University of the Witwatersrand

Masters Dissertation

Hints of Multi-Matrix Eigenvalue Dynamics

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Declaration

I, David Gossman, declare that except where acknowledged in the customary manner, the material presented in this dissertation is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Date: 23 May 2017 **Signature:** 

Acknowledgments

I would like to thank my supervisor Prof. R. de Mello Koch for his absolute dedication to my learning and understanding. I have yet to meet someone as passionate about physics and teaching it to the next generation as he is. I am exceedingly grateful for his patient and dedicated supervision over the course of my MSc.

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Abstract

After studying matrix model quantum mechanics for one and two matrices and the eigenvalue dynamics of the one matrix model, evidence is found that there is a sector of the two matrix model that can be reduced to eigenvalue dynamics. This sector is defined by the $SU(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory. Evidence is found by doing explicit computations of correlation functions using an interesting generalization of the usual Van der Monde determinant. The observables we study are the BPS operators of the $SU(2)$ sector and include traces of products of both matrices, which are genuine multi matrix observables. These operators are associated to supergravity solutions of string theory.

Chapter 1

Introduction

By the end of the 19th century physics was widely thought to be able to explain all fundamental aspects of nature and that all the laws of physics were discovered, with Classical Mechanics explaining the motion of objects, Thermodynamics explaining temperature and heat transfer and Maxwell's equations explaining optics and electromagnetism. But at the turn of the 20th century this had to be questioned with the emergence of problems that physics at that time could not answer, such as the ultraviolet catastrophe, the photoelectric effect and the seemingly bizarre prediction of Maxwell's equations: that the speed of light is constant regardless of observer. These questions saw the emergence of modern physics and what are thought of as the pillars of modern physics: Quantum Mechanics (QM) which explains the interactions of fundamental particles and explains the ultraviolet catastrophe and the photoelectric effect and General Relativity (GR) which is the logical generalisation of Special Relativity which was birthed out of the fact that the speed of light is a constant for any observer. QM explains three of the four forces seen, the Coulomb force, the weak nuclear force and the strong nuclear force, as the interactions between fundamental particles. GR explains the fourth force: gravity, as the warping of space-time by mass and energy. These two theories explain beautifully the things in their domain: QM for very small objects and GR for very large objects. But what happens when we need to study objects that fall in both domains like an object that is very massive but also very small, a black hole or the very early universe for instance? For these objects we need a combination of these theories—a theory of Quantum Gravity. If, however, we try and write down a quantum field theory for gravity we run into difficulties because gravity is an irrelevant (in the sense of the renormalisation group) interaction. What this means is that at low energies the gravitational interaction will be weak, as we see in nature, but it will grow with energy scale so that at large energies the interaction will diverge and calculations are impossible. This is analogous to the Fermi interaction which was an interaction proposed by Enrico Fermi to explain Beta decay: the process in which a neutron decays into a proton, electron and neutrino. Fermi proposed a four point interaction to describe this process but the coupling for this process also turned out to be irrelevant so it worked for low energies but diverged at large energies. These kinds of divergences tell us that there is something, some physics, not being accounted for. In the case of Beta decay what was missing was the W and Z bosons proposed by Sheldon Glashow, Abdus Salam, and Steven Weinberg – eventually winning them the Nobel Prize in 1979. The missing element in the Quantum Gravity is yet to be discovered. Just like the Fermi interaction and the problems which were seen at the turn of the 20th century this problem indicates to us that we probably need a new theory.

The theory with the most promise in answering these questions seems to be string theory. In it we define the fundamental particles as one dimensional strings. What this means is that we can no longer probe distances smaller than the size of a string. We would expect the size of the string to be smaller than the smallest distances we have been able to probe thus far. We can make a more quantitative argument for the size of the string. The length of the string $l_s$would be a fundamental constant just like the speed of light c, which is a fundamental speed, and $\hbar$, which is a fundamental uncertainty. We suspect that the string length will be related to the other fundamental constants. In the theory of gravity we also have the gravitational constant G. Out of these constants we can make a length as follows

$$l_P = \sqrt{\frac{G\hbar}{c^3}} \tag{1.1}$$

This is known as the Plank length and it is on the order of 10^{-36}m. This is very small and we expect l_s will be proportional to this. In string theory all fundamental particles are the same object: the string, and their properties are determined by the vibrational mode of the string thus this theory unifies theories of fermions and bosons into a theory of strings.

The problem is that string theory is extremely difficult to do. So difficult in fact that the scientific community has not yet been able to write down any sort of governing equations for the theory. This is in fact not an uncommon predicament physicists find themselves in. In this kind of situation, where the problem at hand seems too difficult to solve, a good strategy is to try and find a simpler problem that encodes as much of the physics of the original problem as possible. Then if this problem is solvable it will give us insight into the original problem. This strategy has been employed by physicists throughout history with great success. A good example is the Bohr Model of the hydrogen atom; it is a simple and solvable model of the hydrogen atom that made a monumental contribution to our understanding of atomic physics and QM. Of course now our understanding of the Hydrogen atom is much better and it can be given a detailed quantum mechanical description but it all started with the Bohr model. Even though this model may be considered primitive with respect to our current understanding it is still useful and powerful in its simplicity. In order to fully describe the Hydrogen atom one needs a good grasp on QM and the details may actually obscure an intuitive understanding. But the Bohr Model gives an intuitive understanding from which to explore the details. This is in essence the main goal of this work: to find a problem simple enough to solve that encodes as much of the physics of string theory as possible.

We are by no means the first to want to do this but we aim to build on the work done by physicists before us in this regard and hopefully present a simpler sector of the String Theory that can be solved and studied to gain understanding and intuition for the full theory. Since the inception of string theory physicists have been trying to come up with regimes in which it can be solved and there have been many note worthy strides forward to this end. One is worth mentioning here as it is the foundation for this work. It is the link between matrix models and string theory. In 1973 ’t Hooft’s proposed that the large N expansions of matrix model quantum mechanics are equivalent to perturbative expansions in terms of topologies of worldsheets in string theory [1]. This will be further illustrated later. These matrix model quantum mechanics represent a sector of $\mathcal{N} = 4$ Super Yang-Mills Theory (SYM), a conformal field theory (CFT), and in 1997 Maldacena proposed a duality between String theories on AdS spaces and CFT’s on the boundary of the spaces [2] known as the AdS/CFT correspondence. This solidified an equivalence between matrix model quantum mechanics at large N and String theory. This specific example of the correspondence involving $\mathcal{N} = 4$ SYM is probably the most famous within the full AdS/CFT correspondence. It postulates that type IIB String theory on the space $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ SYM on the four dimensional boundary. This means that we can study things in a matrix model quantum mechanics defined by $\mathcal{N} = 4$ SYM and learn things about the dual String theory. This is quite a break through because we know a lot more about matrix models than string theory.

When studying the large N expansions of matrix models we usually study the planar limit where classical operator dimensions are held fixed as we take $N \to \infty$, there are non-planar large N limits of the theory [3] defined by considering operators with a bare dimension that is allowed to scale with N as we take $N \to \infty$. These limits are also relevant for the AdS/CFT correspondence. Indeed, operators with a dimension that scales as N include operators relevant for the description of giant graviton branes[4, 5, 6] while operators with a dimension of order N^2 include operators that correspond to new geometries in supergravity[7, 8, 9]. These convincing motivations have motivated sustained study of large N field theory. However, carrying out the large N expansion for most matrix models is still beyond our current capabilities. What we want to do is study the dynamics of the eigenvalues of these matrices only with the hope that it is dual to a simpler but still interesting sector of the string theory. We can do the large N expansion for the singlet sector of matrix quantum mechanics of a single hermitian matrix[10] and in this sector the eigenvalue dynamics has been successfully studied. The eigenvalue dynamics actually reproduces the entire matrix model in this case because we can always work in a basis in which this matrix is diagonal. So it is in essence just a rewriting of the theory. This reduction to eigenvalue dynamics has proven to be quite insightful. For example, one can formulate the physics of the planar limit by using the density of eigenvalues as a dynamical variable. The resulting collective field theory defines a field theory that explicitly has $1/N$ as the loop expansion parameter[11, 12]. It has found both application in the context of the $c = 1$ string[13, 14, 15] and in descriptions of the LLM geometries[16].

Standard arguments show that eigenvalue dynamics corresponds to a familiar system: non-interacting fermions in an external potential[10]. This makes the description extremely convenient because the fermion dynamics is rather simple. This eigenvalue dynamics is also a natural description of the large N but non-planar limits discussed above. Giant graviton branes which have expanded into the AdS_5 of the spacetime correspond to highly excited fermions or, equivalently, to single highly excited eigenvalues: the giant graviton is an eigenvalue[5, 9]. Giant graviton branes which have expanded into the S^5 of the spacetime correspond to holes in the Fermi sea, and hence to collective excitations of the eigenvalues where many eigenvalues are excited[9]. Half-BPS geometries also have a natural interpretation in terms of the eigenvalue dynamics: every fermion state can be identified with a particular supergravity geometry[8, 9]. The map between the two descriptions was discovered by Lin, Lunin and Maldacena in [7]. The fermion state can be specified by stating which states in phase space are occupied by a fermion, so we can divide phase space up into occupied and unoccupied states. By requiring regularity of the corresponding supergravity solution exactly the same structure arises: the complete set of regular solutions are specified by boundary conditions obtained by dividing a certain plane into black (identified with occupied states in the fermion phase space) and white (unoccupied states) regions. See [7] for the details.

The main goal in this work is to ask if a similar eigenvalue description can be constructed for a two matrix model. The dynamics of two complex matrices is of interest because the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory on $R\times S^3$ admits a matrix model quantum mechanics of two complex matrices. This means that if an eigenvalue construction exists for a two matrix model, it may have a natural AdS/CFT interpretation. Work with a similar motivation but focusing on a different set of questions has appeared in[17, 18, 19, 20, 21]. The difficulty is that if we work with more than one matrix, the matrices of the theory are in general independent and so cannot be simultaneously diagonalized. This means that we cannot rewrite our theory in terms of eigenvalues alone and if we decide to study the dynamics of the eigenvalues only, we know that it will not reproduce the dynamics of the full matrix model. In looking for a sector of the two matrix model that could be reproduced by eigenvalue dynamics we look for aspects of the one matrix model that allowed us to reduce the theory to eigenvalue dynamics in the hope that they can be generalised to the two matrix model. In the one matrix model we found that the eigenvalue dynamics reproduced a system of non-interacting fermions and that the state of these fermions is described by Schur polynomials ($\chi_R(Z)$) where each row in the Young diagram (R) labeling the Schur polynomial corresponds to an eigenvalue. The Schur polynomials can be generalised to the two matrix case with the restricted Schur polynomials ($\chi_{R,(r,s)\alpha\beta}(Z,Y)$) [22, 23, 24] but we do not know if the same interpretation between rows and eigenvalues can be made because the restricted Schur polynomials are labeled by three Young diagrams $R\vdash n+m, r\vdash n$ and $s\vdash m$ and two multiplicity labels α and β. Where r is the diagram associated with the Z matrices and is obtained by removing m boxes from R, s is constructed from the removed boxes and is associated with the Y matrices, and the multiplicity labels label the specific copy of s since there may be multiple ways of constructing s from m boxes. The (r,s) diagrams and multiplicity labels obscure the idea of rows being associated with eigenvalues since the Zs and Ys mix in the rows of R. If, however, we consider operators built using many Z fields and only a few Y fields at least a rough outline of the one matrix physics should be visible and we may be able to generalise the idea of rows being associated with eigenvalues. In the limit in which R has order 1 rows (or columns), $m \ll n$ and n is of order N, operators of a definite dimension are labeled by Gauss graphs composed of nodes that are traversed by oriented edges [25, 26]. Examining these operators with insight from the AdS/CFT correspondence we see that the nodes correspond to giant graviton branes and the oriented edges correspond to strings stretching between the branes. The constraint that the number of edges ending on a node equals the number of edges emanating from the node is simply encoding the Gauss law on the brane world volume. In order to obtain a system of non-interacting giant graviton branes as we had for the one matrix model, we need only to consider Gauss graphs that have no directed edges stretching between nodes. These in fact all correspond to BPS operators. The fact that we can find a system in the two matrix model that corresponds to non-interacting giant graviton branes with the Gauss graph operators is remarkable and concrete evidence that there is a sector of the two matrix model which can be reproduced by eigenvalue dynamics.

The layout of this dissertation will be as follows: in Chapter 2 matrix model quantum mechanics will be defined and some details of how calculations are done is given, in Chapter 3 eigenvalue dynamics is defined and the possibilities of two matrix model eigenvalue dynamics is studied, then in Chapter 4 some concluding remarks are given and the outlook for this work is discussed.

Chapter 2

Matrix Model Quantum Mechanics

2.1 Matrix Models and String Theory

The goal of this section is to introduce matrix models and provide an argument for their equivalence to string theories.

2.1.1 Free theory

Start with a $N \times N$ Hermitian matrix M and define a free action:

$$S = -\frac{w}{2}\mathrm{Tr}(M^2) \tag{2.1}$$

then expectation values are given by:

$$\langle M_{ij}...M_{nm}\rangle_0 = \mathcal{N}\int [dM] e^{-\frac{w}{2}\mathrm{Tr}(M^2)} M_{ij}...M_{nm}, \tag{2.2}$$

where the subscript 0 stands for free and the integral over $[dM]$ is understood as a functional integral over each element of the matrix M, making this N^2 integrals in total. We choose the normalisation $\mathcal{N}$ such that:

$$\langle 1\rangle_0 = \mathcal{N}\int [dM] e^{-\frac{w}{2}\mathrm{Tr}(M^2)} = 1. \tag{2.3}$$

This implies for a general N:

$$\mathcal{N} = \left[\frac{1}{2}\left(\frac{\omega}{\pi}\right)^N\right]^{\frac{N}{2}}. \tag{2.4}$$

By defining a generating function:

$$Z_0[J] = \mathcal{N}\int [dM] e^{-\frac{w}{2}\mathrm{Tr}(M^2)+\mathrm{Tr}(JM)} = e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}, \tag{2.5}$$

correlators can be represented as follows:

$$\langle M_{ij}...M_{nm}\rangle_0 = \frac{d}{dJ_{ji}}...\frac{d}{dJ_{mn}} Z_0[J]\bigg|_{J=0}. \tag{2.6}$$

Gauge invariant operators are traces of powers of matrices so we should study correlators of traces of powers of M. For example:

$$\langle \mathrm{Tr}(M^2)\rangle_0 = \langle M_{ij}M_{ji}\rangle = \frac{N^2}{\omega}, \tag{2.7}$$

$$\langle \mathrm{Tr}(M^4) \rangle_0 = \langle M_{ij} M_{jk} M_{kl} M_{li} \rangle = \frac{1}{\omega^2} \left[2N^3 + N \right], \tag{2.8}$$

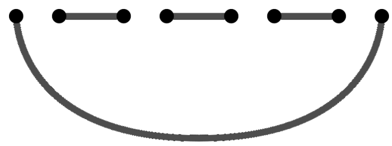
and

$$\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^2) \rangle_0 = \langle M_{ij} M_{ji} M_{kl} M_{lk} \rangle = \frac{1}{\omega^2} \left[N^4 + 2N^2 \right]. \tag{2.9}$$

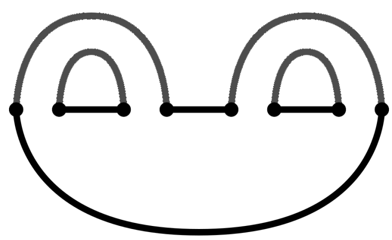
The Feynman rules for these correlators are as follows (illustrated with $\langle \mathrm{Tr}(M^4) \rangle$):

1. Draw two dots for each pair of indices on the operators

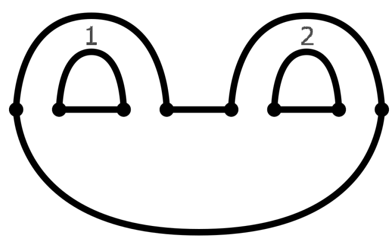
2. Connect the contracted indices below



3. Connect the pairs of indices above in ribbons, each different way of connecting is a different diagram

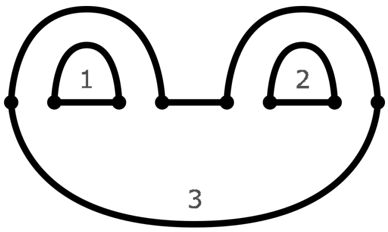


4. Each ribbon comes with a factor of $1/\omega$



$$\equiv \frac{1}{\omega^2} \cdots$$

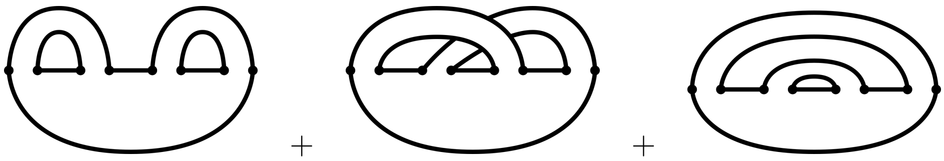
5. Each closed loop comes with a factor of N



$\equiv \frac{N^3}{\omega^2}$

Computing $\langle \text{Tr}(M^4) \rangle_0$ with ribbon graphs gives:

$\langle \text{Tr}(M^4) \rangle_0 =$



$= \frac{N^3}{\omega^2} + \frac{N}{\omega^2} + \frac{N^3}{\omega^2} = \frac{1}{\omega^2}\left[2N^3 + N\right]$

2.1.2 Interacting theory

Now we would like to add interactions to the theory and we will do this by putting an interaction term into the action, quartic in M. The generating function becomes

$$\begin{aligned} Z[J] =& \mathcal{N} \int [dM] e^{-w/2\text{Tr}(M^2) - g\text{Tr}(M^4) + \text{Tr}(JM)} && (2.10) \\ =& \mathcal{N} \sum_{n=0}^{\infty} \frac{1}{n!} (-g)^n \int [dM] e^{-w/2\text{Tr}(M^2) + \text{Tr}(JM)} \text{Tr}(M^4)^n \\ =& \sum_{n=0}^{\infty} \frac{1}{n!} \left[-g \frac{d}{dJ_{ji}} \frac{d}{dJ_{kj}} \frac{d}{dJ_{lk}} \frac{d}{dJ_{il}} \right]^n Z_0[J] \\ =& \sum_{n=0}^{\infty} \frac{1}{n!} \left[-g \frac{d}{dJ_{ji}} \frac{d}{dJ_{kj}} \frac{d}{dJ_{lk}} \frac{d}{dJ_{il}} \right]^n e^{\frac{1}{2\omega}\text{Tr}(J^2)} && (2.11) \\ \Rightarrow Z[J=0] =& \sum_{n=0}^{\infty} \frac{1}{n!} (-g)^n \left\langle \left(\text{Tr}(M^4) \right)^n \right\rangle_0 && (2.12) \end{aligned}$$

The correlators in the interacting theory are given by:

$$\langle M_{ij} ... M_{nm} \rangle = \frac{d}{dJ_{ji}} ... \frac{d}{dJ_{mn}} Z[J] \bigg|_{J=0} \tag{2.13}$$

The interaction is put into the diagrams by a 4-point vertex of the following form:

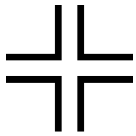
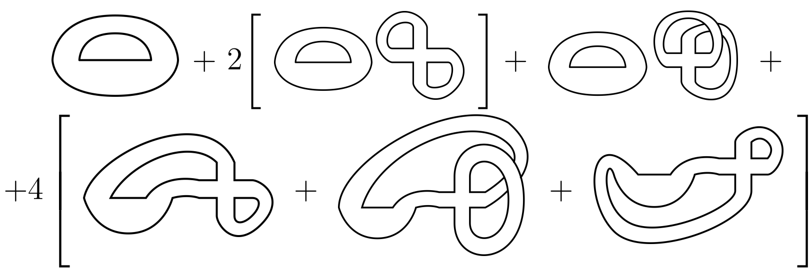


Figure 2.1: 4-point vertex

Each vertex comes with a factor of $(-g)$. Correlators in the interacting theory can now be calculated perturbatively in g. Consider $\langle \mathrm{Tr}(M^2)\rangle$ to first order in g.

$$\begin{aligned}
\langle \mathrm{Tr}(M^2)\rangle &= \frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}}\sum_{n=0}^{1}\frac{1}{n!}\left[-g\frac{d}{dJ_{ji}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}\right]^n e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}\Bigg|_{J=0} \\
&=\langle \mathrm{Tr}(M^2)\rangle_0 - g\,\frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}}\frac{d}{dJ_{ji}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}\Bigg|_{J=0} \\
&=\frac{N^2}{\omega} - g\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\rangle_0 \\
&=\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5 + 9N^3 + 4N\right]
\end{aligned} \tag{2.14}$$

This can also be done diagrammatically:
$\langle \mathrm{Tr}(M^2)\rangle =$



$$\begin{aligned}
&=\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5 + N^3 + 4\left(N^3 + N + N^3\right)\right] \\
&=\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5 + 9N^3 + 4N\right]
\end{aligned}$$

2.1.3 Normalisation

Now it is convenient and more physically insightful to normalise such that the vacuum contributions are removed. This is achieved by normalising as follows:

$$\begin{aligned} Z_{nn}[J] =& \frac{Z[J]}{Z[J=0]} \\ =& \frac{Z[J]}{\sum_{n=0}^{\infty} \frac{1}{n!} (-g)^n \langle (\mathrm{Tr}(M^4))^n \rangle_0} \\ =& \frac{Z[J]}{1 - \frac{g}{\omega^2}(2N^2+N) + \frac{g^2}{2\omega^4}(4N^6+40N^4+61N^2) + \mathcal{O}(g^3)}, \end{aligned} \tag{2.15}$$

where the subscript nn stands for new normalisation. To illustrate this consider $\langle \mathrm{Tr}(M^2) \rangle_{nn}$.

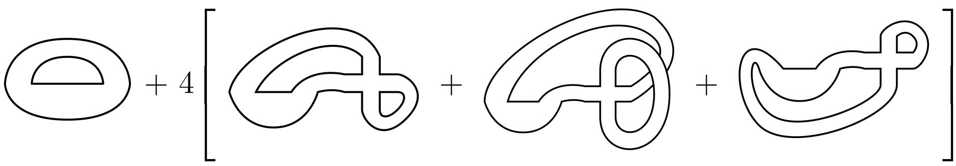
$$\begin{aligned} \langle \mathrm{Tr}(M^2) \rangle_{nn} =& \frac{\langle \mathrm{Tr}(M^2) \rangle}{1 - \frac{g}{\omega^2}(2N^2+N) + \frac{g^2}{2\omega^4}(4N^6+40N^4+61N^2) + \mathcal{O}(g^3)} \\ =& \frac{\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5+9N^3+4N\right] + \mathcal{O}(g^2)}{1 - \frac{g}{\omega^2}(2N^2+N) + \frac{g^2}{2\omega^4}(4N^6+40N^4+61N^2) + \mathcal{O}(g^3)} \end{aligned} \tag{2.16}$$

Assuming g is small and expanding to first order in g gives:

$$\begin{aligned} \langle \mathrm{Tr}(M^2) \rangle_{nn} \cong& \left[\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5+9N^3+4N\right] + \mathcal{O}(g^2)\right]\left[1 + \frac{g}{\omega^2}(2N^2+N) + \mathcal{O}(g^2)\right] \\ =& \frac{N^2}{\omega} - \frac{g}{\omega^3}\left[8N^3+4N\right] \end{aligned} \tag{2.17}$$

Comparing this to adding up the diagrams excluding the diagrams with vacuum graphs.

$\langle \mathrm{Tr}(M^2) \rangle_{nn} =$



$$\begin{aligned} &= \frac{N^2}{\omega} - \frac{4g}{\omega^3}\left[N^3+N+N^3\right] \\ &= \frac{N^2}{\omega} - \frac{g}{\omega^3}\left[8N^3+4N\right] \end{aligned} \tag{2.18}$$

This is the same result and henceforth, this new normalisation will be used.

2.1.4 t' Hooft expansion

Here we would like to consider the double scaling limit $N \to \infty$ and $g \to 0$ such that $\lambda = gN$ is a fixed small number. Consider $\langle \mathrm{Tr}(M^2) \rangle_{nn}$ up to second order in g.

$$\begin{aligned}\langle \mathrm{Tr}(M^2) \rangle_{nn} =& \frac{N^2}{\omega} - \frac{g}{\omega^3}\left[8N^3 + 4N\right] + \frac{g^2}{\omega^5}\left[144N^4 + 224N^2\right] \\ =& \frac{N^2}{\omega} - \frac{N^2\lambda}{\omega^3}\left[8 + \frac{4}{N^2}\right] + \frac{N^2\lambda^2}{\omega^5}\left[144 + \frac{224}{N^2}\right] + \mathcal{O}(\lambda^3)\end{aligned} \tag{2.19}$$

Here we see an expansion in λ and $\frac{1}{N^2}$ which are both small quantities in this limit. These small quantities are analogous to $\hbar$ in a QFT and so imply that this theory has two sources of intrinsic uncertainty. Because we are aiming to show the equivalence to a string theory we can interpret the extra source of uncertainty as coming from the fact that a string is 1 dimensional with a finite size and so cannot probe position with the precision that a point particle could. In other words on top of the fuzzy uncertainty that comes from the quantum nature of the theory, the finite size of the string will give rise to another uncertainty in how precisely position can be measured. This is because the probes in this theory are strings and their finite nature ensures that positions can never be known to a precision smaller than the order of the length of the string. Intuitively this follows since it is uncertain which part of the string has interacted.

Now it is clear that the terms with a N to the power of 2 are the main contributors in the $N \to \infty$ limit so we would like to restructure the tools of the theory so that we can easily write down correlators as a perturbative expansion in $\frac{1}{N^2}$ and λ instead of just g. In doing this we will also find a clear link between this theory and a string theory. Let us put λ in and change variables to:

$$M = \sqrt{N}M' \tag{2.20}$$

$$\Rightarrow -\frac{\omega}{2}\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4) = -\frac{\omega N}{2}\mathrm{Tr}(M'^2) - N\lambda \mathrm{Tr}(M'^4) \tag{2.21}$$

Since this is just a scaling factor and the measure goes over all M's the measure will not change.

$$\Rightarrow Z[J'] = \mathcal{N}\int [dM'] e^{-\frac{\omega N}{2}\mathrm{Tr}(M'^2) - N\lambda\mathrm{Tr}(M'^4) + \mathrm{Tr}(J'M')} \tag{2.22}$$

This changes our Feynman rules so that each ribbon comes with a $\frac{1}{\omega N}$ and each vertex comes with a $-\lambda N$. Now we would like to be able to understand the N dependencies of each diagram. Let the number of ribbons be E, the number of closed loops be F and the number of vertices be V. Then the N dependance of a diagram will be N^{F-E+V}. Let us shift our perspective and imagine that the ribbon diagrams triangulate surfaces and try and understand these surfaces.

In this view the ribbons correspond to edges and the closed loops correspond to faces, and the quantity $F - E + V = \chi$ is a topological invariant known as the Euler characteristic of a surface. It is also given by $\chi = 2 - 2h - b$ where h is the number of handles and b is the number of boundaries. This is interesting because it implies that the diagrams with a specific N dependence correspond to the triangulation of a surface of specific topology. Now let us look at a diagram with a N^2 dependance and determine what surface it triangulates. Consider:

Figure 2.2: Figure of eight diagram

This diagram has two ribbons or edges, three closed loops or faces and one vertex. $\Rightarrow \chi = F - E + V = 2$. As it turns out this value corresponds to a sphere as can be illustrated by this drawing:

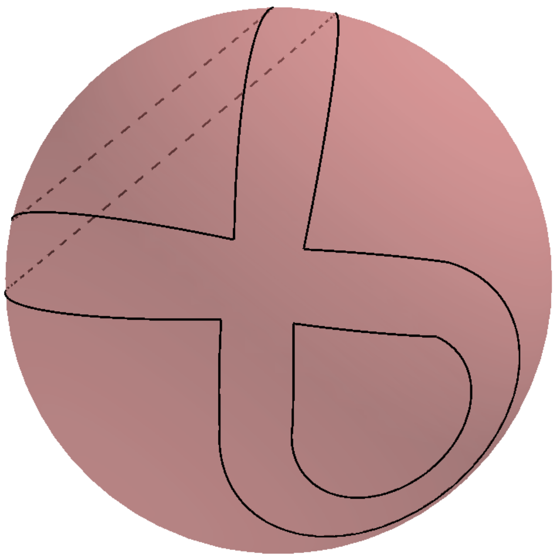


Figure 2.3: Triangulation of a sphere

This implies that all diagrams which triangulate spheres come with a N^2 dependance. Since we have seen that N^2 is the highest power of N a diagram can have we can conclude that the dominant diagrams will always triangulate a sphere. These are also referred to as the planar diagrams. Now what about the lower power diagrams? We have seen that the N dependance drops by two powers in our expansion (Equation 2.19) but how does this fit into the surfaces picture? Consider the situation where we have a surface being triangulated by some ribbon diagram with a certain $\chi = F - E + V$ value with a section of it looking as follows:

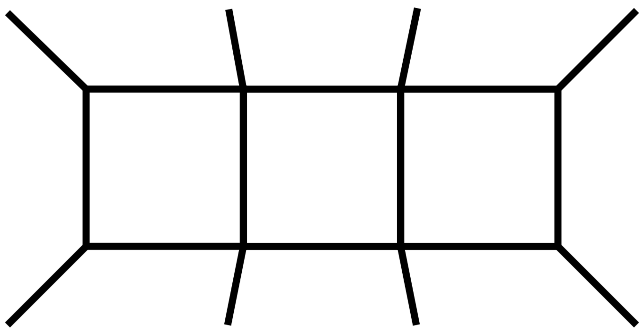


Figure 2.4: Section of a triangulated surface

Where each line corresponds to a ribbon. Now imagine we cut two holes (Figure 2.5) and glue the edges of the holes together so that a new surface is formed.

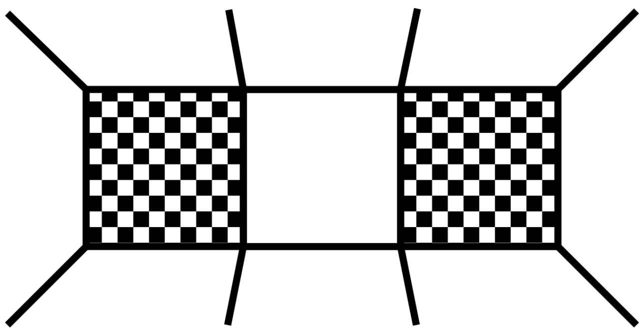


Figure 2.5: Section of a triangulated surface with two holes

This process reduces the number of edges and vertices by 4 and the number of faces by 2.

$$
\begin{aligned}
E' &= E - 4 \\
F' &= F - 2 \\
V' &= V - 4
\end{aligned}
\tag{2.23}
$$

$$\Rightarrow \chi' = F' - E' + V' = F - E + V - 2 = \chi - 2 \tag{2.24}$$

It corresponds to adding a handle to the surface and does indeed correspond to the desired reduction in the power of N that we were looking for. Topologically adding a handle is equivalent to increasing the genus of a surface. Adding a handle to a sphere (genus 0) creates a torus (genus 1), adding a handle to a torus makes a pretzel (genus 2) and so on. So that the genus of a surface is equivalent to the number of handles. This implies that our expansion can be understood as an expansion in the genuses of surfaces. This is very interesting because in a string theory the classical contribution to a string being created at x_1 and being detected at x_2 would map out a surface of genus 0 and the first quantum correction corresponding to one loop would map out a genus 1 surface, two loops a genus 2 surface and so on.

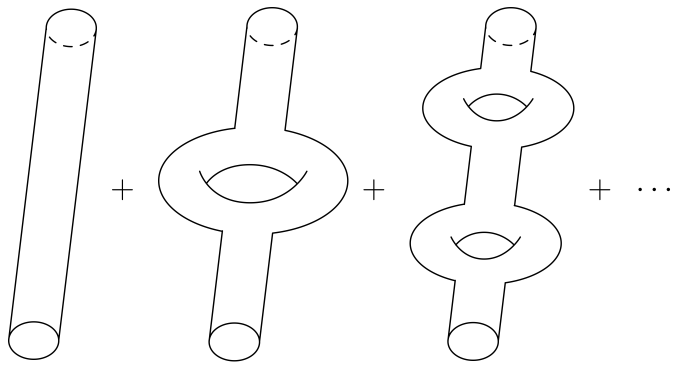


Figure 2.6: Perturbation expansion in terms of topologies of worldsheets in string theory

This has a significant implication: if we ignore the higher order corrections and only consider the planar contributions to correlators we should find the classical limit of the theory. In the classical limit one finds that expectation values of products of observables are equivalent to the product of the expectation values of the individual observables. We describe this by saying that the theory exhibits factorization in this limit. Let this be illustrated by some arbitrary system in state i with the probability of being in state i given by μ_i.

$$\Rightarrow \sum_i \mu_i = 1 \tag{2.25}$$

$$\mu_i \geq 0, \forall \mu_i \tag{2.26}$$

In this system there are a set of observables O_I whose value in state i is $O_I(i)$.

$$\Rightarrow \langle O_I \rangle = \sum_i \mu_i O_I(i) \tag{2.27}$$

Now this theory exhibits factorization in the classical limit because the expectation values of the observables in this limit behave in the following manner:

$$\langle O_{I_1} O_{I_2} ... O_{I_n} \rangle = \langle O_{I_1} \rangle \langle O_{I_2} \rangle ... \langle O_{I_n} \rangle \tag{2.28}$$

$$\Rightarrow \sum_i \mu_i O_{I_1}(i) O_{I_2}(i) ... O_{I_n}(i) = \sum_{i_1} \mu_{i_1} O_{I_1}(i_1) \sum_{i_2} \mu_{i_2} O_{I_2}(i_2) ... \sum_{i_n} \mu_{i_n} O_{I_n}(i_n) \tag{2.29}$$

This is only possible if there is only one accessible state $i*$,

$$\Rightarrow \mu_i = \begin{cases} 1 & i = i^* \\ 0 & i \neq i^* \end{cases} \tag{2.30}$$

implying that this is indeed the classical limit of the theory.

Now consider the following correlators in the matrix model in the large N limit.

$$\langle \mathrm{Tr}(M^2) \rangle_{nn} = \frac{N^2}{\omega} - \frac{N^2 \lambda}{\omega^3} \left[8 + \frac{4}{N^2} \right] + \mathcal{O}(\lambda^2) \tag{2.31}$$

$$\cong \frac{N^2}{\omega} - \frac{8N^2 \lambda}{\omega^3} + \mathcal{O}(\lambda^2) \tag{2.32}$$

and

$$\langle \mathrm{Tr}(M^2) \mathrm{Tr}(M^2) \rangle_{nn} = \frac{N^4}{\omega^2} \left[1 + \frac{2}{N^2} \right] - \frac{\lambda N^4}{\omega^4} \left[8 + \frac{52}{N^2} + \frac{24}{N^4} \right] + \mathcal{O}(\lambda^2) \tag{2.33}$$

$$\cong \frac{N^4}{\omega^2} - \frac{8\lambda N^4}{\omega^4} + \mathcal{O}(\lambda^2) \tag{2.34}$$

$$= \langle \mathrm{Tr}(M^2) \rangle_{nn} \langle \mathrm{Tr}(M^2) \rangle_{nn} \tag{2.35}$$

This shows that factorization is exhibited in the large N limit and the theory reduces to a classical one. This discussion, together with our previous discussion of ribbon graphs, is substance for the claim of the equivalence between matrix models and a string theory.

What we have done so far corresponds to the singlet sector of matrix quantum mechanics of a single hermitian matrix[10]. We can also consider a complex matrix model as long as we restrict ourselves to potentials that are analytic in Z (summed with the dagger of this which needs to be added to get a real potential) and observables constructed out of traces of a product of Zs or out of a product of $Z^\dagger$s [27]. In these situations we can reduce the problem to eigenvalue dynamics.

2.2 Number of Fields Comparable to N

We have studied a matrix model with one matrix field and observables built using a number of fields that does not grow parametrically with N (the planar limit). So let us quickly motivate why one cannot use the techniques developed for studying these observables at large N for observables whose number of fields grows parametrically with N (the non-planar large N limits of the theory [3]) and then outline the tools needed for studying these non-planar operators.

2.2.1 No mixing of trace structure

When studying a complex matrix model with one complex field Z one can define physical operators $\mathcal{O}_i$ according to the trace structure of the operator. Let i correspond to the number of traces such that:

$$\begin{array}{rll} \mathcal{O}_1(J) & \text{proportional to} & \mathrm{Tr}(Z^J) \\ \mathcal{O}_2(J_1,J_2) & \text{proportional to} & \mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J_2}) \\ \mathcal{O}_3(J_1,J_2,J_3) & \text{proportional to} & \mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J_2})\mathrm{Tr}(Z^{J_3}) \\ & \vdots & \end{array} \tag{2.36}$$

If one defines the operators so that the $\mathcal{O}_{A_1}$ is normalised to have a unit two point function in the large N limit, they have the following form:

$$\begin{array}{c} \mathcal{O}_1(n) = \frac{1}{\sqrt{nN^n}}\mathrm{Tr}(Z^n) \\ \mathcal{O}_2(n,m) = \mathcal{O}_1(n)\mathcal{O}_1(m) = \frac{1}{\sqrt{nmN^{n+m}}}\mathrm{Tr}(Z^n)\mathrm{Tr}(Z^m) \\ \vdots \end{array} \tag{2.37}$$

This implies that in the large N limit one has:

$$\left\langle \mathcal{O}_1(n)\mathcal{O}_1^\dagger(m)\right\rangle = \delta_{nm}\left(1+O\left(\frac{1}{N^2}\right)\right), \tag{2.38}$$

and

$$\left\langle \mathcal{O}_1(n)\mathcal{O}_1(m)\mathcal{O}_2^\dagger(n,m)\right\rangle = \frac{\sqrt{nm(m+n)}}{N}\left(1+O\left(\frac{1}{N^2}\right)\right). \tag{2.39}$$

From this one can clearly see that the right hand side of the latter result vanishes in the large N limit if the number of fields (n and m) are fixed as $N \longrightarrow \infty$. This is in fact a general result that different types of trace structures do not mix in the large N limit. This means that the set of observable quantities in the theory reduces to correlators of the following form:

$$\left\langle \mathcal{O}_i \mathcal{O}_i^\dagger \right\rangle \tag{2.40}$$

This however, would obviously not hold when n and m grow comparably to N and so all trace structures must be considered together in the large N limit.

2.2.2 Genus expansion and planar limit

Let us now consider correlators of this form:

$$\left\langle \mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J}) \right\rangle \tag{2.41}$$

The cylinder contribution is:

$$\left\langle \mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J}) \right\rangle = JN^J + \text{higher genus contributions,} \tag{2.42}$$

and it can be shown that the genus h contribution is given by:

$$\frac{1 \cdot 3 \cdot 5 \cdots (4h-1)}{2h+1} J \frac{J(J-1)(J-2)\cdots(J-4h+1)}{(4h)!} N^{J-2h} \tag{2.43}$$

$$\approx \frac{1 \cdot 3 \cdot 5 \cdots (4h-1)}{2h+1} \frac{J^{1+4h} N^{J-2h}}{(4h)!}. \tag{2.44}$$

From this it is clear that for J's that grow more slowly than $\sqrt{N}$ only planar (genus 0) diagrams will contribute in the large N limit. This drastically reduces the amount of work that needs to be done to calculate correlators because one only needs to sum planar ribbon graphs. This is not necessarily trivial in some cases but it is doable. If however, we let the number of fields grow parametrically with N such that $J = N^p$ the genus 0 contribution will grow like:

$$N^{p+J}, \tag{2.45}$$

and the genus h contributions will grow like:

$$N^{p+4ph+J-2h}. \tag{2.46}$$

So one can see that for $p < \frac{1}{2}$ the genus 0 contributions will dominate but if $p \geq \frac{1}{2}$ then the higher genus contributions will be the same as the genus 0 contributions at $p = \frac{1}{2}$ and actually dominate when $p > \frac{1}{2}$. This implies that in order to get any sort of realistic prediction one has to sum all diagrams and because there are N fields and $N \longrightarrow \infty$ there will also be an infinite number of diagrams. Thus, the drawing of ribbon graphs can not be used to calculate correlators anymore and one needs a new technique. Luckily there is a technique that will aid us in our endeavor. It is the Schur polynomials which have their roots in group theory.

2.2.3 Schur polynomials and group theory

In this section the goal is to show how the Schur polynomials can be used to sum all possible ribbon graphs, not just planar ones and that any multi-trace operator can be represented as a linear combination of Schur polynomials. Thus, using the Schur polynomials any correlator can be calculated with finite or large N. To this end it is necessary to recall some group theory results [28] and define some notation.

Group theory results and notation The symmetric group S_n is the group of all permutations of n distinct objects. Cycle notation is used to represent the group elements. For example for the S_3 permutation of the group element (a,b,c) to (c,a,b) is written as (1 2 3) in cycle notation and is understood as the element in position 1 goes to position 2, the element in position 2 goes to position 3 and the element in position 3 goes to position 1. The rearrangement (c,a,b) is achieved by the group element (1 2 3).

Representations: A representation ($\Gamma(g)$) of a group G is a set of matrices, one for each element of the group, such that

$$\Gamma(g_1) \cdot \Gamma(g_2) = \Gamma(g_1 \cdot g_2) \quad \forall g_1, g_2 \in G \tag{2.47}$$

We call the space on which these matrices act the carrier space of the representation. There are infinite number of representations for a group but most are equivalent or reducible representations. An equivalent representation ($\Gamma_2(g)$) of $\Gamma_1(g)$ is related to $\Gamma_1(g)$ as follows:

$$\Gamma_2(g) = M\Gamma_1(g)M^{-1} \tag{2.48}$$

where M is some invertible matrix. A quick way to check whether two representations are equivalent is to check if their characters are the same for all elements of the group. The character of the representation r ($\chi_r(g)$) is defined as

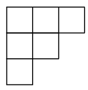
$$\chi_r(g) = \mathrm{Tr}(\Gamma_r(g)) \tag{2.49}$$

So the check of equivalence between two group representations ($\Gamma_r(g)$ and $\Gamma_s(g)$) of group G becomes

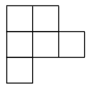
$$\chi_r(g) = \chi_s(g) \quad \forall g \in G \tag{2.50}$$

A representation is said to be reducible if it is equivalent to a block diagonal representation. An equivalent statement is: a representation is irreducible if it has no invariant subspaces. We call the list of inequivalent, irreducible representation of a group the irreps of a group. The list of irreps of a group is finite for a finite group and infinite for an infinite group.

Labeling the irreps of a group: The irreps of the symmetric group S_n are labeled by Young diagrams that are n boxes arranged in a specific order. The rules for the boxes are: all boxes must be arranged in rows starting from a common vertical line and there can not be more boxes in subsequent rows, than previous rows. An example of a valid diagram for S_6 is:



An example of an invalid diagram for S_6 is:



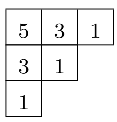
Similarly labels for the irreps of the unitary group ($U(N)$) can be defined. The diagrams look the same and have the same rules except that the diagrams can have any total number of boxes (instead of only n boxes as with S_n) but may only have a maximum of N rows.

These Young diagrams can be used to determine the dimensions of the representations and the states of the carrier space of the representations. One gets the dimension of reps as follows: For a representation of S_n one notates the dimension of representation R as d_R. It is found by dividing $n!$ by the product of the hook lengths (hooks). The hooks are defined as follows:

1. Draw a line starting from below the Young diagram into a box then make a perpendicular elbow from that line out of the diagram to the right.

2. The hook length of the box with the elbow is the number of boxes the line goes through in its path through the diagram.

The following shows the hook lengths of a diagram and the corresponding dimension of the representation.



$$d_{\tiny\yng(3,2,1)} = \frac{6!}{5 \cdot 3 \cdot 3 \cdot 1 \cdot 1 \cdot 1} = 16 \tag{2.51}$$

Now when working with the unitary group the dimension of a representation is notated as Dim_R and it is calculated by dividing the product of the factors (f_R) by the product of the hooks. The factors are defined as follows:

> Start at the top, leftmost box. Its factor is N. Then every consecutive box to the right has a factor 1 larger than the box to its left and every consecutive box down has a factor of 1 less then the box above it.

The factors of a representation of $U(N)$ are shown here with the corresponding dimension.

$$\begin{array}{|c|c|c|}\hline N & N+1 & N+2 \\ \hline N-1 & N & \\ \cline{1-2} N-2 & & \\ \cline{1-1}\end{array}$$

$$\text{Dim}_R = \frac{f_R}{\prod \text{hooks}} = \frac{N \cdot N \cdot (N-2)(N-1)(N+1)(N+2)}{45} \tag{2.52}$$

These diagrams can now be used to find the actual irreps themselves. We do this with Young-Yamanouchi (YY) patterns. YY patterns are constructed by writing all the possible ordered ways of dropping boxes from a Young diagram that leave a valid Young diagram after each drop. For instance consider the irrep of S_4 labeled by $\yng(3,1)$ ($\Gamma_{\yng(3,1)}(\sigma)$), the YY patterns will be

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 & & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 & & \\ \cline{1-1}\end{array} \tag{2.53}$$

where the numbers represent the order of dropping boxes. These YY patterns are considered states that when acted on by $\Gamma_{\yng(3,1)}(\sigma)$, where $\sigma \in$ adjacent 2-cycles of S_4, behave in a specific way. Let a specific state of the irrep R of S_n be $|R\rangle$ and $|R_{(i,j)}\rangle$ be the state were box i and j have been swaped. Then

$$\Gamma_R((i,i+1))|R\rangle = \frac{1}{c_i - c_{i+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}}|R_{(i,i+1)}\rangle \tag{2.54}$$

where c_i is the content of box i. The content of a diagram is defined in much the same way as the factors of the irreps of $U(N)$ are defined except that one starts with 0 in the top left box instead of N. For example the content of $\yng(3,2,1)$ is

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 & 0 & \\ \cline{1-2} -2 & & \\ \cline{1-1}\end{array} \tag{2.55}$$

Doing this for all the states of the irrep R gives equations that can be solved to find the matrix elements of the adjacent 2-cycles of $\Gamma_R(\sigma)$. Then all the other elements can be written as a product of the adjacent 2-cycles of the group.

<u>Notation:</u> Just like in special relativity where the position of an index tells us how a quantity transforms under the action of a Lorentz group we want to use the position of the indices here to tell us how quantities transform under the action of the unitary group. Quantities with

an upper index transform with U while quantities with lower indices transform with $U^\dagger$. So column vectors (notated as kets $|v\rangle$) will transform with U and row vectors (notated as bras $\langle v|$) will transform with $U^\dagger$. So the action of the unitary group will look as follows:

$$\Rightarrow U|v\rangle \equiv U^i{}_j v^j \tag{2.56}$$

$$\langle v|U^\dagger \equiv U^*{}_i{}^j v_j \tag{2.57}$$

Now Z is an $N \times N$ complex matrix and we declare that the action of the unitary group on Z will be

$$Z \longrightarrow UZU^\dagger \tag{2.58}$$

which implies that Z must be written with upper and lower indices (Z^i_j) so that

$$\Rightarrow U^i_j Z^j_k (U^\dagger)^k_l \longrightarrow Z^i_l \tag{2.59}$$

Z is an endomorphism of the N dimensional vector space V_N. Tensoring n Z's produces an operator $Z^{\otimes n} = Z \otimes Z \otimes \cdots \otimes Z$ that acts on $V_N^{\otimes n}$. The following index notation is used:

$$(Z^{\otimes n})^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} \tag{2.60}$$

The symmetric group S_n has a natural action on $V_N^{\otimes n}$given by:

$$(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_1}_{j_{\sigma(2)}} \cdots \delta^{i_n}_{j_{\sigma(n)}} \quad \sigma \in S_n \tag{2.61}$$

Using this notation:

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = (\sigma)^I_J (Z^{\otimes n})^J_I = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} \tag{2.62}$$

Consider $n = 4$ with $\sigma = (1)(2)(34)$:

$$\mathrm{Tr}((1)(2)(34)Z^{\otimes 4}) = Z^{i_1}_{i_1} Z^{i_2}_{i_2} Z^{i_3}_{i_4} Z^{i_4}_{i_3} = \mathrm{Tr}(Z)^2 \mathrm{Tr}(Z^2) \tag{2.63}$$

This is thus a unified notation for representing multi-trace operators of n fields in the space $V_N^{\otimes n}$.

The power of this notation can be illustrated with free field correlators. Consider the following correlator:

$$\left\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} Z^{\dagger k_1}_{l_1} \cdots Z^{\dagger k_n}_{l_n} \right\rangle = \left\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \right\rangle \tag{2.64}$$

There are a total of $n!$ Wick contractions that need to be summed. The sum of Wick contractions can however be represented as a sum of actions by S_n. To illustrate this consider the following Wick contraction with $n = 3$

$$
\begin{aligned}
&\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{i_3}_{j_3} (Z^\dagger)^{k_1}_{l_1} (Z^\dagger)^{k_2}_{l_2} (Z^\dagger)^{k_3}_{l_3} \rangle \\
&= \delta^{i_1}_{l_3} \delta^{i_2}_{l_1} \delta^{i_3}_{l_2} \delta^{k_3}_{j_1} \delta^{k_1}_{j_2} \delta^{k_2}_{j_3} \\
&= ((132))^I_L \left((132)^{-1}\right)^K_J
\end{aligned} \tag{2.65}
$$

Thus, one can deduce that

$$
\left\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \right\rangle = \sum_{\sigma \in S_n} (\sigma)^I_L (\sigma^{-1})^K_J \tag{2.66}
$$

Projection operators Now that the results and notation necessary have been defined we can begin to work towards our goal. To that end define an operator

$$
(\hat{P}_R)^I_J = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)(\sigma)^I_J \tag{2.67}
$$

where R labels the irrep and d_R is the dimension of the irrep; put there for normalisation. This operator acts on $V_N^{\otimes n}$ and is in fact a projection operator as can be seen from the following.

Working in S_2 with irrep $R = \Box\Box$. Consider the operator above acting on $(v \otimes w)$ where $v, w \in V_N$

$$
\begin{aligned}
(\hat{P}_{\Box\Box})^I_J (v \otimes w)^J =& (\hat{P}_{\Box\Box})^{i_1 i_2}_{j_1 j_2} v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[\chi_{\Box\Box}(\mathbb{I})(\mathbb{I})^{i_1 i_2}_{j_1 j_2} + \chi_{\Box\Box}((12))((12))^{i_1 i_2}_{j_2 j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[1 \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} + 1 \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[v^{i_1} w^{i_2} + v^{i_2} w^{i_1} \right]
\end{aligned} \tag{2.68}
$$

This has clearly projected onto the symmetric part of $(v \otimes w)$. This is in fact always the case when the irrep of the projector is a row of boxes. If on the other hand the irrep is a column of boxes it projects onto the antisymmetric part as can be seen if we choose $R = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$.

$$
\begin{aligned}
(\hat{P}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}})^I_J (v \otimes w)^J =& (\hat{P}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}})^{i_1 i_2}_{j_1 j_2} v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[\chi_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}(\mathbb{I})(\mathbb{I})^{i_1 i_2}_{j_1 j_2} + \chi_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}((12))((12))^{i_1 i_2}_{j_2 j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[1 \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} + (-1) \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[v^{i_1} w^{i_2} - v^{i_2} w^{i_1} \right]
\end{aligned} \tag{2.69}
$$

Now if one acts with the same projector twice it should be the same as acting once.

$$\begin{aligned}(\hat{P}_{\square\square})^I_J(\hat{P}_{\square\square})^J_K =&\frac{1}{4}\left[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}+\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right]\left[\delta^{j_1}_{k_1}\delta^{j_2}_{k_2}+\delta^{j_1}_{k_2}\delta^{j_2}_{k_1}\right]\\ =&\frac{1}{4}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}+\delta^{i_1}_{k_2}\delta^{i_2}_{k_1}+\delta^{i_1}_{k_2}\delta^{i_2}_{k_1}+\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\right]\\ =&\frac{1}{2}\left[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}+\delta^{i_1}_{k_2}\delta^{i_2}_{k_1}\right]\\ =&(\hat{P}_{\square\square})^I_K\end{aligned} \tag{2.70}$$

This is a defining property of a projector. It holds for all irreps. Projectors of different irreps are also orthogonal. Consider acting with two projectors one of irrep R and the other S.

$$\begin{aligned}(\hat{P}_R)^I_J(\hat{P}_S)^J_K =&\frac{d_R}{n!}\sum_{\sigma_1\in S_n}\chi_R(\sigma_1)(\sigma_1)^I_J\frac{d_S}{n!}\sum_{\sigma_2\in S_n}\chi_S(\sigma_2)(\sigma_2)^J_K\\ =&\frac{d_Rd_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\chi_R(\sigma_1)\chi_S(\sigma_2)(\sigma_1)^I_J(\sigma_2)^J_K\\ =&\frac{d_Rd_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\chi_R(\sigma_1)\chi_S(\sigma_2)(\sigma_2\sigma_1)^I_K\end{aligned} \tag{2.71}$$

change variables:

$$\sigma_1,\sigma_2 \longrightarrow \sigma_1,\psi=\sigma_2\sigma_1 \tag{2.72}$$

$$\Rightarrow (\psi)^I_K=(\sigma_2\sigma_1)^I_K=(\sigma_1)^I_J(\sigma_2)^J_K \tag{2.73}$$

$$\Rightarrow (\sigma_1^{-1})^L_I(\psi)^I_K=(\sigma_1^{-1})^L_I(\sigma_1)^I_J(\sigma_2)^J_K=(\sigma_2)^L_K \tag{2.74}$$

$$\Rightarrow (\sigma_2)^L_K=(\psi\sigma_1^{-1})^L_K, \tag{2.75}$$

because S_n is a finite group the sum over ψ and $\psi\sigma_1^{-1}$ both sum every element in the group. So we can write this as follows

$$\begin{aligned}\Rightarrow (\hat{P}_R)^I_J(\hat{P}_S)^J_K =&\frac{d_Rd_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}\chi_R(\sigma_1)\chi_S(\psi\sigma_1^{-1})(\psi)^I_K\\ =&\frac{d_Rd_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}(\Gamma_R(\sigma_1))^A_A(\Gamma_S(\psi\sigma_1^{-1}))^B_B(\psi)^I_K\\ =&\frac{d_Rd_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}(\Gamma_R(\sigma_1))^A_A(\Gamma_S(\psi))^B_C(\Gamma_S(\sigma_1^{-1}))^C_B(\psi)^I_K\\ =&\frac{d_Rd_S}{n!\,n!}\sum_{\sigma_1\in S_n}(\Gamma_R(\sigma_1))^A_A(\Gamma_S(\sigma_1^{-1}))^C_B\sum_{\psi\in S_n}(\Gamma_S(\psi))^B_C(\psi)^I_K\end{aligned} \tag{2.76}$$

Now the fundamental orthogonality relation states that

$$\sum_{g\in G}\Gamma_R(g^{-1})^A_B\Gamma_S(g)^C_D=\frac{|G|}{d_R}\delta_{RS}(\delta)^A_D(\delta)^C_B \tag{2.77}$$

Applying this to (2.76) where $|S_n|=n!$ gives

$$\begin{aligned}(\hat{P}_R)^I_J(\hat{P}_S)^J_K=&\frac{d_R d_S}{n!\,n!}\frac{n!}{d_R}\delta_{RS}(\delta)^A_B(\delta)^C_A\sum_{\psi\in S_n}(\Gamma_S(\psi))^B_C(\psi)^I_K\\ =&\delta_{RS}\frac{d_S}{n!}\sum_{\psi\in S_n}(\Gamma_S(\psi))^B_B(\psi)^I_K\\ =&\delta_{RS}(\hat{P}_S)^I_K\\ =&\delta_{RS}(\hat{P}_R)^I_K\end{aligned} \tag{2.78}$$

Now it will be useful to know if $\hat{P}_R$ commutes with the elements of the symmetric group ($\psi\in S_n$). So consider the following

$$(\hat{P}_R)^I_J(\psi)^J_K=\frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J(\psi)^J_K \tag{2.79}$$

Change σ to its conjugate τ related as follows

$$(\sigma)^I_J=\psi^I_L(\tau)^L_K(\psi^{-1})^K_J=(\psi^{-1}\tau\psi)^I_J \tag{2.80}$$

$$\begin{aligned}\Rightarrow(\hat{P}_R)^I_J(\psi)^J_K=&\frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_B(\psi^{-1})^B_J(\psi)^J_K\\ =&\frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_K\end{aligned} \tag{2.81}$$

Now the characters of the elements of a conjugacy class are the same since the trace has a cyclic symmetry. Consider the conjugacy class defined by $h=\sigma g\sigma^{-1}$. Then the characters of h and g are the same.

$$\begin{aligned}\chi_R(h)=&\mathrm{Tr}(\Gamma_R(\sigma g\sigma^{-1}))\\ =&\mathrm{Tr}(\Gamma_R(\sigma^{-1})\Gamma_R(g)\Gamma_R(\sigma))\\ =&\mathrm{Tr}(\Gamma_R(\sigma)\Gamma_R(\sigma^{-1})\Gamma_R(g))\\ =&\mathrm{Tr}(\Gamma_R(g)) && (2.82)\\ =&\chi_R(g) && (2.83)\end{aligned}$$

$$\Rightarrow (\hat{P}_R)^I_J(\psi)^J_K = (\psi)^I_L \frac{d_R}{n!} \sum_{\tau \in S_n} \chi_R(\tau)(\tau)^L_K = (\psi)^I_L (\hat{P}_R)^L_K \tag{2.84}$$

$$\Rightarrow \hat{P}_R \psi = \psi \hat{P}_R \tag{2.85}$$

So $\hat{P}_R$ commutes with the elements of the symmetric group.

Schur Polynomials Now we are in a position where we can define the Schur polynomials. The Schur polynomials for irrep R ($\chi_R(Z)$) are defined as:

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}(\sigma Z^{\otimes n}) = \mathrm{Tr}(P_R Z^{\otimes n}) \tag{2.86}$$

where $P_R = \frac{\hat{P}_R}{d_R}$ in the un-normalized projection operator. Let us examine this for $R = \square\square\square$ for S_3.

$$\chi_{\square\square\square}(Z) = \frac{1}{6} \sum_{\sigma \in S_n} \chi_{\square\square\square}(\sigma) \mathrm{Tr}(\sigma Z^{\otimes 3}) \tag{2.87}$$

$$= \frac{1}{6}\left[Z^{i_1}_{i_1} Z^{i_2}_{i_2} Z^{i_3}_{i_3} + Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} + Z^{i_1}_{i_1} Z^{i_2}_{i_3} Z^{i_3}_{i_2} + Z^{i_1}_{i_3} Z^{i_2}_{i_2} Z^{i_3}_{i_1} + Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} + Z^{i_1}_{i_3} Z^{i_2}_{i_1} Z^{i_3}_{i_2} \right] \tag{2.88}$$

$$= \frac{1}{6}\left[\mathrm{Tr}(Z)^3 + \mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + \mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + \mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + \mathrm{Tr}(Z^3) + \mathrm{Tr}(Z^2) \right] \tag{2.89}$$

$$= \frac{1}{6}\left[\mathrm{Tr}(Z)^3 + 3\mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + 2\mathrm{Tr}(Z^3) \right] \tag{2.90}$$

So we see that the Schur polynomials are linear combinations of multi-trace operators and it should be noted that these multi-trace operators are related to the conjugacy classes of S_3. The conjugacy classes of the symmetric group are determined by the cycle structure. For instance the cycle structure of (1)(23), (3)(12) and (2)(13) are all $1^1 2^1$ in other words 1 times 1-cycle and 1 times 2-cycle. The conjugacy classes of S_3 are 1^3, $1^1 2^1$ and 3^1 and there is one element in 1^3, three elements in $1^1 2^1$ and 2 elements in 3^1. So we see that each conjugacy class corresponds to an operator in which a 1-cycle corresponds to the trace of a single Z, a 2-cycle corresponds to the trace of a product of two Z's and so on.

Now we would like to calculate the two point correlation function [8]

$$
\begin{aligned}
\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle =& \frac{1}{d_R d_S}(\hat{P}_R)^I_J(\hat{P}_S)^K_L\left\langle (Z^{\otimes n})^J_I(Z^{\dagger\otimes n})^L_K\right\rangle \\
=& \frac{1}{d_R d_S}\sum_{\sigma\in S_n}(\hat{P}_R)^I_J(\hat{P}_S)^K_L(\sigma)^L_I(\sigma^{-1})^J_K \\
=& \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}(\hat{P}\sigma^{-1}\hat{P}_S\sigma)
\end{aligned}
\tag{2.91}
$$

But we have shown above that $\hat{P}_S$ and σ commute so the σ's cancel and we are left with

$$
\begin{aligned}
\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle =& \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}(\hat{P}_R\sigma^{-1}\hat{P}_S\sigma) \\
=& \frac{1}{d_R d_S}\mathrm{Tr}(\hat{P}_R\hat{P}_S)\sum_{\sigma\in S_n} \\
=& \frac{n!}{d_R d_S}\delta_{RS}\mathrm{Tr}(\hat{P}_R)
\end{aligned}
\tag{2.92}
$$

Now $\mathrm{Tr}(\hat{P}_R)$ is just the dimension of the subspace that is projected to which in this case is $d_R\mathrm{Dim}_R$. So we have

$$
\begin{aligned}
\left\langle \chi_R(Z)\chi_S^\dagger(Z)\right\rangle =& \frac{n!}{d_R d_S}\delta_{RS}d_R\mathrm{Dim}_R = \delta_{RS}\frac{n!}{d_R}\mathrm{Dim}_R \\
=& \delta_{RS}f_R
\end{aligned}
\tag{2.93}
$$

This is diagonal so forms an orthogonal basis. This was first achieved in [8].

For example the result for $R = S = \square\square\square$ will simply be

$$
\left\langle \chi_{\square\square\square}(Z)\chi_{\square\square\square}^\dagger(Z)\right\rangle = N(N+1)(N+2) \tag{2.94}
$$

which is in exact agreement with what is obtained by summing all ribbon graphs of

$$
\begin{aligned}
\left\langle \chi_{\square\square\square}(Z)\chi_{\square\square\square}^\dagger(Z)\right\rangle = \frac{1}{36}\Big\langle \big[\mathrm{Tr}(Z)^3 + 3\mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + 2\mathrm{Tr}(Z^3)\big]\,\big[\mathrm{Tr}(Z^\dagger)^3 \\
+ 3\mathrm{Tr}(Z^\dagger)\mathrm{Tr}(Z^{\dagger 2}) + 2\mathrm{Tr}(Z^{\dagger 3})\big]\Big\rangle
\end{aligned}
\tag{2.95}
$$

<u>Group of translations and the Fourier transform:</u> Consider the group of translations $T(x)$ that translate by a distance x. They can be represented by plane waves since we have

$$
T(a)T(b) = T(a+b) \tag{2.96}
$$

and

$$e^{ika}e^{ikb} = e^{ik(a+b)} \tag{2.97}$$

These representations are labeled by k and each different k is a different irreducible, inequivalent representation as can be seen by the following. Consider a representation of $T(x)$ labeled by k and an invertible matrix M then

$$Me^{ikx}M^{-1} = e^{ikx}MM^{-1} = e^{ikx} \tag{2.98}$$

which implies that a representation can only be equivalent to itself. Also, they are irreducible since they are 1 dimensional representations. Now consider three group elements of $T(\cdot)$: $T(a)$, $T(b)$ and $T(c)$. If $T(a)$ and $T(b)$ are related as follows

$$T(a) = T(c)T(b)T^{-1}(c) = T(c)T(b)T(-c) = T(b)T(c)T(-c) = T(b) \tag{2.99}$$

This shows that the group of translations is abelian and that each element is its own conjugacy class. This is the case for every abelian group. Now because these representations are 1 dimensional they are also their own character.

$$\chi_k(x) = e^{ikx} \tag{2.100}$$

We can show that the characters of different representations are orthogonal as was shown before by summing the characters of two different irreps over all the group elements. Is this case the space of group elements is continuous and so our sum is an integral.

$$\int_{-\infty}^{\infty} dx \chi_k(x)\chi_{k'}(x) = \int_{-\infty}^{\infty} dx e^{ikx}e^{ik'x} = 2\pi\delta(k-k') \tag{2.101}$$

This is a well known identity for plane waves and when viewed from the perspective of group theory is quite insightful. Now another well known identity for plane waves is

$$\int_{-\infty}^{\infty} dk \chi_k(x)\chi_k(x') = \int_{-\infty}^{\infty} dk e^{ikx}e^{ikx'} = 2\pi\delta(x-x') \tag{2.102}$$

What this identity shows is that plane waves span the space. Another well known mathematical tool involving plane waves is plane wave decomposition or the Fourier transform. It shows that any arbitrary function ($F(k)$) can be represented as a "linear combination" of plane waves

$$F(k) = \int_{-\infty}^{\infty} dx e^{ikx} f(x) \tag{2.103}$$

where $f(x)$ is an arbitrary function of x (which is the conjugacy class) and can be thought of as the coefficients of the "linear combination." The inverse of this is also true

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} F(k) \tag{2.104}$$

This shows that functions that take their values on the conjugacy classes (called class functions) can be represented as a sum of characters.

Now we would like to try and apply these ideas for the characters of the symmetric group. In other words we would like to develop the notion of a Fourier transform and inverse Fourier transform for the symmetric group characters. Consider the Schur polynomials

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}(\sigma Z^{\otimes n}) \tag{2.105}$$

Here $\chi_R(Z)$ is in fact an arbitrary function of the irrep by its dependance on R and as we have seen $\mathrm{Tr}(\sigma Z^{\otimes n})$ is a function on the conjugacy class. So the Schur polynomials are in fact the notion of a Fourier transform for the symmetric group. So can we find the inverse? In other words does this hold

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = A \sum_{R \vdash n} \chi_R(\sigma^{-1}) \chi_R(Z) \tag{2.106}$$

and for what A does it hold? Let us put $\mathrm{Tr}(\sigma Z^{\otimes n})$back into (2.105).

$$\begin{aligned} \chi_R(Z) =& \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) A \sum_{R' \vdash n} \chi_{R'}(\sigma^{-1}) \chi_{R'}(Z) \\ =& \frac{A}{n!} \sum_{R' \vdash n} \chi_{R'}(Z) \sum_{\sigma \in S_n} \chi_R(\sigma) \chi_{R'}(\sigma^{-1}) \end{aligned} \tag{2.107}$$

Taking the traces of the fundamental orthogonality relation gives

$$\sum_{\sigma \in S_n} \chi_{R'}(\sigma^{-1}) \chi_R(\sigma) = \frac{|S_n|}{d_R} \delta_{RR'} (\delta)^A_B (\delta)^B_A = \frac{n!}{d_R} \delta_{RR'} d_R = n!\, \delta_{RR'} \tag{2.108}$$

$$\begin{aligned} \Rightarrow \chi_R(Z) =& \frac{A}{n!} \sum_{R' \vdash n} \chi_{R'}(Z) n!\, \delta_{RR'} \\ =& A \sum_{R' \vdash n} \chi_{R'}(Z) \delta_{RR'} \\ =& A \chi_R(Z) \end{aligned} \tag{2.109}$$

$$\Rightarrow A = 1 \tag{2.110}$$

So we have

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \sum_{R \vdash n} \chi_R(\sigma^{-1})\chi_R(Z) \tag{2.111}$$

and since we know the left hand side can be any multi-trace operator this equation proves that any multi-trace operator can be represented as a linear combination of Schur polynomials.

2.3 Two Matrices

Here we will consider the dynamics of two general complex matrices. Expectation values in this model are computed as follows

$$\langle \cdots \rangle = \int [dZ][dZ^\dagger][dY][dY^\dagger] e^{-\mathrm{Tr}(ZZ^\dagger) - \mathrm{Tr}(YY^\dagger)} \cdots \tag{2.112}$$

So we will have

$$\left\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \right\rangle = \sum_{\sigma \in S_n} (\sigma)^I_L (\sigma^{-1})^K_J = \left\langle (Y^{\otimes n})^I_J (Y^{\dagger \otimes n})^K_L \right\rangle \tag{2.113}$$

$$\left\langle (Z^{\otimes n})^I_J (Z^{\otimes n})^K_L \right\rangle = \left\langle (Y^{\otimes n})^I_J (Y^{\otimes n})^K_L \right\rangle = \left\langle (Z^{\otimes n})^I_J (Y^{\otimes n})^K_L \right\rangle = 0 \tag{2.114}$$

$$\left\langle (Z^{\otimes n})^I_J (Y^{\dagger \otimes n})^K_L \right\rangle = \left\langle (Z^{\dagger \otimes n})^I_J (Y^{\otimes n})^K_L \right\rangle = 0 \tag{2.115}$$

and

$$\left\langle (Z^{\otimes n})^I_J (Y^{\otimes m})^K_L (Z^{\dagger \otimes n})^M_N (Y^{\dagger \otimes m})^O_P \right\rangle = \sum_{\sigma \in S_n} (\sigma)^I_N (\sigma^{-1})^M_J \sum_{\psi \in S_m} (\psi)^K_P (\psi^{-1})^O_L \tag{2.116}$$

We will use the following notation

$$\left(Z^{\otimes n} \otimes Y^{\otimes m}\right)^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} Y^{i_{n+1}}_{j_{n+1}} \cdots Y^{i_{n+m}}_{j_{n+m}} \tag{2.117}$$

where this is understood as acting on $V_N^{\otimes(n+m)}$ and

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}} \tag{2.118}$$

As before this produces multi-trace operators but we need to try and figure out what operators are produced by different permutations. Before only permutations from different conjugacy classes produced distinct observables but what permutations produce distinct observables in this case? In order to answer this let us distill why only permutations from different conjugacy classes produced distinct observables when we had one matrix. Consider operators A, B, C, D acting on V_N so that

$$(A \otimes B \otimes C \otimes D)^I_J = A^{i_1}_{j_1} B^{i_2}_{j_2} C^{i_3}_{j_3} D^{i_4}_{j_4} \tag{2.119}$$

acts on $V_N^{\otimes 4}$. Now consider the following permutations

$$(\sigma)^I_J = ((1432))^I_J = \delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3} \tag{2.120}$$

$$\Rightarrow (\sigma^{-1})^I_J = ((1234))^I_J = \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_4}\delta^{i_4}_{j_1} \tag{2.121}$$

and the following

$$\begin{aligned}(\sigma^{-1})^I_J (A \otimes B \otimes C \otimes D)^J_K (\sigma)^K_L =&(\sigma \cdot A \otimes B \otimes C \otimes D \cdot \sigma^{-1})^I_L \\ =&\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_4}\delta^{i_4}_{j_1} A^{j_1}_{k_1} B^{j_2}_{k_2} C^{j_3}_{k_3} D^{j_4}_{k_4} \delta^{k_1}_{l_4}\delta^{k_2}_{l_1}\delta^{k_3}_{l_2}\delta^{k_4}_{l_3} \\ =&A^{i_4}_{l_4} B^{i_1}_{l_1} C^{i_2}_{l_2} D^{i_3}_{l_3} \\ =&B^{i_1}_{l_1} C^{i_2}_{l_2} D^{i_3}_{l_3} A^{i_4}_{l_4} \\ =&(B \otimes C \otimes D \otimes A)^I_L \end{aligned} \tag{2.122}$$

We see that the operators $ABCD$ are rearranged according to the permutation σ. When we do this to an operator $Z^{\otimes n}$ or $Y^{\otimes n}$ we have

$$(\sigma \cdot Z^{\otimes n} \cdot \sigma^{-1})^I_L = (Z^{\otimes n})^I_L \tag{2.123}$$

and

$$(\sigma \cdot Y^{\otimes n} \cdot \sigma^{-1})^I_L = (Y^{\otimes n})^I_L \tag{2.124}$$

because the Z's are indistinguishable as are the Y's. This is because these fields are bosonic. This and the cyclic property of the trace ensures that

$$\mathrm{Tr}(\rho Z^{\otimes n}) = \mathrm{Tr}(\rho \cdot \sigma \cdot Z^{\otimes n} \cdot \sigma^{-1}) = \mathrm{Tr}(\sigma^{-1} \cdot \rho \cdot \sigma \cdot Z^{\otimes n}) \tag{2.125}$$

for $\forall \rho, \sigma \in S_n$. This is why when we had one matrix all permutations from the same conjugacy class produced the same observable. But now what happens when we have Y's and Z's such as

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}} \tag{2.126}$$

Clearly, we cannot swap a Z and a Y and produce the same observable. In other words

$$\begin{aligned}\mathrm{Tr}(\rho Z^{\otimes n} \otimes Y^{\otimes m}) =&\mathrm{Tr}(\rho \cdot \sigma \cdot Z^{\otimes n} \otimes Y^{\otimes m} \cdot \sigma^{-1}) \\ =&\mathrm{Tr}(\sigma^{-1} \cdot \rho \cdot \sigma \cdot Z^{\otimes n} \otimes Y^{\otimes m}) \end{aligned} \tag{2.127}$$

only holds if the permutation σ only permutes Z's with Z's and Y's with Y's but not Z's with Y's. This implies that for (2.127) to hold we can only have permutations where $\sigma \in S_n \times S_m$ and the classes of permutations defined by

$$g = \sigma h \sigma^{-1} \tag{2.128}$$

where $g, h \in S_{(n+m)}$ and $\sigma \in S_n \times S_m$, are called restricted conjugacy classes [29]. So we now know that only permutations from different restricted conjugacy classes produce distinct observables. We can now rewrite equation (2.116) as follows

$$\left\langle (Z^{\otimes n} Y^{\otimes m})^I_J (Z^{\dagger \otimes n} Y^{\dagger \otimes m})^K_L \right\rangle = \sum_{\sigma \in S_n \times S_m} (\sigma)^I_L (\sigma^{-1})^K_J \tag{2.129}$$

Our goal now is to work towards defining restricted Schur polynomials that will do for us what the normal Schur polynomials did with one matrix. So we need to define a trace on this restricted space $S_n \times S_m \subset S_{n+m}$. So that we can define the characters $\chi(\sigma) \ \ \forall \sigma \in S_n \times S_m$. At this point it is necessary to digress and talk about branching rules.

Branching rules: When restricting to a subgroup we see that an irrep of the group decomposes into a number of irreps of the subgroup and more than one of each may appear. Consider the irrep of S_4, $T = \yng(3,1)$. There are three states in this irrep

$$\young(421,3) \quad \young(431,2) \quad \young(432,1) \tag{2.130}$$

If we want to restrict this to the subgroup $S_2 \subset S_4$ of permutations that leave 1 and 2 fixed. The subgroup will be

$$S_2 = \{1, (34)\} \tag{2.131}$$

This implies that none of the states will mix with each other and after the restriction we will have

$$\yng(3,1) \longrightarrow \yng(1,1) \oplus \yng(2) \oplus \yng(2) \tag{2.132}$$

So we see that after a restriction of S_p to S_{p-q} the irrep $T \vdash p$ will subduce every representation that can be obtained by dropping q boxes. We write

$$\yng(1,1) \oplus \yng(2) \oplus \yng(2) \tag{2.133}$$

as

$$\yng(1,1) \oplus 2\yng(2) \tag{2.134}$$

So we have had obtained two representations of the same type ($\square\square$) that do not mix. These two copies have come from the YY patterns

$$\begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \tag{2.135}$$

In general when multiple copies of an irrep appear they need to be labeled so we introduce a multiplicity index which can label them. In this case we have used the partially labeled YY patterns as multiplicity indices.

Restricted trace and Schur polynomials: Now we are restricting from S_{n+m} to $S_n \times S_m$ so we need an irrep for S_{n+m}, $R \vdash n+m$, an irrep for S_n, $r \vdash n$, and an irrep for S_m, $s \vdash m$. We also need a multiplicity index α to tell us which copy of the irrep (r,s) of $S_n \times S_m$ we have and a state label (I) to tell us which state of $(r,s)\alpha$ we have. Thus, the states are denoted by $|R,(r,s)\alpha; I\rangle$. We now define the restricted character as follows [29]

$$\chi_{R,(r,s)\alpha\beta}(\sigma) = \mathrm{Tr}_{R,(r,s)\alpha\beta}((\sigma)) \tag{2.136}$$

$$= \sum_I \langle R,(r,s)\alpha; I|\Gamma_R(\sigma)|R,(r,s)\beta; I\rangle \tag{2.137}$$

Now we can choose a basis of the carrier space of R such that the different matrix elements for different copies of $\psi \in S_n \times S_m$ are equal:

$$\Gamma_{(r,s)\alpha}(\psi) = \Gamma_{(r,s)\beta}(\psi) \qquad \psi \in S_n \times S_m \quad \& \quad \forall \alpha, \beta \tag{2.138}$$

This basis will have the following form

$$|R,(r,s)\beta; I\rangle = \begin{pmatrix} |0\rangle \\ \vdots \\ |(r,s)\beta; I\rangle \\ \vdots \\ |0\rangle \end{pmatrix} \tag{2.139}$$

where every entry is a zero column vector except for the βth entry and

$$|(r,s)\alpha; I\rangle = |(r,s)\beta; I\rangle \qquad \forall \alpha, \beta \tag{2.140}$$

The states $|(r,s)\alpha; I\rangle$ represent a basis for the carrier space of $\Gamma_{(r,s)\alpha}(\psi)$ and since $|(r,s)\alpha; I\rangle = |(r,s)\beta; I\rangle$ we can choose $\Gamma_{(r,s)\alpha}(\psi) = \Gamma_{(r,s)\beta}(\psi)$ so that we can drop the multiplicity label:

$$\Gamma_{(r,s)\alpha}(\psi) = \Gamma_{(r,s)\beta}(\psi) = \Gamma_{(r,s)}(\psi) \tag{2.141}$$

This implies that for $\psi \in S_n \times S_m$ we have

$$\Gamma_R(\psi) = \begin{pmatrix} \Gamma_{(r,s)1}(\psi) & 0 & \cdots & 0 \\ 0 & \Gamma_{(r,s)2}(\psi) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \Gamma_{(r,s)g(r,s,R)}(\psi) \end{pmatrix} \tag{2.142}$$

$$= \begin{pmatrix} \Gamma_{(r,s)}(\psi) & 0 & \cdots & 0 \\ 0 & \Gamma_{(r,s)}(\psi) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \Gamma_{(r,s)}(\psi) \end{pmatrix} \tag{2.143}$$

where $g(r,s,R)$ is a Littlewood-Richardson number. Thus

$$\Gamma_R(\psi)|R,(r,s)\alpha;I\rangle = \Gamma_{(r,s)}(\psi)|R,(r,s)\alpha;I\rangle \qquad \psi \in S_n \times S_m \tag{2.144}$$

We can now show that the restricted characters are functions on the restricted conjugacy class. To show this we must show that

$$\chi_{R,(r,s)\alpha\beta}(\sigma) = \chi_{R,(r,s)\alpha\beta}(\rho) \tag{2.145}$$

for

$$\rho = \psi^{-1}\sigma\psi \qquad \psi \in S_n \times S_m. \tag{2.146}$$

Consider

$$\begin{aligned} \chi_{R,(r,s)\alpha\beta}(\rho) &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\rho)|R,(r,s)\beta;I\rangle \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\psi^{-1}\sigma\psi)|R,(r,s)\beta;I\rangle \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\psi^{-1})\Gamma_R(\sigma)\Gamma_R(\psi)|R,(r,s)\beta;I\rangle \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\Gamma_{(r,s)}(\psi)|R,(r,s)\beta;I \end{aligned} \tag{2.147}$$

We can use the linear algebra identity

$$\mathrm{Tr}(|i\rangle\langle j|A) = \langle j|A|i\rangle \tag{2.148}$$

where A is a matrix that acts on the set of states $|i\rangle$, to rewrite (2.147) as follows.

$$\begin{aligned}\chi_{R,(r,s)\alpha\beta}(\rho) &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\Gamma_{(r,s)}(\psi)|R,(r,s)\beta;I\rangle \\ &= \sum_I \mathrm{Tr}(|R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\Gamma_{(r,s)}(\psi)) \\ &= \mathrm{Tr}\left(\Gamma_{(r,s)}(\psi)\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\right) \end{aligned} \tag{2.149}$$

Now in the basis we have chosen

$$\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I| = \sum_I \begin{pmatrix} |0\rangle \\ \vdots \\ |(r,s)\beta;I\rangle \\ \vdots \\ |0\rangle \end{pmatrix} \begin{pmatrix} \langle 0| & \cdots & \langle (r,s)\alpha;I| & \cdots & \langle 0| \end{pmatrix} \tag{2.150}$$

Since we have $|(r,s)\beta;I\rangle = |(r,s)\alpha;I\rangle$, then

$$\sum_I |(r,s)\beta;I\rangle\langle (r,s)\alpha;I| = \mathbb{I} \tag{2.151}$$

where $\mathbb{I}$ is the identity matrix.

$$\Rightarrow \sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I| = \mathbb{I}\delta_{\beta\alpha} \tag{2.152}$$

where $\delta_{\beta\alpha}$ here is a matrix where all entries are zero except for the entry at row position β and column position α at which the entry is one. This tells us that

$$\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right] \tag{2.153}$$

will change the multiplicity label of something that acts on it from β to α. Then

$$\begin{aligned}&\Gamma_{(r,s)\beta}(\psi)\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right] \\ &= \left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_{(r,s)\alpha}(\psi)\end{aligned} \tag{2.154}$$

Since we have already dropped the multiplicity label, we have

$$\begin{aligned}
\chi_{R,(r,s)\alpha\beta}(\rho) &= \mathrm{Tr}\left(\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_{(r,s)}(\psi)\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\right)\\
&= \mathrm{Tr}\left(\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_R(\sigma)\right)\\
&= \sum_I \mathrm{Tr}\left(|R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\Gamma_R(\sigma)\right)\\
&= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\sigma)|R,(r,s)\beta;I\rangle\\
&= \chi_{R,(r,s)\alpha\beta}(\sigma)
\end{aligned} \tag{2.155}$$

This proves that the restricted characters are functions on the restricted conjugacy classes. We can then show that the restricted characters are orthogonal.

$$\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1})$$

$$= \sum_{\sigma\in S_{n+m}} \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\sigma)|R,(r,s)\beta;I \sum_J \langle T,(t,u)\gamma;J|\Gamma_T(\sigma^{-1})|T,(t,u)\delta;J\rangle$$

$$\begin{aligned}
= \sum_{\sigma\in S_{n+m}} \sum_I (\langle R,(r,s)\alpha;I|)_N(\Gamma_R(\sigma))^N_K(|R,(r,s)\beta;I\rangle)^K \times\\
\times \sum_J (\langle T,(t,u)\gamma;J|)_L(\Gamma_T(\sigma^{-1}))^L_M(|T,(t,u)\delta;J\rangle)^M
\end{aligned}$$

$$\begin{aligned}
= \sum_{\sigma\in S_{n+m}} (\Gamma_R(\sigma))^N_K(\Gamma_T(\sigma^{-1}))^L_M \sum_I (\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K \times\\
\times \sum_J (\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M
\end{aligned} \tag{2.156}$$

Using the fundamental orthogonality relation

$$\sum_{g\in G} \Gamma_R(g^{-1})^A_B\Gamma_S(g)^C_D = \frac{|G|}{d_R}\delta_{RS}(\delta)^A_D(\delta)^C_B \tag{2.157}$$

we get

$$\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1}) =$$

$$\frac{(n+m)!}{d_R}\delta_{RT}(\delta)^N_M(\delta)^L_K \sum_I (\langle R,(r,s)\alpha; I|)_N(|R,(r,s)\beta; I\rangle)^K \times$$
$$\times \sum_J (\langle T,(t,u)\gamma; J|)_L(|T,(t,u)\delta; J\rangle)^M \tag{2.158}$$

$$= \frac{(n+m)!}{d_R}\delta_{RT}(\delta)^N_M(\delta)^L_K \sum_I (\langle R,(r,s)\alpha; I|)_N(|R,(r,s)\beta; I\rangle)^K \times$$
$$\times \sum_J (\langle T,(t,u)\gamma; J|)_L(|T,(t,u)\delta; J\rangle)^M$$

$$= \frac{(n+m)!}{d_R}\delta_{RT} \sum_I \sum_J \langle R,(r,s)\alpha; I|T,(t,u)\delta; J\rangle\langle T,(t,u)\gamma; J|R,(r,s)\beta; I\rangle$$
$$= \frac{(n+m)!}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma} \sum_I \sum_J \delta_{IJ} \tag{2.159}$$

$$\Rightarrow \sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1}) = \frac{(n+m)!}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}d_r d_s \tag{2.160}$$

We now define an operator $P_{R,(r,s)\alpha\beta}$ such that

$$P_{R,(r,s)\alpha\beta} = \frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\sigma \tag{2.161}$$

These will have the same role as the projection operators in the one matrix case but these are not in general projection operators. We can now check what $P_{R,(r,s)\alpha\beta}\cdot\psi$ is for $\psi\in S_n\times S_m$

$$(P_{R,(r,s)\alpha\beta})^I_J(\psi)^J_K = \frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)(\sigma)^I_J(\psi)^J_K \tag{2.162}$$

Change σ to its conjugate τ related as follows

$$(\sigma)^I_J = \psi^I_L(\tau)^L_K(\psi^{-1})^K_J = (\psi^{-1}\tau\psi)^I_J \tag{2.163}$$

This implies

$$
\begin{aligned}
(P_{R,(r,s)\alpha\beta})^I_J(\psi)^J_K =& \frac{1}{n!\,m!}\sum_{\tau\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_B(\psi^{-1})^B_J(\psi)^J_K \\
=& \frac{1}{n!\,m!}\sum_{\tau\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_K
\end{aligned}
\tag{2.164}
$$

Now since the restricted characters are functions on the restricted conjugacy class we have

$$
\begin{aligned}
(P_{R,(r,s)\alpha\beta})^I_J(\psi)^J_K =& (\psi)^I_L\frac{1}{n!\,m!}\sum_{\tau\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\tau)(\tau)^L_K \\
=& (\psi)^I_L(P_{R,(r,s)\alpha\beta})^L_K
\end{aligned}
\tag{2.165}
$$

$$
\Rightarrow P_{R,(r,s)\alpha\beta}\cdot\psi = \psi\cdot P_{R,(r,s)\alpha\beta} \tag{2.166}
$$

We are now in a position where we can define the restricted Schur polynomials [29] ($\chi_{R,(r,s)\alpha\beta}(Z,Y)$) or "Fourier transform [24]"

$$
\chi_{R,(r,s)\alpha\beta}(Z,Y) \equiv \mathrm{Tr}(P_{R,(r,s)\alpha\beta}Z^{\otimes n}\otimes Y^{\otimes m}) \tag{2.167}
$$

$$
\begin{aligned}
=& \frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}}\mathrm{Tr}(\chi_{R,(r,s)\alpha\beta}(\sigma)\sigma\cdot\ Z^{\otimes n}\otimes Y^{\otimes m}) \\
=& \frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\mathrm{Tr}(\sigma\cdot\ Z^{\otimes n}\otimes Y^{\otimes m})
\end{aligned}
\tag{2.168}
$$

So we would again like to find the inverse Fourier transform by solving for A in the following

$$
\mathrm{Tr}(\sigma\cdot\ Z^{\otimes n}\otimes Y^{\otimes m}) = A\sum_{R,(r,s)\alpha\beta}\chi_{R,(r,s)\alpha\beta}(\sigma^{-1})\chi_{R,(r,s)\alpha\beta}(Z,Y) \tag{2.169}
$$

Putting this back into (2.168) we get

$$
\begin{aligned}
\chi_{R,(r,s)\alpha\beta}(Z,Y) =& \frac{A}{n!\,m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\sum_{T,(t,u)\gamma\delta}\chi_{T,(t,u)\gamma\delta}(\sigma^{-1})\chi_{T,(t,u)\gamma\delta}(Z,Y) \\
=& \frac{A}{n!\,m!}\sum_{T,(t,u)\gamma\delta}\chi_{T,(t,u)\gamma\delta}(Z,Y)\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1})
\end{aligned}
\tag{2.170}
$$

We know

$$
\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1}) = \frac{(n+m)!}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}d_rd_s \tag{2.171}
$$

$$\Rightarrow \chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{A(n+m)!\, d_r d_s}{d_R n!\, m!} \sum_{T,(t,u)\gamma\delta} \chi_{T,(t,u)\gamma\delta}(Z,Y)\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}$$
$$= \frac{A(n+m)!\, d_r d_s}{d_R n!\, m!} \chi_{R,(r,s)\alpha\beta}(Z,Y) \tag{2.172}$$

This implies

$$A = \frac{d_R n!\, m!}{d_r d_s (n+m)!} \tag{2.173}$$

and so the inverse Fourier transform [24] is given by

$$\mathrm{Tr}(\sigma \cdot \; Z^{\otimes n} \otimes Y^{\otimes m}) = \frac{d_R n!\, m!}{d_r d_s (n+m)!} \sum_{R,(r,s)\alpha\beta} \chi_{R,(r,s)\alpha\beta}(\sigma^{-1}) \chi_{R,(r,s)\alpha\beta}(Z,Y) \tag{2.174}$$

This again tells us that any multi-trace operator can be represented as a linear combination of restricted Schur polynomials. Now we would like to find an expression for the 2 point function. This was first done in [23].

$$\langle \chi_{R,(r,s)\alpha\beta}(Z,Y)\chi_{T,(t,u)\gamma\delta}(Z^\dagger, Y^\dagger)\rangle$$
$$= \langle \mathrm{Tr}(P_{R,(r,s)\alpha\beta} Z^{\otimes n} \otimes Y^{\otimes m}) \mathrm{Tr}(P_{T,(t,u)\gamma\delta} Z^{\dagger \otimes n} \otimes Y^{\dagger \otimes m})\rangle$$
$$= (P_{R,(r,s)\alpha\beta})^I_J (P_{T,(t,u)\gamma\delta})^K_L \langle (Z^{\otimes n} \otimes Y^{\otimes m})^J_I (Z^{\dagger\otimes n} \otimes Y^{\dagger\otimes m})^L_K \rangle$$
$$= \sum_{\sigma\in S_n \otimes S_m} (P_{R,(r,s)\alpha\beta})^I_J (P_{T,(t,u)\gamma\delta})^K_L (\sigma)^J_K (\sigma^{-1})^L_I$$
$$= \sum_{\sigma\in S_n \otimes S_m} \mathrm{Tr}(P_{R,(r,s)\alpha\beta} \sigma P_{T,(t,u)\gamma\delta} \sigma^{-1})$$
$$= \sum_{\sigma\in S_n \otimes S_m} \mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta})$$
$$= n!\, m!\, \mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta}) \tag{2.175}$$

Now we just need to know $\mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta})$. Let us consider

$$P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta}$$
$$= \frac{1}{(n!\, m!\,)^2} \sum_{\sigma\in S_{n+m}} \sum_{\psi\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \chi_{T,(t,u)\gamma\delta}(\psi)(\sigma)^I_J (\psi)^J_K$$
$$= \frac{1}{(n!\, m!\,)^2} \sum_{\sigma\in S_{n+m}} \sum_{\psi\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \chi_{T,(t,u)\gamma\delta}(\psi)(\psi\sigma)^I_K \tag{2.176}$$

Change variables

$$\sigma, \psi \longrightarrow \sigma, \tau = \psi\sigma \tag{2.177}$$

$$\Rightarrow (\tau)^I_K = (\psi\sigma)^I_K = (\sigma)^I_J(\psi)^J_K \tag{2.178}$$

$$\Rightarrow (\sigma^{-1})^L_I(\tau)^I_K = (\psi)^L_K \tag{2.179}$$

$$\Rightarrow (\psi)^L_K = (\tau\sigma^{-1})^L_K \tag{2.180}$$

So we have

$$P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta} = \frac{1}{(n!\,m!\,)^2}\sum_{\sigma\in S_{n+m}}\sum_{\tau\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\tau\sigma^{-1})(\tau)^I_K$$

$$= \frac{1}{(n!\,m!\,)^2}\sum_{\tau\in S_{n+m}}\sum_{\sigma\in S_{n+m}}(\Gamma_R(\sigma))^N_K(\Gamma_T(\tau\sigma^{-1}))^L_M\sum_I(\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K\times$$
$$\times\sum_J(\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M(\tau)^I_K$$

$$= \frac{1}{(n!\,m!\,)^2}\sum_{\tau\in S_{n+m}}(\Gamma_T(\tau))^L_O\sum_{\sigma\in S_{n+m}}(\Gamma_R(\sigma))^N_K(\Gamma_T(\sigma^{-1}))^O_M\times$$
$$\times\sum_I(\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K\sum_J(\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M(\tau)^I_K$$

$$= \frac{1}{(n!\,m!\,)^2}\frac{(n+m)!}{d_R}\sum_{\tau\in S_{n+m}}(\Gamma_T(\tau))^L_O\delta_{RT}(\delta)^N_M(\delta)^O_K\sum_I(\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K\times$$
$$\times\sum_J(\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M(\tau)^I_K$$

$$= \frac{1}{(n!\,m!\,)^2}\frac{(n+m)!}{d_R}\sum_{\tau\in S_{n+m}}\delta_{RT}\sum_I\sum_J\langle R,(r,s)\alpha;I|T,(t,u)\delta;J\rangle\times$$
$$\times\langle T,(t,u)\gamma;J|\Gamma_T(\tau)|R,(r,s)\beta;I\rangle(\tau)^I_K$$

$$= \frac{1}{(n!\,m!\,)^2}\frac{(n+m)!}{d_R}\sum_{\tau\in S_{n+m}}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\sum_I\sum_J\delta_{IJ}\langle R,(r,s)\gamma;J|\Gamma_T(\tau)|R,(r,s)\beta;I\rangle(\tau)^I_K$$

$$=\frac{(n+m)!}{(n!\,m!\,)^2}\frac{1}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\sum_{\tau\in S_{n+m}}\chi_{T,(t,u)\gamma\beta}(\tau)(\tau)^I_K$$
$$=\frac{(n+m)!}{n!\,m!}\frac{1}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}P_{T,(t,u)\gamma\beta} \tag{2.181}$$

Now it can further be shown that [23, 29]

$$\mathrm{Tr}(P_{T,(t,u)\gamma\beta}) = \frac{1}{n!\,m!} d_r d_s f_R \delta_{\gamma\beta} \tag{2.182}$$

$$\begin{aligned}\Rightarrow \mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta}) =& \frac{(n+m)!}{n!\,m!}\frac{1}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\mathrm{Tr}(P_{T,(t,u)\gamma\beta}) \\ =& \frac{(n+m)!}{(n!\,m!\,)^2}\frac{d_r d_s}{d_R} f_R \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\gamma\beta}\end{aligned} \tag{2.183}$$

This implies that the two point function is given by

$$\begin{aligned}\langle \chi_{R,(r,s)\alpha\beta}(Z,Y)\chi_{T,(t,u)\gamma\delta}(Z^\dagger,Y^\dagger)\rangle =& \frac{(n+m)!}{n!\,m!}\frac{d_r d_s}{d_R} f_R \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\gamma\beta} \\ =& \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s} f_R \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\gamma\beta}\end{aligned} \tag{2.184}$$

Which is diagonal and allows us to easily sum all ribbon graphs.

2.4 Displaced Corners and Gauss Graphs

Within the AdS/CFT correspondence the set of states in the $\mathcal{N}=4$ SYM defined by the Schur polynomials $\chi_R(Z)$ are dual to systems of giant gravitons in the string theory on $AdS_5 \times S^5$ [3, 8]. If the Young diagram R has p rows of length $O(N)$, where p is order one, they are dual to p giants expanded in the AdS_5 space and if R has p columns of length $O(N)$, where p is order one, they are dual to p giants expanded in the S^5 space. The examination of the reduction to eigenvalue dynamics of one matrix shows that the Schur polynomials are in fact associated with non-interacting fermion states or equivalently non-interacting eigenvalue states. This tells us that giant graviton branes expanded in the AdS_5 correspond to single highly excited eigenvalues which leads us to the conclusion that the giant graviton is an eigenvalue[5, 9]. We can also interpret giant graviton branes expanded in the S^5 as holes in the Fermi sea, and hence to collective excitations of the eigenvalues where many eigenvalues are excited[9]. This link to fermions provided by eigenvalue dynamics also allows us to link half-BPS geometries with the Schur polynomials. Lin, Lunin and Maldacena, in [7], discovered a map between half-BPS geometries and fermion states where every fermion state can be identified with a particular half-BPS geometry[8, 9]. Since the Schur polynomials define the fermion state, this leads us to the conclusion every Schur polynomial can also be identified with a particular half-BPS geometry. Now it seems that the restricted Schur polynomials $\chi_{R,(r,s)\alpha\beta}(Z,Y)$ might also have a similar link to giant graviton states and in fact they do [26]. Instead of having giant graviton branes on their own the Y matrices alow us to attach open strings to the giants thus allowing the giants to interact.

It is of interest for us to try and see if we can apply the insight provided by the eigenvalue dynamics of a one matrix model to the restricted Schur polynomials since this is a two matrix operator and our goal is to find eigenvalue dynamics for a two matrix model. If we want to do this then it is logical to start by constructing a picture as close to the one we have for the one matrix model. We consider a restricted Schur polynomial $\chi_{R,(r,s)\alpha\beta}(Z,Y)$ labeled by three Young diagrams, $R \vdash n+m$, $r \vdash n$ and $s \vdash m$, and impose the condition that the row lengths in R differ by $O(N)$ boxes. We have $O(1)$ Y matrices at the ends of these rows so that we have much fewer Y matrices than Z matrices in the large N limit. This is known as the displaced corners approximation. It was found in [25] that the dilatation operator, when acting on the restricted Schur polynomials in this approximation, factorizes into a separate action on the r labels and on the s labels meaning these operators have a definite scaling dimension. In [25, 26] a basis for these operators was given by what are called Gauss graph operators. They are labeled Young diagrams R, r and Gauss graphs. These graphs are composed of nodes that are traversed by oriented edges[25, 26]. There is one node for each row of R and the directed edges start and end on the nodes. There is one edge for each Y field and the number of oriented edges ending on a node must equal the number of oriented edges emanating from a node, see Fig. 2.7 as an example. The constraint that the number of edges ending on a node equals the number of edges emanating from the node has a natural interpretation in the dual string theory: it reproduces the Gauss law on the brane world volume hence the name Gauss graphs. The brane world volume is topologically an S^3 and so compact, meaning that the Gauss law requires the charge on the giant's world volume to vanish. Since the ends of strings are charged this translates into the requirement that the number of strings emanating from a giant must equal the number of strings terminating on that giant.

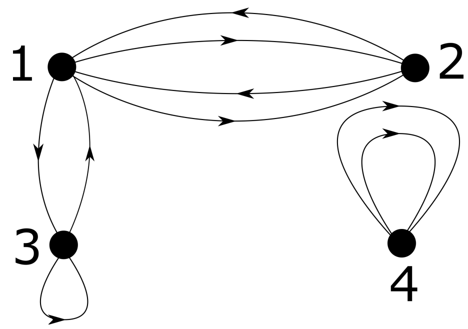


Figure 2.7: An example of a Gauss graph labeling an operator with a definite scaling dimension.

These operators represent a system of interacting giants with the open strings between the giants representing the interactions. This is not what we had with one matrix; there we had non-interacting giants. We need to impose another condition: that the operators be BPS. For that we need to consider the action of the dilatation operator

$$D = D_0 + \hbar D_1 + \hbar^2 D_2 + \cdots \tag{2.185}$$

which measures the scaling dimensions of operators. BPS states are known to have a non-corrected scaling dimension which means that we must have that the action of D_1 is zero. From [25, 26] we see that D_1 is of the following form

$$D_1 \sim \sum_{i>j} n_{ij} \Delta_{ij} \tag{2.186}$$

where n_{ij} is an integer telling us how many strings are stretched between branes i and j and Δ_{ij} is a quantity related to swapping Zs between rows i and j of r [26]. This implies that the only BPS configurations of Gauss graphs are ones in which dots are only connected to themselves, see Fig. 2.8 as an example.

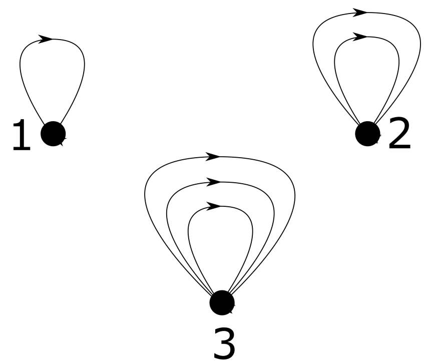


Figure 2.8: An example of a graph labeling a BPS operator.

The Gauss graph operators that are BPS are defined as

$$O^{BPS}_{R,r,\vec{m}}(Z,Y) = \frac{|H|}{\sqrt{m!}} \sum_{s \vdash m} \sum_{\mu} \sqrt{d_s} O_{R,(r,s)\mu\mu}(Z,Y) \tag{2.187}$$

where $O_{R,(r,s)\alpha\beta}(Z,Y)$ are normalized versions of the restricted Schur polynomials

$$O_{R,(r,s)\alpha\beta}(Z,Y) = \sqrt{\frac{\text{hooks}_r \text{hooks}_s}{\text{hooks}_R f_R}} \chi_{R,(r,s)\alpha\beta}(Z,Y) \tag{2.188}$$

and $\vec{m}$ is the vector telling us how many Y matrices are on each row of R. For instance, given a row i of R the number of Y matrices on that row will be given by m_i. H is defined as

$$H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p} \tag{2.189}$$

Thus, we can write $O^{BPS}_{R,r,\vec{m}}(Z,Y)$ as

$$O^{BPS}_{R,r,\vec{m}}(Z,Y) = \frac{|H|}{\sqrt{m!\,n!}}\sqrt{\frac{\text{hooks}_r}{\text{hooks}_R f_R}}\sum_{\sigma\in S_{n+m}} \text{Tr}(P^{\vec{m}}_{R,r}\Gamma^{(R)}(\sigma))\text{Tr}(\sigma Y^{\otimes m}Z^{\otimes n}) \tag{2.190}$$

where

$$P^{\vec{m}}_{R,r} = \sum_{s\vdash m}\sum_{\mu} P_{R,(r,s)\mu\mu} \tag{2.191}$$

This is remarkable because these BPS Gauss graph operators represent a system of non-interacting giant graviton branes where the branes again represent rows in a Young diagram, exactly the same system that we could reduce to eigenvalues for the one matrix model. This is very strong evidence that we should be able to find an eigenvalue description for a two matrix model in the sector in which operators are BPS and have definite scaling dimension.

Chapter 3

Eigenvalue Dynamics

3.1 Motivation for Eigenvalue Dynamics

To start let us look at eigenvalue dynamics for a matrix model with one matrix. The classical limit of eigenvalue dynamics for one matrix corresponds to the large N limit of the Matrix Models. As can be seen by the following. Consider taking a general path integral and replacing it with its classical configuration

$$\langle\cdots\rangle = \int [d\phi](\cdots)e^{-S} \longrightarrow (\cdots)\Big|_{\text{classical configuration}} \tag{3.1}$$

In the path integral there are fluctuations of order $\hbar$ around the classical value meaning we will make an error of order $\hbar$ by this replacement. In the matrix model the role of $\hbar$ is played by $\frac{1}{N^2}$ so if we replace the matrix model path integral with its classical configuration

$$\langle\cdots\rangle = \int [dM](\cdots)e^{-S} \longrightarrow (\cdots)\Big|_{\text{classical configuration}} \tag{3.2}$$

We will make an error of order $\frac{1}{N^2}$ for each integral. Recalling that this represents N^2 integrals we see that the total error made for this correlator will be of order $\frac{1}{N^2} \times N^2$ which is order 1 and so will not vanish in the $N \longrightarrow \infty$ limit. This means that the classical limit of the matrix model does not correspond to the large N limit of the theory. We need to formulate a theory whose classical limit does correspond to the large N limit of the matrix model. This theory will have to have an error that is suppressed by $\frac{1}{N^a}$ where $a > 0$ so that in the large N limit it vanishes. We can do this by writing the theory in terms of eigenvalues.

We can express all the gauge invariant observables ($\text{Tr}(M^n)$) in terms of the eigenvalues of M_{ij} as follows

$$\text{Tr}(M^n) = \sum_{i=0}^{N} \lambda_i^n \tag{3.3}$$

where $\lambda_i,\ i = 1, 2, ..., N$ are the eigenvalues of M_{ij}. If we now rewrite all of the dynamics in terms of the eigenvalues we will be able to write the general correlator as

$$\langle\cdots\rangle = \int [d\lambda_i](\cdots)e^{-S_{eff}} \tag{3.4}$$

where S_{eff} is the effective action that results from changing to eigenvalues. This now represents N integrals each of whose fluctuations about the classical value is of order $\frac{1}{N^2}$. So that if we replace the path integral with its classical configuration

$$\langle \cdots \rangle = \int [d\lambda_i](\cdots)e^{-S_{eff}} \longrightarrow (\cdots)\Big|_{\text{classical configuration}} \tag{3.5}$$

The total error made for this correlator will be of order $\frac{1}{N^2} \times N = \frac{1}{N}$ which will vanish in the large N limit and so if formulated correctly the large N limit will give the classical limit of the matrix model. See Appendix A.1 for an example of how to formulate eigenvalue dynamics.

3.2 Motivation for Studying the Eigenvalue Dynamics of two Matrices

Now let us turn to the main goal of this work and try to give motivation for studying the eigenvalue dynamics of a two matrix system. We will consider the dynamics of two complex matrices, corresponding to the $SU(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills theory. Further, we consider the theory on $R \times S^3$ and expand all fields in spherical harmonics of the S^3. We will consider only the lowest s-wave components of these expansions so that the matrices are constant on the S^3. The reduction to the s-wave will be motivated below. In this way we find a matrix model quantum mechanics of two complex matrices. Recall that with the dynamics of two complex matrices, expectation values are computed as follows

$$\langle \cdots \rangle = \int [dZdZ^\dagger dYdY^\dagger]e^{-S} \cdots \tag{3.6}$$

At first sight it appears that any attempts to reduce (3.6) to an eigenvalue description are doomed to fail: the integral in (3.6) runs over two independent complex matrices Z and Y which will almost never be simultaneously diagonalizable. However, perhaps there is a class of questions, generalizing the singlet sector of a single hermitian matrix model, that can be studied using eigenvalue dynamics. To explore this possibility, let's review the arguments that lead to eigenvalue dynamics for a single complex matrix Z (see Appendix A.1). We can use the Schur decomposition[27, 30, 31],

$$Z = U^\dagger DU \tag{3.7}$$

with U a unitary matrix and D an upper triangular matrix, to explicitly change variables. Since we only consider observables that depend on the eigenvalues (the diagonal elements of D) we can integrate U and the off diagonal elements of D out of the model, leaving only the eigenvalues. The result of the integrations over U and the off diagonal elements of D is a non trivial Jacobian. Denoting the eigenvalues of Z by z_i, those of $Z^\dagger$ are given by complex conjugation, $\bar{z}_i$. The resulting Jacobian is[27]

$$J = \Delta(z)\Delta(\bar{z}) \tag{3.8}$$

where

$$\Delta(z) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} = \prod_{j>k}^{N}(z_j - z_k) \quad (3.9)$$

is the usual Van der Monde determinant. A standard argument now maps this into non-interacting fermion dynamics[10]. Trying to apply a very direct change of variables argument to the two matrix model problem appears difficult. There is however an approach which both agrees with the above non-interacting fermion dynamics and can be generalized to the two matrix model. The idea is to construct a basis of operators that diagonalizes the inner product of the free theory. This is given by the Schur polynomials. Recall that the two point function of Schur polynomials is

$$\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle = f_R\delta_{RS} \quad (3.10)$$

where all spacetime dependence in the correlator has been suppressed. This dependence is trivial as it is completely determined by conformal invariance. Remarkably there is an immediate and direct connection to non-interacting fermions: the fermion wave function can be written as

$$\psi_R(\{z_i, \bar{z}_i\}) = \chi_R(Z)\Delta(z)e^{-\frac{1}{2}\sum_i z_i\bar{z}_i} \quad (3.11)$$

This relation can be understood as a combination of the state operator correspondence (we associate a Schur polynomial operator on R^4 to a wave function on $R \times S^3$) and the reduction to eigenvalues (which is responsible for the $\Delta(z)$ factor)[9]. In this map the number of boxes in each row of R determines the amount by which each fermion is excited. In this way, each row in the Young diagram corresponds to a fermion and hence to an eigenvalue. Having one very long row corresponds to exciting a single fermion by a large amount, which corresponds to a single large (highly excited) eigenvalue. In the dual AdS gravity, a single long row is a giant graviton brane that has expanded in the AdS_5 spacetime. Having one very long column corresponds to exciting many fermions by a single quantum, which corresponds to many eigenvalues excited by a small amount. In the dual AdS gravity, a single long column is a giant graviton brane that has expanded in the S^5 space.

The first questions we should tackle when approaching the two matrix problem should involve operators built using many Z fields and only a few Y fields. In this case at least a rough outline of the one matrix physics should be visible, and experience with the one matrix model will prove to be valuable.

For the case of two matrices the restricted Schur polynomials $\chi_{R,(r,s)ab}(Z,Y)$ diagonalize the free field two point function [22, 23, 24]. With the two point function given by

$$\langle \chi_{R,(r,s)ab}(Z,Y)\chi_{T,(t,u)cd}(Z^\dagger,Y^\dagger)\rangle = f_R \frac{\text{hooks}_R}{\text{hooks}_r \text{hooks}_s}\delta_{RT}\delta_{rt}\delta_{su}\delta_{ac}\delta_{bd} \tag{3.12}$$

These operators do not have a definite dimension. However, they only mix weakly under the action of the dilatation operator and they form a convenient basis in which to study the spectrum of anomalous dimensions[32]. This action has been diagonalized in a limit in which R has order 1 rows (or columns), $m \ll n$ and n is of order N. Operators of a definite dimension are labeled by graphs composed of nodes that are traversed by oriented edges[25, 26]. There is one node for each row, so that each node corresponds to an eigenvalue. The directed edges start and end on the nodes. There is one edge for each Y field and the number of oriented edges ending on a node must equal the number of oriented edges emanating from a node. See Figure 2.7 for an example of a graph labeling an operator. This picture, derived in the Yang-Mills theory, has an immediate and compelling interpretation in the dual gravity: each node corresponds to a giant graviton brane and the directed edges are open string excitations of these branes. The constraint that the number of edges ending on a node equals the number of edges emanating from the node is simply encoding the Gauss law on the brane world volume, which is topologically an S^3. For this reason the graphs labeling the operators are called Gauss graphs. If we are to obtain a system of non-interacting eigenvalues, we should only consider Gauss graphs that have no directed edges stretching between nodes. See Figure 2.8 for an example. In fact, these all correspond to BPS operators. We thus arrive at a very concrete proposal:

If there is a free fermion description arising from the eigenvalue dynamics of the two matrix model, it will describe the BPS operators of the $SU(2)$ sector.

The BPS operators are associated to supergravity solutions of string theory. Indeed, the only one-particle states saturating the BPS bound in gravity are associated to massless particles and lie in the supergravity multiplet. Thus, eigenvalue dynamics will reproduce the supergravity dynamics of the gravity dual.

The BPS operators are all constructed from the s-wave of the spherical harmonic expansion on S^3[9]. This is our motivation for only considering operators constructed using the s-wave of the fields Y and Z. One further comment is that it is usually not consistent to simply restrict to a subset of the dynamical degrees of freedom. Indeed, this is only possible if the subset of degrees of freedom dynamically decouples from the rest of the theory. In the case that we are considering this is guaranteed to be the case, in the large N limit, because the Chan-Paton indices of the directed edges are frozen at large N [25].

We should mention that eigenvalue dynamics as dual to supergravity has also been advocated by Berenstein and his collaborators[33, 34, 35, 36, 37, 38, 39]. See also [40, 41, 42, 43] for related studies. Using a combination of numerical and physical arguments, which are rather different to the route we have followed, compelling evidence for this proposal has already been found. The basic idea is that at strong coupling the commutator squared term in the action forces the Higgs fields to commute and hence, at strong coupling, the Higgs fields of the theory should be simultaneously diagonalizable. In this case, an eigenvalue description is possible. Notice that our argument is a weak coupling large N argument argument, based on diagonalization of the one loop dilatation operator, that comes to precisely the same conclusion. In this work we will make some exact analytic statements that agree with and, in our opinion, refine some of the physical picture of the above studies. For example, we will start to make precise statements about what eigenvalue dynamics does and does not correctly reproduce.

3.3 Eigenvalue Dynamics for AdS$_5\times$S^5

To motivate our proposal for eigenvalue dynamics, we will review the $\frac{1}{2}$-BPS sector stressing the logic that we will subsequently use. The way in which a direct change of variables is used to derive the eigenvalue dynamics can be motivated by considering correlation functions of arbitrary observables $\cdots$ that are functions only of the eigenvalues. Because we are considering BPS operators, correlators computed in the free field theory agree with the same computations at strong coupling[44], so that we now work in the free field theory. Performing the change of variables we find

$$\begin{aligned}\langle\cdots\rangle &= \int [dZdZ^\dagger] e^{-\mathrm{Tr}ZZ^\dagger}\cdots \\ &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\sum_k z_k\bar{z}_k}\Delta(z)\Delta(\bar{z})\cdots \\ &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i |\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\})|^2\cdots\end{aligned}$$

where the groundstate wave function is given by

$$\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\}) = \Delta(z) e^{-\frac{1}{2}\sum_i z_i\bar{z}_i} \tag{3.13}$$

We will shortly qualify the adjective "groundstate". Under the state-operator correspondence, this wave function is the state corresponding to the identity operator. The above transformation is equivalent to the identification

$$[dZ]e^{-\frac{1}{2}\mathrm{Tr}(ZZ^\dagger)} \leftrightarrow c\prod_{i=1}^{N} dz_i\,\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\}) \tag{3.14}$$

where c is a constant that arises from integrating over $U, U^\dagger$ and the off diagonal elements of D in (A.6). The role of each of the elements of the wave function is now clear:

1. Under the state operator correspondence, dimensions of operators map to energies of states. The dimensions of BPS operators are not corrected, i.e. they take their free field values. This implies an evenly spaced spectrum and hence a harmonic oscillator wave function. This explains the $e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}$ factor. It also suggests that the wavefunction will be a polynomial times this Gaussian factor.

2. There is a gauge symmetry $Z \to UZU^\dagger$ that is able to permute the eigenvalues. Consequently we are discussing identical particles. Two matrices drawn at random from the complex Gaussian ensemble will not have degenerate eigenvalues, so we choose the particles to be fermions. This matches the fact that the wave function is a Slater determinant.

3. Under the transformation $Z \to e^{i\theta}Z$, dZ transforms with charge N^2. Since $\prod_i dz_i$ has charge N, $c\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\})$ must have charge $N(N-1)$. The constant c is obtained by integrating over the off diagonal elements of D in (A.6). Thus, c has charge $\frac{1}{2}N(N-1)$ and $\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\})$ itself has the same charge[1].

4. If we assign the dimension $[Z] = L$ it is clear that both $\psi_{\mathrm{gs}}(\{z_i,\bar{z}_i\})$ and c must have dimension $\frac{1}{2}N(N-1)$.

The wave function (3.13) satisfies these properties. Further, if we require that the wavefunction is a polynomial in the eigenvalues z_i times the exponential $e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}$, then (3.13) is the state of lowest energy (we did not write down a Hamiltonian, but any other wave function has more nodes and hence a higher energy) so it deserves to be called the ground state. The wave function (3.13) is the state corresponding to the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime in the $\frac{1}{2}$-BPS sector.

The above discussion can be generalized to write down a wave function corresponding to the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime in the $SU(2)$ sector. The equation (3.14) is generalized to

$$[dZdY]e^{-\frac{1}{2}\mathrm{Tr}(ZZ^\dagger)-\frac{1}{2}\mathrm{Tr}(YY^\dagger)} \to c^2\prod_{i=1}^{N} dz_i dy_i\,\Psi_{\mathrm{gs}}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}) \tag{3.15}$$

[1]We are assuming that any non-trivial measure depends only on the eigenvalues. This is a guess and we do not know a proof of this. We will make this assumption for the two matrix model as well.

where c^2 is again a constant coming from integrating the non-eigenvalue variables out. The wave function must obey the following properties:

1. Our wave functions again describe states that correspond to BPS operators. The dimensions of the BPS operators take their free field values, implying an evenly spaced spectrum and hence a harmonic oscillator wave function. This suggests the wave function is a polynomial times the Gaussian factor $e^{-\frac{1}{2}\sum_i z_i\bar{z}_i - \frac{1}{2}\sum_i y_i\bar{y}_i}$ factor.

2. There is a gauge symmetry $Z \to UZU^\dagger$ and $Y \to UYU^\dagger$ that is able to permute the eigenvalues. Consequently we are discussing N identical particles. Matrices drawn at random will not have degenerate eigenvalues, so we choose the particles to be fermions. Thus we expect the wave function is a Slater determinant.

3. Under the transformation $Z \to e^{i\theta}Z$ and $Y \to Y$ the measure $dZdY$ transforms with charge N^2. Since $\prod_i dz_i dy_i$ has charge N and c^2 has charge $\frac{1}{2}N(N-1)$, the wave function $\Psi_{\text{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$ must have charge $\frac{1}{2}N(N-1)$. Similarly, under the transformation $Z \to Z$ and $Y \to e^{i\theta}Y$ the measure $dZdY$ transforms with charge N^2. Since $\prod_i dz_i dy_i$ has charge N and again c^2 has charge $\frac{1}{2}N(N-1)$, the wave function $\Psi_{\text{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$ should have charge $\frac{1}{2}N(N-1)$.

4. If we assign the dimension $[Z] = L = [Y]$ it is clear that both $\Psi_{\text{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$ and c^2 must have dimension $N(N-1)$.

5. The probability density associated to a single particle $\rho_{\text{gs}}(z_1, \bar{z}_1, y_1, \bar{y}_1)$ must have an $SO(4)$ symmetry, i.e. it should be a function of $|z_i|^2+|y_i|^2$.

The single particle probability density referred to in point 5 above is given, for any state $\Psi(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$ as usual, by

$$\rho(z_1, \bar{z}_1, y_1, \bar{y}_1) = \int \prod_{i=2}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2 \tag{3.16}$$

There is a good reason why the single particle probability density is an interesting quantity to look at: at short distances the eigenvalues feel a repulsion from the Slater determinant, which vanishes when two eigenvalues are equal. At long distances the confining harmonic oscillator potential dominates, ensuring the eigenvalues are clumped together in some finite region and do not wander off to infinity. In the end we expect that at large N the locus where the eigenvalues lie defines a specific surface, generalizing the idea of a density of eigenvalues for the single matrix model. This large N surface is captured by $\rho(z_1, \bar{z}_1, y_1, \bar{y}_1)$. We will make this connection more explicit in a later section.

There appears to be a unique wave function singled out by the above requirements. It is given by

$$\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}) = \mathcal{N}\Delta(z,y)e^{-\frac{1}{2}\sum_k z_k\bar{z}_k-\frac{1}{2}\sum_k y_k\bar{y}_k} \tag{3.17}$$

where

$$\begin{aligned}\Delta(z,y) &= \begin{vmatrix} y_1^{N-1} & y_2^{N-1} & \cdots & y_N^{N-1} \\ z_1 y_1^{N-2} & z_2 y_2^{N-2} & \cdots & z_N y_N^{N-2} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ z_1^{N-2}y_1 & z_2^{N-2}y_2 & \cdots & z_N^{N-2}y_N \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} \\ &= \prod_{j>k}^{N}(z_j y_k - y_j z_k) \end{aligned} \tag{3.18}$$

generalizes the usual Van der Monde determinant and $\mathcal{N}$ is fixed by normalizing the wave function. Normalizing the wave function in the state picture corresponds to choosing a normalization in the original matrix model so that the expectation value of 1 is 1. In the next section we will discuss the proposal (3.17) with a special emphasis on the symmetries realized by this wavefunction. As we will review, a wave function given as a product of Van der Monde determinants is also a natural guess. We will argue that (3.17) realizes more symmetries than a product of Van der Monde determinants does. We will then use the wave function to compute correlators. Surprisingly, for a large class of correlators the wave function (3.17) gives the exact answer.

3.4 Symmetries of the AdS$_5\times$S^5 Wavefunction

The original two (complex) matrix model enjoys an $SO(4)\simeq SU(2)_L\times SU(2)_R$ symmetry. Indeed, the generators

$$\begin{aligned}
J_3^R &= Z_{ij}\frac{\partial}{\partial Z_{ij}} - Z^\dagger_{ij}\frac{\partial}{\partial Z^\dagger_{ij}} + Y_{ij}\frac{\partial}{\partial Y_{ij}} - Y^\dagger_{ij}\frac{\partial}{\partial Y^\dagger_{ij}} \\
J_+^R &= Y_{ij}\frac{\partial}{\partial Z^\dagger_{ij}} - Z_{ij}\frac{\partial}{\partial Y\dagger_{ij}} \qquad J_-^R = Z^\dagger_{ij}\frac{\partial}{\partial Y_{ij}} - Y^\dagger_{ij}\frac{\partial}{\partial Z_{ij}} \\
J_3^L &= Z_{ij}\frac{\partial}{\partial Z_{ij}} - Z^\dagger_{ij}\frac{\partial}{\partial Z^\dagger_{ij}} - Y_{ij}\frac{\partial}{\partial Y_{ij}} + Y^\dagger_{ij}\frac{\partial}{\partial Y^\dagger_{ij}} \\
J_+^L &= Y^\dagger_{ij}\frac{\partial}{\partial Z^\dagger_{ij}} - Z_{ij}\frac{\partial}{\partial Y_{ij}} \qquad J_-^L = Z^\dagger_{ij}\frac{\partial}{\partial Y^\dagger_{ij}} - Y_{ij}\frac{\partial}{\partial Z_{ij}}
\end{aligned} \tag{3.19}$$

annihilate $\mathrm{Tr}(ZZ^\dagger) + \mathrm{Tr}(YY^\dagger)$. The above $SO(4)$ symmetry can also be realized at the level of the eigenvalues. In this case, the generators are

$$\begin{aligned}
J_3^R &= z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} + y_i \frac{\partial}{\partial y_i} - \bar{y}_i \frac{\partial}{\partial \bar{y}_i} \\
J_+^R &= y_i \frac{\partial}{\partial \bar{z}_i} - z_i \frac{\partial}{\partial \bar{y}_i} \qquad J_-^R = \bar{z}_i \frac{\partial}{\partial y_i} - \bar{y}_i \frac{\partial}{\partial z_i} \\
J_3^L &= z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} - y_i \frac{\partial}{\partial y_i} + \bar{y}_i \frac{\partial}{\partial \bar{y}_i} \\
J_+^L &= \bar{y}_i \frac{\partial}{\partial \bar{z}_i} - z_i \frac{\partial}{\partial y_i} \qquad J_-^L = \bar{z}_i \frac{\partial}{\partial \bar{y}_i} - y_i \frac{\partial}{\partial z_i}
\end{aligned} \tag{3.20}$$

It is simple to verify that

$$J_3^L \Psi_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = J_+^L \Psi_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = J_-^L \Psi_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = 0 \tag{3.21}$$

so that the wave function is manifestly invariant under $SU(2)_L$. Further, since

$$J_3^R \Psi_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = N(N-1)\, \Psi_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) \tag{3.22}$$

it transforms covariantly under $U(1) \subset SU(2)_R$ generated by J_3^R. Thus, in summary, out of the original $SO(4)$ symmetry, the wave function is invariant under $SU(2)_L$ and covariant under a $U(1) \subset SU(2)_R$. Since we will restrict to the subset of BPS operators that are holomorphic in Y and Z, this is the biggest symmetry we should expect.

A few comments are in order. If the interaction is switched off, the system is invariant under separate $U(N)$ actions on Z and Y. Thus, in this case, the model has a $U(N) \times U(N)$ symmetry. If we restrict ourselves to correlators of operators that never have Ys and Zs in the same trace, the wave function

$$\Psi_{\mathrm{VdM}} = c^2 \Delta(z) \Delta(y) e^{-\frac{1}{2}\sum_j (z_j \bar{z}_j + y_j \bar{y}_j)} \tag{3.23}$$

will reproduce the exact values for all correlators. Notice that this wave function is covariant under $U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R$ generated by J_3^L and J_3^R, i.e. it has less symmetry than (3.17). Further, if we consider correlators of operators that include products of Z and Y matrices the symmetry is broken to $U(N)$. The integration over the non-eigenvalue degrees of freedom is nontrivial, but the result will again be a polynomial in the eigenvalues. The precise form of the polynomial will depend on the choice of operators in the correlator and we will not get a simple rule for translating a specific operator. In the next section it will be shown that using (3.17), we will in fact obtain a simple rule for translating a specific operator into the eigenvalue language and the translation will not depend on the choice of the other operators in the correlator. For these reasons, Ψ_{VdM} is not discussed further.

To end this section consider the location of the zeros of (3.17). For each eigenvalue we have a vector with coordinates (z_i, y_i) on $\mathbb{C}^2$. Physically we expect that the wave function must vanish whenever $n > 1$ eigenvalues coincide, leading to an enhanced symmetry of the joint eigenvalue configuration[33]. The wave function vanishes whenever the vectors associated to two distinct eigenvalues are parallel, i.e. whenever $(z_i, y_i) = \lambda(z_j, y_j)$. If $\lambda \neq 1$ the eigenvalues are not coincident, there is no enhanced symmetry of the joint eigenvalue configuration and physically there is no reason why such an eigenvalue configuration should be weighted with zero. Thus, there are more zeroes than expected. Clearly then (3.17) will get various things wrong, but given that it realize more symmetries that $\Psi_{\rm VdM}$, it may be good enough for some computations. We will confirm this in the next section by showing that this wave function reproduces the correct exact answer for a large class of matrix model correlators.

Finally, note that it is useful to think of the wave function as a function of two points in $\mathbb{C}P^1 \times \mathbb{C}^*$, with (z_i, y_i) simultaneoulsy the coordinates of a point and the affine coordinates of the projective shphere base. With this interpretation, the singularities are associated with points cooinciding in the base which is physically more sensible.

3.5 Correlators

In this section we will provide detailed tests of this wave function by computing correlators with the wave function and comparing them to the exact results from the matrix model. The comparison is accomplished by using the equation

$$\int [dYdZdY^\dagger dZ^\dagger] e^{-{\rm Tr}(ZZ^\dagger)-{\rm Tr}(YY^\dagger)} \cdots = \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2 \cdots \qquad (3.24)$$

to compute correlators of observables (denoted by $\cdots$ above) that depend only on the eigenvalues. We have already argued above that we expect that the observables that are correctly computed using eigenvalue dynamics are the BPS operators of the CFT. As a first example, consider correlators of traces $O_J = {\rm Tr}(Z^J)$. These can be computed exactly in the matrix model, using a variety of different techniques - see for example [27, 45, 30]. The result is

$$\langle {\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})\rangle = \frac{1}{J+1}\Big[\frac{(J+N)!}{(N-1)!} - \frac{N!}{(N-J-1)!}\Big] \qquad (3.25)$$

if $J < N$ and

$$\langle {\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})\rangle = \frac{1}{J+1}\frac{(J+N)!}{(N-1)!} \qquad (3.26)$$

if $J \geq N$. These expressions could easily be expanded to generate the $1/N$ expansion if we wanted to do that. We would now like to consider the eigenvalue computation. It is useful to write the wave function as

$$\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1 a_2\cdots a_n}\frac{z_{a_1}^0 y_{a_1}^{N-1}}{\sqrt{0!\,(N-1)!}}\cdots\frac{z_{a_k}^{k-1}y_{a_k}^{N-k}}{\sqrt{(k-1)!\,(N-k)!}}\cdots \\ \cdots\frac{z_{a_N}^{N-1}y_{a_N}^0}{\sqrt{(N-1)!\,0!}}e^{-\frac{1}{2}\sum_q z_q\bar{z}_q-\frac{1}{2}\sum_q y_q\bar{y}_q} \tag{3.27}$$

The gauge invariant observable in this case is given by

$$\mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J}) = \sum_{i=1}^N z_i^J\sum_{j=1}^N \bar{z}_j^J \tag{3.28}$$

It is now straightforward to find

$$\int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i|\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\sum_{i=1}^N z_i^J\sum_{j=1}^N\bar{z}_j^J = \frac{1}{J+1}\frac{(J+N)!}{(N-1)!} \tag{3.29}$$

When evaluating the above integral, only the terms with $i=j$ contribute. From this result we see that we have not reproduced traces with $J<N$ correctly - we don't even get the leading large N behavior right. We have, however, correctly reproduced the exact answer (to all orders in $1/N$) of the two point function for all single traces of dimension N or greater. For $J>N$ there are trace relations of the form

$$\mathrm{Tr}(Z^J) = \sum_{i,j,\dots,k} c_{ij\dots k}\mathrm{Tr}(Z^i)\mathrm{Tr}(Z^j)\cdots\mathrm{Tr}(Z^k) \tag{3.30}$$

$i,j,...,k \leq N$ and $i+j+\cdots+k=J$. The fact that we reproduce two point correlators of traces with $J>N$ exactly implies that we also start to reproduce sums of products of traces of less than N fields. This suggests that the important thing is not the trace structure of the operator, but rather the dimension of the state.

The fact that we only reproduce observables that have a large enough dimension is not too surprising. Indeed, supergravity can't be expected to correctly describe the back reaction of a single graviton or a single string. To produce a state in the CFT dual to a geometry that is different from the AdS vacuum one needs to allow a number of giant gravitons (eigenvalues) to condense. The eigenvalue dynamics is correctly reproducing the two point function of traces when their energy is greater than that required to blow up into a giant graviton.

With a very simple extension of the above argument we can argue that we also correctly reproduce the correlator $\langle \mathrm{Tr}(Y^J)\mathrm{Tr}(Y^{\dagger J})\rangle$ with $J \geq N$. A much more interesting class of observables to consider are mixed traces, which contain both Y and Z fields. To build BPS operators using both Y and Z fields we need to construct symmetrized traces. A very convenient way to perform this construction is as follows

$$\mathcal{O}_{J,K} = \frac{J!}{(J+K)!}\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \mathrm{Tr}(Z^{J+K}) \tag{3.31}$$

The normalization up front is just the inverse of the number of terms that appear. With this normalization, the translation between the matrix model observable and an eigenvalue observable is

$$\mathcal{O}_{J,K} \leftrightarrow \sum_i z_i^J y_i^K \tag{3.32}$$

Since we could not find this computation in the literature, we will now explain how to evaluate the matrix model two point function exactly, in the free field theory limit. Since the dimension of BPS operators are not corrected, this answer is in fact exact. To start, perform the contraction over the $Y, Y^\dagger$ fields

$$\begin{aligned}\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle &= \left(\frac{J!}{(J+K)!}\right)^2 \langle \mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \mathrm{Tr}(Z^{J+K})\mathrm{Tr}\left(Y^\dagger\frac{\partial}{\partial Z^\dagger}\right)^K \mathrm{Tr}(Z^{\dagger\, J+K})\rangle \\ &= \left(\frac{J!}{(J+K)!}\right)^2 K!\,\langle \mathrm{Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\right)^K \mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger\, J+K})\rangle \end{aligned} \tag{3.33}$$

Given the form of the matrix model two point function

$$\langle Z_{ij} Z^\dagger_{kl}\rangle = \delta_{il}\delta_{jk} \tag{3.34}$$

we know that we can write any free field theory correlator as

$$\langle \cdots \rangle = e^{\mathrm{Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^\dagger}\right)} \cdots \Big|_{Z=Z^\dagger=0} \tag{3.35}$$

Using this identity we now find

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \left(\frac{J!}{(J+K)!}\right)^2 K!\,\frac{(J+K)!}{J!}\langle \mathrm{Tr}(Z^{J+K})\mathrm{Tr}(Z^{\dagger\, J+K})\rangle \tag{3.36}$$

Thus, the result of the matrix model computation is

$$\langle \mathcal{O}_{J,K}\mathcal{O}^\dagger_{J,K}\rangle = \frac{J!\,K!}{(J+K+1)!}\left[\frac{(J+K+N)!}{(N-1)!} - \frac{N!}{(N-J-K-1)!}\right] \tag{3.37}$$

if $J+K<N$ and

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle = \frac{J!\,K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!} \tag{3.38}$$

if $J+K\geq N$. Notice that for these two matrix observables we again get a change in the form of the correlator as the dimension of the trace exceeds N.

Next, consider the eigenvalue computation. We need to perform the integral

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle = \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm gs}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \sum_{k=1}^{N} z_k^J y_k^K \sum_{j=1}^{N} \bar{z}_j^J \bar{y}_j^K \tag{3.39}$$

After some straightforward manipulations we have

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle = \pi^{-2N}\int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i \frac{|z_1|^0|y_1|^{2N-2}}{0!\,(N-1)!}\cdots\frac{|z_k|^{2k-2}|y_k|^{2N-2k}}{(k-1)!\,(N-k)!}\cdots \\ \frac{|z_N|^{2N-2}|y_N|^0}{(N-1)!\,0!}\times e^{-\sum_q z_q\bar{z}_q-\sum_q y_q\bar{y}_q}\sum_{k,j=1}^{N} z_k^J y_k^K \bar{z}_j^J \bar{y}_j^K \tag{3.40}$$

Only terms with $k=j$ contribute so that

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle = \sum_{k=1}^{N}\frac{(N-k+K)!}{(N-k)!}\frac{(J+k-1)!}{(k-1)!} = \frac{K!\,J!}{(K+J+1)!}\frac{(J+K+N)!}{(N-1)!} \tag{3.41}$$

Thus, we again correctly reproduce the exact (to all orders in $1/N$) answer for the two point function of single trace operators of dimension N or greater. Inspecting (3.19) we notice that we have obtained $\mathcal{O}_{J,K}$ from $\mathcal{O}_{J+K}$ by applying J_-^L, that is, by applying an $SU(2)_L$ rotation. Since both the original matrix description and the eigenvalue description enjoy $SU(2)_L$ symmetry, the agreement of the $\langle \mathcal{O}^{\dagger}_{J,K}\mathcal{O}_{J,K}\rangle$ correlator is not independent of the agreement of the $\langle \mathcal{O}^{\dagger}_{J+K}\mathcal{O}_{J+K}\rangle$ correlator.

It is also interesting to consider multi trace correlators. We will start with the correlator between a double trace and a single trace and we will again start with the matrix model computation

$$\langle O_{J_1,K_1}O_{J_2,K_2}O^{\dagger}_{J_1+J_2,K_1+K_2}\rangle = \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}\times \\ \langle {\rm Tr}\left(Y\frac{\partial}{\partial Z}\right)^{K_1}{\rm Tr}(Z^{J_1+K_1}){\rm Tr}\left(Y\frac{\partial}{\partial Z}\right)^{K_2}{\rm Tr}(Z^{J_2+K_2}){\rm Tr}\left(Y^{\dagger}\frac{\partial}{\partial Z^{\dagger}}\right)^{K_1+K_2}{\rm Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle \tag{3.42}$$

We could easily set $K_1 = K_2 = 0$ and obtain traces involving only a single matrix. Begin by contracting all $Y, Y^\dagger$ fields to obtain

$$\begin{aligned}\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}(K_1+K_2)! \times \\ &\langle \frac{\partial}{\partial Z_{i_1 j_1}} \cdots \frac{\partial}{\partial Z_{i_{K_1} j_{K_1}}} \mathrm{Tr}(Z^{J_1+K_1}) \frac{\partial}{\partial Z_{i_{K_1+1} j_{K_1+1}}} \cdots \frac{\partial}{\partial Z_{i_{K_1+K_2} j_{K_1+K_2}}} \mathrm{Tr}(Z^{J_2+K_2}) \\ &\frac{\partial}{\partial Z^\dagger_{j_1 i_1}} \cdots \frac{\partial}{\partial Z^\dagger_{j_{K_1+K_2} i_{K_1+K_2}}} \mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle\end{aligned} \tag{3.43}$$

It is now useful to integrate by parts with respect to $Z^\dagger$, using the identity

$$\langle \frac{\partial}{\partial Z_{ij}} f(Z)\, g(Z)\, \frac{\partial}{\partial Z^\dagger_{ji}} h(Z^\dagger)\rangle = n_f \langle f(Z)\, g(Z)\, h(Z^\dagger)\rangle \tag{3.44}$$

where $f(Z)$ is of degree n_f in Z. Repeatedly using this identity, we find

$$\begin{aligned}\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \frac{J_1!}{(J_1+K_1)!}\frac{J_2!}{(J_2+K_2)!}\frac{(J_1+J_2)!}{(J_1+K_1+J_2+K_2)!}(K_1+K_2)! \times \\ &\frac{(J_1+K_1)!}{J_1!}\frac{(J_2+K_2)!}{J_2!}\langle \mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle \\ &= \frac{(J_1+J_2)!\,(K_1+K_2)!}{(J_1+K_1+J_2+K_2)!}\langle \mathrm{Tr}(Z^{J_1+K_1})\mathrm{Tr}(Z^{J_2+K_2})\mathrm{Tr}(Z^{\dagger J_1+K_1+J_2+K_2})\rangle\end{aligned} \tag{3.45}$$

This last correlator is easily computed. For example, if $J_1 + K_1 < N$ and $J_2 + K_2 < N$ we have

$$\begin{aligned}\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \frac{(J_1+J_2)!\,(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\Big[\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!} \\ &+ \frac{N!}{(N-J_1-K_1-J_2-K_2-1)!} - \frac{(N+J_1+K_1)!}{(N-J_2-K_2-1)!} \\ &- \frac{(N+J_2+K_2)!}{(N-J_1-K_1-1)!}\Big]\end{aligned} \tag{3.46}$$

and if $J_1 + K_1 \geq N$ and $J_2 + K_2 \geq N$ we have

$$\langle O_{J_1,K_1} O_{J_2,K_2} O^\dagger_{J_1+J_2,K_1+K_2}\rangle = \frac{(J_1+J_2)!\,(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!} \tag{3.47}$$

It is a simple exercise to check that, in terms of eigenvalues, we have

$$\begin{aligned}\langle \mathcal{O}_{J_1,K_1}\mathcal{O}_{J_2,K_2}\mathcal{O}^\dagger_{J_1+J_2,K_1+K_2}\rangle &= \int\prod_{i=1}^N dz_i d\bar z_i dy_i d\bar y_i |\Psi_{\rm gs}(\{z_i,\bar z_i,y_i,\bar y_i\})|^2 \\ &\quad\times\sum_{k=1}^N z_k^{J_1}y_k^{K_1}\sum_{l=1}^N z_l^{J_2}y_l^{K_2}\sum_{j=1}^N \bar z_j^{J_1+J_2}\bar y_j^{K_1+K_2} \\ &= \frac{(J_1+J_2)!\,(K_1+K_2)!}{(J_1+K_1+J_2+K_2+1)!}\frac{(J_1+K_1+J_2+K_2+N)!}{(N-1)!}\end{aligned} \tag{3.48}$$

so that once again we have reproduced the exact answer as long as the dimension of each trace is not less than N. The agreement that we have observed for multi trace correlators continues as follows: as long as the dimension of each trace is greater than $N-1$ the matrix model and the eigenvalue descriptions agree and both give

$$\langle O_{J_1,K_1}O_{J_2,K_2}\cdots O_{J_n,K_n}O^\dagger_{J,K}\rangle = \frac{J!\,K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!}\delta_{J_1+\cdots+J_n,J}\delta_{K_1+\cdots+K_n,K} \tag{3.49}$$

for the exact value of this correlator. We have limited our selves to a single daggered observable in the above expression for purely technical reasons: it is only in this case that we can compute the matrix model correlator using the identity (3.44). It would be interesting to develop analytic methods that allow more general computations.

Finally, we can also test multi trace correlators with a dimension of order N^2. A particularly simple operator is the Schur polynomial labeled by a Young diagram R with N rows and M columns. For this R we have

$$\chi_R(Z) = (\det Z)^M = z_1^M z_2^M\cdots z_N^M \tag{3.50}$$

$$\chi_R(Z^\dagger) = (\det Z^\dagger)^M = \bar z_1^M \bar z_2^M\cdots \bar z_N^M \tag{3.51}$$

The dual LLM geometry is labeled by an annulus boundary condition that has an inner radius of $\sqrt{M}$ and an outer radius of $\sqrt{M+N}$. The two point correlator of this Schur polynomial is

$$\begin{aligned}\langle\chi_R(Z)\chi_R(Z^\dagger)\rangle &= \int\prod_{i=1}^N dz_i d\bar z_i dy_i d\bar y_i \chi_R(Z)\chi_R(Z^\dagger)|\Psi_{\rm gs}(\{z_i,\bar z_i,y_i,\bar y_i\})|^2 \\ &= \pi^{-2N}\int\prod_{i=1}^N dz_i d\bar z_i dy_i d\bar y_i \frac{|z_1|^{0+2M}|y_1|^{2N-2}}{0!\,(N-1)!}\cdots\frac{|z_k|^{2k-2+2M}|y_k|^{2N-2k}}{(k-1)!\,(N-k)!}\end{aligned}$$

$$
\begin{aligned}
\times & \quad \cdots \frac{|z_N|^{2N-2+2M}|y_N|^0}{(N-1)!\,0!} \times e^{-\sum_q z_q \bar{z}_q - \sum_q y_q \bar{y}_q} \\
= & \quad \prod_{i=1}^{N} \frac{(i-1+M)!}{(i-1)!}
\end{aligned}
\tag{3.52}
$$

which is again the exact answer for this correlator.

After this warm up example we will now make a few comments that are relevant for the general case. The details are much more messy, so we will not manage to make very precise statements. We have however included this discussion as it does provide a guide as to when eigenvalue dynamics is applicable. A Schur polynomial labeled with a Young diagram R that has row lengths r_i is given in terms of eigenvalues as (our labeling of the rows is defined by $r_1 \geq r_2 \geq \cdots \geq r_N$)

$$
\chi_R(Z) = \frac{\epsilon_{a_1 a_2 \cdots a_N} z_{a_1}^{N-1+r_1} z_{a_2}^{N-2+r_2} \cdots z_{a_N}^{r_N}}{\epsilon_{b_1 b_2 \cdots b_N} z_{b_1}^{N-1} z_{b_2}^{N-2} \cdots z_{b_{N-1}}}
\tag{3.53}
$$

Using this expression, we can easily write the exact two point function as follows

$$
\begin{aligned}
\langle \chi_R(Z) \chi_R(Z^\dagger) \rangle &= \frac{1}{N!\,\pi^N} \prod_{j=0}^{N-1} \frac{1}{j!} \int \prod_{i=1}^{N} dz_i d\bar{z}_i \epsilon_{a_1 a_2 \cdots a_N} z_{a_1}^{N-1+r_1} z_{a_2}^{N-2+r_2} \cdots z_{a_N}^{r_N} \\
&\quad \times \epsilon_{b_1 b_2 \cdots b_N} \bar{z}_{b_1}^{N-1+r_1} \bar{z}_{b_2}^{N-2+r_2} \cdots \bar{z}_{b_N}^{r_N} e^{-\sum_k z_k \bar{z}_k} \\
&= \prod_{j=0}^{N-1} \frac{(j+r_{N-j})!}{j!} = f_R
\end{aligned}
\tag{3.54}
$$

Using our wave function we can compute the two point function of Schur polynomials. The result is

$$
\begin{aligned}
\langle \chi_R(Z) \chi_R(Z^\dagger) \rangle &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i \chi_R(Z) \chi_R(Z^\dagger) |\Psi_{\rm gs}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2 \\
&= \frac{1}{\pi^N} \prod_{j=0}^{N-1} \frac{1}{j!} \int \prod_{i=1}^{N} dz_i d\bar{z}_i \; |z_{a_1}|^{2N-2} |z_{a_2}|^{2N-4} \cdots |z_{a_{N-1}}|^2 \\
&\quad \times \frac{\epsilon_{b_1 b_2 \cdots b_N} z_{b_1}^{N-1+r_1} z_{b_2}^{N-2+r_2} \cdots z_{b_N}^{r_N}}{\epsilon_{c_1 c_2 \cdots c_N} z_{c_1}^{N-1} z_{c_2}^{N-2} \cdots z_{c_{N-1}}} \\
&\quad \times \frac{\epsilon_{d_1 d_2 \cdots d_N} \bar{z}_{d_1}^{N-1+r_1} \bar{z}_{d_2}^{N-2+r_2} \cdots \bar{z}_{d_N}^{r_N}}{\epsilon_{e_1 e_2 \cdots e_N} \bar{z}_{e_1}^{N-1} \bar{z}_{e_2}^{N-2} \cdots \bar{z}_{e_{N-1}}} e^{-\sum_k z_k \bar{z}_k}
\end{aligned}
\tag{3.55}
$$

When the integration over the angles θ_i associated to $z_i = r_i e^{i\theta_i}$ are performed, a non-zero result is only obtained if powers of the z_i match the powers of the $\bar{z}_i$. The difference between the above expression and the exact answer is simply that in the eigenvalue expression these powers are separately set to be equal in the measure and in the product of Schur polynomials - there are two matchings, while in the exact answer the power of z_i arising from the product of the measure and the product of Schur polynomials is matched to the power of $\bar{z}_i$ from the product of the measure and the product of Schur polynomials - there is a single matching happening. Thus, the eigenvalue computation may miss some terms that are present in the exact answer[2]. For Young diagrams with a few corners and $O(N^2)$ boxes (the annulus above is a good example) the eigenvalues clump into groupings, with each grouping collecting eigenvalues of a similar size corresponding to rows with a similar row length[43]. This happens because the product of the Gaussian fall off $e^{-z\bar{z}}$ and a polynomial of fixed degree $|z^2|^n$ is sharply peaked at $|z|= n$. Thus, for example if $r_i \approx M_1$ for $i = 1, 2, \cdots, \frac{N}{2}$ and $r_i \approx M_2$ for $i = 1 + \frac{N}{2}, 2 + \frac{N}{2}, \cdots, N$ with M_1 and M_2 well separated $(M_1 - M_2 \geq O(N))$, under the integral we can replace

$$\frac{\epsilon_{b_1 b_2 \cdots b_N} z_{b_1}^{N-1+r_1} z_{b_2}^{N-2+r_2} \cdots z_{b_N}^{r_N}}{\epsilon_{c_1 c_2 \cdots c_N} z_{c_1}^{N-1} z_{c_2}^{N-2} \cdots z_{c_{N-1}}} \to \prod_{i=1}^{\frac{N}{2}} z_{a_i}^{M_1} z_{a_{i+\frac{N}{2}}}^{M_2} \tag{3.56}$$

After making a replacement of this type, we recover the exact answer. This replacement is not exact - we need to appeal to large N to justify it. It would be very interesting to explore this point further and to quantify in general (if possible) what the corrections to the above replacement are. For Young diagrams with many corners, row lengths are not well separated and there is no similar grouping that occurs, so that the eigenvalue description will not agree with the exact result, even at large N. A good example of a geometry with many corners is the superstar[46]. The corresponding LLM boundary condition is a number of very thin concentric annuli, so that we effectively obtain a gray disk, signaling a singular supergravity geometry. It is then perhaps not surprising that the eigenvalue dynamics does not correctly reproduce this two point correlator.

Having discussed the two point function of Schur polynomials in detail, the product rule

$$\chi_R(Z)\chi_S(Z) = \sum_T f_{RST}\chi_T(Z) \tag{3.57}$$

with f_{RST} a Littlewood-Richardson coefficient, implies that there is no need to consider correlation functions of products of Schur polynomials.

3.6 Other Backgrounds

In the $\frac{1}{2}$ BPS sector there is a wave function corresponding to every LLM geometry. The (not normalized) wave function has already been given in (3.11). In this section we consider

[2]This is the reason why (3.29) only captures one of the terms present in the two point function for $J < N$.

the problem of writing eigenvalue wave functions that correspond to geometries other than AdS$_5$×S^5. The simplest geometry we can consider is the annulus geometry considered in the previous section, where we argued that the eigenvalue dynamics reproduces the exact correlator of the Schur polynomials dual to this geometry. Our proposal for the state that corresponds to this LLM spacetime is

$$\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1a_2\cdots a_n}\frac{z_{a_1}^M y_{a_1}^{N-1}}{\sqrt{M!\,(N-1)!}}\cdots\frac{z_{a_k}^{k-1+M}y_{a_k}^{N-k}}{\sqrt{(k-1+M)!\,(N-k)!}} \cdots\frac{z_{a_N}^{N-1+M}y_{a_N}^{0}}{\sqrt{(N-1+M)!\,0!}}e^{-\frac{1}{2}\sum_q z_q\bar{z}_q-\frac{1}{2}\sum_q y_q\bar{y}_q} \qquad (3.58)$$

This is simply obtained by multiplying the ground state wave function by the relevant Schur polynomial and normalizing the resulting state. The connection between matrix model correlators and expectation values computed using the above wave function is the following[3]

$$\begin{aligned}\langle\,\cdots\rangle_{\rm LLM} &= \frac{\langle\,\cdots\;\chi_R(Z)\chi_R(Z^\dagger)\rangle}{\langle\chi_R(Z)\chi_R(Z^\dagger)\rangle}\\ &= \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i|\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\,\cdots \end{aligned}\qquad (3.59)$$

We can use this wave function to compute correlators that we are interested in. Traces involving only Zs for example lead to

$$\begin{aligned}\langle{\rm Tr}(Z^J){\rm Tr}(Z^{\dagger J})\rangle_{\rm LLM} &= \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i|\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\sum_{k=1}^N z_k^J\sum_{l=1}^N \bar{z}_l^J\\ &= \sum_{k=0}^{N-1}\frac{(J+k+M)!}{(k+M)!}\\ &= \frac{1}{J+1}\left[\frac{(J+M+N)!}{(M+N-1)!}-\frac{(J+M)!}{(M-1)!}\right]\end{aligned}\qquad (3.60)$$

which agrees with the exact result, as long as $J > N-1$. Thus, in this background, eigenvalue dynamics is correctly reproducing the same set of correlators as in the original AdS$_5$×S^5 background. Traces involving only Y fields are also correctly reproduced

$$\langle{\rm Tr}(Y^J){\rm Tr}(Y^{\dagger J})\rangle_{\rm LLM} = \int\prod_{i=1}^N dz_i d\bar{z}_i dy_i d\bar{y}_i|\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2\sum_{k=1}^N y_k^J\sum_{l=1}^N \bar{y}_l^J$$

[3]The new normalization for matrix model correlators is needed to ensure that the identity operator has expectation value 1. This matches the normalization adopted in the eigenvalue description.

$$= \frac{1}{J+1}\frac{(J+N)!}{(N-1)!} \qquad (3.61)$$

where $J \geq N$. Notice that these results are again exact, i.e. we reproduce the matrix model correlators to all orders in $1/N$. Finally, let's consider the most interesting case of traces involving both matrices. The LLM wave function we have proposed does not reproduce the exact matrix model computation. The matrix model computation gives

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\rm LLM} &= \left(\frac{J!}{(J+K)!}\right)^2 \langle {\rm Tr}\left(Y\frac{\partial}{\partial Z}\right)^K {\rm Tr}(Z^{J+K}){\rm Tr}\left(Y^{\dagger}\frac{\partial}{\partial Z^{\dagger}}\right)^K {\rm Tr}(Z^{\dagger\, J+K})\rangle_{\rm LLM} \\
&= \left(\frac{J!}{(J+K)!}\right)^2 K!\,\langle {\rm Tr}\left(\frac{\partial}{\partial Z}\frac{\partial}{\partial Z^{\dagger}}\right)^K {\rm Tr}(Z^{J+K}){\rm Tr}(Z^{\dagger\, J+K})\rangle_{\rm LLM} \\
&= \left(\frac{J!}{(J+K)!}\right)^2 K!\,\frac{(J+K)!}{J!}\langle {\rm Tr}(Z^{J+K}){\rm Tr}(Z^{\dagger\, J+K})\rangle_{\rm LLM} \\
&= \frac{J!\,K!}{(J+K+1)!}\left[\frac{(J+K+M+N)!}{(M+N-1)!} - \frac{(J+K+M)!}{(M-1)!}\right] \qquad (3.62)
\end{aligned}$$

if $J+K \geq N$. Next, consider the eigenvalue computation. We need to perform the integral

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\rm LLM,eigen} &= \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm LLM}(\{z_i,\bar{z}_i,y_i,\bar{y}_i\})|^2 \sum_{k=1}^{N} z_k^J y_k^K \sum_{j=1}^{N} \bar{z}_j^J \bar{y}_j^K \\
&= \sum_{k=1}^{N} \frac{(N-k+K)!}{(N-k)!}\frac{(J+M+k-1)!}{(M+k-1)!} \qquad (3.63)
\end{aligned}$$

It is not completely trivial to compare (3.62) and (3.63), but it is already clear that they do not reproduce exactly the same answer. To simplify the discussion, let's consider the case that $M = O(\sqrt{N})$. In this case, in the large N limit, we can drop the second term in (3.62) to obtain

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\rm LLM} = \frac{J!\,K!}{(J+K+1)!}\frac{(J+K+M+N)!}{(M+N-1)!}(1+\cdots) \qquad (3.64)$$

where $\cdots$ stand for terms that vanish as $N \to \infty$. In the sum appearing in (3.63), change variables from k to $k'-M$ and again appeal to large N to write

$$\begin{aligned}
\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle_{\rm LLM,eigen} &= \sum_{k'=M+1}^{M+N} \frac{(N+M-k'+K)!}{(N+M-k')!}\frac{(J+k'-1)!}{(k'-1)!} \\
&= \sum_{k'=1}^{M+N} \frac{(N+M-k'+K)!}{(N+M-k')!}\frac{(J+k'-1)!}{(k'-1)!}(1+\cdots)
\end{aligned}$$

$$= \frac{J!\,K!}{(J+K+1)!}\frac{(J+K+M+N)!}{(M+N-1)!}(1+\cdots) \tag{3.65}$$

In the last two lines above $\cdots$ again stands for terms that vanish as $N\to\infty$. Thus, we find agreement between (3.62) and (3.63). It is again convincing to see genuine multi matrix observables reproduced by the eigenvalue dynamics. Notice that in this case the agreement is not exact, but rather is realized to the large N limit. This is what we expect for the generic situation - the $\mathrm{AdS}_5\times\mathrm{S}^5$ case is highly symmetric and the fact that eigenvalue dynamics reproduces so many observables exactly is a consequence of this symmetry. We only expect eigenvalue dynamics to reproduce classical gravity, which should emerge from the CFT at $N=\infty$.

Much of our intuition came from thinking about the Gauss graph operators constructed in [25, 26]. It is natural to ask if we can write down wave functions dual to the Gauss graph operators. The simplest possibility is to consider a Gauss graph operator obtained by exciting a single eigenvalue by J levels, and then attaching a total of K Y strings to it. The extreme simplicity of this case follows because we can write the (normalized) Gauss graph operator in terms of a familiar Schur polynomial as

$$\hat{O} = \sqrt{\frac{J!}{K!\,(J+K)!}\frac{(N-1)!}{(N+J+K-1)!}}\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \chi_{(J+K)}(Z) \tag{3.66}$$

where we have used the notation (n) to denote a Young diagram with a single row of n boxes. Consider the correlator

$$\begin{aligned}\langle \hat{O}\mathrm{Tr}(Y^\dagger)^K\mathrm{Tr}(Z^{\dagger J})\rangle &= \langle \mathrm{Tr}\left(\frac{\partial}{\partial Y}\right)^K \hat{O}\mathrm{Tr}(Z^{\dagger J})\rangle \\ &= \sqrt{\frac{J!\,K!}{(J+K)!}\frac{(N+J+K-1)!}{(N-1)!}}\end{aligned} \tag{3.67}$$

This answer is exact, in the free field theory. In what limit should we compare this answer to eigenvalue dynamics? Our intuition is coming from the $\frac{1}{2}$- BPS sector where we know that rows of Schur polynomials correspond to eigenvalues and we know exactly how to write the corresponding wave function. If we only want small perturbations of this picture, we should keep $K\ll J$. In this case we should simplify

$$\begin{aligned}\frac{J!}{(J+K)!} &\to \frac{1}{J^K} \\ \frac{(N+J+K-1)!}{(N-1)!} &= \frac{(N+J+K-1)!}{(N+J-1)!}\frac{(N+J-1)!}{(N-1)!} \\ &\to (N+J-1)^K\frac{(N+J-1)!}{(N-1)!}\end{aligned} \tag{3.68}$$

How should we scale J as we take $N \to \infty$? The Schur polynomials are a sum over all possible matrix trace structures. We want these sums to be dominated by traces with a large number of matrices (N or more) in each trace. To accomplish this we will scale $J = O(N^{1+\epsilon})$ with $\epsilon > 0$. In this case, at large N, we can replace

$$\frac{1}{J^K}(N+J-1)^K \to 1 \tag{3.69}$$

and hence, the result that should be reproduced by the eigenvalue dynamics is given by

$$\langle \hat{O}\mathrm{Tr}(Y^\dagger)^K \mathrm{Tr}(Z^{\dagger J})\rangle = \sqrt{K!\,\frac{(N+J-1)!}{(N-1)!}} \tag{3.70}$$

In the eigenvalue computation, we will use the wave function of the ground state and the wave function of the Gauss graph operator ($\Psi_{\mathrm{GG}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$) to compute the amplitude

$$\int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i \Psi^*_{\mathrm{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})(\sum_i \bar{y}_i)^K \sum_j \bar{z}_j^J \Psi_{\mathrm{GG}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) \tag{3.71}$$

We expect the amplitude (3.71) to reproduce (3.70). Our proposal for the wave function corresponding to the above Gauss graph operator is

$$\begin{aligned}\Psi_{GG}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1 a_2 \cdots a_n}\frac{z^0_{a_1} y^{N-1}_{a_1}}{\sqrt{0!\,(N-1)!}}\cdots\frac{z^{k-1}_{a_k} y^{N-k}_{a_k}}{\sqrt{(k-1)!\,(N-k)!}}\cdots\\ \cdots\frac{z^{N-2}_{a_{N-1}} y_{a_{N-1}}}{\sqrt{(N-2)!\,1!}}\frac{z^{J+N-1}_{a_N} y^K_{a_N}}{\sqrt{(J+N-1)!\,K!}}e^{-\frac{1}{2}\sum_q z_q\bar{z}_q - \frac{1}{2}\sum_q y_q\bar{y}_q}\end{aligned} \tag{3.72}$$

The eigenvalue with the largest power of z (i.e. z_{a_N}) was the fermion at the very top of the Fermi sea. It has been excited by J powers of z and K powers of y. It is now trivial to verify that (3.71) does indeed reproduce (3.70).

Finally, the state with three eigenvalues excited by $J_1 > J_2 > J_3$ and with $K_1 > K_2 > K_3$ strings attached to each eigenvalue is given by

$$\begin{aligned}\Psi_{GG}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\}) = \frac{\pi^{-N}}{\sqrt{N!}}\epsilon^{a_1 a_2 \cdots a_n}\frac{z^0_{a_1} y^{N-1}_{a_1}}{\sqrt{0!\,(N-1)!}}\cdots\frac{z^{k-1}_{a_k} y^{N-k}_{a_k}}{\sqrt{(k-1)!\,(N-k)!}}\cdots\\ \cdots\frac{z^{N-4}_{a_{N-3}} y^3_{a_{N-3}}}{\sqrt{(N-4)!\,3!}}\frac{z^{J_3+N-3}_{a_{N-2}} y^{2+K_3}_{a_{N-2}}}{\sqrt{(J_3+N-3)!\,(2+K_3)!}}\frac{z^{J_2+N-2}_{a_{N-1}} y^{K_2+1}_{a_{N-1}}}{\sqrt{(J_2+N-2)!\,(K_2+1)!}}\\ \times\frac{z^{J_1+N-1}_{a_N} y^{K_1}_{a_N}}{\sqrt{(J_1+N-1)!\,K_1!}}e^{-\frac{1}{2}\sum_q z_q\bar{z}_q - \frac{1}{2}\sum_q y_q\bar{y}_q}\end{aligned} \tag{3.73}$$

The generalization to any Gauss graph operator is now clear.

3.7 Connection to Supergravity

In this section we would like to explore the possibility that the eigenvalue dynamics of the $SU(2)$ sector has a natural interpretation in supergravity. The relevant supergravity solutions have been considered in [47, 48, 49, 50].

There are 6 adjoint scalars in the $\mathcal{N} = 4$ super Yang-Mills theory that can be assembled into the following three complex combinations

$$Z = \phi^1 + i\phi^2 \qquad Y = \phi^3 + i\phi^4 \qquad X = \phi^5 + i\phi^6 \tag{3.74}$$

The operators we consider are constructed using only Z and Y so that they are invariant under the $U(1)$ which rotates ϕ^5 and ϕ^6. Further, since our operators are BPS they are built only from the s-wave spherical harmonic components of Y and Z, so that they are invariant under the $SO(4)$ symmetry which acts on the S^3 of the $R \times S^3$ spacetime on which the CFT is defined. Local supersymmetric geometries with $SO(4) \times U(1)$ isometries have the form[47, 50]

$$ds^2_{10} = -h^{-2}(dt+\omega)^2 + h^2\Big[\frac{2}{Z+\frac{1}{2}}\partial_a\bar{\partial}_b K dz^a d\bar{z}^b + dy^2\Big] + y(e^G d\Omega_3^2 + e^{-G}d\psi^2) \tag{3.75}$$

$$d\omega = \frac{i}{y}\Big(\partial_a\bar{\partial}_b\partial_y K dz^a d\bar{z}^b - \partial_a Z dz^a dy + \bar{\partial}_a Z d\bar{z}^a dy\Big) \tag{3.76}$$

Here z^1 and z^2 is a pair of complex coordinates and K is a Kahler potential which may depend on y, z^a and $\bar{z}^a$. y^2 is the product of warp factors for S^3 and S^1. Thus we must be careful and impose the correct boundary conditions at the $y = 0$ hypersurface if we are to avoid singularities. The $y = 0$ hypersurface includes the four dimensional space with coordinates given by the z^a. These boundary conditions require that when the S^3 contracts to zero, we need $Z = -\frac{1}{2}$ and when the ψ-circle collapses we need $Z = \frac{1}{2}$[47, 50]. There is a surface separating these two regions, and hence, defining the supergravity solution. So far the discussion given closely matches what is found for the $\frac{1}{2}$-BPS supergravity solutions. In that case the $y = 0$ hypersurface includes a two dimensional space which is similarly divided into two regions, giving the black droplets on a white plane. The edges of the droplets are completely arbitrary, which is an important difference from the case we are considering. The surface defining local supersymmetric geometries with $SO(4) \times U(1)$ isometries is not completely arbitrary - it too has to satisfy some additional constraints as spelled out in [50]. It is natural to ask if the surface defining the supergravity solution is visible in the eigenvalue dynamics?

To answer this question we will now review how the surface defining the local supersymmetric geometries with $SO(4) \times U(1)$ isometries corresponding to the $\frac{1}{2}$-BPS LLM geometries is constructed. According to [50], the boundary condition for these geometries have walls between the two boundary conditions determined by the equation[4]

$$z^2\bar{z}^2 = e^{-2\hat{D}(z^1,\bar{z}^1)} \tag{3.77}$$

where $\hat{D}(z^1,\bar{z}^1)$ is determined by expanding the function D as follows (it is the y coordinate that we set to zero to get the LLM plane)

$$D = \log(y) + \hat{D}(z,\bar{z}) + O(y) \tag{3.78}$$

The function D is determined by the equations

$$y\partial_y D = \frac{1}{2} - Z \qquad V = -i(dz\partial_z - d\bar{z}\partial_{\bar{z}})D \tag{3.79}$$

where $Z(y,z^1,\bar{z}^1)$ is the function obeying Laplace's equation that determines the LLM solution and $V(y,z^1,\bar{z}^1)$ is the one form appearing in the combination $(dt+V)^2$ in the LLM metric.

Consider an annulus that has an outer edge at radius $M+N$ and an inner edge at a radius M. This solution has (these solutions were constructed in the original LLM paper [7])

$$\begin{aligned}
Z(y,z^1,\bar{z}^1) = -\frac{1}{2}\Bigg(& \frac{|z^1|^2+y^2-M}{\sqrt{(|z^1|^2+y^2+M)^2-4|z^1|^2M}} \\
& +\frac{|z^1|^2+y^2-M-N}{\sqrt{(|z^1|^2+y^2+M+N)^2-4|z^1|^2(M+N)}}\Bigg) \\
V(y,z^1,\bar{z}^1) = \frac{d\phi}{2}\Bigg(& \frac{|z^1|^2+y^2+M}{\sqrt{(|z^1|^2+y^2+M)^2-4|z^1|^2M-}} \\
& +\frac{|z^1|^2+y^2+M+N}{\sqrt{(|z^1|^2+y^2+M+N)^2-4|z^1|^2(M+N)}}\Bigg)
\end{aligned}$$

Evaluating at $y=0$, the second of (3.79) says

$$V = -i(dz\partial_z - d\bar{z}\partial_{\bar{z}})\hat{D} \tag{3.80}$$

Setting $z^1 = re^{-i\phi}$ and assuming that $\hat{D}$ depends only on r we find

$$r\frac{\partial\hat{D}}{\partial r} = -\frac{M+N}{r^2-M-N} + \frac{M}{r^2-M} \tag{3.81}$$

[4]This next equation is (6.35) of [50]. We will relate z^1 and z^2 to z_i (the eigenvalues of Z) and y_i (the eigenvalues of Y) when we make the correspondence to eigenvalues.

which is solved by

$$\hat{D} = \frac{1}{2} \log \frac{|z^1 \bar{z}^1 - M|}{|z^1 \bar{z}^1 - M - N|} \tag{3.82}$$

Thus, the wall between the two boundary conditions is given by

$$|z^2|^2 = \frac{M + N - z^1 \bar{z}^1}{z^1 \bar{z}^1 - M} \tag{3.83}$$

The same analysis applied to the $AdS_5 \times S^5$ solution gives

$$|z^1|^2 + |z^2|^2 = N \tag{3.84}$$

For the pair of geometries described above, we know the wave function in the eigenvalue description. We will now return to the eigenvalue description and see how these surfaces are related to the eigenvalue wave functions.

At large N, since fluctuations are controlled by $1/N^2$, we expect a definite eigenvalue distribution. These eigenvalues will trace out a surface specified by the support of the single fermion probability density

$$\rho(z_1, \bar{z}_1, y_1, \bar{y}_1) = \int \prod_{i=2}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2 \tag{3.85}$$

Denote the points lying on this surface using coordinates z, y.

Using the wave function $\Psi_{\text{gs}}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$ corresponding to the $AdS_5 \times S^5$ spacetime, the probability density for a single eigenvalue is

$$\rho(z, \bar{z}, y, \bar{y}) = \frac{1}{N\pi^2} \sum_{i=0}^{N-1} \frac{(z\bar{z})^i}{i!} \frac{(y\bar{y})^{N-i-1}}{(N-i-1)!} e^{-z\bar{z} - y\bar{y}} \tag{3.86}$$

As y and z vary, the dominant contribution comes from a term with a specific value for i. When the ith term dominates the sum, the value of the eigenvalue coordinate is given by

$$\begin{aligned} \frac{(z\bar{z})^i}{i!} &= 1 \qquad & |z|^{2i} &= i! \approx i^i \\ \frac{(y\bar{y})^{N-i-1}}{(N-i-1)!} &= 1 \qquad & |y|^{2(N-i-1)} &= (N-i-1)! \approx (N-i-1)^{N-i-1} \end{aligned} \tag{3.87}$$

This leads to the following points

$$|z_{(i)}|^2 = i \qquad |y_{(i)}|^2 = N - i \qquad i = 1, 2, ..., N \tag{3.88}$$

Thus, if we identify the points $z_{(i)}, y_{(i)}$ and the supergravity coordinate z^1, z^2 as follows

$$z^2 = y_{(i)} \qquad z^1 = z_{(i)} \tag{3.89}$$

we find

$$|z^1|^2 + |z^2|^2 = i + (N - i) = N \tag{3.90}$$

so that the eigenvalues condense on the surface that defines the wall between the two boundary conditions.

Let's now compute the positions of our eigenvalues, using $\Psi_{\rm LLM}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})$. The probability density for a single eigenvalue is easily obtained by computing the following integral

$$\begin{aligned}\rho(z_1, \bar{z}_1, y_1, \bar{y}_1) &= \int \prod_{i=2}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi_{\rm LLM}(\{z_i, \bar{z}_i, y_i, \bar{y}_i\})|^2 \\ &= \frac{1}{N\pi^2} \sum_{i=0}^{N-1} \frac{(z_1\bar{z}_1)^{M+i}}{(M+i)!} \frac{(y_1\bar{y}_1)^{N-i-1}}{(N-i-1)!} e^{-z_1\bar{z}_1 - y_1\bar{y}_1}\end{aligned} \tag{3.91}$$

Following the analysis we performed above, we find that the complete set of points on the eigenvalue surface is given by

$$|z_{(i)}|^2 = (M + i) \qquad |y_{(i)}|^2 = N - i \qquad i = 1, 2, ..., N - 1 \tag{3.92}$$

Thus, if we identify the points $z_{(i)}, y_{(i)}$ and the supergravity coordinate z^1, z^2 as follows

$$z^2 = \frac{y_{(i)}}{\sqrt{|z_{(i)}|^2 - M}} \qquad z^1 = z_{(i)} \tag{3.93}$$

we find that (3.83) gives

$$\frac{|y^{(i)}|^2}{i} = \frac{M + N - |z^{(i)}|^2}{|z^{(i)}|^2 - M} \tag{3.94}$$

in complete agreement with where our wave function is localized. This again shows that the eigenvalues are collecting on the surface that defines the wall between the two boundary conditions. Although these examples are rather simple, they teach us something important: the map between the eigenvalues and the supergravity coordinates depends on the specific geometry we consider.

The fact that eigenvalues condense on the surface that defines the wall between the two boundary conditions is something that was already anticipated by Berenstein and Cotta in [35]. The proposal of [35] identifies the support of the eigenvalue distribution with the degeneration locus of the three sphere in the full ten dimensional metric. Our results appear to be in perfect accord with this proposal.

Chapter 4

Outlook

There are a number of definite conclusions resulting from our study. One of our key results is that we have found substantial evidence for the proposal that there is a sector of the two matrix model that is described (sometimes exactly) by eigenvalue dynamics. This is rather non-trivial since, as we have already noted, it is simply not true that the two matrices can be simultaneously diagonalized. The fact that we have reproduced correlators of operators that involve products of both matrices in a single trace is convincing evidence that we are reproducing genuine two matrix observables. The observables we can reproduce correspond to BPS operators. In the dual gravity these operators map to supergravity states corresponding to classical geometries. The local supersymmetric geometries with $SO(4) \times U(1)$ isometries are determined by a surface that defines the boundary conditions needed to obtain a non-singular supergravity solution. At large N where we expect classical geometry, the eigenvalues condense on this surface. In this way the supergravity boundary conditions appear to match the large N eigenvalue description perfectly.

The eigenvalue dynamics appears to provide some sort of a coarse grained description. Correlators of operators dual to states with a very small energy are not reproduced correctly: for example the energy of states dual to single traces has to be above some threshold (N) before they are correctly reproduced. For complicated operators with a detailed multi trace structure we would thus expect to get the gross features correct, but we may miss certain finer details - see the discussion after (3.55). Developing this point of view, perhaps using the ideas outlined in [40], may provide a deeper understanding of the eigenvalue wave functions.

The eigenvalue description we have developed here is explicit enough that we could formulate the dynamics in terms of the density of eigenvalues. This would provide a field theory that has $1/N$ appearing explicitly as a coupling. It would be very interesting to work out, for example, what the generalization of the Das-Jevicki Hamiltonian[51] is.

The picture of eigenvalue dynamics that we are finding here is almost identical to the proposal discussed by Berenstein and his collaborators[33, 34, 35, 36, 37, 38, 39], developed using numerical methods and clever heuristic arguments. The idea of these works is that the eigenvalues represent microscopic degrees of freedom. At large N one can move to collective degrees of freedom that represent the 10 dimensional geometry of the dual gravitational description. This is indeed what we are seeing. They have also considered cases with reduced supersymmetry and orbifold geometries[52, 53, 54]. These are natural examples to consider using the ideas and methods we have developed in this work. Developing other examples of eigenvalue dynamics will allow us to further test the proposals for wave functions and the large N distributions of eigenvalues that we have put forward in this work.

An important question that should be tackled is to ask how one could derive (and not guess) the wave functions we have described. Progress with this question is likley to give some insights into how it is even possible to have a consistent eigenvalue dynamics. One would like to know when an eigenvalue description is relevant and to what classes of observables it is applicable.

Another important question is to consider the extension to more matrices, including gauge and fermion degrees of freedom. The Gauss graph labeling of operators continues to work when we include gauge fields and fermions[55, 56], so that our argument goes through without modification and we again expect that eigenvalue dynamics in these more general settings will be an effective approach to compute these more general correlators of BPS operators. Another important extension is to consider the eigenvalue dynamics, perturbed by off diagonal elements, which should allow one to start including stringy degrees of freedom. Can this be done in a controlled systematic fashion? In this context, the studies carried out in [29, 57, 58], will be relevant.

Appendix A

Eigenvalue Dynamics Formulation

A.1 One Complex Matrix

In order to make an explicit change to eigenvalues we will use the Schwinger-Dyson equation in the matrix model

$$0 = \int [dZdZ^\dagger] \frac{d}{dZ_{ij}} \left[(Z^n)_{ij} e^{-\mathrm{Tr}(ZZ^\dagger)} \mathrm{Tr}(Z^{\dagger n-1}) \right] \tag{A.1}$$

and then write this equation in terms of eigenvalues and compare it to the Schwinger-Dyson equation for the eigenvalues

$$0 = \int [dzd\bar{z}] \sum_{i=1}^{N} \frac{d}{dz_i} \left[J(z,\bar{z})(z^n)_i e^{-z_i\bar{z}_i} \bar{z}_i^{n-1} \right] \tag{A.2}$$

in order to solve for $J(z,\bar{z})$. We could also consider the equations with $Z \longleftrightarrow Z^\dagger$ and $z \longleftrightarrow \bar{z}$ but because the theory is symmetric in this swap we will be able to infer the part of the solution that would have come from this swap.

Starting with (A.1) we need

$$\begin{aligned} \frac{d}{dZ_{ij}}(Z^n)_{ij} &= \sum_{r=0}^{n-1} (Z^r)_{is} \frac{dZ_{st}}{dZ_{ij}} (Z^{n-r-1})_{tj} \\ &= \sum_{r=0}^{n-1} (Z^r)_{is} \delta_{si} \delta_{tj} (Z^{n-r-1})_{tj} \\ &= \sum_{r=0}^{n-1} \mathrm{Tr}(Z^r)\mathrm{Tr}(Z^{n-r-1}) \end{aligned} \tag{A.3}$$

and

$$\frac{d}{dZ_{ij}} \mathrm{Tr}(ZZ^\dagger) = \delta_{ki}\delta_{sj} Z^\dagger_{sk} = Z^\dagger_{ji} \tag{A.4}$$

$$\Rightarrow 0 = \int [dZdZ^\dagger] \left[\sum_{r=0}^{n-1} \mathrm{Tr}(Z^r)\mathrm{Tr}(Z^{n-r-1})\mathrm{Tr}(Z^{\dagger n-1}) - \mathrm{Tr}(Z^n Z^\dagger)\mathrm{Tr}(Z^{\dagger n-1}) \right] e^{-\mathrm{Tr}(ZZ^\dagger)}$$
$$= \left\langle \sum_{r=0}^{n-1} \mathrm{Tr}(Z^r)\mathrm{Tr}(Z^{n-r-1})\mathrm{Tr}(Z^{\dagger n-1}) - \mathrm{Tr}(Z^n Z^\dagger)\mathrm{Tr}(Z^{\dagger n-1}) \right\rangle \quad \text{(A.5)}$$

Now we can easily write most of the terms in terms of eigenvalues only using the Schur decomposition[27, 30, 31],

$$Z = U^\dagger D U \quad \text{(A.6)}$$

with U a unitary matrix and D an upper triangular matrix with eigenvalues on its diagonal. The only term that will still depend on the off diagonal terms is $\mathrm{Tr}(Z^n Z^\dagger)$ since

$$\begin{aligned} \mathrm{Tr}(Z^n) &= \mathrm{Tr}((U^\dagger D U)^n) \\ &= \mathrm{Tr}(D^n) \\ &= \sum_{i=1}^{N} z_i^n \end{aligned} \quad \text{(A.7)}$$

but

$$\begin{aligned} \mathrm{Tr}(Z^n Z^\dagger) &= \mathrm{Tr}((U^\dagger D U)^n (U^\dagger D^\dagger U)) \\ &= \mathrm{Tr}(D^n D^\dagger) \end{aligned} \quad \text{(A.8)}$$

This will evidently still depend on the off diagonal terms. So we need to integrate them off.

$$\int [dZdZ^\dagger] \mathrm{Tr}(Z^n Z^\dagger)\mathrm{Tr}(Z^{\dagger n-1}) e^{-\mathrm{Tr}(ZZ^\dagger)}$$
$$= \int [dzd\bar{z}] J(z,\bar{z}) \sum_{k=1}^{N} \bar{z}_k^{n-1} e^{-z_j \bar{z}_j} \int [\text{off diagonals}] \mathrm{Tr}(D^n D^\dagger) e^{-\sum |\text{off diagonals}|^2} \quad \text{(A.9)}$$

Doing the off diagonal integrals numerically we find

$$\int [\text{off diagonals}] \mathrm{Tr}(D^n D^\dagger) e^{-\sum |\text{off diagonals}|^2}$$
$$= (N - \frac{n}{2}) \sum_{i=1}^{N} z_i^{n-1} + \frac{1}{2} \sum_{r=1}^{n-2} \sum_{i=1}^{N} z_i^r \sum_{j=1}^{N} z_j^{n-r-1} + \sum_{i=1}^{N} z_i^n \bar{z}_i \quad \text{(A.10)}$$

$$\Rightarrow 0 = \int [dz_i d\bar{z}_i] J(z,\bar{z}) \left[\sum_{r=0}^{n-1} \sum_{i=1}^{N} z_i^r \sum_{j=1}^{N} z_j^{n-r-1} + \right.$$

$$\left. -(N-\frac{n}{2}) \sum_{i=1}^{N} z_i^{n-1} - \frac{1}{2} \sum_{r=1}^{n-2} \sum_{i=1}^{N} z_i^r \sum_{j=1}^{N} z_j^{n-r-1} - \sum_{i=1}^{N} z_i^n \bar{z}_i \right] \sum_{k=1}^{N} \bar{z}_k^{n-1} e^{-z_j \bar{z}_j}$$

$$= \int [dz_i d\bar{z}] J(z,\bar{z}) \left[\frac{1}{2} \sum_{r=0}^{n-1} \sum_{i=1}^{N} z_i^r \sum_{j=1}^{N} z_j^{n-r-1} + \frac{n}{2} \sum_{i=1}^{N} z_i^{n-1} - \sum_{i=1}^{N} z_i^n \bar{z}_i \right] \sum_{k=1}^{N} \bar{z}_k^{n-1} e^{-z_j \bar{z}_j}$$

$$= \int [dz_i d\bar{z}] J(z,\bar{z}) \left[\frac{1}{2} \sum_{r=0}^{n-1} \sum_{i=1}^{N} z_i^r \sum_{j=1, j\neq i}^{N} z_j^{n-r-1} + n \sum_{i=1}^{N} z_i^{n-1} - \sum_{i=1}^{N} z_i^n \bar{z}_i \right] \sum_{k=1}^{N} \bar{z}_k^{n-1} e^{-z_j \bar{z}_j} \tag{A.11}$$

Now let us look at (A.2)

$$0 = \int [dz d\bar{z}] \sum_{i=1}^{N} \frac{d}{dz_i} \left[J(z,\bar{z}) (z^n)_i e^{-z_i \bar{z}_i} \bar{z}_i^{n-1} \right]$$

$$= \int [dz d\bar{z}] J(z,\bar{z}) \left[\sum_{i=1}^{N} z_i^n \frac{d \log(J)}{dz_i} + n \sum_{i=1}^{N} z_i^{n-1} - \sum_{i=1}^{N} z_i^n \bar{z}_i \right] \sum_{k=1}^{N} \bar{z}_k^{n-1} e^{-z_j \bar{z}_j} \tag{A.12}$$

Comparing this to (A.11) we see that

$$\sum_{i=1}^{N} z_i^n \frac{d \log(J(z,\bar{z}))}{dz_i} = \frac{1}{2} \sum_{r=0}^{n-1} \sum_{i=1}^{N} z_i^r \sum_{j=1, j\neq i}^{N} z_j^{n-r-1}$$

$$= \frac{1}{2} \sum_{j=1, j\neq i}^{N} \sum_{i=1}^{N} z_j^{n-1} \sum_{r=0}^{n-1} \left(\frac{z_i}{z_j} \right)^r \tag{A.13}$$

The last sum is a geometric series given by

$$\sum_{r=0}^{n-1} \left(\frac{z_i}{z_j} \right)^r = \frac{\left(\frac{z_i}{z_j} \right)^n - 1}{\frac{z_i}{z_j} - 1}$$

$$= \frac{z_j (z_i^n - z_j^n)}{z_j^n (z_i - z_j)} \tag{A.14}$$

$$\begin{aligned}
\Rightarrow \sum_{i=1}^{N} z_i^n \frac{d\log(J(z,\bar{z}))}{dz_i} &= \frac{1}{2}\sum_{j=1,j\neq i}^{N}\sum_{i=1}^{N} z_j^n \frac{(z_i^n - z_j^n)}{z_j^n(z_i - z_j)} \\
&= \frac{1}{2}\sum_{j=1,j\neq i}^{N}\sum_{i=1}^{N} \frac{z_i^n - z_j^n}{z_i - z_j} \\
&= \frac{1}{2}\sum_{j=1,j\neq i}^{N}\sum_{i=1}^{N} \frac{z_i^n}{z_i - z_j} - \frac{1}{2}\sum_{i=1,i\neq j}^{N}\sum_{j=1}^{N} \frac{z_i^n}{z_j - z_i} \\
&= \frac{1}{2}\sum_{j=1,j\neq i}^{N}\sum_{i=1}^{N} \frac{z_i^n}{z_i - z_j} - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \frac{z_i^n}{z_j - z_i} \\
&= \sum_{i=1}^{N} z_i^n \sum_{j=1,j\neq i}^{N} \frac{1}{z_i - z_j} \qquad \text{(A.15)}
\end{aligned}$$

$$\Rightarrow \frac{d\log(J(z,\bar{z}))}{dz_i} = \sum_{j=1,j\neq i}^{N} \frac{1}{z_i - z_j} \qquad \text{(A.16)}$$

$$\Rightarrow \log(J(z,\bar{z})) = \sum_{i,j=1,j\neq i,z_i>z_j} \log(z_i - z_j) + A(\bar{z}) \qquad \text{(A.17)}$$

where $A(\bar{z})$ is an arbitrary constant of integration that is a function of $\bar{z}$ only. We can arrange the eigenvalues so that $z_i > z_j$ for $i > j$

$$\Rightarrow \log(J(z,\bar{z})) = \sum_{i,j=1,j\neq i,i>j} \log(z_i - z_j) + A(\bar{z}) \qquad \text{(A.18)}$$

$$\Rightarrow J(z,\bar{z}) = B(\bar{z})\prod_{i>j}(z_i - z_j) = B(\bar{z})\Delta(z) \qquad \text{(A.19)}$$

The measure must be real which implies

$$B(\bar{z}) = \prod_{i>j}(\bar{z}_i - \bar{z}_j) = \Delta(\bar{z}) \qquad \text{(A.20)}$$

So that

$$J(z,\bar{z}) = \Delta(z)\Delta(\bar{z}) \qquad \text{(A.21)}$$

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