Symmetries, Conservation Laws and Reductions of Schrödinger Systems of equations

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A thesis submitted to the Faculty of Science, University of the Witwatersrand, in fulfilment of the requirements for the degree of Doctor of Philosophy.

Johannesburg, 2014.

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand in Johannesburg. It has not been submitted before for any degree to any other institution.

Phetogo Masemola

This ____________ day of ____________ 2014.

Acknowledgements

Firstly I thank God for continuing to bless me and give me strength. I would also like to extend my gratitude to my supervisor Prof A.H Kara for guidance and support over the past three years in my post graduate studies. I would like to make a special mention to Ingrid Eitzen for all the help with latex.

I am also grateful to everybody whose work is cited in my thesis as well as the NRF without whose financial support this thesis would not be possible.

Lastly I thank my dearest mum and dad for unfailing support throughout my studies.

Abstract

One of the more recently established methods of analysis of differentials involves the invariance properties of the equations and the relationship of this with the underlying conservation laws which may be physical. In a variational system, conservation laws are constructed using a well known formula via Noether's theorem. This has been extended to non variational systems too. This association between symmetries and conservation laws has initiated the double reduction of differential equations, both ordinary and, more recently, partial. We apply these techniques to a number of well known equations like the damped driven Schrödinger equation and a transformed PT symmetric equation(with Schrödinger like properties), that arise in a number of physical phenomena with a special emphasis on Schrödinger type equations and equations that arise in Optics.

Contents

Introduction

In the 19th century, symmetry was one of the main doctrines. Symmetries cover many mathematical models, especially those formulated in terms of differential equations. The discipline which encompasses symmetries of differential equations is the Lie group theory.

Lie symmetry analysis of differential equations was pioneered by the prominent Norwegian mathematician Marius Sophus Lie (1842-1899). One of his greatest achievements was the discovery that continous transformation groups are better understood by making the respective differential equation linear. With his theory of transformation groups he showed that most of the intergration methods could be used altogether using group theory. He continued to make great strides in his field publishing his last work in 1896, 'Geometrie der Berührungstransformationen' [66]. His work impacted many mathematicians including Hermann Weyl and Emmy Noether. The latter worked with Felix Klein, who was a close friend of Sophus Lie.

In the paper, Invariante Varlations probleme [67], Emmy Noether proved two theorems (later named after her). These theorems cemented the vital relationship between symmetries and conservation laws in physics. Noether's theorem allows for the construction of conservation laws for Euler-Lagrange equations, provided the

Noether symmetries exist. There are also methods which provide conservation laws independent of a Lagrangian, however such methods can be quite cumbersome and can be accelerated by the use of computer software.

Having the basis of Lie group analysis and Noether's theorem, we study the theory of double reduction. When applying a Lie point symmetry generator to a conserved vector, either a conservation law is associated with that symmetry or the conservation law may be trivial. However, if the conservation law and the symmetry are associated, a double reduction is possible for partial differential equations with two independent variables. The use of one symmetry associated with a conservation law results in two reductions. Firstly, a reduction in the number of independent variables and the second being in the order of the differential equation using Lie point symmetries. This method is applied to various classes of the nonlinear Schrödinger equation. In her papers, Sjöberg says that generalizing the double reduction theory to partial differential equations of higher dimensions is an open problem. Much work is still needed to generalize the theory of double reduction to more than two independent variables although it has bee touched on in [4]. In this thesis however, we provide some double reductions that have previously not been performed.

Brief outline of the chapters:
In the first chapter we introduce definitions and theorems needed to perform the respective calculations. In particular, the theorem of double reduction. This technique is first applied to the nonlinear Schrödinger equation with damping and driving terms [10, 11] and exact solutions are given in chapter two. In chapter three, we show how exact solutions are calculated for the density-dependent Fitzhugh-Nagumo equation and a specific case of the Fisher equation [32, 33, 34, 35, 36, 37] via the method of double reduction.In chapter four we will address the Nonlinear Schrödinger equation using Kerr and Power law nonlinearity [42, 7]. In order to find

solutions using soliton theory, we need the conserved vectors using the multipliers and consequaently the conserved quantities are calculated. In the fifth chapter we investigate the compressional dispersive Alfven (CDA) [52, 53] waves. We see if we can obtain symmetries and conservation laws and check if they are perhaps Noether symmetries which will in turn allow us to use Noether's Theorem. Furthermore, one case of double reduction is mentioned via the association of a conserved vector with a Noether symmetry (with zero gauge). Finally in the last chapter, we study the conservation laws and lie symmetries of a PT symmetric coupler with gain in one waveguide and loss in another. A transformation in the PT system [59, 60] results in a scalar cubic Schrödinger equation. We investigate a Lagrangian, corresponding Noether symmetries, conserved vectors and exact solutions.

The results of our investigations appeared in a number of articles, viz [43, 63, 64, 65]

Chapter 1

Groundwork

1.1 Introduction

We introduce some basic principles necessary for the subsequent chapters.

1.2 Definitions and Theorems

Let $\mathcal{B} = \bigcup_{r=0}^{p} \Omega_0^r(\mathcal{U})$ for some $p < \infty$. Then $\mathcal{B}$ is the universal space of differential functions of finite orders.

Definition: A function $f(x, \phi, \phi_{(1)}, \dots, \phi_{(k)})$ of a finite number of variables is called a differential function of order k.

Definition: $\phi_{(1)}, \phi_{(2)}, \dots, \phi_{(k)}$ denotes the collections of all first, second,..., k^{th}

order partial derivatives, that is, $\phi^\alpha_l = D_l(\phi^\alpha), \phi^\alpha_{lc} = D_c D_l(\phi^\alpha)$,... respectively, with the total differentiation operator with respect to x^l given by

$$D_l = \frac{\partial}{\partial x^l} + \phi^\alpha_l \frac{\partial}{\partial \phi^\alpha} + \phi^\alpha_{lc} \frac{\partial}{\partial \phi^\alpha_c} + \cdots, \ l = 1, \ldots, m \ . \tag{1.1}$$

Sometimes (1.1) is written as a summation.

If we consider an array of partial differential equations of order k having m independent variables $x = (x^1, x^2, \ldots, x^m)$ and n dependent variables $\phi = (\phi^1, \phi^2, \ldots, \phi^n)$ then,

$$Q^\mu(x, \phi, \phi_{(1)}, \ldots, \phi_{(k)}) = 0, \quad \phi = 1, \ldots, \tilde{n}. \tag{1.2}$$

The Lie operator is written as

$$T = \theta^l \frac{\partial}{\partial x^l} + \omega^\alpha \frac{\partial}{\partial \phi^\alpha} \theta^l, \ \omega^\alpha \in \mathcal{B}, \tag{1.3}$$

The function (1.3) is a condensed version of the infinite formal sum given by

$$T = \theta^l \frac{\partial}{\partial x^l} + \omega^\alpha \frac{\partial}{\partial \phi^\alpha} + \sum_{r \geq 1} \pi^\alpha_{l_1 l_2 \ldots l_r} \frac{\partial}{\partial \phi^\alpha_{l_1 l_2 \ldots l_r}}. \tag{1.4}$$

The coefficients θ and ω are prolonged using a formula given below to arrive at π.

$$\begin{aligned} \pi^\alpha_l &= D_l(X^\alpha) + \theta^c \phi^\alpha_{lc} \\ \pi^\alpha_{l_1 \ldots l_r} &= D_{l_1} \ldots D_{l_r}(X^\alpha) + \theta^c \phi^\alpha_{c l_1 \ldots l_r}, \ r > 1. \end{aligned} \tag{1.5}$$

Where X^α is a Lie characteristic function given by:

$$X^\alpha = \omega^\alpha - \theta^c \phi^\alpha_c. \tag{1.6}$$

A Lie operator T is a Noether symmetry with Lagrangian $P \in \mathcal{B}$, if there exists a vector $A^l = (A^1, \ldots, A^m)$, $A^l \in \mathcal{B}$, such that

$$T(P) + P D_l(\theta^l) = D_l(A^l). \tag{1.7}$$

If $A^l = 0, (l = 1, \ldots, m)$, then T is said to be a strict Noether symmetry with Lagrangian $P \in \mathcal{B}$. This case is also obtained by setting the Lie derivative on the m-form $Pdx^1 \wedge \ldots \wedge dx^n$ in the direction of T to zero, i.e.,

$$\mathcal{L}_T P dx^1 \wedge \ldots \wedge dx^m = T(P dx^1 \wedge \ldots \wedge dx^m) = 0, \tag{1.8}$$

where $\mathcal{L}$ is the Lie derivative operator.

In view of the above discussions and definitions, the Noether theorem (Noether, 1918) is formulated as follows [67].

Noether's Theorem. For any Noether symmetry T corresponding to a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $S^l = (S^1, \ldots, S^m), \quad S^l \in \mathcal{A}$, defined by

$$S^l = B^l - N^l(L) , \qquad l = 1, \ldots, m , \tag{1.9}$$

which is a conserved current of the Euler-Lagrange equations $\dfrac{\delta L}{\delta \phi^\alpha} = 0$, $\alpha = 1, \ldots, n$, where $\dfrac{\delta}{\delta \phi^\alpha}$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta \phi^\alpha} = \frac{\partial}{\partial \phi^\alpha} + \sum_{r \geq 1} (-1)^r D_{l_1} \cdots D_{l_r} \frac{\partial}{\partial \phi^\alpha_{l_1 \cdots l_r}} , \quad \alpha = 1, \ldots, n. \tag{1.10}$$

and the Noether operator associated with a Lie operator T is given by

$$N^l = \theta^l + X^\alpha \frac{\delta}{\delta \phi^\alpha_l} + \sum_{r \geq 1} D_{l_1} \cdots D_{l_r}(X^\alpha) \frac{\delta}{\delta \phi^\alpha_{l l_1 \cdots l_r}} , \quad l = 1, \ldots, m, \tag{1.11}$$

where the Euler-Lagrange operators with respect to derivatives of ϕ^α are determined from (1.10) by substituting ϕ^α with the respective derivatives, e.g.,

$$\frac{\delta}{\delta \phi^\alpha_l} = \frac{\partial}{\partial \phi^\alpha_l} + \sum_{r \geq 1} (-1)^r D_{c_1} \cdots D_{c_r} \frac{\partial}{\partial \phi^\alpha_{l c_1 \cdots c_r}} \quad l = 1, \ldots, m, \quad \alpha = 1, \ldots, n . \tag{1.12}$$

An important part of partial differential equations is conservation laws. Further work on this can be found in [6].

For a current $S = (S^1, \ldots, S^m)$ to be conserved it must satisfy the following condition

$$D_l S^l = 0 \tag{1.13}$$

along the solutions of (1.2).

It has been proved that that every admitted conserved flow is derived from multipliers $G_\mu(x, \phi, \phi_{(1)}, \ldots)$ such that

$$G_\mu Q^\mu = D_l S^l. \tag{1.14}$$

This is true on the whole solution space.

If our system of equations is variational, then the corresponding multipliers are also variational. In order to find the multipliers and consequently the conservation laws, we solve for G from the determining system in (1.14). The method employed is the multiplier approach.

We determine the conservation laws by the homotopy operator (see [7, 21]).

Definition 1 [22] A Lie symmetry T defined in (1.3) is said to be associated with a conservation law S of the system (1.2) provided that T and S satisfy the relation below

$$T(S^l) + S^l D_k(\theta^k) - S^k D_k(\theta^l) = 0,\ l = 1, \ldots, m. \tag{1.15}$$

Theorem 1.4 [4] Suppose that $D_l S^l = 0$ is a conservation law of the partial differential equation system (1.2). Then under a contact transformation, there exists functions $\tilde{S}^l$ such that $E\ D_l S^l = \tilde{D}_l \tilde{S}^l$, where $\tilde{S}^l$ is given as

$$\begin{pmatrix} \tilde{S}^1 \\ \tilde{S}^2 \\ \vdots \\ \tilde{S}^m \end{pmatrix} = E(U^{-1})^T \begin{pmatrix} S^1 \\ S^2 \\ \vdots \\ S^m \end{pmatrix}, \quad E \begin{pmatrix} S^1 \\ S^2 \\ \vdots \\ S^m \end{pmatrix} = U^S \begin{pmatrix} \tilde{S}^1 \\ \tilde{S}^2 \\ \vdots \\ \tilde{S}^m \end{pmatrix} \tag{1.16}$$

where

$$U = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_m \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_m \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_m x_1 & \tilde{D}_m x_2 & \cdots & \tilde{D}_m x_m \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} D_1 \tilde{x_1} & D_1 \tilde{x_2} & \cdots & D_1 \tilde{x_m} \\ D_2 \tilde{x_1} & D_2 \tilde{x_2} & \cdots & D_2 \tilde{x_m} \\ \vdots & \vdots & \vdots & \vdots \\ D_m \tilde{x_1} & D_m \tilde{x_2} & \cdots & D_m \tilde{x_m} \end{pmatrix} \tag{1.17}$$

and $E = \det(U)$.

Theorem 1.5 [4] **(fundamental theorem on double reduction)** Suppose that $D_l S^l = 0$ is a conserved vector of the partial differential system (1.2). Then under a similarity transformation of a symmetry T defined as in (1.4) for the partial differential system, there exist functions $\tilde{S}^l$ such that T is still a symmetry for the partial differential equation (pde) $\tilde{D}_l\tilde{S}^l = 0$ and

$$\begin{pmatrix} T\tilde{S}^1 \\ T\tilde{S}^2 \\ \vdots \\ T\tilde{S}^m \end{pmatrix} = E(U^{-1})^T \begin{pmatrix} [S^1, T] \\ [S^2, T] \\ \vdots \\ [S^m, T] \end{pmatrix}, \tag{1.18}$$

in which

$$U = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_m \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_m \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_m x_1 & \tilde{D}_m x_2 & \cdots & \tilde{D}_m x_m \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} D_1 \tilde{x_1} & D_1 \tilde{x_2} & \cdots & D_1 \tilde{x_m} \\ D_2 \tilde{x_1} & D_2 \tilde{x_2} & \cdots & D_2 \tilde{x_m} \\ \vdots & \vdots & \vdots & \vdots \\ D_m \tilde{x_1} & D_m \tilde{x_2} & \cdots & D_m \tilde{x_m} \end{pmatrix} \tag{1.19}$$

and $E = \det(U)$.

Chapter 2

Reducing the nonlinear Schrödinger equation with damping and driving parameters

2.1 Introduction and Background

The Nonlinear Schrödinger Equation (NLSE) has been studied for decades in the world of physics and mathematics and has been used in the modelling of many phenomena. In this chapter we are interested in the NLSE with 'damping' and 'driving' terms. These are also investigated in papers [10, 11] to name a few.

To quote from [10], 'the application of a resonant driving force is an efficient way of compensating dissipative losses in a soliton bearing system. If the dissipation coefficient and driving strength are weak, and the driving frequency is just below the phonon band, the amplitude of the arising oscillating soliton is governed by the nonlinear Schrödinger equation with damping and driving terms. The damped-driven nonlinear Schrödinger equations exhibit localised solutions with a variety of temporal behaviours, from stationary to periodic and chaotic. There is a whole range of analytical and numerical approaches to the study of stationary and steadily travelling solitary waves'.

In this chapter we consider the double reduction of the damped-driven NLSE and try to see if we can calculate some exact solutions, viz.,

$$iq_t + q_{xx} + 2|q|^2 q - q = h\bar{q} - i\gamma q, \tag{2.1}$$

where q is the complex valued dependent variable. In (2.1), $\gamma > 0$ is the damping coefficient and h the amplitude of the parametric driver, which can also be assumed positive. Equation (2.1) was used to model a variety of resonant phenomena in nonlinear dispersive media, including the nonlinear Faraday resonance in a vertically oscillating water trough [12, 13], formation of oscillons in granular materials and suspensions [14], synchronization in parametrically excited pendula arrays [15], phasesensitive amplification of light pulses in optical fibers [16] and propagation of magnetization waves in an easy-plane ferromagnet placed in a microwave field [17, 18]. The same equation governs the amplitude of breathers in a variety of systems reducible to the parametrically driven damped sine-Gordon and the ϕ^4 equation.

When we split the equation (2.1) into the respective real and imaginary components, we have the following system

$$\begin{aligned} \phi_t + \psi_{xx} + 2\psi(\phi^2 + \psi^2) - \psi + h\psi + \gamma\phi = 0,\\ -\psi_t + \phi_{xx} + 2\phi(\phi^2 + \psi^2) - \phi + h\phi - \gamma\psi = 0. \end{aligned} \tag{2.2}$$

Therefore, in the sequel, we will consider the system of PDEs (2.2) as all the analysis and corresponding results of the system of equations (2.2) would, equivalently, impact on the class of equations (2.1). The results of this chapter appear in [43]

2.2 Symmetries, reductions and conservation laws of the equation (2.2)

The Lie symmetry approach on differential equations is well known; for details see e.g., [19, 40, 68]. In this section, we list a summary of these and explore the notion of a 'double reduction' in order to obtain symmetry invariant (exact) solutions.

2.2.1 Symmetries and reductions

A one parameter Lie group of transformations that leave invariant the system (2.2) will be written as a vector field

$$T = \tau(t,x,\phi,\psi)\partial_t + \theta(t,x,\phi,\psi)\partial_x + \omega^1(t,x,\phi,\psi)\partial_\phi + \omega^2(t,x,\phi,\psi)\partial_\psi. \tag{2.3}$$

This would be a generator of point symmetry of the system. The calculations(using the software Mathematica and Maple) reveal the following classes and symmetries.

(a) $\gamma \neq 0, \quad h \neq 0$

$$\begin{aligned} &T_1 = \phi\partial_\psi - \psi\partial_\phi, \qquad T_2 = (-1+h)\phi\partial_\psi + \partial_t + (\psi - h\psi)\partial_\phi, \\ &T_3 = \partial_x, \qquad T_4 = 2t\partial_x + \phi x\partial_\psi - \psi x\partial_\phi \end{aligned}$$

(b) $\gamma = 0, \quad h \neq 0$

$$\begin{aligned}
&Y_1 = \phi\partial_\psi - \psi\partial_\phi, \qquad Y_2 = -\partial_t + \phi\partial_\psi - \psi\partial_\phi, \\
&Y_3 = \partial_x, \qquad Y_4 = 2t\partial_x + \phi x\partial_\psi - \psi x\partial_\phi, \\
&Y_5 = 2t\partial_t + (-2t\phi - \psi)\partial_\psi + (-\phi + 2t\psi)\partial_\phi + x\partial_x
\end{aligned}$$

(c) $\gamma \neq 0, \quad h = 0$

$$\begin{aligned}
&Z_1 = \partial_t - \phi\partial_\psi + \psi\partial_\phi, \qquad Z_2 = \phi\partial_\psi - \psi\partial_\phi, \\
&Z_3 = \partial_x, \qquad Z_4 = 2t\partial_x + \phi x\partial_\psi - \psi x\partial_\phi
\end{aligned}$$

2.2.2 Conservation Laws

For us to calculate the conserved vector we use the multiplier approach (see [7]). Therefore, if (S^t, S^x) is a conserved vector with a conservation law, then

$$D_t S^t + D_x S^x = 0$$

along the solutions of the differential equation ($P(x, t, \phi) = 0$).

Moreover, if there exists a nontrivial differential function G, called a 'multiplier', such that

$$E_g[G(P(x, t, \phi))] = 0,$$

then $G(P(x, t, \phi))$ is a total divergence, i.e.,

$$G(P(x, t, \phi)) = D_t S^t + D_x S^x,$$

for some (conserved) vector (S^t, S^x) and E_g is the respective Euler-Lagrange operator. Thus, a knowledge of each multiplier G leads to a conserved vector determined by, inter alia, a Homotopy operator. See details and references in [21, 7].

For a system $P^1(x,t,\phi)=0$ & $P^2(x,t,\phi)=0$, $G=(g^1,q^2)$, say, so that

$$g^1(P^1(x,t,\phi))+g^2(P^2(x,t,\phi))=D_tS^t+D_xS^x,$$

and

$$E_{(\phi,\psi)}[D_tS^t+D_xS^x]=0.$$

In each case, S^t is the *conserved density.*

The lengthy calculations for the system (2.2), lead to the following multipliers and corresponding conserved vectors.

(a) $\gamma\neq 0,\quad h\neq 0$

$(g^1,g^2)=(e^{2\gamma t}\psi_x,e^{2\gamma t}\phi_x)$

$$\begin{aligned} S^x &= e^{2\gamma t}(\phi^4+(-1+h)\psi^2+\psi^4+\phi^2(-1+h+2\psi^2)+\psi\phi_t-\phi\psi_t+{\phi_x}^2+{\psi_x}^2),\\ S^t &= e^{2\gamma t}(-\psi\phi_x+\phi\psi_x). \end{aligned}$$

Thus, the conserved density for the scalar equation is

$$\Phi^t=-\frac{i}{2}e^{2\gamma t}(q_x-q_x).$$

where i is an imaginary number

(b) $\gamma=0,\quad h\neq 0$

(i) $(g^1,g^2)=(\psi_t,\phi_t)$

$$\begin{aligned} S^x &= \tfrac{1}{2}(\phi_t\phi_x+\psi_t\psi_x-\phi\phi_{xt}-\psi\psi_{xt}),\\ S^t &= \tfrac{1}{2}(\phi^4+\phi^2(-1+h+2\psi^2)+\phi\phi_{xx}+\psi((-1+h)\psi+\psi^3+\psi_{xx})). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = \frac{1}{2}|q|^4 + \frac{1}{2}(h-1)|q|^2 + q_{xx} + q_{xx}.$$

(ii) $(g^1, g^2) = (\psi_x, \phi_x)$

$$\begin{aligned} S^x &= \tfrac{1}{2}(\phi^4 + (-1+h)\psi^2 + \psi^4 + \phi^2(-1+h+2\psi^2) + \psi\phi_t - \phi\psi_t + {\phi_x}^2 + {\psi_x}^2), \\ S^t &= \tfrac{1}{2}(-\psi\phi_x + \phi\psi_x). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{i}{4}(q_x - q_x).$$

(iii) $(g^1, g^2) = (-\phi, \psi)$

$$\begin{aligned} S^x &= \psi\phi_x - \phi\psi_x, \\ S^t &= \tfrac{1}{2}(-\phi^2 - \psi^2). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{1}{2}|q|^2.$$

(iv) $(g^1, g^2) = (-\frac{1}{2}x\phi + t\psi_x, \frac{1}{2}x\psi + t\phi_x)$

$S^x = \frac{1}{2}(t\phi^4 + (-1+h)t\psi^2 + t\psi^4 + t\phi^2(-1+h+2\psi^2) + \psi(t\phi_t + x\phi_x) - \phi(t\psi_t +$
$x\psi_x) + t({\phi_x}^2 + {\psi_x}^2)),$
$S^t = \frac{1}{4}(-x\phi^2 - \psi(x\psi + 2t\phi_x) + 2t\phi\psi_x).$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{1}{4}x|q|^2 - it(q_x - q_x).$$

(c) $\gamma \neq 0, \quad h = 0$

In addition to the case (a), we have

(i) $(g^1, g^2) = (-x^{2\gamma t} + 2te^{2\gamma t}\psi_x, x\psi e^{2\gamma t} + 2te^{2\gamma t}\phi_x)$

$$\begin{aligned} S^x &= e^{2t\gamma}(t\phi^4 - t\psi^2 + t\psi^4 + t\phi^2(-1 + 2\psi^2) + \psi(t\phi_t + x\phi_x) - \phi(t\psi_t + x\psi_x) + t({\phi_x}^2 + {\psi_x}^2)), \\ S^t &= -\tfrac{1}{2}e^{2t\gamma}(x\phi^2 + \psi(x\psi + 2t\phi_x) - 2t\phi\psi_x). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{1}{2}xe^{2\gamma t}|q|^2 - \frac{i}{2}te^{2\gamma t}(q_x - q_x).$$

(ii) $(g^1, g^2) = (e^{2\gamma t}\phi, -e^{2\gamma t}\psi)$

$$\begin{aligned} S^x &= e^{2t\gamma}(-\psi\phi_x + \phi\psi_x), \\ S^t &= \tfrac{1}{2}e^{2t\gamma}(\phi^2 + \psi^2). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = \frac{1}{2}e^{2\gamma t}|q|^2.$$

Our original system is equivalent to

$$sys_1 = \begin{cases} {g_1}^1 Q^1 + {g_1}^2 Q^2 = 0, \\ {g_1}^1 Q^1 - {g_1}^2 Q^2 = 0. \end{cases} \tag{2.4}$$

This system can be rewritten as

$$\begin{aligned} D_t{S_1}^t + D_x{S_1}^x &= 0, \\ {g_1}^1 Q^1 - {g_1}^2 Q^2 &= 0. \end{aligned} \tag{2.5}$$

Reducing the damped-driven NLSE by $< T_1, T_3 >$

We first show that T_1 and T_3 are associated with $S_1 = (S_1^t, S_1^x)$ using the following version of equation (1.16) for $l = 1, 2$

$$S^* = T\begin{pmatrix} S^t \\ S^x \end{pmatrix} - \begin{pmatrix} D_t\theta^t & D_x\theta^t \\ D_t\theta^x & D_x\theta^x \end{pmatrix}\begin{pmatrix} S^t \\ S^x \end{pmatrix} + (D_t\theta^t + D_x\theta^x)\begin{pmatrix} S^t \\ S^x \end{pmatrix}, \tag{2.6}$$

where S^* is our new conservation law. First we calculate $T_1^{[1]}$ from the prolongation formulae.

$$T_1^{[1]} = -\psi\frac{\partial}{\partial\phi} + \phi\frac{\partial}{\partial\psi} - \psi_t\frac{\partial}{\partial\phi_t} - \psi_x\frac{\partial}{\partial\phi_x} + \phi_t\frac{\partial}{\partial\psi_t} + \phi_x\frac{\partial}{\partial\psi_x}, \quad T_3^{[1]} = \frac{\partial}{\partial x}.$$

We obtain from equation (2.6)

$$\begin{aligned} \begin{pmatrix} S_1^{*t} \\ S_1^{*x} \end{pmatrix} &= T_1^{[1]}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} + (0)\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}e^{2\gamma t}(-\psi\psi_x - \phi\phi_x + \psi\psi_x + \phi\phi_x) \\ \frac{1}{2}e^{2\gamma t}\Big(-4\phi\psi(\phi^2+\psi^2) - 2(h-1)\phi\psi + \psi\psi_t + 4\phi\psi(\phi^2+\psi^2) + 2(h-1)\phi\psi \\ \qquad + \phi\phi_t - \psi\psi_t - 2\phi_x\psi_x - \phi\phi_t + 2\phi_x\psi_x\Big) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} S_1^{*t} \\ S_1^{*x} \end{pmatrix} = T_3^{[1]}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} + (0)\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix}$$

$$
= \begin{pmatrix} \dfrac{\partial}{\partial x}\left[\dfrac{1}{2}e^{2\gamma t}(\phi\psi_x - \psi\phi_x)\right] \\ \dfrac{\partial}{\partial x}\left[\dfrac{1}{2}e^{2\gamma t}\left((\phi^2+\psi^2)^2 + (h-1)(\phi^2+\psi^2) + \psi\phi_t - \phi\psi_t + \phi_x^2 + \psi_x^2\right)\right] \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

Thus, T_1 and T_3 are both associated with S_1. This implies that the first equation of the array of equations in (2.1) can be reduced since T_1 and T_3 are both symmetries associated with S_1.

We write both T_1 and T_3 as a linear combination. Thus we get $T = T_3 + fT_1($ f is a constant), and after transforming the operator to a canonical format we get $Z = \frac{\partial}{\partial j}$, where we assume that this generator is of the form $Z = 0\frac{\partial}{\partial i} + \frac{\partial}{\partial j} + 0\frac{\partial}{\partial u} + 0\frac{\partial}{\partial v}$.

From $T(i) = 0$, $T(j) = 1$, $T(u) = 0$ and $T(w) = 0$, we get

$$
\frac{dt}{0} = \frac{dx}{1} = \frac{d\phi}{-f\psi} = \frac{d\phi}{f\phi} = \frac{di}{0} = \frac{dj}{1} = \frac{du}{0} = \frac{dv}{0}. \qquad (2.7)
$$

Equations (2.7) are solved using variational methods. A summary of the results is given in the table that follows.

n_4, n_5, n_6 and n_7 are arbitrary functions all dependent on n_1, n_2 and n_3.

By choosing $n_4 = n_1$, $n_5 = 0$, $n_6 = \sqrt{n_2}$ and $n_7 = n_3$, we get the following canonical coordinates

$$
r = t, \quad s = x, \quad u = \sqrt{\phi^2 + \psi^2}, \quad v = \arctan\left(\frac{\psi}{\phi}\right) - fj, \qquad (2.8)
$$

where $u = u(i)$ and $v = v(i)$ since $Z = \frac{\partial}{\partial j}$.

$\frac{dt}{0}$	$n_1 = t$
$\frac{d\phi}{-f\psi} = \frac{d\psi}{f\phi}$	$n_2 = \phi^2 + v^2$
$\frac{dx}{1} = \frac{d\psi}{f\phi}$	$n_3 = \arctan\left(\frac{\psi}{\phi}\right) - fx$
$\frac{di}{0}$	$n_4 = i$
$\frac{dx}{1} = \frac{dj}{1}$	$n_5 = j - x$
$\frac{du}{0}$	$n_6 = u$
$\frac{dv}{0}$	$n_7 = v$

From equation(1.17), we compute U and $(U^{-1})^T$

$$U = \begin{pmatrix} D_i t & D_i x \\ D_j t & D_j x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} D_t i & D_t j \\ D_x i & D_x j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (U^{-1})^T,$$

where $E = \det(U) = 1$.

Calculating the inverse coordinates from the system (2.8) we get

$$t = i, \quad x = j, \quad \phi = u(i)\cos\left(v(i) + fj\right), \quad \psi = u(i)\sin\left(v(i) + fj\right). \tag{2.9}$$

Note:

$$v(i) = \arctan(\frac{\psi}{\phi}) - fj$$

$$
\begin{aligned}
v(i) &= \arcsin(\frac{\psi}{\sqrt{\phi^2+\psi^2}}) - fj \\
v(i) + fj &= \arcsin(\frac{\psi}{\sqrt{\phi^2+\psi^2}}) \\
\sin(v(i)+fj) &= \frac{\psi}{\sqrt{\phi^2+\psi^2}} \\
\sin(v(i)+fj) &= \frac{\psi}{u(i)} \\
\psi &= u(i)\sin(v(i)+fj),
\end{aligned}
$$

and similarly $\phi = u(i)\cos(v(i)+fj)$

We calculate the first and second-order partial derivatives of ϕ and ψ from equations in (2.9)

$$
\begin{aligned}
\phi_t = \frac{\partial\phi}{\partial i}\frac{\partial i}{\partial t} + \frac{\partial\phi}{\partial j}\frac{\partial j}{\partial t} &= \left(\frac{d}{di}u(i)\right)\cos\left(v(i)+fj\right) - u(i)\left(\frac{d}{di}v(i)\right)\sin\left(v(i)+fj\right) \\
\phi_x = \frac{\partial\phi}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\phi}{\partial j}\frac{\partial j}{\partial x} &= -fu(i)\sin\left(v(i)+fj\right) \\
\psi_t = \frac{\partial\psi}{\partial i}\frac{\partial i}{\partial t} + \frac{\partial\psi}{\partial j}\frac{\partial j}{\partial t} &= \left(\frac{d}{di}u(i)\right)\sin\left(v(i)+fj\right) + u(i)\left(\frac{d}{di}v(i)\right)\cos\left(v(i)+fj\right) \\
\psi_x = \frac{\partial\psi}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\psi}{\partial j}\frac{\partial j}{\partial x} &= fu(i)\cos\left(v(i)+fj\right) \\
\phi_{xx} = \frac{\partial\phi_x}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\phi_x}{\partial j}\frac{\partial j}{\partial x} &= -f^2u(i)\cos\left(v(i)+fj\right) \\
\psi_{xx} = \frac{\partial\psi_x}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\psi_x}{\partial j}\frac{\partial j}{\partial x} &= -f^2u(i)\sin\left(v(i)+fj\right). \qquad (2.10)
\end{aligned}
$$

In order to get a reduced conserved form, we use the formula from equation(1.18) with $l = 1, 2$

$$
\begin{pmatrix} S_1^i \\ S_1^j \end{pmatrix} = E(U^{-1})^T \begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix}. \qquad (2.11)
$$

By substituting E, $(U^{-1})^T$ and the inverse canonical coordinates into equation (2.11), to get

$$\begin{pmatrix} S_1^i \\ S_1^j \end{pmatrix} = \left(\begin{array}{c} \dfrac{1}{2}ce^{2\gamma i}u(i)^2 \\ \dfrac{1}{2}e^{2\gamma i}\left[u(i)^4 + (h-1)u(i)^2 - \left(\dfrac{d}{di}v(i)\right)u(i)^2 + c^2u(i)^2\right] \end{array} \right). \tag{2.12}$$

Recall from chapter 1 that the reduced conserved form is

$$D_iS_1^i = 0. \tag{2.13}$$

Differentiating S_1^i totally with respect to i is zero, therefore

$$e^{2\gamma i}u(i)^2 = p \tag{2.14}$$

where p is a constant, or equivalently

$$u(i)^2 = ke^{-2\gamma i}. \tag{2.15}$$

We use implicit differentiation with respect to i in equation (2.15) to get

$$2\gamma e^{2\gamma i}u(i)^2 + 2e^{2\gamma i}u(i)\frac{d}{di}u(i) = 0.$$

We find that $2e^{2\gamma i}u(i)$ is a common factor

$$\gamma u(i) + \frac{d}{di}u(i) = 0. \tag{2.16}$$

In the system (2.1), the second equation from the system is : ${g_1}^1Q^1 - {g_1}^2Q^2 = 0$. This implies that

$$\begin{aligned} &e^{2\gamma t}\psi_x\Big[\phi_t + \psi_{xx} + 2(\phi^2+\psi^2)\psi + (h-1)\psi + \gamma\phi\Big] \\ &\quad - e^{2\gamma t}\phi_x\Big[-\psi_t + \phi_{xx} + 2(\phi^2+\psi^2)\phi + (h-1)\phi - \gamma\psi\Big] = 0. \end{aligned} \tag{2.17}$$

Substituting the partial derivatives for ϕ and ψ from equation (2.10) as well as the inverse canonical coordinates, we get:

$$\begin{aligned} -2fe^{2\gamma i}u(i)^2 \left[\frac{d}{di}v(i) + f^2 - 2u(i)^2 - (h-1)\right] \cos\left(v(i) + fj\right) \sin\left(v(i) + fj\right) \\ + fe^{2\gamma i}u(i)\left[\cos 2(v(i) + fj)\right]\left[\gamma u(i) + \frac{d}{di}u(i)\right] = 0. \end{aligned} \tag{2.18}$$

We now substitute equations (2.14) and (2.16) into equation (2.18).

After dividing both sides by $-2fp$, this results in the following ordinary differential equation

$$\frac{d}{di}v(i) = 2u(i)^2 + h - 1 - f^2. \tag{2.19}$$

Substituting equation (2.15) into equation (2.19) and then finding the integral with respect to i results in

$$v(i) = \frac{-p}{\gamma}e^{-2\gamma i} + (h - 1 - f^2)i + \alpha, \tag{2.20}$$

where α is an integration constant.

Therefore substituting into equation (2.9) we get the exact solution:

$$\begin{aligned} \phi(t,x) &= \sqrt{p}e^{-\gamma t}\cos\left(\frac{-p}{\gamma}e^{-2\gamma t} + (h-1-f^2)t + \alpha + fx\right) \\ \psi(t,x) &= \sqrt{p}e^{-\gamma t}\sin\left(\frac{-p}{\gamma}e^{-2\gamma t} + (h-1-f^2)t + \alpha + fx\right). \end{aligned} \tag{2.21}$$

That is

$$q = \sqrt{p}e^{-\gamma t}exp\Big\{i\Big[\frac{-p}{\gamma}e^{2\gamma t} + (h-1-f^2)t + fx\Big]\Big\}, \tag{2.22}$$

because equation(2.21) is invariant under translation in space and rotation in (ϕ, ψ). This corresponds to equation(2.22) being invariant under space translation with scale invariance in q

Case 2: A double reduction when $\gamma \neq 0, h = 0$

Recalling from above and relabelling, the equation also admits the algebra of Lie point symmetries generated by

$$\begin{aligned}
T_1 &= -\psi\frac{\partial}{\partial\phi} + \phi\frac{\partial}{\partial\psi}, \\
T_2 &= \frac{\partial}{\partial t} + \psi\frac{\partial}{\partial\phi} - \phi\frac{\partial}{\partial\psi}, \\
T_3 &= \frac{\partial}{\partial x}, \\
T_4 &= 2t\frac{\partial}{\partial x} - x\psi\frac{\partial}{\partial\phi} + x\phi\frac{\partial}{\partial\psi}. \qquad (2.23)
\end{aligned}$$

and the conserved vectors

$$\begin{aligned}
S_1 &= \Bigg[-\frac{1}{2}e^{2\gamma t}(x(\phi^2+\psi^2) + 2t(\psi\phi_x - \phi\psi_x)), e^{2\gamma t}(t\phi^4 - t\psi^2 + t\psi^4 + t\phi^2(-1+2\psi^2) \\
&\quad + \psi(t\phi_t + x\phi_x) - \phi(t\psi_t + x\psi_x) + t(\phi_x^2 + \psi_x^2)\Bigg], \\
S_2 &= \left[\frac{1}{2}e^{2\gamma t}(\phi^2 + psi^2), e^{2\gamma t}(-\psi\phi_x + \phi\psi_x)\right] \qquad (2.24)
\end{aligned}$$

with the associated multipliers

$$\begin{aligned}
G_1 = &\ [-x^{2\gamma t}\psi_x + 2te^{2\gamma t}\psi_x, x\psi e^{2\gamma t} + 2te^{2\gamma t}\phi_x], \\
G_2 = &\ [e^{2\gamma t}\phi, -e^{2\gamma t}\psi].
\end{aligned} \qquad (2.25)$$

We now show that T_4 is associated with $S_2 = (S_2^t, S_2^x)$ using the formula (2.6).

If the prolongation of $T_4^{[1]}$ is

$$T_4^{[1]} = 2t\frac{\partial}{\partial x} - x\psi\frac{\partial}{\partial\phi} + x\phi\frac{\partial}{\partial\psi} - (x\psi_t + 2\phi_x)\frac{\partial}{\partial\phi_t} - (x\psi_x + \psi)\frac{\partial}{\partial\phi_x} + (x\phi_t - 2\psi_x)\frac{\partial}{\partial\psi_t} + (x\phi_x + \phi)\frac{\partial}{\partial\psi_x}$$

then using the following version of equation(1.16) for $l = 1, 2$ we recall equation (2.6), $\mathrm{S}^* = T\binom{S^t}{S^x} - \begin{pmatrix} D_t\theta^t & D_x\theta^t \\ D_t\theta^x & D_x\theta^x \end{pmatrix}\binom{S^t}{S^x} + (D_t\theta^t + D_x\theta^x)\binom{S^t}{S^x}$,
where S^* is our new conservation law. We obtain

$$\begin{aligned}
\begin{pmatrix} S_2^{*t} \\ S_2^{*x} \end{pmatrix} &= T_4^{[1]}\begin{pmatrix} S_2^t \\ S_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} S_2^t \\ S_2^x \end{pmatrix} + (0)\begin{pmatrix} S_2^t \\ S_2^x \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}e^{2\gamma t}(-2x\phi\psi + 2x\phi\psi) \\ e^{2\gamma t}\Big(- x\psi\psi_x - x\phi\phi_x + x\psi\psi_x + \psi^2 + x\phi\phi_x + \phi^2 - \phi^2 - \psi^2\Big) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

Therefore, T_4 is associated with S_2.

This implies that the first equation of the array of equations in (2.1) can be reduced since T_4 is an associated symmetry with S_2.

After transforming the operator T_4 to $Z = \frac{\partial}{\partial j}$, assuming that this generator is also of the form $Z = 0\frac{\partial}{\partial i} + \frac{\partial}{\partial j} + 0\frac{\partial}{\partial u} + 0\frac{\partial}{\partial v}$.

From $T(i) = 0$, $T(j) = 1$, $T(u) = 0$ and $T(v) = 0$, we obtain

$$\frac{dt}{0} = \frac{dx}{2t} = \frac{d\phi}{-x\psi} = \frac{d\psi}{x\phi} = \frac{di}{0} = \frac{dj}{1} = \frac{du}{0} = \frac{dv}{0}. \tag{2.26}$$

Equations (2.26) are solved using variational methods. A summary of the results is given in the table below.

n_4, n_5, n_6 and n_7 are arbitrary functions all dependent on n_1, n_2 and n_3.

The choice $n_4 = n_1$, $n_5 = 0$, $n_6 = \sqrt{n_2}$ and $n_7 = n_3$, results in the following the

$\dfrac{dt}{0}$	$n_1 = t$
$\dfrac{d\phi}{-x\psi} = \dfrac{d\psi}{x\phi}$	$n_2 = \phi^2 + \psi^2$
$\dfrac{dx}{2t} = \dfrac{d\psi}{x\phi}$	$n_3 = \arctan\left(\dfrac{\psi}{\phi}\right) - \dfrac{x^2}{4t}$
$\dfrac{di}{0}$	$n_4 = i$
$\dfrac{dx}{2t} = \dfrac{dj}{1}$	$n_5 = j - \dfrac{x}{2t}$
$\dfrac{du}{0}$	$n_6 = u$
$\dfrac{dv}{0}$	$n_7 = v$

canonical coordinates

$$i = t, \quad j = \frac{x}{2t}, \quad u = \sqrt{\phi^2 + \psi^2}, \quad v = \arctan\left(\frac{\psi}{\phi}\right) - \frac{x^2}{4t}, \tag{2.27}$$

where $u = u(i)$ and $v = v(i)$ since $Z = \frac{\partial}{\partial j}$.

Using equation(1.17), we compute U and $(U^{-1})^T$

$$U = \begin{pmatrix} D_i t & D_i x \\ D_j t & D_j x \end{pmatrix} = \begin{pmatrix} 1 & 2j \\ 0 & 2i \end{pmatrix}$$

and

$$(U^{-1})^T = \begin{pmatrix} D_t i & D_t j \\ D_x i & D_x j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{j}{i} & \dfrac{1}{2i} \end{pmatrix}$$

where $E = \det(U) = 2i$.

Calculating the inverse coordinates from equation (2.27) we get the following

$$t = i, \quad x = 2ij, \quad \phi = u(i)\cos\left(v(i) + ij^2\right), \quad \psi = u(i)\sin\left(v(i) + ij^2\right). \tag{2.28}$$

Note:

$$\begin{aligned}
v(i) &= \arctan(\frac{\psi}{\phi}) - ij^2 \\
v(i) &= \arcsin(\frac{\psi}{\sqrt{\phi^2 + \psi^2}}) - ij^2 \\
v(i) + ij^2 &= \arcsin(\frac{\psi}{\sqrt{\phi^2 + \psi^2}}) \\
\sin(v(i) + ij^2) &= \frac{\psi}{\sqrt{\phi^2 + \psi^2}} \\
\sin(v(i) + ij^2) &= \frac{\psi}{u(i)} \\
\psi &= u(i)\sin(v(i) + ij^2),
\end{aligned}$$

and similarly $\phi = u(i)\cos(v(i) + ij^2)$

We calculate the respective derivatives of ϕ and ψ from equation (2.27)

$$\begin{aligned}
\phi_t = \frac{\partial\phi}{\partial i}\frac{\partial i}{\partial t} + \frac{\partial\phi}{\partial j}\frac{\partial j}{\partial t} &= \left(\frac{d}{di}u(i)\right)\cos\left(v(i) + ij^2\right) - u(i)\left(\frac{d}{di}v(i)\right)\sin\left(v(i) + ij^2\right) \\
&\quad + j^2 u(i)\sin\left(v(i) + ij^2\right) \\
\phi_x = \frac{\partial\phi}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\phi}{\partial j}\frac{\partial j}{\partial x} &= -ju(i)\sin\left(v(i) + ij^2\right) \\
\psi_t = \frac{\partial\psi}{\partial i}\frac{\partial i}{\partial t} + \frac{\partial\psi}{\partial j}\frac{\partial j}{\partial t} &= \left(\frac{d}{di}u(i)\right)\sin\left(v(i) + ij^2\right) + u(i)\left(\frac{d}{di}v(i)\right)\cos\left(v(i) + ij^2\right) \\
&\quad - j^2 u(i)\cos\left(v(i) + ij^2\right) \\
\psi_x = \frac{\partial\psi}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\psi}{\partial j}\frac{\partial j}{\partial x} &= ju(i)\cos\left(v(i) + ij^2\right) \\
\phi_{xx} = \frac{\partial\phi_x}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\phi_x}{\partial j}\frac{\partial j}{\partial x} &= -\frac{u(i)\sin\left(v(i) + ij^2\right)}{2i} - j^2 u(i)\cos\left(v(i) + ij^2\right) \\
\psi_{xx} = \frac{\partial\psi_x}{\partial i}\frac{\partial i}{\partial x} + \frac{\partial\psi_x}{\partial j}\frac{\partial j}{\partial x} &= \frac{u(i)\cos\left(v(i) + ij^2\right)}{2i} - j^2 u(i)\sin\left(v(i) + ij^2\right). \qquad (2.29)
\end{aligned}$$

We now apply the formula from equation (1.18) with $l = 1, 2$ to obtain the reduced conserved form

$$\begin{pmatrix} S_2^i \\ S_2^j \end{pmatrix} = E(U^{-1})^T \begin{pmatrix} S_2^t \\ S_2^x \end{pmatrix}. \tag{2.30}$$

By substituting E, $(U^{-1})^T$ and the inverse canonical coordinates and into equation (2.30), to get

$$\begin{pmatrix} S_2^i \\ S_2^j \end{pmatrix} = \begin{pmatrix} ie^{2\gamma i}u(i)^2 \\ \\ 0 \end{pmatrix}, \tag{2.31}$$

where the reduced conserved form is also given by

$$D_i S_2^i = 0. \tag{2.32}$$

Since the total derivative of S_2^i with respect to i is zero, this implies that

$$ie^{2\gamma i}u(i)^2 = p \tag{2.33}$$

or equivalently

$$u(i)^2 = \frac{pe^{-2\gamma i}}{i}, \tag{2.34}$$

where p is a constant.

Differentiating equation (2.34) implicitly with respect to i results in

$$\Big[e^{2\gamma i} + 2ie^{2\gamma i}\Big]u(i)^2 + 2iu(i)e^{2\gamma i}\frac{d}{di}u(i) = 0.$$

Taking out a common factor of $e^{2\gamma i}u(i)$ results in

$$2i\frac{d}{di}u(i) + (2\gamma i + 1)u(i) = 0 \tag{2.35}$$

or equivalently after dividing both sides by $2i$

$$\frac{d}{di}u(i) + \frac{u(i)}{2i} + \gamma u(i) = 0. \tag{2.36}$$

The second equation of sys_2 from equation (6.11) is given by

$$e^{2\gamma t}\Big[\phi\phi_t - \psi\psi_t + \phi\psi_{xx} + \psi\phi_{xx} + 2\phi\psi(2(\phi^2+\psi^2)-1) + \gamma(\phi^2-\psi^2)\Big] = 0. \quad (2.37)$$

After transforming equation (6.41) using equations (6.33) and (6.34), we obtain

$$\begin{aligned} e^{2\gamma r}\Big[-2w(r)\left(\frac{d}{dr}p(r)\right) + 4w(r)^3 - 2w(r)\Big]\cos\left(p(r)+rs^2\right)\sin\left(p(r)+rs^2\right) \\ + e^{2\gamma r}\left(\frac{d}{dr}w(r) + \frac{w(r)}{2r} + \gamma w(r)\right)\left(\cos 2(p(r)+rs^2)\right) = 0. \qquad (2.38) \end{aligned}$$

After multiplying both sides of equation (2.38) by $-\dfrac{u(i)}{2}$ and then substituting equations (2.34) and (2.36), this results in the ordinary differential equation

$$\frac{d}{di}v(i) = 2v(i)^2 - 1. \qquad (2.39)$$

If we substitute equation (2.34) into equation (2.39) and then integration with respect to i results in the following

$$v(i) = 2p\int \frac{e^{-2\gamma i}}{i}di - i, \qquad (2.40)$$

where $\int \frac{e^{-2\gamma i}}{i}dr = \ln i + \sum\limits_{l=1}^{\infty}\frac{(-1)^l(2\gamma i)^l}{l l!} + \alpha.$

Therefore substituting into equation (2.27) we get the exact solution

$$\begin{aligned} \phi(t,x) &= \sqrt{\frac{p}{t}}e^{-\gamma t}\cos\left(v(i)+\frac{x^2}{4t}\right) \\ \psi(t,x) &= \sqrt{\frac{p}{t}}e^{-\gamma t}\sin\left(v(i)+\frac{x^2}{4t}\right), \end{aligned} \qquad (2.41)$$

where $v(i) = 2p\Big(\ln t + \sum\limits_{l=1}^{\infty}\frac{(-1)^l(2\gamma t)^l}{l l!} + \alpha\Big) - t.$

2.2.3 Conserved quantities

In order to calculate the conserved quantities from the conservred densities, the following 1-soliton solution is used:

$$q(x,t) = A\,\mathrm{sech}[B(x - vt)]e^{i(-\kappa x+\omega t+\theta)}$$

where A is the amplitude of the soliton and B is the inverse width of the soliton while v is the velocity of the soliton. In the phase component, κ represents the soliton frequency, ω is the soliton wave number and θ is the phase constant.

Therefore the conservation laws are respectively given by

$$I_1 = \int_{-\infty}^{\infty} \Phi^t dx = -\frac{i}{2}e^{2\gamma t}\int_{-\infty}^{\infty}(q_x - q_x)dx = -\frac{2\kappa A^2}{B}e^{2\gamma t}$$

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} \Phi^t dx = \int_{-\infty}^{\infty}\left\{\frac{1}{2}|q|^4 + \frac{1}{2}(h-1)|q|^2 + q_{xx} + q_{xx}\right\}dx \\ &= \frac{A^2}{3B}\left\{2\left(A^2 - 2B^2 - 6\kappa^2\right) + 3(h-1)\right\} \end{aligned}$$

$$I_3 = \int_{-\infty}^{\infty} \Phi^t dx = -\frac{i}{4}\int_{-\infty}^{\infty}(q_x - q_x)dx = -\frac{\kappa A^2}{B}$$

$$I_4 = \int_{-\infty}^{\infty} \Phi^t dx = -\frac{1}{2}\int_{-\infty}^{\infty}(q_x - q_x)dx = -\frac{A^2}{B}$$

$$I_5 = \int_{-\infty}^{\infty} \Phi^t dx = -\int_{-\infty}^{\infty}\left\{\frac{1}{4}x|q|^2 + it(q_x - q_x)\right\}dx = \frac{4\kappa t A^2}{B}$$

$$I_6 = \int_{-\infty}^{\infty} \Phi^t dx = -\int_{-\infty}^{\infty}\left\{\frac{1}{2}xe^{2\gamma t}|q|^2 + \frac{i}{2}te^{2\gamma t}(q_x - q_x)\right\}dx = -\frac{2\kappa t e^{2\gamma t}A^2}{B}$$

and

$$I_7 = \int_{-\infty}^{\infty} \Phi^t dx = \frac{1}{2}e^{2\gamma t}\int_{-\infty}^{\infty}|q|^2 dx = -\frac{e^{2\gamma t}A^2}{B}$$

2.3 Discussion and Conclusion

In this chapter we showed that by using the relationship between Lie symmetries and conserved vectors, exact solutions can be constructed for the Nonlinear Schrödinger equation. In particular, that with damping and driving parameters. Although new exact solutions were found, the method of double reduction still recognizes well known solutions (for example) travelling wave solutions.

Chapter 3

Reducing the Nagumo and Fisher equations of density dependence

3.1 Opening Remarks

It is well known that symmetry analysis of differential equations (DE's) provides a formal mechanism for the reduction of the DE's. Furthermore, a knowledge of the conservation laws do not only provide an insight into the physical aspects of the model like conservations of momentum, energy, etc., but also allows one to mathematically analyse the problem and play a role in the reduction of the of the DE's. That is, the conserved form derived from the conservation law is itself a reduction of the DE's. Thus, a combined symmetry and conservation law approach suggests that a double reduction of the DE's under investigation may be achieved. This is because the conservation law reduced system inherits the associated symmetry. We show how this is achieved for the FitzHugh-Nagumo and Fisher models and obtain some solutions.

When considering the FitzHugh-Nagumo equation, our goal was to focus on the mathematicl properties. It is an equation derived in 1952 from the Hodgkin-Huxley equation. Furthermore, the FitzHugh-Nagumo equation has fewer variables(two independent and one dependent than the Hodgkin-Huxley equation which has four. For details, the reader is referred to [24, p. 106], and [25, pp. 10-14]. Most of the applications for the Nagumo equation occur in physics ([26, 27]. In this chapter, exact solutions are investigated for a particular case of the Nagumo equation being density dependent.

Furthermore, we consider the Fisher model, sometimes referred to as the Fisher-Kolmogorov equation. Like the Nagumo case, we take a look at a more general case. For background and some methods applied to the Fisher equation and the density dependent versions, we refer the reader to [28, 29, 30, 31] and [32, 33, 34, 35, 36, 37], respectively, and references therein. We note here that the Nagumo and Fisher equations belong to a class of equations arising in biology with modified source terms. The source term which, inter alia, describes pattern formation are either quadratic or cubic.

The method of analysis that is employed is summarized as follows. We determine the Lie point symmetry generators, the conserved vectors via the 'multiplier' approach of the partial differential equation under investigation and then exploit the association between these two concepts to perform a double reduction of the equation. More details may be found in, inter alia, the references [4, 9, 22, 23]. The procedure is general and is applicable to all classes of equations that admit symmetries and associated conservation laws. The results of this chapter appear in [63].

3.2 Transformed Nagumo equation

The Nagumo equation is given by

$$\phi_t = (\phi^c \phi_x)_x + \phi(1-\phi)(\phi-\alpha), \tag{3.1}$$

where α is a parameter for density and $\phi(1-\phi)(\phi-\alpha)$ illustrates the reaction term [26, 27, 53, 39]. When $c = 0$, equation (3.1) becomes the regular Nagumo equation and has been solved using non-classical methods [69] and references therein. Usually equation (3.1) results only in travelling wave solutions with no conserved vectors. However, we find that for a particular 'c', conserved vectors may be calculated. That is for $c = 2$ and $\alpha = -1$. We investigate this using the method of double reduction, although there are other methods which can be employed.

Direct double reduction

For $c = 2, \alpha = -1$, (3.1) becomes

$$\phi_t = 2\phi\phi_x^2 + \phi^2\phi_{xx} - \phi^3 + \phi. \tag{3.2}$$

By the standard procedure (see for example, [40, 41], chapter 2 in both cases), it can be shown that the equation admits the Lie algebra of Lie point symmetries generated by

$$T_1 = \frac{\partial}{\partial x}, \quad T_2 = \frac{\partial}{\partial t}, \quad T_3 = -\frac{1}{2}e^{-2t}\frac{\partial}{\partial t} - \frac{1}{2}e^{-2t}\phi\frac{\partial}{\partial \phi} \tag{3.3}$$

and the conserved vectors

$$\begin{aligned} S_1 &= \left(e^{-\sqrt{3}x-t}\phi, e^{-\sqrt{3}x-t}\left(-\phi_x\phi^2 - \frac{\sqrt{3}}{3}\phi^3\right)\right), \\ S_2 &= \left(e^{\sqrt{3}x-t}\phi, e^{\sqrt{3}x-t}\left(-\phi_x\phi^2 - \frac{\sqrt{3}}{3}\phi^3\right)\right) \end{aligned} \tag{3.4}$$

with associated multipliers

$$G_1 = e^{-\sqrt{3}x-t}, \qquad G_2 = e^{\sqrt{3}x-t}. \tag{3.5}$$

T_3 is associated with $S_1 = (S_1^t, S_1^x)$ since

$$T_3^{[1]}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} - \left(\begin{array}{cc} e^{-2t} & 0 \\ 0 & 0, \end{array} \right) \begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} + e^{-2t}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.6}$$

where

$$T_3^{[1]} = -\frac{1}{2}e^{-2t}\frac{\partial}{\partial t} - \frac{1}{2}e^{-2t}\phi\frac{\partial}{\partial \phi} - \frac{1}{2}e^{-2t}\phi_x\frac{\partial}{\partial \phi_x}.$$

The calculations show that the other cases only yield the translation symmetries with no associated conservation laws. In this case, the method cannot be used and the exact solutions are travelling wave type ones or solutions obtained via some numerical approach. The notion of 'association' between symmetry and conservation laws has been investigated and established in [22] and is identified by the relationship defined in equation (3.6).

A symmetry T in variables x, t and ϕ may be transformed to Z in variables i, j and u by $Z = T(i)\partial_i + T(j)\partial_j + T(u)\partial_u$. In particular, a desirable transformation is translation in j, that is, $Z = \partial_j$.

We now can get a reduced conserved form for the equation equation (3.2). We transform the generator T_3 to its canonical form $Z = \frac{\partial}{\partial j}$, where we assume that this generator is of the form $Z = 0\frac{\partial}{\partial i} + \frac{\partial}{\partial j} + 0\frac{\partial}{\partial u}$. From $T(i) = 0$, $T(j) = 1$ and $T(u) = 0$, we obtain

$$\frac{-2dt}{e^{-2t}} = \frac{dx}{0} = \frac{-2d\phi}{\phi e^{-2t}} = \frac{di}{0} = \frac{dj}{1} = \frac{du}{0}. \tag{3.7}$$

The system leads to 'integration constants' (n_l) which are the invariants (dependent and independent) relating the variables x, t and ϕ to the transformed ones i, j and

u. Solving equation (3.7) leads to the set of invariants $n_1 = \phi e^{-t}$, $n_2 = x$, $n_3 = i$, $n_4 = j + e^{2t}$ and $n_5 = u$, where n_3, n_4, and n_5 are arbitrary functions dependent on n_1 and n_2. That is, $n_3 = n_3(n_1, n_2)$, $n_4 = n_4(n_1, n_2)$ and $n_5 = n_5(n_1, n_2)$. A convenient choice is $n_3 = n_2$, $n_4 = 0$, and $n_5 = n_1$. A different choice for n_3 or n_4 would lead to a different set of canonical variables and the equations obtained may be different from those obtained here but equivalent after substitution to the original variables. A particular case of the canonical coordinates are

$$i = x, \quad j = -e^{2t}, \quad u = \phi e^{-t}, \tag{3.8}$$

where $u = u(i)$ and since $Z = \frac{\partial}{\partial j}$. Since

$$U \;=\; \begin{pmatrix} D_i t & D_i x \\ D_j t & D_j x \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ \frac{1}{2j} & 0 \end{pmatrix}$$

we get

$$U^{-1} \;=\; \begin{pmatrix} D_t i & D_t j \\ D_x i & D_x j \end{pmatrix} \;=\; \begin{pmatrix} 0 & 2j \\ 1 & 0 \end{pmatrix}$$

and, therefore,

$$(U^{-1})^T \;=\; \begin{pmatrix} 0 & 1 \\ 2j & 0 \end{pmatrix}$$

where $E = \det(U) = -1/2j$ and D_l is the total derivative operator with respect to the lth variable. From equation (3.8), we calculate the inverses to get

$$t = \frac{\ln(-j)}{2}, \quad \phi = u(-j)^{\frac{1}{2}}. \tag{3.9}$$

The respective derivatives of ϕ are

$$\phi_t = u(-j)^{\frac{1}{2}}, \quad \phi_x = u_i(-j)^{\frac{1}{2}}, \quad \phi_{xx} = u_{ii}(-j)^{\frac{1}{2}}. \tag{3.10}$$

We now let

$$\begin{pmatrix} S_1^i \\ S_1^j \end{pmatrix} = E(U^{-1})^{\mathrm{S}} \begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix}. \tag{3.11}$$

We substitute equations (3.9) and (3.10) into equation (3.11), to get

$$\begin{pmatrix} S_1^i \\ S_1^j \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}, \tag{3.12}$$

where

$$J_1 = -\frac{1}{2}e^{-\sqrt{3}i}u^2\left(u_i + \frac{\sqrt{3}}{3}u\right)$$

and

$$J_2 = -S_1^t.$$

Showing the conserved form in a reduced state

$$D_i S_1^i = 0. \tag{3.13}$$

The next step in the double reduction procedure is shown as

$$p = -\frac{1}{2}e^{-\sqrt{3}i}u^2\left(u_i + \frac{\sqrt{3}}{3}u\right) \tag{3.14}$$

so that

$$u = \frac{e^{-\sqrt{3}i}}{2}\left(\left(e^{2\sqrt{3}i}4p\sqrt{3} + 8q\right)e^{2\sqrt{3}i}\right)^{\frac{1}{3}}, \tag{3.15}$$

where q is an integration constant. Substitute equation (3.15) into equation (3.9), to get the exact solution (group invariant solution)

$$\phi = \frac{e^{-\sqrt{3}x+t}}{2}\left(\left(e^{2\sqrt{3}x}4p\sqrt{3} + 8q\right)e^{2\sqrt{3}x}\right)^{\frac{1}{3}}, \tag{3.16}$$

where p is a constant.

It is clear that this is not a travelling wave solution and has not been given elsewhere (to the best of the our knowledge). The previous analytical solutions are travelling wave solutions.

3.3 Transformed Fisher-Kolmogorov equation

The general Fisher equation is shown by

$$\phi_t = (\phi^c \phi_x)_x + \phi(1-\phi). \tag{3.1}$$

Like the Nagumo model, for $c = 0$, conserved vectors do not exist but for $c = 1$, we get two conserved vectors, i.e. equation (3.1) becomes:

$$\phi_t = \phi_x^2 + \phi\phi_{xx} - \phi^2 + \phi. \tag{3.2}$$

With the mathematical packages Maple and Mathematica we generate the following Lie symmetries:

$$T_1 = \frac{\partial}{\partial x}, \quad T_2 = \frac{\partial}{\partial t}, \quad T_3 = e^{-t}\frac{\partial}{\partial t} - e^{-t}\phi\frac{\partial}{\partial \phi} \tag{3.3}$$

and the conserved vectors

$$\begin{aligned} S_1 &= \left(e^{-\sqrt{2}x-t}\phi, e^{-\sqrt{2}x-t}\left(-\phi_x\phi - \frac{\sqrt{2}}{2}\phi^2\right)\right), \\ S_2 &= \left(e^{\sqrt{2}x-t}\phi, e^{\sqrt{2}x-t}\left(-\phi_x\phi - \frac{\sqrt{2}}{2}\phi^2\right)\right) \end{aligned} \tag{3.4}$$

with associated multipliers

$$G_1 = e^{-\sqrt{2}x-t}, \quad G_2 = e^{\sqrt{2}x-t}. \tag{3.5}$$

The generator T_3 is associated with $S_1 = (S_1^t, S_1^x)$ since

$$T_3^{[1]}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} - \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} + (e^{-t})\begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$T_3^{[1]} = e^{-t}\frac{\partial}{\partial t} - e^{-t}\phi\frac{\partial}{\partial \phi} - e^{-t}\phi_x\frac{\partial}{\partial \phi_x}.$$

After transforming the operator T_3 to its canonical form $Z = \frac{\partial}{\partial j}$, we assume that the lie operator is of the form $Z = 0\frac{\partial}{\partial i} + \frac{\partial}{\partial j} + 0\frac{\partial}{\partial u}$. See the explanation above for the details. From $T(i) = 0$, $T(j) = 1$ and $T(u) = 0$, we obtain

$$\frac{-dt}{e^{-t}} = \frac{dx}{0} = \frac{-d\phi}{\phi e^{-t}} = \frac{di}{0} = \frac{dj}{1} = \frac{du}{0}. \tag{3.6}$$

As before, equation (3.6) leads to the invariants $n_1 = \phi e^{-t}$, $n_2 = x$, $n_3 = i$, $n_4 = j+e^t$ and $n_5 = u$ where n_3, n_4, and n_5 are arbitrary functions all dependent on n_1 and n_2. The comment regarding the particular choice of n_3, n_4 and n_5 made in the previous section is relevant here too. By choosing $n_3 = n_2$, $n_4 = 0$, and $n_5 = n_1$, we obtain the canonical coordinates

$$i = x, \quad j = -e^t, \quad u = \phi e^{-t}, \tag{3.7}$$

where $u = u(i)$ and since $Z = \frac{\partial}{\partial j}$. We compute U and $(U^{-1})^T$

$$U \; = \; \begin{pmatrix} D_i t & D_i x \\ D_j t & D_j x \end{pmatrix} \; = \; \begin{pmatrix} 0 & 1 \\ \frac{1}{j} & 0 \end{pmatrix}$$

and

$$U^{-1} \; = \; \begin{pmatrix} D_t i & D_t j \\ D_x i & D_x j \end{pmatrix} \; = \; \begin{pmatrix} 0 & j \\ 1 & 0. \end{pmatrix}$$

So

$$(U^{-1})^T \; = \; \begin{pmatrix} 0 & 1 \\ j & 0 \end{pmatrix}$$

where $E = \det(U) = -1/j$. Calculating the inverses from equation (3.7) we get,

$$t = \frac{\ln(-j)}{2}, \quad \phi = -uj, \tag{3.8}$$

Thus,

$$\phi_t = -uj, \quad \phi_x = -u_i j, \quad \phi_{xx} = -u_{ii}. \tag{3.9}$$

Let

$$\begin{pmatrix} S_1^i \\ S_1^j \end{pmatrix} = E(U^{-1})^{\mathrm{S}} \begin{pmatrix} S_1^t \\ S_1^x \end{pmatrix}. \tag{3.10}$$

By substituting equations (3.8) and (3.9) into equation (3.10), we obtain

$$\begin{pmatrix} S_1^j \\ S_1^s \end{pmatrix} = \begin{pmatrix} J_1 \\ \\ J_2 \end{pmatrix}, \tag{3.11}$$

$$J_1 = e^{\sqrt{2}i} u \left(-u_i + \frac{\sqrt{2}}{2} u \right)$$

and

$$J_2 = -S_1^t$$

Showing the conserved form in a reduced state

$$D_i S_1^i = 0. \tag{3.12}$$

Further reduction leads to

$$p = e^{-\sqrt{2}i} u \left(-u_i + \frac{\sqrt{2}}{2} u \right) \tag{3.13}$$

and, therefore,

$$u = \pm \frac{1}{2} \left(2e^{-\sqrt{2}i} p\sqrt{2} + 4qe^{\sqrt{2}i} \right)^{\frac{1}{2}}, \tag{3.14}$$

where q is an integration constant. Substitute equation (3.14) into equation (3.8), to get

$$\phi = \pm \frac{e^t}{2} \left(2e^{-\sqrt{2}x} p\sqrt{2} + 4qe^{\sqrt{2}x} \right)^{\frac{1}{2}}, \tag{3.15}$$

where p is a constant.

As before, this is not a travelling wave solution and has not been given elsewhere (to the best of our knowledge). The previous analytical solutions are travelling wave solutions.

3.4 Concluding Remarks

We have determined the Lie point symmetry generators as well as the conservation laws using the multiplier approach for both the particular cases of the Nagumo and Fisher equations. The association between the Lie point symmetries and conservation laws led to a double reduction and consequantly new exact solutions [4] and references therein.

Chapter 4

Approaching the Schrödinger's Equation using soliton theory

4.1 Introduction

The dynamics of optical solitons propagating through optical fibers for trans-contenental and trans-oceanic distances is governed by the nonlinear Schrödinger's equation (NLSE) [42, 43, 7, 44, 45, 46, 47, 48, 58]. The derivation of this particular case of the NLSE comes from Maxwell's equation in electromagnetics. Although the use of the NLSE is common in soliton theory, in this chapter we will apply it to the well known Kerr Law nonlinearity. To investigate other kinds of nonlinearity, we refer the reader to [42]. Furthermore we investigate or rather generalize the Kerr Law nonlinearity to Power law nonlinarity. We also obtain solitons and in order to do this we calculate conserved vectors using the multiplier approach from a previous chapter. Thereafter we calculate conserved quantities. The results of this chapter are referred to in [64].

4.2 Governing Equation

We study the nonlinear Schrodinger's equation (NLSE).There is also a form of the NLSE with zero diminsion given by [42, 43]

$$i\psi_t + a\psi_{xx} + bF\left(|\psi|^2\right)\psi - d^2\psi + i\alpha\psi = c, \tag{4.1}$$

where x and t are the spatial and temporal variables, . The first term is the evolution term. The next term is the group velocity dispersion (GVD) and the third term represents the nonlinearity where the function G describes the nonlinearity under investigation. The coefficient α is also known as linear attenuation. Lastly, d is also a real valued constant and the constant c is the driving term. The dependent variable ψ represents the wave profile and is a complex valued function. If a balance between GVD and nonlinearity is established, we get solitons.

In equation (4.1), G is a real-valued algebraic nonlinear function and it is necessary to have the smoothness of the complex function $G\left(|\psi|^2\right)\psi : C \mapsto C$. We consider the complex plane C as a two-dimensional linear space R^2, the function $G\left(|\psi|^2\right)\psi$ is d times continuously differentiable, so that [42, 7]

$$G\left(|\psi|^2\right)\psi \in \bigcup_{r,s=1}^{\infty} C^d\left((-s,s)\times(-r,r);R^2\right). \tag{4.2}$$

To solve equation (4.1), we use the ansatz method. In addition, in order to integrate equation (4.1), we set $\alpha = c = 0$. For non-trivial values α and c the Painleve test of integrability fails. Thus the soliton solution for equation (4.1) using the ansatz method is:

$$\psi(x,t) = D\,\mathrm{sech}^f \tau e^{i(-\kappa x+\omega t+\theta)} \tag{4.3}$$

where

$$\tau = E(x - vt) \tag{4.4}$$

In equation (4.3), D represents the soliton amplitude, while E and v represent the inverse width of the soliton and velocity respectively. The unknown index f will fall away while deriving the 1-soliton solution. Furthermore κ is the soliton wave number, ω represents the frequency of the solitons and θ is a phase constant.

If we substitute equation (4.3) into equation (4.1) and decompose into the respective real and imaginary parts we get

$$\begin{aligned} & a\left\{f^2 DE^2 \mathrm{sech}^f \tau - f(f+1)DE^2 \mathrm{sech}^{f+2}\tau\right\} \\ - & \left(a\kappa^2 + d^2\right) D\mathrm{sech}^f\tau + bD\,\mathrm{sech}^f\tau G\left(D^2\,\mathrm{sech}^{2f}\tau\right) = 0 \end{aligned} \tag{4.5}$$

and

$$(v + 2a\kappa)\ \mathrm{sech}^f\tau \tanh\tau = 0 \tag{4.6}$$

Using equation (4.6) we can say that

$$v = -2a\kappa \tag{4.7}$$

we see that the soliton velocity stays constant regardless of the nonlinearity law under investigation. In the next sections, we study the real part of equation (4.5) which does depend on the nonlinearity used.

4.2.1 Kerr Law

The Kerr law is a consequence of the nonlinear responses of the light wave in an optical fiber stemming from the non-harmonic motion of electrons bound in molecules from an external electric field [42, 7]. Here $G(z) = z$. Thus, in this case, equation (4.1) becomes

$$i\psi_t + a\psi_{xx} + b|\psi|^2\psi - d^2\psi + i\alpha\psi = 0, \tag{4.8}$$

Thus equation (4.1), for $\alpha = c = 0$ becomes

$$\begin{aligned} & a\left\{f^2 DE^2 \text{sech}^f\tau - f(f+1)DE^2\text{sech}^{f+2}\tau\right\} \\ - & \left(a\kappa^2 + d^2\right) D\text{sech}^f\tau + bD^3\,\text{sech}^{3f}\tau = 0 \end{aligned} \tag{4.9}$$

We get solitons by balancing dispersion and nonlinearity, thus we can equate the exponents $3f$ and $f+2$ against each other using this balancing principle. This yields

$$f = 1 \tag{4.10}$$

In equation (4.9), if we set the coefficients of the linearly independent functions $\text{sech}^{f+j}\tau$ to zero for $j = 1, 2$ results in

$$\omega = a\left(E^2 - \kappa^2\right) - d^2 \tag{4.11}$$

and

$$E = \sqrt{\frac{b}{2a}}D \tag{4.12}$$

This results in the following to the 1-soliton solution

$$\psi(x,t) = D\,\text{sech}[E(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \tag{4.13}$$

where equation (4.12) represents the amplitude-width relation and equation (4.11) stands for the soliton frequency. The velocity of the soliton is shown in equation (4.7). We point out that the terms involving d^2 make equation (4.8) integrable and this results in a phase shift. This observation is being made for the first time in this chapter, at least by the ansatz method.

4.2.2 Power Law

In power law nonlinearity $G(z) = z^s$ where the parameter s represents the power law parameter. This nonlinearity law has been used in a number of materials in particular, semiconductors semiconductors [42, 7]. Furthermore, this law has been investigated in plasmas when the problem of small K-condensation in weak turbulence theory is solved. Power law nonlinearity is stable when $0 < s < 2$ and especially when $s \neq 2$. Here, the NLSE becomes

$$i\psi_t + a\psi_{xx} + b|\psi|^{2s}\psi - d^2\psi + i\alpha\psi = 0, \tag{4.14}$$

where the real part of equation (4.1) becomes

$$\begin{aligned} & a\left\{f^2DE^2\mathrm{sech}^f\tau - f(f+1)DE^2\mathrm{sech}^{f+2}\tau\right\} \\ - & \left(a\kappa^2 + d^2\right)D\mathrm{sech}^f\tau + bD^{2s+1}\,\mathrm{sech}^{(2s+1)f}\tau = 0 \end{aligned} \tag{4.15}$$

Once again we balance dispersion and nonlinearity in equation (4.15), and we equate the exponents $(2s+1)f$ with $f+2$ that implies

$$f = \frac{1}{s} \tag{4.16}$$

If we equate the coefficients of the linearly independent functions to zero, we get equation (4.17). This was done earlier for the case of Kerr law nonlinearity.

$$E = s\sqrt{\frac{b}{(s+1)a}}D^s \tag{4.17}$$

and

$$\omega = \frac{1}{s^2}\left\{aD^2 - s^2\left(a\kappa^2 + d^2\right)\right\} \tag{4.18}$$

Therefore the 1-soliton solution for the NLSE with power law nonlinearity is shown as

$$\psi(x,t) = D\,\mathrm{sech}^{\frac{1}{s}}[E(x - vt)]e^{i(-\kappa x + \omega t + \theta)}. \tag{4.19}$$

In equation (4.17) represents the amplitude-width relation and equation (4.18) is the soliton frequency. In equation (4.7) represents the soliton velocity and remains unchanged. Lastly, we need to point out that when $s = 1$, the results of the power law nonlinearity reduce to a case of Kerr law nonlinearity.

4.3 Conserved vectors

In this chapter, the vector equation is decomposed into two equations. One representing the imaginary part and the other the real part. From the conserved vector, the density is given by Φ.

We find that the conserved densities are given by [43]. Also see chapter 2, only here we rename some of the constants

$$\Phi_1 = -\frac{i}{4}e^{2t\alpha}\left(\psi^*\psi_x - \psi\psi_x^*\right) \tag{4.20}$$

$$\Phi_2 = -\frac{1}{2}e^{2t\alpha}|\psi|^2 \tag{4.21}$$

$$\Phi_3 = -\frac{1}{2}e^{2t\alpha}\left\{x|\psi|^2 + it\left(\psi^*\psi_x - \psi\psi_x^*\right)\right\} \tag{4.22}$$

and for $\alpha = 0$, we obtain also

$$\Phi_4 = \frac{1}{2}\left\{|\psi|^4 - k^2|\psi|^2 + \frac{1}{2}\left(\psi^*\psi_{xx} + \psi\psi_{xx}^*\right)\right\} \tag{4.23}$$

In the next sections we will turn our attention to conserved quantities. However, to calculate these, we need to take the types of nonlinear fibre into account.

4.3.1 Kerr Law

We apply soliton theory in equation (4.13) that is, the 1-sotliton solution, and consequently find the conserved quantities.

$$J_1 = \int_{-\infty}^{\infty} \Phi_1 dx = -\int_{-\infty}^{\infty} \frac{i}{4} e^{2t\alpha} \left(\psi^* \psi_x - \psi \psi_x^*\right) dx = -\frac{\kappa D^2}{E} e^{2t\alpha} \tag{4.24}$$

$$J_2 = \int_{-\infty}^{\infty} \Phi_2 dx = -\int_{-\infty}^{\infty} \frac{1}{2} e^{2t\alpha} |\psi|^2 dx = -\frac{D^2}{E} e^{2t\alpha} \tag{4.25}$$

$$J_3 = \int_{-\infty}^{\infty} \Phi_3 dx = -\int_{-\infty}^{\infty} \frac{1}{2} e^{2t\alpha} \left\{x|\psi|^2 + it\left(\psi^* \psi_x - \psi \psi_x^*\right)\right\} dx = -\frac{2\kappa t D^2}{E} e^{2t\alpha} \tag{4.26}$$

$$\begin{aligned} J_4 &= \int_{-\infty}^{\infty} \Phi_4 dx = \frac{1}{2} \int_{-\infty}^{\infty} \left\{|\psi|^4 - d^2|\psi|^2 + \frac{1}{2}\left(\psi^* \psi_{xx} + \psi \psi_{xx}^*\right)\right\} dx \\ &= \frac{D^2}{6E}\left\{4D^2 - 2E^2 - 6\left(d^2 + \kappa^2\right)\right\} \end{aligned} \tag{4.27}$$

Therefore for J_1 and J_2 to be conserved quantities, we must set $\alpha = 0$. J_4 is also a conserved quantity given that $\alpha = 0$, but J_3 is a conserved quantity when time is fixed and also when $\alpha = 0$ like the previous quantities.

4.3.2 Power Law

We apply the same approach for finding the conserved quantities in power law as we did in Kerr Law.

$$J_1 = \int_{-\infty}^{\infty} \Phi_1 dx = -\int_{-\infty}^{\infty} \frac{i}{4} e^{2t\alpha} \left(\psi^* \psi_x - \psi \psi_x^*\right) dx = -\frac{\kappa D^2}{E} e^{2t\alpha} \frac{\Gamma\left(\frac{1}{s}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{s}\right)} \tag{4.28}$$

$$J_2 = \int_{-\infty}^{\infty} \Phi_2 dx = -\int_{-\infty}^{\infty} \frac{1}{2} e^{2t\alpha} |\psi|^2 dx = -\frac{D^2}{2E} e^{2t\alpha} \frac{\Gamma\left(\frac{1}{s}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{s}\right)} \tag{4.29}$$

$$
\begin{aligned}
J_3 &= \int_{-\infty}^{\infty} \Phi_3 dx = -\int_{-\infty}^{\infty} \frac{1}{2} e^{2t\alpha} \left\{ x|\psi|^2 + it\left(\psi^* \psi_x - \psi \psi_x^*\right) \right\} dx \\
&= -\frac{\kappa t D^2}{E} e^{2t\alpha} \frac{\Gamma\left(\frac{1}{s}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{s}\right)} \qquad (4.30)
\end{aligned}
$$

$$
\begin{aligned}
J_4 &= \int_{-\infty}^{\infty} \Phi_4 dx = \frac{1}{2} \int_{-\infty}^{\infty} \left\{ |\psi|^4 - d^2|\psi|^2 + \frac{1}{2}\left(\psi^* \psi_{xx} + \psi \psi_{xx}^*\right) \right\} dx \\
&= \frac{D^2}{2E} \left[\frac{\Gamma\left(\frac{2}{s}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{s}\right)} - \left\{ \left(d^2+\kappa^2\right) + \frac{E^2}{s(s+2)} \right\} \frac{\Gamma\left(\frac{1}{s}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{s}\right)} \right] \\
& \qquad (4.31)
\end{aligned}
$$

Here the same argument holds for power law nonlinearity for these to be conserved quantities.

4.4 Discussion and Conclusions

In this chapter we obtained solutions to the NLSE using soliton theory. There is another term that is owing to d^2 which only produces a phase shift in the soliton solutions. It has been shown that the NLSE can only be integrated when the driving term and linear attenuation terms are both equal to zero. As a result, when we use the multiplier approach, our Schrödinger system admits several conserved vectors. We also calculated the conserved quantities utilizing the 1-soliton approach. The results found for Kerr law and power law are both new.

Chapter 5

Symmetry properties and solitons of equations of compressional dispersive Alvén waves

5.1 Introduction

In this section of the thesis we look at compressional dispersive Alvén (CDA) waves. In the past there have been two main experiments conducted regarding CDA waves. The first [56] looked at the relationship between the CDA waves and the pertubations in a low plasma and the second [57] looked at the amplitude of the waves in an magnetic electron-positron plasma. In both the experiments, it was concluded that the system of equations under investigation can be conveniently written as one. We get a wave equation of order four

$$\phi_{tt} - (3a^2 + c^2)\phi_{xx} - \delta^2\phi_{xxxx} - \delta^2\phi_{xxtt} = 0. \tag{5.1}$$

This is the equation we will analyse.

In order to get an envelope of CDA waves we neen an interaction a CDA pump and and a quasi stationary compressional magnetic field [56]. This interaction ought to be nonlinear. When the CDA envelope evolves we get a cubic nonlinear Schrödinger equation (NLSE) written as,

$$iq_t + \beta iq_x - \gamma q_{xx} + \delta q|q|^2 = 0, \tag{5.2}$$

where q is the complex valued dependent variable, and γ is the coefficient of the GVD, δ is the self-phase modulation (SPM) owing to Kerr law β is the inter-modal dispersion (IMD). We find that if we balance SPM and GVD, we get solitons. This type of balance was referred to in the previous chapter.

Equation (5.1) has been derived from the ion continuity equation, ion momentum equation and Faraday's law of electromagnetic induction. In [52, 53] probable solutions have been discussed for equation (5.2) , although they are stationary and are soliton type solutions. However, in the sections that follow, we investigate if perhaps we can calculate an exact solution using the theory of double reduction. The results of this chapter appear in [65]

5.2 Noether symmetries and conserved vectors of equation (5.1)

The Lagrangian of equation (5.1), is given as

$$L = -\frac{1}{2}\phi_t^2 + \frac{1}{2}(3a^2 + c^2)\phi_x^2 - \frac{1}{2}\delta^2\phi_{xx}^2 - \frac{1}{2}\delta^2\phi_{xt}^2, \tag{5.3}$$

therefore we get the follwing Noether symmetries with zero gauge,

$$T_1 = \partial_\phi, \qquad T_2 = \partial_t, \qquad T_3 = \partial_x. \tag{5.4}$$

and conservation laws (S_j^t, S_j^x) where $D_tS_j^t + D_xS_j^x = 0$ along equation (5.1) are shown as

$$\begin{aligned} S_1^t &= \phi_t - \frac{1}{2}\delta^2\phi_{xxt}, \\ S_1^x &= -\frac{1}{2}\delta^2\phi_{xtt} - \delta^2\phi_{xxx} - (3a^2 + c^2)\phi_x, \end{aligned}$$

$$\begin{aligned} S_2^t &= -\frac{1}{2}\phi_t^2 - \frac{3}{2}a^2\phi_x^2 - \frac{1}{2}c^2\phi_x^2 + \frac{1}{2}\delta^2\phi_{xx}^2 + \frac{1}{2}\delta^2\phi_t\phi_{xxt}, \\ S_2^x &= -\frac{1}{2}\delta^2\phi_{tt}\phi_{xt} - \delta^2\phi_{xt}\phi_{xx} + \frac{1}{2}\delta^2\phi_t\phi_{xtt} + \delta^2\phi_t\phi_{xxx} + (3a^2 + c^2)\phi_x\phi_t, \end{aligned}$$

$$\begin{aligned} S_3^t &= -\frac{1}{2}\delta^2\phi_{xx}\phi_{xt} + \frac{1}{2}\delta^2\phi_x\phi_{xxt} - \phi_x\phi_t, \\ S_3^x &= \frac{1}{2}\phi_t^2 + \frac{3}{2}a^2\phi_x^2 + \frac{1}{2}c^2\phi_x^2 - \frac{1}{2}\delta^2\phi_{xx}^2 + \frac{1}{2}\delta^2\phi_x\phi_{xtt} + \delta^2\phi_x\phi_{xxx}. \end{aligned}$$

The corresponding multipliers are as follows

$$G_1 = 1, \qquad G_2 = -\phi_t, \qquad G_3 = -\phi_x. \tag{5.5}$$

We note that a double reduction of equation (5.1) by $< T_1, T_2, T_3 >$ on S_2 yields the following exact solution.

$$\begin{aligned}
\phi &= s_2 \cos\left(\sqrt{\frac{(r^2-d^2(3a^2+c^2))}{-\delta^2(r^2+d^2)}}\left(x-\frac{r}{d}t\right)\right) \\
&\quad + s_3 \sin\left(\sqrt{\frac{(r^2-d^2(3a^2+c^2))}{-\delta^2(r^2+d^2)}}\left(x-\frac{r}{d}t\right)\right) \\
&\quad + \frac{1}{(r^2-d^2(3a^2+c^2))}\left(s_4\left(x-\frac{r}{d}t\right)+s_5\right)+\frac{1}{r}x. \qquad (5.6)
\end{aligned}$$

where s_2, s_3, s_4, s_5, r and d are constants

5.3 Conservation laws and conserved quantities of equation (5.2)

In this section, we consider equation (5.2). If we split the system (5.2) into two equations, being the real and imaginary components respectively, we get the system,

$$\begin{aligned}
\phi_t + \beta\phi_x - \gamma\psi_{xx} + \delta\psi(\phi^2+\psi^2) = 0, \\
-\psi_t - \beta\psi_x - \gamma\phi_{xx} + \delta\phi(\phi^2+\psi^2) = 0. \qquad (5.7)
\end{aligned}$$

Thus we analyse the system (5.7).

5.3.1 Conservation Laws

To find multipliers and conservation laws, the condition below must hold and we use Maple and Mathematica to solve.

$$G^1(\phi_t+\beta\phi_x-\gamma\psi_{xx}+\delta\psi(\phi^2+\psi^2))+G^2(-\psi_t-\beta\psi_x-\gamma\phi_{xx}+\delta\phi(\phi^2+\psi^2)) = D_tS^t+D_xS^x.$$

Setting

$$\frac{\delta}{\delta\phi}[G^1(\phi_t+\beta\phi_x-\gamma\psi_{xx}+\delta\psi(\phi^2+\psi^2))+G^2(-\psi_t-\beta\psi_x-\gamma\phi_{xx}+\delta\phi(\phi^2+\psi^2))]=0$$

yields the folwoing results.

(i) $(G^1,G^2)=(\psi_x,\phi_x)$

$$\begin{aligned} S^x &= \tfrac{1}{4}\left(\delta\phi^4+2\delta\phi^2\psi^2+\delta\psi^4+2\psi\phi_t-2\phi\psi_t-2\gamma\left({\phi_x}^2+{\psi_x}^2\right)\right), \\ S^t &= \tfrac{1}{2}\left(-\psi\phi_x+\phi\psi_x\right) \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t=\frac{i}{4}(q\bar{q}_x-\bar{q}q_x).$$

(ii) $(G^1,G^2)=(\phi,-\psi)$

$$\begin{aligned} S^x &= \tfrac{1}{2}\left(\beta\phi^2+\psi\left(\beta\psi+2\gamma\phi_x\right)-2\gamma\phi\psi_x\right), \\ S^t &= \tfrac{1}{2}\left(\phi^2+\psi^2\right). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t=\frac{1}{2}|q|^2.$$

(iii) $(G^1,G^2)=(\psi_t,\phi_t)$

$$\begin{aligned} S^x &= \tfrac{1}{2}\left(-\gamma\left(\phi_t\phi_x+\psi_t\psi_x\right)+\phi\left(\beta\psi_t+\gamma\phi_{xt}\right)+\psi\left(-\beta\phi_t+\gamma\psi_{xt}\right)\right), \\ S^t &= \tfrac{1}{4}\left(\delta\phi^4+2\delta\phi^2\psi^2-2\phi\left(\beta\psi_x+\gamma\phi_{xx}\right)+\psi\left(\delta\psi^3+2\beta\phi_x-2\gamma\psi_{xx}\right)\right). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t=\frac{\delta}{4}|q|^4+\frac{i\beta}{4}(\bar{q}q_x-q\bar{q}_x)-\frac{\gamma}{4}(q\bar{q}_{xx}+\bar{q}q_{xx}).$$

(iv) $(G^1, G^2) = (-\beta t\phi + x\phi + 2\gamma t\psi_x, \beta t\psi - x\psi + 2\gamma t\phi_x)$

$$\begin{aligned} S^x &= \tfrac{1}{2}[t\gamma\delta\phi^4 + \beta(x-t\beta)\psi^2 + t\gamma\delta\psi^4 + \phi^2\left(\beta(x-t\beta) + 2t\gamma\delta\psi^2\right) \\ &\quad +2\gamma\psi\left(t\phi_t + (x-t\beta)\phi_x\right) - 2\gamma\phi\left(t\psi_t + (x-t\beta)\psi_x\right) - 2t\gamma^2\left({\phi_x}^2 + {\psi_x}^2\right)], \\ S^t &= \tfrac{1}{2}\left((x-t\beta)\phi^2 + \psi\left((x-t\beta)\psi - 2t\gamma\phi_x\right) + 2t\gamma\phi\psi_x\right). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = \frac{1}{2}(x-t\beta)|q|^2 + \frac{it\gamma}{2}(q\bar{q}_x - \bar{q}q_x).$$

5.3.2 Conserved quantities

We want the conserves quantities for equation (5.2), therefore we need the 1-soliton solution shown as [51, 58]

$$q(x,t) = D\,\mathrm{sech}[E(x-vt)]e^{i(-\kappa x+\omega t+\theta)}, \tag{5.8}$$

where D represents the amplitude of the soliton, E is the inverse width and v represents the velocity. κ is the frequency of the soliton, and ω represents the wave number and lastly θ is the phase constant. The velocity of the soliton is written as [51, 58]

$$v = \beta + 2\gamma\kappa. \tag{5.9}$$

We calculate the conserved quantities from the conserved densities in the section above.

$$J_1 = \int_{-\infty}^{\infty} \Phi_1^t dx = \frac{i}{4}\int_{-\infty}^{\infty}\left(\bar{q}q_x - q\bar{q}_x\right)dx = -\frac{\kappa D^2}{E} \tag{5.10}$$

$$J_2 = \int_{-\infty}^{\infty} \Phi_2^t dx = \frac{1}{2}\int_{-\infty}^{\infty} |q|^2 \, dx = \frac{D^2}{E} \tag{5.11}$$

$$\begin{aligned} J_3 &= \int_{-\infty}^{\infty} \Phi_3^t dx = \int_{-\infty}^{\infty} \left\{ \frac{\delta}{4}|q|^4 + \frac{i\beta}{4}(\bar{q}q_x - q\bar{q}_x) - \frac{\gamma}{4}(q\bar{q}_{xx} + \bar{q}q_{xx}) \right\} dx \\ &= \frac{D^2}{3E}\left\{\delta + \gamma\left(E^2 + 3\kappa^2\right) + 3\beta\kappa\right\} \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} J_4 &= \int_{-\infty}^{\infty} \Phi_4^t dx = \frac{1}{2}\int_{-\infty}^{\infty} \left\{(x - t\beta)|q|^2 + it\gamma\left(q\bar{q}_x - \bar{q}q_x\right)\right\} dx \\ &= -t\left(\beta + \gamma\kappa\right)\frac{D^2}{E}. \end{aligned} \tag{5.13}$$

We can say that J_4 is a conserved quantity if

$$\beta = -\gamma\kappa. \tag{5.14}$$

These quantites can be interpreted as follows: J_1 and J_2 are the linear momentum and energy of the solitons respectively. J_3 is the Hamiltonian and J_4 is only a conserved quantity if equation (5.14) is satisfied.

5.4 Discussion and Conclusion

We used Noether's theorem to obtain Noether symmetries for trivial and nontrivial gauge terms for the wave equation of order four. In order to find an exact solution we used the theory of double reduction.

Next we considered a cubic nonlinear Shrödinger equation and calculated the multipliers and conservation laws. Thereafter the conserved quantities were obtained using soliton theory.

Chapter 6

On PT symmetry systems: invariance, conservation laws and reductions

6.1 Introduction

Physical systems exhibiting parity-time (PT) symmetry have been the subject of much investigation in recent years and is now extensively considered in diverse areas of physics namely, quantum field theories, non Hermitian Anderson models, complex Lie Algebras just to name a few [62, 10, 11, 12, 13, 14, 15, 16, 17, 18]. We know that even a single PT cell can exhibit unconventional features, it follows that one may wish to investigate what new behaviour and properties can be expected from PT symmetric lattices [60, 61].

In optics, it has recently been discovered that there is a class of optical systems, of which elements consist of gain and loss, that can be interpreted as an optics equivalent of the PT symmetry in quantum mechanics [59, 60]. The underlying equations describing the effects of pulse dispersion [59] have the following form:

$$\begin{aligned} iU_t + U_{xx} + 2|U|^2U &= -V + i\gamma U, \\ iV_t + V_{xx} + 2|V|^2V &= -U - i\gamma V. \end{aligned} \tag{6.1}$$

To analyse the solutions of this equation we make a change of variables

$$U(x,t) = e^{i(\omega t-\theta)}\alpha(x,t), \qquad V(x,t) = e^{i\omega t}\beta(x,t), \tag{6.2}$$

where θ is a constant angle satisfying

$$\sin\theta = \gamma$$

and ω is an arbitrary real parameter. As a result of the transformation, equation (6.1) becomes

$$\begin{aligned} i\alpha_t + \alpha_{xx} - \omega\alpha + 2|\alpha|^2\alpha &= -\cos\theta\beta + i\gamma(\alpha-\beta), \\ i\beta_t + \beta_{xx} - \omega\beta + 2|\beta|^2\beta &= -\cos\theta\alpha + i\gamma(\alpha-\beta). \end{aligned} \tag{6.3}$$

The system (6.3) admits a reduction $\alpha = \beta = q$ to the following scalar cubic Schrödinger equation:

$$iq_t + q_{xx} - a^2q + 2|q|^2q = 0, \tag{6.4}$$

where q is the complex valued dependent variable and $a^2 = \omega - \cos\theta$.

Equation (6.4) has a family of stationary soliton solutions, however, we focus on the theory of double reduction for which we will need conservation laws and calculate exact solutions. This will be done by decomposing equation (6.4) into real and imaginary parts to obtain the following system of partial differential equations (PDEs) if $q = \phi + i\psi$ then we have the following system:

$$\begin{aligned}
\phi_t + \psi_{xx} - a^2\psi + 2\psi(\phi^2 + \psi^2) &= 0, \\
-\psi_t + \phi_{xx} - a^2\phi + 2\phi(\phi^2 + \psi^2) &= 0.
\end{aligned} \tag{6.5}$$

In light the of symmetries, conservation laws and double reduction to exact solutions, we will briefly consider the system of PDEs in equation (6.1) but the bulk of the analysis will centre around equation (6.4) via the system (6.5). The results of the findings have been submitted to the Journal of Computational and Applied Mathematics.

6.2 On the conservation laws of equation (6.1)

To calculate the conservation laws we use the multiplier approach. So if (S^t, S^x) is a conserved vector corresponding to a conservation law, then

$$D_t S^t + D_x S^x = 0$$

along the solutions of the differential equation ($G(x, t, q, q_{(i)}, \ldots) = 0$, say).

If the system (6.1) is split into real and imaginary parts with $U = \mu + i\nu$ and $V = \hat{\eta} + i\delta$ and replacing γ with 'general' parameters, we get

$$\begin{aligned}
\mu_t + \nu_{xx} + 2(\mu^2 + \nu^2)\nu + \delta - g_1\mu &= 0, \\
-\nu_t + \mu_{xx} + 2(\mu^2 + \nu^2)\mu + \hat{\eta} + g_1\nu &= 0, \\
\hat{\eta}_t + \delta_{xx} + 2(\hat{\eta}^2 + \delta^2)\delta + \nu + g_3\hat{\eta} &= 0, \\
\delta_t + \hat{\eta}_{xx} + 2(\hat{\eta}^2 + \delta^2)\hat{\eta} + \mu - g_4\delta &= 0.
\end{aligned} \tag{6.6}$$

The system (6.6), it turns out only admits nontrivial conservation laws (two) for $g_2 = -g_1$ and $g_4 = -g_3$ corresponding to *multipliers* (see below) $Q = (\nu_t, \mu_t, \delta_t, \hat{\eta}_t)$ and $Q = (\nu_x, \mu_x, \delta_x, \hat{\eta}_x)$ which in turn corresponds to time and space translations, respectively. In this case, equation (6.6) becomes

$$\begin{aligned}
\mu_t + \nu_{xx} + 2(\mu^2 + \nu^2)\nu + \delta - g_1\mu &= 0,\\
-\nu_t + \mu_{xx} + 2(\mu^2 + \nu^2)\mu + \hat{\eta} - g_2\nu &= 0,\\
\hat{\eta}_t + \delta_{xx} + 2(\hat{\eta}^2 + \delta^2)\delta + \nu + g_3\hat{\eta} &= 0,\\
\delta_t + \hat{\eta}_{xx} + 2(\hat{\eta}^2 + \delta^2)\hat{\eta} + \mu + g_3\delta &= 0.
\end{aligned} \tag{6.7}$$

Thus, equation (6.1) has no non trivial conservation laws even though the system is invariant under time and space translations.

We thus do a detailed study of the special case given in equation (6.5) instead.

6.3 Symmetries, reductions and conservation laws of equation (6.5)

The Lie symmetry approach on differential equations is well known; for details see e.g., [19, 40]. In this section, we list a summary of these and explore the notion of a ‘double reduction’ in order to obtain symmetry invariant (exact) solutions.

6.3.1 Symmetries and reductions

A one parameter Lie group of transformations that leave invariant equation (6.5) will be written as a vector field

$$T=\tau(t,x,\phi,\psi)\partial_t+\theta(t,x,\phi,\psi)\partial_x+\omega^1(t,x,\phi,\psi)\partial_\phi+\omega^2(t,x,\phi,\psi)\partial_\psi. \tag{6.8}$$

This would be a generator of point symmetries of the system. We get the algebra generated by

$$\begin{aligned}
T_1&=\partial_t\\
T_2&=\partial_x\\
T_3&=\phi\partial_\psi-\psi\partial_\phi\\
T_4&=2t\partial_x+\phi x\partial_\psi-\psi x\partial_\phi\\
T_5&=\left(-2a^2t\phi-\psi\right)\partial_\psi+2t\partial_t+\left(-\phi+2a^2t\psi\right)\partial_\phi+x\partial_x
\end{aligned} \tag{6.9}$$

6.3.2 Conservation Laws

The lengthy calculations for the system (6.5), lead to the following multipliers and corresponding conserved vectors.

(i) $(Q^1, Q^2) = (\psi_t, \phi_t)$

$$\begin{aligned} S^x &= \tfrac{1}{2}\left(\phi_t\phi_x + \psi_t\psi_x - \phi\phi_{xt} - \psi\psi_{xt}\right), \\ S^t &= \tfrac{1}{2}\left(\phi^4 - \phi^2\left(a^2 - 2\psi^2\right) + \phi\phi_{xx} + \psi\left(-a^2\psi + \psi^3 + \psi_{xx}\right)\right). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = \frac{1}{2}|q|^4 + \frac{1}{2}(h-1)|q|^2 + q_{xx} + q_{xx}.$$

(ii) $(Q^1, Q^2) = (\psi_x, \phi_x)$

$$\begin{aligned} S^x &= \tfrac{1}{2}\left(\phi^4 - a^2\psi^2 + \psi^4 - \phi^2\left(a^2 - 2\psi^2\right) + \psi\phi_t - \phi\psi_t + {\phi_x}^2 + {\psi_x}^2\right), \\ S^t &= \tfrac{1}{2}\left(-\psi\phi_x + \phi\psi_x\right). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{i}{4}(q_x - q_x).$$

(iii) $(Q^1, Q^2) = (\phi, -\psi)$

$$\begin{aligned} S^x &\quad -\psi\phi_x + \phi\psi_x, \\ S^t &= \tfrac{1}{2}\left(\phi^2 + \psi^2\right). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{1}{2}|q|^2.$$

(iv) $(Q^1, Q^2) = (-\frac{1}{2}x\phi + t\psi_x, \frac{1}{2}x\psi + t\phi_x)$

$$\begin{aligned} S^x &= \tfrac{1}{2}\left(t\phi^4 - a^2t\psi^2 + t\psi^4 - t\phi^2\left(a^2 - 2\psi^2\right) + \psi\left(t\phi_t + x\phi_x\right) - \phi\left(t\psi_t + x\psi_x\right) + t\left({\phi_x}^2 + {\psi_x}^2\right)\right), \\ S^t &= \tfrac{1}{4}\left(-x\phi^2 - \psi\left(x\psi + 2t\phi_x\right) + 2t\phi\psi_x\right). \end{aligned}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{1}{4}x|q|^2 - it(q_x - q_x).$$

3.2.2. The system (6.5) admits a Lagrangian

$$L = -\frac{1}{2}{\phi_x}^2 - \frac{1}{2}{\psi_x}^2 + \frac{1}{2}\phi_t\psi - \frac{1}{2}\psi_t\phi - \frac{1}{2}a^2(\phi^2+\psi^2) + \frac{1}{2}\phi^4 + \frac{1}{2}\psi^4 + \phi^2\psi^2 \quad (6.10)$$

so that the corresponding Lagrangian for the Schrödinger equation (6.4) is $\mathcal{L} = -\frac{1}{2}|q_x|^2 - \frac{1}{2}a^2|q|^2 + \frac{1}{2}|q|^4 + \frac{i}{4}(q_t - q_t)$. The Noether symmetries, that is, the one parameter Lie groups of transformations, that leave invariant the functional $\int\int L\mathrm{d}x\mathrm{d}t$ with zero gauge are the translations

$$T_1 = \partial_t, \qquad T_2 = \partial_x$$

with corresponding conserved vectors

$$\begin{aligned} S^x &= \psi_x\psi_t + \phi_x\phi_t, \\ S^t &= -\tfrac{1}{2}{\phi_x}^2 - \tfrac{1}{2}{\psi_x}^2 - \tfrac{1}{2}a^2\phi^2 - \tfrac{1}{2}a^2\psi^2 + \tfrac{1}{2}\phi^4 + \tfrac{1}{2}\psi^4 + \phi^2\psi^2. \end{aligned}$$

with density of equation (6.4) given by

$$\Phi^t = -\frac{1}{2}|q_x|^2 - \frac{1}{2}a^2|q|^2 + \frac{1}{2}|q|^4$$

and

$$\begin{aligned} S^x &= \tfrac{1}{2}{\phi_x}^2 + \tfrac{1}{2}{\psi_x}^2 + \tfrac{1}{2}\psi\phi_t - \tfrac{1}{2}\phi\psi_t - \tfrac{1}{2}a^2\phi^2 - \tfrac{1}{2}a^2\psi^2 + \tfrac{1}{2}\phi^4 + \tfrac{1}{2}\psi^4 + \phi^2\psi^2, \\ S^t &= \tfrac{1}{2}(\psi_x\phi - \phi_x\psi). \end{aligned}$$

with density of equation (6.4) given by

$$\Phi^t = -\frac{i}{4}(q_x - q_x),$$

respectively.

6.3.3 Double Reduction

Our original system is equivalent to

$$sys_1 = \begin{cases} {q_1}^1 G^1 + {q_1}^2 G^2 = 0, \\ {q_1}^1 G^1 - {q_1}^2 G^2 = 0. \end{cases} \tag{6.11}$$

The system (6.11) can be rewritten as

$$\begin{aligned} D_t {S_1}^t + D_x {S_1}^x &= 0, \\ {q_1}^1 G^1 - {q_1}^2 G^2 &= 0. \end{aligned} \tag{6.12}$$

Case 1: A double reduction of equation (6.5) by $< T_2, T_3 >$

We first show that T_2 and T_3 are associated with $S_1 = (S_2^t, S_2^x)$ using the following version of equation (1.16) for $i = 1, 2$

$$S^* = T\begin{pmatrix} S^t \\ S^x \end{pmatrix} - \begin{pmatrix} D_t\theta^t & D_x\theta^t \\ D_t\theta^x & D_x\theta^x \end{pmatrix}\begin{pmatrix} S^t \\ S^x \end{pmatrix} + (D_t\theta^t + D_x\theta^x)\begin{pmatrix} S^t \\ S^x \end{pmatrix}. \tag{6.13}$$

We obtain

$$\begin{aligned} \begin{pmatrix} S_2^{*t} \\ S_2^{*x} \end{pmatrix} &= T_3^{[1]}\begin{pmatrix} S_2^t \\ S_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_2^t \\ S_2^x \end{pmatrix} + (0)\begin{pmatrix} S_2^t \\ S_2^x \end{pmatrix} \\ &= \begin{pmatrix} (-\psi\psi_x - \phi\phi_x + \psi\psi_x + \phi\phi_x) \\ \left(\frac{1}{2}(\phi_t\phi_x + \psi_t\psi_x - \phi\phi_{xt} - \psi\psi_{xt})\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} S_2^{*t} \\ S_2^{*x} \end{pmatrix} &= T_2^{[1]}\begin{pmatrix} S_2^{t} \\ S_2^{x} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_2^{t} \\ S_2^{x} \end{pmatrix} + (0)\begin{pmatrix} S_2^{t} \\ S_2^{x} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\partial}{\partial x}\left[\dfrac{1}{2}(\phi\psi_x - \psi\phi_x)\right] \\ \dfrac{\partial}{\partial x}\left[\dfrac{1}{2}\left(\phi_t\phi_x + \psi_t\psi_x - \phi\phi_{xt} - \psi\psi_{xt}\right)\right] \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},\end{aligned}$$

where

$$T_3^{[1]} = -\psi\frac{\partial}{\partial\phi} + \phi\frac{\partial}{\partial\psi} - \psi_t\frac{\partial}{\partial\phi_t} - \psi_x\frac{\partial}{\partial\phi_x} + \phi_t\frac{\partial}{\partial\psi_t} + \phi_x\frac{\partial}{\partial\psi_x}, \quad T_2^{[1]} = \frac{\partial}{\partial x}.$$

Thus, T_2 and T_3 are both associated with S_2.

This implies that the first equation of the array of equations in equation (6.11) can be reduced, since T_2 and T_3 are both associated symmetries with S_2.

We write both T_2 and T_3 as a linear combination as follows, $T = T_2 + cT_3$, and transform the operator to its canonical form $Z = \frac{\partial}{\partial j}$, where we assume that this generator is of the form $Z = 0\frac{\partial}{\partial i} + \frac{\partial}{\partial j} + 0\frac{\partial}{\partial u} + 0\frac{\partial}{\partial v}$.

From $T(i) = 0$, $T(j) = 1$, $T(u) = 0$ and $T(v) = 0$, we get

$$\frac{dt}{0} = \frac{dx}{1} = \frac{d\phi}{-c\psi} = \frac{d\psi}{c\phi} = \frac{di}{0} = \frac{dj}{1} = \frac{du}{0} = \frac{dv}{0}. \tag{6.14}$$

Equations (6.14)are solved using variational methods. A summary of the results is given below

$$n_1 = t, \quad n_2 = \phi^2+\psi^2, \quad n_3 = \arctan\left(\frac{\psi}{\phi}\right)-cx, \quad n_4 = i, \quad n_5 = j-x, \quad n_6 = u, \quad n_7 = v, \tag{6.15}$$

where n_4, n_5, n_6 and n_7 are arbitrary functions all dependent on n_1, n_2 and n_3.

By choosing $n_4 = n_1$, $n_5 = 0$, $n_6 = \sqrt{n_2}$ and $n_7 = n_3$, we obtain the canonical coordinates

$$i = t, \quad j = x, \quad u = \sqrt{\phi^2 + \psi^2}, \quad v = \arctan\left(\frac{\psi}{\phi}\right) - cj, \tag{6.16}$$

where $u = u(i)$ and $v = v(i)$, since $Z = \frac{\partial}{\partial j}$.

From equation (1.17), we compute U and $(U^{-1})^T$

$$U \ = \ \begin{pmatrix} D_i t & D_i x \\ D_j t & D_j x \end{pmatrix} \ = \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$U^{-1} \ = \ \begin{pmatrix} D_t i & D_t j \\ D_x i & D_x j \end{pmatrix} \ = \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ = \ (U^{-1})^T,$$

where $E = \det(U) = 1$.

From equation (6.16), calculating the inverses we get the following

$$t = i, \quad x = j, \quad \phi = u(i)\cos\left(v(i) + cj\right), \quad \psi = u(i)\sin\left(v(i) + cj\right). \tag{6.17}$$

We calculate the respective derivatives of ϕ and ψ from equation (6.17)

$$\begin{aligned} \psi_t &= \left(\frac{d}{di}u(i)\right)\cos\left(v(i) + cj\right) - u(i)\left(\frac{d}{di}v(i)\right)\sin\left(v(i) + cj\right), \\ \phi_x &= -cu(i)\sin\left(v(i) + cj\right), \\ \psi_t &= \left(\frac{d}{di}u(i)\right)\sin\left(v(i) + cj\right) + u(i)\left(\frac{d}{di}v(i)\right)\cos\left(v(i) + cj\right), \\ \psi_x &= cu(i)\cos\left(v(i) + cj\right), \\ \phi_{xx} &= -c^2 u(i)\cos\left(v(i) + cj\right), \\ \psi_{xx} &= -c^2 u(i)\sin\left(v(i) + cj\right). \end{aligned} \tag{6.18}$$

We now apply the formula from equation (1.18) with $l = 1, 2$ to obtain the reduced conserved form

$$\begin{pmatrix} S_2^i \\ S_2^j \end{pmatrix} = E(U^{-1})^T \begin{pmatrix} S_2^t \\ S_2^i \end{pmatrix}. \tag{6.19}$$

By substituting equations (6.17) and (6.18) into equation (6.19), we obtain

$$\begin{pmatrix} S_2^i \\ S_2^j \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} c u(i)^2 \\ \dfrac{1}{2}\left[u(i)^4 + a^2 u(i)^2 - \left(\dfrac{d}{di} v(i)\right) u(i)^2 + c^2 u(i)^2 \right] \end{pmatrix}, \tag{6.20}$$

where we write the conserved vector in a reduced form

$$D_i S_2^i = 0. \tag{6.21}$$

The next step of double reduction is shown as

$$c u(i)^2 = k \tag{6.22}$$

or

$$u(i)^2 = k, \tag{6.23}$$

where k is a constant.

We use implicit differentiation in equation (6.22) with respect to i to get

$$\frac{d}{di} u(i) = 0. \tag{6.24}$$

The second equation of sys_1 from equation (6.11) is given by

$$\begin{aligned} &\psi_x\Big[\phi_t + \psi_{xx} - a^2\psi + 2(\phi^2 + \psi^2)\psi + \Big] \\ &\qquad - \phi_x\Big[- \psi_t + \phi_{xx} - a^2\phi + 2(\phi^2 + \psi^2)\phi\Big] = 0. \end{aligned} \tag{6.25}$$

After transforming equation (6.25) using equations (6.17) and (6.18), we obtain

$$\begin{aligned}-2cu(i)^2\frac{d}{di}v(i)\left[\cos\left(v(i)+cj\right)\sin\left(v(i)+cj\right)\right]& \\ +\,4u(i)^4\left[\cos\left(v(i)+cj\right)\sin\left(v(i)+cj\right)\right]=0.& \qquad (6.26)\end{aligned}$$

We now substitute equations (6.22) and (6.24) into equation (6.26) and simplify.

This results in the ordinary differential equation (ODE)

$$\frac{d}{di}v(i)=\frac{2u(i)^2}{c}. \qquad (6.27)$$

If we substitute equation (6.23) into equation (6.27) and find the integral with respect to i we get

$$v(i)=\frac{2ki}{c}+m, \qquad (6.28)$$

where m is an integration constant.

Using equation (6.17), we get the exact solution to the system (6.5) as

$$\begin{aligned}\phi(t,x) &= \sqrt{k}\cos\left(\frac{2kt}{c}+m+cx\right),\\ \psi(t,x) &= \sqrt{k}\sin\left(\frac{2kt}{c}+m+cx\right). \qquad (6.29)\end{aligned}$$

Thus, $q=\sqrt{k}e^{i(\frac{2kt}{c}+m+cx)}$.

Case 2: A reduction of equation (6.5) by T_4 on S_3

We now show that T_4 is associated with $S_3=(S_3^t,S_3^x)$ using the formula in equation (6.13).

We obtain

$$\begin{aligned}\begin{pmatrix} S_3^{*t} \\ S_3^{*x} \end{pmatrix} &= T_4^{[1]}\begin{pmatrix} S_3^{t} \\ S_3^{x} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} S_3^{t} \\ S_3^{x} \end{pmatrix} + (0)\begin{pmatrix} S_3^{t} \\ S_3^{x} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(-2x\phi\psi + 2x\phi\psi) \\ \Big(- x\psi\psi_x - x\phi\phi_x + x\psi\psi_x + \psi^2 + x\phi\phi_x + \phi^2 - \phi^2 - \psi^2\Big) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},\end{aligned}$$

where

$$T_4^{[1]} = 2t\frac{\partial}{\partial x} - x\psi\frac{\partial}{\partial \phi} + x\phi\frac{\partial}{\partial \psi} - (x\psi_t + 2\phi_x)\frac{\partial}{\partial \phi_t} - (x\psi_x + \psi)\frac{\partial}{\partial \phi_x} + (x\phi_t - 2\psi_x)\frac{\partial}{\partial \psi_t} + (x\phi_x + \phi)\frac{\partial}{\partial \psi_x}.$$

Thus, T_4 is associated with S_3.

Transforming the operator T_4 to its canonical form $Z = \frac{\partial}{\partial j}$.

From $T(i) = 0$, $T(j) = 1$, $T(u) = 0$ and $T(v) = 0$, we obtain

$$\frac{dt}{0} = \frac{dx}{2t} = \frac{d\phi}{-x\psi} = \frac{d\psi}{x\phi} = \frac{di}{0} = \frac{dj}{1} = \frac{du}{0} = \frac{dv}{0}. \tag{6.30}$$

The results from solving equation (6.30) are summarized below

$$n_1 = t, \quad n_2 = \phi^2 + \psi^2, \quad n_3 = \arctan\left(\frac{\psi}{\phi}\right) - \frac{x^2}{4t}, \quad n_4 = i, \quad n_5 = j - \frac{x}{2t}, \quad n_6 = u, \quad n_7 = v \tag{6.31}$$

where n_4, n_5, n_6 and n_7 are arbitrary functions all dependent on n_1, n_2 and n_3.

With the choice $n_4 = n_1$, $n_5 = 0$, $n_6 = \sqrt{n_2}$ and $n_7 = n_3$, we obtain the canonical coordinates

$$i = t, \quad j = \frac{x}{2t}, \quad u = \sqrt{\phi^2 + \psi^2}, \quad v = \arctan\left(\frac{\psi}{\phi}\right) - \frac{x^2}{4t}. \tag{6.32}$$

From equation (1.17), we compute U and $(U^{-1})^T$

$$U = \begin{pmatrix} D_i t & D_i x \\ D_j t & D_j x \end{pmatrix} = \begin{pmatrix} 1 & 2j \\ 0 & 2i \end{pmatrix}$$

and

$$(U^{-1})^T = \begin{pmatrix} D_t i & D_t j \\ D_x i & D_x j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{j}{i} & \dfrac{1}{2i} \end{pmatrix}$$

where $E = \det(U) = 2i$.

Calculating the inverses from equation (6.32), we get

$$t = i, \quad x = 2ij, \quad \phi = u(i)\cos(v(i) + ij^2), \quad \psi = u(i)\sin(v(i) + ij^2). \tag{6.33}$$

The respective derivatives of ϕ and ψ from equation (6.33) are:

$$\begin{aligned}
\phi_t &= \left(\frac{d}{di}u(i)\right)\cos(v(i) + ij^2) - u(i)\left(\frac{d}{di}v(i)\right)\sin(v(i) + ij^2) \\
&\quad + j^2 u(i)\sin(v(i) + ij^2), \\
\phi_x &= -su(i)\sin(v(i) + ij^2), \\
\psi_t &= \left(\frac{d}{di}u(i)\right)\sin(v(i) + ij^2) + u(i)\left(\frac{d}{di}v(i)\right)\cos(v(i) + ij^2) \\
&\quad - j^2 u(i)\cos(v(i) + ij^2), \\
\psi_x &= ju(i)\cos(v(i) + ij^2), \\
\phi_{xx} &= -\frac{u(i)\sin(v(i) + ij^2)}{2i} - j^2 u(i)\cos(v(i) + ij^2), \\
\psi_{xx} &= \frac{u(i)\cos(v(i) + ij^2)}{2i} - j^2 u(i)\sin(v(i) + ij^2).
\end{aligned} \tag{6.34}$$

By substituting equations (6.33) and (6.34) into equation (6.19), we obtain

$$\begin{pmatrix} S_3^i \\ S_3^j \end{pmatrix} = \begin{pmatrix} iu(i)^2 \\ 0 \end{pmatrix}, \tag{6.35}$$

where we write the conserved vector in a reduced form

$$D_i S_3^i = 0. \tag{6.36}$$

The next step of double reduction can be written as

$$iu(i)^2 = k \tag{6.37}$$

or equivalently

$$u(i)^2 = \frac{k}{i}, \tag{6.38}$$

where k is a constant.

If we use implicit differentiation in equation (6.37) with respect to i we get

$$2iu\frac{d}{di}u(i) + u(i)^2 = 0 \tag{6.39}$$

or equivalently after dividing both sides by $2i$ we get

$$u\frac{d}{di}u(i) + \frac{u(i)^2}{2i} = 0. \tag{6.40}$$

The second equation of sys_2 from equation (6.11) is given by

$$\phi\phi_t - \psi\psi_t + \phi\psi_{xx} + \psi\phi_{xx} - 2a^2\phi\psi + 4\phi^3\psi + 4\psi^3\phi = 0. \tag{6.41}$$

After transforming equation (6.41) using equations (6.33) and (6.34) and multiplying both sides $2i$, we get the ODE

$$\frac{d}{di}v(i) = 2k - a^2. \tag{6.42}$$

If we substitute equation (6.38) into equation (6.42) and then find the integral with respect to i we get

$$v(i) = 2ki - ia^2 + m, \tag{6.43}$$

Using equation (6.33), we get the exact solution for the system (6.5) as

$$\begin{aligned} \phi(t,x) &= \sqrt{\frac{k}{t}}\cos\left(2kt - ta^2 + m + \frac{x^2}{4t}\right), \\ \psi(t,x) &= \sqrt{\frac{k}{t}}\sin\left(2kt - ta^2 + m + \frac{x^2}{4t}\right), \end{aligned} \qquad (6.44)$$

so that $q = \sqrt{\frac{k}{t}}e^{i(2kt - ta^2 + m + \frac{x^2}{4t})}$.

6.4 Discussion and Conclusion

We have constructed conservation laws for the scalar cubic Schrödinger equation via the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates total divergence. Interestingly enough, the scalar cubic Schrödinger equation admits a Lagrangian resulting in Noether symmetries. Furthermore, two cases of double reduction were successfully performed and exact solutions were calculated.

Conclusion

In the thesis, a large emphasis on Schrödinger type equations was placed. Although these have been studied in the past, and continuing to be studied, it is our hope that the new properties of the known Schrödinger equations will be of interest to the world of physics and mathematics.

In chapter two, we investigated the nonlinear damped-driven Schrödinger equation with γ (damping coefficient) and h (the amplitude or parametric driver). To proceed, we considered a system of equations from the decomposition of the main equation into real and imaginary parts. In order to analyse this equation, the Lie symmetry approach was applied to obtain symmetries. However, three cases emerged, namely the damped-driven, driven-undamped and damped-undriven where each case has its corresponding set of Lie symmetries, conservation laws and multipliers (based on the well known result that the Euler-Lagrange operator annihilates a total divergence).

In the first case (damped-driven) the double reduction of a rotation symmetry and a translation symmetry is performed where a new and exact solution is calculated. The second case (damped-undriven), here the symmetry chosen has variables in x, t and u, although interesting at first glance, properties of this symmetry are still under investigation. Regarding the reduction of this symmetry however, an exact and new solution were computed. The cases shown in this thesis are by no means exhaustive but only provide some impetus of what can be done in future work especially in the theory of double reduction. Lastly, in this chapter we found some conserved quantities from the calculated conservation laws. This chapter, we could say was a model for the chapters to follow as there are some recurring themes.

In the next chapter, we turned our attention to the Fitzhugh-Nagumo and Fisher equation. However, only specific cases of these equations were considered as these were the only cases found to yield non-trivial symmetries and conservation laws. We started with the Nagumo equation where $c = 2$ and $\alpha = -1$. To find symmetries, conservation laws and corresponding multipliers, we enlisted the assistance of mathematical packages Mathematica and Maple, otherwise the calculations become tedious. We calculated three symmetries and two conservation laws from which a double reduction is performed, where a new exact solution was found (not the standard travelling wave solution). We would like to point out that although the theory of double reduction was used to obtain an exact solution, there are other variational methods that could have been employed. For the Fisher equation, a similar approach was adopted where the case $c = 1$, was the only case that yielded symmetries. Once again, three symmetries and two conserved vectors were found. Applying double reduction led to two new exact solutions, one being the conjugate of the other.

In chapter four, once again we referred to the nonlinear damped-driven Schrödinger equation though we investigate optical solitons instead. Here we used the Kerr Law and power Law to obtain the 1-soliton solutions. We found that the results of the power law nonlinearity collapses to the case of Kerr law nonlinearity on setting $s = 1$. Kerr Law and power law were used a second time in the chapter to find conserved quantities from the conserved vectors calculated in chapter two.

In chapter five we analysed a fourth order wave equation. It was found that this equation admits a Lagrangian, therefore by Noether's theorem, we were able to find Noether symmetries. The Noether symmetries found, turned out to be translation symmetries (with zero gauge). Taking a linear combination of the translation symmetries, a double reduction was performed. However, there were also symmetries with non-zero gauge possibly leading to solutions which haven't been investigated. Next, if we have an evolution of the compressional dispersive Alvén envelope, this gives rise to a cubic nonlinear Schroödinger equation. However, for the Schrödinger equation, association between symmetries and conservation laws was not established (necessary for double reduction). Nevertheless, four conservation laws, conserved quantities and a 1-soliton solution was calculated.

In the final chapter, we considered a PT symmetric system. However, for the purpose of the thesis, we transformed the PT symmetric equation, thus the equation was no longer PT symmetric in nature, but showed Schrödinger like properties. Once again with the help of Maple and Mathematica, we found five symmetries and four conservation laws using the homotopy operator. The system (after decomposing into real and imaginary parts) admits a Lagrangian where two of the symmetries are Noether symmeries. Namely, the translation symmetries in x and t. We considered two cases for double reduction. The first being a linear combination of the rotation and translation symmetries and the second being a double reduction of a symmetry similar to the one reduced in the second chapter. In both cases, exact solutions were found.

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