The Algebra and Geometry of Continued Fractions with Integer Quaternion Coefficients [1]

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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This _____________ day of February 2015, at Johannesburg, South Africa.

Abstract

We consider continued fractions with coefficients that are in $\mathbb{K}$, the quaternions. In particular we consider coefficients in the Hurwitz integers $\mathcal{H}$ in $\mathbb{K}$. These continued fractions are expressed as compositions of Möbius maps in $M\left(\mathbb{R}^4_\infty\right)$ that act, by Poincaré extension, as isometries on $\mathbb{H}^5$.

This dissertation explores groups of 2×2 matrices over $\mathbb{K}$ and two particular determinant type functions acting on these groups. On the one hand we find $M\left(\mathbb{R}^4_\infty\right)$, the group of orientation preserving Möbius transformations acting on $\mathbb{R}^4_\infty$ in terms of a determinant $\mathcal{D}$ [19],[38]. On the other hand $\mathbb{K}$ may be considered as a Clifford algebra C_3 based on two generators i and j, or more generally i_1 and i_2, where $ij = k$ or $i_1 i_2 = k$. It is shown this group of matrices over C_4 defined in terms of a pseudo-determinant Δ [1],[37] can also be used to establish $M\left(\mathbb{R}^4_\infty\right)$. Through this relationship we are able to connect the determinant $\mathcal{D}$ to the pseudo-determinant Δ when acting on the matrices that generate $M\left(\mathbb{R}^4_\infty\right)$.

We explore and build on the results of Schmidt [30] on the subdivision of a Farey simplex into 31 Farey simplices. These results are reinterpreted in $\mathbb{H}^5$ with boundary $\mathbb{K}_\infty$ using the group of Möbius transformations on $\mathbb{R}^4_\infty$ [19], [38]. We investigate the unimodular group $G = PS_{\mathcal{D}}L(2, \mathbb{K})$ with its generators and derive a fundamental domain for this group in $\mathbb{H}^5$. We relate this domain to the 24-cells $\mathcal{P}_{\mathcal{U}}$ and ∇ that tessellate $\mathbb{K}$. We define the concepts of Farey neighbours, Farey geodesics and Farey simplices in the Farey tessellation of $\mathbb{H}^5$. This tessellation of $\mathbb{H}^5$ by a Farey pentacross under a discrete subgroup G of $M\left(\mathbb{R}^4_\infty\right)$ is analogous to the Farey tessellation by Farey triangles of $\mathbb{H}^2$ under the modular group [31]. The result in Schmidt [30], that for each quaternion ξ there is a chain of Farey simplices that converge to ξ, is reinterpreted as a continued fraction, with entries from $\mathcal{H}$, that converges to ξ. We conclude with a review of Pringsheim's theorem on convergence of continued fractions in higher dimensions [5].

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Introduction

1.1 Historical Background

Continued fractions have been used in mathematics since the 16th century mainly as a tool for evaluating or approximating real numbers. Bombelli (1526-72) used continued fractions to find approximations for the square root of a positive non-square integer [22]. Since each non-square integer x can be written as $x = b^2 + a$, with $a, b \in \mathbb{Z}$ we have

$$\sqrt{x} = b + \cfrac{a}{2b + \cfrac{a}{2b + \cfrac{a}{2b + \cdots}}}.$$

The value chosen for b is either of the whole numbers whose squares x lies between, thus for $\sqrt{19}$ we have $\sqrt{19} = 4 + \cfrac{3}{8 + \cfrac{3}{8 + \cfrac{3}{8 + \cdots}}}$ or $\sqrt{19} = 5 - \cfrac{6}{10 - \cfrac{6}{10 - \cfrac{6}{10 - \cdots}}}$.

Bombelli did not, however, consider the question of whether the expansion converged to the number it was supposed to represent [22].

Lord Brouckner (1659) gave the first infinite continued fraction expansion for $\frac{4}{\pi}$ [9].

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

In the nineteenth century mathematicians worked on general continued fractions of the form

$$b_0 + \mathbf{K}(a_n|b_n) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{1.1}$$

where the $\{a_i\}$ and $\{b_j\}$ are sequences of complex numbers [21],[29]. Since the complex numbers have the reals as a subfield and are in turn a "subfield" of the skew non commutative field of real quaternions, we would like to extend the notion of continued fractions to include quaternion entries. However, even though it is known that each non zero quaternion is invertible, the fact that the quartenions are non commutative causes great problems.

In the Proceedings of the Royal Irish Academy of 1852 Hamilton [13] addressed this notion of a continued fraction with quaternion entries. Essentially, Hamilton considered the mapping $t(x) = j(x+i)^{-1}$ and looked at its fixed points being the roots of the quadratic equation $x^2 + xi - j = 0$ in $\mathbb{K}$, the ring of quaternions. He established that there are two real quaternion roots and four non-real quaternion roots of this equation. The fact that the quadratic equation with solutions in the quaternions (real and non-real) may have six solutions, rather than the usual two over the reals, had already been established as early as 1843. Hamilton ventured the name *Biquaternions* for this theory of roots in quadratic quaternion equations. In the above example he noted that if $T_n(x) = t^n(x)$ then we have the following convergents:

$$T_1(0) = k, \quad T_2(0) = \frac{1}{2}(k-i), \quad T_3(0) = k - i, \quad T_4(0) = -i,$$

$$T_5(0) = \infty, \quad T_6(0) = 0, \quad T_7(0) = k \text{ etc.}$$

He further considered the continued fraction given by the mapping $t(x) = 10j(x+5i)^{-1}$ and its iterations. Once again he established the fixed points of this map by looking at the solutions of the quadratic equation $x^2 + x5i - 10j = 0$. He considered the real quaternion roots $2k - i$ and $2k - 4i$ of this equation and noted that the continued fraction $T_n(c) = t^n(c)$ converges to the root $2k - i$ for all c except at the root $2k - 4i$.

Hamilton [13] also showed that when considering a continued fraction based on the mapping $t(x) = b(x + a)^{-1}$, where a and b are given real quaternions, if the composition $T_n(0) = t^n(0)$ converges to a limit u, then u is equal to one of the two real quaternion roots of the quadratic equation $x^2 + ax = b$, whichever has the lesser "tensor". Hamilton states as a theorem the following: If the real quaternion c is not a root of the quadratic equation $x^2 + ax = b$, then the value of the continued fraction $T_n(c) = t^n(c)$ will converge indefinitely towards one of the real quadratic roots of the quadratic, whichever has the lesser tensor.

These results are very evocative of the results in continued fractions using Möbius maps and considering the periodicity of these continued fraction. There is also strong resonance with the known results on the limits of the iteration of loxodromic maps [8].

1.2 Möbius Transformations from an Abstract Point of View

Beardon [6] suggests that from an abstract point of view a traditional continued fraction is simply a pair of sequences $\{a_n\}$ and $\{b_n\}$ of complex numbers, where no a_n is zero. Given such sequences, we may consider a sequence of Möbius maps defined by $t_n(z) = \frac{a_n}{b_n+z}$ with associated matrix $\begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}$ in $M_2(\mathbb{C})$ such that $t_n(\infty) = 0$ for all n. Conversely any Möbius map g with $g(\infty) = 0$ can be expressed as $g(z) = \frac{a}{b+z}$ with corresponding matrix $\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$ for a and b in $\mathbb{C}$.

Thus, in general, we are able to identify a class of continued fractions with a class of sequences of Möbius maps which must satisfy the condition $t_n(\infty) = 0$. A truncated continued fraction can be identified with the finite composition $t_1 t_2 \cdots t_n$ of such Möbius maps where $T_n(\infty) = t_1 t_2 \cdots t_n(\infty)$. The convergents to this continued fraction are the terms $t_1(\infty), t_1 t_2(\infty), \cdots, t_1 t_2 \cdots t_n(\infty)$.
Convergence of continued fractions thus depends on

$$\lim_{n\to\infty} T_n(\infty).$$

As Möbius maps exist in all dimensions, the identification of continued fractions with the composition of Möbius maps provides us with the concept of a continued fraction in all dimensions. The continued fraction is dependent on the iteration of translations and inversions in unit spheres and thus in higher dimensions there is a greater dependence on geometric rather than algebraic arguments [6].

To consider, in this manner, continued fractions with integer coefficients and with Gaussian integer coefficients, we need to explore the modular group acting on $\mathbb{H}^2$ and the Picard group acting on $\mathbb{H}^3$ respectively as subgroups of $PSL(2, \mathbb{C})$ [8]. The modular group is given by $\left\{ z \mapsto \dfrac{ax+b}{cx+d} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ and the Picard group is given by $\left\{ z \mapsto \dfrac{ax+b}{cx+d} : a, b, c, d \in \mathbb{Z}[i], ad - bc = 1 \right\}$ with $\mathbb{Z}$ and $\mathbb{Z}[i]$ the integers and Gaussian integers respectively. We note that both $\mathbb{Z}$ and $\mathbb{Z}[i]$ are Euclidean domains and specifically, they are both commutative rings. The continued fractions in these cases are sequence of Möbius maps of the form $t_n(z) = \dfrac{a_n}{b_n + z}$ where a_n and b_n are in $\mathbb{Z}$ and $\mathbb{Z}[i]$ respectively and where $T_n = t_1 t_2 \cdots t_n$. Convergence of these continued fractions is again established by considering

$$\lim_{n\to\infty} T_n(\infty).$$

In the above discussions, the Möbius maps act as isometries on $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively, with the appropriate hyperbolic metrics given by $ds = \dfrac{|dz|}{y}$, with $z = x + iy$, $y > 0$, and $ds = \dfrac{|dx|}{x_3}$ where $x = (x_1, x_2, x_3)$ and $x_3 > 0$.

We notice that a circle or a line in $\mathbb{C}_\infty$ is the boundary of a hyperbolic plane in $\mathbb{H}^3$, and conversely the boundary of a hyperbolic plane in $\mathbb{H}^3$ is a circle or a line in $\mathbb{C}_\infty$. We can thus see the effects of Möbius maps on circles in $\mathbb{C}$ when applied as an isometry in $\mathbb{H}^3$. We note that each Möbius map g with $g(z) = (ax + b)/(cx + d)$ acting on $\mathbb{C}$ extends naturally to act on $\mathbb{R}^3_\infty$ in the following way. We express g as the composition $\alpha_1\alpha_2\cdots\alpha_n$, of a finite number of inversions in circles C_j in $\mathbb{C}_\infty$. As noted C_j is the equator of a unique sphere $\sum_j$ in $\mathbb{R}^3$ where $\sum_j$ is orthogonal to $\mathbb{C}_\infty$. Thus we can also regard α_j as inversions in $\sum_j$. In this way every Möbius map extends to a bijection on $\mathbb{R}^3_\infty$ and this extension leaves both $\mathbb{C}_\infty$ and $\mathbb{H}^3$ invariant. It can be shown that the action on $\mathbb{H}^3$ of this Poincaré extension of g is uniquely determined by the action of g on $\mathbb{C}_\infty$ [7].

There is also an algebraic way to carry out the extension of g acting on $\mathbb{C}_\infty$. We can identify the point (x_0, x_1, x_2) in $\mathbb{R}^3$ with the quaternion $x_0 + x_1 i + x_2 j + 0k$ where $\{1, i, j, k\}$ is the usual basis of the quaternions $\mathbb{K}$ when regarded as a 4-dimensional vector space over $\mathbb{R}$. We may write $x_0 + x_1 i + x_2 j = (x_0 + x_1 i) + x_2 j = z + tj$ with $z = x_0 + x_1 i \in \mathbb{C}$ and $t = x_2 \in \mathbb{R}$. If $(x_0, x_1, t) \in \mathbb{H}^3$ then $t > 0$. Thus the hyperbolic plane has the model $\mathbb{H}^3 = \{(z, t) : z \in \mathbb{C}, t > 0\}$ with the metric derived from the differential given as $ds^2 = \dfrac{|dx|^2}{x_3} = \dfrac{|dz|^2 + t^2}{t^2}$. We now define g acting on $\mathbb{H}^3$ by the formula $g(z + tj) = \{a(z + tj) + b\}\{c(z + tj) + d\}^{-1}$ where the calculations is carried out in the quaternions [6]. We note that when we refer to a quaternion we will mean a real quaternion. These developments lead to the important theorem:

Theorem 1.2.1. *The Möbius group acting on $\mathbb{C}_\infty$ is the group of all conformal orientation preserving isometries of $\mathbb{H}^3$.*

Thus it appears that the most effective way of studying the Möbius group is the study of its action in $\mathbb{H}^3$, even if we are only interested in its action on $\mathbb{C}_\infty$, its boundary. We recall, importantly, that all Möbius maps preserve circles, spheres, inverse points and cross ratios. We emphasize that a Möbius map acting in $\mathbb{R}^n$ is a finite composition $\gamma_1\gamma_2\cdots\gamma_k$ of reflections (inversions) in $n-1$ dimensional hyperspheres and hyperplanes.

1.3 Möbius Transformations with Quaternion Entries

In this thesis we are interested in extending the above ideas to higher dimensions, specifically to the Möbius maps acting on $\mathbb{R}^4$, and hence on $\mathbb{H}^5$. To this end we first note that $\mathbb{H}^5 = \left\{x = (x_0, x_1, x_2, x_3, x_4) : x_p \in \mathbb{R}, p = 0, 1, 2, 3, 4 \text{ and } x_4 > 0\right\}$ is the upper half space of $\mathbb{R}^5$, with boundary $\mathbb{R}^4_\infty$ or $\mathbb{K}_\infty$ and equipped with a hyperbolic metric derived from the line segment $ds = \dfrac{|dx|}{x_4}$. We note that $\mathbb{R}^4$ can be embedded in $\mathbb{R}^5$ by identifying (x_0, x_1, x_2, x_3) with $(x_0, x_1, x_2, x_3, 0)$. Here we adjoin ∞ in the usual way to both $\mathbb{R}^4$ or $\mathbb{R}^5$ to get $\mathbb{R}^4_\infty$ and $\mathbb{R}^5_\infty$ respectively.

In Chapter 2 we introduce Clifford algebras C_n with their associated Clifford groups Γ_n, the linear space of vectors V^n and the extended space V^n_∞. We introduce three important involutions on C_n. In particular we consider the case with $n = 3$ where $C_3 = \Gamma_3 \cup \{0\} = \mathbb{K}$. In this case we define the trace function S and norm function N acting on the quaternions. This work is drawn from Ahlfors [1], Kellerhals [19], Schmidt [30], Waterman [37] and Wilker [38].

In Chapter 3 we introduce the Hurwitz integers $\mathcal{H}$, and show that they form a right (left) Euclidean ring. We consider the multiplicative group of units $\mathcal{U}$ and connect these units to the 24-cell $\mathcal{P}_{\mathcal{U}}$. Finally we consider the Dirichlet region ∇ with respect to $\mathcal{H}$ and its subset ∇^*. This development is largely due to the work of Mahler [23] and Schmidt [30].

In Chapter 4 we begin by considering the general ring of 2×2 matrices over C_n that contains the Clifford matrices. We recall that the General Möbius group $GM(\mathbb{R}^n_\infty)$ acting on $\mathbb{R}^n_\infty$ is the group generated by inversions in hyperplanes and hyperspheres, while the normal subgroup of $GM(\mathbb{R}^n_\infty)$ derived as the composition of an even number of inversions is denoted by $M(\mathbb{R}^n_\infty)$ and is called the group of Möbius transformations.

We introduce the rings of matrices $GL(2, C_n)$ and $GL(2, \Gamma_n)$ following Waterman [37] and Ahlfors [1] respectively, and define a pseudo-determinant Δ acting on these matrices. We consider the bijective action of these groups on both V^n_∞ and V^{n+1}_∞ and establish that these groups are in fact equal.

More specifically, the chapter continues with the examination of the 2×2 matrices over the quaternions $\mathbb{K}$. In this case we introduce a determinant $\mathcal{D}$ of a matrix that is defined in terms of the norm N and trace S of the entries of the matrix. This development is drawn from the works of Schmidt [30], Kellerhals [19] and Wilker [38].
Finally, we show that $M(\mathbb{R}^4_\infty)$ is derived from both the groups

$$G_{\mathcal{D}}L(2, \mathbb{K}) \quad \text{and} \quad GL(2, \Gamma_4).$$

Chapter 5 introduces the hyperbolic space $\mathbb{H}^5$ and its metric derived from the line segment $ds = \frac{|dx|}{x_5}$. Lines (geodesics) and planes in $\mathbb{H}^5$ are defined followed by the definitions of inversions in hyperbolic planes which are either hyperspheres in $\mathbb{H}^5$ centered on $\mathbb{R}^4$ or hyperplanes in $\mathbb{H}^5$ orthogonal to $\mathbb{R}^4$. The exploration of $Iso^+\mathbb{H}^5$ is included in this chapter. The reconciliation of $Iso^+\mathbb{H}^5$ as $PS_{\mathcal{D}}L(2, \mathbb{K})$ [19], [38] or as $PSL(2, \Gamma_4)$ [1], [37] is achieved through the use of Poincarè extensions and isometric spheres in both $\mathbb{R}^4$ and $\mathbb{R}^5$.

In Chapter 6 we introduce the unimodular group G that is derived from matrices with Hurwitz integer coefficients. This group is a discrete subgroup of $Iso^+\mathbb{H}^5$. We find the fundamental region for this group as it acts on $\mathbb{H}^5$, we introduce the special point v that is a cusp of this fundamental domain and its corresponding v-cell denoted by $\mathcal{N}(v)$. These results are drawn from the work of Vulakh [35]. The tessellation of $\mathbb{H}^5$ by copies of $\mathcal{N}(v)$ under G introduces us to the Farey tessellation of $\mathbb{H}^5$. The establishment of $\mathcal{N}(v)$ as a Farey pentacross follows in the next chapter.

Chapter 7 and 8 follow the paper of Schmidt [30] closely in discussing the Farey subdivision of Farey simplices and Farey polytopes. The notion of Farey neighbours is given and this leads to the definition of Farey geodesics, Farey simplices and Farey polytopes as well as the Farey pentacross. The results in Schmidt [30] are reinterpreted by distinguishing between a Farey simplex in $\mathbb{H}^5$ and its projection, the Farey polytope in $\mathbb{K}$, thus allowing us to work in $\mathbb{H}^5$ rather than on its boundary $\mathbb{K}_\infty$. The latter approach is taken by Schmidt [30]. Our use of $\mathbb{H}^5$ results in the theorem that every Farey simplex is the common face between exactly two Farey pentacross and that these pentacross are inverses with respect to the circumsphere of the Farey simplex. This chapter includes the theorem of Schmidt [30] that for any quaternion we can find a chain of Farey polytopes converging to it.

In Chapter 9 we investigate continued fractions in higher dimensions and specifically continued fractions with Hurwitz integer coefficients. The derivation of these continued fractions results from the work of Schmidt [30] on subdivision of Farey simplices (polytopes). We conclude with a review of continued fractions in higher dimensions as given in Beardon [5]. We consider Pringsheim's Theorem and a condition for convergence for continued fractions in higher dimensions.

Finally, we recall Beardon [5], who notes that the real strength of these ideas is not that the algebra of continued fractions generalizes to higher dimensions, but that the theory of continued fractions may be regarded as a part of the theory of inversive geometry in any dimension.

Chapter 2

Clifford Algebras

Our discussion will focus on the real quaternions, which form a Clifford algebra. To facilitate this we examine Clifford algebras, their elements, the Clifford numbers, and the associated Clifford group. We explore the algebraic and the geometric aspects of the arithmetic in the Clifford algebras. Multiplication in the Clifford algebras is not commutative and this fact has a major bearing on the arithmetic of the Clifford algebra.

2.1 The Clifford Algebras C_n and their associated Clifford Groups Γ_n

2.1.1 Clifford Algebras and Clifford Numbers

We use the notation for Clifford algebras taken from Ahlfors [1] and define a Clifford algebra as follows.

Definition 2.1.1. *The Clifford algebra C_n is the associative algebra over $\mathbb{R}$, generated by exactly $n-1$ elements $i_1, i_2 \ldots i_{n-1}$. These elements satisfy the relations $i_s i_t = -i_s i_t$ for $s \neq t$*

and $i_t^2 = -1$, with $s, t = 1, \cdots, n-1$, and no other. Every element of C_n can be expressed uniquely in the form $a = \sum a_I I$, where the sum is over all products of generating elements $I = i_{v_1} i_{v_2} \dots i_{v_p}$ with $1 \leq v_1 < \cdots < v_p \leq n-1$ and $a_I \in \mathbb{R}$. The empty product $I = \emptyset$ is interpreted as the real number 1 or i_0 with the coefficient a_0. Elements in C_n are called Clifford numbers.

The Clifford algebra C_n can be regarded as a real vector space of dimension $m = 2^{n-1}$ [1] [37]. If $a \in C_n$ we denote by $\underline{a}$ the corresponding element in $\mathbb{R}^m$.

Example 2.1.2. $\mathbb{R} = C_1$ is the $2^0 = 1$-dimensional real vector space with basis $\{1\}$.
$\mathbb{C} = C_2$ is the $2^1 = 2$-dimensional real vector space with basis $\{1, i\}$.
$\mathbb{K} = C_3$ is the $2^2 = 4$-dimensional real vector space with basis $\{1, i_1, i_2, i_1 i_2\}$.
$\mathbb{K} \oplus \mathbb{K} = C_4$ [3] is the $2^3 = 8$-dimensional real vector space with basis

$$\{1, i_1, i_2, i_3, i_1 i_2, i_2 i_3, i_1 i_3, i_1 i_2 i_3\}.$$

We discuss particular properties of Clifford numbers.

Definition 2.1.3. *For each $a, b \in C_n$, $a = \sum a_I I$, $b = \sum b_I I$ as given above, we have:*

1. *the Euclidean norm $N(a) = |a|^2 = \sum a_I^2$,*
2. *the scalar product $a \cdot b = \sum a_I b_I$,*
3. *$Re(a) = a_0$ and $Ve(a)$ is the sum of all the other terms,*
4. *a is a vector in C_n if $a = a_0 + a_1 i_1 + \cdots + a_{n-1} i_{n-1}$ where $a_t \in \mathbb{R}$, with $t = 0, \dots, n-1$. The vectors in C_n form an n-dimensional vector space V^n over $\mathbb{R}$ with basis $\{1, i_1, \dots, i_{n-1}\}$.*

We note that the expression of a vector in C_n has no mixed terms of the form $i_s i_t$, $s \neq t$. In fact, the vector $a = a_0 + a_1 i_1 + \cdots + a_{n-1} i_{n-1} \in V^n$ is often identified with the n-tuple $\underline{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{R}^n$ without further explanation. We let $V^n_\infty \cong \mathbb{R}^n_\infty = \mathbb{R}^n \cup \{\infty\}$ with the role of ∞ in $\mathbb{R}^n_\infty$ being analogous to the role of ∞ in $\mathbb{C}_\infty$. We note that $\mathbb{R}$ is contained in the centre of C_n. That is $\lambda a = a\lambda$ for all $a \in C_n$ and $\lambda \in \mathbb{R}$.

2.1.2 Involutions in C_n

The Clifford algebras C_n support three distinct involutions described as follows by their actions.

(a) The involution $*$: Each term $I = i_{v_1} \dots i_{v_p}$ of $a \in C_n$ is replaced by $I^* = i_{v_p} \dots i_{v_1}$, reversing the order of the factors of I. To return the factors $i_{v_p} \dots i_{v_1}$ to the original order requires $\dfrac{p(p-1)}{2}$ switches. Since $i_s i_t = -i_t i_s$ each switch introduces an additional factor of -1, thus $I^* = (-1)^{\frac{p(p-1)}{2}} I$.

We note that $(a+b)^* = a^* + b^*$, while $(ab)^* = b^* a^*$ for all $a, b \in C_n$. Thus the involution $*$ is an anti-automorphism on C_n.

(b) The involution $'$: Each factor in the term $I = i_{v_1} \dots i_{v_p}$ is replaced by its negative, so $I' = (-i_{v_1})(-i_{v_2}) \dots (-i_{v_p}) = (-1)^p I$.

We note that $(a+b)' = a' + b'$ and $(ab)' = a'b'$ for all $a, b \in C_n$. Thus the involution $'$ is an automorphism on C_n.

(c) The involution $\bar{\ }$: The composition of involutions $'$ and $*$ results in the involution $\bar{\ }$. That is $\overline{a} = (a')^* = (a^*)'$ and $\overline{I} = (-1)^{\frac{p(p+1)}{2}} I$.

We note $\overline{a+b} = \overline{a} + \overline{b}$, while $\overline{ab} = \overline{b}\,\overline{a}$ for all $a, b \in C_n$. Thus the involution $\bar{\ }$ is an anti-automorphism.

2.1.3 The Clifford Group

We remark at this point that if x is a vector (in V^n) then $x^* = x$ and $\overline{x} = x'$. We note that while $V^n \cong \mathbb{R}^n$, the property of $x^* = x$ for all $x \in V^n$ does not hold in real n-dimensional space. For any $x \in V^n$ we have that

$$x\overline{x} = (x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1})(x_0 - x_1 i_1 - \cdots - x_{n-1} i_{n-1}) = \sum_{t=0}^{n-1} x_t^2 = x \cdot x = |x|^2.$$

Similarly $\overline{x}x = |x|^2$. This important result establishes that all non-zero vectors have multiplicative inverses. That is $x^{-1} = \dfrac{\overline{x}}{|x|^2}$ for all $x \in V^n$, $x \neq 0$. This remark leads to the following definition.

Definition 2.1.4. *Let Γ_n be the set of products of non-zero vectors in V_n.*
That is $\Gamma_n = \left\{x_1 x_2 \ldots x_p : x_t \in V^n, x_t \neq 0, 1 \leq t \leq p \text{ and } t, p \in \mathbb{N}\right\}$.

If x is a product of vectors $x_1 x_2 \ldots x_p$ then $x^{-1} = (x_1 x_2 \ldots x_p)^{-1} = x_p^{-1} x_{p-1}^{-1} \ldots x_1^{-1}$. Thus each non-zero vector has an inverse and the products of non-zero vectors are also invertible. The set of products of non-zero vectors is closed under multiplication. Γ_n has a multiplicative identity $1 = 1 + 0 \sum a_I I$. We have established the following Theorem.

Theorem 2.1.5. *Γ_n as described above is a multiplicative group, called the Clifford group of C_n.*

Example 2.1.6. We note that $\Gamma_1 = \mathbb{R} \setminus \{0\}$, $\Gamma_2 = \mathbb{C} \setminus \{0\}$, $\Gamma_3 = \mathbb{K} \setminus \{0\}$, but in general, $\Gamma_n \subset C_n \setminus \{0\}$ for $n \geq 4$.

2.1.4 Properties of the Clifford Group Γ_n in C_n

We recall, if $a, b \in \mathbb{R}^m$ with $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{b} = (b_1, \ldots, b_m)$, $m \in \mathbb{N}$, then the standard scalar product is given by $a \cdot b = \sum_{t=1}^{m} a_t b_t$ and the norm of a is given by $N(a) = |a|^2 = a \cdot a = \sum_{t=1}^{m} a_t^2$. The following general lemma relates the scalar product and norm of $\mathbb{R}^m$ to the scalar product of Clifford numbers and their norms. In particular if $a, b \in C_n$ then $a \cdot b = \underline{a} \cdot \underline{b}$ where $\underline{a}, \underline{b} \in \mathbb{R}^m$ and $a \cdot b \in \mathbb{R}$. We follow [1] and [37] with the next lemma.

Lemma 2.1.7. *If $a, b \in C_n$ and $x, y \in V^n$ then*

1. $a \cdot b = \sum a_I b_I = Re\left(a\overline{b}\right) = Re\left(\overline{a}b\right)$ *and* $x \cdot y = \sum_{t=0}^{n-1} x_t y_t = Re\left(x\overline{y}\right) = Re\left(\overline{x}y\right)$
2. $|a|^2 = Re\left(a\overline{a}\right) = Re\left(\overline{a}a\right)$ *for* $a \in C_n$, *and* $|x|^2 = x\overline{x} = \overline{x}x = x'x^* = x^*x'$ *for* $x \in \Gamma_n$
3. $|ab| = |a||b|$ *if either* $a \in \Gamma_n$ *or* $b \in \Gamma_n$ *and*
4. $x \cdot y = \frac{1}{2}(x\overline{y} + y\overline{x}) \in \mathbb{R}$ *if* $x, y \in V^n$

Proof. 1. In the product $a\overline{b}$ the terms that yield elements in $\mathbb{R}$ are the products of the terms in a and b with corresponding I's, where $I = i_{v_1}\dots i_{v_p}$. We note that

$$I\overline{I} = i_{v_1}\dots i_{v_p}\overline{i_{v_1}\dots i_{v_p}} = i_{v_1}\dots i_{v_p}(-i_{v_p})\dots(-i_{v_1}) = 1.$$

Thus $\sum a_I I\overline{b_I I} = \sum a_I \overline{b_I} I\overline{I} = \sum a_I b_I$ is precisely $Re\left(a\overline{b}\right)$. Similarly $Re\,(\overline{a}b) = a\cdot b$. The proof of $x\cdot y = (x\overline{y})_{\mathbb{R}}$ is by direct calculation.

2. We use part 1 above and let $b = a$ to get $a\cdot a = |a|^2 = Re\,(a\overline{a}) = Re\,(\overline{a}a)$ for $a \in C_n$. For $x = x_1x_2\dots x_p \in \Gamma_n$, with $x_t \in V^n$ for $t = 1,\dots,p$, we have

$$x\overline{x} = (x_1x_2\dots x_p)\overline{(x_1x_2\dots x_p)} = x_1x_2\dots x_p\overline{x_p}\dots\overline{x_2}\,\overline{x_1} = |x_1|^2|x_2|^2\dots|x_p|^2 = |x|^2.$$

Also, since $x^* = x$ and $x' = \overline{x}$, we have

$$x^*x' = (x_1x_2\dots x_p)^*(x_1x_2\dots x_p)' = x_p^*\dots x_2^*x_1^*x_1'x_2'\dots x_p' = x_p\dots x_2x_1\overline{x_1}\,\overline{x_2}\dots\overline{x_p}$$

as required. The proofs for $\overline{x}x$ and $x'x^*$ are similar.

3. Assume $a \in \Gamma_n$ then a has an inverse $a^{-1} = \dfrac{\overline{a}}{|a|^2}$ and, using part 2 above, we get

$$|ab|^2 = Re\left((\overline{ab})(ab)\right) = Re\left(\overline{b}\overline{a}ab\right) = |a|^2Re\left(\overline{b}a^{-1}ab\right) = |a|^2Re\left(\overline{b}b\right) = |a|^2|b|^2$$

4. Applying the properties of norm and scalar product to the sum of vectors $x+y$ gives $|x+y|^2 = (x+y)(\overline{x+y}) = (x+y)(\overline{x}+\overline{y}) = |x|^2+|y|^2+x\overline{y}+y\overline{x}$.
Also $|x+y|^2 = (x+y)\cdot(x+y) = x\cdot x + x\cdot y + y\cdot x + y\cdot y = |x|^2+|y|^2+x\cdot y+y\cdot x$.
Thus $x\overline{y}+y\overline{x} = x\cdot y + y\cdot x = 2(x\cdot y) \in \mathbb{R}$, since $x\cdot y = y\cdot x \in \mathbb{R}$

□

The following result is most important in the development of an arithmetic in V^n.

Lemma 2.1.8. *If $a, b \in \Gamma_n$ then ab^{-1} and a^*b are simultaneously in V^n. Similarly, $a^{-1}b$ and ab^* are simultaneously in V^n.*

Proof. If $x = ab^{-1} \in V^n$ then x is a vector and $x^* = x$. Now $(b^*xb)^* = b^*x^*b = b^*xb$ for any $b \in \Gamma_n$, so b^*xb is a vector in V^n. Thus $b^*xb = b^*(ab^{-1})b = b^*a$ is a vector, so $(b^*a)^* = a^*b$ is also a vector.

Conversely, if $y = a^*b \in V^n$, then $y^* = y$. For any $b \in \Gamma_n$, $(b^*)^{-1} = \dfrac{\overline{b^*}}{|b^*|^2} = \dfrac{\left(\overline{b}\right)^*}{|b|^2} = \left(b^{-1}\right)^*$. Thus $(b^{*-1}yb^{-1})^* = b^{*-1}y^*b^{-1} = b^{*-1}yb^{-1}$ for any $b \in \Gamma_n$, so $(b^{*-1}yb^{-1})^*$ is a vector in V^n. Thus $b^{*-1}yb^{-1} = b^{*-1}(a^*b)b^{-1} = b^{*-1}a^* = (ab^{-1})^*$ is a vector, therefore ab^{-1} is a vector. Similarly $a^{-1}b$ and ab^* are simultaneously in V^n. □

2.1.5 Conjugate Products

We follow [1] and [37] introducing a conjugate product of elements in C_n.

Definition 2.1.9. *For $a \in \Gamma_n$ and $x \in V^n$, the product axa'^{-1} is called the conjugate product of x with respect to a.*

From [1] we establish that V^n is closed under conjugate products with respect to elements in V^n.

Lemma 2.1.10. *If $x, y \in V^n$ then $yxy'^{-1} \in V^n$.*

Proof. Let $x, y \in V^n$ then from Lemma 2.1.7(4) we know that $x\bar{y} + y\bar{x} = 2(x \cdot y) \in \mathbb{R}$. Multiplying on the right by y gives $x\bar{y}y + y\bar{x}y = 2(x \cdot y)y \in V^n$, hence

$$y\bar{x}y = 2(x \cdot y)y - |y|^2 x \in V^n.$$

Since $x \in V^n$ we have $\bar{x} \in V^n$ so interchanging x and $\bar{x}$ we get

$$yxy = 2(\bar{x} \cdot y)y - |y|^2 \bar{x} \in V^n.$$

Further, since $y \in V^n$ so $y^* = y$, thus $yxy^* \in V^n$, and since $\dfrac{y^*}{|y|^2} = y'^{-1}$, we have $yxy'^{-1} \in V^n$. □

We use the above result to establish that V^n is closed under conjugate products with respect to elements in Γ_n [1].

Theorem 2.1.11. *If $a \in \Gamma_n$ and $x \in V^n$ then $axa'^{-1} \in V^n$.*

Proof. Since $a \in \Gamma_n$ we have $a = a_0a_1 \ldots a_p$, where $a_t \in V^n$ for all $0 \leq t \leq p$. So

$$\begin{aligned} axa'^{-1} &= (a_0a_1 \ldots a_p)x(a_p'^{-1}a_{p-1}'^{-1} \ldots a_0'^{-1}) \\ &= a_0a_1 \ldots a_{p-1}(a_pxa_p'^{-1})a_{p-1}'^{-1} \ldots a_0'^{-1} \\ &= \left(\frac{1}{|a_p|^2}\right)a_0a_1 \ldots a_{p-1}(x_p)a_{p-1}'^{-1} \ldots a_0'^{-1} \end{aligned}$$

where $x_p = a_pxa_p'^{-1} \in V^n$ by Lemma 2.1.10. Continue this process for each t from $p-1$ to 1. Each $x_t = a_tx_{t+1}a_t'^{-1}$, for $0 \leq t \leq p-1$, is in V^n by Lemma 2.1.10, so $axa'^{-1} \in V^n$. □

2.2 The Geometry of Clifford numbers

According to Waterman [37], multiplication by a non-zero Clifford number in C_n is related to an orthogonal transformation in both V^n and the full algebra C_n. Each orthogonal transformation in $\mathbb{R}^m$ can be represented as a matrix in $O(m, \mathbb{R})$ acting on $\mathbb{R}^m$ where $O(m, \mathbb{R}) = \left\{A \in M_m(\mathbb{R}) : A^{-1} = A^T\right\}$.

We recall from Beardon [7], that the equation of a hyperplane in $\mathbb{R}^m$ through 0 orthogonal to $a \in \mathbb{R}^m$ is given by $P(a, 0) = \{x \in \mathbb{R}^m : x \cdot a = 0\}$. Reflection in the plane $P(a, 0)$ is given by the mapping $R_a(x) = x + \lambda a$ where $\lambda = \dfrac{-2(x \cdot a)}{|a|^2}$ and this ensures that $\frac{1}{2}(x + R_a(x))$ lies on the plane $P(a, 0)$. That is $\frac{1}{2}(x + R_a(x)) \cdot a = 0$.

We note from Beardon [4] the definition of an orthogonal map:

Definition 2.2.1. *A linear transformation $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ is an orthogonal transformation if $\alpha(e_1), \ldots, \alpha(e_m)$ is an orthonormal basis of $\mathbb{R}^m$, where $\{e_1, \ldots, e_m\}$ is the standard unit basis in $\mathbb{R}^m$.*

From Beardon [4], equivalent conditions for which a transformation is orthogonal are stated without proof in the following theorem.

Theorem 2.2.2. *Let $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation. Then the following are equivalent:*

1. *α is an orthogonal transformation;*

2. *α preserves scalar products, that is $\alpha(x) \cdot \alpha(y) = x \cdot y$ for all x and y;*

3. *α preserves lengths of all vectors. That is $|\alpha(x)| = |x|$ for all x;*

4. *if $v_1, \dots v_m$ is an orthonormal basis, then so is $\alpha(v_1), \dots \alpha(v_m)$.*

Following [37], we show that conjugate products of vectors in V^n by elements in Γ_n yield orthogonal transformations:

Theorem 2.2.3. 1. *If $a \in \Gamma_n$ and $x \in V^n$ then $\rho_a : x \to ax(a')^{-1}$ is an orthogonal transformation.*

2. *If $a \in V^n - \{0\}$ and $x \in V^n$ then $\rho_a : x \to ax(a')^{-1}$ is comprised of*

   - *R_1: Reflection in the plane through the origin which is perpendicular to the line through $x = 0$ and $x = 1$ and*

   - *R_a: Reflection in the plane through the origin which is perpendicular to the line through $x = 0$ and $x = a$.*

3. *If $a \in C_n$ and $\rho_a(i_t) = ai_t(a')^{-1} = i_t$, for all $t = 1 \dots n-1$ then $a \in \mathbb{R}$.*

Proof. 1. By Theorem 2.1.11, if $a \in \Gamma_n$ and $x \in V^n$ then $ax(a')^{-1} \in V^n$. Then

$$|\rho_a(x)|^2 = \left|ax(a')^{-1}\right|^2 = \left[ax(a')^{-1}\right]\left[ax\,(a')^{-1}\right]' = ax\,(a')^{-1}\,a'x'\left(a'^{-1}\right)' = ax\bar{x}a^{-1} = |x|^2.$$

Thus ρ_a preserves lengths of all vectors in V^n and ρ_a is an orthogonal transformation.

2. Let $P(a, 0)$ be the hyperplane through 0 orthogonal to a. We note that if $a = 1$ then $R_1(x) = -\bar{x}$. Reflection in $P(a, 0)$ is given by

$$\begin{aligned} R_a(x) &= x - 2(x \cdot a)\frac{a}{|a|^2} = x - (x\bar{a} + a\bar{x})\frac{a}{|a|^2} = -a\bar{x}\frac{a}{|a|^2} \\ &= a(-\bar{x})(a')^{-1} = a[R_1(x)](a')^{-1} = \rho_a(R_1(x)) \end{aligned}$$

for all $x \in V^n$. So $R_a = \rho_a R_1$ hence $\rho_a = R_a R_1$ as $R_1^{-1} = R_1$.

3. Assume $\rho_a(i_t) = a i_t (a')^{-1} = i_t$. Then $a i_t = i_t a'$.

   We note that if $i_{v_1} \dots i_{v_p}$ is a basis element appearing in the expansion of $a \in C_n$, then $(i_{v_1} \dots i_{v_p}) \cdot i_t = i_t (-1)^p (i_{v_1} \dots i_{v_p}) = i_t (i_{v_1} \dots i_{v_p})'$ if i_t does not appear in the expression. Without loss of generality, if i_t does appear in the expression put $v_1 = t$. Now

$$(i_{v_1} \dots i_{v_p}) \cdot i_t = i_{v_1} i_t (-1)^{p-1} (i_{v_2} \dots i_{v_p}) = i_t (-1)^{p-1} (i_{v_1} i_{v_2} \dots i_{v_p}) = -i_t (i_{v_1} i_{v_2} \dots i_{v_p})'$$

   This contradicts the assumption, unless $a \in \mathbb{R}$. Thus there are no non-real terms in a. Hence $a \in \mathbb{R}$.

□

We note that since $a^{-1} = \dfrac{\overline{a}}{N(a)}$ for all $a \in \Gamma_n$ we have that $\rho_a(x) = ax(a')^{-1} = ax\dfrac{a^*}{N(a)} = \dfrac{1}{N(a)} axa^*$. Hence $axa^* = N(a)\rho_a(x)$.

A geometric interpretation of the conjugate product in C_n may be described in terms of the effect of orthogonal matrices acting in $\mathbb{R}^n$ where $\mathbb{R}^n \cong V^n$.

Let $Q_a = (a_{st}) = \dfrac{1}{|a|^2} \begin{pmatrix} a_0^2 & a_0 a_1 & \cdots & a_0 a_{n-1} \\ a_1 a_0 & a_1^2 & \cdots & a_1 a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} a_0 & a_{n-1} a_1 & \cdots & a_{n-1}^2 \end{pmatrix}$ where $a_{st} = \dfrac{a_s a_t}{|a|^2}$ for $0 \leq s, t \leq n-1$.

Then $R_a(x) = \rho_a(-x')$ corresponds to $(I - 2Q_a)\,\underline{x}^T$ with $x = x_0 + x_1 i_1 + \cdots + x_{n-1} i_{n-1}$ and $\underline{x} = (x_0, x_1, \cdots, x_{n-1})$.

We note in particular that $Q_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ and $I - 2Q_1 = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.

Hence $(I - 2Q_1)\,\underline{x}^T = \begin{pmatrix} -x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = -\underline{(x')}^T$ is the reflection $R_1(x)$ of x in the plane through 0 orthogonal to the line through $x = 0$ and $x = 1$. In general $(I - 2Q_a)$ acting on $\mathbb{R}^n$ is the equivalent to R_a acting in V^n where R_a is the reflection in a plane through 0 orthogonal to a. So $(I - 2Q_a)\,\underline{x}^T$ is equivalent to $R_a(x)$ where $a \in \Gamma_n$ and $x \in V^n$. The matrix product $(I - 2Q_1)\,(I - 2Q_a)\,x^T$ is equivalent to the map $R_a R_1(x) = \rho_a(x)$. We prove a geometric characterisation of Γ_n [37].

Theorem 2.2.4. *1. The map $\phi : \Gamma_n \to O(n, \mathbb{R})$, given by $\phi(a) = \rho_a$ where ρ_a is identified with $(I - 2Q_a)\,(I - 2Q_1)$ in $O(n, \mathbb{R})$, is a group epimorphism from Γ_n to $SO(n, \mathbb{R})$ with $ker\phi = \mathbb{R} - \{0\}$. Thus $\Gamma_n / Ker\phi \cong Im\phi \cong SO(n, \mathbb{R})$.*

2. $\Gamma_n = \{a \in C_n : a\bar{a} = |a|^2, a \neq 0$ and $axa^* \in V^n$ for all $x \in V^n\}$.

3. $\Gamma_n = \{a \in C_n \setminus \{0\} : axa^* \in V^{n+1}$ for all $x \in V^{n+1}\}$.

Proof. 1. By Theorem 2.2.3(1) we know that $\rho_a \in O(n, \mathbb{R})$ for all $a \in \Gamma_n$. Consider the product of an even number of reflections in $SO(n, \mathbb{R})$. Theorem 2.2.3(2) allows us to write ρ_a as a product of 2 reflections. Orientation preserving maps are generated by an even number of reflections so a product of an even number of reflections, with v_n even, can be written as

$$\begin{aligned} R_{v_1} R_{v_2} \dots R_{v_n} &= (R_{v_1} R_1)(R_1 R_{v_2}) \dots (R_1 R_{v_n}) = \rho_{v_1} \rho_{v_2}^{-1} \dots \rho_{v_n}^{-1} = \rho_{v_1 v_2^{-1} \dots v_n^{-1}} \\ &= \phi(v_1 v_2^{-1} \dots v_n^{-1}) \end{aligned}$$

which is the product of orthogonal transformations ρ_{v_i}, and this is in $SO\,(n, \mathbb{R})$.

$$\begin{aligned} Ker\phi &= \{a \in \Gamma_n : \phi(a) = \rho_a = 1_{map}\} \\ &= \{a \in \Gamma_n : \rho(x) = x \text{ for all } x \in V^n\} \\ &= \{a \in \Gamma_n : axa'^{-1} = x \text{ for all } x \in V^n\} \\ &= \{a \in \Gamma_n : ax = xa' \text{ for all } x \in V^n\}. \end{aligned}$$

By Theorem 2.2.3(3) if $a \in Ker\phi$ then $ai_t = i_t a'$, for $t = 1 \ldots n-1$, so $a \in \mathbb{R}$, therefore $Ker\phi = \mathbb{R} \setminus \{0\}$.

2. For $a \in \Gamma_n$ we write $a = a_0 a_1 \ldots a_p$, where the a_t, $0 \le t \le p$, are all non-zero vectors. So $a^* = (a_0 a_1 \ldots a_p)^* = a_p^* a_{p-1}^* \ldots a_0^* = a_p a_{p-1} \ldots a_0 = (a_0^{-1} a_1^{-1} \ldots a_p^{-1})^{-1}$ $= \dfrac{(\overline{a_0}\ \overline{a_1} \ldots \overline{a_p})^{-1}}{(|a_0|^2 |a_1|^2 \ldots |a_p|^2)} = \dfrac{(a_0' a_1' \ldots a_p')^{-1}}{|a|^2} = \dfrac{(a')^{-1}}{|a|^2}$, so $axa^* = \dfrac{1}{|a|^2} ax(a')^{-1}$.
   By Theorem 2.1.11, if $a \in \Gamma_n$ and $x \in V^n$ then $ax(a')^{-1} \in V^n$. From Theorem 2.2.3(1) the mapping $x \mapsto \rho_a(x)$ is bijective. Thus if $a \neq 0$ then a^{-1} exists and hence $ax(a')^{-1} \in V^n$ so $axa^* \in V^n$.
   Thus $a \in \{a \in C_n : a\bar{a} = |a|^2 \neq 0 \text{ and } axa^* \in V^n \text{ for all } x \in V^n\}$.
   Conversely, if $|a|^2 \neq 0$ and $|a|^2 = a\bar{a}$ then $a\dfrac{\bar{a}}{|a|^2} = 1$. So $a^{-1} = \dfrac{\bar{a}}{|a|^2}$. Now $a'a^* = a'\overline{a'} = a'\bar{a}' = (a\bar{a})' = |a|^2$ so $(a')^{-1} = \dfrac{a^*}{|a|^2}$. For $x \in V^n$, $axa^* \in V^n$ so $\dfrac{axa^*}{|a|^2} \in V^n$ and $ax(a')^{-1} = \rho_a(x) \in V^n$. By part 1 above $\rho_a \in SO(n, \mathbb{R})$ and so $a \in \Gamma_n$.

3. By part (2) we know $\Gamma_n = \{a \in C_n : a\bar{a} = |a|^2, a \neq 0 \text{ and } axa^* \in V^n \text{ for all } x \in V^n\}$.
   Consider the new generating element $i_n \in V^{n+1}$: Now if $a \in \Gamma_n$ then a is a product of vectors, say $a = a_0 a_1 \ldots a_p$ where $a_t \in V^n$ for $t = 1, \ldots, p$.
   Let $a_t = x_0 + x_1 i_1 + \ldots x_{n-1} i_{n-1} \in V^n$ then

$$a_t i_n = (x_0 + x_1 i_1 + \ldots x_{n-1} i_{n-1}) i_n = i_n x_0 - i_n x_1 i_1 - \cdots - i_n x_{n-1} i_{n-1} = i_n a_t',$$

   thus

$$a i_n = (a_0 a_1 \ldots a_{p-1})(a_p i_n) = (a_0 a_1 \ldots a_{p-1})(i_n a_p') = \cdots = i_n (a_0' a_1' \ldots a_p') = i_n a'.$$

   So $a i_n a^* = i_n a' a^* = i_n |a|^2 \in V^{n+1}$, since $a'a^* = |a|^2$ for $a \in \Gamma_n$.
   Thus if $a \in \Gamma_n$ then $a \in \{a \in C_n - \{0\} : axa^* \in V^{n+1} \text{ for all } x \in V^{n+1}\}$.
   Conversely: Let $a \in \{a \in C_n - \{0\} : axa^* \in V^{n+1} \text{ for all } x \in V^{n+1}\}$. If $axa^* \in V^n$, then $ax\dfrac{a^*}{|a|^2} \in V^n$ and so $ax(a')^{-1} \in V^n$ for all $x \in V^n$. Hence $\rho_a(x) = ax(a')^{-1} \in V^n$ and $\rho_a \in SO(n, \mathbb{R}) \cong \Gamma_n / \{\mathbb{R} \setminus \{0\}\}$. Thus $a \in \Gamma_n$.

□

Example 2.2.5. *Any* $a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 + a_4 i_1 i_2 + a_5 i_1 i_3 + a_6 i_2 i_3 + a_7 i_1 i_2 i_3 \in C_4$ *is an element in* Γ_4 *if and only if* $a_7 a_0 + a_5 a_2 = a_6 a_1 + a_4 a_3$*, using parts 2 and 3 of Theorem 2.2.4.*

2.3 The Quaternions as a Clifford Algebra and a Clifford Group

We consider the Clifford algebra C_3 and its associated Clifford group Γ_3. We will write $C_3 = \mathbb{K}$. We note that since V^4 and $\mathbb{K}$ are both 4-dimensional vector spaces, we have $V^4 \cong \mathbb{K} \cong \mathbb{R}^4$ as vector spaces. However $\mathbb{K}$ cannot represent the vectors of C_4 since $k = ij$, with $k^* = -k$, and $i_3 \neq i_1 i_2$, with $i_3^* = i_3$.

Each element $a \in \mathbb{K}$ can be written as $a = (a_0 + a_1 i + a_2 j) + a_3 k = a_v + a_3 ij$, where $a_v = a_0 + a_1 i + a_2 j$ is a vector, as seen in Definition 2.1.3(4). Thus $\mathbb{K}$ is identified with $\mathbb{R}^3 \times \mathbb{R}$.

Further for $a \in \mathbb{K}$, we may write $a = a_0 + (a_1 i + a_2 j + a_3 k) = Re(a) + Ve(a)$ where $Re(a) = a_0 \in \mathbb{R}$ and the pure quaternion part $Ve(a) = a_1 i + a_2 j + a_3 k$ is identified with $\underline{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$. Thus a is represented by $(a_0, \underline{a}) \in \mathbb{R} \times \mathbb{R}^3$.

2.3.1 Multiplication of Quaternions

From above we identify a quaternion a with an ordered pair of an element in $\mathbb{R}$ and an element in $\mathbb{R}^3$. That is $\mathbb{K} \cong \mathbb{R} \times \mathbb{R}^3$. We show a striking link between quaternion multiplication and the scalar product and vector product in $\mathbb{R}^3$ [4]. These links allow us to show that the Clifford group Γ_3 is in fact equal to the non-zero quaternions in Corollary 2.3.4.

Theorem 2.3.1. *The product of quaternions* $a = a_0 + a_1 i + a_2 j + a_3 k$ *and* $b = b_0 + b_1 i + b_2 j + b_3 k$ *is given by the formula*

$$ab = (a_0, \underline{a})(b_0, \underline{b}) = (a_0 b_0 - \underline{a} \cdot \underline{b}, a_0 \underline{b} + b_0 \underline{a} - \underline{a} \times \underline{b}),$$

where $\underline{a} \cdot \underline{b}$ and $\underline{a} \times \underline{b}$ are the scalar and vector products respectively of $\underline{a}$ and $\underline{b}$ in $\mathbb{R}^4$.

Proof. Let $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$ be identified with $(a_0, (a_1, a_2, a_3)) = (a_0, \underline{a})$ and $(b_0, (b_1, b_2, b_3)) = (b_0, \underline{b}) \in \mathbb{R} \times \mathbb{R}^3$ respectively.
Then $ab = c_0 + c_1 i + c_2 j + c_3 k$ is determined by

$$\begin{aligned} & (a_0, (a_1, a_2, a_3))(b_0, (b_1, b_2, b_3)) \\ = \; & (a_0 b_0 - (a_1, a_2, a_3) \cdot (b_1, b_2, b_3), b_0(a_1, a_2, a_3) + a_0(b_1, b_2, b_3) + (a_1, a_2, a_3) \times (b_1, b_2, b_3)) \\ = \; & (a_0 b_0 - \underline{a} \cdot \underline{b}, b_0 \underline{a} + a_0 \underline{b} + \underline{a} \times \underline{b}), \end{aligned}$$

where $c_0 = a_0 b_0 - \underline{a} \cdot \underline{b} \in \mathbb{R}$ and $(c_1, c_2, c_3) = b_0 \underline{a} + a_0 \underline{b} + \underline{a} \times \underline{b}$ □

From Porteous [27] we see that each quaternion can be expressed as a product of pure quaternions (see Corollary 2.3.4) and as a product of vectors (see Corollary 2.3.3).

Theorem 2.3.2. *Let $q \in \mathbb{K}$, then there exists a non-zero pure quaternion b such that qb is a pure quaternion*

Proof. : Let $q = (q_0, (q_1, q_2, q_3))$ and $b = (0, (b_1, b_2, b_3))$ where
$(q_1, q_2, q_3) \cdot (b_1, b_2, b_3) = q_1 b_1 + q_2 b_2 + q_3 b_3 = 0$ then

$$\begin{aligned} qb &= (0, (q_0(b_1, b_2, b_3) + (q_1, q_2, q_3) \times (b_1, b_2, b_3))) \\ &= (0, (q_0 b_1 + q_2 b_3 - q_3 b_2, q_0 b_2 - q_1 b_3 + q_3 b_1, q_0 b_3 + q_1 b_2 - q_2 b_3)) \end{aligned}$$

which is pure quaternion. □

Corollary 2.3.3. *Each $q \in \mathbb{K}$ is expressible as a product of a pair of pure quaternions.*

Proof. For every quaternion q there exists a non-zero pure quaternion b such that qb is a pure quaternion. If $d = qb$ then $q = db^{-1}$, hence q is a product of pure quaternions. □

Corollary 2.3.4. *Each $q \in \mathbb{K}$ is expressible as a product of a pair of vectors and $\mathbb{K}\backslash\{0\} = \Gamma_3$.*

Proof. By the Corollary 2.3.3 each quaternion q is a product of a pair of pure quaternions, b and c, where $b = Ve(b) = b_1 i + b_2 j + b_3 k$ and $c = Ve(c) = c_1 i + c_2 j + c_3 k$ say. Now $bk = -b_3 + b_2 i - b_1 j$ and $kc = -c_3 - c_2 i + c_1 j$ are vectors. Thus if $q = bc$ then $q = -bk^2 c = -(bk)(kc)$. Thus q is a product of two vectors.
Since each non-zero quaternion is a product of vectors we have that $\mathbb{K} \setminus \{0\} = \Gamma_3$. □

It is useful to be able to recognise when certain quaternion products are vectors [1]. We note that for $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{K}$, $a \in V^3$ implies $a_3 = 0$ and $a^* = a$. Conversely if $a^* = a$ then $a_3 = 0$ and $a \in V^3$.

Lemma 2.3.5. *For all $a, b \in \mathbb{K}$, $a^* b = b^* a$ if and only if $a^* b \in V^3$.*

Proof. $(a^* b)^* = (b^*)(a^*)^* = b^* a$ from properties of involutions.
So if $a^* b = b^* a$ then $ab^* \in V^3$. Conversely, if $ab^* \in V^3$ then $ab^* = (ab^*)^*$, so $ab^* = ba^*$ □

2.3.2 The Norm and Trace of Quaternions

For any $a \in \mathbb{K}$, with $a = a_0 + a_1 i + a_2 j + a_3 k$, $a_t \in \mathbb{R}$ for $t = 0, \ldots, 3$, we may write $\alpha_1 = a_0 + a_1 i$ and $\alpha_2 = a_2 + a_3 i$ as elements of $\mathbb{C}$. Let $\mathbf{K} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$. We know from Stillwell [33] that $\mathbb{K} \cong \mathbf{K}$ as rings where $a = a_0 + a_1 i + a_2 j + a_3 k = \alpha_1 + \alpha_2 j$ is related to $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} \end{pmatrix}$. Certainly $\mathbf{K} \subseteq M_2(\mathbb{C})$.
We know that there is a correspondence between the modulus, conjugate and inverse of non-zero complex numbers and the determinant, transpose and inverse of the corresponding invertible matrices in $M_2(\mathbb{R})$ respectively. That is, if
$z = x + iy \neq 0$ in $\mathbb{C}$ and $Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in M_2(\mathbb{R})$ then $|z|^2 = x^2 + y^2 = det \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$;

$\overline{z} = x - yi$ corresponds to $Z^T = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$; and $z^{-1} = \frac{1}{|z|^2}\overline{z}$ corresponds to $Z^{-1} = \frac{1}{detZ}Z^T$,
with $\mathbf{C} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x, y \in \mathbb{R} \right\} \cong \mathbb{C}$ as rings.

$M_2(\mathbb{C})$ is a ring of matrices over $\mathbb{C}$ with a well defined trace and a well defined determinant. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{C})$ then $detA = \alpha\delta - \beta\gamma$ and $trA = \alpha + \delta$. These trace and determinant functions are linked by definition to the trace and norm functions of quaternions [30] for all $a \in \mathbb{K}$, as follows:

Definition 2.3.6. *Let $a = a_0 + a_1 i + (a_2 + a_3 i)\, j = \alpha_1 + \alpha_2 j \in \mathbb{K}$ as given above. We define $S(a)$, the trace of a, as* $tr\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} \end{pmatrix} = \alpha_1 + \overline{\alpha_1} = 2a_0$.

So $S(a)$ is actually the trace of the matrix A in $M_2(\mathbb{C})$ associated with $a \in \mathbb{K}$. We note that $det\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} \end{pmatrix} = \alpha_1\overline{\alpha_1} + \alpha_2\overline{\alpha_2} = |\alpha_1|^2 + |\alpha_2|^2 = |a|^2$ is given as the norm of a, $N(a)$, in Definition 2.1.3(1), so $N(a)$ is the determinant of the matrix A in $M_2(\mathbb{C})$ associated with $a \in \mathbb{K}$. We also note that

$$a^{-1} = \frac{1}{N(a)}\overline{a} = \frac{1}{N(a)}(a_0 - a_1 i - a_2 j - a_3 k) = \frac{1}{N(a)}[(a_0 - a_1 i) - (a_2 j + a_3 k)] = \frac{1}{N(a)}[\overline{\alpha_1} - \alpha_2 j],$$

so a^{-1} corresponds to the matrix $\frac{1}{N(a)}\begin{pmatrix} \overline{\alpha_1} & -\alpha_2 \\ \overline{\alpha_2} & \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} \end{pmatrix}^{-1} = A^{-1}$. As in the case for complex numbers, there is a correspondence between $N(a)$, $S(a)$ and a^{-1}, $a \neq 0$ in $\mathbb{K}$ with $detA$, trA and A^{-1} in $\mathbf{K}$ where a and A are as given above.

We establish some properties of the norm and trace of quaternions [30].

Theorem 2.3.7. *Let $a, b \in \mathbb{K}$ and $t \in \mathbb{R}$, then:*

1. $N(ab) = N(a)N(b)$,
2. $S(a \pm b) = S(a) \pm S(b)$,
3. $N(a \pm b) = N(a) + N(b) \pm S(a\bar{b})$,
4. $S(ab) = S(ba)$ *and*
5. $S(ta) = tS(a)$.

Proof.

1. $N(ab) = |ab|^2 = ab(\overline{ab}) = ab\bar{b}\bar{a} = aN(b)\bar{a} = N(a)N(b)$.

2. Let $a = a_0 + a_1 i + a_2 j + a_3 k, b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{K}$

$$\begin{aligned} S(a \pm b) &= S\left[(a_0 + a_1 i + a_2 j + a_3 k) \pm (b_0 + b_1 i + b_2 j + b_3 k)\right] \\ &= S\left[(a_0 \pm b_0) + (a_1 \pm b_1)i + (a_2 \pm b_2)j + (a_3 \pm b_3)k\right] \\ &= 2(a_0 \pm b_0) \\ &= S(a) \pm S(b). \end{aligned}$$

3. Firstly, $S(a\bar{b}) = a\bar{b} + \overline{(a\bar{b})} = a\bar{b} + b\bar{a} = 2(a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3)$, so

$$\begin{aligned} N(a \pm b) &= N\left[(a_0 + a_1 i + a_2 j + a_3 k) \pm (b_0 + b_1 i + b_2 j + b_3 k)\right] \\ &= N\left[(a_0 \pm b_0) + (a_1 \pm b_1)i + (a_2 \pm b_2)j + (a_3 \pm b_3)k\right] \\ &= (a_0 \pm b_0)^2 + (a_1 \pm b_1)^2 + (a_2 \pm b_2)^2 + (a_3 \pm b_3)^2 \\ &= (a_0^2 \pm 2a_0b_0 + b_0^2) + (a_1^2 \pm 2a_1b_1 + b_1^2) + (a_2^2 \pm 2a_2b_2 + b_2^2) + (a_3^2 \pm 2a_3b_3 + b_3^2) \\ &= (a_0^2 + a_1^2 + a_2^2 + a_3^2) \pm 2(a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3) + (b_0^2 + b_1^2 + b_2^2 + b_3^2) \\ &= N(a) + N(b) \pm S(a\bar{b}). \end{aligned}$$

4. $S(ab) = 2Re(ab) = 2(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) = 2Re(ba) = S(ba)$.

5. $S(ta) = 2Re(ta) = 2ta_0 = t(2a_0) = tS(a)$.

□

2.3.3 The Centre of the Quaternions

We conclude this section with a discussion on the centre of $\mathbb{K}$.

Definition 2.3.8. *The center of* $\mathbb{K}$*, denoted by* $Z(\mathbb{K})$*, is the set* $\{a \in \mathbb{K} : ab = ba \text{ for all } b \in \mathbb{K}\}$.

Certainly, if $a \in Z(\mathbb{K})$ then a must commute with i, j, and k.

Lemma 2.3.9. $Z(\mathbb{K}) = \mathbb{R}$

Proof. If $a \in \mathbb{R}$ and $b \in \mathbb{K}$ then

$$\begin{aligned} ab &= a(b_0 + b_1 i + b_2 j + b_3 k) \\ &= ab_0 + ab_1 i + ab_2 j + ab_3 k \\ &= b_0 a + b_1 ia + b_2 ja + b_3 ka \\ &= ba. \end{aligned}$$

Therefore $a \in Z(\mathbb{K})$.

If $a \in Z(\mathbb{K})$ and $a = a_0 + a_1 i + a_2 j + a_3 k$ then $ai = ia$ so $a_0 i - a_1 - a_2 k + a_3 j = a_0 i - a_1 + a_2 k - a_3 j$ so $a_2 = a_3 = 0$. Further $aj = ja$ so $a_0 j + a_1 k = a_0 j - a_1 k$ therefore $a_1 = 0$, so $a \in \mathbb{R}$.

Thus $Z(\mathbb{K}) = \mathbb{R}$ as required. □

Chapter 3

Hurwitz Integers and Rationals in $\mathbb{K}$

In this chapter we define the Hurwitz integers and establish that the Hurwitz rationals are "quotients" of Hurwitz integers, for non-zero denominators.

3.1 Hurwitz Integers

We follow Mahler [23] and define the set $\mathcal{H}$ of Hurwitz integers as follows:

Definition 3.1.1. *The Hurwitz integers $\mathcal{H}$ in $\mathbb{K}$ are defined as the set*

$$\mathcal{H} = \left\{ h_0 + h_1 i + h_2 j + h_3 \omega : h_0, h_1, h_2, h_3 \in \mathbb{Z}, \omega = \frac{1 + i + j + k}{2} \right\}.$$

Mahler notes that the Hurwitz integers form a lattice in the quaternion space which is generated by 1, i, j and ω. This lattice is the four dimensional analogue to the centered cube lattice in ordinary three-space.

Lemma 3.1.2. *Every h in $\mathcal{H}$ can be written as a quaternion $a = a_0 + a_1 i + a_2 j + a_3 k$ with coefficients $a_t, t = 0, \cdots, 3$ all in either $\mathbb{Z}$ or in $\frac{1}{2} + \mathbb{Z} = \left\{ x + \frac{1}{2} : x \in \mathbb{Z} \right\}$.*

Proof. Let $h \in \mathcal{H}$, then

$$\begin{aligned} h &= h_0 + h_1 i + h_2 j + h_3\omega, h_t \in \mathbb{Z}, t = 0,1,2,3 \\ &= h_0 + h_1 i + h_2 j + h_3\left(\frac{1+i+j+k}{2}\right) \\ &= \left(h_0 + \frac{h_3}{2}\right) + \left(h_1 + \frac{h_3}{2}\right)i + \left(h_2 + \frac{h_3}{2}\right)j + \left(\frac{h_3}{2}\right)k \\ &= a_0 + a_1 i + a_2 j + a_3 k \end{aligned}$$

with $a_t = h_t + \dfrac{h_3}{2}$ for $t = 0,1,2$ and $a_3 = \frac{h_3}{2}$. If h_3 is even then $a_t \in \mathbb{Z}$ for all t and if h_3 is odd then $a_t \in \frac{1}{2} + \mathbb{Z}$ for all t as required.

□

Theorem 3.1.3. *$\mathcal{H}$ is a non-commutative ring with multiplicative identity. Further if $a \in \mathcal{H}$ then $\overline{a}$, a' and a^* are in $\mathcal{H}$.*

Proof. $\mathcal{H}$ is certainly an abelian group under addition and $0 \in \mathcal{H}$. Since the products

$$\omega^2 = \omega - 1, \;\; i\omega = -\omega + i + k, \;\; \omega i = -\omega + i + j, \;\; j\omega = -\omega + i + j,$$

$$\omega j = -\omega + j + k, \;\; k\omega = -\omega + j + k, \;\; \omega k = -\omega + i + k$$

are all Hurwitz integers, $\mathcal{H}$ is closed under multiplication. This multiplication is distributive over addition, but not commutative. Finally $1 + 0i + 0j + 0k$ is the multiplicative identity in $\mathcal{H}$. We note $\overline{\omega} = \overline{\frac{1}{2}(1 + i + j + k)} = \frac{1}{2}(1 - i - j - k) = -\omega + 1$, $\omega' = \omega - i - j$ and $\omega^* = \omega - k$. So $\overline{\omega}$, ω' and ω^* are in $\mathcal{H}$ and so $\overline{a}$, a' and a^* are in $\mathcal{H}$ if $a \in \mathcal{H}$, since the involutions are automorphisms (antiautomorphism) of $\mathbb{K}$ leaving the reals fixed.

□

From Schmidt [30] we have the following.

Definition 3.1.4. *A quaternion a in $\mathbb{K}$ is said to be rational in $\mathbb{K}$ if $a = a_0 + a_1 i + a_2 j + a_3 k$ where a_0, a_1, a_2 and a_3 are rationals in $\mathbb{Q}$.*

Lemma 3.1.5. *If $h \in \mathcal{H}$ then h is rational in $\mathbb{K}$ with norm $N(h) \in \mathbb{Z}^+ \cup \{0\}$ and with trace $S(h) \in \mathbb{Z}$*

Proof. Let $h \in \mathcal{H}$ then $h = h_0 + h_1 i + h_2 j + h_3 \omega$ with h_0, h_1, h_2, h_3 in $\mathbb{Z}$.
If $h = 0$, then h is rational, $N(h) = 0$ and $S(h) = 2h_0 = 0$. Consider $h \neq 0$, with $h = h_0 + h_1 i + h_2 j + h_3 \omega$. By Lemma 3.1.2, $h = a_0 + a_1 i + a_2 j + a_3 k$ with $a_t = h_t + \frac{h_3}{2}$, for $t = 0 \cdots 2$ and $a_3 = \frac{h_3}{2}$. Thus $a_t \in \mathbb{Q}$ as required.
We consider $N(h)$ and $S(h)$ for the cases where h_3 is even and h_3 is odd.
For h_3 even, $a_3 = \frac{h_3}{2} \in \mathbb{Z}$ so then a_0, a_1 and a_2 are in $\mathbb{Z}$ and $N(h)$ and $S(h)$ are in $\mathbb{Z}$.
For h_3 odd:

$$\begin{aligned} N(h) &= a_0^2 + a_1^2 + a_2^2 + a_3^2 \\ &= \sum_{i=0}^{2} \left(h_i + \frac{h_3}{2}\right)^2 + \left(\frac{h_3}{2}\right)^2 \\ &= h_0^2 + h_1^2 + h_2^2 + h_3^2 + (h_0 + h_1 + h_2)h_3 \in \mathbb{Z} \end{aligned}$$

and $S(h) = 2a_0 = 2h_0 + h_3 \in \mathbb{Z}$ □

Lemma 3.1.6. *The quaternion rationals are exactly the set the set of Hurwitz rationals given by* $\{ab^{-1} : a, b \in \mathcal{H}, b \neq 0\}$ *or* $\{a^{-1}b : a, b \in \mathcal{H}, a \neq 0\}$.

Proof. Let $\mathcal{Q}$ be the quaternion rationals. Then we can easily write

$$\mathcal{Q} = \left\{ \frac{h_0 + h_1 i + h_2 j + h_3 \omega}{r} : h_0, h_1, h_2, h_3, r \in \mathbb{Z}, \omega = \frac{1 + i + j + k}{2}, r \neq 0 \right\}.$$

Conversely, for $a, b \in \mathcal{H}$, $ab^{-1} = ab^{-1}\left(\left(\overline{b}\right)^{-1}\overline{b}\right) = a\left(\overline{b}b\right)^{-1}\overline{b} = \frac{a\overline{b}}{|b|^2}$ and $|b|^2 \in \mathbb{Z}$. Thus ab^{-1} is in $\mathcal{Q}$ since $ab^{-1} \in \mathcal{H}$ and $|b|^2 \in \mathbb{Z}^+$.
Similarly $a^{-1}b = \frac{\overline{a}b}{|a|^2}$ is of the required form, with $|a|^2 \in \mathbb{Z}^+$. □

Analogous to $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ we write $\mathcal{Q}_\infty = \mathcal{Q} \cup \{\infty\}$ where ∞ can be regarded as $1(0)^{-1}$.

3.2 The Group of Hurwitz Units

We follow Schmidt [30] in our development.

Definition 3.2.1. *$a \in \mathcal{H}$ is a unit if there exists a quaternion $b \in \mathcal{H}$ such that $ab = ba = 1$. b is called the inverse of a.*

It is easy to see that if a is a unit, with inverse b, then b is unique and is also a unit. We write $b = a^{-1}$.

Lemma 3.2.2. *a is a unit in $\mathcal{H}$ if and only if $N(a) = 1$.*

Proof. Let a be a unit. Then a^{-1} exists and $aa^{-1} = a^{-1}a = 1$.
Thus $N(aa^{-1}) = N(a)N(a^{-1}) = N(1) = 1$. Hence $N(a) = N(a^{-1}) = 1$ since $N(a) \in \mathbb{Z}^+ \cup \{0\}$.
Conversely, if $N(a) = 1$ then $a\overline{a} = 1$ so $\overline{a} = a^{-1}$ and a is a unit and $\overline{a} \in \mathcal{H}$. □

Theorem 3.2.3. *The 24 units of $\mathcal{H}$ are $\pm1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm1 \pm i \pm j \pm k)$ and they form a group under multiplication with 1 as multiplicative identity.*

Proof. Let u be a unit in $\mathcal{H}$. Thus $N(u) = u_0^2+u_1^2+u_2^2+u_3^2 = 1$ where $u = u_0+u_1i+u_2j+u_3k$.
We have shown that $u_0, u_1, u_2, u_3 \in \mathbb{Z}$ or $u_0, u_1, u_2, u_3 \in \frac{1}{2} + \mathbb{Z}$.
If $u_0, u_1, u_2, u_3 \in \mathbb{Z}$ then $u_0^2+u_1^2+u_2^2+u_3^2 = 1$ implies that $u_t = \pm1$ for one value of $t = 0, \cdots, 3$ and $u_s = 0$ for $s \neq t$. There are 8 units generated this way. They are $\pm1, \pm i, \pm j, \pm k$.
If $u_0, u_1, u_2, u_3 \in \frac{1}{2} + \mathbb{Z}$, then $u_0^2 + u_1^2 + u_2^2 + u_3^2 = \left(\frac{1}{2} + r_0\right)^2 + \left(\frac{1}{2} + r_1\right)^2 + \left(\frac{1}{2} + r_2\right)^2 + \left(\frac{1}{2} + r_3\right)^2$, where $u_t = \frac{1}{2} + r_t$ for $r_t \in \mathbb{Z}$, $t = 0, \cdots, 3$. Assume $r_t < -1$ for some $t = 0, \cdots, 3$ and the other $r_s = 0$ or -1, for $s \neq t$. Then $u_t = \frac{1}{2} + r_t < \frac{1}{2} - 1 = \frac{-1}{2}$, and $u_t^2 > \frac{1}{4}$. Finally $u_0^2 + u_1^2 + u_2^2 + u_3^2 > 1$ which contradicts $N(u) = 1$.
Similarly, if we assume $r_t > 0$ for some $t = 0, \cdots, 3$ and the other $r_s = 0$ or -1, again $N(u) > 1$, a contradiction.
If $u_t \in \frac{1}{2} + \mathbb{Z}$ then $u_t = \pm\frac{1}{2}$. This gives us the 16 units $\frac{1}{2}(\pm1 \pm i \pm j \pm k)$.
Since $N(uv) = N(u)N(v)$ for all $u, v \in \mathcal{H}$ the units

$$\pm1, \quad \pm i, \quad \pm j, \quad \pm k, \quad \frac{1}{2}(\pm1 \pm i \pm j \pm k)$$

form a multiplicative group and 1 is unity. □

We denote the group of units of $\mathcal{H}$ by $\mathcal{U}$.

Definition 3.2.4. *Let a be a quaternion. If $\epsilon \in \mathcal{H}$ is a unit then ϵa and $a\epsilon$ are called associates of a. If $b = \epsilon a$ then we write $a \approx b$ or $b \approx a$.*

It is easily seen that $\approx$ is an equivalence relation on $\mathbb{K}$ and $N(a) = N(b)$ if $a \approx b$.

Lemma 3.2.5. *Let $\mathcal{U}$ be the multiplicative group of units of $\mathcal{H}$. On $\mathcal{U}$ we define $\equiv$ as follows: $a \equiv b$ if there exists $m \in \mathcal{U}$ such that $a = m^{-1}bm$. $\mathcal{U}$ can be written as the union of distinct disjoint equivalence classes under $\equiv$ as follows:*

$$\mathcal{U} = \{[1], [-1], [i], [\omega], [1-\omega], [-\omega], [\omega - 1]\}$$

where

$$[1] = \{1\}, \quad [-1] = \{-1\}, \quad [i] = \{\pm i, \pm j, \pm k\},$$

$$[\omega] = \{\omega, \omega - i - j, \omega - j - k, \omega - i - k\},$$

$$[1-\omega] = [\overline{\omega}] = \left[\omega^{-1}\right] = \{1-\omega, \omega - i, \omega - j, \omega - k\},$$

$$[-\omega] = \{-\omega, i + j - \omega, j + k - \omega, i + k - \omega\}$$

$$\text{and } [\omega - 1] = [-\overline{\omega}] = \{\omega - 1, i - \omega, j - \omega, k - \omega\}$$

.

Proof. Certainly $\equiv$ is an equivalence relation on $\mathcal{U}$. Each product of the form $m^{-1}um$ is computed, to establish the equivalence class for each $u \in \mathcal{U}$. The elements of each equivalence class can be established by laborious calculation or using Table 9.1. □

Corollary 3.2.6. *Let $\pi \in \mathcal{U}$, then $S(\pi) = 1$ if and only if $\pi \in [\omega] \cup [\overline{\omega}]$.*

Proof. The trace of m is $S(m) = m + \overline{m} = 2Re(m)$. All elements with $Re(m) = \frac{1}{2}$ have $S(m) = 1$. These are precisely the elements in $[\omega] \cup [\overline{\omega}]$. □

To facilitate the next results let us represent $[\omega]$ and $[\overline{\omega}]$ as follows.

$$[\omega] = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

$$\text{with } \alpha_1 = \omega, \quad \alpha_2 = \omega - j - k, \quad \alpha_3 = \omega - i - k, \quad \alpha_4 = \omega - i - j$$

$$\text{and } [\overline{\omega}] = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

$$\text{with } \beta_1 = \omega^{-1} = \overline{\omega} = 1 - \omega = \omega - i - j - k, \quad \beta_2 = \omega - i, \quad \beta_3 = \omega - j, \quad \beta_4 = \omega - k.$$

The following two results from Schmidt [30] will be useful later.

Corollary 3.2.7. *For any $\pi, \mu \in \mathcal{U}$, $S(\pi\overline{\mu}) = S(\pi) = S(\mu) = 1$ if and only if π and μ are a pair of units of the form $(\pi, \mu) = (\alpha_l, \beta_m)$ or $(\pi, \mu) = (\beta_m, \alpha_l)$, $l \neq m$, $1 \leq l, m \leq 4$, where $\alpha_l \in [\omega]$ and $\beta_m \in [\overline{\omega}]$.*

Proof. Using Table 9.1 we see that for any product $\alpha_l\overline{\beta_m}$, with $l \neq m$, $1 \leq l, m \leq 4$, $Re\left(\alpha_l\overline{\beta_m}\right) = \frac{1}{2}$ so $S\left(\alpha_l\overline{\beta_m}\right) = 1$. A similar proof for any product of the form $\beta_m\overline{\alpha_l}$. Conversely, by Corollary 3.2.6, $\pi\overline{\mu}, \pi, \mu$ are in $[\omega] \cup [\overline{\omega}]$ and hence satisfy the requirements. □

Example 3.2.8. Consider $\alpha_4 = \omega - i - j \in [\omega]$ and $\beta_2 = \omega - i \in [\overline{\omega}]$.
The product $\alpha_4\overline{\beta_2} = (\omega - i - j)(\overline{\omega - i}) = (\omega - i - j)(\omega - j - k) = \omega - j$, where $S(\omega - j) = 2Re(\omega - j) = 2\left(\frac{1}{2}\right) = 1$.

Theorem 3.2.9. *For any pair $\pi, \mu \in \mathcal{U}$ with $S(\pi\bar{\mu}) = S(\pi) = S(\mu) = 1$ we can find a unit $\epsilon \in \mathcal{H}$ such that $\epsilon^{-1}\pi\epsilon = \omega$ and $\epsilon^{-1}\mu\epsilon = \omega - k$ (or $\epsilon^{-1}\pi\epsilon = \omega - k$ and $\epsilon^{-1}\mu\epsilon = \omega$).*

Proof. Assume that $\pi \in [\omega]$. By Corollary 3.2.5 we can find $\epsilon_1 \in \mathcal{H}$ such that $\epsilon_1^{-1}\pi\epsilon_1 = \alpha_1 = \omega$. Then, by Corollary 3.2.7, $\epsilon_1^{-1}\mu\epsilon_1$ is equal to β_m with $m = 2, 3$ or 4 as given above. Take $\epsilon = \epsilon_1\omega$, $\epsilon_1(1 - \omega)$ or ϵ_1 according as $m = 2, 3$ or 4 respectively.
For $m = 2$: $\epsilon^{-1}\pi\epsilon = (\epsilon_1\omega)^{-1}\pi(\epsilon_1\omega) = \omega^{-1}\left[\epsilon_1^{-1}\pi\epsilon_1\right]\omega = \omega^{-1}[\omega]\,\omega = \omega$ and

$$\epsilon^{-1}\mu\epsilon = (\epsilon_1\omega)^{-1}\mu(\epsilon_1\omega) = \omega^{-1}\left[\epsilon_1^{-1}\mu\epsilon_1\right]\omega = \omega^{-1}[\beta_2]\,\omega = (1 - \omega)[\omega - i]\,\omega = \omega - k$$

For $m = 3$: $\epsilon^{-1}\pi\epsilon = (\epsilon_1(1 - \omega))^{-1}\pi(\epsilon_1(1 - \omega)) = (1 - \omega)^{-1}\left[\epsilon_1^{-1}\pi\epsilon_1\right](1 - \omega) = \omega\,[\omega]\,\omega^{-1} = \omega$

and

$$\begin{aligned}
\epsilon^{-1}\mu\epsilon &= (\epsilon_1(1-\omega))^{-1}\mu(\epsilon_1(1-\omega)) \\
&= (1-\omega)^{-1}\left[\epsilon_1^{-1}\mu\epsilon_1\right](1-\omega) \\
&= (1-\omega)^{-1}\left[\beta_3\right](1-\omega) \\
&= \omega\left[\omega - j\right](1-\omega) \\
&= \omega - k
\end{aligned}$$

For $m = 4$: $\epsilon^{-1}\pi\epsilon = \epsilon_1^{-1}\pi\epsilon_1 = \omega$ and

$$\epsilon^{-1}\mu\epsilon = \epsilon_1^{-1}\mu\epsilon_1 = \beta_4 = \omega - k$$

Thus for any $\pi = \alpha_l$, $l = 1, \cdots, 4$, we can find ϵ such that

$$\epsilon^{-1}\pi\epsilon = \omega, \epsilon^{-1}\mu\epsilon = \omega - k.$$

In a similar way if $\pi \in [\overline{\omega}]$ and $\pi = \beta_m$, $m = 1, \cdots, 4$, we find analogously a unit ϵ such that

$$\epsilon^{-1}\pi\epsilon = \omega - k, \epsilon^{-1}\mu\epsilon = \omega.$$

□

3.3 Greatest Common Divisors and the Division Algorithm in $\mathcal{H}$

We define left-hand (right-hand) divisors in $\mathcal{H}$ and subsequently show that $\mathcal{H}$ is a left (right) Euclidean ring. These results are taken from Stokesbary [32].

Definition 3.3.1. *If $a, b, c \in \mathcal{H}$ and $c = ab$ then we say a is the left hand divisor and b is the right hand divisor of c and write $a \rceil c$ and $b \lceil c$.*

To prove the division algorithm in $\mathcal{H}$ we need the following result.

Theorem 3.3.2. *Suppose $c \in \mathcal{H}$ and $m \in \mathbb{Z}$. Then there exists $\lambda \in \mathcal{H}$ such that $N(c-m\lambda) < m^2$ and $N(c-\lambda m) < m^2$.*

Proof. Let $m \in \mathbb{Z}$. Then $N(m) = m^2$. We need to establish that $N(c - m\lambda) < N(m)$.
Let $c = c_0 + c_1 i + c_2 j + c_3 k$ and $\lambda = l_0 + l_1 i + l_2 j + l_3 k$, then

$$c - m\lambda = (c_0 - ml_0) + (c_1 - ml_1)\, i + (c_2 - ml_2)\, j + (c_3 - ml_3)\, k.$$

Now $N(c - m\lambda) < m^2$ holds if $(c_0 - ml_0)^2 + (c_1 - ml_1)^2 + (c_2 - ml_2)^2 + (c_3 - ml_3)^2 < m^2$, so $(c_t - ml_t)^2 < \frac{m^2}{4}$ for $t = 0, \cdots, 3$. Equivalently $|c_t - ml_t| < \left|\frac{m}{2}\right| \Leftrightarrow \left|\frac{c_t}{m} - l_t\right| < \frac{1}{2} \Leftrightarrow \frac{c_t}{m} - \frac{1}{2} < l_t < \frac{c_t}{m} + \frac{1}{2}$. Since $\left[\frac{c_t}{m} + \frac{1}{2}\right] - \left[\frac{c_t}{m} - \frac{1}{2}\right] = 1$ we can find at least one λ in $\mathcal{H}$, such that all $l_t \in \mathbb{Z}$ or all $l_t + \frac{1}{2} \in \mathbb{Z}$, $0 \le t \le 3$. Finding each coordinate l_0, l_1, l_2, l_3 of λ in this manner ensures that $\lambda \in \mathcal{H}$ exists such that $N(c - \lambda m) < m^2$ holds with $m \in \mathbb{Z}$. The proof for $N(c - \lambda m) < m^2$ follows since $m\lambda = \lambda m$ for all $m \in \mathbb{Z}$, $\lambda \in \mathcal{H}$. □

Using the above theorem we can now prove the existence of a division algorithm for $\mathcal{H}$ [16], [32].

Theorem 3.3.3. *Suppose $a, b \in \mathcal{H}$, and $b \neq 0$, then there exists $\lambda, \gamma \in \mathcal{H}$ such that $0 \le N(a - \lambda b) < N(b)$ and $0 \le N(a - b\gamma) < N(b)$.*

Proof. Let $c = a\bar{b}$ and let $m = N(b) > 0$.
From Theorem 3.3.2 there exists $\lambda \in \mathcal{H}$ such that $N(c - m\lambda) < m^2$. Then

$$(a - \lambda b)\bar{b} = a\bar{b} - \lambda b\bar{b} = c - m\lambda,$$

and

$$N\left\{(a - \lambda b)\bar{b}\right\} = N\{a - \lambda b\}\, N\left\{\bar{b}\right\} = N\{c - m\lambda\} < m^2.$$

Since $N\left(\bar{b}\right) = m$, we have $N(a-\lambda b) < m = N(b)$. Note that if $N(a-\lambda b) = 0$ then $a-\lambda b = 0$. Similarly, if $c = \bar{b}a$ and $m = N(b)$, we see that $N\left\{\bar{b}(a - b\gamma)\right\} = N\left\{\bar{b}\right\} N\{a - b\gamma\} < m^2$, and the result follows. Again, if $N(a - b\gamma) = 0$ then $a - b\gamma = 0$. □

Definition 3.3.4. *Let $a, b \in \mathcal{H}$. Then a and b have a greatest common left-hand (right-hand) divisor d, denoted $gcd_l(a, b)$ ($gcd_r(a, b)$), if*

1. *d is a left-hand (right-hand) divisor of both a and b, and*

2. *every left-hand (right-hand) divisor of a and b is also a left-hand (right-hand) divisor of d.*

Note that if $gcd_l(a, b) = c$ then cu is also a $gcd_l(a, b)$ for u a unit, and where $cu \approx c$. Thus the greatest common divisors on the left and on the right are not unique.

Following Stokesbary [32] we show that a greatest common divisor of a and b, with $a, b \in \mathcal{H}$, can be written as a linear combination of a and b for both left and right divisors over $\mathcal{H}$.

Theorem 3.3.5. *Given $a, b \in \mathcal{H}$, where at least one of a and b is non-zero, then a and b have a greatest common left-hand (right-hand) divisor d, which is unique up to associates and can be written in the form*

$$d = am + bn \qquad (d = ma + nb),$$

where $m, n \in \mathcal{H}$

Proof. Let X denote the set of positive norms of linear combinations of a and b. That is,

$$X = \{N(am + bn) : N(am + bn) > 0, m, n \in \mathcal{H}\}.$$

Since $N(am + bn) > 0$ for any $m, n \in \mathcal{H}$, we conclude by Lemma 3.1.5 that $X \subseteq \mathbb{Z}^+$. Furthermore taking $m = \bar{a}$ and $n = \bar{b}$, $N(am + bn) = N(N(a) + N(b)) = N(a) + N(b) > 0$, so $X \neq \emptyset$. Since $\mathbb{Z}$ is well ordered, there exists a $g_0 = N(am_0 + bn_0)$, with $m_0, n_0 \in \mathcal{H}$, such that $g_o \leq g$ for all $g \in X$. Let $d = am_0 + bn_0$, and $g_0 = N(d)$. Then any common left-hand divisor of a and b is a left-hand divisor of d. We show that d is a greatest common left-hand divisor of a and b. By the division algorithm on $\mathcal{H}$, for any $a, d \in \mathcal{H}$, $d \neq 0$ there exists $q \in \mathcal{H}$ such that $0 \leq N(a - dq) < N(d)$.

Let $r = a - dq = a - (am_0 + bn_0)q = a(1 - m_0q) - b(n_0q)$, so $N(r) = N(a - dq) \in X$. This is a contradiction, unless $r = 0$ and $d \rceil a$. Similarly $d \rceil b$. Thus $d = gcd_l(a, b)$.

The proof of the case for the right hand divisor is similar. $\square$

Corollary 3.3.6. *If* $a = bq + r$ *then* $gcd_l(a, b) \approx gcd_l(b, r)$ *for* $a, b, q, r \in \mathcal{H}$. *Similarly if* $a = qb + r$*, then* $gcd_r(a, b) \approx gcd_r(b, r)$.

Proof. Write $d = gcd_l(a, b)$ and $m = gcd_l(b, r)$. Then $m \rceil b$ and $m \rceil r$ and so m also divides $a = bq + r$ on the left. Thus m is a common left divisor of a and b so $m \rceil d$. Similarly, using $r = -bq + a$ we can show that $d \rceil m$, so $d \approx m$. The proof for the common right hand divisor is similar. □

Theorem 3.3.7. $gcd_r(a, c) = u$*,* u *a unit, if and only if we can find* $s, t \in \mathcal{H}$ *such that* $sa + tc = 1$. *(For the left-hand case:* $gcd_l(a, b) = u$*,* u *a unit, if and only if we can find* $s, t \in \mathcal{H}$ *such that* $as + bt = 1$.*)*

Proof. If $gcd_r(a, c) = u$, $u \in \mathcal{H}$ a unit, then we can find $m, n \in \mathcal{H}$ such that $ma + nc = u$ by Theorem 3.3.5. Since u is a unit, for $s = u^{-1}m$ and $t = u^{-1}n$, $sa + tc = u^{-1}ma + u^{-1}nc = 1$ as required.

Conversely let $a, c \in \mathcal{H}$ and assume $sa + tc = 1$ for some $s, t \in \mathcal{H}$. Let $gcd_r(a, c) = p$ then $a = r_1 p$ and $c = r_2 p$. Thus $sa + tc = sr_1 p + tr_2 p = (sr_1 + tr_2)p = 1$, so p is a right divisor of 1 and so p must be a unit.

The proof for the left hand divisor case is similar. □

Corollary 3.3.8. $\mathcal{H}$ *forms a left (right) principal ideal domain.*

Proof. Let I be a left ideal of $\mathcal{H}$. That is, for each $x, y \in I$ and $r \in \mathcal{H}$ we have that $x + y \in I$ and $rx \in I$. If $I = \{0\}$ then $I = \langle 0 \rangle$ and we have our result. If $I \neq \{0\}$, then there is $x \in I$ such that $x \neq 0$. Choose $d \neq 0$ in I to be an element in I of smallest positive norm $N(d)$. Then certainly $rd \in I$ for all $r \in \mathcal{H}$. Hence the ideal generated by d is contained in I, $\langle d \rangle \subseteq I$.

Consider any $x \in I$. By the left Euclidean division algorithm, since $d \neq 0$ we can find $q, r \in \mathcal{H}$ such that $x = qd + r$, $0 \leq N(r) < N(d)$. Thus $r = x - qd$ and $r \in I$. By choice of d, we must have $N(r) = 0$, so $r = 0$. Thus $x = qd$ and I is the left ideal generated by d. □

We note that the Lipschitz integers $\{a_0 + a_1 i + a_2 j + a_3 k : a_i \in \mathbb{Z}\}$, that seem more intuitive than the Hurwitz integers, do not form a left or right Euclidean ring in the quaternions, as the division algorithm cannot be established for this set.

Definition 3.3.9. *ac^{-1} is a reduced quaternion rational if and only if $gcd_r(a, c) = u$, u a unit in $\mathcal{H}$.*

Example 3.3.10. $(1 - i)^{-1}$ is the reduced form of $\dfrac{1+i}{2}$.
We assume that $\infty = 1(0)^{-1}$ is reduced and 0 is reduced as $0(1)^{-1}$.

3.4 The Convex Hull $\mathcal{P}_{\mathcal{U}}$

Let $\mathcal{P}_{\mathcal{U}}$ denote the convex hull of the points in set $\mathcal{U}$, the 24 units in $\mathcal{H}$ [30]. $\mathcal{P}_{\mathcal{U}}$ is a *24-cell* that exists in 4-dimensions, and it is unique to 4-dimensional space as it has no analogues in higher or lower dimensions. $\mathcal{P}_{\mathcal{U}}$ is a regular polytope of Schläfli symbol $(3, 4, 3)$. The Schläfli symbol $(3, 4)$ is an octahedron with 4 triangles (3-sided figures) at each vertex. The Schläfli symbol $(3, 4, 3) = ((3, 4), 3)$ establishes that there are 3 octahedrons ($(3, 4)$ polytopes) around each edge. Thus the boundary of $\mathcal{P}_{\mathcal{U}}$ consists of 24 vertices, 96 edges of unit length, 96 regular triangles and 24 regular octahedra. We represent the 24 units as points in $\mathbb{R}^4$ with coordinates $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$, $(0, 0, 0, \pm 1)$ and $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. One of the octahedral faces of $\mathcal{P}_{\mathcal{U}}$ has vertices $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, $\omega = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\omega - k = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, $\omega - j - k = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ and $\omega - j = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$, as shown in Figure 3.1. This octahedral face lies in the 3-space or hyperplane in $\mathbb{R}^4$ generated by the orthogonal vectors $(1, -1, 0, 0)$, $(0, 0, 1, 1)$ and $(0, 0, 1, -1)$, and is orthogonal to the vector $(1, 1, 0, 0)$ in $\mathbb{R}^4$. We note that all units lie on a hypersphere centre 0 radius 1, but $\dfrac{1+i}{2}$ lies on a hypersphere centre 0 radius $\frac{1}{\sqrt{2}}$ in $\mathbb{R}^4$.

We let $I = \{z \in \mathcal{H} : N(z) \equiv 0(mod 2)\}$. That is, I is the set of Hurwitz integers with even norms [30]. The units do not belong to I.

Theorem 3.4.1. *I is the two sided ideal of $\mathcal{H}$ generated by $1 + i$.*

Proof. First we show that I is a subring of $\mathcal{H}$. If $a, b \in I$ then $N(a) = 2k_1, N(b) = 2k_2$, for $k_1, k_2 \in \mathbb{Z}$. So $N(a \pm b) = N(a) + N(b) \pm S\left(a\overline{b}\right) = 2k_1 + 2k_2 \pm 2Re\left(a\overline{b}\right)$ and $N(ab) =$

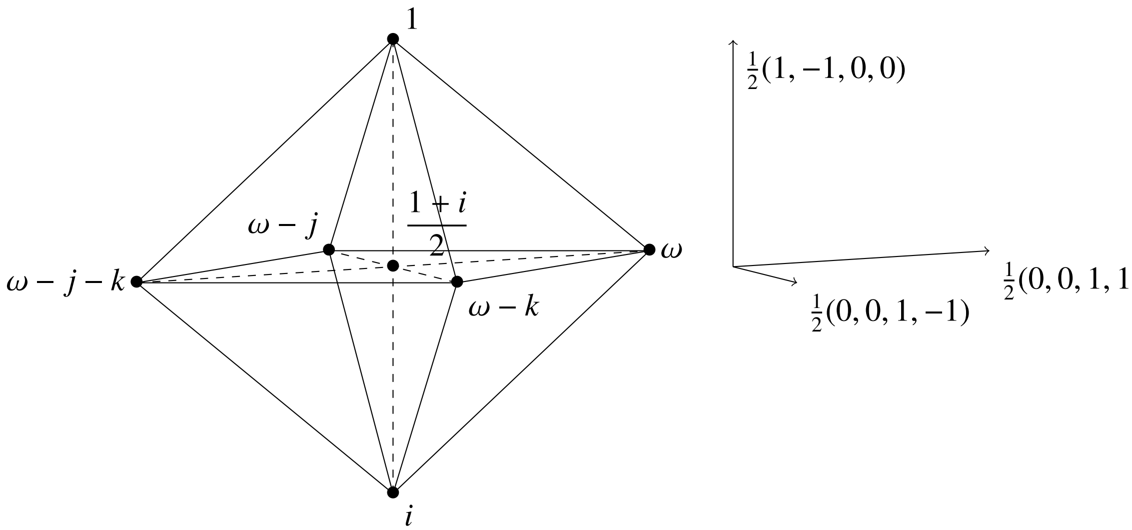

Figure 3.1: The octahedral face of $\mathcal{P}_{\mathcal{U}}$ centered at $\frac{1+i}{2}$.

$N(a)N(b)$. Thus $a \pm b \in I$ and $ab \in I$. Certainly $0 \in I$ and so I is a subring of $\mathcal{H}$.
Let $z \in \mathcal{H}$ and $a \in I$ then $N(za) = N(z)N(a) = N(a)N(z) = 2k_1N(z) = N(az)$ so $za, az \in I$ and I is a left and right ideal of $\mathcal{H}$. Certainly $1 + i \in I$ and so $\langle 1 + i \rangle \subseteq I$.
We show $\langle 1+i \rangle = I$. Let $z \in I$. Since $\mathcal{H}$ is a right (left) Euclidean ring we can find $q, r \in \mathcal{H}$ such that $z = (1 + i)q + r$ and $N(r) < N(1 + i) = 2$ or $r = 0$. Thus $r = z - (1 + i)q$ and

$$\begin{aligned} N(r) &= N(z) + N(1+i)N(q) - S\left(z\left(\overline{1+i}\right)q\right) \\ &= 2k + 2N(q) - 2Re\left(z\left(\overline{1+i}\right)q\right) \text{ for } k \in \mathbb{Z} \end{aligned}$$

So $N(r) \neq 1$ and $r = 0$. Therefore $z = (1 + i)q$ and $I = \langle 1 + i \rangle$. □

The following example shows that if $(1 + i)$ is a right multiple of a quaternion it is also a left multiple of the same quaternion.

Example 3.4.2. If $1 + i$ is a left divisor of $h \in \mathcal{H}$, then it is also a right divisor of $h \in \mathcal{H}$. That is $(a-b)+(a+b)i+(c-d)j+(c+d)k = (1+i)(a+bi+cj+dk) = (a+bi-dj+ck)(1+i)$ [12].

Since I is a two sided ideal in $\mathcal{H}$ we know that $\mathcal{H}/I$ is a ring and $\mathcal{H} = \bigcup_{z \in T} (z + I)$, where T is a transversal consisting of exactly one element from each distinct coset modulo I. Choose

$T = \{0, 1\}$, then $\mathcal{H} = I \cup (1 + I)$.

Theorem 3.4.3. *$\mathbb{K}$ is tessellated by $\{\mathcal{P}_{\mathcal{U}} + z : z \in I\}$.*

Proof. We note that $z_1 + I = z_2 + I$ if and only if $z_1 - z_2 \in I$ or equivalently $N(z_1 - z_2)$ is even. Since $N(z_1 - z_2) = N(z_1) + N(z_2) - S(z_1\overline{z_2}) = N(z_1) + N(z_2) - 2Re(z_1\overline{z_2})$ (Proposition 2.3.7(3)) we see that $z_1 + I = z_2 + I$ if and only if the norms of z_1 and z_2 have the same parity. That is $N(z_1)$ and $N(z_2)$ are both congruent to 0 or both congruent to 1 modulo 2.

All the norms of the units u in $\mathcal{U}$ are of odd parity. But $N(u - 1) = N(u) + N(-1) + S\left(u\left(\overline{-1}\right)\right) = 1 + 1 + 2Re(u) \equiv 0 \pmod 2$, so $u - 1 \in I$. By the geometry of $\mathcal{P}_{\mathcal{U}}$, for each $c \in \mathcal{P}_{\mathcal{U}}$ we can find a unit u such that $c + u \in \mathcal{P}_{\mathcal{U}}$ and $u + I = 1 + I$ as $u - 1 \in I$ and $N(u - 1)$ is even.

Let $z \in \mathbb{K}$ be any quaternion. Since $\mathcal{H}$ is a "centered cube" lattice in $\mathbb{K}$ we can find $h \in \mathcal{H}$ nearest to z. Let $z' = z - h$. Then $|z'| = |z - h| \leq |z - h_1|$ for all $h_1 \in \mathcal{H}$.

Thus $z' \in \mathcal{P}_{\mathcal{U}}$ where $z' = z - h$ and $h \in \mathcal{H}$. Thus $z = z' + h$ and $z \in \{\mathcal{P}_{\mathcal{U}} + h : h \in \mathcal{H}\}$. But $\mathcal{H} = I \cup \{1 + I\}$ and so $z \in \{\mathcal{P}_{\mathcal{U}} + h : h \in I\}$ or $z \in \{\mathcal{P}_{\mathcal{U}} + h : h \in 1 + I\}$. Equivalently, $z \in \{\mathcal{P}_{\mathcal{U}} + h : N(h) \equiv 0 \pmod 2\}$ or $z \in \{\mathcal{P}_{\mathcal{U}} + h : N(h) \equiv 1 \pmod 2\}$.

If $h \in I$ we are done.

If $h \in 1 + I$ then since $z' \in \mathcal{P}_{\mathcal{U}}$ we can find u_0 in $\mathcal{U}$ such that $z'' = z' + u_0 \in \mathcal{P}_{\mathcal{U}}$. So $z'' = z - h + u_0 = z - (h - u_o)$ and $N(h - u_0) = N(h) + N(u_0) + 2Re(h\overline{u_o})$ is even. So $\{\mathcal{P}_{\mathcal{U}} + z : z \in I\}$ constitutes a regular tessellation of $\mathbb{K}$. □

3.5 The Dirichlet Region ∇ in $\mathbb{K}$

The set of all Hurwitz integer quaternions forms a lattice L, in the space $\mathbb{K}$, which is generated by the four points i, j, k and ω, where $\omega = \dfrac{1 + i + j + k}{2}$. Lattice L is the four-dimensional analogue to the centered cube lattice in three dimensional space. Two points x and y in $\mathbb{K}$ are called congruent if their difference lies in L.

For any point in the lattice L we can find a set of points in $\mathbb{K}$ that are closer to this point than to any other lattice points. This set of points is known as the Dirichlet region for the point in the lattice.

In particular, let $T = \langle \tau_\lambda \rangle = \{x \mapsto x + \lambda : \lambda \in \mathcal{H}\}$. Then T acts on $\mathbb{K}$ by translating each point x in $\mathbb{K}$ through $\lambda \in \mathcal{H}$. Since $\mathbb{K} \cong \mathbb{R}^4$ with Euclidean metric, we can define a Dirichlet region for the discrete group T acting on $\mathbb{K}$.

Definition 3.5.1. *Let $g \in T$ and $g \neq 1$, and let 0 be the origin in $\mathbb{K}$, not fixed by any $g \neq 1 \in T$. Then $\nabla = \{x \in \mathbb{K} : |x| \leq |x - g(0)|, g \neq 1 \in T\}$. This is the Dirichlet region for the point 0 in the lattice L with respect to T [24].*

We see that ∇ lies inside of $\mathcal{P}_{\mathcal{U}}$. The nature of ∇ is made explicit through the following results from [23]:

Lemma 3.5.2. *Suppose that the m linear inequalities with real coefficients*

$$l_r(x) = a_{r1}x_1 + a_{r2}x_2 + a_{r3}x_3 + a_{r4}x_4 + a_r \geq 0, \quad r = 1, 2, \cdots, m$$

define a bounded set S of points $x = (x_1, x_2, x_3, x_4)$ in 4-dimensional Euclidean space. Then the maximum of

$$f(x) = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{\frac{1}{2}},$$

the distance from x to the origin, is assumed only in points of S in which at least 4 of the functions are zero simultaneously.

Proof. Since $f(x)$ is continuous and bounded, there is at least one point $x' = \left(x_1', x_2', x_3', x_4'\right)$ of S in which $f(x')$ takes its largest value. If less than 4 of the numbers l_r are zero, then we can find a positive number ϵ and 4 numbers x_1'', x_2'', x_3'' and x_4'', not all zero, such that all points $x = \left(x_1' + tx_1'', x_2' + tx_2'', x_3' + tx_3'', x_4' + tx_4''\right)$ with $-\epsilon \leq t \leq \epsilon$ belong to S. For these points $(f(x))^2 = \alpha t^2 + \beta t + \gamma = \phi(t)$ with real coefficients α, β, γ, of which $\frac{1}{2}\phi''(0) = \alpha = (f(x''))^2 > 0$. Hence this function is not maximum for $t = 0$, contrary to the hypothesis. □

Following [23] we use Lemma 3.5.2 to prove Theorem 3.5.3.

Theorem 3.5.3. *Let $x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{K}$ then*

1. *∇ is defined by the linear inequalities*

$$|x_0| \leq \frac{1}{2}, \quad |x_1| \leq \frac{1}{2}, \quad |x_2| \leq \frac{1}{2}, \quad |x_3| \leq \frac{1}{2}, \quad |x_0| + |x_1| + |x_2| + |x_3| \leq 1 \tag{3.1}$$

2. *For all points of ∇,*

$$|x| \leq \frac{1}{\sqrt{2}}, \tag{3.2}$$

with equality if and only if x is one of the 24 points

$$\frac{\pm 1 \pm i}{2}, \quad \frac{\pm 1 \pm j}{2}, \quad \frac{\pm 1 \pm k}{2}, \quad \frac{\pm i \pm j}{2}, \quad \frac{\pm i \pm k}{2}, \quad \frac{\pm j \pm k}{2}. \tag{3.3}$$

3. *To every quaternion there is a congruent quaternion in ∇.*

Proof. 1. If x satisfies the inequalities in equation (3.1) then it satisfies the inequalities in equation (3.2).

The formulas in equation (3.1) can be written as $\pm x_r + \frac{1}{2} \geq 0$ for $r = 0, 1, 2, 3$ and $\pm x_0 \pm x_1 \pm x_2 \pm x_3 + 1 \geq 0$. Hence by Lemma 3.5.2, the maximum of x is attained in a point in which at least 4 of these inequalities hold in the stronger form with the sign of equality. This is clearly only possible in the points given in equation (3.3), and these are all of absolute value $\frac{1}{\sqrt{2}}$.

2. The set in equation (3.1) is identical with ∇.

If $a = h_0 + h_1 i + h_2 j + h_3 \omega$ is an integral quaternion, then the inequality $|x| \leq |x - a|$ can be written as $\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 \leq \Gamma$, where

$$\gamma_0 = \frac{2h_0 + h_3}{2|a|}, \quad \gamma_1 = \frac{2h_1 + h_3}{2|a|}, \quad \gamma_2 = \frac{2h_2 + h_3}{2|a|}, \quad \gamma_3 = \frac{h_3}{2|a|}, \quad \Gamma = \frac{|a|}{2},$$

and therefore $\gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$. Hence the distance Γ of the hyperplane $\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 = \Gamma$ from the origin is $\frac{1}{2}$ if a is one of the 24 units $\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)$ of $\mathcal{H}$, and it is greater than or equal to $\dfrac{1}{\sqrt{2}}$, if $a \neq 0$ is any other integral quaternion.

3. If x is an arbitrary quaternion, there is an integral quaternion p such that $|x - p| \leq |x - p - a|$ for all integral quaternions a. Hence $x' = x - p \equiv x$ satisfies (1) and therefore lies in ∇.

□

For any quaternion a, let $\nabla(a)$ be the set of all points x for which $x - a$ belongs to ∇. In particular $\nabla(0) = \nabla$. $\nabla(a)$ can also be defined as the set of all points whose distance from a is not larger than that from any other point congruent to a. If a runs over all elements of L, then the sets $\nabla(a)$ together just tessellate the space $\mathbb{K}$ without interiors overlapping.

Definition 3.5.4. *A closed region $F \subset X$, for X any metric space, is defined to be a fundamental region for group H [18] if*

(a) $\bigcup_{g \in H} g(F) = X$

(b) $\overset{\circ}{F} \cap g\left(\overset{\circ}{F}\right) = \emptyset$ *for all $g \neq 1_{map}$ in H, where $\overset{\circ}{F}$ is the interior of F.*

The family $\{g(F) : g \in H\}$ tessellates X. We note that $\nabla \subset \mathbb{K}$ is a fundamental region for the group $T = \langle \tau_\lambda \rangle$ acting in $\mathbb{K}$.

$\mathcal{P}_{\mathcal{U}}$ and ∇ are both 24-cells. They are convex hulls of 24 vertices with 24 regular octahedral faces. The centre of each octahedral face of ∇ is one of the vertices of $\mathcal{P}_{\mathcal{U}}$, so ∇ and $\mathcal{P}_{\mathcal{U}}$ are dual of each other. The 24-cell is considered to be self-dual as interchanging octahedral faces with vertices and vertices with octahedral faces results in another 24-cell. The vertices of $\mathcal{P}_{\mathcal{U}}$ lie on the sphere of radius 1 centered at 0, while the vertices ot ∇ lie on the sphere of radius $\frac{1}{\sqrt{2}}$ centered at 0 [11].

3.6 Singular Vertices and ∇^*

The 24 points listed in equation 3.3 on the boundary of ∇ are called singular vertices, or briefly, the set SV [23]. If ϵ_0 denotes the special vertex $\frac{1+i}{2}$ then all SV of ∇ given by equation 3.3 can be written as $\epsilon = u\epsilon_0$, where u is a unit in $\mathcal{H}$. The 24 transformations of $\mathbb{K}$, $x \mapsto ux$, leaves the set SV invariant. These transformations leave all distances and the lattice L invariant, and therefore also the set ∇.

Definition 3.6.1. *Let* ∇^* *be the subset of* ∇ *whose points satisfy the 24 inequalities* $|x-\epsilon_0| \leq |x-\epsilon|$ *for all* $\epsilon \in SV$.

Mahler [23] shows that ∇^* is determined by the points $x = x_0 + x_1 i + x_2 j + x_3 k$ in $\mathbb{K}$ that satisfy the inequalities

$$0 \leq x_0 \leq \frac{1}{2}, \quad 0 \leq x_1 \leq \frac{1}{2}, \quad \max(|x_2|, |x_3|) \leq \min(x_0, x_1), \quad x_0 + x_1 + |x_2| + |x_3| \leq 1$$

Theorem 3.6.2. *To every* $x \in \nabla$ *there is a unit* $u \in \mathcal{H}$ *such that* $u^{-1}x$ *lies in* ∇^*.

Proof. If x lies in ∇, then let $u\epsilon_0$ be the SV closest to it. Then ϵ_0 is the SV nearest to $u^{-1}x$ and therefore $u^{-1}x$ lies in ∇^*. □

For any quaternion x we can find a $\lambda \in \mathcal{H}$ such that $x - \lambda \in \nabla$ and we can find $u \in \mathcal{U}$ such that $u^{-1}(x-\lambda) \in \nabla^*$, where $u \in \mathcal{U}$.

Chapter 4

Clifford Matrices and $M\left(\mathbb{R}^4_\infty\right)$

4.1 Matrices in $M_2(C_n)$

Consider the ring of matrices $M_2(C_n) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in C_n\right\}$, the set of 2×2 matrices with entries in C_n. Let A and $B \in M_2(C_n)$. We define $\sim$ on $M_2(C_n)$ by the following condition: $A \sim B$ if and only if $B = \lambda A$ for $\lambda \in \mathbb{R} \setminus \{0\}$. We note that $\sim$ is an equivalence relation on $M_2(C_n)$ and each element in $M_2(C_n)$ is contained in an equivalence class $\widetilde{A} = \{\lambda A : \lambda \in \mathbb{R} \setminus \{0\}\}$. Let $\widetilde{M_2}(C_n)$ be the set of equivalence classes of matrices. On $\widetilde{M_2}(C_n)$ we have a product $\widetilde{A}\,\widetilde{B} = (\widetilde{AB})$. We will consider $\widetilde{\mathcal{I}_2} = \left\{\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{R} \setminus \{0\}\right\}$ and see that $\widetilde{\mathcal{I}_2}$ is the multiplicative identity in $\widetilde{M_2}(C_n)$, where $\widetilde{\mathcal{I}_2}\widetilde{A} = \widetilde{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \widetilde{\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}} = \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\widetilde{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}} = \widetilde{A}\,\widetilde{\mathcal{I}_2} = \widetilde{A}$ for all $\widetilde{A} \in \widetilde{M_2}(C_n)$ and $\lambda \in \mathbb{R} \setminus \{0\}$. A matrix $\widetilde{A}$ in $\widetilde{M_2}(C_n)$ is invertible if we can find $\widetilde{B}$ in $\widetilde{M_2}(C_n)$ such that $\widetilde{A}\,\widetilde{B} = \widetilde{B}\widetilde{A} = \widetilde{\mathcal{I}_2}$. We may represent the equivalence class $\widetilde{A}$ by A, when the understanding is clear.

In what follows, we first consider the subrings of $M_2(C_n)$ given by $GL(2, C_n)$ and $GL(2, \Gamma_n)$ and a pseudo-determinant Δ. These are the rings of Clifford Matrices. Following this we consider the special subring $M_2(\mathbb{K})$ or $M_2(C_3)$ and a determinant $\mathcal{D}$. The first development is informed by [1] and [37], while the latter is informed by [19] and [38]. We will reconcile these two independent approaches relating Δ to $\mathcal{D}$. We see that $M\left(\mathbb{R}^4_\infty\right)$, the group of orientation preserving Möbius maps acting on $\mathbb{R}^4_\infty$, corresponds to both $PSL(2, \Gamma_4)$ and $PS_{\mathcal{D}}L(2, \mathbb{K})$.

4.2 Matrices in $GL(2, C_n)$ and Fractional Linear Maps

In this section we consider a subring of $M_2(C_n)$ called the Clifford Matrices. We will consider V^n, the n-dimensional linear space of vectors in C_n, with basis $\{1, i_1, \cdots, i_{n-1}\}$. Following [37] we have:

Definition 4.2.1. *Let* $GL(2, C_n) =$
$$\left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in C_n, A \text{ induces the bijection } x \mapsto (ax+b)(cx+d)^{-1} \text{ on } V^{n+1}_\infty\right\}$$

We note that if $a = b = c = d = 0$ then the action $x \mapsto (ax+b)(cx+d)^{-1}$ is not well defined, so A is not a zero matrix.

Lemma 4.2.2. *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, C_n)$ *induces the* 1_{map} *on* V^{n+1}_∞, *then* $A = \lambda \mathcal{I}_2$, $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, C_n)$ induce the 1_{map} then $g(x) = (ax+b)(cx+d)^{-1} = x$, so $ax + b = x(cx + d)$ for all $x \in V^{n+1}_\infty$. Thus

- If $x = 0$ then $g(0) = 0$ and $b = 0$
- If $x = \infty$ then $g(\infty) = \infty$ and $c = 0$, and

- If $x = 1$ then $g(1) = 1$ and $a = d$.

Since A is a non-zero matrix we know $a \neq 0$. Thus $g(x) = axa^{-1} = x$. That is $ax = xa$ for any $x \in V^{n+1}_\infty$. So $ax = xa$ for $x = i_t$, where $t = 1, \cdots, n-1, n$. Further if $a = \Sigma a_I I$ we must have $Ix = xI$ for each product I in the sum. Hence $ai_n = i_n a$ and $ai_t = i_t a$, where $t = 1, \cdots, n-1$.
Since $a = \Sigma a_I I \in C_n$ with $I = i_{v_1} \cdots i_{v_p}$, where $1 \leq v_s \leq n-1$ for all $s = 1, \cdots, p$, $Ii_n = (-1)^p i_n I = i_n I$ for p even only. Consider i_r, $r = v_1$. $Ii_r = i_r(-1)^{p-1} I = -i_r I \neq i_r I$ for p even. Hence we have a contradiction. Thus $a \in \mathbb{R} \setminus \{0\}$. □

From [37] we have the following theorem.

Theorem 4.2.3. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, C_n)$ *then* $A^* = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ *induces the inverse bijection of* A *on* V^{n+1}_∞ *and* $bd^{-1}, ac^{-1}, a^{-1}b$ *and* $c^{-1}d$ *are vectors in* C_n. *Furthermore* $ad^* - bc^* \in \mathbb{R} \setminus \{0\}$.

Proof. Let $g : V^{n+1}_\infty \to V^{n+1}_\infty$ be the bijection given by $g(x) = (ax+b)(cx+d)^{-1}$ corresponding to the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We note the following equivalences:

$$\begin{aligned} y = g(x) \quad &\Leftrightarrow \quad y = (ax+b)(cx+d)^{-1} \\ &\Leftrightarrow \quad y(cx+d) = ax+b \\ &\Leftrightarrow \quad ycx - ax = -yd + b \\ &\Leftrightarrow \quad (a - yc)x = yd - b. \end{aligned}$$

If $x, y \in V^{n+1}_\infty$ they are vectors of C_{n+1} and so $x = x^*$ and $y = y^*$. Thus

$$\begin{aligned} y = g(x) \quad &\Leftrightarrow \quad \{(a - yc)x\}^* = \{yd - b\}^* \\ &\Leftrightarrow \quad x(a^* - c^* y) = d^* y - b^* \\ &\Leftrightarrow \quad x = (d^* y - b^*)(-c^* y + a^*)^{-1}. \end{aligned}$$

Thus the matrix $A^* = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ induces the inverse of g. Further $A^*A = AA^* = \lambda I_2$, $\lambda \in \mathbb{R} \setminus \{0\}$.

Thus $\begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $d^*b - b^*d = 0$. It follows that $b^*d = d^*b = (b^*d)^*$. Since $b^*d \in C_n$ we know b^*d is a vector in C_n and hence is in V^n_∞. By Lemma 2.1.8 we know that bd^{-1} is also a vector in V^n_∞. Similarly we can show that $a^*c \in V^n_\infty$ and thus $ac^{-1} \in V^n_\infty$. Considering $AA^* = \lambda I_2$ we see that $a^{-1}b$ and $c^{-1}d$ are also vectors in V^n_∞. Finally, from the product AA^*, we see that $ad^* - bc^* = \lambda \in \mathbb{R} \setminus \{0\}$. Furthermore, from the product A^*A, $da^* - b^*c = \lambda$ and so $\lambda = ad^* - bc^* = d^*a - b^*c = da^* - cb^* = a^*d - c^*b$. □

We establish that the action of composition of bijections corresponds to the multiplication of associated matrices in $GL(2, C_n)$.

Lemma 4.2.4. *Multiplication of matrices A and $A_1 \in GL(2, C_n)$ induces the composition of maps f and f_1 associated with A and A_1 respectively.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ be in $GL(2, C_n)$, with corresponding maps

$$f(x) = (ax + b)(cx + d)^{-1} \text{ and } f_1(x) = (a_1x + b_1)(c_1x + d_1)^{-1}$$

acting on V^{n+1}_∞. The composition of the maps yields the map

$$\begin{aligned} f_1(f(x)) &= \left[a_1(ax + b)(cx + d)^{-1} + b_1\right]\left[c_1(ax + b)(cx + d)^{-1} + d_1\right]^{-1} \\ &= [a_1(ax + b) + b_1(cx + d)]\,(cx + d)^{-1}(cx + d)\,[c_1(ax + b) + d_1(cx + d)]^{-1} \\ &= [(a_1a + b_1c)x + (a_1b + b_1d)]\,[(c_1a + d_1c)x + (c_1b + d_1d)]^{-1}. \end{aligned}$$

This map is associated with the matrix product given by

$$\begin{pmatrix} a_1a + b_1c & a_1b + b_1d \\ c_1a + d_1c & c_1b + d_1d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_1A.$$

□

It follows that $GL(2, C_n)$ is a group under multiplication and the associated bijections acting on V^{n+1}_∞ form the group $PGL(2, C_n)$ under composition, where

$$PGL(2, C_n) \cong GL(2, C_n)/\{\pm\lambda I_2\}, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

4.3 The Pseudo-Determinant Δ and $M\left(\mathbb{R}^n_\infty\right)$

We define the pseudo-determinant Δ for matrices in $M_2(C_n)$ in accordance with [1] and [37] as follows.

Definition 4.3.1. *For any matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(C_n)$ *the pseudo-determinant of* A *is* $\Delta A = ad^* - bc^*$.

We note from Theorem 4.2.3 that if $A \in GL(2, C_n)$ then $\Delta A = ad^* - bc^* \in \mathbb{R} \setminus \{0\}$. Further we note that Δ does not act as a determinant on $M_2(C_n)$. Yet on $GL(2, C_n)$ it satisfies the multiplicative property of determinants ($\Delta AB = \Delta A \Delta B$) since in this case Δ is a non-zero real number.

From [2] we show that each map in $PGL(2, C_n)$ has a formal decomposition.

Lemma 4.3.2. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, C_n)$ *be a matrix with associated map* $g : V^{n+1}_\infty \to V^{n+1}_\infty$ *given by* $g : x \to (ax + b)(cx + d)^{-1}$*, then*

1. $g(x) = ac^{-1} - \Delta c^{*-1}(x + c^{-1}d)^{-1}c^{-1}$, $c \neq 0$

2. $g(x) = \left(\frac{1}{\Delta}\right) axa^* + bd^{-1}$, $c = 0$ *and* $\Delta \neq 0$.

Proof. 1. For $c \neq 0$, we observe that since $\Delta = ad^* - bc^*$ and $c^{-1}d = \left(c^{-1}d\right)^* = d^*c^{*-1} \in$

V^n we may write $b = ac^{-1}d - \Delta c^{*-1}$.

$$\begin{aligned} g(x) &= (ax+b)(cx+d)^{-1} \\ &= (ac^{-1}cx + ac^{-1}d + b - ac^{-1}d)(cx+d)^{-1} \\ &= (ac^{-1}(cx+d) + (b - ac^{-1}d))(cx+d)^{-1} \\ &= ac^{-1} + (b - ac^{-1}d)(cx+d)^{-1} \\ &= ac^{-1} + (ac^{-1}d - \Delta(c^*)^{-1} - ac^{-1}d)(cx+d)^{-1} \\ &= ac^{-1} - \Delta(c^*)^{-1}(cx+d)^{-1} \\ &= ac^{-1} - \Delta c^{*-1}(x + c^{-1}d)^{-1}c^{-1} \end{aligned}$$

2. For $c = 0$ we have $\Delta = ad^* - bc^* = ad^* = da^*$ since Δ is real. Thus $1 = d\left(\frac{1}{\Delta}a^*\right)$ and $d^{-1} = \frac{1}{\Delta}a^*$. So $g(x) = (ax+b)d^{-1} = \left(\frac{1}{\Delta}\right)axa^* + bd^{-1}$.

□

Lemma 4.3.3. *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathcal{C}_n)$*, then* $a, b, c, d \in \Gamma_n \cup \{0\}$.

Proof. From Lemma 4.3.2 we know that if $c \neq 0$ we can write

$$g(x) = ac^{-1} - \Delta c^{*-1}\left(x + c^{-1}d\right)^{-1} c^{-1} = \left(c^{-1}\right)^* \left\{c^*a - \Delta\left(x - c^{-1}d\right)^{-1}\right\} c^{-1}$$

where g is bijective and $c^*g(x)c = \left\{c^*a - \Delta\left(x - c^{-1}d\right)^{-1}\right\}$. If $x \in V^{n+1}$ then

$$\left\{c^*a - \Delta\left(x + c^{-1}d\right)^{-1}\right\}^* = \{c^*a\}^* - \Delta\left\{\left(x + c^{-1}d\right)^*\right\}^{-1} = c^*a - \Delta\left(x + c^{-1}d\right)^{-1},$$

since $x^* = x$ and $c^*a, c^{-1}d \in V^n$ (Lemma 4.2.3).

Thus $c^*a - \Delta\left(x + c^{-1}d\right)^{-1}$ is a vector in $V^{n+1}$and hence c and c^* are in Γ_n by Theorem 2.2.4(3).

Since $a^{-1}c \in V^n$ we have $a = ac^{-1}c \in \Gamma \cup \{0\}$. Similarly $c\left(c^{-1}d\right) = d \in \Gamma \cup \{0\}$ and if $d \neq 0$, $\left(bd^{-1}\right)d = b \in \Gamma \cup \{0\}$, since $bd^{-1} \in V^n$.

If $c = 0$, then $\Delta = ad^* \neq 0$ so $a \neq 0$. By Lemma 4.3.2

$$g(x) = \left(\frac{1}{\Delta}\right) axa^* + bd^{-1} = a\left\{\frac{1}{\Delta}x + a^{-1}bd^{-1}(a^*)^{-1}\right\}a^*$$

so $a^{-1}g(x)a^{*-1} = \dfrac{1}{\Delta}x + a^{-1}bd^{-1}\,(a^*)^{-1}$ and $bd^{-1} \in V^n$. If $x \in V^{n+1}$ then

$$\left\{\frac{1}{\Delta}x + a^{-1}bd^{-1}(a^*)^{-1}\right\}^* = \left\{\frac{1}{\Delta}x + a^{-1}\left(bd^{-1}\right)^*(a^*)^{-1}\right\} = \frac{1}{\Delta}x + a^{-1}bd^{-1}(a^*)^{-1}$$

so $\dfrac{1}{\Delta}x+a^{-1}bd^{-1}(a^*)^{-1} \in V^{n+1}$ and $a \in \Gamma_n$ (Theorem 2.2.4(3)). Again $a^{-1}b \in \mathbb{R}^n$ so $a\left(a^{-1}b\right) = b \in \Gamma_n$ and $b^{-1}\left(bd^{-1}\right) = d^{-1} \in \Gamma_n$ if $b \neq 0$.

Thus $a, b, c, d \in \Gamma_n \cup \{0\}$.

□

The following theorem is immediate from Theorem 4.2.3 since $\Gamma_n \subseteq C_n$ for all n [1], [37] .

Theorem 4.3.4. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *where* $a, b, c, d \in \Gamma_n \cup \{0\}$. *If A induces bijections of* $V^n_\infty \to V^n_\infty$ *and* $V^{n+1}_\infty \to V^{n+1}_\infty$ *then:*

1. $ab^*, cd^*, c^*a, d^*b \in V^n$ *and*

2. $\Delta A = ad^* - bc^* \in \mathbb{R} \setminus \{0\}$.

The converse of Theorem 4.3.4 follows as the next theorem [1].

Theorem 4.3.5. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *where* $a, b, c, d \in \Gamma_n \cup \{0\}$, $\Delta A = ad^* - bc^* \in \mathbb{R} \setminus \{0\}$ *and* $ab^*, cd^*, c^*a, d^*b \in V^n$. *Then A induces bijections of* $V^n_\infty \to V^n_\infty$ *and* $V^{n+1}_\infty \to V^{n+1}_\infty$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ induce a mapping g given by $g(x) = (ax + b)(cx + d)^{-1}$. We show that:

1. g is well defined,

2. A^{-1} exists with $A^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ and

3. if $y = g(x)$ and $x \in V^n_\infty$ (or V^{n+1}_∞) then $y \in V^n_\infty$ (or V^{n+1}_∞)

1. If $c = 0$ then $g(x) = (ax+b)(d)^{-1} = axd^{-1} + bd^{-1}$ and is unique.

   If $c \neq 0$ then $cx + d = c(x + c^{-1}d)$. Since $cd^* \in V^n$ we know $c^{-1}d \in V^n$. Certainly if $x \in V^n$ (or V^{n+1}) then $cx + d \in \Gamma_n \cup \{0\}$ (or $\Gamma_{n+1} \cup \{0\}$). If $(cx+d) = (ax+b) = 0$ then $x = -c^{-1}d = -\left(c^{-1}d\right)^* = -d^*c^{*-1}$, so $ax + b = a(-d^*c^{*-1}) + b = 0$. That is $(-ad^* + bc^*)c^{*-1} = 0$. Since $c \neq 0$ this yields $ad^* - bc^* = 0$ and contradicts our assumption. Thus $ax+b$ and $cx+d$ cannot simultaneously be zero, so g is well defined.

2. $\Delta A = ad^* - bc^* \in \mathbb{R} \setminus \{0\}$ then $\overline{\Delta A} = \Delta A$ and

$$\begin{aligned}
(d^*c')\,\overline{(d^*b)} &= (d^*c')\,(d^*b)' \;\text{ since } d^*b \in V^n \\
&= d^*c'd^{*\prime}b' = d^*\,(cd^*)'\,b' \\
&= d^*\overline{(cd^*)}b' \text{ since } cd^* \in V^n \\
&= d^*d'\overline{c}b'.
\end{aligned}$$

Therefore, since $d \in \Gamma_n$ and $|d|^2 = d'd^*$ (Lemma 2.1.7(2)) we have

$$\begin{aligned}
\Delta A|d|^2 &= \Delta A(d^*d') = d^*\Delta A d' = d^*\left(\overline{\Delta A}\right)d' \\
&= d^*\left\{\overline{d^*}\,\overline{a} - \overline{c^*}\,\overline{b}\right\}d' = d^*\left\{d'\overline{a} - c'\overline{b}\right\}d' \\
&= d^*d'\overline{a}d' - d^*c'\overline{b}d' = |d|^2\overline{a}d' - (d^*c')\,\overline{(d^*b)} \\
&= |d|^2\overline{a}d' - d^*d'\overline{c}b' = |d|^2\left(\overline{a}d' - \overline{c}b'\right) \\
&= |d|^2\overline{(d^*a - b^*c)}
\end{aligned}$$

If $d \neq 0$ we have $\Delta A = ad^* - bc^* = \overline{d^*a - b^*c}$. Thus $\overline{\Delta A} = \Delta A = d^*a - b^*c$. Let $A_1 = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$. Then $\Delta A_1 = d^*(a^*)^* - (-b^*)(-c^*)^* = d^*a - b^*c$.

So $\Delta A = \Delta A_1 \in \mathbb{R} \setminus \{0\}$ by the assumption, and $(\Delta A)^* = \Delta A = \Delta A_1 = (\Delta A_1)^*$. We note that $AA_1 = \begin{pmatrix} \Delta A & 0 \\ 0 & (\Delta A)^* \end{pmatrix}$ and $A_1A = \begin{pmatrix} \Delta A_1 & 0 \\ 0 & (\Delta A_1)^* \end{pmatrix}$, so $AA_1 = A_1A = \lambda I_2$ where $\lambda = \Delta A \in \mathbb{R} \setminus \{0\}$.

Let g_1 be given by $g_1(x) = (d^*x - b^*)(-c^*x + a^*)^{-1}$. Certainly g_1 is well defined and $g_1 g = g g_1 = 1_{map}$. So g is bijective with $y = g(x)$ if and only if $x = g_1(y)$

3. Assume that $x \in V^n$ (or V^{n+1}). Then $x = x^*$. We show if $y = g(x)$ then $y^* = y$. We note the equivalences:

$$\begin{aligned} y = y^* = g(x) \quad & iff \quad y^* = (ax+b)(cx+d)^{-1} = \left[(ax+b)(cx+d)^{-1}\right]^* \\ & iff \quad (ax+b)(cx+d)^{-1} = (xc^* + d^*)^{-1}(xa^* + b^*) \\ & iff \quad (xc^* + d^*)(ax+b) = (xa^* + b^*)(cx+d) \\ & iff \quad x(a^*d - c^*b) = (d^*a - b^*c)x \text{ since } a^*c = c^*a \text{ and } b^*d = d^*b. \end{aligned}$$

Now $\Delta A = ad^* - bc^* = d^*a - b^*c = \Delta A_1$

$(\Delta A_1)^* = (d^*a - b^*c)^* = a^*d - c^*b = \Delta A \in \mathbb{R} \setminus \{0\}$

Thus $x(\Delta A_1)^* = (\Delta A_1)x$ and so $y = y^*$.

Thus $g : V^n_\infty \to V^n_\infty$ bijectively and $g : V^{n+1}_\infty \to V^{n+1}_\infty$ bijectively, where g is the map given by $g(x) = (ax+b)(cx+d)^{-1}$.

□

From [1], we denote the set of matrices in $M_2(C_n)$ that induce bijections on V^n_∞ and V^{n+1}_∞ by $GL(2, \Gamma_n)$ which we define formally as follows.

Definition 4.3.6. $GL(2, \Gamma_n) =$

$$\left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \Gamma_n \cup \{0\}, \Delta A = ad^* - bc^* \in \mathbb{R} \setminus \{0\} \text{ and } ab^*, cd^*, c^*a, d^*b \in V^n \right\}.$$

Hence, from Theorems 4.3.4 and 4.3.5 we have proved the following theorem. This result corresponds to Theorem A in Ahlfors [1].

Theorem 4.3.7. *The matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $a, b, c, d \in \Gamma_n$ *induces bijective mappings* $V^n_\infty \to V^n_\infty$ *and* $V^{n+1}_\infty \to V^{n+1}_\infty$ *if and only if* $A \in GL(2, \Gamma_n)$.

Corollary 4.3.8. $GL(2, C_n) = GL(2, \Gamma_n)$.

Proof. By Theorem 4.3.5, $A \in GL(2, C_n)$ for all $A \in GL(2, \Gamma_n)$. Conversely Theorem 4.2.3 and Lemma 4.3.3 ensure that each $A \in GL(2, C_n)$ is in $GL(2, \Gamma_n)$ □

This corollary reconciles the approaches to Clifford Matrices of [1], [36] and [37]. The matrices in $GL(2, \Gamma_n)$ act bijectively on V^n_∞ and V^{n+1}_∞ by the rule

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow (ax + b)(cx + d)^{-1} = g(x).$$

While $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \Gamma_n)$, we know that the corresponding map is $g \in PGL(2, \Gamma_n)$. We thus have:

$$PGL(2, \Gamma_n) \cong GL(2, \Gamma_n) / \{\lambda I : \lambda \in \mathbb{R} \setminus \{0\}\}$$

If

$$SL(2, \Gamma_n) = \{A \in GL(2, \Gamma_n) : \Delta A = ad^* - bc^* = 1\}$$

then

$$PSL(2, \Gamma_n) \cong SL(2, \Gamma_n) / \{\pm I\}.$$

We recall from Beardon [7] the definition of a Möbius transformation on $\mathbb{R}^n_\infty$.

Definition 4.3.9. *A Möbius transformation acting in* $\mathbb{R}^n_\infty$ *is a finite composition of reflections in hyperplanes and hyperspheres in* $\mathbb{R}^n$.

Definition 4.3.10. *The group of Möbius transformations acting in* $\mathbb{R}^n_\infty$ *is called the General Möbius Group* $GM(\mathbb{R}^n_\infty)$. $M(\mathbb{R}^n_\infty)$ *is the group of orientation preserving transformations in* $GM(\mathbb{R}^n_\infty)$. $M(\mathbb{R}^n_\infty)$ *is called the Möbius group acting in* $\mathbb{R}^n_\infty$.

We prove the following theorem from [1], [37].

Theorem 4.3.11. *The group $PSL(2,\Gamma_n)$ is isomorphic to the group $M(\mathbb{R}^n_\infty)$ of orientation preserving Möbius transformations. It is generated by the transformations with associated matrices* $\begin{pmatrix} p & 0 \\ 0 & p^{*-1} \end{pmatrix}$, $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ *with p a unit in Γ_n, $t \in V^n$ and $\lambda \in \mathbb{R}^+$.*

Proof. Assume first that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\Gamma_n)$, with $c \neq 0$. Then $bc^* = ad^* - 1$ and $\left(c^{-1}d\right)^* = d^*c^{*-1} = c^{-1}d$. Since $c^* \in \Gamma_n$ and $c^{*-1} \in \Gamma_n$, $b = -c^{*-1} + ac^{-1}d$.
Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -c^{*-1} \\ c & d \end{pmatrix}$ leading to the factorization

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -c^{*-1} & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix},$$

with $\begin{pmatrix} -c^{*-1} & 0 \\ 0 & -c \end{pmatrix} = \begin{pmatrix} 1/|c| & 0 \\ 0 & |c| \end{pmatrix}\begin{pmatrix} -c^{*-1}|c| & 0 \\ 0 & -c/|c| \end{pmatrix}$, which is a product of matrices in $SL(2,\Gamma_n)$.
For $c = 0$, $ad^* = 1$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$,
again with $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix} = \begin{pmatrix} |a| & 0 \\ 0 & 1/|a| \end{pmatrix}\begin{pmatrix} a/|a| & 0 \\ 0 & a^{*-1}|a| \end{pmatrix}$, which is a product of matrices in $SL(2,\Gamma_n)$.
This proves that the transformations associated with these matrices generate $PSL(2,\Gamma_n)$.
We note that the matrix $\begin{pmatrix} p & 0 \\ 0 & p^{*-1} \end{pmatrix}$ corresponds to the map

$$x \mapsto px\left(p^{*-1}\right)^{-1} = pxp^* = |p|^2 pxp'^{-1}$$

which was shown to be an orthogonal transformation in Theorem 2.2.3, followed by magnification in the ratio $|p|^2$. The matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is a parallel translation in the direction of t, t a vector in C_n, and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ induces the map $-x^{-1}$ which represents reflection in the unit sphere followed by reflection in the hyperplane $x_0 = 0$. All these maps are orientation preserving in V^n_∞.

The converse, that all orientation preserving Möbius transformations are obtainable by repetition of these steps, is essentially a matter of definition, since a pair of inversions is either a parallel translation, an orthogonal transformation or of the form $g(x) = -x^{-1}$. □

We draw attention to the result in Lemma 4.3.2 in which maps in $PGL(2, C_n)$ are decomposed. We see that the decompositions are equivalent as $GL(2, \Gamma_n) = GL(2, C_n)$. We further recall that mapping $\rho_p(x) = pxp^*$, $p \in \Gamma_n$, corresponding to the matrix $\begin{pmatrix} p & 0 \\ 0 & p^{*-1} \end{pmatrix}$ for $p \in \Gamma_n$, leaves V^n invariant (Theorem 2.2.4(2)). We note that $\begin{pmatrix} p & 0 \\ 0 & p^{*-1} \end{pmatrix} \in GL(2, \Gamma_n)$ if $p \in \Gamma_n$ and p is not a unit.

4.4 $M_2(\mathbb{K})$ and the Determinants $\mathcal{D}$

We now consider the special case of matrices in $M_2(\mathbb{K})$ or $M_2(C_3)$. We define a trace function and a determinant function of each M in $M_2(\mathbb{K})$ that are related to similar functions of matrices in $M_4(\mathbb{C})$. Each quaternion $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{K}$ corresponds to a matrix A in $M_2(\mathbb{C})$. That is $\mathbb{K} \cong \mathbf{K} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$ as rings where $\alpha = a_0 + a_1 i$ and $\beta = a_2 + a_3 i$ [33]. Thus we can associate each $M \in M_2(\mathbb{K})$ with a matrix $\mathcal{M} \in M_4(\mathbb{C})$ as follows.

Let a, b, c, d be quaternions with the expansions $a = a_0 + a_1 i + a_2 j + a_3 k$, $b = b_0 + b_1 i + b_2 j + b_3 k$, $c = c_0 + c_1 i + c_2 j + c_3 k$ and $d = d_0 + d_1 i + d_2 j + d_3 k$ where all a_t, b_t, c_t, d_t are real for $t = 0, \cdots, 3$. Set

$$\alpha_1 = a_0 + a_1 i, \quad \alpha_2 = a_2 + a_3 i, \quad \beta_1 = b_0 + b_1 i, \quad \beta_2 = b_2 + b_3 i,$$

$$\gamma_1 = c_0 + c_1 i, \quad \gamma_2 = c_2 + c_3 i, \quad \delta_1 = d_0 + d_1 i, \quad \delta_2 = d_2 + d_3 i \in \mathbb{C}.$$

Let $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} \end{pmatrix}$, $B = \begin{pmatrix} \beta_1 & \beta_2 \\ -\overline{\beta_2} & \overline{\beta_1} \end{pmatrix}$, $C = \begin{pmatrix} \gamma_1 & \gamma_2 \\ -\overline{\gamma_2} & \overline{\gamma_1} \end{pmatrix}$ and $D = \begin{pmatrix} \delta_1 & \delta_2 \\ -\overline{\delta_2} & \overline{\delta_1} \end{pmatrix}$ in **K** be the complex 2×2 matrices associated with $a, b, c, d \in \mathbb{K}$ respectively.

Then $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $M_4(\mathbb{C})$ is the complex block matrix that corresponds to the matrix

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M_2(\mathbb{K})$.

$M_4(\mathbb{C})$ is a well defined matrix ring with well defined determinant and the trace functions. The matrices $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with A, B, C and D as above, form a subring of $M_4(\mathbb{C})$ and we can consider $det\mathcal{M}$ and $tr\mathcal{M}$ in the standard way. Thus if

$$\mathcal{M} = \begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} & -\overline{\beta_2} & \overline{\beta_1} \\ \gamma_1 & \gamma_2 & \delta_1 & \delta_2 \\ -\overline{\gamma_2} & \overline{\gamma_1} & -\overline{\delta_2} & \overline{\delta_1} \end{pmatrix}$$

then $tr\mathcal{M} = \alpha_1 + \overline{\alpha_1} + \delta_1 + \overline{\delta_1} = 2Re(\alpha_1) + 2Re(\delta_1) = 2Re(\alpha_1 + \delta_1) = 2(a_0 + d_0) = S(a+d)$. This relationship motivates the following definition [19].

Definition 4.4.1. *Let $M \in M_2(\mathbb{K})$ be associated with $\mathcal{M} \in M_4(\mathbb{C})$ as above. Then $TrM = \frac{1}{2}tr(\mathcal{M})$. Certainly $TrM = a_0 + d_0 = \frac{1}{2}(tr(\mathcal{M}))$ where $tr(\mathcal{M}) = S(a+d)$, with $a, d \in \mathbb{K}$.*

We define a determinant $\mathcal{D}$ of $M \in M_2(\mathbb{K})$ using the corresponding matrix $\mathcal{M} \in M_4(\mathbb{C})$ and its well defined determinant [19], [38]. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$ correspond to

$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as given above. We consider three special cases in the development of an expression of $\mathcal{D}(M)$, a determinant of M.

Case (i). Assume $c = 0$, so $M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Thus $C = 0$ and $\mathcal{M} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. We use elementary row operations on $\mathcal{M} \in M_4(\mathbb{C})$ to find

$$
\begin{aligned}
det\mathcal{M} &= det\begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} & -\overline{\beta_2} & \overline{\beta_1} \\ 0 & 0 & \delta_1 & \delta_2 \\ 0 & 0 & -\overline{\delta_2} & \overline{\delta_1} \end{pmatrix} \\
&= \frac{1}{\overline{\delta_2}\delta_1} det\begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} & -\overline{\beta_2} & \overline{\beta_1} \\ 0 & 0 & \overline{\delta_2}\delta_1 & \overline{\delta_2}\delta_2 \\ 0 & 0 & 0 & |\delta_1|^2 + |\delta_2|^2 \end{pmatrix} \\
&= \frac{|\delta_1|^2 + |\delta_2|^2}{\overline{\delta_2}\delta_1} det\begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 \\ -\overline{\alpha_2} & \overline{\alpha_1} & -\overline{\beta_2} \\ 0 & 0 & \overline{\delta_2}\delta_1 \end{pmatrix} \\
&= \left(|\delta_1|^2 + |\delta_2|^2\right)\left(|\alpha_1|^2 + |\alpha_2|^2\right) = det(D)det(A) = det(DA) = det(A)det(D),
\end{aligned}
$$

with $A, D \in M_2(\mathbb{C})$, where $det(A)$ and $det(D)$ are well defined.

Case(ii). Assume $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and A^{-1} exists, where $\mathcal{M} \in M_4(\mathbb{C})$ and $A \in M_2(\mathbb{C})$. Using elementary row operations on $\mathcal{M} \in M_4(\mathbb{C})$ (following Wilker, [38]) we get

$$
\begin{aligned}
det\mathcal{M} &= det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = Adet\begin{pmatrix} A^{-1}A & A^{-1}B \\ C & D \end{pmatrix} = Adet\begin{pmatrix} I & A^{-1}B \\ C & D \end{pmatrix} \\
&= Adet\begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} = det\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}.
\end{aligned}
$$

Thus we have reduced Case(ii), by elementary row operations, to Case (i) above.

$$
det\mathcal{M} = detAdet\left(D - CA^{-1}B\right) = det\left(AD - ACA^{-1}B\right).
$$

Case(iii). Consider $\mathcal{M}$ with the from $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$. By using elementary row operations on $\mathcal{M} \in M_4(\mathbb{C})$ we get

$$\begin{aligned} det\mathcal{M} &= det\begin{pmatrix} 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & -\overline{\beta_2} & \overline{\beta_1} \\ \gamma_1 & \gamma_2 & \delta_1 & \delta_2 \\ -\overline{\gamma_2} & \overline{\gamma_1} & -\overline{\delta_2} & \overline{\delta_1} \end{pmatrix} \\ &= (-1)^2 det\begin{pmatrix} \gamma_1 & \gamma_2 & \delta_1 & \delta_2 \\ -\overline{\gamma_2} & \overline{\gamma_1} & -\overline{\delta_2} & \overline{\delta_1} \\ 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & -\overline{\beta_2} & \overline{\beta_1} \end{pmatrix} \\ &= det\begin{pmatrix} C & D \\ 0 & B \end{pmatrix} \\ &= det(C)det(B) = det(CB) \text{ by Case (i).} \end{aligned}$$

With elementary checks, we can establish that $det(CB) = det(-CB)$, where C and B are as given above [38].

Putting the three cases together we conclude the following. We know either $a = 0$ or a^{-1} exists for each $a \in \mathbb{K}$. By the correspondence between elements in $\mathbb{K}$ and matrices in $M_2(\mathbb{C})$ we know either $A = 0$ or A^{-1} exists where A^{-1} corresponds to a^{-1}.

If $a = 0$ and $A = 0$, by Case (iii) above,

$$det\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = det(-CB) = det(0 - CB) = det\left(AD - \left(ACA^{-1}\right)B\right),$$

where $A = 0$, provided we interpret conjugation by zero as the identity as suggested by [38]. We may assume $ACA^{-1} = C$ for $A = 0$. At least 0 does commute with all quaternions, so $0C = C0$ as matrices.

If $a \neq 0$ and $A \neq 0$, then A^{-1} exists and so $det\mathcal{M} = det\left(AD - ACA^{-1}B\right)$ by Case (ii).

So for $a = 0$ and $a \neq 0$ we know that

$$det\mathcal{M} = det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = det\left(AD - ACA^{-1}B\right) = detAdet\left(D - CA^{-1}B\right).$$

We are now ready to define a determinant $\mathcal{D}$ on $M_2(\mathbb{K})$.

Definition 4.4.2. *Let* $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(\mathbb{C})$ *be the complex block matrix corresponding to* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{K})$ *then* $\mathcal{D}(M) = det\mathcal{M} = detAdet\left(D - CA^{-1}B\right)$.

By the determinant property in $M_4(\mathbb{C})$ we have $\mathcal{D}(P_1P_2) = \mathcal{D}(P_1)\mathcal{D}(P_2)$ for P_1, P_2 in $M_2(\mathbb{K})$.

4.4.1 Properties of $\mathcal{D}$ acting on $M_2(\mathbb{K})$

Schmidt [30] and Parker and Short [26] define a 'determinant' of $M \in \mathcal{M}_2(\mathbb{K})$ as $N(a)N(d) + N(b)N(c) - S\left(a\bar{c}d\bar{b}\right)$ in accordance with the work of Study [34]. We show that this determinant is equal to $\mathcal{D}$ defined above.

Theorem 4.4.3. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$ *have corresponding complex block matrix* $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(\mathbb{C})$. *Then* $\mathcal{D}(M) = N(a)N(d) + N(b)N(c) - S\left(a\bar{c}d\bar{b}\right) = N(ad - aca^{-1}b)$ *and* $\mathcal{D}(M) \in \mathbb{R}$.

Proof. We defined $\mathcal{D}(M) = detAdet(D - CA^{-1}B)$, where $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the corresponding complex block matrix of $M \in M_2(\mathbb{K})$. Let

$$\begin{aligned} q &= ad - aca^{-1}b \in \mathbb{K} \\ &= u_0 + u_1 i + u_2 j + u_3 k \text{ with } u_0, u_1, u_2, u_3 \in \mathbb{R} \\ &= \mu_1 + \mu_2 j \end{aligned}$$

where $\mu_1 = u_0 + u_1 i, \mu_2 = u_2 + u_3 i \in \mathbb{C}$.

The quaternion q corresponds to the matrix $\begin{pmatrix} \mu_1 & \mu_2 \\ -\overline{\mu_2} & \overline{\mu_1} \end{pmatrix} \in M_2(\mathbb{C})$ (Section 2.3.2).

Further $N(q) = det \begin{pmatrix} \mu_1 & \mu_2 \\ -\overline{\mu_2} & \overline{\mu_1} \end{pmatrix} = |\mu_1|^2 + |\mu_2|^2$.

By the correspondence between $\mathbb{K}$ and $\mathbf{K}$ given above, and by matrix multiplication we have that $q = ad - aca^{-1}b \in \mathbb{K}$ corresponds to $AD - ACA^{-1}B \in \mathbf{K}$ as well.

So $AD - ACA^{-1}B = \begin{pmatrix} \mu_1 & \mu_2 \\ -\overline{\mu_2} & \overline{\mu_1} \end{pmatrix} \in M_2(\mathbb{C})$.

Thus $N(q) = det \begin{pmatrix} \mu_1 & \mu_2 \\ -\overline{\mu_2} & \overline{\mu_1} \end{pmatrix} = det\left(AD - ACA^{-1}B\right) = detAdet\left(D - CA^{-1}B\right)$.

Now if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then

$$\begin{aligned} \mathcal{D}(M) &= det(AD - ACA^{-1}B) = det \begin{pmatrix} \mu_1 & \mu_2 \\ -\overline{\mu_2} & \overline{\mu_1} \end{pmatrix} = N(q) = q\overline{q} \\ &= \left(ad - aca^{-1}b\right)\overline{(ad - aca^{-1}b)} \\ &= N(a)N(d) + N(b)N(c) - \left[ad\overline{b}\,\overline{a^{-1}}\overline{c}\,\overline{a} + aca^{-1}b\overline{d}\overline{a}\right] \end{aligned}$$

Now $\overline{a^{-1}} = \dfrac{a}{|a|^2}$, and by Proposition 2.3.7(4) we have that $S(ab) = S(ba)$, so

$$ad\overline{b}\,\overline{a^{-1}}\overline{c}\,\overline{a} - aca^{-1}b\overline{d}\overline{a} = S\left[(ad\overline{b})(\overline{a^{-1}}\overline{c}\,\overline{a})\right] = S\left[a\overline{c}d\overline{b}\right]$$

So $\mathcal{D}(M) = N(a)N(d) + N(b)N(c) - S\left(a\overline{c}d\overline{b}\right) \in \mathbb{R}$. □

From [38, p.107] we note that calculations we have performed can be varied, using column operations instead, to produce a total of eight forms for $\mathcal{D}(M)$. By setting

$$l_{11} = da - dbd^{-1}c, \;\; l_{12} = bdb^{-1}a - bc, \;\; l_{21} = cac^{-1}d - cb, \;\; l_{22} = ad - aca^{-1}b$$

and

$$r_{11} = ad - bd^{-1}cd, \;\; r_{12} = db^{-1}ab - cb, \;\; r_{21} = ac^{-1}dc - bc, \;\; r_{22} = da - ca^{-1}ba,$$

we get $\mathcal{D}(M) = N\left(l_{ij}\right) = N\left(r_{ij}\right)$ for $i, j = 1, 2$.

From [30] the following properties of $\mathcal{D}(M)$ are established.

Theorem 4.4.4. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$*, then the following hold:*

1. $\mathcal{D}(M) \geq 0$ *for all M,*
2. $\mathcal{D}\begin{pmatrix} \zeta a & \zeta b \\ \eta c & \eta d \end{pmatrix} = N(\zeta)N(\eta)\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\zeta, \eta \in \mathbb{K}$,
3. $\mathcal{D}\begin{pmatrix} a\zeta & b\eta \\ c\zeta & d\eta \end{pmatrix} = N(\zeta)N(\eta)\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\zeta, \eta \in \mathbb{K}$,
4. $N(c)N(d)N(ac^{-1} - bd^{-1}) = \mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *for* $c, d \neq 0$.

Further, $\mathcal{D}$ is invariant under a left row operation (a left multiple of one row is added to the other row) and under a right column operation.

Proof. 1. From Theorem 4.4.3 we have established that $\mathcal{D}(M) = N(q)$ where $q = ad - aca^{-1}b$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$. So $\mathcal{D}(M) \geq 0$.

2. We note that $S\left(\zeta a\overline{\eta c}\eta d\overline{\zeta b}\right) = N(\eta)N(\zeta)S\left(a\bar{c}\,d\bar{b}\,\right)$ using properties 4 and 5 of Proposition 2.3.7, so

$$\begin{aligned}
\mathcal{D}\begin{pmatrix} \zeta a & \zeta b \\ \eta c & \eta d \end{pmatrix} &= N(\zeta a)\,N(\eta d) + N(\zeta b)\,N(\eta c) - S\left(\zeta a\overline{\eta c}\eta d\overline{\zeta b}\right) \\
&= N(\zeta)\,N(a)\,N(\eta)\,N(d) + N(\zeta)\,N(b)\,N(\eta)\,N(c) - N(\eta)N(\zeta)S\left(a\bar{c}\,d\bar{b}\,\right) \\
&= N(\eta)N(\zeta)\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\end{aligned}$$

3. The proof is similar to the proof for Part 2.

4. By Properties 1 and 3 of Proposition 2.3.7

$$\begin{aligned}
N(c)N(d)N(ac^{-1} - bd^{-1}) &= N(ac^{-1} - bd^{-1})N(cd) \\
&= N(ad - bd^{-1}cd) \\
&= N(ad) + N(bd^{-1}cd) - S\left(ad\,\bar{d}\,\bar{c}\,\overline{d^{-1}}\,\bar{b}\right) \\
&= N(a)N(d) + N(b)N(c) - S\left(a\bar{c}d\bar{b}\right) \\
&= \mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

□

4.5 Matrices in $M_2(\mathbb{K})$ and their Inverses

From [38] and [19] we have the following result.

Lemma 4.5.1. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$ *be such that* $\mathcal{D}(M) \neq 0$*, then* M^{-1} *exists and*

$$M^{-1} = \begin{pmatrix} l_{11}^{-1}d & -l_{12}^{-1}b \\ -l_{21}^{-1}c & l_{22}^{-1}a \end{pmatrix} = \begin{pmatrix} dr_{11}^{-1} & -br_{12}^{-1} \\ -cr_{21}^{-1} & ar_{22}^{-1} \end{pmatrix}$$

where

$$l_{11} = da - dbd^{-1}c,\;\; l_{12} = bdb^{-1}a - bc,\;\; l_{21} = cac^{-1}d - cb,\;\; l_{22} = ad - aca^{-1}b$$

and

$$r_{11} = ad - bd^{-1}cd, \;\; r_{12} = db^{-1}ab - cb, \;\; r_{21} = ac^{-1}dc - bc, \;\; r_{22} = da - ca^{-1}ba.$$

Proof. If $\mathcal{D}(M) \neq 0$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$, we have that $\mathcal{D}(M) = N(q)$ where $q = ad - aca^{-1}b$ by Theorem 4.4.3. Thus $q \neq 0$. We find that

$$\begin{pmatrix} l_{11}^{-1}d & -l_{12}^{-1}b \\ -l_{21}^{-1}c & l_{22}^{-1}a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} l_{11}^{-1}da - l_{12}^{-1}bc & l_{11}^{-1}db - l_{12}^{-1}bd \\ -l_{21}^{-1}ca + l_{22}^{-1}ac & -l_{21}^{-1}cb + l_{22}^{-1}ad \end{pmatrix} = (x_{st}),$$

for $1 \leq s, t \leq 2$.

Similarly we see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} dr_{11}^{-1} & -br_{12}^{-1} \\ -cr_{21}^{-1} & ar_{22}^{-1} \end{pmatrix} = (y_{st}) \text{ for } 1 \leq s, t \leq 2.$$

Now

$$\begin{aligned} x_{11} &= (da - dbd^{-1}c)^{-1}da - (bdb^{-1}a - bc)^{-1}bc \\ &= (1 - a^{-1}bd^{-1}c)^{-1} - (c^{-1}db^{-1}a - 1)^{-1} \\ &= \left(\overline{1 - a^{-1}bd^{-1}c}\right)/|1 - a^{-1}bd^{-1}c|^2 - \left(\overline{c^{-1}db^{-1}a - 1}\right)/|c^{-1}db^{-1}a - 1|^2 = 1 \end{aligned}$$

by tedious but elementary calculation. Similarly $x_{22} = 1$.

$$\begin{aligned} x_{12} &= (da - dbd^{-1}c)^{-1}db - (bdb^{-1}a - bc)^{-1}bd \\ &= \left(b^{-1}a - d^{-1}c)\right)^{-1} - \left(b^{-1}a - d^{-1}c)\right)^{-1} \\ &= 0 \end{aligned}$$

Similarly $x_{21} = 0$.

The proof for the second product (y_{st}) for $1 \leq s, t \leq 2$ is similar.

We show that $(x_{st}) = (y_{st})$ for $1 \leq s, t \leq 2$. Certainly $l_{11}^{-1}d = dr_{11}^{-1}$ or equivalently $l_{11}d = dr_{11}$. That is

$$l_{11}d = \left(da - dbd^{-1}c\right)d = dad - dbd^{-1}cd$$

and

$$dr_{11} = d\left(ad - bd^{-1}cd\right) = dad - dbd^{-1}cd.$$

Similarly $l_{12}^{-1}b = br_{12}^{-1}$, $l_{21}^{-1}c = cr_{21}^{-1}$ and $l_{22}^{-1}a = ar_{22}^{-1}$ as required. □

Definition 4.5.2. *Let* $G_{\mathcal{D}}L(2,\mathbb{K}) = \{A \in M_2(\mathbb{K}) : \mathcal{D}(A) \neq 0\}$ *and* $PG_{\mathcal{D}}L(2,\mathbb{K}) = G_{\mathcal{D}}L(2,\mathbb{K})/\{\lambda I_2\}$, $\lambda \in \mathbb{R}\setminus\{0\}$, *be the corresponding group of maps. Let* $S_{\mathcal{D}}L(2,\mathbb{K}) = \{A \in M_2(\mathbb{K}) : \mathcal{D}(A) = 1\}$ *and* $PS_{\mathcal{D}}L(2,\mathbb{K}) = S_{\mathcal{D}}L(2,\mathbb{K})/\{\pm I_2\}$ *be the corresponding group of maps.*

We note that $S_{\mathcal{D}}L(2,\mathbb{K})$ is a normal subgroup of $G_{\mathcal{D}}L(2,\mathbb{K})$.

4.6 The Correspondence between $\mathcal{D}$ and Δ

We note the following theorem linking the pseudo-determinant $\Delta M = ad^* - bc^*$ [1], [37] and the determinant $\mathcal{D}(M)$ [19], [38], where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{K}$ and $\mathbb{K} \cong C_3$.

Theorem 4.6.1. *Let* $M \in GL(2, C_3)$ *and* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then* $\mathcal{D}(M) = N(\Delta M)$ *where* $\Delta M = ad^* - bc^*$ *and* $N(a) = a\bar{a}$.

Proof. Let $M \in GL(2, C_3)$ such that $\mathcal{D}(M) \neq 0$. Thus by Lemma 4.5.1, M^{-1} exists and M induces a bijection g on V^4_∞ with $g(x) = (ax+b)(cx+d)^{-1}$. Thus from Theorem 4.2.3 we know for $n = 3$, $ac^{-1}, bd^{-1}, -c^{-1}d, -a^{-1}b$ in V^3, the vectors of $\mathbb{K}$ or C_3. By Lemma 2.1.8, cd^* and $c^{-1}d$ are simultaneously vectors of C_3 in V^3. For $c^{-1}d \in V^3$, we have that $\bar{c}d \in V^3$.

Then $\overline{c}d = (\overline{c}d)^* = (d^*(\overline{c})^*) = (d^*c') = d^*c'$. Hence:

$$\begin{aligned} N(\Delta M) &= (ad^* - bc^*)(\overline{ad^* - bc^*}) \\ &= (ad^* - bc^*)(\overline{d^*}\,\overline{a} - \overline{c^*}\,\overline{b}) \\ &= N(a)N(d) + N(b)N(c) - S(ad^*\overline{c^*}\,\overline{b}) \\ &= N(a)N(d) + N(b)N(c) - S(a[\overline{c}d]^*\overline{b}) \\ &= N(a)N(d) + N(b)N(c) - S(a\overline{c}d\overline{b}) \\ &= \mathcal{D}(M) \end{aligned}$$

□

We note that $G_{\mathcal{D}}L(2, C_3) = G_{\mathcal{D}}L(2, \Gamma_3) \subsetneq PG_{\mathcal{D}}L(2, \mathbb{K})$. Certainly $\mathcal{D}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = 1$, but $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \notin G_{\mathcal{D}}L(2, C_3)$ as $\Delta\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = k \notin \mathbb{R}$.

4.7 $G_{\mathcal{D}}L(2, \mathbb{K})$, $S_{\mathcal{D}}L(2, \mathbb{K})$ and $M(\mathbb{R}^4_\infty)$

We show that each matrix $M \in G_{\mathcal{D}}L(2, \mathbb{K})$ can be normalised to a matrix M' in $S_{\mathcal{D}}L(2, \mathbb{K})$.

Theorem 4.7.1. *Let $M \in G_{\mathcal{D}}L(2, \mathbb{K})$. Then we can find $M' \in S_{\mathcal{D}}L(2, \mathbb{K})$ such that M and M' induce the same bijection of $\mathbb{K}_\infty$, with $M = M'\lambda$ and $\lambda^4 = \mathcal{D}(M)$, $\lambda \in \mathbb{R}$. Thus $PG_{\mathcal{D}}L(2, \mathbb{K}) = PS_{\mathcal{D}}L(2, \mathbb{K})$.*

Proof. Let $\mathcal{D}(M) = N(a)N(d) + N(b)N(c) - S\left(a\overline{c}d\overline{b}\right) > 0$. Let $\lambda^4 = \mathcal{D}(M)$, $\lambda \in \mathbb{R}^+$ [25]. Since λ is a positive real number $\lambda x = x\lambda$ for all $x \in \mathbb{K}$ and $\lambda^{-1} \in \mathbb{R}^+$.
Let $M' = \begin{pmatrix} a\lambda^{-1} & b\lambda^{-1} \\ c\lambda^{-1} & d\lambda^{-1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\lambda^{-1}$.

Then

$$\begin{aligned}\mathcal{D}(M') &= N(\lambda^{-2})\,(N(a)N(d)+N(b)N(c)) - \frac{1}{\lambda^4}S\left(a\bar{c}d\bar{b}\right)\\ &= \frac{1}{\lambda^4}\left(N(a)N(d)+N(b)N(c) - S\left(a\bar{c}d\bar{b}\right)\right)\\ &= \frac{1}{\lambda^4}\lambda^4 = 1.\end{aligned}$$

The map induced by M' is given by

$$\begin{aligned}g'(x) &= (a\lambda^{-1}x+b\lambda^{-1})(c\lambda^{-1}x+d\lambda^{-1})^{-1}\\ &= (ax+b)\lambda^{-1}\lambda(cx+d)^{-1}\\ &= (ax+b)(cx+d)^{-1}\\ &= g(x) \text{ for all } x\in\mathbb{K}\end{aligned}$$

where g is induced by M. So $PG_{\mathcal{D}}L(2,\mathbb{K}) = PS_{\mathcal{D}}L(2,\mathbb{K})$ as required.

□

Consider transformations of the form $g : x \to (ax+b)(cx+d)^{-1}$, where $M = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in G_{\mathcal{D}}L(2,\mathbb{K})$. Since the arithmetic of quaternions is continuous and the norm of quaternions is continuous, the transformations are also continuous [38, p.110]. The maps are all continuous with continuous inverses.

From [23] we have the following:

Lemma 4.7.2. *Each matrix $A \in G_{\mathcal{D}}L(2,\mathbb{K})$ can be decomposed into a product of matrices of the form $\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix}$, $b\in\mathbb{K}$, $\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}$, $a\neq 0$ in $\mathbb{K}$. Correspondingly, each $g \in PS_{\mathcal{D}}L(2,\mathbb{K})$ can be decomposed into the composition of maps τ_b, ϕ and δ_a, where $\tau_b(x) = x+b$, $\phi(x) = x^{-1}$ and $\delta_a(x) = ax$*

Proof. We will use τ_b, ϕ and δ_a to represent either the maps or the corresponding matrices. We note that τ_b, ϕ and δ_a are all in $G_{\mathcal{D}}L(2,\mathbb{K})$, as are their inverses where $(\tau_b)^{-1} = \tau_{-b}$,

$\phi^{-1} = \phi$ and $(\delta_a)^{-1} = \delta_{a^{-1}}$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathcal{D}}L(2, \mathbb{K})$. We consider the cases where $c \neq 0$ and where $c = 0$.

Case 1. Let $c \neq 0$, then by direct multiplication of the matrices we see that

$$A = \tau_{b_1}\phi\delta_{a_1}\phi\delta_{a_2}\phi\tau_{b_2},$$

where $a_1 = c$; $a_2 = b - ac^{-1}d$; $b_1 = ac^{-1}$ and $b_2 = c^{-1}d$.

Case 2. Let $c = 0$ then again by direct multiplication of the matrices we see that

$$A = \delta_a\tau_{a^{-1}bd^{-1}}\phi\delta_d\phi,$$

where $a \neq 0$ and $d \neq 0$, since $c = 0$ and $\mathcal{D}(A) \neq 0$. □

Corollary 4.7.3. *The generators of* $G_{\mathcal{D}}L(2, \mathbb{K})$ *can alternatively be given as* $\tau_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{K}$, $\varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $R_r = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, $r \in \mathbb{R} \setminus \{0\}$ *and* $\delta_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, $u \neq 0$ *a unit quaternion.*

Proof. Since $a = -1 \in \mathbb{K}$ we know that $\delta_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a generator so $\varphi = \delta_{-1}\phi$ and $\phi = \delta_{-1}\varphi$ since $(-1)^{-1} = -1$. For $a \in \mathbb{K}$, $a \neq 0$, let $r = |a|^2 \neq 0$, then $a = r\frac{a}{r}$. So $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{a}{r} & 0 \\ 0 & 1 \end{pmatrix}$, where $N\left(\frac{a}{r}\right) = 1$ and $\frac{a}{r} \neq 0$ in $\mathbb{K}$. □

A conformal map or transformation is a transformation $w = f(z)$ that preserves local angles. A map $f : z \to w$ is conformal (or angle preserving) at a point u_0 if it preserves oriented angles between curves through u_0 with respect to their orientation.

Theorem 4.7.4. $PS_{\mathcal{D}}L(2, \mathbb{K}) = M(\mathbb{R}^4_\infty)$.

Proof. We consider the generators τ_b, R_r, φ and δ_u, where $b \in \mathbb{K}$, $r \in \mathbb{R}^+$ and u is a unit quaternion. As noted in Lemma 4.3.11, τ_b and φ are orientation preserving maps in

$PS_{\mathcal{D}}L(2,\mathbb{K})$. Certainly R_r is a magnification (contraction) and is sense preserving. Consider $u = u_0 + u_1 i + u_2 j + u_3 k$ with $|u|^2 = u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1$. Then we can write $u = \cos\alpha + I\sin\alpha$ for $\alpha \in [0, 2\pi)$, where $\cos\alpha = u_0$, $0 \leq |u_0| < 1$ and $I = \frac{u_1}{\sin\alpha}i + \frac{u_2}{\sin\alpha}j + \frac{u_3}{\sin\alpha}k$. We see that I is pure quaternion and is a unit. That is $\left(\frac{u_1}{\sin\alpha}\right)^2 + \left(\frac{u_2}{\sin\alpha}\right)^2 + \left(\frac{u_3}{\sin\alpha}\right)^2 = \frac{1}{\sin^2\alpha}\left(u_1^2 + u_2^2 + u_3^2\right) = \frac{1}{\sin^2\alpha}\left(1 - u_0^2\right) = \frac{1}{\sin^2\alpha}\left(1 - \cos^2\alpha\right) = 1$ [10], [33].

We know that any sense preserving (direct, orientation preserving) orthogonal transformation acting in $\mathbb{R}^4$ can be expressed using a suitable pair of unit quaternions as $x \longmapsto u_1 x \overline{u_2}$. Thus $\delta_u(x) = ux$ is an orthogonal transformation that preserves orientation. So δ_u can be regarded as a rotation through 2α about I. Thus the generators, and hence all elements of $PS_{\mathcal{D}}L(2,\mathbb{K})$, are orientation preserving and conformal, since all inversions are conformal. Finally, we note from [38] that all maps $g : x \to (ax+b)(cx+d)^{-1}$ in $PS_{\mathcal{D}}L(2,\mathbb{K})$ preserve cross ratios and are therefore all Möbius transformations. Thus $PS_{\mathcal{D}}L(2,\mathbb{K}) = M(\mathbb{R}^4_\infty)$. □

Wilker [38] further establishes that any $g \in GM(\mathbb{R}^4)$ is of the form $g : x \to (ax+b)(cx+d)^{-1}$ with $a, b, c, d \in \mathbb{K}$ (orientation preserving) or $g : x \to (a\overline{x} + b)(c\overline{x} + d)^{-1}$ with $a, b, c, d \in \mathbb{K}$ (orientation reversing), where $x \to \overline{x}$ is a product of three reflections in hyperplanes orthogonal to i, j and k.

Jones and Singerman [17] have noted that $PSL(n, F) = PGL(n, F)$ if and only if F is a field that allows the n^{th} root of any element. While $\mathbb{K}$ is not a field, it does allow the n^{th} root of any element in $\mathbb{K}$ to be determined [25]. Hence we have $PS_{\mathcal{D}}L(2,\mathbb{K}) = PG_{\mathcal{D}}L(2,\mathbb{K})$ as above. We thus observe the analogy to the case $C_2 = \mathbb{C}$ where $PS_{\mathcal{D}}L(2,\mathbb{C}) = PG_{\mathcal{D}}L(2,\mathbb{C})$ and where each element of $M\left(\mathbb{R}^2_\infty\right)$ can be associated with a (normalised) matrix of determinant 1 in $M_2(\mathbb{C})$.

Chapter 5

$\mathbb{H}^5$ and the isometries of $\mathbb{H}^5$

5.1 The Geometry of $\mathbb{R}^n$

In $\mathbb{R}^n$ the components of dimensions 0, 1, and $n-1$ form the points, lines and hyperplanes of $\mathbb{R}^n$ respectively [35]. We let $z = (x_1, \cdots, x_n)$, $x_t \in \mathbb{R}$ for $t = 1 \cdots n$, be a general point in $\mathbb{R}^n$. Following Beardon [7] hyperplanes and hyperspheres in $\mathbb{R}^n$ are defined as follows:

Definition 5.1.1. *The set $P(a, d)$ is a hyperplane in $\mathbb{R}^n$ given by*

$$\{z \in \mathbb{R}^n : (z \cdot a) = d\} \cup \{\infty\},$$

where $a \neq 0 \in \mathbb{R}^n$ is the normal to the hyperplane, where $(z \cdot a)$ is the usual scalar product $\Sigma z_j a_j$ and d is real.

Definition 5.1.2. *The hypersphere $S(a, r)$ in $\mathbb{R}^n$ is given by*

$$\{z \in \mathbb{R}^n : |z - a| = r\},$$

where $a \in \mathbb{R}^n$ and $r > 0$.

Reflections in hyperplanes and hyperspheres in $\mathbb{R}^n$ are defined as follows [7]:

Definition 5.1.3. *The inversion (reflection) in hypersphere $S(a, r)$ in $\mathbb{R}^n_\infty$ is given by the bijection g defined by $g(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x - a)$, $x \neq a$ and $g(\infty) = a$ and $g(a) = \infty$.*

Definition 5.1.4. *The reflection in hyperplane $P(a,t)$ in $\mathbb{R}^n_\infty$ is given by the bijection g defined by $g(x) = x - 2[(x \cdot a) - t]\left(\frac{a}{|a|^2}\right)$ when $x \in \mathbb{R}^n$ and $g(\infty) = \infty$.*

Example 5.1.5. *1. The reflection of $z = (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5$ in the unit hypersphere $|z| = 1$ is given by*

$$I(z) = \frac{z}{|z|^2}, \qquad |z|^2 = N(z) = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

2. Reflection of $z = (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5$ about the plane through 0 normal to $a = (0, 1, -1, 0, 0)$ is given by

$$g(x_0, x_1, x_2, x_3, x_4) = (x_0, x_1, x_2, x_3, x_4) - 2(x_1 - x_2)\frac{(0, 1, -1, 0, 0)}{2} = (x_0, x_2, x_1, x_3, x_4).$$

Alternatively, using Clifford algebra, with $a = i - j$, the reflection is given by

$$J(z) = -a\bar{z}a'^{-1},$$

where $\bar{z} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_1 i_2 - x_4 i_3$.

We now describe the natural analogues of a familiar class of figures in elementary geometry [11]. In $\mathbb{R}^2$, the convex hull of a 2-dimensional finite set of points is a closed polygonal region, and its boundary is a convex polygon. In $\mathbb{R}^3$, the convex hull of a 3-dimensional finite set of points is a closed polyhedral region, and its boundary is a convex polyhedron. From [7],[20] we have the following definitions:

Definition 5.1.6. *A set S is a convex set in $\mathbb{R}^n$ if $x, y \in S$ imply that $[x : y] \subset S$, where $[x : y]$ is the closed Euclidean line segment joining x and y.*

Definition 5.1.7. *The convex hull of a finite set P in $\mathbb{R}^n$ is the intersection of all the convex sets that contain P and is denoted by $CH(P)$.*

We note from [20] that a set P in $\mathbb{R}^n$ is a convex set if and only if $P = CH(P)$.

In the following definitions we generalize the triangle and square in $\mathbb{R}^2$ and the tetrahedron and octahedron in $\mathbb{R}^3$ to simplices and cross-polytopes in $\mathbb{R}^n$ respectively.

Definition 5.1.8. *The convex hull of $n+1$ independent points $x_1, \cdots, x_{n+1}$ in $\mathbb{R}^n$ is an n-dimensional simplex. The points $A_1, \cdots, A_{n+1}$ are called the vertices of the simplex. The convex hull of any n of the $n+1$ vertices is a face of the simplex (an $(n-1)$-dimensional simplex) and is said to be opposite the remaining vertex. The convex hull of each pair of vertices is an edge of the simplex.*

Example 5.1.9. The basic n-dimensional simplices in $\mathbb{R}^n$ are as follows.

1. In $\mathbb{R}$, the 1-dimensional simplex is a line segment between 2 vertices.
2. In $\mathbb{R}^2$, the 2-dimensional simplex is the convex hull of a triangle.
3. In $\mathbb{R}^3$, the 3-dimensional simplex is the convex hull of a tetrahedron. The edges are the line segments between any pair of the 4 vertices and the faces are the triangles formed by any 3 of the 4 vertices.
4. In $\mathbb{R}^4$, the 4-dimensional simplex is the convex hull of 5 vertices. The edges are the line segments between any pair of the 5 vertices and the faces are the 3-dimensional simplices formed by any 4 of the 5 vertices.
5. In $\mathbb{R}^5$, the 5-dimensional simplex is the convex hull of 6 vertices. The edges are the line segments between any pair of the 6 vertices and the faces are the 4-dimensional simplices formed by any 5 of the 6 vertices.

We note that the n-dimensional simplex in $\mathbb{R}^n$ is generated from the $(n-1)$-dimensional simplex in $\mathbb{R}^{n-1}$ by the addition of one vertex in a dimension orthogonal to $\mathbb{R}^{n-1}$, and linking this vertex to all vertices of the $(n-1)$-dimensional simplex with an edge.

Definition 5.1.10. *Consider $x = (x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$ such that $|x| = 1$ and $x_t \in \mathbb{Z}$, for $t = 1, \cdots, n$. The convex hull of these $2n$ vertices is an n-dimensional cross-polytope centered at 0. A diagonal is a line segment that passes through the origin and links two non-adjacent vertices. The line segment linking any two adjacent vertices are the edges of the n-dimensional cross-polytope. The convex hull of each pair of vertices is either an edge* *or a diagonal. The faces of the n-dimensional cross-polytope are the simplices of $\mathbb{R}^{n-1}$, each with any n of the 2n vertices.*

Example 5.1.11. The basic n-dimensional cross-polytopes in $\mathbb{R}^n$ are as follows.

1. In $\mathbb{R}$, the 1-dimensional cross-polytope is a line segment, which is an edge, between 2 vertices.

2. In $\mathbb{R}^2$, the 2-dimensional cross-polytope is a convex quadrilateral. It has 2 pairs of non-adjacent vertices so it has 2 diagonals. There are 4 pairs of adjacent vertices linked by edges.

3. In $\mathbb{R}^3$, the 3-dimensional cross-polytope is the convex hull of an octahedron. It has 3 pairs of non-adjacent vertices so it has 3 diagonals. There are 12 pairs of adjacent vertices linked by edges.

4. In $\mathbb{R}^4$, the 4-dimensional cross-polytope is the convex hull of 8 vertices. Since it has 4 pairs of non-adjacent vertices, it has 4 diagonals. There are 24 pairs of adjacent vertices linked by edges. There are 16 simplectic faces (3-dimensional) of the 4-dimensional cross-polytope. This is a tetracross in $\mathbb{R}^4$.

5. In $\mathbb{R}^5$, the 5-dimensional cross-polytope is the convex hull of 10 vertices. Since it has 5 pairs of non-adjacent vertices, it has 5 diagonals. There are 40 pairs of adjacent vertices linked by edges. There are 32 simplectic faces (4-dimensional) of the 5-dimensional cross-polytope. This is a pentacross in $\mathbb{R}^5$.

We note that the n-dimensional cross-polytope in $\mathbb{R}^n$ is generated from the $(n-1)$-dimensional cross-polytope in $\mathbb{R}^{n-1}$ by the addition of two vertices in a dimension orthogonal to $\mathbb{R}^{n-1}$, each on opposite sides of $\mathbb{R}^{n-1}$. Each of these 2 vertices is linked to all vertices in the $(n-1)$-dimensional cross-polytope with an edge. The 2 new vertices are linked with a diagonal.

All simplices and cross-polytopes with lower dimensions exist as defined in the higher dimensions. We will write 'tetracross' and 'pentacross' for either the singular or plural cases.

5.2 Isometric Spheres in $\mathbb{R}^5$

Consider any Möbius map $T : \mathbb{R}^5_\infty \to \mathbb{R}^5_\infty$ such that $T(\infty) \neq \infty$. From Beardon ([7], p41) we can find a sphere in $\mathbb{R}^5$, $S(a,t)$ with $T^{-1}(a) = \infty$, such that $|T(x) - T(y)| = \dfrac{t|x-y|}{|x-a||y-a|}$. In fact $\lim_{y\to x} \dfrac{|T(y) - T(x)|}{|y-x|}$ is greater than, equal to, or less than 1 according as x is inside, on, or outside $S(a,t)$. We call $S(a,t)$ the isometric sphere of T and write $S(a,t) = I_T$ Thus the isometric sphere divides $\mathbb{R}^5_\infty$ into two regions. The outside region contains ∞, while the inside region contains $a = T^{-1}(\infty)$. In fact, by the above, I_T is the set of points for which the differential $T'(z)$ satisfies $|T'(z)| = 1$. The outside of I_T corresponds to points where $|T'(z)| < 1$,while the inside corresponds to points where $|T'(z)| > 1$.
If $g \in PSL(2,\Gamma_4)$ is identified with matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $g(z) = (\alpha x + \beta)(\gamma x + \delta)^{-1}$ then

$$g(z) - g(w) = (\alpha z + \beta)(\gamma z + \delta)^{-1} - (\alpha w + \beta)(\gamma w + \delta)^{-1} = ((\gamma z + \delta)^*)^{-1}(z - w)(\gamma w - \delta)^*$$

for $z, w \in \mathbb{R}^5$. In infinitesimal form we have $dg(z) = ((\gamma z + \delta)^*)^{-1} dz(\gamma z - \delta)^{-1}$. From this we deduce that $g'(z)$ is the matrix with operator norm $|g'(z)| = |\gamma z + \delta|^{-2}$. Ahlfors [2] defines the isometric sphere I_g as the set of points $|g'(z)| = 1$. That is, the hypersphere in $\mathbb{R}^5$, $|\gamma z + \delta| = 1$ with centre $-\gamma\delta^{-1} = \widetilde{g}^{-1}(\infty)$, $\gamma \neq 0$ and radius $|\gamma|^{-1}$.
Let $g \in PSL(2,\Gamma_4)$ have isometric sphere I_g. The intersection of I_g with $\mathbb{R}^4_\infty$ is a sphere in $\mathbb{R}^4_\infty$ with centre cd^{-1}, $c \neq 0$ and radius $|c|^{-1}$. The sphere given by $|cx + d| = 1$ is the 'equator' of the sphere in $\mathbb{R}^5$ given by $|cz + d| = 1$. The two spheres have the same centre and the same radius. That is $|\gamma|^{-1} = |c|^{-1}$ and $-cd^{-1}$ and $-\gamma\delta^{-1}$ are the same point in $\mathbb{R}^4$, $c \neq 0$ and $\gamma \neq 0$. We note that the upper hemisphere of $I_{\widetilde{g}}$ is a hyperplane in $\mathbb{H}^5$ for each $\widetilde{g} \in Iso^+\mathbb{H}^5$ if $\widetilde{g}(\infty) \neq \infty$.

5.3 Hyperbolic 5-space $\mathbb{H}^5$

We work in the model of hyperbolic 5-space $\mathbb{H}^5$ that is the upper half of the Euclidean space $\mathbb{R}^5$ with a metric defined as follows.

Definition 5.3.1. *Hyperbolic 5-space is given by the set*

$$\mathbb{H}^5 = \{(x_0, x_1, x_2, x_3, t) : x_p \in \mathbb{R}, p = 0 \cdots 3, t > 0\}$$

together with the associated hyperbolic metric

$$ds^2 = \frac{|d\zeta|^2}{t^2} = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dt^2}{t^2},$$

with $\zeta = (x_0, x_1, x_2, x_3, t)$.

It is convenient to identify the Euclidean point $\zeta = (x_0, x_1, x_2, x_3, t)$ with an ordered pair in $\mathbb{K} \times \mathbb{R}^+$. That is $\zeta = x + ti_3$ where $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{K}$ with $i = i_1$, $j = i_2$, $k = ij$ and $i_3 = (0, 0, 0, 0, 1)$, with $i_3^2 = -1$, which is orthogonal to any vector in $\mathbb{R}^4$. We note that $i_3 \in C_4$. Thus

$$\mathbb{H}^5 = \{(x, t) : x \in \mathbb{K}, t > 0\},$$

with hyperbolic (Riemannian) metric given by

$$|ds|^2 = \frac{|dx|^2 + dt^2}{t^2}.$$

We note that $\mathbb{K}_\infty$ is the boundary of $\mathbb{H}^5$. We recall that $\mathbb{K}$ is isomorphic to $\mathbb{R}^4$ as a vector space with basis $\{1, i, j, k\}$ and in fact to V^4 with basis vectors $\{1, i_1, i_2, i_3\}$.
Given two distinct points ζ_1 and ζ_2 in $\mathbb{H}^5$, we let Λ be a piecewise continuously differentiable path in $\mathbb{H}^5$ between ζ_1 and ζ_2.

Definition 5.3.2. *The hyperbolic length of* Λ *is defined as* $h(\Lambda) = \int_\Lambda \frac{|d\zeta|}{t}$ *with* $\zeta \in \mathbb{H}^5$.

Definition 5.3.3. *We define a metric* $\sigma : \mathbb{H}^5 \times \mathbb{H}^5 \to \mathbb{R}^+ \cup \{0\}$ *by* $\sigma(\zeta_1, \zeta_2) = \inf_\Lambda h(\Lambda)$ *where* Λ *ranges over all piecewise continuously differentiable paths between* ζ_1 *and* ζ_2.

From [7] we give, without proof, an explicit formula for the hyperbolic metric σ in $\mathbb{H}^5$.

Theorem 5.3.4. *If $\alpha = a + ti_3$ and $\beta = b + si_3 \in \mathbb{H}^5$ with $t, s > 0$, and $a, b \in \mathbb{K}$ then*

$$\sigma(\alpha, \beta) = \frac{2|\alpha - \beta|}{(1 + |\alpha|^2)^{1/2}(1 + |\beta|^2)^{1/2}}.$$

and hence equivalently

$$\sinh^2 \frac{\sigma(\alpha, \beta)}{2} = \frac{|\alpha - \beta|^2}{4ts}.$$

We note that there exists a path Λ in $\mathbb{H}^5$ which realises $\sigma(\zeta_1, \zeta_2)$. This is the hyperbolic line segment in $\mathbb{H}^5$ between ζ_1 and ζ_2. We call Λ the geodesic segment passing through ζ_1 and ζ_2 and we denote the geodesic segment between ζ_1 and ζ_2 as $\Lambda_{\zeta_1,\zeta_2}$ or $[\zeta_1, \zeta_2]$. Λ is a segment of a circle orthogonal to $\mathbb{R}^4$ and centered on $\mathbb{R}^4$ or part of a 'vertical' line segment orthogonal to $\mathbb{R}^4$, that joins ζ_1 and ζ_2. We note that $[\zeta_1, \zeta_2] = [\zeta_2, \zeta_1]$.

Definition 5.3.5. *Geodesics or hyperbolic lines in $\mathbb{H}^5$ are the Euclidean rays and the Euclidean semicircles with centers in $\mathbb{R}^4$ with the following properties:*

(a) They are orthogonal to $\mathbb{R}^4$,

(b) Their endpoints are in $\mathbb{R}^4_\infty$.

Definition 5.3.6. *$\mathbb{H}$-planes in $\mathbb{H}^5$ are defined as follows:*

1. *Let $P(a, d)$ be the hyperplane in $\mathbb{R}^5$ orthogonal to $\mathbb{R}^4$, where $a \in \mathbb{R}^4$. The upper half of this Euclidean plane is an $\mathbb{H}$-plane in $\mathbb{H}^5$.*

2. *Let $S(a, r)$ be the hypersphere in $\mathbb{R}^5$ orthogonal to $\mathbb{R}^4$. The upper hemisphere is an $\mathbb{H}$-plane in $\mathbb{H}^5$.*

From [7],[20] we have the following definitions:

Definition 5.3.7. *A set S is a convex set in $\mathbb{H}^5$ if $x, y \in S$ imply that $[x : y] \subset S$, where $[x : y]$ is the closed geodesic segment joining x and y.*

Definition 5.3.8. *The convex hull of a finite set P in $\mathbb{H}^5$ is the intersection of all the convex sets that contain P and is denoted by $CH(P)$.*

We note from [20] that a set P is a convex set in $\mathbb{H}^5$ if and only if $P = CH(P)$.
A 4-dimensional hyperbolic simplex and a 5-dimensional hyperbolic cross-polytope in $\mathbb{H}^5$ are defined as follows.

Definition 5.3.9. *A 4-dimensional simplex in $\mathbb{H}^5$ is a convex hull of 5 vertices v_1, v_2, v_3, v_4 and v_5 in $\mathbb{K}_\infty$ with 10 geodesic edges represented by $[v_l : v_m]$, $1 \leq l < m \leq 5$ in $\mathbb{H}^5$.*

We call the 4-dimensional hyperbolic simplex in $\mathbb{H}^5$ a simplex in $\mathbb{H}^5$ when the meaning is clear.

Definition 5.3.10. *The 5-dimensional hyperbolic cross-polytope in $\mathbb{H}^5$ is derived from any 4-dimensional cross-polytope in $\mathbb{R}^4$ by including as vertices the centre of this cross-polytope in $\mathbb{R}^4$ and it image (reflection) with respect to the $\mathbb{H}$-plane through the 8 vertices. Thus the vertices all lie on $\partial\mathbb{H}^5 = \mathbb{K}_\infty$. The convex hull in $\mathbb{H}^5$ of these 10 vertices is the 5-dimensional cross-polytope in $\mathbb{H}^5$. The edges of a 5-dimensional hyperbolic cross-polytope in $\mathbb{H}^5$ are geodesics in $\mathbb{H}^5$.*

We may call the 4-dimensional hyperbolic cross-polytope in $\mathbb{R}^4$ the tetracross in $\mathbb{H}^5$ and the 5-dimensional hyperbolic cross-polytope in $\mathbb{H}^5$ the pentacross in $\mathbb{H}^5$, when the situation is clear. As in the Euclidean case for the 5-dimensional cross-polytope in Example 5.1.11(5), the pentacross has 40 geodesic edges and 32 simplectic faces in $\mathbb{H}^5$.

Definition 5.3.11. *A reflection in a hyperplane in $\mathbb{H}^5$ is the Euclidean reflection in hyperspheres and hyperplanes in $\mathbb{R}^5$ that leave $\mathbb{H}^5$ invariant.*

We note from Beardon [7] that every reflection in a hyperplane is orientation-reversing and conformal.
For any Euclidean sphere, $S(a, r) = \{x \in \mathbb{R}^n : |x - a| = r\}$, where $a \in \mathbb{R}^n$ and $r > 0$, the following theorem is true:

Theorem 5.3.12. *If g is an inversion in the sphere $S(a, r)$ in $\mathbb{R}^n$ with radius r and centre a, then*

$$\frac{|g(y) - g(x)|}{|y - x|} = \frac{r^2}{|x - a||y - a|}.$$

Proof.

$$|g(y) - g(x)| = r^2 \left(\frac{1}{|y-a|^2} - \frac{2(x-a)\cdot(y-a)}{|x-a|^2|y-a|^2} + \frac{1}{|x-a|^2} \right)^{\frac{1}{2}} = \frac{r^2|y-x|}{|x-a||y-a|}.$$

□

If $y - x = h > 0$, then

$$\lim_{x\to y} \frac{|g(y) - g(x)|}{y - x} = \lim_{h\to 0} \frac{|g(x+h) - g(x)|}{h} = \frac{r^2}{|x-a|^2}.$$

This limit measures the local magnification of g at x.

5.4 The Poincaré Extensions in $\mathbb{H}^5$ of elements in $M\left(\mathbb{R}^4_\infty\right)$

We identify $\mathbb{R}^5$ with $\mathbb{R}^4 \times \mathbb{R}$ and $\mathbb{H}^5$ with $\mathbb{K} \times \mathbb{R}^+$.

Definition 5.4.1. *A map $f : \mathbb{H}^5 \to \mathbb{H}^5$ is an isometry if it preserves hyperbolic distances. That is, f is an isometry if $\sigma(f(z), f(w)) = \sigma(z, w)$ for every z and w in $\mathbb{H}^5$, where σ is the hyperbolic metric in $\mathbb{H}^5$.*

Definition 5.4.2. *$Iso^+\mathbb{H}^5$ is the group of orientation preserving isometries acting on $\mathbb{H}^5$.*

We show that $M\left(\mathbb{R}^4_\infty\right)$ is isomorphic to $Iso^+\mathbb{H}^5$, where $M\left(\mathbb{R}^4_\infty\right)$ is the group of orientation preserving transformations on $\mathbb{R}^4_\infty$. To this end we show that each $g \in M\left(\mathbb{R}^4_\infty\right)$ can to be uniquely extended to act in $\mathbb{H}^5$.

Poincaré noted that any g acting on $\mathbb{R}^4_\infty$ can be uniquely extended to a map $\widetilde{g}$ acting on $\mathbb{R}^5_\infty$ in the following way [7], [8]. Firstly, we note that $\mathbb{R}^4_\infty$ can be embedded in $\mathbb{R}^5_\infty$ by

$$x \mapsto \widetilde{x} = (x_0, x_1, x_2, x_3, 0), \text{ where } x = (x_0, x_1, x_2, x_3)$$

If $x \in \mathbb{R}^4_\infty$ and $y = g(x)$ for some $g \in M\left(\mathbb{R}^4_\infty\right)$, then $\widetilde{g}$ is defined to act on $\mathbb{R}^5_\infty$ as

$$\widetilde{g}(x_0, x_1, x_2, x_3, 0) = (y_0, y_1, y_2, y_3, 0) = \widetilde{g(x)}$$

where $y = (y_0, y_1, y_2, y_3)$ is in $\mathbb{R}^4_\infty$. It is in this sense that $\widetilde{g}$ is regarded as an extension of g with $\widetilde{g}(x, 0) = (g(x), 0)$.

For each reflection or inversion g acting in $\mathbb{R}^4_\infty$, we can thus define a reflection $\widetilde{g}$ acting in $\mathbb{R}^5$ as follows.

Definition 5.4.3. *1. Let g be the inversion of $x = (x_0, x_1, x_2, x_3)$ in the sphere $S(a, r)$ of radius r centred at $a = (a_0, a_1, a_2, a_3)$ in $\mathbb{R}^4_\infty$. Then $\widetilde{g}$ is the inversion of $z = (x_0, x_1, x_2, x_3, x_4)$ in the sphere $S(\widetilde{a}, r)$, where $\widetilde{a} = (a, 0)$ in $\mathbb{R}^5$, with formula*

$$\widetilde{g}(z) = \widetilde{g}(x, x_4) = (a, 0) + \left(\frac{r^2}{|(x - a, x_4)|^2}\right)(x - a, x_4)$$

2. Let g be a reflection of $x = (x_0, x_1, x_2, x_3)$ in the plane

$$P(a, d) = \{x \in \mathbb{R}^4 : (x \cdot a) = d\} \cup \{\infty\}$$

in $\mathbb{R}^4$. Then $\widetilde{g}$ is the inversion of $z = (x_0, x_1, x_2, x_3, x_4)$ in the plane

$$P(\widetilde{a}, d) = \{x \in \mathbb{R}^5 : (x \cdot \widetilde{a}) = d\} \cup \{\infty\}$$

in $\mathbb{R}^5$, where $\widetilde{a} = (a, 0)$, with formula

$$\widetilde{g}(z) = \widetilde{g}(x, x_4) = (x, x_4) - 2\left[(x, x_4) \cdot (a, 0) - d\right] \frac{(a, 0)}{|(a, 0)|^2}.$$

In both parts 1 and 2 of Definition 5.4.3 we call $\widetilde{g}$ the Poincaré extension of g.

We see from Definition 5.4.3 that $\widetilde{g}$ leaves the plane $x_4 = 0$ invariant. That is $\mathbb{K}_\infty \cong \mathbb{R}^4_\infty$ is left invariant by $\widetilde{g}$. Also the half-spaces $x_4 > 0$ and $x_4 < 0$ are left respectively invariant under the action of $\widetilde{g}$. This invariance proves that a Poincaré extension of g exists and is unique [7].

We now focus on the action of the Poincaré extension $\widetilde{g}$ in $\mathbb{H}^5$ as described by Beardon [7]. Let $\alpha = a + ti_3$ and $\beta = b + si_3 \in \mathbb{H}^5$, where $a, b \in \mathbb{K}$ and $s, t > 0$. Firstly, if $\widetilde{g}$ is the reflection in the sphere $S(\widetilde{a}, r)$, $a \in \mathbb{K}$, then by Theorem 5.3.12, we have

$$\frac{|\widetilde{g}(\alpha) - \widetilde{g}(\beta)|}{|\alpha - \beta|} = \frac{r^2}{|\beta - \widetilde{a}||\alpha - \widetilde{a}|}.$$

Let $[\widetilde{g}(\beta)]_5$ represent the 5th component of $\widetilde{g}(\beta)$. As

$$\widetilde{g}(\beta) = \widetilde{a} + \left(\frac{r^2(\beta - \widetilde{a})}{|\beta - \widetilde{a}|^2}\right),$$

we find that

$$[\widetilde{g}(\beta)]_5 = 0 + \frac{r^2 s}{|\beta - \widetilde{a}|^2}$$

and this shows that

$$\frac{|\widetilde{g}(\alpha) - \widetilde{g}(\beta)|^2}{[\widetilde{g}(\alpha)]_5\,[\widetilde{g}(\beta)]_5} = \frac{|\alpha - \beta|^2}{st}.$$

Hence $\dfrac{|\alpha - \beta|^2}{st}$ is invariant under $\widetilde{g}$ and $\widetilde{g}$ is an isometry of $\mathbb{H}^5$.

The reflection $\widetilde{g}$ in the plane $P(\widetilde{a}, d)$, $a \in \mathbb{R}^4$, is a Euclidean isometry and moreover,

$$[\widetilde{g}(\beta)]_5 = s$$

thus $\dfrac{|\alpha - \beta|^2}{st}$ is also invariant under this reflection. We conclude that $\dfrac{|\alpha - \beta|^2}{st}$ is invariant under all Poincaré extensions. It is the direct consequence of this invariance that the Poincaré extension of any g in $GM\left(\mathbb{R}^4_\infty\right)$ is an isometry of the space $\mathbb{H}^5$ endowed with the Riemannian metric σ given by

$$ds = \frac{|d\zeta|}{t}.$$

Further we observe that if g_i and f_j are inversions in $\mathbb{R}^4_\infty$, then $\widetilde{g_i f_j} = \widetilde{g_i}\widetilde{f_j}$. Thus if g and f are in $GM\left(\mathbb{R}^4_\infty\right)$ with $g = g_1 g_2 \ldots g_n$ and $f = f_1 f_2 \ldots f_m$ where the g_i and f_j are inversions in $\mathbb{K}_\infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, then $\widetilde{gf} = (\widetilde{g_1}\widetilde{g_2} \ldots \widetilde{g_n}\widetilde{f_1}\widetilde{f_2} \ldots \widetilde{f_m}) = \widetilde{g}\widetilde{f}$, so the map $f \mapsto \widetilde{f}$ is an injective homomorphism of $GM\left(\mathbb{R}^4_\infty\right)$ into $GM\left(\mathbb{R}^5_\infty\right)$.

We note that if $g \in M\left(\mathbb{R}^4_\infty\right)$ is the composition of an even number of reflections in hyperplanes and hyperspheres in $\mathbb{K}_\infty$ then $\widetilde{g}$ is the composition of an even number of reflections in hyperplanes and hyperspheres in $\mathbb{R}^5_\infty$, where each such hyperplane is orthogonal to $\mathbb{R}^4_\infty$ and each such hypersphere $S(a, r)$ is centered at $a \in \mathbb{R}^4$ and is orthogonal to $\mathbb{R}^4_\infty$ at its equator which is formed by $S(a, r) \subset \mathbb{R}^4$.

The upper halves of the hyperplanes and hyperspheres in $\mathbb{R}^5_\infty$ that are orthogonal to $\mathbb{R}^4_\infty$ as described above form $\mathbb{H}$-planes in $\mathbb{H}^5$. Since the Poincaré extension of every orientation preserving Möbius transformation in $\mathbb{K}_\infty$ to a Möbius transformation acting in $\mathbb{R}^5_\infty$ leaves the space $x_4 = 0$ and the half-space $x_4 > 0$ invariant, the Poincaré extensions are in fact orientation preserving isometries of $\mathbb{H}^5 \cong \mathbb{K} \times \mathbb{R}^+$.

5.5 Orientation Preserving Isometries of $\mathbb{H}^5$

Ahlfors [1], Waterman [37], Kellerhals [19] and Wilker [38] note this close relationship between $PS_{\mathcal{D}}L(2,\mathbb{K})$ and $Iso^+(\mathbb{H}^5)$, the orientation preserving isometries of $\mathbb{H}^5$ with the boundary $\mathbb{K}_\infty$ as described above. The orientation preserving isometries of $\mathbb{H}^5$ may also be given by the restrictions of Möbius transformations in $M\left(\mathbb{R}^5_\infty\right)$ that stabilise the upper half space and leave the boundary fixed.

Ahlfors [1][2] and Waterman [37] further establish $Iso^+\mathbb{H}^5$ as follows.

Let C_4 be the Clifford algebra generated by i_1, i_2, i_3 with the usual properties and let Γ_4 be the Clifford group of products of vectors in V^4, where $V^4 \cong \mathbb{R}^4$ as a vector space over $\mathbb{R}$. We note again that $\mathbb{R} = C_1$, $\mathbb{C} = C_2$ and $\mathbb{K} = C_3$. We have seen that $GL(2, C_4) = GL(2,\Gamma_4)$. We note that $SL(2,\Gamma_4) = \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, C_4) : \alpha\delta^* - \beta\gamma^* = 1\right\}$ and that $SL(2,\Gamma_4)$ preserves hyperbolic 5–space $\mathbb{H}^5$ where

$$\mathbb{H}^5 = \left\{x \in \mathbb{R}^5 : x = (x_0, x_1, x_2, x_3, x_4) \text{ and } x_4 > 0\right\}$$

with metric induced from the differential $ds = \dfrac{|dx|}{x_4}$.

[1] and [37] establish that $PSL(2, C_4) \cong SL(2, C_4)/\{\pm I_2\}$ is $Iso^+\mathbb{H}^5$, the full group of orientation preserving isometries of $\mathbb{H}^5$ since they leave both V^4 and V^5 invariant (See Theorem 4.3.7).

In conclusion we note that each map $g \in PS_{\mathcal{D}}L(2,\mathbb{K})$ given by $g(x) = (ax+b)(cx+d)^{-1}$ can be extended by using the Poincaré method to a bijection $\widetilde{g} : \mathbb{H}^5_\infty \to \mathbb{H}^5_\infty$ given by $\widetilde{g}(z) = (\alpha z + \beta)(\gamma z + \delta)^{-1}$ where $\alpha\beta^*, \gamma\delta^*, \delta^*\beta$ and $\gamma^*\alpha \in \mathbb{R}^4$, $\alpha\delta^* - \beta\gamma^* \in \mathbb{R} \setminus \{0\}$ and $\alpha, \beta, \gamma, \delta \in \Gamma_4$.

Conversely, each map $g \in PSL(2,\Gamma_4)$ when restricted to $\mathbb{K}_\infty$ can be represented as $g(x) = (ax+b)(cx+d)^{-1}$, where $a, b, c, d \in \mathbb{K}$ and $\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. That is each map $g \in PSL(2,\Gamma_4)$ that leaves $\mathbb{R}^5_\infty$ and $\mathbb{R}^4_\infty$ invariant is a Poincaré extension of some map in $PS_{\mathcal{D}}L(2,\mathbb{K})$.

Chapter 6

Unimodular Matrices and the Unimodular Group

We make extensive use of 2×2 matrices with entries in $\mathbb{K}$. We recall that, for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$, we have defined $\mathcal{D}(A) = N(a)N(d) + N(b)N(c) - S(a\bar{c}d\bar{b})$. We note that $g : \mathbb{K}_\infty \to \mathbb{K}_\infty$ is given by $g(z) = (az + b)(cz + d)^{-1}$ and $g \in M\left(\mathbb{R}^4_\infty\right)$. In what follows we make use of the particular matrices $\tau_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\delta_\mu = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$ in $M_2(\mathbb{K})$ as generators of an arbitrary $g \in M\left(\mathbb{R}^4_\infty\right)$, where λ and μ are non zero quaternions.

6.1 Unimodular Matrices and the associated Unimodular Group

Definition 6.1.1. *A matrix in $M_2(\mathbb{K})$ is called integral if its entries are in $\mathcal{H}$, the Hurwitz integers.*

Definition 6.1.2. *A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{K})$ is called unimodular if A is integral and*

$\mathcal{D}(A) = 1$. Let $U = \{A \in M_2(\mathbb{K}) : \mathcal{D}(A) = 1$ and A is integral$\}$ be the set of such unimodular matrices.

Lemma 6.1.3. *Let $\tau_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\delta_\mu = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$, with $\lambda \in \mathcal{H}$ and μ a unit in $\mathcal{H}$. Then τ_λ, ϕ and δ_μ are in U and so are their inverses.*

Proof. $\mathcal{D}(\tau_\lambda) = 1$ and $\mathcal{D}(\phi) = 1$. Also $\phi^{-1} = \phi$ and $\mathcal{D}\left(\tau_\lambda^{-1}\right) = 1$. So τ_λ and ϕ and their inverses are in U. $\mathcal{D}\left(\delta_\mu\right) = \mathcal{D}\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} = N(\mu) = 1$, so $\delta_\mu \in U$. Further $\delta_\mu^{-1} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and $\mu^{-1} = \overline{\mu}$ is a unit. □

According to Mahler [23], the unimodular matrices form a multiplicative group. The proof of this theorem involves showing that $A^{-1} \in U$ for each $A \in U$. To do this it suffices to show that each $A \in U$ can be factorised into a product of a finite number of matrices τ_λ, ϕ and δ_μ, each of which has an inverse in U. Following [23] we establish the next lemma.

Lemma 6.1.4. *Each element $A \in U$ can be written as a finite product of matrices τ_λ, ϕ and δ_μ as given above.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$, where $a, b, c, d \in \mathcal{H}$ and $\mathcal{D}(A) = 1$.

If $c = 0$ then $\mathcal{D}(A) = 1$ and $N(a)N(d) = 1$. Hence $N(a) = N(d) = 1$ and a and d are units. Thus a^{-1} and $d^{-1} \in \mathcal{H}$. In this case $A = \delta_a \tau_{a^{-1}bd^{-1}} \phi \delta_d \phi$ and the result holds for $c = 0$.

If $c \neq 0$, then by Theorem 3.5.3(3) we can find $\lambda \in \mathcal{H}$ such that $c^{-1}d - \lambda$ lies in ∇ since each quaternion is congruent to a quaternion in ∇. Therefore

$$\left|c^{-1}d - \lambda\right| \leq \frac{1}{\sqrt{2}} \text{ and then } N(d - c\lambda) \leq \frac{1}{2}N(c).$$

Let $A_1 = A\tau_\lambda^{-1}\phi = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ or $A = A_1\phi\tau_\lambda$ and $c_1 = d - c\lambda$. Thus $0 \leq N(c_1) \leq \frac{1}{2}N(c)$ and $N(d - c\lambda) \in \mathbb{Z}$.

If $c_1 \neq 0$ we find an integral quaternion λ_1 such that $A_1 = A_2\phi\tau_{\lambda_1}$ and $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, where $0 \leq N(c_2) \leq \frac{1}{2}N(c_1) \leq \frac{1}{4}N(c)$ and $N(c_2) \in \mathbb{Z}$.

Repeating this process, if necessary, we finally find a matrix A_p with coefficient $c_p = 0$.

So $A = A_1\phi\tau_\lambda = A_2\phi\tau_{\lambda_1}\phi\tau_\lambda = \cdots = A_p\phi\tau_{\lambda_{p-1}}\phi\tau_{\lambda_{p-2}}\cdots\phi\tau_{\lambda_1}\phi\tau_\lambda$ and A_p can be decomposed as the case where $c = 0$ above. □

Theorem 6.1.5. *The unimodular matrices form a multiplicative group.*

Proof. $U \subseteq M_2(\mathbb{K})$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U$, so $U \neq \emptyset$. Further, for each $A \in U$ we have $A^{-1} \in U$ by Lemma 6.1.4. Since $\mathcal{D}(AB) = \mathcal{D}(A)\mathcal{D}(B)$, U is also closed under multiplication. □

If $gcd_r(a, c) = u$, $u \in \mathcal{U}$ and c not a unit then there exist $p \neq 0$ and q in $\mathcal{H}$ such that $u = pa - qc$. Since u is unit we write $1 = \bar{u}pa - \bar{u}qc = \bar{u}p(ac^{-1} - p^{-1}q)c$. Thus $N(1) = 1 = N(\bar{u}p)N(c)N(ac^{-1} - p^{-1}q)$. Let $t = p^{-1}qp$. Then $\mathcal{D}\begin{pmatrix} a & t \\ c & p \end{pmatrix} = 1$.

If $c \neq 0$ is a unit then $\mathcal{D}\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix} = 1$.

Conversely, if $\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ with $c, d \neq 0$ then $N(c)N(d)N(ac^{-1} - bd^{-1}) = N(da - dbd^{-1}c) = 1$. So $da - dbd^{-1}c = u$, $u \in \mathcal{U}$ and hence $gcd_r(a, c) = u$.
If $d = 0$ then b and c are units and $gcd_r(a, c) = u$, $u \in \mathcal{U}$.
Similarly if $c = 0$.

Hence we have established the following result from [30]:

Theorem 6.1.6. *Let $a, c \in \mathcal{H}$. $\begin{pmatrix} a \\ c \end{pmatrix}$ can occur as a column in a unimodular matrix if and only if $gcd_r(a, c) = u$, u a unit.*

A similar result holds for a row of a unimodular matrix. That is for $a, b \in \mathcal{H}$, $\begin{pmatrix} a & b \end{pmatrix}$ can occur as a row in a unimodular matrix if and only if $gcd_l(a, b) = u$, u a unit.

6.2 The Fundamental Domain of Unimodular Group G

We have shown that the set

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathcal{H} \text{ and } \mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}$$

is a multiplicative group. We have noted that

$$\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = N(a)N(d) + N(b)N(c) - S(a\bar{c}d\bar{b}).$$

This group may be denoted by $S_{\mathcal{D}}L((2, \mathcal{H})$, or the special linear matrices over $\mathcal{H}$ where the $\mathcal{D}$-determinant is 1. Each matrix $A \in U$ is identified with the bijection $g : z \to (az + b)(cz + d)^{-1}$ of $\mathcal{Q}_\infty$ onto itself. Here $\mathcal{Q}_\infty$ is regarded as the set of all reduced rationals over $\mathcal{H}$, including ∞. If $ac^{-1} \in \mathcal{Q}_\infty$ and ac^{-1} is not reduced then $gcd_r(a, c) = t$ and t not a unit. Let $a = a_1 t$ and $c = c_1 t$ and $gcd_r(a_1, c_1) = u$, u a unit, so $a_1 c_1^{-1}$ is reduced. Now $a_1 c_1^{-1} = a_1 t t^{-1} c_1^{-1} = (a_1 t)(c_1 t)^{-1} = ac^{-1}$. So every rational in $\mathcal{Q}_\infty$ can be written in reduced form.

For the bijection g we note that $g(\infty) = ac^{-1}$, $g(0) = bd^{-1}$, while $g(-c^{-1}d) = \infty$ and $g(-ca^{-1}) = 0$. We represent the group of these associated bijective maps on $\mathcal{Q}_\infty$ as
$PS_{\mathcal{D}}L(2, \mathcal{H}) = G$, the unimodular group of homographic maps. So

$$G = PS_{\mathcal{D}}L(2, \mathcal{H}) = \left\{ z \mapsto (az + b)(cz + d)^{-1} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U \right\}$$

and $PS_{\mathcal{D}}L(2, \mathcal{H}) \cong S_{\mathcal{D}}L(2, \mathcal{H})/\{\pm I_2\}$. G is a geometrically finite discrete subgroup of $PS_{\mathcal{D}}L(2, \mathbb{K})$ which we have established is the group of orientation preserving isometries of $\mathbb{H}^5$ (Section 5.5).

6.2.1 The Orbit of ∞ under G and the Stabilizer of ∞ under G

Let $g \in G$ with $g(z) = (az + b)(cz + d)^{-1}$. Certainly, $g(\infty) = ac^{-1} \in \mathcal{Q}_\infty$. Thus the orbit of ∞ under G is contained in $\mathcal{Q}_\infty$. Consider any reduced rational mn^{-1} in $\mathcal{Q}_\infty$. By Theorem 6.1.6, we can find $s, t \in \mathcal{H}$ such that $A = \begin{pmatrix} m & s \\ n & t \end{pmatrix}$ and $\mathcal{D}(A) = 1$. So $A \in U$ and $g : z \to (mz + s)(nz + t)^{-1}$ is its corresponding unimodular map. Finally $g(\infty) = mn^{-1}$ and thus mn^{-1} is in the orbit of ∞ under G. If $mn^{-1} \in \mathcal{Q}_\infty$ is not reduced, then $gcd_r(m, n) = t$, t not a unit. Let $m = m_1 t$ and $n = n_1 t$, $gcd_r(m_1, n_1) = u$, u a unit, where $m_1 n_1^{-1} = (m_1 t)(n_1 t)^{-1} = mn^{-1}$. Now since $gcd_r(m_1, n_1) = u$, there exists $g \in G$ with $g(x) = (m_1 x + b)(n_1 x + d)^{-1}$. Then $g(\infty) = m_1 n_1^{-1} = mn^{-1}$ and mn^{-1} is in the orbit of ∞ under G.

Thus the orbit of ∞ under G is exactly $\mathcal{Q}_\infty$. We have established the following lemma.

Lemma 6.2.1. *The orbit of ∞ under G is exactly $\mathcal{Q}_\infty$.*

We know that $G_\infty = \{g \in G : g(\infty) = \infty\}$ is a subgroup of G and G is generated by elements with matrix form $\tau_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ where $\lambda \in \mathcal{H}$, $\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\delta_\mu = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$ where $\mu \in \mathcal{U}$. Certainly $\tau_\lambda(\infty) = \infty$ for all $\lambda \in \mathcal{H}$ and $\delta_\mu(\infty) = \infty$ for all $\mu \in \mathcal{U}$, thus we have the following lemma.

Lemma 6.2.2. $G_\infty = \langle \tau_\lambda, \delta_\mu \rangle$ *for $\lambda \in \mathcal{H}$ and $\mu \in \mathcal{U}$.*

We do note that $\delta_\mu(0) = 0$ and ∞ and 0 are both fixed by δ_μ. Let T be a subgroup of elements in G generated by generated by τ_λ, $\lambda \in \mathcal{H}$ only. So $T = \langle \tau_\lambda \rangle \leq G_\infty \leq G$.

6.2.2 Dirichlet Polygons and Fundamental Regions

We revisit the Dirichlet region ∇ introduced in Section 3.5, where

$$\nabla = \{x \in \mathbb{K} : |x| \leq |x - a|, a \in \mathcal{H}\}.$$

Let $\overset{\circ}{\nabla}$ is the interior of ∇ where $\partial\nabla = \nabla\backslash\overset{\circ}{\nabla}$ is the boundary of ∇. Thus ∇, the closed polygon in $\mathbb{K}$, is a fundamental region for T where $\mathbb{K}$ has the Euclidean metric. We consider the region $\overset{\circ}{\nabla}_\infty = \left\{(z,t) \in \mathbb{H}^5 : z \in \overset{\circ}{\nabla}, t > 0\right\}$. We think of $\overset{\circ}{\nabla}_\infty$ as a 'tower' whose footprint in $\mathbb{K}$ is $\overset{\circ}{\nabla}_\infty$. We know from Section 5.2 that each unimodular map, $g \in G$, with associated matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in U, has an isometric hypersphere defined by $|cx + d| = 1$ if $c \neq 0$. These isometric hyperspheres can be represented by $I_g = \left\{z \in \mathbb{R}^5 : |g'(z)| = 1\right\}$ and denotes the complete locus of points in the neighbourhood of which lengths and areas are unaltered in magnitude by the map g. Here $g'(z)$ stands for the Jacobian of the conformal transformation $g : z \rightarrow (az + b)(cz + d)^{-1}$ considered as a vector function of $\mathbb{R}^5$ onto $\mathbb{R}^5$. The Jacobian is the matrix

$$J = \frac{\partial(u_0, u_1, u_2, u_3)}{\partial(z_0, z_1, z_2, z_3)}$$

with $z = (z_0, z_1, z_2, z_3, z_4)$, $g(z) = (u_0, u_1, u_2, u_3, u_4)$ and $|g'(z)|^2 = detJ$. The region outside the isometric hypersphere I_g has $|g'(z)| < 1$.

From Vulakh [35] we learn that the region $D_0 = \overset{\circ}{\nabla}_\infty \cap \left\{z \in \mathbb{H}^5 : |g'(z)| < 1, g \in G\right\}$ is a fundamental region for G in $\mathbb{H}^5$. So D_0 is the intersection of the 'tower' $\overset{\circ}{\nabla}_\infty$ with the outside of all isometric spheres of g, $g \in G$ where $g(\infty) \neq \infty$.

Let $K := K(\infty) = G_\infty\left(\overline{D_0}\right)$, the union of all images of $\overline{D_0}$ under G_∞, with $\overline{D_0}$ as the hyperbolic closure of D_0 in $\mathbb{H}^5$. For any $g \in G$ let $gK(\infty) := K(g(\infty))$. Let ∂K be the boundary of K. Following [35] we say that the floor of the fundamental region D_0 is $\partial K \cap \overline{D_0}$. We will be interested in the components of ∂K of dimension 0 (vertices or cusps), of dimension 1 (edges) and of dimension 4 (faces). The vertices and edges of K that belong to $\overline{D_0}$ are called the vertices and edges of D_0. We note that the sections of vertical faces of ∇_∞ that lie in K become vertical faces of D_0. (Here ∇_∞ is the 'tower' whose footprint in $\mathbb{K}$ is ∇.)

We consider the point v in $\mathbb{H}^5$ given by $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{\sqrt{2}}\right)$. This point lies at the intersection of the 8 unit isometric hyperspheres of elements in G whose centers are the lattice points

0, 1, i, $1+i$, ω, $\omega-k$, $\omega-j-k$ and $\omega-j$. The hypersphere in $\mathbb{R}^5$ centered at $\frac{1+i}{2}$ of radius $\frac{1}{\sqrt{2}}$ also passes through v. Further all hyperplanes in $\mathbb{R}^5$ through $\frac{1+i}{2}$ orthogonal to $\mathbb{R}^4$ will also pass through v. To be specific, $\frac{1+i}{2}$ is the projection from ∞ of v to $\mathbb{R}^4 \cong \mathbb{K}$. Let $v_{\mathbb{K}} = \frac{1+i}{2}$. The point v lies in $\overline{D_0}$ (and K) and v is called a vertex of D_0.

We note that the orientation preserving isometries of $\mathbb{H}^5$ that leave v fixed are the composition of an even number of reflections in $\mathbb{H}$-planes in $\mathbb{H}^5$ via the Poincare extension. These $\mathbb{H}$-planes are part of hyperspheres or hyperplanes in $\mathbb{R}^5$ that are orthogonal to $\mathbb{R}^4$. To leave v fixed, these $\mathbb{H}$-planes must lie on the hyperspheres and hyperplanes given above and must pass through v.

6.3 The Structure of $\mathcal{N}(v)$ and the Farey Tessellation of $\mathbb{H}^5$

We noted above that $v = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{\sqrt{2}}\right)$ lies on the 8 unit isometric hyperspheres in $\mathbb{R}^5$ that have centres at the 8 points 0, 1, i, $1+i$, ω, $\omega-k$, $\omega-j-k$ and $\omega-j$. The surfaces of these hyperspheres form part of the floor of K. We denote by ϕ_t the part of the isometric hypersphere of $g_t \in G$ that is part of the floor of K. The centre of the isometric hypersphere is given by $u_t = g_t(\infty)$. The projection of face ϕ_t from ∞ onto $\mathbb{R}^4 \cong \mathbb{K}$ is denoted by the polytope $p(u_t)$. In particular, the face of K that lies on the unit hypersphere with centre the origin that is on the isometric hypersphere of $g(x) = \frac{x}{|x|^2}$ and is denoted be ϕ_0. The projection of the face ϕ_0 from ∞ to $\mathbb{K}$ is the polytope $p(u_0) = \nabla$ (Section 3.5) with $u_0 = 0$. ∇ and its images under T tessellate $\mathbb{K}$ (Theorem 3.5.3(3)), so the polytopes $p(u_t)$ tessellate $\mathbb{K}$ under the action of $T \subseteq G_\infty$.

The point $v_{\mathbb{K}} = \frac{1+i}{2}$ is a common vertex of the polytopes $p(u_t)$ where

$$u_t \in \{0, 1, i, 1+i, \omega, \omega-k, \omega-j-k, \omega-j\} = \mathcal{H}_v.$$

Let $A_{v_{\mathbb{K}}}$ be the convex hull of the points in $\mathcal{H}_v$. Thus $A_{v_{\mathbb{K}}}$ is a tetracross in $\mathbb{R}^4$ (Example

5.1.11(4)), centred at $\frac{1+i}{2}$, with opposite pairs of points given in the sets

$$\{0, 1+i\}, \quad \{1, i\}, \quad \{\omega, \omega - j - k\}, \quad \{\omega - k, \omega - j\}.$$

Let $P(v_{\mathbb{K}})$ be the 'tower' in $\mathbb{H}^5$ whose footprint in $\mathbb{K}$ is $A_{v_{\mathbb{K}}}$, that is

$$P(v_{\mathbb{K}}) = \left\{(z, t) : z \in A_{v_{\mathbb{K}}}, t > 0\right\}.$$

We denote by $\mathcal{A}(v)$, the intersection of $P(v_{\mathbb{K}})$ with K. We call $\mathcal{A}(v)$ a v-component. Then $g\left(\mathcal{A}(v)\right) := \mathcal{A}\left(g(v)\right)$ are all v-components. The v-components tessellate $\mathbb{H}^5$ under G since the sets $g\left(D_0\right)$ tessellate $\mathbb{H}^5$ under G.

Regarding $\mathbb{H}^5 = \{(x, t) : x \in \mathbb{K}, t > 0\}$, this v-component $\mathcal{A}(v)$ is the convex hull of 34 vertices in $\mathbb{H}^5$. The 34 vertices include ∞ and the vertex $v = \left(\frac{1+i}{2}, \frac{\sqrt{2}}{2}\right)$ which projects onto the point $v_{\mathbb{K}} = \frac{1+i}{2} \in \mathbb{K}$ from ∞. There are 8 vertices where $t = 1$. The projection of these 8 points from ∞ onto $\mathbb{K}$ give the vertices of the tetracross centered at $v_{\mathbb{K}} = \frac{1+i}{2}$ in $\mathbb{K}$ with its vertices in $\mathcal{H}_v$. The remaining 24 points have $t = \frac{\sqrt{3}}{2}$. The projection of these 24 points from ∞ onto $\mathbb{K}$ lie on a sphere with radius $\frac{1}{2}$ and are the vertices of the 24-cell centered at $v_{\mathbb{K}} = \frac{1+i}{2}$.

Definition 6.3.1. *The v-cell of v denoted by $\mathcal{N}(v)$ is given by $\bigcup_{g \in G_v} g\left(\mathcal{A}(v)\right) = \bigcup_{g \in G_v} \mathcal{A}\left(g(v)\right)$, where G_v is the stabiliser of v in G.*

Since the stabiliser G_v is generated by reflections in an even number of $\mathbb{H}$-planes passing through v, we find that $\mathcal{N}(v)$ has vertices in the set $\mathcal{H}_v \cup \left\{\frac{1+i}{2}, \infty\right\}$. By Definition 5.3.10, $\mathcal{N}(v)$ is a pentacross in $\mathbb{H}^5$ derived from the tetracross $A_{v_{\mathbb{K}}}$ with vertices in $\mathcal{H}_v$, including the centre of the tetracross, $\frac{1+i}{2}$, and ∞, as the pair of new vertices. $\mathcal{N}(v)$ is a 5-dimensional cross-polytope with 40 geodesic edges and 32 simplectic faces.

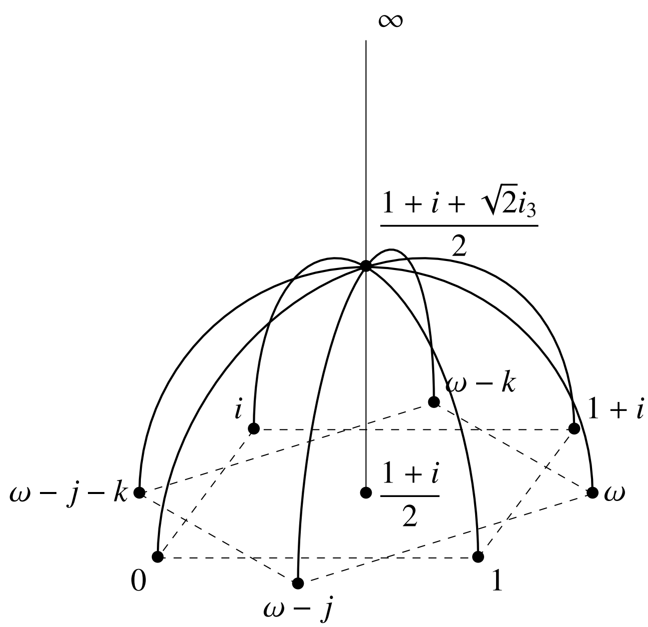

Figure 6.1: The v-cell $\mathcal{N}(v)$ centered at $\frac{1+i}{2}$.

Figure 6.1 illustrates the 5 diagonals of the pentacross, each joining a pair of opposite vertices. The two squares that form the base are orthogonal in $\mathbb{K}$ and only intersect at the point $\frac{1+i}{2} = (1-i)^{-1}$.

The creation of $\mathcal{N}(v)$ above is analogous to the creation of the ideal triangle $\{0, 1, \infty\}$ as the union of the images of the fundamental domain of the Modular group under the stabilizer of $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the Modular group [31].

Chapter 7

Farey Simplices

The next two chapters develop the ideas of Farey simplices in the space of quaternions given in Schmidt [30]. We note that Schmidt develops all the results in $\mathbb{K}$. We re-interpret his results in $\mathbb{H}^5$ with boundary $\mathbb{K}_\infty$. We distinguish a particular class of geodesics in $\mathbb{H}^5$ and hence particular classes of simplices and pentacross in $\mathbb{H}^5$.

7.1 Farey Neighbours and Farey Geodesics

We know

$$\mathcal{Q}_\infty = \{ab^{-1} : a, b \in \mathcal{H}, b \neq 0\} \cup \{\infty\} = \left\{ \frac{h_1 + h_2 i + h_3 j + h_4 \omega}{r} : h_1, h_2, h_3, h_4, r \in \mathbb{Z} \right\}$$

is the orbit of ∞ under G. We recall a rational ac^{-1} is reduced if $gcd_r(a, c) = u, u \in \mathcal{U}$. We assume that $\infty = 1(0)^{-1}$ is reduced. Further note that if ac^{-1} is reduced with $N(c) = 1$, then c is a unit and thus $ac^{-1} = a\bar{c} \in \mathcal{H}$ since $c\bar{c} = N(c) = 1$.

Definition 7.1.1. *Two reduced rationals, ac^{-1} and bd^{-1} in $\mathcal{Q}_\infty$ are said to be Farey neighbours if $\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. We write $ac^{-1} \sim bd^{-1}$.*

In fact, each reduced rational ac^{-1} has infinitely many Farey neighbours. Thus, since $\mathcal{D}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = 1$ for all $b \in \mathcal{H}$, we have that $\infty \sim b$ for all Hurwitz integers b. We note that $\mathcal{D}\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} = 1$ if and only if $N(c) = 1$ if and only if c is a unit if and only if $ac^{-1} = a\overline{c} \in \mathcal{H}$.

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$ with corresponding map $g \in G$ and $g(x) = (ax + b)(cx + d)^{-1}$ then $g(\infty) = ac^{-1}$ and $g(0) = bd^{-1}$ and $ac^{-1} \sim bd^{-1}$.

We prove the following important results about Farey neighbours.

Lemma 7.1.2. *If $ac^{-1} \sim bd^{-1}$ then $h\left(ac^{-1}\right) \sim h\left(bd^{-1}\right)$ for any $h \in G$.*

Proof. Since $ac^{-1} \sim bd^{-1}$, we know $\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.

Let $g \in G$ with $g(x) = (ax + b)(cx + d)^{-1}$. Thus $h\left(ac^{-1}\right) = hg(\infty)$ and $h\left(bd^{-1}\right) = hg(0)$ with $h, g \in G$. Hence we can find a_1, b_1, c_1, d_1 with $hg(x) = (a_1x + b_1)(c_1x + d_1)^{-1}$, such that $hg \in G$ with $hg(\infty) = a_1c_1^{-1}$, $hg(0) = b_1d_1^{-1}$ and $\mathcal{D}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = 1$. Thus we have that $a_1c_1^{-1} \sim b_1d_1^{-1}$ or $h\left(ac^{-1}\right) \sim h\left(bd^{-1}\right)$ as required. □

Definition 7.1.3. *A geodesic in $\mathbb{H}^5$ is called a Farey geodesic if its endpoints on $\mathbb{K}_\infty$ are Farey neighbours.*

If ac^{-1} and bd^{-1} are the endpoints of a geodesic we may write the geodesic as $\left[ac^{-1} : bd^{-1}\right] = \left[bd^{-1} : ac^{-1}\right]$. In particular, $\left[1(0)^{-1} : 0(1)^{-1}\right]$ is a Farey geodesic, and we refer to it as the fundamental Farey geodesic, denoted by $\mathbb{I}_0$. It follows that if Λ is a Farey geodesic with endpoints ac^{-1} and bd^{-1} in $\mathbb{K}_\infty$ and $h \in G$, then $h(\Lambda)$ has endpoints $h\left(ac^{-1}\right)$ and $h\left(bd^{-1}\right)$ which are Farey neighbours and thus $h(\Lambda)$ is a Farey geodesic.

In fact we have:

Lemma 7.1.4. *The set of all Farey geodesics in $\mathbb{H}^5$ is exactly the orbit of $\mathbb{I}_0$ under G.*

Proof. Let $g \in G$. Then $g[\mathbb{I}_0] = [g(\infty), g(0)] = \left[ac^{-1} : bd^{-1}\right]$ where $g(x) = (ax+b)(cx+d)^{-1}$. So $g[\mathbb{I}_0]$ is a Farey geodesic with endpoints ac^{-1} and bd^{-1}, since $ac^{-1} \sim bd^{-1}$.
Conversely, any Farey geodesic $\left[ac^{-1} : bd^{-1}\right]$ with $\mathcal{D}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ can be written as $g[\mathbb{I}_0]$, where $g(x) = (ax+b)(cx+d)^{-1}$ and $g \in G$. □

We note that the 40 geodesic edges of the pentacross $\mathcal{N}(v)$ are all Farey geodesics in $\mathbb{H}^5$. Thus we can state the following lemma.

Lemma 7.1.5. *The geodesics in the Farey tessellation of $\mathbb{H}^5$ by $\mathcal{N}(v)$ under G are in the orbit of $\mathbb{I}_0$ under G and are all the Farey geodesics. In fact each $g(\mathcal{N}(v)) := \mathcal{N}(g(v))$, $g \in G$, is composed of 40 geodesic edges.*

These Farey geodesics play a central role in the development of the Farey tessellation of $\mathbb{H}^5$ by simplices.

7.2 Farey Simplices in $\mathbb{H}^5$

The simplices in $\mathbb{H}^5$, whose vertices are pairwise Farey neighbours, form a special set of simplices.

Definition 7.2.1. *A hyperbolic simplex in $\mathbb{H}^5$ is a Farey simplex $\mathbb{FS}$ if the bounding geodesics are all Farey geodesics. Thus a Farey simplex is a simplex in $\mathbb{H}^5$ whose vertices $p_t q_t^{-1}$, $p_t, q_t \in \mathcal{H}$, $1 \leq t \leq 5$ in pairs are Farey neighbours with $p_m q_m^{-1} \sim p_n q_n^{-1}$ for $1 \leq m < n \leq 5$. This Farey simplex, denoted by $\mathbb{FS}\left(p_1q_1^{-1}, p_2q_2^{-1}, p_3q_3^{-1}, p_4q_4^{-1}, p_5q_5^{-1}\right)$, is associated with the Farey matrix*

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix}.$$

If $\mathbb{FS}$ is a Farey simplex then $\overline{\mathbb{FS}}$ is its closure. Thus $\overline{\mathbb{FS}} = \mathbb{FS} \cup \Lambda_{\infty,0} \cup \Lambda_{0,1} \cup \Lambda_{1,\infty} \cup \cdots \cup \Lambda_{\omega,\omega-k}$, where $\Lambda_{a,b}$ is the geodesic with endpoints $a, b \in \mathbb{K}_\infty$.

Definition 7.2.2. *A Farey simplex with ∞ as a vertex is called degenerate. A Farey simplex where ∞ is not a vertex is called non-degenerate.*

Definition 7.2.3. *The fundamental simplex $\mathbb{FS}_0$ in $\mathbb{H}^5$ is defined to be the Farey simplex with vertex set $\{\infty, 0, 1, \omega, \omega - k\}$. It is a degenerate Farey simplex in $\mathbb{H}^5$ since ∞ is a vertex and all edges are Farey geodesics.*

From Theorem 4.4.4 we note that $\mathcal{D}\begin{pmatrix} p_m & p_n \\ q_m & q_n \end{pmatrix} = 1$, $m \neq n$, is invariant under right multiplication of columns by units. That is $\mathcal{D}\begin{pmatrix} p_m u & p_n \\ q_m u & q_n \end{pmatrix} = \mathcal{D}\begin{pmatrix} p_m & p_n v \\ q_m & q_n v \end{pmatrix} = 1$ for any $u, v \in \mathcal{U}$. Since $p_m q_m^{-1} \sim p_n q_n^{-1}$, any matrix obtained from $\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix}$ by permuting the columns or multiplying any column on the right by a unit is also associated with the Farey simplex $\mathbb{FS}\left(p_1 q_1^{-1}, p_2 q_2^{-1}, p_3 q_3^{-1}, p_4 q_4^{-1}, p_5 q_5^{-1}\right)$.

We note that for any Farey simplex a denominator q_t may be zero for at most one t, $1 \leq t \leq 5$. Also if $q_t = 0$ for one t value, $1 \leq t \leq 5$, then ∞ is a cusp of the Farey Simplex and the other cusps are Hurwitz integers. We see that if $p_m q_m^{-1} \sim p_n q_n^{-1}$ then $p_m q_m^{-1} \sim (p_n u)(q_n u)^{-1}$ for any unit $u \in \mathcal{H}$.

If ∞ is not a vertex of the Farey simplex $\mathbb{FS}$ then the projection of $\mathbb{FS}$ from ∞ onto $\mathbb{K}$ is a simplex in $\mathbb{K}$ with the same vertices as $\mathbb{FS}$. We call this simplex in $\mathbb{K}$ a Farey polytope (to distinguish it from the Farey simplex $\mathbb{FS}$) and denote it by $\mathbb{PFS}$. The vertices of the non-degenerate Farey simplex $\mathbb{FS}$ are the vertices of the corresponding Farey polytope $\mathbb{PFS}$. The geodesic edges of $\mathbb{FS}$ are projected to Euclidean line segments in $\mathbb{K}$ so $\mathbb{PFS}$ lies in $\mathbb{K}$.

From [30], we have the result that every Farey simplex is in the orbit of $\mathbb{FS}_0$ under G.

Theorem 7.2.4. *Any Farey simplex is associated with a Farey matrix of the form*

$$A\begin{pmatrix} 1 & 0 & 1 & \omega & \omega-k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where A is a unimodular matrix in U corresponding to $g \in G$.

Proof. Let the Farey simplex have associated matrix of the form $\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix}$ (Definition 7.2.1). Then

$$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^{-1} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \pi_3 & \pi_4 & \pi_5 \\ 0 & 1 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix},$$

where $A = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \in U$, $g \in G$. We know that $p_s q_s^{-1} \sim p_t q_t^{-1}$, $s \neq t$ and so by Lemma 7.1.2 we know that $g(p_s q_s^{-1}) \sim g(p_t q_t^{-1})$ for each $A \in U$, $g \in G$.
Consider A^{-1} in U corresponding to $g^{-1} \in G$.
With $s = 1$ and $t = 3$ we have $\infty \sim \pi_3 \mu_3^{-1}$. Thus $\mathcal{D}\begin{pmatrix} 1 & \pi_3 \\ 0 & \mu_3 \end{pmatrix} = 1$ and $N(\mu_3) = 1$, so μ_3 is a unit.
Further with $s = 2$ and $t = 3$ we have $0 \sim \pi_3 \mu_3^{-1}$. Thus $\mathcal{D}\begin{pmatrix} 0 & \pi_3 \\ 1 & \mu_3 \end{pmatrix} = 1$ and $N(\pi_3) = 1$, so π_3 is a unit. By similar arguments for $t = 4$ and $t = 5$, we find that π_t and μ_t are units. We know that a matrix obtained from $\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{pmatrix}$ by permuting the columns or multiplying any column on the right by a unit is associated with the same Farey simplex $\mathbb{FS}\left(p_1 q_1^{-1}, p_2 q_2^{-1}, p_3 q_3^{-1}, p_4 q_4^{-1}, p_5 q_5^{-1}\right)$. Hence for any unit $\epsilon \in \mathcal{H}$ the Farey matrix is associated with

$$\begin{pmatrix} p_1\pi_3\epsilon & p_2\mu_3\epsilon & p_3\epsilon & p_4\mu_4^{-1}\mu_3\epsilon & p_5\mu_5^{-1}\mu_3\epsilon \\ q_1\pi_3\epsilon & q_2\mu_3\epsilon & q_3\epsilon & q_4\mu_4^{-1}\mu_3\epsilon & q_5\mu_5^{-1}\mu_3\epsilon \end{pmatrix}$$

$$= \begin{pmatrix} p_1\pi_3\epsilon & p_2\mu_3\epsilon \\ q_1\pi_3\epsilon & q_2\mu_3\epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \epsilon^{-1}\pi\epsilon & \epsilon^{-1}\mu\epsilon \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{7.1}$$

where $\pi = \pi_3^{-1}\pi_4\mu_4^{-1}\mu_3$ and $\mu = \pi_3^{-1}\pi_5\mu_5^{-1}\mu_3$ are units.

Using Lemma 7.1.2, with $s = 3$ and $t = 4$ we see that $1 \sim \epsilon^{-1}\pi\epsilon$. Now

$$\begin{aligned}\mathcal{D}\begin{pmatrix} 1 & \epsilon^{-1}\pi\epsilon \\ 1 & 1 \end{pmatrix} &= N(1) + N(\epsilon^{-1}\pi\epsilon) - S(\overline{\epsilon^{-1}\pi\epsilon}) \text{ by Theorem 4.4.3} \\ &= N(1) + N(\epsilon^{-1}\pi\epsilon) - S(\epsilon^{-1}\overline{\pi}\epsilon) \text{ since } \epsilon^{-1} = \overline{\epsilon} \text{ and } \overline{\pi} = \pi^{-1} \\ &= 1 + 1 - S(\overline{\pi})\end{aligned}$$

Since $\mathcal{D}\begin{pmatrix} 1 & \epsilon^{-1}\pi\epsilon \\ 1 & 1 \end{pmatrix} = 1$ we have $S(\overline{\pi}) = S(\pi) = 1$. Similarly $S(\overline{\mu}) = S(\mu) = 1$.

Further, $\mathcal{D}\begin{pmatrix} \epsilon^{-1}\pi\epsilon & \epsilon^{-1}\mu\epsilon \\ 1 & 1 \end{pmatrix} = \mathcal{D}\begin{pmatrix} \pi & \mu \\ 1 & 1 \end{pmatrix} = N(\pi) + N(\mu) - S(\pi\overline{\mu}) = 1$. Thus $S(\pi\overline{\mu}) = 1$.

So by Theorem 3.2.9 we can find a unit $\epsilon \in \mathcal{H}$ such that $\epsilon^{-1}\pi\epsilon = \omega$ and $\epsilon^{-1}\mu\epsilon = \omega - k$ or $\epsilon^{-1}\pi\epsilon = \omega - k$ and $\epsilon^{-1}\mu\epsilon = \omega$ as required.

By equation 7.1, $\begin{pmatrix} 1 & 0 & 1 & \epsilon^{-1}\pi\epsilon & \epsilon^{-1}\mu\epsilon \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$ is thus the fundamental Farey simplex, so the result follows. □

Consider, in particular, the Farey simplex $\mathbb{FS}_1 = \mathbb{FS}\left((1-i)^{-1}, 1+i, i, \omega - j - k, \omega - j\right)$, where $\dfrac{1+i}{2}$ is written in reduced form as $(1-i)^{-1}$ in the Farey simplex. We have noted that the 10 points of the simplices $\mathbb{FS}_0$ and $\mathbb{FS}_1$ are the vertices of the pentacross that constitutes the v-cell $\mathcal{N}(v)$, where $v = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{H}^5$. We also note that the 10 vertices can be paired into diagonals of the pentacross $\mathcal{N}(v)$ in $\mathbb{H}^5$. The diagonals of $\mathcal{N}(v)$, $\left[\infty : (1-i)^{-1}\right]$, $[0 : 1+i]$, $[1 : i]$, $[\omega : \omega - j - k]$ and $[\omega - k : \omega - j]$, intersect in the point v in $\mathbb{H}^5$. From this point we refer to $\mathcal{N}(v)$ as the fundamental pentacross and write $\mathbb{FP}_0$.

Using Theorem 7.2.4 we find $g \in G$ such that $g(\mathbb{FS}_0) = \mathbb{FS}_1$.

Example 7.2.5. Consider the matrices

$$\begin{pmatrix} 1 & 0 & 1 & \omega & \omega - k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1+i & i & \omega - j - k & \omega - j \\ 1-i & 1 & 1 & 1 & 1 \end{pmatrix}$$

corresponding to the two Farey simplices $\mathbb{FS}_0$ and $\mathbb{FS}_1$ above.

We let $A = \begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix} \in M_2(\mathbb{K})$, then $\mathcal{D}(A) = N(1)N(1) + N(1+i)N(1-i) - S\left(1\left(\overline{1-i}\right)1\left(\overline{1+i}\right)\right) = 5 - S(2) = 1$. Using the procedure of Theorem 7.2.4 we note that

$$\begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1+i & i & \omega-j-k & \omega-j \\ 1-i & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \pi_3 & \pi_4 & \pi_5 \\ 0 & 1 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix},$$

where $N(\pi_t) = N(\mu_t) = 1$, for $3 \leq t \leq 5$. We also know that any Farey matrix that corresponds to a Farey simplex is associated with a matrix obtained by permuting columns and/or right multiplication by a unit in $\mathcal{H}$. Thus

$$\begin{aligned} &\begin{pmatrix} 1 & 1+i & i & \omega-j-k & \omega-j \\ 1-i & 1 & 1 & 1 & 1 \end{pmatrix} \\ =& \begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \pi_3 & \pi_4 & \pi_5 \\ 0 & 1 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} \\ =& \begin{pmatrix} 1 & 1+i & \pi_3+(1+i)\mu_3 & \pi_4+(1+i)\mu_4 & \pi_5+(1+i)\mu_5 \\ 1-i & 1 & (1-i)\pi_3+\mu_3 & (1-i)\pi_4+\mu_4 & (1-i)\pi_5+\mu_5 \end{pmatrix} \end{aligned} \tag{7.2}$$

or

$$\begin{aligned} &\begin{pmatrix} 1 & 1+i & i & \omega-j-k & \omega-j \\ 1-i & 1 & 1 & 1 & 1 \end{pmatrix} \\ =& \begin{pmatrix} 1\pi_3\epsilon & (1+i)\mu_3\epsilon & i\epsilon & (\omega-j-k)\mu_4^{-1}\mu_3\epsilon & (\omega-j)\mu_5^{-1}\mu_3\epsilon \\ (1-i)\pi_3\epsilon & 1(\mu_3\epsilon) & 1(\epsilon) & 1(\mu_4^{-1}\mu_3\epsilon) & 1(\mu_5^{-1}\mu_3\epsilon) \end{pmatrix} \\ =& \begin{pmatrix} 1\pi_3\epsilon & (1+i)\mu_3\epsilon \\ (1-i)\pi_3\epsilon & 1(\mu_3\epsilon) \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & \epsilon^{-1}\pi\epsilon & \epsilon^{-1}\mu\epsilon \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \end{aligned} \tag{7.3}$$

where $\pi = \pi_3^{-1}\pi_4\mu_4^{-1}\mu_3$ and $\mu = \pi_3^{-1}\pi_5\mu_5^{-1}\mu_3$ are units. Let $A_1 = \begin{pmatrix} 1\pi_3\epsilon & (1+i)\mu_3\epsilon \\ (1-i)\pi_3\epsilon & 1(\mu_3\epsilon) \end{pmatrix}$.

From Equation 7.2 we have

$$i = (\pi_3 + (1+i)\mu_3)\left((1-i)\pi_3 + \mu_3\right)^{-1}, \tag{7.4}$$

$$\omega - j - k = (\pi_4 + (1+i)\mu_4)\left((1-i)\pi_4 + \mu_4\right)^{-1} \tag{7.5}$$

and

$$\omega - j = (\pi_5 + (1+i)\mu_5)\left((1-i)\pi_5 + \mu_5\right)^{-1} \tag{7.6}$$

From Equations 7.4, 7.5 and 7.6 we get $i = \mu_3\pi_3^{-1}$, $j + k - \omega = \pi_4\mu_4^{-1}$ and $j - \omega = \pi_5\mu_5^{-1}$. Since $\mu_3 = i\pi_3$, $\pi = \pi_3^{-1}\{\omega - i - k\}\pi_3$ and $\mu = \pi_3^{-1}\{1-\omega\}\pi_3$. So $\pi\mu = \pi_3^{-1}\{-i\}\pi_3$.
Further for some $\epsilon \in \mathcal{U}$, $\pi = \epsilon\omega\epsilon^{-1}$ and $\mu = \epsilon(\omega - k)\epsilon^{-1}$. Thus $\pi\mu = \pi_3^{-1}\{-i\}\pi_3 = \epsilon(j)\epsilon^{-1}$, then $-i\pi_3\epsilon = \pi_3\epsilon j$. Let $p = \pi_3\epsilon$, so $-ip = pj$. From Table 9.1 we see that $p = \omega - i$. That is $-ip = -i(\omega - i) = j - \omega$ and $pj = (\omega - i)j = j - \omega$.
With $A = \begin{pmatrix} 1\pi_3\epsilon & (1+i)\mu_3\epsilon \\ (1-i)\pi_3\epsilon & 1(\mu_3\epsilon) \end{pmatrix} = \begin{pmatrix} \omega - i & i - j \\ j - i & \omega - j \end{pmatrix}$, the corresponding map is
$g(z) = ((\omega - i)z + i - j)\left((j - i)z + \omega - j\right)^{-1}$ and $g \in G$ with $g(\mathbb{FS}_0) = \mathbb{FS}_1$.

7.3 Properties of Farey Polytopes in $\mathbb{K}$

From [30] we have the following result.

Lemma 7.3.1. *Let Γ be a sphere in $\mathbb{K}$ with radius $\sqrt{2}$ and arbitrary centre O. Then O is an interior point of a polytope $\mathcal{P}$ with vertices $x_1, x_2, \cdots, x_m$ where m depends on O, $8 \le m \le 48$, such that $\mathcal{P}$ satisfies the following conditions:*

1. *$x_l \in \mathcal{H}$, $1 \le l \le m$*
2. *all the n faces of $\mathcal{P}$, $16 \le n \le 192$ are regular tetrahedra of edge length 1*
3. *all the n spheres through O and the vertices of a face of $\mathcal{P}$ lie in the closed ball bounded by Γ*

Proof. We suppose that O is the point 0 in $\mathbb{K}$. $\mathcal{P}_{\mathcal{U}}$, the convex hull of the 24 units in $\mathcal{U}$, has 24 octahedral faces (Section 3.4), each with 6 vertices in $\mathcal{U}$. Each of the 24 permutations of the point $(\pm1, \pm1, 0, 0)$ is symmetric to 0 in one of the 24 octahedral faces of $\mathcal{P}_{\mathcal{U}}$.

A sphere centre 0 and radius $\sqrt{2}$ will include all the 24 units in $\mathcal{U}$ and the 24 symmetric vertices. So $m = 48$. Say these vertices are labeled $z_1, z_2, \cdots, z_{48}$. We assume that $z_1, z_2, \cdots, z_{24}$ are the 24 units in $\mathcal{U}$, while $z_{25}, \cdots, z_{48}$ are the symmetric vertices with norm 2. The polytope $\mathcal{P}$, with the 48 vertices, has $n = 192 = 24 \times 8$ regular tetrahedral faces of edge length 1. Each tetrahedral face has one vertex z_l, with $25 \leq l \leq 48$ and the other 3 vertices are from one of the 8 triangular face of the corresponding octahedral face in $\mathcal{P}_{\mathcal{U}}$. So the total number of tetrahedral faces of $\mathcal{P}$ is $\frac{24.6.4.1}{3} = 24.8 = 192$. Hence the conditions 1. and 2. are satisfied.

Each of the n spheres passes through 0 and the vertices of a given tetrahedral face. Since there are 192 tetrahedral faces there may be 192 spheres, but each sphere will occur 8 times. Thus there are just 24 distinct spheres all lying inside Γ, each having diameter $\sqrt{2}$ or radius $\frac{1}{\sqrt{2}}$, thus condition 3. is satisfied.

If the point O is a Hurwitz integer, but not 0, then the result follows after a translation by some h in $\mathcal{H}$.

If $O \notin \mathcal{H}$ then, since $\{\mathcal{P}_{\mathcal{U}} + z : z \in I\}$ is a regular tessellation of $\mathbb{K}$, we may assume that $O \in \mathcal{P}_{\mathcal{U}}$. Now the polytope $\mathcal{P}$ defined above in the special case where $O = 0$ may be subdivided into 24 tetracross of edge length 1. Each tetracross is a double pyramid with an octahedral face of $\mathcal{P}_{\mathcal{U}}$ as a base and it has 0 and the element in the set of symmetric vertices opposite 0 with respect to the tetracross as the two other vertices.

We define the polytope $\mathcal{P}$ as the union of those tetracross for which O is an interior point of the corresponding circumscribed spheres. Since $O \notin \mathcal{H}$, this union is nonempty. By construction $\mathcal{P}$ satisfies the conditions 1., 2. and 3. as well as the inequalities on m and n stated in the theorem. □

The following theorem [30] relates a quaternion ζ to a Farey polytope with a vertex, a rational $p_0 q_0^{-1}$ in $\mathcal{Q}_\infty$, that approximates ζ. This result is an analogue of a result that any real number ζ lies in a Farey interval whose endpoints are rationals approximating ζ [8]. This result emphasizes the importance of Farey polytopes as a tool to approximate $\zeta \in \mathbb{K}$.

Theorem 7.3.2. *Let ζ be a quaternion satisfying the inequality* $\left|\zeta - p_0 q_0^{-1}\right| \leq \frac{1}{\sqrt{2}N(q_0)}$, *where* $p_0, q_0 \in \mathcal{H}$, $q_0 \neq 0$ *and* $gcd_r(p_0, q_0) = u \in \mathcal{U}$. *Then ζ belongs to a Farey polytope* $\mathbb{PFS}$ *having* $p_0 q_0^{-1}$ *as one of its vertices. The constant* $\sqrt{2}$ *is the least possible.*

Proof. We consider a closed ball in $\mathbb{K}$ bounded by a hypersphere C with centre at $p_0 q_0^{-1}$ and radius $\frac{1}{\sqrt{2}N(q_0)}$. We need to show that this closed ball in $\mathbb{K}$ is covered by a set of Farey polytopes having $p_0 q_0^{-1}$ as a vertex. We will show that at most 192 such $\mathbb{PFS}$ are sufficient to cover the ball bounded by C.

Since $gcd_r(p_0, q_0) = u$, using Theorem 6.1.6 we can find a map

$$g : z \to (p_0 z + P)(q_0 z + Q)^{-1} = w$$

where $\mathcal{D}\begin{pmatrix} p_0 & P \\ q_0 & Q \end{pmatrix} = 1$ and $g(\infty) = p_0 q_0^{-1}$ and $g(-q_0^{-1}Q) = \infty$. Since g is a Möbius map it maps hyperspheres to hyperspheres. Similarly g^{-1} exists and maps hyperspheres to hyperspheres.

If $Q \neq 0$ and $q_0 \neq 0$ we have $N\left(p_0 q_0^{-1} - PQ^{-1}\right) N(Q) N(q_0) = 1$ by Theorem 4.4.4(4), hence $N\left(Pq_0 - p_0 q_0^{-1} Q q_0\right) = 1$. Further, with $w = g(z)$,

$$\begin{aligned} \left(P - p_0 q_0^{-1} Q\right)(q_0 z + Q)^{-1} q_0 &= \left(p_0 z + P - p_0 q_0^{-1}(q_0 z + Q)\right)(q_0 z + Q)^{-1} q_0 \\ &= (p_0 z + P)(q_0 z + Q)^{-1} q_0 - p_0 \\ &= w q_0 - p_0. \end{aligned}$$

Since $N\left(Pq_0 - p_0 q_0^{-1} Q q_0)\right) = 1$ we have

$$\begin{aligned} N(wq_0 - p_0) &= N\left\{\left(P - p_0 q_0^{-1} Q\right)(q_0 z + Q)^{-1} q_0\right\} \\ &= N\left(P - p_0 q_0^{-1} Q\right) N(q_0) N\left((q_0 z + Q)^{-1}\right) \\ &= N\left(Pq_0 - p_0 q_0^{-1} Q q_0\right) N\left((q_0 z + Q)^{-1}\right) \\ &= N\left((q_0 z + Q)^{-1}\right) \text{ since } N\left(P - p_0 q_0^{-1} Q\right) = 1. \end{aligned}$$

Hence we have $|wq_0 - p_0| = |q_0 z + Q|^{-1}$.

$$\begin{aligned}
\left|w - p_0 q_0^{-1}\right| \leq \frac{1}{\sqrt{2}N(q_0)} &\Longleftrightarrow |wq_0 - p_0| \left|q_0^{-1}\right| \leq \frac{1}{\sqrt{2}N(q_0)} \\
&\Longleftrightarrow |wq_0 - p_0| \leq \frac{1}{\sqrt{2}\,|q_0|} \text{ since } q_0^{-1} = \frac{\overline{q_0}}{N(q_0)} \text{ and } |q_0| = |\overline{q_0}| \\
&\Longleftrightarrow |q_0 z + Q|^{-1} \leq \frac{1}{\sqrt{2}\,|q_0|} \\
&\Longleftrightarrow \left|q_0 \left(z + q_0^{-1} Q\right)\right|^{-1} \leq \frac{1}{\sqrt{2}\,|q_0|} \\
&\Longleftrightarrow \left|z + q_0^{-1} Q\right|^{-1} |q_0|^{-1} \leq \frac{1}{\sqrt{2}\,|q_0|} \\
&\Longleftrightarrow \left|z + q_0^{-1} Q\right|^{-1} \leq \frac{1}{\sqrt{2}} \\
&\Longleftrightarrow \left|z + q_0^{-1} Q\right| \geq \sqrt{2}
\end{aligned}$$

Consequently, $g^{-1}(C) = \Gamma$ where $g\left(-q_0^{-1}Q\right) = \infty$ and Γ is the hypersphere in $\mathbb{K}$ with radius $\sqrt{2}$ and centre $-q_0^{-1}Q$. Hence by Lemma 7.3.1 the closed ball in $\mathbb{K}$ bounded by Γ, with centre $O = -q_0^{-1}Q$, is covered by the n Farey polytopes denoted by

$$\mathbb{PFS}\left(-q_0^{-1}Q, g^{-1}\left(z_{v_1}\right), g^{-1}\left(z_{v_2}\right), g^{-1}\left(z_{v_3}\right), g^{-1}\left(z_{v_4}\right)\right)$$

where $1 \leq v \leq n$ for $16 \leq n \leq 192$. The points $g^{-1}\left(z_{v_1}\right), g^{-1}\left(z_{v_2}\right), g^{-1}\left(z_{v_3}\right), g^{-1}\left(z_{v_4}\right) \in \mathcal{H}$ are the vertices of the n tetrahedral faces of the polytope $\mathcal{P}$ of Lemma 7.3.1.

Now, the interior of Γ corresponds by g to the exterior of C. So the application of g to these n Farey polytopes result in the exterior of C being covered by the n images of the Farey polytopes under g. The image of each Farey polytope is a polytope with vertex set $\{\infty, z_{v_1}, z_{v_2}, z_{v_3}, z_{v_4}\}$ for1 $\leq v \leq n$ for $16 \leq n \leq 192$. The vertices $z_{v_1}, z_{v_2}, z_{v_3}, z_{v_4}$ are Hurwitz integers since they are Farey neighbours of ∞. The vertices z_{v_t} lie inside the hypersphere C since the vertices $g^{-1}\left(z_{v_t}\right)$ lie outside of Γ, $1 \leq t \leq 4$.

The image under g of the polytope with vertex set $\{\infty, z_{v_1}, z_{v_2}, z_{v_3}, z_{v_4}\}$ are the n Farey polytopes

$$\mathbb{PFS}\left(p_0 q_0^{-1}, g\left(z_{v_1}\right), g\left(z_{v_2}\right), g\left(z_{v_3}\right), g\left(z_{v_4}\right)\right)$$

where $g(\infty) = p_0q_0^{-1}$ is a vertex, as these images have $g(z_{v_t}) \neq \infty$ as vertices, where $1 \leq v \leq n$ for $16 \leq n \leq 192$ and $1 \leq t \leq 4$.

If $Q = 0$ and $q_0 \neq 0$ then $PQ^{-1} = \infty$ and $p_0q_0^{-1} \in \mathcal{H}$ since $gcd_r(p_0, q_0) = u$. So $q_0 \in \mathcal{U}$ and $N(q_0) = 1$. If $\left|\zeta - p_0q_0^{-1}\right| \leq \dfrac{1}{\sqrt{2}N(q_0)} = \dfrac{1}{\sqrt{2}}$ where $p_0q_0^{-1} \in \mathcal{H}$ then ζ is in a Farey polytope of the form $h(\mathbb{PFS}_1)$ for $h \in G$, when $\mathbb{PFS}_1$ is the Farey polytope for $\mathbb{FS}_1 = \left\{(1-i)^{-1}, 1+i, i, \omega - j - k, \omega - j\right\}$ from Theorems 3.5.3(3) and 3.6.2.

The constant $\sqrt{2}$ in the statement is the best possible when $\zeta = \dfrac{1+i}{2}$ and $p_0q_0^{-1} = 0$. □

Chapter 8

Farey Subdivision

8.1 The Norm of Farey Simplices and Farey Polytopes

We recall that $\mathcal{P}_{\mathcal{U}}$ is a 24-cell composed of 24 octahedral cells (Section 3.4). The 24-cell has 24 neighbours under the action of $I = \langle 1+i \rangle$ with neighbours sharing an octahedral face. In particular, the octahedral face given by the 6 vertices $A_1 = 1, A_2 = i, B_1 = \omega, B_2 = \omega - j - k$ and $C_1 = \omega - k, C_2 = \omega - j$, has diagonals $[A_1 : A_2]$, $[B_1 : B_2]$, and $[C_1 : C_2]$ (See Figure 3.1). The centre of the octahedron is $\dfrac{1+i}{2} = (1-i)^{-1}$. This octahedron is the common octahedral face between $\mathcal{P}_{\mathcal{U}}$ and $\mathcal{P}_{\mathcal{U}} + (1+i)$. We further note that the vertices A_1, A_2, B_1, B_2, C_1 and C_2 together with 0 and $1+i$ yield the 8 centers of the isometric spheres that form the base of the fundamental region of G given in Section 6.2.2. These 8 points all lie in $\mathbb{K}$ and the convex hull of these 8 vertices form a tetracross in $\mathbb{K}$.

Finally we recall that a Farey simplex in $\mathbb{H}^5$ is the convex hull of points

$$\left\{p_1q_1^{-1}, p_2q_2^{-1}, p_3q_3^{-1}, p_4q_4^{-1}, p_5q_5^{-1}\right\}$$

in $\mathbb{H}^5$ where $p_i, q_i \in \mathcal{H}$ and these points are pairwise Farey neighbours. That is, $p_mq_m^{-1} \sim p_nq_n^{-1}$ or $\mathcal{D}\begin{pmatrix} p_m & p_n \\ q_m & q_n \end{pmatrix} = 1$ for $m \neq n$ and $1 \leq m, n \leq 5$. If $q_m \neq 0$ for $1 \leq m \leq 5$ then the vertices of the Farey simplex all lie in $\mathbb{K}$. The projection of this non-degenerate Farey simplex into $\mathbb{K}$ is a simplex in $\mathbb{R}^4$ called a Farey polytope $\mathbb{PFS}$.

We have established in Theorem 7.2.4 that each Farey simplex in $\mathbb{H}^5$ is in the orbit of $\mathbb{FS}_0$ under G. Hence G acts transitively on the Farey simplices and the Farey polytopes.

Definition 8.1.1. *The norm of the Farey simplex* $\mathbb{FS}\left(p_1q_1^{-1}, p_2q_2^{-1}, p_3q_3^{-1}, p_4q_4^{-1}, p_5q_5^{-1}\right)$ *(or the corresponding Farey polytope) is* $\mathbf{N}(\mathbb{FS}) = \sum_{t=1}^{5} N(q_t)$.

Since ∞ is a Farey neighbour of only the Hurwitz integers in $\mathcal{H}$ the norm of a Farey simplex with ∞ as a vertex is always 4. The norm of a non-degenerate Farey simplex (or its corresponding Farey polytope) is thus always greater than or equal to 6. We may write the simplex with vertices $p_iq_i^{-1}$, $i = 1, 2, 3, 4, 5$ with $N(q_1) \geq N(q_2) \geq N(q_3) \geq N(q_4) \geq N(q_5)$.

Definition 8.1.2. *The diameter of a Farey polytope is the length of its longest side.*

From Schmidt [30] we have the following result.

Lemma 8.1.3. *The* $\mathbb{PFS}$ *of norm* 6 *constitute a tessellation of the space of quaternions* $\mathbb{K}$.

Proof. We have shown that $\{\mathcal{P}_{\mathcal{U}} + z : z \in I\}$ is a regular tessellation of $\mathbb{K}$ (Theorem 3.4.3), where $I = \langle 1 + i \rangle$ is a 2-sided ideal of $\mathcal{H}$. We show that $\mathcal{P}_{\mathcal{U}}$ is tessellated by all $\mathbb{PFS}$ of norm 6.

Consider $\mathcal{P}_{\mathcal{U}}$ and subdivide it into 24 pyramids with apex at 0 and base one of the 24 octahedral faces on its boundary. Each of the 24 pyramids is divided into 8 $\mathbb{PFS}$ each of norm 6. In particular we consider the pyramid from 0 with the face that has vertex set $\{A_1, A_2, B_1, B_2, C_1, C_2\}$ (given above) with $\frac{1+i}{2}$ as its centre. This pyramid is subdivided into the 8 $\mathbb{PFS}$ with vertex sets given by $\left\{0, A_{l_1}, B_{l_2}, C_{l_3}, (1 - i)^{-1}\right\}$ with $l_v = 1, 2$ and $1 \leq v \leq 3$. Thus $\mathcal{P}_{\mathcal{U}}$ is subdivisible into $24 \times 8 = 192$ $\mathbb{PFS}$, each of norm 6. Thus $\mathbb{K}$ is tessellated by $\mathbb{PFS}$ of norm 6 under the action of I.

Secondly, we show that each $\mathbb{PFS}$ of norm 6 occurs in the tessellation constructed above. The norms of the 'denominators' of the vertices of the $\mathbb{PFS}$ of norm 6 can only be 1, 1, 1, 1 and 2. That is, if the $\mathbb{PFS}$ has 0 as a vertex it must have 3 units in $\mathcal{H}$ as vertices, where 0 and the 3 unit vertices span a regular tetrahedron of side 1. The fifth vertex is then determined

as it is one of the two Farey neighbours to the vertices of the tetrahedron, where these 2 points are symmetric in the hyperplane through the tetrahedron.
Consider the $\mathbb{PFS}$ with 0, 1, ω and $\omega - k$ as the first 4 vertices, then the points $\frac{1+i}{2}$ and $\frac{1+j}{2}$ are symmetric to each other about the plane through the 4 vertices 0, 1, ω and $\omega - k$ and they are both Farey neighbours to each of these 4 vertices. The convex hulls of either of the points $\frac{1+i}{2}$ and $\frac{1+j}{2}$ and the 4 given vertices result in two $\mathbb{PFS}$.
So for each equilateral triangle on the boundary of $\mathcal{P}_{\mathcal{U}}$, when including the vertex 0, we can find two $\mathbb{PFS}$. Since $\mathcal{P}_{\mathcal{U}}$ has 96 regular triangles of side 1 on the boundary there are at most $192 = 96 \times 2$ $\mathbb{PFS}$ of norm 6 having 0 as a vertex.
Hence the tessellation constructed above, which contains exactly $192 = 24 \times 8$ $\mathbb{PFS}$ of norm 6 with 0 as vertex must contain all such $\mathbb{PFS}$.
In an arbitrary Farey polytope of norm 6, four of the vertices are in $\mathcal{H}$ and span a regular tetrahedron of edge 1. Hence the four vertices in $\mathcal{H}$ are in different residue classes modulo I, where $\mathcal{H}/I$ is a ring with exactly 4 cosets given by I, $1+I$, $\omega+I$ and $\omega-k+I$. Thus one of the vertices considered is in I. Consequently an arbitrary $\mathbb{PFS}$ of norm 6 is the translate by $z \in I$ of a $\mathbb{PFS}$ of norm 6 with 0 as a vertex. This implies that the tessellation constructed above contains all $\mathbb{PFS}$ of norm 6. □

Lemma 8.1.4. *The convex hull of the tetracross given by the vertices* $\{0, A_1, A_2, B_1, B_2, C_1, C_2, 1+i\}$ *is covered by* 16 $\mathbb{PFS}$ *of norm* 6*.*

Proof. By Lemma 8.1.3 we know that the pyramid with apex 0 and base with vertex set $\{A_1, A_2, B_1, B_2, C_1, C_2\}$ subdivides into 8 $\mathbb{PFS}$ of norm 6 with vertices

$$\left\{0, A_{l_1}, B_{l_2}, C_{l_3}, (1-i)^{-1}\right\}$$

with $l_v = 1, 2$ and $1 \leq v \leq 3$.
Similarly the pyramid with apex $1+i$ and base $\{A_1, A_2, B_1, B_2, C_1, C_2\}$ subdivides into 8 $\mathbb{PFS}$ of norm 6 with vertices

$$\left\{1+i, A_{l_1}, B_{l_2}, C_{l_3}, (1-i)^{-1}\right\}$$

with $l_v = 1, 2$ and $1 \leq v \leq 3$. Thus the convex hull of the tetracross with vertices

$\{0, A_1, A_2, B_1, B_2, C_1, C_2, 1+i\}$, where 0 and $1+i$ are opposite vertices, is covered by 16 $\mathbb{PFS}$ of norm 6. □

Theorem 8.1.5. *The base of* $\mathbb{FP}_0$ *consists of* 16 *Farey simplex faces each of norm* 6*, each with* $(1-i)^{-1}$ *as a vertex. There are* 16 *faces of* $\mathbb{FP}_0$ *that have* ∞ *as a vertex and each such face is a Farey simplex of norm* 4*.* $\mathbb{FP}_0$ *is a pentacross in* $\mathbb{H}^5$ *derived from the 4-dimensional cross polytope in* $\mathbb{K}$ *with the 8 vertices* $\{0, A_1, A_2, B_1, B_2, C_1, C_2, 1+i\}$.

Proof. The result follows from Lemma 8.1.3 and Lemma 8.1.4. □

8.2 The Hyperplane $\mathbb{H}^\perp$ in $\mathbb{H}^5$ Orthogonal to $\mathbb{K}$

From [30] we show that there is a subdivision of a Farey simplex in $\mathbb{H}^5$ into a finite number of Farey simplices. The subdivision is taken to be a combinatorial rather than geometric. We note that Schmidt uses the term Farey simplex for our Farey polytope. We represent a Farey polytope in $\mathbb{K}$ by $\mathbb{PFS}$ where $\mathbb{FS}$ represents its associated Farey simplex in $\mathbb{H}^5$. The fundamental Farey simplex will be written as $\mathbb{FS}_0$, while $\mathcal{N}(v)$, the fundamental pentacross will be written $\mathbb{FP}_0$. A general Farey pentacross in the Farey tessellation of $\mathbb{H}^5$ will be written as $\mathbb{FP}$. From Schmidt [30, p.43] we have the theorem:

"Every Farey simplex **FS** with $\boldsymbol{N}(\mathbf{FS}) > 6$ is in two different ways subdivisible into 31 Farey simplices. The vertices of each subdivision all lie on one side of the circumscribed sphere (hyperplane) of **FS**, and the vertices of the two subdivisions are inverse (symmetric) with respect to this sphere (hyperplane).
Every Farey simplex **FS** with $\boldsymbol{N}(\mathbf{FS}) = 6$ is in one way subdivisible into 31 Farey simplices. The vertices of this subdivision all lie inside the circumscribed sphere of **FS**.
The graph of the subdivision of **FS** together with **FS** itself is isomorphic to the graph of the boundary of the 5-dimensional crosspolytope."

We reformulate this statement in the following theorem.

Theorem 8.2.1. *Every Farey simplex $\mathbb{FS}$ is the common face between exactly two Farey pentacross $\mathbb{FP}$ in the Farey tessellation of $\mathbb{H}^5$. These pentacross are inverse to each other with respect to the circumsphere of the Farey simplex $\mathbb{FS}$.*

In order to prove this theorem we need the following results.

Consider the fundamental Farey simplex $\mathbb{FS}_0 = \mathbb{FS}(\infty, 0, 1, \omega, \omega - k)$. The hyperplane in $\mathbb{H}^5$ that contains these 5 points is generated by 4 linearly independent elements and it is orthogonal to $\mathbb{K}$ in $\mathbb{H}^5$.

Consider the tetrahedron in $\mathbb{K}$ with vertices 0, 1, ω and $\omega - k$ in Figure 8.1. Two of the edges of the tetrahedron are determined by the vector $\xi_1 = (1, 0, 0, 0, 0)$ between 0 and 1 and the vector $\xi_2 = (0, 0, 0, 1, 0)$ between $\omega - k$ and ω, giving two independent generators. The third independent generator is the vector $\frac{1}{2}\xi_3$, where $\xi_3 = (0, 1, 1, 0, 0)$. We note that $\frac{1}{2}\xi_3$ is a vector joining $\frac{1}{2}$ to $\frac{1+i+j}{2}$, the midpoints of the edges determined by ξ_1 and ξ_2. We let $\xi_4 = (0, 1, -1, 0, 0)$ and $\xi_5 = (0, 0, 0, 0, 1)$. We note we may write $\xi_1 = 1$, $\xi_2 = k$, $\xi_3 = i + j$ and $\xi_4 = i - j$ as quaternions. The vectors $\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\}$ are linearly independent and generate $\mathbb{R}^5 = \mathbb{K} \times \mathbb{R}$.

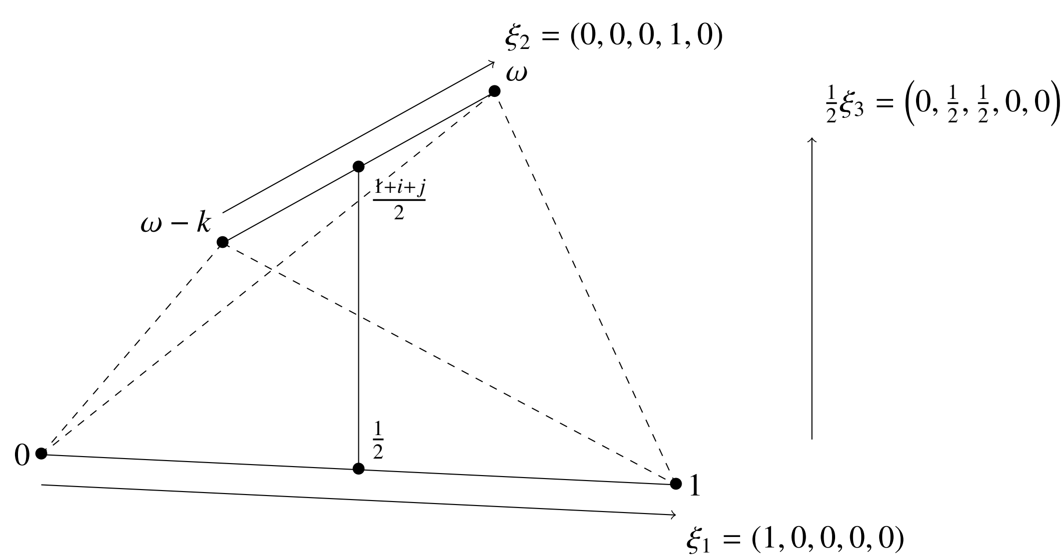

Figure 8.1: The tetrahedron with vertices 0, 1, ω and $\omega - k$.

The hyperplane orthogonal to ξ_4 in $\mathbb{R}^5$ is given by

$$P(\xi_4, 0) = \left\{z \in \mathbb{R}^5 : z \cdot (0, 1, -1, 0, 0) = 0\right\} \cup \{\infty\} = \left\{\sum_{t=1}^{5} a_t \xi_t : a_t \in \mathbb{R}, a_4 = 0\right\} \cup \{\infty\}.$$

Let

$$\mathbb{H}^{\perp} = P(\xi_4, 0) \cap \mathbb{H}^5 = \left\{\sum_{t=1}^{5} a_t \xi_t : a_t \in \mathbb{R}, a_4 = 0, a_5 > 0\right\} \cup \{\infty\}.$$

Thus $\mathbb{H}^{\perp}$ is the hyperplane in $\mathbb{H}^5$ that contains the fundamental Farey simplex $\mathbb{FS}_0 = \mathbb{FS}(\infty, 0, 1, \omega, \omega - k)$.

Lemma 8.2.2. *Each Farey simplex $\mathbb{FS}$ is circumscribed by a hypersphere that separates $\mathbb{H}^5$ into an interior and an exterior region.*

Proof. The hyperplane $\mathbb{H}^{\perp}$ is the circumsphere in $\mathbb{H}^5$ about the simplex $\mathbb{FS}_0$. It separates $\mathbb{H}^5$ into an inner section with $a_4 > 0$ and an outer section with $a_4 < 0$.

By Theorem 7.2.4, we know that every Farey simplex in $\mathbb{H}^5$ is in the orbit of $\mathbb{FS}_0$ under G. Since each element g of G is a Möbius map we know that g maps hyperspheres/planes to hyperspheres/planes and preserve Farey neighbours. Thus for any $\mathbb{FS}$ in $\mathbb{H}^5$ we can find $g \in G$ such that $\mathbb{FS} = g(\mathbb{FS}_0)$ and $g(\mathbb{H}^{\perp})$ is a hypersphere circumscribing $\mathbb{FS}$ that separates $\mathbb{H}^5$ into an inner section and an outer section as required. □

Lemma 8.2.3. *$\mathbb{FS}_0$ is the common face between two Farey pentacross which are images of each other with respect to $\mathbb{H}^{\perp}$ given above.*

Proof. Consider the vertices of $\mathbb{FS}_0 = \mathbb{FS}(\infty, 0, 1, \omega, \omega - k)$ and the sets of points

$$A = \left\{(1 - i)^{-1}, 1 + i, i, \omega - j - k, \omega - j\right\}$$

and

$$B = \left\{(1 - j)^{-1}, 1 + j, j, \omega - i - k, \omega - i\right\}.$$

We express the elements of set A as linear combinations of the ξ_t, $t = 1 \cdots 5$, given above:

$$(1 - i)^{-1} = \frac{1 + i}{2} = \frac{1}{2}\xi_1 + \frac{1}{4}\xi_3 + \frac{1}{4}\xi_4,$$

$$1 + i = \xi_1 + \frac{1}{2}\xi_3 + \frac{1}{2}\xi_4,$$
$$i = \frac{1}{2}\xi_3 + \frac{1}{2}\xi_4,$$
$$\omega - j - k = \frac{1}{2}\xi_1 - \frac{1}{2}\xi_2 + \frac{1}{2}\xi_4 \text{ and}$$
$$\omega - j = \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 + \frac{1}{2}\xi_4,$$

The coefficients of ξ_4 for all the points of A is positive and hence all these points lie in the inner subdivision of $\mathbb{H}^5$ by $\mathbb{H}^\perp$. Reflection in $P(\xi_4, 0)$ is given by $\Phi(x) = x - 2(x \cdot \xi_4)\frac{\xi_4}{2}$. It is easily seen that Φ maps the points $(1-i)^{-1}, 1+i, i, \omega - j - k, \omega - j$ to $(1-j)^{-1}, 1+j, j, \omega - i - k, \omega - i$ respectively. So the points in B are the image of the points in A under reflection in $\mathbb{H}^\perp$. It follows that each point in B lies in the outer subdivision of $\mathbb{H}^5$ by $\mathbb{H}^\perp$. It is seen that the Farey pentacross in $\mathbb{H}^5$ with vertices $\left\{\infty, 0, 1, \omega, \omega - k, (1-i)^{-1}, 1+i, i, \omega - j - k, \omega - j\right\}$ is the image of the Farey pentacross with vertices $\left\{\infty, 0, 1, \omega, \omega - k, (1-j)^{-1}, 1+j, j, \omega - i - k, \omega - i\right\}$ with respect to $\mathbb{H}^\perp$, with $\mathbb{FS}_0$ as the common face. □

Lemma 8.2.4. *The fundamental simplex $\mathbb{FS}_0$ is subdivided into 31 Farey simplices in exactly two ways. The subdivisions are inverse to each other with respect to the circumsphere $\mathbb{H}^\perp$, one lying inside and one lying outside of $\mathbb{H}^\perp$.*

Proof. The 10 vertices $\{\infty, 0, 1, \omega, \omega - k\}$ and $\left\{(1-i)^{-1}, 1+i, i, \omega - j - k, \omega - j\right\}$ form the vertices of the Farey pentacross $\mathbb{FP}_0$ where $v = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{H}^5$. In order, these vertices are on opposite ends of diagonals of the Farey pentacross in $\mathbb{H}^5$, that pass through $v = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{\sqrt{2}}\right)$. The vertices $\left\{(1-i)^{-1}, 1+i, i, \omega - j - k, \omega - j\right\}$ all lie inside $\mathbb{H}^\perp$. We can see that $\mathbb{FS}_0$ is a face of $\mathbb{FP}_0$ and that the other 31 faces of $\mathbb{FP}_0$ are Farey simplices obtained by replacing 1,2,3,4 or 5 elements in $\mathbb{FS}_0$ by the diametrically opposite vertex of the pentacross from set A. That is $\sum_{r=0}^{5} \frac{5!}{r!(5-r)!} = 2^5 = 32$ Farey simplices can be constructed (including $\mathbb{FS}_0$). This is the inner subdivision of $\mathbb{FS}_0$. We know by Lemma 8.2.2 that A reflected in $\mathbb{H}^\perp$ is B, and by the same argument we have that $\mathbb{FS}_0$ is a face of the 5-dimensional crosspolytope obtained by reflection of $\mathbb{FP}_0$ in $\mathbb{H}^\perp$. The outer subdivision of $\mathbb{FS}_0$ is the image of the inner subdivision with respect to $\mathbb{H}^\perp$. □

The next corollary follows immediately from Theorem 7.2.4.

Corollary 8.2.5. *Each of the 31 Farey simplices in the subdivision can be written as* $g(\mathbb{FS}_0)$*, for some* $g \in G$*.*

We have seen that $\Phi(x) = x - (x \cdot \xi_4)\,\xi_4$ maps the vertices of the inner subdivision to the vertices of the outer subdivision and visa versa. But $\Phi \notin G$ as it is orientation reversing. Consider the inversion in $S(0,1)$ given by $\Phi_0(x) = \dfrac{x}{N(x)}$. Then $\Phi_0\Phi = \Phi\Phi_0$ is in G and this composition interchanges the inner and outer subdivisions. Let $g_0 = \Phi_0\Phi = \Phi\Phi_0$. Using g_0, where necessary, we can ensure that $g \in G$ maps an inner subdivision to an inner subdivision of a Farey simplex.

The proof of Theorem 8.2.1 now follows.

Proof. Since every Farey simplex is in the orbit of $\mathbb{FS}_0$ under G and since each $g \in G$ leaves Farey neighbours invariant, Lemmas 8.2.2, 8.2.3 and 8.2.4 yield the required result. □

8.3 Subdivision of Farey Simplices

We have defined a relationship between a non-degenerate Farey simplex $\mathbb{FS}$ and its Farey polytope $\mathbb{PFS}$. The vertices of the Farey simplex are the vertices of the corresponding Farey polytope. The geodesic edges of the Farey simplex are projected to Euclidean line segments in $\mathbb{K}$ so the Farey polytope lies in $\mathbb{K}$. If $\mathbb{FS}$ is the Farey simplex then $\mathbb{PFS}$ is the corresponding Farey polytope. Certainly $\boldsymbol{N}(\mathbb{PFS}) = \boldsymbol{N}(\mathbb{FS})$.
The next result is from Schmidt [30].

Lemma 8.3.1. *Let* $\xi \in \mathbb{K}$ *belong to the Farey polytope* $\mathbb{PFS}$*. Then there is a non-degenerate Farey simplex* $\mathbb{FS}_I$ *among the Farey simplices in the inner subdivision of* $\mathbb{FS}$ *such that* $\xi \in \mathbb{PFS}_I$ *and* $\boldsymbol{N}(\mathbb{FS}_I) > \boldsymbol{N}(\mathbb{FS})$*.*

Proof. Let $\mathbb{FS}$ be a non degenerate Farey simplex, so $\mathbb{FS}\left(p_1q_1^{-1}, p_2q_2^{-1}, p_3q_3^{-1}, p_4q_4^{-1}, p_5q_5^{-1}\right)$ has $q_t \neq 0$ for $t = 1,2,3,4,5$. Let $g(z) = (p_1uz + p_2v)(q_1uz + q_2v)^{-1}$ where $g \in G$, with corresponding matrix $\begin{pmatrix} p_1u & p_2v \\ q_1u & q_2v \end{pmatrix} \in U$ and with $g(\mathbb{FS}_0) = \mathbb{FS}$ and u, v units in the Hurwitz integers. Then

$$g(\infty) = (p_1u)(q_1u)^{-1} = p_1(q_1)^{-1} = A$$
$$g(0) = (p_2v)(q_2v)^{-1} = p_2(q_2)^{-1} = B$$
$$g(1) = (p_1u + p_2v)(q_1u + q_2v)^{-1} = p_3(q_3)^{-1} = C$$
$$g(\omega) = (p_1u\omega + p_2v)(q_1u\omega + q_2v)^{-1} = p_4(q_4)^{-1} = D$$
$$g(\omega - k) = (p_1u(\omega - k) + p_2v)(q_1u(\omega - k) + q_2v)^{-1} = p_5(q_5)^{-1} = E$$

The points that yield the inner Farey subdivision of $\mathbb{FS}$ are thus

$$g\left((1-i)^{-1}\right) = A_1 = p_6(q_6)^{-1}$$
$$g(1+i) = B_1 = p_7(q_7)^{-1}$$
$$g(i) = C_1 = p_8(q_8)^{-1}$$
$$g(\omega - j - k) = D_1 = p_9(q_9)^{-1}$$
$$g(\omega - j) = E_1 = p_{10}(q_{10})^{-1}$$

Note, if necessary, we may consider $gg_0 \in G$, where $g_0 = \Phi_0\Phi = \Phi\Phi_0$, to ensure that the inside of spheres are mapped to inside of spheres, and the inner subdivision is mapped to the inner subdivision as above.

Let $C = g(\mathbb{H}^{\perp})$ be a hypersphere in $\mathbb{H}^5$ and $g^{-1}(\infty) = -u^{-1}q_1^{-1}q_2v \in \mathbb{K}$.

Now $g\left(-u^{-1}q_1^{-1}q_2v\right) = \infty$ and ∞ lies outside of C. Thus $-u^{-1}q_1^{-1}q_2v$ lies in the open half space bounded by $\mathbb{H}^{\perp}$ that does not include $(1-i)^{-1}$.

By the above $p_6(q_6)^{-1} = A_1 = g\left((1-i)^{-1}\right) = (p_1u + p_2v(1-i))(q_1u + q_2v(1-i))^{-1}$. So $N(q_6) = N(q_1u + q_2v(1-i)) = N(q_1)N(1-i)N\left((1-i)^{-1} + (q_1u)^{-1}q_2v\right)$.

The following equivalences hold:

$$
\begin{aligned}
& N(q_6) \leq N(q_1) \\
\Longleftrightarrow & N\left((1-i)^{-1} + (q_1 u)^{-1} q_2 v\right) \leq \frac{1}{2} \\
\Longleftrightarrow & N\left((1-i)^{-1} - \left(-(q_1 u)^{-1} q_2 v\right)\right) \leq \frac{1}{2}
\end{aligned}
$$

$\Longleftrightarrow -u^{-1}q_1^{-1}q_2v$ lies in (or on the boundary of) $S\left((1-i)^{-1}, \frac{1}{\sqrt{2}}\right)$, the hypersphere with centre $(1-i)^{-1}$ and radius $\frac{1}{\sqrt{2}}$ that passes through $0, 1, \omega, \omega - k$ and $1+i, i, \omega - j - k, \omega - j$.

$\Longleftrightarrow -u^{-1}q_1^{-1}q_2v$ lies in the sectoral region of $S\left((1-i)^{-1}, \frac{1}{\sqrt{2}}\right)$ containing $\mathbb{FS}\left((1-i)^{-1}, 0, 1, \omega, \omega - k\right)$ but on the opposite side of $\mathbb{H}^{\perp}$ from $(1-i)^{-1}$.

$\Longleftrightarrow$ Hypersphere K_1 through $-(q_1u)^{-1}q_2v, 0, 1, \omega$ and $\omega - k$ contains $(1-i)^{-1}, i, 1+i, \omega - j - k$ and $\omega - j$ in its interior and ∞ outside.

$\Longleftrightarrow$ Hyperplane $g(K_1)$ through $\infty, g(0), g(1)g(\omega)$ and $g(\omega - k)$ (hyperplane BCDE) has $g\left((1-i)^{-1}\right) = A_1, g(1+i) = B_1, g(i) = C_1, g(\omega - j - k) = D_1$ and $g(\omega - j) = E_1$ on one side and $g(\infty) = A$ on the other side.

$\Longleftrightarrow A_1, B_1, C_1, D_1$ and E_1 lie in the halfspace bounded by $g(K_1)$ (hyperplane BCDE) that does not include $A = g(\infty)$ but inside $C = g\left(\mathbb{H}^{\perp}\right)$.

Consequently, if $N(q_6) \leq N(q_1)$, none of the 16 Farey polytopes in the subdivision of the $\mathbb{PFS}$ that have $A_1 = p_6(q_6)^{-1}$ as a vertex can contain ξ, and these Farey polytopes can be ignored.

That is, if $N(q_6) \leq N(q_1)$, we need not consider vertex $A_1 = p_6(q_6)^{-1}$.

We have assumed that $N(q_1) \geq N(q_2) \geq N(q_3) \geq N(q_4) \geq N(q_5)$. We know that $B = g(0)$, $C = g(1)$, $D = g(\omega)$ and $E = g(\omega - k)$ are Farey neighbours. For any Farey neighbours, $p_s q_s^{-1} \sim p_t q_t^{-1}$ and $\mathcal{D}\begin{pmatrix} p_s & p_t \\ q_s & q_t \end{pmatrix} = 1 = N(q_s)N(q_t)N\left(p_s q_s^{-1} - p_t q_t^{-1}\right)$ since $q_s \neq 0, q_t \neq 0$

(Theorem 4.4.4(4)), so $N\left(p_s q_s^{-1} - p_t q_t^{-1}\right) = \dfrac{1}{N(q_s)N(q_t)}$.

Since $g \in G$ can be chosen so that $g(\infty)$, $g(0)$, $g(1)$ $g(\omega)$ and $g(\omega-k)$ can be any permutation of A, B, C, D and E we know that if $N(q_t) \leq N(q_1)$, $t = 6, 7, 8, 9, 10$ we can ignore a Farey simplex with vertex $p_t(q_t)^{-1}$ in the subdivision of the Farey simplex $\mathbb{FS}$.

Thus in all the allowable Farey polytopes in the subdivision of the Farey polytope $\mathbb{PFS}$ we have $N(q_t) > N(q_1)$. Consequently, the given Farey polytope $\mathbb{PFS}$ is covered by Farey polytopes in the inner subdivision of $\mathbb{PFS}$ with norms strictly greater than the norm of $\mathbb{FS}$. □

Definition 8.3.2. *A chain of $\mathbb{PFS}$ containing $\xi \in \mathbb{K}$ is an infinite sequence of Farey polytopes: $\mathbb{PFS}^{(0)}$, $\mathbb{PFS}^{(1)}$, $\mathbb{PFS}^{(2)}$,..., $\mathbb{PFS}^{(n)}$, ... such that*

1. $\xi \in \mathbb{PFS}^{(n)}$ *for all* $n \geq 0$,
2. $\mathbb{PFS}^{(n+1)}$ *is one of the Farey simplices in the inner subdivision of* $\mathbb{PFS}^{(n)}$, $n \geq 0$,
3. $\mathbf{N}\left(\mathbb{FS}^{(0)}\right) = 6$,
4. $\mathbf{N}\left(\mathbb{FS}^{(n+1)}\right) > \mathbf{N}\left(\mathbb{FS}^{(n)}\right)$ *for all* $n \geq 0$.

From Schmidt [30] we now have the following theorem.

Theorem 8.3.3. *For every quaternion $\xi \in \mathbb{K}$ and any Farey polytope $\mathbb{PFS}$ containing ξ there exists a chain of Farey polytopes containing ξ such that $\mathbb{PFS} = \mathbb{PFS}^{(n)}$, for some $n \geq 0$. For every chain of Farey polytopes containing $\xi \in \mathbb{K}$ we have $\mathbf{N}\left(\mathbb{PFS}^{(n)}\right) \to \infty$ as $n \to \infty$, diameter of $\mathbb{PFS}^{(n)} \to 0$ as $n \to \infty$ and the vertices $p_t^n\,(q_t^n)^{-1} \to \xi$ as $n \to \infty$, for $t = 1, \cdots, 5$.*

Proof. By Lemma 8.3.1 we can find the chain of $\mathbb{PFS}$ with strictly increasing norms, so $\mathbf{N}\left(\mathbb{PFS}^{(n)}\right) \to \infty$ as $n \to \infty$. Diameter $\mathbb{PFS}^{(n)}$ is the length of the longest side of $\mathbb{PFS}^{(n)}$, and diam $\mathbb{PFS}^{(n)}$ is inversely proportional to $N(q_t^n)$, for $t = 1, 2, 3, 4, 5$, so diam $\mathbb{PFS}^{(n)} \to 0$ as $\mathbf{N}\left(\mathbb{PFS}^{(n)}\right) \to \infty$. Thus the radius of the circumspheres of $\mathbb{FS}^{(n)}$ tend to 0 and thus the vertices converge to ξ. □

Chapter 9

Continued Fractions in Higher Dimensions

9.1 Introduction to Continued Fractions with Quaternion Coefficients

We recall that continued fractions with complex coefficients can be written as

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}, \quad a_n \neq 0.$$

This algebraic expression, where $\{a_n\}$ and $\{b_n\}$ are sequences of complex numbers, has a convenient geometric form given in terms of Möbius transformations acting on $\mathbb{C}$. Let $t_n(z) = \dfrac{a_n}{z + b_n}$, $n = 1, 2, \cdots$ where $t_n(\infty) = 0$, $t_0(z) = z + b_0$ and no a_n is zero. Assume that the given continued fraction converges to $\alpha \in \mathbb{C}$, then the convergents to α can be expressed as

$$T_n(\infty) = t_0 t_1 t_2 \cdots t_n(\infty) = \frac{p_n}{q_n}$$

where p_n and q_n are complex numbers.

Here $T_n = t_0 t_1 t_2 \cdots t_n$ is a composition of a finite number of Möbius maps.

From an abstract point of view a continued fraction with quaternion entries may be viewed as two sequences of quaternions, $\{a_n\}$ and $\{b_n\}$, that give rise to a sequence of Möbius maps in $M\left(\mathbb{R}^4_\infty\right) = PS_{\mathcal{D}}L(2, \mathbb{K})$ acting on $\mathbb{K}_\infty$ (Section 5.5) of the form $t_n(x) = a_n(z + b_n)^{-1}$ where $t_n(\infty) = 0$ and where no a_n is zero.

We note that the matrix associated with each t_n, $n \geq 1$, is of the form $A_n = \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}$ where a_n and b_n are quaternions and $\mathcal{D}(A_n) > 0$.

We know from [23, p.453] that elements in $G_{\mathcal{D}}L(2, \mathbb{K})$ are generated by matrices of the forms:

$$\tau_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta_a = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

where λ and $u \neq 0$ are quaternions. The corresponding maps are $\tau_\lambda : x \to x+\lambda$, $\phi : x \to x^{-1}$ $\delta_a : x \to ax$ for all $x \in \mathbb{K}$, where $\tau_\lambda, \phi, \delta_a \in PS_{\mathcal{D}}L(2, \mathbb{K})$.

We note the important representation of t_n in terms of τ_λ, ϕ and δ_u that will be used in the sequel as

$$t_n = \delta_{a_n} \phi \tau_{b_n}$$

since $a_n \neq 0$ for all n.

Thus in $\mathbb{K}_\infty$ the operation of division is essentially replaced by inversion in a unit hypersphere and the arguments become geometric rather that algebraic. The convergence of continued fractions is dependent on the convergence of a composition of Möbius maps

$$T_n(\infty) = t_0 t_1 t_2 \cdots t_n(\infty)$$

with quaternion entries and these values can be calculated in $\mathbb{K}$.

That is

$$\lim_{n \to \infty} T_n(\infty)$$

exists in $\mathbb{K}_\infty$ and is unique. The study of numerical convergence of continued fractions might better be replaced by a study of convergence of corresponding Möbius maps. An infinite continued fraction is thus the limit of a function evaluated at a certain point [8, p.8].

We have noted that the group of Möbius transformations acting on $\mathbb{K}$ has a discrete subgroup G, the unimodular group generated by τ_λ, δ_u and ϕ, where $\lambda \in \mathcal{H}$ and $u \in \mathcal{U}$. This group is composed of maps associated with 2×2 matrices over the Hurwitz integers with $\mathcal{D}$ - determinant equal to 1. In this case the continued fraction sequences, $\{a_n\}$ and $\{b_n\}$, are sequences of Hurwitz units and Hurwitz integers respectively that give rise to a sequence of unimodular maps with associated matrices $A_n = \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}$, where $\mathcal{D}(A_n) = N(a_n) = 1$ and a_n is unit for all n. Thus $t_n = \delta_{a_n}\phi\tau_{b_n}$ where $a_n \in \mathcal{U}$ and $b_n \in \mathcal{H}$. The role of the unimodular group in continued fraction theory with coefficients from $\mathbb{K}$ is analogous to the role of the modular group in continued fractions theory with coefficients from $\mathbb{R}$. The latter yield integer continued fractions [14].

In the sequel we will represent both matrices and their corresponding maps with the same symbol. For example the matrix $\delta_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ has corresponding map $\delta_u(x) = ux$, the matrix $\tau_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ has corresponding map given by $\tau_\lambda(x) = x + \lambda$ while the matrix $\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ corresponds to the mapping $\phi(x) = x^{-1}$, where $\lambda \in \mathcal{H}$ and $u \in \mathcal{U}$.

We note the following identities that will be useful in reducing the composition of mappings. For $\lambda_1, \lambda_2, a, b \in \mathcal{H}$,

1. $\tau_{\lambda_1}\tau_{\lambda_2} = \tau_{\lambda_1+\lambda_2}$ while $\delta_a\delta_b = \delta_{ab}$,
2. $\phi^2 = I$ and
3. $\delta_a\tau_\lambda = \tau_{a\lambda}\delta_a$ and $\tau_\lambda\delta_a = \delta_a\tau_{a^{-1}\lambda}$ where $a^{-1} = \overline{a}$ if a is a unit.

We know that a general Möbius map

$$g : x \to (ax+b)(cx+d)^{-1}$$

either moves ∞ or fixes ∞. From Lemma 4.7.2 we know that in the former case $g = \tau_{\lambda_1}\phi\delta_{u_1}\phi\delta_{u_2}\phi\tau_{\lambda_2}$ where $u_1 = c$, $u_2 = b - ac^{-1}d$, $\lambda_1 = ac^{-1}$ and $\lambda_2 = c^{-1}d$ while in the latter case $g = \delta_a\tau_{a^{-1}bd^{-1}}\phi\delta_d\phi = \tau_{bd^{-1}}\delta_a\phi\delta_d\phi$.

If the entries are all Hurwitz integers and $c = 0$ with $\mathcal{D}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = N(ad) = 1$, then both a and d are units. On the other hand when $c \neq 0$, $g = g_1\phi\tau_{\lambda_1}\phi\tau_{\lambda_2}\cdots\phi\tau_{\lambda_n}$ where $g_1(\infty) = \infty$ and $\lambda_1, \lambda_2, \cdots \lambda_n$ are in $\mathcal{H}$ by Lemma 6.1.4.

9.2 The Farey Simplectic Faces of $\mathbb{FP}_0$ as Images of $\mathbb{FS}_0$

The results in the following sections are recorded in Hockman [15]. We can represent the 32 simplectic faces of $\mathbb{FP}_0$ in two sets. The first 16, referred to as the sides of $\mathbb{FP}_0$ all have ∞ as a vertex and the second 16, referred to as the floor of $\mathbb{FP}_0$ all have $(1-i)^{-1}$ as a vertex. We have seen that each face is the image of the fundamental simplex $\mathbb{FS}_0 = \{\infty, 0, 1, w, w-k\}$ under the unimodular group G. $\mathbb{FS}_0$ can be represented by the 2×5 matrix $\begin{pmatrix} 1 & 0 & 1 & w & w-k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$, and so the faces of $\mathbb{FP}_0$ that have ∞ as a vertex are found by the product

$$\begin{pmatrix} u & pv \\ 0 & v \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & w & w-k \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

where u and v are unit Hurwitz integers (Theorem 7.2.4).

By the above, the matrices $A = \begin{pmatrix} u & pv \\ 0 & v \end{pmatrix}$ can be decomposed as

$$A = \delta_u \tau_{u^{-1}pvv^{-1}} \phi \delta_v \phi = \tau_p \delta_u \phi \delta_v \phi$$

where u and v are units and $p \in \mathcal{H}$ and $A\,(\mathbb{FS}_0)$ has ∞ as a vertex.

We see that both $\tau_p(x) = x + p$ and $\delta_u \phi \delta_v \phi(x) = uxv^{-1}$ are Möbius maps that leave ∞ fixed. We further note that any two matrices A_1 and A_2 of the above form have a product

$$\begin{pmatrix} u_1 & p_1 v_1 \\ 0 & v_1 \end{pmatrix}\begin{pmatrix} u_2 & p_2 v_2 \\ 0 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 u_2 & u_1 p_2 v_2 + p_1 v_1 v_2 \\ 0 & v_1 v_2 \end{pmatrix}$$

where $u_1 u_2$ and $v_1 v_2$ are units and $u_1 p_2 v_2 + p_1 v_1 v_2 \in \mathcal{H}$.

Consider now the matrix $\sigma = \begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix}$ or $\sigma = \phi \tau_{1-i} \delta_{-1} \phi$ in G. We note that σ interchanges ∞ and $(1-i)^{-1}$ while leaving the other vertices of $\mathbb{FP}_0$ fixed as a set. Hence the 16 faces of $\mathbb{FP}_0$ that have $(1-i)^{-1}$ as a vertex can be written as $\sigma A(\mathbb{FS}_0)$ where A ranges over the 16 maps that take $\mathbb{FS}_0$ to the Farey simplectic faces with ∞ as a vertex.

This map σ is in fact given by inversion in the isometric sphere centered at 1 followed by inversion in the isometric sphere centered at i. So σ fixes $\mathbb{FP}_0$ but is not unique in this role. Any $g \in Stab_G(v)$ will fix $\mathbb{FP}_0$.

The role of the mapping σ is most significant in the development of a continued fraction expansion for a $\xi \in \mathbb{K}$. As we have seen it leaves the vertices of $\mathbb{FP}_0$ invariant as a set, while interchanging ∞ with $(1-i)^{-1}$. We note that the 8 vertices of the 4-dimensional tetracross in $\mathbb{K}$ given by the set $\{0, 1, 1+i, i, w, w-j-k, w-k, w-j\}$ all lie on the hypersphere S, with centre $(1-i)^{-1}$ and radius $\frac{1}{\sqrt{2}}$ in $\mathbb{K}$. The inside of this tetracross is thus mapped to the outside of the hypersphere S by σ and visa versa. The role of σ with respect to the fundamental Farey pentacross $\mathbb{FP}_0$ and the Farey tessellation of $\mathbb{H}^5$ under the unimodular group G is thus analogous to the role of ϕ where $\phi(x) = 1/x$ with respect to the fundamental Farey triangle $\{\infty, 0, 1\}$ in the Farey tessellation of $\mathbb{H}^2$ under the action of the modular group [31].

9.3 Farey Simplices in the Farey Tessellation of $\mathbb{H}^5$

Theorem 9.3.1. *Every Farey simplex* $\mathbb{FS}$ *in the Farey tessellation of* $\mathbb{H}^5$ *can be written as*

$$\mathbb{FS} = A_1\sigma A_2\sigma A_3\sigma A_4\cdots\sigma A_n(\mathbb{FS}_0)$$

where $A_t = \begin{pmatrix} u_t & \lambda_t \\ 0 & v_t \end{pmatrix}$ *for* $t = 1, 2, \cdots, n$, *where* u_t *and* v_t *are units and* $\lambda_t \in \mathcal{H}$ *and* $\sigma = \begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix}$.

Proof. In Section 9.2 we showed that the sides and the floor of the fundamental pentacross $\mathbb{FP}_0$ can be written in the required form. We note that A_t may be the identity matrix.

Let $\mathbb{FP}_1$ be a pentacross adjacent to $\mathbb{FP}_0$ by a face $\mathbb{FS}_1$ of the form $\sigma A(\mathbb{FS}_0)$ (in the floor of $\mathbb{FP}_0$). The vertices of $\mathbb{FP}_1$ are established by the inner Farey subdivision of $\mathbb{FS}_1$, the common face of $\mathbb{FP}_0$ and $\mathbb{FP}_1$. The inner Farey subdivision of $\mathbb{FS}_0$ is given by the maps $A_t(\mathbb{FS}_0)$ or $\sigma A_t(\mathbb{FS}_0)$ for $t = 1, 2, \cdots, 16$. Thus the inner Farey subdivision of $\mathbb{FS}_1$ is given by $\sigma A(A_t(\mathbb{FS}_0))$ or by $\sigma A(\sigma A_t(\mathbb{FS}_0))$ for $t = 1, 2, \cdots, 16$. Since $AA_t = \begin{pmatrix} u_t & \lambda_t \\ 0 & v_t \end{pmatrix}$ for some λ_t in $\mathcal{H}$ with u_t and v_t units, the result holds in this case.

Thus the result follows, by induction, for any Farey simplectic face of $\mathbb{FP}_m$ that has vertices in the convex hull of the tetracross in $\mathbb{K}$ given by the set of vertices $\{0, 1, i, 1+i, w, w-k-j, w-k, w-j\}$.

Finally consider any $\mathbb{FP}$ in the Farey tessellation of $\mathbb{H}^5$ that does not lie over the tetracross in $\mathbb{K}$ given by the set of vertices $\{0, 1, i, 1+i, w, w-k-j, w-k, w-j\}$. Then $\mathbb{FP} = A(\mathbb{FP}_m)$ where $A = \begin{pmatrix} u & \lambda \\ 0 & v \end{pmatrix}$ or $A = \tau_\lambda\delta_u\phi\delta_v\phi$ for u and v units and λ a Hurwitz integer and $\mathbb{FP}_m$ lies over the tetracross given above. Thus any Farey simplex in the Farey tessellation can be written as

$$\mathbb{FS} = A_1\sigma A_2\sigma A_3\sigma A_4\cdots\sigma A_n(\mathbb{FS}_0)$$

where $A_t = \begin{pmatrix} u_t & \lambda_t \\ 0 & v_t \end{pmatrix}$ and where each A_t in the expansion leaves ∞ fixed. □

Theorem 9.3.2. *Every chain $A_1\sigma A_2\sigma A_3\sigma\cdots A_n$, where the A_t and σ are of the form given above, gives rise to a continued fraction expansion of some Hurwitz rational.*

Proof. Let $A_t = \begin{pmatrix} u_t & \lambda_t \\ 0 & v_t \end{pmatrix}$ where u_t and v_t are units and λ_t is in $\mathcal{H}$ for $t = 1, 2, \cdots, n$. Recall that $A_t = \tau_{\lambda_t}\delta_{u_t}\phi\delta_{v_t}\phi$ and $\sigma = \phi\tau_{1-i}\delta_{-1}\phi$. Thus,

$$\begin{aligned} A_1\sigma A_2\sigma A_3\sigma\cdots A_n &= \{\tau_{\lambda_1}\delta_{u_1}\phi\delta_{v_1}\phi\}\{\phi\tau_{1-i}\delta_{-1}\phi\}\cdots\{\tau_{\lambda_{n-1}}\delta_{u_{n-1}}\phi\delta_{v_{n-1}}\phi\}\{\phi\tau_{1-i}\delta_{-1}\phi\}\{\tau_{\lambda_n}\delta_{u_n}\phi\delta_{v_n}\phi\} \\ &= \tau_{\lambda_1}\{\delta_{u_1}\phi\tau_{v_1(1-i)}\}\{\delta_{-v_1}\phi\tau_{\lambda_2}\}\{\delta_{u_2}\phi\tau_{v_2(1-i)}\}\cdots\{\delta_{-v_{n-1}}\phi\tau_{\lambda_n}\}\{\delta_{u_n}\phi\delta_{v_n}\phi\}. \end{aligned}$$

Thus,

$$\begin{aligned} A_1\sigma A_2\sigma A_3\sigma\cdots A_n(\infty) &= \tau_{\lambda_1}\{\delta_{u_1}\phi\tau_{v_1(1-i)}\}\{\delta_{-v_1}\phi\tau_{\lambda_2}\}\{\delta_{u_2}\phi\tau_{v_2(1-i)}\}\cdots\{\delta_{-v_{n-1}}\phi\tau_{\lambda_n}\}\{\delta_{u_n}\phi\delta_{v_n}\phi\}(\infty) \\ &= \tau_{\lambda_1}\{\delta_{u_1}\phi\tau_{v_1(1-i)}\}\{\delta_{-v_1}\phi\tau_{\lambda_2}\}\{\delta_{u_2}\phi\tau_{v_2(1-i)}\}\cdots\{\delta_{-v_{n-1}}\phi\tau_{\lambda_n}\}(\infty) \end{aligned}$$

since $\{\delta_{u_n}\phi\delta_{v_n}\phi\}(\infty) = \infty$.

That is, we can find sequences $\{a_k\}$ and $\{b_k\}$ of Hurwitz integers that give rise to a sequence of Möbius maps of the form $t_k(x) = a_k(x + b_k)^{-1}$ for $k = 2, \cdots, 2n-1$ and $t_1(x) = x + \lambda_1$, where $\lambda_1 \in \mathcal{H}$, $a_k \in \mathcal{U}$ and $b_k \in \mathcal{H}$. We note that $t_k(\infty) = 0$ for each $k = 2, \cdots, 2n-1$ and that

$$A_1\sigma A_2\sigma A_3\sigma\cdots A_n(\infty) = T_{2n-1}(\infty) = t_1 t_2\cdots t_{2n-1}(\infty) = p_{2n-1}q_{2n-1}^{-1} = ab^{-1}$$

where $a = p_{2n-1}$ and $b = q_{2n-1}$ are in $\mathcal{H}$.

Thus we have generated a continued fraction expansion for ab^{-1}. □

We call this continued fraction a *regular σ-continued fraction* for ab^{-1}. In Theorem 8.3.3 we established that for each $\xi \in \mathbb{K}$ we can find a chain of Farey polytopes $\mathbb{PFS}^{(n)}$ in $\mathbb{K}$, each containing ξ , such that

$$\lim_{n\to\infty} p_l^{\,n}\,(q_l^n)^{-1} = \xi$$

for $1 \leq l \leq 5$. The vertices of these Farey polytopes coincide with the vertices for the Farey simplices when ∞ is not a vertex. Farey simplices with ∞ as a vertex can be be found as a preliminary chain to the chain of Farey polytopes.

We can finally state the following theorem that is a result of the theorems above.

Theorem 9.3.3. *For each $\xi \in \mathbb{K}$ we can find a regular σ-continued fraction expansion where the convergents to ξ are $A_1\sigma A_2\sigma A_3\sigma \cdots A_n(\infty)$, as given in the chain of Farey polytopes above.*

9.4 An Algorithm for finding the Regular σ-Continued Fraction Expansion for $\xi \in \mathbb{K}$.

Consider any $\xi \in \mathbb{K}$. From Theorems 3.5.3(3) and 3.6.2 we can find $\lambda_1 \in \mathcal{H}$ and a unit $u_1 \in \mathcal{U}$ such that $\tau_{-\lambda_1}(\xi) \in \nabla$ and $\delta_{u_1^{-1}}\tau_{-\lambda_1}(\xi) \in \nabla^*$. This Hurwitz integer λ_1 will be a nearest Hurwitz integer to our $\xi \in \mathbb{K}$ with respect to the Euclidean metric in $\mathbb{K} \cong \mathbb{R}^4$.

Now ∇^* lies in the tetracross in $\mathbb{K}$ with vertex set $\{0, 1+i, 1, i, w, w-j-k, w-k, w-j\}$. This set of vertices lies on the hypersphere centered at $(1-i)^{-1}$ with radius $\frac{1}{\sqrt{2}}$. Thus we know that $\xi_1 = \sigma\delta_{u^{-1}}\tau_{-\lambda}(\xi)$ lies outside of ∇^*. Let $A_1^{-1} = \delta_{u_1^{-1}}\tau_{-\lambda_1}$ and $\xi_1 = \sigma A_1^{-1}(\xi)$ or $\xi = A_1\sigma(\xi_1)$. We repeat the process to find a sequence of matrices (mappings) $A_2, A_3, \cdots, A_k$ such that $\xi = A_1\sigma A_2\sigma \cdots A_k\sigma(\xi_k)$ and $A_k = \tau_{\lambda_k}\delta_{u_k}$ where $\lambda_k \in \mathcal{H}$ and $u_k \in \mathcal{U}$ for all k and ξ_k lies outside of ∇^*. We note again that $A_k = \begin{pmatrix} u_k & \lambda_k \\ 0 & 1 \end{pmatrix}$ and $A_k(\infty) = \infty$ for each k. If ξ is a non rational quaternion, this process will continue without end. If ξ is a rational quaternion the process will stop. Rational convergents to ξ are of the form $p_k q_k^{-1} = A_1\sigma A_2\sigma \cdots A_k\sigma(\infty)$.

9.5 Möbius Transformations in Higher Dimensions and Clifford Matrices

We recall that a Möbius transformation acting on $\mathbb{R}^n_\infty$ is a finite composition of inversions in hyperplanes and hyperspheres in $\mathbb{R}^n$. Following [1] (Theorems A and B) and [37], we have shown that the orientation preserving Möbius transformations on $\mathbb{R}^n_\infty$ are exactly of the form $g(x) = (ax + b)(cx + d)^{-1}$ where

1. a, b, c, d are in $\Gamma_n \cup \{\infty\}$
2. $ad^* - bc^* \in \mathbb{R} \setminus \{0\}$
3. ab^*, cd^*, c^*a, d^*b are vectors in V^n.

That is, g corresponds to a Clifford matrix in $GL(2, \Gamma_n)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and it acts bijectively on both V^n_∞ and V^{n+1}_∞. The calculation of $g(x) = (ax + b)(cx + d)^{-1}$ is made in the Clifford algebras C_n and C_{n+1} respectively. It is seen that the group $PSL(2, \Gamma_n)$ is isomorphic to the orientation preserving Möbius mappings on $\mathbb{R}^n_\infty$ and the full group of orientation preserving isometries of $\mathbb{H}^{n+1}$.

Thus when $n = 4$ we have $Iso^+\mathbb{H}^5$ represented by $PSL(2, \Gamma_4)$ and as the Poincarè extensions of the group $PS_{\mathcal{D}}L(2, \mathbb{K})$ as established in Section 5.5.

Once again, from an abstract point of view a continued fraction with Clifford number entries may be viewed as two sequences of Clifford numbers, $\{a_k\}$ and $\{b_k\}$, that give rise to a sequence of Möbius maps in $PSL(2, \Gamma_n) \cong M\left(\mathbb{R}^4_\infty\right)$ acting on V^n_∞ and V^{n+1}_∞ of the form $t_k(z) = a_k(z + b_k)^{-1}$ where $t_k(\infty) = 0$ and where no a_k is zero.

We note that the matrix in $GL(2,\Gamma_n)$ associated with each t_k is of the form $A = \begin{pmatrix} 0 & a_k \\ 1 & b_k \end{pmatrix}$ and $\Delta\begin{pmatrix} 0 & a_k \\ 1 & b_k \end{pmatrix} = -a_k \in \mathbb{R} \setminus \{0\}$.

9.6 Pringsheim's Theorem in Higher Dimensions

The following results are all drawn from Beardon [5, p.304]. In this paper an analogue of Pringsheim's Theorem is given in higher dimensions. We consider $\mathbb{B}$ to be the open unit ball in $\mathbb{R}^n$ with boundary $\partial\mathbb{B}$ and closure $\bar{\mathbb{B}}$.

Following [5] we establish a general result on convergence in $\mathbb{R}^n$. In order to do this we prove the next lemma.

Lemma 9.6.1. *Let S and S' be two disjoint spheres of radii r and r' respectively, such that S' separates S from ∞. Suppose that x lies between the spheres S and S', and let $\hat{x}$ be the inverse point of x with respect to the inner sphere S. Then $|\hat{x} - x| \leq 4(r' - r)$.*

Proof. We may assume that S has center at the origin. Then

$$|\hat{x} - x| = \left|\frac{r^2 x}{|x|^2} - x\right| = \left(1 + \frac{r}{|x|}\right)(|x| - r) \leq 2\,(|x| - r)$$

since $|x| > r$. Now let C be the chord of S' passing through 0 and x. Then S partitions C into three segments, one which is the diameter of S of length $2r$ and another which has length at least $|x| - r$. Then $|x| - r \leq 2r' - 2r$. Thus $|\hat{x} - x| \leq 4(r' - r)$ as required. □

Theorem 9.6.2. *Suppose that $t_1, t_2, \cdots$ is a sequence of Möbius transformations acting on $\mathbb{R}^n_\infty$ such that for each k, $t_k(\infty) = 0$ and $t_k(\mathbb{B}) \subset \mathbb{B}$. Then the sequence $t_1 t_2 t_3 \cdots t_k(0)$ converges.*

Proof. Let $K_0 = \overline{\mathbb{B}}$, the sphere $S_0 = \partial K_0$ and $x_0 = 0$. For $n \geq 1$ let $T_k = t_1 t_2 \cdots t_k$, $x_k = T_k(0)$, $K_k = T_k(\overline{\mathbb{B}}) = T_k(K_0)$ and spheres $S_k = \partial K_k$. As t_j maps $\overline{\mathbb{B}}$ into itself we have the chain $\mathbb{B}_0 = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \cdots$ so that S_k forms a set of nested spheres. As ∞ and 0 are inverse points with respect to the sphere $S_0 = \partial\mathbb{B}$ we see that $T_{k+1}(0)$ and $T_{k+1}(\infty)$ are inverse points with respect to the sphere S_{k+1}. But $T_{k+1}(\infty) = T_k t_{k+1}(\infty) = T_k(0) = x_k$. We deduce that x_k and x_{k+1} are inverse points with respect to the sphere S_{k+1}. Further as $x_k = T_k(0) = T_{k+1}(\infty)$ we see that $x_k \in K_k \setminus K_{k+1}$. So for each k, x_k lies between S_k and S_{k+1}.

Finally we note that the radii r_k of the spheres S_k form a decreasing sequence and hence are convergent. We see that x_k lies between S_k and S_{k+1} and x_{k+1} is the inverse of x_k with respect to the sphere S_{k+1}. From Lemma 9.6.1 we thus have that $|x_{nk+1} - x_k| \leq 4(r_k - r_{k+1})$. It follows that $\Sigma_k |x_{k+1} - x_k|$ converges and hence that x_k converges. This establishes the convergence of the '*continued fraction*' $t_1 t_2 t_3 \cdots t_k$ and completes the proof. □

The next result establishes an important equivalence that allows us to interpret Pringsheim's theorem in terms of Clifford matrices.

Lemma 9.6.3. *A Möbius map g acting on $\mathbb{R}^n$ satisfies $g(\mathbb{B}) \subset \mathbb{B}$ and $g(\infty) = 0$ if and only if it can be expressed in the form $g(x) = b(cx + d)^{-1}$, where $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ is a Clifford matrix with $|d| \geq |b| + |c|$.*

Proof. Assume that g is a Möbius map acting on $\mathbb{R}^n$ with $g(\mathbb{B}) \subset \mathbb{B}$ and $g(\infty) = 0$. From Section 4.2, we know that g corresponds to a Clifford matrix $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$. In particular we have $bc^* \in \mathbb{R} \setminus \{0\}$ and cd^* is a vector in V^n. Thus we simultaneously have $c^{-1}d$ is in V^* (Lemma 2.1.8) and that $-c^{-1}d = g^{-1}(\infty)$. If x is a vector other than $-c^{-1}d$ then $g(x)$ is also a vector. We recall from Lemma 2.1.7(3) that for elements in the Clifford group Γ_n we have $|ab| = |a||b|$. Thus if $g(x) = (ax + b)(cx + d)^{-1}$ we have

$$|b| = |g(x)(cx + d)| = |g(x)||cx + d| \geq |g(x)|(|d| - |cx|) = |g(x)|(|d| - |c||x|).$$

Since $g(\mathbb{B}) \subset \mathbb{B}$ we know that $g^{-1}(\infty) = -c^{-1}d \notin \mathbb{B}$ so that $|c^{-1}d| > 1$. So $|d| > |c| > 0$. Let $x = -(|c|/|d|)cd^{-1}$ where $|c|/|d| < 1$. Then $|x| = 1$ and since $c^{-1}d$ is a vector then both x and subsequently $g(x)$ are also vectors. Since $g(x) \in \mathbb{B}$ we have $|g(x)| \leq 1$. Also

$$|cx + d| = |-(|c|/|d|)d + d| = |d||1 - (|c|/|d|)| = |d| - |c|$$

and so $|b| = |g(x)||cx + d| \leq |d| - |c|$ with $|d| \geq |b| + |c|$.

Conversely assume $|d| \geq |b| + |c|$. Then $|d| > |c|$. Since $g^{-1}(\infty) = -c^{-1}d$, we have $|g^{-1}(\infty)| = |-c^{-1}d| = |d|/|c|$ where c and d are in $\Gamma_n \cup \{0\}$. So $|g^{-1}(\infty)| > 1$ and hence $g^{-1}(\infty) \notin \bar{\mathbb{B}}$. It follows that if $g(x) = b(cx + d)^{-1}$ that $g(x)$ is a vector if x is a vector.

From the above inequality we have $|b| \geq |g(x)||(|d| - |c||x|)$. If $|x| \leq 1$ then $|b| \geq |g(x)|(|d| - |c|) \geq |g(x)||b|$. Thus $|g(x)| \leq 1$ and $g(\mathbb{B}) \subset \mathbb{B}$.

It follows the result holds whenever x is a vector with $|x| \leq 1$, so that $|b| \geq |g(x)||b|$, and as $b \neq 0$ we see that $g(\mathbb{B}) \subset \mathbb{B}$. □

We can now use the above results to establish a general version of Pringsheim's theorem. The original theorem [28] states:

Theorem 9.6.4. *Suppose that for all n, $a_n \neq 0$ and $|a_n| + 1 \leq |b_n|$ where $\{a_n\}$ and $\{b_n\}$ are sequences of complex numbers. Then the continued fraction*

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}, \quad a_n \neq 0$$

converges.

The general version of *Pringsheim's Theorem* in higher dimensions is stated as follows:

Theorem 9.6.5. *Consider a sequence of Clifford matrices $\begin{pmatrix} 0 & b_k \\ c_k & d_k \end{pmatrix}$, for $k = 1, 2, \cdots$, where for each k, $|d_k| \geq |b_k| + |c_k|$, and let these act on $\mathbb{R}^n_\infty$ by the rule*

$$t_k(x) = b_k(c_k x + d_k)^{-1},$$

where this is calculated in the Clifford Algebra C_n. Then the sequence $t_1 t_2 \cdots t_k(0)$ converges.

Proof. We note that the proof of *Pringsheim's Theorem* follows from Lemma 9.6.3 since the condition $g(\mathbb{B}) \subset \mathbb{B}$ and $g(\infty) = 0$ in the lemma establishes the convergence. □

9.7 Continued Fractions with Hurwitz Integer Coefficients

We remark that when $n = 3$, $C_3 = \mathbb{K} = \Gamma_3 \cup \{0\}$. The Clifford matrix $\begin{pmatrix} 0 & b_k \\ c_k & d_k \end{pmatrix}$ will have b_k, c_k and $d_k \in \mathbb{K}$ and $b_k c_k^* \in \mathbb{R} \setminus \{0\}$. In particular, if $c_k = 1$ then b_k is a non zero real number. In this case d_k is a vector in $\mathbb{K} = C_3$.
When $n = 4$ and $c_k = 1$ we see that d_k will be in $v^4 \cong \mathbb{K}$ as a linear space.
Then Prinsheim's theorem yields the condition for convergence that the sequence $\begin{pmatrix} 0 & b_k \\ 1 & d_k \end{pmatrix}$ of Clifford matrices with corresponding maps $g(x) = b_k(x+d_k)^{-1} = \delta_{b_k}\phi\tau_{d_k}(x)$ is convergent if $|d_k| \geq |b_k|+1$. If we further assume that b_k and d_k are Hurwitz integers, then this condition implies that d_k are all non unit.

Finally Beardon [5] notes that while the above theorem may be proved by purely algebraic means, the real strength of these ideas is not that the algebra of continued fractions generalizes to higher dimensions, but that the theory of continued fractions may be regarded as a part of the theory of inversive geometry in any dimension.

	1	i	j	k	ω	$\omega - i - j$	$\omega - j - k$	$\omega - i - k$	$\omega - k$	$\omega - i$	$\omega - 1$	$\omega - j$
1	1	i	j	k	ω	$\omega - i - j$	$\omega - j - k$	$\omega - i - k$	$\omega - k$	$\omega - i$	$\omega - 1$	$\omega - j$
i	i	-1	k	$-j$	$i + k - \omega$	$\omega - j - k$	$i + j - \omega$	ω	$\omega - 1$	$\omega - j$	$k - \omega$	$i - \omega$
j	j	$-k$	-1	i	$i + j - \omega$	ω	$\omega - i - k$	$j + k - \omega$	$j - \omega$	$\omega - 1$	$i - \omega$	$\omega - k$
k	k	j	$-i$	-1	$j + k - \omega$	$i + k - \omega$	ω	$\omega - i - j$	$\omega - i$	$k - \omega$	$j - \omega$	$\omega - 1$
ω	ω	$i + j - \omega$	$j + k - \omega$	$i + k - \omega$	$\omega - 1$	$\omega - j$	$\omega - k$	$\omega - i$	j	k	-1	i
$\omega - i - j$	$\omega - i - j$	ω	$\omega - i - k$	$j + k - \omega$	$\omega - i$	$k - \omega$	$\omega - j$	$1 - \omega$	1	$-i$	j	k
$\omega - j - k$	$\omega - j - k$	$i + k - \omega$	ω	$\omega - i - j$	$\omega - j$	$1 - \omega$	$i - \omega$	$\omega - k$	i	1	k	$-j$
$\omega - i - k$	$\omega - i - k$	$\omega - j - k$	$i + j - \omega$	ω	$\omega - k$	$\omega - i$	$1 - \omega$	$j - \omega$	$-k$	j	i	1
$\omega - k$	$\omega - k$	$i - \omega$	$\omega - 1$	$\omega - j$	i	1	$-k$	j	$i + j - \omega$	ω	$i + k - \omega$	$\omega - j - k$
$\omega - i$	$\omega - i$	$\omega - k$	$j - \omega$	$\omega - 1$	j	k	1	$-i$	$\omega - i - k$	$j + k - \omega$	$i + j - \omega$	ω
$\omega - 1$	$\omega - 1$	$j - \omega$	$k - \omega$	$i - \omega$	-1	i	j	k	$j + k - \omega$	$1 + k - \omega$	$-\omega$	$i + j - \omega$
$\omega - j$	$\omega - j$	$\omega - 1$	$\omega - i$	$k - \omega$	k	$-j$	i	1	ω	$\omega - i - j$	$j + k - \omega$	$i + k - \omega$

Table 9.1: A Multiplication Table for the Hurwitz Integer Units

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