Ideals in Stone-Čech compactifications

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Contents

Declaration

I declare that this thesis was composed by myself while registered as a part-time student at the University of the Witwatersrand at Johannesburg between the years 2009 and 2012. The two results presented in this thesis were objects of the talks I gave at the weekly seminars at the school of Mathematics of the University of the Witwatersrand at Johannesburg. These results were also presented at the annual congress of the South African Mathematical Society between the years 2009 and 2012 before their publications respectively in two prestigious scientific journals (Proceedings of the American Mathematical Society in 2010 and Topology and its Applications in 2012).

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Abstract

Let S be an infinite discrete semigroup and βS the Stone-Čech compactification of S. The operation of S naturally extends to βS and makes βS a compact right topological semigroup with S contained in the topological center of βS. The aim of this thesis is to present the following new results.

1. If S embeddable in a group, then βS contains $2^{2^{|S|}}$ pairwise incomparable semiprincipal closed two-sided ideals.

2. Let S be an infinite cancellative semigroup of cardinality κ and $U(S)$ the set of uniform ultrafilters on S. If $\kappa > \omega$, then there is a closed left ideal decomposition of $U(S)$ such that the corresponding quotient space is homeomorphic to $U(\kappa)$. If $\kappa = \omega$, then for any connected compact metric space X, there is a closed left ideal decomposition of $U(S)$ with the quotient space homeomorphic to X.

Introduction

The operation of a discrete semigroup S naturally extends to the Stone-Čech compactification βS of S making βS a compact right topological semigroup with S contained in its topological center. That is, for each $a \in S$, the left translation

$$\beta S \ni x \mapsto ax \in \beta S$$

is continuous, and for each $q \in \beta S$, the right translation

$$\beta S \ni x \mapsto xq \in \beta S$$

is continuous.

We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. The topology of βS is generated by taking as a base the subsets

$$\overline{A} = \{p \in \beta S \,:\, A \in p\}$$

where $A \subseteq S$. For $p, q \in \beta S$, the ultrafilter pq has a base consisting of subsets of the form

$$\bigcup_{x \in A} xB_x$$

where $A \in p$ and $B_x \in q$.

The fact that the operation of S can be extended to βS was implicitly established by M. Day [12] using multiplication of the second conjugate of a Banach algebra, in this case $\ell^1(S)$, first introduced by R. Aren [24] for arbitrary Banach algebras. P. Civin and B. Yood [23] explicitly stated that if S is a group, then the above operation produced an operation on βS, viewed as a subspace of that second dual. R. Ellis [26] carried out the extension in βS viewed as the space of ultrafilters, again assuming that S is a group.

Algebraic properties of βS have been a useful tool in Ramsey Theory. The first example of such an application was provided by Hindman's Theorem, known also as the Finite Sum Theorem. It says that whenever $\mathbb{N}$ is finitely colored, there is a sequence $(x_n)_{n=1}^{\infty}$ with monochrome $FS((x_n)_{n=1}^{\infty})$. Here

$$FS((x_n)_{n=1}^{\infty}) := \{\sum_{n\in F} x_n \,:\, F \text{ is a finite nonempty subset of } \mathbb{N}\}.$$

The original proof of this theorem was one of enormous complexity [17]. In 1975, F. Galvin and S. Glazer came up with a very simple proof based on the fact that $\beta\mathbb{N}$ has an idempotent (see [19], Section 4.2).
The semigroup βS also has important applications to topological dynamics and to topological groups (see books [19] and [37]).
As any compact Hausdorff right topological semigroup, βS has idempotents ([10] and [25]) and the smallest two-sided ideal $K(\beta S)$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. The intersection of a minimal right ideal and a minimal left ideal is a group and all these groups are isomorphic. The precise description of $K(\beta S)$ is given by the Rees-Suschkewitsch Theorem (see [19], Theorem 1.64 or [37], Theorem 6.23).
It has long been known that the semigroup $\beta\mathbb{N}$ has $2^{2^{\omega}}$ minimal left ideals (C. Chou [1] or [2]), $2^{2^{\omega}}$ minimal right ideals (J. Baker and P. Miles [11]), and the structure group of $\beta\mathbb{N}$ contains copies of the free group on $2^{2^{\omega}}$ generators (N. Hindman and J. Pym [18]). Later on these results have been extended by showing that for every infinite cancellative semigroup S, βS has $2^{2^{|S|}}$ minimal left ideals (N. Hindman and D. Strauss [19], Theorem 6.42), and for every infinite Abelian group G, βG has $2^{2^{|G|}}$ minimal right ideals (Y. Zelenyuk [34]) and the structure group of $K(\beta G)$ contains copies of the free group on $2^{2^{|G|}}$ generators (S. Ferri, N. Hindman, and D. Strauss [28]; Y. Zelenyuk and Yu. Zelenyuk [35]).
I. Protasov and O. Protasova [6] raised a problem of counting closed two-sided ideals of βS and showed that for every infinite Abelian group G, βG contains $2^{2^{|G|}}$ closed two-sided ideals. M. Filali, E. Lutsenko, and I. Protasov [13] extended this result to an arbitrary countably infinite group. However, the question whether this is true for all infinite groups remained open.
The above-mentioned result about the number of minimal left ideals of βS, where S is a cancellative semigroup, is a consequence of the following stronger theorem: for every infinite cancellative semigroup S of cardinality κ, the ideal $U(S)$ consisting of all the uniform ultrafilters on S can

be decomposed into 2^{2^κ} left ideals of βS (E. van Douwen, published in [3]). Relatively recently, this theorem has been strengthened by showing that $U(S)$ can be decomposed into 2^{2^κ} closed left ideals of βS. First this was done in the case where κ is a regular cardinal by I. Protasov [7] and then for all κ by M. Filali and P. Salmi [14]. The proof was complicated, based on balleans and slowly oscillating functions. Another direct proof was given by Y. Zelenyuk [39]. The fact that there exist decompositions of $U(S)$ into closed left ideals of βS raised the question of what the quotient spaces of $U(S)$ corresponding to such decompositions are.
The aim of this thesis is to present two new results.

1. For every infinite semigroup S embeddable in a group, βS contains $2^{2^{|S|}}$ pairwise incomparable semiprincipal closed two-sided ideals [29]. In particular, for every infinite group G, βG contains $2^{2^{|G|}}$ closed two-sided ideals. This extends significantly the results from [6] and [13].

2. Let S be an infinite cancellative semigroup of cardinality κ. If $\kappa > \omega$, then there is a closed left ideal decomposition of $U(S)$ such that the corresponding quotient space is homeomorphic to $U(\kappa)$. If $\kappa = \omega$, then for any connected compact metric space X, there is a closed left ideal decomposition of $U(S)$ with the quotient space homeomorphic to X. This is an interesting complement to the results from [7], [14], and [39].

The results (1) and (2) are published in [29] and [30], respectively.
This thesis is organized as follows.
Chapters 1 through 3 have a preliminary character containing the definitions and results that will be needed in chapters 4 and 5. There we discuss compact right topological semigroups, ultrafilters, extending the operation from S to βS, and other things. In Chapter 4 we present result (1), and in Chapter 5 result (2).

Chapter 1

Background

1.1 Semigroups

We recall in this section the basic concepts of semigroups and ideals that will be needed to introduce the algebraic structure of Stone-Čech compactifications and their properties. The highlight of this section is the so-called *Structure Theorem* or the *Rees-Suschkewitsch Theorem*. Most of these concepts reviewed below are taken from [19].

Definition 1.1.1 *Let S be a nonempty set. We say that there exists a (closed) binary operation in S if there exists an application*

$$* : S \times S \ni (x, y) \mapsto x * y \in S.$$

We have the following axioms.

1. *The operation $*$ is* ***associative*** *if all $x, y, z \in S$ satisfy the following condition:*
$$x * (y * z) = (x * y) * z.$$

2. *The operation $*$ is* ***commutative*** *if all $x, y \in S$ satisfy the following condition:*
$$x * y = y * x.$$

3. *If S endowed with the operation $*$ satisfies axiom (1), we say that S is a* ***semigroup****. If S satisfies also axiom (2), the semigroup S is said to be* ***commutative*** *(or* ***Abelian****).*

4. *A nonempty set T is a **subsemigroup** of the semigroup S if and only if $T \subseteq S$ and T is a semigroup under the restriction of the operation $*$ to T.*

We denote the set S endowed with the operation $*$ by a pair $(S, *)$, or simply by S, if there is no risk of confusion and we say $(S, *)$ forms an algebraic structure. We will often denote $x * y$ by the multiplicative notation $x \cdot y$, or simply xy. If the structure is commutative, we will use the additive notation $x + y$.

Definition 1.1.2 *Let A be a nonempty set.*

1. *The **free semigroup on the alphabet** A is the set*

$$\begin{aligned} S = \{ & f : f \text{ is a function and } range(f) \subseteq A \text{ and there is some} \\ & n \in \mathbb{N} \text{ such that } domain(f) = \{0, 1, \dots, n-1\}\}. \end{aligned}$$

2. *Given f and g in S, the operation $f \frown g$ is called **concatenation** and is defined as follows. Assume*

$$domain(f) = \{0, 1, \dots, n-1\} \text{ and } domain(g) = \{0, 1, \dots, m-1\}.$$

Then

$$domain(f \frown g) = \{0, 1, \dots, m+n-1\}$$

and given $i \in \{0, 1, \dots, m+n-1\}$,

$$f \frown g = \begin{cases} f(i) & \text{if } i < n \\ g(i-n) & \text{if } i \geq n. \end{cases}$$

*The **free semigroup with identity on the alphabet** A is $S \cup \{\varnothing\}$ where S is the free semigroup on the alphabet A. Given $f \in S \cup \{\varnothing\}$ one defines $f \frown \varnothing = \varnothing \frown f = f$.*

3. *The elements of a free semigroup are called **words** and written by listing the values of the function in order. The **length** of a word is n where the domain of the word is $\{0, 1, \dots, n-1\}$ (and the length of $\varnothing$ is 0).*

Definition 1.1.3 *Let $(S, *)$ be a semigroup. Let $e \in S$.*

1. *The element e is a **left identity** of S if for all $x \in S$, we have $e * x = x$.*

2. *The element e is a* ***right identity*** *of S if for all $x \in S$, we have $x * e = x$.*

3. *The element e is a* ***two-sided identity****, or simply an* ***identity****, of S if and only if it is both a left identity and a right identity of S.*

A semigroup $(S, *)$ containing an element $e \in S$ and satisfying condition (3) is then called a ***monoid****.*

Notice that if $(S, *)$ is a semigroup, then there exists a monoid $(M, \cdot)$, with e the identity, such that $S = M \setminus \{e\}$ and for all $x, y \in M \setminus \{e\}$, $x \cdot y = x * y$, $x \cdot e = x$, and $e \cdot x = x$. So, there is a slight difference between a semigroup and a monoid.

Definition 1.1.4 *Let $(S, *)$ be a semigroup.*

1. *If e is an identity of S and $x \in S$, an element y of S is called a* ***left*** e***-inverse*** *of x if $y * x = e$.*

2. *If e is an identity of S and $x \in S$, an element y of S is called a* ***right*** e***-inverse*** *of x if $x * y = e$.*

3. *If e is an identity of S and $x \in S$, an element y of S is called a* e***-inverse****, or simply an* ***inverse****, of x if it is both a left e-inverse and a right e-inverse of x in S.*

The inverse of x will be denoted by x^{-1} when using the multiplicative notation, and by $-x$ when using the additive notation.

Definition 1.1.5 *Let $(S, *)$ be a semigroup.*

1. *An element $x \in S$ is* ***right cancelable*** *if and only if, for all $y, z \in S$,*
$$y * x = z * x \text{ implies } y = z.$$

2. *An element $x \in S$ is* ***left cancelable*** *if and only if, for all $y, z \in S$,*
$$x * y = x * z \text{ implies } y = z.$$

3. *An element is* ***cancelable*** *if and only if it is both right cancelable and left cancelable.*

4. *S is **right cancellative** if and only if every element of S is right cancelable.*

5. *S is **left cancellative** if and only if every element of S is left cancelable.*

6. *S is **cancellative** if and only if S is both left cancellative and right cancellative.*

A monoid M is not cancellative in general. Nevertheless, if an element a has an inverse a^{-1}, then a is cancelable.

Definition 1.1.6 *Let $(S, *)$ be a semigroup.*

1. *The pair $(S, *)$ is a **group** if it is a cancellative monoid.*

2. *$(T, \cdot)$ is called a **subgroup** of $(S, *)$ if and only if $T \subseteq S$ and $(T, \cdot)$ is a group where $\cdot$ is the restriction of $*$ to T.*

In the following, a semigroup will be frequently denoted by S and a group by G.

Definition 1.1.7 *Let $(S, *)$ be a semigroup.*

1. *An element e of S is called an **idempotent** if $e * e = e$.*

2. *The set of all idempotents in S is denoted by $E(S)$.*

3. *Let $e \in E(S)$. Then*

$$H(e) = \bigcup \{G \,:\, G \text{ is a subgroup of } S \text{ and } e \in G\}.$$

Lemma 1.1.1 *Let G be a group with identity e. Then $E(G) = \{e\}$.*

Proof. Let $f \in E(G)$. Then $ff = f = fe$. Thus, $f = e$ by multiplying on the left by the inverse of f. So, for any $f \in E(G)$, $f = e$. That is, $E(G) = \{e\}$. $\square$

Theorem 1.1.1 *Let S be a semigroup and let $e \in E(S)$. Then $H(e)$ is the largest subgroup of S with e as identity.*

Proof. See Proof of Theorem 1.18 in [19]. $\square$

Definition 1.1.8 *Let S be a semigroup and let $e \in E(S)$. The group $H(e)$ is called a* ***maximal group*** *of S.*

Definition 1.1.9 *Let $(S, *)$ be a semigroup. Let $z \in S$.*

1. *The element z is called a* ***left zero*** *of S if for all $x \in S$, we have $z * x = z$.*
2. *The element z is called a* ***right zero*** *of S if for all $x \in S$, we have $x * z = z$.*
3. *The element z is called a* ***zero*** *of S if and only if it is both a left zero and a right zero of S.*

A semigroup $(S, *)$ satisfying condition (1) (respectively, condition (2), and condition (3)) is called ***left zero semigroup*** *(respectively,* ***right zero semigroup****, and* ***zero semigroup****).*

Recall that a zero, if it exists, is unique since, if z_1 and z_2 are two zeros, then $z_1 = z_1 * z_2 = z_2$.

There are many relationships between different semigroups' structures.

Definition 1.1.10 *Let $(S, *)$ and $(T, \cdot)$ be two semigroups.*

1. *A* ***homomorphism*** *from S to T is a function $\varphi : S \to T$ such that $\varphi(x * y) = \varphi(x) \cdot \varphi(y)$ for all $x, y \in S$.*
2. *An* ***isomorphism*** *from S to T is a homomorphism from S to T which is both one-to-one and onto T.*
3. *The semigroups $(S, *)$ and $(T, \cdot)$ are said to be* ***isomorphic*** *and, written $S \approx T$, if and only if there exists an isomorphism from S to T.*

Each monoid is also a semigroup. Given a monoid G, it has a semigroup structure which is the underlying semigroup of G. Given two monoids $(M, *)$ with identity e and $(M', \cdot)$ with identity e', a homomorphism from

M to M' is a homomorphism φ between the underlying semigroups such that $\varphi(e) = e'$. A homomorphism between a monoid and a group is a homomorphism of monoids since a group is cancellative. So a homomorphism of groups is a homomorphism of underlying monoids, where $\varphi(x^{-1}) = \varphi(x)^{-1}$.

Example 1.1.1 *1.* $(\mathbb{N}, +)$ *is an Abelian semigroup and* $(\mathbb{N}, \cdot)$ *is an Abelian monoid with 1 as the identity, where* $\mathbb{N}$ *is the set* $\{1, 2, 3, \ldots\}$ *of positive integers.*

2. *Let* ω *be the set* $\mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ *of nonnegative integers.* $(\omega, +)$ *and* $(\omega, \cdot)$ *are Abelian monoids with the identities 0 and 1 respectively.*

3. *The set* C_n *defined as follows:*

$$C_n = \{e = a^0, a = a^1, a^2, \ldots, a^{n-1}\},$$

with the operation defined by

$$a^i * a^j = a^{i+j \, (mod \, n)},$$

is a group called the cyclic group of n *elements. The infinite cyclic group is* $(\mathbb{Z}, +)$.

4. *Given a set* X *with* $|X| > 1$. *Then,* $(X^X, \circ)$ *is a noncommutative monoid, where* X^X *is the set of all functions from* X *to* X *and* $\circ$ *represents the natural composition of functions, with the identity function as the identity element. The bijective functions in* X^X *form a group. An element* g *of* X^X *is an idempotent if and only if the set of fixed points of* g *equals the range of* g. *The identity function of* X *and constant functions on* X *are typical examples of idempotents in* $(X^X, \circ)$.

Definition 1.1.11 *Let* $(G, *)$ *be a group. Recall that a subset* $H \subset G$ *is a* ***subgroup*** *of* G *if* H*, endowed with the restriction of* $*$ *to* H*, is a group. Equivalently,* H *is a subgroup of* G *if and only if* H *is nonempty and* $h_1 * h_2^{-1} \in H$ *for all* $h_1, h_2 \in H$.

Let us recall that a *binary relation* R between two sets A and B is a subset R of $A \times B$ and, for all $a \in A$ and $b \in B$, we write aRb if $(a, b) \in R$.

Definition 1.1.12 *An **equivalence relation** on a set X is a binary relation R that satisfies the following conditions:*

1. *reflexivity: for all $x \in X$, xRx;*
2. *symmetry: for all $x, y \in X$, if xRy, then yRx;*
3. *transitivity: for all $x, y, z \in X$, if xRy and yRz, then xRz.*

*The **equivalence class** of an element $x \in X$ is a subset of X denoted by $[x]$ and defined by*

$$[x] = \{y \in X \ : \ yRx\}.$$

*The **quotient set** of X by the equivalence relation R, denoted by X/R, is the set of all the equivalence classes of X by R.*

We recall the following properties, for all $x, y \in X$, $[x] = [y]$ if and only if $y \in [x]$ and if $[x] \neq [y]$, then $[x] \cap [y] = \emptyset$.

Let G be a group and $H \subset G$ a subgroup. One can define an equivalence relation on G with $[g] = gH$ with the following properties:

1. $hH = H$ if and only if $h \in H$;
2. If $h_1H \neq h_2H$, then $h_1H \cap h_2H = \emptyset$;
3. $\displaystyle\bigcup_{g \in G} gH = G$.

Hence we obtain a quotient set of G, denoted by G/H, from the equivalence relation with respect to H. This quotient will be a group (called the *quotient group*) if the equivalence classes are all normal making the canonical projection of G on G/H a typical example of a (surjective) homomorphism. Recall that a subgroup H of G is normal if for every $g \in G$, $gH = Hg$.

Let $(G, *)$ be a group and X a set. We say that G acts on X, if there exists an application

$$G \times X \ni (x, g) \mapsto x \cdot g \in X,$$

such that

1. $(g * h) \cdot x = g \cdot (h \cdot x)$, for all $x \in X$ and $g, h \in G$;
2. $e \cdot x = x$, for all $x \in X$.

Obviously, any group acts on itself. The set $O_x = \{g \cdot x \ : \ g \in G\}$ is called the *orbit* of the element $x \in X$. The relation on the elements of X defined by

$$x \sim y \text{ if and only if } x \in O_y$$

is an equivalence relation on X. The quotient set denoted by X/G consists of all the orbits of X.

Definition 1.1.13 *1. Let I be an infinite set of indices and consider an indexed family of semigroups $(S_i, *_i)_{i \in I}$. The* ***direct product*** *of these semigroups is the semigroup $S = \bigotimes_{i \in I} S_i$, where*

$$\bigotimes_{i \in I} S_i = \{(x_i)_{i \in I} \ : \ x_i \in S_i, \text{ for all } i \in I\},$$

endowed with the operation $*$ defined by

$$(x * y)_i = x_i *_i y_i,$$

for all $x, y \in S$.

2. The ***direct sum*** *of a family of semigroups $(S_i, *_i)$, where each S_i has a two-sided identity e_i, is the subsemigroup of the direct product, given by the following set*

$$\bigoplus_{i \in I} S_i = \{(x_i)_{i \in I} \in \bigotimes_{i \in I} S_i \ : \ \{i \in I \ : \ x_i \neq e_i\} \text{ is finite } \}.$$

Let us recall that if G is a group and S a subset of G, then the intersection of all the subgroups of G containing S is also a subsgroup of G. This subgroup is the smallest subgroup of G containing S. It is called the subgroup generated by S and denoted by $\langle S \rangle$ or $\langle x_1, x_2, \ldots, x_n \rangle$ if $S = \{x_1, x_2, \ldots, x_n\}$.

Definition 1.1.14 *(See [19]) Let S be the free semigroup with identity on the alphabet A and let*

$$\begin{aligned} G \ = \ & \{g \in S \ : \ \text{there do not exist } t, t+1 \in domain(g), a \in A \text{ and} \\ & i \in \{1, -1\} \text{ for which } g(t) = a^i \text{ and } g(t+1) = a^{-i}\}. \end{aligned}$$

Given $f, g \in G \setminus \{\varnothing\}$ *with*

$$domain(f) = \{0, 1, \cdots, n-1\} \text{ and } domain(g) = \{0, 1, \cdots, m-1\},$$

define

$$f \cdot g = f \frown g$$

unless there exist $a \in A$ *and* $i \in \{1, -1\}$ *with*

$$f(n-1) = a^i \text{ and } g(0) = a^{-i}.$$

In the latter case, pick the largest $k \in \mathbb{N}$ *such that for all* $t \in \{1, 2, \cdots, k\}$*, there exist* $b \in A$ *and* $j \in \{1, -1\}$ *such that*

$$f(n-t) = b^j \text{ and } g(t-1) = b^{-j}.$$

If $k = m = n$*, then* $f \cdot g = \varnothing$*. Otherwise,*

$$domain(f \cdot g) = \{0, 1, \cdots, n+m-2k-1\}$$

and for $t \in \{0, 1, \cdots, n+m-2k-1\}$*,*

$$(f \cdot g)(t) = \begin{cases} f(t) & \text{if } t < n-k \\ g(t+2k-n) & \text{if } t \geq n-k. \end{cases}$$

Then $(G, \cdot)$ *is the* ***free group generated by*** A*.*

The following universal property characterizes the free groups up to isomorphism and is sometimes used as an alternative definition of a free group.

Lemma 1.1.2 *Let* A *be a set, let* G *be the free group generated by* A*, let* H *be an arbitrary group, and let* $\phi : A \to H$ *be any mapping. There is a unique homomorphism* $\tilde{\phi} : G \to H$ *for which* $\tilde{\phi}(g) = \phi(g)$ *for every* $g \in A$*.*

$$\begin{tikzcd} G \arrow[dr, "\tilde{\phi}"] & \\ A \arrow[u, "\iota"] \arrow[r, "\phi"] & H \end{tikzcd}$$

Proof. We may think of the mapping ι as sending each symbol of A to a word in G consisting of that symbol. Notice that $\tilde{\phi}$ sends the empty word $\varnothing$ to the identity of H and $\tilde{\phi}(g) = \phi(g)$ for every $g \in A$.

For more than one symbol, take a word of length n, $f = a_0^{i_0} a_1^{i_1} \dots a_{n-1}^{i_{n-1}}$, where $a_r \in A$ and $i_r \in \{-1, +1\}$. We construct $\tilde{\phi}(f)$ by induction on n and then $\tilde{\phi}$ extends to any word of G.

$$\begin{aligned}\tilde{\phi}(f) &= \tilde{\phi}(a_0^{i_0} a_1^{i_1} \dots a_{n-1}^{i_{n-1}}) \\ &= \tilde{\phi}(a_0^{i_0} a_1^{i_1} \dots a_{n-2}^{i_{n-2}}) \cdot \tilde{\phi}(a_{n-1}^{i_{n-1}}),\end{aligned}$$

is an element of H since $\tilde{\phi}(a_0^{i_0} a_1^{i_1} \dots a_{n-2}^{i_{n-2}}) \in H$, by the induction hypothesis, $\tilde{\phi}(a_{n-1}^{+1}) = \phi(a_{n-1}) \in H$, and by taking $\tilde{\phi}(a_{n-1}^{-1})$ as the inverse of $\phi(a_{n-1})$ in H. □

One can show that, for any set A, the free group generated by A always exists and is unique.

Theorem 1.1.2 *Let A be a set, let G be the free group generated by A, and let $g \in G \setminus \{\varnothing\}$. There exists a finite group F and a homomorphism $\hat{\phi} : G \to F$ such that $\hat{\phi}(g)$ is not the identity of F.*

Proof. See Proof of Theorem 1.23 in [19]. □

The cyclic group of n elements C_n is a group generated by an element g:

$$C_n = \{g \,:\, g^n = 1\}.$$

Each infinite cyclic group is isomorphic to $(C_\infty, \cdot) \cong (\mathbb{Z}, +)$. Each finite cyclic group of order n is isomorphic to $(C_n, \cdot) \cong (\mathbb{Z}_n, +)$. Hence, two cyclic groups of the same order are isomorphic and all the cyclic groups are Abelian groups.

Definition 1.1.15 *Let S be a semigroup.*

1. *A nonempty subset L of S is called a **left ideal** of S if $SL \subseteq L$.*

2. *A nonempty subset R of S is called a **right ideal** of S if $RS \subseteq R$.*

3. *A **two-sided ideal**, or simply an **ideal**, is a subset which is both a left and right ideal of S.*

All the ideals of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for $n \in \mathbb{Z}$.

Proposition 1.1.1 *Let $h : S \to T$ be a homomorphism of semigroups S and T. Then*

1. *if J is a left ideal (resp. right ideal, resp. subsemigroup) of T, then $h^{-1}(J) = \{s \in S : h(s) \in J\}$ is a left ideal (resp. right ideal, resp. subsemigroup) of S;*

2. *if h is surjective, and I is a left ideal (resp. right ideal) of S, then $h(I) = \{h(x) : x \in I\}$ is a left ideal (resp. right ideal, resp. subsemigroup) of T.*

Proof. For (1), we first show that if J is a left ideal of T, then $h^{-1}(J)$ is a left ideal of S. We must show that $Sh^{-1}(J) \subseteq h^{-1}(J)$. To this end, let $s \in S$ and $x \in h^{-1}(J)$. Then $h(s), h(x) \in J$ and since h is a homomorphism, $h(s)h(x) = h(sx)$ so that $sx \in h^{-1}(J)$. Thus $Sh^{-1}(J) \subseteq h^{-1}(J)$ showing that h^{-1} is a left ideal of S. If J is a right ideal of T then by a similar argument above we may show that $h^{-1}(J)$ is a right ideal of S. As a consequence, we may then show that $h^{-1}(J)h^{-1}(J) \subseteq h^{-1}(J)$ which shows that $h^{-1}(J)$ is a subsemigroup of S whenever J is a subsemigroup of T.
For (2), let h be surjective. We show that if I is a left ideal of S then $h(I)$ is a left ideal of T. The proofs of the remaining then follows as in (1) with appropriate modifications. Let $t \in T$ and $h(x) \in h(I)$. Since $h : S \to T$ is surjective there is $r \in S$ such that $h(r) = t$. Since I is a left ideal of T, $rx \in I$. Since h is a homomorphism, $th(x) = h(r)h(x) = h(rx)$ so that $th(x) \in h(I)$. Thus $Th(I) \subseteq h(I)$ showing that $h(I)$ is a left ideal of T. $\square$

Definition 1.1.16 *Let S be a semigroup.*

1. *L is a* ***minimal left ideal*** *of S if and only if L is a left ideal of S and whenever J is a left ideal of S and $J \subseteq L$ one has $J = L$.*

2. *R is a* ***minimal right ideal*** *of S if and only if R is a right ideal of S and whenever J is a right ideal of S and $J \subseteq R$ one has $J = R$.*

3. *S is* ***left simple*** *if and only if S is a minimal left ideal of S.*

4. *S is* ***right simple*** *if and only if S is a minimal right ideal of S.*

5. *S is* ***simple*** *if and only if the only ideal of S is S.*

Lemma 1.1.3 *Let S be a semigroup.*

1. *Let L_1 and L_2 be left ideals of S. Then $L_1 \cap L_2$ is a left ideal of S if and only if $L_1 \cap L_2 \neq \emptyset$.*

2. *Let L be a left ideal of S and let R be a right ideal of S. Then $L \cap R \neq \emptyset$.*

Proof. To prove (1),

($\Rightarrow$) It follows by Definition 1.1.15(1).

($\Leftarrow$) Pick $x \in L_1 \cap L_2$ and let $s \in S$. Then $x \in L_1$ and $x \in L_2$. Since L_1 and L_2 are left ideals of S, $xs \in L_1$ and $xs \in L_2$. So, $xs \in L_1 \cap L_2$. Thus, $L_1 \cap L_2$ is a left ideal of S.

To prove (2), let $x \in L$ and $y \in R$. Then $yx \in L$ since L is left ideal of S and $yx \in R$ since R is a right ideal of S. So, $yx \in L \cap R$. Thus, $L \cap R \neq \emptyset$. $\square$

Lemma 1.1.4 *Let S be a semigroup and let $x \in S$.*

1. *Then Sx is a left ideal, xS is a right ideal and SxS is a two-sided ideal of S.*

2. *Let $e \in E(S)$. Then e is a left identity of eS, a right identity of Se, and an identity of eSe.*

Proof. The proof of (1) is immediate, $S(Sx) = (SS)x \subseteq Sx$, $(xS)S = x(SS) \subseteq xS$, and $S(SxS)S = (SS)x(SS) \subseteq SxS$.
To prove (2), let $e \in E(S)$, that is $ee = e$. Let $x \in eS$, we can find $t \in S$ such that $x = et$. Then $ex = eet = et = x$ to conclude e is a left identity of eS. The same goes for the rest of the proof. $\square$

Theorem 1.1.3 *Let S be a semigroup.*

1. *If S is left simple and $e \in E(S)$, then e is a right identity for S.*

2. *If L is a left ideal of S and $s \in L$, then $Ss \subseteq L$.*

3. *Let $\emptyset \neq L \subseteq S$. Then L is a minimal left ideal of S if and only if for each $s \in L$, $Ss = L$*

Proof.

1. By Lemma 1.1.4(1), Se is a left ideal of S. Since S is simple, $Se = S$. Thus, by Lemma 1.1.4(2), e is a right identity for S.

2. $Ss \subseteq SL \subseteq L$. Thus, $Ss \subseteq L$.

3. ($\Rightarrow$) By Lemma 1.1.4(1), Ss is a left ideal and by (2), $Ss \subseteq L$. So, $Ss = L$ since L is minimal.
   ($\Leftarrow$) Since $L = Ss$ for some $s \in L$, L is a left ideal. Let J be a left ideal of S with $J \subseteq L$ and pick $s \in J$. Then by (2), $Ss \subseteq J$. So, $J \subseteq L = Ss \subseteq J$. Hence, $L = J$. Thus, L is a minimal left ideal of S.

$\square$

Definition 1.1.17 *Let S be a semigroup.*

1. *The smallest two-sided ideal of S which contains a given element $x \in S$ is called the* ***principal two-sided ideal generated by*** *x.*

2. *The smallest left ideal of S which contains a given element $x \in S$ is called the* ***principal left ideal generated by*** *x.*

3. *The smallest right ideal of S which contains a given element $x \in S$ is called the* ***principal right ideal generated by*** *x.*

Theorem 1.1.4 *Let S be a semigroup and let $x \in S$.*

1. *The principal two-sided ideal generated by x is $SxS \cup xS \cup Sx \cup \{x\}$.*

2. *If S has an identity, then the principal ideal generated by x is SxS.*

3. *The principal left ideal generated by x is $Sx \cup \{x\}$ and the principal right ideal generated by x is $xS \cup \{x\}$.*

Proof. For (1) , $SxS \cup xS \cup Sx \cup \{x\}$ is certainly a two-sided ideal containing x. Now suppose that I is any two-sided ideal containing x. We show that $SxS \cup xS \cup Sx \cup \{x\} \subseteq I$. We need only to show that $SxS \subseteq I$. Let $s_1xs_2 \in SxS$. Since I is a left ideal $SI \subseteq I$ so that $s_1x \in I$. Since I is a right ideal $IS \subseteq I$ so that $(s_1x)s_2 \in I$. Thus $s_1xs_2 \in I$, whence

$SxS \subseteq I$. Thus $SxS \cup xS \cup Sx \cup \{x\} \subseteq I$.

For (2), if S has identity, then $SxS \cup xS \cup Sx \cup \{x\} = SxS$ which is a two-sided ideal (Lemma 1.1.4(1)) and the result follows as in (1).

For (3), $S(Sx \cup \{x\}) \subseteq Sx \cup \{x\}$. Let $s \in S$ and $t \in Sx \cup \{x\}$. If $t \in Sx$, then since Sx is a left ideal $st \in S(Sx) \subseteq Sx$ so that $st \in Sx \cup \{x\}$. Otherwise $t = x$ and thus $st = sx \in Sx \cup \{x\}$. Thus $Sx \cup \{x\}$ is indeed a left ideal containing x. If I is any left ideal containing x. Then $SI \subseteq I$ so that $Sx \cup \{x\} \subseteq I$. Hence $Sx \cup \{x\}$ is the principal left ideal generated by x. The proof that $xS \cup \{x\}$ is the principal right ideal containing x follows by a similar argument. □

Definition 1.1.18 *Let S be a semigroup and let $x \in S$.*

1. *The ideal SxS of S is called the* ***semiprincipal two-sided ideal generated by*** *x.*
2. *The ideal Sx of S is called the* ***semiprincipal left ideal generated by*** *x.*
3. *The ideal xS of S is called the* ***semiprincipal right ideal generated by*** *x.*

Note that the semiprincipal two-sided (respectively, left and right) ideal generated by x is equal to the principal two- sided (respectively, left and right) ideal of S generated by x if and only if $x \in SxS$ (respectively, $x \in Sx$ and $x \in xS$).

Definition 1.1.19 *Let S be a semigroup. Let $\leq$ be a relation in $E(S)$ defined by $e \leq f$ if and only if $e = ef$, for all $e, f \in E(S)$. Then a minimal element in $E(S)$ with respect to the order $\leq$ is called a* ***minimal idempotent****.*

Theorem 1.1.5 *Let S be a semigroup. The following statements are equivalent.*

1. *S is cancellative and simple and $E(S) \neq \emptyset$.*
2. *S is both left simple and right simple.*

3. *For all a and b in S, the equations $ax = b$ and $ya = b$ have solutions x, y in S.*

4. *S is a group.*

Proof. See Proof of Theorem 1.39 in [19]. □

Theorem 1.1.6 *Let S be a semigroup and e be a left identity for S such that for each $x \in S$ there is some $y \in S$ with $xy = e$. Let $Y = E(S)$ and let $G = Se$. Then Y is a right zero semigroup and G is a group and $S = GY \approx G \times Y$.*

Proof. Let $x \in Y$ and $y \in S$. Pick $z \in S$ such that $xz = e$. Then $xe = xxz = xz = e$. Therefore $xy = x(ey) = ey = y$. Then it follows that for all $x, y \in Y$, $xy = y$, and $Y \neq \emptyset$ because $e \in Y$. Y is also closed, since $xy = y \in Y$ for all $x, y \in Y$. Hence Y is a right zero semigroup.
By assumption, $G = Se$. By Lemma 1.1.4(2), e is a right identity for G. By assumption, every element of S has a right e-inverse in S. So every element of G has a right e-inverse in S. One needs only to show that every element of G has a right e-inverse in G. In fact, let $x \in G$ and pick $y \in S$ such that $xy = e$. Then $ye \in Se = G$ and $xye = ee = e$. So, ye is a right e-inverse of x in G. Also, $GG = SeSe \subseteq SSSe \subseteq Se = G$. So, G is closed. Thus, G is a group.
Define $\varphi : G \times Y \to S$ by $\varphi(g, y) = gy$. To prove that φ is a homomorphism, let $(g_1, y_1), (g_2, y_2) \in G \times Y$. Then, one may show that $\varphi(g_1, y_1)\varphi(g_2, y_2) = \varphi(g_1 g_2, y_1 y_2)$ using the fact that Y is a right zero semigroup (see above).
To show that φ is surjective, let $s \in S$. Then $se \in Se = G$ which is a group, and so there exists $x \in Se$ such that $x(se) = (se)x = e$. Also, $xsxs = xsexs = xes = xs$ since $x \in G$, $ex = x$ and $xe = x$. Thus, $xs \in Y = E(S)$. Hence, $(se, xs) \in G \times Y$ and $\varphi(se, xs) = sexs = es = s$. Since φ is onto S, $S = GY$.
To show that φ is one-to-one, let $(g, y) \in G \times Y$ and let $s = \varphi(g, y)$. One may show that $g = se$ and $y = xs$ where x is the (unique) inverse of se in Se using the fact that $Y = E(S)$ is a right zero semigroup. □

Theorem 1.1.7 *Let S be a semigroup and assume that there is a minimal left ideal L of S which has an idempotent e. Then $L = XG \approx X \times G$ where X is the (left zero) semigroup of idempotents of L, and $G = eL = eSe$ is a group. All maximal groups in L are isomorphic to G.*

Proof. Given $x \in L$, Lx is a left ideal of S and $Lx \subseteq L$. So $Lx = L$ and hence there is some $y \in L$ such that $yx = e$. By Lemma 1.1.4(2), e is a right identity for $Le = L$. Therefore, Theorem 1.1.4 applies with the right-left switch and replacing S by L. One may show that the maximal groups of $X \times G$ are sets of the form $\{x\} \times G$. □

Theorem 1.1.8 *Let S be a semigroup, let L be a left ideal of S, and let T be a left ideal of L. The following statements hold.*

1. *For all $t \in T$, Lt is a left ideal of S and $Lt \subseteq T$.*

2. *If L is a minimal left ideal of S, then $T = L$. (Meaning that minimal left ideals are also left simple.)*

3. *If T is a minimal left ideal of L, then T is a left ideal of S.*

Proof. Assume that L is a left ideal of S and T a left ideal of L.

1. Let $t \in T$. Since L is a left ideal of S, $S(Lt) = (SL)t \subseteq Lt$. And, since T is a left ideal of L, $Lt \subseteq LT \subseteq T$.

2. Let $t \in T$. By (1), Lt is a left ideal of S and $Lt \subseteq T \subseteq L$. Assuming that L is a minimal left ideal of S, so $Lt = L$. So $T = L$.

3. Let $t \in T$. By (1), Lt is a left ideal of S, so Lt is a left ideal of L. Since $Lt \subseteq T$ and T is a minimal left ideal of L, $Lt = T$. Therefore, $ST = S(Lt) = (SL)t \subseteq Lt = T$ with the fact that L is a left ideal of S.

□

The statements above imply that if we let L be a left ideal of S and T a left ideal of L and either L is minimal in S or T is minimal in L, then T is a left ideal of S. The same holds for the switch to the right. That is, if R be a right ideal of S and T be a right ideal of L and either L is minimal in S or T is minimal in L, then T is a right ideal of S.

Example 1.1.2 *Let X be an arbitrary set.*

1. *The collection of all constant maps from X to X is a (minimal) left ideal as well as a right ideal of X^X. If $Y \subsetneqq X$ is a nonempty closed set, then $\{g : X \to X \ : \ g$ is constant on $Y\}$ is a left ideal of X^X but not a right ideal, and $\{g : X \to X \ : \ g[X] \subset Y\}$ is a right ideal but not a left ideal.*

2. *In X^X, the collection of all constant maps from X to X is a minimal left ideal.*

3. *In particular, the minimal idempotents in X^X are precisely the constant maps.*

Lemma 1.1.5 *Let S be a semigroup, let I be an ideal of S and let L be a minimal left ideal of S. Then $L \subseteq I$.*

Proof. For any $x \in I$, $xL \subseteq L \cap I$. So $L \cap I$, being nonempty, is a left ideal contained in L. It follows that $L \cap I = L$. □

Theorem 1.1.9 *Let S be a semigroup, let L be a minimal left ideal of S, and let $T \subseteq S$. Then T is a minimal left ideal of S if and only if there is some $a \in S$ such that $T = La$*

Proof. Let S be a semigroup, let L be a minimal left ideal of S, and let $T \subseteq S$.

($\Rightarrow$) Let T be a minimal left ideal of S and let $a \in T$. Then $SLa \subseteq La$, since L is a left ideal of S, and $La \subseteq ST \subseteq T$, since $a \in T$ and T is a left ideal of S. So La is a left ideal of S contained in T. Thus $La = T$, since T is assumed minimal in S.

($\Leftarrow$) Assume that $T = La$ for some $a \in S$. Since L is a left ideal of S,

$$ST = S(La) = (SL)a \subseteq La = T$$

shows that T is a left ideal of S. Now suppose that B is a left ideal of S with $B \subseteq T$. Let $A = \{s \in L : sa \in B\}$. If $b \in B \subseteq T = La$, there is $s \in L$ such that $b = sa$. Thus $A \neq \emptyset$. Now let $t \in S$ and $s \in A$. Since L is a left ideal we have that $ts \in SL \subseteq L$. Furthermore, $sa \in B$ and B is a left ideal so that $tsa \in SB \subseteq B$. Thus $tsa \in B$ so that $ts \in A$. Hence $SA \subseteq A$ showing that $A \subseteq L$ is a left ideal in S. Since L is minimal we have that $A = L$ so that $T = La \subseteq B$. Consequently, $B = T$ and thus T is minimal.

□

Corollary 1.1.1 *Let S be a semigroup. If S has a minimal left ideal, then every left ideal of S contains a minimal left ideal.*

Proof. Let L be a minimal left ideal of S and let J be a left ideal of S. Pick $a \in J$, since $J \neq \emptyset$. Then by Theorem 1.1.5, La is a minimal left ideal which is contained in J. $\square$

Definition 1.1.20 *Let S be a semigroup. The minimal two-sided ideal of S is denoted by $K(S)$ and is called the **smallest ideal** of S.*

Theorem 1.1.10 *Let S be a semigroup. If S has a minimal left ideal, then $K(S)$ exists and $K(S) = \bigcup\{L : L \text{ is a minimal left ideal of } S\}$.*

Proof. Let $L = \bigcup\{L : L$ is a minimal left ideal of $S\}$. By Lemma 1.1.5, if J is any ideal of S, then $I \subseteq J$. So, it suffices to show that I is an ideal of S. By the definition of I, $I \neq \emptyset$. So, let $x \in I$ and $s \in S$. Pick a minimal left ideal L of S such that $x \in L$. Then $xs \in L \subseteq I$. Thus, $xs \in I$. Also, by Theorem 1.1.6, Ls is a minimal left ideal of S. So, $Ls \subseteq I$ while $xs \in Ls$. $\square$

Lemma 1.1.6 *Let S be a semigroup.*

1. *Let L be a left ideal of S. Then L is minimal if and only if $Lx = L$ for every $x \in L$.*

2. *Let I be an ideal of S. Then I is the smallest ideal if and only if $IxI = I$ for every $x \in I$.*

Proof.

1. Let L be a minimal left ideal of S and $x \in L$. Then Lx is a left ideal of S and $Lx \subseteq L$. Since L is considered minimal, $Lx = L$. Now assume that L is a left ideal of S such that $Lx = L$ for every $x \in L$. Let J be a left ideal of S with $J \subseteq L$. Pick $x \in J$. Then $L = Lx \subseteq LJ \subseteq J \subseteq L$. So $J = L$. Meaning that L is minimal.

2. Let I be the smallest ideal of S and let $x \in I$. Then IxI is a two-sided ideal of S and $IxI \subseteq I$. But I is the smallest ideal of S. So $IxI = I$. Now assume that I is a two-sided ideal of S such that $I = IxI$ for every $x \in I$. Let J be a two-sided ideal of S with $J \subseteq I$. Pick $x \in J$. Then $I = IxI \subseteq IJI = (IJ)I \subseteq JI \subseteq J \subseteq I$. Thus, $J = I$. that is, I is the smallest ideal of S.

□

Theorem 1.1.11 *Let S be a semigroup. If L is a minimal left ideal of S and R is a minimal right ideal of S, then $K(S) = LR$.*

Proof. $\emptyset \neq LR \subseteq SS \subseteq S$ and $SLRS = (SL)(RS) \subseteq LR$ since L and R are respectively left and right ideals of S. Thus, LR is a two-sided ideal of S. Let $x \in LR$. Then $LRxL$ is a left ideal of S which is contained in L. So, $LRxL = L$ and hence $LRxLR = LR$. By Lemma 1.1.6(2), $K(S) = LR$. □

Theorem 1.1.12 *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent, and let $e \in E(S)$. Then the following statements are equivalent:*

1. *Se is a minimal left ideal.*
2. *eSe is a group.*
3. *$eSe = H(e)$.*
4. *eS is a minimal right ideal.*
5. *e is a minimal idempotent.*
6. *$e \in K(S)$.*
7. *$K(S) = SeS$.*

Proof. Let S be a semigroup with a minimal left ideal of S which has an idempotent, and let $e \in E(S)$.

(1) $\Rightarrow$ (2): $(eSe)(eSe) = eS(ee)Se = eSeSe \subseteq eSe$. Hence, eSe is closed. From Lemma 1.1.4(2), e is an identity of eSe. Let $x \in eSe$. Pick $s \in S$ such that $x = ese$. One has $x \in Se$. So, Sx is a left ideal of Se and consequently, $Sx = Se$, since Se is a minimal left ideal of S. Thus, $e \in Sx$. So, pick $y \in S$ such that $e = yx$. Then $eye \in eSe$ and, since e is an identity of eSe, $eyex = eyx = ee = e$. So, x has a left e-inverse in eSe. Hence, eSe is a group.

(2) $\Rightarrow$ (3): By assumption eSe is a group and $e \in eSe$. Then, $eSe \subseteq H(e)$. On the other hand, by Theorem 1.1.1, e is an identity of $H(e)$. Let $x \in H(e)$. So, $x = exe \in eSe$. Hence, $H(e) \subseteq eSe$.

(3) $\Rightarrow$ (1): Let L be a left ideal of S with $L \subseteq Se$. Pick $t \in L$. Then, $t \in Se$ implies that $et \in eSe$. Pick $x \in eSe$ such that $x(et) = e$. Then, $xt = (xe)t = x(et) = e$. So, $e \in L$. Thus, $Se \subseteq SL \subseteq L$.

(1) $\Rightarrow$ (5): Let $f \in E(S)$ with $f \leq e$. Then, by Definition 1.1.16, $f = fe$. Clearly, $f \in Se$. By the assumptions, $Se = Sf$. So, $e \in Sf$. By Lemma 1.1.4(2), $e = ef$. Thus, $e = ef = f$. Hence, e is a minimal idempotent.

(5) $\Rightarrow$ (1): Let L be a left ideal with $L \subseteq Se$. Pick an idempotent $t \in L$, and let $f = et$. Then, $f \in L$. Since $t \in Se$, $t = te$. Thus, $f = et = ete$. Therefore, $ff = etet = (ete)t = ett = et = f$. So, $f \in E(S)$. Also, $ef = eete = ete = f$ and $fe = etee = ete = f$. So, $f \leq e$. So, $f = e$ and hence $e \in L$. Thus, $Se \subseteq L$.

(1) $\Rightarrow$ (6): Let Se be a minimal idempotent. By Theorem 1.1.6, $e \in K(S)$.

(6) $\Rightarrow$ (1): Since $e \in K(S)$, by Theorem 1.1.6, pick a minimal left ideal L of S with $e \in L$. By Lemma 1.4.1(1), Se is a left ideal of S. From Definition 1.1.15(1), $Se \subseteq L$. Then $Se = L$, since L is taken minimal.

(6) $\Rightarrow$ (7): Since SeS is an ideal, $K(S) \subseteq SeS$. Since $e \in K(S)$, $SeS \subseteq K(S)$.

(7) $\Rightarrow$ (6): $e = eee \in SeS = K(S)$.

By left-right duality and the fact that (2) and (3) are two-sided statements, (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows. $\square$

Theorem 1.1.13 *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Then*

1. *Every minimal left ideal of S has an idempotent.*
2. *There is a minimal right ideal of S which has an idempotent.*

Proof. To prove (1), let L be a minimal left ideal with an idempotent e and let J be a minimal left ideal. By Theorem 1.1.6, there is some $x \in S$ such that $J = Lx$. By Theorem 1.1.12(2), $eL = eSe$ is a group. So let $y = eye$ be the inverse of exe in this group. Then $yx \in Lx = J$

and $yxyx = (ye)x(ey)x = y(exe)yx = eyx = yx$. Hence, J has an idempotent.
To prove (2), pick a minimal left ideal L of S and an idempotent $e \in L$. Then, $L = Se$. Thus, by Theorem 1.1.7(4), eS is a minimal right ideal of S and e is an idempotent in eS. $\square$

Theorem 1.1.14 *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Given any minimal left ideal L of S and any minimal right ideal R of S, there is an idempotent $e \in R \cap L$ such that $R \cap L = RL = eSe$ and eSe is a group.*

Proof. Let R and L be given. Pick an idempotent $f \in K(S)$ such that $L = Sgf$. By Theorem 1.1.7(2), fSf is a group. Pick $a \in R$ and let x be an inverse of faf in fSf. Then $x \in Sf = L$ so $ax \in R \cap L$. By Theorem 1.1.6, $ax \in K(S)$. Also, one can show that $axax = ax$. Let $e = ax$. Then $eSe \subseteq Sx \subseteq L$ and $eSe \subseteq aS \subseteq R$. So, $eSe \subseteq R \cap L$. To see that $R \cap L \subseteq eSe$, let $b \in R \cap L$. By Lemma 1.1.4, $L = Se$ and $R = eS$ and $b = eb = be$. Thus, $b = eb = ebe \in eSe$.
Now, $RL = eSSe \subseteq eSe \subseteq RL$. So, $RL = eSe$ which is a group by Theorem 1.1.7(2) with $e \in K(S)$. $\square$

Theorem 1.1.15 *Let X be a left zero semigroup, let Y be a right zero semigroup, and let G be a group. Let e be the identity of G, fix $u \in X$ and $v \in Y$ and let $[\,,\,] : Y \times X \to G$ be a function such that $[y, u] = [v, x] = e$ for all $y \in Y$ and all $x \in X$. Let $S = X \times G \times Y$ and define an operation $\cdot$ on S by $(x, g, y) \cdot (x', g', y') = (x, g[y, x']g', y')$. Then S is a simple group (so that $K(S) = S = X \times G \times Y$) and each of the following statements holds.*

1. *For every $(x, y) \in X \times Y$, $(x, [y, x]^{-1}, y)$ is an idempotent (where the inverse is taken in G) and all idempotents are of this form. In particular, the idempotents in $X \times G \times \{v\}$ are of the form (x, e, v) and the idempotents in $\{u\} \times G \times Y$ are of the form (u, e, y).*

2. *For every $y \in Y$, $X \times G \times \{y\}$ are minimal left ideals of S and all minimal left ideals of S are of this form.*

3. *For every $x \in X$, $\{x\} \times G \times Y$ are minimal right ideals of S and all minimal right ideals of S are of this form.*

4. *For every $(x, y) \in X \times Y$, $\{x\} \times G \times \{y\}$ are maximal groups in S and all maximal groups in S are of this form.*

5. *The minimal left ideal of $X \times G \times \{v\}$ is the direct product of X, G, and $\{v\}$ and the minimal right ideal of $\{u\} \times G \times Y$ is the direct product of $\{v\}$, G, and Y.*

6. *All maximal groups in S are isomorphic to G.*

7. *All minimal left ideals of S are isomorphic to $X \times G$ and all minimal right ideals of S are isomorphic to $G \times Y$.*

Proof. See Proof of Theorem 1.63 in [19]. $\square$

The following theorem, called the *Structure Theorem* due to A. Suschkewitsch in 1928 and D. Rees in 1940, gives the precise description of a completely simple semigroup.

Theorem 1.1.16 *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let R be a minimal right ideal of S, let L be a minimal left ideal of S, let $X = E(L)$, let $Y = E(R)$, and let $G = RL$. Define an operation $\cdot$ on $X \times G \times Y$ by $(x, g, y) \cdot (x', g', y') = (x, gyx'g', y')$. Then $X \times G \times Y$ satisfies the conclusions of Theorem 1.1.15 (where $[y, x] = yx$) and $K(S) \approx X \times G \times Y$. In particular:*

1. *The minimal right ideals of S partition $K(S)$ and the minimal left ideals of S partition $K(S)$.*

2. *The maximal groups in $K(S)$ partition K(S).*

3. *All minimal right ideals of S are isomorphic and all minimal left ideals of S are isomorphic,*

4. *All maximal groups in $K(S)$ are isomorphic.*

Proof. By Theorem 1.1.13, there exists a minimal right ideal R of S which has an idempotent. By Theorem 1.1.14, RL is a group and, by Theorem 1.1.7, X is a left zero semigroup and Y is a right zero semigroup. Let e be the identity of $RL = R \cap L$ and let $u = v = e$. Given $y \in Y$ one has, since Y is zero right semigroup, that $[y, u] = yu = u = e$. Similarly, given $x \in X$, $[v, x] = e$. Thus, the hypotheses of Theorem 1.1.15 are

satisfied.
Define $\varphi : X \times G \times Y \to S$ by $\varphi(x, g, y) = xgy$. We now prove that φ is an isomorphism onto $K(S)$. For every $(x, g, y), (x', g', y') \in X \times G \times Y$,

$$\begin{aligned}
\varphi((x, g, y) \cdot (x', g', y')) &= \varphi(x, gyx'g', y') \\
&= x(gyx'g')y' \\
&= (xgy)(x'g'y') \\
&= \varphi(x, g, y)\varphi(x', g', y').
\end{aligned}$$

So, φ is a homomorphism. By Theorem 1.1.7, $L = XG$ and $R = GY$. By Theorem 1.1.14, $K(S) = LR = XGGY = XGY = \varphi[X \times G \times Y]$. thus it suffices to produce the inverse of φ.
For each $t \in K(S)$, let $\gamma(t)$ be the inverse of ete in $eSe = G$. Then $t\gamma(t) = t\gamma(t)e \in Se = L$ and

$$\begin{aligned}
t\gamma(t)t\gamma(t) &= t\gamma(t)ete\gamma(t) \\
&= te\gamma(t) \\
&= t\gamma(t),
\end{aligned}$$

so $t\gamma(t) \in X$. Similarly, $\gamma(t)t \in Y$.
Define $\tau : K(S) \to X \times G \times Y$ by $\tau(t) = (t\gamma(t), ete, \gamma(t)t)$. We now prove that $\tau = \varphi^{-1}$. So let $(x, g, y) \in X \times G \times Y$. Then

$$\begin{aligned}
xgy\gamma(xgy) &= xxgy\gamma(xgy), \text{ since } x = xx \\
&= xexgye\gamma(xgy), \text{ since } x = xe \text{ and } \gamma(xgy) = e\gamma(xgy) \\
&= xe, \text{ since } \gamma(xgy) \text{ is the inverse of } exgye \text{ in } G \\
&= x.
\end{aligned}$$

Similarly, $\gamma(xgy)xgy = y$. Also, since X is a left zero semigroup and Y is a right zero semigroup, $exgye = ege = g$. Hence, for every $(x, g, y) \in X \times G \times Y$,

$$\begin{aligned}
\tau(\varphi(x, g, y)) &= \tau(xgy) \\
&= (xgy\gamma(xgy), exgye, \gamma(xgy)xgy) \\
&= (x, g, y).
\end{aligned}$$

That is, $\tau = \varphi^{-1}$. $\square$

Theorem 1.1.17 *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let T be a subsemigroup of S and assume also that T has a minimal left ideal with an idempotent. If $K(S) \cap T \neq \emptyset$, then $K(T) = K(S) \cap T$.*

Proof. Theorem 1.1.10 guarantees the existence of $K(T)$. Since $K(S)\cap T$ is an ideal of T, $K(T)\subseteq K(S)\cap T$. For the reverse inclusion, let $x\in K(S)\cap T$ be given. Then Tx is a left ideal of T. So, by Corollary 1.1.1 and Theorem 1.1.13(1), Tx contains a minimal left ideal Te of T for some idempotent $e\in T$. Now $x\in K(S)$. So by Theorem 1.1.10, pick a minimal left ideal L of S with $x\in L$. Then $L=Sx$ and $e\in Tx\subseteq Sx$. So, $L=Se$. So, $x\in Se$. So, by Lemma 1.1.4, $x=xe\in Te\subseteq K(T)$. Hence $K(S)\cap T\subseteq K(T)$. □

Theorem 1.1.18 *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $e,f\in E(K(S))$. If g is the inverse of efe in eSe, then the function $\varphi: eSe\to fSf$ defined by $\varphi(x)=fxgf$ is an isomorphism.*

Proof. To prove that φ is a homomorphism, let $x,y\in eSe$. Then

$$\begin{aligned}
\varphi(x)\varphi(y) &= fxgffygf \\
&= fxgfygf && (f \text{ is an idempotent})\\
&= fxgefeygf && (ge=g, ey=y)\\
&= fxeygf && (g \text{ is the inverse of } efe)\\
&= fxygf && (ey=y)\\
&= \varphi(xy).
\end{aligned}$$

To prove that φ is one-to-one, let x be in the kernel of φ. Then $\varphi(x)=f$ implies

$$\begin{aligned}
fxgf &= f \\
efxgfe &= efe && (f \text{ is an idempotent})\\
efexgefe &= efe && (ex=x, ge=g)\\
efexe &= efe && (g \text{ is the inverse of } efe)\\
efex &= efe && (xe=x)\\
efex &= efee && (e \text{ is an idempotent})\\
x &= e && (\text{ left cancellation in } eSe).
\end{aligned}$$

To prove that φ is onto fSf, let $y\in fSf$ and let h and k be the inverses of fgf and fef respectively in fSf. Then $ekyhe\in eSe$ and

$$\begin{aligned}
\varphi(ekyhe) &= fekyhegf \\
&= fefkyhgf && (fk=k, eg=g)\\
&= fyhgf && (fefk=f)\\
&= fyhfgf && (h=hf)\\
&= fyf && (hfgf=f)\\
&= y && (y\in fSf).
\end{aligned}$$

□

Theorem 1.1.19 *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $s \in S$. The following statements are equivalent.*

1. $s \in K(S)$.
2. *For all* $t \in S$, $s \in Sts$.
3. *For all* $t \in S$, $s \in stS$.
4. *For all* $t \in S$, $s \in stS \cap Sts$.

Proof.

(1) $\Rightarrow$ (4): Pick, by Theorem 1.1.10 and Lemma 1.1.13(2), a minimal left ideal L of S and a minimal right ideal R of S with $s \in L \cap R$. Let $t \in S$. Then $ts \in L$. So, Sts is a left ideal contained in L. So, $Sts = L$. Similarly, $stS = R$.

Clearly, (4) $\Rightarrow$ (3) and (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Let $t \in K(S)$. Then $s \in Sts \subseteq StS \subseteq K(S)$. That is, $s \in K(S)$.

Similarly, (3) $\Rightarrow$ (1).

□

Lemma 1.1.7 *Let S and T be semigroups and let $h : S \to T$ be a surjective homomorphism . If S has a smallest ideal, then T also has a smallest ideal and $K(T) = h[K(S)]$.*

Proof. See Exercise 1.7.3 in [19]. □

The following Lemma and its proof are taken from [35].

Lemma 1.1.8 *Let S and T be semigroups and let $h : S \to T$ be a surjective homomorphism. Suppose that $K(S)$ exists and is a completely simple semigroup. Then for every minimal right ideal R of S, $h[R]$ is a minimal right ideal of T, and for every minimal right ideal R' of T, there is a minimal right ideal R of S with $h[R] = R'$. The same holds also for minimal left ideals and for maximal subgroups from the smallest ideals.*

Proof. By Lemma 1.1.7, $h[K(S)]$ is a two-sided ideal of T. If K' is any ideal of T, then $h^{-1}[K']$ is an ideal of S, so contains $K(S)$ by minimality. Thus $h[K(S)] \subseteq K'$, whence $h[K(S)]$ is the smallest ideal of T. Similarly, each right ideal R' of T contains a right ideal of the form $h[R]$ with R a right ideal of S, so the minimal right ideals of T are of the form $h[R]$. The same holds for the left ideals. Finally, for any minimal right ideal R and minimal left ideal L, RL is the structural group of $K(S)$, and then $h[RL] = h[R]h[L]$ is the structural group of $h[K(S)]$. $\square$

1.2 Fundamentals on topology

We recall some fundamental notions of topology such as neighborhoods, denseness, connectedness, separation properties, and compactness which are important for the understanding of the topological structure of Stone-Čech compactifications and their points, the ultrafilters. The main reference of this section is [16].

Definition 1.2.1 *A **topology** τ on a set X is a collection of subsets of X satisfying:*

1. *$\emptyset, X \in \tau$.*
2. *if $A_i \in \tau$, for each $i = 1, \ldots, n$, then $\bigcap_{i=1}^{n} A_i \in \tau$.*
3. *if $A_i \in \tau$, for each $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \tau$.*

Definition 1.2.2 *A nonempty set X equipped with a topology τ is called a **topological space**, and is denoted (X, τ), (or simply X when there is no confusion). A member of τ is called an **open set** in X. The complement of an open set is called a **closed set**. A set that is both closed and open is called a **clopen set**.*

One can prove, by using the two definitions above, that if X is a topological space, then

1. $\emptyset$ and X are closed sets,
2. the intersection of any number of closed sets is a closed set and
3. the union of any finite number of closed sets is a closed set.

Let X be a topological space. A subset A of X is called an **F_σ-set** if it is the union of a countable number of closed sets. A subset B of X is called a **G_δ-set** if it is the intersection of a countable number of open sets. It follows that F_σ-sets and G_δ-sets are complement of one another.

Definition 1.2.3 *Let X, Y be two topological spaces. A mapping $f : X \to Y$ is said to be **open** (resp. **closed**) if the image under f of each open (resp. closed) set of X is open (resp. closed) in Y.*

Definition 1.2.4 *A subfamily $\mathcal{B}$ of a topology τ is called a **basis** for τ if every $U \in \tau$ is a union of members of $\mathcal{B}$.*

Equivalently, $\mathcal{B}$ is a basis for τ if for every element x of X and every open set U containing x, there is an open set $V \in \mathcal{B}$ satisfying $x \in V \subset U$. Conversely, if $\mathcal{B}$ is a family of sets that is closed under finite intersection and $\bigcup_{B\in\mathcal{B}} B = X$, then the family τ of all unions of members of $\mathcal{B}$ is a topology for which $\mathcal{B}$ is a basis. This is important, since instead of trying to describe all of the open sets, one can define topologies by simply writing down a basis.

Definition 1.2.5 *A subfamily σ of a topology τ is called a **subbasis** for τ if the collection of all finite intersections of members of σ is a basis for τ.*

Definition 1.2.6 *A topology is said to be **zero-dimensional** if it has a basis of clopen sets.*

Definition 1.2.7 *Let X be a nonempty set.*

1. *The family of all subsets of a set is a topology called the **discrete topology**. In a discrete space, every subset is both open and closed (that is, clopen).*
2. *If Y is a subset of a topological space (X, τ), then the collection $\tau_Y = \{V \cap Y : V \in \tau\}$ of subsets of Y is a topology on Y called the **topology induced** by τ on Y. Then Y, equipped with the induced topology, is called a **(topological) subspace** of X.*

In a discrete space (X, τ), the collection $\mathcal{B} = \{\{x\} : x \in X\}$ forms a basis for the topology τ. For any topological space (X, τ), $\mathcal{B} := \tau$ is a basis for the topology τ. For example, the set of all subsets of X is a basis for the discrete topology on X.
It is also important to remember that there can be many bases for the same topology.

Definition 1.2.8 *Let (X_i, τ_i), $i \in I$, be topological spaces and $\pi_i : X \to X_i$ continuous canonical projections. The **product topology** τ on the product set $X = \prod_{i \in I} X_i$ is the topology having the family of $\pi_i^{-1}(V_i)$ with $V_i \in V_i$ for each $i \in I$, as a subbasis. The pair (X, τ) is then called the **product of spaces** (X_i, τ_i), with $i \in I$.*

Definition 1.2.9 *Let (X, τ) be a topological space, and let A be any subset of X.*

1. *The largest (with respect to inclusion) open set included in A, denoted $int(A)$ or $\mathring{A}$, is called the **interior** of A. It is the union of all open subsets of A. The interior of a nonempty set may be empty.*
2. *The smallest closed set that includes A, denoted $\overline{A}$ or $c\ell_X(A)$ (or simply $c\ell(A)$ if there is no confusion), is called the **closure** of A into X. It is the intersection of all closed sets that include A. If B is closed set such that $A \subset B$, then $\overline{A} \subset B$.*

One can conclude that $A \subset B$ implies $\mathring{A} \subset \mathring{B}$ and $\overline{A} \subset \overline{B}$. Also a set A is open if and only if $A = \mathring{A}$ and a set B closed if and only if $B = \overline{B}$.

Definition 1.2.10 *The elements of the set X will be referred to as points. Any set V that contains a point x in its interior is called a **neighborhood** of x. In this case the point x is said to be to be an **interior point** of V.*

Equivalently, a neighborhood of a point $x \in X$ is a subset of X containing an open set containing x. Also, a neighborhood of a subset A of a topological space X is a subset of X containing an open set containing A. But a point x can be identified with the singleton set $\{x\}$, meaning a neighborhood of x is also a neighborhood of $\{x\}$. Thus, the concept of neighborhood of a point in a set can be used to give an axiomatic definition of topological spaces. Let $x \in X$. We have the following axioms to characterize the topological space X:

1. Every neighborhood of x contains x.

2. Any finite intersection of neighborhoods of x is a neighborhood of x.

3. If a subset A of X contains a neighborhood of x, then A is a neighborhood of x.

4. If V is a neighborhood of x, there exists a neighborhood W of x such that V is a neighborhood of each point of W.

Following this definition, a subset U of X is open if and only if it is a neighborhood of each of its points.

Definition 1.2.11 *A* ***neighborhoods system*** *of a point x of X is any family $\mathcal{N}_x$ of neighborhoods of x such that any neighborhood of x contains a neighborhood from $\mathcal{N}_x$.*

A neighborhoods system can also be referred to as a neighborhoods base.

Definition 1.2.12 *Let A be a nonempty subset of a topological space.*

1. *A point x is called a* ***limit point*** *(or an* ***accumulation point*** *or a* ***cluster point****) of A if for each open neighborhood V of x, we have $(V \setminus \{x\}) \cap A \neq \emptyset$.*

2. *A point $x \in A$ is called an* ***isolated point*** *of A if there is an open neighborhood V of x with $(V \setminus \{x\}) \cap A = \emptyset$.*

Let X be a discrete space and A a subset of X. Then A has no limit points, since for each $x \in X$, the singleton set $\{x\}$ is open but contains no point of A different from x. Thus, all the points of a subset of a discrete space are isolated points.
The notion of limit point may lead to a better understanding of the notion of closed set. One can prove that that a set is closed if and only if it contains all its limit points. This provides a useful way to prove if a set is closed or not.
Let A be a subset of a topological space X. If A' denotes the set of all limit points of A, then $A \cup A'$ is a closed set. Hence, $A \cup A' = \overline{A}$.

Definition 1.2.13 *A subset D of a topological space X is said to be* ***dense*** *in X if $\overline{D} = X$.*

In other words, a set D is dense if and only if every nonempty open subset of X contains a point in D. In particular, if D is dense in X and x belongs to X, then every neighborhood of x contains a point in D. This means that any point in X can be approximated arbitrarily well by points in D.
In a discrete space X, the only dense subset is X itself, since each other subset of X is its own closure. One classical example, $\mathbb{Q}$ is dense in $\mathbb{R}$.
Let S and T be nonempty subsets of a topological space X with $S \subseteq T$. Then, we have the following:

1. A limit point of S is also a limit point of T.

2. If S is dense in X, then T is also dense in X, since $\overline{S} \subseteq \overline{T}$.

From (2), it is possible to deduce that the real line $\mathbb{R}$ has an uncountable number of distinct uncountable dense subsets.

Definition 1.2.14 *A subset A of a topological space X is said to be* ***nowhere dense*** *in X if there is no neighborhood in X on which A is dense.*

In other words, a nowhere dense set has the interior of its closure empty.

Definition 1.2.15 *Let X be a topological space.*

1. *X is said to be* ***connected*** *if it is not the union of two disjoint non-empty closed sets.*

2. *X is said to be* ***locally connected*** *if every point has a base of open connected neighborhoods.*

3. *X is said to be* ***totally disconnected*** *if the only connected subsets are the singleton sets.*

4. *X is said to be* ***extremally disconnected*** *if the closure of every open subspace is open.*

Equivalently, we can also say that X is connected if the only clopen subsets are X and $\emptyset$. One such example is the usual topological space $\mathbb{R}$. It follows from this definition that a topological space X is disconnected if and only if we can find two nonempty open sets A and B such that

$A \cap B = \emptyset$ and $A \cup B = X$. One can show that any infinite discrete space is disconnected since each singleton set is a clopen set.

We have also the property that the product $\prod_{i \in I} X_i$ is connected if and only if, for each $i \in I$, the space X_i is connected.

Definition 1.2.16 *Suppose that X and Y are topological spaces, that $A \subseteq X$ and that $f : A \to Y$. Let $a \in c\ell_X(A)$ and $y \in Y$. We define*

$$\lim_{x \to a} f(x) = y$$

if and only if, for every neighborhood V of y, there is a neighborhood U of a such that $f[A \cap U] \subseteq V$.

Definition 1.2.17 *A function $f : X \to Y$ between two topological spaces is said to be **continuous** if $f^{-1}(U)$ is open in X for each open set U in Y. We say that f is **continuous at the point** x of Y if $f^{-1}(V)$ is a neighborhood of x whenever V is an open neighborhood of $f(x)$ in Y.*

We can show that any continuous image of a connected set is connected.

Theorem 1.2.1 *Let $f : X \to Y$ be a function between two topological spaces.*

1. *f is continuous if and only if f is continuous at every point of X.*
2. *f is continuous at a point a of $A \subseteq X$ if and only if $\lim_{x \to a} f(x) = f(a)$.*

Proof. The proof can be found in any book of general topology. $\square$

Definition 1.2.18 *Two topological spaces X and Y are said to be **homeomorphic** if there is a one-to-one continuous function f from X onto Y such that f^{-1} is continuous too. The function f is then called a **homeomorphism**.*

Any two nonempty open intervals of $\mathbb{R}$ are homeomorphic. Also, $\mathbb{R}$ is homeomorphic to the open interval $(-1, 1)$ with the usual topology by defining a homeomorphism $f : (-1, 1) \to \mathbb{R}$ by $f(x) = \frac{x}{1-|x|}$. It follows that any open interval (a, b), with $a < b$, is homeomorphic to $\mathbb{R}$.
We have that any topological space homeomorphic to a connected space is connected.

Definition 1.2.19 *A mapping $f : X \to Y$ between two topological spaces is called an **inclusion map** or **embedding** if $f : X \to f[X]$ is a homeomorphism. It is sometimes denoted by ι.*

That is, X, as a topological subspace of Y, can be identified with its direct image $f[X]$.
We may also add the following definition. Let X and Y be topological spaces. A map $f : X \to Y$ is called a *local homeomorphism* if each point $x \in X$ has an open neighborhood U such that $f_{|_U}$ is a homeomorphism onto an open subspace of Y. We say that X is locally homeomorphic to Y.

Definition 1.2.20 *Let X be a topological space. A subset A of X is said to be C^***-embedded** in X if each map $f : A \to [0,1]$ can be extended to a map $f : X \to [0,1]$.*

In the following, we recall some separation properties. The separation properties are preserved by homeomorphisms.

Definition 1.2.21 *A topological space is said to be **separable** if it includes a countable dense subset.*

For example, every countable topological space is separable. In particular, $\mathbb{R}$ is separable since $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 1.2.22 *A topology on X is called **Hausdorff** (or **separated**) if any two distinct points can be separated by disjoint neighborhoods of the points. Hausdorff spaces are also known as T_2**-spaces**.*

That is, for each pair $x, y \in X$ with $x \neq y$ there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $U \cap V = \emptyset$. For example, discrete spaces and metric spaces are Hausdorff.

Definition 1.2.23 *A topological space X is said to be a T_1**-space** if every singleton set $\{x\}$ is closed in X.*

For example, a discrete space is a T_1-space.

Definition 1.2.24 *A topological space X is said to be a T_0**-space** if for each pair of distinct points a, b in X, either there exists an open set containing a and not b, or there exists an open set containing b and not a.*

One can see that every T_1-space is a T_0-space, but the converse is not true.

Definition 1.2.25 *A topological space X is said to be **regular** if it is Hausdorff and for any closed subset F of X and any point $x \notin F$ there is a neighborhood of x and a neighborhood subset of F which do not intersect. A regular T_1-space is called a T_3-space.*

Obviously, a T_3-space is a T_2-space. The converse is false.

Theorem 1.2.2 *Every subspace of a regular space is regular.*

Proof. See the proof on page 80 of [16]. □

Definition 1.2.26 *Two subsets A and B of a space X are said to be **completely separated** in X if there exists a real-valued continuous mapping f on X such that $f(a) = 0$ for all a in A and $f(b) = 1$ for all b in B.*

Definition 1.2.27 *A space X is said to be **completely regular** if every closed subspace F of X is completely separated from one point x not in F and if each point is closed. If X is also Hausdorff, then X is called a **Tychonoff space** or a $T_{3\frac{1}{2}}$**-space**.*

Every metric space is a Tychonoff space. In particular, the closed interval $[0, 1]$ is a Tychonoff space. For any set X, the cube $[0, 1]^X$ is a Tychonoff space.

Theorem 1.2.3 ***(Tychonoff's theorem)*** *The completely regular spaces are precisely those spaces which can be embedded in a product of copies of the closed unit interval I.*

One can deduce that any Tychonoff space is homeomorphic to a subspace of a cube. Thus, a topological space can be embedded in a cube if and only if it is a Tychonoff space.

Theorem 1.2.4 *For any topological space X, there exists a completely regular space ρX which is a continuous image of X such that any real-valued mapping from X factors through ρX.*

Definition 1.2.28 *A topological space X is said to be a* ***normal space*** *if for each pair of disjoint closed sets A and B, there exist a pair of disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. If the normal space X is also Hausdorff, then X is called a* ***T_4-space****.*

Theorem 1.2.5 ***(Urysohn's Lemma)*** *Let X be a topological space. Then X is normal if and only if for each pair of disjoint closed sets A and B in X there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$, and $f(b) = 1$ for all $b \in B$.*

It follows that every normal Hausdorff space is a Tychonoff space; that is, every T_4-space is a $T_{3\frac{1}{2}}$-space. As a consequence, every normal Hausdorff space is homeomorphic to a subspace of a cube.

Definition 1.2.29 *A subset D of a topological space X is said to be* ***strongly discrete*** *if and only if there is a family $(U_a)_{a\in D}$ of pairwise disjoint open subsets of X such that $a \in U_a$ for every $a \in D$.*

Notice that if D is countable and X is regular, then D is strongly discrete if and only if it is discrete.

Definition 1.2.30 *A subset K of a topological space is* ***compact*** *if every family of open sets V_i, $i \in I$, satisfying $K \subset \bigcup_{i\in I} V_i$ there exists a finite subfamily $V_{i_1}, \ldots, V_{i_n}$ such that $K \subset \bigcup_{j=1}^{n} V_{i_j}$. The family of V_i, $i \in I$, is called an* ***open cover*** *of K and the finite subfamily $V_{i_1}, \ldots, V_{i_n}$ is called a* ***finite subcover*** *of K.*

One can prove that every compact Hausdorff space is normal. Consequently, every compact Hausdorff space can be embedded in a cube. We need to note that only finite sets are compact in a discrete topology. The notion of compactness can be viewed as a topological generalization of finiteness.
We have the following properties of compactness:

1. A continuous image of a compact space is compact.

2. Every closed subset of a compact space is compact.

3. A compact subset of a Hausdorff topological space is closed.

4. If $K \subset Y \subset X$, then K is a compact subset of X if and only if K is a compact subset of Y (in the induced topology).

5. Every continuous function between topological spaces carries compact sets to compact sets.

6. Finite unions of compact sets are compact.

The well-known *Heine Borel Theorem* states that any subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. But this not true in more general topological spaces. For instance, consider infinite sets with the discrete topology. But in general, a compact subset of a metric space is closed and bounded.

Definition 1.2.31 *A topological space X is said to be* ***locally compact*** *if each point in X has at least one neighborhood which is compact.*

As an example, every discrete space is locally compact. It can be proved that every compact space is locally compact.

Definition 1.2.32 *A family of sets has the* ***finite intersection property*** *if every finite subfamily has a nonempty intersection.*

Compactness can also be characterized in terms of the finite intersection property as follows. A topological space is compact if and only if every family of its closed subsets with the finite intersection property has a nonempty intersection.

In brief, we have the following hierarchy of separation properties:

$$\text{Compact Hausdorff} \Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

All the topological spaces mentioned in the rest of this work are presumed Hausdorff.

1.3 Compact right topological semigroups and ideals

Since the Stone-Čech compactification of a discrete semigroup is a compact Hausdorff right topological semigroup, we review in this section the general notions of compact Hausdorff right topological semigroups. For instance, every compact Hausdorff right topological semigroup has an idempotent. Consequently, every compact Hausdorff right topological semigroup has a smallest two-sided ideal which is a completely simple semigroup given by the *Rees-Suschkewitsch Theorem.* The main references are [20] and [37].

Definition 1.3.1 *1. A **right topological semigroup** is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for all $x \in S$, $\rho_x : S \to S$, defined by $\rho_x(y) = y \cdot x$, is continuous. The map ρ_x is called **right translation**.*

2. *A **left topological semigroup** is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for all $x \in S$, $\lambda_x : S \to S$, defined by $\lambda_x(y) = x \cdot y$, is continuous. The map λ_x is called **left translation**.*

3. *A **semitopological semigroup** is a right topological semigroup which is also a left topological semigroup.*

4. *A **topological semigroup** is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and $\cdot : S \times S \to S$ is continuous.*

5. *A **topological group** is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a group, (S, τ) is a topological space, $\cdot : S \times S \to S$ is continuous, and $In : S \to S$ is continuous (where $In(x)$ is the inverse of x in S).*

Example 1.3.1 *Let X be a compact Hausdorff topological space.*

1. *X^X is the space of all maps (continuous or not) from X to X. We can write X^X as $X^X = \prod_{x \in X} X_x$, where $X_x = X$ for every $x \in X$. That is, a map $g : X \to X$ is identified with the element $(g(x))_{x \in X} \in \prod_{x \in X} X_x$. Then, X^X with the product topology is compact by Tychonoff's Theorem.*

2. *The topology on X^X is the topology of pointwise convergence. This means the following: '$g_n \to g$ in X^X if and only if, for each $x \in X$, $g_n(x) \to g(x)$ in X'.*

3. *Let τ be the product topology on X^X. The composition of functions is an associative binary operation on X^X and the identity map of X belongs to X^X making X^X nonempty. Thus, $(X^X, \circ, \tau)$ is a compact right topological semigroup.*

Definition 1.3.2 *Let S be a right topological semigroup. The set*

$$\Lambda(S) = \{x \in S : \lambda_x \text{ is continuous}\}$$

*is called the **topological center** of S.*

The following theorem shows that every compact right topological semigroup has an idempotent.

Theorem 1.3.1 *Let S be a compact right topological semigroup. Then $E(S) \neq \emptyset$.*

Proof. Let

$$\mathcal{A} = \{T \subseteq S : T \neq \emptyset, T \text{ is compact, and } T \cdot T \subseteq T\}.$$

That is, $\mathcal{A}$ is the set of compact subsemigroups of S. We show that $\mathcal{A}$ has a minimal member using Zorn's Lemma. Since $S \in \mathcal{A}$, we have $\mathcal{A} \neq \emptyset$. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Then $\mathcal{C}$ is a collection of closed subsets of the compact space S with the finite intersection property, so $\bigcap \mathcal{C} \neq \emptyset$ and $\bigcap \mathcal{C}$ is trivially compact and a semigroup. Thus $\bigcap \mathcal{C} \in \mathcal{C}$, so we may pick a minimal member A of $\mathcal{A}$.
Pick $x \in A$. We shall show that $xx = x$. We start by showing that $Ax = A$. Let $B = Ax$. Since $A \neq \emptyset$, we have $B \neq \emptyset$ as well. Since $B = \rho_x[A]$, B is the continuous direct image of a compact space, hence B is compact. Also $BB = AxAx \subseteq AAAx \subseteq AAx \subseteq Ax = B$. Thus $B \in \mathcal{A}$. Since $B = Ax \subseteq AA \subseteq A$ and A is minimal, we have $B = A$.
Let $C = \{y \in A : yx = x\}$. Since $x \in A = Ax$, we have $C \neq \emptyset$. Also $C = A \cap \rho_x^{-1}[\{x\}]$. So, C is closed and hence compact. Given $y, z \in C$, one has $yz \in AA \subseteq A$ and $yzx = yx = x$. So $yz \in C$. Thus $C \in \mathcal{A}$. Since $C \subseteq A$ and A is minimal, we have $C = A$. So $x \in C$ and thus $xx = x$ as required. □

Definition 1.3.3 *Let S be a semigroup which is also a topological space. A **semigroup compactification** of S is a pair (φ, T) where T is a compact right topological semigroup, $\varphi : S \to T$ is a continuous homomorphism, $\varphi[S] \subseteq \Lambda(T)$, and $\varphi[S]$ is dense in T.*

Corollary 1.3.1 *Let S be a compact right topological semigroup. Then every left ideal of S contains a minimal left ideal. Minimal left ideals of S are closed, and each minimal left ideal has an idempotent.*

Proof. Let L be an ideal of S and $x \in L$. Then $Sx = \rho_x[S]$ is a continuous image of a compact space, hence compact. Also Sx is a left ideal of S and $Sx \subseteq L$. Consequently any minimal left ideal is closed. By Theorem 1.3.1, any minimal left ideal contains an idempotent. Thus one need only to show that any left ideal of S contains a minimal left ideal. So let L be a left ideal of S and let

$$\mathcal{A} = \{T : T \text{ is a closed left ideal of } S \text{ and } T \subseteq L\}.$$

Applying Zorn's Lemma to $\mathcal{A}$, one gets a left ideal M minimal among all closed left ideals contained in L. But since every left ideal contains a closed left ideal, M is a minimal left ideal. □

Theorem 1.3.2 *Let S be a compact right topological semigroup.*

1. *Every right ideal of S contains a minimal right ideal which has an idempotent.*
2. *Let $T \subseteq S$. Then T is a minimal left ideal of S if and only if there is some $e \in E(K(S))$ such that $T = Se$*
3. *Let $T \subseteq S$. Then T is a minimal right ideal of S if and only if there is some $e \in E(K(S))$ such that $T = eS$*
4. *Given any minimal left ideal L of S and any minimal right ideal R of S, there is an idempotent $e \in R \cap L$ such that $R \cap L = eSe$ and eSe is a group.*

Proof.

1. From Corollary 1.3.1, every minimal left ideal of S has an idempotent. Then, by Theorem 1.1.13(2), there is a minimal right ideal of S which has an idempotent. Considering a right version of Corollary 1.1.1, every right ideal of S contains a minimal right ideal. So by the right version of Theorem 1.1.13(1), every minimal right ideal has an idempotent.
2. Corollary 1.3.1 guarantees the existence of a minimal left ideal of S which has an idempotent. Then one may use the equivalence (1) and (6) of Theorem 1.1.12.
3. Same as above using the equivalence (4) and (6) of Theorem 1.1.12.

4. Corollary 1.3.1 guarantees the existence of a minimal left ideal of S which has an idempotent. Then use Theorem 1.1.14.

$\square$

Theorem 1.3.3 *Let S be a compact right topological semigroup.*

1. *Then S has the smallest two-sided ideal $K(S)$ which is the union of all minimal left ideals of S and also the union of all minimal right ideals of S.*

2. *Each of $\{Se : e \in E(K(S))\}$, $\{eS : e \in E(K(S))\}$, and $\{eSe : e \in E(K(S))\}$ are partitions of $K(S)$.*

Proof. By Theorem 1.3.2, all the minimal left ideals of S are of the form Se, all the minimal right ideals of S are of the form eS, and $eSe = H(e)$ are maximal groups of S with $e \in E(K(S))$.

1. By Theorem 1.1.10, $K(S)$ exists and is the union of all minimal left ideals of S. By taking the right version of Theorem 1.1.10, $K(S)$ is also the union of all minimal right ideals of S.

2. The partitions of $K(S)$ are obtained using Theorem 1.1.16(1),(2).

$\square$

Theorem 1.3.4 *Let S be a compact right topological semigroup.*

1. *All maximal subgroups of $K(S)$ are (algebraically) isomorphic.*

2. *Maximal subgroups of $K(S)$ which lie in the same minimal right ideal are topologically and algebraically isomorphic.*

3. *All minimal left ideals of S are homeomorphic. In fact, if L and L' are minimal left ideals of S and $z \in L'$, then $\rho_{z|L}$ is a homeomorphism from L onto L'.*

Proof.

1. Corollary 1.3.1 guarantees the existence of a minimal left ideal of S which has an idempotent. Using Theorem 1.1.18 and the last part of Theorem 1.3.3(2), one obtains isomorphisms between all the maximal subgroups of $K(S)$.

2. Let R be a minimal right ideal of S and $e, f \in E(R)$ guaranteed by Corollary 1.3.1 and Theorem 1.1.13(2). Then eS and fS are right ideals of S contained in R. Then by Lemma 1.1.4(2), $ef = f$ and $fe = e$ by taking respectively e a left identity of eS and f a left identity of fS. Let g be the inverse of efe in the group eSe and define a map $\varphi : eSe \to fSf$ by $\varphi(x) = fxgf$ as in Theorem 1.1.18. Then φ is an isomorphism from eSe onto fSf. Now we show that φ is continuous. Let $x \in eSe$. Then

$$\begin{aligned}
\varphi(x) &= fxgf \\
&= fexgf, \text{ since } x = ex \\
&= exgf, \text{ since } fe = e \\
&= xgf, \text{ since } ex = x \\
&= \rho_{gf}(x).
\end{aligned}$$

So φ is the restriction of ρ_{gf} to eSe and thus continuous. To show that φ^{-1} is continuous, consider h and k to be the inverses in fSf of fgf and fef respectively. Let $y \in fSf$. Then

$$\begin{aligned}
\varphi^{-1}(x) &= ekyhe \text{ see the last part of the proof of Theorem 1.1.18} \\
&= fefkyhe, \text{ since } fk = k \text{ and } fe = e \\
&= fyhe, \text{ since } fefk = f \\
&= yhe, \text{ since } fy = y \\
&= \rho_{he}(y).
\end{aligned}$$

So φ^{-1} is the restriction of ρ_{he} to fSf and hence is continuous. Thus eSe and fSf are topologically and algebraically homeomorphic.

3. Let L and L' be minimal left ideals of S and let $z \in L'$. By Theorem 1.3.2(2), pick $e \in E(K(S))$ such that $L = Se$. Then $\rho_{z|L}$ is a continuous function from Se to $Sz = L'$ and $\rho_z[Se] = L'$ since Sez is a left ideal of S which is contained in L'. To see that ρ_z is one-to-one on Se, let g be the inverse of eze in eSe. For $x \in Se$,

$$\begin{aligned}
\rho_g(\rho_z(x)) &= \rho_g(xz) \\
&= xzg \\
&= xezeg, \text{ since } x = xe \text{ and } g = eg \\
&= xe \\
&= x.
\end{aligned}$$

Since $\rho_{z|L}$ is one-to-one and continuous and L is compact (by the fact that L is closed as a minimal left ideal of S and S is compact), $\rho_{z|L}$ is a homeomorphism.

□

Theorem 1.3.5 *Let S be a compact right topological semigroup and let R be a right ideal of S. Then $c\ell(R)$ is a right ideal of S.*

Proof. Let $x \in S$. $(c\ell(R))x = \rho_x[c\ell(R)] \subseteq c\ell(\rho_x[R]) = c\ell(Rx) \subseteq c\ell(R)$, since R is a right ideal of S. □

Theorem 1.3.6 *Let S be a compact right topological semigroup, and let T be a subset of the topological center of S. Then $c\ell(T)$ is a semigroup if T is a semigroup.*

Proof. One can see that $c\ell(T) \neq \emptyset$ since $\emptyset \neq T \subseteq c\ell(T)$. Now let $x, y \in c\ell(T)$, we need to show that $xy \in c\ell(T)$. For this, consider an open neighborhood U of xy. Take a neighborhood V of x such that $Vy = \rho_y[V] \subseteq U$ (since $c\ell(T) \subseteq S$ and ρ_y is continuous). Pick $a \in V \cap T \subseteq V \cap \Lambda(S)$. Then $ay = \lambda_a(y) \in U$ (since λ_a is continuous for any $a \in \Lambda(S)$) so pick a neighborhood W of y such that $aW \subseteq U$. Pick $b \in W \cap T$. Then $ab \in aW \cap T \subseteq U \cap T$ (since $aW \subseteq U$ and T is a semigroup). Thus $xy \in c\ell(T)$. □

Consider a right topological semigroup S and a right ideal R of S. We have that $c\ell(R)$ is also a right ideal of S, but this is not always true in the case of a left ideal of S [19]. One needs an additional hypothesis.

Theorem 1.3.7 *Let S be a compact right topological semigroup and assume that $\Lambda(S)$ is dense in S. Let L be a left ideal of S. Then $c\ell(L)$ is a left ideal of S.*

Proof. Let $x \in c\ell(L)$ and let $y \in S$. To see that $yx \in c\ell(L)$, let U be an open neighborhood of yx. Pick a neighborhood V of y such that $Vx = \rho_x[V] \subseteq U$ and pick $z \in \Lambda(S)$ (since $\Lambda(S)$ is dense in S). Then $zx = \lambda_z(x) \in U$ so pick a neighborhood W of x such that $zW \subseteq U$. Pick $w \in W \cap L$ (since $x \in c\ell(L)$). Then, since L is a left ideal of S, $zw \in zW \cap L \subseteq U \cap L$ so $zw \in U \cap L$. Indeed $yx \in c\ell(L)$. □

Theorem 1.3.8 *Let S be a compact right topological semigroup with dense center. We have the following statements:*

1. *If R is a right ideal of S, then $c\ell(R)$ is a two-sided ideal of S.*
2. *If $e \in E(K(S))$, then $c\ell(eSe) = Se$.*

Proof. Let A be the center of S.

1. By Theorem 1.3.5, $c\ell(R)$ is a right ideal of S. Now we prove that $c\ell(R)$ is also a left ideal of S. Let $y \in c\ell(R)$. For any $x \in A$ one has $\rho_y(x) = xy = yx \in (c\ell(R))x \subseteq c\ell(R)$ (since A is the center of S and $c\ell(R)$ is a right ideal of S). Thus $\rho_y[A] \subseteq c\ell(R)$. Since A is dense in S, so $\rho_y[S] = \rho_y[c\ell(A)] \subseteq c\ell(\rho_y[A]) \subseteq c\ell(R)$. That is $Sy \subseteq c\ell(R)$. Hence $S \cdot (c\ell(R)) \subseteq c\ell(R)$.
2. Since $Se = \rho_e[S]$ is closed (a continuous image of a compact space) and is a (minimal) left ideal of S, one has $c\ell(eSe) \subseteq Se$. On the other hand, since ρ_e is continuous and A a dense center of S, $Se = (c\ell(A))e = \rho_e[c\ell(A)] \subseteq c\ell(\rho_e[A]) = c\ell(Ae) = c\ell(eAe) \subseteq c\ell(eSe)$ ($Ae = eA$ implies $eAe = eeA = eA$ since e is an idempotent). Hence $c\ell(eSe) \subseteq Se$ and $Se \subseteq c\ell(eSe)$ implies $c\ell(eSe) = Se$.

□

Let S be a semitopological semigroup and let T be a subsemigroup of S. Then $c\ell(T)$ is a subsemigroup of S. But this is not true for right topological semigroups in general. For instance, assuming $X = \mathbb{N} \cup \{\infty\}$ and

$$T = \{f \in X^X \,:\, f \text{ is one-to-one }\}$$

then $c\ell(T)$ is not a semigroup (see [19], Theorem 2.28). However, if each f in T is continuous, then $c\ell(T)$ is a semigroup.

Theorem 1.3.9 *Let S be a compact right topological semigroup with dense center. Assume that S has some minimal right ideal R which is closed. Then $R = K(S)$ and all maximal subgroups of S are closed and pairwise algebraically and topologically isomorphic.*

Proof. By Theorem 1.3.8(1), $c\ell(R)$ is an ideal of S so $K(S) \subseteq c\ell(R) = R \subseteq K(S)$ (since R is closed and minimal). Thus $R = K(S)$. Let $e \in$

$E(K(S))$. By Theorem 1.1.12(3) and Theorem 1.3.2(4), $H(e) = eSe = R \cap Se$ so $H(e)$ is closed (being the intersection of two closed sets R, by assumption, and Se, see Corollary 1.3.1(2)). Any two maximal subgroups of $K(S)$ lie in the same minimal right ideal and so are algebraically and topologically isomorphic by Theorem 1.3.4(2). □

Theorem 1.3.10 *Let S be a compact right topological semigroup with dense topological center. The following statements are equivalent:*

1. *$K(S)$ is a minimal right ideal of S.*
2. *All maximal subgroups of $K(S)$ are closed.*
3. *Some maximal subgroups of $K(S)$ are closed.*

Proof.

(1) $\Rightarrow$ (2). Let $e \in E(K(S))$. Then by Theorem 1.1.12(1),(4) Se is a minimal left ideal and eS is a minimal right ideal so by Theorem 1.3.2(4), $eSe = eS \cap Se$. Since $K(S)$ is a minimal right ideal, $eS = K(S)$. Since $Se \subseteq K(S)$, $eSe = eS \cap Se = K(S) \cap Se = Se$, and Se is closed by Corollary 1.3.1(2). Thus $K(S)$ is closed.

It is trivial that (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Pick $e \in E(K(S))$ such that eSe is closed. Let $R = eS$. By Theorem 1.3.8(2), $Se = c\ell(eSe) = eSe = eS \cap Se \subseteq eS = R$. Now any other minimal right ideal of S would be disjoint from R so would miss Se, contradicting Lemma 1.1.3(2). Thus R is the only minimal right ideal of $K(S)$, which is the union of all minimal right ideals by Theorem 1.1.10, so $K(S) = R$.

□

Theorem 1.3.11 *Let A be a set and let G be the free group generated by A. Then G can be embedded in a compact topological semigroup. This means that there is a compact group H and a one-to-one homomorphism $\varphi : G \to H$.*

Proof. Let $G' = G \setminus \{\emptyset\}$, where $\emptyset$ is the identity of G. For each $g \in G'$, using Theorem 1.1.2, pick a finite group F_g and a homomorphism $\phi_g : G \to F$ such that $\phi_g(g)$ is not the identity of F_g. Let each F_g have the discrete topology. Then $H = \bigotimes_{g \in G'} F_g$ is a compact topological group. Define a homomorphism $\varphi : G \to H$ by stating that $\varphi(h)_g = \phi_g(h)$. Then, if $g \in G'$, we know that $\varphi(g)_g$ is not the identity of F_g. So the kernel of φ is $\emptyset$ and hence φ is one-to-one as required. $\square$

Chapter 2

Ultrafilters and Stone-Cěch compactifications

In this chapter, we define the notion of an ultrafilter on a set. The notion of an ultrafilter was first introduced by F. Riesz in 1909 to capture the sense of largeness. We introduce the notion of filters since these are useful in their own right, for instance in the characterization of Stone-Cěch compactifications. We outline some properties and some examples of ultrafilters. We also give some insights of the Stone-Cěch compactifications of $\mathbb{N}$ and $\mathbb{R}$. The main references of the notions reviewed in this section are [19], [37], [16], [27], [9], and [31].

2.1 Ultrafilters

Definition 2.1.1 *Let X be any nonempty set. A **filter** on X is a family $\mathcal{F}$ of subsets of X satisfying the following properties:*

1. *$\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$;*
2. *If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; and*
3. *If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.*

In brief, we would say that a filter on X is a nonempty family of nonempty subsets of X closed under finite intersections and supersets.

Let us recall that every filter has the finite intersection property.

Example 2.1.1 *Here are some examples of filters.*

1. *The collection* $\mathcal{N}_x$ *of all neighborhoods of a point* x *in a topological space* X *is a classic example of a filter called the* ***neighborhood filter*** *of* x.

2. *Let* X *be an arbitrary set, and let* S *be a nonempty subset of* X. *Then the collection of sets*

$$\mathcal{F}(S) = \{A \subseteq X \, : \, S \subseteq A\}$$

   is a filter on X.

3. *Let* X *be an infinite set and consider the collection* $\mathcal{F}$ *of cofinite sets. That is,*

$$\mathcal{F} = \{A \subseteq X \, : \, X \setminus A \text{ is a finite set}\}.$$

   Then $\mathcal{F}$ *is a filter on* X.

Definition 2.1.2 *Let* X *be a nonempty set. A nonempty family* $\mathcal{B} \subseteq \mathcal{P}(X)$ *is called a* ***filter base*** *if it satisfies the following properties:*

1. $\emptyset \notin \mathcal{B}$, *and*

2. *for every* $A, B \in \mathcal{B}$ *there is* $C \in \mathcal{B}$ *such that* $C \subseteq A \cap B$.

Similarly, a nonempty family $\mathcal{B} \subseteq \mathcal{P}(X)$ *is a* ***filter base*** *if the set*

$$\mathcal{F}_{\mathcal{B}} = \{A \subseteq X \, : \, A \supseteq B \text{ for some } B \in \mathcal{B}\}$$

is a filter, and in this case $\mathcal{B}$ *is called a* ***base*** *for* $\mathcal{F}_{\mathcal{B}}$. *We also have that if* $\mathcal{F}_{\mathcal{B}}$ *is a filter, then* $\mathcal{B} \subseteq \mathcal{F}_{\mathcal{B}}$ *is a base for* $\mathcal{F}_{\mathcal{B}}$ *if and only if for each* $A \in \mathcal{F}_{\mathcal{B}}$ *there is* $B \in \mathcal{B}$ *such that* $B \subseteq A$. *Then the filter* $\mathcal{F}_{\mathcal{B}}$ *is called the* ***filter generated by*** $\mathcal{B}$.

Notice that each filter is a filter base. The open neighborhoods of a point x of a topological space form a filter base $\mathcal{B}$ satisfying $\mathcal{F}_{\mathcal{B}} = \mathcal{N}_x$.

Definition 2.1.3 *Let* X *be a nonempty set.*

1. *The filters* $\mathcal{F}$ *and* $\mathcal{G}$ *on* X *are said to be* ***incompatible*** *if there are* $A \in \mathcal{F}$ *and* $B \in \mathcal{G}$ *such that* $A \cap B = \emptyset$.

2. *A filter $\mathcal{F}$ on X is said to be **open** if $\mathcal{F}$ has a base of open sets of the topology of X.*

Definition 2.1.4 *An **ultrafilter** is a filter which is not properly contained in any other filter.*

That is, an ultrafilter is simply a maximal filter or, $\mathcal{U}$ is an ultrafilter if $\mathcal{U} \subseteq \mathcal{G}$ for a filter $\mathcal{G}$ implies $\mathcal{U} = \mathcal{G}$.

Definition 2.1.5 *A **principal ultrafilter** corresponding to x on a set X is an ultrafilter of the form*

$$\mathcal{U}_x = \{A \subseteq X \,:\, x \in A\}$$

*for a fixed $x \in X$. Ultrafilters which are not principal are called **non-principal ultrafilters**.*

For example, the collection $\mathcal{N}_x$ all of neighborhoods of a point x in a topological space is a principal ultrafilter corresponding to x.

Theorem 2.1.1 *Let X be a set and let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on X.*

1. *If B is such that $A \cap B \neq \emptyset$ for all $A \in \mathcal{U}$ then $B \in \mathcal{U}$.*
2. *If A and B are such that $A \cup B \in \mathcal{U}$ then at least one of A or B belongs to $\mathcal{U}$.*
3. *If $\mathcal{U} \neq \mathcal{V}$ then there are $A \in \mathcal{U}$, $B \in \mathcal{V}$ with $A \cap B = \emptyset$.*

Proof.

To prove (1), notice that the family $\mathcal{A} = \{A \cap B \,:\, A \in \mathcal{U}\}$ is nonempty and has the finite intersection property, thus it extends to an ultrafilter $\mathcal{F}$. We have $B \in \mathcal{F}$ since $B \in \mathcal{A}$. Let $A \in \mathcal{U}$. We have $A \cap B \in \mathcal{A}$, so $A \cap B \in \mathcal{F}$. Since $A \cap B \subseteq A$ and $\mathcal{F}$ is an ultrafilter, $A \in \mathcal{F}$. Thus $\mathcal{U} \subseteq \mathcal{F}$ and so $\mathcal{U} = \mathcal{F}$ and indeed $B \in \mathcal{U}$.

To prove (2), suppose both $A, B \notin \mathcal{U}$. Then by (1), pick $C, D \in \mathcal{U}$ such that $A \cap C = \emptyset$ and $B \cap D = \emptyset$, so $(A \cup B) \cap (C \cap D) = \emptyset$. Since $C \cap D \in \mathcal{U}$, we have $A \cup B \in \mathcal{U}$.

To prove (3), pick $B \in \mathcal{V}$ such that $B \notin \mathcal{U}$. By (1), there exist some $A \in \mathcal{U}$ such that $A \cap B = \emptyset$.

$\square$

Corollary 2.1.1 *If $\mathcal{U}$ is an ultrafilter and $\bigcup_{i=1}^{n} A_i \in \mathcal{U}$, where the A_i are disjoint, then $A_i \in \mathcal{U}$ for exactly one i.*

Proof. It follows from the general case of Theorem 2.1.1(2) by induction: if $\bigcup_{i=1}^{n} A_i \in \mathcal{U}$ then $A_i \in \mathcal{U}$ for some i. If the A_i are disjoint, then $A_i \in \mathcal{U}$ for exactly one i. $\square$

We give in the following some characterizations of ultrafilters.

Theorem 2.1.2 *Let X be a nonempty set and let $\mathcal{U} \subseteq \mathcal{P}(X)$. Then the following statements are equivalent:*

1. *$\mathcal{U}$ is an ultrafilter on X,*
2. *$\mathcal{U}$ is a maximal family with the finite intersection property, and*
3. *$\mathcal{U}$ is a filter on X and for every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Proof.

(1) $\Rightarrow$ (3): If $\mathcal{U}$ is an ultrafilter on X then it is a filter on X and so satisfies the first part of (3). The second part is a special case of the previous corollary.

(3) $\Rightarrow$ (2): Let $A \subseteq X$ and $A \notin \mathcal{U}$. Then $X \setminus A \in \mathcal{U}$. Since $A \cap (X \setminus A) = \emptyset$, $\mathcal{U} \cup \{A\}$ has no finite intersection property.

(2) $\Rightarrow$ (1): Since $\mathcal{U}$ has the finite intersection property, the family

$$\mathcal{A} = \{A \subseteq X \,:\, A \supseteq \bigcap \mathcal{F} \text{ for some finite } \mathcal{F} \subseteq X\}$$

is a filter on X and has the finite intersection property as well. Since $\mathcal{U}$ is maximal, $\mathcal{A} = \mathcal{U}$. Then $\mathcal{U}$ is a filter, and consequently, an ultrafilter.

$\square$

It follows that a filter $\mathcal{F}$ on X is an ultrafilter if and only if exactly one of A or $X \setminus A$ belongs to $\mathcal{F}$ for all subsets A of X. This characterization captures the notion of an ultrafilter as the set of "large" subsets of a set. Meaning that one can bring in the notion of a finitely additive measure on X.

Definition 2.1.6 *A* $\{0,1\}$***-valued finite additive measure*** *on* X *is a function* $\mu : \mathcal{P}(X) \to \{0,1\}$ *such that*

1. $\mu(X) = 1$
2. *If* $A_1, \ldots, A_n$ *are pairwise disjoint subsets of* X*, then*
$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

Lemma 2.1.1 *Let* $\mathcal{F}$ *be an ultrafilter on* X*. The function* $\mu_{\mathcal{F}}$ *on* $\mathcal{P}(X)$ *defined by*
$$\mu_{\mathcal{F}}(A) = \begin{cases} 1 & \text{if } A \in \mathcal{F} \\ 0 & \text{if } A \notin \mathcal{F} \end{cases}$$
determines a $\{0,1\}$*-valued finite additive measure on* $\mathcal{P}(X)$*. Conversely, any such measure determines an ultrafilter.*

Proof. First, we show that $\mu_{\mathcal{F}}$ is a finite additive measure. In fact, for the first condition, $\mu_{\mathcal{F}}(X) = 1$ since $X \in \mathcal{F}$. The second one follows by induction if we prove $\mu_{\mathcal{F}}(A \cup B) = \mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B)$ for two disjoint subsets of X. If $A \in \mathcal{F}$ but $B \notin \mathcal{F}$, or vice versa, then $\mu_{\mathcal{F}}(A) = 1$ and $\mu_{\mathcal{F}}(B) = 0$. Thus,
$$\mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B) = 1 = \mu_{\mathcal{F}}(A \cup B),$$
since $A \cup B \in \mathcal{F}$ (the superset property of $\mathcal{F}$). We cannot have both $A, B \in \mathcal{F}$ since A and B are disjoint. Now consider the case $A, B \notin \mathcal{F}$. Thus $\mu_{\mathcal{F}}(A) = \mu_{\mathcal{F}}(B) = 0$ and $A \cup B \notin \mathcal{F}$ since $(A \cup B)' = A' \cap B' \in \mathcal{F}$. So,
$$\mu_{\mathcal{F}}(A \cup B) = 0 = \mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B).$$
To see the converse, define
$$\mathcal{F}_\mu = \{A \subseteq X \,:\, \mu(A) = 1\},$$
where μ is a $\{0,1\}$-valued finite additive measure on $\mathcal{P}(X)$. The set $\mathcal{F}_\mu$ as defined is indeed an ultrafilter on X. $\square$

Lemma 2.1.2 *Let X be a nonempty set and let $\mathcal{U}$ be an ultrafilter on X. Then the following statements are equivalent:*

1. *$\mathcal{U}$ is a nonprincipal ultrafilter on X,*
2. *$\bigcap \mathcal{U} = \emptyset$,*
3. *for every $A \in \mathcal{U}$, $|A| \geq \omega$',*

Proof.

(1) $\Rightarrow$ (2): Suppose $\bigcap \mathcal{U} \neq \emptyset$. Pick $x \in \bigcap \mathcal{U}$, so

$$\mathcal{U} \subseteq \{A \subseteq X \,:\, x \in A\},$$

and thus $\mathcal{U} = \{A \subseteq X \,:\, x \in A\}$ since $\mathcal{U}$ is a maximal filter. Hence $\mathcal{U}$
is a principal ultrafilter on X.

(2) $\Rightarrow$ (3): Suppose that some $A \in \mathcal{U}$ is finite. Then, by Theorem 2.1.2, pick $x \in A$ such that $\{x\} \in \mathcal{U}$. It follows that $\mathcal{U} = \{A \subseteq X \,:\, x \in A\}$.

(3) $\Rightarrow$ (1): See the proof of (1) $\Rightarrow$ (2) above.

□

One concludes from the previous statements that for a nonprincipal filter $\mathcal{F}$ on a set X, we have $\mu_{\mathcal{F}}(A) = 0$ for any finite set A.

The following theorem for the existence of nonprincipal ultrafilters is called the **ultrafilter theorem** and it involves the **Zorn's Lemma**. We hereby recall the lemma: if every chain in a partially ordered set X has an upper bound, then X has a maximal element.

Theorem 2.1.3 *Every filter on X can be extended to an ultrafilter on X.*

Proof. Let $\mathcal{F}$ be a filter on a set X, and let $\mathcal{C}$ be the nonempty collection of all subfilters of $\mathcal{F}$. That is,

$$\mathcal{C} = \{\mathcal{G} \,:\, \mathcal{G} \text{ is a filter and } \mathcal{F} \subset \mathcal{G}\}.$$

The collection $\mathcal{C}$ is partially ordered by inclusion. Given a chain $\mathcal{B}$ in $\mathcal{C}$, the family $\{A : A \in \mathcal{G} \text{ for some } \mathcal{G} \in \mathcal{B}\}$ is a filter that is an upper bound for $\mathcal{B}$ in $\mathcal{C}$. Thus the hypotheses of Zorn's Lemma are satisfied, so $\mathcal{C}$ has a maximal element. Note that every maximal element of $\mathcal{C}$ is an ultrafilter including $\mathcal{F}$. $\square$

Definition 2.1.7 *Let X be a set and let $\mathcal{U}$ be an ultrafilter on X. The norm on $\mathcal{U}$ is defined as follows*

$$\|\mathcal{U}\| = \min\{|A| : A \in \mathcal{U}\}.$$

Note that from the fact above, if $\mathcal{U}$ is an ultrafilter, then $\|\mathcal{U}\|$ is either 1 or infinite.

Definition 2.1.8 *Let X be a set and let κ be an infinite cardinal.*

1. *A κ-**uniform ultrafilter** on X is an ultrafilter $\mathcal{U}$ on X such that $\|\mathcal{U}\| \geq \kappa$.*
2. *$U_\kappa(X) := \{\ \mathcal{U} : \mathcal{U}$ is a κ-uniform ultrafilter on $X\ \}$.*
3. *A **uniform ultrafilter** on X is a κ-uniform ultrafilter on X, where $\kappa = |X|$.*

Equivalently, an ultrafilter $\mathcal{U}$ is uniform if for every $A \in \mathcal{U}$, we have $|A| = |X|$ and the set of uniform ultrafilters on X will simply be denoted by $U(X)$ (without the script κ), whenever there is no risk of confusion.

Corollary 2.1.2 *There are uniform ultrafilters on any infinite set. Let X be any set and let $\mathcal{A}$ be a family of subsets of X. We have the following statements:*

1. *If the intersection of every finite subfamily of $\mathcal{A}$ is infinite, then $\mathcal{A}$ is contained in an ultrafilter on X all of whose members are infinite.*
2. *More generally, if κ is an infinite cardinal and if the intersection of every finite subfamily of $\mathcal{A}$ has cardinality at least κ, then $\mathcal{A}$ is contained in a κ-uniform ultrafilter on X.*

Proof. See Proof of Corollary 3.14 in [19]. $\square$

Recall that if the intersection of every finite subfamily of $\mathcal{A}$ is infinite, $\mathcal{A}$ is said to have the **infinite finite intersection property**. Thus, the above result suggests that if $\mathcal{A}$ has the infinite finite intersection property, then there is a nonprincipal ultrafilter $\mathcal{U}$ on X with $\mathcal{A} \subseteq \mathcal{U}$.

But most of the time we will use the following simplified version from [37] if there is no confusion at all.

Definition 2.1.9 *An ultrafilter $\mathcal{U}$ on X is said to be **uniform** if for every $A \in \mathcal{U}$, $|A| = |X|$.*

Corollary 2.1.3 *There are uniform ultrafilters on any infinite set.*

Proof. Let X be any infinite set and let

$$\mathcal{F} = \{A \subseteq X : |X \setminus A| < |X|\}.$$

By Theorem 2.1.2, there is an ultrafilter $\mathcal{U}$ on X containing $\mathcal{F}$. Suppose $|A| < |X|$ for $A \subseteq X$, then $X \setminus A \in \mathcal{F} \subseteq \mathcal{U}$, so $A \notin \mathcal{U}$. Hence we take $|A| = |X|$, then $A \in \mathcal{U}$. That is, $\mathcal{U}$ is uniform. $\square$

Lemma 2.1.3 *Let $\mathcal{F}$ be a filter on X. Let $T \subseteq X$ such that $T \cap A \neq \emptyset$ for each $A \in \mathcal{F}$.*

1. *Then the set $\mathcal{F}_{|_T} = \{T \cap A : A \in \mathcal{F}\}$ is a filter on T.*
2. *If $\mathcal{F}$ is an ultrafilter, $\mathcal{F}_{|_T}$ is also an ultrafilter.*

Proof.

Let prove (1). Clearly, $\emptyset \notin \mathcal{F}_{|_T}$ since $T \cap A \neq \emptyset$ for each $A \in \mathcal{F}$. And indeed, with the assumptions on T, $T = T \cap X \in \mathcal{F}_{|_T}$. Let $A, B \in \mathcal{F}_{|_T}$. Then we can find $A_1, B_1 \in \mathcal{F}$ such that $A = T \cap A_1$ and $B = T \cap B_1$. $A \cap B = (T \cap A_1) \cap (T \cap B_1) = T \cap (A_1 \cap B_1) \in \mathcal{F}_{|_T}$ since $A_1 \cap B_1 \in \mathcal{F}$. Let $A \subseteq B$ and $A \in \mathcal{F}_{|_T}$. Pick $A_1 \in \mathcal{F}$ such that $A = T \cap A_1$. Since $A \subseteq B$ and $B \subseteq T$, there is $B_1 \subseteq T$ such that $B = T \cap B_1$ (otherwise $B = \emptyset$ or $B = T$). Obviously $A_1 \subseteq B_1$ since $A \subseteq B$, so $B_1 \in \mathcal{F}$. Hence $B \in \mathcal{F}_{|_T}$.

The proof of (2) follows using Theorem 2.1.2.

$\square$

Definition 2.1.10 *Let $\mathcal{F}$ be a filter on X and $T \subseteq X$. The filter $\mathcal{F}_{|T}$ is called the **trace** of $\mathcal{F}$ on T.*

Lemma 2.1.4 *Let $\mathcal{F}$ be a filter on X and let a function $f : X \to Y$.*

1. *Then the set $f(\mathcal{F}) = \{f(A) : A \in \mathcal{F}\}$ is a filter base on Y.*
2. *If f is surjective, then $f(\mathcal{F})$ is a filter.*

Proof.

To prove (1), let's check the conditions of Definition 2.1.3. Since $\emptyset \notin \mathcal{F}$, $\emptyset \notin f(\mathcal{F})$. Let $A, B \in f(\mathcal{F})$. Pick $A_1, B_1 \in \mathcal{F}$ such that $A = f(A_1)$ and $B = f(B_1)$. Since $\mathcal{F}$ is a filter base on X, pick $C_1 \in \mathcal{F}$ such that $C_1 \subseteq A_1 \cap B_1$. So $f(C_1) \in f(\mathcal{F})$ and $f(C_1) \subseteq A \cap B$.

To prove (2), let $A \in f(\mathcal{F})$ and $A \subseteq B$. Pick $A_1 \in \mathcal{F}$ such that $A = f(A_1)$. Since f is surjective and $f(A_1) \subseteq B$, we can find $B_1 \in \mathcal{F}$ with $f(B_1) = B$. Thus, $B \in f(\mathcal{F})$.

$\square$

Definition 2.1.11 *Let $\mathcal{F}$ be a filter on X and let a function $f : X \to Y$. The filter base on Y is called the **image** of $\mathcal{F}$ with respect to f.*

Lemma 2.1.5 *If $\mathcal{F}$ is an ultrafilter, then $f(\mathcal{F})$ is an ultrafilter base.*

Proof. Let $B \subseteq Y$ and let $A = f^{-1}(B)$. Since $\mathcal{F}$ is an ultrafilter, by Theorem 2.1.2, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. Then either $B \supseteq f(A) \in f(\mathcal{F})$ or $Y \setminus B \supseteq f(X \setminus A) \in f(\mathcal{F})$. $\square$

Definition 2.1.12 *A filter base $\mathcal{B}$ on a space X **converges** to a point $x \in X$ if every neighborhood U of x, there is $A \in \mathcal{B}$ such that $A \subseteq U$. In another words, A filter base $\mathcal{B}$ on a space X **converges** to a point $x \in X$ if the filter generated by $\mathcal{B}$ converges to x.*

Notice that if $\mathcal{B}$ is a filter, then $\mathcal{B}$ converges to x if and only if $\mathcal{B}$ includes the neighborhood filter of x, that is, $\mathcal{N}_x \subseteq \mathcal{B}$. Also, $\mathcal{N}_x$ converges to x for each $x \in X$.

Theorem 2.1.4 *Let $f : X \to Y$ be a function between two topological spaces and let $x \in X$. The following statements are equivalent:*

1. *The function f is continuous at x.*
2. *If a filter $\mathcal{F}$ on X converges to x, then $f(\mathcal{F})$ converges to $f(x)$ in Y.*

Proof.

(1) $\Rightarrow$ (2). Let $\mathcal{F}$ be a filter on X convergent to $x \in X$. That is, $\mathcal{N}_x \subseteq \mathcal{F}$. Since f is continuous at x, $f^{-1}(V) \in \mathcal{N}_x$ for each $V \in \mathcal{N}_{f(x)}$. Hence $f^{-1}(V) \in \mathcal{F}$ for each $V \in \mathcal{N}_{f(x)}$. But then from $f(f^{-1}(V)) \subseteq V$, we have that $\mathcal{N}_{f(x)}$ is included in the filter generated by $f(\mathcal{F})$. Thus $f(\mathcal{F})$ converges to $f(x)$.

(2) $\Rightarrow$ (1). Assume (2) and on the contrary that f is not continuous at x. Then there is an open neighborhood V of $f(x)$ such that $f^{-1}(V)$ is not a neighborhood of x. That is, $x \notin int(f^{-1}(V))$ which implies $x \in c\ell([f^{-1}(V)]') = c\ell([f^{-1}(V')])$. So we can find a sequence $\{x_\alpha\}$ in $f^{-1}(V')$ that converges to x. If we define $F_\alpha = \{x_\beta : \beta \geq \alpha\}$, then $\mathcal{B} = \{F_\alpha : \alpha \in \mathbb{N}\}$ converges to x, so by hypothesis, $f(\mathcal{B})$ converges to $f(x)$. Since $f(F_\alpha) \subseteq V'$, we can find $\alpha_0 \geq \alpha$ such that $f(x_{\alpha_0}) \in V'$. But since V' is closed, $f(x) \in V'$ which is a contradiction.

$\square$

Theorem 2.1.5 *Let X be a topological space. The following statements are equivalent:*

1. *X is compact.*
2. *Every family of closed subsets of X with the finite intersection property has a nonempty intersection.*
3. *Every ultrafilter on X is convergent.*

Proof.

(1) $\Leftrightarrow$ (2). Suppose X compact, and let $\mathcal{E}$ be the family of closed subsets of X. If $\bigcap_{E\in\mathcal{E}} E = \emptyset$, then $X = \bigcup_{E\in\mathcal{E}} E'$ (with $E' = X \setminus E$). Therefore the family $\mathcal{C} = \{E' : E \in \mathcal{E}\}$ forms an open cover of X. Since X is compact, we can extract $E'_1, \ldots, E'_n$ from $\mathcal{C}$ with $E_1, \ldots, E_n \in \mathcal{E}$ such that $X = \bigcup_{i=1}^n E'_i$. This implies $\bigcap_{i=1}^n E_i = \emptyset$, so $\mathcal{E}$ does not have the finite intersection property. Thus, if $\mathcal{E}$ possesses the finite intersection property, then $\bigcap_{E\in\mathcal{E}} E \neq \emptyset$.
Conversely, suppose (2) is true and that $\mathcal{V}$ is an open cover of X. Then $\cap_{V\in\mathcal{V}} V' = \emptyset$, so the finite intersection property must be violated. That is, there exist $V_1, \ldots, V_n \in \mathcal{V}$ satisfying $\bigcap_{j=1}^n V'_j$. Thus, $X = \bigcup_{j=1}^n V_j$ proving that X is compact.

(1) $\Leftrightarrow$ (3). Let X be a compact space and let $\mathcal{U}$ be an ultrafilter on X. Assume on the contrary that for every point $x \in X$, there is a neighborhood U_x of x such that $U_x \notin \mathcal{U}$. One can take U_x open. Then, since $\mathcal{U}$ is an ultrafilter on X, $X \setminus U_x \in \mathcal{U}$ by Theorem 2.1.2. The open sets U_x, where $x \in X$, cover X. Since X is compact, there is a finite subcover $\{U_{x_i} : i < n\}$ of $\{U_x : x \in X\}$. But then $\emptyset = \bigcap_{i<n} (X \setminus U_{x_i}) \in \mathcal{U}$, a contradiction since $\mathcal{U}$ is an ultrafilter. Thus, if X is compact then every neighborhood U_x of $x \in X$ belongs to $\mathcal{U}$. That is, $\mathcal{U}$ converges.
Conversely, suppose that every ultrafilter on X is convergent and let $\mathcal{E}$ be a family of closed subsets of X with the finite intersection property. Assume on the contrary that $\bigcap \mathcal{E} = \emptyset$. By Theorem 2.1.3, there is an ultrafilter $\mathcal{U}$ such that $\mathcal{E} \subseteq \mathcal{U}$. We claim that $\mathcal{U}$ is not convergent. Indeed, let $x \in X$. Since $\bigcap \mathcal{E} = \emptyset$, there is $E_x \in \mathcal{E}$ such that $x \notin E_x$. Then $U_x = X \setminus E_x$ is a neighborhood of x and $U_x \notin \mathcal{U}$. Hence $\mathcal{U}$ is not convergent, which is a contradiction.

$\square$

2.2 The space βX

Definition 2.2.1 *Let X be a nonempty set and let βX denote the set of all ultrafilters on X. Given $A \subseteq X$, we define $\overline{A} \subseteq \beta X$ by*

$$\overline{A} = \{p \in \beta X : A \in p\}.$$

So, for every $A \subseteq X$ and $p \in \beta X$, $p \in \overline{A}$ if and only if $A \in p$.

Lemma 2.2.1 *Let X be a set and let $A, B \subseteq X$.*

1. $\overline{A \cap B} = \overline{A} \cap \overline{B}$;
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
3. $\overline{X \setminus A} = \beta X \setminus \overline{A}$;
4. $\overline{A} = \emptyset$ *if and only if* $A = \emptyset$;
5. $\overline{A} = \beta X$ *if and only if* $A = X$;
6. $\overline{A} = \overline{B}$ *if and only if* $A = B$.

Proof. See pages 24-25 in [16]. □

Theorem 2.2.1 *Let X be a nonempty set. Then the family*

$$\{\overline{A} : A \subseteq X\}$$

forms a base for the topology on βX.

Proof. Let $A, B \subseteq X$. $\overline{A} \cap \overline{B} = \overline{A \cap B}$. Since $A \cap B \subseteq X$, the family $\{\overline{A} : A \subseteq X\}$ is closed under finite intersections. Hence it forms a base for a topology on βX. □

For every pair of distinct ultrafilters $p, q \in \beta X$, there exist $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$, and so $\emptyset = \overline{A \cap B} = \overline{A} \cap \overline{B}$ making the above-mentioned topology Hausdorff.

Definition 2.2.2 *Let X be a set and let $x \in X$. Then we define the following:*

1. $e(x) = \{A \subseteq X : x \in A\}$.
2. $A^* = \overline{A} \setminus e[A]$.

For each $x \in X$, we note that $e(x)$ is the principal ultrafilter on X corresponding to x.

Theorem 2.2.2 *Let X be any nonempty set.*

1. *βX is a compact Hausdorff space.*

2. *The sets of the form $\overline{A}$ are the clopen subsets of βX.*

3. *For every $A \subseteq X$, $c\ell_{\beta X}(A) = c\ell_{\beta X}(e[A])$.*

4. *For any $A \subseteq X$ and any $p \in \beta X$, $p \in c\ell_{\beta X}(e[A])$ if and only if $A \in p$.*

5. *The mapping e is injective and $e[X]$ is a dense subset of βX whose points are precisely the isolated points of βX.*

6. *If U is an open subset of βX, $c\ell_{\beta X}(U)$ is also open.*

Proof.

1. Consider two distinct elements of βX. If $A \in p \setminus q$, then $X \setminus A \in q$. So $\overline{A}$ and $\overline{X \setminus A}$ are disjoint open subsets of βX containing p and q respectively. Thus βX is Hausdorff (see also the comment after Theorem 2.2.1).
   The sets of the form $\overline{A}$ are also a base for the closed sets, since $\beta X \setminus \overline{A} = \overline{X \setminus A}$. Thus, to show that βX is compact, consider a family $\mathcal{A}$ of sets of the form $\overline{A}$ with the finite intersection property and we show that $\mathcal{A}$ has nonempty intersection. Let $\mathcal{B} = \{A \subseteq X : \overline{A} \in \mathcal{A}\}$. If $\mathcal{F}$ is a finite nonempty subset of $\mathcal{B}$, then there is some $p \in \bigcap_{A \in \mathcal{F}} \overline{A}$ and so $\bigcap \mathcal{F} \in p$ and thus $\bigcap \mathcal{F} \neq \emptyset$. That is, $\mathcal{B}$ has the finite intersection property, so by Theorem 2.1.2 pick $q \in \beta X$ with $\mathcal{B} \subseteq q$. Hence $q \in \bigcap \mathcal{A}$.

2. In the proof of (1) each set $\overline{A}$ was taken closed as well as open. Let C be a clopen subset of βX and let $\mathcal{A} = \{\overline{A} : A \subseteq X \text{ and } \overline{A} \subseteq C\}$. Since C is open, $\mathcal{A}$ is an open cover of C. Since C is closed, it is compact by (1) so pick a finite subfamily $\mathcal{F}$ of $\mathcal{P}(X)$ such that $C = \bigcup_{A \in \mathcal{F}} \overline{A}$. Then by Lemma 2.2.1(2), $C = \overline{\bigcup \mathcal{F}}$.

3. For each $a \in A$, one clearly has $e(a) \in \overline{A}$. Therefore $c\ell_{\beta X}(e[A]) \subseteq \overline{A}$. To prove the reverse inclusion, let $p \in \overline{A}$. If $\overline{B}$ denotes a basic neighborhood of p, then $A \in p$ and $B \in p$ and so $A \cap B \neq \emptyset$. Choose any $a \in A \cap B$. Since $e(a) \in e[A] \cap \overline{B}$, then $e[A] \cap \overline{B} \neq \emptyset$ and thus $p \in c\ell_{\beta X}(e[A])$.

4. By (3) and the definition of $\overline{A}$,

$$\begin{aligned} p \in c\ell_{\beta X}\ e[A] &\Leftrightarrow p \in \overline{A} \\ &\Leftrightarrow A \in p. \end{aligned}$$

5. If $a, b \in X$ are distinct, $X \setminus \{a\} \in e(b) \setminus e(a)$ and so $e(a) \neq e(b)$.
   If $\overline{A}$ is a nonempty basic open subset of βX, then $A \neq \emptyset$ by Lemma 2.2.1(4). Any $a \in A$ satisfies $e(a) \in e[X] \cap \overline{A}$ and so $e[X] \cap \overline{A} \neq \emptyset$. Thus $e[X]$ is dense in βX.
   For any $a \in X$, $e(a)$ is isolated in βX because $\overline{\{a\}}$ is an open subset of βX whose only member is $e(a)$. Conversely if p is an isolated point of βX, then $\{p\} \cap e[X] \neq \emptyset$ and so $p \in e[X]$.

6. If $U = \emptyset$, the conclusion is trivial and so let us assume that $U \neq \emptyset$. Put $A = e^{-1}[U]$. So let $p \in U$ and let $\overline{B}$ be a basic neighborhood of p. Then $U \cap \overline{B}$ is a nonempty open set and so by (5), $U \cap \overline{B} \cap e[X] \neq \emptyset$. So pick $b \in B$ with $e(b) \in U$. Then $e(b) \in \overline{B} \cap e[A]$ and so $\overline{B} \cap e[A] \neq \emptyset$. Thus $U \subseteq c\ell_{\beta X}(e[A])$.
   Also $e[A] \subseteq U$ and hence $U \subseteq c\ell_{\beta X}(e[A]) \subseteq c\ell_{\beta X}(U)$. Thus

$$c\ell_{\beta X}(U) = c\ell_{\beta X}(e[A]) = \overline{A}$$

   by the condition (3) above, and so $c\ell_{\beta X}\ U$ is open in βX.

□

Note that βX is an extremally disconnected space, but X^* is not. See [19].

Definition 2.2.3 *Let X be a nonempty set and let $\mathcal{F}$ be a family of subsets of X. We define $\overline{\mathcal{F}} \subseteq \beta X$ by*

$$\overline{\mathcal{F}} = \bigcap_{A \in \mathcal{F}} \overline{A}.$$

Theorem 2.2.3 *Let X be a nonempty set.*

1. *For every filter $\mathcal{F}$ on X, $\overline{\mathcal{F}}$ is a nonempty closed subset of βX consisting of all $p \in \beta X$ such that $\mathcal{F} \subseteq p$.*

2. *Conversely, for every nonempty subset $U \subset \beta X$, the intersection of all ultrafilters from U is a filter $\mathcal{F}$ on X such that $\overline{\mathcal{F}} = U$.*

Proof.

> To prove (1), let $\mathcal{F}$ be a filter on X. Pick $p \in \overline{\mathcal{F}}$. Then, for all $A \in \mathcal{F}$ we have $p \in \overline{A}$. That is, $A \in p$. Hence $\overline{\mathcal{F}} \subseteq p$.
>
> To prove (2), take $\mathcal{F}$ as the intersection of all ultrafilters from U which is a filter on X. Now let $p \in \overline{\mathcal{F}}$. Assume on the contrary that $p \notin U$. Then there is $A \in p$ such that $\overline{A} \cap U = \emptyset$. It follows that for each $q \in U$, $X \setminus A \in q$. Hence $X \setminus A \in \mathcal{F}$. Using (1), $\mathcal{F} \subseteq p$. Hence $D \setminus A \in p$, which is a contradiction. Thus, each $p \in \overline{\mathcal{F}}$ implies $p \in U$. That is, $\overline{\mathcal{F}} \subseteq U$. Finally, $\overline{\mathcal{F}} \subseteq U$.

□

Definition 2.2.4 *Let X be a nonempty set and let $A \subseteq X$. Then $A^* = \overline{A} \setminus A$ and $c\ell_{\beta X}(A) = \overline{A}$.*

Consequently, the principal ultrafilters will be identified with the elements of X. And, $X^* = \beta X \setminus X$ is the set of nonprincipal ultrafilters on X.

Theorem 2.2.4 *Let X be a nonempty set, let $p \in \beta X$ and let U be a subset of βX. If U is a neighborhood of p in βX, then $e^{-1}[U] \in p$.*

Proof. Suppose U is a neighborhood of p. Then there is a basic open subset $\overline{A}$ of βX for which $p \in \overline{A} \subseteq U$ with $A \subseteq X$. This implies that $A \in p$. Since $A \subseteq e^{-1}[U]$ and p is an ultrafilter on X, we have $e^{-1}[U] \in p$ as well. □

Definition 2.2.5 *Let X be a topological space. A* ***compactification*** *of X is a pair (φ, K) such that K is a compact space, φ is an embedding of X into K, and $\varphi[X]$ is dense in K.*

Theorem 2.2.5 *Every completely regular space X has a Hausdorff compactification βX in which it is C^*-embedded.*

Proof. See Proof of Theorem 1.9 in [31]. □

Corollary 2.2.1 *If X is a compact space, βX is homeomorphic to X.*

Proof. See Proof of Corollary 1.10 in [31]. □

Theorem 2.2.6 *Every completely regular space X has a compactification βX such that any mapping from X to a compact space K extends uniquely to βX.*

Proof. See Proof of Theorem 1.11 in [31]. □

Corollary 2.2.2 *Any compactification of X is a continuous image of βX under a mapping which leaves points of X fixed.*

Proof. This is a special case of the theorem above. □

Corollary 2.2.3 *Any compactification of X to which every mapping of X to a compact space has an extension, is homeomorphic to βX under a homeomorphism which leaves points of X fixed.*

Proof. See the proof of Corollary 1.13 in [31]. □

Theorem 2.2.7 ***(Cěch Theorem)****. βX is that compactification of a space X in which completely separated subsets of X have disjoint closures.*

Proof. For the proof, please refer to the proof of Theorem 1.14 in [31]. □

Corollary 2.2.4 *When X is a normal space, βX is that compactification of X in which disjoint closed subsets of X have disjoint closures.*

Proof. Recall that in a normal space, two disjoint closed sets are completely separated. The proof is obtained by applying the theorem above. □

Definition 2.2.6 *Let X be a completely regular topological space. A* ***Stone-Čech compactification*** *of X is pair (φ, Z) such that*

1. *Z is a compact space,*

2. *φ is an embedding of X into Z,*

3. *$\varphi[X]$ is dense in Z, and*

4. *given any compact space Y and any continuous function $f : X \to Y$ there exits a unique continuous function $g : Z \to Y$ such that $g \circ \varphi = f$.*

If f is continuous mapping from a completely regular space X into a compact space Y, we use $\tilde{f}$ to denote the continuous mapping from βX to Y which extends f.

The following theorem summarize the characteristic properties of the Stone-Čech compactification of a completely regular space.

Theorem 2.2.8 *Every completely regular space X has a unique compactification βX which has the following equivalent properties:*

1. *X is C^*-embedded in βX.*

2. *Every mapping of X into a compact space extends uniquely to βX.*

3. *Every point of βX is the limit of an ultrafilter on X.*

4. *If Z_1 and Z_2 are zero-sets in X, then*

$$c\ell_{\beta X}(Z_1) \cap c\ell_{\beta X}(Z_2) = c\ell_{\beta X}(Z_1 \cap Z_2).$$

5. *Disjoint zero-sets in X have disjoint closures in βX.*

6. *Completely separated sets in X have disjoint closures in βX.*

7. *βX is a maximal in the partially ordered set of compactifications of X.*

Proof. The proof follows the proof of Theorem 1.46 in [31]. □

Notice that if $T \subseteq X$, then $p \in c\ell_{\beta X}(T)$ (which is an ultrafilter on X) shall be identified with the ultrafilter $\{A \cap T \ : \ A \in p\}$ (which is an ultrafilter on T).

Theorem 2.2.9 *Let T be a subspace of X. Then T is C^*-embedded in X if and only if $\beta T = c\ell_{\beta X}(T)$.*

Proof. See the proof of Proposition 1.48 in [31]. $\square$

Theorem 2.2.10 *If T is a subspace of βX containing X, then βT is βX.*

Proof. See the proof of Proposition 1.49 in [31]. $\square$

Any Boolean algebra B is associated with a totally disconnected compact Hausdorff space, called its Stone space, whose points could be described as the ultrafilters on B. All totally disconnected compact Hausdorff spaces arise in this way, as any such space can be identified with the Stone space associated with the Boolean algebra formed by its clopen subsets. This theory displays the category of totally disconnected compact Hausdorff spaces as being the dual of the category of Boolean algebras. The extremally disconnected compact spaces are those corresponding to complete Boolean algebras. If D is a discrete space, βD could be described as the Stone space of the Boolean algebra $\mathcal{P}(D)$, while D^* could be described as the Stone space of the quotient of $\mathcal{P}(D)$ by the ideal of finite subsets of D. See [19]. We mention this construction because it is relevant to the Stone-Čech compactification as used in [6], [7], and [13].

Before giving the definition of a Boolean algebra, let us recall the notion of partially ordered set.

Definition 2.2.7 *Let X be a nonempty set.*

1. *The set X together with a binary relation $\leq$ defined on X is called a* ***partially ordered set*** *if it satisfies the following conditions for arbitrary elements x, y, and z of X:*

   (a) *$\leq$ is* ***reflexive****: $x \leq x$ for all x.*

   (b) *$\leq$ is* ***transitive****: $x \leq y$ and $y \leq z$ implies that $x \leq z$.*

   (c) *$\leq$ is* ***antisymmetric****: $x \leq y$ and $y \leq x$ implies that $x = y$.*

2. *An* ***upper bound*** *for a subset A of X is an element x such that $a \leq x$ for all a in A, and a* ***supremum*** *for the set A is an upper bound x such that for any other upper bound y, $x \leq y$. The supremum of a family $\{x_\alpha\}$ is written $\bigvee_\alpha x_\alpha$.*

3. *A* ***lower bound*** *for a subset A of X is an element z such that $z \leq a$ for all a in A, and an* ***infimum*** *for the set A is an upper bound z such that for any other lower bound w, $w \leq z$. The infimum of a family $\{x_\alpha\}$ is written $\bigwedge_\alpha x_\alpha$.*

We recall in the following the definition of a Boolean algebra.

Definition 2.2.8 1. *A* ***lattice*** *is a partially ordered set in which every pair of elements has a supremum and an infimum.*

2. *A lattice is said to be* ***complete*** *if every set has a supremum and an infimum.*

3. *A lattice is said to be* ***distributive*** *if the operations supremum and infimum satisfy the two identities:*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

4. *A lattice is said to be* ***complemented*** *if it contains two elements 0 and 1 such that $0 \leq x \leq 1$ for all elements x and to each element element x is assigned an element x' such that $x \vee x' = 1$ and $x \wedge x' = 0$.*

5. *A* ***Boolean algebra*** *is a complemented distributive lattice.*

6. *A Boolean algebra is said to be* ***complete*** *if it is complete as a lattice.*

We list a couple of examples of Boolean algebras.

Example 2.2.1 1. *The family of all subsets of a set and the family of clopen subsets of a space are Boolean algebras with the operations of set-theoretic complementation, union, and intersection.*

2. *The family of all regular closed subsets of a topological space X is a complete Boolean algebra with the following operations:*

   (a) *$A \leq B$ if and only if $A \subseteq B$.*

   (b) *$\bigvee_\alpha A_\alpha = c\ell(\cup_\alpha int(A_\alpha))$.*

(c) $\bigwedge_\alpha A_\alpha = c\ell(\cap_\alpha int(A_\alpha))$.

(d) $A' = c\ell(int(X \setminus A))$.

Definition 2.2.9 *A subset F of a space X is called a* ***regular closed set*** *if $F = c\ell(int(F))$. Similarly, a subset G of a space X is said to be a* ***regular open set*** *if $G = int(c\ell(G))$.*

We define the notion of set-theoretical (Boolean algebra) ideal and filter on a set X as follows:

Definition 2.2.10 *Let X be a set. An* ***ideal*** *$\mathcal{I}$ on X is a nonempty set of subsets of X such that*

1. *$X \notin \mathcal{I}$,*
2. *If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$, and*
3. *If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.*

Notice that the relationship between ideals and filters in a Boolean algebra is apparent by comparing their definitions and observing that a Boolean algebra satisfies the *de Morgan Laws*:

$$A \cup B = (A' \cap B')' \text{ and } A \cap B = (A' \cup B')'.$$

Thus, the complements of members of an ideal form a filter and conversely. So $\mathcal{I}$ is an ideal on X if and only if the set $\{X \setminus A : A \in \mathcal{I}\}$ is a filter on X. For this reason, an ideal in a Boolean algebra is the dual of a filter.

Definition 2.2.11 *A* ***field of sets*** *is a family of subsets of a set X which is closed under finite unions, finite intersections, and complementation.*

An example of a field of sets is the family of clopen subsets of any topological space. It is clear that any field is a Boolean algebra. The converse of this statement is known as the Stone Representation Theorem which states that every Boolean algebra can be represented as the field of clopen subsets of some totally disconneted compact space.

Definition 2.2.12 *(See [31]) Let L be a Boolean algebra, $S(L)$ the set of maximal filters of L, and $\mathcal{E}$ the Boolean algebra of clopen subsets of $S(L)$. The set $S(L)$ with the topology generated by $\mathcal{E}$ is called the* ***Stone space*** *of the Boolean algebra L.*

Theorem 2.2.11 *Let D be a discrete space and let e be the embedding of D into βD. Then $(e, \beta D)$ is the Stone-Čech compactification of D.*

Proof. Let us prove the conditions of Definition 2.2.6. For any set D, βD is a compact space by Theorem 2.2.2(1) so the condition (1) holds. By Theorem 2.2.2(5), e is an embedding and $e[D]$ is dense in βD so the conditions (2) and (3) hold. Now to prove the condition (4), let Y be a compact space and let $f : D \to Y$ be a continuous function. For each $p \in \beta D$, let

$$\mathcal{A}_p = \{c\ell_Y(f[A]) \,:\, A \in p\}.$$

To show that $\mathcal{A}_p$ has the finite intersection property, let $\mathcal{G} \in \mathcal{P}_f(\mathcal{A}_p)$ such that $\bigcap \mathcal{G} \neq \emptyset$. Define

$$\mathcal{F} = \{A \in p \,:\, c\ell_Y(f[A]) \in \mathcal{G}\}.$$

Then $\mathcal{F}$ is a finite nonempty subset of p. Since $p \in \beta D$, $\bigcap \mathcal{F} \neq \emptyset$ by Theorem 2.1.2. Pick $A \in \mathcal{F}$ such that $c\ell_Y(f[A]) \in \mathcal{G}$. Let consider some $r \in \bigcap_{A\in\mathcal{F}} c\ell_Y(f[A])$ so $r \in \bigcap \mathcal{G}$. Thus $\bigcap \mathcal{G} \neq \emptyset$. Hence, for each $p \in \beta D$, $\mathcal{A}_p$ has the finite intersection property. Since Y is compact, by Theorem 2.1.5, $\mathcal{A}_p$ has a nonempty intersection. Choose $g(p) \in \bigcap \mathcal{A}_p$. We prove that $g \circ e = f$ and that g is continuous.
For the first assertion, let $x \in D$. Since $e(x)$ is the principal ultrafilter generated by x, then $\{x\} \in e(x)$. So

$$g(e(x)) \in c\ell_Y(f[\{x\}]) = c\ell_Y(\{f(x)\}) = \{f(x)\},$$

by the fact that $g(p) \in \bigcap \mathcal{A}_p$ with $p = e(x)$, $A = \{x\}$ and f is continuous. Thus $g(e(x)) = f(x)$ for any $x \in D$. That is $g \circ e = f$ as required.
To see that g is continuous, let $p \in \beta D$ and let U be a neighborhood of $g(p)$ in Y. Since Y is regular, pick a neighborhood V of $g(p)$ with $c\ell_Y(V) \subseteq U$ and let $A \in f^{-1}[V]$. To prove that $A \in p$, consider the contrary. Suppose instead that $D \setminus A \in p$. Then $g(p) \in c\ell_Y(f[D \setminus A])$ and V is a neighborhood of $g(p)$ so $V \cap f[D \setminus A] \neq \emptyset$, contradicting the fact that $A = f^{-1}[V]$. Thus $\overline{A}$ is a neighborhood of p. To prove that $g[\overline{A}] \subseteq U$, consider the contrary. So pick $q \in \overline{A}$ such that $g(q) \notin U$.

Then $Y \setminus c\ell_Y(V)$ is a neighborhood of $g(p)$ and $g(p) \in c\ell_Y(f[A])$. So $(Y \setminus c\ell_Y(V)) \cap f[A] \neq \emptyset$, again contradicting the fact that $A = f^{-1}[V]$. $\square$

Theorem 2.2.12 *Let D be an infinite discrete space. We have the following statements:*

1. *The points of D are identified with the principal ultrafilters in βD generated by those points.*
2. *$D \subseteq \beta D$.*
3. *D is dense in βD, and*
4. *given any compact space K and any function $f : D \to K$ there exists a continuous function $\tilde{f} : \beta D \to K$ such that $\tilde{f}_{|D} = f$.*

Proof. The proof uses Definition 2.2.6 and Theorem 2.2.11 above. $\square$

Definition 2.2.13 *Let X be a topological space. A subset Z of X is called a **zero-set** if $Z = f^{-1}[\{0\}]$ for some continuous function $f : X \to [0,1]$. The complement of a zero-set is called a **cozero-set**.*

Note that if D is an infinite discrete space, the clopen sets and the zero-sets of D coincide and the Stone space of the algebra of clopen sets of D is βD.

Recall that a F_σ**-set**, also known as σ**-compact space**, is one which is the union of countably many compact subspaces and a G_δ**-set** is one which is the intersection of countably many open sets.

Theorem 2.2.13 *If D is any infinite set, every non-empty G_δ-subset of D^* has a non-empty interior in D^*.*

Proof. See Proof of Theorem 3.36 in [19]. $\square$

Corollary 2.2.5 *Let D be any set. Any countable union of nowhere dense subsets of D^* is again nowhere dense in D^*.*

Proof. See Proof of Corollary 3.37 in [19]. $\square$

Definition 2.2.14 *A space X is called an* ***F-space*** *if each cozero-set in X is C^*-embedded in X.*

Extremally disconnected spaces are examples of F-spaces and also, one can prove that if X is locally compact and σ-compact, then X^* is an F-space.

The Stone-Čech compactification of the discrete space $\mathbb{N}$ of natural numbers is an object of interest because of its complex properties but still useful. $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ contains exactly c pairwise disjoint clopen subspaces. It should be noted that every nonempty clopen subset of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$ and $\mathbb{N}^*$ contains 2^c disjoint copies of itself.

Theorem 2.2.14 *Every infinite compact F-space contains a copy of $\beta\mathbb{N}$.*

Proof. See the proof of Proposition 1.64 in [31]. $\square$

Definition 2.2.15 *A point in a topological space is called a* ***P-point*** *if every G_δ containing the point is a neighborhood of the point.*

Recall that every zero-set is a G_δ since it can be written as an intersection of countably many open sets.

Theorem 2.2.15 *The continuum Hypothesis implies that P-points exist in $\mathbb{N}^*$.*

Proof. See the proof of Theorem 3.38 in [19]. $\square$

Theorem 2.2.16 *Let D be a discrete space and let A and B be σ-compact subsets of βD. If $A \cap c\ell(B) = c\ell(A) \cap B = \emptyset$, then $c\ell(A) \cap c\ell(B) = \emptyset$.*

Proof. For the proof, see the proof of Theorem 3.40 in [19] or the proof of Theorem 2.22 in [34].
Write $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$ where A_n and B_n are compact for each n. Since βD is a compact (Hausdorff) space, it is normal. For each $n \in \mathbb{N}$, A_n and $c\ell(B)$ are disjoint closed sets and $c\ell(A)$ and B_n are disjoint closed sets. So pick open sets T_n, U_n, V_n and W_n such that

$T_n \cap U_n = V_n \cap W_n = \emptyset$, $A_n \subseteq T_n$, $c\ell(B) \subseteq U_n$, $c\ell(A) \subseteq V_n$ and $B_n \subseteq W_n$. For each $n \in \mathbb{N}$, let $G_n = T_n \cap \bigcap_{k=1}^n V_k$ and let $H_n = W_n \cap \bigcap_{k=1}^n U_k$. Then for each n, one has $A_n \subseteq G_n$ and $B_n \subseteq H_n$. Let $n, m \in \mathbb{N}$. If $m \geq n$, then

$$G_n \cap H_m \subseteq T_n \cap \bigcap_{k=1}^{m} U_k \subseteq T_n \cap U_n = \emptyset.$$

Or if $m \leq n$, then

$$G_n \cap H_m \subseteq \bigcap_{k=1}^{n} V_k \cap W_m \subseteq V_m \cap W_m = \emptyset$$

Thus, for any $n, m \in \mathbb{N}$, $G_n \cap H_m = \emptyset$.
Let $C = \bigcup_{n=1}^{\infty} G_n$ and let $D = \bigcup_{n=1}^{\infty} H_n$. Then C and D are disjoint open sets. So $D \cap c\ell(C) = \emptyset$, $A \subseteq C$, and $B \subseteq D$. Since $c\ell(C)$ is open by Theorem 2.2.2(6), $c\ell(B) \cap c\ell(C) = \emptyset$. Since $c\ell(A) \subseteq c\ell(C)$, $c\ell(A) \cap c\ell(B) = \emptyset$ as required. $\square$

Corollary 2.2.6 *Let D be a discrete space.*

1. *Let A and B be σ-compact subsets of D^*. If $A \cap c\ell(B) = c\ell(A) \cap B = \emptyset$, then $c\ell(A) \cap c\ell(B) = \emptyset$.*
2. *Let A and B be countable subsets of βD. If $A \cap c\ell(B) = c\ell(A) \cap B = \emptyset$, then $c\ell(A) \cap c\ell(B) = \emptyset$.*
3. *Let A and B be countable subsets of D^*. If $A \cap c\ell(B) = c\ell(A) \cap B = \emptyset$, then $c\ell(A) \cap c\ell(B) = \emptyset$.*

Proof. See the proof of Corollary 3.42 in [19] or the proof of Corollary 2.23 in [34]. $\square$

As a consequence of this theorem, if D is a discrete space then every compact subset of βD is an F-space. Thus D^* is an F-space, although it need not to be extremally disconnected unlike βD (see Theorem 2.2.2(6)). It should be noted that every closed subspace of a compact F-space is an F-space.

Theorem 2.2.17 *Let $f : D \to E$ be a mapping of discrete spaces and let $\tilde{f} : \beta D \to \beta E$. Then $\tilde{f}$ is injective if f is injective, surjective if f is surjective and a homeomorphism if f is bijective.*

Proof. Exercise 3.4.1 in [19]. $\square$

Theorem 2.2.18 *Let D be any discrete space. Then every separable subspace of βD is extremally disconnected.*

Proof. Exercise 3.4.7 in [19]. $\square$

Definition 2.2.16 *Let D be a discrete space, let $p \in \beta D$, let $(x_s)_{s\in D}$ be an indexed family in a topological space X, and let $y \in X$. Then*

$$p - \lim_{s\in D} x_s = y$$

if and only if for every neighborhood U of y,

$$\{s \in D \,:\, x_s \in U\} \in p.$$

Definition 2.2.17 *Suppose that X and Y are topological spaces, that $A \subseteq X$ and that $f : A \to Y$. Let $a \in c\ell_X(A)$ and $y \in Y$. We write*

$$\lim_{x\to a} f(x) = y$$

if and only if, for every neighborhood V of y, there is a neighborhood U of a such that $f[A \cap U] \subseteq V$.

Theorem 2.2.19 *Let D be a discrete space, let Y be a topological space, and let $p \in \beta D$ and $y \in Y$. If $A \in p$ and $f : A \to Y$, then*

$$p - \lim_{x\in A} f(x) = y \text{ if and only if } \lim_{x\to p} f(x) = y.$$

Proof. Consider $p - \lim\limits_{x\in A} f(x) = y$ and V a neighborhood of y. Then, $f^{-1}[V] \in p$. Let $B = f^{-1}[V]$. By Theorem 2.2.2, $\overline{B}$ is a neighborhood of p. Let $U = \overline{B}$. Since $B = f^{-1}[V] \subseteq A$, then $f[A \cap U] = f[B] \subseteq V$. Thus, $\lim\limits_{x\to p} f(x) = y$.

Conversely, consider that $\lim\limits_{x\to p} f(x) = y$ and V a neighborhood of y. Then, there is a neighborhood U of p such that $f[A \cap U] \subseteq V$. Now $U \cap A \in p$ and so, since $U \cap A \subseteq f^{-1}[V]$, it follows that $f^{-1}[V] \in p$ (the fact that p is an ultrafilter on D). Thus, $p - \lim\limits_{x\in A} f(x) = y$. $\square$

Let us note that $\lim\limits_{x \to a} f(x)$, if it exists, is unique. The concepts of p-limits and limits coincide for functions defined on βD, we use them interchangeably.

The cardinality of the set of natural numbers $\mathbb{N}$ is the first infinite cardinal and is denoted by ω. The cardinality of the set of real numbers $\mathbb{R}$ is called the cardinality of **continuum** and is denoted by c. Therefore, $c = 2^\omega = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

Theorem 2.2.20 *Let D be a countably infinite set. There is a $c \times c$ matrix $\left\langle \langle A_{\sigma,\delta} \rangle_{\sigma<c} \right\rangle_{\delta<c}$ of subsets of D satisfying the following two statements.*

1. *Given $\sigma, \delta, \tau < c$ with $\delta \neq \tau$, $|A_{\sigma,\delta} \cap A_{\delta,\tau}| < \omega$*
2. *Given any $F \in \mathcal{P}_f(c)$ and any $g : F \to c$, $|\bigcap_{\sigma \in F} A_{\sigma,g(\sigma)}| = \omega$*

Proof. See the proof of Theorem 3.56 in [19]

$\square$

Corollary 2.2.7 *Let D be a countably infinite discrete space. Then $|\beta D| = 2^{2^\omega}$.*

Proof. Since $\beta D \subseteq \mathcal{P}(\mathcal{P}(D))$, one has that

$$|\beta D| \leq |\mathcal{P}(\mathcal{P}(D))| = 2^{2^{|D|}} = 2^{2^\omega}.$$

That is, $|\beta D| \leq 2^{2^\omega}$. Let $\left\langle \langle A_{\sigma,\delta} \rangle_{\sigma<c} \right\rangle_{\delta<c}$ be as in Theorem 2.2.20, with $c = 2^\omega$. For each $f : c \to \{0, 1\}$, let

$$\mathcal{A}_f = \{A_{\sigma,f(\sigma)} : \sigma < c\}.$$

Then by Theorem 2.2.20(2), for each f, $\mathcal{A}_f$ has the property that all finite intersections are infinite, so pick by Corollary 2.1.2(1) a nonprincipal ultrafilter p_f on D such that $\mathcal{A}_f \subseteq p_f$. If f and g are distinct functions from c to $\{0, 1\}$, then since p_f and p_g are nonprincipal, one has by Theorem 2.2.21(1) that $p_f \neq p_g$. Since there are 2^{2^ω} such functions, $\beta D \geq 2^{2^\omega}$. $\square$

Corollary 2.2.8 *The cardinality of $\beta\mathbb{N}$ is $2^{2^\omega} = 2^c$.*

Proof. One needs to take $D = \mathbb{N}$ in Corollary 2.2.7 above. $\square$

Theorem 2.2.21 *The cardinality of each infinite closed subset of $\beta\mathbb{N}$ is 2^c.*

Proof. See the proof of Theorem 3.3 in [31], by considering the fact that every infinite closed subspace of $\beta\mathbb{N}$ contains a copy of $\beta\mathbb{N}$ and therefore has cardinality 2^c. $\square$

Corollary 2.2.9 *Every uncountable open subset of $\beta\mathbb{N}$ has cardinality 2^c.*

Proof. Since every uncountable open subset of $\beta\mathbb{N}$ must contain an infinite closed subset, by Theorem 2.2.21 above, it has also cardinality 2^c. $\square$

Corollary 2.2.10 *Every countable subspace of $\beta\mathbb{N}$ is C^*-embedded.*

Proof. The proof is contained in the proof of Theorem 3.3 in [31]. $\square$

This is an interesting and important property of $\beta\mathbb{N}$ that implies that the closure in $\beta\mathbb{N}$ of any discrete, countably infinite subspace of $\beta\mathbb{N}$ is homeomorphic with $\beta\mathbb{N}$.

Theorem 2.2.22 *If D is an infinite discrete space of cardinality κ, then $|U_\kappa(D)| = |\beta D| = 2^{2^\kappa}$.*

Proof. See the proof of Theorem 3.58 in [19] and also the proof of Theorem 2.26 in [34].

$\square$

Theorem 2.2.23 *Let D be an infinite discrete space and let A be an infinite closed subset of βD. Then A contains a topological copy of $\beta\mathbb{N}$. In particular $|A| \geq 2^{2^\omega}$.*

Proof. See the proof of Theorem 3.59 in [19].

$\square$

We list in the following some properties that characterize the space $\mathbb{N}^*$ in the presence of the Continuum Hypothesis. The Continuum Hypothesis helps to index the c clopen sets of $\mathbb{N}^*$ by the countable ordinals. In the absence of using the Continuum Hypothesis, $\mathbb{N}^*$ maps continuously onto every compact space having weight at most c. See [9] and [31] for more details.

Theorem 2.2.24 *Every zero-set in βX which misses X contains a copy of $\mathbb{N}^*$ and therefore its cardinality is at least 2^c.*

Proof. See the proof of Theorem 3.6 in [31]. $\square$

Lemma 2.2.2 *A compact space is zero-dimensional if and only if it is totally disconnected.*

Proof. See the proof of Proposition 2.4 in [31]. $\square$

Theorem 2.2.25 *$\beta\mathbb{N}$ is totally disconnected and therefore is also zero-dimensional.*

Proof. Let p and q be distinct points of $\beta\mathbb{N}$ and choose $A \subseteq \mathbb{N}$ such that $A \in p$ but $A \notin q$. Then $\overline{A}$ is a clopen neighborhood of p which misses q. $\beta\mathbb{N}$ is therefore totally disconnected since the only connected subspaces of $\beta\mathbb{N}$ are the singletons and is zero-dimensional by Lemma 2.2.2. See the proof of Proposition 3.9 in [31] for more details. $\square$

Theorem 2.2.26 *Every clopen subspace of $\beta\mathbb{N}$ is of the form $c\ell_{\beta\mathbb{N}}(A)$ for some subset A of $\mathbb{N}$.*

Proof. Let U be a clopen subset of $\beta\mathbb{N}$. Then U is compact (as a closed subset of a compact space). Hence if U is contained in $\mathbb{N}$, then U is finite and $U = c\ell_{\beta\mathbb{N}}(U)$. If U meets $\mathbb{N}^*$, then $U \cap \mathbb{N} \neq \emptyset$ since $\mathbb{N}$ is dense in $\beta\mathbb{N}$. So $c\ell_{\beta\mathbb{N}}(U \cap \mathbb{N}) \subseteq U$ since U is closed. Since U is open, $U \setminus c\ell_{\beta\mathbb{N}}(U \cap \mathbb{N})$ is open in $\beta\mathbb{N}$ and misses $\mathbb{N}$. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, this difference must be empty. Thus, $U = c\ell_{\beta\mathbb{N}}(U \cap \mathbb{N})$. See [31] for the details about this proof. $\square$

By Theorem 2.2.9, if A is an infinite subset of $\mathbb{N}$, then $c\ell_{\beta\mathbb{N}}(A)$ is homeomorphic to $\beta\mathbb{N}$ and A^* is homeomorphic to $\mathbb{N}^*$. Further, by Theorem 2.2.27 above, $c\ell_{\beta\mathbb{N}}\ A$ is an infinite clopen subspace of $\beta\mathbb{N}$ and hence is a copy of of $\beta\mathbb{N}$.

Corollary 2.2.11 *Every open set of $\beta\mathbb{N}$ which meets $\mathbb{N}^*$ has cardinality 2^c.*

Proof. See [31]. Since $\beta\mathbb{N}$ has a base of clopen subsets, the proof is an improvement of Corollary 2.2.9. $\square$

Definition 2.2.18 *A point in a topological space is called a κ-**point** if it is the limit of a sequence of distinct points of the space.*

Theorem 2.2.27 $\mathbb{N}^*$ *is a compact Hausdorff space containing no isolated points and no κ-points.*

Proof. See the proof of Proposition 3.12 in [31]. $\mathbb{N}^*$ is clearly a compact Hausdorff space since it is a closed subspace of $\beta\mathbb{N}$. $\mathbb{N}^*$ cannot contain an isolated point since, by Corollary 2.2.11, an open subset of $\beta\mathbb{N}$ which meets $\mathbb{N}^*$ contains 2^c points. If a point p of $\mathbb{N}^*$ is a limit of the sequence $\{x_n\}$, then $\{x_n\} \cup \{p\}$ is a countably infinite closed subspace of $\beta\mathbb{N}$ which is impossible by Theorem 2.2.21. $\square$

Theorem 2.2.28 *Every clopen subset of $\mathbb{N}^*$ is of the form A^* for some infinite subset A of $\mathbb{N}$. The sets A^* form a base for both the open and the closed sets of $\mathbb{N}^*$.*

Proof. See the proof of Proposition 3.16 in [31]. $\square$

Definition 2.2.19 *The **weight** of a space is the least cardinal number of a base for the space.*

Corollary 2.2.12 *$\beta\mathbb{N}$ and $\mathbb{N}^*$ each have a base consisting of c clopen sets. $\beta\mathbb{N}$ and $\mathbb{N}^*$ have weight c.*

Proof. Since $\mathbb{N}$ has c infinite subsets and only c subsets in all, the above results establish the cardinalities of bases for both $\beta\mathbb{N}$. See [31]. $\square$

Theorem 2.2.29 *If A^* and B^* are proper clopen subspaces of $\mathbb{N}^*$, there exists an automorphism of $\beta\mathbb{N}$ and hence of $\mathbb{N}^*$ carrying A^* onto B^*.*

Proof. See the proof of Proposition 3.19 in [31]. $\square$

Let S be a semigroup and $w \in S^*$. We recall that the set $\mathcal{O}(w) := \{sw : s \in S\}$ is called the *orbit* of w.

Corollary 2.2.13 *The orbit of any point of* $\mathbb{N}^*$ *is a dense subspace of* $\mathbb{N}^*$.

Proof. The proof is immediate since the sets A^* form a base for the open sets of $\mathbb{N}^*$ if A be an infinite subset of $\mathbb{N}$ [31]. $\square$

Definition 2.2.20 *A family of sets is said to be* ***almost disjoint*** *if the intersection of any two of the sets is finite.*

Lemma 2.2.3 $\mathbb{N}$ *admits a family of* c *almost disjoint infinite subsets.*

Proof. See the proof of Proposition 3.21 in [31]. $\square$

Definition 2.2.21 *The cellularity of a topological space* Y *is the smallest cardinal number* λ *for which each pairwise disjoint family of open sets of* Y *has* λ *or fewer members.*

Theorem 2.2.30 *The cellularity of* $\mathbb{N}^*$ *is* c.

Proof. See the proof of Theorem 3.22 in [31]. Lemma 2.2.3 guarantees the existence of the family $\mathcal{E}$ of c almost disjoint infinite subsets of $\mathbb{N}$. For distinct members E and F of $\mathcal{E}$, $c\ell_{\beta\mathbb{N}}(E) \cap c\ell_{\beta\mathbb{N}}(F) \cap \mathbb{N}^* = \emptyset$ since any point in this intersection would belong to $c\ell_{\beta\mathbb{N}}(E) \cap c\ell_{\beta\mathbb{N}}(F) = c\ell_{\beta\mathbb{N}}(E \cap F) = E \cap F$ since $E \cap F$ is finite. Thus, $\{E^* : E \in \mathcal{E}\}$ is a family of c pairwise disjoint open sets of $\mathbb{N}^*$. On the other hand, there can be no such family of larger cardinality since the power set of $\mathbb{N}$ has cardinality c. $\square$

Corollary 2.2.14 *Any dense subset of* $\mathbb{N}^*$ *must contain at least* c *points.*

Proof. It follows from Theorem 2.2.30 above. $\square$

Corollary 2.2.15 $\mathbb{N}^*$ *contains a copy of* $\beta\mathbb{N}$.

Proof. By Theorem 2.2.30 above, $\mathbb{N}^*$ contains a pairwise disjoint family of c clopen subsets. Thus, a copy of $\mathbb{N}$ may be obtained in $\mathbb{N}^*$ by choosing a point from each of countably many members of such a family of clopen sets. By Corollary 2.2.10, the copy of $\mathbb{N}$ just obtained is C^*-embedded in $\mathbb{N}^*$ and its closure is homeomorphic to $\beta\mathbb{N}$. Thus, $\mathbb{N}^*$ contains a copy of $\beta\mathbb{N}$. $\square$

Note that every infinite closed subset of $\beta\mathbb{N}$ contains a copy of $\beta\mathbb{N}$. In fact, one can show that every infinite closed subset of an F-space contains a copy $\beta\mathbb{N}$.

Theorem 2.2.31 *Two disjoint cozero sets of* $\mathbb{N}^*$ *are separated by a partition and hence* $\mathbb{N}^*$ *is an F-space.*

Proof. See the proof of Proposition 3.24 in [31] for the details. Since $\mathbb{N}$ is both σ-compact and locally compact, its growth $\mathbb{N}^*$ is an F-space. However, $\mathbb{N}^*$ actually satisfies a stronger condition in that disjoint cozero sets are separated by a partition of $\mathbb{N}^*$, i.e. they are contained in disjoint clopen sets whose union is all of $\mathbb{N}^*$.
Since sets which are separated by a partition are completely separated, disjoint cozero sets of $\mathbb{N}^*$ are completely separated and $\mathbb{N}^*$ is therefore an F-space. $\square$

The following theorem can also be deduced from Theorem 2.2.13.

Theorem 2.2.32 *If a sequence of clopen subsets of* $\mathbb{N}^*$ *has the finite intersection property, then the intersection of the sequence of sets contains a nonempty clopen set.*

Proof. See the proof of Theorem 3.26 in [31]. Let $\{A_n^*\}$ be a sequence of clopen subsets of $\mathbb{N}^*$ having the finite intersection property. Then assuming that

$$A_1^* \supset A_2^* \supset \cdots A_n^* \supset \cdots ,$$

we have that $|A_{n+1} \setminus A_n| < \omega$ for all n. Pick a sequence of distinct points $A = \{x_n\}$ such that x_n belongs to $\bigcap\ \{A_i\ :\ i \leq n\}$. Then $|A \setminus A_n| \leq n-1$ for each n so that $A^* \subseteq A_n^*$ and therefore $A^* \subseteq \bigcap\ A_n^*$. $\square$

Corollary 2.2.16 *Every nonempty* G_δ *in* $\mathbb{N}^*$ *has nonempty interior.*

Proof. Recall that G_δ is an intersection of countably open sets. Since an open set is also clopen and assuming that G_δ is nonempty, the preceding result applies to G_δ. $\square$

Corollary 2.2.17 *The zero-sets of* $\mathbb{N}^*$ *are regular closed sets.*

Proof. See the proof of Corollary 3.28 in [31]. Using the preceding corollary, a nonempty zero-set of $\mathbb{N}^*$ has non-empty interior since a zero-set is a G_δ. Now assume the contrary that there is a zero-set Z_1 of $\mathbb{N}^*$ that is not regular closed. That is, there is $p \in Z_1 \setminus c\ell_{\mathbb{N}^*}(int(Z_1))$. So there exists a zero-set neighborhood Z_2 of p such that Z_2 misses $int(Z_1)$.

Thus, $Z_1 \cap Z_2$ is a nonempty zero-set of $\mathbb{N}^*$ which is contained in the boundary of Z_1 and hence has empty interior. But this contradicts the fact that a non-empty zero-set of $\mathbb{N}^*$ has non-empty interior. $\square$

According to Corollary 2.2.10, each countable subspace of $\mathbb{N}^*$ is C^*-embedded in $\mathbb{N}^*$. Further, $\mathbb{N}^*$ contains many copies of $\mathbb{N}^*$ and $\beta\mathbb{N}$, each of which is C^*-embedded. However, in the following theorem, no dense subspace of $\mathbb{N}^*$ is C^*-embedded and therefore, $\mathbb{N}^*$ is not the Stone-Čech compactification of any of its dense subspaces. The following results use the Continuum Hypothesis, see [31] and [9].

Theorem 2.2.33 *Dense subsets of $\mathbb{N}^*$ are not C^*-embedded.*

Proof. By Theorem 2.2.10, it is sufficient to prove that $\mathbb{N}^* \setminus \{p\}$ is not C^*-embedded in $\mathbb{N}^*$ for any $p \in \beta\mathbb{N}$. If $\mathbb{N}^* \setminus \{p\}$ is the union of disjoint open sets A and B containing each p in its closure, then there exists a two-valued mapping on $\mathbb{N}^* \setminus \{p\}$ which does not extend continuously to $\mathbb{N}^*$.
Assuming the Continuum Hypothesis, the base of c zero-set neighborhoods of p can be indexed by κ and written $\{Z_\alpha : \alpha < \kappa\}$. By transfinite induction, assume for a given $\alpha < \kappa$ that cozero sets A_σ and B_σ have been defined for all $\sigma < \alpha$ such that

$$p \notin A_\gamma \cup B_\gamma \text{ and } A_\tau \cap B_\tau = \emptyset$$

for all $\gamma, \tau < \alpha$. Since a countable union of cozero sets is a cozero set, then by Theorem 2.2.31 there exists complementary clopen sets A'_α and B'_α such that

$$\bigcup_{\sigma<\alpha} A_\sigma \subset A'_\alpha \text{ and } \bigcup_{\sigma<\alpha} B_\sigma \subset B'_\alpha.$$

The set $Z_\alpha \cap (\bigcap\{\mathbb{N}^* \setminus (A_\sigma \cup B_\sigma) : \sigma < \alpha\})$ contains p and thus is a non-void zero-set of $\mathbb{N}^*$ and has a non-void interior by Corollary 2.2.17. Since $\mathbb{N}^*$ contains no isolated points, the set

$$Z_\alpha \cap (\bigcap\{\mathbb{N}^* \setminus (A_\sigma \cup B_\sigma) : \sigma < \alpha\}) \setminus \{p\}$$

contains disjoint non-void cozero-sets A''_α and B''_α. Now define

$$A_\alpha = (A'_\alpha \setminus Z_\alpha) \cup A''_\alpha \text{ and } B_\alpha = (B'_\alpha \setminus Z_\alpha) \cup B''_\alpha.$$

Then A_α and B_α are disjoint cozero sets both of which fail to contain p and the induction hypothesis is satisfied for all $\sigma, \tau \leq \alpha$. Now define

$$A = \bigcup_{\alpha<\omega_1} A_\alpha \text{ and } B = \bigcup_{\alpha<\omega_1} B_\alpha.$$

A and B are disjoint open sets, and neither contains p. If $q \in \mathbb{N}^* \setminus \{p\}$, then there is a neighborhood Z_α of p that misses q such that $q \in A_\alpha \cup B_\alpha$ by construction. Hence, $A \cup B = \mathbb{N}^* \setminus \{p\}$. Since each basic neighborhood Z_α of p contains $A''_\alpha \cup B''_\alpha$, every neighborhood of p meets both A and B so that $p \in c\ell_{\mathbb{N}^*}(A) \cap c\ell_{\mathbb{N}^*}(B)$. Thus, A and B are disjoint open sets of $\mathbb{N}^*$ which are not separated by a partition and have disjoint closures. □

Theorem 2.2.34 *A totally disconnected compact F-space without isolated points and having weight c and such that every zero-set is a regular closed set is homeomorphic with* $\mathbb{N}^*$.

Proof. See the proof of Theorem 3.31 in [31]. □

Another particular interesting example is $\beta\mathbb{R}$ that we briefly present below. See [31] for more details about $\beta\mathbb{R}$.

Theorem 2.2.35 *An infinite compact F-space contains at least* 2^c *non-P-points.*

Proof. By Theorem 2.2.14, every infinite compact F-space contains a copy of $\mathbb{N}^*$. It was shown in Proposition 4.31 of [31] that $\mathbb{N}^*$ contains 2^c non-P-points. The result follows since the non-P-points of the embedded copy of $\mathbb{N}^*$ must also be non-P-points of the compact F-space. □

Corollary 2.2.18 $\mathbb{R}^*$ *has* 2^c *non-P-points.*

Proof. Since $\mathbb{R}$ is both locally compact and σ-compact, $\mathbb{R}^*$ is an F-space. So the proof uses the result of Theorem 2.2.35. □

Definition 2.2.22 *A* ***remote point*** *of* βX *is a point which does not belong to the closure of any discrete subspace of* X. *Thus, any remote point must lie in* X^* *.*

Let R and P denote the sets of remote points of $\beta\mathbb{R}$ and of P-points of $\mathbb{R}^*$, respectively. Then R' and P' will denote the non-remote points and the non-P-points.

Theorem 2.2.36 *The sets* $P \cap R$, $P' \cap R$, $P \cap R'$ *and* $P' \cap R'$ *are each dense subsets of* $\mathbb{R}^*$ *and each has cardinal* 2^c.

Proof. See the proof of Theorem 4.46 in [31]. $\square$

Definition 2.2.23 *Let $\mathcal{A}$ be a set of sets and let κ be an infinite cardinal. Then $\mathcal{A}$ has the **κ-uniform finite intersection property** if and only if whenever $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, one has $|\bigcap \mathcal{F}| \geq \kappa$.*

Lemma 2.2.4 *Let D be an infinite set with cardinality κ and let $\mathcal{A}$ be a set of at most κ subsets of D with the κ-uniform finite intersection property. There is a set $\mathcal{B}$ of κ pairwise disjoint subsets of D such that for each $B \in \mathcal{B}$, $\mathcal{A} \cup \{B\}$ has the κ-uniform finite intersection property.*

Proof. See the proof of Corollary 3.61 in [19]. $\square$

Theorem 2.2.37 *Let D be an infinite set with cardinality κ and let $\mathcal{A}$ be a set of at most κ subsets of D with the κ-uniform finite intersection property. Then*

$$|\{p \in U_\kappa(D) \,:\, \mathcal{A} \subseteq p\}| = 2^{2^\kappa}.$$

Proof. See the proof of Theorem 3.62 in [19]. $\square$

Because of the correspondence between ultrafilters and measures, we may view an element of an ultrafilter on a set X as a "large" subset of X. This leads to another connection with the infinite version of Ramsey Theory using ultrafilters for the proof of Ramsey's Theorem. We recall first the classical form of Ramsey's Theorem without proof (the classical proof does not use the notion of ultrafilter).

Theorem 2.2.38 *Whenever the edges of an infinite complete graph are colored with finitely many colors, there is an infinite complete monochromatic subgraph.*

One particular case of Ramsey's Theorem is the pigeon hole principle: "whenever $\mathbb{N}$ is partitioned into finitely many classes, one of those classes is infinite".

Definition 2.2.24 *Let X be a set and let κ be a cardinal number.*

1. $[X]^\kappa = \{A \subseteq X \,:\, |A| = \kappa\}$.
2. $[X]^{<\kappa} = \{A \subseteq X \,:\, |A| < \kappa\}$.

Lemma 2.2.5 *Let X be a set, let $p \in X^*$, let $k, r \in \mathbb{N}$, and let $[X]^k = \bigcup_{i=1}^{r} A_i$. For each $i \in \{1, \cdots, r\}$, each $t \in \{1, \cdots, k\}$, and each $E \in [X]^{t-1}$, define $B_t(E, i)$ by downward induction on t:*

1. *For each $E \in [X]^{k-1}$, $B_k(E, i) = \{y \in X \setminus E : E \cup \{y\} \in A_i\}$.*
2. *For $t \in \{1, 2, \cdots, k-1\}$ and $E \in [X]^{t-1}$,*

$$B_t(E, i) = \{y \in X \setminus E : B_{t+1}(E \cup \{y\}, i) \in p\}.$$

Then for each $t \in \{1, 2, \ldots, k\}$ and each $E \in [X]^{t-1}$, $X \setminus E = \bigcup_{i=1}^{r} B_t(E, i)$.

Proof. See the proof of Lemma 18.1 in [19].
The proof proceeds by downward induction on t. If $t = k$, then for each $y \in X \setminus E$. Since $E \in [X]^{k-1}$, one has $E \cup \{y\} \in [X]^k = \bigcup_{i=1}^{r} A_i$. So $E \cup \{y\} \in A_i$ for some i. Thus $y \in B_k(E, i)$ for some i.
So let $t \in \{1, 2, \ldots, k-1\}$ and let $E \in [X]^{t-1}$. Then given $y \in X \setminus E$, one has by the induction hypothesis that $X \setminus (E \cup \{y\}) = \bigcup_{i=1}^{r} B_{t+1}(E \cup \{y\}, i)$. Pick i such that $B_{t+1}(E \cup \{y\}, i) \in p$ since $E \cup \{y\} \in [X]^t$. That is $y \in B_t(E, i)$ for i. $\square$

Ramsey's Theorem can also be reformulated following [19] with the proof using the notion of nonprincipal ultrafilter.

Theorem 2.2.39 *Let X be an infinite set and let $k, r \in \mathbb{N}$. If $[X]^k = \bigcup_{i=1}^{r} A_i$, then there exist $i \in \{1, \cdots, r\}$ and an infinite subset B of X with $[B]^k \subseteq A_i$.*

Proof. See the proof of Theorem 18.2 in [19]. The case $k = 1$ is the pigeon hole principle, so assume that $k \geq 2$.
Let p be any nonprincipal ultrafilter on X. Define $B_t(E, i)$ as in the statement of Lemma 2.2.5. Then $X = \bigcup_{i=1}^{r} B_1(\emptyset, i)$. So there is some $i \in \{1, 2, \ldots, r\}$ such that $B_1(\emptyset, i) \in p$, that is, $B_1(\emptyset, i)$ is not empty. Pick $x_1 \in B_1(\emptyset, i)$ so that $B_2(\{x_1\}, i) \in p$.
Inductively, let $n \in \mathbb{N}$ and choose $\langle x_m \rangle_{m=1}^{n}$ such that if $t \in \{1, 2, \ldots, k-1\}$ and $m_1 < m_2 < \cdots < m_t \leq n$, one has

$$B_{t+1}(\{x_{m_1}, x_{m_2}, \ldots, x_{m_t}\}, i) \in p.$$

Set

$$\mathcal{B} = B_1(\emptyset, i) \setminus \{x_1, x_2, \ldots, x_n\}$$

and
$\mathcal{C} = \bigcap\{B_{t+1}(\{x_{m_1}, x_{m_2}, \ldots, x_{m_t}\}, i) \ : \ t \in \{1, 2, \ldots, k-1\}$ and $m_1 < m_2 < \cdots < m_t \leq n\}$.
Choose $x_{n+1} \in \mathcal{B} \cap \mathcal{C}$. To see that $B_{t+1}(\{x_{m_1}, x_{m_2}, \ldots, x_{m_t}\}, i) \in p$ whenever $t \in \{1, 2, \ldots, k-1\}$ and $m_1 < m_2 < \cdots \leq n+1$, assume that such t and $m_1, m_2, \ldots, m_t$ are indeed given. If $m_t \leq n$, then the induction hypothesis applies, so let $m_t = n+1$. If $t = 1$, the conclusion holds since $x_{n+1} \in B_1(\emptyset, i)$. If $t > 1$, the conclusion holds since $x_{n+1} \in B_t(\{x_{m_1}, x_{m_2}, \ldots, x_{m_{t-1}}\}, i)$.
The sequence $\langle x_n \rangle_{n=1}^{\infty}$ having been chosen, let $m_1 < m_2 < \cdots < m_k$. Then $x_{m_k} \in B_k(\{x_{m_1}, x_{m_2}, \ldots, x_{m_{k-1}}\}, i)$. So $\{x_{m_1}, x_{m_2}, \ldots, x_{m_k}\} \in A_i$.
□

More about generalizations of Ramsey's Theorem can be found in [19], Section 18.1. The following theorem draws a rather strong connection between Ramsey Theory and ultrafilters. Using Ramsey Theory language, the statement (1) can be also read as follow: "whenever X is finitely coloured, there is a part of X which is monochrome".

Theorem 2.2.40 *Let X be a set and let $\mathcal{G} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. The following statements are equivalent.*

1. *Whenever $r \in \mathbb{N}$ and $X = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $G \in \mathcal{G}$ such that $G \subseteq A_i$.*
2. *There is an ultrafilter p on X with the property that for each $A \in p$, there exists $G \in \mathcal{G}$ with $G \subseteq A$.*

Proof. See also the proof of Theorem 5.7 in [19].

(1) implies (2). Let $\mathcal{A} = \{B \subseteq X \ : \ \text{for every } G \in \mathcal{G}, B \cap G \neq \emptyset\}$. Then $\mathcal{A}$ has the finite intersection property. In fact, suppose the contrary, $\{B_1, B_2, \ldots, B_r\} \subseteq \mathcal{A}$ with $\bigcap_{i=1}^{r} B_i = \emptyset$. Then $X = \bigcup_{i=1}^{r} (X \setminus B_i)$. So there exist some $i \in \{1, 2, \ldots, r\}$ and some $G \in \mathcal{G}$ with $G \cap B_i = \emptyset$, which is a contradiction. Pick some ultrafilter p on X with $\mathcal{A} \subseteq p$. Then given any $A \in p$, $X \setminus A \notin \mathcal{A}$. So there is some $G \in \mathcal{G}$ with $G \cap (X \setminus A) = \emptyset$.

(2) implies (1). For some $i \in \{1, 2, \ldots, r\}$, $A_i \in p$.

□

The following theorem is called Schur's Theorem which can be considered as a consequence of Ramsey's Theorem even though it was first proved before Ramsey's Theorem [19].

Theorem 2.2.41 *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There there exist $i \in \{1, \cdots, r\}$ and x and y in $\mathbb{N}$ with $\{x, y, x+y\} \subseteq A_i$.*

Proof. See the comment in the proof of Theorem 5.3 in [19]. $\square$

Definition 2.2.25 *Given a set X. $\mathcal{P}_f(X)$ denotes the set of finite nonempty subsets of X.*

Definition 2.2.26 *Let S be an additive semigroup. Given an infinite sequence $(x_n)_{n=1}^{\infty}$ in S, the set defined by*

$$FS((x_n)_{n=1}^{\infty}) = \left\{ \sum_{n \in F} x_n \,:\, F \in \mathcal{P}_f(\mathbb{N}) \right\}$$

*is the set of **finite sums** of the sequence. In the same way, if S is a multiplicative semigroup then the set of **finite products** is defined by*

$$FP((x_n)_{n=1}^{\infty}) = \left\{ \prod_{n \in F} x_n \,:\, F \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

The following result called Hindman's Theorem or Finite Sums Theorem is the generalization of the Schur's Theorem [19].

Theorem 2.2.42 *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There there exist $i \in \{1, \cdots, r\}$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS((x_n)_{n=1}^{\infty}) \subseteq A_i$.*

A simple proof of this theorem using ultrafilters was given by Glazer and Galvin as a consequence of the fact that $\beta\mathbb{N}$ has an idempotent element. It was the first application of the algebraic structure of $\beta\mathbb{N}$ to Ramsey Theory. Later, a second simpler proof with fewer lines also using the algebraic structure of $\beta\mathbb{N}$ was given by Hindman. See [19] for more details.

A relatively easy consequence is obtained by substituting FS by FP in the previous theorem by defining a homomorphism $\varphi : (\mathbb{N}, +) \to (\mathbb{N}, \cdot)$ defined by $\varphi(n) = 2^n$.

Theorem 2.2.43 *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There there exist $i \in \{1, \cdots, r\}$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FP((x_n)_{n=1}^{\infty}) \subseteq A_i$.*

These two results raised a natural question as formulated in the following theorem.

Theorem 2.2.44 *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There there exist $i \in \{1, \cdots, r\}$ and sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $\mathbb{N}$ with*

$$FS((x_n)_{n=1}^{\infty}) \cup FP((y_n)_{n=1}^{\infty}) \subseteq A_i.$$

The proof of this theorem uses again the algebraic structure of $\beta\mathbb{N}$ by the fact that the closure of the set of idempotents in $(\beta\mathbb{N}, +)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$. Since then many questions arise from the algebraic structures of $\beta\mathbb{N}$ which most are notoriously very difficult due to the complex and strange behaviour of $\beta\mathbb{N}$. One may consider among others the research papers on the algebraic structure of $\beta\mathbb{N}$ in [4].

More of the algebraic properties of $\beta\mathbb{N}$ extend to βS, where S is any infinite discrete semigroup. We study in more details the algebraic structure of βS in the next chapters.

Chapter 3

The semigroup βS and its ideals

3.1 Algebraic structure on βS

Here the semigroup $(S, \cdot)$ is endowed with the discrete topology and S is embedded in βS. In the following results, the operation of S is extended to βS. We repeat the results from [19] with their proof, slightly modified for the sake of our work. See also [37].

Theorem 3.1.1 *Let S be a discrete space and let $\cdot$ be a binary operation defined on S. There is a unique binary operation $* : \beta S \times \beta S \to \beta S$ satisfying the following three conditions:*

1. *For every $s, t \in S$, $s * t = s \cdot t$.*
2. *For each $q \in \beta S$, the function $\rho_q : \beta S \to \beta S$ defined by $\rho_q(p) = p * q$ is continuous.*
3. *For each $s \in S$, the function $\lambda_s : \beta S \to \beta S$ defined by $\lambda_s(q) = s * q$ is continuous.*

Proof. Given $s \in S$, define $\ell_s : S \to S \subseteq \beta S$ by $\ell_s(t) = s \cdot t$. Then by Theorem 2.2.12(4), there is a continuous function $\lambda_s : \beta S \to \beta S$ such that $\lambda_{s|S} = \ell_s$. Then define $*$ on $S \times \beta S$ by $s * q = \lambda_s(q)$ for $s \in S$ and $q \in \beta S$. Then (3) holds and so does (1), since λ_s extends ℓ_s. Furthermore, the extension λ_s is unique since continuous functions agreeing on a dense

subspace are equal. So this is the only possible definition of $*$ satisfying (1) and (3).
To extend $*$ to the rest of $\beta S \times \beta S$, given $q \in \beta S$, define $r_q : S \to \beta S$ by $r_q(s) = s * q$. Then by Theorem 2.2.12(4), there is a continuous function $\rho_q : \beta S \to \beta S$ such that $\rho_{q|S} = r_q$. For $p \in \beta S \setminus S$, define $p*q = \rho_q(p)$ and note that if $s \in S$, $\rho_q(s) = r_q(s) = s * q$. So for all $p \in \beta S$, $\rho_q(p) = p * q$. Thus (2) holds. Again, by the uniqueness of continuous extensions, this is the only possible definition which satisfies the required conditions. $\square$

We denote the operation on βS by the same symbol used for the operation on S. The operation on S can be extended to βS using the following characterization.

Definition 3.1.1 *Let $\cdot$ be a binary operation on a discrete space S.*

1. *If $s \in S$ and $q \in \beta S$, then $s \cdot q = \lim\limits_{t \to q} s \cdot t$.*
2. *If $p, q \in \beta S$, then $p \cdot q = \lim\limits_{s \to p}(\lim\limits_{t \to q} s \cdot t)$, or*
3. *Let $p, q \in \beta S$, let $P \in p$ and $Q \in q$. Then $p \cdot q = p - \lim_{s \in P}(q - \lim_{t \in Q} s \cdot t)$.*

The proof of the following theorem is made longer with more details on purpose of using all the necessary properties of βS. However, the shorter version of this proof can be found in [19] and [37].

Theorem 3.1.2 *Let $(S, \cdot)$ be a semigroup. The extended operation of S to βS is associative.*

Proof. Let $p, q, r \in \beta S$. Consider $\lim\limits_{x \to p} \lim\limits_{y \to q} \lim\limits_{z \to r} (x \cdot y) \cdot z$, where $x, y, z \in S$. One has

$$\begin{aligned} \lim_{z \to r} (x \cdot y) \cdot z &= \lim_{z \to r} \lambda_{x \cdot y}(z) \\ &= \lambda_{x \cdot y}(\lim_{z \to r} z) \quad \text{(since } \lambda_{x \cdot y} \text{ is continuous)} \\ &= \lambda_{x \cdot y}(r) \\ &= (x \cdot y) \cdot r, \end{aligned}$$

$$\begin{aligned}
\lim_{y\to q}(x\cdot y)\cdot r &= \lim_{y\to q}(\lambda_x(y))\cdot r \\
&= \lim_{y\to q}\rho_r(\lambda_x(y)) \\
&= \lim_{y\to q}(\rho_r\circ\lambda_x)(y) \\
&= (\rho_r\circ\lambda_x)(\lim_{y\to q}y) \quad \text{(since } \rho_r\circ\lambda_x \text{ is continuous)} \\
&= (\rho_r\circ\lambda_x)(q) \\
&= \rho_r(\lambda_x(q)) \\
&= \rho_r(x\cdot q) \\
&= (xq)r,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to p}(x\cdot q)\cdot r &= \lim_{x\to p}\rho_r(\rho_q(x)) \\
&= \lim_{x\to p}(\rho_r\circ\rho_q)(x) \\
&= (\rho_r\circ\rho_q)(\lim_{x\to p}x) \quad \text{(since } \rho_r\circ\rho_q \text{ is continuous)} \\
&= (\rho_r\circ\rho_q)(p) \\
&= \rho_r(\rho_q(p)) \\
&= \rho_r(p\cdot q) \\
&= (p\cdot q)\cdot r.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to p}\lim_{y\to q}\lim_{z\to r}(x\cdot y)\cdot z &= \lim_{x\to p}\lim_{y\to q}(x\cdot y)\cdot r \\
&= \lim_{x\to p}(x\cdot q)\cdot r \\
&= (p\cdot q)\cdot r.
\end{aligned}$$

Also

$$
\begin{aligned}
\lim_{z\to r} x\cdot(y\cdot z) &= \lim_{z\to r} x\cdot(\lambda_y(z)) \\
&= \lim_{z\to r} \lambda_x(\lambda_y(z)) \\
&= \lim_{z\to r} (\lambda_x\circ\lambda_y)(z) \\
&= (\lambda_x\circ\lambda_y)(\lim_{z\to r} z) \quad \text{(since } \lambda_x\circ\lambda_y \text{ is continuous)} \\
&= (\lambda_x\circ\lambda_y)(r) \\
&= \lambda_x(\lambda_y(r)) \\
&= \lambda_x(y\cdot r) \\
&= x\cdot(y\cdot r),
\end{aligned}
$$

$$
\begin{aligned}
\lim_{y\to q} x\cdot(y\cdot r) &= \lim_{y\to q} x\cdot(\rho_r(y)) \\
&= \lim_{y\to q} \lambda_x(\rho_r(y)) \\
&= \lim_{y\to q} (\lambda_x\circ\rho_r)(y) \\
&= (\lambda_x\circ\rho_r)(\lim_{y\to q} y) \quad \text{(since } \lambda_x\circ\rho_r \text{ is continuous)} \\
&= (\lambda_x\circ\rho_r)(q) \\
&= \lambda_x(\rho_r(q)) \\
&= \lambda_x(q\cdot r) \\
&= x\cdot(q\cdot r),
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{x\to p} x\cdot(q\cdot r) &= \lim_{x\to p} \rho_{q\cdot r}(x) \\
&= \rho_{q\cdot r}(\lim_{x\to p} x) \quad \text{(since } \rho_{q\cdot r} \text{ is continuous)} \\
&= \rho_{q\cdot r}(p) \\
&= p\cdot(q\cdot r).
\end{aligned}
$$

So,

$$
\begin{aligned}
\lim_{x\to p}\lim_{y\to q}\lim_{z\to r} x\cdot(y\cdot z) &= \lim_{x\to p}\lim_{y\to q} x\cdot(y\cdot r) \\
&= \lim_{x\to p} x\cdot(q\cdot r) \\
&= p\cdot(q\cdot r).
\end{aligned}
$$

Furthermore, since the operation $\cdot$ is associative on S,

$$
(p\cdot q)\cdot r = \lim_{x\to p}\lim_{y\to q}\lim_{z\to r}(x\cdot y)\cdot z = \lim_{x\to p}\lim_{y\to q}\lim_{z\to r} x\cdot(y\cdot z) = p\cdot(q\cdot r).
$$

Hence, the extended operation is associative on βS. $\square$

As a consequence of the two theorems above, we conclude that $(\beta S, \cdot)$ is a compact right topological semigroup making then all the previous results in Section 1.2 apply here.

Lemma 3.1.1 *Let K and T be compact right topological semigroups, let S be a dense subsemigroup of K such that $S \subseteq \Lambda(K)$, and let $\varphi : K \to T$ be a continuous mapping such that*

1. *$\varphi[S] \subseteq \Lambda(T)$ and*
2. *$\varphi_{|S}$ is a homomorphism.*

Then φ is a homomorphism.

Proof. See the proof of Lemma 6.3 in [37].

First let $x \in S$ and $q \in K$. Then

$$\begin{array}{rcll}
\varphi(xq) & = & \varphi(\lambda_x(q)) & \text{by definition of } \lambda_x \\
 & = & \varphi(\lambda_x(\lim_{y \to q} y)) & \text{where } y \in S \\
 & = & \varphi(\lim_{y \to q} \lambda_x(y)) & \text{because } \lambda_x \text{ is continuous} \\
 & = & \varphi(\lim_{y \to q} xy) & \text{by definition of } \lambda_x \\
 & = & \lim_{y \to q} \varphi(xy) & \text{because } \varphi \text{ is continuous} \\
 & = & \lim_{y \to q} \varphi(x)\varphi(y) & \text{because } \varphi_{|S} \text{ is a homomorphism} \\
 & = & \lim_{y \to q} \lambda_{\varphi(x)}\varphi(y) & \text{by definition of } \lambda_{\varphi(x)} \\
 & = & \varphi(x) \lim_{y \to q} \varphi(y) & \text{because } \lambda_{\varphi(x)} \text{ is continuous} \\
 & = & \varphi(x)\varphi(q) & \text{because } \varphi \text{ is continuous.}
\end{array}$$

Now let $p, q \in K$. Then

$$\begin{array}{rcll}
\varphi(pq) & = & \varphi(\rho_q(p)) & \text{by definition of } \rho_q \\
& = & \varphi(\rho_q(\lim\limits_{x\to p} x)) & \text{where } x \in S \\
& = & \varphi(\lim\limits_{x\to p} \rho_q(x)) & \text{because } \rho_q \text{ is continuous} \\
& = & \varphi(\lim\limits_{x\to p} xq) & \text{by definition of } \rho_q \\
& = & \lim\limits_{x\to p} \varphi(xq) & \text{because } \varphi \text{ is continuous} \\
& = & \lim\limits_{x\to p} \varphi(x)\varphi(q) & \text{because } \varphi(xq) = \varphi(x)\varphi(q) \\
& = & \lim\limits_{x\to p} \rho_{\varphi(q)}(\varphi(x)) & \text{by definition of } \rho_{\varphi(q)} \\
& = & \rho_{\varphi(q)}(\lim\limits_{x\to p} \varphi(x)) & \text{because } \rho_{\varphi(q)} \text{ is continuous} \\
& = & (\lim\limits_{x\to p} \varphi(x))\varphi(q) & \text{by definition of } \rho_{\varphi(q)} \\
& = & \varphi(\lim\limits_{x\to p} x)\varphi(q) & \text{because } \varphi \text{ is continuous} \\
& = & \varphi(p)\varphi(q). &
\end{array}$$

□

Theorem 3.1.3 *Let S be a discrete semigroup, and let $\iota : S \to \beta S$ be the inclusion map.*

1. *$(\iota, \beta S)$ is a semigroup compactification of S.*
2. *If T is a compact right topological semigroup and $\varphi : S \to T$ is a continuous homomorphism with $\varphi[S] \subseteq \Lambda(T)$, then there is a continuous homomorphism $\tilde{\varphi} : \beta S \to T$ such that $\tilde{\varphi}_{|S} = \varphi$ making the following diagram commutative.*

$$\begin{tikzcd}
\beta S \arrow[dr, "\tilde{\varphi}"] & \\
S \arrow[u, "\iota"] \arrow[r, "\varphi"'] & T
\end{tikzcd}$$

Proof.

1. According to Theorem 2.2.11, $(\iota, \beta S)$ is a compactification of S. From Theorem 3.1.1 and Theorem 3.1.2, βS is a semigroup, since S is a semigroup.

2. Since T is a compact right topological semigroup and φ a continuous function then, by Theorem 2.2.12(4), pick $\tilde{\varphi} : \beta S \to T$ a continuous function such that $\tilde{\varphi}_{|S} = \varphi$. By Theorem 2.2.12(3), S is a dense subsemigroup of βS and, by Theorem 3.1.1(3), $S \subseteq \Lambda(\beta S)$. Furthermore, since $\tilde{\varphi}_{|S} = \varphi$ is a homomorphism and clearly $\tilde{\varphi}[S] = \varphi[S] \subseteq \Lambda(T)$ so , by Lemma 3.1.1, $\tilde{\varphi}$ is a homomorphism.

□

Definition 3.1.2 *Let S be a multiplicative semigroup, let $A \subseteq S$, and let $s \in S$.*

1. $s^{-1}A = \{t \in S : st \in A\}$
2. $As^{-1} = \{t \in S : ts \in A\}$

Note that $s^{-1}A$ is simply an alternative notation for $\lambda_S^{-1}[A]$.

Definition 3.1.3 *Let S be a semigroup. For every $A, B \subseteq S$,*

$$A^{-1}B = \bigcup_{x \in A} x^{-1}B \text{ and } AB^{-1} = \bigcup_{x \in B} Ax^{-1}$$

Definition 3.1.4 *Let S be an additive semigroup, let $A \subseteq S$, and let $s \in S$.*

1. $-s + A = \{t \in S : s + t \in A\}$
2. $A - s = \{t \in S : t + s \in A\}$

Theorem 3.1.4 *Let S be a semigroup and let $A \subseteq S$.*

1. *For any $s \in S$ and $q \in \beta S$, $A \in sq$ if and only if $s^{-1}A \in q$*
2. *For any $p, q \in \beta S$, $A \in pq$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$.*

Proof. See the proof of Theorem 4.12 in [19] and the proof of Lemma 6.6 in [37].

1. Necessity. Let $A \in sq$. Then $\overline{A}$ is a neighborhood of $sq = \lambda_s(q)$. So, since λ_s is continuous, pick $Q \in q$ such that $\lambda_s[\overline{Q}] \subseteq \overline{A}$. Since $Q \subseteq s^{-1}A$ and $Q \in q$, then $s^{-1}A \in q$.
   Sufficiency. Assume $s^{-1}A \in q$ and suppose on the contrary that $A \notin sq$. Consequently $S \setminus A \in sq$. Then, from the necessity above, $s^{-1}(S \setminus A) \in q$. Then,
   $$\emptyset = s^{-1}A \cap s^{-1}(S \setminus A) \in q,$$
   that is, $\emptyset \in q$. This is a contradiction since q is an ultrafilter on S. Thus, we have indeed $A \in sq$.

2. Necessity. Let $A \in pq$. Then $\overline{A}$ is a neighborhood of $pq = \rho_q(p)$. Since ρ_q is continuous, there is $P \in p$ such that $\rho_q[\overline{P}] \subseteq \overline{A}$. Then for every $s \in P$, $A \in sq$ and so by (1), $s^{-1}A \in q$. Hence, $\{s \in S : s^{-1}A \in q\} \in p$.
   Sufficiency. Let $\{s \in S : s^{-1}A \in q\} \in p$. Suppose on the contrary that $A \notin pq$. Consequently, $S \setminus A \in pq$. Then, from the necessity above, $\{s \in S : s^{-1}(S \setminus A) \in q\} \in p$. But $(s^{-1}A) \cap (s^{-1}(S \setminus A)) = \emptyset$ for each $s \in S$. Then
   $$\{s \in S : s^{-1}A \in q\} \cap \{s \in S : s^{-1}(S \setminus A) \in q\} = \emptyset.$$
   That is, $\emptyset \in p$ which is a contradiction since q is an ultrafilter on S. Thus, we have indeed $A \in pq$.

$\square$

Corollary 3.1.1 *Let S be a semigroup, $s \in S$ and $p, q \in \beta S$. Then*

1. *the ultrafilter sq has a base consisting of subsets of the form sQ where $Q \in q$, and*

2. *the ultrafilter pq has a base consisting of subsets of the form $\bigcup_{s \in P} sQ_s$ where $P \in p$ and $Q_s \in q$.*

Theorem 3.1.5 *Let S be a semigroup, let $s \in S$, let $q \in \beta S$, and let $A \subseteq S$.*

1. *If $A \in q$, then $sA \in sq$.*

2. *If S is left cancellative and $sA \in sq$ then $A \in q$.*

Proof. See the proof of Lemma 4.16 in [19].

1. Let $A \subseteq S$ and $s \in S$. One has $A \subseteq s^{-1}(sA)$. Since $A \in q$, one has $s^{-1}(sA) \in q$. Using Theorem 3.1.4(1), $sA \in sq$.

2. Let S be left cancellative. Then $s^{-1}(sA) = A$ and the condition (1) above implies $A \in q$.

$\square$

Theorem 3.1.6 *Let S be a discrete semigroup, let (φ, T) be a semigroup compactification of S, and let $A \subseteq B \subseteq S$. Suppose that B is a subsemigroup of S.*

1. *$c\ell(\varphi[B])$ is a subsemigroup of T.*

2. *If A is a left ideal of B, then $c\ell(\varphi[A])$ is a left ideal of $c\ell(\varphi[B])$.*

3. *If A is a right ideal of B, then $c\ell(\varphi[A])$ is a right ideal of $c\ell(\varphi[B])$.*

Proof. See the proof of Theorem 4.17 in [19].

1. Since φ is an inclusion map and T is a compact semigroup, one needs to use Exercise 2.3.2 in [19].

2. Suppose that A is an ideal of B. Let $x \in c\ell(\varphi[B])$ and $y \in c\ell(\varphi[A])$. Then
$$xy = \lim_{\varphi(s)\to x} \lim_{\varphi(t)\to y} \varphi(s)\varphi(t),$$
where $s \in B$ and $t \in A$. Since $s \in B$ and $t \in A$, then $st \in A$. Thus $\varphi(s)\varphi(t) = \varphi(st) \in \varphi[A]$ (φ is a homomorphism) and so $xy \in c\ell(\varphi[A])$.

3. The proof of (3) is similar to (2).

$\square$

Corollary 3.1.2 *Let S be a subsemigroup of the discrete semigroup T. Then $c\ell(S)$ is a subsemigroup of βT. If S is a right or left ideal of T, then $c\ell(S)$ is respectively a right or left ideal of βT.*

Proof. See the proof of Corollary 4.18 in [19]. The proof follows from Theorem 3.1.6 since, by Theorem 3.1.3, $(\iota, \beta T)$ is a semigroup compactification. $\square$

Remark 3.1.1 *If S is a subsemigroup of a discrete semigroup T, then βS is also a subsemigroup of βT. For example, $\beta\mathbb{N}$ is a subsemigroup of $\beta\mathbb{Z}$.*

Theorem 3.1.7 *Let S be a semigroup and $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property. If for each $A \in \mathcal{A}$ and each $x \in A$, there exists $B \in \mathcal{A}$ such that $xB \subseteq A$, then $\bigcap_{A\in\mathcal{A}} c\ell_{\beta S}(A)$ is a subsemigroup of βS.*

Proof. See the proof of Theorem 4.20 in [19].
Let $T = \bigcap_{A\in\mathcal{A}} c\ell_{\beta S}(A)$. Since $\mathcal{A}$ has the finite intersection property, $T \neq \emptyset$. Let $p, q \in T$ and let $A \in \mathcal{A}$. Given $x \in A$, there is some $B \in \mathcal{A}$ such that $xB \in A$ and hence $x^{-1}A \in q$. Thus $A \subseteq \{x \in S : x^{-1}A \in q\}$ so $\{x \in S : x^{-1}A \in q\} \in p$. By Theorem 3.1.5(2), $A \in p \cdot q$. $\square$

Theorem 3.1.8 *Let $(S, \cdot)$ be a semigroup and let $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property. Let $(T, \cdot)$ be a compact right topological semigroup and let $\varphi : S \to T$ satisfy $\varphi[S] \subseteq \Lambda(T)$. Assume there is some $A \in \mathcal{A}$ such that for each $x \in A$, there is $B \in \mathcal{A}$ for which $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ for every $y \in B$. Then for all $p, q \in \bigcap_{A\in\mathcal{A}} c\ell_{\beta S}(A)$, $\tilde{\varphi}(p \cdot q) = \tilde{\varphi}(p) \cdot \tilde{\varphi}(q)$.*

Proof. Let $p, q \in \bigcap_{A\in\mathcal{A}} c\ell_{\beta S}(A)$. For each $x \in A$, one has

$$\begin{aligned}
\tilde{\varphi}(x \cdot q) &= \tilde{\varphi}(x \cdot \lim_{y\to q} y) \\
&= \lim_{y\to q} \varphi(x \cdot y) && \text{because } \tilde{\varphi} \circ \lambda_x \text{ is continuous} \\
&= \lim_{y\to q} \varphi(x) \cdot \varphi(y) && \text{because } \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \text{ on a member of } q \\
&= \varphi(x) \cdot \lim_{y\to q} \varphi(y) && \text{because } \varphi(x) \in \Lambda(T) \\
&= \varphi(x) \cdot \tilde{\varphi}(q).
\end{aligned}$$

Since $A \in p$ one then has

$$
\begin{aligned}
\tilde{\varphi}(p \cdot q) &= \tilde{\varphi}((\lim_{x \to p} x) \cdot q) \\
&= \lim_{x \to p} \tilde{\varphi}(x \cdot q) && \text{because } \tilde{\varphi} \circ \rho_q \text{ is continuous} \\
&= \lim_{x \to p} (\varphi(x) \cdot \tilde{\varphi}(q)) \\
&= (\lim_{x \to p} \varphi(x)) \cdot \tilde{\varphi}(q) && \text{by continuity of } \rho_{\tilde{\varphi}(q)} \\
&= \tilde{\varphi}(p) \cdot \tilde{\varphi}(q).
\end{aligned}
$$

□

Corollary 3.1.3 *Let S be a semigroup and let $\varphi : S \to T$ be a homomorphism to a compact right topological semigroup T such that $\varphi[S] \subseteq \Lambda(T)$. Then $\tilde{\varphi}$ is a homomorphism from βS to T.*

Proof. See Proof of Theorem 3.1.8 by taking $\mathcal{A} = \{S\}$. □

Theorem 3.1.9 *Let S be a semigroup. Then S^* is a subsemigroup of βS if and only if for any $A \in \mathcal{P}_f(S)$ and for any infinite subset B of S there exists $F \in \mathcal{P}_f(B)$ such that $\bigcap_{x \in F} x^{-1}A$ is finite.*

Proof. See the proof of Theorem 4.28 in [19].
Necessity. Let $A \in \mathcal{P}_f(S)$ and B an infinite subset of S. Suppose that for each $F \in \mathcal{P}_f(B)$, $\bigcap_{x \in F} x^{-1}A$ is infinite. Then $\{x^{-1}A : x \in B\}$ has the property that all its finite intersections are infinite. So, by Corollary 1.4.1(1), we may pick $p \in S^*$ such that $\{x^{-1}A : x \in B\} \subseteq p$. Pick $q \in S^*$ such that $B \in q$. Then $A \in qp$ and A is finite so by Theorem 1.4.3, $qp \in S$. That is $qp \notin S^*$, a contradiction (since S^* is assumed a subsemigroup of βS).
Sufficiency. Let $p, q \in S^*$ be given and suppose that $qp \notin S^*$. That is $qp \in S$. Let $y \in S$ such that $qp = y$ (qp is the principal ultrafilter generated by y). Take $A = \{y\}$ and $B = \{x \in S : x^{-1}A \in p\}$. Then $B \in q$ while for each $F \in \mathcal{P}_f(B)$, one has $\bigcap_{x \in F} x^{-1}A \in p$ so that $\bigcap_{x \in F} x^{-1}A$ is infinite, a contradiction (since A was taken finite). □

Corollary 3.1.4 *Let S be a semigroup. If S is either right or left cancellative then S^* is a subsemigroup of βS.*

Proof. It is obvious that $\emptyset \neq S^*$ is a subset of βS. Let $p, q \in S^*$. To see that $pq \in S^*$, let $A \in pq$. By Theorem 3.1.4 and that S is left cancellative, one has that A is infinite. □

Definition 3.1.5 *Let S be a semigroup.*

1. *S is said to be **weakly left cancellative** if and only if for all $u, v \in S$, $\{x \in S : ux = v\}$ is finite.*

2. *Similarly, S is said to be **weakly right cancellative** if and only if for all $u, v \in S$, $\{x \in S : xu = v\}$ is finite.*

Theorem 3.1.10 *Let S be an infinite semigroup. Then S^* is a left ideal of βS if and only if S is weakly left cancellative.*

Proof. See the proof of Theorem 4.31 in [19].
Necessity. Let $x, y \in S$ be given, let $A = y^{-1}x$ and suppose that A is infinite. Pick $p \in S^* \cap c\ell(A)$. Then $y \cdot p = x \in S$, a contradiction.
Sufficiency. Since S is infinite, $S^* \neq \emptyset$. Let $p \in S^*$, let $q \in \beta S$ and suppose that $q \cdot p \notin S^*$. That is $q \cdot p \in S$. Pick $x \in S$ such that $q \cdot p = x$. Then $\{x\} \in q \cdot p$ so $\{y \in S : y^{-1}\{x\} \in p\} \in q$ and is hence nonempty. So pick $y \in S$ such that $y^{-1}\{x\} \in p$. But $y^{-1}\{x\} = \lambda_y^{-1}[\{x\}]$ so $\lambda_y^{-1}[\{x\}]$ is infinite, a contradiction. □

Lemma 3.1.2 *Let $(S, \cdot)$ be a semigroup, let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences in S, and let $p, q \in \beta S$. If $\{\{x_n : n > k\} : k \in \mathbb{N}\} \subseteq q$ and $\{y_k : k \in \mathbb{N}\} \in p$, then $\{y_k \cdot x_n : k, n \in \mathbb{N}$ and $k < n\} \in p \cdot q$.*

Proof. It is a consequence of Corollary 3.1.1(2). □

Theorem 3.1.11 *Let S be an infinite semigroup. The following statements are equivalent.*

1. *S^* is a right ideal of βS.*

2. *Given any finite subset A of S, any sequence $(z_n)_{n=1}^{\infty}$ in S, and any one-to-one sequence $(x_n)_{n=1}^{\infty}$ in S, there exist $n < m$ in $\mathbb{N}$ such that $x_n \cdot z_m \notin A$.*

3. *Given any $a \in S$, any sequence $(z_n)_{n=1}^{\infty}$ in S, and any one-to-one sequence $(x_n)_{n=1}^{\infty}$ in S, there exist $n < m$ in $\mathbb{N}$ such that $x_n \cdot z_m \neq a$.*

Proof. See the proof of Corollary 4.32 in [19].

(1) implies (2). Suppose $\{x_n \cdot z_m : n, m \in \mathbb{N} \text{ and } n < m\} \subseteq A$. Pick $p \in \beta S$ such that $\{\{z_m : m > n\} : n \in \mathbb{N}\} \subseteq p$ and pick $q \in S^*$ such that $\{x_n : n \in \mathbb{N}\} \in q$ since $\{x_n : n \in \mathbb{N}\}$ is infinite. Then by Lemma 3.1.2, $A \in q \cdot p$ so by Theorem 1.4.3, $q \cdot p \in S$. That is $q \cdot p \notin S^*$, a contradiction.

(2) implies (3). Take $A = \{a\}$ and use (2).

(3) implies (1). Since S is infinite, $S^* \neq \emptyset$. Let $p \in \beta S$, let $q \in S^*$, and suppose that $q \cdot p = a \in S$. Then $\{s \in S : s^{-1}\{a\} \in p\} \in q$ so choose a one-to-one sequence $(x_n)_{n=1}^{\infty}$ such that

$$\{x_n : n \in \mathbb{N}\} \subseteq \{s \in S : s^{-1}\{a\} \in p\}.$$

Inductively choose a sequence $(z_n)_{n=1}^{\infty}$ in S such that for each $m \in \mathbb{N}$, $z_m \in \bigcap_{n=1}^{m} x_n^{-1}\{a\}$ since $\bigcap_{n=1}^{m} x_n^{-1}\{a\} \in p$. Then for each $n < m$ in $\mathbb{N}$, $x_n \cdot z_m = a$, a contradiction.

□

Corollary 3.1.5 *Let S be an infinite semigroup.*

1. *If S is left cancellative, then S^* is a left ideal of βS.*
2. *If S is right cancellative, then S^* is a right ideal of βS.*

Proof. See the proof of Lemma 6.8 in [37].

To prove (1), suppose that S is left cancellative. Let $p \in \beta S$ and $q \in S^*$. To see that $pq \in S^*$, let $A \in pq$. Then there exist $x \in S$ and $B \in q$ such that $xB \subseteq A$. Since S is cancellative and B is infinite, it follows that A is infinite. So pq is not principal.

To prove (2), suppose that S is right cancellative. Let $p \in S^*$ and $q \in \beta S$. Assume on the contrary that $pq = a \in S$. Then there exist $A \in p$ and, for each, $x \in A$, $B_x \in q$ such that $xB_x = \{a\}$. Pick distinct $x, y \in A$ and any $z \in B_x \cap B_y$. Then $xz = a = yz$. Since S is right cancellative, $x = y$ which is a contradiction. So $pq \notin S$ for any $p \in S^*$ and $q \in \beta S$. Hence indeed $pq \in (\beta S \setminus S) = S^*$.

□

Corollary 3.1.6 *Let S be an infinite semigroup. If S^* is a right ideal of βS, then S is weakly right cancellative.*

Proof. See also the proof of Corollary 4.34 in [19].
Let $a, y \in S$ and suppose that $\rho_y^{-1}[\{a\}]$ is infinite. Choose a one-to-one sequence $(x_n)_{n=1}^{\infty}$ in $\rho_y^{-1}[\{a\}]$ and for each $m \in \mathbb{N}$ let $z_m = y$. Then $(z_n)_{n=1}^{\infty}$ is a sequence in S such that $x_n \cdot z_m = a$ for all $n < m$ in $\mathbb{N}$, a contradiction. $\square$

Corollary 3.1.7 *Let S be an infinite semigroup. If S is weakly right cancellative and for all but finitely many $y \in S$, λ_y is finite-to-one, then S^* is a right ideal of βS.*

Proof. This is the proof from Corollary 4.35 in [19].
Suppose S^* is not a right ideal of βS and pick by Theorem 3.1.11(3) some $a \in S$, a one-to-one sequence $(x_n)_{n=1}^{\infty}$ in S, and a sequence $(z_n)_{n=1}^{\infty}$ in S such that $x_n \cdot z_m = a$ for all $n < m$ in $\mathbb{N}$. Now if $\{z_m : m \in \mathbb{N}\}$ is infinite, then for all $n \in \mathbb{N}$, $\lambda_{x_n^{-1}}[\{a\}]$ is infinite, a contradiction. Thus $\{z_m : m \in \mathbb{N}\}$ is finite so we may pick b such that $\{m \in \mathbb{N} : z_m = b\}$ is infinite. Then given $n \in \mathbb{N}$ one may pick $m > n$ such that $z_m = b$ and conclude that $x_n \cdot b = a$. Consequently, $\rho_b^{-1}[\{a\}]$ is infinite, a contradiction. $\square$

Theorem 3.1.12 *Let S be an infinite semigroup. Then S^* is an ideal of βS if and only if S is both weakly left cancellative and weakly right cancellative.*

Proof. See the comments in the proof of Theorem 4.36 in [19]. $\square$

It will be sometimes easier to work with S^* instead of βS because of the following result.

Theorem 3.1.13 *Let S be an infinite semigroup. If S^* is an ideal of βS, then the minimal left ideals, the minimal right ideals, and the smallest ideals of S^* and βS are the same.*

Proof. This is the proof of Theorem 4.37 in [19].
First assume L is a minimal left ideal of βS. Since S^* is an ideal of βS, then $L \subseteq S^*$ and hence L is a left ideal of S^*. To see that L is a minimal left ideal of S^*, let L' be a left ideal of S^* with $L' \subseteq L$. Then by Lemma

1.1.8(2), $L' = L$.
Now assume L is a minimal left ideal of S^*. Then by Lemma 1.1.8(c), L is a left ideal of βS. If L' is a left ideal of βS with $L' \subseteq L$, then L' is a left ideal of S^* and consequently $L' = L$.
The proof for minimal right ideals is established the same way. Since the smallest ideal is the union of minimal left ideals (and of all minimal right ideals), the proof is completed. $\square$

Definition 3.1.6 *Let S be a semigroup.*

1. *A set $A \subseteq S$ is **right syndetic** if and only if there exists some $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in G} t^{-1}A$.*

2. *A set $A \subseteq S$ is **right piecewise syndetic** if and only if there is some $G \in \mathcal{P}_f(S)$ such that $S = \{a^{-1}(\bigcup_{t\in G} t^{-1}A) : a \in S\}$ has a finite intersection property.*

Theorem 3.1.14 *Let S be a semigroup and let $p \in \beta S$. The following statements are equivalent.*

1. *$p \in K(\beta S)$.*

2. *For all $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is right syndetic.*

3. *For all $q \in \beta S$, $p \in \beta S \cdot q \cdot p$.*

Proof.

(1) implies (2). Let $A \in p$ and let $B = \{x \in S : x^{-1}A \in p\}$. Let L be the minimal left ideal of βS for which $p \in L$. For every $q \in L$, we have $p \in \beta S \cdot q = c\ell_{\beta S}(S \cdot q)$. Since $\overline{A}$ is a neighborhood of p in βS, we have $t \cdot q \in \overline{A}$ for some $t \in S$ and so $q \in \overline{t^{-1}A}$. Thus the sets of the form $\overline{t^{-1}A}$ cover the compact set L and hence $L \subseteq \bigcup_{t\in G} \overline{t^{-1}A}$ for some finite subset G of S.
To see that $S \subseteq \bigcup_{t\in G} t^{-1}B$, let $a \in S$. Then $a \cdot p \in L$ so pick $t \in G$ such that $a \cdot p \in \overline{t^{-1}A}$. Then $t^{-1}A \in a \cdot p$ so that $(ta)^{-1}A = a^{-1}t^{-1}A \in p$ and so $ta \in B$ and thus $a \in t^{-1}B$.

(2) implies (3). Let $q \in \beta S$ and suppose that $p \notin \beta S \cdot q \cdot p$. Pick $A \in p$ such that $\overline{A} \cap \beta S \cdot q \cdot p = \emptyset$. Let $B = \{x \in S : x^{-1}A \in p\}$ and pick $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}B$. Pick $t \in G$ such that $t^{-1}B \in q$. Then $B \in tq$. that is, $\{x \in S : x^{-1}A \in p\} \in tq$ so $A \in tqp$, a contradiction.

(3) implies (1). This is Theorem 1.1.19. In fact, let $q \in K(\beta S)$. Then $p \in \beta S \cdot q \cdot p \subseteq \beta S \cdot q \cdot \beta S \subseteq K(\beta S)$.

$\square$

Theorem 3.1.15 *Let S be a semigroup and let $A \subseteq S$. Then $\overline{A} \cap K(\beta S) \neq \emptyset$ if and only if A is right piecewise syndetic.*

Proof. Let $p \in \overline{A} \cap K(\beta S)$ and let $B = \{x \in S : x^{-1}A \in p\}$. Then by Theorem 3.1.14, B is syndetic and so $S = \bigcup_{t \in G} t^{-1}B$ for some $G \in \mathcal{P}_f(S)$. For each $a \in S$, $a \in t^{-1}B$ for some $t \in G$ and so $a^{-1}(t^{-1}A) = (ta)^{-1}A \in p$. It follows that $a^{-1}(\bigcup_{t \in G} t^{-1}A) \in p$ and hence $\{a^{-1}(\bigcup_{t \in G} t^{-1}A) : a \in S\}$ has the finite intersection property.
Sufficiency. Assume that A is right piecewise syndetic and pick $G \in \mathcal{P}_f(S)$ such that $\{a^{-1}(\bigcup_{t \in G} t^{-1}A) : a \in S\}$ has the finite intersection property. Pick $q \in \beta S$ such that $\{a^{-1}(\bigcup_{t \in G} t^{-1}A) : a \in S\} \subseteq q$. That is, $\bigcup_{t \in G} t^{-1}A \in a \cdot q \subseteq S \cdot q$. Then $S \cdot q \subseteq \overline{\bigcup_{t \in G} t^{-1}A}$. This implies that $(\beta S) \cdot q \subseteq \overline{\bigcup_{t \in G} t^{-1}A}$. We can choose $y \in K(\beta S) \cap (\beta S \cdot q)$. Then $y \in \overline{t^{-1}A}$ for some $t \in G$ and so $t \cdot y \in \overline{A} \cap K(\beta S)$. $\square$

Corollary 3.1.8 *Let S be a semigroup and let $p \in \beta S$. Then $p \in c\ell(K(\beta S))$ if and only if every $A \in p$ is right piecewise syndetic.*

Proof. This is an immediate consequence of Theorem 3.1.15. $\square$

Definition 3.1.7 *Let S be a semigroup and let $A \subseteq S$. Then A is* ***central*** *in S if and only if there exists $p \in E(K(\beta S))$ such that $A \in p$.*

Theorem 3.1.16 *Let S be a semigroup and let $A \subseteq S$. The following statements are equivalent:*

1. *A is right piecewise syndetic.*
2. *The set $\{x \in S : x^{-1}A$ is central$\}$ is right syndetic.*

3. There is some $x \in S$ such that $x^{-1}A$ is central.

Proof. See the proof of Theorem 4.43 in [19].

> (1) implies (2). By Theorem 3.1.15, pick some $p \in K(\beta S)$ with $A \in p$. Now $K(\beta S)$ is the union of all minimal left ideals of βS, by Theorem 1.3.3. So pick a minimal left ideal L of βS with $p \in L$ and pick an idempotent $e \in L$. Then $p = p \cdot e$ and so $A \in p \cdot e$. By Theorem 3.1.4(2), pick $y \in S$ such that $y^{-1}A \in e$.
> Now by Theorem 3.1.14, $B = \{z \in S : z^{-1}(y^{-1}A) \in e\}$ is right syndetic, so pick finite $G \subseteq S$ such that $S = \bigcup_{t \in G} t^{-1}B$. Let $D = \{x \in S : x^{-1}A \text{ is central }\}$. We claim that $S = \bigcup_{t \in y \cdot G} t^{-1}D$. Indeed, let $x \in S$ be given and pick $t \in G$ such that $t \cdot x \in B$. Then $(t \cdot x)^{-1}(y^{-1}A) \in e$ so, by Definition 3.1.7, $(t \cdot x)^{-1}(y^{-1}A)$ is central. But $(t \cdot x)^{-1}(y^{-1}A) = (y \cdot t \cdot x)^{-1}A$. Thus $y \cdot t \cdot x \in D$ so $x \in (y \cdot t)^{-1}D$ as required.
>
> (2) implies (3). Let $B = \{x \in S : x^{-1}A \text{ is central}\}$. We can find some $t \in G$ with G a finite subset of S such that $t \cdot x \in B$. Indeed $B \neq \emptyset$.
>
> (3) implies (1). Pick $x \in S$ such that $x^{-1}A$ is central and pick an idempotent $p \in K(\beta S)$ such that $x^{-1}A \in p$. Then $A \in x \cdot p$ and $x \cdot p \in K(\beta S)$ so, by Theorem 3.1.15, A is right piecewise syndetic.

□

Theorem 3.1.17 *Let S be a discrete semigroup and let T be a subsemigroup of S. Then T is central if $c\ell_{\beta S}(T) \cap K(\beta S) \neq \emptyset$.*

Proof. See Exercise 4.4.8 in [19]. □

3.2 The semigroup $\beta\mathbb{N}$ and applications

We consider the particular case $S = \mathbb{N}$. One can then see that the additive operation on $\mathbb{N}$ can be extended to $\beta\mathbb{N}$ as follows. For all $p, q \in \beta\mathbb{N}$,

$$A \in p + q \Longleftrightarrow \{n \in \mathbb{N} : -n + A \in q\} \in p.$$

This operation is indeed closed and associative in $\beta\mathbb{N}$. And for each $p \in \beta\mathbb{N}$, the map given by $q \mapsto p+q$ from $\beta\mathbb{N}$ to itself is continuous. The same conclusions are true for the multiplicative operation. So $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ are then compact right topological semigroups. As a consequence they admit idempotent elements. That is, there exist $p \in \beta\mathbb{N}$ with $p+p = p$.

The algebraic structure of $\beta\mathbb{N}$ is complicated and many applications rely in the understanding of $\beta\mathbb{N}$. It was showed in [4] that if the map $\phi : \beta\mathbb{N} \to \mathbb{N}^*$ is a continuous homomorphism, then the direct image $\phi[\beta\mathbb{N}]$ is finite. As a consequence, $\mathbb{N}^*$ does not contain an algebraic copy of $\beta\mathbb{N}$. Nonetheless, the algebraic structure of $\beta\mathbb{N}$ is very rich as we will see it with the following applications.

3.2.1 Hindman's Theorem

We now review the proof of Hindman's Theorem using the algebraic structure of $\beta\mathbb{N}$.

The following lemma is an exercise in [19], see Exercise 4.1.4, where a caution is given that even though very easy, the reason of the result is not because of $s^{-1}t^{-1} = (ts)^{-1}$.

Lemma 3.2.1 *Let S be a semigroup, let $s, t \in S$, and let $A \subseteq S$. Then $s^{-1}(t^{-1}A) = (ts)^{-1}A$.*

Proof. Let $x \in s^{-1}(t^{-1}A)$. By Definition 3.1.2, $sx \in t^{-1}A$ and also $(ts)x = t(sx) \in A$. That is $x \in (ts)^{-1}A$. The implication is sufficient. $\square$

Definition 3.2.1 *Let $A \subseteq \mathbb{N}$ and let $p \in \beta\mathbb{N}$. Then*

$$A^*(p) = \{n \in A : -n + A \in p\}.$$

By Theorem 3.1.4(2), we have that p is an idempotent in $\beta\mathbb{N}$ if and only if for every $A \in p$, $A^*(p) \in p$. And, if $A \in p$ and $n \in A^*(p)$, then $-n + A \in p$. As a consequence we have the following lemma from [19].

Lemma 3.2.2 *Let p be an idempotent in $\beta\mathbb{N}$ and let $A \subseteq \mathbb{N}$. For each $n \in A^*(p)$, $-n + A^*(p) \in p$.*

Proof. See the proof of Lemma 4.14 in [19].
Let $n \in A^*(p)$, and let $B = -n + A$. Then, by Definition, $B \in p$, and, since p is an idempotent, $B^*(p) \in p$. We only need to show that $B^*(p) \subseteq -n + A^*(p)$. So let $m \in B^*(p)$. Then $m \in B$ so $n + m \in A$. We have also $-m + B \in p$. That is, by Lemma 3.2.1, $-(n + m) \in p$. Since $n + m \in A$ and $-(n + m) \in p$, then $n + m \in A^*(p)$. That is, $m \in -n + A^*(p)$. So, we have shown that indeed $B^*(p) \subseteq -n + A^*(p)$. By supersets closedness of p, $-n + A^*(p) \in p$. □

Theorem 3.2.1 *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There there exist $i \in \{1, \dots, r\}$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS((x_n)_{n=1}^{\infty}) \subseteq A_i$.*

Proof. Let p be an idempotent in $\beta\mathbb{N}$. Pick $A = A_i \in p$. Choose any $x_1 \in A^*(p)$. Assume that $x_1, x_2, \dots, x_n$ have been chosen so that $FS(x_1, \dots, x_n) \subseteq A^*(p)$. We can choose $x_{n+1} \in A^*(p)$ such that $x_{n+1} \in -y + A^*(p)$ for every $y \in FS(x_1, \dots, x_n)$. By Lemma 3.2.2, $-y + A^*(p) \in p$. Then $y + x_{n+1} \in A^*(p)$. This implies $FS(x_1, \dots, x_n, x_{n+1}) \subseteq A^*(p)$.
□

3.2.2 The duals of Banach algebras

The Stone-Cěch compactification $\beta\mathbb{N}$ can be used to characterize the Banach space of all bounded sequences in the scalar field $\mathbb{R}$ or $\mathbb{C}$, with its supremum norm, and its dual space.

Recall that, for $1 \leq p < \infty$, the set of all sequences $x = (x_n)_{n=1}^{\infty}$ for which $\sum_{n=1}^{\infty} |x_n|^p < \infty$ is denoted by $\ell^p(\mathbb{N})$ (or simply by ℓ^p, or again by ℓ_p). The norm of the sequence x is the number

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

Now $\ell^\infty(\mathbb{N})$ is the set of all sequences $x = (x_n)_{n=1}^{\infty}$ satisfying $\sup\{|x_n|\} < \infty$ with the norm

$$\|x\|_\infty = \sup\{|x_n|\},$$

and indeed, $\|x\|_\infty = \lim\limits_{p\to\infty} \|x\|_p$.

Let a be a sequence in $\ell^\infty(\mathbb{N})$, there exists a closed subset E of the scalar field that contains the image of a then making it a function from $\mathbb{N}$ to E.

The function a is continuous since $\mathbb{N}$ is discrete and E is compact Hausdorff. Following the universal property, there exists a unique extension $\beta a : \beta\mathbb{N} \to E$. This extension does not depend on the set E taken above.

Let $C(\beta\mathbb{N})$ denote the space of continuous functions on $\beta\mathbb{N}$. Then we have defined an extension map from $\ell^\infty(\mathbb{N})$ to $C(\beta\mathbb{N})$. This map is bijective since every function in $C(\beta\mathbb{N})$ must be bounded and can be restricted to a bounded scalar sequence. If we consider the sup norm defined on both spaces, then the extension map is an isometry. Thus, $\ell^\infty(\mathbb{N})$ can be identified with $C(\beta\mathbb{N})$.

It follows that the dual of $\ell^1(\mathbb{N})$ is $C(\beta\mathbb{N})$ and the dual of $C(\beta\mathbb{N})$ is the space $M(\beta\mathbb{N})$ of complex-valued regular Borel measures on $\beta\mathbb{N}$. That is, the bi-dual of $\ell^1(\mathbb{N})$ can be identified with $M(\beta\mathbb{N})$.

Let us recall that given a Banach algebra A and its bi-dual A'' where $A \subset A''$, there is a natural product on A'' (called the *Arens product*) defined, for all $M, N \in A''$, by

$$M \Box N = \lim_\alpha \lim_\beta a_\alpha b_\beta$$

whenever (a_α) and (b_β) are nets in A with $\lim_\alpha a_\alpha = M$ and $\lim_\beta b_\beta = N$.

Consider the Banach algebra $(\ell^1(\mathbb{N}), \star)$. For every $\varphi, \psi \in \beta\mathbb{N}$, $\delta_\varphi, \delta_\psi \in M(\beta\mathbb{N})$, we define

$$\delta_\varphi \Box \delta_\psi = \lim_\alpha \lim_\beta \delta_{s_\alpha} \star \delta_{t_\beta} = \lim_\alpha \lim_\beta \delta_{s_\alpha t_\beta} = \delta_{\varphi \Box \psi},$$

where s_α and t_β are nets in $\mathbb{N}$ with $\lim_\alpha s_\alpha = \varphi$ and $\lim_\beta t_\beta = \psi$.

Hence, the compact right topological semigroup $(\beta\mathbb{N}, \Box)$ is a subsemigroup of $(M(\beta\mathbb{N}), \Box)$. The properties of $(M(\beta\mathbb{N}), \Box)$ are related to those of $(\beta\mathbb{N}, \Box)$. More details can be found in [5].

3.2.3 Topological Dynamics

Let (X, f) be a dynamical system and let $E(X, f)$ be its enveloping semigroup. Then, there is a natural surjective morphism from $\beta\mathbb{N}$ onto $E(X, f)$, defined by $p \mapsto f^p$, where f^p can be described as follows: fix $x \in X$, the map defined from the discrete space $\mathbb{N}$ to the compact Hausdorff space X, by $n \mapsto f^n(x)$, is continuous and has a unique extension as a map from $\beta\mathbb{N}$ to X, and $f^p(x)$ is nothing but the image of $p \in \beta\mathbb{N}$ under this extended continuous map. Equivalently, for $p \in \beta\mathbb{N}$ and $x \in X$, the collection

$$\{\{f^n(x) \,:\, n \in A\} \,:\, A \in p\}$$

forms an ultrafilter base on X, and the corresponding ultrafilter converges to a unique point in the compact Hausdorff space X. This unique limit is $f^p(x)$.

In brief, it can be proved that the map defined from $\beta\mathbb{N}$ to $E(X, f)$ by $p \mapsto f^p$ is surjective, continuous and is a group homomorphism (see Theorem 19.11 in [19]), and $f^p(x) = p - \lim_{n \in \mathbb{N}} f^n(x)$ (see Remark 19.13 in [19]).

In this setting, we have the following results from topological dynamics:

- Let $x \in X$. Then a point y is in the closure of the orbit of x if and only if there is $p \in \beta\mathbb{N}$ such that $f^p(x) = y$.
- Let $x, y \in X$. Then x and y are proximal if and only if there is $p \in \beta\mathbb{N}$ such that $f^p(x) = f^p(y)$.
- The minimal points in (X, f) are given by the set

  $$\{f^p(x) \,:\, x \in X \text{ and } p \text{ is a minimal idempotent in } \beta\mathbb{N}\}.$$

- Let $x \in X$. Then the limit points of the orbit of x are $f^p(x)$ for $p \in \beta\mathbb{N} \setminus \mathbb{N}$.
- Let $x, y \in X$. Then x and y are asymptotic if and only if $f^p(x) = f^p(y)$ for every $p \in \beta\mathbb{N} \setminus \mathbb{N}$.

The "bigness" of the enveloping semigroup $E(X, f)$ is a measure of the complexity of the dynamical system (X, f). For a simple dynamical system, the corresponding enveloping semigroup can be computed, hence "small". But the maximum complexity of (X, f) is limited by the morphism from $\beta\mathbb{N}$ to $E(X, f)$. That is, $E(X, f)$ cannot be more complicated that $\beta\mathbb{N}$. If the morphism $p \mapsto f^p$ is injective, then $E(X, f)$ is isomorphic to $\beta\mathbb{N}$. This is the case when X is the circle group $\mathbb{T}$ as the quotient space $\mathbb{R}/\mathbb{Z}$ under addition. The action $\varphi : \mathbb{N} \to \mathbb{T}^{\mathbb{T}}$ is defined by $\varphi(n) = f^n$, where $f^n(\mathbb{Z} + x) = \mathbb{Z} + a^n x$ for each $n \in \mathbb{N}$ and $a \in \{2, 3, 4, \ldots\}$. Then $\beta\mathbb{N}$ is the enveloping semigroup of the dynamical system $(\mathbb{T}, (f^n)_{n \in \mathbb{N}})$. See [19] for more details.

3.3 Ultrafilter semigroups

Definition 3.3.1 *Let G be an arbitrary infinite group. A topology τ on G is said to be **left invariant** if for every $U \in \tau$ and $a \in G$, then $aU \in \tau$.*

Equivalently, τ is a left invariant topology if for every $a \in G$, the left translation $\lambda_a : G \to G$, $x \mapsto ax$, is continuous in τ. Thus, left invariant topologies on G are those that make G a left topological group.

Definition 3.3.2 *Let τ be a left invariant topology on G. The subset of βG defined by*

$$Ult(\tau) = \{p \in G^* : p \text{ converges to the identity } e \in G \text{ in } \tau\}$$

*is called the **ultrafilter semigroup** of τ.*

Equivalently,

$$Ult(\tau) = \bigcap_{V \in \mathcal{F}} \overline{V \setminus \{e\}},$$

where $\mathcal{F}$ denotes the neighborhood filter of the identity e. One can prove that the ultrafilter semigroup $Ult(\tau)$ is a closed subsemigroup of G^*, see [33].

Example 3.3.1 *In the following are some examples of ultrafilter semigroups taken from [19], [32], [33], and [36]:*

1. *The subsemigroup $\mathbb{H}$ of $\beta\mathbb{N}$ defined by*

$$\mathbb{H} = \bigcap_{n \in \mathbb{N}} \overline{2^n \mathbb{N}}.$$

2. *For every infinite cardinal κ, the subsemigroup $\mathbb{H}_\kappa$ of $\beta(\bigoplus_\kappa \mathbb{Z}_2)$ defined by*

$$\mathbb{H}_\kappa = \bigcap_{\alpha < \kappa} \overline{V_\alpha \setminus \{0\}},$$

where $V_\alpha = \{x \in \bigoplus_\kappa \mathbb{Z}_2 : x(\gamma) = 0 \text{ for all } \gamma < \alpha\}$. Note that if we take $\kappa = \omega$, $\mathbb{H}_\omega$ is topologically and algebraically isomorphic to $\mathbb{H}$.

3. *The subsemigroup 0^+ of $\beta\mathbb{R}_d$ (where $\mathbb{R}_d$ is the additive group of real numbers reendowed with the discrete topology) defined by*

$$0^+ = \bigcap_{n\in\mathbb{N}} \bar{I}_n,$$

where I_n denotes the interval $(0, \frac{1}{n})$.

A natural way of producing homomorphisms of ultrafilter groups is via local homomorphisms.

Definition 3.3.3 *Let G be an arbitrary infinite group, let τ be a left invariant topology on G, let e be the identity of G, and let T be a semigroup. A mapping $f : (G, \tau) \to T$ is a* ***local homomorphism*** *if for every $x \in G \setminus \{e\}$, there is a neighborhood U_x of e in τ such that $f(xy) = f(x)f(y)$ for all $y \in U_x \setminus \{e\}$.*

Lemma 3.3.1 *See [35]. Let $f : (G, \tau) \to T$ be a local homomorphism into a compact right topological semigroup T such that $f[G] \subseteq \Lambda(T)$, let $\tilde{f} : \beta G \to T$ be the continuous extension of f, and let $f^* = \tilde{f}_{|Ult(\tau)}$. Then $f^* : Ult(\tau) \to T$ is a continuous homomorphism.*

Theorem 3.3.1 *See [35]. Let G be an infinite group with cardinality κ embeddable into a sum of countable groups and let p be an idempotent in G^*. Then there is a zero-dimensional left invariant topology τ on G and a discrete subset D of (G, τ) such that*

1. *$p \in Ult(\tau)$*
2. *$|D| = \kappa$ and $D^* \subset Ult(\tau)$*
3. *every mapping from D into a commutative semigroup T can be extended to a local homomorphism $f : (G, \tau) \to T$.*

Recall that if S is a compact right topological semigroup then, the set of idempotents in S, $E(S) \neq \emptyset$.

Corollary 3.3.1 *Let G be an infinite group and let τ be a left invariant topology on G. Then*

1. *$Ult(\tau)$ contains all the idempotents of G^* and, in particular, the idempotents of $K(\beta G)$;*

2. *$K(Ult(\tau)) = K(\beta G) \cap Ult(\tau)$.*

Proof. For (1), see the proof of Lemma 3 in [32] and also Theorem 3.3.1. Since G^* is an ideal of βG, we have that $K(\beta G) \subseteq G^*$. That is, $Ult(\tau) \cap K(\beta G) \neq \emptyset$. Then the proof of (2) is obtained by Theorem 1.1.17. $\square$

Theorem 3.3.2 *Let G be an infinite group and let τ be a left invariant topology on G. Let $S = Ult(\tau)$ and U a nonempty open subset in (G, τ), we have that $\bar{U} \cap S$ is a right ideal of S.*

Proof. See Lemma 2.5 and the proof of Lemma 2.4 in [33]. $\square$

Corollary 3.3.2 *Let G be an infinite group, let τ be a left invariant topology on G, and let $S = Ult(\tau)$. If U and V are disjoint open subsets of S, then $\overline{U} \cap S$ and $\overline{V} \cap S$ are two disjoint right ideals of S.*

Proof. The proof follows from Theorem 3.3.1. See also Lemma 7.1 in [37]. $\square$

3.4 Number of ideals in βS

It was established in [1] and [11] that $\beta\mathbb{N}$ contains 2^{2^ω} minimal left ideals and 2^{2^ω} minimal right ideals, respectively. It was showed in [19] that for every infinite cancellative semigroup S, βS contains $2^{2^{|S|}}$ minimal left ideals and at least 2^{2^ω} minimal right ideals. And this last result on the number of minimal right ideals was extended in [34] to any abelian group. In the following, we review these results in detail.

We recall that

$$\mathbb{H} = \bigcap_{n \in \mathbb{N}} \overline{2^n\mathbb{N}}$$

and that for every $n \in \mathbb{N}$, we can find $F \in \mathcal{P}_f(\omega)$ such that $n = \sum_{i \in F} 2^i$. Fixing $n \in \mathbb{N}$, we denote F by $supp(n)$. [19]

Lemma 3.4.1 *Let $(T, \cdot)$ be a compact right topological semigroup and let $\varphi : \mathbb{N} \to T$ with $\varphi[\mathbb{N}] \subseteq \Lambda(T)$. Assume that there is some $k \in \omega$ such that whenever $x, y \in \mathbb{N}$ and $\max supp(x) + k < minsupp(y)$ one has $\varphi(x+y) = \varphi(x) \cdot \varphi(y)$. Then for each $p, q \in \mathbb{H}$, $\tilde{\varphi}(p+q) = \tilde{\varphi}(p) \cdot \tilde{\varphi}(q)$.*

Proof. See comments in Proof of Lemma 6.3 in [19]. $\square$

The following theorem, Theorem 6.7 in [19], shows how to obtain homomorphisms on $\mathbb{H}$ which extended to $\beta\mathbb{N}$ might not be homomorphisms.

Theorem 3.4.1 *Let T be a compact right topological semigroup with a countable dense topological center. Then T is the image of $\mathbb{H}$ under a continuous homomorphism.*

Proof. See the proof of Theorem 6.4 in [19].
Enumerate $\Lambda(T)$ as $\{t_i : i \in I\}$ since it is countable, where I is either ω or $\{0, 1, 2, \ldots, k\}$ for some $k \in \omega$. Then choose a disjoint partition $\{A_i : i \in I\}$ of $\{2^n : n \in \omega\}$, with each being infinite.
Define a mapping $\tau : \mathbb{N} \to T$ by first stating that $\tau(2^n) = t_i$ if $2^n \in A_i$. Then extend τ to $\mathbb{N}$ by putting

$$\tau(n) = \prod_{i \in supp(n)} \tau(2^i),$$

with the terms in this product occuring in the order of increasing i.
It follows from Lemma 3.4.1, that $\tau : \tilde{\mathbb{N}} \to T$ is a homomorphism on $\mathbb{H}$. To show that $\tilde{\tau}[\mathbb{H}] = T$, let $i \in I$ and let $x \in A_i^*$. Then $x \in \mathbb{H}$ and $\tilde{\tau}(x) = t_i$. If $t_i \in \Lambda(T)$, then $t_i = \tau(2^i)$ with $2^i \in A_i$. It implies $t_i = \tilde{\tau}(x)$, for $x \in A_i^*$ ($A_i \in x$). Thus, $t_i \in \underline{\tilde{\tau}[\mathbb{H}].}$ So $\Lambda(T) \subseteq \tilde{\tau}[\mathbb{H}]$ and hence, since $\Gamma(T)$ is dense in T, we have $T = \overline{\Lambda(T)} \subseteq \tilde{\tau}[\mathbb{H}]$. Obviously, by definition of τ, we have finally $\tilde{\tau}[\mathbb{H}] \subseteq \overline{\Lambda(T)} = T$. $\square$

Corollary 3.4.1 *Every finite (discrete) semigroup is the image of $\mathbb{H}$ under a continuous homomorphism.*

Proof. See comments in Proof of Corollary 6.5 in [19]. $\square$

Lemma 3.4.2 *Let p be an idempotent in $(\beta\mathbb{N}, +)$. Then for every $n \in \mathbb{N}$, $n\mathbb{N} \in p$.*

Proof. See the proof of Lemma 6.6 in [19]. Let $n \in \mathbb{N}$ and $\gamma : \mathbb{N} \to \mathbb{Z}_n$ denote the canonical homomorphism. Then $\tilde{\gamma} : \beta\mathbb{N} \to \mathbb{Z}_n$ is also a homomorphism, by Corollary 3.1.3. Thus $\tilde{\gamma}(p) = \tilde{\gamma}(p+p) = \tilde{\gamma}(p) + \tilde{\gamma}(p)$ (p is an idempotent in $(\beta\mathbb{N}, +)$) and, since $\mathbb{Z}_n$ is cancellative, $\tilde{\gamma}(p) = 0$. It follows that $\tilde{\gamma}^{-1}[\{0\}]$ is a neighborhood of p, and hence that $\tilde{\gamma}^{-1}[\{0\}] \cap \mathbb{N} = n\mathbb{N} \in p$. $\square$

Definition 3.4.1 *We define $\phi : \mathbb{N} \to \omega$ and $\theta : \mathbb{N} \to \omega$ by stating that $\phi(n) = \max(supp(n))$ and $\theta(n) = \min(supp(n))$*

Lemma 3.4.3 *The set $\mathbb{H}$ is a compact subsemigroup of $(\beta\mathbb{N}, +)$ which contains all the idempotents of $(\beta\mathbb{N}, +)$. Furthermore, for any $p \in \beta\mathbb{N}$ and any $q \in \mathbb{H}$, $\tilde{\phi}(p+q) = \tilde{\phi}(q)$ and $\tilde{\theta}(p+q) = \tilde{\theta}(p)$*

Proof. See the proof of Lemma 6.8 in [19]. $\mathbb{H}$ is compact since it is closed in $\beta\mathbb{N}$. Now, if $n \in 2^k\mathbb{N}$ and if $r > \phi(n)$,

$$n + 2^r\mathbb{N} \subseteq 2^k\mathbb{N} + 2^r\mathbb{N} \subseteq 2^k\mathbb{N}$$

since $k = \min(supp(n)) < \max(supp(n)) < r$. It follows from Example 3.3.1(1) and Theorem 3.1.7 that $\mathbb{H}$ is a subsemigroup of $\beta\mathbb{N}$.
By the Lemma 3.2.2, $\mathbb{H}$ contains all the idempotents of $(\beta\mathbb{N}, +)$.
Now take $p \in \beta\mathbb{N}$ and $q \in \mathbb{H}$. Given any $m \in \mathbb{N}$, choose r satisfying $r > \phi(m)$. Then, if $n \in 2^r\mathbb{N}$, $\phi(m+n) = \phi(n)$ and $\theta(m+n) = \theta(n)$. Hence since $2^r\mathbb{N} \in q$,

$$\begin{aligned} \tilde{\phi}(m+q) &= q - \lim_{n \in 2^r\mathbb{N}} \phi(m+n) \\ &= q - \lim_{n \in 2^r\mathbb{N}} \phi(n) \\ &= \tilde{\phi}(q). \end{aligned}$$

It follows that

$$\tilde{\phi}(p+q) = \lim_{m \to p} \tilde{\phi}(m+q) = \tilde{\phi}(q).$$

Similarly, $\tilde{\theta}(p+q) = \tilde{\theta}(p)$. $\square$

The reason we keep repeating the properties of $\beta\mathbb{N}$ is because it makes it something of a prototype. The properties are first examined and the problems solved in this simple case of $\beta\mathbb{N}$, and the arguments are adjusted and extended to achieve more general solutions in the case of βS where S is any infinite discrete semigroup.

Theorem 3.4.2 $(\beta\mathbb{N}, +)$ *contains* 2^c *minimal left ideals and* 2^c *minimal right ideals. Each of these contains* 2^c *idempotents.*

Proof. See the proof of Theorem 6.9 in [19].
Let $A = \{2^n : n \in \mathbb{N}\}$. We have by Definition 3.4.3, $\phi(2^n) = \theta(2^n) = n$ for any $n \in \mathbb{N}$. Thus, $\phi_{|_A} = \theta_{|_A} : A \to \mathbb{N}$ is bijective. So by Theorem 2.2.17, $\tilde{\phi}_{|_A} = \tilde{\theta}_{|_A} : \bar{A} \to \beta\mathbb{N}$ is bijective as well. Assuming that for any $q_1, q_2 \in A^*$,

$$(\beta\mathbb{N} + q_1) \cap (\beta\mathbb{N} + q_2) \neq \emptyset.$$

We can find $r, s \in \beta\mathbb{N}$ such that $r + q_1 = s + q_2$. By Lemma 3.4.3, then

$$\tilde{\phi}(q_1) = \tilde{\phi}(r + q_1) = \tilde{\phi}(s + q_2) = \tilde{\phi}(q_2).$$

Since $\tilde{\phi}_{|_{\bar{A}}}$ is bijective, it implies that $q_1 = q_2$. Thus, by contraposition, if $q_1 \neq q_2$ in A^* then

$$(\beta\mathbb{N} + q_1) \cap (\beta\mathbb{N} + q_2) = \emptyset.$$

By Corollary 2.2.9, $|A^*| = 2^c$. Which implies that $\mathbb{N}$ has 2^c pairwise disjoint left ideals and, by Corollary 1.3.1, each contains a minimal left ideal.
Let $q \in A^*$. By Theorem 1.3.2(1) and Theorem 1.1.12(5), $q + \mathbb{H}$ contains an idempotent $e(q)$ which is minimal in $\mathbb{H}$. Since $\mathbb{H}$ meets $K(\beta\mathbb{N})$, $e(q)$ is also minimal in $\beta\mathbb{N}$. Otherwise, there is an idempotent f of $\beta\mathbb{N}$ with $f < e(q)$. But $\mathbb{H}$ contains all the idempotents of $\beta\mathbb{N}$ according to Lemma 3.4.3 and so it contains also f. This contradicts the minimality of $e(q)$ in $\mathbb{H}$. Thus by Theorem 1.1.12(4), $e(q) + \beta\mathbb{N}$ is a minimal right ideal of $\beta\mathbb{N}$. We claim that if $q_1 \neq q_2$ in A^*, then

$$e(q_1) + \beta\mathbb{N} \neq e(q_2) + \beta\mathbb{N}.$$

For otherwise, $e(q_2) \in e(q_1) + \beta\mathbb{N}$ and so $e(q_2) = e(q_1) + e(q_2)$ by Lemma 1.1.4(2). However, by Lemma 3.4.3,

$$\tilde{\theta}(e(q_2)) \in \tilde{\theta}[q_2 + \mathbb{H}] = \tilde{\theta}[\{q_2\}] = \{q_2\}$$

implies $\tilde{\theta}(e(q_2)) = q_2$. In fact,

$$\tilde{\theta}(q_2) = q_2 - \lim_{n \in A} \theta(n) = q_2 - \lim_{k \in \mathbb{N}} \theta(2^k) = q_2 - \lim_{k \in \mathbb{N}} k = q_2.$$

We have also

$$\tilde{\theta}(e(q_2)) = \tilde{\theta}(e(q_1) + e(q_2)) = \tilde{\theta}(e(q_1)) = q_1$$

hence yielding a contradiction.
If L is any minimal left ideal and R is any minimal right ideal of $(\beta\mathbb{N}, +)$ then, by Lemma 1.1.14, $L \cap R$ contains an idempotent. It follows that L and R both contain $2^{\mathfrak{c}}$ idempotents. $\square$

Lemma 3.4.4 *Let S be a discrete group and let s be a cancelable element of S. For every $t \in S$ and $p \in \beta S$*

1. *if S is right cancellative and $sp = tp$, then $s = t$, and*
2. *if S is left cancellative and $ps = pt$, then $s = t$.*

Proof. See the proof of Lemma 6.28 in [19]. $\square$

Lemma 3.4.5 *Let S be an infinite weakly left cancellative semigroup and let I be the set of right identities of S. Let κ be an infinite cardinal with $\kappa \leq |S|$, and let $\langle s_\lambda \rangle_{\lambda<\kappa}$ be a one-to-one κ-sequence in S. If T is a subset of S with cardinality κ, then there exists a one-to-one κ-sequence $\langle t_\lambda \rangle_{\lambda<\kappa}$ in T such that*

1. *for every $\mu < \kappa$, $t_\mu \notin FP(\langle t_\lambda \rangle_{\lambda<\mu})$,*
2. *for every $\lambda < \mu < \kappa$ and every $u, v \in I \cup \{s_\iota : \iota < \mu\} \cup FP(\langle t_\lambda \rangle_{\lambda<\mu})$, $u \neq vt_\mu$ and $ut_\lambda \neq vt_\mu$, and*
3. *$I \cap FP(\langle t_\lambda \rangle_{\lambda<\mu}) = \emptyset$.*

Proof. See the proof of Lemma 6.29 in [19]. $\square$

Theorem 3.4.3 *Let S be an infinite discrete semigroup which is weakly left cancellative. Let κ be an infinite cardinal with $\kappa \leq |S|$ and let R and T be subsets of S of cardinality κ. Then there is a subset V of T with cardinality κ such that the set P of uniform ultrafilters on V has the following properties:*

1. *$|P| = 2^{2^\kappa}$,*
2. *for each pair of distinct elements $p, q \in P$, $c\ell_{\beta S}(Rp)$ and $c\ell_{\beta S}(Rq)$ are disjoint, and*

3. *if S is also right cancellative, then for each $p \in P$, $ap \neq bp$ whenever $a \neq b$ in R and Rp is strongly discrete in βS.*

Proof. The proof uses Lemma 3.4.5 and Theorem 2.2.23. The details can be found in [19], the proof of Theorem 6.30. □

Theorem 3.4.4 *There is a subset L of $\mathbb{N}^*$ such that $|L| = 2^c$ and for all $p \neq q$ in L and all $f : \mathbb{N} \to \mathbb{N}$, $\tilde{f}(p) \neq q$.*

Proof. See the comments in the proof of Theorem 6.36 in [19]. □

Lemma 3.4.6 *Let $p, q \in \mathbb{N}^*$. If there is a one-to-one function $f : \mathbb{N} \to \beta\mathbb{N}$ such that $f[\mathbb{N}]$ is discrete and $\tilde{f}(p) = q$, then there is a function $g : \mathbb{N} \to \mathbb{N}$ such that $\tilde{g}(q) = p$.*

Proof. See the proof of Lemma 6.37 in [19]. □

Definition 3.4.2 *Two points x and y in a topological space X are said to have the same* ***homeomorphism type*** *if and only if there is a homeomorphism f from X onto X such that $f(x) = y$.*

Theorem 3.4.5 *Let D be a discrete space and let X be an infinite compact subset of βD. Then X has at least 2^c distinct homeomorphism types.*

Proof. See the proof of Theorem 6.38 in [19]. It uses Corollary 2.2.5, Theorem 2.2.13, Theorem 2.2.24, Theorem 3.4.4, and Lemma 3.4.6 in [19]. □

Theorem 3.4.6 *Let S be a discrete semigroup. If the minimal left ideals of βS are infinite, βS contains at least 2^c minimal right ideals.*

Proof. See the proof of Theorem 6.39 in [19] which uses Theorem 1.1.14, Theorem 1.3.4(3) and Theorem 3.4.5 in [19]. □

Theorem 3.4.7 *Let S be an infinite discrete semigroup which is weakly left cancellative. If $|S| = \kappa$, then there are 2^{2^κ} pairwise disjoint left ideals of βS.*

Proof. The proof is an immediate consequence of Theorem 3.4.3, with $R = S$. $\square$

Theorem 3.4.8 *Let S be an infinite cancellative discrete semigroup. Then βS contains at least 2^c minimal left ideals and at least 2^c minimal right ideals. Each minimal left ideal and each minimal right ideal contains at least 2^c idempotents.*

Proof. The proof uses Theorem 3.4.7 and Theorem 3.4.6 in [19]. $\square$

As we presented above, the problem of counting the minimal right ideals of βS turned out to be more challenging than counting the minimal left ideals of βS. We saw that βS contains at least 2^{2^ω} minimal right ideals for any infinite discrete cancellative semigroup S. We recall that the proof uses the fact that every minimal left ideal L of βS is infinite and the points of L belong to at least 2^{2^ω} different homeomorphism classes. Two points of L which belong to R belong to the same homeomorphism class in L. In [34], this result is extended to any infinite Abelian group G using a different approach which we describe in the following lines. More details can also be found in [37], from page 136.

Definition 3.4.3 *Let G be a topological group. The* ***Bohr compactification*** *of G is a compact topological group, denoted bG, together with a continuous homomorphism $e : G \to bG$ satisfying the following properties:*

1. *$e[G]$ is dense in bG, and*
2. *the universal property: For every continuous homomorphism $h : G \to K$ from G to a compact topological group K there is a continuous homomorphism $h^b : bG \to K$ such that $h = h^b \circ e$.*

Note that if $\hat{G}$ is the dual group of an Abelian group G and $\hat{G}_d$ is the group $\hat{G}$ reendowed with the discrete topology, then bG is the dual of $\hat{G}_d$. The mapping $e : G \to bG$ is then given by $e(x)(\chi) = \chi(x)$, where $x \in G$ and $\chi \in \hat{G}_d$. This mapping is injective, please refer to [34].

The complete version of the following lemma can be found in [34] or in [37].

Lemma 3.4.7 *Let G be an infinite Abelian group of cardinality κ. Then G admits a homomorphism onto $\bigoplus_\kappa \mathbb{Z}_p$, where p is a prime number and $\mathbb{Z}_p$ is the cyclic group of order p.*

Proof. See the proof of Lemma 1(2) in [34] or the proof of Lemma 9.22 in [37]. The proof generalizes a well known fact on infinite Abelian groups. $\square$

Lemma 3.4.8 *For every infinite discrete Abelian group G of cardinality κ, bG admits a continuous homomorphism onto $\prod_{2^\kappa} \mathbb{Z}_p$, where $\prod_{2^\kappa} \mathbb{Z}_p$ is endowed with the product topology.*

Proof. The proof is similar to the proof of Lemma 2 in [34] or to the proof of Lemma 9.23. The proof uses Lemma 3.4.7 and the Pontrjagin duality. That is, using both cases $\kappa = \omega$ and $\kappa > \omega$, if G admits a homomorphism onto $\bigoplus_\kappa \mathbb{Z}_p$, then the extension is a continuous homomorphism from the Bohr compactification bG onto the Pontrjagin dual $\prod_{2^\kappa} \mathbb{Z}_p$ of $\bigoplus_\kappa \mathbb{Z}_p$. $\square$

Lemma 3.4.9 *Let κ be an infinite cardinal. For each $\alpha < \kappa$, let X_α be a space having at least two disjoint nonempty open sets, and let $X = \prod_{\alpha<\kappa} X_\alpha$. Then there are at least 2^κ many pairwise incompatible open filters on X converging to the same point.*

Proof. See the proof of Lemma 3 in [34] or the proof of Lemma 9.24 in [37] including the comments before these proofs. One needs to define a filter on X considering the fact that X is not extremally disconnected. Then the filters hereby obtained are open and pairwise incompatible converging to an element of X. $\square$

Theorem 3.4.9 *For every infinite discrete Abelian group G of cardinality κ, there are 2^{2^κ} many pairwise incompatible open filters on bG converging to zero.*

Proof. Let $f : G \to \bigoplus_{2^\kappa} \mathbb{Z}_p$ be the homomorphism guaranteed by Lemma 3.2.1. Then, by Lemma 3.2.2, there is a continuous surjective homomorphism $\tilde{f} : bG \to \prod_{2^\kappa} \mathbb{Z}_p$. By Lemma 3.2.3, there are pairwise incompatible open filters $\mathcal{F}_\alpha$ $(\alpha < 2^{2^\kappa})$ on $\prod_{2^\kappa} \mathbb{Z}_p$ converging to zero. For each $(\alpha < 2^{2^\kappa})$, let $\mathcal{H}_\alpha$ be the filter on bG with a base consisting of subsets of the form $\tilde{f}^{-1}(A) \cap U$, where $A \in \mathcal{F}_\alpha$ and U runs over neighborhoods of zero. Then $\mathcal{H}_\alpha$ $(\alpha < 2^{2^\kappa})$ are pairwise incompatible open filters on bG converging to zero. $\square$

The following theorem is the main result found in [34].

Theorem 3.4.10 *For every infinite discrete Abelian group G of cardinality κ, βG contains 2^{2^κ} many minimal right ideals.*

Proof. See the proof of Theorem 9.20 in [37] or the proof of Theorem 1 in [34].
Let τ denote the Bohr compactification on the infinite discrete Abelian group G, that is, the one induced by the mapping $e : G \to bG$. By Theorem 3.4.9, there are 2^{2^κ} pairwise incompatible open filters on bG converging to zero. Restricting these filters to $e(G)$, there are α pairwise incompatible open filters $\mathcal{F}_\alpha$ on (G, τ) converging to zero with $\alpha < 2^{2^\kappa}$. Let $S = Ult(\tau)$ and, for each $\alpha < 2^{2^\kappa}$, let $J_\alpha = \bigcap_{U \in \mathcal{F}_\alpha} \overline{U \setminus \{0\}}$. Then, by Theorem 3.3.1, each J_α is a closed right ideal of S, and since the filters $\mathcal{F}_\alpha$ are pairwise incompatible, the ideals J_α are pairwise disjoint (see Theorem 3.3.2). Furthermore, by Corollary 3.3.1(1), S contains all the idempotents of G^* since (G, τ) is a subgroup of a compact group. In particular, S contains all the idempotents of $K(\beta G)$. It follows that $S \cap K(\beta G) \neq \emptyset$. But then, by Corollary 3.3.1(2), $K(S) = S \cap K(\beta G)$. Thus, every minimal right ideal R of S is contained in a minimal right ideal R' of βG, and the correspondence $R \mapsto R'$ obtained is injective. Consequently, the number of minimal right ideals of βG is greater than or equal to 2^{2^κ}, the number of minimal right ideals of S. Hence, the number of minimal right ideals of βG is 2^{2^κ}. $\square$

Chapter 4

Closed two-sided ideals in βS

In this chapter, we study the closure of two-sided ideals and compute the number of two-sided ideals of βS. In the first section, we present the results from [38], [36] and [35] about the closure of the smallest ideal of βS. In the second section, we review in detail the approach used in [6] and [13] to compute the number of closed two-sided ideals of βG, where G is any countably infinite group. In the third section, we present an alternative approach to obtain a similar result to the previous one. We show that βS contains $2^{2^{|S|}}$ pairwise incomparable semiprincipal closed two-sided ideals, where S is any infinite discrete semigroup. The result was published in [29].

4.1 Closure of the smallest ideals

We recall that for any compact Hausdorff right topological semigroup S, βS has the smallest two-sided ideal $K(\beta S)$. $K(\beta S)$ is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. The ultrafilters from $K(\beta S)$ and $c\ell(K(\beta S))$ have nice combinatorial characterizations, see [19]. From Theorem 3.1.14, $p \in K(\beta S)$ if and only if, for all $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is syndetic if and only if, for all $q \in \beta S$, $p \in \beta S \cdot q \cdot p$. Also, from Corollary 3.1.8, $\overline{A} \cap K(\beta S) \neq \emptyset$ if and only if A is piecewise syndetic for any $A \subseteq S$.

We remind the reader that for every compact Hausdorff right topological semigroup T, $c\ell(K(T))$ is a right ideal, but not always a left ideal, see Example 2.16 in [19]. But we have the following result from [19], Theorem

4.44.

Theorem 4.1.1 *For any semigroup S, $c\ell(K(\beta S))$ is a two-sided ideal of βS.*

Proof. Recall that the set $\Lambda(\beta S) = \{p \in \beta S : \lambda_p \text{ is continuous }\}$ is the topological center of βS. By Theorem 3.1.1(3), $S \subseteq \Lambda(\beta S)$. Since S is dense in βS and, obviously, βS is compact, we have that $\Lambda(\beta S)$ is dense in βS. Using Theorem 1.3.8(1), $c\ell(K(\beta S))$ is an ideal of βS. □

The above combinatorial characterizations were extended in [21], [22], and [36] to an arbitrary closed subsemigroup T of βS, with S an arbitrary discrete semigroup. There it was showed that, for a large class of closed subsemigroups T of βS, $c\ell(K(T))$ is not a left ideal of T.

We recall (see Lemma 6.28 in [19]) that if an element s is cancelable in the semigroup S, for every $t \in S$ and $p \in \beta S$, it follows that

1. if S is right cancellative and $sp = tp$, then $s = t$, and
2. if S is left cancellative and $ps = pt$, then $s = t$.

Then one has the following results from [19].

Lemma 4.1.1 *Let S be a discrete cancellative semigroup. If $s_1, s_2 \in S$ with $s_1 \neq s_2$, then, for every $p \in \beta S$, $s_1 p \neq s_2 p$ and $ps_1 \neq ps_2$.*

Proof. See the proof of Lemma 6.28 in [19]. □

Theorem 4.1.2 *Let S be an infinite semigroup, let $p \in \beta S$, and let $T \subseteq S$ with $|T| = \omega$. Then, for every $r_1, r_2 \in c\ell(T)$ and $r_1 \neq r_2$, $r_1 p \neq r_2 p$ if and only if for every $A \subseteq T$ there is some $B \subseteq S$ such that*

$$A = \{s \in T : s^{-1}B \in p\}.$$

In particular, p is right cancelable in βS if and only if for every $A \subseteq S$ there is some $B \subseteq S$ such that $A = \{s \in S : s^{-1}B \in p\}$.

Proof. See the proof of Theorem 8.7 in [19]. □

Corollary 4.1.1 *Let S be an infinite semigroup and let $p \in S^*$. Then p is right cancelable in S^* if and only if for every $A \subseteq S$ there is some $B \subseteq S$ such that $|A \Delta \{s \in S : s^{-1}B \in p\}| < \omega$.*

Proof. The proof follows as a modification of the proof of the previous theorem. See Exercise 8.2.1 in [19]. $\square$

Lemma 4.1.2 *Let S be a discrete semigroup, let $T \subseteq S$ with $|T| = |S|$, and let $p \in \beta S$ such that for every $s_1, s_2 \in T$ and $s_1 \neq s_2$, $s_1 p \neq s_2 p$.*

1. *If, for every $r_1, r_2 \in c\ell(T)$ and $r_1 \neq r_2$, $r_1 p \neq r_2 p$, then Tp is discrete in βS.*

2. *If Tp is strongly discrete in βS, then $r_1 p \neq r_2 p$ whenever $r_1, r_2 \in c\ell(T)$ and $r_1 \neq r_2$.*

Proof. The proof uses Theorem 4.1.2 to conclude (2), while (1) is proved by negation. For more details, see the proof of Lemma 8.9 in [19]. $\square$

Theorem 4.1.3 *Let S be an infinite discrete right cancellative and weakly left cancellative semigroup with cardinality κ. Then the set of right cancelable elements of βS contains dense open subsets of $U_\kappa(S)$. In particular, S^* contains 2^{2^κ} elements which are right cancelable in βS.*

Proof. The proof uses Theorem 3.4.3 with $R = S$ and Lemma 4.1.2 to show that there are right cancelable uniform ultrafilters in βS with a dense open subset. The cardinality of these uniform ultrafilters is obtained using Theorem 2.2.32. For more details, see the proof of Theorem 8.10 in [19]. $\square$

The following results from [19] give more properties of right cancelability in βS when S is a countably infinite semigroup.

Theorem 4.1.4 *Let S be a semigroup, let $p \in \beta S$, and let $T \subseteq S$ with $|T| = \omega$. The following statements are equivalent.*

1. *For every $s_1, s_2 \in T$ and $s_1 \neq s_2$, $s_1 p \neq s_2 p$ and Tp is strongly discrete.*

2. *There is a function $f : S \to S$ such that for every $q \in c\ell(T)$, $\tilde{f}(q \cdot p) = q$, where $\tilde{f}$ is the extension of f to βS.*

3. *Whenever r_1 and $r_2 \in c\ell(T)$ and $r_1 \neq r_2$, then $r_1 p \neq r_2 p$.*

4. *For every $A \subseteq T$ there exists $B \subseteq S$ such that $A = \{s \in T : s^{-1}B \in p\}$.*

5. *For every $A, B \subseteq T$ and $A \cap B = \emptyset$, $c\ell(Ap) \cap c\ell(Bp) = \emptyset$.*

6. *The left translation $\rho_{p|_{c\ell(T)}} : c\ell(T) \to (c\ell(T))p$ is a homeomorphism.*

7. *The left translation $\rho_{p|_T} : T \to Tp$ is a homeomorphism.*

8. *For every $s_1, s_2 \in T$ and $s_1 \neq s_2$, $s_1 p \neq s_2 p$ and Tp is discrete.*

9. *For every $s \in T$ and every $r \in c\ell(T) \setminus \{s\}$, $sp \neq rp$.*

Proof. See the proof of Theorem 8.11 in [19]. □

Lemma 4.1.3 *Let S be a weakly left cancellative semigroup and let $p \in \beta S$. If p is right cancelable in βS, then $p \notin S^* p$.*

Proof. See the proof of Lemma 8.15 in [19]. □

The following theorems complete Theorem 4.1.4 when S is taken as a countable group.

Theorem 4.1.5 *Let S be either $(\mathbb{N}, +)$ or a countable group. For every $p \in S^*$ the following statements are equivalent.*

1. *p is right cancelable in βS.*

2. *$p \notin S^* p$.*

Proof. See the proof of the first two conditions of Theorem 8.18 in [19]. It uses Lemma 4.1.3 and Corollary 3.1.5 because a group is a cancellative monoid. □

Theorem 4.1.6 *Let S be either $(\mathbb{N}, +)$ or a countable group. An element $p \in \beta S$ is right cancelable in βS if and only if there exists $B \in p$ such that, for every $s \in S$ distinct from the identity, $s^{-1}B \notin p$.*

Proof. See the proof of Theorem 8.19 in [19]. It uses Theorem 4.1.2 to guarantee the existence of B and also uses Theorem 4.1.5 for the sufficient condition. □

Theorem 4.1.7 *Let S be a discrete countably infinite cancellative semigroup and let T be an infinite subsemigroup of S. Then there exists $p \in c\ell(E(K(\beta T)))$ such that $Sp \cap S^* S^* = \emptyset$.*

Proof. See the proof of Theorem 8.22 in [19]. $\square$

Corollary 4.1.2 *Let S be a discrete countably infinite cancellative semigroup and let T be an infinite subsemigroup of S. There exist $p \in c\ell(E(K(\beta T)))$ such that $p \notin S^*S^*$.*

Proof. See the proof of Corollary 8.23 in [19]. The proof follows from Theorem 4.1.7. $\square$

Corollary 4.1.3 *Let S be a discrete countably infinite cancellative semigroup. Then $K(\beta S)$ is not closed in βS.*

Proof. It is a consequence of Theorem 4.1.7 by taking $S = T$ and the fact that, by Theorem 3.1.12, $K(\beta S) \subseteq S^*S^*$ since S^*S^* is a two-sided ideal of βS. See the proof of Corollary 8.25 in [19]. $\square$

The following result, see Corollary 8.26 in [19], shows that right cancelability occurs in the smallest two-sided ideal of βS.

Corollary 4.1.4 *Let S be a discrete countably infinite cancellative semigroup and let T be an infinite subsemigroup of S. There are right cancelable elements of βS in $c\ell(E(K(\beta T)))$.*

Proof. The proof follows from Theorem 4.1.7, Lemma 4.1.1 and Theorem 4.1.4(9). $\square$

Theorem 4.1.8 $c\ell(K(\mathbb{H}))$ *is not a left ideal of* $\mathbb{H}$.

Proof. See the proof of Theorem 8.30 in [19]. $\square$

Corollary 4.1.5 $c\ell(K(\mathbb{H})) \subsetneq \mathbb{H} \cap c\ell(K(\beta\mathbb{N}))$.

Proof. The proof is obtained from the fact that $\mathbb{H} \cap c\ell(K(\beta\mathbb{N}))$ is an ideal of $\mathbb{H}$, since $c\ell(K(\beta\mathbb{N}))$ is a two-sided ideal of $\beta\mathbb{N}$ and $\mathbb{H}$ is a closed subsemigroup of $\beta\mathbb{N}$ (see Theorem 1.3.5 and Theorem 1.3.7). Thus, $K(\mathbb{H}) \subset \mathbb{H} \cap c\ell\ K(\beta\mathbb{N})$. But by Theorem 4.1.8, this inclusion is strict, that is $c\ell(K(\mathbb{H})) \subsetneq \mathbb{H} \cap c\ell(K(\beta\mathbb{N}))$. $\square$

These results are true for a large class of closed subsemigroups of βS, namely 0^+ and $\mathbb{H}_\kappa$ respectively closed subsemigroups of $\beta\mathbb{R}_d$ and $\beta(\bigoplus_\kappa \mathbb{Z})$

as defined in Example 3.3.1.

Recall that if (S, τ) is a left topological semigroup with identity 1 and τ is T_1, then the ultrafilter semigroup $Ult(\tau)$ is the set of all ultrafilters on S^* converging to 1 in τ. Now let $T = Ult(\tau)$. We have that T is a closed subsemigroup of βS and that it contains all the idempotents of S^* if (S, τ) is topologically and algebraically embeddable into a compact group. In particular, T contains all the idempotents of $K(\beta S)$ (see Lemma 1 and Lemma 2 in [38]). Hence, following Theorem 1.1.17, the idempotents of $K(T)$ are the same as those of $K(\beta S)$. Then, we have the following results from [36], [37] and [38] to strengthen the results above.

Theorem 4.1.9 *Let T be as defined above. There exists $p \in c\ell(E(K(T)))$ but $p \notin (c\ell(K(T)))T$.*

Proof. See the proof of Theorem 4 in [38] or the proof of Theorem 9.33 in [37]. □

Theorem 4.1.10 *Let S be an infinite discrete semigroup algebraically embeddable into a compact group. Then both $K(\beta S)$ and $E(c\ell(K(\beta S)))$ are not closed.*

Proof. The proof of this result was the object of the paper [38] using Theorem 4.1.9 above. One may consult the proof of the similar result in [37] on page 145. □

Lemma 4.1.4 *Let τ be an invariant topology on an infinite group G with the identity 1 and let λ denote the character of τ. Then the following conditions are equivalent.*

1. *There is a neighborhood base $\{W_\alpha : \alpha < \lambda\}$ at 1 such that for $\gamma < \alpha < \lambda$, $W_\alpha \subseteq W_\gamma$ and $W_\alpha = \bigcap_{\gamma<\alpha} W_\gamma$ whenever α is a limit ordinal.*
2. *There is a neighborhood base $\{W_\alpha : \alpha < \lambda\}$ at 1 such that for $\gamma < \alpha < \lambda$, $W_\alpha \subseteq W_\gamma$.*
3. *If $\mathcal{A}$ is a subfamily of τ with $|\mathcal{A}| < \lambda$, then $\bigcap \mathcal{A} \in \tau$.*

If τ satisfies the conditions (1), (2), and (3) above, then λ is regular.

Proof. See the proof of Lemma 3.1 in [36]. $\square$

Theorem 4.1.11 *Suppose that τ satisfies the conditions of Lemma 4.1.4 and let $T = Ult(\tau)$. Then $c\ell(K(T))$ is not a left ideal of T.*

Proof. See the proof in [36], Theorem 3.2. $\square$

Corollary 4.1.6 *$c\ell(K(0^+))$ and $c\ell(K(\mathbb{H}_\kappa))$ are not (respectively) left ideals of 0^+ and $\mathbb{H}_\kappa$.*

Proof. This is an immediate consequence of Theorem 4.1.11. $\square$

4.2 Boolean algebras and closed ideals of βG

In this section, we review how to obtain a closed two-sided ideal of the Stone-Čech compactification βG of an infinite group G from a Boolean group ideal on G. Recall from Section 2.2 that, if D is a discrete space, then the Stone-Čech compactification βD is alternatively defined from the Stone space of a Boolean algebra $\mathcal{P}(D)$. Using this approach, it was shown in [6] that, if G is Abelian, then βG contains $2^{2^{|G|}}$ distinct closed two-sided ideals. In [13], this result was extended to any countably infinite group G.

Definition 4.2.1 *A **Boolean group ideal** is an ideal $\mathcal{I}$ on a group G such that*

1. *finite subsets of G are members of $\mathcal{I}$;*
2. *if $A \in \mathcal{I}$, then $A^{-1} \in \mathcal{I}$; and*
3. *if $A, B \in \mathcal{I}$, then $AB \in \mathcal{I}$.*

We recall that, given a filter $\mathcal{F}$ on a set X, the closure of $\mathcal{F}$ is defined by $\hat{\mathcal{F}} = \{p \in \beta X : \mathcal{F} \subseteq p\}$.

The following theorem establishes a relationship between a filter and a Boolean group ideal.

Theorem 4.2.1 *Let G be an infinite group, let $\mathcal{I}$ be a Boolean group ideal on G, and let $\mathcal{F} = \{G \setminus A : A \in \mathcal{I}\}$. Then $\hat{\mathcal{F}}$ is a closed two-sided ideal of the semigroup βG.*

Proof. The proof can be found in [7]. $\square$

By computing the number of Boolean group ideals, one can deduce the number of the corresponding closed two-sided ideals.

Corollary 4.2.1 *Let G be an infinite Abelian group with $|G| = \kappa$. There are 2^{2^κ} distinct closed two-sided ideals in βG.*

Proof. The proof was the object of [6] using mainly Theorem 4.2.1 above. $\square$

The following theorem is the countable case of Corollary 4.2.1 dropping the commutativity from the hypothesis.

Theorem 4.2.2 *Let G be a countably infinite group. There are 2^{2^ω} distinct Boolean group ideals on G. Consequently, there are 2^{2^ω} distinct closed two-sided ideals of βG.*

Proof. It uses the fact that if $\mathcal{F}$ and $\mathcal{G}$ are two distinct filters on G with respect to Boolean group ideals on G, then $\hat{\mathcal{F}} \neq \hat{\mathcal{G}}$. Also the fact that G contains $2^{2^{|G|}}$ distinct Boolean group ideals. Then βG contains $2^{2^{|G|}}$ distinct closed two-sided ideals of βG. The detailed proof can be found in [13] and also in [20]. $\square$

4.3 Semiprincipal closed two-sided ideals of βS

Our result published in [29] extended the result in Theorem 4.2.2 above to any infinite semigroup S embeddable into a group by using a different approach, relatively simple and short. The approach was inspired from [1] and [19]. We first need to introduce the concept of thin subsets of a group.

Definition 4.3.1 *Let G be a group. A subset A of G with the identity e is said to be*

1. ***left thin*** *if $gA \cap A$ is finite for every g in $G \setminus \{e\}$.*

2. ***right thin*** *if $Ag \cap A$ is finite for every g in $G \setminus \{e\}$.*

3. ***thin*** *if $g_1A \cap g_2A$ is finite for each pair of distinct elements g_1, g_2 in G.*

Example 4.3.1 *The examples are from [15].*

1. *The sets $\{n! : n \in \mathbb{N}\}$ and $\{2^n : n \in \mathbb{N}\}$ are infinite thin subsets of $\mathbb{Z}$.*

2. *Consider the infinite set Abelian periodic group $G = \bigoplus\limits_{\omega} \mathbb{Z}_2$. Let A be an infinite subset of G such that for each n, the set of all $x = (x_k) \in A$ with $x_k \neq 0$ for some $k \leq n$ is finite. Now for each pair of distinct elements $g_1, g_2 \in G$, $|(g_1 + A) \cap (g_2 + A)|$ needs to be calculated.*
   But $g_1 \neq g_2$ if and only if $g = g_1 - g_2 \neq 0_\kappa$, so it is enough to calculate $|A \cap (g + A)|$ if $g \neq 0_\kappa$. For any $g = (x_1, x_2, x_3, \ldots, x_k, \ldots)$ where the first nonzero component appears in the k-th position, denote by A_1 the set of all $x = (x_k) \in A$ with $x_k \neq 0$. By assumption, A_1 is a finite subset of A. Put $A = A_1 \cup A_2$, a disjoint union.

$$\begin{aligned} A \cap (g + A) &= (A_1 \cup A_2) \cap (g + A) \\ &= (A_1 \cap (g + A)) \cup (A_2 \cap (g + A)) \\ &= (A_1 \cap (g + A)) \cup (A_2 \cap (g + (A_1 \cup A_2))) \\ &= (A_1 \cap (g + A)) \cup (A_2 \cap (g + A_1)) \cup (A_2 \cap (g + A_2)). \end{aligned}$$

   On the right hand side, the cardinalities of the first and the second term are finite and the last term is empty. Then $|A \cap (g + A)|$ is finite, which implies A is a thin subset of G.

Definition 4.3.2 *Let G be a group. A subset A of G with the identity e is said to be*

1. ***left k-thin*** *for $k \in \mathbb{N}$, if $|gA \cap A| \leq k$ for every g in $G \setminus \{e\}$.*

2. ***right k-thin*** *for $k \in \mathbb{N}$, if $|Ag \cap A| \leq k$ for every g in $G \setminus \{e\}$.*

3. ***k-thin*** *for $k \in \mathbb{N}$, if $|g_1A \cap g_2A| \leq k$ for each pair of distinct elements g_1, g_2 in G.*

Recall that ω-thin sets were used in [1] to prove that every infinite left amenable group G admits $2^{2^{|G|}}$ distinct left invariant Banach measures. Chou in [1] used the following two results that were reformulated in [8].

Lemma 4.3.1 *Let G be an infinite group and let A be an ω-thin subset of G such that $|A| = |G|$. Then $p, q \in A^*$, $p \neq q$ implies $(\beta G)p \cap (\beta G)q = \emptyset$.*

Proof. Let $|A| = \kappa$, where κ is regular, and let $|G| = \kappa$ such that $|gA \cap A| \leq \omega$ for $g \neq e$ (e the identity of G). Take $x \in G$. We can find $A_x \subset A$ such that $|A \setminus A_x| < \omega$. We construct $G = \{g\alpha : \alpha < \kappa\}$. For every $\alpha < \kappa$, there exist $A_\alpha \subset A$ such that $|A \setminus A_\alpha| < \kappa$ and $\{g_\alpha A_\alpha : \alpha < \kappa\}$ are pairwise disjoint. Let $p, q \in \mathcal{U}(A)$, $A_\alpha = P_\alpha \cup Q_\alpha$, where $P_\alpha \in p$ and $Q_\alpha \in q$. Since $P \cap Q = \emptyset$, for $P \in p$ and $Q \in q$, and if we take $P_\alpha = P \cap A_\alpha$ and $Q_\alpha = Q \cap A_\alpha$, then

$$\left(\bigcup_{\alpha<\kappa} g_\alpha P_\alpha\right) \bigcap \left(\bigcup_{\alpha<\kappa} g_\alpha Q_\alpha\right) = \emptyset.$$

And, we have

$$\overline{\bigcup_{\alpha<\kappa} g_\alpha P_\alpha} \supseteq \beta G p \text{ and } \overline{\bigcup_{\alpha<\kappa} g_\alpha Q_\alpha} \supseteq \beta G q.$$

Thus, βGp and βGq are disjoint. $\square$

The following theorem is from [8].

Theorem 4.3.1 *Let G be an infinite group. Then the semigroup βG has $2^{2^{|G|}}$ pairwise disjoint closed left ideals.*

Proof. The proof is an immediate consequence of Lemma 4.3.1 above. $\square$

The following lemma without condition (2) is a well known fact from [1].

Lemma 4.3.2 *Let G be a group, $B \subseteq G$, and let $|G| = |B| = \kappa \geq \omega$. Then there is $A \subseteq B$ with $|A| = \kappa$ such that*

(1) A is a left 3-thin subset of G

(2) for every $g, h \in G$, $|\{a \in A : a \neq gah \in A\}| < \kappa$

Proof. Enumerate G as $\{g_\alpha : \alpha < \kappa\}$. Construct recursively a κ-sequence $(a_\alpha)_{\alpha<\kappa}$ in B such that

$$a_\alpha \in B \setminus (A_\alpha A_\alpha^{-1} A_\alpha \cup C_\alpha A_\alpha C_\alpha),$$

where

$$A_\alpha = \{a_\beta : \beta < \alpha\} \text{ and } C_\alpha = \{g_\beta^{\pm 1} : \beta < \alpha\}.$$

The set $A = \{a_\alpha : \alpha < \kappa\}$ is as required and $|A| = \kappa$.

To prove (1), assume instead that one has $g \in G \setminus \{1\}$ and distinct α, β, and γ such that $\{a_\alpha, a_\beta, a_\gamma\} \subseteq gA$. Pick distinct δ, μ, and ν such that $a_\alpha = ga_\delta$, $a_\beta = ga_\mu$, and $a_\gamma = ga_\nu$. Consider two cases:

(a) $\max\{\alpha, \beta, \gamma, \delta, \mu, \nu\} \in \{\alpha, \beta, \gamma\}$. Then without lost of generality, this maximum is α. Also $\alpha > \delta$. Since $\mu \neq \nu$, without lost of generality we have $\alpha > \mu$ so that $a_\alpha = ga_\delta = a_\beta a_\mu^{-1} a_\delta \in A_\alpha A_\alpha^{-1} A_\alpha$, a contradiction.

(b) $\max\{\alpha, \beta, \gamma, \delta, \mu, \nu\} \notin \{\alpha, \beta, \gamma\}$. Then without lost of generality that maximum is δ and ona has that $\delta > \alpha$, $\delta > \beta$, and $\delta > \mu$. Thus $a_\delta = g^{-1} a_\alpha = a_\mu a_\beta^{-1} a_\alpha \in A_\delta A_\delta^{-1} A_\delta$, a contradiction.

To prove (2), assume instead that we have some β and δ such that

$$|\{a \in A : a \neq g_\beta a g_\delta \in A\}| = \kappa.$$

Pick $\alpha > \max\{\beta, \delta\}$ such that $a_\alpha \neq g_\beta a_\alpha g_\delta \in A$ and pick δ such that $g_\beta a_\alpha g_\delta = a_\gamma$. If $\gamma > \alpha$, then $a_\gamma \in C_\gamma A_\gamma C_\gamma)$. If $\gamma < \alpha$, then $a_\alpha = g_\beta^{-1} a_\gamma g_\delta^{-1} \in C_\alpha A_\alpha C_\alpha$. In either case we have a contradiction.

□

We list in the following some results from [19] that inspire the approach of some of our results and their proofs.

Theorem 4.3.2 *Let S be a countably infinite discrete semigroup embedded in a countable discrete group G. Let $q \in K(\beta S)$ and let $B \in q$. Let $n \in \mathbb{N}$ and let $p_1, p_2, \ldots, p_n \in S^* \setminus K(\beta S)$. Then there exists $A \subseteq B$ with $|A| = \omega$ such that, for every $r \in A^*$ and every $i, j \in \{1, 2, \ldots, n\}$, $p_i \notin (\beta G) r p_j$.*

Proof. See the proof of Theorem 6.55 in [19]. □

Theorem 4.3.3 *Let S be a countably infinite discrete semigroup embedded in a countable discrete group G. Let $p \in S^* \setminus K(\beta S)$, let $q \in K(\beta S)$ and let $B \in q$. Let $n \in \mathbb{N}$. There exists $A \subseteq B$ with $|A| = \omega$ such that whenever $r_1, r_2 \in A^*$ and $r_1 \neq r_2$, one has $(\beta G)r_1 p \cap (\beta G)r_2 p = \emptyset$.*

Proof. See the proof of Theorem 6.56 in [19]. □

Theorem 4.3.4 *Let S be a countable discrete semigroup embedded in a countable discrete group G. Let $n \in \mathbb{N}$ and let $p_1, p_2, \ldots, p_n \in S^* \setminus K(\beta S)$. Suppose that, in addition, for every distinct $i, j \in \{1, 2, \ldots, n\}$ $p_i \notin Gp_j$. Let $A \subseteq S$ with $|A| = \omega$ such that, for every $r \in A^*$ and $i, j \in \{1, 2, \ldots, n\}$, $p_i \notin (\beta G)rp_j$. Then, for every $r \in A^*$ and every distinct $i, j \in \{1, 2, \ldots, n\}$, one has $(\beta G)rp_i \cap (\beta G)rp_j = \emptyset$.*

Proof. See the proof of Theorem 6.57 in [19]. □

Theorem 4.3.5 *Let $p \in \mathbb{N}^* \setminus K(\beta\mathbb{N})$. Let $A \subseteq \mathbb{N}$ with $|A| = \omega$ such that, for every $r \in A^*$, $p \notin \beta\mathbb{Z} + r + p$, and for every $r_1, r_2 \in A^*$ and $r_1 \neq r_2$, $(\beta\mathbb{Z} + r_1 + p) \cap (\beta\mathbb{Z} + r_2 + p) = \emptyset$. Then, for every $r \in A^*$, $\mathbb{N}^* + r + p \subsetneq \mathbb{N}^* + p$.*

Proof. See the proof of Theorem 6.59 in [19]. □

Corollary 4.3.1 *Let $p \in \mathbb{N}^* \setminus K(\beta\mathbb{N})$. Then $\mathbb{N}^* + p$ contains 2^{2^ω} disjoint semiprincipal left ideals.*

Proof. The proof follows from Theorem 4.3.5 with $A \subseteq \mathbb{N}$ and $|A^*| = 2^{2^\omega}$ (see Theorem 2.2.21). □

Corollary 4.3.2 *Let $p \in \mathbb{N}^* \setminus K(\beta\mathbb{N})$. Then $\mathbb{N}^* + p$ belongs to a decreasing sequence of semiprincipal left ideals of $\mathbb{N}^*$, each maximal subject to being strictly contained in its predecessor.*

Proof. The proof uses Theorem 4.3.3 and Theorem 4.3.5. See the proof of Corollary 6.61 in [19] for more details. □

Recall that, from Definition 1.1.17 and Theorem 1.1.4(3), the principal right ideal of a semigroup S generated by $x \in S$ is $xS \cup \{x\}$. Then, by Theorem 1.3.8(1), $c\ell(xS)$ is a two-sided ideal of S. By Definition 1.1.17, if $c\ell(xS)$ contains x, then it is called the principal two-sided ideal generated by x.

Definition 4.3.3 *Let G be a commutative group and let $p \in \beta G$. Then $c\ell(p + \beta G)$ is the* ***principal closed two-sided ideal generated by*** *p.*

Theorem 4.3.6 *There is a sequence $(p_n)_{n=1}^{\infty}$ in $\mathbb{N}^*$ such that $(p_n + \beta\mathbb{Z})_{n=1}^{\infty}$ is a strictly increasing chain of principal right ideals of $\beta\mathbb{Z}$ and $(c\ell(p_n + \beta\mathbb{Z}))_{n=1}^{\infty}$ is a strictly increasing chain of principal closed two-sided ideals of $\beta\mathbb{Z}$.*

Proof. See the proof of Theorem 6.74 in [19]. □

Similarly, we define the notion of a semiprincipal closed two-sided ideal as the closure of a two-sided ideal of βS.

Definition 4.3.4 *Given a semigroup S and $p \in \beta S$, let*

$$I(S, p) = c\ell((\beta S)p(\beta S)).$$

We call $I(S, p)$ the ***semiprincipal closed two-sided ideal generated by*** *p.*

Theorem 4.3.7 *Let S be an infinite discrete semigroup and let $p \in \beta S$. Then $I(S, p)$ is a closed two-sided ideal of βS.*

Proof. Let S be an infinite discrete semigroup. The extended operation makes βS a compact right topological semigroup with $S \subseteq \Lambda(\beta S)$. The closure inclusion property implies then $c\ell(S) \subseteq c\ell(\Lambda(\beta S))$. We have also that $c\ell_{\beta S}(S) = \beta S$. Since βS is compact, $\Lambda(\beta S) \subseteq \beta S$ implies that $c\ell(\Lambda(\beta S)) \subseteq \beta S$. Thus, $c\ell(\Lambda(\beta S)) = \beta S$. That is, $\Lambda(\beta S)$ is dense in βS. Let $p \in \beta S$. By Lemma 1.1.4, $(\beta S)p(\beta S)$ is a two-sided ideal of βS. Hence, $(\beta S)p(\beta S)$ is a right ideal of βS. Using Theorem 1.3.8, the closure $I(S, p) = c\ell((\beta S)p(\beta S))$ is a two-sided ideal of βS. $I(S, p)$ is closed as a closure in βS. □

Note that $I(S, p) \cup \{p\}$ is the smallest closed two-sided ideal of βS containing p. But the two-sided ideal $I(S, p)$ is not a principal one since it need not contain p. If it does, then it is the smallest closed two-sided ideal of βS containing p (see Definition 1.1.14 and Theorem 1.1.4).

Theorem 4.3.8 *Let G be a group, let $B \subseteq G$, and let $|B| = |G| = \kappa \geq \omega$. Then there is $A \subseteq B$ with $|A| = \kappa$ such that for every $p \in U(A)$, one has $I(G, p) \cap \overline{A} = \{p\}$.*

Proof. Let A be a subset of B guaranteed by Lemma 4.3.2. We claim that for every $p \in U(A)$, one has $I(G,p) \cap \overline{A} = \{p\}$. Since $I(G,p) = c\ell(Gp(\beta G))$, it suffices to prove that $(Gp(\beta G)) \cap \overline{A} = \{p\}$.

We first show that $(G^*G^*) \cap \overline{A} = \emptyset$.

Assume the contrary that $qr \in \overline{A}$ for some $q, r \in G^*$. Let

$$Q = \{x \in G : x^{-1}A \in r\}.$$

Then $Q \in q$ and $Q \ni x \mapsto R_x \in r$ such that $xR_x \subseteq A$. Pick any distinct x and y in Q and let $R = R_x \cap R_y$. Then $R \subseteq (x^{-1}A) \cap (y^{-1}A)$. Since $R \in r$, it follows that $(x^{-1}A) \cap (y^{-1}A) \in r$. That is infinite, since $r \in G^*$. Consequently, by the fact that G is a group, we also have $(xy^{-1}A) \cap A$ is infinite. But this contradicts condition (1) of Lemma 4.3.2.

From $(G^*G^*) \cap \overline{A} = \emptyset$ and the fact that G^* is a left ideal of βG (see Corollary 3.1.5), we obtain that $(Gp(\beta G)) \cap \overline{A} = (GpG) \cap \overline{A}$. Hence, in order to finish the proof, it remains to show that $(GpG) \cap \overline{A} = \{p\}$.

Assume on the contrary that $gph = q$ for some $g, h \in G$ and $q \in \overline{A} \setminus \{p\}$. Then there is $P \in p$ such that $P \subseteq A, gPh \subseteq A$ and $P \cap (gPh) = \emptyset$. It follows that $\{a \in A : a \neq gah \in A\} \in p$. Since $p \in U(A)$, this contradicts condition (2) of Lemma 4.3.2. □

Theorem 4.3.9 *Let S be a semigroup embeddable into a group, let $B \subseteq S$, and let $|B| = |S| = \kappa \geq \omega$. There is $A \subseteq B$ with $|A| = \kappa$ such that whenever $p, q \in U(A)$ and $p \neq q$, one has $I(S,p) \setminus I(S,q) \neq \emptyset$ and $I(S,q) \setminus I(S,p) \neq \emptyset$.*

Proof. Let G be a group containing S and let $|G| = \kappa$. By Theorem 4.3.8, there is $A \subseteq B$ with $|A| = \kappa$ such that for every $p \in U(A)$, one has $I(G,p) \cap \overline{A} = \{p\}$. Now let $p, q \in U(A)$ and $p \neq q$. We have also $I(G,q) \cap \overline{A} = \{q\}$. Then

$$I(G,p) \cap (GqG) = \emptyset \text{ and } I(G,q) \cap (GpG) = \emptyset.$$

Indeed, if $g, h \in G$ and $gqh \in I(G,p)$, then $q \in g^{-1}I(G,p)h^{-1} \subseteq I(G,p)$. It follows that,

$$I(S,p) \cap (SqS) = \emptyset \text{ and } I(S,q) \cap (SpS) = \emptyset,$$

and so

$$I(S,p) \setminus I(S,q) \neq \emptyset \text{ and } I(S,q) \setminus I(S,p) \neq \emptyset,$$

since $I(S,p)$ and $I(S,q)$ are nonempty as ideals of βS. $\square$

Corollary 4.3.3 *For every infinite discrete semigroup S embeddable into a group, βS contains $2^{2^{|S|}}$ pairwise incomparable semiprincipal closed ideals.*

Proof. Since $|U(A)| = 2^{2^{|A|}}$ and for any two distinct elements p and q in $U(A)$ implies two distinct ideals $I(S,p)$ and $I(S,q)$, then we obtain from Theorem 4.3.9 that βS contains $2^{2^{|S|}}$ pairwise incomparable semiprincipal closed ideals. $\square$

Chapter 5

Decomposition of $U(S)$ into closed left ideals of βS

Recall that, given an infinite cancellative discrete semigroup S, $U(S)$ the set of all uniform ultrafilters is a closed two-sided ideal of βS. In this chapter, we present the results about the decomposition of $U(S)$ into closed left ideals of βS.

In the first section, we review with details two main results of the decomposition theorems. The first result is from [3] and [19] which shows that, for every infinite cancellative discrete semigroup S of cardinality κ, $U(S)$ can be decomposed into 2^{2^κ} left ideals of βS. And the second one is from [14] and it is a stronger version of the previous one. It shows that $U(S)$ can be decomposed into 2^{2^κ} *closed* left ideals of βS when S is an infinite *weakly* cancellative discrete semigroup of cardinality κ.

In the remaining part of the chapter, we present our result from [30] about the study of the quotient spaces of $U(S)$ corresponding to the decompositions into closed left ideals of βS.

5.1 Decomposition Theorems

Definition 5.1.1 *(See [19]) Let S be a discrete semigroup with cardinality κ. Define a binary relation R on $U_\kappa(S)$ by stating that*

$$pRq \text{ if } (c\ell(Sp) \cup c\ell(pS)) \cap (c\ell(Sq) \cup c\ell(qS)) \neq \emptyset.$$

The relation R can be extended to an equivalence relation $\sim$ on $U_\kappa(S)$ as follows: for every $p, q \in U_\kappa(S)$,

$$p \sim q \text{ if and only if there exists some } n \in \mathbb{N} \text{ with } pR^n q.$$

The notation R^n means the composition of R with itself n times. Thus, given $p, q \in U_\kappa(S)$, $p \sim q$ if and only if there exist elements $x_0, x_1, x_2, \ldots, x_n \in U_\kappa(S)$ such that $p = x_0$, $q = x_n$ and $x_i R x_{i+1}$ for every $i \in \{0, 1, 2, \ldots, n-1\}$.

Recall that the equivalence class of an element $p \in U_\kappa(S)$ is denoted by $[p]$.

Lemma 5.1.1 *Let S be an infinite cancellative discrete semigroup of cardinality κ and let E be a subset of S of cardinality κ. Then there exist an enumeration $(a_\sigma)_{\sigma<\kappa}$ of S, an increasing sequence $(m(\sigma))_{\sigma<\kappa}$ in κ and a nondecreasing function $f : \kappa \to \kappa$ such that,*

1. *if σ is a limit, then $m(\sigma) = \sup_{\delta<\sigma} m(\delta)$,*
2. *if $\sigma < \omega$, then $|m(\sigma)| < \omega$,*
3. *if $\omega \leq \sigma < \kappa$, then $|m(\sigma)| = |\sigma|$,*
4. *for all $\sigma < \kappa$, $m(f(\sigma)) \leq \sigma < m(f(\sigma) + 1)$, and*
5. *for all $\sigma < \kappa$, $a_{m(\sigma)} \in E$.*

   Further, if $T = \{(a_\nu, a_\tau) :$ there exist $\gamma < f(\tau), \mu < f(\nu)$, and $z \in S$ with (i) $z = a_\tau a_\gamma$ or $z = a_\nu a_\tau$ and (ii) $z = a_\nu a_\mu$ or $z = a_\mu a_\nu\}$ define $B_o(y) = \{y\}$ and $B_n(y) = \{x \in S : xT^n y\}$ for $n > 0$, then

6. *if $\sigma < \kappa$, $\tau < m(\sigma)$, $n < \omega$, $\nu < \kappa$, and $a_\nu \in B_n(a_\tau)$, then $\nu < m(\sigma + n)$.*

Proof. See the proof of Lemma 3.1 in [3]. $\square$

Lemma 5.1.2 *Let S be an infinite cancellative discrete semigroup of cardinality κ and let E be a subset of S of cardinality κ. Let $(a_\sigma)_{\sigma<\kappa}$, $(m(\sigma))_{\sigma<\kappa}$, $f : \kappa \to \kappa$, T and $B_n(y)$ be as in Lemma 5.1.1. Let $p, q \in U_\kappa(S)$ such that $p \sim q$. Then for each $A \in q$, $\bigcup_{x \in A} B_n(x) \in p$.*

Proof. See the proof of Lemma 3.2 in [3]. □

Lemma 5.1.3 *Let S be an infinite cancellative discrete semigroup of cardinality κ and let E be a subset of S of cardinality κ. Let $(a_\sigma)_{\sigma<\kappa}$, $(m(\sigma))_{\sigma<\kappa}$, $f:\kappa\to\kappa$, T and $B_n(y)$ be as in Lemma 5.1.1.*

1. *Assume $\kappa=\omega$ and for each $n<\omega$, let $y_n=a_{m(n^2+2n)}$. If $t<n<m$ then*
   - *(a) if $a_l\in\bigcup_{k=0}^{2n}B_k(y_t)$, then $l<m(n^2+2n)$ and*
   - *(b) $(\bigcup_{k=0}^{n}B_k(y_n))\cap(\bigcup_{k=0}^{t}B_k(y_t))=\emptyset$.*
2. *Assume $\kappa>\omega$ and for each $\sigma<\kappa$ let $y_\sigma=a_{m(\omega\cdot\sigma)}$. Then for $\sigma<\nu<\kappa$,*
$$(\bigcup_{n<\omega}B_n(y_\sigma))\cap(\bigcup_{n<\omega}B_n(y_\nu))=\emptyset.$$

Proof. See the proofs of Lemma 3.4 and Lemma 3.3 in [3] respectively. □

The following theorem is also called *The Van Douwen's Right Ideal Theorem* (see [3]) as a tribute to its author. Unfortunately, he passed away without providing the proof. The proof was provided by Davenport and Hindman that we repeat because of its importance to the approaches used in this chapter.

Theorem 5.1.1 *If S is an infinite cancellative discrete semigroup of cardinality κ, then there exists a decomposition $\mathcal{I}$ of $U_\kappa(S)$ into pairwise disjoint subsets such that*

1. *$|\mathcal{I}|=2^{2^\kappa}$,*
2. *for each $I\in\mathcal{I}$, I is nowhere dense in $U_\kappa(S)$.*
3. *for each $I\in\mathcal{I}$, I is a right ideal of βS, and*
4. *for all $I\in\mathcal{I}$ and all $p\in I$, $c\ell(Sp)\subseteq I$.*

Proof.

To prove (1), choose $(a_\sigma)_{\sigma<\kappa}$, $(m(\sigma))_{\sigma<\kappa}$, and $(y_\sigma)_{\sigma<\kappa}$ all as defined in Lemma 5.1.1. Suppose $\kappa > \omega$ and for each $\sigma < \kappa$, let $y_\sigma = a_{m(\omega\cdot\sigma)}$. Consider $\mathcal{I} = \{[p] : p \in U_\kappa(S)\}$. From Theorem 2.2.23, we have $|U_\kappa(S)| = 2^{2^\kappa}$. Now let $Y = \{y_\sigma : \sigma < \kappa\}$. For any two distinct elements p and q of $\overline{Y} \cap U_\kappa(S)$, show that $[p] \neq [q]$. Then using Lemma 5.1.3 to the cases $\kappa = \omega$ and $\kappa > \omega$, Lemma 5.1.2 yields a contradiction.

To prove (2), take any $p \in U_\kappa(S)$ and show that $[p]$ is nowhere dense in $U_\kappa(S)$. Consider the contrary, that is, the closure of $[p]$ has nonempty interior in $U_\kappa(S)$, and taking E a subset of S of cardinality κ with $\overline{E} \cap U_\kappa(S)$ a subset of the closure of $[p]$. Choose $(a_\sigma)_{\sigma<\kappa}$, $(m(\sigma))_{\sigma<\kappa}$, $f : \kappa \to \kappa$, T and $B_n(y)$ as in Lemma 5.1.1. Then using Lemma 5.1.3 to the cases $\kappa = \omega$ and $\kappa > \omega$, Lemma 5.1.2 yields a contradiction.

To prove (3) , take $p \in U_\kappa(S)$ and show that $[p]$ is a right ideal of βS. That is, let $q \in [p]$ and $r \in \beta S$ and show that $qr \in [p]$. But first, one needs to make sure that qr is a uniform ultrafilter on S, that is, $qr \in U_\kappa(S)$. In fact, let $A \in qr$. It implies, by definition, $\{x \in S : Ax^{-1} \in q\} \in r$. Choose $x \in S$ with $Ax^{-1} \in q$. Thus, $|Ax^{-1}| = \kappa$, since q is a uniform ultrafilter on S. Then, since S is right cancellative, ρ_x takes Ax^{-1} one-to-one into A. That is, $|A| = \kappa$. So, we have indeed $qr \in U_\kappa(S)$. Now $q \in [p]$ and $r \in \beta S$ imply $qr \in q\beta S$ and $(qr)r \in (qr)\beta S$. We have then $qrr \in q\beta S \cap (qr)\beta S$. Thus, $q\beta S \cap (qr)\beta S \neq \emptyset$. So, $(qr)Rq$. That is, $qr \sim q$. Hence, $qr \in [p]$.

To prove (4), let $p \in U_\kappa(S)$. For each $q \in [p]$, let $r \in c\ell(Sq)$. Then let $v \in c\ell(Sr)$ and let $A \in v$. Take $x \in S$ such that $xr \in \overline{A}$. Since λ_x is continuous, there exist $B \in r$ such that $x\overline{B} \subseteq \overline{A}$. Since $r \in c\ell(Sq)$, we can find $y \in S$ such that $yq \in \overline{B}$. So $xyq \in \overline{A}$. Since x, y are taken in S, we have $\overline{A} \cap Sq \neq \emptyset$. This implies $c\ell(Sr) \cap c\ell(Sq) \neq \emptyset$. Thus, $r \sim q$. Hence $r \sim p$ since $q \in [p]$. This concludes that indeed $r \in [p]$.

$\square$

Lemma 5.1.4 *Let S be a discrete infinite weakly left cancellative semigroup with cardinality κ and let T be a subset of S of cardinality κ. Then there exists a function $f : S \to \kappa$ such that*

1. *$f[T] = \kappa$,*

2. *for every $\lambda < \kappa$, $f^{-1}[\lambda]$ is finite if λ is finite and $|f^{-1}[\lambda]| \leq |\lambda|$ if λ is infinite, and*

3. *for every $s, s' \in S$, if $f(s) + 1 < f(s')$, then*

$$f(ss') \in \{f(s') - 1, f(s'), f(s') + 1\}$$

if $f(s')$ is not a limit ordinal then

$$f(ss') \in \{f(s'), f(s') + 1\}$$

otherwise.

Proof. See the proof of Lemma 6.47 in [19] for more details. Rearrange S as a κ-sequence $(s_\alpha)_{\alpha<\kappa}$. Then construct an increasing κ-sequence $(E_\alpha)_{\alpha<\kappa}$ of subsets of S that satisfies the following conditions for $\gamma < \kappa$:

1. $s_\gamma \in E_\gamma$;

2. $T \cap \bigcup_{\alpha<\gamma} E_\alpha \subsetneq T \cap E_\gamma$;

3. if $\alpha < \gamma$, then $E_\alpha E_\alpha \subseteq E_\gamma$;

4. if $\gamma = \alpha + 1$, $s \in E_\alpha$, and $ss' \in E_\alpha$, then $s' \in E_\gamma$;

5. if $\gamma < \omega$, then $|E_\alpha| < \omega$; and

6. if $\gamma \geq \omega$, then $|E_\gamma| \leq \gamma$.

Choose any $t \in T$ and put $E_0 = \{s_0, t\}$. Then assume that $0 < \gamma < \kappa$ and that E_α has been defined for every $\alpha < \gamma$.

Let $U = \bigcup_{\alpha<\gamma} E_\alpha$ and notice that $|U| < \omega$ if $\gamma < \omega$ and that $|U| \leq |\gamma|$ if $\gamma \geq \omega$. If γ is a limit ordinal, let $V = \emptyset$. If $\gamma = \alpha + 1$, let

$$V = E_\alpha E_\alpha \cup \{t \in S \,:\, E_\alpha t \cap E_\alpha \neq \emptyset\}$$

and note that, since S is weakly left cancellative, $|V| < \omega$ if $\gamma < \omega$ and $|V| \leq |\gamma|$ if $\gamma \geq \omega$. Pick $t \in T \setminus U$ and let $E_\gamma = U \cup V \cup \{s_\gamma, t\}$. One can verify that the induction hypotheses are satisfied.

Now define $f : S \to \kappa$ by putting $f(s) = \min\{\alpha < \kappa : s \in E_\alpha\}$.

For every $\beta < \kappa$ there exists $t \in T \cap E_\beta \setminus \bigcup_{\alpha<\beta} E_\alpha$. Since $f(t) = \beta$, it follows that $f[T] = \kappa$. For every $\lambda < \kappa$, $f^{-1}[\lambda] \subseteq E_\lambda$. So $f^{-1}[\lambda]$ is finite if λ is finite and $|f^{-1}[\lambda]| \leq |\lambda|$ if λ is infinite.

Finally, suppose that $f(s) = \alpha$, $f(s') = \beta$ and $\alpha+1 < \beta$. Let $\gamma = f(ss')$. Then $ss' \in E_{\beta+1}$, since $s, s' \in E_\beta$, and so $\gamma \leq \beta + 1$. We show that $\gamma+1 \geq \beta$. Suppose instead that $\gamma+1 < \beta$ and let $\mu = \max\{\alpha, \gamma\}$. Then $s, ss' \in E_\mu$ and so $s' \in E_{\mu+1}$. Thus $\beta \leq \mu + 1 < \beta$, a contradiction. $\square$

Lemma 5.1.5 *Let S be a discrete infinite weakly left cancellative semigroup with cardinality κ and let T be a given subset of S with cardinality κ. Let f be the function guaranteed by Lemma 5.1.4 above and let $\tilde{f} : \beta S \to \beta\kappa$ be its extension.*

1. *If $\kappa = \omega$, we put $A = \{2^m : m \in \mathbb{N}\}$.*
2. *If $\kappa > \omega$, we put A equal to the set of limit ordinals in κ.*

If B and C are disjoint subsets of A with cardinality κ and if $B \in \tilde{f}(p)$ and $C \in \tilde{f}(q)$ for some $p, q \in U_\kappa(S)$, then $p \not\sim q$.

Proof. See the proof of Lemma 6.49 in [19] for more details.

1. Suppose first that $\kappa = \omega$ and that $A = \{2^m : m \in \mathbb{N}\}$. We show by induction that, for every $n \in \mathbb{N}$, every $x, y \in S^*$ and every $X \subseteq \mathbb{N}$, if $X \in \tilde{f}(x)$ and $xR^n y$, then $\bigcup_{i=-2n}^{2n}(i + X) \in \tilde{f}(y)$.

   Suppose first that $n = 1$. If xRy then $ux = vy$ for some $u, v \in \beta S$. Let

   $$U = \{ss' : s, s' \in S, f(s) + 1 < f(s'), \text{ and } f(s') \in X\}.$$

   Then $U \in ux$, by Corollary 3.4.1, and $f[U] \subseteq (X-1) \cup X \cup (X+1)$ by Lemma 5.1.4(3). So $(X - 1) \cup X \cup (X + 1) \in \tilde{f}(ux)$. Similarly, if $Y \in \tilde{f}(y)$, then $(Y - 1) \cup Y \cup (Y + 1) \in \tilde{f}(vy)$. So $Y \cap \bigcup_{i=-2}^{2}(i + X) \neq \emptyset$. Thus $\bigcup_{i=-2}^{2}(i + X) \in \tilde{f}(y)$.
   Now assume that the claim holds for $n \in \mathbb{N}$ and we verify for $n+1$.

Suppose that $xR^{n+1}y$. Then $xR^n z$ and zRy for some $z \in S^*$. Let $Z = \bigcup_{i=-2n}^{2n}(i+X)$. By the inductive assumption, $Z \in \tilde{f}(z)$. By the step $n=1$, $\bigcup_{i=-2}^{2}(i+Z) = \bigcup_{i=-2(n+1)}^{2(n+1)}(i+X) \in \tilde{f}(y)$. So the proof by induction is completed.

Now observe that, for every $n \in \mathbb{N}$, $B \cap (\bigcup_{i=-2n}^{2n}(i+C))$ is finite and so $\bigcup_{i=-2n}^{2n}(i+C) \notin \tilde{f}(p)$. Thus one cannot have $pR^n q$.

2. Now suppose that $\kappa > \omega$ and that A is the set of limit ordinals in κ. Let $B' = \{\lambda + n : \lambda \in B \text{ and } n \in \omega\}$. We show that, if $x, y \in U_\kappa(S)$, $B' \in \tilde{f}(x)$, and $xR^n y$, then $B' \in \tilde{f}(y)$.

   Suppose first that xRy. Then $ux = vy$ for some $u, v \in \beta S$. Now, if

   $$U = \{ss' : s, s' \in S, f(s) + 1 < f(s'), \text{ and } f(s') \in B'\},$$

   then $U \in ux$ (see Corollary 3.4.1). By Lemma 5.1.4(3), $f[U] \subseteq B'$. So $B' \in \tilde{f}(ux)$. Similarly, if $\kappa \setminus B' \in \tilde{f}(y)$, then $\kappa \setminus B' \in \tilde{f}(vy)$. This contradicts the assumption that $ux = vy$ and $B' \in \tilde{f}(y)$.

   It follows immediately, by induction, that $xR^n y$ implies that $B' \in \tilde{f}(y)$.

   Since $B' \in \tilde{f}(p)$ and $B' \notin \tilde{f}(q)$, it follows that $p \nsim q$.

$\square$

Except for its weaker hypotheses as reformulated in [19], the following theorem is a special case of *Van Douwen's Right Ideal Theorem.*

Theorem 5.1.2 *Let S be a discrete infinite weakly left cancellative semigroup with cardinality κ. Then there exits a decomposition $\mathcal{I}$ of $U_\kappa(S)$ with the following properties:*

1. *Each $I \in \mathcal{I}$ is a left ideal of βS.*
2. *Each $I \in \mathcal{I}$ is nowhere dense in $U_\kappa(S)$.*
3. *$|\mathcal{I}| = 2^{2^\kappa}$.*

Proof. The detail of the proof can be found in the proof of Theorem 6.53, [19]. Let $\mathcal{I} = \{[p] : p \in U_\kappa(S)\}$.

To prove (1), take $p \in U_\kappa(S)$ and $r \in \beta S$. We have $pRrp$, by the fact that $U_\kappa(S)$ is a left ideal of βS. That is $rp \in [p]$. Thus $[p]$ is a left ideal of βS.

To prove (2), suppose the contrary. That is there is some subset T of S of the same cardinality κ such that $U_\kappa(T) \subseteq c\ell(\{q \in U_\kappa(S) : q \sim p\})$. Take f as guaranteed by Lemma 5.1.4. If $\kappa = \omega$, put $A = \{2^m : m \in \mathbb{N}\}$. If $\kappa > \omega$, put A equal to the set of limit ordinals in κ. Let B and C be two disjoint subsets of A of cardinality κ for both cases. Using Lemma 5.1.5, $c\ell(\{q \in U_\kappa(S) : q \sim p\})$ is disjoint from $c\ell(f^{-1}[B]) \cap U_\kappa(S)$ or from $c\ell(f^{-1}[C]) \cap U_\kappa(S)$. So either $c\ell(f^{-1}[B]) \cap U_\kappa(T)$ or $c\ell(f^{-1}[B]) \cap U_\kappa(T)$ is a nonempty subset of $U_\kappa(T)$ disjoint from $\{q \in S^* : q \sim p\}$. Thus a contradiction.

To prove (3), take f as in Lemma 5.1.4, with $T = S$, and its continuous extension $\tilde{f} : \beta S \to \kappa$. Since f is surjective, $\tilde{f}$ is also surjective. Furthermore, $\tilde{f}^{-1}[U_\kappa(\kappa)] \subseteq U_\kappa(S)$. If $\kappa = \omega$, put $A = \{2^m : m \in \mathbb{N}\}$. If $\kappa > \omega$, put A equal to the set of limit ordinals in κ. For each $u \in U_\kappa(A)$, since $\tilde{f}$ is also surjective, choose $p_u \in U_\kappa(S)$ such that $\tilde{f}(p_u) = u$. If u and v are distinct elements of $U_\kappa(A)$, there is then two disjoint subsets B and C of A such that $B \in u$ and $C \in v$. Thus $B \in \tilde{f}(p_u)$ and $C \in \tilde{f}(p_v)$. It follows from Lemma 5.1.5 that $[p_u] \neq [p_v]$. Now $|A| = \kappa$ implies $|U_\kappa(A)| = 2^{2^\kappa}$, by Theorem 2.2.23. Thus $|\mathcal{I}| = 2^{2^\kappa}$.

□

The decomposition theorem above has been strengthened in [14] using slowly oscillating functions.

Definition 5.1.2 *Let S be an infinite cancellative discrete semigroup of cardinality κ. Let f be a bounded left uniformly continuous real-valued function on S. The function f is said* ***slowly oscillating*** *if, for every $\epsilon > 0$ and for every compact set F in S, there exists a subset A of S with $|A| < \kappa$ such that*

$$|f(st) - f(t)| < \epsilon \text{ and } |f(ts) - f(t)| < \epsilon \text{ whenever } s \in F \text{ and } t \in S \setminus A.$$

We repeat hereafter (without any change) the main lemma and its proof from [14] for the sake of comparison between the approaches used in this chapter.

Lemma 5.1.6 *Suppose that S is a discrete weakly cancellative semigroup of cardinality κ and let X be a subset of S of cardinality κ. There exists a subset T of X of cardinality κ such that the points in $\overline{T}$ can be separated by slowly oscillating functions.*

Proof. The details of the proof can be found in [14]. Suppose first that $\kappa = \omega$. Since S is weakly cancellative, there is a cover $(K_n)_{n=1}^{\infty}$ of S consisting of finite sets satisfying

$$K_n \cup K_n^2 \cup K_n^{-1}K_n \cup K_nK_n^{-1} \subseteq K_{n+1}$$

for every positive integer n. It follows that, for example,

$$K_n^{-1}K_m \subseteq K_{\max\{n,m\}}^{-1}K_{\max\{n,m\}} \subseteq K_{n+m}.$$

For convenience, put $K_n = \emptyset$ for very $n = 0, -1, -2, \ldots$ and notice that

$$K_nK_m \subseteq K_{n+m},\ K_n^{-1}K_m \subseteq K_{n+m},\ \text{and } K_nK_m^{-1} \subseteq K_{n+m}$$

for every integers n and m. There exits a subset $T = (t_n)_{n=1}^{\infty}$ of X such that

$$K_nt_nK_n \cap K_mt_mK_m = \emptyset \text{ whenever } n \neq m.$$

Indeed, suppose that we have chosen the points $t_1, t_2, \ldots, t_{n-1}$. Then $\bigcup_{m=1}^{n-1} K_n^{-1}K_mt_mK_mK_n^{-1}$ is finite because S is weakly cancelative, so we can pick t_n from $X \setminus (\bigcup_{m=1}^{n-1} K_n^{-1}K_mt_mK_mK_n^{-1})$. For every $n \geq 2$, define

$$f_n = \begin{cases} 1 & \text{on } K_1t_nK_1, \\ 1 - \frac{1}{n-1} & \text{on } (K_2t_nK_2) \setminus (K_1t_nK_1), \\ 1 - \frac{2}{n-1} & \text{on } (K_3t_nK_3) \setminus (K_2t_nK_2), \\ \vdots & \\ \frac{1}{n-1} & \text{on } (K_{n-1}t_nK_{n-1}) \setminus (K_{n-2}t_nK_{n-2}), \\ 0 & \text{off } K_{n-1}t_nK_{n-1}. \end{cases}$$

Let $f = \sum_{n \in I} f_n$, where I is any subset of $\{2, 3, \ldots\}$. The function f is bounded, so it has a continuous extension to βS. The function of the same form as f separate points in $\overline{T}$ because

$$f(t_n) = \begin{cases} 1 & \text{if } n \in I, \\ 0 & \text{if } n \notin I \end{cases}$$

for every $n \geq 2$.

To complete the countable case, it remains to show that f is slowly oscillating. Let $0 < \epsilon < 1$ and let F be a finite subset of S. Then $F \subseteq K_m$ for some m. Choose a positive integer l such that $m/l < \epsilon$ and put

$$A = \bigcup_{j=1}^{l} (K_m^{-1} K_j t_j K_j) \cup (K_j t_j K_j) \cup (K_j t_j K_j K_m^{-1}).$$

Then A is finite. Fix s in $F \subseteq K_m$ and t in $S \setminus A$.

If $f(st) \neq 0$, then $st \in K_{n-1} t_n K_{n-1}$ for some n. Therefore $t \in K_m^{-1} K_{n-1} t_n K_{n-1}$ and the choice of A implies that $n > l > m$. Therefore $K_m^{-1} K_{n-1} \subseteq K_n$ and $t \in K_n t_n K_n$. If $f(t) \neq 0$, then t and st are in $K_n t_n K_n$ for some $n > l$. The remaining case is trivial so we can assume that both t and st are in $K_n t_n K_n$ for some $n > l$.

There is a unique k in $\{0, 1, \ldots, n-1\}$ such that $t \in K_{k+1} t_n K_{k+1} \setminus K_k t_n K_k$. Then $st \in K_{m+k+1} t_n K_{k+1} \setminus K_{k-m} t_n K_k$, and so

$$1 - \frac{m+k}{n-1} \leq f(st) \leq 1 - \frac{k-m}{n-1}.$$

Since $f(t) = 1 - k/(n-1)$, it follows that

$$-\frac{m}{n-1} \leq f(st) - f(t) \leq \frac{m}{n-1}.$$

Thus $|f(st) - f(t)| < m/l < \epsilon$. The other requirement for slow oscillation is confirmed similarly.

Now suppose that $\kappa > \omega$. Construct a cover $(S_\alpha)_{\alpha<\kappa}$ of S such that

$$S_\alpha^2 \subseteq S_\alpha,\ S_\alpha^{-1} S_\alpha \subseteq S_\alpha,\ S_\alpha S_\alpha^{-1} \subseteq S_\alpha,$$

$$|S_\alpha| \leq \max\{\omega, |\alpha|\} \text{ and } \bigcup_{\beta<\alpha} S_\beta \subseteq S_\alpha$$

for every $\alpha < \kappa$. Let $(S_\alpha)_{\alpha<\kappa}$ be an enumeration of S. For each $\alpha < \kappa$, define S_α as follows. Put $Y_1 = (s_\beta)_{\beta\leq\alpha}$, and for every $n = 1, 2, \ldots$ put

$$Y_{n+1} = Y_n \cup Y_n^2 \cup Y_n^{-1} Y_n \cup Y_n Y_n^{-1}.$$

Finally, put $S_\alpha = \cup_{n=1}^{\infty} Y_n$. The required properties of $\{S_\alpha\}_{\alpha<\kappa}$ are easily confirmed. The application of transfinite induction gives a subset $T = \{t_\alpha\}_{\alpha<\kappa}$ of X such that

$$S_\alpha t_\alpha S_\alpha \cap S_\beta t_\beta S_\beta = \emptyset \text{ whenever } \alpha \neq \beta.$$

For each $\alpha < \kappa$, let f_α be the characteristic function of $S_\alpha t_\alpha S_\alpha$. It suffices to show that for each subset I of κ the function $f = \sum_{\alpha\in I} f_\alpha$ is slowly oscillating.

Let $F \subseteq S$ be finite, and let $\alpha < \kappa$ be such that $F \subseteq S_\alpha$. Put

$$A = \bigcup_{\gamma<\kappa} (S_\alpha^{-1} S_\gamma t_\gamma S_\gamma) \cup (S_\gamma t_\gamma S_\gamma) \cup (S_\gamma t_\gamma S_\gamma S_\gamma^{-1}),$$

and note that $|A| \leq \max\{\omega, |\alpha|\} < \kappa$. Let $s \in F$ and $t \in S \setminus A$. If $f(st) = 1$, then $t \in S_\alpha^{-1} S_\beta t_\beta S_\beta$ for some β in I. By the choice of A, we have $\beta \geq \alpha$, and so $t \in S_\beta t_\beta S_\beta$. Therefore also $f(t) = 1$. On the other hand, if $f(t) = 1$, then $f(st) = 1$. A similar argument prove also the second requirement for slow oscillation. $\square$

We recall from [16] the following definition.

Definition 5.1.3 *An equivalence relation R on a topological space X is said to be closed if the canonical mapping of X onto X/R is closed.*

Then we give the following proposition taken from [16] without proof.

Theorem 5.1.3 *Let X, Y two topological spaces, let $f : X \to Y$ be a continuous mapping, let R be the equivalence relation $f(x) = f(y)$ on X, and let $X \xrightarrow{p} X/R \xrightarrow{h} f(X) \xrightarrow{i} Y$ be the canonical decomposition of f. Then the following three statements are equivalent:*

1. *f is a closed map.*
2. *The mappings p, h, i are closed.*
3. *The equivalence relation R is closed, h is a homeomorphism, and $f(X)$ is also a closed subset of Y.*

Proof. See the proof in [16]. $\square$

Definition 5.1.4 *A set A is said to be **saturated** with respect to R if and only if it is the union of a set of equivalence classes with respect to R; or equivalently, if for every $x \in A$ the equivalence class of x is contained in A.*

The closed subsets of X/R are thus the canonical images of the subsets B of X which are saturated with respect to R.

The following theorem provides a stronger result with respect to Theorem 5.1.2. The proof uses Lemma 5.1.6 above and some techniques from Functional Analysis. The rest of the proof proceeds as in the proof of Theorem 5.1.2 by usind the partitions obtained from the slowly oscillating functions on S.

Theorem 5.1.4 *Let S be a discrete weakly cancellative semigroup of cardinality $\kappa \geq \omega$. There is a decomposition $\mathcal{I}$ of $U_\kappa(S)$ into a pairwise disjoint sets such that:*

1. *each member of $\mathcal{I}$ is a closed left ideal in βS,*
2. *if $I \in \mathcal{I}$ and $x \in I$, then $c\ell(xS) \subseteq I$,*
3. *each member of $\mathcal{I}$ has an empty interior in $U(S)$,*
4. *$|\mathcal{I}| = 2^{2^\kappa}$.*

Proof. The detail of the proof can be found in [14]. Let S be a discrete weakly cancellative semigroup of cardinality $\kappa \geq \omega$. Define an equivalence relation on $U_\kappa(S)$ such that the family $\mathcal{I}$ of the equivalence classes satisfies the above properties.

For every x and y in $U_\kappa(S)$, define a binary relation

$$x \sim y \text{ if } f(x) = f(y) \text{ for every slowly oscillating function } f \text{ on } S.$$

One can prove that $\sim$ is a closed equivalence relation on $U_\kappa(S)$, then each equivalence class is closed. For every $x \in U_\kappa(S)$, $[x]$ is the equivelence class containing x. If $y \in U_\kappa(S)$, then

$$f(sy) = f(y) \text{ and } f(ys) = f(y)$$

for every slowly oscillating function f and every $s \in S$. Therefore

$$Sy \subseteq [x] \text{ and } yS \subseteq [x]$$

whenver $y \in [x]$. Since $[x]$ is closed, so the property (2) is immediate. Since the right traslation by y is continuous and $[x]$ closed, $(\beta S)y \subseteq [x]$.

It remains to show that there are exactly 2^{2^κ} equivalence classes. This statement follows from Lemma 5.1.5 above. Applying this lemma, it follows that distinct points in $\overline{T} \cap U_\kappa(S)$ are in a distinct equivalence classes. The points in $\overline{T} \cap U_\kappa(S)$ correspond to uniform ultrafilters on the set T, so there is at least 2^{2^κ} distinct equivalence classes (see Theorem 2.2.23). That 2^{2^κ} is the exact number of the equivalence classes can be seen by the following argument. □

Now, we introduce an alternative approach and equivalent to the previous one using the following concept from [37].

Definition 5.1.5 *A mapping $f : T \to X$ of a semigroup T into a set X is a **right zero homomorphism** if $f(pq) = f(q)$ for all $p, q \in T$.*

The following two lemmas provide a framework to study the quotient spaces of $U(S)$ obtained from the decompositions into closed left ideals of βS.

Lemma 5.1.7 *If S is a cancellative semigroup, X a compact Hausdorff space, and $\pi : U(S) \to X$ a continuous surjective right zero homomorphism, then*

1. *$\{\pi^{-1}(x) : x \in X\}$ is a decomposition of $U(S)$ into closed left ideals of βS, and*
2. *the corresponding quotient space of $U(S)$ is canonically homeomorphic to X.*

Proof. Let $x \in X$, $q \in \pi^{-1}(x)$ and $y \in \beta S$. Suppose the contrary that $yq \notin \pi^{-1}(x)$. Then $\pi(yq) \neq x$. Since π is a continuous surjective right zero homomorphism, obtain $\pi(q) \neq x$. This implies $q \notin \pi^{-1}(q)$, which is a contradiction. Thus, each member of $\{\pi^{-1}(x) : x \in X\}$ is a left ideal of βS. □

Lemma 5.1.8 *Let S be a cancellative semigroup. If $\mathcal{I}$ is a decomposition of $U(S)$ into the left ideals of $U(S)$, then every member of $\mathcal{I}$ is also a left ideal of βS.*

Proof. Assume the contrary. Then there are distinct $I, J \in \mathcal{I}$, $p \in I$ and $q \in \beta S$ such that $qp \in J$. From this we obtain that $p(qp) \in J$ (since J is a left ideal of $U(S)$) and $(pq)p \in I$ (since I is a left ideal of $U(S)$), since $pqp \in U(S)$, a contradiction. $\square$

5.2 Uncountable case and reduction to $\mathbb{N}^*$

Lemma 5.2.1 *Let S be a cancellative semigroup and let A be an infinite subset of S. Then there is a subsemigroup Q of S such that $A \subseteq Q$, $|Q| = |A|$, $Q^{-1}Q \subseteq Q$, and $QQ^{-1} \subseteq Q$.*

Proof. Define inductively a sequence $(A_n)_{n<\omega}$ of subsets of S with $A_0 = A$ by

$$A_{n+1} = A_n \cup A_n^2 \cup A_n^{-1}A_n \cup A_nA_n^{-1}$$

and let

$$Q = \bigcup_{n<\omega} A_n.$$

Indeed, we have $\emptyset \neq Q \subseteq S$ since $\emptyset \neq A = A_0 \subset Q$ and each A_n is a subset of S. Thus $Q \subseteq S$ as an arbitrary union of A_n. The set Q is closed under the product of S and the proof is completed by the way Q is constructed above. $\square$

Lemma 5.2.2 *Let S be a cancellative semigroup of cardinality $\kappa > \omega$. Then there is a surjective function $f : S \to \kappa$ such that*

1. *for every $\alpha < \kappa$, $|f^{-1}(\alpha)| < \kappa$, and*
2. *whenever $x, y \in S$ and $f(x) < f(y)$, one has $f(xy) = f(yx) = f(y)$.*

Proof. Using Lemma 5.2.1, construct inductively a κ-sequence $(S_\alpha)_{\alpha<\kappa}$ of subsemigroups of S such that

1. for every $\alpha < \kappa$, $|S_\alpha| < \kappa$,

2. for every $\alpha < \kappa$, $S_\alpha^{-1} S_\alpha \subseteq S_\alpha$ and $S_\alpha S_\alpha^{-1} \subseteq S_\alpha$,

3. for every $\alpha < \kappa$, $S_\alpha \subset S_{\alpha+1}$,

4. for every limit ordinal $\alpha < \kappa$, $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$, and

5. $\bigcup_{\alpha<\kappa} S_\alpha = S$.

Note that S is a disjoint union of nonempty sets $S_{\alpha+1} \setminus S_\alpha$, where $\alpha < \kappa$, and S_0. Define $f : S \to \kappa$ by

$$f(x) = \begin{cases} \alpha & \text{if } x \in S_{\alpha+1} \setminus S_\alpha \\ 0 & \text{if } x \in S_0 \end{cases}$$

Clearly, f is surjective since $S_{\alpha+1} \setminus S_\alpha \subset S_{\alpha+1}$, $S_{\alpha+1} \subset S$ and $S_0 \subset S$, and then satisfies (1) since $f^{-1}(\alpha) \subset S_{\alpha+1}$ and $|S_{\alpha+1}| < \kappa$.

To check (2), let $x, y \in S$ and $f(x) < f(y)$. Then $x \in S_\beta$ and $y \in S_{\alpha+1} \setminus S_\alpha$ for some $\beta \leq \alpha < \kappa$. It follows from (2), (3) and the fact each S_α is a semigroup that both xy and yx also belong to $S_{\alpha+1} \setminus S_\alpha$. Hence, $f(xy) = f(yx) = f(y)$ by definition of f. $\square$

Theorem 5.2.1 *Let S be a cancellative semigroup of cardinality $\kappa > \omega$. Then there is a continuous surjective right zero homomorphism $\psi : U(S) \to U(\kappa)$ such that for every $x \in U(\kappa)$,*

1. *$I_x := \psi^{-1}(x)$ is nowhere dense in $U(S)$, and*

2. *$I_x \cdot S \subseteq I_x$.*

Proof. Let $f : S \to \kappa$ be a function guaranteed by Lemma 5.2.2 and let $\bar{f} : \beta S \to \beta\kappa$ be the continuous extension of f. Then

1. $\bar{f}(U(S)) = U(\kappa)$ and $\bar{f}^{-1}(U(\kappa)) = U(S)$,

2. $\bar{f}(qp) = \bar{f}(p)$ for all $p \in U(S)$ and $q \in \beta S$,

3. $\bar{f}(px) = \bar{f}(p)$ for all $p \in U(S)$ and $x \in S$, and

4. for every $u \in U(\kappa)$, $\bar{f}^{-1}(u)$ is nowhere dense in $U(S)$.

Indeed, (1) follows from surjectivity of f and condition (1) of Lemma 5.2.2.

To see (2), let $A \in p$. For every $x \in S$, let $A_x = A \setminus \{y \in S : f(y) \leq f(x)\}$. Then $A_x \in p$ and by condition (2) of Lemma 5.2.2, $f(xy) = f(y) \in f(A)$ for all $y \in A_x$. Consequently, $B = \bigcup_{x \in S} xA_x \in qp$ and $f(B) \subseteq f(A)$. Hence, $\bar{f}(qp) = \bar{f}(p)$. The check of (3) is similar.

Finally, to see (4), let $A \subseteq S$ and suppose that $U(A) \cap \bar{f}^{-1}(u) \neq \emptyset$. Then $E = f(A) \in u$. Pick $D \subseteq E$ such that $|D| = \kappa$ and $D \notin u$, and let $B = f^{-1} \cap A$. Then $B \subseteq A$, $U(B) \neq \emptyset$, but $f(B) \notin u$, and so $U(B) \cap \bar{f}^{-1}(u) = \emptyset$. Hence, $\bar{f}^{-1}(u)$ is nowhere dense in $U(S)$.

Now define $\psi : U(S) \to U(\kappa)$ by $\psi = \bar{f}_{|U(S)}$. It then follows from the conditions (1)-(4) above that ψ is as required. □

Corollary 5.2.1 *Let S be a cancellative semigroup. The decomposition $\mathcal{I}$ of $U(S)$ is such that, for every $I \in \mathcal{I}$,*

1. *I is nowhere dense in $U(S)$, and*
2. *$I \cdot S \subseteq I$.*

Proof. The proof uses Lemma 5.2.1 and Theorem 5.2.1. □

Lemma 5.2.3 *Let S be a countably infinite cancellative semigroup. Then there is a surjective finite-to-one function $f : S \to \mathbb{N}$ such that whenever $x, y \in S$ and $f(x) + 1 < f(y)$, one has*

$$f(xy), f(yx) \in \{f(y) - 1, f(y), f(y) + 1\}.$$

Proof. Construct inductively a strictly increasing sequence $(A_n)_{1 \leq n < \omega}$ of finite subsets of S such that $S = \bigcup_{1 \leq n < \omega} A_n$, and for every $n \in \mathbb{N}$,

1. $A_n^2 \subseteq A_{n+1}$, and
2. $A_n^{-1} A_n \subseteq A_{n+1}$ and $A_n A_n^{-1} \subseteq A_{n+1}$,

Define $f : S \to \mathbb{N}$ by

$$f(x) = \begin{cases} n+1 & \text{if } x \in A_{n+1} \setminus A_n \\ 1 & \text{if } x \in A_n \end{cases}$$

Clearly, f is surjective and finite-to-one. Let $x, y \in S$ and $f(x)+1 < f(y)$. Then $f(x) \leq n$ and $f(y) = n+2$ for some $n \in \mathbb{N}$, so $x \in A_n$ and $x \in A_{n+2} \setminus A_{n+1}$. But then by (1), $xy, yx \in A_{n+3}$, and by (2), $xy, yx \notin A_n$. Indeed, to see that $xy \notin A_n$, assume the contrary. Then $y \in x^{-1}A_n \subseteq A_n^{-1}A_n \subseteq A_{n+1}$, a contradiction. Hence,

$$f(xy), f(yx) \in \{n+1, n+2, n+3\} = \{f(y)-1, f(y), f(y)+1\}.$$

□

Proposition 5.2.1 *Let S be a countably infinite cancellative semigroup. Then there is a continuous surjective mapping $\psi : S^* \to \mathbb{N}^*$ such that whenever $\pi : \mathbb{N}^* \to X$ is a continuous right zero homomorphism, so is $\pi \circ \psi : S^* \to X$. Furthermore, for every $x \in X$,*

1. *if $J_x := \pi^{-1}(x)$ is nowhere dense in $\mathbb{N}^*$, $I_x := \psi^{-1}(J_x)$ is nowhere dense in S^*, and*
2. *$I_x \cdot S \subseteq I_x$.*

Proof. Let $f : S \to \mathbb{N}$ be a function guaranteed by Lemma 5.2.3 and let $\bar{f} : \beta S \to \beta\mathbb{N}$ be the continuous extension of f. Then

1. $\bar{f}(S^*) = \mathbb{N}^*$ and $\bar{f}^{-1}(\mathbb{N}^*) = S^*$,
2. $\bar{f}(qp) \in \{\bar{f}(p)-1, \bar{f}(p), \bar{f}(p)+1\}$ for all $p \in S^*$ and $q \in \beta S$,
3. $\bar{f}(px) \in \{\bar{f}(p)-1, \bar{f}(p), \bar{f}(p)+1\}$ for all $p \in S^*$ and $x \in S$, and
4. for every nowhere dense $Z \subseteq \mathbb{N}^*$, $\bar{f}^{-1}(Z)$ is nowhere dense in S^*.

The check of the conditions (1)-(4) above is similar to that of Theorem 5.2.1.

Now define $\psi : S^* \to \mathbb{N}^*$ by $\psi = \bar{f}_{|S^*}$ and let $\pi : \mathbb{N}^* \to X$ be a continuous right zero homomorphism. Then every $p, q \in S^*$,

$$\psi(qp) \in \{\psi(p)-1, \psi(p), \psi(p)+1\},$$

and consequently, $\pi(\psi(qp)) = \pi(\psi(p))$. Hence, $\pi(\psi(px)) = \pi(\psi(p))$ for every $p \in S^*$ and $x \in S$, which shows the property (2). And the property (1) follows from the condition (4) above. □

5.3 Continuous right zero homomorphisms of $\mathbb{N}^*$

Proposition 5.3.1 *If $\pi : \mathbb{N}^* \to X$ is a continuous surjective right zero homomorphism, then X is connected.*

Proof. Assume on the contrary that X can be partitioned into two nonempty clopen sets U_0 and U_1. Then $\{\pi^{-1}(U_i) : i < 2\}$ is a partition of $\mathbb{N}^*$ into nonempty clopen sets. It follows that there is a partition $\{A_i : i < 2\}$ of $\mathbb{N}$ such that $A_i^* = \pi^{-1}(U_i)$ for each $i < 2$. Since both A_0^* and A_1^* are left ideals of $\beta\mathbb{N}$, we obtain that for each $i < 2$ and every $x \in \mathbb{N}$, $\{y \in A_i : x + y \notin A_i\}$ is finite. In particular, the set $F = \{y \in A_0 : 1 + y \notin A_0\}$ is finite. Let $n = \min(A_0 \setminus F)$. Then $n + 1 \in A_0$, $n + 2 \in A_0$, and so on. Hence $A_1 \subseteq \{1, 2, \ldots, n\}$, a contradiction. $\square$

In the following definition, we reformulate Definition 5.1.2.

Definition 5.3.1 *Let $X = (X, d)$ be a compact metric space. A function $f : \mathbb{N} \to X$ is **slowly oscillating** if for every $u \in \mathbb{N}$ and for every $\epsilon > 0$, $|f(v + u) - f(v)| < \epsilon$ for all but finitely many $v \in \mathbb{N}$.*

The following proposition establishes sort of relationship between the slowly oscillating functions and the continuous right zero homomorphisms.

Proposition 5.3.2 *Let $X = (X, d)$ be a compact metric space. If $f : \mathbb{N} \to X$ is slowly oscillating function and $\bar{f} : \beta\mathbb{N} \to X$ the continuous extension of f, then $\pi = \bar{f}_{|\mathbb{N}^*} : \mathbb{N}^* \to X$ is a continuous right zero homomorphism.*

Proof. The proof uses Definition 5.1.5 and Definition 5.3.1. $\square$

Lemma 5.3.1 *Let $X = (X, d)$ be a compact metric space. Then X is connected if and only if for every $x, y \in X$ and for every $\epsilon > 0$, there exist $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in X$ with $x_0 = x$ and $x_n = y$ such that $d(x_i, x_{i+1}) < \epsilon$ for each $i < n$.*

Proof. This is Exercise 6.1.D in [27]. $\square$

Lemma 5.3.2 *Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence in $\mathbb{N}$ such that*

$$\lim_{n\to\infty}(a_{n+1}-a_n)=\infty$$

and let $A=\{a_n : n\in\mathbb{N}\}$. Let X be a connected compact metric space and let x_0, x_1 be any distinct points of X. Then for any partition of A into two subsets A_0, A_1, there is a slowly oscillating function $f:\mathbb{N}\to X$ such that

1. *$f(\mathbb{N})$ is dense in X and $f^{-1}(x)$ is infinite for every $x\in f(\mathbb{N})$, and*
2. *$f(A_i)=\{x_i\}$ for each $i<2$.*

Proof. Using Lemma 5.3.1, for every $m\in\mathbb{N}$, construct a finite sequence $(x_{m,j})_{j=0}^{l_m}$ in X with $x_{m,0}=x_0$ and $x_{m,l_m}=x_1$ satisfying the following contitions:

1. $d(x_{m,j},x_{m,j+1})<\frac{1}{m}$ for each $j<l_m$,
2. $X_m=\{x_{m,j} : j\le l_m\}$ is an $\frac{1}{m}$-net in X, that is, for every $x\in X$, there is $j\le l_m$ such that $d(x,x_{m,j})<\frac{1}{m}$, and
3. $X_{m-1}\subseteq X_m$ (for $m>1$).

Choose an increasing sequence $(n_m)_{m<\omega}$ in $\mathbb{N}$ such that for every $m<\omega$ and for every $i\in[n_m,n_{m+1})$, $a_i-a_{i-1}>2l_m$. For every such i, put $\mu(i)=m$. Then the intervals $[a_i-l_{\mu(i)},a_i+l_{\mu(i)}]$, where $n_0\le i<\omega$, are pairwise disjoint.

Now define $f:\mathbb{N}\to X$ as follows. For every $i\ge n_0$ such that $a_i\in A_0$ and for every $j\le l_{\mu(i)}$, put $f(a_i\pm j)=x_{\mu(i),j}$. For every $i<n_0$ such that $a_i\in A_0$, put $f(a_i)=x_0$. For all others $a\in\mathbb{N}$, put $f(a)=x_1$.

Thus the function $f:\mathbb{N}\to X$ so defined is as required. $\square$

Theorem 5.3.1 *Let $(a_n)_{n=1}^{\infty}$ be an increasing sequence in $\mathbb{N}$ such that*

$$\lim_{n\to\infty}(a_{n+1}-a_n)=\infty$$

and let $A=\{a_n : n\in\mathbb{N}\}$. Let X be a connected compact metric space and let x_0, x_1 be any distinct points of X. Then for any partition of A into two infinite subsets A_0, A_1, there is a continuously surjective right zero homomorphism $\pi:\mathbb{N}^*\to X$ such that $\pi(A_i^*)=\{x_i\}$ for each $i<2$.

Proof. Let $f : \mathbb{N} \to X$ be a slowly oscillating function guaranteed by Lemma 5.3.2, let $\bar{f} : \beta\mathbb{N} \to X$ be the continuous extension of f, and let $\pi = \bar{f}_{|\mathbb{N}^*}$. Then $\pi : \mathbb{N}^* \to X$ is a continuous right zero homomorphism, π is surjective by the property (1) of Lemma 5.3.2, and $\pi(A_i^*) = \{x_i\}$ by the property (2) of Lemma 5.3.2. □

Corollary 5.3.1 *Let S be a countably infinite cancellative semigroup. Then there exist a connected compact Hausdorff space X of cardinality 2^{2^ω} and a continuous surjective right zero homomorphism $\psi : S^* \to X$ such that for every $x \in X$,*

1. *$I_x := \psi^{-1}(x)$ is nowhere dense in S^*, and*
2. *$I_x \cdot S \subseteq I_x$.*

Proof. By Proposition 5.2.1, it suffices to show that there exist a connected compact Hausdorff space X of cardinality 2^{2^ω} and a continuous surjective right zero homomorphism $\pi : \mathbb{N}^* \to X$ such that for every $x \in X$, $\pi^{-1}(x)$ is nowhere dense in $\mathbb{N}^*$.

Let $\{\pi_\alpha : \alpha \in \mathcal{A}\}$ be the family of all continuous surjective right zero homomorphisms of $\mathbb{N}^*$ onto the unit interval $[0, 1]$. Define

$$\pi : \mathbb{N}^* \to \prod_{\alpha \in \mathcal{A}} [0, 1]_\alpha$$

by $\pi(p) = (\pi_\alpha(p))_{\alpha \in \mathcal{A}}$ and let $X = \pi(\mathbb{N}^*)$, so $\pi : \mathbb{N}^* \to X$ is a continuous surjective right zero homomorphism. Clearly, X is compact, and by Proposition 5.3.1, X is connected.

Now every infinite subset of $\mathbb{N}$ contains an infinite subset $A = \{a_n : n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} (a_{n+1} - a_n) = \infty$, and by Theorem 5.3.1, for any partition of A into two infinite subsets A_0, A_1, there is $\alpha \in \mathcal{A}$ such that $\pi_\alpha(A_i^*) = \{i\}$ for each $i = 0, 1$. It follows that π separates points of A^* and for every $x \in X$, $|\pi^{-1}(x) \cap A^*| \leq 1$, so $\pi^{-1}(x)$ is nowhere dense in $\mathbb{N}^*$. □

To complete this section, we include the following recent results from [37] and [39]. If G is an infinite discrete group, then for every $p \in U(G)$ the set $I_p \subseteq \beta G$ defined by

$$I_p = \bigcap_{A \in p} c\ell(G(U(A))),$$

where $U(A) = \{p \in U(G) : A \in p\}$, is a closed left ideal of βG contained in $U(G)$. Moreover, if $|G|$ is a regular cardinal, then

$$\mathcal{I} := \{I_p : p \in U(G)\}$$

is the finest decomposition of $U(G)$ of βG such that the corresponding quotient space of $U(G)$ is Hausdorff. We have the following decomposition theorem.

Theorem 5.3.2 *([37]) If G is an infinite discrete group of cardinality $\kappa > \omega$, then there exists a decomposition $\mathcal{I}$ of $U(G)$ into a closed left ideals of βG satisfying the following properties:*

1. *the corresponding quotient space of $U(G)$ is homeomorphic to $U(\kappa)$,*
2. *for every $I \in \mathcal{I}$, $IG \subseteq I$, and*
3. *for every $I \in \mathcal{I}$, I is nowhere dense in $U(G)$.*

Proof. See the proof of Theorem 9.16 in [37]. $\square$

As a consequence, the cardinality of $\mathcal{I}$ is $2^{2^{|G|}}$ and $\Lambda(\beta G) = G$. See [37] for more details.

Conclusion

Given an infinite discrete semigroup S, the set βS of ultrafilters on S is the Stone-Čech compactification of S. The operation of S extended to βS makes βS a compact right topological semigroup with S contained in the topological center of βS. The set $U(S)$ of uniform ultrafilters on S is a two-sided ideal of βS.
In this thesis, we showed the following main results:

1. If S is embeddable in a group, then βS contains $2^{2^{|S|}}$ pairwise incomparable closed two-sided ideals. In particular, for every infinite group G, βG contains $2^{2^{|G|}}$ closed two-sided ideals.

2. If S is an infinite cancellative semigroup of cardinality κ, then

   (a) If $\kappa > \omega$, then there is a closed left ideal decomposition of $U(S)$ such that the corresponding quotient space is homeomorphic to $U(\kappa)$.

   (b) If $\kappa = \omega$, then there is a closed left ideal decomposition of $U(S)$ with the quotient space homeomorphic to any connected compact metric space.

The first result extends significantly the results from [6] and [13] and the second result complements the well known results from [7], [14], and [39]. These two results also add more to the knowledge of algebraic properties of βS and, as a consequence, they might have implications for the study of the asymptotic behavior of topological dynamics and other applications.

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