $\mathbb{R}$, therefore $\{f(x)\} \in \beta_{\mathbb{R}}$, $\forall\ f \in F_X$. That is $\{f(x)\} \subset \mathbb{R}$ is bounded $\forall\ f \in F_X$. This implies that $\{x\} \in \beta_{F_X}$, $\forall\ x \in X$, that is β_{F_X} covers X.
Let $U_1, U_2, \cdots, U_n \in \beta_{F_X}$, then $\forall\ i = 1, 2, \cdots, n$ we have that $U_i \subseteq X$ and $f(U_i) \subset \mathbb{R}$ is bounded, $\forall\ f \in F_X$. That is $f(U_i) \in \beta_{\mathbb{R}}$, $\forall\ f \in F_X$ and $\forall\ i = 1, 2, \cdots, n$. Therefore $f\left(\bigcup_{i=1}^{n} U_i\right) = \bigcup_{i=1}^{n} f(U_i) \in \beta_{\mathbb{R}}$, $\forall\ f \in F_X$, since $\beta_{\mathbb{R}}$ is closed under finite union. Since $\bigcup_{i=1}^{n} U_i \subseteq X$ therefore $\bigcup_{i=1}^{n} U_i \in \beta_{F_X}$. That is $U_1, U_2, \cdots, U_n \in \beta_{F_X}$ implies that $\bigcup_{i=1}^{n} U_i \in \beta_{F_X}$. Thus β_{F_X} is closed under finite union.
Let $U \subseteq V$ and $V \in \beta_{F_X}$, then $U \subseteq V \subseteq X$ and $f(V) \subset \mathbb{R}$ is bounded, $\forall\ f \in F_X$. That is $f(V) \in \beta_{\mathbb{R}}$, $\forall\ f \in F_X$. And $U \subseteq V$ implies that $f(U) \subseteq f(V) \in \beta_{\mathbb{R}}$, $\forall\ f \in F_X$. Therefore $f(U) \in \beta_{\mathbb{R}}$, $\forall\ f \in F_X$, since $\beta_{\mathbb{R}}$ is closed under inclusion. That is $f(U) \subset \mathbb{R}$ is bounded, $\forall\ f \in F_X$. This implies that $U \in \beta_{F_X}$. That is $U \subseteq V \in \beta_{F_X}$ implies that $U \in \beta_{F_X}$. Thus β_{F_X} is closed under inclusion. □

Definition 3.1.1 *Let (X, C_X, F_X) be a Frölicher space. The bornology*

$$\beta_{F_X} := \{U \subseteq X \mid f(U) \subseteq \mathbb{R} \textit{ is bounded},\ \forall\ f \in F_X\}$$

on X is called the functional bornology as it is induced from structure functions on X, that is induced from F_X.

Remark 3.1.1 *Any subset of X is bounded if and only if $f : X \longrightarrow \mathbb{R}$ is bounded under the functional bornology on X, $\forall\ f \in F_X$.*

3.1.2 Bounded maps and bounded sets.

This section focuses on bounded maps between functional bornological sets and bounded sets under structure curves.

Lemma 3.1.2 *Let β_{F_X} be the functional bornology on X and (X, β_{F_X}) be a bornological set. The identity map $id_{F_X} : (X, \beta_{F_X}) \longrightarrow (X, \beta_{F_X})$ is bounded.*

Proof: Let $id_{F_X} : (X, \beta_{F_X}) \longrightarrow (X, \beta_{F_X})$ be an identity map. Let $U \in \beta_{F_X}$, that is $U \subseteq X$ such that $f(U) \subset \mathbb{R}$ is bounded, $\forall\ f \in F_X$. But $f \circ id_{F_X} = f$, therefore $f(id_{F_X}(U)) = f(U) \subseteq \mathbb{R}$ is bounded $\forall\ f \in F_X$, which implies $id_{F_X} : (X, \beta_{F_X}) \longrightarrow (X, \beta_{F_X})$ is bounded. □

Identity maps between bornological sets are bounded (see [45] and [51]).

Lemma 3.1.3 *Let (X, C_X, F_X) be a Frölicher space, β_{F_X} be the functional bornology on X and (X, β_{F_X}) a bornological set. If the mapping $\phi : (X, \beta_{F_X}) \longrightarrow (X, \beta_{F_X})$ is Frölicher smooth then it is bounded.*

Proof: Let $\phi : (X, \beta_{F_X}) \longrightarrow (X, \beta_{F_X})$ be Frölicher smooth, that is

$$\{f \circ \phi \mid f \in F_X\} \subseteq F_X$$

and

$$\{\phi \circ c \mid c \in C_X\} \subseteq C_X.$$

Let $U \in \beta_{F_X}$, then $U \subseteq X$ such that $f(U) \subset \mathbb{R}$ is bounded, $\forall\ f \in F_X$. Since $f \circ \phi \in F_X$, $\forall\ f \in F_X$, and $U \in \beta_{F_X}$ therefore $(f \circ \phi)(U) = f(\phi(U)) \subset \mathbb{R}$ is bounded, $\forall\ f \in F_X$. This implies that $\phi(U) \in \beta_{F_X}$. Thus $U \in \beta_{F_X}$ implies that $\phi(U) \in \beta_{F_X}$, therefore $\phi : (X, \beta_{F_X}) \longrightarrow (X, \beta_{F_X})$ is bounded. □

Lemma 3.1.4 *Let (X, C_X, F_X) and (Y, C_Y, F_Y) be Frölicher spaces, β_{F_X} and β_{F_Y} be functional bornologies on X and Y respectively, and (X, β_{F_X}) and (Y, β_{F_Y}) be bornological sets. If the mapping $j : (X, \beta_{F_X}) \longrightarrow (Y, \beta_{F_Y})$ is Frölicher smooth then it is bounded.*

Proof: Let $j : (X, \beta_{F_X}) \longrightarrow (Y, \beta_{F_Y})$ be Frölicher smooth, therefore we have that $g \circ j \in F_X$, $\forall\ g \in F_Y$, and that $j \circ c \in C_Y$, $\forall\ c \in C_X$. We will use the condition that $g \circ j \in F_X$, $\forall\ g \in F_Y$, for this proof. Let $U \in \beta_{F_X}$, then $U \subseteq X$ and $f(U) \subset \mathbb{R}$ is bounded, $\forall\ f \in F_X$. Since $g \circ j \in F_X$, $\forall\ g \in F_Y$, therefore $(g \circ j)(U) = g(j(U)) \subset \mathbb{R}$ is bounded, $\forall\ g \in F_Y$. This implies that $j(U) \in \beta_{F_Y}$. That is $U \in \beta_{F_X}$ implies that $j(U) \in \beta_{F_Y}$, therefore the mapping $j : (X, \beta_{F_X}) \longrightarrow (Y, \beta_{F_Y})$ is bounded. □

Lemma 3.1.5 *Let (X, C_X, F_X) be a Frölicher space. If $U_1, U_2, \cdots, U_n \subseteq X$ such that $c^{-1}(U_i) \subset \mathbb{R}$ is bounded $\forall\ i = 1, 2, \cdots, n$ then $c^{-1}\left(\bigcup_{i=1}^{n} U_i\right) \subset \mathbb{R}$ is bounded, $\forall\ c \in C_X$.*

Proof: Let $U_1, U_2, \cdots, U_n \subseteq X$ such that $\forall\ i = 1, 2, \cdots, n$ we have that $c^{-1}(U_i) \subset \mathbb{R}$ is bounded, $\forall\ c \in C_X$. That is, $\forall\ c \in C_X$ we have that $c^{-1}(U_i) \in \beta_{\mathbb{R}}$, $\forall\ i = 1, 2, \cdots, n$. Since $\beta_{\mathbb{R}}$ is closed under finite union therefore

$$\beta_{\mathbb{R}} \ni \bigcup_{i=1}^{n} c^{-1}(U_i) = c^{-1}\left(\bigcup_{i=1}^{n} U_i\right), \ \forall\ c \in C_X.$$

Thus $c^{-1}\left(\bigcup_{i=1}^{n} U_i\right) \subset \mathbb{R}$ is bounded $\forall\ c \in C_X$. □

Lemma 3.1.6 *Let (X, C_X, F_X) be a Frölicher space. For all $c \in C_X$, if $U \subseteq V$ and $c^{-1}(V) \subset \mathbb{R}$ is bounded then $c^{-1}(U)$ is bounded.*

Proof: Let $U \subseteq V$ such that $c^{-1}(V) \subset \mathbb{R}$ is bounded, $\forall\ c \in C_X$. That is $c^{-1}(V) \in \beta_{\mathbb{R}}$, $\forall\ c \in C_X$. Since $U \subseteq V$ then it follows that $c^{-1}(U) \subseteq c^{-1}(V)\beta_{\mathbb{R}}$, $\forall\ c \in C_X$. This implies that $c^{-1}(U) \in \beta_{\mathbb{R}}$, $\forall\ c \in C_X$, since $\beta_{\mathbb{R}}$ is closed under inclusion. Therefore $c^{-1}(U) \subset \mathbb{R}$ is bounded $\forall\ c \in C_X$. □

Corollary 3.1.1 *Let (X, C_X, F_X) be a Frölicher space. The collection*

$$\beta_{C_X} := \{U \subseteq X \mid c^{-1}(U) \subset \mathbb{R} \text{ is bounded, for all } c \in C_X\}$$

is closed under finite union and inclusion.

Proof: By Lemma 3.1.5 we have that $U_1, U_2, \cdots, U_n \in \beta_{C_X}$ implies that $\bigcup_{i=1}^{n} U_i \in \beta_{C_X}$, therefore β_{C_X} is closed under finite union. Also, we have that $U \subseteq V \in \beta_{C_X}$ implies that $U \in \beta_{C_X}$ by Lemma 3.1.6, therefore β_{C_X} is closed under inclusion. □

Remark 3.1.2 *Let $e : \mathbb{R} \longrightarrow X$ be a mapping defined by $e(r) = t$, $\forall\ r \in \mathbb{R}$, where t is fixed in X. That is e is a constant function. Then $\forall\ f \in F_X$ we have that $(f \circ e)(r) = f(e(r)) = f(t) \in \mathbb{R}$, therefore $f \circ e \in C^{\infty}(\mathbb{R}, \mathbb{R})$. This implies that $e \in C_X$. For*

$$\beta_{C_X} := \{U \subseteq X \mid c^{-1}(U) \subset \mathbb{R} \text{ is bounded, for all } c \in C_X\}$$

to cover X we must have that $\{x\} \in \beta_{C_X}$, $\forall\ x \in X$. That is, $\forall\ x \in X$, we must have that $c^{-1}(\{x\}) \subset \mathbb{R}$ is bounded, $\forall\ c \in C_X$. However, $\{t\} \notin \beta_{C_X}$ since $e^{-1}(\{t\}) = \mathbb{R}$, and $\mathbb{R}$ is not bounded. This is sufficient that β_{C_X} does not cover X, therefore it is not a bornology on X. Based on this and on the nature of structure curves, we cannot attain, or induce, canonically, a bornology from structure curves.

3.2 Initial bornology.

There are standard methods in Functional Analysis that consists of forming a structure from an existing structure (see [51]). In the category of Frölicher spaces, structures, whether initial or final structures, can be built from existing Frölicher structures. Frölicher subspace, Frölicher product, Frölicher quotient and Frölicher coproduct are built from existing Frölicher structures in the category of Frölicher spaces (see [8], [13], [14], [29], [35], [86], [88] and [89]). Similarly bornologies can be constructed from existing bornologies (see [51]). We hereby start with initial bornologies.

Consider the family $\{(X_i, \beta_i) \mid i \in I\}$ of bornological sets and the family $\{g_i : X \longrightarrow X_i \mid i \in I\}$ of maps, with X being a non-empty set. Then the subset $U \subset X$ is bounded if and only if $g_i(U) \subset X_i$ is bounded, $i \in I$. Such bounded subsets $U \subset X$ form a bornology on X called the initial bornology (see [35] and [51]).

Lemma 3.2.1 *Let $\{(X_i, \beta_i) \mid i \in I\}$ be a collection of bornological sets, X be a non-empty set and $\{g_i : X \longrightarrow X_i \mid i \in I\}$ be a collection of maps. The collection*

$$\mathcal{N} := \{U \subset X \mid g_i(U) \subset X_i \text{ is bounded}, \ i \in I\}$$

is a bornology on X.

Proof: Let $U_1, U_2, \cdots, U_n \in \mathcal{N}$, then $U_j \subset X$ and $g_i(U_j) \subset X_i$ is bounded $\forall\ j = 1, 2, \cdots, n$, that is, $g_i(U_j) \in \beta_i$, $\forall\ j = 1, 2, \cdots, n$, $i \in I$. Therefore $\displaystyle\bigcup_{j=1}^{n} g_i(U_j) = g_i\left(\bigcup_{j=1}^{n} U_j\right) \in \beta_i$ since β_i is closed under finite union as it is a bornology on X_i. That is $g_i\left(\displaystyle\bigcup_{j=1}^{n} U_j\right) \subset X_i$ is bounded, $i \in I$, and therefore $\displaystyle\bigcup_{j=1}^{n} U_j \in \mathcal{N}$. Thus $\mathcal{N}$ is closed under finite union.
Let $U \subseteq V \in \mathcal{N}$, then $U \subseteq V \subset X$ and $g_i(V) \subset X_i$ is bounded, that is $g_i(V) \in \beta_i$, $i \in I$. Since $U \subseteq V \subset X$ then $g_i(U) \subseteq g_i(V) \subset X_i$, therefore $g_i(U) \in \beta_i$ since β_i is closed under inclusion. That is $g_i(U) \subset X_i$ is bounded, therefore $U \in \mathcal{N}$. Thus $\mathcal{N}$ is closed under inclusion.
Consider the singleton $\{x\} \subset X$. Since β_i covers X_i, that is $X_i = \displaystyle\bigcup_{B \in \beta_i} B$, then $\forall\ x \in X$ we have that $g_i(\{x\}) = \{g(x)\} \in \beta_i$, $i \in I$. That is $g_i(\{x\}) \subset X_i$ is bounded, therefore $\{x\} \in \mathcal{N}$, $\forall\ x \in X$. Thus $\mathcal{N}$ covers X.
Hence $\mathcal{N}$ is a bornology on X since it covers X, is closed under inclusion and is closed under finite union. $\square$

Definition 3.2.1 ***Initial bornology.***
Let X be a non-empty set, $\{(X_i, \beta_i) \mid i \in I\}$ be a family of bornological sets and $\{g_i : X \longrightarrow X_i \mid i \in I\}$ be a family of maps, then the bornology

$$\mathcal{N} = \{U \subset X \mid g_i(U) \subset X_i \text{ is bounded}, \ i \in I\}$$

is called an initial bornology on X.

Lemma 3.2.2 *Let X be a non-empty set, $\{(X_i, \beta_i) \mid i \in I\}$ be a family of bornological sets and $\{g : X \longrightarrow X_i \mid i \in I\}$ be a family of maps. The collection*

$$\mathcal{N}_0 := \bigcap_{i \in I} \{g_i^{-1}(U) \mid U \in \beta_i\}$$

is a base of the initial bornology

$$\mathcal{N} = \{U \subset X \mid g_i(U) \subset X_i \text{ is bounded}, \ i \in I\}.$$

Proof: Let $U \in \mathcal{N}_0$, then $g_i(U) \subset X_i$ is bounded, that is $g_i(U) \in \beta_i$, $i \in I$. Then there exists $B_i \in \beta_i$ such that $g_i(U) \subset B_i$ since β_i is closed under inclusion. And $g_i(U) \subset B_i \implies U \subset g_i^{-1}(B_i)$ and $g_i^{-1}(B_i) \in \mathcal{N}$. Let $V \in \mathcal{N}_0$, then $V = g_i^{-1}(U) \subset X$ where $U \in \beta_i$, $i \in I$. Then $g_i(V) = U \in \beta_i$, therefore $g_i(V) \subset X_i$ is bounded and hence $V \in \mathcal{N}$. Thus we have that $\mathcal{N}_0 \subset \mathcal{N}$ and every element of $\mathcal{N}$ is contained in an element of $\mathcal{N}_0$, therefore $\mathcal{N}_0$ is a base of the initial bornology $\mathcal{N}$. □

3.3 Bornological comparison on Frölicher subspace.

3.3.1 Subspace bornology.

Let (X, C_X, F_X) be a Frölicher space, $S \subseteq X$, and (S, C_S, F_S) be the Frölicher subspace of (X, C_X, F_X). Consider the inclusion $\imath : S \hookrightarrow X$, which is a Frölicher smooth map (see Section 2.3.4).

Lemma 3.3.1 *Let $\mathcal{B}$ be a bornology on X, that is $(X, \mathcal{B})$ be a bornological set, $S \subseteq X$ and $\imath : S \hookrightarrow X$. The collection*

$$\beta_S := \{U \subseteq S \mid \imath(U) \subset X \text{ is bounded}\}$$

is a bornology on S.

Proof: Let $U_1, U_2, \cdots, U_n \in \beta_S$, then $U_j \subseteq S$ and $\imath(U_j) \subset X$ is bounded $\forall\ j = 1, 2, \cdots, n$, that is $\imath(U_j) \in \mathcal{B}$, $\forall\ j = 1, 2, \cdots, n$. Therefore

$$\mathcal{B} \ni \bigcup_{j=1}^{n} \imath(U_j) = \imath\left(\bigcup_{j=1}^{n} U_j\right)$$

since $\mathcal{B}$ is closed under finite union as it is a bornology on X. This implies that $\bigcup_{j=1}^{n} U_j \in \beta_S$, thus β_S is closed under finite union.
Let $A \subseteq B \in \beta_S$, then $A \subseteq B \subseteq S$ and $\imath(B) \subset X$ is bounded, that is $\imath(B) \in \mathcal{B}$. Since $A \subseteq B \subseteq S$ then $\imath(A) \subseteq \imath(B) \in \beta$, therefore $\imath(A) \in \beta$ since $\mathcal{B}$ is closed under inclusion. This implies that $A \in \beta_S$ and thus β_S is closed under inclusion.
Let $\{s\} \subset S$, then $\{s\} \subset X$ since $S \subset X$. Therefore $\mathcal{B} \ni \{s\} = \{\imath(s)\} = \imath(\{s\})$ since $\mathcal{B}$ covers X. This implies that $\forall\ s \in S$, $\{s\} \in \beta_S$, therefore β_S covers S.
Hence β_S is a bornology on S since it covers S, is closed under inclusion and is closed under finite union. □

Definition 3.3.1 ***Subspace bornology.***
Let (X, C_X, F_X) be a Frölicher space, $S \subset X$, (S, C_S, F_S) be a Frölicher subspace of (X, C_X, F_X), $(X, \mathcal{B})$ be a bornological set and $\imath : S \hookrightarrow X$ be an inclusion map, then the bornology

$$\beta_S := \{U \subseteq S \mid \imath(U) \subset X \text{ is bounded}\}$$

on S is called the subspace bornology.

Proposition 3.3.1 *Let the subspace bornology β_S be defined as usual and $(X, \mathcal{B})$ be a bornological set, then $\beta_0 := \{U \cap S \mid U \in \beta\}$ is a base of the subspace bornology.*

Proof: We have to show that $\beta_0 \subset \beta_S$ and that every element of β_S is contained in an element of β_0. Let $A \in \beta_0$, then $A = U \cap S = \imath^{-1}(U) \subset S$ where $U \in \mathcal{B}$. Therefore $\imath(A) = \imath(U \cap S) = \imath(\imath^{-1}(U)) = U \in \mathcal{B}$, which implies that $\imath(A) \subset X$ is bounded and thus $A \in \beta_S$. That is $A \in \beta_0 \implies U \in \beta_S$, therefore $\beta_0 \subset \beta_S$.
Let $V \in \beta_S$, then $V \subseteq S$ and $\imath(V) \subset X$ is bounded, that is $\imath(V) \in \mathcal{B}$. Therefore there exists $B \in \mathcal{B}$ such that $\imath(V) \subset B$ since $\mathcal{B}$ is closed under inclusion. But $\imath(V) \subset B \implies V \subset \imath^{-1}(B) = B \cap S \in \beta_0$. That is every element of β_S is contained in an element of β_0. Hence β_0 is a base of β_S. □

Remark 3.3.1 *Let β_{F_X} be a functional bornology on X (see Definition 3.1.1), if $\imath(U) \in \beta_{F_X}$ then $U \subset S$ is bounded since β_{F_X} is a bornology on X. This follows from the structure definition of subspace bornology. Therefore the collection*

$$\beta_{F_X}(S) := \{U \subseteq S \mid \imath(U) \in \beta_{F_X}\}$$

is a subspace bornology. The subspace bornology $\beta_{F_X}(S)$ will be used in the bornological comparison on Frölicher subspace. It is a special case of the subspace bornology defined in Definition 3.3.1.

Definition 3.3.2 ***Functional subspace bornology.***
Let (X, C_X, F_X) be a Frölicher space, $S \subseteq X$, (S, C_S, F_S) be a Frölicher subspace of (X, C_X, F_X), β_{F_X} be the functional bornology on X and $\imath : S \hookrightarrow X$ be an inclusion map. The bornology

$$\beta_{F_X}(S) := \{U \subseteq S \mid \imath(U) \in \beta_{F_X}\}$$

on S is called the functional subspace bornology.

3.3.2 Bornology from structure functions of Frölicher subspace.

Let (X, C_X, F_X) be a Frölicher space, $S \subseteq X$ and (S, C_S, F_S) be a Frölicher subspace of (X, C_X, F_X). In Section 3.1.1 we induced a bornology from the structure functions of a Frölicher space. That is the functional bornology. Recall that for the Frölicher structure (C_S, F_S) on S we have that

$$C_S = \{c : \mathbb{R} \longrightarrow S \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_S\}$$

and

$$F_S = \{f : S \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C_S\}.$$

That is C_S and F_S determine each other. The bornology induced from structure functions of Frölicher subspace is the collection

$$\beta_{F_S} := \{U \subseteq S \mid f(U) \subset \mathbb{R} \textit{ is bounded},\ \forall\ f \in F_S\}.$$

It is a functional bornology on S (see Section 3.1.1 and Definition 3.1.1).

3.3.3 Bounded maps and bornological comparison.

Let (X, C_X, F_X) be a Frölicher space, $S \subseteq X$ and (S, C_S, F_S) be a Frölicher subspace of (X, C_X, F_X). We have induced two bornologies from the Frölicher subspace. One bornology induced from structure functions, that is from F_S, and that is the functional bornology

$$\beta_{F_S} := \{U \subseteq S \mid f(U) \subset \mathbb{R} \textit{ is bounded},\ \forall\ f \in F_S\}$$

on S. The other bornology is induced canonically on S, and that is the functional subspace bornology

$$\beta_{F_X}(S) := \{U \subseteq S \mid \imath(U) \in \beta_{F_X}\}$$

where $\imath : S \hookrightarrow X$ is an inclusion map and β_{F_X} is the functional bornology on X. In this section we compare the bornologies β_{F_S}, $\beta_{F_X}(S)$ and β_{F_X} and the bounded maps and sets they form.

Lemma 3.3.2 *Let β_{F_S} be the functional bornology on S, $\beta_{F_X}(S)$ be the functional subspace bornology, and (S, β_{F_S}) and $(S, \beta_{F_X}(S))$ be bornological sets. The identity maps $id_1 : (S, \beta_{F_S}) \longrightarrow (S, \beta_{F_S})$ and $id_2 : (S, \beta_{F_X}(S)) \longrightarrow (S, \beta_{F_X}(S))$ are bounded.*

Proof: This is trivial, identity maps between bornological sets are bounded (see [45] and [51]). □

Corollary 3.3.1 *Let β_{F_S} be the functional bornology on S and (S, β_{F_S}) be a bornological set. The mapping $\phi : (S, \beta_{F_S}) \longrightarrow (S, \beta_{F_S})$ is bounded if it is Frölicher smooth.*

Proof: Since β_{F_S} is a functional bornology on S then it follows from Lemma 3.1.3 that if the mapping $\phi : (S, \beta_{F_S}) \longrightarrow (S, \beta_{F_S})$ is Frölicher smooth then it is bounded. □

Corollary 3.3.2 *Let $S_1, S_2 \subseteq X$, (X, C_X, F_X) be a Frölicher space and (S_1, C_{S_1}, F_{S_1}) and (S_2, C_{S_2}, F_{S_2}) be Frölicher subspaces, $\beta_{F_{S_1}}$ and $\beta_{F_{S_2}}$ be the functional bornology on S_1 and S_2 respectively, and $(S_1, \beta_{F_{S_1}})$ and $(S_2, \beta_{F_{S_2}})$ be bornological sets. If the mapping $j : (S_1, \beta_{F_{S_1}}) \longrightarrow (S_2, \beta_{F_{S_2}})$ is Frölicher smooth then it is bounded.*

Proof: Since the Frölicher subspaces (S_1, C_{S_1}, F_{S_1}) and (S_2, C_{S_2}, F_{S_2}) are Frölicher spaces then by Lemma 3.1.4 we have that the mapping $j : (S_1, \beta_{F_{S_1}}) \longrightarrow (S_2, \beta_{F_{S_2}})$ is bounded if it is Frölicher smooth. □

Lemma 3.3.3 *Let $S \subseteq X$, (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X), β_{F_X} be the functional bornology on X, $\beta_{F_X}(S)$ be the functional subspace bornology, and $(S, \beta_{F_X}(S))$ and (X, β_{F_X}) be bornological sets. The inclusion $\imath : (S, \beta_{F_X}(S)) \hookrightarrow (X, \beta_{F_X})$ is bounded.*

Proof: Let $U \in \beta_{F_X}(S)$ therefore $\imath(U) \in \beta_{F_X}$ by definition. Therefore by definition $\imath : (S, \beta_{F_X}(S)) \hookrightarrow (X, \beta_{F_X})$ is bounded. □

Lemma 3.3.4 *Let $S \subseteq X$, (S, C_S, F_S) be a Frölicher subspace of the Frölicher space (X, C_X, F_X), β_{F_X} be the functional bornology on X, β_{F_S} be the functional bornology on S, and (S, β_{F_S}) and (X, β_{F_X}) be bornological spaces. The inclusion $\imath : (S, \beta_{F_S}) \hookrightarrow (X, \beta_{F_X})$ is bounded.*

Proof: Since (S, C_S, F_S) and (X, C_X, F_X) are Frölicher spaces then $\imath : (S, \beta_{F_S}) \hookrightarrow (X, \beta_{F_X})$ is a morphism in the category of Frölicher spaces since it is Frölicher smooth, that is $\{f \circ \imath \mid f \in F_X\} \subseteq F_S$ and $\{\imath \circ c \mid c \in C_S\} \subseteq C_X$. Let $V \in \beta_{F_S}$, then $V \subseteq S$ and $g(V) \subseteq \mathbb{R}$ is bounded $\forall\ g \in F_S$. Since $f \circ \imath \in F_S$, $\forall\ f \in F_X$, and $\imath(V) \subseteq X$ therefore $(f \circ \imath)(V) = f(\imath(V)) \subseteq \mathbb{R}$ is bounded, $\forall\ f \in F_X$. This implies that $\imath(V) \in \beta_{F_X}$. That is $V \in \beta_{F_S} \implies \imath(V) \in \beta_{F_X}$. Hence $\imath : (S, \beta_{F_S}) \hookrightarrow (X, \beta_{F_X})$ is bounded. □

Theorem 3.3.1 *Let (S, C_S, F_S) be a Frölicher subspace of the Frölicher space (X, C_X, F_X) with $S \subseteq X$, $\beta_{F_X}(S)$ be the functional subspace bornology, β_{F_S} be the functional bornology on S, and (S, β_{F_S}) and $(S, \beta_{F_X}(S))$ be bornological sets. Then $\beta_{F_S} \subset \beta_{F_X}(S)$, thus the identity map $id : (S, \beta_{F_S}) \longrightarrow (S, \beta_{F_X}(S))$ is bounded.*

Proof: Let $U \in \beta_{F_S}$, then $U \subseteq S$ and $f(U) \subset \mathbb{R}$ is bounded $\forall\ f \in F_S$. Consider the inclusion $\imath : S \hookrightarrow X$ then $\imath(U) \subset X$ and $g(\imath(U)) = (g \circ \imath)(U) \subset \mathbb{R}$, $\forall\ g \in F_X$. But $g \circ \imath \in F_S$, $\forall\ g \in F_X$, since $\imath : S \hookrightarrow X$ is Frölicher smooth. Now since $f(U) \subset \mathbb{R}$ is bounded $\forall\ f \in F_S$ therefore $g(\imath(U)) = (g \circ \imath)(U) \subset \mathbb{R}$ is bounded $\forall\ g \in F_X$. This implies that $\imath(U) \in \beta_{F_X}$, therefore $U \in \beta_{F_X}(S)$. That is $U \in \beta_{F_S} \implies U \in \beta_{F_X}(S)$, hence $\beta_{F_S} \subset \beta_{F_X}(S)$. Then it follows from definition that the identity map $id : (S, \beta_{F_S}) \longrightarrow (S, \beta_{F_X}(S))$ is bounded since $\beta_{F_S} \subset \beta_{F_X}(S)$. □

Lemma 3.3.5 *Let $S \subseteq X$, (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X), β_{F_X} be the functional bornology on X and $\beta_{F_X}(S)$ be the functional subspace bornology. Then $\beta_{F_X}(S) \subset \beta_{F_X}$.*

Proof: Let $U \in \beta_{F_X}(S)$, then $U \subset S \subseteq X$ and $\imath(U) \in \beta_{F_X}$, which implies that $f(\imath(U)) = f(U) \subset \mathbb{R}$ is bounded $\forall\, f \in F_X$, therefore $U \in \beta_{F_X}$. That is $U \in \beta_{F_X}(S) \implies U \in \beta_{F_X}$, therefore $\beta_{F_X}(S) \subset \beta_{F_X}$. □

Lemma 3.3.6 *Let $S \subseteq X$, (S, C_S, F_S) be a Frölicher subspace of the Frölicher space (X, C_X, F_X), $\beta_{F_X}(S)$ be the functional subspace bornology and β_{F_S} be the functional bornology on S. If $\forall\, f \in F_S\ \exists\, g \in F_X$ such that $f = g \circ \imath$ then $\beta_{F_X}(S) \subset \beta_{F_S}$.*

Proof: Let $U \in \beta_{F_X}(S)$, then $U \subset S$ and $\imath(U) \in \beta_{F_X}$, which implies $g(\imath(U)) \subset \mathbb{R}$ is bounded $\forall\, g \in F_X$. But since (S, C_S, F_S) is a Frölicher subspace of the Frölicher space (X, C_X, F_X) and $\imath : S \hookrightarrow X$ is Frölicher smooth then $g \circ \imath = g|_S \in F_S$, $\forall\, g \in F_X$. That is $g(\imath(U)) = (g \circ \imath)(U) \subset \mathbb{R}$ is bounded and $g \circ \imath \in F_S$. Since $\forall\, f \in F_S\ \exists\, g \in F_X$ such that $f = g \circ \imath$, therefore $f(U) \subset \mathbb{R}$ is bounded $\forall\, f \in F_S$. Thus $U \in \beta_{F_S}$. That is we have that $U \in \beta_{F_X}(S) \implies U \in \beta_{F_S}$, hence $\beta_{F_X}(S) \subset \beta_{F_S}$. □

Let
$$\beta_{F_X}|_S := \{U \subseteq S \mid f(U) \subset \mathbb{R} \textit{ is bounded},\ \forall\, f \in F_X\}.$$
Note that $\beta_{F_X}|_S$ is the bornology β_{F_X} restricted on $S \subseteq X$. The collection $\beta_{F_X}|_S$ is a subbornology induced by the functional bornology β_{F_X}, (see [76]). That is, $\beta_{F_X}|_S$ is a bornology and $\beta_{F_X}|_S \subset \beta_{F_X}$.

Lemma 3.3.7 *Let $S \subseteq X$, (S, C_S, F_S) be a Frölicher subspace of the Frölicher space (X, C_X, F_X), β_{F_X} be the functional bornology on X, $\beta_{F_X}|_S$ be the subbornology of β_{F_X} and $\beta_{F_X}(S)$ be the functional subspace bornology. Then $\beta_{F_X}|_S = \beta_{F_X}(S)$.*

Proof: Let $U \in \beta_{F_X}|_S$, then $U \subseteq S$ and $f(U) \subset \mathbb{R}$ is bounded $\forall\, f \in F_X$. Consider the canonical inclusion $\imath : S \hookrightarrow X$ then $U = \imath(U)$ since $U \subseteq S$. Therefore $f(U) = f(\imath(U)) \subset \mathbb{R}$ is bounded $\forall\, f \in F_X$, this implies that $\imath(U) \in \beta_{F_X}$ and therefore $U \in \beta_{F_X}(S)$. That is $U \in \beta_{F_X}|_S \implies U \in \beta_{F_X}(S)$, therefore $\beta_{F_X}|_S \subset \beta_{F_X}(S)$.
Now we have to show that $\beta_{F_X}(S) \subset \beta_{F_X}|_S$. Let $V \in \beta_{F_X}(S)$, then $V = \imath(V) \in \beta_{F_X}$ since $V \subseteq S$. With $V \in \beta_{F_X}$, $V \subseteq S$ and that $f(V) \subset \mathbb{R}$ is bounded $\forall\, f \in F_X$, therefore $V \in \beta_{F_X}|_S$. That is $V \in \beta_{F_X}(S) \implies V \in \beta_{F_X}|_S$, therefore $\beta_{F_X}(S) \subset \beta_{F_X}|_S$. Hence $\beta_{F_X}|_S = \beta_{F_X}(S)$ since $\beta_{F_X}(S) \subset \beta_{F_X}|_S$ and $\beta_{F_X}|_S \subset \beta_{F_X}(S)$. □

Lemma 3.3.8 *Let $S \subseteq X$, (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X), β_{F_X} be the functional bornology on X, $\beta_{F_X}|_S$ be the restriction of β_{F_X} on S and β_{F_S} be the subspace functional bornology. Then $\beta_{F_S} \subset \beta_{F_X}|_S$.*

Proof: Let $U \in \beta_{F_S}$, then $U \subseteq S$ and $g(U) \subset \mathbb{R}$ is bounded $\forall\ g \in F_S$. Consider the canonical inclusion $\imath : S \hookrightarrow X$ then $f \circ \imath \in F_S,\ \forall\ f \in F_X$, since $\imath : S \hookrightarrow X$ is Frölicher smooth, and $U = \imath(U)$ since $U \subseteq S$. Then $g(U) \subset \mathbb{R}$ being bounded $\forall\ g \in F_S$ implies that $(f \circ \imath)(U) = f(\imath(U)) = f(U) \subset \mathbb{R}$ is bounded $\forall\ f \in F_X$, therefore $U \in \beta_{F_X}|_S$. That is $U \in \beta_{F_S} \implies U \in \beta_{F_X}|_S$, hence $\beta_{F_S} \subset \beta_{F_X}|_S$. $\square$

3.4 Bornological comparison on Frölicher product.

3.4.1 Product bornology.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces and $\mathfrak{X} := \prod_{i=1}^{n} X_i$ be the Cartesian product in the category of sets. Then we have the Frölicher struture $(C_\mathfrak{X}, F_\mathfrak{X})$ on $\mathfrak{X}$ where

$$C_\mathfrak{X} := \{c : \mathbb{R} \longrightarrow \mathfrak{X} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_\mathfrak{X}\}$$

and

$$F_\mathfrak{X} = \{f : \mathfrak{X} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C_\mathbb{X}\}.$$

That is, $C_\mathfrak{X}$ and $F_\mathfrak{X}$ determine each other, making the triple $(\mathfrak{X}, C_\mathfrak{X}, F_\mathfrak{X})$ a Frölicher space - the Frölicher product (see Subsection 2.3.5). The mapping $p_i : \mathfrak{X} \longrightarrow X_i$ is a canonical projection, $\forall\ i = 1, 2, \cdots, n$. Let $\{(X_i, \beta_i) \mid i = 1, 2, \cdots, n\}$ be a family of bornological sets, the initial bornology on $\mathfrak{X}$ gives a bornology called the product bornology, (see [51]). A subset of $\mathfrak{X}$ is bounded if and only if its image under $p_i : \mathfrak{X} \longrightarrow X_i$ is bounded, $\forall\ i = 1, 2, \cdots, n$. That is, $U \subset \mathfrak{X}$ is bounded if and only if $p_i(U) \subset X_i$ is bounded $\forall\ i = 1, 2, \cdots, n$ (see [45] and [51]).

Let $\{(Y_j, \beta_{Y_j}) \mid j = 1, 2, \ldots, m\}$ be a family of prebornological spaces. A prebornology is a bornology that is closed under finite non-disjoint union (see Definition 2.4.3). That is, if β is a prebornology and $A, B \in \beta$ such that $A \cap B \neq \phi$ then $A \cup B \in \beta$. The product prebornology on $\prod_{j=1}^{m} Y_j$ is given by the collection

$$\left\{ U \subseteq \prod_{j=1}^{m} Y_j \mid \pi_j(U) \in \beta_{Y_j},\ \forall\ j = 1, 2, \cdots, m \right\},$$

where $\pi_j : \prod_{j=1}^{m} Y_j \longrightarrow Y_j$ is a canonical projection (see [55]). Keeping the same structure, we use a bornological set instead of a prebornological space to attain the product bornology on $\mathfrak{X} = \prod_{i=1}^{n} X_i$.

Lemma 3.4.1 *Let* $\mathfrak{X} := \prod_{i=1}^{n} X_i$ *be the Cartesian product and* $p_i : \mathfrak{X} \longrightarrow X_i$ *be a canonical projection,* $\forall\ i = 1, 2, \cdots, n$. *The collection*

$$\beta_{\mathfrak{X}} := \{U \subseteq \mathfrak{X} \mid p_i(U) \subseteq X_i \textit{ is bounded}, \ \forall\ i = 1, 2, \cdots, n\}$$

is a bornology on $\mathfrak{X}$.

Proof: Let $U_1, U_2, \cdots, U_m \in \beta_{\mathfrak{X}}$, then $U_j \subseteq \mathfrak{X}$ and $p_i(U_j) \subseteq X_i$ is bounded $\forall\ j = 1, 2, \cdots, m$ and $\forall\ i = 1, 2, \cdots, n$. That is $p_i(U_j) \in \beta_i$, $\forall\ j = 1, 2 \cdots, m$ and $\forall\ i = 1, 2, \cdots, n$, where β_i is a bornology on X_i. Then since β_i is closed under finite union as it is a bornology, and $\bigcup_{j=1}^{m} U_j \subseteq \mathfrak{X}$, then we have that

$$\beta_i \ni \bigcup_{j=1}^{m} p_i(U_j) = p_i\left(\bigcup_{j=1}^{m} U_j\right), \ \forall\ i = 1, 2, \cdots, n.$$

Therefore $p_i\left(\bigcup_{j=1}^{m} U_j\right)$ is bounded, $\forall\ i = 1, 2, \cdots, n$, thus $\bigcup_{j=1}^{m} U_j \in \beta_{\mathfrak{X}}$. Thus $\beta_{\mathfrak{X}}$ is closed under finite union.
Let $A \subseteq B \in \beta_{\mathfrak{X}}$, then $A \subseteq B \subseteq \mathfrak{X}$ and $p_i(B) \subseteq X_i$ is bounded, that is $p_i(B) \in \beta_i$, $\forall\ i = 1, 2, \cdots, n$. Since $A \subseteq B \subseteq \mathfrak{X}$ then $p_i(A) \subseteq p_i(B) \subseteq \beta_i$, therefore $p_i(A) \in \beta_i$ since β_i is closed under inclusion, $\forall\ i = 1, 2, \cdots, n$. This implies that $A \in \beta_{\mathfrak{X}}$. That is, $A \subseteq B \in \beta_{\mathfrak{X}} \implies A \in \beta_{\mathfrak{X}}$, therefore $\beta_{\mathfrak{X}}$ is closed under inclusion.
Let $x_i \in X_i$, $\forall\ i = 1, 2, \cdots, n$, and $\{(x_1, x_2, \ldots, x_n)\} \subset \mathfrak{X}$ be a singleton. Then $p_i(\{(x_1, x_2, \ldots, x_n)\}) = \{x_i\} \subset X_i$ is bounded since $\{x_i\} \in \beta_i$ as β_i covers X_i, $\forall\ i = 1, 2, \cdots, n$. Therefore $\{(x_1, x_2, \ldots, x_n)\} \in \beta_{\mathfrak{X}}$ and thus $\beta_{\mathfrak{X}}$ covers $\mathfrak{X}$.
Hence $\beta_{\mathfrak{X}}$ is a bornology on $\mathfrak{X}$ since it covers $\mathfrak{X}$, is closed under finite union and closed under inclusion. □

Definition 3.4.1 ***Product bornology.***

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ *be a family of Frölicher spaces,* $\mathfrak{X} = \prod_{i=1}^{n} X_i$ *be a Cartesian product in the category of sets,* $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ *be a Frölicher product,*

$\{(X_i, \beta_i) \mid i = 1, 2, \cdots, n\}$ be a family of bornological sets and $p_i : \mathfrak{X} \longrightarrow X_i$ be a canonical projection, $\forall\ i = 1, 2, \cdots, n$. The initial bornology

$$\beta_{\mathfrak{X}} := \{U \subseteq \mathfrak{X} \mid p_i(U) \subseteq X_i \textit{ is bounded } \forall\ i = 1, 2, \cdots, n\}$$

on $\mathfrak{X}$ is called the product bornology.

Proposition 3.4.1 *Let the product bornology $\beta_{\mathfrak{X}}$ be defined as usual and $\{(X_i, \beta_i) \mid i = 1, 2, \cdots, n\}$ be a family of bornological sets, then the collection*

$$\beta_0 := \left\{ \prod_{i=1}^{n} U_i \mid U_i \in \beta_i,\ \forall\ i = 1, 2 \cdots, n \right\}$$

is a base of $\beta_{\mathfrak{X}}$.

Proof: We have to show that $\beta_0 \subset \beta_{\mathfrak{X}}$ and that every element of $\beta_{\mathfrak{X}}$ is contained in an element of β_0. Let $A \in \beta_0$, then $A = \prod_{i=1}^{n} U_i$ where $U_i \in \beta_i$, $\forall\ i = 1, 2, \cdots, n$. But $U_i \in \beta_i$ implies that $U_i \subseteq X_i$ since β_i is a bornology on X_i, $\forall\ i = 1, 2, \cdots, n$. Therefore $A = \prod_{i=1}^{n} U_i \subseteq \prod_{i=1}^{n} X_i = \mathfrak{X}$, and $p_i(A) = p_i\left(\prod_{i=1}^{n} U_i\right) = U_i \in \beta_i$, $\forall\ i = 1, 2, \cdots, n$. Thus $\forall\ i = 1, 2, \cdots, n$ we have that $p_i(A) \subseteq X_i$ is bounded, which implies that $A \in \beta_{\mathfrak{X}}$. That is $A \in \beta_0 \implies A \in \beta_{\mathfrak{X}}$, therefore $\beta_0 \subset \beta_{\mathfrak{X}}$.
Let $V_i \subseteq X_i$, $\forall\ i = 1, 2, \cdots, n$, and let $B = \prod_{i=1}^{n} V_i$, therefore

$B = \prod_{i=1}^{n} V_i \subset \prod_{i=1}^{n} X_i = \mathfrak{X}$. Let $B \in \beta_{\mathfrak{X}}$ then $p_i(B) = p_i\left(\prod_{i=1}^{n} V_i\right) = V_i \subseteq X_i$ is bounded $\forall\ i = 1, 2, \cdots, n$, that is $p_i(B) = V_i \in \beta_i$, $\forall\ i = 1, 2, \cdots, n$. Thus $\forall\ i = 1, 2, \cdots, n\ \exists\ K_i \in \beta_i$ such that $p_i(B) = V_i \in K_i$ since β_i is closed under inclusion, as it is a bornology. Therefore $B = \prod_{i=1}^{n} V_i \subset \prod_{i=1}^{n} K_i \in \beta_0$. That is, every element of $\beta_{\mathfrak{X}}$ is contained in an element of β_0. Hence β_0 is the base of the product bornology $\beta_{\mathfrak{X}}$. □

Remark 3.4.1 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces, $\beta_{F_{X_i}}$ be a functional bornology on X_i. If $U \subseteq \mathfrak{X} = \prod_{i=1}^{n} X_i$ is such that $p_i(U) \in \beta_{F_{X_i}}$ then $p_i(U)$ is bounded in X_i, $\forall\ i = 1, 2, \cdots, n$, since $\beta_{F_{X_i}}$ is a bornology on X_i. This follows from the structure and definition of the product bornology. Therefore the collection*

$$\rho_{\mathfrak{X}} := \{U \subseteq \mathfrak{X} \mid p_i(U) \in \beta_{F_{X_i}},\ \forall\ i = 1, 2 \cdots, n\}$$

is also a product bornology. We will use the product bornology $\rho_{\mathfrak{X}}$ in the bornological comparison on Frölicher product instead of the general product bornology.

Definition 3.4.2 ***Functional product bornology.***
Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2 \cdots, n\}$ *be a family of Frölicher spaces,* $\mathfrak{X} = \prod_{i=1}^{n} X_i$ *be a Cartesian product in the category of sets,* $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ *be the Frölicher product,* $\beta_{F_{X_i}}$ *be the functional bornology on* X_i *and* $p_i : \mathfrak{X} \longrightarrow X_i$ *be a canonical projection. The product bornology*

$$\rho_{\mathfrak{X}} := \{U \subseteq \mathfrak{X} \mid p_i(U) \in \beta_{F_{X_i}},\ \forall\ i = 1, 2, \cdots, n\}$$

on $\mathfrak{X}$ *is called the functional product bornology.*

3.4.2 Bornology from structure functions of Frölicher product.

In Subsection 3.1.1, given a Frölicher space, a bornology was induced from structure functions. Recall that $\forall\ i = 1, 2, \cdots, n$ the Frölicher structure (C_{X_i}, F_{X_i}) is such that

$$F_{X_i} := \{f_i : X_i \longrightarrow \mathbb{R} \mid f_i \circ c_i \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ c_i \in C_{X_i}\}$$

and

$$C_{X_i} := \{c_i : \mathbb{R} \longrightarrow X_i \mid f_i \circ c_i \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ f_i \in F_{X_i}\}.$$

For all $i = 1, 2, \cdots, n$ the bornology

$$\beta_{F_{X_i}} := \{U \subset X_i \mid f_i(U) \subset \mathbb{R} \textit{ is bounded},\ \forall\ f_i \in F_{X_i}\}$$

is from structure functions on X_i, that is from F_{X_i}, thus it is the functional bornology on X_i. Since the Frölicher product $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$, where $\mathfrak{X} := \prod_{i=1}^{n} X_i$ a Cartesian product in the category of sets, is a Frölicher space then from Subsection 3.1.1 we have a bornology from structure functions on $\mathfrak{X}$. Recall that $C_{\mathfrak{X}}$ and $F_{\mathfrak{X}}$ determine each other in the following way:

$$F_{\mathfrak{X}} := \{f : \mathfrak{X} \longrightarrow \mathbb{R} \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C_{\mathfrak{X}}\},$$

$$C_{\mathfrak{X}} := \{c : \mathbb{R} \longrightarrow \mathfrak{X} \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_{\mathfrak{X}}\}.$$

The bornology from $F_{\mathfrak{X}}$, that is from structure functions on $\mathfrak{X}$, is given by

$$\beta_{F_{\mathfrak{X}}} := \{U \subseteq \mathfrak{X} \mid f(U) \subset \mathbb{R} \textit{ is bounded},\ \forall\ f \in F_{\mathfrak{X}}\}.$$

3.4.3 Bounded maps and bornological comparison.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces, $\mathfrak{X} = \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets, $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product and $p_i : \mathfrak{X} \longrightarrow X_i$ be the canonical projection $\forall\ i = 1, 2, \ldots, n$. We have the functional bornology $\beta_{F_{\mathfrak{X}}}$ on $\mathfrak{X}$, induced from struture functions on $\mathfrak{X}$, and the functional product bornology $\rho_{\mathfrak{X}}$. In this subsection we determine the relationship between these bornologies and their associated bounded maps.

Theorem 3.4.1 *Let $\rho_{\mathfrak{X}}$ be the functional product bornology, $\beta_{F_{\mathfrak{X}}}$ be the functional bornology on $\mathfrak{X}$, and $(\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ and $(\mathfrak{X}, \rho_{\mathfrak{X}})$ be bornological sets. Then $\beta_{F_{\mathfrak{X}}} \subset \rho_{\mathfrak{X}}$ and thus the identity map $id : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (\mathfrak{X}, \rho_{\mathfrak{X}})$ is bounded.*

Proof: Let $U \in \beta_{F_{\mathfrak{X}}}$, then $U \subseteq \mathfrak{X}$ and $g(U) \subset \mathbb{R}$ is bounded $\forall\ g \in F_{\mathfrak{X}}$. Consider the canonical projection $p_i : \mathfrak{X} \longrightarrow X_i$, then $p_i(U) \subset X_i$, $\forall\ i = 1, 2, \cdots, n$. Let $f_i \in F_{X_i}$, $\forall\ i = 1, 2, \cdots, n$, then $f_i \circ p_i \in F_{\mathfrak{X}}$, $\forall\ i = 1, 2 \cdots, n$, since $p_i : \mathfrak{X} \longrightarrow X_i$ is Frölicher smooth $\forall\ i = 1, 2, \cdots, n$. Therefore $\forall\ i = 1, 2, \cdots, n$ we have that $(f_i \circ p_i)(U) \subset \mathbb{R}$ is bounded $\forall\ f_i \in F_{X_i}$ since $U \in \beta_{F_{\mathfrak{X}}}$. That is $f_i(p_i(U)) = (f_i \circ p_i)(U) \subset \mathbb{R}$ is bounded $\forall\ f_i \in F_{X_i}$, $\forall\ i = 1, 2, \cdots, n$. Therefore $p_i(U) \in \beta_{F_{X_i}}$, $\forall\ i = 1, 2 \cdots, n$, which implies that $U \in \rho_{\mathfrak{X}}$. Thus $U \in \beta_{F_{\mathfrak{X}}} \implies U \in \rho_{\mathfrak{X}}$, hence $\beta_{F_{\mathfrak{X}}} \subset \rho_{\mathfrak{X}}$. It follows that the identity map $id : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (\mathfrak{X}, \rho_{\mathfrak{X}})$ is bounded by definition since $\beta_{F_{\mathfrak{X}}} \subset \rho_{\mathfrak{X}}$. □

Lemma 3.4.2 *Let $\beta_{F_{\mathfrak{X}}}$ be the functional bornology on $\mathfrak{X}$, $\rho_{\mathfrak{X}}$ be the functional product bornology, and $(\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ and $(\mathfrak{X}, \rho_{\mathfrak{X}})$ be bornological sets. The identity maps $id_1 : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ and $id_2 : (\mathfrak{X}, \rho_{\mathfrak{X}}) \longrightarrow (\mathfrak{X}, \rho_{\mathfrak{X}})$ are bounded.*

Proof: This is trivial, identity maps between bornological sets are bounded (see [45] and [51]). □

Corollary 3.4.1 *Let $\beta_{F_{\mathfrak{X}}}$ be the functional bornology on $\mathfrak{X}$ and $(\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ be a bornological set. The mapping $\phi : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ is bounded if it is Frölicher smooth.*

Proof: Since $\beta_{F_{\mathfrak{X}}}$ is a functional bornology on $\mathfrak{X}$ then it follows from Lemma 3.1.3 that if the mapping $\phi : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ is Frölicher smooth then it is bounded. □

Corollary 3.4.2 *Let $(\mathfrak{X}_1, C_{\mathfrak{X}_1}, F_{\mathfrak{X}_1})$ and $(\mathfrak{X}_2, C_{\mathfrak{X}_2}, F_{\mathfrak{X}_2})$ be Frölicher products, $\beta_{F_{\mathfrak{X}_1}}$ and $\beta_{F_{\mathfrak{X}_2}}$ be the functional bornology on $\mathfrak{X}_1$ and $\mathfrak{X}_2$ respectively, and $(\mathfrak{X}_1, \beta_{F_{\mathfrak{X}_1}})$ and $(\mathfrak{X}_2, \beta_{F_{\mathfrak{X}_2}})$ be bornological sets. If the mapping $j : (\mathfrak{X}_1, \beta_{F_{\mathfrak{X}_1}}) \longrightarrow (\mathfrak{X}_2, \beta_{F_{\mathfrak{X}_2}})$ is Frölicher smooth then it is bounded.*

Proof: Since the Frölicher products $(\mathfrak{X}_1, C_{\mathfrak{X}_1}, F_{\mathfrak{X}_1})$ and $(\mathfrak{X}_2, C_{\mathfrak{X}_2}, F_{\mathfrak{X}_2})$ are Frölicher spaces then by Lemma 3.1.4 we have that the mapping $j : (\mathfrak{X}_1, \beta_{F_{\mathfrak{X}_1}}) \longrightarrow (\mathfrak{X}_2, \beta_{F_{\mathfrak{X}_2}})$ is bounded if it is Frölicher smooth. □

Lemma 3.4.3 *Let $\rho_{\mathfrak{X}}$ be the functional product bornology, $\beta_{F_{\mathfrak{X}}}$ be the functional bornology on $\mathfrak{X}$, and $(\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ and $(\mathfrak{X}, \rho_{\mathfrak{X}})$ be bornological sets. If $\forall\ i = 1, 2, \cdots, n$ we have that $\forall\ g \in F_{\mathfrak{X}}\ \exists\ f_i \in X_i$ such that $g = f_i \circ p_i$ then $\rho_{\mathfrak{X}} \subset \beta_{F_{\mathfrak{X}}}$, thus the identity map $id : (\mathfrak{X}, \rho_{\mathfrak{X}}) \longrightarrow (\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ is bounded.*

Proof: Let $U \in \rho_{\mathfrak{X}}$, then $U \subseteq \mathfrak{X}$ and $p_i(U) \in \beta_{F_{X_i}},\ \forall\ i = 1, 2, \cdots, n$. But $p_i(U) \in \beta_{F_{X_i}}\ \forall\ i = 1, 2, \cdots, n$, implies that $p_i(U) \subset X_i$ and $(f_i \circ p_i)(U) = f_i(p_i(U)) \subset \mathbb{R}$ is bounded $\forall\ f_i \in F_{X_i}$, $\forall\ i = 1, 2, \cdots, n$. We have that $\forall\ i = 1, 2, \cdots, n$, $f_i \circ p_i \in F_{\mathfrak{X}}$ since the Frölicher product $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is an initial structure and (X_i, C_{X_i}, F_{X_i}) is a Frölicher space. That is $(f_i \circ p_i)(U) = f_i(p_i(U)) \subset \mathbb{R}$ is bounded and $f_i \circ p_i \in F_{\mathfrak{X}}$, $\forall\ i = 1, 2, \cdots, n$. Since $\forall\ g \in F_{\mathfrak{X}}\ \exists\ f_i \in F_{X_i}$ such that $g = f_i \circ p_i$, $\forall\ i = 1, 2, \cdots, n$, therefore $g(U) \subset \mathbb{R}$ is bounded $\forall\ g \in F_{\mathfrak{X}}$. Thus $U \in \beta_{F_{\mathfrak{X}}}$. That is, $U \in \rho_{\mathfrak{X}} \implies U \in \beta_{F_{\mathfrak{X}}}$, hence $\rho_{\mathfrak{X}} \subset \beta_{F_{\mathfrak{X}}}$. It follows from definition that the identity map $id : (\mathfrak{X}, \rho_{\mathfrak{X}}) \longrightarrow (\mathfrak{X}, \beta_{F_{\mathfrak{X}}})$ is bounded since $\rho_{\mathfrak{X}} \subset \beta_{F_{\mathfrak{X}}}$. □

Lemma 3.4.4 *Let $\beta_{F_{X_i}}$ be the functional bornology on X_i, $\rho_{\mathfrak{X}}$ be the functional product bornology, and $(\mathfrak{X}, \rho_{\mathfrak{X}})$ and $(X_i, \beta_{F_{X_i}})$ be bornological sets. Then $p_i : (\mathfrak{X}, \rho_{\mathfrak{X}}) \longrightarrow (X_i, \beta_{F_{X_i}})$ is bounded, $\forall\ i = 1, 2, \cdots, n$.*

Proof: Let $U \in \rho_{\mathfrak{X}}$ then $U \subseteq \mathfrak{X}$ and $p_i(U) \in \beta_{F_{X_i}},\ \forall\ i = 1, 2, \cdots, n$. Therefore by definition of $\rho_{\mathfrak{X}}$ we have that $p_i : (\mathfrak{X}, \rho_{\mathfrak{X}}) \longrightarrow (X_i, \beta_{F_{X_i}})$ is bounded, $\forall\ i = 1, 2, \cdots, n$. □

Lemma 3.4.5 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces, $\mathfrak{X} = \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets, $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product, then $p_i : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (X_i, \beta_{F_{X_i}})$ is bounded $\forall\ i = 1, 2, \ldots, n$.*

Proof: With $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ and $(X_i, C_{X_i}, F_{X_i})_{i=1}^{n}$ being Frölicher spaces then $p_i : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (X_i, \beta_{F_{X_i}})$ is a morphism in the category of Frölicher spaces since p_i is Frölicher smooth. Thus $\forall\ i = 1, 2, \cdots, n$ we have that $\{f_i \circ p_i \mid f_i \in F_{X_i}\} \subseteq F_{\mathfrak{X}}$ and $\{p_i \circ c \mid c \in C_{\mathfrak{X}}\} \subseteq C_{X_i}$. Let $U \in \beta_{F_{\mathfrak{X}}}$, then $U \subseteq \mathfrak{X}$ and $f(U) \subset \mathbb{R}$ is bounded $\forall\ f \in F_{\mathfrak{X}}$. Since $f_i \circ p_i \in F_{\mathfrak{X}},\ \forall\ f_i \in F_{X_i},\ \forall\ i = 1, 2, \cdots, n$, therefore $(f_i \circ p_i)(U) = f_i(p_i(U)) \subset \mathbb{R}$ is bounded $\forall\ f_i \in F_{X_i},\ \forall\ i = 1, 2, \cdots, n$. Thus $p_i(U) \in \beta_{F_{X_i}},\ \forall\ i = 1, 2, \cdots, n$. That is, $U \in \beta_{F_{\mathfrak{X}}} \implies p_i(U) \in \beta_{F_{X_i}}$, hence $p_i : (\mathfrak{X}, \beta_{F_{\mathfrak{X}}}) \longrightarrow (X_i, \beta_{F_{X_i}})$ is bounded $\forall\ i = 1, 2, \cdots, n$. □

3.5 Final bornology.

Let $\{(X_i, \beta_i) \mid i \in I\}$ be a family of bornological sets, X be a non-empty set and $\{h_i : X_i \longrightarrow X \mid i \in I\}$ be a family of maps. The final bornology on X

is generated by the family $\mathcal{T} := \bigcup_{i \in I} h_i(\beta_i)$ (see [51]). The bornology generated by a set, say the set U, is the intersection of all bornologies containing U, [51]. Therefore the final bornology, which we denote by $\mathcal{B}_F$, is such that $\mathcal{B}_F = \bigcap_{j \in J} \mathcal{P}_j$ where $\{\mathcal{P}_j \mid j \in J\}$ is a family of bornologies on X such that $\mathcal{T} = \bigcup_{i \in I} h_i(\beta_i) \subset \mathcal{P}_j$, $\forall\ j \in J$. That is, $\mathcal{B}_F$ is the smallest bornology containing $\mathcal{T}$. We now use the characterization in [35] to characterize the final bornology on X on the following lemma.

Lemma 3.5.1 *Let*

$$\mathcal{D} = \{h_i(A_i) \mid A_i \in \beta_i, i \in I\} \cup \{\{x\} \mid x \in X\},$$

$J \subset I$ be finite and

$$\mathcal{B}_F := \left\{ \bigcup_{j \in J} H_j \mid H_j \in \mathcal{D}, \forall\ j \in J \right\},$$

that is $\mathcal{B}_F$ is a finite union of elements of $\mathcal{D}$. Then $\mathcal{B}_F$ is a bornology on X.

Proof: Let $x \in X$ then $\{x\} \in \mathcal{D}$ by definition of $\mathcal{D}$, therefore $\{x\} \in \mathcal{B}_F$. That is, $\mathcal{B}_F$ covers X.
Let $U_1, U_2, \cdots, U_n \in \mathcal{B}_F$, that is U_i is a finite union of elements of $\mathcal{D}$, $\forall\ i = 1, 2, \cdots, n$. Therefore, with the finite union of finite sets being finite, we have that $\bigcup_{i=1}^{n} U_i$ is a finite union of elements of $\mathcal{D}$. Hence $\bigcup_{i=1}^{n} U_i \in \mathcal{B}_F$.
Let $A_j \in \beta_j$, $\forall\ j \in J$, then $\bigcup_{j \in J} h_j(A_j) \in \mathcal{B}_F$. But since $\forall\ j \in J$ we have that β_j is closed under inclusion and $A_j \in \beta_j$, $\forall\ j \in J$, then $\forall\ j \in J\ \exists\ C_j \in \beta_j$ such that

$$C_j \subset A_j \implies h_j(C_j) \subset h_j(A_j) \implies \bigcup_{j \in J} h_j(C_j) \subset \bigcup_{j \in J} h_j(A_j)$$

and $\bigcup_{j \in J} h_j(C_j) \in \mathcal{B}_F$. Let $U \subset \bigcup_{j \in J} h_j(A_j)$ then $U = \bigcup_{j \in J} h_j(C_j)$ where $C_j \subset A_j$ or $U = \bigcup_{k \in K} h_k(A_k)$, $K \subset J$. Now let $x_1, x_2, \cdots, x_n \in X$ then $\{x_i\} \in \mathcal{D}$, $\forall\ i = 1, 2, \cdots, n$, therefore $\bigcup_{i=1}^{n} \{x_i\} \in \mathcal{B}_F$. Let $M \subset \bigcup_{i=1}^{n} \{x_i\}$, then $M = \bigcup_{\alpha \in L} \{x_\alpha\}$ where $L = \{1, 2, \cdots, n\}$. Thus $M = \bigcup_{\alpha \in L} \{x_\alpha\}$ is finite and $\{x_\alpha\} \in \mathcal{D}$ for $\alpha \in L$, therefore $M \in \mathcal{B}_F$. Hence if $U \subset V$ and $V \in \mathcal{B}_F$ then $U \in \mathcal{B}_F$. Thus $\mathcal{B}_F$ is closed under inclusion.
That is, we have that $\mathcal{B}_F$ covers X, is closed under finite union and is closed under inclusion, therefore $\mathcal{B}_F$ is a bornology on X. $\square$

Definition 3.5.1 ***Final bornology.***
Let $\{(X_i, \beta_i) \mid i \in I\}$ *be a family of bornological sets,* X *be a non-empty set,*

$$\{h_i : X_i \longrightarrow X \mid i \in I\}$$

be a family of maps. Let $\mathcal{B}_F$ *be the bornology generated by* $\mathcal{T} := \bigcup_{i \in I} h_i(\beta_i)$ *or equivalently* $\mathcal{B}_F$ *be the collection of finite union of elements of*

$$\mathcal{D} := \{h_i(A_i) \mid A_i \in \beta_i,\ i \in I\} \cup \{\{x\} \mid x \in X\}.$$

The bornology $\mathcal{B}_F$ *is called the final bornology on* X*.*

Remark 3.5.1 *From the characterization of the final bornology* $\mathcal{B}_F$ *we have that* $U \subset X$ *is bounded in the final bornology* $\mathcal{B}_F$ *if and only if* U *is a finite union of subsets of*

$$\mathcal{D} := \{h_i(A_i) \mid A_i \in \beta_i,\ i \in I\} \cup \{\{x\} \mid x \in X\}.$$

3.6 Bornological comparison on Frölicher quotient.

3.6.1 Quotient bornology.

Let (X, C_X, F_X) and (Y, C_Y, F_Y) be Frölicher spaces and

$$g : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$$

be a morphism in the category of Frölicher spaces. Consider the quotient set $Q := X/_\sim$ where $\sim$ is an equivalence relation on X defined as: $x_1 \sim x_2$ if and only if $g(x_1) = g(x_2)$, $\forall\ x_1, x_2 \in X$. We call the equivalence relation $\sim$ a kernel equivalence of g (see Definition 2.3.3). Then we have a Frölicher structure (C_Q, F_Q) on Q where C_Q is a set of structure curves into Q and F_Q is a set of functions on Q. Thus C_Q and F_Q determine each other as follows:

$$C_Q := \{c : \mathbb{R} \longrightarrow Q \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall\ f \in F_Q\}$$

$$F_Q := \{f : Q \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C_Q\}.$$

Then we have the Frölicher space (Q, C_Q, F_Q) called the Frölicher quotient (see Subsection 2.3.7).

Given a surjection map $v : X \longrightarrow Y$ where (X, β) is a bornological set then the final bornology on Y is the bornology generated by $v(\beta)$ (see [51]). That is $v(\beta)$ is the base of the final bornology on Y since $v : X \longrightarrow Y$ is onto Y. Now consider the canonical projection $q : X \longrightarrow Q$ with (X, β) being a bornological set, then $q(\beta)$ generates the final bornology on Q. The bornological set made of Q endowed with the final bornology generated by $q(\beta)$ is referred to as the bornological quotient of (X, β) (see [51]).

Lemma 3.6.1 *Let Q be a quotient set on X and β be a bornology on X. The image of the bornology β under $q : X \longrightarrow Q$, that is $q(\beta) = \{q(B) \mid B \in \beta\}$, is a bornology on Q.*

Proof: Let $U_1, U_2, \cdots, U_n \in q(\beta)$, that is, $U_i = q(B_i)$ where $B_i \in \beta$, $\forall\ i = 1, 2, \cdots, n$. Then $\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} q(B_i) = q\left(\bigcup_{i=1}^{n} B_i\right) \in q(\beta)$ since $\bigcup_{i=1}^{n} B_i \in \beta$ as β is closed under finite union as it is a bornology. Thus $q(\beta)$ is closed under finite union.
Let $B \in \beta$ then $q(B) \in q(\beta)$. Since β is closed under inclusion and $B \in \beta$ then there exists $A \in \beta$ such that $A \subset B \implies q(A) \subset q(B)$, and $q(A) \in q(\beta)$. Let $U \subset q(B)$ then $q^{-1}(U) \subset B \in \beta$ which implies that $q^{-1}(U) \in \beta$ since β is closed under inclusion. Thus there exists $A \in \beta$ such that $U = q(A)$, therefore $U \in q(\beta)$. Hence if $U \subset V$ and $V \in q(\beta)$ then $U \in q(\beta)$. That is $q(\beta)$ is closed under inclusion.
Let $\{[x]\} \subset Q$ be a singleton $\forall\ x \in X$, where $[x]$ is the equivalence class of x. We have that $\{x\} \in \beta$, $\forall\ x \in X$, since β covers X as it is a bornology on X. And $q(\{x\}) = \{q(x)\} = \{[x]\}$, $\forall\ x \in X$, therefore $\{[x]\} \in q(\beta)$, $\forall\ [x] \in Q$. Thus $q(\beta)$ covers X.
That is we have that $q(\beta)$ is closed under finite union, is closed under inclusion and covers X, hence $q(\beta)$ is a bornology on Q. □

Definition 3.6.1 ***Quotient bornology.***
Let (X, β) be a bornological set, $g : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$ be a morphism in the category of Frölicher spaces, $\sim$ be the kernel equivalence of g, $Q := X/_\sim$ be a quotient set and $q : X \longrightarrow Q$ be a canonical projection. The bornology $q(\beta)$ on Q is called the quotient bornology.

Remark 3.6.1 *Let (X, C_X, F_X) be Frölicher space and (Q, C_Q, F_Q) be a Frölicher quotient and $q : X \longrightarrow Q$ be a canonical projection. Since (X, C_X, F_X) is a Frölicher space then we have the functional bornology β_{F_X} on X. Since β_{F_X} is a bornology on X then the image of β_{F_X} under q, that is $q(\beta_{F_X})$, is a quotient bornology. This follows from the definition of a quotient bornology, thus $q(\beta_{F_X})$ is the special case of the quotient bornology in Definition 3.6.1. We'll use the quotient bornology $q(\beta_{F_X})$ in the bornological comparison on Frölicher quotient.*

Definition 3.6.2 ***Functional quotient bornology.***
Let (X, C_X, F_X) be a Frölicher space, β_{F_X} be a functional bornology on X, (Q, C_Q, F_Q) be a Frölicher quotient and $q : X \longrightarrow Q$ be a canonical projection. The bornology $q(\beta_{F_X})$ is called the functional quotient bornology.

3.6.2 Bornology from structure functions of Frölicher quotient.

Given the Frölicher quotient (Q, C_Q, F_Q) we have canonical bornology induced from the structure functions of Frölicher quotient. That is the functional bornology on Q, denoted β_{F_Q}, where

$$\beta_{F_Q} := \{U \subset Q \mid f(U) \subset \mathbb{R} \text{ is bounded}, \ \forall\ f \in F_Q\},$$

induced from F_Q. This is so since for every Frölicher space there is a functional bornology from structure functions (see Section 3.1.1), and the Frölicher quotient is a Frölicher space (see Subsection 2.3.7).

3.6.3 Bounded maps and bornological comparison.

Let (X, C_X, F_X) and (Y, C_Y, F_Y) be Frölicher spaces,

$$g : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$$

be a morphism in the category of Frölicher spaces, $\sim$ be a kernel equivalence of g, that is, $x_1 \sim x_2 \iff g(x_1) = g(x_1),\ \forall\ x_1, x_2 \in X$, $Q := X/_\sim$ be a quotient set, (Q, C_Q, F_Q) be the Frölicher quotient and $q : X \longrightarrow Q$ be the canonical projection. We have the quotient bornology $q(\beta)$, where β is a bornology on X. Let β_{F_X} be the functional bornology on X then we have the functional quotient bornology $q(\beta_{F_X})$. We also have the functional bornology β_{F_Q} on Q. In this subsection we determine the relationship between the aforementioned bornologies and their associated bounded maps.

Theorem 3.6.1 *Let $q(\beta_{F_X})$ be the functional quotient bornology, β_{F_Q} be the functional bornology on Q, and $(Q, q(\beta_{F_X}))$ and (Q, β_{F_Q}) be bornological sets. Then $q(\beta_{F_X}) \subset \beta_{F_Q}$ and thus the identity map $id : (Q, q(\beta_{F_X})) \longrightarrow (Q, \beta_{F_Q})$ is bounded.*

Proof: Let $A \in q(\beta_{F_X})$, then $A = q(U) \subset Q$ for some $U \in \beta_{F_X}$. Therefore $U \subseteq X$ and $f(U) \subset \mathbb{R}$ is bounded $\forall\ f \in F_X$ since $U \in \beta_{F_X}$. Let $h \in F_Q$, then we have that $h(A) = h(q(U)) = (h \circ q)(U)$. But $h \circ q \in F_X,\ \forall\ h \in F_Q$, since $q : X \longrightarrow Q$ is Frölicher smooth, (see Subsection 2.3.7). Therefore
$(h \circ q)(U) = h(q(U)) = h(A) \subset \mathbb{R}$ is bounded $\forall\ h \in F_Q$ since $U \in \beta_{F_X}$. Therefore $A = q(U) \in \beta_{F_Q}$. That is, $A \in q(\beta_{F_X}) \implies A \in \beta_{F_Q}$, hence $q(\beta_{F_X}) \subset \beta_{F_Q}$. Then we have that the identity map $id : (Q, q(\beta_{F_X})) \longrightarrow (Q, \beta_{F_Q})$ is bounded by definition. □

Lemma 3.6.2 *Let $q(\beta_{F_X})$ be the functional quotient bornology, β_{F_Q} be the functional bornology on Q, and (Q, β_{F_Q}) and $(Q, q(\beta_{F_X}))$ be bornological sets. If $\forall\ g \in F_X\ \exists\ f \in F_Q$ such that $g = f \circ q$ then $\beta_{F_Q} \subset q(\beta_{F_X})$ and thus the identity map $id : (Q, \beta_{F_Q}) \longrightarrow (Q, q(\beta_{F_X}))$ is bounded.*

Proof: Let $U \in \beta_{F_Q}$, then $U \subset Q$ and $f(U) \subset \mathbb{R}$ is bounded $\forall\, f \in F_Q$. Now since $U \subset Q$ with Q a quotient set of X and $q : X \longrightarrow Q$ is a surjection then there exists $A \subset X$ such that $U = q(A)$. Therefore $f(U) = f(q(A)) = (f \circ q)(A) \subset \mathbb{R}$ is bounded $\forall\, f \in F_Q$ since $q(A) = U \in \beta_{F_Q}$. But $f \circ q \in F_X$, $\forall\, f \in F_Q$, and since $\forall\, g \in F_X\ \exists\, f \in F_Q$ such that $g = f \circ q$ then $(f \circ q)(A) \subset \mathbb{R}$ is bounded $\forall\, f \in F_Q$, which implies that $g(A) \subset \mathbb{R}$ is bounded $\forall\, g \in F_X$. Therefore $A \in \beta_{F_X}$, which implies that $U = q(A) \in q(\beta_{F_X})$. That is, $U \in \beta_{F_Q} \implies U \in q(\beta_{F_X})$, hence $\beta_{F_Q} \subset q(\beta_{F_X})$. By definition it follows that the identity map
$id : (Q, \beta_{F_Q}) \longrightarrow (Q, q(\beta_{F_X}))$ is bounded. □

Lemma 3.6.3 *Let β_{F_X} be a functional bornology on X, $q(\beta_{F_X})$ be the functional quotient bornology, and (X, β_{F_X}) and $(Q, q(\beta_{F_X}))$ be bornological sets. Then $q : (X, \beta_{F_X}) \longrightarrow (Q, q(\beta_{F_X}))$ is bounded.*

Proof: By definition of $q(\beta_{F_X})$ we have that $U \in \beta_{F_X} \implies q(U) \in q(\beta_{F_X})$, therefore $q : (X, \beta_{F_X}) \longrightarrow (Q, q(\beta_{F_X}))$ is bounded. □

Lemma 3.6.4 *Let β_{F_X} be a functional bornology on X, β_{F_Q} be the functional bornology on Q, and (X, β_{F_X}) and (Q, β_{F_Q}) be bornological sets. Then $q : (X, \beta_{F_X}) \longrightarrow (Q, \beta_{F_Q})$ is bounded.*

Proof: Let $U \in \beta_{F_X}$, then $U \subseteq X$ and $f(U) \subset \mathbb{R}$ is bounded $\forall\, f \in F_X$. Let $h \in F_Q$ be arbitrary. Then $h \circ q \in F_X$ since the Frölicher quotient (Q, C_Q, F_Q) is a final structure in the category of Frölicher spaces. Therefore $(h \circ q)(U) \subset \mathbb{R}$ is bounded since $h \circ q \in F_X$ and $U \in \beta_{F_X}$. Thus, since $h \in F_Q$ is arbitrary and $U \in \beta_{F_X}$ then $(h \circ q)(U) = h(q(U)) \subset \mathbb{R}$ is bounded $\forall\, h \in F_Q$, therefore $q(U) \in \beta_{F_Q}$. That is, $U \in \beta_{F_X} \implies q(U) \in \beta_{F_Q}$, hence $q : (X, \beta_{F_X}) \longrightarrow (Q, \beta_{F_Q})$ is bounded. □

Lemma 3.6.5 *Let β_{F_Q} be the functional bornology on Q, $q(\beta_{F_X})$ be the functional quotient bornology, and (Q, β_{F_Q}) and $(Q, q(\beta_{F_X}))$ be bornological sets. The identity maps $id_1 : (Q, \beta_{F_Q}) \longrightarrow (Q, \beta_{F_Q})$ and $id_2 : (Q, q(\beta_{F_X})) \longrightarrow (Q, q(\beta_{F_X}))$ are bounded.*

Proof: This is trivial, identity maps between bornological sets are bounded (see [45] and [51]). □

Corollary 3.6.1 *Let β_{F_Q} be the functional bornology on Q and (Q, β_{F_Q}) be a bornological set. The mapping $\phi : (Q, \beta_{F_Q}) \longrightarrow (Q, \beta_{F_Q})$ is bounded if it is Frölicher smooth.*

Proof: Since β_{F_Q} is a functional bornology on Q then it follows from Lemma 3.1.3 that if the mapping $\phi : (Q, \beta_{F_Q}) \longrightarrow (Q, \beta_{F_Q})$ is Frölicher smooth then it is bounded. □

Corollary 3.6.2 *Let* (Q_1, C_{Q_1}, F_{Q_1}) *and* (Q_2, C_{Q_2}, F_{Q_2}) *be Frölicher quotients,* $\beta_{F_{Q_1}}$ *and* $\beta_{F_{Q_2}}$ *be the functional bornology on* Q_1 *and* Q_2 *respectively, and* $(Q_1, \beta_{F_{Q_1}})$ *and* $(Q_2, \beta_{F_{Q_2}})$ *be bornological sets. If the mapping* $j : (Q_1, \beta_{F_{Q_1}}) \longrightarrow (Q_2, \beta_{F_{Q_2}})$ *is Frölicher smooth then it is bounded.*

Proof: Since the Frölicher quotients (Q_1, C_{Q_1}, F_{Q_1}) and (Q_2, C_{Q_2}, F_{Q_2}) are Frölicher spaces then by Lemma 3.1.4 we have that the mapping $j : (Q_1, \beta_{F_{Q_1}}) \longrightarrow (Q_2, \beta_{F_{Q_2}})$ is bounded if it is Frölicher smooth. □

3.7 Bornological comparison on Frölicher coproduct.

3.7.1 Coproduct bornology.

Let $\{(X_j, C_{X_j}, F_{X_j}) \mid j \in J\}$ be a family of Frölicher spaces and $\mathcal{N} := \coprod_{j\in J} X_j$ be the coproduct of the sets $\{X_j \mid j \in J\}$ in the category of sets and $\varphi_j : X_j \longrightarrow \mathcal{N}$ be a canonical projection, $j \in J$. Then we have a Frölicher structure $(C_{\mathcal{N}}, F_{\mathcal{N}})$ on $\mathcal{N}$ generated by

$$\{\varphi_j \circ c \mid c \in C_{X_j},\ j \in J\}$$

where

$$C_{\mathcal{N}} = \{c : \mathbb{R} \longrightarrow \mathcal{N} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\, f \in F_{\mathcal{N}}\}$$

and

$$F_{\mathcal{N}} = \{f : \mathcal{N} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\, c \in C_{\mathcal{N}}\}.$$

The triple $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ is a Frölicher space called the Frölicher coproduct, (see Subsection 2.3.6).

We construct the notion of coproduct bornology from the concept of coproduct topology. Given the canonical projection $\varphi_j : X_j \longrightarrow \mathcal{N}$ we have that the coproduct topology is the finest topology on $\mathcal{N}$ for which $\varphi_j : X_j \longrightarrow \mathcal{N}$ is continuous, $j \in J$. That is, $U \subset \mathcal{N}$ is open if and only if $\varphi_j^{-1}(U) \subset X_j$ is open, $j \in J$ (see [3] and [70]). From this concept we construct the coproduct bornology.

Lemma 3.7.1 *Let* $\{(X_j, \beta_j) \mid j \in J\}$ *be a family of bornological sets,* $\mathcal{N} := \coprod_{j\in J} X_j$ *and* $\varphi_j : X_j \longrightarrow \mathcal{N}$ *be a canonical projection,* $j \in J$*. The collection*

$$\beta_{\coprod} := \{U \subset \mathcal{N} \mid \varphi_j^{-1}(U) \in \beta_j,\ j \in J\}$$

is a bornology on $\mathcal{N}$*.*

Proof: Let $U_1, U_2, \cdots, U_n \in \beta_{\coprod}$, then $\forall\ i = 1, 2, \cdots, n$ we have that $U_i \subset \mathcal{N}$ and $\varphi_j^{-1}(U_i) \in \beta_j$, $j \in J$. Since β_j is a bornology on X_j then it is closed under finite union, therefore $\bigcup_{i=1}^{n} \varphi_j^{-1}(U_i) = \varphi_j^{-1}\left(\bigcup_{i=1}^{n} U_i\right) \in \beta_j$, $j \in J$. Which implies that $\bigcup_{i=1}^{n} U_i \in \beta_{\coprod}$. That is $U_i \in \beta_{\coprod}$, $\forall\ i = 1, 2, \ldots, n \implies \bigcup_{i=1}^{n} U_i \in \beta_{\coprod}$, therefore $\beta_{\coprod}$ is closed under finite union.
Let $A \subseteq B$ and $B \in \beta_{\coprod}$. Then we have that $B \subset \mathcal{N}$ and $\varphi_j^{-1}(B) \in \beta_j$, $j \in J$, since $B \in \beta_{\coprod}$. And $A \subseteq B \subset \mathcal{N} \implies \varphi_j^{-1}(A) \subseteq \varphi_j^{-1}(B) \in \beta_j$, $j \in J$. Therefore $\varphi_j^{-1}(A) \in \beta_j$ since β_j is closed under inclusion as it is a bornology, $j \in J$. Thus $A \in \beta_{\coprod}$ since $A \subset \mathcal{N}$ and $\varphi_j^{-1}(A) \in \beta_j$, $j \in J$. That is, $A \subseteq B$ and $B \in \beta_{\coprod}$ implies that $A \in \beta_{\coprod}$, therefore $\beta_{\coprod}$ is closed under inclusion.
Let $j \in J$ be fixed and $x \in X_j$ also be fixed. Consider the singleton $\{(x, j)\} \subset \mathcal{N}$. By definition of $\varphi_j : X_j \longrightarrow \mathcal{N}$ we have that $\varphi_j^{-1}(\{(x, j)\}) = \{x\}$. The singleton $\{x\} \subset X$ is such that $\{x\} \in \beta_j$ since β_j covers X_j as β_j is a bornology on X_j. That is $\{x\} = \varphi_j^{-1}(\{(x, j)\}) \in \beta_j$, therefore $\{(x, j)\} \in \beta_{\coprod}$. Thus $\beta_{\coprod}$ covers $\mathcal{N}$.
Hence $\beta_{\coprod}$ is a bornology on $\mathcal{N}$ since it covers $\mathcal{N}$, is closed under inclusion and is closed under finite union. □

Definition 3.7.1 ***Coproduct bornology.***
Let $\{(X_j, C_{X_j}, F_{X_j}) \mid j \in J\}$ be a family of Frölicher spaces, $\mathcal{N} := \coprod_{j \in J} X_j$ be a coproduct in the category of sets, $\{(X_j, \beta_j) \mid j \in J\}$ be a family of bornological sets, $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be a Frölicher coproduct and $\varphi_j : X_j \longrightarrow \mathcal{N}$ be a canonical projection. The bornology

$$\beta_{\coprod} := \{U \subset \mathcal{N} \mid \varphi_j^{-1}(U) \in \beta_j,\ j \in J\}$$

is called the coproduct bornology.

Remark 3.7.1 *Let $\{(X_j, C_{X_j}, F_{X_j}) \mid j \in J\}$ be a family of Frölicher spaces, $\mathcal{N} := \coprod_{j \in J} X_j$ be a coproduct in the category of sets, $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be a Frölicher coproduct. The collection*

$$\mathcal{B}_{\mathcal{N}} := \{U \subset \mathcal{N} \mid \varphi_j^{-1}(U) \in \beta_{F_{X_j}},\ j \in J\}$$

is a coproduct bornology since the functional bornology $\beta_{F_{X_j}}$ on X_j is a bornology on X_j, $j \in J$. For the purpose of bornological comparison on Frölicher coproduct we will use the coproduct bornology $\mathcal{B}_{\mathcal{N}}$.

Definition 3.7.2 ***Functional coproduct bornology.***
Let $\{(X_j, C_{X_j}, F_{X_j}) \mid j \in J\}$ be a family of Frölicher spaces, $\mathcal{N} := \coprod_{j \in J} X_j$ be

a coproduct in the category of sets, $\varphi_j : X_j \longrightarrow \mathcal{N}$ be a canonical projection, $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct and $\beta_{F_{X_j}}$ be a functional bornology on X_j, $j \in J$. The coproduct bornology

$$\mathcal{B}_{\mathcal{N}} := \{U \subset \mathcal{N} \mid \varphi_j^{-1}(U) \in \beta_{F_{X_j}},\ j \in J\}$$

is called the functional coproduct bornology.

3.7.2 Bornology from structure functions of Frölicher coproduct.

Consider the Frölicher coproduct $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$. Since the Frölicher coproduct is a Frölicher space (see Subsection 2.3.6) then we have a bornology induced from the structure functions of Frölicher coproduct (See Section 3.1.1). That is, the functional bornology

$$\beta_{F_{\mathcal{N}}} := \{U \subset \mathcal{N} \mid f(U) \subset \mathbb{R} \textit{ is bounded},\ \forall\ f \in F_{\mathcal{N}}\}$$

on $\mathcal{N}$.

3.7.3 Bounded maps and bornological comparison.

Let $\{(X_j, C_{X_j}, F_{X_j}) \mid j \in J\}$ a family of Frölicher spaces, $\mathcal{N} := \coprod_{j\in J} X_j$ a coproduct in the category of sets, $\varphi : X_j \longrightarrow \mathcal{N}$ be the canonical projection and $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct. From the Frölicher coproduct we have induced two bornologies. That is the functional bornology, denoted $\beta_{F_{\mathcal{N}}}$, on $\mathcal{N}$, and the functional coproduct bornology, denoted $\mathcal{B}_{\mathcal{N}}$. We also have a bornology from the structure functions on X_j, that is the functional bornology, denoted $\beta_{F_{X_j}}$, on X_j, $j \in J$. In this subsection we determine the relationship between the aforementioned bornologies and their associated bounded maps and bounded sets.

Lemma 3.7.2 *Let $\beta_{F_{\mathcal{N}}}$ be the functional bornology on $\mathcal{N}$, $\mathcal{B}_{\mathcal{N}}$ be the functional coproduct bornology, and $(\mathcal{N}, \beta_{F_{\mathcal{N}}})$ and $(\mathcal{N}, \mathcal{B}_{\mathcal{N}})$ be bornological sets. The identity maps $id_1 : (\mathcal{N}, \beta_{F_{\mathcal{N}}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ and $id_2 : (\mathcal{N}, \mathcal{B}_{\mathcal{N}}) \longrightarrow (\mathcal{N}, \mathcal{B}_{\mathcal{N}})$ are bounded.*

Proof: This is trivial, identity maps between bornological sets are bounded (see [45] and [51]). □

Corollary 3.7.1 *Let $\beta_{F_{\mathcal{N}}}$ be the functional bornology on $\mathcal{N}$ and $(\mathcal{N}, \beta_{F_{\mathcal{N}}})$ be a bornological set. The mapping $\phi : (\mathcal{N}, \beta_{F_{\mathcal{N}}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ is bounded if it is Frölicher smooth.*

Proof: Since $\beta_{F_{\mathcal{N}}}$ is a functional bornology on $\mathcal{N}$ then it follows from Lemma 3.1.3 that if the mapping $\phi : (\mathcal{N}, \beta_{F_{\mathcal{N}}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ is Frölicher smooth then it is bounded. $\square$

Corollary 3.7.2 *Let $(\mathcal{N}_1, C_{\mathcal{N}_1}, F_{\mathcal{N}_1})$ and $(\mathcal{N}_2, C_{\mathcal{N}_2}, F_{\mathcal{N}_2})$ be Frölicher coproducts, $\beta_{F_{\mathcal{N}_1}}$ and $\beta_{F_{\mathcal{N}_2}}$ be the functional bornology on $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, and $(\mathcal{N}_1, \beta_{F_{\mathcal{N}_1}})$ and $(\mathcal{N}_2, \beta_{F_{\mathcal{N}_2}})$ be bornological sets. If the mapping $j : (\mathcal{N}_1, \beta_{F_{\mathcal{N}_1}}) \longrightarrow (\mathcal{N}_2, \beta_{F_{\mathcal{N}_2}})$ is Frölicher smooth then it is bounded.*

Proof: Since the Frölicher coproducts $(\mathcal{N}_1, C_{\mathcal{N}_1}, F_{\mathcal{N}_1})$ and $(\mathcal{N}_2, C_{\mathcal{N}_2}, F_{\mathcal{N}_2})$ are Frölicher spaces then by Lemma 3.1.4 we have that the mapping $j : (\mathcal{N}_1, \beta_{F_{\mathcal{N}_1}}) \longrightarrow (\mathcal{N}_2, \beta_{F_{\mathcal{N}_2}})$ is bounded if it is Frölicher smooth. $\square$

Theorem 3.7.1 *Let $\mathcal{B}_{\mathcal{N}}$ be the functional coproduct bornology, $\beta_{F_{\mathcal{N}}}$ be the functional bornology on $\mathcal{N}$, and $(\mathcal{N}, \mathcal{B}_{\mathcal{N}})$ and $(\mathcal{N}, \beta_{F_{\mathcal{N}}})$ be bornological sets. Then $\beta_{\mathcal{N}} \subset \beta_{F_{\mathcal{N}}}$ and thus the identity map $id : (\mathcal{N}, \mathcal{B}_{\mathcal{N}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ is bounded.*

Proof: Let $U \in \mathcal{B}_{\mathcal{N}}$, then $U \subset \mathcal{N}$ and $\varphi_j^{-1}(U) \in \beta_{F_{X_j}}$, $j \in J$. And $\varphi_j^{-1}(U) \in \beta_{F_{X_j}} \implies U \in \varphi_j(\beta_{F_{X_j}})$, $j \in J$. With $U \in \varphi_j(\beta_{F_{X_j}})$, this implies that for some $j \in J \ \exists \ A \in \beta_{F_{X_j}}$ such that $U = \varphi_j(A)$. Since $A \in \beta_{F_{X_j}}$ then $f_j(A) \subset \mathbb{R}$ is bounded $\forall \ f_j \in F_{X_j}$. But $\forall \ g \in F_{\mathcal{N}}$, $j \in J$, we have that $g \circ \varphi_j \in F_{X_j}$ since $\varphi_j : X_j \longrightarrow \mathcal{N}$ is Frölicher smooth. Therefore $(g \circ \varphi_j)(A) = g(\varphi_j(A)) = g(U) \subset \mathbb{R}$ is bounded $\forall \ g \in F_{\mathcal{N}}$ since $A \in \beta_{F_{X_j}}$. This gives that $U = \varphi_j(A) \in \beta_{F_{\mathcal{N}}}$. That is, $U \in \mathcal{B}_{\mathcal{N}} \implies U \in \beta_{F_{\mathcal{N}}}$, hence $\mathcal{B}_{\mathcal{N}} \subset \beta_{F_{\mathcal{N}}}$. Therefore it follows from definition that the identity map $id : (\mathcal{N}, \mathcal{B}_{\mathcal{N}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ is bounded. $\square$

Lemma 3.7.3 *Let $\beta_{F_{X_j}}$ be the functional bornology on X_j, $\beta_{F_{\mathcal{N}}}$ be the functional bornology on $\mathcal{N}$, and $(X_j, \beta_{F_{X_j}})$ and $(\mathcal{N}, \beta_{F_{\mathcal{N}}})$ be bornological sets. Then $\varphi_j : (X_j, \beta_{F_{X_j}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ is bounded, $j \in J$.*

Proof: Let $U \in \beta_{F_{X_j}}$, then $U \subset X_j$ and $f(U) \subset \mathbb{R}$ is bounded $\forall \ f \in F_{X_j}$. Since $\varphi_j : (X_j, \beta_{F_{X_j}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ is Frölicher smooth then we have that $g \circ \varphi_j \in F_{X_j}$, $\forall \ g \in F_{\mathcal{N}}$. Then $(g \circ \varphi_j)(U) = g(\varphi_j(U)) \subset \mathbb{R}$ is bounded $\forall \ g \in F_{\mathcal{N}}$ since $U \in \beta_{F_{X_j}}$. Therefore $\varphi_j(U) \in \beta_{F_{\mathcal{N}}}$. Thus $U \in \beta_{F_{X_j}} \implies \varphi_j(U) \in \beta_{F_{\mathcal{N}}}$, $j \in J$. Hence $\varphi_j : (X_j, \beta_{F_{X_j}}) \longrightarrow (\mathcal{N}, \beta_{F_{\mathcal{N}}})$ is bounded, $j \in J$. $\square$

Chapter 4

Relatively-compact bornologies.

4.1 Relatively-compact bornology from Frölicher topology.

The previous chapter entails bornologies induced canonically from the structure functions, and the initial and final bornologies. We now look at the bornology induced from the Frölicher topology. A Frölicher space (X, C_X, F_X) is a topological space - a bitopological space since we have two topologies, one topology induced from structure functions, that is

$$\tau_{F_X} := \{U \subseteq X \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_X\},$$

and the other topology induced from structure curves, that is

$$\tau_{C_X} := \{U \subseteq X \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_X\},$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$ (see [8], [10], [13], [14], [45], [86], [88] and [89]). The topology τ_{F_X} is coarser than the topology τ_{C_X}. If $\tau_{F_X} = \tau_{C_X}$ then the Frölicher space (X, C_X, F_X) is said to be a balanced space (see [27]). The topology τ_{F_X} is the coarsest topology in which all smooth maps are continuous. Due to that, the topology τ_{F_X} is considered as the Frölicher topology (see [14]). The basis and subbasis of τ_{F_X} are given by the collections $\{f^{-1}(0, +\infty) \mid f \in F_X\}$ and $\{f^{-1}(0, 1) \mid f \in F_X\}$, respectively (see [36] and [45]). If $\forall\ x_1, x_2 \in X$ such that $x_1 \neq x_2$ we have that there exists $f \in F_X$ such that $f(x_1) \neq f(x_2)$ then τ_{F_X} is Hausdorff (see [89]).

This section focuses on relatively-compact bornology (also called compact bornology, see [51]) induced from the Frölicher topology of a general Frölicher space. The relatively-compact subsets of a separated (Hausdorff) topological space from a bornology, and the family of compact subsets is the base of such a bornology (see [45] and [51]).

Definition 4.1.1 τ_{F_X}***-compact set.***
We say that a set is τ_{F_X}-compact if it is compact (see Definition 2.2.6) with respect to the (open sets of the) topology τ_{F_X}. That is, a set $U \subseteq X$ is said to be τ_{F_X}-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_{F_X}$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_i} \mid i = 1, 2, \cdots, n\}$, where $O_{\alpha_i} \in \tau_{F_X}$, $\forall\ i = 1, 2, \cdots, n$.

Lemma 4.1.1 *Let (X, C_X, F_X) be a Frölicher space. The collection*

$$\mathcal{Q}_{\tau_{F_X}} := \{V \subseteq X \mid V \textit{ is } \tau_{F_X} - \textit{compact}\}$$

is a base of a bornology on X.

Proof: We have to show that $\mathcal{Q}_{\tau_{F_X}}$ covers X and that every finite union of τ_{F_X}-compact subsets of X is contained in a τ_{F_X}-compact subset of X. Consider the singleton $\{x\} \subset X$. Since $f \in F_X$ is such that $f : X \longrightarrow \mathbb{R}$ is a function then there exists $\{r\} \subset \mathbb{R}$ such that $\{x\} = f^{-1}(\{r\})$. There exists $W \in \tau_{\mathbb{R}}$ such that $\{r\} \subset W$. That is, there exists $W \in \tau_{\mathbb{R}}$ such that $\{x\} = f^{-1}(\{r\}) \subset f^{-1}(W)$, thus $\{x\} \subset f^{-1}(W) \in \tau_{F_X}$. This suffices that $\{x\}$ is τ_{F_X}-compact, $\forall\ x \in X$. That is, $\mathcal{Q}_{\tau_{F_X}}$ covers X.
Let $V_1, V_2, \cdots, V_n \in \mathcal{Q}_{\tau_{F_X}}$. That is $V_i \subseteq X$ is τ_{F_X}-compact $\forall\ i = 1, 2, \cdots, n$. The finite union of compact sets is compact (see Theorem 2.2.1) therefore $\bigcup_{i=1}^{n} V_i \subseteq X$ is τ_{F_X}-compact. That is, $\mathcal{Q}_{\tau_{F_X}}$ is closed under finite union. Since $\mathcal{Q}_{\tau_{F_X}}$ covers X then there exists $K \in \mathcal{Q}_{\tau_{F_X}}$ such that $\bigcup_{i=1}^{n} V_i \subset \bigcup_{i=1}^{n} (V_i \cup K) \in \mathcal{Q}_{\tau_{F_X}}$. That is, every finite union of τ_{F_X}-compact subsets of X is contained in a τ_{F_X}-compact subset of X. □

Definition 4.1.2 ***Relatively*** τ_{F_X}***-compact set.***
Let $U \subseteq X$, then we say that U is relatively τ_{F_X}-compact if it is relatively compact (see Definition 2.2.10) with respect to (the open sets of) τ_{F_X}.

Lemma 4.1.2 *Let X be a Hausdorff space and (X, C_X, F_X) be a Frölicher space. The collection*

$$\beta_{\tau_{F_X}} := \{U \subseteq X \mid U \textit{ is relatively } \tau_{F_X} - \textit{compact}\}$$

is a bornology on X.

Proof: Let X be a Hausdorff space. We have to show that $\beta_{\tau_{F_X}}$ covers X, is closed under finite union and is closed under inclusion.
Since

$$\mathcal{Q}_{\tau_{F_X}} := \{V \subseteq X \mid V \textit{ is } \tau_{F_X} - \textit{compact}\}$$

covers X (see Lemma 4.1.1) therefore the singleton $\{x\}$ is τ_{F_X}-compact, $\forall\, x \in X$. But since X is Hausdorff then by Theorem 2.2.7 we have that $\{x\}$ is relatively τ_{F_X}-compact, $\forall\, x \in X$. Therefore $\{x\} \in \beta_{\tau_{F_X}}$, $\forall\, x \in X$, thus $\beta_{\tau_{F_X}}$ covers X.
Let $U_1, U_2, \cdots, U_n \in \beta_{\tau_{F_X}}$, that is, $U_i \subseteq X$ is relatively τ_{F_X}-compact $\forall\, i = 1, 2, \cdots, n$. By definition, $\forall\, i = 1, 2, \cdots, n$, the closure of U_i, that is $Cl(U_i)$, is τ_{F_X}-compact since U_i is relatively τ_{F_X}-compact $\forall\, i = 1, 2, \cdots, n$. Therefore $\bigcup_{i=1}^{n} Cl(U_i) = Cl\left(\bigcup_{i=1}^{n} U_i\right)$ is τ_{F_X}-compact since by Theorem 2.2.1 the finite union of compact sets is compact. With $Cl\left(\bigcup_{i=1}^{n} U_i\right)$ τ_{F_X}-compact then by definition $\bigcup_{i=1}^{n} U_i \subseteq X$ is relatively τ_{F_X}-compact. Therefore $\bigcup_{i=1}^{n} U_i \in \beta_{\tau_{F_X}}$, thus $\beta_{\tau_{F_X}}$ is closed under finite union.
Let $A \subseteq B$ and $B \in \beta_{\tau_{F_X}}$, then $A \subseteq B \subseteq X$ and B is relatively τ_{F_X}-compact. Since B is relatively τ_{F_X}-compact then the closure of B, that is $Cl(B)$, is τ_{F_X}-compact. But $A \subseteq B \implies Cl(A) \subseteq Cl(B)$, giving that every open cover of $Cl(B)$ is the open cover of $Cl(A)$. Therefore $Cl(A)$ is τ_{F_X}-compact which implies that A is relatively τ_{F_X}-compact, which implies that $A \in \beta_{\tau_{F_X}}$. That is, $A \subseteq B \in \beta_{\tau_{F_X}} \implies A \in \beta_{\tau_{F_X}}$, thus $\beta_{\tau_{F_X}}$ is closed under inclusion. □

Definition 4.1.3 ***Functional relatively-compact bornology.***
Let (X, C_X, F_X) be a Hausdorff Frölicher space, the bornology

$$\beta_{\tau_{F_X}} := \{U \subseteq X \mid U \textit{ is relatively } \tau_{F_X} - \textit{compact}\}$$

is called the functional relatively-compact bornology.

Lemma 4.1.3 *Let X be a Hausdorff space and (X, C_X, F_X) be a Frölicher space. The base of the functional relatively-compact bornology*

$$\beta_{\tau_{F_X}} := \{U \subseteq X \mid U \textit{ is relatively } \tau_{F_X} - \textit{compact}\}$$

on X is the collection

$$\mathcal{Q}_{\tau_{F_X}} := \{V \subseteq X \mid V \textit{ is } \tau_{F_X} - \textit{compact}\}.$$

Proof: We have to show that every element of $\beta_{\tau_{F_X}}$ is contained in an element of $\mathcal{Q}_{\tau_{F_X}}$ and that $\mathcal{Q}_{\tau_{F_X}} \subset \beta_{\tau_{F_X}}$. Let $U \in \beta_{\tau_{F_X}}$, that is $U \subseteq X$ is relatively τ_{F_X}-compact. The closure of U, that is $Cl(U)$, is τ_{F_X}-compact since U is relatively τ_{F_X}-compact (see Definition 2.2.10) and by Proposition 2.2.1 we have that $U \subseteq Cl(U)$. That is, $\forall\, U \in \beta_{\tau_{F_X}}$ there exists the closure $Cl(U) \in \mathcal{Q}_{\tau_{F_X}}$ of U, such that $U \subseteq Cl(U)$.
Let $V \in \mathcal{Q}_{\tau_{F_X}}$, that is $V \subseteq X$ is τ_{F_X}-compact. But by Theorem 2.2.6 we have that $V \subseteq X$ is closed since X is Hausdorff, and $V = Cl(V)$ by Theorem 2.2.2. Therefore $Cl(V)$ is τ_{F_X}-compact. This implies that V is relatively τ_{F_X}-compact, thus $V \in \beta_{F_X}$, therefore $\mathcal{Q}_{\tau_{F_X}} \subset \beta_{\tau_{F_X}}$. □

4.2 Relatively-compact bornologies from Frölicher subspace.

Let X be a Hausdorff space, (X, C_X, F_X) be a Frölicher space with $S \subseteq X$ and (S, C_S, F_S) be a Frölicher subspace of (X, C_X, F_X) (see Subsection 2.3.4). From the Frölicher subspace (S, C_S, F_S) we have the topologies induced from the Frölicher structure (C_S, F_S), that is, the functional topology

$$\tau_{F_S} := \{U \subseteq S \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_S\}$$

on S and the curvaceous topology

$$\tau_{C_S} := \{U \subseteq S \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_S\}$$

on S, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$ (see Subsection 2.3.3). The topology τ_{F_S} is coarser than the topology τ_{C_S} (see Lemma 2.3.3). If $\tau_{F_S} = \tau_{C_S}$ then the Frölicher subspace (S, C_S, F_S) is said to be a balanced space. The topology τ_{F_S} is the coarsest topology in which all smooth maps are continuous. Due to that, the topology τ_{F_S} is considered as the Frölicher topology from Frölicher subspace. The basis and subbasis of τ_{F_S} is given by the collections $\{f^{-1}(0, +\infty) \mid f \in F_S\}$ and $\{f^{-1}(0, 1) \mid f \in F_S\}$, respectively (see [36] and [45]). Also, we have the canonical topology induced on S, and that is the subspace topology

$$\tau_{F_X}(S) := \{S \cap U \mid U \in \tau_{F_X}\}$$

on S.

4.2.1 Functional relatively-compact bornology.

Definition 4.2.1 ***τ_{F_S}-compact set.***
We say that a set is τ_{F_S}-compact if it is compact (see Definition 2.2.6) with respect to (the open sets of) the topology τ_{F_S}. That is, a set $U \subseteq S$ is said to be τ_{F_S}-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_{F_S}$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_i} \mid i = 1, 2, \cdots, n\}$, where $O_{\alpha_i} \in \tau_{F_S}$, $\forall\ i = 1, 2, \cdots, n$.

Definition 4.2.2 ***Relatively τ_{F_S}-compact set.***
Let $U \subseteq S$, then we say that U is relatively τ_{F_S}-compact if it is relatively compact (see Definition 2.2.10) with respect to (the open sets of) τ_{F_S}.

Lemma 4.2.1 *Let X be a Hausdorff space, (X, C_X, F_X) be a Frölicher space, $S \subseteq X$ and (S, C_S, F_S) be a Frölicher subspace. The collection*

$$\beta_{\tau_{F_S}} := \{U \subseteq S \mid U \text{ is relatively } \tau_{F_S} - \text{compact}\}$$

is a bornology on S and its base is

$$P_{\tau_{F_S}} := \{U \subseteq S \mid U \text{ is } \tau_{F_S} - \text{compact}\}.$$

Proof: Frölicher subspaces are Frölicher spaces (see Subsection 2.3.4), therefore the Frölicher subspace (S, C_S, F_S) is a Frölicher space. It then follows, by Lemma 4.1.1, that $P_{\tau_{F_S}}$ is a base of a bornology on S. By Theorem 2.2.3, since $S \subseteq X$ and X is Hausdorff therefore S is also Hausdorff, therefore by Lemma 4.1.2 we have that $\beta_{\tau_{F_S}}$ is a bornology on S. Hence $P_{\tau_{F_S}}$ is a base of $\beta_{\tau_{F_S}}$ by Lemma 4.1.3. $\square$

Definition 4.2.3 *Let (S, C_S, F_S) be the Frölicher subspace. The bornology*

$$\beta_{\tau_{F_S}} := \{U \subseteq S \mid U \text{ is relatively } \tau_{F_S} - \text{compact}\}$$

is called the functional relatively-compact bornology on S.

4.2.2 Subspace relatively-compact bornology.

Definition 4.2.4 $\tau_{F_X}(S)$***-compact set.***
We say that a set is $\tau_{F_X}(S)$-compact if it is compact (see Definition 2.2.6) with respect to (the open sets of) the topology $\tau_{F_X}(S)$. That is, a set $U \subseteq S$ is said to be $\tau_{F_X}(S)$-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_{F_X}(S)$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_i} \mid i = 1, 2, \cdots, n\}$, where $O_{\alpha_i} \in \tau_{F_X}(S)$, $\forall\, i = 1, 2, \cdots, n$.

Lemma 4.2.2 *Let (S, C_S, F_S) be a Frölicher subspace. The collection*

$$\mathcal{P}_{\tau_{F_X}}(S) := \{U \subseteq S \mid U \text{ is } \tau_{F_X}(S) - \text{compact}\}$$

is a base of a bornology on S.

Proof: We have to show that $\mathcal{P}_{\tau_{F_X}}(S)$ covers S and that every finite union of $\tau_{F_X}(S)$-compact subsets of S is contained in a $\tau_{F_X}(S)$-compact subset of S. Let $s \in S$, since $S \subseteq X$ then we have that $s \in X$. For $f \in F_X$, we have that $f : X \longrightarrow \mathbb{R}$ is a function, then $f(s) \in \mathbb{R}$. Therefore there exists $V \in \tau_{\mathbb{R}}$ such that $f(s) \in V$. That is $\forall\, \{s\} \subset S \subseteq X\ \exists\, V \in \tau_{\mathbb{R}}$ such that $\{s\} \subset f^{-1}(V)$. Thus there exists $U \in \tau_{F_X}$ such that $\{s\} \subset U$. Therefore there exists $(S \cap U) \in \tau_{F_X}(S)$ such that $\{s\} \subset S \cap U$. This is sufficient that every open cover of the singleton $\{s\} \subset S \subseteq X$ has a finite subcover. That is, every singleton $\{s\} \subset S \subseteq X$ is $\tau_{F_X}(S)$-compact. Therefore $\{s\} \in \mathcal{P}_{\tau_{F_X}}(S)$, that is $\mathcal{P}_{\tau_{F_X}}(S)$ covers S.
Let $U_1, U_2, \cdots, U_n \in \mathcal{P}_{\tau_{F_X}}(S)$, that is $U_i \subseteq X$ is $\tau_{F_X}(S)$-compact $\forall\, i = 1, 2, \cdots, n$. Since the finite union of compact sets is compact (see Theorem 2.2.1) therefore $\bigcup_{i=1}^{n} U_i \subseteq S$ is $\tau_{F_X}(S)$-compact. Since $\mathcal{P}_{\tau_{F_X}}(S)$ covers S therefore there exists $K \in \mathcal{P}_{\tau_{F_X}}(S)$ such that $\bigcup_{i=1}^{n} U_i \subset \bigcup_{i=1}^{n}(K \cup U_i) \in \mathcal{P}_{\tau_{F_X}}(S)$. That is, every finite union of $\tau_{F_X}(S)$-compact subsets of S is contained in a $\tau_{F_X}(S)$-compact subset of S. $\square$

Definition 4.2.5 ***Relatively*** $\tau_{F_X}(S)$***-compact set.***
Let $U \subseteq S$, then we say that U is relatively $\tau_{F_X}(S)$-compact if it is relatively compact (see Definition 2.2.10) with respect to (the open sets of) $\tau_{F_X}(S)$.

Lemma 4.2.3 *Let X be Hausdorff, $S \subseteq X$ and (S, C_S, F_S) be a Frölicher subspace of the Frölicher space (X, C_X, F_X). The collection*

$$\mathcal{B}_{\tau_{F_X}}(S) := \{U \subseteq S \mid U \text{ is relatively } \tau_{F_X}(S) - \text{compact}\}$$

is a bornology on S.

Proof: We have to show that $\mathcal{B}_{\tau_{F_X}}(S)$ covers S, is closed under finite union and is closed under inclusion. Let $\{s\} \subset S$ be arbitrary. Since $\tau_{F_X}(S)$-compact subsets of S form a base on S then the singleton $\{s\}$ is $\tau_{F_X}(S)$-compact. The subset $S \subseteq X$ is a Hausdorff space since X is Hausdorff, by Theorem 2.2.3. Then by Theorem 2.2.7 we have that $\{s\}$ is relatively $\tau_{F_X}(S)$-compact since $\{s\} \subset S$ and S is Hausdorff. That is $\{s\} \subset S$ is relatively $\tau_{F_X}(S)$-compact $\forall\ s \in S$. Therefore $\{s\} \in \mathcal{B}_{\tau_{F_X}}(S)$, $\forall\ s \in S$, thus $\mathcal{B}_{\tau_{F_X}}(S)$ covers S.
Let $U_1, U_2, \cdots, U_n \in \mathcal{B}_{\tau_{F_X}}(S)$, that is $U_i \subseteq S$ is relatively $\tau_{F_X}(S)$-compact $\forall\ i = 1, 2, \cdots, n$. Then by definition $Cl(U_i)$, that is the closure of U_i, is $\tau_{F_X}(S)$-compact $\forall\ i = 1, 2, \cdots, n$. Since the finite union of compact sets is compact, by Theorem 2.2.1, then $\bigcup_{i=1}^{n} Cl(U_i) = Cl\left(\bigcup_{i=1}^{n} U_i\right)$ is $\tau_{F_X}(S)$-compact. Therefore $\bigcup_{i=1}^{n} U_i$ is relatively $\tau_{F_X}(S)$-compact, by definition. That is $\bigcup_{i=1}^{n} U_i \in \mathcal{B}_{\tau_{F_X}}(S)$, thus $\mathcal{B}_{\tau_{F_X}}(S)$ is closed under finite union.
Let $A \subseteq B$ and $B \in \mathcal{B}_{\tau_{F_X}}(S)$, then $A \subseteq B \subseteq S$ and B is relatively $\tau_{F_X}(S)$-compact. That is $Cl(B)$, the closure of B, is $\tau_{F_X}(S)$-compact by definition. But $A \subseteq B \implies Cl(A) \subset Cl(B)$, giving that the open cover of $Cl(B)$ is also an open cover of $Cl(A)$. Therefore $Cl(A)$ is also $\tau_{F_X}(S)$-compact. This implies that A is relatively $\tau_{F_X}(S)$-compact, therefore $A \in \mathcal{B}_{\tau_{F_X}}(S)$. That is, $A \subseteq B \in \mathcal{B}_{\tau_{F_X}}(S)$ $\implies A \in \mathcal{B}_{\tau_{F_X}}(S)$, therefore $\mathcal{B}_{\tau_{F_X}}(S)$ is closed under inclusion. □

Definition 4.2.6 ***Subspace relatively-compact bornology.***
Let X be Hausdorff, (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X), and $\tau_{F_X}(S)$ be the subspace topology on S. Then we call the compact bornology

$$\mathcal{B}_{\tau_{F_X}}(S) := \{U \subseteq S \mid U \text{ is relatively } \tau_{F_X}(S) - \text{compact}\}$$

the subspace relatively-compact bornology.

Lemma 4.2.4 *Let X be a Hausdorff space, $S \subseteq X$ and (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X). The collection*

$$\mathcal{P}_{\tau_{F_X}}(S) = \{U \subseteq S \mid U \text{ is } \tau_{F_X}(S) - \text{compact}\}$$

is the base of the subspace relatively-compact bornology

$$\mathcal{B}_{\tau_{F_X}}(S) = \{U \subseteq S \mid U \textit{ is relatively } \tau_{F_X}(S) - \textit{compact}\}.$$

Proof: We have to show that $\mathcal{P}_{\tau_{F_X}}(S) \subset \mathcal{B}_{\tau_{F_X}}(S)$ and that every element of $\mathcal{B}_{\tau_{F_X}}(S)$ is contained in an element of $\mathcal{P}_{\tau_{F_X}}(S)$. Since a compact subset of a Hausdorff space is relatively compact (see Theorem 2.2.7) therefore $\mathcal{P}_{\tau_{F_X}}(S) \subset \mathcal{B}_{\tau_{F_X}}(S)$. By definition, for $U \in \mathcal{B}_{\tau_{F_X}}(S)$ we have that the closure of U, that is $Cl(U)$, is such that $Cl(U) \in \mathcal{P}_{\tau_{F_X}}(S)$, and by Proposition 2.2.1 $U \subseteq Cl(U)$. Hence the collection $\mathcal{B}_{\tau_{F_X}}(S)$ is a bornology on S and $\mathcal{P}_{\tau_{F_X}}(S)$ is its base. □

4.2.3 Bornological comparison.

Let (X, C_X, F_X) be a Frölicher space, $S \subseteq X$ and (S, C_S, F_S) be a Frölicher subspace. We have the Frölicher topology on S, that is the topology τ_{F_S}, from structure functions. We also have the subspace topology $\tau_{F_X}(S)$ on S. The relationship between $\tau_{F_X}(S)$ and τ_{F_S} is such that $\tau_{F_X}(S) \subset \tau_{F_S}$ (see Lemma 2.3.5). We have the subspace relatively-compact bornology $\mathcal{B}_{\tau_{F_X}}(S)$ induced from the subspace topology $\tau_{F_X}(S)$ and the functional relatively-compact bornology $\beta_{\tau_{F_S}}$ on S induced from the topology τ_{F_S}. In this subsection we compare the bornologies $\mathcal{B}_{\tau_{F_X}}(S)$ and $\beta_{\tau_{F_S}}$.

Theorem 4.2.1 *Let X be Hausdorff, $S \subseteq X$, (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X), $\mathcal{B}_{\tau_{F_X}}(S)$ be the subspace relatively-compact bornology and $\beta_{\tau_{F_S}}$ be the functional relatively-compact bornology on S. Then $\mathcal{B}_{\tau_{F_X}}(S) \subset \beta_{\tau_{F_S}}$.*

Proof: Let $U \in \mathcal{B}_{\tau_{F_X}}(S)$, that is, $U \subseteq S$ such that U is relatively $\tau_{F_X}(S)$-compact. That is the closure of U, denoted $Cl(U)$, is $\tau_{F_X}(S)$-compact. Thus every open cover $\{O_\alpha \mid \alpha \in I\} \subset \tau_{F_X}(S)$ of $Cl(U)$ has a finite subcover. Now since $\tau_{F_X}(S) \subset \tau_{F_S}$ then it follows that $\{O_\alpha \mid \alpha \in I\}$ and its finite subcover are subsets of τ_{F_S}. Therefore $Cl(U)$ is τ_{F_S}-compact, which implies that $U \in \beta_{F_S}$. That is we have that $U \in \mathcal{B}_{\tau_{F_X}}(S) \implies U \in \beta_{\tau_{F_S}}$, hence $\mathcal{B}_{\tau_{F_X}}(S) \subset \beta_{\tau_{F_S}}$. □

Corollary 4.2.1 *Let X be Hausdorff, $S \subseteq X$, (S, C_S, F_S) be the Frölicher subspace of the Frölicher space (X, C_X, F_X), $\mathcal{B}_{\tau_{F_X}}(S)$ be the subspace relatively-compact bornology, $\beta_{\tau_{F_S}}$ be the functional relatively-compact bornology on S, and $(S, \mathcal{B}_{\tau_{F_X}}(S))$ and $(S, \beta_{\tau_{F_S}})$ be bornological sets. The identity map $id : (S, \mathcal{B}_{\tau_{F_X}}(S)) \longrightarrow (S, \beta_{\tau_{F_S}})$ is bounded.*

Proof: By Theorem 4.2.1 we have that $\mathcal{B}_{\tau_{F_X}}(S) \subset \beta_{\tau_{F_S}}$, therefore it follows from definition that the identity maps $id : (S, \mathcal{B}_{\tau_{F_X}}(S)) \longrightarrow (S, \beta_{\tau_{F_S}})$ is bounded. □

4.3 Relatively-compact bornologies from Frölicher product.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces with X_i a Haudorff space, $\forall\ i = 1, 2, \cdots, n$, and let $\mathfrak{X} := \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets. Then we have a Frölicher structure $(C_{\mathfrak{X}}, F_{\mathfrak{X}})$ on $\mathfrak{X}$, and thus $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is a Frölicher space called the Frölicher product (see Subsection 2.3.5). We have topologies induced from the Frölicher structure $(C_{\mathfrak{X}}, F_{\mathfrak{X}})$, that is, the functional topology

$$\tau_{F_{\mathfrak{X}}} := \{U \subseteq \mathfrak{X} \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_{\mathfrak{X}}\}$$

on $\mathfrak{X}$ and the curvaceous topology

$$\tau_{C_{\mathfrak{X}}} := \{U \subseteq \mathfrak{X} \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_{\mathfrak{X}}\}$$

on $\mathfrak{X}$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. The topology $\tau_{F_{\mathfrak{X}}}$ is coarser than the topology $\tau_{C_{\mathfrak{X}}}$ (see Lemma 2.3.3). If $\tau_{F_{\mathfrak{X}}} = \tau_{C_{\mathfrak{X}}}$ then the Frölicher product $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ is said to be a balanced space. The topology $\tau_{F_{\mathfrak{X}}}$ is the coarsest topology in which all smooth maps are continuous. Due to that the topology $\tau_{F_{\mathfrak{X}}}$ is considered as the Frölicher topology from Frölicher product. The basis and subbasis of $\tau_{F_{\mathfrak{X}}}$ is given by the collections $\{f^{-1}(0, +\infty) \mid f \in F_{\mathfrak{X}}\}$ and $\{f^{-1}(0, 1) \mid f \in F_{\mathfrak{X}}\}$, respectively (see [36] and [45]). Also we have the canonical product topology on $\mathfrak{X} := \prod_{i=1}^{n} X_i$, denoted by τ_{π}, generated by the basis

$$\hat{\beta} := \left\{ \bigcap_{i=1}^{n} {p_i}^{-1}(U_i) \mid U_i \in \tau_{F_{X_i}}, \forall\ i = 1, 2, \cdots, n \right\}$$

where $p_i : \mathfrak{X} \longrightarrow X_i$ is a canonical projection, $\forall\ i = 1, 2, \cdots, n$.

4.3.1 Functional relatively-compact bornology.

Definition 4.3.1 $\tau_{F_{\mathfrak{X}}}$***-compact set.***
We say that a set is $\tau_{F_{\mathfrak{X}}}$-compact if it is compact (see Definition 2.2.6) with respect to (open sets of) the topology $\tau_{F_{\mathfrak{X}}}$. That is, a set $U \subseteq \mathfrak{X}$ is said to be $\tau_{F_{\mathfrak{X}}}$-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_{F_{\mathfrak{X}}}$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_j} \mid j = 1, 2, \cdots, n\}$, where $O_{\alpha_j} \in \tau_{F_{\mathfrak{X}}}$, $\forall\ j = 1, 2, \cdots, n$.

Definition 4.3.2 ***Relatively*** $\tau_{F_{\mathfrak{X}}}$***-compact set.***
Let $U \subseteq \mathfrak{X}$, then we say that U is relatively $\tau_{F_{\mathfrak{X}}}$-compact if it is relatively-compact (see Definition 2.2.10) with respect to (the open sets of) $\tau_{F_{\mathfrak{X}}}$.

Lemma 4.3.1 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a collection of Frölicher spaces with X_i a Hausdorff space, $\forall\ i = 1, 2, \cdots, n$, $\mathfrak{X} = \prod_{i=1}^{n} X_i$ be the Cartesian product in the category of sets and $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product. Then*

$$\beta_{\tau_{F_{\mathfrak{X}}}} := \{U \subseteq \mathfrak{X} \mid U \text{ is relatively } \tau_{F_{\mathfrak{X}}} - \text{compact}\}$$

is a bornology on $\mathfrak{X}$ and

$$P_{\tau_{F_{\mathfrak{X}}}} := \{U \subseteq \mathfrak{X} \mid U \text{ is } \tau_{F_{\mathfrak{X}}} - \text{compact}\}$$

is its base.

Proof: The collection $P_{\tau_{F_{\mathfrak{X}}}}$ is a base of a bornology on $\mathfrak{X}$, by Lemma 4.1.1. By Theorem 2.2.4 the Cartesian product of Hausdorff spaces is Hausdorff, therefore $\mathfrak{X} = \prod_{i=1}^{n} X_i$ is Hausorff since X_i is Hausdorff $\forall\ i = 1, 2, \cdots, n$. Then by Lemma 4.1.2 we have that $\beta_{\tau_{F_{\mathfrak{X}}}}$ is a bornology on $\mathfrak{X}$. By Lemma 4.1.3 we have that the base of $\beta_{\tau_{F_{\mathfrak{X}}}}$ is $P_{\tau_{F_{\mathfrak{X}}}}$. □

Definition 4.3.3 *Let $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product. The bornology*

$$\beta_{\tau_{F_{\mathfrak{X}}}} := \{U \subseteq \mathfrak{X} \mid U \text{ is relatively } \tau_{F_{\mathfrak{X}}} - \text{compact}\}$$

is called the functional relatively-compact bornology on $\mathfrak{X}$.

4.3.2 Product relatively-compact bornology.

Definition 4.3.4 ***τ_π-compact set.***
We say that a set is τ_π-compact if it is compact (see Defintion 2.2.6) with respect to (open sets of) the topology τ_π. That is, a set $U \subseteq \mathfrak{X}$ is said to be τ_π-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_\pi$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_j} \mid j = 1, 2, \cdots, n\}$, where $O_{\alpha_j} \in \tau_\pi$, $\forall\ j = 1, 2, \cdots, n$.

Lemma 4.3.2 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a collection of Frölicher spaces, $\mathfrak{X} := \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets, $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product and τ_π be the canonical product topology on $\mathfrak{X}$. The collection*

$$\rho_{\tau_\pi} := \{U \subseteq \mathfrak{X} \mid U \text{ is } \tau_\pi - \text{compact}\}$$

is a base of a bornology on $\mathfrak{X}$.

Proof: We have to show that ρ_{τ_π} covers $\mathfrak{X}$ and that every finite union of elements of ρ_{τ_π} is contained in an element of ρ_{τ_π}. Consider the canonical projections $p_i : \mathfrak{X} \longrightarrow X_i$, for $i = 1, 2, \cdots, n$, and recall the basis

$$\hat{\beta} := \left\{ \bigcap_{i=1}^{n} {p_i}^{-1}(U_i) \mid U_i \in \tau_{F_{X_i}}, \forall\, i = 1, 2, \cdots, n \right\}$$

of the product topology τ_π. Let $\{(x_1, x_2, \cdots, x_n)\} \subset \mathfrak{X}$ be a singleton. With the canonical projections $p_i : \mathfrak{X} \longrightarrow X_i$ consider the projections $p_1 : \mathfrak{X} \longrightarrow X_1$ and $p_2 : \mathfrak{X} \longrightarrow X_2$. Let $U_1 \in \tau_{F_{X_1}}$ and $U_2 \in \tau_{F_{X_2}}$, and since $p_1(x_1, x_2, \cdots, x_n) = x_1$ and $p_2(x_1, x_2, \cdots, x_n) = x_2$ then

$$\begin{aligned} {p_1}^{-1}(U_1) &= \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid p_1(x_1, x_2, \cdots, x_n) \in U_1\} \\ &= \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid x_1 \in U_1\} \end{aligned} \tag{4.1}$$

and similarly

$$\begin{aligned} {p_2}^{-1}(U_2) &= \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid p_2(x_1, x_2, \cdots, x_n) \in U_2\} \\ &= \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid x_2 \in U_2\} \end{aligned} \tag{4.2}$$

Therefore

$$\begin{aligned} {p_1}^{-1}(U_1) \cap {p_2}^{-1}(U_2) &= \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid x_1 \in U_1\} \cap \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid x_2 \in U_2\} \\ &= \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid x_1 \in U_1,\ x_2 \in U_2\} \end{aligned} \tag{4.3}$$

Using this as a conjecture, we have that $\forall\, i = 1, 2, \cdots, n$, for $p_i : \mathfrak{X} \longrightarrow X_i$ and $U_i \in \tau_{F_{X_i}}$,

$$\begin{aligned} \hat{\beta} \ni \bigcap_{i=1}^{n} {p_i}^{-1}(U_i) &= \{(x_1, x_2, \cdots, x_n) \in \mathfrak{X} \mid x_i \in U_i,\ \forall\, i = 1, 2, \cdots, n\} \\ &= \{(x_1, x_2, \cdots, x_n) \mid x_i \in U_i,\ \forall\, i = 1, 2, \cdots, n\} \\ &= \prod_{i=1}^{n} U_i \subset \mathfrak{X} = \prod_{i=1}^{n} X_i \end{aligned} \tag{4.4}$$

By definition $U_i \in \tau_{F_{X_i}}$ implies that $\forall\, f_i \in F_{X_i}\ \exists\, V \in \tau_{\mathbb{R}}$ such that $U_i = {f_i}^{-1}(V)$, $\forall\, i = 1, 2, \cdots, n$. And $f_i \in F_{X_i}$ implies that $f_i : X_i \longrightarrow \mathbb{R}$ is a function, therefore $\forall\, x_i \in X_i\ \exists\, V \in \tau_{\mathbb{R}}$ such that $x_i \in {f_i}^{-1}(V) = U_i$. That is, $\forall\, x_i \in X_i\ \exists\, U_i \in \tau_{F_{X_i}}$ such that $x_i \in U_i$, $\forall\, i = 1, 2, \cdots, n$. This implies that $\forall\, (x_1, x_2, \cdots, x_n) \in \mathfrak{X}$ $\exists\, B \in \hat{\beta}$ such that $(x_1, x_2, \cdots, x_n) \in B$. This also follows from

$$\hat{\beta} = \left\{ \bigcap_{i=1}^{n} {p_i}^{-1}(U_i) \mid U_i \in \tau_{F_{X_i}}, \forall\, i = 1, 2, \cdots, n \right\}$$

being a basis of the product topology τ_π on $\mathfrak{X}$. Thus for every singleton $\{(x_1, x_2, \cdots, x_n)\} \subset \mathfrak{X}\ \exists\, B \in \hat{\beta} \subset \tau_\pi$ such that $\{(x_1, x_2, \cdots, x_n)\} \subset B$. This is

sufficient that every singleton $\{(x_1, x_2, \cdots, x_n)\} \subset \mathfrak{X}$ is τ_π-compact, giving that $\{(x_1, x_2, \cdots, x_n)\} \in \rho_{\tau_\pi}$. Thus ρ_{τ_π} covers $\mathfrak{X}$.
Let $A_1, A_2, \cdots, A_n \in \rho_{\tau_\pi}$, that is, $A_i \subseteq \mathfrak{X}$ and A_i is τ_π-compact $\forall\ i = 1, 2, \cdots, n$. Since the finite union of compact sets is compact by Theorem 2.2.1 therefore $\bigcup_{i=1}^{n} A_i \subseteq \mathfrak{X}$ is τ_π-compact. Since ρ_{τ_π} covers $\mathfrak{X}$ then there exists $K \in \rho_{\tau_\pi}$ such that $\bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n} (K \cup A_i) \in \rho_{\tau_\pi}$. That is, every finite union of elements of ρ_{τ_π} is contained in an element of ρ_{τ_π}. □

Lemma 4.3.3 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \ldots, n\}$ be a collection of Frölicher spaces, X_i be Hausdorff $\forall\ i = 1, 2, \cdots, n$, $\mathfrak{X} := \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets, $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product and τ_π be the canonical product topology on $\mathfrak{X}$. The collection*

$$\beta_{\tau_\pi} := \{U \subseteq \mathfrak{X} \mid U \textit{ is relatively } \tau_\pi - \textit{compact}\}$$

is a bornology on $\mathfrak{X}$.

Proof: We have to show that β_{τ_π} is closed under inclusion, is closed under finite union and it covers $\mathfrak{X}$. Since

$$\rho_{\tau_\pi} = \{U \subseteq \mathfrak{X} \mid U \textit{ is } \tau_\pi - \textit{compact}\}$$

is a base of a bornology on $\mathfrak{X}$ then ρ_{τ_π} covers $\mathfrak{X}$. That is every singleton $\{(x_1, x_2, \cdots, x_n)\} \subset \mathfrak{X}$ is τ_π-compact. The Cartesian product $\mathfrak{X} := \prod_{i=1}^{n} X_i$ is Hausdorff by Theorem 2.2.4 since it is a Cartesian product of Hausdorff spaces. Thus by Theorem 2.2.7 we have that $\{(x_1, x_2, \cdots, x_n)\}$ is relatively τ_π-compact since $\mathfrak{X}$ is Hausdorff. That is, $\{(x_1, x_2, \cdots, x_n)\} \subset \mathfrak{X}$ is relatively τ_π-compact $\forall\ (x_1, x_2, \cdots, x_n) \in \mathfrak{X}$, therefore $\{(x_1, x_2, \cdots, x_n)\} \in \beta_{\tau_\pi}$, $\forall\ (x_1, x_2, \cdots, x_n) \in \mathfrak{X}$. Thus β_{τ_π} covers $\mathfrak{X}$.
Let $U_1, U_2, \cdots, U_n \in \beta_{\tau_\pi}$, then $U_i \subseteq \mathfrak{X}$ is relatively τ_π-compact, $\forall\ i = 1, 2, \cdots, n$. Then $Cl(U_i)$, the closure of U_i, is τ_π-compact $\forall\ i = 1, 2, \cdots, n$, by definition. Now since the finite union of compact sets is compact by Theorem 2.2.1 then $\bigcup_{i=1}^{n} Cl(U_i) = Cl\left(\bigcup_{i=1}^{n} U_i\right)$ is τ_π-compact. Since $Cl\left(\bigcup_{i=1}^{n} U_i\right)$ is τ_π-compact, this implies, by definition, that $\bigcup_{i=1}^{n} U_i \subseteq \mathfrak{X}$ is relatively τ_π-compact. Therefore $\bigcup_{i=1}^{n} U_i \in \beta_{\tau_\pi}$, thus β_{τ_π} is closed under finite union.
Let $A \subseteq B$ and $B \in \beta_{\tau_\pi}$. Since $B \in \beta_{\tau_\pi}$ then $B \subseteq \mathfrak{X}$ is relatively τ_π-compact.

Therefore $Cl(B)$, that is the closure of B, is τ_π-compact, by definition. With $Cl(A)$ denoting the closure of A, then $A \subseteq B \implies Cl(A) \subseteq Cl(B)$. This implies that the open cover of $Cl(B)$ is an open cover of $Cl(A)$. Therefore $Cl(A)$ is τ_π-compact since $Cl(B)$ is τ_π-compact. This gives that A is relatively τ_π-compact, therefore $A \in \beta_{\tau_\pi}$. That is $A \subseteq B \in \beta_{\tau_\pi} \implies A \in \beta_{\tau_\pi}$. Thus β_{τ_π} is closed under inclusion. □

Definition 4.3.5 ***Product relatively-compact bornology.***
Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ *be a collection of Frölicher spaces,* X_i *be a Hausdorff space* $\forall\ i = 1, 2, \cdots, n$, $\mathfrak{X} = \prod_{i=1}^{n} X_i$ *be the Cartesian product,* $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ *be the Frölicher product and* τ_π *be the product topology on* $\mathfrak{X}$*. Then we call the relatively-compact bornology*

$$\beta_{\tau_\pi} := \{U \subseteq \mathfrak{X} \mid U \text{ is relatively } \tau_\pi - \text{compact}\}$$

the product relatively-compact bornology on $\mathfrak{X}$*.*

Lemma 4.3.4 *Let* $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ *be a collection of Frölicher spaces,* X_i *be a Hausdorff space* $\forall\ i = 1, 2, \cdots, n$, $\mathfrak{X} = \prod_{i=1}^{n} X_i$ *be the Cartesian product,* $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ *be the Frölicher product and* τ_π *be the product topology on* $\mathfrak{X}$*. The collection*

$$\rho_{\tau_\pi} = \{U \subseteq \mathfrak{X} \mid U \text{ is } \tau_\pi - \text{compact}\}$$

is the base of the product relatively-compact bornology

$$\beta_{\tau_\pi} = \{U \subseteq \mathfrak{X} \mid U \text{ is relatively } \tau_\pi - \text{compact}\}.$$

Proof: We have to show that $\rho_{\tau_\pi} \subset \beta_{\tau_\pi}$ and that every element of β_{τ_π} is contained in an element of ρ_{τ_π}. By Theorem 2.2.7 we've that $\rho_{\tau_\pi} \subset \beta_{\tau_\pi}$ since a compact subset of a Hausdorff space is relatively-compact. Let $A \in \beta_{\tau_\pi}$, therefore $A \subseteq \mathfrak{X}$ is relatively τ_π-compact, that is, the closure of A, denoted $Cl(A)$, is τ_π-compact as $\mathfrak{X}$ is Hausdorff (Theorem 2.2.4). Thus $Cl(A) \in \rho_{\tau_\pi}$. But $A \subseteq Cl(A)$ by Proposition 2.2.1. That is every element of β_{τ_π} is contained in an element of ρ_{τ_π}. Hence the collection β_{τ_π} is a bornology on $\mathfrak{X}$ and the collection ρ_{τ_π} is its base. □

4.3.3 Bornological comparison.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a collection of Frölicher spaces, X_i be a Hausdorff space $\forall\ i = 1, 2, \cdots, n$, $\mathfrak{X} = \prod_{i=1}^{n} X_i$ be the Cartesian product and $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product. The Cartesian product of Hausdorff spaces is Hausdorff by Theorem 2.2.4, therefore $\mathfrak{X}$ is Hausdorff. We have the functional topology $\tau_{F_{\mathfrak{X}}}$ on $\mathfrak{X}$ induced from $F_{\mathfrak{X}}$. Also, we have the canonical product topology τ_π on $\mathfrak{X}$. The relationship between $\tau_{F_{\mathfrak{X}}}$ and τ_π is such that $\tau_{F_{\mathfrak{X}}} = \tau_\pi$ (see Lemma 2.3.7). Then we have the functional relatively-compact bornology $\beta_{\tau_{F_{\mathfrak{X}}}}$ and the product relatively-compact bornology β_{τ_π}, induced from $\tau_{F_{\mathfrak{X}}}$ and τ_π, respectively. In this subsection we determine the relation between $\beta_{\tau_{F_{\mathfrak{X}}}}$ and β_{τ_π} and any bounded maps between them.

Theorem 4.3.1 *Let $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product, β_{τ_π} be the functional relatively-compact bornology on $\mathfrak{X}$ and $\beta_{\tau_{F_{\mathfrak{X}}}}$ be the product relatively-compact bornology, then $\beta_{\tau_\pi} = \beta_{\tau_{F_{\mathfrak{X}}}}$.*

Proof: Let $U \in \beta_{\tau_\pi}$, that is, $U \subseteq \mathfrak{X}$ is relatively τ_π-compact. Thus every open cover $\{O_\alpha \mid \alpha \in I\} \subset \tau_\pi = \tau_{F_{\mathfrak{X}}}$ of the closure of U, that is $Cl(U)$, has a finite subcover, therefore U is relatively $\tau_{F_{\mathfrak{X}}}$-compact. That is $U \in \beta_{\tau_\pi} \implies U \in \mathcal{B}_{\tau_{F_{\mathfrak{X}}}}$. Similarly, since $\tau_\pi = \tau_{F_{\mathfrak{X}}}$, we also have that $U \in \mathcal{B}_{\tau_{F_{\mathfrak{X}}}} \implies U \in \beta_{\tau_\pi}$. Therefore $\beta_{\tau_\pi} = \mathcal{B}_{\tau_{F_{\mathfrak{X}}}}$. □

Corollary 4.3.1 *Let $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be the Frölicher product, β_{τ_π} be the functional relatively-compact bornology on $\mathfrak{X}$ and $\beta_{\tau_{F_{\mathfrak{X}}}}$ be the product relatively-compact bornology, and $(\mathfrak{X}, \beta_{\tau_\pi})$ and $(\mathfrak{X}, \beta_{\tau_{F_{\mathfrak{X}}}})$ be bornological sets. Then the identity maps $id_1 : (\mathfrak{X}, \beta_{\tau_\pi}) \longrightarrow (\mathfrak{X}, \beta_{\tau_{F_{\mathfrak{X}}}})$ and $id_2 : (\mathfrak{X}, \beta_{\tau_{F_{\mathfrak{X}}}}) \longrightarrow (\mathfrak{X}, \beta_{\tau_\pi})$ are bounded.*

Proof: By Theorem 4.3.1 we have that $\beta_{\tau_\pi} = \beta_{\tau_{F_{\mathfrak{X}}}}$, which implies that $\beta_{\tau_\pi} \subset \mathcal{B}_{\tau_{F_{\mathfrak{X}}}}$ and $\mathcal{B}_{\tau_{F_{\mathfrak{X}}}} \subset \beta_{\tau_\pi}$. Then it follows that the identity maps $id_1 : (\mathfrak{X}, \beta_{\tau_\pi}) \longrightarrow (\mathfrak{X}, \beta_{\tau_{F_{\mathfrak{X}}}})$ and $id_2 : (\mathfrak{X}, \beta_{\tau_{F_{\mathfrak{X}}}}) \longrightarrow (\mathfrak{X}, \beta_{\tau_\pi})$ are bounded, by definition. □

4.4 Relatively-compact bornologies from Frölicher quotient.

Let $g : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$ be a morphism in the category of Frölicher spaces, where (X, C_X, F_X) and (Y, C_Y, F_Y) are Frölicher spaces. Consider the equivalence relation $\sim$ on X defined by: $x \sim y$ if and only if $g(x) = g(y)$, $\forall\ x, y \in X$. The equivalence relation $\sim$ is called the kernel equivalence (see [14]). Let $\mathcal{Q} := X/_\sim$ be a quotient set on X, $(C_{\mathcal{Q}}, F_{\mathcal{Q}})$ be a Frölicher stucture on $\mathcal{Q}$, then $(\mathcal{Q}, C_{\mathcal{Q}}, F_{\mathcal{Q}})$ is a Frölicher space - called the Frölicher quotient (see Subsection 2.3.7). We have topologies induced canonically from the Frölicher structure $(C_{\mathcal{Q}}, F_{\mathcal{Q}})$, that is, the functional topology

$$\tau_{F_{\mathcal{Q}}} := \{U \subseteq \mathcal{Q} \mid U = g^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ g \in F_{\mathcal{Q}}\}$$

on $\mathcal{Q}$ and the curvaceous topology

$$\tau_{C_{\mathcal{Q}}} := \{U \subseteq \mathcal{Q} \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_{\mathcal{Q}}\}$$

on $\mathcal{Q}$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. The topology $\tau_{F_\mathcal{Q}}$ is coarser than the topology $\tau_{C_\mathcal{Q}}$ (see Lemma 2.3.3). If $\tau_{F_\mathcal{Q}} = \tau_{C_\mathcal{Q}}$ then the Frölicher quotient $(\mathcal{Q}, C_\mathcal{Q}, F_\mathcal{Q})$ is said to be a balanced space. The topology $\tau_{F_\mathcal{Q}}$ is the coarsest topology in which all smooth maps are continuous. Thus the topology $\tau_{F_\mathcal{Q}}$ is considered as the Frölicher topology from Frölicher quotient. The basis and subbasis of $\tau_{F_\mathcal{Q}}$ are given by the collections $\{f^{-1}(0, +\infty) \mid f \in F_\mathcal{Q}\}$ and $\{f^{-1}(0, 1) \mid f \in F_\mathcal{Q}\}$, respectively (see [36] and [45]). Also we have a canonical topology induced on $\mathcal{Q} = X/_\sim$, that is, the quotient topology

$$\tau_\sim := \{U \subseteq \mathcal{Q} \mid \pi^{-1}(U) \in \tau_{F_X}\}$$

where $\pi : X \longrightarrow \mathcal{Q}$ is a canonical projection and τ_{F_X} is a functional topology on X.

4.4.1 Functional relatively-compact bornology.

Definition 4.4.1 $\tau_{F_\mathcal{Q}}$***-compact set.***
We say that a set is $\tau_{F_\mathcal{Q}}$-compact if it is compact (see Definition 2.2.6) with respect to (open sets of) the topology $\tau_{F_\mathcal{Q}}$. That is, a set $U \subseteq \mathcal{Q}$ is said to be $\tau_{F_\mathcal{Q}}$-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_{F_\mathcal{Q}}$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_j} \mid j = 1, 2, \cdots, n\}$, where $O_{\alpha_j} \in \tau_{F_\mathcal{Q}}$, $\forall\, j = 1, 2, \cdots, n$.

Definition 4.4.2 ***Relatively*** $\tau_{F_\mathcal{Q}}$***-compact set.***
Let $U \subseteq \mathcal{Q}$, then we say that U is relatively $\tau_{F_\mathcal{Q}}$-compact if it is relatively compact (see Definition 2.2.10) with respect to (the open sets of) $\tau_{F_\mathcal{Q}}$.

Lemma 4.4.1 *Let (X, C_X, F_X) be a Frölicher space, $\sim$ be the kernel equivalence, $\mathcal{Q} := X/_\sim$ be the quotient set such that $\mathcal{Q}$ is Hausdorff and $(\mathcal{Q}, C_\mathcal{Q}, F_\mathcal{Q})$ be the Frölicher quotient. Then*

$$\mathcal{B}_{\tau_{F_\mathcal{Q}}} := \{U \subseteq \mathcal{Q} \mid U \text{ is relatively } \tau_{F_\mathcal{Q}} - \text{compact}\}$$

is a bornology on $\mathcal{Q}$ and

$$\mathcal{P}_{\tau_{F_\mathcal{Q}}} := \{U \subseteq \mathcal{Q} \mid U \text{ is } \tau_{F_\mathcal{Q}} - \text{compact}\}$$

is its base.

Proof: The collection $\mathcal{P}_{\tau_{F_\mathcal{Q}}}$ is a base of a bornology on $\mathcal{Q}$, by Lemma 4.1.1. Since the quotient $\mathcal{Q}$ is Hausdorff then by Lemma 4.1.2 the collection $\mathcal{B}_{\tau_{F_\mathcal{Q}}}$ is a bornology on $\mathcal{Q}$. It follows that $\mathcal{P}_{\tau_{F_\mathcal{Q}}}$ is the base of the bornology $\mathcal{B}_{\tau_{F_\mathcal{Q}}}$, by Lemma 4.1.3. □

Definition 4.4.3 *Let $(\mathcal{Q}, C_\mathcal{Q}, F_\mathcal{Q})$ be the Frölicher quotient. The bornology*

$$\mathcal{B}_{\tau_{F_\mathcal{Q}}} := \{U \subseteq \mathcal{Q} \mid U \text{ is relatively } \tau_{F_\mathcal{Q}} - \text{compact}\}$$

is called the functional relatively-compact bornology on $\mathcal{Q}$.

4.4.2 Quotient relatively-compact bornology.

Definition 4.4.4 ***$\tau_\sim$-compact set.***
We say that a set is $\tau_\sim$-compact if it is compact (see Definition 2.2.6) with respect to (open sets of) the topology $\tau_\sim$. That is, a set $U \subseteq \mathcal{Q}$ is said to be $\tau_\sim$-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_\sim$, $\forall\ \alpha \in I$, has a finite subcover, say $\{O_{\alpha_j} \mid j = 1, 2, \cdots, n\}$, where $O_{\alpha_j} \in \tau_\sim$, $\forall\ j = 1, 2, \cdots, n$.

Definition 4.4.5 ***Relatively $\tau_\sim$-compact set.***
Let $U \subseteq \mathcal{Q}$, then we say that U is relatively $\tau_\sim$-compact if it is relatively compact (see Definition 2.2.10) with respect to (the open sets of) $\tau_\sim$.

Lemma 4.4.2 *Let (X, C_X, F_X) be a Frölicher space, $\sim$ be the kernel equivalence on X, $\mathcal{Q} := X/_\sim$ be a quotient set such that $\mathcal{Q}$ is Hausdorff, $(\mathcal{Q}, C_\mathcal{Q}, F_\mathcal{Q})$ be the Frölicher quotient and $\tau_\sim$ be the canonical quotient topology. Then*

$$\beta_{\tau_\sim} := \{U \subseteq \mathcal{Q} \mid U \ \textit{is relatively}\ \tau_\sim - \textit{compact}\}$$

is a bornology on $\mathcal{Q}$ with the base

$$\rho_{\tau_\sim} := \{U \subseteq \mathcal{Q} \mid U \ \textit{is}\ \tau_\sim - \textit{compact}\}.$$

Proof: By Lemma 2.3.12 we have that $\tau_{F_\mathcal{Q}} = \tau_\sim$, which implies that every $\tau_{F_\mathcal{Q}}$-compact set is $\tau_\sim$-compact, vice versa. Subsequently every relatively $\tau_{F_\mathcal{Q}}$-compact set is relatively $\tau_\sim$-compact, vice versa. Since, by Lemma 4.4.1,

$$\mathcal{P}_{\tau_{F_\mathcal{Q}}} := \{U \subseteq \mathcal{Q} \mid U \ \textit{is}\ \tau_{F_\mathcal{Q}} - \textit{compact}\}$$

is a base of the bornology

$$\mathcal{B}_{\tau_{F_\mathcal{Q}}} := \{U \subseteq \mathcal{Q} \mid U \ \textit{is relatively}\ \tau_{F_\mathcal{Q}} - \textit{compact}\}$$

then it follows that $\beta_{\tau_\sim}$ is a bornology on $\mathcal{Q}$ and its base is $\rho_{\tau_\sim}$. □

Definition 4.4.6 ***Quotient relatively-compact bornology.***
Let $(\mathcal{Q}, C_\mathcal{Q}, F_\mathcal{Q})$ be the Frölicher quotient. The bornology

$$\beta_{\tau_\sim} := \{U \subseteq \mathcal{Q} \mid U \ \textit{is relatively}\ \tau_\sim - \textit{compact}\}$$

is called the quotient relatively-compact bornology.

4.4.3 Bornological comparison.

Given the Frölicher quotient $(\mathcal{Q}, C_\mathcal{Q}, F_\mathcal{Q})$ with $\mathcal{Q}$ Hausdorff we have the functional topology $\tau_{F_\mathcal{Q}}$ and the canonical quotient topology $\tau_\sim$. The relationship between $\tau_{F_\mathcal{Q}}$ and $\tau_\sim$ is such that $\tau_{F_\mathcal{Q}} = \tau_\sim$, by Lemma 2.3.12. Then we have the functional relatively-compact bornology $\mathcal{B}_{\tau_{F_\mathcal{Q}}}$ on $\mathcal{Q}$ and the quotient relatively-compact bornology $\beta_{\tau_\sim}$, induced from $\tau_{F_\mathcal{Q}}$ and $\tau_\sim$ respectively. We now determine the relationship between these bornologies and their associated bounded maps.

Theorem 4.4.1 *Let $(\mathcal{Q}, C_{\mathcal{Q}}, F_{\mathcal{Q}})$ be the Frölicher quotient, $\mathcal{B}_{\tau_{F_{\mathcal{Q}}}}$ be the functional relatively-compact bornology on $\mathcal{Q}$ and $\beta_{\tau_\sim}$ be the quotient relatively-compact bornology, then $\mathcal{B}_{\tau_{F_{\mathcal{Q}}}} = \beta_{\tau_\sim}$.*

Proof: Since $\tau_{F_{\mathcal{Q}}} = \tau_\sim$ then it follows that a set that is relatively $\tau_{F_{\mathcal{Q}}}$-compact is also relatively $\tau_\sim$-compact, vice versa. Therefore $\mathcal{B}_{\tau_{F_{\mathcal{Q}}}} = \beta_{\tau_\sim}$. □

Corollary 4.4.1 *Let $(\mathcal{Q}, C_{\mathcal{Q}}, F_{\mathcal{Q}})$ be the Frölicher quotient, $\mathcal{B}_{\tau_{F_{\mathcal{Q}}}}$ be the functional relatively-compact bornology on $\mathcal{Q}$ and $\beta_{\tau_\sim}$ be the quotient relatively-compact bornology, and $(\mathcal{Q}, \mathcal{B}_{\tau_{F_{\mathcal{Q}}}})$ and $(\mathcal{Q}, \beta_{\tau_\sim})$ be bornological sets. The identity maps $id_1 : (\mathcal{Q}, \mathcal{B}_{\tau_{F_{\mathcal{Q}}}}) \longrightarrow (\mathcal{Q}, \beta_{\tau_\sim})$ and $id_2 : (\mathcal{Q}, \beta_{\tau_\sim}) \longrightarrow (\mathcal{Q}, \mathcal{B}_{\tau_{F_{\mathcal{Q}}}})$ are bounded.*

Proof: By Theorem 4.4.1 we have that $\mathcal{B}_{\tau_{F_{\mathcal{Q}}}} = \beta_{\tau_\sim}$, which implies that $\mathcal{B}_{\tau_{F_{\mathcal{Q}}}} \subset \beta_{\tau_\sim}$ and $\beta_{\tau_\sim} \subset \mathcal{B}_{\tau_{F_{\mathcal{Q}}}}$. Therefore by definition it follows that the identity maps $id_1 : (\mathcal{Q}, \mathcal{B}_{\tau_{F_{\mathcal{Q}}}}) \longrightarrow (\mathcal{Q}, \beta_{\tau_\sim})$ and $id_2 : (\mathcal{Q}, \beta_{\tau_\sim}) \longrightarrow (\mathcal{Q}, \mathcal{B}_{\tau_{F_{\mathcal{Q}}}})$ are bounded.
□

4.5 Relatively-compact bornologies from Frölicher coproduct.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a family of Frölicher spaces with X_i a Hausdorff space, $i \in I$, and let $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be a Frölicher coproduct (see Subsection 2.3.6). That is,

$$\mathcal{N} := \coprod_{i \in I} X_i = \bigcup_{i \in I} \{(x_i, i) \mid x_i \in X_i\}$$

is a coproduct in the category of sets, and $(C_{\mathcal{N}}, F_{\mathcal{N}})$ is a Frölicher structure on $\mathcal{N}$. We have the topologies induced from the Frölicher structure $(C_{\mathcal{N}}, F_{\mathcal{N}})$, that is, the functional topology

$$\tau_{F_{\mathcal{N}}} := \{U \subseteq \mathcal{N} \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_{\mathcal{N}}\}$$

on $\mathcal{N}$ and the curvaceous topology

$$\tau_{C_{\mathcal{N}}} := \{U \subseteq \mathcal{N} \mid c^{-1}(U) \in \tau_{\mathbb{R}},\ \forall\ c \in C_{\mathcal{N}}\}$$

on $\mathcal{N}$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. The topology $\tau_{F_{\mathcal{N}}}$ is coarser than the topology $\tau_{C_{\mathcal{N}}}$ (see Lemma 2.3.3). If $\tau_{F_{\mathcal{N}}} = \tau_{C_{\mathcal{N}}}$ then the Frölicher coproduct $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ is said to be a balanced space. The topology $\tau_{F_{\mathcal{N}}}$ is the coarsest topology in which all smooth maps are continuous. Based on that, the topology $\tau_{F_{\mathcal{N}}}$ is considered as the Frölicher topology from Frölicher coproduct. The basis and subbasis of $\tau_{F_{\mathcal{N}}}$ are given by the collections $\{f^{-1}(0, +\infty) \mid f \in F_{\mathcal{N}}\}$ and $\{f^{-1}(0, 1) \mid f \in F_{\mathcal{N}}\}$, respectively (see [36] and [45]). Also we have the canonical coproduct topology on $\mathcal{N}$ given by

$$\tau_{\coprod} := \{U \in \mathcal{N} \mid \varphi_i^{-1}(U) \in \tau_{F_{X_i}},\ i \in I\}$$

where $\varphi_i : X_i \longrightarrow \mathcal{N}$ is a canonical projection and $\tau_{F_{X_i}}$ is a functional topology on X_i, $i \in I$.

4.5.1 Functional relatively-compact bornology.

Definition 4.5.1 $\tau_{F_{\mathcal{N}}}$***-compact set.***
We say that a set is $\tau_{F_{\mathcal{N}}}$-compact if it is compact (see Definition 2.2.6) with respect to (open sets of) the topology $\tau_{F_{\mathcal{N}}}$. That is, a set $U \subseteq \mathcal{N}$ is said to be $\tau_{F_{\mathcal{N}}}$-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_{F_{\mathcal{N}}}$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_j} \mid j = 1, 2, \cdots, n\}$, where $O_{\alpha_j} \in \tau_{F_{\mathcal{N}}}$, $\forall\ j = 1, 2, \cdots, n$.

Definition 4.5.2 ***Relatively*** $\tau_{F_{\mathcal{N}}}$***-compact set.***
Let $U \subseteq \mathcal{N}$, then we say that U is relatively $\tau_{F_{\mathcal{N}}}$-compact if it is relatively compact (see Definition 2.2.10) with respect to (the open sets of) $\tau_{F_{\mathcal{N}}}$.

Lemma 4.5.1 *Let $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct such that $\mathcal{N}$ is Hausdorff. The collection*

$$\mathcal{B}_{\tau_{F_{\mathcal{N}}}} := \{U \subseteq \mathcal{N} \mid U \textit{ is relatively } \tau_{F_{\mathcal{N}}} - \textit{compact}\}$$

is a functional relatively-compact bornology on $\mathcal{N}$ and

$$\mathcal{P}_{\tau_{F_{\mathcal{N}}}} := \{U \subseteq \mathcal{N} \mid U \textit{ is } \tau_{F_{\mathcal{N}}} - \textit{compact}\}$$

is its base.

Proof: The collection $\mathcal{P}_{\tau_{F_{\mathcal{N}}}}$ is a base of a bornology on $\mathcal{N}$, by Lemma 4.1.1. Since the quotient $\mathcal{N}$ is Hausdorff then by Lemma 4.1.2 the collection $\mathcal{B}_{\tau_{F_{\mathcal{N}}}}$ is a bornology on $\mathcal{N}$. It follows that $\mathcal{P}_{\tau_{F_{\mathcal{N}}}}$ is the base of the bornology $\mathcal{B}_{\tau_{F_{\mathcal{N}}}}$, by Lemma 4.1.3. □.

Definition 4.5.3 *Let $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct. The bornology*

$$\mathcal{B}_{\tau_{F_{\mathcal{N}}}} := \{U \subseteq \mathcal{N} \mid U \textit{ is relatively } \tau_{F_{\mathcal{N}}} - \textit{compact}\}$$

is called the functional relatively-compact bornology on $\mathcal{N}$.

4.5.2 Coproduct relatively-compact bornology.

Definition 4.5.4 $\tau_{\coprod}$***-compact set.***
We say that a set is $\tau_{\coprod}$-compact if it is compact (see Definition 2.2.6) with respect to (open sets of) the topology $\tau_{\coprod}$. That is, a set $U \subseteq \mathcal{N}$ is said to be $\tau_{\coprod}$-compact if every open cover $\{O_\alpha \mid \alpha \in I\}$ of U, where $O_\alpha \in \tau_{\coprod}$, $\alpha \in I$, has a finite subcover, say $\{O_{\alpha_j} \mid j = 1, 2, \cdots, n\}$, where $O_{\alpha_j} \in \tau_{\coprod}$, $\forall\ j = 1, 2, \cdots, n$.

Lemma 4.5.2 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a family of Frölicher spaces, $\mathcal{N} := \coprod_{i \in I} X_i$, $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct and $\tau_{\coprod}$ be the canonical coproduct topology. The collection*

$$\rho_{\tau_{\coprod}} := \{U \subseteq \mathcal{N} \mid U \textit{ is } \tau_{\coprod} - \textit{compact}\}$$

is a base of a bornology on $\mathcal{N}$.

Proof: We have to show that $\rho_{\tau_{\coprod}}$ covers $\mathcal{N}$ and that every finite union of elements of $\rho_{\tau_{\coprod}}$ is contained in an element of $\rho_{\tau_{\coprod}}$. Let $x_i \in X_i$, $i \in I$, and $\{(x_i, i)\}$ be a singleton in $\mathcal{N} := \coprod_{i \in I} X_i$. Let $A \in \tau_{\coprod}$, that is, $A \subseteq \mathcal{N}$ and $\varphi_i^{-1}(A) \in \tau_{F_{X_i}}$, $i \in I$. And $\varphi_i^{-1}(A) \in \tau_{F_{X_i}}$ implies that there exists $M \in \tau_{\mathbb{R}}$ such that $\varphi_i^{-1}(A) = f_i^{-1}(M)$, $\forall\, f_i \in F_{X_i}$. Therefore we have that $A = \varphi_i(f_i^{-1}(M))$ with $f_i^{-1}(M) \subseteq X_i$, and since $\varphi_i : (x_i \in X_i) \mapsto ((x_i, i) \in \mathcal{N})$ therefore $A = \varphi_i(f_i^{-1}(M)) = (f_i^{-1}(M), i)$. Since $f_i \in F_{X_i}$ is such that $f_i : X_i \longrightarrow \mathbb{R}$ is a function, then there exists $V \in \tau_{\mathbb{R}}$ such that $x_i \in f_i^{-1}(V)$, which implies that $(x_i, i) \in (f_i^{-1}(M), i) = \varphi_i(f_i^{-1}(M)) \in \tau_{\coprod}$. Thus for each singleton $\{(x_i, i)\}$ there exists $A \in \tau_{\coprod}$ such that $\{(x_i, i)\} \subset A$. This is sufficient that every singleton $\{(x_i, i)\}$ in $\mathcal{N}$ is $\tau_{\coprod}$-compact, therefore $\rho_{\tau_{\coprod}}$ covers $\mathcal{N}$.
Let $U_1, U_2, \cdots, U_n \in \rho_{\tau_{\coprod}}$, that is, $U_j \subseteq \mathcal{N}$ and U_j is $\tau_{\coprod}$-compact $\forall\, j = 1, 2, \cdots, n$. Then $\bigcup_{j=1}^{n} U_j \subseteq \mathcal{N}$ is $\tau_{\coprod}$-compact since the finite union of compact sets is compact (see Theorem 2.2.1). Since $\rho_{\tau_{\coprod}}$ covers $\mathcal{N}$ then there exists $K \in \rho_{\tau_{\coprod}}$ such that $\bigcup_{j=1}^{n} U_j \subset \bigcup_{j=1}^{n} (U_j \cup K) \in \rho_{\tau_{\coprod}}$. Thus every finite union of elements of $\rho_{\tau_{\coprod}}$ is contained in an element of $\rho_{\tau_{\coprod}}$. □

Lemma 4.5.3 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a family of Frölicher spaces with X_i Hausdorff, $\mathcal{N} := \coprod_{i \in I} X_i$, $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct and $\tau_{\coprod}$ be the canonical coproduct topology. The collection*

$$\beta_{\tau_{\coprod}} := \{U \subseteq \mathcal{N} \mid U \text{ is relatively } \tau_{\coprod} - \text{compact}\}$$

is a bornology on $\mathcal{N}$.

Proof: We have to show that $\beta_{\tau_{\coprod}}$ is closed under inclusion, is closed under finite union and it covers $\mathcal{N}$. Since $\rho_{\tau_{\coprod}} = \{U \subseteq \mathcal{N} \mid U \text{ is } \tau_{\coprod} - \text{compact}\}$ is a base of a bornology on $\mathcal{N}$ then $\rho_{\tau_{\coprod}}$ covers $\mathcal{N}$. That is every singleton $\{(x_i, i)\} \subset \mathcal{N}$ is $\tau_{\coprod}$-compact. Therefore $\{(x_i, i)\} \subset \mathcal{N}$ is relatively $\tau_{\coprod}$-compact since $\mathcal{N}$ is Hausdorff (see Theorem 2.2.7). That is, $\{(x_i, i)\} \subset \mathcal{N}$ is relatively $\tau_{\coprod}$-compact $\forall\, (x_i, i) \in \mathcal{N}$, therefore $\{(x_i, i)\} \in \beta_{\tau_{\coprod}}$, $\forall\, (x_i, i) \in \mathcal{N}$. Thus $\beta_{\tau_{\coprod}}$ covers $\mathcal{N}$.
Let $U_1, U_2, \cdots, U_n \in \beta_{\tau_{\coprod}}$, then $U_j \subseteq \mathcal{N}$ is relatively $\tau_{\coprod}$-compact $\forall\, j = 1, 2, \cdots, n$. Then $Cl(U_j)$, the closure of U_j, is $\tau_{\coprod}$-compact $\forall\, j = 1, 2, \cdots, n$, by definition. Now since the finite union of compact sets is compact (Theorem 2.2.1) then $\bigcup_{j=1}^{n} Cl(U_j) = Cl\left(\bigcup_{j=1}^{n} U_j\right)$ is $\tau_{\coprod}$-compact. With $Cl\left(\bigcup_{j=1}^{n} U_j\right)$ $\tau_{\coprod}$-compact, this implies, by definition, that $\bigcup_{j=1}^{n} U_j \subseteq \mathcal{N}$ is relatively $\tau_{\coprod}$-compact. Therefore

$\bigcup_{j=1}^{n} U_j \in \beta_{\tau_\coprod}$, thus $\beta_{\tau_\coprod}$ is closed under finite union.
Let $A \subseteq B$ and $B \in \beta_{\tau_\coprod}$. Since $B \in \beta_{\tau_\coprod}$ then $B \subseteq \mathcal{N}$ is relatively $\tau_\coprod$-compact, therefore $Cl(B)$, that is the closure of B, is $\tau_\coprod$-compact, by definition. With $Cl(A)$ denoting the closure of A, then $A \subseteq B \implies Cl(A) \subseteq Cl(B)$. This implies that the open cover of $Cl(B)$ is an open cover of $Cl(A)$. Therefore $Cl(A)$ is $\tau_\coprod$-compact since $Cl(B)$ is $\tau_\coprod$-compact. This gives that A is relatively $\tau_\coprod$-compact, therefore $A \in \beta_{\tau_\coprod}$. That is $A \subseteq B \in \beta_{\tau_\coprod} \implies A \in \beta_{\tau_\coprod}$. Thus $\beta_{\tau_\coprod}$ is closed under inclusion. □

Definition 4.5.5 ***Coproduct relatively-compact bornology.***
Let $(\mathcal{N}, C_\mathcal{N}, F_\mathcal{N})$ be the Frölicher coproduct and $\tau_\coprod$ be the canonical coproduct topology on $\mathcal{N}$. Then we call the compact bornology

$$\beta_{\tau_\coprod} := \{U \subseteq \mathcal{N} \mid U \textit{ is relatively } \tau_\coprod - \textit{compact}\}$$

the coproduct relatively-compact bornology on $\mathcal{N}$.

Lemma 4.5.4 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a collection of Frölicher spaces, $\mathcal{N} = \coprod_{i\in I} X_i$, $(\mathcal{N}, C_\mathcal{N}, F_\mathcal{N})$ be the Frölicher coproduct. The collection*

$$\rho_{\tau_\coprod} = \{U \subseteq \mathcal{N} \mid U \textit{ is } \tau_\coprod - \textit{compact}\}$$

is the base of the coproduct relatively-compact bornology

$$\beta_{\tau_\coprod} = \{U \subseteq \mathcal{N} \mid U \textit{ is relatively } \tau_\coprod - \textit{compact}\}.$$

Proof: We have to show that $\rho_{\tau_\coprod} \subset \beta_{\tau_\coprod}$ and that every element of $\beta_{\tau_\coprod}$ is contained in an element of $\rho_{\tau_\coprod}$. Let $U \in \rho_{\tau_\coprod}$, that is $U \subseteq \mathcal{N}$ is $\tau_\coprod$-compact, therefore by Theorem 2.2.7, U is relatively $\tau_\coprod$-compact since $\mathcal{N} := \coprod_{i\in I} X_i$ is a Hausdorff space, thus $U \in \beta_{\tau_\coprod}$. This implies that $\rho_{\tau_\coprod} \subset \beta_{\tau_\coprod}$. Let $B \in \beta_{\tau_\coprod}$, then $B \subseteq \mathcal{N}$ is relatively $\tau_\coprod$-compact. By definition of relative-compactness (see Definition 2.2.10) we have that the closure of B, denoted $Cl(B)$, is $\tau_\coprod$-compact, thus $Cl(B) \in \rho_{\tau_\coprod}$. But $B \subseteq Cl(B)$ (by Proposition 2.2.1). That is every element of $\beta_{\tau_\coprod}$ is contained in an element of $\rho_{\tau_\coprod}$. Hence $\beta_{\tau_\coprod}$ is a bornology on $\mathcal{N}$ and $\rho_{\tau_\coprod}$ is its base. □

4.5.3 Bornological comparison.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be the collection of Frölicher space, $\mathcal{N} := \coprod_{i\in I} X_i$ be a coproduct in the category of sets and $(\mathcal{N}, C_\mathcal{N}, F_\mathcal{N})$ be the Frölicher coproduct. We have the functional topology $\tau_{F_\mathcal{N}}$ on $\mathcal{N}$, and the canonical coproduct topology

$\tau_{\coprod}$. From these topologies we have induced the relatively-compact bornologies, that is, the functional relatively-compact bornology $\mathcal{B}_{\tau_{F_{\mathcal{N}}}}$ on $\mathcal{N}$ and the coproduct relatively-compact bornology $\beta_{\tau_{\coprod}}$, induced from $\tau_{F_{\mathcal{N}}}$ and $\tau_{\coprod}$, respectively. We determine the relationship between these bornologies and their associated bounded maps.

Theorem 4.5.1 *Let $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct, $\mathcal{N}$ be Hausdorff, $\mathcal{B}_{\tau_{F_{\mathcal{N}}}}$ be the functional relatively-coproduct bornology on $\mathcal{N}$ and $\beta_{\tau_{\coprod}}$ be the coproduct relatively-compact bornology, then $\mathcal{B}_{\tau_{F_{\mathcal{N}}}} = \beta_{\tau_{\coprod}}$.*

Proof: Based on the definition of $\mathcal{B}_{\tau_{F_{\mathcal{N}}}}$ and $\beta_{\tau_{\coprod}}$, since $\tau_{F_{\mathcal{N}}} = \tau_{\coprod}$ (by Lemma 2.3.9), then it follows that $\mathcal{B}_{\tau_{F_{\mathcal{N}}}} = \beta_{\tau_{\coprod}}$. □

Corollary 4.5.1 *Let $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct, $\mathcal{N}$ be Hausdorff, $\mathcal{B}_{\tau_{F_{\mathcal{N}}}}$ be the functional relatively-coproduct bornology on $\mathcal{N}$ and $\beta_{\tau_{\coprod}}$ be the coproduct relatively-compact bornology, and $(\mathcal{N}, \mathcal{B}_{\tau_{F_{\mathcal{N}}}})$ and $(\mathcal{N}, \beta_{\tau_{\coprod}})$ be bornological sets. Then the identity maps $id_1 : (\mathcal{N}, \mathcal{B}_{\tau_{F_{\mathcal{N}}}}) \longrightarrow (\mathcal{N}, \beta_{\tau_{\coprod}})$ and $id_2 : (\mathcal{N}, \beta_{\tau_{\coprod}}) \longrightarrow (\mathcal{N}, \mathcal{B}_{\tau_{F_{\mathcal{N}}}})$ are bounded.*

Proof: By Theorem 4.5.1 we have that $\mathcal{B}_{\tau_{F_{\mathcal{N}}}} = \beta_{\tau_{\coprod}}$ which implies that $\mathcal{B}_{\tau_{F_{\mathcal{N}}}} \subset \beta_{\tau_{\coprod}}$ and $\beta_{\tau_{\coprod}} \subset \mathcal{B}_{\tau_{F_{\mathcal{N}}}}$. Therefore it follows by definition that the identity maps $id_1 : (\mathcal{N}, \mathcal{B}_{\tau_{F_{\mathcal{N}}}}) \longrightarrow (\mathcal{N}, \beta_{\tau_{\coprod}})$ and $id_2 : (\mathcal{N}, \beta_{\tau_{\coprod}}) \longrightarrow (\mathcal{N}, \mathcal{B}_{\tau_{F_{\mathcal{N}}}})$ are bounded. □

Chapter 5

Compact bornologies.

5.1 Compact bornology from the Frölicher topology.

Let X be a topological space, then the subsets of X contained in a compact set forms a bornology called the compact bornology (see [55]). This implies that a Frölicher space (X, C_X, F_X) naturally inherits a compact bornology, with bounded sets as subsets of X contained in a τ_{F_X}-compact set.

Lemma 5.1.1 *Let (X, C_X, F_X) be a Frölicher space, then every $A \subseteq X$ is τ_{F_X}-compact.*

Proof: Since $f \in F_X$ is such that $f : X \longrightarrow \mathbb{R}$ is a real-valued function then $\forall A \subseteq X \ \exists \ V \in \tau_{\mathbb{R}}$ such that $A \subseteq f^{-1}(V) \in \tau_{F_X}$. In fact, by definition of the standard topology $\tau_{\mathbb{R}}$ on $\mathbb{R}$, $\forall \ A \subseteq X \ \exists \ V_\alpha \in \tau_{\mathbb{R}}$, $\alpha \in I$, such that
$A \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha) \in \tau_{F_X}$. Therefore every $A \subseteq X$ is τ_{F_X}-compact.
□

Lemma 5.1.2 *Let (X, C_X, F_X) be a Frölicher space, the collection*

$$\mathbb{B}_{\tau_{F_X}} := \{U \subseteq X \mid U \text{ is contained in a } \tau_{F_X} - \text{compact set}\}$$

is a bornology on X.

We have to show that $\mathbb{B}_{\tau_{F_X}}$ covers X, is closed under inclusion and closed under finite union. By Lemma 5.1.1 we have that every $A \subseteq X$ is τ_{F_X}-compact, which implies that for every singleton $\{x\} \subset X$ there exists a τ_{F_X}-compact set $A \subseteq X$ such that $\{x\} \subset A$, hence every singleton $\{x\} \subset X$ is contained in a τ_{F_X}-compact. Thus $\{x\} \in \mathbb{B}_{\tau_{F_X}}$, $\forall \ x \in X$, therefore $\mathbb{B}_{\tau_{F_X}}$ covers X.

Let $U_1, U_2, \cdots, U_n \in \mathbb{B}_{\tau_{F_X}}$, that is, $U_i \subseteq X$ is contained in a τ_{F_X}-compact set, $\forall\ i = 1, 2, \cdots, n$. Thus $\forall\ U_i \subseteq X$ there exists a τ_{F_X}-compact set, say $A_i \subseteq X$, such that $U_i \subset A_i$, $\forall\ i = 1, 2, \cdots, n$. Since $U_i \subset A_i \subseteq X$, $\forall\ i = 1, 2, \ldots, n$, then $\bigcup_{i=1}^{n} U_i \subset \bigcup_{i=1}^{n} A_i \subseteq X$ and $\bigcup_{i=1}^{n} A_i$ is τ_{F_X}-compact since the finite union of compact sets is compact by Theorem 2.2.1, therefore $\bigcup_{i=1}^{n} U_i \in \mathbb{B}_{\tau_{F_X}}$. Thus $\mathbb{B}_{\tau_{F_X}}$ is closed under finite union.
Let $M \subseteq N \in \mathbb{B}_{\tau_{F_X}}$, that is $M \subseteq N \subseteq X$ and N is contained in a τ_{F_X}-compact set. This implies that there exists a τ_{F_X}-compact set, say W, such that $N \subset W$. Since we have that $M \subseteq N \subset W \subseteq X$, therefore by transitivity of inclusion of sets we have that $M \subset W$. This implies that $M \in \mathbb{B}_{\tau_{F_X}}$, that is $\mathbb{B}_{\tau_{F_X}}$ is closed under inclusion. □

Definition 5.1.1 ***Functional compact bornology.***
Let (X, C_X, F_X) be a Frölicher space and τ_{F_X} be the functional topology on X. The compact bornology

$$\mathbb{B}_{\tau_{F_X}} := \{U \subseteq X \mid U \textit{ is contained in a } \tau_{F_X} - \textit{compact set}\}$$

is called the functional compact bornology.

5.2 Compact bornologies from Frölicher subspace.

5.2.1 Functional and subspace compact bornology.

Let (X, C_X, F_X) be a Frölicher space, $S \subseteq X$ and (S, C_S, F_S) be a Frölicher subspace of (X, C_X, F_X). Then we have the functional topology

$$\tau_{F_S} = \{U \subseteq S \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_S\}$$

on S, where $\tau_{\mathbb{R}}$ is the standard bornology on $\mathbb{R}$. By Lemma 5.1.2 the collection

$$\mathbb{B}_{\tau_{F_S}} := \{U \subseteq S \mid U \textit{ is contained in a } \tau_{F_S} - \textit{compact set}\}$$

is a functional compact bornology on S. There is a canonical topology induced on the ambient set of Frölicher subspace (S, C_S, F_S), that is on S. That is the subspace topology

$$\tau_{F_X}(S) := \{U \cap S \mid U \in \tau_{F_X}\},$$

where

$$\tau_{F_X} = \{U \subseteq X \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_X\}$$

is a functional topology X.

Lemma 5.2.1 *Let (X, C_X, F_X) be a Frölicher space, $S \subseteq X$, (S, C_S, F_S) be a Frölicher subspace and $\tau_{F_X}(S)$ be the subspace topology. The collection*

$$\mathbb{B}_{\tau_{F_X}}(S) := \{U \subseteq S \mid U \textit{ is contained in a } \tau_{F_X}(S) - \textit{compact set}\}$$

is a bornology.

Proof: We have to show that $\mathbb{B}_{\tau_{F_X}}(S)$ covers S, is closed under finite union and is closed under inclusion. By Lemma 5.1.1 every $A \subseteq X$ is τ_{F_X}-compact. Therefore $\forall\ A \subseteq X\ \exists\ V_\alpha \in \tau_{\mathbb{R}},\ \alpha \in I$, such that $A \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(V_\alpha) \in \tau_{F_X}$. Then $A \cap S \subseteq \left(\bigcup_{\alpha \in I} f^{-1}(V_\alpha)\right) \cap S = \bigcup_{\alpha \in I}(f^{-1}(V_\alpha) \cap S)$. Now since $f^{-1}(V_\alpha) \in \tau_{F_X}$, $\forall\ V_\alpha \in \tau_{\mathbb{R}}\ , \alpha \in I$, then $f^{-1}(V_\alpha) \cap S \in \tau_{F_X}(S)$ which implies that $\bigcup_{\alpha \in I}(f^{-1}(V_\alpha) \cap S) \in \tau_{F_X}(S)$ since $\tau_{F_X}(S)$ is closed under infinite union as it is a topology. Thus $A \cap S \subseteq \bigcup_{\alpha \in I}(f^{-1}(V_\alpha) \cap S) \in \tau_{F_X}(S)$, therefore $\forall\ A \subseteq X$ we have that $A \cap S$ is $\tau_{F_X}(S)$-compact. Now since $S \subseteq X$ then by Lemma 5.1.1 we have that S is τ_{F_X}-compact. That is, there exists an open cover $\{O_j \mid j \in J\} \subset \tau_{F_X}$ with a finite subcover such that $S \subseteq \bigcup_{j \in J} O_j$. Therefore since S is τ_{F_X}-compact then $\forall\ \{s\} \subset S\ \exists\ A \in \tau_{F_X}$ such that $\{s\} \subset A \cap S$. Thus $\{s\} \in \mathbb{B}_{\tau_{F_X}}(S)$, $\forall\ s \in S$, therefore $\mathbb{B}_{\tau_{F_X}}(S)$ covers S.
Let $U_1, U_2, \cdots, U_n \in \mathbb{B}_{\tau_{F_X}}(S)$, that is $U_i \subseteq S$ is contained in a $\tau_{F_X}(S)$-compact set, $\forall\ i = 1, 2, \cdots, n$. Thus $\forall\ U_i \subseteq S$ there exists a $\tau_{F_X}(S)$-compact set, say $A_i \subseteq S$, such that $U_i \subset A_i$, $\forall\, i = 1, 2, \cdots, n$. Since $U_i \subset A_i \subseteq S$, $\forall\, i = 1, 2, \cdots, n$, then $\bigcup_{i=1}^{n} U_i \subset \bigcup_{i=1}^{n} A_i \subseteq S$ and $\bigcup_{i=1}^{n} A_i$ is $\tau_{F_X}(S)$-compact since the finite union of compact sets is compact by Theorem 2.2.1, therefore $\bigcup_{i=1}^{n} U_i \in \mathbb{B}_{\tau_{F_X}}(S)$. Thus $\mathbb{B}_{\tau_{F_X}}(S)$ is closed under finite union.
Let $M \subseteq N \in \mathbb{B}_{\tau_{F_X}}(S)$, that is, $M \subseteq N \subseteq S$ and N is contained in a $\tau_{F_X}(S)$-compact set. This implies that there exists a $\tau_{F_X}(S)$-compact set, say W, such that $N \subset W$. Since we have that $M \subseteq N \subset W \subseteq S$, therefore by transitivity of inclusion of sets we have that $M \subset W$. This implies that $M \in \mathbb{B}_{\tau_{F_X}}(S)$. That is $M \subseteq N \in \mathbb{B}_{\tau_{F_X}}(S) \implies M \in \mathbb{B}_{\tau_{F_X}}(S)$, therefore $\mathbb{B}_{\tau_{F_X}}(S)$ is closed under inclusion. □

Definition 5.2.1 ***Subspace compact bornology.***
Let $S \subseteq X$, (X, C_X, F_X) be a Frölicher space and (S, C_S, F_S) be the Frölicher subspace, $\tau_{F_X}(S)$ be the subspace topology on S. The compact bornology

$$\mathbb{B}_{\tau_{F_X}}(S) := \{U \subseteq S \mid U \textit{ is contained in a } \tau_{F_X}(S) - \textit{compact set}\}$$

is called the subspace compact bornology.

5.2.2 Bornological comparison.

Let $S \subseteq X$, (X, C_X, F_X) be a Frölicher space and (S, C_S, F_S) be the Frölicher subspace. Then we have two topologies from the Frölicher subspace. That is, the canonical subspace topology $\tau_{F_X}(S)$ on S and the functional topology τ_{F_S} on S, and $\tau_{F_X}(S) \subset \tau_{F_S}$, by Lemma 2.3.5. Subsequently we have the subspace compact bornology $\mathbb{B}_{\tau_{F_X}}(S)$ and the functional compact bornology $\mathbb{B}_{\tau_{F_S}}$ on S, induced from $\tau_{F_X}(S)$ and τ_{F_S}, respectively. Since (X, C_X, F_X) is a Frölicher space then we have the functional compact bornology $\mathbb{B}_{\tau_{F_X}}$ on X, induced from the functional topology τ_{F_X} on X. We compare the bornologies $\mathbb{B}_{\tau_{F_X}}(S)$, $\mathbb{B}_{\tau_{F_S}}$ and $\mathbb{B}_{\tau_{F_X}}$ and their associated bounded maps.

Theorem 5.2.1 *Let $\mathbb{B}_{\tau_{F_X}}(S)$ be the subspace compact bornology, $\mathbb{B}_{\tau_{F_S}}$ be the functional compact bornology on S, and $(S, \mathbb{B}_{\tau_{F_X}}(S))$ and $(S, \mathbb{B}_{\tau_{F_S}})$ be bornological sets. Then $\mathbb{B}_{\tau_{F_X}}(S) \subset \mathbb{B}_{\tau_{F_S}}$ and thus the identity map $id : (S, \mathbb{B}_{\tau_{F_X}}(S)) \longrightarrow (S, \mathbb{B}_{\tau_{F_S}})$ is bounded.*

Proof: Let $U \in \mathbb{B}_{\tau_{F_S}}$, then $U \subseteq S$ and U is contained in a $\tau_{F_X}(S)$-compact set. That is there exists $A \subseteq S$ such that A is $\tau_{F_X}(S)$-compact and $U \subset A$. Since $\tau_{F_X}(S) \subset \tau_{F_S}$ (Lemma 2.3.5) and A is $\tau_{F_X}(S)$-compact then every open cover $\{O_j \mid j \in J\} \subset \tau_{F_X}(S) \subset \tau_{F_S}$ of A has a finite subcover $\{O_{j_i} \mid i = 1, 2, \cdots, n\} \subset \tau_{F_X}(S) \subset \tau_{F_S}$. Then it follows that A is τ_{F_S}-compact. Thus $U \subset A$ and A is τ_{F_S}-compact therefore, $U \in \mathbb{B}_{\tau_{F_S}}$. That is, $U \in \mathbb{B}_{\tau_{F_x}}(S) \implies U \in \mathbb{B}_{\tau_{F_S}}$, hence $\mathbb{B}_{\tau_{F_X}}(S) \subset \mathbb{B}_{\tau_{F_S}}$. Then it follows, by definition, that the identity map $id : (S, \mathbb{B}_{\tau_{F_X}}(S)) \longrightarrow (S, \mathbb{B}_{\tau_{F_S}})$ is bounded. □

Lemma 5.2.2 *Let $\mathbb{B}_{\tau_{F_X}}(S)$ be the subspace compact bornology, $\mathbb{B}_{\tau_{F_X}}$ be the functional compact bornology on X, then $\mathbb{B}_{\tau_{F_X}}(S) \subset \mathbb{B}_{\tau_{F_X}}$.*

Proof: Let $U \in \mathbb{B}_{\tau_{F_X}}(S)$, then $U \subseteq S$ and it is contained in a $\tau_{F_X}(S)$-compact set. That is, there exists A such that A is $\tau_{F_X}(S)$-compact and $U \subset A$. Since A is $\tau_{F_X}(S)$-compact then every open cover

$$\{M_\alpha \cap S \mid M_\alpha \in \tau_{F_X}, \alpha \in I\}$$

of A has a finite subcover, say

$$\{M_{\alpha_i} \cap S \mid M_{\alpha_i} \in \tau_{F_X}, i = 1, 2 \cdots, n\}.$$

Thus $A \subseteq \bigcup_{\alpha \in I}(M_\alpha \cap S) = \left(\bigcup_{\alpha \in I} M_\alpha\right) \cap S$ and $A \subseteq \bigcup_{i=1}^{n}(M_{\alpha_i} \cap S) = \left(\bigcup_{i=1}^{n} M_{\alpha_i}\right) \cap S$. But

$$A \subseteq \left(\bigcup_{\alpha \in I} M_\alpha\right) \cap S \implies A \subseteq \bigcup_{\alpha \in I} M_\alpha$$

and

$$A \subseteq \left(\bigcup_{i=1}^{n} M_{\alpha_i} \right) \cap S \implies A \subseteq \bigcup_{i=1}^{n} M_{\alpha_i},$$

where $M_\alpha, M_{\alpha_i} \in \tau_{F_X}$, $\alpha \in I$ and $i = 1, 2, \cdots, n$. Therefore

$$\{M_\alpha \mid M_\alpha \in \tau_{F_X}, \alpha \in I\}$$

is an open cover of A, with finite subcover

$$\{M_{\alpha_i} \mid M_{\alpha_i} \in \tau_{F_X}, i = 1, 2 \cdots, n\}.$$

That is A is τ_{F_X}-compact. Also $U \subseteq S \implies U \subseteq X$ since $S \subseteq X$, and $U \subset A$, thus we have that U is contained in a τ_{F_X}-compact set. Therefore $U \in \mathbb{B}_{\tau_{F_X}}$. That is $U \in \mathbb{B}_{\tau_{F_X}}(S) \implies U \in \mathbb{B}_{\tau_{F_X}}$, hence $\mathbb{B}_{\tau_{F_X}}(S) \subset \mathbb{B}_{\tau_{F_X}}$. □

Proposition 5.2.1 *Let τ_{F_X} be the functional topology on X, $\mathbb{B}_{\tau_{F_X}}(S)$ be the subspace compact bornology and $\mathbb{B}_{\tau_{F_S}}$ be the functional bornology on S. If $S \in \tau_{F_X}$ then $\mathbb{B}_{\tau_{F_X}}(S) = \mathbb{B}_{\tau_{F_S}}$.*

Proof: Let $\tau_{F_X}(S)$ be the subspace topology on S and τ_{F_S} be the functional topology on S, If $S \in \tau_{F_X}$ then by Proposition 2.3.1 we have that $\tau_{F_X}(S) = \tau_{F_S}$. Let $U \in \mathbb{B}_{\tau_{F_X}}(S)$, then there exists A such that A is $\tau_{F_X}(S)$-compact and $U \subset A$. But since $\tau_{F_X}(S) = \tau_{F_S}$ then it follows that A is also τ_{F_S}-compact. That is $U \subset A$ and A is τ_{F_S}-compact, therefore $U \in \mathbb{B}_{\tau_{F_S}}$.
Similarly let $U \in \mathbb{B}_{\tau_{F_S}}$, then there exists A such that A is τ_{F_S}-compact and $U \subset A$. But since $\tau_{F_X}(S) = \tau_{F_S}$ by Proposition 2.3.1 then it follows that A is also $\tau_{F_X}(S)$-compact. That is, $U \subset A$ and A is $\tau_{F_X}(S)$-compact, therefore $U \in \mathbb{B}_{\tau_{F_X}}(S)$.
Thus $U \in \mathbb{B}_{\tau_{F_X}}(S) \implies U \in \mathbb{B}_{\tau_{F_S}}$ and $U \in \mathbb{B}_{\tau_{F_S}} \implies U \in \mathbb{B}_{\tau_{F_X}}(S)$, hence $\mathbb{B}_{\tau_{F_X}}(S) = \mathbb{B}_{\tau_{F_S}}$. □

Corollary 5.2.1 *Let τ_{F_X} be the functional topology on X and $(S, \mathbb{B}_{\tau_{F_X}}(S))$ and $(S, \mathbb{B}_{\tau_{F_S}})$ be bornological sets. If $S \in \tau_{F_X}$ then the identity maps $id_1 : (S, \mathbb{B}_{\tau_{F_X}}(S)) \longrightarrow (S, \mathbb{B}_{\tau_{F_S}})$ and $id_2 : (S, \mathbb{B}_{\tau_{F_S}}) \longrightarrow (S, \mathbb{B}_{\tau_{F_X}}(S))$ are bounded.*

Proof: Let $S \in \tau_{F_X}$ then by Proposition 5.2.1 we have that $\mathbb{B}_{\tau_{F_X}}(S) = \mathbb{B}_{\tau_{F_S}}$. But $\mathbb{B}_{\tau_{F_X}}(S) = \mathbb{B}_{\tau_{F_S}}$ implies that $\mathbb{B}_{\tau_{F_X}}(S) \subset \mathbb{B}_{\tau_{F_S}}$ and $\mathbb{B}_{\tau_{F_S}} \subset \mathbb{B}_{\tau_{F_X}}(S)$. Therefore, it follows that the identity maps $id_1 : (S, \mathbb{B}_{\tau_{F_X}}(S)) \longrightarrow (S, \mathbb{B}_{\tau_{F_S}})$ and $id_2 : (S, \mathbb{B}_{\tau_{F_S}}) \longrightarrow (S, \mathbb{B}_{\tau_{F_X}}(S))$ are bounded, by definition. □

5.3 Compact bornologies from Frölicher product.

5.3.1 Functional and product compact bornology.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces, $\mathfrak{X} := \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets and $(\mathfrak{X}, C_{\mathfrak{X}}, F_{\mathfrak{X}})$ be a Frölicher product. We have the functional topology

$$\tau_{F_{\mathfrak{X}}} := \{U \subseteq \mathfrak{X} \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_{\mathfrak{X}}\}$$

on $\mathfrak{X}$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. We also have the canonical product topology on $\mathfrak{X}$, denoted τ_π, generated by the basis

$$\beta := \left\{\bigcap_{i=1}^{n} p_i^{-1}(U_i) \mid U_i \in \tau_{F_{X_i}},\ p_i : \mathfrak{X} \longrightarrow X_i,\ \forall\ i = 1, 2, \cdots, n\right\}$$

where $\tau_{F_{X_i}}$ is a functional topology on X_i and $p_i : \mathfrak{X} \longrightarrow X_i$ is a canonical projection, $\forall\ i = 1, 2, \cdots, n$. By Lemma 5.1.2 we have that the collection

$$\mathbb{B}_{\tau_{F_{\mathfrak{X}}}} := \{U \subseteq \mathfrak{X} \mid U \text{ is contained in a } \tau_{F_{\mathfrak{X}}} - \text{compact set}\}$$

is the functional compact bornology on $\mathfrak{X}$.

Lemma 5.3.1 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces, $\mathfrak{X} = \prod_{i=1}^{n} X_i$ be a Cartesian product in the category of sets and τ_π be the canonical product topology on $\mathfrak{X}$. The collection*

$$\mathbb{B}_{\tau_\pi} := \{U \subseteq \mathfrak{X} \mid U \text{ is contained in a } \tau_\pi - \text{compact}\}$$

is a bornology on $\mathfrak{X}$.

Proof: We have to show that $\mathbb{B}_{\tau_\pi}$ covers $\mathfrak{X}$, is closed under finite union and is closed under inclusion. Since

$$\beta := \left\{\bigcap_{i=1}^{n} p_i^{-1}(U_i) \mid U_i \in \tau_{F_{X_i}},\ p_i : \mathfrak{X} \longrightarrow X_i,\ \forall\ i = 1, 2, \cdots, n\right\}$$

is a basis for the canonical product topology τ_π then it follows that
$\forall\ (x_1, x_2, \cdots, x_n) \in \mathfrak{X}\ \exists\ B \in \beta$ such that $(x_1, x_2, \cdots, x_n) \in B$. Since $B \in \beta$ then $B = \bigcap_{i=1}^{n} p_i^{-1}(U_i)$, therefore $B \subset p_i^{-1}(U_i),\ \forall\ i = 1, 2, \cdots, n$. Since

$\bigcap_{i=1}^{n} p_i^{-1}(U_i) \subset p_i^{-1}(U_i) \subset \mathfrak{X}, \ \forall\, i = 1, 2, \cdots, n$, and β is a basis of τ_π then it follows that $\forall\ (x_1, x_2, \cdots, x_n) \in p_i^{-1}(U_i)\ \exists\ B \in \beta$ such that $(x_1, x_2, \cdots, x_n) \in B \subset \beta$, therefore $p_i^{-1}(U_i) \in \tau_\pi,\ \forall\ i = 1, 2, \cdots, n$. Also, since $B \in \beta \subset \tau_\pi$ then there exists $O_\alpha \in \tau_\pi$, $\alpha \in I$, such that $\beta \subset \bigcup_{\alpha\in I} O_\alpha \in \tau_\pi \implies B \subset \bigcup_{\alpha\in I} O_\alpha \in \tau_\pi$. For every $i = 1, 2, \cdots, n$ we also have that $\bigcup_{\alpha\in I}(p_i^{-1}(U_i) \cup O_\alpha) \in \tau_\pi$ since $p_i^{-1}(U_i) \in \tau_\pi$, $\forall\, i = 1, 2, \cdots, n$. It then follows that every open cover of B has a finite subcover. Therefore B is τ_π-compact. Now since $\forall\ (x_1, x_2, \cdots, x_n) \in \mathfrak{X}\ \exists\ B \in \beta$ such that $(x_1, x_2, \cdots, x_n) \in B$ therefore $\{(x_1, x_2, \cdots, x_n)\} \subset B,\ \forall\ (x_1, x_2, \cdots, x_n) \in \mathfrak{X}$. Thus $\forall\ (x_1, x_2, \cdots, x_n) \in \mathfrak{X}$ we have that $\{(x_1, x_2, \cdots, x_n)\} \in \mathbb{B}_{\tau_\pi}$. This gives that $\mathbb{B}_{\tau_\pi}$ covers $\mathfrak{X}$.
Let $U_1, U_2, \cdots, U_n \in \mathbb{B}_{\tau_\pi}$, that is, $U_i \subseteq \mathfrak{X}$ is contained in a τ_π-compact set, $\forall\, i = 1, 2, \cdots, n$. Thus $\forall\ U_i \subseteq \mathfrak{X}$ there exists a τ_π-compact set, say A_i, such that $U_i \subset A_i$, $\forall\, i = 1, 2, \ldots, n$. Since $U_i \subset A_i$, $\forall\, i = 1, 2, \cdots, n$, then $\mathfrak{X} \supseteq \bigcup_{i=1}^{n} U_i \subset \bigcup_{i=1}^{n} A_i$ and $\bigcup_{i=1}^{n} A_i$ is τ_π-compact since the finite union of compact sets is compact by Theorem 2.2.1, therefore $\bigcup_{i=1}^{n} U_i \in \mathbb{B}_{\tau_\pi}$. Thus $\mathbb{B}_{\tau_\pi}$ is closed under finite union.
Let $M \subseteq N \in \mathbb{B}_{\tau_\pi}$, that is $M \subseteq N \subseteq \mathfrak{X}$ and N is contained in a τ_π-compact set. This implies that there exists a τ_π-compact set, say W, such that $N \subset W$. Since we have that $M \subseteq N \subset W$, therefore by transitivity of inclusion of sets we have that $M \subset W$. This implies that $M \in \mathbb{B}_{\tau_\pi}$. That is $M \subseteq N \in \mathbb{B}_{\tau_\pi} \implies M \in \mathbb{B}_{\tau_\pi}$ therefore $\mathbb{B}_{\tau_\pi}$ is closed under inclusion. □

Definition 5.3.1 ***Product compact bornology.***
Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces, $\mathfrak{X} := \prod_{i=1}^{n} X_i$, $(\mathfrak{X}, C_\mathfrak{X}, F_\mathfrak{X})$ be a Frölicher product and τ_π be the canonical product topology on $\mathfrak{X}$. The bornology

$$\mathbb{B}_{\tau_\pi} := \{U \subseteq \mathfrak{X} \mid U \textit{ is contained in a } \tau_\pi - \textit{compact}\}$$

is called the product compact bornology.

5.3.2 Bornological comparison.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i = 1, 2, \cdots, n\}$ be a family of Frölicher spaces, $\mathfrak{X} := \prod_{i=1}^{n} X_i$ be the Cartesian product in the category of sets and $(\mathfrak{X}, C_\mathfrak{X}, F_\mathfrak{X})$ be the Frölicher product. We have two topologies from $(\mathfrak{X}, C_\mathfrak{X}, F_\mathfrak{X})$, that is the functional topology

$\tau_{F_{\mathfrak{X}}}$ on $\mathfrak{X}$ and the product topology τ_π, and $\tau_{F_{\mathfrak{X}}} = \tau_\pi$, by Lemma 2.3.7. Subsequently we have the functional compact bornology $\mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$ and the product compact bornology $\mathbb{B}_{\tau_\pi}$, induced from $\tau_{F_{\mathfrak{X}}}$ and τ_π, respectively. The bornologies $\mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$ and $\mathbb{B}_{\tau_\pi}$ are compared and the bounded maps between them are determined.

Theorem 5.3.1 *Let $\mathbb{B}_{\tau_\pi}$ be the product compact bornology and $\mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$ be the functional compact bornology on $\mathfrak{X}$, then $\mathbb{B}_{\tau_\pi} = \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$.*

Proof: Let $U \in \mathbb{B}_{\tau_\pi}$, then $U \subseteq \mathfrak{X}$ and U is contained in a τ_π-compact set. That is there exists A such that A is τ_π-compact and $U \subset A$. With A being τ_π-compact then every open cover $\{O_\alpha \mid \alpha \in I\} \subset \tau_\pi$ of A has a finite subcover. Now since $\tau_\pi = \tau_{F_{\mathfrak{X}}}$ (Lemma 2.3.7) then it follows that A is also $\tau_{F_{\mathfrak{X}}}$-compact. That is $U \subset A$ and A is $\tau_{F_{\mathfrak{X}}}$-compact, therefore $U \in \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$.
Similarly let $U \in \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$, then there exists $B \subseteq \mathfrak{X}$ such that B is $\tau_{F_{\mathfrak{X}}}$-compact and $U \subset B$. Every open cover $\{P_j \mid j \in J\} \subset \tau_{F_{\mathfrak{X}}}$ of B has a finite subcover since B is $\tau_{F_{\mathfrak{X}}}$-compact. It follows that B is τ_π-compact since $\tau_\pi = \tau_{F_{\mathfrak{X}}}$. That is $U \subset B$ and B is τ_π-compact, therefore $U \in \mathbb{B}_{\tau_\pi}$. Thus we have that $U \in \mathbb{B}_{\tau_{F_{\mathfrak{X}}}} \implies U \in \mathbb{B}_{\tau_\pi}$ and $U \in \mathbb{B}_{\tau_\pi} \implies U \in \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$, hence $\mathbb{B}_{\tau_\pi} = \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$. □

Corollary 5.3.1 *Let $\mathbb{B}_{\tau_\pi}$ be the product compact bornology, $\mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$ be the functional compact bornology on $\mathfrak{X}$, and $(\mathfrak{X}, \mathbb{B}_{\tau_\pi})$ and $(\mathfrak{X}, \mathbb{B}_{\tau_{F_{\mathfrak{X}}}})$ be bornological sets. The identity maps $id_1 : (\mathfrak{X}, \mathbb{B}_{\tau_\pi}) \longrightarrow (\mathfrak{X}, \mathbb{B}_{\tau_{F_{\mathfrak{X}}}})$ and $id_2 : (\mathfrak{X}, \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}) \longrightarrow (\mathfrak{X}, \mathbb{B}_{\tau_\pi})$ are bounded.*

Proof: By Theorem 5.3.1 we have that $\mathbb{B}_{\tau_{\mathfrak{X}}} = \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$. But $\mathbb{B}_{\tau_{\mathfrak{X}}} = \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$ implies that $\mathbb{B}_{\tau_\pi} \subset \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$ and $\mathbb{B}_{\tau_{F_{\mathfrak{X}}}} \subset \mathbb{B}_{\tau_\pi}$. By definition, $\mathbb{B}_{\tau_\pi} \subset \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}$ implies that the identity map $id_1 : (\mathfrak{X}, \mathbb{B}_{\tau_\pi}) \longrightarrow (\mathfrak{X}, \mathbb{B}_{\tau_{F_{\mathfrak{X}}}})$ is bounded. Similarly the identity map $id_2 : (\mathfrak{X}, \mathbb{B}_{\tau_{F_{\mathfrak{X}}}}) \longrightarrow (\mathfrak{X}, \mathbb{B}_{\tau_\pi})$ is bounded, since $\mathbb{B}_{\tau_{F_{\mathfrak{X}}}} \subset \mathbb{B}_{\tau_\pi}$. □

5.4 Compact bornologies from Frölicher quotient.

5.4.1 Functional and quotient compact bornology.

Let (Q, C_Q, F_Q) be the Frölicher quotient. That is $Q := X/_\sim$ is a quotient set, $g : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$ is Frölicher smooth where (X, C_X, F_X) and (Y, C_Y, F_Y) are Frölicher spaces, and $\sim$ is the equivalence relation on X defined by: $x \sim y$ if and only if $g(x) = g(y)$, $\forall\ x, y \in X$. From the Frölicher quotient structure (C_Q, F_Q) we have the functional topology

$$\tau_{F_Q} := \{U \subseteq Q \mid U = f^{-1}(V),\ V \in \tau_{\mathbb{R}},\ \forall\ f \in F_Q\}$$

on Q induced from the structure functions of F_Q. Denoted $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. We also have the canonical quotient topology on Q, that is

$$\tau_\sim := \{U \subseteq Q \mid q^{-1}(U) \in \tau_{F_X}\}$$

where $q : X \longrightarrow Q$ is a canonical projection and τ_{F_X} is the functional topology on X. By Lemma 5.1.2 we have that the collection

$$\mathbb{B}_{\tau_{F_Q}} := \{U \subseteq Q \mid U \textit{ is contained in a } \tau_{F_Q} - \textit{compact set}\}$$

is a functional compact bornology on Q.

Lemma 5.4.1 *Let $Q := X/_\sim$ be a quotient set as defined above and $\tau_\sim$ be the canonical qotient topology on Q. The collection*

$$\mathbb{B}_{\tau_\sim} := \{U \subseteq Q \mid U \textit{ is contained in a } \tau_\sim - \textit{compact set}\}$$

is a bornology.

Proof: We have to show that $\mathbb{B}_{\tau_\sim}$ is closed under inclusion, is closed under finite union and covers Q. Since $\rho_{\tau_\sim} = \{U \subseteq Q \mid U \textit{ is } \tau_\sim - \textit{compact}\}$ is a base of a bornology on Q (see Section 4.4) then $\rho_{\tau_\sim}$ covers Q, therefore $\{[x]\} \in \rho_{\tau_\sim}$, $\forall\ [x] \in Q$. That is every singleton $\{[x]\}$ is $\tau_\sim$-compact, $\forall\ [x] \in Q$. Then for every $\tau_\sim$-compact singleton $\{[y]\}$ such that $x \nsim y \iff [x] \neq [y]$ we have that $\{[x]\} \subset \{[x]\} \cup \{[y]\}$,$\forall\ [x] \in Q$, and $\{[x]\} \cup \{[y]\}$ is $\tau_\sim$-compact since the finite union of compact sets is compact by Theorem 2.2.1. Therefore $\forall\ [x] \in Q$, every singleton $\{[x]\}$ is contained in a $\tau_\sim$-compact set. Thus $\{[x]\} \in \mathbb{B}_{\tau_\sim}$, $\forall\ [x] \in Q$, therefore $\mathbb{B}_{\tau_\sim}$ covers Q.
Let $U_1, U_2, \cdots, U_n \in \mathbb{B}_{\tau_\sim}$, that is, $U_i \subseteq Q$ is contained in a $\tau_\sim$-compact set, $\forall\ i = 1, 2, \cdots, n$. Thus $\forall\ U_i \subseteq Q$ there exists a $\tau_\sim$-compact set, say $A_i \subset Q$, such that $U_i \subset A_i$, $\forall\ i = 1, 2, \cdots, n$. Since $U_i \subset A_i$, $\forall\ i = 1, 2, \cdots, n$, then $\bigcup_{i=1}^{n} U_i \subset \bigcup_{i=1}^{n} A_i \subset Q$ and $\bigcup_{i=1}^{n} A_i$ is $\tau_\sim$-compact since the finite union of compact sets is compact by Theorem 2.2.1, therefore $\bigcup_{i=1}^{n} U_i \in \mathbb{B}_{\tau_\sim}$. Thus $\mathbb{B}_{\tau_\sim}$ is closed under finite union.
Let $M \subseteq N \in \mathbb{B}_{\tau_\sim}$, that is, $M \subseteq N \subseteq Q$ and N is contained in a $\tau_\sim$-compact set. This implies that there exists a $\tau_\sim$-compact set, say W, such that $N \subset W$. Since we have that $M \subseteq N \subset W$, therefore by transitivity of inclusion of sets we have that $M \subset W$. This implies that $M \in \mathbb{B}_{\tau_\sim}$. That is $M \subseteq N \in \mathbb{B}_{\tau_\sim} \implies M \in \mathbb{B}_{\tau_\sim}$ therefore $\mathbb{B}_{\tau_\sim}$ is closed under inclusion. □

Definition 5.4.1 ***Quotient compact bornology.***
Let (X, C_X, F_X) and (Y, C_Y, F_Y) be Frölicher spaces, $\sim$ be an equivalence relation on X defined by $x \sim y$ if and only if $g(x) = g(y)$ where
$g : (X, C_X, F_X) \longrightarrow (Y, C_Y, F_Y)$ is Frölicher smooth, $Q = X/_\sim$ be a quotient set and $\tau_\sim$ is the canonical quotient topology. The bornology

$$\mathbb{B}_{\tau_\sim} := \{U \subseteq Q \mid U \textit{ is contained in a } \tau_\sim - \textit{compact set}\}$$

is called the quotient compact bornology.

5.4.2 Bornological comparison.

Let (Q, C_Q, F_Q) be the Frölicher quotient. We have two topologies from (Q, C_Q, F_Q), that is, the functional topology τ_{F_Q} and the quotient topology $\tau_\sim$. By Lemma 2.3.12, $\tau_{F_Q} = \tau_\sim$. Subsequently we have the functional compact bornology $\mathbb{B}_{\tau_{F_Q}}$ on Q and the quotient compact bornology $\mathbb{B}_{\tau_\sim}$. These bornologies are compared, in this subsection.

Theorem 5.4.1 *Let $\mathbb{B}_{\tau_\sim}$ be the quotient compact bornology and $\mathbb{B}_{\tau_{F_Q}}$ be the functional compact bornology on Q, then $\mathbb{B}_{\tau_{F_Q}} = \mathbb{B}_{\tau_\sim}$.*

Proof: Let $U \in \mathbb{B}_{\tau_{F_Q}} \cup \mathbb{B}_{\tau_\sim}$ then there exists B such that B is a τ-compact set and $U \subset B$ where $\tau = \tau_{F_Q} = \tau_\sim$. Therefore $U \in \mathbb{B}_{\tau_{F_Q}} \cap \mathbb{B}_{\tau_\sim}$, then it follows that $\mathbb{B}_{\tau_{F_Q}} = \mathbb{B}_{\tau_\sim}$. □

Corollary 5.4.1 *Let $\mathbb{B}_{\tau_\sim}$ be the quotient compact bornology, $\mathbb{B}_{\tau_{F_Q}}$ be the functional compact bornology on Q, and $(Q, \mathbb{B}_{\tau_\sim})$ and $(Q, \mathbb{B}_{\tau_{F_Q}})$ be bornological sets. The identity maps $id_1 : (Q, \mathbb{B}_{\tau_{F_Q}}) \longrightarrow (Q, \mathbb{B}_{\tau_\sim})$ and $id_2 : (Q, \mathbb{B}_{\tau_\sim}) \longrightarrow (Q, \mathbb{B}_{\tau_{F_Q}})$ are bounded.*

Proof: By Theorem 5.4.1 we have that $\mathbb{B}_{\tau_{F_Q}} = \mathbb{B}_{\tau_\sim}$, which implies that $\mathbb{B}_{\tau_{F_Q}} \subset \mathbb{B}_{\tau_\sim}$ and $\mathbb{B}_{\tau_\sim} \subset \mathbb{B}_{\tau_{F_Q}}$. Therefore it follows, by definition, that the identity maps $id_1 : (Q, \mathbb{B}_{\tau_{F_Q}}) \longrightarrow (Q, \mathbb{B}_{\tau_\sim})$ and $id_2 : (Q, \mathbb{B}_{\tau_\sim}) \longrightarrow (Q, \mathbb{B}_{\tau_{F_Q}})$ are bounded. □

5.5 Compact bornologies from Frölicher coproduct.

5.5.1 Functional and coproduct compact bornology.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a family of Frölicher spaces and $(\mathcal{N}, C_\mathcal{N}, F_\mathcal{N})$ be the Frölicher coprodut. That is

$$\mathcal{N} := \coprod_{i \in I} X_i = \bigcup_{i \in I} \{(x_i, i) \mid x_i \in X_i\}$$

is a coproduct in the category of sets and $(C_\mathcal{N}, F_\mathcal{N})$ is a Frölicher structure on $\mathcal{N}$. We have the functional topology

$$\tau_{F_\mathcal{N}} := \{U \subseteq \mathcal{N} \mid U = f^{-1}(V),\ V \in \tau_\mathbb{R},\ \forall\ f \in F_\mathcal{N}\}$$

on $\mathcal{N}$, where $\tau_\mathbb{R}$ is the standard topology on $\mathbb{R}$. We also have the canonical topology on $\mathcal{N}$, that is, the coproduct topology

$$\tau_\coprod := \{U \subseteq \mathcal{N} \mid \varphi_i^{-1}(U) \in \tau_{F_{X_i}},\ i \in I\},$$

where $\varphi_i : X_i \longrightarrow \mathcal{N}$ is the canonical projection and $\tau_{F_{X_i}}$ is the functional topology on X_i, $i \in I$. By Lemma 5.1.2 we have that the collection

$$\mathbb{B}_{\tau_{F_{\mathcal{N}}}} := \{U \subseteq \mathcal{N} \mid U \textit{ is contained in a } \tau_{F_{\mathcal{N}}} - \textit{compact set}\}$$

is a functional compact bornology on $\mathcal{N}$.

Lemma 5.5.1 *Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a collection of Frölicher spaces, $\mathcal{N} := \coprod_{i\in I} X_i$ be a coproduct in the category of sets and $\tau_{\coprod}$ be the canonical coproduct topology on $\mathcal{N}$. The collection*

$$\mathbb{B}_{\tau_{\coprod}} := \{U \subseteq \mathcal{N} \mid U \textit{ is contained in } \tau_{\coprod} - \textit{compact set}\}$$

is a bornology on $\mathcal{N}$.

Proof: We have to show that $\mathbb{B}_{\tau_{\coprod}}$ is closed under inclusion, is closed under finite union and covers $\mathcal{N}$. Let $k \in I$ be fixed, $x \in X_k$ be arbitrary and fixed, consider the singleton $\{(x, k)\}$. Since $\rho_{\tau_{\coprod}} := \{U \subseteq \mathcal{N} \mid U \textit{ is } \tau_{\coprod} - \textit{compact}\}$ is the base of a bornology on $\mathcal{N}$ (see Lemma 4.5.4) then every singleton $\{(x_i, i)\}$, with $x_i \in X_i$, is $\tau_{\coprod}$-compact for $i \in I$ since $\rho_{\tau_{\coprod}}$ covers $\mathcal{N}$. Therefore $\{(x, k)\}$ is also $\tau_{\coprod}$-compact. Also $\{(x, k)\} \subset \{(y, k) \mid y \in X_k\}$, but

$$\begin{aligned} \{(y,k) \mid y \in X_k\} &= \{(x,k)\} \cup \left(\bigcup_{w\in X_k} \{(w,k)\} \right), w \neq x \\ &= \bigcup_{w\in X_k} (\{(x,k)\} \cup \{(w,k)\}) \end{aligned} \tag{5.1}$$

With $\rho_{\tau_{\coprod}}$ a base of a bornology on $\mathcal{N}$ it follows that $\{(w, k)\}$ is $\tau_{\coprod}$-compact $\forall\ w \in X_k,\ w \neq x$. Now since the finite union of compact sets is compact by Theorem 2.2.1 therefore

$$\bigcup_{w\in X_k} (\{(x,k)\} \cup \{(w,k)\})$$

is $\tau_{\coprod}$-compact, thus $\{(y, k) \mid y \in X_k\}$ is $\tau_{\coprod}$-compact. That is $\{(y, k) \mid y \in X_k\}$ is $\tau_{\coprod}$-compact and $\{(x, k)\} \subset \{(y, k) \mid y \in X_k\}$, therefore $\{(x, k)\} \in \mathbb{B}_{\tau_{\coprod}}$. Now since $x \in X_k$ is arbitrary therefore $\{(x_i, i)\} \in \mathbb{B}_{\tau_{\coprod}}$, $\forall\ (x_i, i) \in \mathcal{N}$. This gives that $\mathbb{B}_{\tau_{\coprod}}$ covers $\mathcal{N}$.

Let $U_1, U_2, \cdots, U_n \in \mathbb{B}_{\tau_{\coprod}}$, that is, $U_j \subseteq \mathcal{N}$ is contained in a $\tau_{\coprod}$-compact set, $\forall\ j = 1, 2, \cdots, n$. Thus $\forall\ U_j \subseteq \mathcal{N}$ there exists a $\tau_{\coprod}$-compact set, say $A_j \subset \mathcal{N}$, such that $U_j \subset A_j$, $\forall\ j = 1, 2, \cdots, n$. Since $U_j \subset A_j$, $\forall\ j = 1, 2, \cdots, n$, then $\bigcup_{j=1}^{n} U_j \subset \bigcup_{j=1}^{n} A_j \subset \mathcal{N}$ and $\bigcup_{j=1}^{n} A_j$ is $\tau_{\coprod}$-compact since the finite union of compact sets is compact by Theorem 2.2.1, therefore $\bigcup_{j=1}^{n} U_j \in \mathbb{B}_{\coprod}$. Thus $\mathbb{B}_{\tau_{\coprod}}$ is closed

under finite union.
Let $M \subseteq N \in \mathbb{B}_{\tau_{\coprod}}$, that is, $M \subseteq N \subseteq \mathcal{N}$ and N is contained in a $\tau_{\coprod}$-compact set. This implies that there exists a $\tau_{\coprod}$-compact set, say W, such that $N \subset W$. Since we have that $M \subseteq N \subset W$, therefore by transitivity of inclusion of sets we have that $M \subset W$. This implies that $M \in \mathbb{B}_{\tau_{\coprod}}$. That is $M \subseteq N \in \mathbb{B}_{\tau_{\coprod}} \implies M \in \mathbb{B}_{\tau_{\coprod}}$ therefore $\mathbb{B}_{\tau_{\coprod}}$ is closed under inclusion. □

Definition 5.5.1 ***Coproduct compact bornology.***
Let $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$ be the Frölicher coproduct and $\tau_{\coprod}$ be the canonical coproduct topology. The bornology

$$\mathbb{B}_{\tau_{\coprod}} := \{U \subseteq \mathcal{N} \mid U \textit{ is contained in } \tau_{\coprod} - \textit{compact set}\}$$

is called the coproduct compact bornology.

5.5.2 Bornological comparison.

Let $\{(X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a family of Frölicher spaces, $\mathcal{N} = \coprod_{i \in I} X_i$ be a coproduct in the category of sets. Consider the Frölicher coproduct $(\mathcal{N}, C_{\mathcal{N}}, F_{\mathcal{N}})$. We have two bornologies from the Frölicher coproduct. That is, the functional compact bornology $\mathbb{B}_{\tau_{F_{\mathcal{N}}}}$ induced from the functional topology $\tau_{F_{\mathcal{N}}}$ on $\mathcal{N}$, and the coproduct compact bornology $\mathbb{B}_{\tau_{\coprod}}$ induced from the coproduct topology $\tau_{\coprod}$. The functional topology $\tau_{F_{\mathcal{N}}}$ is induced from structure functions on $\mathcal{N}$, that is from $F_{\mathcal{N}}$, and the coproduct topology $\tau_{\coprod}$ is the canonical topology on $\mathcal{N}$, and $\tau_{\coprod} = \tau_{F_{\mathcal{N}}}$ by Lemma 2.3.9. The bornologies $\mathcal{B}_{\tau_{F_{\mathcal{N}}}}$ and $\mathbb{B}_{\tau_{\coprod}}$ are compared.

Theorem 5.5.1 *Let $\mathbb{B}_{\tau_{\coprod}}$ be the coproduct compact bornology, $\mathbb{B}_{\tau_{F_{\mathcal{N}}}}$ be the functional bornology on $\mathcal{N}$, then $\mathbb{B}_{\tau_{F_{\mathcal{N}}}} = \mathbb{B}_{\tau_{\coprod}}$.*

Proof: Let $U \in \mathbb{B}_{\tau_{F_{\mathcal{N}}}}$ then $U \subseteq \mathcal{N}$ and U is contained in a $\tau_{F_{\mathcal{N}}}$-compact set. That is there exists $A \subseteq \mathcal{N}$ such that A is $\tau_{F_{\mathcal{N}}}$-compact and $U \subset A$. Since A is $\tau_{F_{\mathcal{N}}}$-compact therefore every open cover, say $\{O_\alpha \mid O_\alpha \in \tau_{F_{\mathcal{N}}}, \alpha \in I\}$, of A has a finite subcover, say $\{O_i \mid O_i \in \tau_{F_{\mathcal{N}}}, i = 1, 2 \ldots, n\}$. Now since $\tau_{F_{\mathcal{N}}} = \tau_{\coprod}$ then it follows that A is also $\tau_{\coprod}$-compact. Since $U \subset A$ and A is $\tau_{\coprod}$-compact, therefore $U \in \mathbb{B}_{\tau_{\coprod}}$. That is $U \in \mathbb{B}_{\tau_{F_{\mathcal{N}}}} \implies U \in \mathbb{B}_{\tau_{\coprod}}$. Similarly, $U \in \mathbb{B}_{\tau_{D_{\mathcal{N}}}} \implies \mathbb{B}_{\tau_{\coprod}}$. Hence $\mathbb{B}_{\tau_{F_{\mathcal{N}}}} = \mathbb{B}_{\tau_{\coprod}}$. □

Corollary 5.5.1 *Let $\mathbb{B}_{\tau_{\coprod}}$ be the coproduct compact bornology, $\mathbb{B}_{\tau_{F_{\mathcal{N}}}}$ be the functional bornology on $\mathcal{N}$, and $(\mathcal{N}, \mathbb{B}_{\tau_{F_{\mathcal{N}}}})$ and $(\mathcal{N}, \mathbb{B}_{\tau_{\coprod}})$ be bornological sets. The identity maps $id_1 : (\mathcal{N}, \mathbb{B}_{\tau_{F_{\mathcal{N}}}}) \longrightarrow (\mathcal{N}, \mathbb{B}_{\tau_{\coprod}})$ and $id_2 : (\mathcal{N}, \mathbb{B}_{\tau_{\coprod}}) \longrightarrow (\mathcal{N}, \mathbb{B}_{\tau_{F_{\mathcal{N}}}})$ are bounded.*

Proof: We have that $\mathbb{B}_{\tau_{F_{\mathcal{N}}}} = \mathbb{B}_{\tau_{\coprod}}$ by Theorem 5.5.1. But $\mathcal{B}_{\tau_{F_{\mathcal{N}}}} = \mathcal{B}_{\tau_{\coprod}}$ implies that $\mathbb{B}_{\tau_{F_{\mathcal{N}}}} \subset \mathbb{B}_{\tau_{\coprod}}$ and $\mathbb{B}_{\tau_{\coprod}} \subset \mathbb{B}_{\tau_{F_{\mathcal{N}}}}$. And $\mathbb{B}_{\tau_{F_{\mathcal{N}}}} \subset \mathbb{B}_{\tau_{\coprod}}$ implies that the identity map $id_1 : (\mathcal{N}, \mathbb{B}_{\tau_{F_{\mathcal{N}}}}) \longrightarrow (\mathcal{N}, \mathbb{B}_{\tau_{\coprod}})$ is bounded, by definition, and $\mathbb{B}_{\tau_{\coprod}} \subset \mathbb{B}_{\tau_{F_{\mathcal{N}}}}$ implies that, by definition, the identity map $id_2 : (\mathcal{N}, \mathbb{B}_{\tau_{\coprod}}) \longrightarrow (\mathcal{N}, \mathbb{B}_{\tau_{F_{\mathcal{N}}}})$ is bounded. □

Chapter 6

Application: Bornologies from Frölicher groups.

6.1 Frölicher group.

We now look at the induction of bornologies from the Frölicher group. They [the bornologies] will be induced as in Chapter 2, Chapter 3 and Chapter 4. This should be possible since Frölicher groups, as we will see later, are Frölicher spaces. In literature there are two terminologies for this kind of groups but with the same definition. The terminologies are "Frölicher group" by M. Laubinger in [57] and [58], and "Frölicher Lie group" by A. Batubenge and P. Ntumba in [10]. We use the terminology by M. Laubinger as it is the latest study on this.

Definition 6.1.1 ***Frölicher group.***
Let $(G, *)$ be a group with the binary operation $*$. Then $(G, *)$ is a Frölicher group if:

1. *(G, C_G, F_G) is a Frölicher space, where*

$$C_G := \{c : \mathbb{R} \longrightarrow G \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \ , \forall \ f \in F_G\}$$

   and

$$F_G := \{f : G \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \ \forall \ c \in C_G\}.$$

2. *The product function $\mu : G \times G \longrightarrow G$ defined by $\mu : (g_1, g_2) \mapsto g_1 * g_2$ and the inverse function $i : G \longrightarrow G$ defined by $i : g \mapsto g^{-1}$, are both Frölicher smooth.*

Remark 6.1.1 *A. Batubenge and P. Ntumba in [10] used the term Frölicher Lie group in an attempt to developing the concept of Lie groups in the setting of Frölicher spaces. A Lie group is a group that is also a smooth manifold such that the product function and the inverse function are smooth with respect to the* *smooth manifold. Structurally, one can think of a Lie group as a group with a smooth structure, in this case a smooth manifold, such that the product function and the inverse function are smooth with respect to the smooth structure. For the Frölicher Lie group by Batubenge and Ntumba the Frölicher structure was used as a smooth structure, and having the product function and the inverse function smooth with respect to the Frölicher structure. However, the Frölicher Lie group by M. Laubinger in [57] and [58] is a Frölicher group (as defined above) endowed with a Lie algebra that is equipped with a Lie bracket.*

Let $(G, *)$ be a Frölicher group, then since G is endowed with a Frölicher structure, as (G, C_G, F_G) is a Frölicher space, then we can also generate a Frölicher structure on the Cartesian product $G \times G$ (see [10], [45] and Subsection 2.3.5). Therefore the product function $\mu : G \times G \longrightarrow G$ is a morphism between Frölicher spaces. The condition that the product function $\mu : G \times G \longrightarrow G$ defined by $\mu : (g_1, g_2) \mapsto g_1 * g_2$ and the inverse function $i : G \longrightarrow G$ defined by $i : g \mapsto g^{-1}$ be both Frölicher smooth is equivalent to the mapping $\sigma : G \times G \longrightarrow G$ defined by $\sigma : (g_1, g_2) \mapsto g_1 * g_2^{-1}$ being Frölicher smooth (see Lemma 2.1 in [10]).

Definition 6.1.2 ***Group homomorphism.***
Let $(G, *)$ be a group under the binary operation $*$ and $(H, \bullet)$ be a group under the binary operation $\bullet$. The mapping $\phi : G \longrightarrow H$ is a group homomorphism if $\phi(g_1 * g_2) = \phi(g_1) \bullet \phi(g_2), \ \forall \ g_1, g_2 \in G$.

Groups and group homomorphisms form a category - the category of groups. Therefore since Frölicher groups are groups then Frölicher groups and the group homomorphisms between them form a category - the category of Frölicher groups (see [10]). We refer to [10] for the properties of the category of Frölicher groups. Here are some of the properties of the category of Frölicher groups:

- Since Frölicher groups are Frölicher spaces then we have that the category of Frölicher groups is a subcategory of the category of Frölicher spaces.
- The category of Frölicher groups is a full subcategory of the category of groups as Frölicher groups are groups.
- The category of Frölicher groups is complete and cocomplete.
- Even though the category of Frölicher spaces is Cartesian-closed the category of Frölicher groups is not Cartesian-closed since the category of groups is not Cartesian-closed.
- There is a canonical forgetful functor from the category of Frölicher groups to the category of groups which allows for initial and final structures in the category of Frölicher groups.

Examples of Frölicher groups.

Example 6.1.1 *Let $(\mathbb{R}^n, +)$ be a group under addition. We have that $(\mathbb{R}^n, C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is a Frölicher space - the canonical n-dimensional Euclidean Frölicher space (see Example 2.3.1). Since there is a Frölicher structure on $\mathbb{R}^n$ then there is a Frölicher structure on $\mathbb{R}^n \times \mathbb{R}^n$ (see Subsection 2.3.5 on Frölicher product), that is,*

$$(C^\infty(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n), C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})).$$

Let $\sigma : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined by

$$\sigma((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{i=1}^{n} (x_i - y_i).$$

Let $c \in C^\infty(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ then it follows that $\sigma \circ c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ and $f \circ \sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. Therefore $\sigma : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is Frölicher smooth. This is equivalent to the product function $\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\mu((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{i=1}^{n} (x_i + y_i)$$

and the inverse function $i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$i(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, \ldots, -x_n)$$

both being Frölicher smooth. Hence $(\mathbb{R}^n, +)$ is a Frölicher group.

Example 6.1.2 *Let M be a smooth manifold. There is a Frölicher structure, that is, $(C^\infty(\mathbb{R}, M), C^\infty(M, \mathbb{R}))$, on M. Thus the triple $(M, C^\infty(\mathbb{R}, M), C^\infty(M, \mathbb{R}))$ is a Frölicher space (see Example 2.3.2). Since M is a smooth manifold then $M \times M$ inherits the smooth atlas, that is, the product function $M \times M \longrightarrow M$ is a map between smooth spaces. Now let $(M, *)$ be a Lie group, then the product function $\mu : M \times M \longrightarrow M$ defined by $\mu : (m_1, m_2) \mapsto m_1 * m_2$ and the inverse function $i : M \longrightarrow M$ defined by $i : m \mapsto m^{-1}$ are both smooth. Therefore $(M, *)$ is a Frölicher group.*

6.2 Bornological comparison and bounded maps.

6.2.1 Functional bornology.

Let $(G, *)$ be a Frölicher group, then it follows that (G, C_G, F_G) is a Frölicher space, where

$$C_G := \{c : \mathbb{R} \longrightarrow G \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_G\}$$

and

$$F_G := \{f : G \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C_G\}.$$

Therefore there is a bornology induced from structure functions on G, that is, from F_G (see Section 3.1.1). That is the functional bornology

$$\beta_{F_G} := \{U \subseteq G \mid f(U) \subseteq \mathbb{R} \textit{ is bounded}, \ \forall\, f \in F_G\}$$

on G. With (G, β_{F_G}) a bornological set then we have that the mapping $\phi : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ is bounded if it is Frölicher smooth. This follows from Lemma 3.1.3 since (G, C_G, F_G) is a Frölicher space. Since the category of Frölicher groups is a subcategory of the category of Frölicher spaces, if $\phi : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ is a group homomorphism then it is a morphism in the category of Frölicher spaces and therefore it is Frölicher smooth. Thus $\phi : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ is bounded if it is a group homomorphism. Note that the mapping $\phi : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ is not necessarily an identity map. Let $id : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ be an identity map then it follows trivially that $id : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ is bounded. Let $(G_1, *)$ and $(G_2, *)$ be Frölicher groups, then it follows that we have the Frölicher spaces (G_1, C_{G_1}, F_{G_1}) and (G_2, C_{G_2}, F_{G_2}). Then we have a functional bornology

$$\beta_{F_{G_1}} := \{U \subseteq G_1 \mid f(U) \subseteq \mathbb{R} \textit{ is bounded}, \ \forall\, f \in F_{G_1}\}$$

on G_1, and a functional bornology

$$\beta_{F_{G_2}} := \{U \subseteq G_2 \mid f(U) \subseteq \mathbb{R} \textit{ is bounded}, \ \forall\, f \in F_{G_2}\}$$

on G_2. With the bornological sets $(G_1, \beta_{F_{G_1}})$ and $(G_2, \beta_{F_{G_2}})$ we have that, by Lemma 3.1.4, if the mapping $(G_1, \beta_{F_{G_1}}) \longrightarrow (G_2, \beta_{F_{G_2}})$ is Frölicher smooth then it is bounded.

Definition 6.2.1 ***Bornological group.***
Let $(G, *)$ be a group with the binary operation $*$, then $(G, *)$ is a bornological group if G is a bornological set such that the product function $\mu : G \times G \longrightarrow G$, defined by $\mu : (g_1, g_2) \mapsto g_1 * g_2$, and the inverse function $i : G \longrightarrow G$, defined by $i : g \mapsto g^{-1}$, are both bounded.

Let $i : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ be an inverse function, that is, it is defined by $i : g \mapsto g^{-1}$. Let $U \in \beta_{F_G}$, then $U \subseteq G$ and $f(U) \subseteq \mathbb{R}$ is bounded, $\forall\, f \in F_G$. We have that

$$i(U) = \{i(g) \mid g \in U\} = \{g^{-1} \mid g \in U\}.$$

Therefore

$$f(i(U)) = \{f(x) \mid x \in i(U)\} = \{f(g^{-1}) \mid g \in U\}.$$

Since $(G, *)$ is a group (a Frölicher group) then $\forall\, g \in G\ \exists\, g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = \epsilon$ where ϵ is the identity element in G. That is $g * \epsilon = \epsilon * g = g,\ \forall\, g \in G$. Therefore $\forall\, g \in G$ we have that $g^{-1} \in G$. This implies that $g^{-1} \in G,\ \forall\, g \in G$, since $U \subseteq G$, therefore $i(U) \subseteq G$. With $f \in F_G$, and $f : G \longrightarrow \mathbb{R}$ is a function, then it follows that $\forall\, g^{-1} \in G$ we have that $f(g^{-1}) \in \mathbb{R}$.

Therefore $\forall\ g^{-1} \in G\ \exists\ r_1, r_2 \in \mathbb{R}$ such that $f(g^{-1}) \in [r_1, r_2]$. Therefore $f(i(U)) \subseteq \mathbb{R}$ is bounded $\forall\ f \in F_G$, thus $i(U) \in \beta_{F_G}$. Hence $i : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$, defined by $i : g \mapsto g^{-1}$, is bounded.

Let $H := G \times G$, then there is a Frölicher structure (C_H, F_H) on H where

$$C_H := \{c : \mathbb{R} \longrightarrow H \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_H\}$$

$$F_H := \{f : H \longrightarrow \mathbb{R} \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_H\}.$$

That is, the triple (H, C_H, F_H) is a Frölicher space - a Frölicher product (see Subsection 2.3.5). Then we have a bornology induced from the structure functions on H, thus, from F_H (see Section 3.1.1). That is, the functional bornology

$$\beta_{F_H} := \{U \subseteq H := G \times G \mid f(U) \subseteq \mathbb{R} \textit{ is bounded},\ \forall\ f \in F_H\}$$

on H. With $H := G \times G$ let $\mu : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ be a product function, that is $\mu : (g_1, g_2) \mapsto g_1 * g_2$. Let $U \in \beta_{F_H}$, then $U \subseteq H := G \times G$ and $f(U) \subseteq \mathbb{R}$ is bounded, $\forall\ f \in F_H$. Now since $(G, *)$ is a Frölicher group then $\mu : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ is Frölicher smooth. Therefore for $U \in \beta_{F_H}$ we have that $(f \circ \mu)(U) = f(\mu(U)) \subseteq \mathbb{R}$ is bounded, $\forall\ f \in F_G$. This implies that $\mu(U) \in \beta_{F_G}$. That is $U \in \beta_{F_H} \implies \mu(U) \in \beta_{F_G}$, therefore the product function $\mu : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ is bounded.

Now since we have that the inverse function $i : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ and the product function $\mu : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ are bounded therefore we have that the Frölicher group $(G, *)$ is a bornological group. The Frölicher smoothness of the product function $\mu : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ and the inverse function $i : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ determines their boundedness. The product function $\mu : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ and the inverse function $i : (G, \beta_{F_G}) \longrightarrow (G, \beta_{F_G})$ can be condensed into the function $\sigma : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ defined by $\sigma : (g_1, g_2) \mapsto g_1 * g_2^{-1}$. Therefore $\sigma : (H, \beta_{F_H}) \longrightarrow (G, \beta_{F_G})$ is bounded.

Definition 6.2.2 ***Frölicher subgroup.***
Let $(G, *)$ be a Frölicher group under the binary operation $*$ then $(S, *)$ is a Frölicher subgroup of $(G, *)$ if $S \subseteq G$ and $(S, *)$ is itself a Frölicher group under the binary operation $*$.

The characterization of the Frölicher subgroup $(S, *)$ of the Frölicher group $(G, *)$ is such that $s_1, s_2 \in S \implies s_1 * s_2^{-1} \in S$ (see [48]). Let $(S, *)$ be a Frölicher subgroup of the Frölicher group $(G, *)$. Then it follows that there is a Frölicher structure (C_S, F_S) on S where

$$C_S := \{c : \mathbb{R} \longrightarrow S \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ f \in F_S\}$$

and

$$F_S := \{f : S \longrightarrow \mathbb{R} \mid f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}),\ \forall\ c \in C_S\}.$$

That is the triple (S, C_S, F_S) is a Frölicher space - a Frölicher subspace of the Frölicher space (G, C_G, F_G) (see Subsection 2.3.4). Then there is a bornology from the structure functions on S, thus from F_S, that is, the functional bornology

$$\beta_{F_S} := \{U \subseteq S \mid f(U) \subseteq \mathbb{R} \textit{ is bounded}, \ \forall \ f \in F_S\}$$

on S. Since $S \subseteq G$ then we have the canonical inclusion $\imath : S \hookrightarrow G$. The canonical inclusion $\imath : S \hookrightarrow G$ is a group homomorphism since $\forall \ s_1, s_2 \in S$ we have that $\imath(s_1 * s_2) = s_1 * s_2 = \imath(s_1) * \imath(s_2) \in G$ since G is closed under the binary operation $*$.

Following from the subsection on Frölicher subspace (Subsection 2.3.4) as (S, C_S, F_S) is a Frölicher subspace of the Frölicher space (G, C_G, F_G) then we have the subspace bornology on S, that is

$$\beta_{F_G}(S) := \{U \subseteq S \mid \imath(U) \in \beta_{F_G}\}$$

where $\imath : S \hookrightarrow G$ is the inclusion map and β_{F_G} is a functional bornology on G. By Theorem 3.3.1 we have that $\beta_{F_S} \subset \beta_{F_G}(S)$, therefore the identity map $id : (S, \beta_{F_S}) \longrightarrow (S, \beta_{F_G}(S))$, with the bornological sets (S, β_{F_S}) and $(S, \beta_{F_G}(S))$, is bounded, by definition.

6.2.2 Relatively-compact bornology.

With $(S, *)$ a Frölicher subgroup it follows that there is a Frölicher topology from the Frölicher structure (C_S, F_S) (see Subsection 2.3.3). That is the functional topology

$$\tau_{F_S} := \{U \subseteq S \mid U = f^{-1}(V), \ V \in \tau_{\mathbb{R}}, \ \forall \ f \in F_S\}$$

on S where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. And since $S \subseteq G$ then there is also the canonical subspace topology

$$\tau_{F_G}(S) := \{U \cap S \mid U \in \tau_{F_G}\}$$

on S. The topological relation on S, that the relationship between the functional topology on S and the canonical subspace topology on S, is such that $\tau_{F_G}(S) \subset \tau_{F_S}$ (see Lemma 2.3.5). The topologies $\tau_{F_G}(S)$ and τ_{F_S} induce the relatively-compact bornologies

$$\beta_{\tau_{F_G}}(S) := \{U \subseteq S \mid U \textit{ is relatively } \tau_{F_G}(S) - \textit{compact}\}$$

and

$$\beta_{\tau_{F_S}} := \{U \subseteq S \mid U \textit{ is relatively } \tau_{F_S} - \textit{compact}\},$$

respectively (see Section 4.2). By Theorem 4.2.1 the bornological relation of the relatively-compact bornologies on S, that is the relationship between the relatively-compact bornologies $\beta_{\tau_{F_G}}(S)$ and $\beta_{\tau_{F_S}}$, is such that $\beta_{\tau_{F_G}}(S) \subset \beta_{\tau_{F_S}}$.

6.2.3 Compact bornology.

The subsets of a $\tau_{F_G}(S)$-compact set and the subsets of a τ_{F_S}-compact set form compact bornologies as in Section 5.2. That is the compact bornologies

$$\mathbb{B}_{\tau_{F_G}}(S) := \{U \subseteq S \mid U \textit{ is contained in a } \tau_{F_G}(S) - \textit{compact set}\}$$

and

$$\mathbb{B}_{\tau_{F_S}} := \{U \subseteq S \mid U \textit{ is contained in a } \tau_{F_S} - \textit{compact set}\},$$

induced respectively from $\tau_{F_G}(S)$ and τ_{F_S}. Since $\tau_{F_X}(S) \subset \tau_{F_S}$ (Lemma 2.3.5) then we have that $\mathbb{B}_{\tau_{F_G}}(S) \subset \mathbb{B}_{\tau_{F_S}}$ (Theorem 5.2.1).

Conclusion.

Chapter 3, 4 and 5 are the main work of this thesis. Chapter 1 and 2 were on introduction and preliminaries, respectively. Herewith is the contribution of this thesis - the summary and the main results:

Chapter 3 focused on the induction of and comparison of bornologies from the Frölicher structure of a general Frölicher space and from the Frölicher structure of the structures in the category of Frölicher spaces. The structures in the category of Frölicher spaces, under the study of this thesis, are Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient. The induction and comparison of the bornologies from the Frölicher structure was inspired by the induction of topologies from the Frölicher structure and the underlying set of a Frölicher space, and the topological comparison thereof in [13], [14] and [45]. For a general Frölicher space there is a bornology induced canonically from structure functions, called the functional bornology. Due to the structure and definition of structure curves, there is no bornology induced canonically from structure curves. Since the structures in the category of Frölicher spaces are Frölicher spaces then for each of them there is a functional bornology. We also showed that Frölicher smooth maps, that is, the morphisms in the category of Frölicher spaces, are bounded under the functional bornology (see Lemma 3.1.2, Lemma 3.1.3 and Lemma 3.1.4). For each of the structures, that is, Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient, a canonical bornology was induced on the underlying set (see Lemma 3.3.1, Lemma 3.4.1, Lemma 3.6.1 and Lemma 3.7.1). The canonical bornology induced from the underlying set of Frölicher subspace and from the underlying set of Frölicher product is an initial bornology and that from the underlying set of Frölicher coproduct and Frölicher quotient is a final bornology. Frölicher subspace and Frölicher product are initial structures in the category of Frölicher spaces, and Frölicher coproduct and Frölicher quotient are final structures in the category of Frölicher spaces. That is, for each of these structures there are two bornologies - the bornology induced from the underlying set and the bornology induced from structure functions. For the initial structures the initial bornology is finer than the functional bornology (see Theorem 3.3.1 and Theorem 3.4.1). Dually, for the final structures the final bornology is coarser than the functional bornology (see Theorem 3.6.1 and Theorem 3.7.1).

Chapter 4 and 5 focused on the induction of bornologies from the topologies induced from Frölicher space. For a general Frölicher space there is a topology induced from the structure functions, called the functional topology. In chapter 4 we induced the relatively-compact bornology and in chapter 5 we induced the compact bornology. The relatively-compact bornology, in this case, is a collection of relatively-compact sets and the compact bornology is the collection of subsets of compact sets. Since Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient are Frölicher spaces then for each of them we have the functional topology. Furthermore, for each of these Frölicher structures, we have a canonical topology induced on the underlying set. Thus, for each of these structures we have two relatively-compact bornologies - one induced from the canonical topology and the other induced from the functional topology, as shown in chapter 4. Similarly, as shown in chapter 5, for each of these structures we have two compact bornologies - one induced from the canonical topology and the other induced from the functional topology. The bornological comparison of these bornologies, that is, the relationship between the bornologies, for each Frölicher structure, follows from the relationship between the canonical topology and the functional topology, on that structure. For the initial structures, that is, Frölicher subspace and Frölicher product, the canonical topology is contained or equals to the functional topology. Therefore, for initial structures, the relatively-compact bornology induced from the canonical topology is contained or equals to the relatively-compact bornology induced from the functional topology (see Theorem 4.2.1 and Theorem 4.3.1). Similarly, for initial structures, the compact bornology induced from the canonical topology is contained or equals to the compact bornology induced from the functional topology (see Theorem 5.2.1 and Theorem 5.3.1). For final structures, that is, Frölicher coproduct and Frölicher quotient, the canonical topology and the functional topology are equal. Thus, for final structures the relatively-compact bornology induced from the canonical topology and the relatively-compact bornology induced from the functional topology are equal (see Theorem 4.4.1 and Theorem 4.5.1), and the compact bornology induced from the canonical topology and the compact bornology induced from the functional topology are equal (see Theorem 5.4.1 and Theorem 5.5.1).

For application we applied the bornological comparison in chapter 3, 4 and 5 on Frölicher groups as covered in chapter 6. Since a Frölicher group is a group that is also a Frölicher space (see [10], [25] and [58]) then we have the functional bornology from Frölicher group, induced from the structure functions on the Frölicher group, as in chapter 3. For a Frölicher subgroup we have the subspace bornology, which is an initial bornology, and the functional bornology. They were compared as in chapter 3. We also have the subspace topology on the underlying subgroup, which is a canonical topology, and the functional topology induced from the structure functions on the subgroup. Relatively-compact and compact bornologies were induced from these topologies and were compared as in chapter 4 and 5 respectively.

The aim of this thesis was to introduce the concept of bornology on Frölicher spaces. That is, the induction and comparative analysis of bornologies on Frölicher space, and this was met. In chapter 3 we induced and compared bornologies from the Frölicher structure with the initial/final bornology (initial bornology for initial structures and final bornology for final structures). In chapter 4 we induced and compared relatively-compact bornology induced from the canonical topology on the underlying set and the relatively-compact bornology induced from the functional topology, for each structure. In chapter 5 we induced and compared compact bornology from the canonical topology on the underlying set and the compact bornology from the functional topology, for each structure. These are the contributions and the main results of this thesis. A. Batubenge and M. Tshilombo studied the topological relation between the canonical topology induced on the underlying set of a Frölicher space and the Frölicher topologies in [13] and [14]. In this thesis we studied the bornological relation or comparison, which is an advancement in the theory of Frölicher spaces as a developing theory.

The study of this thesis was limited to relatively-compact and compact bornologies as bornologies induced from topology. In future we will explore other bornologies induced from topology besides the relatively-compact and compact bornologies. The bornological comparison was also limited to Frölicher subspace, Frölicher product, Frölicher coproduct and Frölicher quotient as structures or subobjects in the category of Frölicher spaces. One of the main investigative questions was whether we can induce a bornology from the Frölicher topology and vice versa? We have shown that bornologies can be induced from the Frölicher topology, and furthermore from the canonical topology induced on the underlying set of the Frölicher space. In future research we will look at what topologies does the bornologies induced from Frölicher space induce. Bornologies have a lot of application in functional analysis - we will look into the application of these bornologies that we have induced from Frölicher space in our future research. Though we induced, canonically, a bornology from structure functions of Frölicher spaces, we could not induce a bornology from structure curves of Frölicher spaces due to the nature and structure of structure curves. This is something we also wish to look into in our future research. That is, looking at the conditions to impose, if any, on structure curves and/or Frölicher space in order to induce or characterize a bornology from structure curves of Frölicher space. In future research we will also look into applying the bornological comparison that we have deduced on this thesis on the product, coproduct and quotient of Frölicher groups, on Frölicher algebra and convenient spaces.

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