Analysis of some convergence results for inertial variational inequalities problem and its application

by

Thembinkosi Eezy Kunene, 1177166
University of the Witwatersrand, Johannesburg South Africa

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School of Mathematics, Faculty of Science, University of the Witwatersrand
Johannesburg South Africa.

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As the supervisor of the candidate, I have granted approval for the submission of this dissertation.

Dr. C. C. Okeke

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Dedication

This project is dedicated to my parents who have inspired me to achieve excellence and always persevere. They have provided me with the necessary motivation and tools to carry on in the face of difficulty. As well as my siblings since they have always supported me and always know how to help me feel better. So I dedicate this to my family because without them I would not have made it this far.

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Abstract

Some core aspects of nonlinear analysis, which is a major branch of mathematics, are the optimization problems, fixed point theory and its applications. These concepts, that is, optimization theory, fixed point theory and its applications are widely applied in several fields of science such as networking, inventory control, engineering, economics, policy modelling, transportation and mathematical sciences to mention but a few. Due to its relevance to different fields, the theory of optimization and fixed point has been a popular field of research for a long time. Given its expansive nature, researchers continue to make new discoveries and advancements, contributing to its enduring significance across various disciplines. The goal of this dissertation is to explore some convergence iterative methods for approximating optimization problems. We propose a new modified projection and contraction algorithm for approximating solutions of a variational inequality problem involving a quasi-monotone and Lipschitz continuous mapping in real Hilbert spaces. We incorporate the technique of two-step inertial into a single projection and contraction method and prove a weak convergence theorem for the proposed algorithm. The weak convergence theorem proved requires neither the prior knowledge of the Lipschitz constant nor the weak sequential continuity of the associated mapping. Under additional strong pseudomonotonicity, the R-linear convergence rate of the two-step inertial algorithm is presented. Finally, some numerical examples are given to illustrate the effectiveness and competitiveness of the proposed algorithm in comparison with some existing algorithms in the literature.

Contents

Declaration

The author affirms that the work presented is original, and in cases where the work of others has been utilized, appropriate references have been provided. This dissertation, whether in its entirety, or in part has not been previously presented to this institution or any other for the intention of acquiring a degree.

Thembinkosi Eezy Kunene

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Contributed papers from the dissertation

The major part of this dissertation has been submitted/accepted in the following accredited journals indexed by Science Citation Index.

1. **Thembinkosi Eezy Kunene**, C. C. Okeke A Adamu, and D. Uzun Ozsahin, Two-step inertial projection and contraction method for variational inequality with quasi-monotonicity, *J. Inequalities and Applications* (2023), **(under review)**.

Chapter 1

General Introduction

1.1 Background of Study

An increasingly crucial domain of study in nonlinear analysis revolves around fixed point and optimization theory. This is due to its fruitful applications in almost all disciplines and mathematical sciences. It is common knowledge that solving fixed point or optimization problems analytically is exceedingly challenging or nearly impossible, necessitating the consideration of approximation methods of solution. As a result, researchers in this field have proposed a variety of techniques for solving fixed point and optimization problems. To name a few, there are proximal-like methods, fixed point methodologies, auxiliary principles, decomposition strategies, extragradient methods, subgradient and extragradient methods, projection contraction methods, and normal map equations, see [4, 5, 6, 33, 37, 38, 39, 40] and the references therein. Fixed point problem has to do with finding a solution to an equation of the form

$$x = Tx, \tag{1.1.1}$$

where a single valued nonlinear operator T is defined on a nonempty set X. The theory of fixed points have been one of the most developed areas of study within the realm of nonlinear analysis and its practical application. Due to its successful applications in almost all fields of study, this development has consistently captured the attention of numerous researchers worldwide. As mentioned above, fixed point methodologies have been extensively used within the fields that includes engineering, physics, biology, chemistry, game theory, economics, mathematical sciences, signal processing, inverse problems, and many others(see [8, 11, 10, 9, 12, 42, 43] and the references therein). In 1986, Poincare in [47] introduced the field of fixed point theory. Thereafter, Brouwer in [15] established some fixed point results. Furthermore, Kakutani in [34] extended and generalised the results obtained by Brouwer. Iterative processes for approximating the fixed point of a nonlinear mapping were first introduced and investigated by the Polish mathematician Stefan Banach in [13]. The most useful and applicable finding in nonlinear analysis is the Banach fixed point theorem. The metric space's completeness is the only prerequisite in terms of space for proving the Banach fixed point theorem. In addition, the Banach contraction result is simple to prove since it employs iterative techniques. It can also be quickly applied to a computer system to locate the fixed point of the contractive mapping because it generates approximations with any desired level of accuracy. The Volterra integral equations, dynamical programming, nonlinear integro-differential equations, game theory, random, ordinary and partial differential equations, and others have all found use for the Banach fixed point theorem to prove their existence and uniqueness.

Definition 1.1.1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a contraction if there exists a constant $\delta \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) \quad \forall x, y \in X. \tag{1.1.2}$$

Theorem 1.1.2 ([13])**.** *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping. Then, T has a unique fixed point $x^* \in X$, and for any $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

Thus, our study will not only be limited to fixed point problems but will also cover a certain optimization problems.

1.1.1 Variational Inequality Problem

Let C be a nonempty closed and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $A : H \to H$ be a nonlinear operator. The classical Variational Inequality Problem (VIP) for A on C is defined as follows: find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{1.1.3}$$

The solution of VIP is denoted by $VIP(A, C)$. The variational inequality theory was first introduced in 1966 by Hartman and Stampacchia in [51] as a tool for studying partial differential equations with applications mainly related to mechanics. Subsequently, Stampacchia expanded the variational inequality theory in several papers. Over the years, other researchers have studied and introduced different methods for solving fixed point, optimization problems and VIPs. These methodologies, encompassing proximal-like methods, fixed point methodologies, auxiliary principles, decomposition strategies, extragradient methods, subgradient and extragradient methods, projection contraction methods, and normal map equations, offer diverse approaches to finding approximate solutions to optimization VIPs. This is crucial as solving these problems analytically often proves to be expensive, difficult, or even impossible. For example, Goldstein, in [28], introduced an iterative technique characterized in the following manner:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda A x_n), \end{cases} \tag{1.1.4}$$

for each and every $n \in \mathbb{N}$, where $\lambda \in (0, \frac{2\alpha}{L^2})$, A is both L-Lipschitz continuous and α-strongly monotone and the mapping from H onto C is a metric projection P_C. The author has demonstrated the convergence of the iterative technique (1.1.4) to the set of solution of the VIP (1.1.3). Nevertheless, it has been noted that when we have the possession of both L-Lipschitz continuity and monotonicity by A, the iterative method (1.1.4) might not converge to $VIP(A, C)$, as discussed in [29] and the references therein for further details. Moreover, the computation of the value λ may pose significant challenges, and in certain instances, it may be highly challenging or even impossible. Given these limitations, Korpelevich in [35] introduced and examined the Extragradient Method (EM). The definition of this method is as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.1.5}$$

for every $n \geq 1$, where $\lambda_n \in (0, \frac{1}{L^2})$, A is both L-Lipschitz continuous and monotone and the mapping from H onto C is a metric projection P_C. While the above method successfully addressed the question of weakening the cost operator, the challenge of computing λ_n persists. An additional limitation of this approach is the need for computing two projections onto the feasible set C in each iteration, incurring significant costs, especially when C has a complex structure. As the introduction of the Extragradient Method (EM), several researchers have proposed, adapted, and investigated various versions of EM with different properties of the cost operator A, such as monotonicity and pseudomonotonicity. For instance, researchers like He *et al.* in [30], He *et al.* in [29], Apostol *et al.* in [7], Ceng *et al.* [?], Nadezhkina and Takahashi in [45], Ceng *et al.* in [17], and different authors have explored various aspects of this method. To overcome the limitations of the Extragradient Method (EM), the Subgradient Extragradient Method (SGEM) was presented and scrutinized in [18], by the author Censor et al., which is defined in the manner below:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ T_n = \{w \in H : \langle x_n - \lambda_n A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.1.6}$$

where $\lambda_n \in (0, \frac{1}{L})$ for every $n \geq 1$, A is both L-Lipschitz continuous and monotone and a mapping from H onto C is a metric projection P_C. They demonstrated that the iterative approach (1.1.6) gives convergence to the solution of VIP (1.1.3). Nevertheless, one drawback of this iterative method is the computation of λ_n.

1.1.2 Split Variational Inequality Problem

The author Censor in *et al.* [21] presented and examined an idea of Split Variational Inequality Problem as a generalization of the VIP (1.1.3) and it is depicted as

$$\text{Get } x^* \in \text{C that solves } \langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{1.1.7}$$

So that $y^* = Tx^* \in Q$ resolves

$$\langle fy^*, y - y^* \rangle \geq 0, \forall y \in Q. \tag{1.1.8}$$

Here, C and Q represents a subset of H_1 and H_2, respectively, where H_1 and H_2 are the real Hilbert spaces. These subsets are nonempty, closed, and convex. Now, we form two operators $A : H_1 \to H_1$and $f : H_2 \to H_2$ and an operator $T : H_1 \to H_2$ is linear and bounded. By observation, we can note that the SVIP (1.1.7)-(1.1.8) is an amalgamation of the widely recognized Split Feasibility Problem (SFP) and the classical VIP (1.1.3), as presented by Censor along with Elfving in [21]. This involves finding $x^* \in C$

$$Tx^* = y^* \in \ Q. \tag{1.1.9}$$

The author Censor *et al.* in [20] and [19], aimed to estimate the answers of SVIP (1.1.7)-(1.1.8). To achieve this, they transformed SVIP (1.1.7)-(1.1.8) into a manageable VIP (1.1.3) within the product space $H_1 \times H_2$. Subsequently, they employed the SGEM to address the equivalent SVIP (1.1.7)-(1.1.8) problem. Using this method, they encountered the challenge of mapping the new product subspaces back into H_1 and H_2. Furthermore, it was noticed that this approach does not possess the inherent splitting structure found in the SVIP(1.1.7)-(1.1.8), and the methodology does not possess the flexibility to be applied effectively to real-world problems, as evident in [20] and related references. In response to these limitations, several researchers have explored and proposed various iterative techniques for addressing the SVIP (1.1.7)-(1.1.8). An example is Tian and Jiang in [53], who proposed and examined the subsequent iterative approach.

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \gamma_n T^*(I - P_Q(I - \nu A))Tx_n), \\ t_n = P_C(y_n - \lambda_n B y_n), \\ x_{n+1} = P_C(y_n - \lambda_n B t_n), \end{cases} \tag{1.1.10}$$

for $n \in \mathbb{N}$, where $\{\gamma_n\} \subset [a, b]$, for some $a, b \in (0, \frac{1}{\|T^2\|}), \{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, \frac{1}{L}), \nu \in (0, 2\alpha)$, a linear operator $T : H_1 \to H_2$ is bounded, A is both Lipschitz continuous and $\alpha-$inversely strongly monotone, B is both Lipschitz continuous and monotone. They demonstrated that the suggested iterative approach converges weakly to the set of solution of the SVIP (1.1.7)-(1.1.8). Additionally, Pham *et al.* in [46] proposed a Halpern-type iterative method in real Hilbert spaces for solving the SVIP (1.1.7)-(1.1.8). Furthermore, they confirmed that for strong convergence to the set of solution of the SVIP (1.1.7)-(1.1.8) requires the iterative approach. As with the VIP, the research area focused on approximating solutions to Split Variational Inequality Problems (SVIP) has garnered significant interest. However, the techniques that have been considered in this literature requires $\alpha-$inversely strongly monotone or pseudomonotone to be the underlying operators. Therefore, there is a pressing need to introduce an iterative approach with weaker monotonicity conditions on cost operators and improved convergence rates. This forms a central focus of the dissertation.

1.1.3 Bilevel Variational Inequality Problem

The notion of Bilevel Variational Inequality Problem (BVIP) was initiated by Mainge in [41] which he defined as follows: Find $x^* \in VIP(A, C)$ in a manner that

$$\langle Gx^*, x - x^* \rangle \geq \ 0 \ \forall \ x \in VIP(A, C) \tag{1.1.11}$$

where $G : H \to H$ is $\gamma-$strongly monotone and L-Lipschitz continuous. It is easy to see that the BVIP (1.1.11) is a problem that is made up of the VIP (1.1.3) as a constraint. He proposed the following extragradient technique:

$$\begin{cases} u_0 \in C \\ v_n = P_C(u_n - \lambda_n A u_n) \\ t_n = P_C(u_n - \lambda A v_n) \\ u_{n+1} = t_n - \alpha_n G t_n, \end{cases} \tag{1.1.12}$$

where $\{\lambda_n\} \subset [a.b] \in \left(0, \frac{1}{L}\right)$ and $\alpha_n \subset (0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\lim\limits_{n\to\infty} \alpha_n = 0$. It was confirmed that the sequence $\{x_n\}$ generated converges strongly to a unique solution to the BVIP (1.1.11). It is readily apparent that the iterative technique (1.1.12) has at least drawbacks which are the facts that $\{\lambda_n\} \subset [a, b] \subset \left(0, \frac{1}{L}\right)$ and the double metric projection (P_C). To overcome these drawbacks, researchers have introduced the Tseng type iterative technique, the projection contraction iterative technique and the subgradient extragradient iterative technique that is self-adaptive, see [50, 53, 56, 59, 60] and the reference therein contains further details. Specifically, Tan *et al.* in [53] presented and investigated the following iterative technique:

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ T_n = \{x \in H : \langle w_n - \lambda_n A w_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(w_n - \lambda_n A y_n), \\ x_{n+1} = z_n - \alpha_n \gamma G z_n, \end{cases} \tag{1.1.13}$$

for all $n \in \mathbb{N}$, and

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = y_n - \lambda_n(A w_n - A y_n), \\ x_{n+1} = z_n - \alpha_n \gamma G z_n, \end{cases} \tag{1.1.14}$$

for all $n \in \mathbb{N}$, where A is L-Lipschitz continuous, pseudomonotone and sequentially weakly continuous, G is $\alpha-$strongly monotone and L_1-Lipschitz continuous. They established that the iterative techniques (1.1.13) and (1.1.14) converge strongly to a unique solution of the BVIP (1.1.11) using some standard assumptions.

Over time, these iterative techniques have been improved upon using different approaches such as the introduction of inertial extrapolation techniques which was first introduced by Polyak in [48] although these techniques have been improved upon by different researchers. The inertial technique required the first two initial terms of the iterative technique while the next iterate is defined by making use of the previous two iterates. The BVIP (1.1.11) has many applications in areas such as equilibrium constraints, bilevel convex programming models, minimum-norm problems with the solution set of variational inequalities, bilevel linear programming, image restoration and many more (see [4, 24, 40, 26, 25, 55, 58]) and the references therein. Due to these applications, many authors have introduced different iterative techniques for solving the BVIP in the framework of Hilbert spaces (see [4, 1, 2, 3, 41, 44, 56] and the references therein). The crucial roles played by the underlying cost operators in real applications of these iterative methods are widely recognized. Hence, the goal of this research is to initiate an iterative method with less restrictive cost operators, that does not require the knowledge of the Lipschitz constant during implementation of the algorithm and has a better rate of convergence. To achieve this, we introduced the incorporation of a modified inertial iterative technique that have a step size that is auto-adaptive to approximate the solutions of the quasimonotone BVIP (1.1.11). In addition, a modified inertial approach to enhance the convergence speed of the suggested methods used numerical experiments to justify that these methods are good compare to presented methods in the literature for solving the BVIP (1.1.11).

In the light of the above facts, the open question remains, can we construct an iterative method that converges faster, more efficient, and more successful in estimating the solutions of fixed point problems and optimization problems as compared to existing iterative approaches in the literature? Moreover, can we also further generalize existing optimization problems?

1.2 Research Problems and Literature Review

Consider H being a real Hilbert space endowed to have an inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ as the corresponding induced norm. Take $C \subset H$ which contains the properties of a set being nonempty, closed, and convex. Also, consider a continuous operator $A : C \to H$. The variational inequality problem, denoted as $VI(A, C)$, seeks a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.2.1}$$

The solution set of $VI(A, C)$ (1.2.1), we shall denote it as $\mathcal{S}$,. Various challenges in economics, engineering, mechanics, mathematical programming, transportation, and other fields can be cast in the form of $VI(A, C)$ (1.2.1) (refer to, e.g., [63, 65, 82]). If $\mathcal{S}_D$ represent the solution set of the dual variational inequality problem, defined as
$\mathcal{S}_D := \{x^* \in C | \langle Ay, y - x^* \rangle \geq 0, \ \forall \ y \in C\}$. Then, $\mathcal{S}_D$ forms a subset of C that is closed and convex . Considering the convexity of C and the continuity of A, it follows that $\mathcal{S}_D$ is contained within $\mathcal{S}$.

The convergence analysis of algorithms addressing variational inequality problems is significantly influenced by the monotonicity of operators. In recent research, scholars have successfully tackled variational inequality problems by relaxing the conditions of the cost operator in their constructed algorithms, incorporating pseudomonotonicity or quasimonotonicity (see, for instance, [81, 102, 87, 101]). In the literature, the extradgradient method is a prominent method for approximating solutions of the $VI(A, C)$ (1.2.1). This method was first introduced by Korpelevich [84] for solving saddle point problems and has been well applied extensively (see, e.g. [76, 80, 93, 73, 74]). The extragradient method involves computing double projections onto the feasible set C twice per each iteration. However, if the set C is not simple, the calculation of the projection onto C can be very complicated and thus impedes the usage of the extragradient method. Considerable efforts have been made by many researchers to modify and improve this method; see, e.g. [95, 69, 70, 71, 85]. One of the prominent modifications of the method of extragradient is the method of subgradient extragradient initiated by Censor et al. (see [73, 74]). In this method, the authors replaced the second projection onto C with a projection onto a half space which can be calculated explicitly. More so, the subgradient extragradient method requires evaluating at two points in each iteration the value of the cost operator. Efficiency is also compromised when the cost operator exhibits a complex structure.

Another adaptation of the extragradient method is the modification proposed by Tseng in [100]. Tseng's method is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - \lambda A x_n), \quad \forall\, n \neq 0, \end{cases} \tag{1.2.2}$$

where $\lambda \in (0, 1/L)$. Tseng's extragradient technique for giving answers to $VI(A, C)$ (1.2.1) has garnered significant attention from various authors, as evidenced by studies such as in [66, 96, 103] and the relevant references.

From a different point of view, another method for solving the $VI(A, C)$ (1.2.1) is the method of projection and contraction, as investigated by various authors in [83, 99], follows a specific following form:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ d(x_n, y_n) = (x_n - y_n) - \lambda(A x_n - A y_n), \\ x_{n+1} = x_n - \gamma \beta_n d(x_n, y_n), \quad \forall\, n \geq 0, \end{cases} \tag{1.2.3}$$

where $\gamma \in (0, 2)$, $\lambda \in (0, 1/L)$, and

$$\begin{cases} \beta_n := \frac{\phi(w_n, y_n)}{\|d(w_n, y_n)\|^2}, \\ \phi(w_n, y_n) := \langle w_n - y_n, d(w_n, y_n) \rangle, \quad \forall n \geq 0. \end{cases}$$

Recently, the projection and contraction method has garnered significant attention from various authors, leading to enhancements and modifications (refer to, for instance, [68, 78, 79] and the cited literature).

Remark 1.2.1. *We remark here that the aforementioned methods require the computation of only one projection onto C per iteration. This may reduce the computational cost of the algorithms compared to the extragradient and subgradient extragradient algorithms and their modifications.*

Next, we introduce an algorithm with an inertial nature, derived from the discrete representation of a dissipative dynamical system of second-order in [62, 64]. This algorithm serves as an approach to enhance convergence properties, as discussed in [61, 88, 90]. In 2001, Alvarez and Attouch in [61] employed the inertial technique to develop an inertial proximal approach. This method was designed to address the challenge of getting the zero of a monotone operator that is maximal, as outlined below: for every $x_{n-1}, x_n \in H$ and two parameters $\theta_n \in [0,1)$, $\lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \big(\lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\big), \quad \forall n \geq 0,$$

which can be written equivalently as follows:

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})), \quad \forall\ n \geq 0,$$

where the term $\theta_n(x_n - x_{n-1})$ induces the inertial and the resolvent of A with parameter λ_n is $J^A_{\lambda_n}$.

The convergence speed of the method of projection and contraction to be enhanced, Dong et al. investigated an inertial variant of this approach for $VI(A, C)$ (1.2.1) in [77]. They introduced the following algorithm:

Algorithm 1.2.2.

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda A w_n), \\ d_n = w_n - y_n - \lambda(A w_n - A y_n), \\ x_{n+1} = w_n - \gamma \eta_n d_n, \end{cases} \tag{1.2.4}$$

where

$$\eta_n = \begin{cases} \frac{\langle w_n - y_n, d_n}{\|d_n\|^2}, & d_n \neq 0, \\ \theta, & d_n = 0, \end{cases}$$

$0 \leq \theta_n \leq \theta_{n+1} \leq \theta < 1$ *and* $\sigma, \delta > 0$ *such that*

1. $\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}$;
2. $0 < \gamma \leq \frac{2[\delta - \theta[\theta(1+\theta)+\theta\delta+\sigma]]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}$.

Then, they demostrated that the sequence $\{x_n\}$ generated by (1.2.4), under the conditions (1) and (2), converges weakly to a solution of $VI(A, C)$ (1.2.1).

Furthermore, when the inertial factor $\{\theta_n\}$ is chosen with $0 \leq \theta_n \leq \overline{\theta_n} < 1$, where

$$\overline{\theta_n} = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1} \\ \theta, & x_n = x_{n-1} \end{cases} \tag{1.2.5}$$

with $\theta \in [0,1)$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$, converges results of (1.2.4) to a solution $VI(A,C)$ (1.2.1) have been obtained in [75, 77]. The findings in [75, 77, 97] all broaden the outcomes of He in [83], particularly when $\theta_n = 0$. Numerical demonstrations indicate an enhanced convergence speed compared to the studied projection and contraction method in [68, 83].

Remark 1.2.3. *We remark here that implementing the results of [75, 77, 97] requires having the exact value or an estimate of A the cost operator's Lipschitz constant. Computing the Lipchitz constant is very challenging in practice.*

To address the concern raised in the foregoing remark, Shehu *et al.* [94] introduced an inertial projection and contraction method and an inertial forward-backward-forward method to address $VI(A,C)$ (1.2.1), where A is Lipschitz continuous and monotone mapping in Hilbert spaces. The inertial factors in the methods proposed by Shehu *et al.* [94] are relaxed and selected in a manner that allows the inertial factor to be taken as 1 if needed. They obtained weak convergence results of the following algorithm under some mild conditions:

Algorithm 1.2.4.

(S0) *Choose* $\theta \in [0,1)$, $\mu \in (0,1)$, $\lambda_1 > 0$, $\gamma \in (0,2)$ *and* $\alpha \in \left(0, \frac{2(1-\theta)^2}{\gamma\theta(1+\theta)+\gamma(1-\theta)^2}\right)$. *Let* $x_0, x_1 \in H$ *be a given starting point. Set* $n := 1$.

(S1) *compute*

$$\begin{cases} w_n = x_n + \theta(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda_n A w_n). \end{cases} \tag{1.2.6}$$

If $w_n = y_n$, *STOP. Otherwise*

(S2) *Compute*

$$d_n = (w_n - y_n) - \lambda_n(Aw_n - Ay_n), \quad \forall n \geq 1.$$

(S3) *Compute*

$$x_{n+1} = (1-\alpha)w_n + \alpha(w_n - \gamma\rho_n d_n), \ n \geq 1,$$

where

$$\rho_n = \begin{cases} \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0 \\ 0, & d_n = 0 \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|\mathcal{F}w_n - \mathcal{F}y_n\|}, \lambda_n\right\}, & Aw_n \neq Ay_n \\ \lambda_n, & \textit{otherwise.} \end{cases} \tag{1.2.7}$$

(S4) Make $n \longleftarrow n+1$, *and return to* $(S1)$.

Observe that all the variants of the projection and contraction method mentioned above only guarantee weak convergence of the sequences to a solution of $VI(A,C)$ (1.2.1). To obtain a strong convergence theorem, Thong *et al.* [97] used the viscosity approximation technique in the method of projection and contraction to initiate a new inertial viscosity method of projection and contraction for solving $VI(A,C)$ (1.2.1). Their algorithm is the following:

Algorithm 1.2.5.

Initialization: Let $\lambda \in (0, 1/L)$ *and* $\alpha > 0$. *Choose two positive sequences* $\{\epsilon_n\} \subset [0,\infty)$, $\{\beta_n\} \subset (0,1)$ *satisfying*

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty,$$

$$\sum_{n=1}^{\infty} \epsilon_n < \infty, \quad \epsilon_n = o(\beta_n).$$

Let $x_0, x_1 \in H$ *be arbitrary.*

Step 1. Given the iterates x_{n-1} *and* x_n $(n \geq 1)$, *choose* α_n *such that*

$$\alpha_n = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if} \quad x_n \neq x_{n-1}, \\ \alpha, & \quad\quad \text{otherwise.} \end{cases} \tag{1.2.8}$$

Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ *and compute*

$$y_n = P_C(w_n - \lambda A w_n).$$

If $y_n = w_n$ *then stop and* y_n *is a solution of the problem* $VI(A,C)$ (1.2.1). *Otherwise, go to* **Step 2.**

Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)(w_n - \theta_n d_n),$$

where

$$d_n := w_n - y_n - \lambda(A w_n - A y_n)$$

and

$$\theta_n := (1 - \lambda L)\frac{\|w_n - y_n\|^2}{\|d_n\|^2}.$$

Set $n := n+1$ *and go to* **Step 1.**

They proved that the produced sequence by their algorithm converges weakly to asolution of $VI(A, C)$ (1.2.1).

Recent research have shown that the one step inertial method had some shortcomings which affected its efficiency to provide acceleration. An example in [92, Section 4] illustrated that the one-step inertial extrapolation that have the form

$$w_n = x_n + \theta(x_n - x_{n-1}),\ \theta \in [0, 1)$$

might encounter a failure to yield acceleration. It was noted in [86, Chapter 4] that incorporating the inertial effect of two points or more, namely x_n and x_{n-1}, may lead to acceleration. In particular, the two-step inertial extrapolation of the form

$$y_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) \tag{1.2.9}$$

such that $\delta < 0$ and $\theta > 0$ have an ability to produce acceleration. Shortcomings in ADMM were addressed too in [91, Section 3] of the one-step inertial acceleration, prompting the proposal of adaptive acceleration for ADMM instead. The scholar, Polyak in [89] also addressed that multi-step inertial techniques have the ability to improve the rate of optimization methods, although neither the rate results of such multi-step inertial techniques nor the convergence were established in [89]. Recent studies have delved into results on multi-step inertial methods, as discussed in [72].

1.3 Objectives

The goal of this dissertation is to enhance the method of inertial projection and contraction further for solving $VI(A, C)$ (1.2.1). This will be achieved through the following objectives:

- To propose a two-step inertial for projection and contraction method to solve $VI(A, C)$ (1.2.1).
- To test the efficiency of our proposed method through numerical experiments with comparative analysis with other existing methods in the literature.

1.4 Organization of the dissertation

We will organize our dissertation in the manner below:

Chapter 1: In this chapter, we provide a concise overview related to our focus of study. Additionally, we explore the research problem, discuss the motivation behind our study, and outline the study's objectives. For the dissertation, we conclude by presenting a comprehensive structure designed to capture the key findings and contributions that are brought by our research.

Chapter 2: In this chapter, we introduce essential terms, provide definitions, discuss critical concepts, and present results relevant to our study. Furthermore, we provide an in-depth literature review of recent and noteworthy past works that contribute to our study.

Chapter 3: The research efforts of this chapter are to present and study an iterative technique for solving quasimonotone variational inequality problems in the framework of real Hilbert spaces. In addition, we establish that the proposed iterative technique converges weakly and we also establish the problem of R-linear convergence and for comparison of present methods in the literature we provide numerical illustrations. These comparisons serve to substantiate our strong convergence result. The findings of this chapter is under review in the following journal:

> **Thembinkosi Eezy Kunene**, C. C. Okeke A Adamu, and D. Uzun Ozsahin, Two-step inertial projection and contraction method for variational inequality with quasi-monotonicity, *J. Inequalities and Applications* (2023), **(under review)**.

Chapter 4: In this chapter, we draw conclusions from our study and emphasize its contributions to thee existing body of knowledge. Furthermore, we identify and discuss potential avenues for of research for the future.

Chapter 2

Preliminaries and Literature Review

2.1 Preliminaries

The section contains definition of concepts and discussions of significant results that will be useful in every part of our study in this dissertation. In addition, we provide an extensive literature review of previous works that align with the results explored in this study.

2.2 Hilbert Spaces

Hilbert spaces are Banach spaces with a norm derived from inner product. They have an extra feature which makes them unique from other Banach spaces. The introduction of Hilbert spaces is in connection to David Hilbert and Erhard Schmidt in [98] during their thorough study of integral equations (between 1862 and 1943). However, John von Neumann gave it the name–Hilbert space. Their study revealed that two square-integrable real-valued functions $\mathcal{A}$ and $\mathcal{T}$ on an interval [a, b] have an inner product.

$$\langle \mathcal{A}, \mathcal{T} \rangle = \int_a^b \mathcal{A}(x)\mathcal{T}(x)dx$$

which shares similar properties of the Euclidean dot product. In fact, Hilbert spaces are the closest generalization of the Euclidean spaces to infinite dimensional spaces. Hilbert spaces are indispensable tools in quantum mechanics, probability theory, fluid mechanics, Fourier analysis (these include applications to signal processing and heat transfer), PDEs, etc. In this dissertation, we explore the fine properties of Hilbert spaces as a support in establishing our main results.

Definition 2.2.1. *Let H be a set that is nonempty. A function $\langle \cdot, \cdot \rangle$ is an inner product on H defined on $H \times H$ with values in $K = \mathbb{R}$ or $\mathbb{C}$ such that the following conditions hold:*

I_1. $\langle aq_1 + bq_2,\, r\rangle = a\langle q_1,\, r\rangle + b\langle q_2,\, r\rangle \quad \forall\ q_1,\, q_2,\, r \in H$ *and* $a,\, b \in K$;

I_2. $\langle q,\, r\rangle = \overline{\langle r, q\rangle}$ *(the bar represents the complex conjugate);*

I_3. $\langle q,\, q\rangle \geq 0$ and $\langle q,\, q\rangle = 0 \iff q = 0$.

Remark 2.2.2. *$(H,\, \langle\cdot,\cdot\rangle)$ is called a complex inner product space. We give the following observation:*

(i) combining I_1 and I_2, we have
$\langle q,\, cr_1 + dr_2\rangle = \overline{\langle cr_1 + dr_2, q\rangle} = \overline{c\langle r_1, q\rangle + d\langle r_2, q\rangle} = \bar{c}\langle q,\, r_1\rangle + \bar{d}\langle q,\, r_2\rangle$;

(ii) by induction, I_1 and I_2, we obtain the generalization formula:
$\left\langle \sum_i a_i q_i, \sum_j b_j r_j \right\rangle = \sum_i \sum_j a_i \overline{b_j}\, \langle q_i, r_j\rangle$;

(iii) $I_1 \implies \langle 0,\, 0\rangle = \langle 0r,\, 0\rangle = 0\langle r,\, 0\rangle = 0$.
Hence, I_1, I_2, I_3 are equivalent to I_1, I_2, and if I_3 is replaced by I_3' (where I_3' is I_3 but with $q \neq 0$) then $\langle q,\, q\rangle$ is positive. A function satisfying I_1, I_2, I_3' is a complex inner product on H.

Remark 2.2.3. *If $\langle r,\, q\rangle = \overline{\langle r, q\rangle}$, then $(H,\, \langle\cdot,\cdot\rangle)$ is called a real inner product space. We give the following observation:*

(i) combining I_1 and I_2, we have
$\langle q,\, cr_1 + dr_2\rangle = \langle cr_1 + dr_2,\, q\rangle = c\langle r_1,\, q\rangle + d\langle r_2,\, q\rangle = c\langle q,\, r_1\rangle + d\langle q,\, r_2\rangle$;

(ii) by induction, I_1 and I_2, we obtain the generalization formula:
$\left\langle \sum_i a_i q_i, \sum_j b_j r_j \right\rangle = \sum_i \sum_j a_i b_j\, \langle q_i, r_j\rangle$;

(iii) for real inner product space, a similar property as Remark 2.2.2(iii) is obtainable.

Remark 2.2.4. *A direct consequence of I_3 is that the square root of $\langle q,\, q\rangle$ exists since $\langle q,\, q\rangle$ is nonnegative. We denote it as*

$$\|q\| = \sqrt{\langle q, q\rangle}.$$

$\|q\|$ is called the norm or length of q.

Definition 2.2.5. *An inner product space $(H,\, \langle\cdot,\cdot\rangle)$ is said to be complete if every Cauchy sequence in H converges to a point in H and a complete inner product space is called a Hilbert space.*

Proposition 2.2.6. *(Cauchy Schwartz inequality) Let $(H,\, \langle\cdot,\cdot\rangle)$ be an inner product space. For $q,\, r \in H$*

$$|\langle q, r\rangle|^2 \leq \langle q, q\rangle\langle r, r\rangle \quad \text{or} \quad |\langle q, r\rangle| \leq \|q\|\|r\|.$$

Remark 2.2.7. *For $q \neq 0$, $r \neq 0$, $\langle q,\, r\rangle = \|q\|\|r\| \iff q = \beta r$ or $r = \beta q$ for some $\beta \in \mathbb{R}_+$.*

2.2.1 Examples of Hilbert Spaces

(i) Let a space H = $\mathbb{R}^n$ endowed with the inner product

$$\langle q, r\rangle = q_1 r_1 + q_1 r_1 + \cdots + q_n r_n = \sum_{i=1}^{n} q_i r_i,$$

where

$$q = (q_1, q_2, \cdots, q_n) \quad \text{and} \quad r = (r_1, r_2, \cdots, r_n) \quad \text{are in} \quad \mathbb{R}^n,$$

in this case, $(H, \langle\cdot\rangle)$ forms a (real) Hilbert space.

(ii) The spaces $W^{s,2}(\Omega)$ or $H^s(\Omega)$ (Sobolev spaces) are Hilbert spaces with inner product

$$\langle \mathcal{A}, \mathcal{T}\rangle = \int_\Omega \mathcal{A}(q)\overline{\mathcal{T}(q)}dq + \int_\Omega D\mathcal{A}(q)\cdot D\overline{\mathcal{T}(q)}dq + \cdots + \int_\Omega D^s\mathcal{A}(q)\cdot D^s\overline{\mathcal{T}(q)}dq,$$

where s is a non-negative integer, $\Omega \subset \mathbb{R}^n$, and $\mathcal{A}$ and $\mathcal{T}$ are $L^2(\Omega)$ functions whose weak derivatives of order up to s are also $L^2(\Omega)$. The dot indicates the dot product in the Euclidean space of partial derivatives of each order.

(iii) The space C[a, b] of all complex-valued continuous function on a closed interval with inner product

$$\langle \mathcal{A}, \mathcal{T}\rangle = \int_a^b \overline{\mathcal{A}(q)}\mathcal{T}(q)dq,$$

where $\mathcal{A}, \mathcal{T}$: [a, b] $\to \mathbb{C}$ are continuous is not a Hilbert space; consider for example, for the interval [-1, 1] the sequence of continuous "step" functions, $\{\mathcal{A}_m\}_m$, characterized by the following definition

$$\mathcal{A}_m(q) = \begin{cases} 0 & q \in [-1, 0], \\ 1 & q \in \left[\frac{1}{m}, 1\right], \\ mq & q \in (0, \frac{1}{m}). \end{cases}$$

This sequence is a Cauchy sequence for the norm induced by the preceeding inner product which does not converge to a continuous function. However, the completion of C[a, b] with respect to the associated norm

$$\|\mathcal{A}\| = \left(\int_a^b |\mathcal{A}(q)|^2 dq\right)^{\frac{1}{2}}$$

is a Hilbert space and it is denoted by $L^2([a, b])$.

Remark 2.2.8. *The spaces $L^p([a, b])$ are Banach spaces but they are not Hilbert spaces when $p \neq 2$.*

2.2.2 Geometric properties of Hilbert Spaces

Hilbert spaces have one of the finest geometric structures. As a result, it makes most problems posed with this space more manageable as compared to spaces such as Banach spaces [22]. Occurrance of $\langle \cdot, \cdot \rangle$ the inner product along with the circumstance that the proximity mapping from H onto C where H is the (real) Hilbert space and a closed convex set C is a contained in H, is said to be Lipschitzian with a constant of one, and some exciting equalities and inequalities defined on H play a crucial role in the development of our main results. In what follows, we state some of these relations.

$$\|q + r\|^2 = \|q\|^2 + 2\langle q, r\rangle + \|q\|^2 \tag{2.2.1}$$

and

$$\|\beta q + (1-\beta)r\|^2 = \beta\|q\|^2 + (1-\beta)\|r\|^2 - \beta(1-\beta)\|q - r\|^2 \tag{2.2.2}$$

which hold for all q, $r \in$ H and $\beta \in (0, 1)$. We remark that (2.2.1) can be written as

$$2\langle q, r\rangle = \|q\|^2 + \|r\|^2 - \|q + r\|^2 = \|q + r\|^2 - \|q\|^2 - \|r\|^2 \quad \forall\, q, r \in H \tag{2.2.3}$$

and

$$2\|q\|^2 + 2\|r\|^2 = \|q - r\|^2 + \|q + r\|^2, \tag{2.2.4}$$

where (2.2.4) is the well-known parallelogram identity.

2.2.3 Basic Identities and Inequalities in Hilbert Spaces

Lemma 2.2.9. *Let q, r, and s be in real Hilbert space H. Then we have the following:*

(i)
$$\|q \pm r\|^2 = \|q\|^2 \pm 2\langle q, r\rangle + \|q\|^2 \tag{2.2.5}$$

(ii)
$$2\|q\|^2 + 2\|r\|^2 = \|q - r\|^2 + \|q + r\|^2, \tag{2.2.6}$$

(iii)
$$4\langle q, r\rangle = \|q + r\|^2 - \|q + r\|^2 \tag{2.2.7}$$

(iv)
$$\|q - r\|^2 = 2\|s - q\|^2 + 2\|s - r\|^2 - 4\left\|s - \left(\frac{q + r}{2}\right)\right\|^2 \tag{2.2.8}$$

(v)
$$\|q + r\|^2 \leq \|q\|^2 + 2\langle r, q + r\rangle \tag{2.2.9}$$

Proof. (i)

$$\begin{aligned} \|q \pm r\|^2 &= \langle q \pm r, q \pm r\rangle \\ &= \langle q, q\rangle \pm 2\langle q, r\rangle + \langle r, r\rangle \\ &= \|q\|^2 \pm 2\langle q, r\rangle + \|q\|^2. \end{aligned} \tag{2.2.10}$$

(ii) From (2.2.5), we have

$$\|q+r\|^2 = \|q\|^2 + 2\langle q, r\rangle + \|q\|^2 \tag{2.2.11}$$

and

$$\|q-r\|^2 = \|q\|^2 - 2\langle q, r\rangle + \|q\|^2 \tag{2.2.12}$$

addition of (2.2.11) and (2.2.12) gives (2.2.6).

(iii) Subtraction of (2.2.12) from (2.2.11) gives (2.2.7).

(iv) Applying (2.2.6) to these points $\frac{s-q}{2}$ and $\frac{s-r}{2}$, we have

$$\begin{aligned} 2\left\|\frac{s-q}{2}\right\|^2 + 2\left\|\frac{s-r}{2}\right\|^2 &= \left\|\frac{s-q}{2} + \frac{s-r}{2}\right\|^2 + \left\|\frac{s-q}{2} - \frac{s-r}{2}\right\|^2 \\ \implies 4\left\|s - \left(\frac{q+r}{2}\right)\right\|^2 &= 2\|s-q\|^2 + 2\|s-r\|^2 - \|q-r\|^2. \end{aligned}$$

(v)

$$\begin{aligned} \|q+r\|^2 &= \langle q+r, q+r\rangle \\ &= \langle q, q\rangle + 2\langle q, r\rangle + \langle r, r\rangle \\ &\leq \langle q, q\rangle + 2\langle q, r\rangle + \langle r, r\rangle + \langle r, r\rangle \\ &= \|q\|^2 + 2\langle r, q+r\rangle. \end{aligned}$$

□

Lemma 2.2.10. *Let q, $r \in H$ and $\beta \in [0,\ 1]$, we have*

$$\|\beta q + (1-\beta)r\|^2 = \beta\|q\|^2 + (1-\beta)\|r\|^2 - \beta(1-\beta)\|q-r\|^2.$$

Proof.

$$\begin{aligned} \|\beta q + (1-\beta)r\|^2 &= \langle \beta q + (1-\beta)r, \beta q + (1-\beta)r\rangle \\ &= \beta^2\langle q, q\rangle + \beta(1-\beta)\langle q, r\rangle + \beta(1-\beta)\langle r, q\rangle + (1-\beta)^2\langle r, r\rangle \\ &= \beta^2\langle q, q\rangle + 2\beta(1-\beta)\langle q, r\rangle + (1-\beta)^2\langle r, r\rangle. \end{aligned} \tag{2.2.13}$$

Applying (2.2.11) to (2.2.13) gives

$$\|\beta q + (1-\beta)r\|^2 = \beta\|q\|^2 + (1-\beta)\|r\|^2 - \beta(1-\beta)\|q-r\|^2.$$

□

Lemma 2.2.11 ([36])**.** *For s, t, q, $r \in H$, we have*

$$2\langle s-t, q-r\rangle = \|s-r\|^2 + \|t-q\|^2 - \|s-q\|^2 - \|t-r\|^2.$$

We also have

$$\begin{aligned} \|s-t+q-r\|^2 &= \|s-t\|^2 + \|q-r\|^2 + 2\langle s-t, q-r\rangle \\ &= \|s-t\|^2 + \|q-r\|^2 + \|s-r\|^2 + \|t-q\|^2 - \|s-q\|^2 - \|t-r\|^2. \end{aligned}$$

2.2.4 Some Definitions and Important Results

In this section, we recall some basic definitions of functions, recall important lemmas and propositions that are relevant to the rest of this study.

In H the real Hilbert space, we say $F(T) = \{x \in H : Tx = x\}$ represents the set of a nonlinear mapping $T : H \to H$ of fixed points. We use "$\to$" to indicate strong convergence and "$\rightharpoonup$" for weak convergence. For every x and y in H, and for every α in the interval $[0, 1]$, it is widely recognized that

Lemma 2.2.12. [14, 27] *Let H be a real Hilbert space. Then for each and every $x, y \in H$ and $\alpha \in (0, 1)$*

1. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
2. $\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x - y\rangle$;
3. $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
4. $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Definition 2.2.13. *Let $T : H \to H$ be an operator. Then the operator T is called*

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for each and every $x, y \in H$. If $L = 1$, then T is called nonexpansive. If $y \in F(T)$, and

$$\|Tx - y\| \leq \|x - y\|,$$

for each and every $x \in H$. Then T is called quasinonexpansive.

(b) monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in H;$$

(c) pseudomonotone, if

$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, y - x\rangle \geq 0, \quad \forall x, y \in H;$$

(d) α- strongly monotone, if there exists $\alpha > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall\ x, y \in H;$$

(e) quasimonotone,

$$\langle Tx, x - y\rangle > 0 \Rightarrow \langle Ty, x - y\rangle \geq 0 \ \ \forall\ x, y \in H;$$

(f) *firmly nonexpansive,*

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \ \forall\ x, y \in H$$

or equivalently,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2 \ \forall\ x, y \in H;$$

(g) *directed (also called to be firmly quasi-nonexpansive) if* $F(T) \neq \emptyset$ *and*

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - b\|^2 \ \forall x \in H \text{ and } b \in F(T);$$

(h) *sequentially weakly continuous if for each sequence* $\{x_n\}$*, we obtain* $\{x_n\}$ *converges weakly to* x *implies that* Tx_n *converges weakly to* Tx*.*

Remark 2.2.14. *It is widely acknowledged that* α*-strongly monotone* $\Rightarrow$ *monotone* $\Rightarrow$ *pseudomonotone* $\Rightarrow$ *quasimonotone. However, it is important to note that the converses of these implications are not generally true.*

For any $u \in H$ in C where H contains C and C has the properties of being nonempty, closed, and convex, there is a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\| \quad \forall y \in C.$$

P_C denotes the metric projection of H onto C. It is established in the literature that P_C is both a nonexpansive mapping and fulfills

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{2.2.14}$$

for every $x, y \in H$.

Lemma 2.2.15. [31, 32] *Let* H_1 *and* H_2 *be two Hilbert spaces and suppose that* $A : H_1 \to H_2$ *is uniformly continuous on a bounded subset of* H_1 *and* C *is a bounded subset of* H_1*. Then,* $A(C)$ *is bounded.*

Lemma 2.2.16. [23] *Let* C *be a nonempty closed and convex subset of* H *and* $A : C \to H$ *be a continuous pseudomonotone mapping and* $x \in C$*, then*

$$x \in VI(C, A) \qquad if \text{ and only } if \qquad \langle Ay, y - x\rangle \geq\ 0\ \forall\ y \in C$$

Lemma 2.2.17. [4] *Let* C *be nonempty closed convex subset of a real Hilbert space* H*. For any* $x \in H$ *and* $z \in C$*, we have* $z = P_C x$ *if and only if* $\langle x - z, y - z\rangle \leq 0\ \forall\ y \in C$*.*

Lemma 2.2.18. [52] *Let* C *be nonempty closed convex subset of a real Hilbert space* H*. For any* $x \in H$ *and* $y \in C$*, we have* $y = P_C x$ *if and only if* $\langle x - y, y - b\rangle \leq\ 0\ \forall\ b \in C$*.*
A well-known fact is that for every nonexpansive mapping T*, we get a set that is closed and convex containing the fixed points of* T*. In addition, the inequality that is satisfied by* T *is shown below:*

$$\langle (x - Tx) - (y - Ty), Ty - Tx\rangle \leq \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2 \ \forall x, y \in H. \tag{2.2.15}$$

Thus, for all $x \in H$ *and* $x^* \in F(T)$*, we have that*

$$\langle x - Tx, x^* - Tx\rangle \leq \frac{1}{2}\|Tx - x\|^2, \ \forall\ x, y \in H. \tag{2.2.16}$$

Lemma 2.2.19. [57] *Let $T : H \to H$ be an operator. Then the following statements are equivalent:*

1. *T is directed; there holds the relation*

$$\|x - Tx\|^2 \leq \langle x - b, x - Tx \rangle \ \forall \ b \in F(T), x \in H; \tag{2.2.17}$$

2. *there holds the relation*

$$\|Tx - b\| \leq \|x - b\|^2 - \|x - Tx\|^2 \ \forall \ b \in F(T), x \in H. \tag{2.2.18}$$

Lemma 2.2.20. [49] *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim\limits_{k\to\infty} a_n = 0$.

Lemma 2.2.21. *Let H be a real Hilbert space, for $i \leq 1 \leq m, a_i \in H$ and $\alpha_i \in (0, 1)$ such that $\sum\limits_{i=1}^{m} \alpha_i = 1$. Then,*

$$\|\sum_{i=1}^{m} \alpha_i a_i\|^2 = \sum_{i=1}^{m} \alpha_i \|a_i\|^2 - \sum_{i,j=1,i\neq j}^{m} \alpha_i \alpha_j \|a_i - a_j\|^2.$$

Remark 2.2.22. *For any β-inverse strongly monotone operator A, the result is that A is $\frac{1}{\beta}$-Lipschitz, i.e., A is $L = \frac{1}{\beta}$-Lipschitz continuous.*

2.3 Metric Projection

Consider C and H such that $C \subset H$ where the set C is said to be nonempty, closed, and convex along with that H depicts the real Hilbert space. Suppose given x as the member of H, there is a distinctive point that is a member of C, which we will note it as $P_C x$, which is the closest point to x within C such that

$$\|x - P_C x\| \leq \|x - y\| \ \forall y \in C$$

We note that the metric projection of H onto C is denoted as operator P_C. We provide few illustration of the metric projection. For a more detailed illustration, refer to [16, 27].

1. Let C be a close ball, that is $C := \{x \in H : \|x - \alpha\| < r\}$ centred at α with radius $r > 0$, then

$$P_C x = \begin{cases} \alpha + \frac{r(x-\alpha)}{\|x-\alpha\|}, & \text{if } x \notin C, \\ x, & \text{if } x \in C. \end{cases} \tag{2.3.1}$$

2. Let $C = [\alpha, \beta]$ be a closed ball in the rectangle in $\mathbb{R}^n$ with $\alpha = (\{\alpha_i\}_1^n)^T$ and $\beta = (\{\beta_i\}_1^n)^T$, then for $1 \le i \le n$, $P_C x$ has the i^{th} coordinate given by

$$(P_C x)_i = \begin{cases} \alpha_i, & \text{if } x_i < \alpha_i \\ x_i, & \text{if } x_i \in [\alpha_i, \beta_i] \\ \beta_i, & \text{if } x_i > \beta_i. \end{cases}$$

3. Let $C = \{y \in H : \langle x, y \rangle = \alpha\}$, that is C is a hyperplane with $\alpha \neq 0$ and $j \in \mathbb{R}$, then

$$P_C x = a - \frac{\langle \alpha, x \rangle - j}{\|\alpha\|^2} \alpha.$$

4. Let C be the closed halfspace, that is $C = \{y \in H : \langle \alpha, y \rangle \le \alpha\}$ with $\alpha \neq 0$ and $j \in \mathbb{R}$, then

$$P_C x = \begin{cases} x - \frac{x - \langle \alpha, x \rangle}{\|\alpha\|^2} & \text{if } \langle \alpha, x \rangle > j \\ a, & \text{if } \langle \alpha, x \rangle \le j \end{cases}$$

5. Let C be the range of $m \times n$ matrix A with full column rank, then $P_C x = A(A^* A)^{-1} A^* x$ where A^* is the adjoint of A.

Proposition 2.3.1. [27] *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $a \in H$. Then,*

1. $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle, \quad \forall\ y \in H.$
2. $\|x - P_C y\|^2 \le \|x - y\|^2 - \|y - P_C y\|^2, \quad \forall\ y \in C.$
3. $\|(I - P_C)x - (I - P_C)y\|^2 \le \langle (I - P_C)x - (I - P_C)y, x - y \rangle\ \forall\ x \in H$ *and* $y \in C.$
4. $b = P_C y \iff \langle x - b, y - x \rangle \le 0, \ \forall y \in C.$

Lemma 2.3.2. [29, 60] *Let C be a nonempty, closed and convex subset of a Hilbert space H and $A : H \to H$ be a L-Lipschitzian and quasimonotone operator. Suppose that $y \in C$ and for some $p \in C$, we have $\langle Ay, p - y \rangle \ge 0$, then at least one of the following holds*

$$\langle Ap, p - y \rangle \ge 0 \quad \text{or} \quad \langle Ay, q - y \rangle \le 0$$

for all $q \in C$.

Lemma 2.3.3. [4] *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if $\langle x - z; y - z \rangle \le 0\ \forall\ y \in C$.*

Lemma 2.3.4. [4] *Let H be a Hilbert space and $F : H \to H$ be a $\tau-$strongly monotone and L-Lipschitz continuous operator on H. Let $\alpha \in (0,1)$ and $\gamma \in (0, \frac{2\tau}{L^2})$. Then for any nonexpansive operator $T : H \to H$, we can associate $T^\gamma : H \to H$ defined by $T^\gamma x = (1 - \alpha\gamma F)Tx$ for all $x \in H$. Then T^γ is a contraction. That is,*

$$\|T^\gamma x - T^\gamma y\| \leq (1 - \alpha\nu)\|x - y\|,$$

for all $x, y \in H$, where $\nu = 1 - \sqrt{1 - \gamma(2\tau - \gamma L^2)} \in (0,1)$.

Lemma 2.3.5. *[104] Suppose either*

(a) T is pseudomonotone on C and $\mathcal{S} \neq \emptyset$;

(b) T is the gradient of G, where G is a differential quasiconvex function on an open set $K \supset C$ and attains its global minimum on C;

(c) T is quasimonotone on C, $T \neq 0$ on C and C is bounded;

(d) T is quasimonotone on C, $T \neq 0$ on C and there exists a positive number r such that, for every $x \in C$ with $\|x\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle \mathcal{F}x,\ y - x\rangle \leq 0$;

(e) T is quasimonotone on C, int $C \neq \emptyset$ and there exists $x^* \in \mathcal{S}$ such that $Tx^* \neq 0$,

then $\mathcal{S}_D$ is nonempty.

Chapter 3

Main Results

3.1 Weak Convergence Theorem

In this section, we introduce the method of projection and contraction of our preference featuring a two-step inertial extrapolation. Initially, we articulate the essential assumptions required for our convergence analysis.

Assumption 3.1.1.

1. *(a)* $S_D \neq \emptyset$,
   (b) A is Lipschitz continuous on C with constant $L > 0$,
   (c) A satisfies the following condition: whenever $\{x_n\} \subset C$ *and* $x_n \rightharpoonup v^*$, *one has* $\|Av^*\| \leq \liminf_{n\to\infty} \|Ax_n\|$,
   (d) A is quasi-monotone on H.
2. *(i)*
$$\delta \leq 0 < 1-\theta-\frac{2\theta}{\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}} \iff \delta \leq 0 < 1-\theta, \quad 0<\rho<1/\beta.$$
   (ii)
$$\frac{\theta(1+\theta)}{(1-\theta)^2} < \frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho} \iff 0<\rho<\frac{2(1-\theta)^2}{\beta\theta(1+\theta)+\beta(1-\theta)^2}$$
   (iii)
$$2+\theta+(1-\theta)\left(\frac{1-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right) > \left(\frac{1-\beta}{\beta\rho}+\frac{1-\rho}{\rho}-1\right)\delta$$
$$\iff (2+\theta)\beta\rho+(1-\theta)(2-\beta\rho) > (2-2\beta\rho)\delta.$$

Note that if

$$0<\rho<\frac{2(1-\theta)^2}{\beta\theta(1+\theta)+\beta(1-\theta)^2}$$

then $0 < \rho < \beta$ *since*

$$\frac{2(1-\theta)^2}{\beta\theta(1+\theta)+\beta(1-\theta)^2} < \frac{1}{\beta}.$$

Algorithm 3.1.2. *Two-step Inertial for Projection and Contraction method*

1. *Choose the iterative parameters* $\beta \in (0,2)$, $\mu \in (0,1)$ *and* $\lambda_1 > 0$. *Choose* $x_{-1}, x_0, x_1 \in H$ *be given starting points. Set* $n := 1$.

2. *Compute*

$$\begin{cases} w_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) \\ y_n = P_C(w_n - \lambda_n A w_n). \end{cases} \tag{3.1.1}$$

If $w_n = y_n$, *STOP. Otherwise*

3. *Compute*

$$d_n = (w_n - y_n) - \lambda_n(Aw_n - Ay_n), \quad \forall n \geq 1.$$

4. *Compute*

$$x_{n+1} = (1-\rho)w_n + \rho(w_n - \beta\rho_n d_n), \quad n \geq 1,$$

where

$$\rho_n = \begin{cases} \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0 \\ 0, & d_n = 0 \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\ \lambda_n + a_n\right\}, & Aw_n \neq Ay_n \\ \lambda_n + a_n, & Aw_n = Ay_n. \end{cases} \tag{3.1.2}$$

5. *Set* $n \longleftarrow n+1$, *and go to* (2)

Lemma 3.1.3. *[102] Assume that* A *is L-Lipschiptz continuous on* H. *Let* $\{\lambda_n\}$ *be the sequence generated by* (3.1.2). *Then*

$$\lim_{n\to\infty} \lambda_n = \lambda \quad with \quad \lambda \in \left[\min\left\{\lambda_1, \frac{\mu}{L}\right\}, \lambda_1 + \alpha\right],$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$. *Moreover*

$$\|Aw_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|. \tag{3.1.3}$$

Lemma 3.1.4. *If Assumption 3.1.1 holds and* $u \in \mathcal{S}_D$. *Then* $\{x_n\}$ *generated by Algorithm 3.1.2 is bounded.*

Proof. Let $u_n := w_n - \beta\rho_n d_n$, $\forall n \geq 1$. By Lemma 2.2.12, we obtain

$$\begin{aligned}\|u_n - u\|^2 &= \|(w_n - u) - \beta\rho_n d_n\|^2 \\ &= \|w_n - u\|^2 - 2\beta\rho_n\langle w_n - u, d_n\rangle + \beta^2\rho_n^2\|d_n\|^2. \end{aligned} \tag{3.1.4}$$

Observe the following

$$\langle w_n - u, d_n\rangle = \langle w_n - y_n, d_n\rangle + \langle y_n - u, d_n\rangle. \tag{3.1.5}$$

According to $y_n = P_C(w_n - \lambda_n A w_n)$ and $u \in \mathcal{S}_D \subseteq \mathcal{S} \subseteq \mathcal{C}$, then, according to Lemma 2.3.1, we get

$$\langle y_n - u, w_n - y_n - \lambda_n A w_n\rangle \geq 0. \tag{3.1.6}$$

Observed that because $y_n \in C$ and $u \in \mathcal{S}_D$, it result in getting that $\langle Ay_n, y_n - u\rangle \geq 0$, for every $n \geq 0$. Hence

$$\langle \lambda_n A y_n, y_n - u\rangle \geq 0. \tag{3.1.7}$$

Summing (3.1.6) and (3.1.7), we are getting

$$\langle y_n - u, w_n - y_n - \lambda_n A w_n + \lambda_n A y_n\rangle \geq 0.$$

Thus,

$$\langle y_n - u, d_n\rangle \geq 0. \tag{3.1.8}$$

Combining (3.1.5) and (3.1.8), we are getting

$$\langle w_n - u, d_n\rangle \geq \langle w_n - y_n, d_n\rangle. \tag{3.1.9}$$

Putting (3.1.9) into (3.1.4), we get $\left(\text{with } \rho_n := \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}\right)$

$$\begin{aligned}\|u_n - u\|^2 &\leq \|w_n - u\|^2 - 2\beta\rho_n\langle w_n - y_n, d_n\rangle + \beta^2\rho_n^2\|d_n\|^2 \\ &= \|w_n - u\|^2 - 2\beta\rho_n\langle w_n - y_n, d_n\rangle + \beta^2\rho_n\langle w_n - y_n, d_n\rangle \\ &= \|w_n - u\|^2 - \beta(2-\beta)\rho_n\langle w_n - y_n, d_n\rangle. \end{aligned} \tag{3.1.10}$$

From the definition of u_n, we get

$$\begin{aligned}\rho_n\langle w_n - y_n, d_n\rangle &= \|\rho_n d_n\|^2 \\ &= \frac{1}{\beta^2}\|u_n - w_n\|^2. \end{aligned} \tag{3.1.11}$$

Combining (3.1.10) and (3.1.11), we get

$$\|u_n - u\|^2 \leq \|w_n - u\|^2 - \frac{2-\beta}{\beta}\|u_n - w_n\|^2. \tag{3.1.12}$$

Now, using Algorithm 3.1.2, we get

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|(1-\rho)(w_n-u)+\rho(u_n-u)\|^2 \\ &= (1-\rho)\|w_n-u\|^2+\rho\|u_n-u\|^2-\rho(1-\rho)\|w_n-u_n\|^2, \end{aligned} \quad (3.1.13)$$

which implies that

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\rho)\|w_n-u\|^2+\rho\|w_n-u\|^2 \\ &\quad -\rho\frac{(2-\beta)}{\beta}\|u_n-w_n\|^2-\rho(1-\rho)\|w_n-u_n\|^2. \end{aligned} \quad (3.1.14)$$

Note that

$$x_{n+1}=(1-\rho)w_n+\rho u_n$$

and it implies

$$u_n-w_n=\frac{1}{\rho}(x_{n+1}-w_n), \quad \forall n. \quad (3.1.15)$$

Now using (3.1.15) in (3.1.14), we get

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\rho)\|w_n-u\|^2+\rho\|w_n-u\|^2 \\ &\quad -\frac{(2-\beta)}{\rho\beta}\|x_{n+1}-w_n\|^2-\frac{(1-\rho)}{\rho}\|x_{n+1}-w_n\|^2 \\ &= \|w_n-u\|^2-\left[\frac{(2-\beta)}{\rho\beta}+\frac{(1-\rho)}{\rho}\right]\|x_{n+1}-w_n\|^2. \end{aligned} \quad (3.1.16)$$

Observe that

$$\begin{aligned}w_n-u &= x_n+\theta(x_n-x_{n-1})+\delta(x_{n-1}-x_{n-2})-u \\ &= (1+\theta)(x_n-u)-(\theta-\delta)(x_{n-1}-u)-\delta(x_{n-2}-u).\end{aligned}$$

Observe that for every $\alpha,\beta\in\mathbb{R}$ and $x,y,z\in H$ we have

$$\begin{aligned}\|(1+\alpha)x-(\alpha-\beta)y-\beta z\|^2 &= (1+\alpha)\|x\|^2-(\alpha-\beta)\|y\|^2-\beta\|z\|^2+(1+\alpha)(\alpha-\beta)\|x-y\|^2 \\ &\quad +\beta(1+\alpha)\|x-z\|^2-\beta(\alpha-\beta)\|y-z\|^2.\end{aligned}$$

Consequently, we have from Algorithm 3.1.2 that

$$\begin{aligned}\|w_n-u\|^2 &= \|(1+\theta)(x_n-u)-(\theta-\delta)(x_{n-1}-u)-\delta(x_{n-2}-u)\|^2 \\ &= (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n+1}-u\|^2-\delta\|x_{n-2}-u\|^2 \\ &\quad +(1+\theta)(\theta-\delta)\|x_n-x_{n-1}\|^2+\delta(1+\theta)\|x_n-x_{n-2}\|^2 \\ &\quad -\delta(\theta-\delta)\|x_{n-1}-x_{n-2}\|^2. \end{aligned} \quad (3.1.17)$$

Furthermore,

$$\begin{aligned}
\|x_{n+1}-w_n\|^2 &= \|x_{n+1}-(x_n+\theta(x_n-x_{n-1})+\delta(x_{n-1}-x_{n-2}))\|^2\\
&= \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1}-x_{n-2})\|^2\\
&= \|x_{n+1}-x_n\|^2-2\theta\langle x_{n+1}-x_n,x_n-x_{n-1}\rangle\\
&\quad -2\delta\langle x_{n+1}-x_n,x_{n-1}-x_{n-2}\rangle+\theta^2\|x_n-x_{n-1}\|^2\\
&\quad +2\delta\theta\langle x_n-x_{n-1},x_{n-1}-x_{n-2}\rangle+\delta^2\|x_{n-1}-x_{n-2}\|^2\\
&\geq \|x_{n+1}-x_n\|^2-2\theta\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|-2|\delta|\|x_{n+1}-x_n\|\|x_{n-1}-x_{n-2}\|\\
&\quad +\theta^2\|x_n-x_{n-1}\|^2-2\theta|\delta|\|x_{n-1}-x_n\|\|x_{n-1}-x_{n-2}\|^2+\delta^2\|x_{n-1}-x_{n-2}\|^2\\
&\geq \|x_{n+1}-x_n\|^2-\theta\|x_{n+1}-x_n\|^2-\theta\|x_n-x_{n-1}\|^2\\
&\quad -|\delta|\|x_{n+1}-x_n\|^2-|\delta|\|x_{n-1}-x_{n-1}\|^2+\theta^2\|x_n-x_{n-1}\|^2\\
&\quad -|\delta|\theta\|x_{n-1}-x_n\|^2-|\delta|\theta\|x_{n-1}-x_{n-2}\|^2+\delta^2\|x_{n-1}-x_{n-2}\|^2\\
&= (1-|\delta|-\theta)\|x_{n+1}-x_n\|^2+(\theta^2-|\theta|-|\delta|\theta)\|x_n-x_{n-1}\|^2\\
&\quad +(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2.
\end{aligned} \tag{3.1.18}$$

Using (3.1.17) and (3.1.18) in (3.1.16), we get

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\leq (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\
&\quad +(1+\theta)(\theta-\delta)\|x_n-x_{n-1}\|^2+\delta(1+\theta)\|x_n-x_{n-2}\|^2\\
&\quad -\delta(\theta-\delta)\|x_{n-1}-x_{n-2}\|^2-\left(\frac{(2-\beta)}{\beta\rho}+\frac{1-\rho}{\rho}\right)(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2\\
&\quad -\left(\frac{(2-\beta)}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2\\
&\quad -\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-\theta-|\delta|\theta)\|x_n-x_{n-1}\|^2\\
&= (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\
&\quad +\left[(1+\theta)(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\theta^2-\theta-|\delta|\theta)\right]\|x_n-x_{n-1}\|^2\\
&\quad +\left[\delta(1+\theta)\|x_n-x_{n-2}\|^2-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\right]\|x_{n+1}-x_n\|^2\\
&\quad -\left[\delta(\theta-\delta)+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2\\
&\leq (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\
&\quad +\left[(1+\theta)(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\theta^2-\theta-|\delta|\theta)\right]\|x_n-x_{n-1}\|^2\\
&\quad +2\delta(1+\theta)\|x_n-x_{n-1}\|^2-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2\\
&\quad -\left[\delta(\theta-\delta)+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2\\
&\quad +2\delta(1+\theta)\|x_{n-1}-x_{n-2}\|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|x_{n+1}-u\|^2-\theta\|x_n-u\|^2-\delta\|x_{n-1}-u\|^2+\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2\\
&\leq \|x_n-u\|^2-\theta\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2+\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2\\
&+\left((\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\right)\|x_n-x_{n-1}\|^2\\
&+\left[2\delta(1+\theta)-\delta(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2. \qquad (3.1.19)
\end{aligned}$$

For each, $n\geq 0$, define

$$\begin{aligned}
\Gamma_n &:= \|x_n-u\|^2-\theta\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\
&\quad+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2.
\end{aligned}$$

We first demonstrate that $\Gamma_n\geq 0\ \forall n\geq 0$. Note that

$$\|x_{n-1}-u\|^2\leq 2\|x_n-x_{n-1}\|^2+2\|x_n-u\|^2.$$

Hence,

$$\begin{aligned}
\Gamma_n &= \|x_n-u\|^2-\theta\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\
&\quad+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2\\
&\geq \|x_n-u\|^2-2\theta\|x_n-x_{n-1}\|^2-2\theta\|x_n-u\|^2\\
&\quad-\delta\|x_{n-2}-u\|^2+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2\\
&= (1-2\theta)\|x_n-u\|^2+\left[\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)-2\theta\right]\|x_n-x_{n-1}\|^2\\
&\quad-\delta\|x_{n-2}-u\|^2. \qquad (3.1.20)
\end{aligned}$$

Note that from Assumption 3.1.1(2) (i), we get

$$|\delta|\leq 1-\theta-\frac{2\theta}{\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)}\iff \frac{2\theta}{\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)}+\theta-1\leq\delta\leq 1-\theta-\frac{2\theta}{\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)} \qquad (3.1.21)$$

We then obtain from (3.1.20) and (3.1.21) that $\Gamma_n\geq 0\ \ \forall n\geq 0$. From (3.1.19), we obtain

$$\begin{aligned}
\Gamma_{n+1}-\Gamma_n &\leq \left((\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\right)\|x_n-x_{n-1}\|^2\\
&\quad+\left[2\delta(1+\theta)-\delta(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2.
\end{aligned}$$

Now, observe that

$$(\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)<0.$$

Since $|\delta|=-\delta$ and $\delta\le 0$, we obtain

$$\begin{aligned}\iff \theta(1+\theta)+\delta(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta+1)\\ -\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)\delta(1+\theta)<0\end{aligned}$$

$$\iff \delta<\frac{\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta-1)^2-\theta(1+\theta)}{(1+\theta)\left(1-\frac{2-\beta}{\beta\rho}-\frac{1-\rho}{\rho}\right)} \tag{3.1.22}$$

(3.1.22) is obiviously true since $\delta\le 0$ and

$$\frac{\theta(1+\theta)}{(\theta-1)^2}<\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho},\quad \text{by}\quad \text{(iii)}.$$

Futhermore, from (3.1.19) again, we get

$$2\delta(1+\theta)-\delta(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\le 0$$

$$\Leftrightarrow \delta\left[2(1+\theta)-(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(|\delta|-1-\theta)\right]\le 0. \tag{3.1.23}$$

By Assumption 3.1.1(2) (iii), we obtain that (3.1.23) holds. Hence, from (3.1.19), we obtain that $\{\Gamma_n\}$ is non-increasing and therefore convergent. That is, $\lim_{n\to\infty}\Gamma_n$ exists. Furthermore, (3.1.19) also implies

$$\begin{aligned}\lim_{n\to\infty}\Bigg(-(\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\Bigg)\|x_n-x_{n-1}\|^2\\ -\Bigg(2\delta(1+\theta)-\delta(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}-\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\Bigg)\|x_{n-1}-x_{n-2}\|^2=0\end{aligned}$$

Consequently

$$\begin{aligned}0 &\le -\Bigg((\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\Bigg)\|x_n-x_{n-1}\|^2\\ &\le -\Bigg((\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\Bigg)\|x_n-x_{n-1}\|^2\\ &\quad -\Bigg(2\delta(1+\theta)-\delta(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\Bigg)\|x_{n-1}-x_{n-2}\|^2.\end{aligned} \tag{3.1.24}$$

Therefore, from (3.1.24), we get

$$\lim_{n\to\infty} -\left[(\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\right]\|x_n-x_{n-2}\|^2=0.$$

Thus,

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0.$$

Additionally,

$$\begin{aligned}\|x_{n+1}-w_n\| &= \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1}-x_{n-2})\|\\ &\le \|x_{n+1}-x_n\|+\theta\|x_n-x_{n-1}\|+|\delta|\|x_{n-1}-x_{n-2}\|\to 0,\ n\to\infty.\end{aligned}$$

Also,

$$\|u_n-w_n\|=\frac{1}{\rho}\|x_{n+1}-w_n\|\to 0,\ n\to\infty, \tag{3.1.25}$$

and

$$\|u_n-x_{n+1}\|\le\|u_n-w_n\|+\|w_n-x_{n+1}\|\to 0,\ n\to\infty, \tag{3.1.26}$$

and

$$\|u_n-x_n\|\le\|u_n-x_{n+1}\|+\|x_{n+1}-x_n\|\to 0,\ n\to\infty. \tag{3.1.27}$$

Since $\lim_{n\to\infty}\Gamma_n$ exists along with the $\lim_{n\to\infty}\|x_n-x_{n-1}\|=0$, it immediately informs us of $\{x_n\}$ being bounded from (3.1.20). □

Remark 3.1.5. *The clarity lies in the evident fact that the boundedness of the sequence x_n implies the boundedness of w_n, y_n, Ay_n, and d_n.*

Lemma 3.1.6. *If Assumption 3.1.1 is fulfilled and $\{x_n\}$ is produced by Algorithm 3.1.2. Then, $\lim_{n\to\infty}\|w_n-y_n\|=0$.*

Proof. By Algorithm (3.1.2), we get

$$\begin{aligned}\|d_n\| &= \|w_n-y_n-\lambda_n(Aw_n-Ay_n)\|\\ &\le \|w_n-y_n\|+\lambda_n\|Aw_n-Ay_n\|\\ &\le \left(1+\frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|w_n-y_n\|.\end{aligned}$$

So,

$$\frac{1}{\|d_n\|}\ge\frac{1}{\left(1+\frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|w_n-y_n\|}. \tag{3.1.28}$$

Then,

$$\begin{aligned}\langle w_n - y_n, d_n\rangle &= \langle w_n - y_n, w_n - y_n - \lambda_n(A - Ay_n)\rangle \\ &= \|w_n - y_n\|^2 - \langle w_n - y_n, \lambda_n(Aw_n - Ay_n)\rangle \\ &\geq \|w_n - y_n\|^2 - \lambda_n\|Aw_n - Ay_n\|\|w_n - y_n\| \\ &\geq \|w_n - y_n\|^2 - \frac{\lambda_n\mu}{\lambda_{n+1}}\|w_n - y_n\|^2 \\ &= \left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|w_n - y_n\|^2. \end{aligned} \tag{3.1.29}$$

By (3.1.28) and (3.1.29),

$$\begin{aligned}\|u_n - w_n\| &= \beta\rho_n\|d_n\| \\ &= \beta\frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|} \\ &\geq \beta\left[\frac{\left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right)}{\left(1 + \frac{\lambda_n\mu}{\lambda_{n+1}}\right)}\right]\|w_n - y_n\|. \end{aligned} \tag{3.1.30}$$

By (3.1.25), we get from (3.1.30) (noting that $\lim_{n\to\infty}\lambda_n = \lambda$.)

$$\lim_{n\to\infty}\|w_n - y_n\| = 0.$$

□

Lemma 3.1.7. *Consider a sequence $\{x_n\}$ produced by Algorithm 3.1.2 such that Assumption 3.1.1 is satisfied. Suppose $\lim_{n\to\infty}\|w_n - y_n\| = 0$. If v^* is one of the weak cluster points of $\{x_n\}$, then we have at least one of the following : $v^* \in \mathcal{S}_D$ or $Av^* = 0$.*

Proof. The sequence $\{x_n\}$ is bounded according to Lemma 3.1.4. Therefore, we make v^* be a weak cluster point of $\{x_n\}$. Thus, an ability is granted to select a subsequence of $\{x_n\}$, that will be noted as $\{x_{n_k}\}$ with $x_{n_k} \rightharpoonup v^* \in C$.

Now the requirement is to examine the two potential cases that will follow.
Case I: If the $\limsup_{n\to\infty}\|Ax_{n_k}\| = 0$. Then, $\lim_{k\to\infty}\|Ax_{n_k}\| = \liminf_{k\to\infty}\|Ax_{n_k}\| = 0$. Thus, from Assumption we get that

$$0 < \|Av^*\| \leq \liminf_{k\to\infty}\|Ax_{n_k}\| = 0. \tag{3.1.31}$$

This signifies that $Av^* = 0$.

Case II: If the $\limsup_{n\to\infty}\|Ax_{n_k}\| > 0$. Without loss of generality, we have the ability to select a subsequence of $\{Ax_{n_k}\}$ and denote it as $\{Ax_{n_k}\}$ such that $\lim_{k\to\infty}\|Ax_{n_k}\| = M_1 > 0$. Now, using Proposition 2.3.1, we found for each and every $y \in C$ that

$$\langle w_{n_k} - \lambda_{n_k}Aw_{n_k} - y_{n_k}, y - y_{n_k}\rangle \leq 0. \tag{3.1.32}$$

Adding $\lambda_{n_k}\langle Ay_{n_k}, y - y_{n_k}\rangle$ to both sides of (3.1.32), we get

$$\lambda n_k \langle Ay_{n_k}, y - y_{n_k}\rangle \geq \langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle Ay_{n_k} - Aw_{n_k}, y - y_{n_k}\rangle. \quad (3.1.33)$$

Since $\lambda_{n_k} > 0$, from (3.1.33), we deduce that

$$\langle Ay_{n_k}, y - y_{n_k}\rangle \geq \frac{1}{\lambda_{n_k}}\langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle Ay_{n_k} - Aw_{n_k}, y - y_{n_k}\rangle. \quad (3.1.34)$$

Since $\lim_{n\to\infty}\|y_n - w_n\| = 0$ and also having A to be Lipschitz continuous on H, we find that $\lim_{n\to\infty}\|Ay_n - Aw_n\| = 0$. Furthermore, we obtain from (3.1.34)

$$0 \leq \liminf_{k\to\infty}\langle Ax_{n_k}, y - x_{n_k}\rangle \leq \limsup_{k\to\infty}\langle Ax_{n_k}, y - x_{n_k}\rangle < \infty, \quad \forall\ y \in C. \quad (3.1.35)$$

In light of the information presented by (3.1.35), we examine two cases within the context of *Case II:*

Case A: Suppose that $\limsup_{k\to\infty}\langle Ax_{n_k}, y - x_{n_k}\rangle > 0, \quad \forall y \in C$. Then an ability is granted to select a subsequence of $\{x_{n_k}\}$ that will be noted as $\{x_{n_{k_j}}\}$ such that $\lim_{j\to\infty}\langle Ax_{n_{k_j}}, y - x_{n_{k_j}}\rangle > 0$. Thus, there exists $j_0 \geq 1$ such that $\langle Ax_{n_{k_j}}, y - x_{n_{k_j}}\rangle > 0, \ \forall j \geq j_0$, so by quasimonotonicity of A on C, gives $\langle A(y), y - x_{n_{k_j}}\rangle \geq 0, \ \forall y \in C, \ j \geq j_0$. Hence $\langle Ay, y - v^*\rangle \geq 0, \ \forall y \in C$, by making $j \to \infty$. Thus, $v^* \in \mathcal{S}_D$.

Case B: Suppose that $\limsup_{k\to\infty}\langle Ax_{n_k}, y - x_{n_k}\rangle = 0, \ \forall y \in C$. Then, by (3.1.35),

$$\lim_{k\to\infty}\langle Ax_{n_k}, y - x_{n_k}\rangle = 0, \quad \forall y \in C, \quad (3.1.36)$$

from which we get that

$$\langle Ax_{n_k}, y - x_{n_k}\rangle + |\langle Ax_{n_k}, y - x_{n_k}\rangle| + \frac{1}{k+1} > 0, \quad \forall y \in C. \quad (3.1.37)$$

Also, because $\lim_{k\to\infty}\|Ax_{n_k}\| = M_1 > 0$, we can find $k_0 \geq 1$ such that $\|Ax_{n_k}\| > \frac{M_1}{2}, \ \forall k \geq k_0$. Hence, we can set $b_{n_k} = \frac{Ax_{n_k}}{\|Ax_{n_k}\|^2}, \quad \forall k \geq k_0$. Thus, $\langle Ax_{n_k}, b_{n_k}\rangle = 1, \ \forall k \geq k_0$. Therefore, by (3.1.37), we get

$$\left\langle Ax_{n_k}, y + b_{n_k}\left[|\langle Ax_{n_k}, y - x_{n_k}\rangle| + \frac{1}{k+1}\right] - x_{n_k}\right\rangle > 0.$$

Utilizing the fact that A on H is quasimonotone to obtain

$$\Big\langle A\left(y + b_{n_k}\left[|\langle Ax_{n_k}, y - x_{n_k}\rangle|\frac{1}{k+1}\right]\right),$$
$$y + b_{n_k}\left[|\langle Ax_{n_k}, y - x_{n_k}\rangle| + \frac{1}{k+1}\right] - x_{n_k}\Big\rangle \geq 0.$$

This implies that

$$\begin{aligned}
&\left\langle Ay, y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]-x_{n_k}\right\rangle \\
\geq\ &\left\langle Ay - A\left(y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]\right),\right. \\
&\left. y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]-x_{n_k}\right\rangle \\
\geq\ & -\left\|Ay - A\left(y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]\right)\right\| \cdot \\
&\left\|y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]-x_{n_k}\right\| \\
\geq\ & -L\left\|b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]\right\| \\
&\left\|y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]-x_{n_k}\right\| \\
=\ & \frac{-L}{\|Ax_{n_k}\|}\left(|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right) \\
&\left\|y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]-x_{n_k}\right\| \\
\geq\ & \frac{-2L}{M_1}\left(|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right)M_2, \qquad (3.1.38)
\end{aligned}$$

for a certain positive constant $M_2 > 0$, where the presence of M_2 is derived from the bounded nature of

$$\left\{y+b_{n_k}\left[|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right]-x_{n_k}\right\}.$$

Now, note that (3.1.36) implies that

$$\lim_{k\to\infty}\left(|\langle Ax_{n_k}, y-x_{n_k}\rangle|+\frac{1}{k+1}\right)=0.$$

Therefore, as $k\to\infty$ in (3.1.38), we obtain that $\langle Ay, y-v^*\rangle \geq 0, \quad \forall y \in C$. From which we get $v^* \in \mathcal{S}_D$. □

Theorem 3.1.8. *Suppose $\{x_n\}$ is a sequence produced by Algorithm 3.1.2. Then within Assumptions 3.1.1 and $\mathcal{F}x \neq 0, \forall x \in C$. Then $\{x_n\}$ converges weakly to an element of $S_D \subset S$.*

Proof. Consider $w_\omega(x_n)$ as the representation of the set of weak cluster points of x_n. Next, we demonstrate that

$$w_\omega(x_n) \subset S_D.$$

Select $v^* \in w_\omega(x_n)$. Then, there exists a subsequence $\{x_{n_k}\}$ that is a subset of $\{x_n\}$ such that for $k\to\infty$,

$x_{n_k} \rightharpoonup v^*$. As C is weakly closed, it follows that $v^* \in C$. Since $Ax \neq 0$, $\forall x \in C$, we have $Av^* \neq 0$. Given Lemma, we get $v^* \in S_D$. Therefore, $w_\omega(x_n) \subset S_D$. Since $\lim_{n\to\infty} \Gamma_n$ exists and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have that

$$\lim_{n\to\infty} [\|x_n - u\|^2 - \theta\|x_{n-1} - u\|^2 - \delta\|x_{n-2} - u\|^2]$$

exists for all $u \in S_D$.

Suppose now that there exists $\{x_{n_j}\} \subset \{x_n\}$ and $\{x_{n_m}\} \subset \{x_n\}$ such that for $j \to \infty$, $x_{n_j} \rightharpoonup v^*$ and for $m \to \infty$, $x_{n_m} \rightharpoonup x^*$. We have to show that $v^* = x^*$. Observe that

$$2\langle x_n, x^* - v^*\rangle = \|x_n - v^*\|^2 - \|x_n - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2 \tag{3.1.39}$$

$$2\langle x_{n-1}, x^* - v^*\rangle = \|x_{n-1} - v^*\|^2 - \|x_{n-1} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2$$

and

$$2\langle x_{n-2}, x^* - v^*\rangle = \|x_{n-2} - v^*\|^2 - \|x_{n-2} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2$$

Therefore

$$2\langle -\theta x_{n-1}, x^* - v^*\rangle = -\theta\|x_{n-1} - v^*\|^2 + \theta\|x_{n-1} - x^*\|^2 + \theta\|v^*\|^2 - \theta\|x^*\|^2. \tag{3.1.40}$$

and

$$\begin{aligned} 2\langle -\delta x_{n-2}, x^* - v^*\rangle &= -\delta\|x_{n-2} - v^*\|^2 + \delta\|x_{n-2} - x^*\|^2 \\ &\quad +\delta\|v^*\|^2 - \delta\|x^*\|^2. \end{aligned} \tag{3.1.41}$$

In addition to (3.1.39), (3.1.40) and (3.1.41) we arrive to

$$\begin{aligned} 2\langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle &= \big(\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \delta\|x_{n-2} - v^*\|^2\big) \\ &\quad -(\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 \\ &\quad -\delta\|x_{n-2} - x^*\|^2) \\ &\quad +(1 - \theta - \delta)(\|x^*\| - \|v^*\|^2). \end{aligned} \tag{3.1.42}$$

Since $\lim_{n\to\infty} \Gamma_n$ exists, we get

$$\lim_{n\to\infty} [\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 - \delta\|x_{n-2} - x^*\|^2] \tag{3.1.43}$$

exists and

$$\lim_{n\to\infty} [\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \delta\|x_{n-2} - v^*\|^2] \tag{3.1.44}$$

exists. This outcome came from (3.1.42) that the $\lim_{n\to\infty} \langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle$ exists. As a result,

$$\begin{aligned} \langle v^* - \theta v^* - \delta v^*, x^* - v^*\rangle &= \lim_{j\to\infty} \langle x_{n_j} - \theta x_{n_{j-1}} - \delta x_{n_{j-2}}, x^* - v^*\rangle \\ &= \lim_{n\to\infty} \langle x_{n-1} - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle \\ &= \lim_{m\to\infty} \langle x_{n_m} - \theta x_{n_{m-1}} - \delta x_{n_{m-2}}, x^* - v^*\rangle \\ &= \langle x^* - \theta x^* - \delta x^*, x^* - v^*\rangle. \end{aligned}$$

Hence

$$(1-\theta-\delta)\|x^* - v^*\|^2 = 0.$$

Given that $\delta \leq 0 < 1-\theta$, we come to a conclusion of $x^* = v^*$. Hence, a sequence $\{x_n\}$ weakly converges to a point in $\mathcal{S}_D$. From which, the proof is concluded. □

3.2 Linear Convergence

Here, for achieving linear convergence we outline the algorithm, operating under the following set of assumptions:

Assumption 3.2.1.

(a) A is ϱ-strongly pseudo-monotone and L-Lipschitz continuous on H.

(b) Let $\mu \in (0,1)$, $\lambda_1 > \frac{\mu}{L}$ and θ, δ, ρ be three real number satisfying

$$0 \leq \rho \leq \frac{1}{2}, \quad 0 \leq \theta \leq \delta \leq \min\left\{\frac{1-2\rho}{1-\rho}, \frac{\rho}{1-\rho}, \frac{\rho\tau}{1-\rho\tau}\right\},$$

where $\tau := \min\left\{\frac{(1-\mu)^2}{2(1+\mu)^2}, \varrho\lambda\frac{(1-\mu)}{(1+\mu)^2}\right\}$.

Algorithm 3.2.2. *Linear double Step Inertial Projection and Contraction Method*

1. *Given starting points $x_0, x_1 \in H$. Set $n := 1$.*

2. *Compute*

$$\begin{cases} z_n = x_n + \delta(x_n - x_{n-1}), \\ w_n = x_n + \theta(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \end{cases}$$

If $w_n = y_n = x_n$, STOP. Otherwise

3. *Compute*

$$d_n = (w_n - y_n) - \lambda_n(Aw_n - Ay_n), \quad \forall n \geq 1.$$

4. *Compute*

$$x_{n+1} = (1-\rho)z_n + \rho(w_n - \beta\rho_n d_n), \quad n \geq 1,$$

where

$$\rho_n = \begin{cases} \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0 \\ 0, & d_n = 0 \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\ \lambda_n + a_n\right\}, & Aw_n \neq Ay_n \\ \lambda_n + a_n, & \text{otherwise.} \end{cases}$$

5. *Make* $n \longleftarrow n+1$, *and return to* (2).

Theorem 3.2.3. *Assume that A is ϱ-strongly pseudo-monotone and L-Lipschitz continuous. Then $\{x_n\}$ be produced by Algorithm (3.2.2) under the Assumptions (3.2.1) converges to a distinctive element of $\mathcal{S}$ linearly.*

Proof. Let u be the distinctive element in $\mathcal{S}$. Then, we get $\langle Au, x - u\rangle \geq 0, \quad \forall x \in \mathcal{S}$. In particula, choose $y_n \in \mathcal{S}$ one obtains

$$\lambda_n\langle Au, y_n - u\rangle \geq 0.$$

because A is ϱ-strongly pseudo-monotone, we obtain

$$\lambda_n\langle Ay_n, y_n - u\rangle \geq \lambda_n\varrho\|y_n - u\|^2. \tag{3.2.1}$$

Adding of (3.1.6) and (3.2.1) gives

$$\langle y_n - u, w_n - y_n - \lambda_n Aw_n + \lambda_n Ay_n\rangle \geq \varrho\lambda_n\|y_n - u\|^2.$$

Thus,

$$\langle y_n - u, d_n\rangle \geq \varrho\lambda_n\|y_n - u\|^2, \tag{3.2.2}$$

we then obtain from (3.1.5) and (3.2.2) that

$$\langle w_n - u, d_n\rangle \geq \langle w_n - y_n, d_n\rangle + \varrho\lambda_n\|y_n - u\|^2. \tag{3.2.3}$$

Also, substituting (3.2.3) into (3.1.4) (when $\beta = 1$) gives

$$\begin{aligned} \|u_n - u\|^2 &\leq \|w_n - u\|^2 + \rho_n^2\|d_n\|^2 - 2\rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\varrho\lambda_n\|y_n - u\|^2 \\ &= \|w_n - u\|^2 + \rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\varrho\lambda_n\|y_n - u\|^2 \\ &= \|w_n - u\|^2 - \rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\varrho\lambda_n\|y_n - u\|^2. \end{aligned} \tag{3.2.4}$$

Adding (3.1.11) and (3.2.4), we obtain

$$\|u_n - u\|^2 \leq \|w_n - u\|^2 - \|u_n - w_n\|^2 - 2\rho_n\varrho\lambda_n\|y_n - u\|^2. \tag{3.2.5}$$

From the definition of ρ_n and (3.1.29), we get for all $n \geq n_0$

$$\rho_n^2\|d_n\|^2 = \langle w_n - y_n, d_n\rangle \geq \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2$$

and

$$\rho_n = \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2} \geq \frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}. \tag{3.2.6}$$

Also, from (3.1.30), we get

$$-\|u_n - w_n\|^2 \leq -\frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}\|w_n - y_n\|^2 \quad \forall n \geq n_0. \tag{3.2.7}$$

Substituting (3.2.7) into (3.2.5) (noting (3.2.6)) one has

$$\begin{aligned}\|u_n - u\|^2 &\leq \|w_n - u\|^2 - \frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}\|w_n - y_n\|^2 - 2\rho_n\varrho\lambda_n\|y_n - u\|^2\\ &\leq \|w_n - u\|^2 - \frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}\|w_n - y_n\|^2\\ &\quad -2\varrho\lambda_n\frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}\|y_n - u\|^2. \end{aligned} \tag{3.2.8}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$, we get

$$\lim_{n\to\infty}\frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2} = \frac{(1-\mu)^2}{(1+\mu)^2} \quad \text{and} \quad \lim_{n\to\infty}\frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2} = \frac{(1-\mu)}{(1+\mu)^2}. \tag{3.2.9}$$

On the other hand,

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|(1-\rho)(z_n - u) + \rho(u_n - u)\|^2\\ &= (1-\rho)\|z_n - u\|^2 + \rho\|u_n - u\|^2 - \rho(1-\rho)\|u_n - z_n\|^2. \end{aligned} \tag{3.2.10}$$

Further, $x_{n+1} = (1-\rho)z_n + \rho u_n$, hence $\frac{1}{\rho}(x_{n+1} - z_n) = u_n - z_n$. Substituting this into (3.2.10), we deduce

$$\|x_{n+1} - u\|^2 \leq (1-\rho)\|z_n - u\|^2 + \rho\|u_n - u\|^2 - \frac{1}{\rho}(1-\rho)\|x_{n+1} - z_n\|^2. \tag{3.2.11}$$

Substituting (3.2.8) and (3.2.9) into (3.2.11), produces

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\rho)\|z_n-u\|^2+\rho\|w_n-u\|^2-\rho\frac{(1-\mu)^2}{(1+\mu)^2}\|w_n-y_n\|^2\\ &\quad -2\varrho\lambda_n\frac{(1-\mu)}{(1+\mu)^2}\|y_n-u\|^2-\frac{1}{\rho}(1-\rho)\|x_{n+1}-z_n\|^2. \qquad (3.2.12)\end{aligned}$$

Recalling that $\tau := \min\left\{\frac{(1-\mu)^2}{2(1+\mu)^2}, \varrho\frac{\mu(1-\mu)}{L(1+\mu)^2}\right\}$, we find

$$\lim_{n\to\infty}\frac{\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}=\frac{(1-\mu)^2}{(1+\mu)^2}\geq 2\tau.$$

Using Lemma 3.1.3 and assuming that $\lambda_1 \geq \frac{\mu}{L}$, we have $\lim_{n\to\infty}\lambda_n=\lambda\geq\min\left\{\lambda_1,\frac{\mu}{L}\right\}=\frac{\mu}{L}$. Thus

$$\lim_{n\to\infty}\varrho\lambda_n\frac{\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}=\varrho\frac{\mu}{L}\frac{(1-\mu)}{(1+\mu)^2}\geq\tau.$$

Using inequality (3.2.12), to conclude that

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\rho)\|z_n-u\|^2+\rho\|w_n-u\|^2-2\rho\tau\|w_n-y_n\|^2-2\rho\tau\|y_n-u\|^2\\ &\quad -\frac{1}{\rho}(1-\rho)\|x_{n+1}-z_n\|^2\\ &\leq (1-\rho)\|z_n-u\|^2+\rho(1-\tau)\|w_n-u\|^2\\ &\quad -\frac{1}{\rho}(1-\rho)\|x_{n+1}-z_n\|^2. \qquad (3.2.13)\end{aligned}$$

Given the definition of z_n,

$$\begin{aligned}\|z_n-u\|^2 &= \|x_n+\delta(x_n-x_{n-1})-u\|^2\\ &= \|(1+\delta)(x_n-u)-\delta(x_{n-1}-u)\|^2\\ &= (1+\delta)\|x_n-u\|^2-\delta\|x_{n-1}-u\|^2+\delta(1+\delta)\|x_n-x_{n-1}\|^2. \qquad (3.2.14)\end{aligned}$$

Given the definition of w_n,

$$\begin{aligned}\|w_n-u\|^2 &= \|x_n+\theta(x_n-x_{n-1})-u\|^2\\ &= \|(1+\theta)(x_n-u)-\theta(x_{n-1}-u)\|^2\\ &= (1+\theta)\|x_n-u\|^2-\theta\|x_{n-1}-u\|^2+\theta(1+\theta)\|x_n-x_{n-1}\|^2. \qquad (3.2.15)\end{aligned}$$

Alternatively, we possess

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &= \|x_{n+1}-x_n-\delta(x_n-x_{n-1})\|^2\\ &= \|x_{n+1}-x_n\|^2+\delta^2\|x_n-x_{n-1}\|^2-2\delta\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle\\ &\geq \|x_{n+1}-x_n\|^2+\delta^2\|x_n-x_{n-1}\|^2-2\delta\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|\\ &\geq (1-\delta)\|x_{n+1}-x_n\|^2+(\delta^2-\delta)\|x_n-x_{n-1}\|^2. \qquad (3.2.16)\end{aligned}$$

Substituting (3.2.14), (3.2.15) and (3.2.16) into (3.2.13), we obtain

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\rho)\left[(1+\delta)\|x_n-u\|^2-\delta\|x_{n-1}-u\|^2+\delta(1+\delta)\|x_n-x_{n-1}\|^2\right]\\ &\quad+\rho(1-\tau)\left[(1+\theta)\|x_n-u\|^2-\theta\|x_{n-1}-u\|^2+\theta(1+\theta)\|x_n-x_{n-1}\|^2\right]\\ &\quad-\frac{1-\rho}{\rho}\left[(1-\delta)\|x_{n+1}-x_n\|^2+(\delta^2-\delta)\|x_n-x_{n-1}\|^2\right]\\ &= \left[(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)\right]\|x_n-u\|^2-\left[(1-\rho)\delta+\rho(1-\tau)\theta\right]\|x_{n-1}-u\|^2\\ &\quad+\left[(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)-\frac{1-\rho}{\rho}(\delta^2-\delta)\right]\|x_n-x_{n-1}\|^2\\ &\quad-\frac{1-\rho}{\rho}(1-\delta)\|x_{n+1}-x_n\|^2.\end{aligned}$$

From which it follows that

$$\begin{aligned}&\|x_{n+1}-u\|^2+\frac{1-\rho}{\rho}(1-\delta)\|x_{n+1}-x_n\|^2\leq\left[(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)\right]\|x_n-u\|^2\\ &+\left[(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)\right]\|x_n-x_{n-1}\|^2\\ &=\mathcal{M}\left(\|x_n-u\|^2+\frac{(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)}{(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)}\|x_n-x_{n-1}\|^2\right),\end{aligned} \tag{3.2.17}$$

where $\mathcal{M}:=\left((1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)\right)$. From Assumptions 3.2.1 we have $\delta\leq\frac{1-2\rho}{1-\rho}$, and hence

$$\frac{1-\rho}{\rho}(1-\delta)\geq 1.$$

Next, we show that

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)<1.$$

Indeed, from $\theta\leq\delta$ we have

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)\leq(1-\rho)(1+\delta)+\rho(1-\tau)(1+\delta). \tag{3.2.18}$$

On the other hand, from $\delta\leq\frac{\rho\tau}{1-\rho\tau}$ we find that

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\delta)=1+\delta-\rho\tau(1+\delta)\leq 1. \tag{3.2.19}$$

Combining (3.2.18) and (3.2.19) we obtain

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)<1.$$

Next, we prove that

$$\frac{(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)}{(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)}<1. \tag{3.2.20}$$

The last inequality is equivalent to

$$(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)<(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta) \tag{3.2.21}$$

That is

$$\frac{1-\rho}{\rho}(\delta-\delta^2) < (1-\delta)(1-\rho)(1+\delta) + \rho(1-\tau)(1-\theta)(1+\theta).$$

This inequality is equivalently to

$$\delta(1-\rho)(1-\delta) < \rho(1-\delta)(1-\rho)(1+\delta) + \rho^2(1-\tau)(1-\theta)(1+\theta). \tag{3.2.22}$$

From $\delta \leq \frac{\rho}{1-\rho}$ we obtain

$$\delta(1-\rho)(1-\delta) < \rho(1-\delta)(1-\rho)(1+\delta). \tag{3.2.23}$$

Combining (3.2.21),(3.2.22) and (3.2.23) we deduce that inequality (3.2.20) holds. Now, we establish the convergence of the sequence x_n to u with an R-linear rate. Indeed, from (3.2.17), we deduce:

$$\begin{aligned}
&\|x_{n+1}-u\|^2 + \|x_{n+1}-x_n\|^2 \leq \|x_{n+1}-u\|^2 + \frac{1-\rho}{\rho}(1-\delta)\|x_{n+1}-x_n\|^2 \\
&\leq \mathcal{M}\left(\|x_n-u\|^2 + \frac{(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)}{(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)}\|x_n-x_{n-1}\|^2\right) \\
&\leq \mathcal{M}\left(\|x_n-u\|^2 + \|x_n-x_{n+1}\|^2\right).
\end{aligned} \tag{3.2.24}$$

Letting $v_n := \|x_n-u\|^2 + \|x_n-x_{n-1}\|^2$ and $\omega := (1-\rho(1-(1-\tau)(1+\delta)))$, from we find

$$\|x_{n+1}-u\|^2 \leq v_{n+1} \leq \omega v_n \leq \omega^{n-N+1} v_N = \frac{\omega}{\omega^N} v^N \omega^n.$$

Hence, the sequence $\{x_n\}$ converges R-linearly to u, as intended. □

3.3 Numerical Illustrations

In this section, we compare the performance of different algorithms to tackle some classical benchmark problems that have been considered by some authors (see, e.g., [67, 97, 94]). We specifically evaluate Algorithm 3.1 of Dong et al. [77], Algorithm 1 of Thong et al. [97], Algorithm 3.2 of Shehu et al. [94], and our proposed Algorithm 3.1.2.

Example 3.3.1. *Define* $A : \mathbb{R}^N \to \mathbb{R}^N$ *by*

$$Ax = \left(e^{-x^TQx} + \alpha\right)(Px + r),$$

where a positive definite matrix is Q*, a positive semi-definite matrix is* P*,* $r \in \mathbb{R}^N$ *and* $\alpha > 0$*. Observe that* A *is Lipschitz continuous and pseudo-monotone but not monotone (see, e.g., [67, Example 2.1]). In MATLAB, we generate* P*,* Q *and* r *randomly and took* $\alpha = 5$*. In addition, we set* $C := \{x \in \mathbb{R}^N : \|x\| \leq 10\}$*. In DCZR, we take* $\alpha_n = \frac{1}{n+1}$*,* $\tau = 10^{-7}$*,* $\gamma = 0.0021$*. In TVC, we take* $\lambda = 10^{-8}$*,* $\beta_n = \frac{1}{10^4 n+1}$*,* $\epsilon_n = \frac{1}{(n+1)^2}$*,* $\alpha = 0.9$*. In SLMD, we take* $\gamma = 0.2$*,* $\theta = 0.9375$*,* $\alpha = 0.0067$*,* $\mu = 10^{-6}$ *and* $\lambda_1 = 0.1$*. and in our proposed Algorithm 3.1.2, we take* $\mu = 10^{-6}$*,* $\lambda_1 = 0.1$*,* $\beta = 0.2$*,* $\theta = 0.9375$*,* $\rho =$

0.0067, $\delta = -0.02$ *and* $\alpha_n = \frac{1}{(n+1)^4}$. *The initial points* x_0, x_1, x_2 *are generated randomly in* $\mathbb{R}^N$. *To test the robustness of the algorithms we consider six (6) different dimensions, i.e.,* $\mathbb{R}^{50}$, $\mathbb{R}^{100}$, $\mathbb{R}^{500}$, $\mathbb{R}^{1000}$, $\mathbb{R}^{3000}$ *and* $\mathbb{R}^{5000}$. *For each case, the simulation is terminated when the number of iterations (Iter for short)* $n = 3001$ *or* $\|x_{n+1} - x_n\| > 10^{-6}$. *The result of the numerical simulations are presented in Figures 3.1, 3.2 and 3.3 below.*

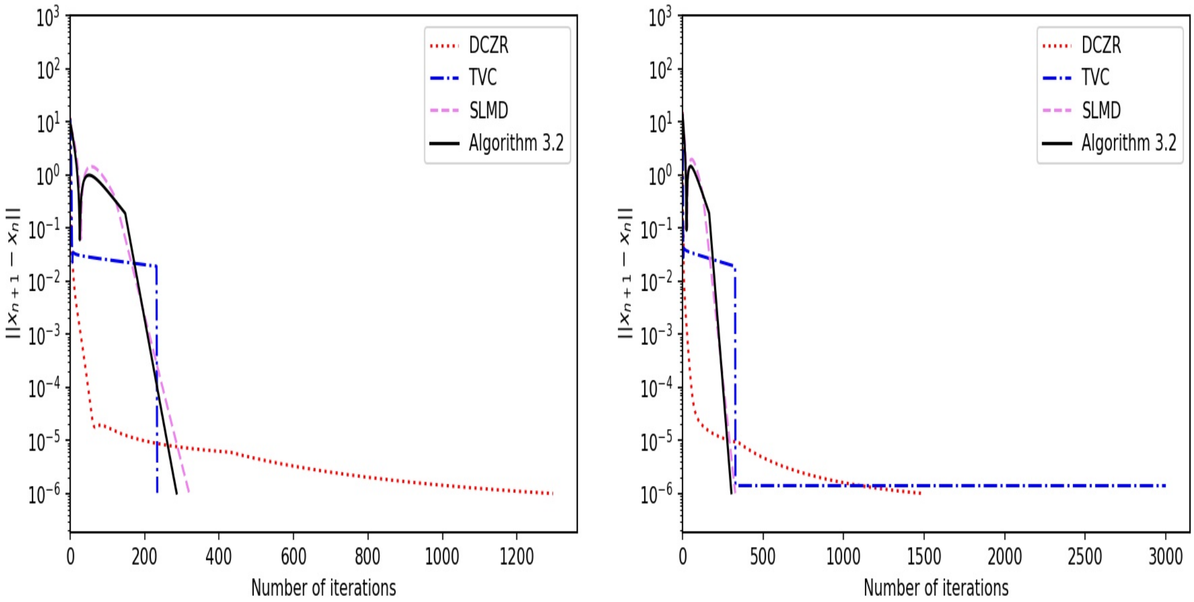

Figure 3.1: Graph the Iterates for Dimensions $N = 50$ and 100

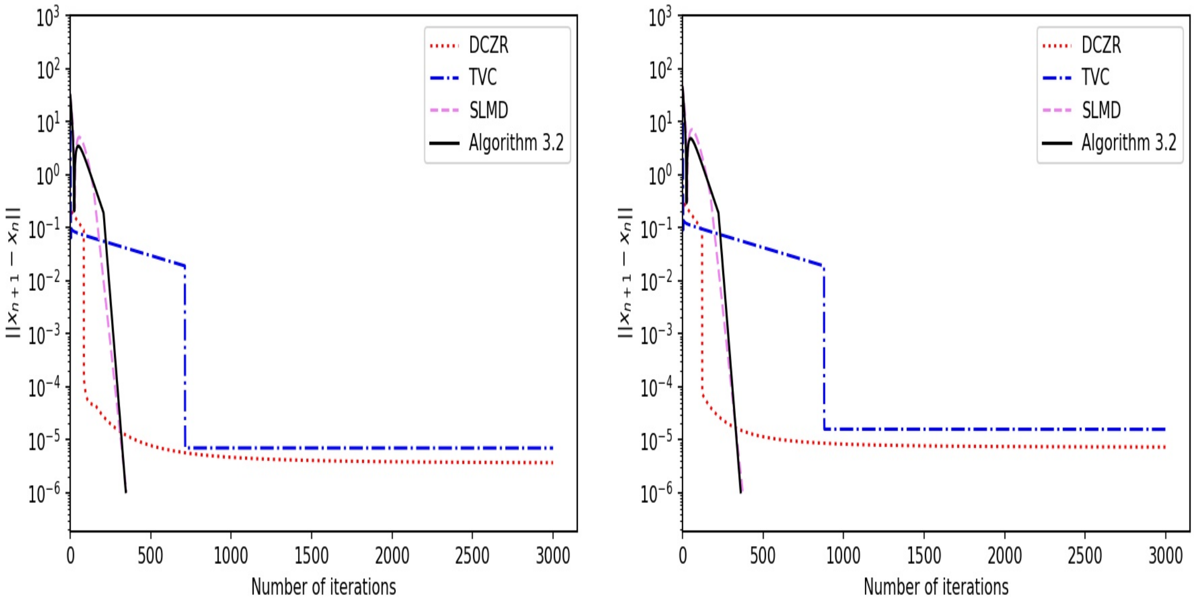

Figure 3.2: Graph the Iterates for Dimensions $N = 500$ and 1000

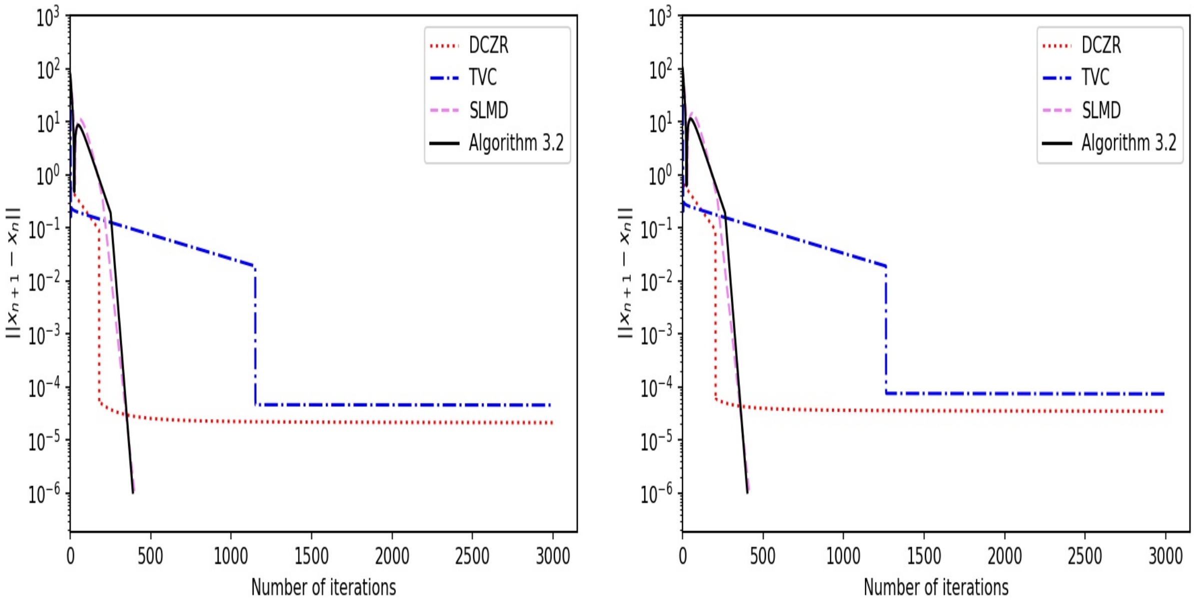

Figure 3.3: Graph the Iterates for Dimensions $N = 3000$ and 5000

Example 3.3.2. *Consider the function $F : \mathbb{R}^N \to \mathbb{R}^N$ defined by $Fx = Ax + b$, where $b \in \mathbb{R}^N$,*

$$A = MM^T + S + D,$$

M is an $N \times N$ matrix, S and D are $N \times N$ skew-symmetric matrix and diagonal matrix, respectively. Then, in A is monotone and Lipschiptz continuous. In MATLAB, we gener-

ated M, S and D randomly with entries in $[1, 100]$. We generated b randomly with entries in $[-100, 0]$. Furthermore, we take

$$C = \{x \in \mathbb{R}^N : -1 \leq x_k \leq 1\}, \quad k = 1, 2, \ldots, N.$$

In DCZR, we take $\alpha_n = \frac{1}{n+1}$, $\tau = 10^{-7}$, $\gamma = 10^{-3}$. In TVC, we take $\lambda = 10^{-8}$, $\beta_n = \frac{1}{10^4 n+1}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\alpha = 0.9$. In SLMD, we take $\gamma = 0.2$, $\theta = 0.5$, $\alpha = 0.25$, $\mu = 10^{-6}$ and $\lambda_1 = 0.1$. and in our proposed Algorithm 3.1.2, we take $\mu = 10^{-6}$, $\lambda_1 = 0.1$, $\beta = 0.2$, $\theta = 0.5$, $\rho = 0.25$, and $\alpha_n = \frac{1}{(n+1)^4}$. The initial points x_0, x_1, x_2 are generated randomly in $\mathbb{R}^N$. To test the robustness of the algorithms we consider six (6) different dimensions, i.e., $\mathbb{R}^{50}$, $\mathbb{R}^{100}$, $\mathbb{R}^{500}$, $\mathbb{R}^{1000}$, $\mathbb{R}^{3000}$ and $\mathbb{R}^{5000}$. For each case, the simulation is terminated when the number of iterations $n = 3001$ or $\|x_{n+1} - x_n\| > 10^{-6}$. The result of the numerical simulations are presented in Figures 3.4, 3.5 and 3.6 below.

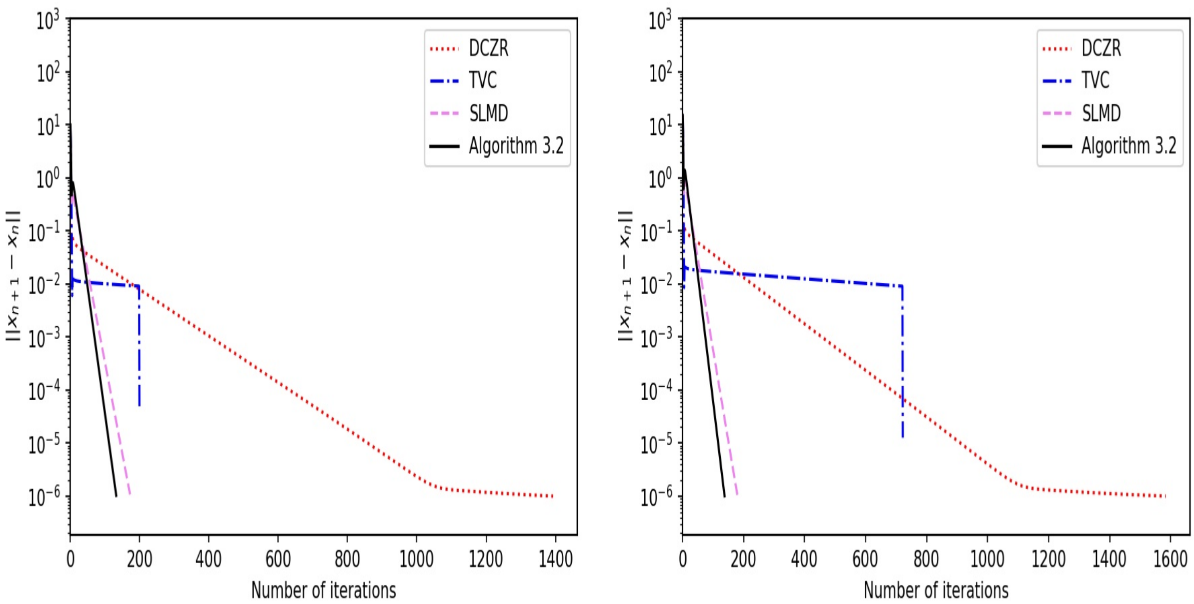

Figure 3.4: Graph the Iterates for Dimensions $N = 50$ and 100

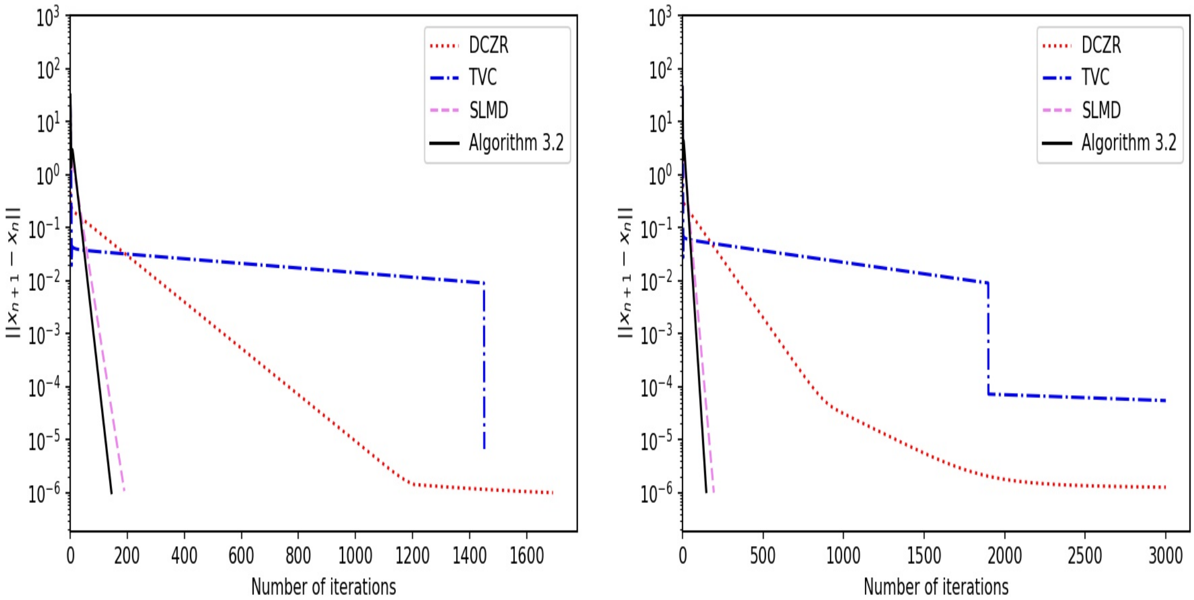

Figure 3.5: Graph the Iterates for Dimensions $N = 500$ and 1000

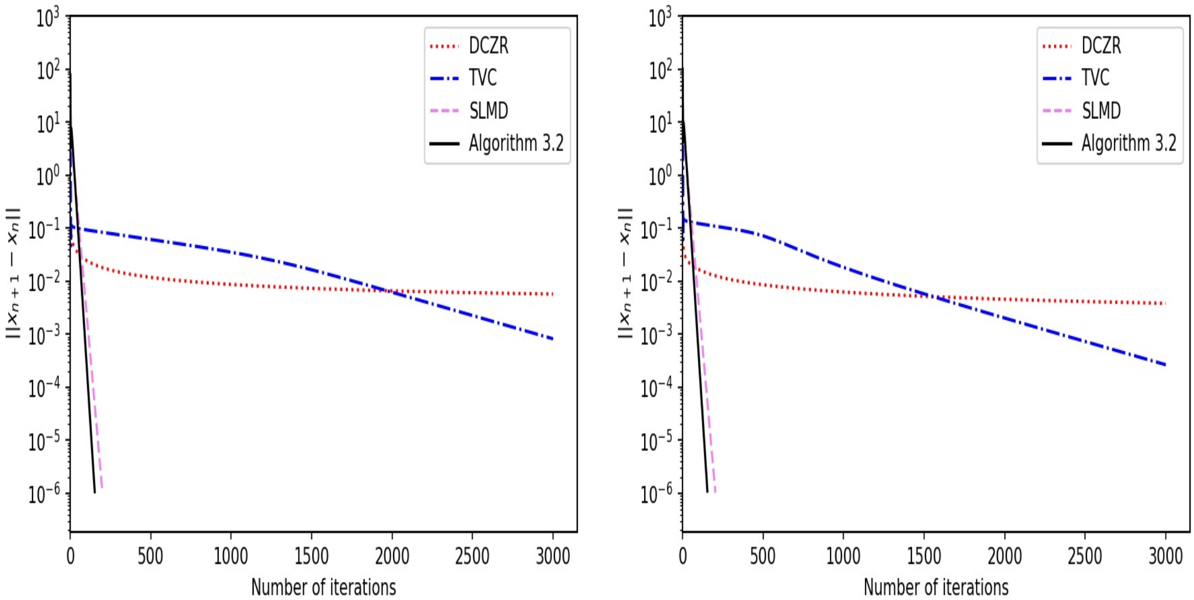

Figure 3.6: Graph the Iterates for Dimensions $N = 3000$ and 5000

Chapter 4

Conclusion, Contribution to Knowledge and Future Research

Within the chapter, we bring our dissertation to a close, summarizing our study and emphasizing its contributions to the current body of knowledge. Additionally, we pinpoint and delve into potential avenues for future research.

4.1 Conclusion

This dissertation provides a thorough analysis of a few fixed point and nonlinear optimization issues within the context of Hilbert spaces. We presented our research logically and divided it into 4 chapters. In Chapter 1, we provided a brief overview of the context of our research and highlighted the research difficulties and motivations that were examined for this dissertation, particularly those that were taken into account in Chapters 3. Then, we emphasized the goals and structure of our research. We defined several fundamental terms and expressions that are crucial to our investigation in Chapter 2. Finally, we recalled many significant findings from our study and offered a thorough evaluation of earlier works that are pertinent to our research. In addition, Chapters 3 of this dissertation were devoted to the analysis of a few fixed point and nonlinear optimization issues within the Hilbert framework. The main conclusions of this investigation are presented in these chapters. Each of these chapters also includes comparisons to other discoveries in the literature, multiple numerical experiments, and applications of our key findings. The primary findings of this dissertation can be summed up in these chapters as new, crucial generalizations and insights into our contribution to the study of fixed point and its applications.

4.2 Contribution to Knowledge

In general, we have introduced a modified iterative methods which generalize, extend, improve and unify existing results in the literature. The outcomes of this dissertation may inspire other researchers in this field to further extend, generalize, and unify various discoveries in the existing literature.. Among other things, we made the following contributions:

The results obtained in Chapter 3, extends, improves and unifies the works of Censor *et al.* in [20, 19], Tian and Jiang in [53], Pham *et al.* in [46] and other works currently present in the literature.

The outcomes acquired in Chapter 4, extends, improves and unifies the works of Tan *et al.* in [54], Mainge in [41] and some other existing works in the literature.

To extend these iterative techniques to multivalued type iterative technique and obtain results similar to those in Chapters 3 and 4.

4.3 Future Research

We plan to investigate the following in our future research:

1. extending the iterative algorithms to Split Inverse Problems (SIPs).
2. extending the iterative algorithms to a more general space like Banach spaces, hyperbolic spaces, and CAT(0) spaces and obtain corresponding results obtained in these chapters;
3. introduce an implicit type of the following iterative schemes in more general spaces.

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